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**2014-03-03**
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# WHY SO NEGATIVE ABOUT NEGATIVES? THE INTENDED, ENACTED AND LIVED OBJECTS OF LEARNING NEGATIVE NUMBERS IN GRADE 7

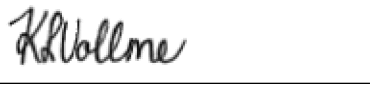
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**A research report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg in partial fulfilment of the requirements for the degree Master of Sciences.**

**Johannesburg, May 2013**

**DECLARATION:**

I declare that this research is my own work and no part of it has been copied from any other source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list, and the sources of ideas not my own have been appropriately referenced.



Kerryn Leigh Vollmer

27 May 2013

**ABSTRACT:**

In this research report, I describe the experiences of a teacher and learners on a sequence of lessons on addition and subtraction of negative numbers. I have used Variation Theory to identify critical features in the intended, enacted and lived object of learning. What emerged from the data was that not one, but two objects of learning were in focus. These objects of learning were *giving meaning to negative numbers and calculations that include negative numbers* and *mastering rules that can be used to do calculations that include negative numbers.* I identify and describe contradictions and tensions caused by these two objects of learning, namely the structure of the number system, the meaning of the minus sign and the commutative property of subtraction. The two objects of learning align themselves with a history of previous research and discussion on conceptual understanding and procedural fluency. The study provides a foundation on which a Learning Study in a South African classroom can be created.

**KEYWORDS:**

Negative number
Minus sign
Variation Theory
Object of learning
Critical features

**ACKOWLEDGEMENTS:**

I can do everything through Him who gives me strength (Phillipians 4:13)

I have truly learned the meaning of this verse over the last two years. I could not have walked this journey alone and I thank God.

My parents, Graham and Dawn were a pillar of strength. They have supported me through all the ups and downs. They have given me time and space to work and have offered words of strong encouragement when I needed it most. Thank you.

My main Cats, Renée and Tracey provided emotional support when I felt like it was all too much. They listened and guided me through difficult decisions that needed to be made and sent more words of encouragement than I can remember. Thank you

My supervisor, Lynn Bowie was always so enthusiastic at our meetings and showed such excitement as the story unfolded. I am very grateful for the time and advice that she gave me. Thank you.

My colleagues at RADMASTE were very supportive and understanding of the journey I was on. I cannot think of a better working environment to study in. Thank you.

## DEDICATION:

There are two women that I owe a huge debt of gratitude. I would not have even started this journey had it not been for their mentoring and support.

Mrs Beverley Nortje was firstly my Standard 8 (Grade 10) Mathematics teacher. She gave me extra time and attention when mathematics was starting to feel like a foreign language to me. Secondly, seven years later, I was privileged to teach with Bev. She was my mentor and friend. I specifically remember planning lessons on negative numbers with her.

Mrs Jackie Scheiber held my hand as I moved from classroom teaching into the fascinating and rewarding world of working with teachers. Working with Jackie has been a transformation. She has gently pushed and moulded while always being encouraging and supportive. I truly believe that I have been under the wing of one of South Africa's greatest authors of children's mathematics text.

I dedicate this work to both of you...

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# CHAPTER 1 - Introduction

## 1.1 Introduction

Why do so many teachers and learners have negative emotions and attitudes about negative numbers? Some researchers argue that negative numbers are so abstract that they lack any connections to the real world (Kilhamn, 2008). Many teachers would describe them as difficult to teach and learners find them difficult to understand. This research report aims to describe how teaching and learning take place during a sequence of lessons on addition and subtraction of negative numbers in a South African classroom in order to identify what may make it difficult to learn and teach in this classroom. I analyse this phenomenon by providing a rich description of the experiences of a Grade 7 Mathematics teacher and her learners.

This research describes these experiences through the lens of Variation Theory (Marton, Runesson & Tsui, 2004). Runesson (2005) explains that variation theory is useful in revealing what is made possible to learn in the classroom. This study is analyzed and conducted within a variation theory framework which emanates from phenomenography. This method of research aims to *describe* people's experiences or conceptions of reality or of a phenomenon (Marton, 1981; Svensson, 1997). Variation theory is a useful tool to use in the analysis as it describes what learning is and how learning is made possible. The main focus of variation theory is on an object of learning – simply the knowledge that is acquired through learning (Marton et al., 2004; Marton & Pang, 2006). The object of learning is considered from three points of view: the intended object of learning which is what the teacher intends to be the object of learning, the enacted object of learning which is the way the object of learning is revealed in the classroom and the lived object of learning which is the way it is experienced by the learners. In setting up my research I considered it important to analyse all three of these aspects of the object of learning and the relationship between them in order to get an in-depth understanding of what was made possible to learn in the class sessions studied.

Previous research categorises three difficulties in teaching and learning negative number. These are: *the meaning of the numerical system and the direction and multitude of number, meaning of arithmetic operations and the meaning of the minus sign* (Altiparmak & Özdoğan, 2010; Kilhamn, 2008). This research will describe how this teacher's difficulties and her learners' difficulties compare to those identified in previous research.

## 1.2 Rationale

South Africa has a history of poor results in school mathematics. As recently as 2011 less than half of the National Senior Certificate candidates who wrote the mathematics exam passed. What is more disturbing is that only 21,0% of all these candidates passed mathematics. Considering that a learner only needs 30% to pass, mathematics education in South Africa is in crisis.

While assisting two Grade 11 mathematics learners with algebra, I became aware of the difficulty that these learners were facing from a poor mathematical foundation. Number concepts that should seem basic were causing problems with more advanced mathematics. Of most particular concern was their poor understanding of operating on negative numbers. These learners correctly followed procedures for multiplying binomials and factorising expressions, but their answers were incorrect. These errors were a direct result of mistakes made when adding, subtracting, multiplying or dividing negative numbers or negative terms.

Gallardo & Rojano (1993) and Gallardo (2002) explain that the extension of number from natural numbers to integers is a crucial part of achieving algebraic competence in the solution of problems and equations. Having a good grounding in negative numbers will enable learners to be more successful in more complex aspects of mathematics, like algebra, calculus and trig. If learners had a better understanding of negative numbers, and therefore a better numerical foundation, they should have a better opportunity to not only pass matric[1] mathematics, but also to obtain a better pass mark. To understand the issues of teaching and learning negative numbers, I have started with this grounding. In South Africa, learners are introduced to negative numbers in Grade 7 and so this is the focus of this study.

## 1.3 Research Problem and Research Questions

According to variation theory, in order to make learning possible, critical features of an object of learning have to be discerned. Making it possible for learners to discern these critical features of negative numbers may not be a simple task. Two factors that need to be considered are firstly, *what* critical features are made possible to discern and secondly, *how* are these critical features made possible to discern?

[1] Matric is the South African school exit level at which national examinations are administered.

Patterns in variation make these features possible to discern (Marton et al., 2006). The aim of this research was to describe these patterns of variation in a sequence of lessons on addition and subtraction of negative numbers in a South African classroom.

In order to identify these patterns of variation, I focussed my research on these four questions:

1. What critical features does the teacher have in mind in the intended object of learning for a sequence of lessons on adding and subtracting negative numbers?
2. What critical features are made possible to discern in the enacted object of learning in a sequence of lessons on adding and subtracting negative numbers?
3. What critical features are evident in the learners' lived object of learning from a sequence of lessons on adding and subtracting negative numbers?
4. How are these critical features made possible to discern in the intended, enacted and lived object of learning?

## 1.4 Outline of chapters to come

In Chapter 2, I describe the Theoretical Framework of my study. I will describe important concepts in Variation Theory and explain their uses in my study.

In Chapter 3, I have discussed literature which supports my study. I will describe how other researchers have described the difficulty of teaching and learning negative number and some of the reasons that they give for this difficulty.

In Chapter 4, I describe the Methodological Framework that I used in this study. I describe the participants and how data was collected. I also discuss the rigour of this research.

In Chapter 5, I describe how the data was analysed. It was unexpected that I found two objects of learning during the data analysis process. Because the lessons were on addition and subtraction of negative numbers, I blindly expected the object of learning in these lessons to be *adding and subtracting negative numbers*. This is essentially the competency that I was expecting to be acquired during the learning in this sequence of lessons. Instead, what was experienced were two contradictory objects of learning, one focussing on meaning of negative numbers and the other focussing on rules. Critical features were identified in the intended, enacted and lived objects of learning, but a far more interesting description of the phenomenon began to emerge when analysing how these two objects of learning contradicted each other.

In Chapter 6, I draw on the literature discussed Chapter 3 to elaborate on the analysis of the data and discuss some of the consequences.

In Chapter 7, I wrap the story together and consider what further research could be done that draws on this research and adds to our understanding of the difficulties in teaching and learning negative numbers.

## 1.5 Summary

Concerned about teacher and learners attitudes to negative number and the poor performance in mathematics of South African learners, I have tried to explain how teachers and learners experience this phenomenon. I have used Variation Theory to explore what was made possible to learn in a sequence of lessons on adding and subtracting negative numbers in a Grade 7 South African classroom.

While looking for critical features that were made possible to discern in the object of learning, I discovered that actually two objects of learning were in focus. The contradictions between these two objects of learning provided insight into features of learning that have been written about Maths education in general, most specifically with regard to learners' mathematical proficiency.

In describing this teacher's experiences and her learners' experiences and how these relate to previous studies, new questions emerge about how we as South African educators and researchers can move towards a phenomenon of thinking positively about negative numbers amongst teachers and learners.

# CHAPTER 2 – Theoretical Framework

## 2.1 Introduction

As outlined in the introduction, the description of this teacher's and learners' experiences is developed using Variation Theory. The variation theory of learning was developed from phenomenography (Runesson, 2005; Kullberg, 2010). This method of research aims to *describe* people's experiences or conceptions of reality or of a phenomenon (Marton, 1981; Svensson, 1997). These phenomena are described in terms of differentiation between the ways they are perceived. Variation theory was a useful tool to use in the analysis as it describes what learning is and how learning is made possible. In this chapter, I outline the theoretical concepts of this framework.

## 2.2 The object of learning

From the perspective of variation theory, learning is seen as the acquisition of knowledge about something or a capability gained through experience and a change in the way that this something is understood by being aware of its critical features (Marton et al., 2004; Kullberg, 2005; Runesson, 2006). The specific knowledge or capability to be acquired through learning is referred to as the *object of learning* (Marton et al., 2004; Marton & Pang, 2006). The capability of the object of learning has general and specific aspects that are different, but work together (Marton et al., 2004). The specific aspect they called the *direct* object of learning. The direct object of learning can be defined as the content (Marton et al., 2006). The general aspect they called the *indirect* object of learning. The indirect object of learning refers to the "capability that learners are supposed to develop" or to the critical features that they should be able to discern (Marton et al., 2006, p. 194). These objects of learning can refer to being able to do addition and subtraction (indirect object of learning) and negative numbers (the direct object of learning).

Marton et al. (2004) point to the fact that in order to get a full picture of the object of learning in a teaching episode, we need to consider what the teacher intends to be the object of learning, called the *intended* object of learning, the way the object of learning is revealed in the classroom, called the *enacted* object of learning and the learning experienced by the learners, called the *lived* object of learning. These different structures of the object of learning seemed to offer me the opportunity to provide a rich description of a teaching and learning episode in a way that could illuminate what

was made possible to learn in relation to negative numbers. These different views of the object of learning were key in my analysis and I expand on and describe each more fully below.

### 2.2.1 The intended object of learning

The intended object of learning is what the teacher is intending to make possible to learn. It is the object of learning as seen from the teacher's perspective (Marton et al., 2004). The intended object of learning is developed before a teaching sequence begins and is the teacher's prediction of the learning she expects to take place in a lesson.

### 2.2.2 The enacted object of learning

The teacher's intended object of learning will be influenced through the interaction with her learners. The enacted object of learning is what the learners actually encounter. It is dependent on what features the learners' discern and focus on in the particular setting (Marton et al., 2004). Neither the intended nor the enacted objects of learning cause learning to happen, but they do make it possible to learn (Marton et al., 2004; Marton et al., 2006). Marton et al. (2004) consider the enacted object of learning to be the object of learning as described by the researcher. It can be described as the extent to which the researcher observes the necessary conditions for a particular object of learning to occur. The research of Marton et al. (2006) describes the enacted object of learning in terms of patterns of variation. Patterns of variation will be described later in this chapter.

### 2.2.3 The lived object of learning

Learners bring their own previous experiences and already constructed knowledge into the learning situation (Runesson et al., 2011) and therefore what the learner actually learns will not be identical to what was made possible to learn in the enacted object of learning. The object of learning as seen from the learner's perspective is the lived object of learning. It is what they actually learn (Marton et al., 2004). It is described as the features of a situation that a learner discerns and focuses on. The lived object of learning can be characterised by learners' answers to written and oral questions after a lesson (Marton et al., 2006).

## 2.3 Critical features and discernment

In order to describe the intended, enacted and lived objects of learning, I focused on what Marton et al. (2004) describe as *critical features*. A core premise in this work is that there are particular critical features of an object of learning that need to be discerned in order for learning to take place. The *critical features* that learners need to discern in order to make learning possible are the aspects or features of an object that are necessary for defining that object (Marton et al., 2004). For example, in order to define a triangle, one would need to discern the critical features of shape, number of sides, size of angles etc. *Discernment* is when features are experienced through how they vary (Marton et al., 2004; Runesson, 2006). For example, a triangle could not be discerned from a square if all shapes were triangles. Identifying and developing the description of these critical features and investigating how various ways of teaching make it possible for them to be discerned are key aspects of the research and development work they propose. In this study I have identified what critical features are made possible to discern during a sequence of lessons on addition and subtraction of negative number.

## 2.4 Learning through variation

From the perspective of Variation Theory, "what varies and what is invariant both constrains learning and makes it possible" (Marton et al., 2006, p. 195). When features vary, learners are able to experience the features that are critical for a particular learning as well as for the development of certain capabilities (Marton et al., 2004). For example, if a teacher was describing congruency and gave the learners several pairs of shapes that were congruent, the learners would not be able to discern congruency unless the teacher included an example of a pair of shapes that were not congruent. In order to understand what is made possible to learn, it is necessary to pay close attention to what varies and what remains unchanged in a learning situation. From the nature of the empirical evidence, patterns of variation were used to illuminate particular core features of negative numbers that were made available to learn and that I considered critical features in relation to conceptual understanding and procedural competence.

## 2.5 Summary

From a Variation Theory framework, learning is being able to discern specific features of an object of learning and in order for these features to be discerned, variation should be noticed (Marton et al., 2004; Kullberg, 2005; Runesson, 2006). There are three views of the object of learning – the *intended*, the *enacted* and the *lived* objects of learning. In the analysis of this data, the intended,

enacted and lived objects of learning are identified and what critical features were made possible to discern in each one. The objects of learning and the critical features associated with them are discussed in relation to other literature about the teaching and learning of negative numbers.

# CHAPTER 3 – Literature Review

## 3.1 Introduction

Altiparmak and Özdoğan (2010) and Kilhamn (2008) refer to previous research and categorise difficulties that are experienced when teaching and learning negative number. Two of these, the meaning of arithmetic operations and the meaning of the minus sign were particularly relevant to my study, so I will discuss the research literature relating to these in greater depth.

Runesson, Kullberg and Maunula (2011) conducted a Learning Study in Sweden that identified four features of negative number that were critical for learners in their study. I identified critical features for the teacher and learners in this study, so it is useful to reflect on their previous study. I also reflect on research done in South Africa, particularly that of Davis (2010) in which I found similarities to my research.

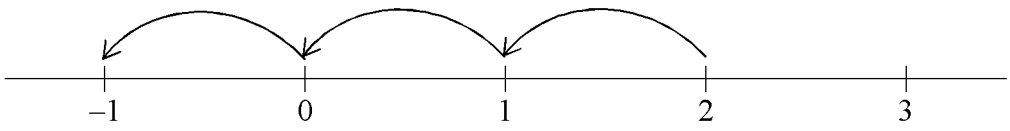
In addition to these areas specific to research on negative numbers I have found Kilpatrick, Swafford and Findell's (2001) work on mathematical proficiency to be particularly useful in understanding maths teaching and learning and so I provide a brief description of this.

## 3.2 Meaning of Arithmetic Operations

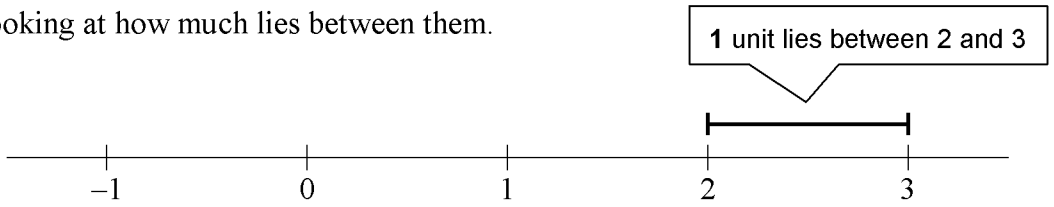
A minus sign as an operational signifier has three uses. Gallardo and Rojano (1994) describe these uses in subtraction, as *taking away*, *completing* and the *difference between two numbers*.

These three uses will be described in terms of 2 – 3.

*Take away* could be described in terms of money, i.e. I have R2, and I owe my mother R3. How much money do I have? Take away could be represented on a number line as:



*Difference* can also be represented on a number line. When finding the difference between 2 and 3, you are looking at how much lies between them.




In this case we also need to consider that subtraction is also from the position of first perspective, so because 2 is smaller than 3, the answer is negative, i.e. 2 – 3 = −1

*Completing* considers 'what is missing from' (Vlassis, 2004). In other words, what is missing from 3 to give 2 or 3 + ? = 2.

Schell and Burns (1962) show that taking away is the easier approach to subtraction (cited in Galbraith, 1974). However, Galbraith (1974) blames the difficulty with negative numbers on this strong association with take away. In Galbraith's (1974) study she analysed three textbooks which approached subtracting negative numbers using a variety of models. She argued that because the order in which we consider the two numbers matters, the relationship between the numbers must be considered before a decision can be made as to whether to use take away, difference or completing.

Different teaching approaches have attempted different models in order to make it possible for learners to be flexible in their approach to negative numbers. Kilhamn (2009) researched how the metaphor "Arithmetic as Motion Along a Path" could be used to teach the addition and subtraction of negative numbers. *The Path* refers to a number line and the *Motion* is either addition or subtraction. Kilhamn describes how the expression a + b = c is represented as *Arithmetic as Motion Along a Path*: a, b and c are all numbers, but only a and c are understood as points on the line. b, however represents a number of steps to move. The number of steps is always positive, but can be in different directions depending on the operation sign (Kilhamn, 2009). This metaphor works well for representing calculations in the form[2] a + b and –a – b. It becomes very complicated when describing how to subtract a negative number, i.e. a – (−b) (Kilhamn, 2009).

[2]In these forms, I use a > 0 and b>0.

From results of the CSMS study, Küchemann suggested that the metaphor of number line be abandoned because the model is so difficult to use for subtraction. He recommended a more consistent approach in which the integers are regarded as discrete entities or objects (Küchemann, 1981). Gallardo (1995) also suggested using discrete models where positive and negative numbers are represented by objects of opposing nature, (for example red counters and blue counters). Linchevski and Williams (1999) paired "authentic situations" with a double abacus which had two wires, one with yellow beads and one with blue beads. Essentially yellow beads tallied negative numbers and blue beads tallied positive numbers. Although, the evidence suggested that this model was mostly successful, especially in that learners could make sense of the 'minus, minus is plus' rule, limitations were acknowledged.

Researchers have attempted to find other metaphors that can be used to illustrate calculations with negative numbers. The metaphor of money (gaining and borrowing) is another metaphor often used. Ball (1993) used both of these metaphors and found that both had limitations. She argued that, "No representations capture all aspects of an idea, nor are all equally useful for particular students" (Ball, 1993, p. 384). Kilhamn (2009) suggested that when "Arithmetic as Motion Along a Path" is used, teachers must be aware of the consequences of extensions to subtraction of negative numbers. This argument is relevant to my study in that the teacher chose to abandon the number line and focus more on the abstract when teaching learners how to subtract negative numbers.

## 3.3 The meaning of the minus sign

Gallardo et al. (1994) and Vlassis (2004; 2008) describe *three functions* of the minus sign. Vlassis (2004) coined the phrase 'negativity' to refer to the multiple functions of the minus sign. They called these functions the *unary* function, the *binary* function and the *symmetric* function (Gallardo et al., 1994; Vlassis, 2004, 2008). The unary function of the minus sign is when it is attached to a number to form a negative number (Vlassis, 2004, 2008). In this context, the minus sign is considered as a 'structural signifier' (Sfard, 2000 cited in Vlassis, 2008). The binary function sees the minus sign as an 'operational signifier' (Sfard, 2000 cited in Vlassis, 2008). It is the function that indicates that the learner should subtract.

The third function described is the symmetric function. The minus sign is also seen as an operational signifier, but indicating that the learner should take the opposite number or the opposite

sum (Vlassis, 2004, 2008). For example 5 – (–6); the minus sign is indicating that 6 should be added (opposite of minus) to 5. There is evidence in Vlassis' (2004) study that learners use this function without understanding, which leads to errors. I do not refer again to this symmetric function as it did not feature in this study.

Vlassis (2008) argues that the difficulties that learners experienced when solving equations with negatives was more to do with the use of symbols than the concept of negative number. This is due to minus sign having to perform several functions. In expressions like those done in the Grade 7 class in my study, the meaning of the minus sign seems clear, for example in the calculation 5 – 7, the minus sign indicates the operation of subtraction, whether it be take away, difference or completing. However, in Vlassis' (2004) study of algebraic equations a minus sign could have more than one function. Her example is $4 + n - 2 = 10$. The purpose of $-2$ could be that 2 should be subtracted from 4, or it could, from an algebraic point of view meaning that one should take the opposite of 2. Although, in this case the functions are different (binary or symmetric), they are both representing *operational signifiers*. Vlassis (2004, 2008) argues that learners do not see the difference between the unary and binary function of the minus sign.

One of the consequences of not identifying the function of the minus sign is that learners may resort to rules. In an early clinical study done by Gallardo and Rojano (1994) they showed that the majority of learners in their study resorted to inventing rules. These rules lead to correct and incorrect results. They describe two cases of errors

1. *Rule of too many signs:* "This sign [$-\overset{\downarrow}{a} - (b)$] is to take away, the other [$-a - \overset{\downarrow}{(-b)}$] is not needed. For example, if learners are asked $-3 - (-7)$, they do $-3 - 7 = -10$.
2. *Multiple rule of signs*: "Minus [$\overset{\downarrow}{-}a - (-b)$] with a minus [$-a \overset{\downarrow}{-} (-b)$] is plus and plus with minus [$-a - \overset{\downarrow}{(-b)}$] is minus. Again, if a learner is asked $-3 - (-7)$, they do $3 + (-7) = 3 - 7 = -4$

In Vlassis (2008) study she found that learner errors seemed to lie in the inadequate use of the minus sign and its change in use from a binary sign to a unary sign. Vlassis (2008) concludes that

> The capacity to take account, according to the context, of the unary, binary and symmetric dimensions and to display considerable flexibility in doing so is vital to students' ability to make sense of these numbers which, above all, obey various formal rules (p. 568).

To assist learners in differentiating between the two functions of the minus sign, researchers have considered using alternate notation. Skemp (1964) described the positive numbers as numbers to the right of zero and negative numbers as numbers to the left of zero. R3 represented 3 and L3 represented –3 (cited in Galbriath, 1974). Ball (1993) introduced adding and subtracting negative numbers modelled on a lift which goes up and down floors of a building. She used a circumflex '^' above the numbers instead of the minus sign to indicate negative numbers. For example, $\hat{5}$ means five floors below ground (zero). The purpose of the circumflex is to allow learners to focus on "the idea of a negative number as a *number*, not as an *operation* (i.e. subtraction) on a positive number" (Ball, 1993, p. 380). The textbook, currently in use in many schools across the United Kingdom is the SMP Interact series[3]. In these textbooks, the unary function of minus is denoted by a small superscript minus sign before the number. For example, 5 subtract negative 4 would be shown as: $5 - {}^{-}4$. In South Africa, this would be represented as $5 - (-4)$ and the teacher in this study considers calculations with brackets as a different "form" and I will show how she teaches a different approach when teaching learners how to do calculations in this form.

The use of minus sign was particularly interesting in my study because in one object of learning, the teacher took account of the unary and binary functions, but did not make these explicit and when focussing on the other object of learning labelled all minus signs as *structural signifiers.*

## 3.4 Mathematical Proficiency

Kilpatrick, Swafford and Findell (2001) provide a useful framework that describes how they see successful mathematics learning. Kilpatrick et al (2001) described what they referred to as *strands* of *mathematical proficiency*. The five strands that they describe provide a framework that can be used when discussing knowledge, skills, abilities and beliefs that constitute mathematical proficiency. Kilpatrick et al. (2001) emphasise that the strands are not independent and should work together in developing mathematical proficiency.

The five strands that Kilpatrick et al. (2001) describe are: *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition*. Conceptual understanding and procedural fluency have long been recognised as important notions in mathematics education research (see for example Skemp, 1976; Hiebert and Lefevre, 1986). In the

[3]Cambridge University Press

lessons that I observed for this study, interplay between conceptual understanding and procedural fluency so I will describe these two strands more fully.

"Conceptual understanding refers to an integrated and functional grasp of mathematical ideas" (Kilpatrick et al., 2001, p. 118). Learners with conceptual understanding have organised their knowledge into a coherent whole and can connect ideas to what they already know which supports retention. Kilpatrick et al. (2001) explain that having conceptual understanding helps learners to avoid many critical errors when solving problems. "Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them flexibly, accurately and efficiently" (Kilpatrick et al., 2001, p. 121).

Kilpatrick et al (2001) explain how these two strands complement each other.

> Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop their understanding (p. 122).

It is also important to note that when learning only focuses on procedures without understanding, it then becomes difficult to help learners understand the reasoning behind the procedures. There is also a danger that practicing procedures without understanding the procedures may be incorrect and more difficult to correct later. This fine balance between conceptual understanding and procedural fluency provided a useful framework for analysing the lessons in this study.

## 3.5 Critical features in teaching and learning negative numbers

Four critical features in the teaching and learning of negative numbers were identified through a Learning Study in Sweden by Runesson, Kullberg and Maunula (2011). The aim of this study was for teachers to find critical features of teaching addition and subtraction of negative numbers and how they should be brought into the learning situation in a way that makes them possible to discern.

The teachers recognised, from past experience that learners struggled to see the difference between the minus sign meaning subtract (i.e. the binary function) and the minus sign signifying a negative number (i.e. the unary function). This was the first critical feature that they intended to make

possible for their students to discern. They did this by using contrasting number patterns for learners to explore what happens in addition when numbers are less than zero (see Figure 1).They brought up different answers to the patterns with the intention of contrasting different ways of understanding. Assessment after this lesson showed that it was successful in most respects, except learners did worse on subtraction tasks with two negative numbers. Another interesting observation was that one learner had pointed out that subtraction could also be seen as *difference*, not only *take away*.

| Pattern A | Pattern B | Pattern C |
|---|---|---|
| 5 + 5 = 10 | 5 + 5 = 10 | 5 + 5 = 10 |
| 5 + 4 = 9 | 5 + 4 = 9 | 5 + 4 = 9 |
| 5 + 3 = 8 | 5 + 3 = 8 | 5 + 3 = 8 |
| 5 + 2 = 7 | 5 + 2 = 7 | 5 + 2 = 7 |
| 5 + 1 = 6 | 5 + 1 = 6 | 5 + 1 = 6 |
| 5 + 0 = 5 | 5 + 0 = 5 | 5 + 0 = 5 |
| 5 + (−1) = 4 | 5 + (−1) = −4 | 5 + (−1) = 6 |
| 5 + (−2) = 3 | 5 + (−2) = −3 | 5 + (−2) = 7 |
| 5 + (−3) = 2 | 5 + (−3) = −2 | 5 + (−3) = 8 |
| 5 + (−4) = 1 | 5 + (−4) = −1 | 5 + (−4) = 9 |
| 5 + (−5) = 0 | 5 + (−5) = 0 | 5 + (−5) = 10 |

**Table1: The contrasting patterns used in lesson 1 for students to explore "What happens in addition when we go below zero?" (Note, pattern B and C are incorrect)**

*Reference: Runesson et al., 2011, p. 269*

The teachers took this critical feature (*subtraction as difference*) as the focus of the next lesson. This metaphor of *difference* was useful when a positive answer was obtained, but not for negative answers. For example, the difference between – 3 and –5: –3 – (–5) = 2, but –5 – (–3) = –2. The teachers realised that they would have to explain to the learners that subtraction is always from the perspective of 'first position'. They used a variety of metaphors to illustrate the difference between 5 – 4 and 4 – 5 for example. Essentially they were pointing out that the commutative law does not apply for subtraction.

The teachers now felt that covering these three critical features were sufficient for learning adding and subtracting negative numbers. However, when the teachers gave learners the following task: "Come up with two negative numbers that in a subtraction equals one, i.e. ____ – ____ = 1", they identified a critical feature that they had perhaps taken for granted. Learners were not familiar with the structure of the number system and were uncertain whether –2 or –1 was the bigger number.

The following four critical features for learning addition and subtraction of negative numbers were identified by the teachers in this Learning Study:

1. The *difference between the two signs for subtraction and for a negative number,*
2. That *subtraction can be both seen as 'take away' and as 'difference between numbers*
3. *Subtraction is always from the perspective of the 'first position'.*
4. *The structure of the number system*

Runesson et al (2011) have described these four critical features as those which were made possible to discern in a sequence of lessons with a particular group of teachers and learners in Sweden. Neither have they argued that these critical features are made possible to discern in *all* classrooms, nor have they claimed that these are the only four critical features of adding and subtracting negative numbers. My study aimed to identify what features are critical for the teachers and learners in the South African class in my study.

## 3.6 Previous studies from South Africa

While the Runesson et al.'s (2011) study provided interesting insights into critical features identified in other classrooms, Davis (2010) described a study that he did in a South African school in the Western Cape. This study was not focussing on critical features of learning, but a teacher in their study took a very similar approach to the teacher in my study that was not evident at all in the study from Sweden.

Davis (2010) considers the question of the constitution of Mathematics in pedagogic contexts and using the Mathematics and Science Education Project[4] is interested in developing the descriptive and analytic resources used to describe the teaching of Mathematics. Previous studies had not provided a resource which enabled a more direct grasping of the articulation of objects and operations. To do this, Davis (2010) focussed on the criteria for the production of mathematical statements that circulate in pedagogic activity and referred to an exchange between a Grade 10 teacher and his learners about adding two integers. Explaining how to calculate $-7 + 5$, the teacher says, "So if the signs are the same, what do you do? You take the common sign and then you add. If the signs are not the same, what do you do? You subtract." (Davis, 2010, p. 384) This *rule* that

[4] See Mathematics and Science Education project. (2007). *MSEP Baseline Survey Report*. University of Cape Town. (Prepared for MSEP by: Lydia Abel, Achmat Adams, Norman Davies, Zain Davis, Cynthia Fakudze, Dave Gilmour, Yusuf Johnson, Rüdiger Laugksch, Emmanuel Mushayikwa, Anthea Roberts & Crain Soudien.) Mimeo. [Embargoed].

this teacher used to explain how to calculate −7 + 5 to his learners was echoed by the teacher in my study.

Davis (2010) argues that this criteria (or *rule*) for adding two integers is unintelligible when integers is the domain of operation for addition. This is because the sign is detached from the integer and seen as a "whole number". When this teacher talks about the bigger number, he is *ignoring the signs*. In other words, he is not saying that 5>−7, but 7>5.

The procedure that this teacher has described to calculate −7 + 5 require that the teachers and learners are able to:

1. Separate the sign from the integer, i.e. substituting integers by whole numbers.
2. The recognition of the smaller or bigger "whole number".
3. Identifying the sign associated with the bigger "whole number'.
4. Subtraction of the smaller "whole number" from the bigger "whole number".
5. Attaching the sign of the larger "whole number" to the calculated difference (Davis, 2010).

Davis (2010) concludes that

> The regulative criteria required by the procedure indicate that the teacher and learners do not operate directly on the mathematical objects and relations being indexed (integer sums). They operate, instead, on more familiar and intuitive objects and relations ("whole number" sums) (p. 385).

Venkat and Adler (2012) considered this data from Davis (2010) in terms of the transformation sequence. They describe the transformation as "steps that produce an interim representation that is not equivalent to the input object (−7 + 5), even though equivalence with the input representation returns at the final stage." (Venkat et al., 2012, p.6) When using a rule like the one used by the teacher described in Davis (2010) study, there is no mathematical connection between the question and the answer.

## 3.7 Summary

From the research literature I have identified known difficulties learners experience with negative numbers. In addition I have drawn on the work of Runesson et al. (2011) who highlight four critical

features in the learning and teaching of negative numbers. These suggest important features to consider in the lessons that I analysed. In the South African context the work of Davis (2010) indicates the possibility of an approach to negative numbers that focuses only on the procedural aspect and avoids (or even negates) a conceptual understanding. His work, together with Kilpatrick et al's (2001) elaboration of procedural fluency and conceptual understanding, provide important insights in understanding the lessons observed in my study.

# CHAPTER 4 - Methodology

## 4.1 Introduction

In this chapter, I explain the methodology and features that guided this research. I will describe the school, teacher and learners who participated in this study. I will explain how the data was collected and analysed, and finally I will discuss issues of rigour and ethics that I was very mindful of when collecting and reporting on this data.

## 4.2 Methodological Orientation

This is a case study of a teacher's experiences and her learners' experiences of the phenomenon of teaching and learning negative numbers analyzed and conducted within a variation theory. The nature of the empirical data collected meant that it was useful to look at how variation illuminated particular core features of the concepts of negative numbers that I considered as critical features in relation to conceptual understanding and procedural fluency.

## 4.3 Selection and description of participants

This research will describe the experiences of a Grade 7 mathematics teacher and a small group of learners. A Grade 7 class was chosen because this is the first time learners are taught addition and subtraction of negative numbers. The participants of this study are described here.

### 4.3.1 The school

The school was selected for two reasons. Firstly, pragmatically, I have good relations with the principal and teachers at the school which facilitates an atmosphere of trust that was necessary to allow such a research study. Secondly the school is well-resourced. The classrooms are spacious, and well furnished with desks and chairs. Teachers and learners have continual access to textbooks. There is an average of 30 learners per class. There are few socio-economic problems and discipline is good. These factors were important for my research in that I wanted to create a rich description of the teaching and learning of negative numbers and did not want lessons disrupted by discipline

issues. This school was selected because teachers and learners interact well and good teaching was expected.

### 4.3.2 The teacher

There is only one Grade 7 mathematics teacher at this school and she was enthusiastic to participate in this study. She has been teaching mathematics for over 20 years, so is well experienced. In this Research Report, in order to protect the anonymity of the teacher I have given her the pseudonym Mrs Murray.

### 4.3.3 Learners

The Grade 7 classes are of mixed abilities; so the choice of class was random. There are 29 learners in this mathematics class, 15 boys and 14 girls.

## 4.4 Data collection techniques

Data that was used to identify the critical features associated with the intended, enacted and lived object of learning had to be collected in different ways.

### 4.4.1 The intended object of learning

The *intended* object of learning is that which the teacher intends to make possible for her learners to discern (Marton et al., 2004). In order to determine what Mrs Murray intended to make possible for the learners to discern, she was interviewed before each lesson.

The questions were worded in such a way that Mrs Murray did not need to be familiar with Variation Theory.

She was asked:

1. What have you taken into consideration when planning this lesson?
2. What do you think is necessary to learn this?
3. What could be critical for the learners to understand?
4. How are you going to make it possible for the learners to learn what you intend?

5. What will you do if they do not learn what you intend them to learn in the lesson?

These interviews took no longer than 10 minutes and were done during Mrs Murray's free time so as not to interrupt normal teaching. The intention of this study was not to influence the normal teaching of negative number. So, at no time during these interviews did I offer my opinion on how the teaching of addition and subtraction with negative numbers might be done. These interviews were audio-taped.

### 4.4.2 The enacted object of learning

The enacted object of learning is what the learners actually encounter. It is dependent on what features the learners' discern and focus on in the particular setting. Marton et al. (2004) consider the enacted object of learning to be the object of learning as described by the researcher. It can be the extent to which the researcher observes the necessary conditions for a particular object of learning to occur. In order to identify the enacted object of learning and those features that were made possible to discern, I observed three consecutive lessons during normal teaching time. Mrs Murray taught the learners about negative numbers and the addition and subtraction in calculations that included negative numbers, at a time appropriate to her year plan. These lessons were video-taped. As observer of these lessons, I did not interact with the teacher or the learners during the lesson.

### 4.4.3 The lived object of learning

The object of learning as seen from the learners' perspective is the lived object of learning (Marton et al., 2004). In order to determine what critical features were discerned in the lived object of learning, I invited the learners to complete a pre- and a post-test and I interviewed a sample of the learners.

The intention of the pre-test was to determine what previous experiences and already constructed knowledge the learner brings into the learning situation. This was necessary because a learner's previous experiences will have an influence on the enacted object of learning. After the sequence of lessons, learners were invited to write a post-test. The purpose of this test was to determine

whether the critical features of the lived object of learning were discerned. At Mrs Murray's request, these tests were written during normal Mathematics lessons.

The test that was used was adapted from a test used in Kullberg's (2010) research in which she wanted to explore how other teachers in new contexts could make use of the four critical features identified by Runesson et al. (2011). The questions in the test were designed to explore learner's understanding of negative numbers and relate to issues in the learning of negative numbers identified in previous research. The test is attached in Appendix 1. The validity and reliability of this test will be discussed in section 4.5

After analysis of these tests, three learners were interviewed. The learners that were invited to be interviewed were those that had explained the way they did a calculation in an interesting way. I wanted to get a deeper understanding of their way of thinking that lead to these explanations. The questions used in the interview were developed from the learners' post-tests and attempted to gently probe which critical features the learners had discerned or focussed on. The learner interview took less than 15 minutes and was audio-taped.

How this data was analysed is described in detail in the next chapter.

## 4.5 Rigour in research

Rigour ensures that the findings of a study reflect the object of the study (Sin, 2010). It is characterised by validity and reliability.

### 4.5.1 Validity

Many writers have critiqued the validity of qualitative research. Maxwell (1992) argues that it is possible that there are different, equally valid accounts from different perspectives. As a researcher, I could not divorce myself from this study. An *interpretive* reading of the transcribed lessons and interviews enabled me to document a version of the story that I thought the data was telling (Mason, 2002, p. 149). In order to ensure my description was not distorted by my own perceptions I sought to confirm my categorisation of data after I had analysed it. My supervisor analysed one of the

three lessons using the original data and the indicators I had established. Her categorisation coincided with mine. This indicated my indicators and analysis were robust.

As in Learning Studies, I used a pre-test to create a description of the knowledge that learners brought in to the learning situation. And, I used the post-test to facilitate a description of the lived object of learning which I could later probe in the interview. The use of a single-group pre-test-post-test design has been criticized by Sanders (1993) who says, "It is a logical fallacy to attribute any changes in the students' test-scores to the specific type of teaching used in research." (p. 5). However, this use of pre- and post- test is not intended to be used as a comparison. They are used purely to *describe* (not measure) what was brought in to the learning situation and what will be taken out. An improvement is expected.

### 4.5.2 Reliability

Reliability is the extent in which the findings of a study can be replicated. Sin (2010) considers the argument that it is problematic when reliability is applied to qualitative research because "the social world is unstable and a particular research setting may change from the experience being studied." (p. 310). Sandberg (1997) suggests that the researcher should acknowledge and explicitly deal with their preconceptions throughout the research process (cited in Sin, 2010). Ashworth and Lucas (2000) recommend that researchers need to deliberately set their presuppositions aside in order to engage fully with the participants' lived experiences (cited in Sin, 2010). As I observed, interviewed and analyzed the data – I considered my own biases and either put these aside or I have acknowledge them in my writing.

### 4.5.3 Generalizability and transferability of findings

The aim of this study was to describe a phenomenon in one school and between one teacher and her 29 learners. It does not claim that the findings from this study be generalized to all Grade 7 classrooms in South Africa, or to any other sequence of lessons in any other school at all. What it does do, is attempt to describe a single sequence of lessons. It is hoped that this description can be used as a comparison to descriptions from other schools' teaching adding and subtracting negative numbers and collectively a description of the phenomenon of teaching addition and subtraction in South Africa can be developed.

## 4.6 Ethical considerations

Permission was requested from all participants. Permission was requested from the parents of the learners because they are all less than 18 years of age. Because research is being conducted in a school, the principal and Chairperson of the School Governing Board have been consulted about the research and are in agreement. See Appendices 2 – 7 for copies of letters sent the teacher, the learners and their parents and to the principal of the school.

The video recordings of the lessons and the audio-recordings of the interviews, including transcripts, have not been shared, nor will they be shared, with anyone other than my supervisor. They will be destroyed on completion of the research report. A summary of the results of the learner test was shared with the teacher. Otherwise, the tests were only seen by me and by my supervisor. These will also be destroyed on completion of this research study.

The participants were guaranteed that their identities will remain anonymous and that they are able to withdraw from the research at any stage. Mrs Murray is a pseudonym for the teacher who participated in this research. All learners' names were also changed in this research report.

A summary of the research report will be shared with the teacher, her school and the Department of Education.

Ethical clearance has been granted from the Department of Education and the University of Witwatersrand (2011ECE156C). Please see Appendices 8 and 9.

## 4.7 Summary

In this chapter, I have described the methodological approach of my research. I have described the participants of the research and how the data was collected. I have also described how rigour and ethical considerations were taken into account. In the next chapter, I provide more detail about how this data was analysed and I start to unravel the story that the data tells about the teacher's and learners' experience of the phenomenon of teaching and learning negative numbers.

# CHAPTER 5 – Analysis and Interpretation of Data

## 5.1 Introduction - How the analysis was done

The transcribed data was read *interpretively* (Mason, 2002, p. 149). An interpretive reading enabled me to document a version of the story that I thought the data was telling. I was interested in finding the critical features that were made possible to discern in the intended, enacted and lived objects of learning. Brown and Dowling describe the research process as one which "begins with vagueness and hesitance and plurality and moves towards precision and coherence" (1998, p. 137). From this reading, it emerged that there was not one, but two objects of learning that were in focus at different times through the sequences of lessons. These two objects of learning have been labelled:

- *Giving meaning to negative numbers and calculations that include negative numbers.*
- *Mastering the rules that can be used to do calculations that include negative numbers.*

As these two objects of learning had emerged inductively from the data, in order to ensure that they adequately described the full data set, I systematically coded the data. The data was categorised according to which object of learning was in focus. I used *categorical indexing* of data. Mason describes categorical indexing as "devising a consistent system for indexing the whole data set according to a set of common principles and measures" (2002, p. 150). Two categories were first applied to the data. These were the two objects of learning that identified above.

The indicators used to categorise data as being focused on *giving meaning to negative numbers and calculations that include negative numbers* were when

- The meaning of the minus and plus sign are used appropriately as either operational or structural signifier.
- There was awareness about the structure of the number system, i.e. the order of integers.
- Adding a positive number indicated that the learner should increase the value and subtracting a positive number indicated that the learner should decrease the value.

The indicators used to categorise data as being focused on *mastering the rules that can be used to do calculations that include negative numbers* were when:

- A method is given without explanation linked to conceptual understanding, for example $- \times - = +$

- The minus and plus sign are only used as a structural signifier.
- The structure of the number system is ignored.

Within each of these objects of learning the critical features that were made possible to discern in the object of learning were identified. "To discern an aspect, the learner must experience potential alternatives, that is, variation in a dimension corresponding to that aspect, against the background of invariance in other aspects of the same object of learning." (Marton et al., 2006, p. 193). The critical features were identified by looking for evidence of variation. The critical features identified in one object of learning were not necessarily in focus in the other object of learning.

### 5.1.1 Analysis of lessons

The lesson observations were divided into *episodes*. I define an episode as the interaction between teacher and learners focusing on a specific concept. An episode starts when a new concept is introduced and ends when another concept is introduced either by the teacher or by the learners.

The example below illustrates how an episode could begin and end. Mrs Murray has just done an activity where learners ordered integers and now she indicates that another feature of negative numbers is to be discerned. The episode starts:

> Mrs Murray: Okay. The next job is we are going to add and subtract positive and negative numbers. Okay. And that is why you have got your little ruler[5] in front of you. Now, when you were in Grade 1, and quite a few Grade 1 teachers actually taught the children to add and subtract using a number line. I don't know whether you remember that, but I am going to draw just a plain number line just with nought, one, two, three, four, five and six [*Draws a number line from 0 to 6 on the board*]. Okay, and when you were in Grade 1 your teacher would say what is two plus three? [*writes 2 + 3 on the board*] So, what you did in Grade 1 is your teacher said just imagine you had a little bunny. And I am not very good at drawing so there's my little bunny. [*Draws a bunny over the number 2 on the number line*] Okay, and it sat on the number two because that is where we started because you said two plus three. And when you added, which way did you move?
>
> **Excerpt 1: Lesson 1, 00:26 – 01:53**

This episode continues with Mrs Murray illustrating how to calculate 6 – 5, –3 + 5 and –5 – 3 using a number line. The concept in focus is moving to the right on the number line when adding and moving left on the number line when subtracting. This same episode continues:

[5] This little *ruler* is a paper number line from –18 to 18.

| | |
|---|---|
| Mrs Murray: | Left. Okay. Don't put your hands up. Let's just see. Minus five and I am going to minus three. Okay, Andrew what's your answer? |
| Andrew: | Minus eight |
| Mrs Murray | Well done. Okay, who got minus eight? Okay. Who did not? Why? |
| *Laughter* | |
| Mark: | Doesn't the bigger number have to be in front? |

**Excerpt 2: Lesson 1, 04:31 – 04:55**

Mark's question, "Doesn't the bigger number have to be in front?" indicates the beginning of the next episode where the focus will be on the commutative property of calculations.

Once the lesson observations had been divided into episodes, each episode was categorised according to which object of learning was in focus. In the episode illustrated above *giving meaning to negative numbers and calculations that include negative numbers* was in focus. This is because the first sign was seen as a structural signifier and the second sign was seen as an operational signifier and the operational signifier was used in a mathematically meaningful way (i.e. add means increasing and subtract means decreasing)[6].

An example of an episode categorised as focusing on *mastering the rules that can be used to do calculations that include negative numbers* is illustrated below. The learners have just completed an activity on their worksheets using their *ruler* to add and subtract negative numbers and Mrs Murray has gone through the answers. The next episode starts:

| | |
|---|---|
| Mrs Murray: | Now, I am going to teach you another method. Okay, are you ready to listen? If both of, say for instance, I've got three plus four [*writes +3 + 4 on the board*], both of those are positive, aren't they? In other words, the signs are the same. And if the signs are the same when you are adding and subtracting integers, the signs are the same – you keep the sign and you add them, so in other words, just like Grade 1 positive three plus four will be positive seven or just seven. If you've got both of them negative – negative three minus four [*writes –3 – 4 = on board*]. The signs are the same, you keep the sign and you add them. Does that make sense? Okay, second method: if the signs are different, [*writes +3 – 4 = on the* board] you take the sign of the largest number, okay, but now I am confusing you a little bit because just now, I said put the numbers in order from smallest to largest. Okay, what you do now is you take sign of the number, ignoring signs in other words, okay, so if I am ignoring the positive and negative sign, four would be my largest number, wouldn't it? So, I take the sign of the largest number, negative [*writes – behind +3 – 4 =*] and because the signs are different, I subtract the numbers [*writes 1 behind +43 – 4 = – sign].* So, if I have negative three plus |

[6]Add means increasing when the number being added is positive and subtract means decreasing when the number being subtracted is positive.

four, I take the sign of the largest number, positive, and I subtract the numbers.

**Excerpt 3: Lesson 1, 35:13 – 36:56**

Continuing in this same episode Mrs Murray does another example, –6 – 8, using the rule that the signs are the same, so you keep the sign and add the numbers. The concept in focus in this episode is recognising that when signs are the same, you add (regardless of whether signs are + or –) and when the signs are different, you subtract. This episode ends and the next episode begins when Mrs Murray tells the learners to check the answer to–6 – 8using their *rulers*. At this point, moving to the right on the number line when adding and moving left on the number line when subtracting comes back into focus.

The reason for this episode being categorised as focusing on *mastering the rules that can be used to do calculations that include negative numbers* is because both signs are used as structural signifiers and the operation depends on whether the signs are the same or different and the order of the integers is ignored. In Excerpt 3 above, Mrs Murray says, "so, if I am ignoring the positive and negative sign, four would be my largest number, wouldn't it?"

While working through the lesson observation transcripts it became apparent that some episodes did not focus on the objects of learning. For example, giving instructions about how to fold and stick the little *ruler*, which book to write in or what content will be in the next test. As these episodes were essentially related to classroom management issues they were not relevant for my analysis and hence were omitted.

Finally, each episode was timed. The reason for the timing was to create a picture of the flow between these objects of learning and how they related to each other. It also gave a sense of which object of learning was a higher priority.

### 5.1.2 Analysis of teacher interviews

The teacher interviews were analysed with the two objects of learning in mind. Dialogue was extracted that provided evidence on which object of learning the teacher was intending to focus and how she intended to make it possible for learners to discern critical features of this object of learning.

### 5.1.3 Analysis of learner tests

The intention of the learner tests was to identify which critical features had been discerned by the learners. The test was developed by Kullberg (2007) to determine whether the learners in her study discerned the same four critical features identified in a previous study by Runesson et al (2011). Kullberg (2007) explains that these four critical features are critical for the specific group that was investigated and not necessarily critical in all teaching and learning of negative numbers. She does suggest though that they may also be critical for other learners. I had not anticipated that there would be two objects of learning in my study. While Kullberg analysed which critical features were discerned, because of the ways the lessons in my study unfolded, my analysis shifted to which object of learning was in the learners focus. Questions that required learners to describe their method of calculation provided this evidence.

### 5.1.4 Analysis of learner interviews

From the twenty-eight tests received, three learners were selected to get a deeper understanding of which object of learning was more in focus for them and whether they were able to use both objects of learning. The three learners were selected on the basis of the reasons they provided for some of the answers. The learners' transcribed interviews were analysed to find evidence of which object of learning dominated in their focus.

## 5.2 The enacted object of learning

The story of the enacted object of learning emerged from the analysis of the lesson observation transcripts. As defined in the beginning of the chapter, the two objects of learning that the lesson focused on were:

- *Giving meaning to negative numbers and calculations that include negative numbers.*
- *Mastering the rules that can be used to do calculations that include negative numbers.*

The lesson transcripts were divided into episodes and each episode was categorised according to which object of learning was in focus. For ease of reference, tables 1, 2 and 3 provide a summary of what was happening in each episode of the story, which object of learning was in focus during that episode and how long each episode took. The objects of learning have been labelled to Meaning and Rules respectively. The amount of time spent on each object of learning was totalled.

**Table 1: Summary of episodes in Lesson 1**

| | | Object of learning | |
|---|---|---|---|
| **Episode** | **Activity** | **Meaning** | **Rules** |
| 1 | Structure of the number line | 0:23 | |
| 2 | Positive and negative temperatures | 3:22 | |
| 3 | Difference between positive and negative temperature | 1:34 | |
| 4 | Ordering integers – ascending (no context) | 1:19 | |
| 5 | Debt, reducing balance | 2:43 | |
| 6 | Ordering integers – descending (no context) | 1:32 | |
| 7 | Adding and subtracting on number line | 4:24 | |
| 8 | Order doesn't matter:[7] $-a + b = b - a$ and $-a - b = -b - a$ | 2:27 | |
| 9 | $-7 + 7$ and $-13 - 1$ | 0:34 | |
| 10 | Learners complete worksheet using *ruler*[8]. | 4:15 | |
| 11 | Learners call out answers for checking | 1:35 | |
| 12 | Rule 1: When the signs are the same, you keep the sign and add the numbers.<br>e.g. $+3 + 4 = +7$ and $-3 - 4 = -7$<br>Rule 2: When the signs are different, you use the sign of the larger number and subtract the numbers.<br>e.g. $+3 - 4 = -1$ and $-3 + 4 = +1$<br>$-6 - 8 = -14$ | | 3:18 |
| 13 | Subtracting from a negative makes a smaller negative number using *ruler* | 0:37 | |
| 14 | Two examples using rules | | 1:08 |
| 15 | Learners complete worksheet using rules. *Ruler* may be used for checking. | 4:49 | |
| 16 | Mrs Murray goes through the answer for each question, and for every answer emphases the rule to be used for each one | | 2:22 |
| 17[9] | These problems can be seen with brackets, e.g. $-4 + (-4)$ , $-3 + (+7)$ and $(+4) + (-8)$ | | 1:14 |
| | TOTAL | 29:34 | 12:51 |

[7] In all that follow, when I use the variables a and b, I will assume $a>0$ and $b>0$ unless I specify otherwise.
[8] The ruler is a paper number line from –12 to 12
[9] This episode was interrupted by the intercom and teaching ceased for the remaining lesson time.

**Table 2: Summary of episodes in Lesson 2**

| | | Object of learning | |
|---|---|---|---|
| **Episode** | **Activity** | **Meaning** | **Rules** |
| 1 | Revise – *If the signs are the same, you keep the sign and add the numbers.* | | 0:51 |
| 2 | Investigating WHY – *If the signs are the same, you keep the sign and add the numbers.* | 01:16 | |
| 3 | Learners answer questions of type +a + b and –a – b without a ruler, i.e. using the rule | | 09:02 |
| 4 | Jessica asks meaning of signs – operational or structural signifiers | 00:27 | |
| 5 | Repeating the rule: *If the signs are the same, you keep the sign and add the numbers* and marking questions of the type +a + b and –a – b | | 02:30 |
| 6 | Revise – *If the signs are the different, you use the sign of the bigger number and you subtract the numbers.* | | 00:43 |
| 7 | Subtract the smaller number from the larger number | | 02:54 |
| 8 | Learner asks about subtracting two negative numbers, i.e. –2 – –3 | 00:38 | |
| 9 | Learners answer questions of type +a – b and –a + b without a ruler, i.e. using the rule | | 12:36 |
| 10 | Repeating the rules: *If the signs are the same, you keep the sign and add the numbers* and *If the signs are the same, you keep the sign and add the numbers* | | 01:16 |
| 11 | Learners call out their answers for marking, repeating the steps of the rules each time. | | 06:27 |
| 12 | Meaning of signs | 00:28 | |
| 13 | Introduces rules to eliminate brackets by multiplication (the secret times sign):<br>Positive × Positive = Positive<br>Negative × Negative = Positive<br>Positive × Negative = Negative<br>Negative × Positive = Negative | | 04:18 |
| | TOTAL | 2:49 | 40:37 |

**Table 3: Analysis of episodes in Lesson 3**

| | | Object of learning | |
|---|---|---|---|
| **Episode** | **Activity** | **Meaning** | **Rules** |
| 1 | Revises rules to eliminate brackets by multiplication (the secret times sign):<br>Positive × Positive = Positive<br>Negative × Negative = Positive<br>Positive × Negative = Negative<br>Negative × Positive = Negative | | 01:27 |
| 2 | Learner exercise – multiplying signs, no numbers. | | 01:16 |
| 3 | Learners call out answers to check exercise. | | 00:48 |
| 4 | The meaning of brackets and signs. | 02:40 | |
| 5 | Combining multiplication rules with addition and subtraction rules. | | 06:16 |
| 6 | Learner activity from worksheet using rules | | 08:30[10] |
| 7 | Reminder of rule: *If the signs are the same, you keep the sign and add the numbers* | | 00:40 |
| 8 | Learners continue their activity using rules | | 00:26 |
| 9 | Mrs Murray marks the activity repeating the rules used for each question. | | 04:26 |
| 10 | Further teacher-led examples using rules | | 02:56 |
| 11 | Learner activity from board using rules | | 08:20 |
| 12 | Mrs Murray goes through board activity repeating rule used for each step. | | 03:54 |
| | TOTAL | 02:40 | 38:11 |

This summary provides evidence of the enacted object of learning moving from *giving meaning to negative numbers and calculations that include negative numbers* to *mastering the rules that can be used to do calculations that include negative numbers.* In the first lesson approximately 25 minutes focussed on *giving meaning to negative numbers and calculations that include negative numbers.* In the second and third lessons less than 3 minutes focussed on *giving meaning to negative numbers and calculations that include negative numbers.*

This transition in focus from *giving meaning to negative numbers and calculations that include negative numbers* to *mastering the rules that can be used to do calculations that include negative numbers* is described in more detail.

[10]There was a break in the recording due to full memory card. This time is an estimate from the observation.

### 5.2.1 Critical features for 'Giving meaning to negative numbers and calculations that include negative numbers'

The critical features that I identified in the analysis of the lesson observations were a) the structure of the number system and b) addition as increasing and subtraction as decreasing.

#### The structure of the number system

Mrs Murray introduces negative numbers by describing real-life applications to the learners. She starts with the context of temperature to give the learners a sense of the order of integers.

Mrs Murray: Alright, we are going to start talking about integers and positive and negative numbers. Now, some of you, you have, here and there, incidentally, come across negative numbers and some of you have coped quite well with them because I popped them in to little bits and pieces and you have discovered them along the wayside. Now, did anybody look at the TV last night or this morning and have a look at the temperature um, today?
Learner: Thirteen
Mrs Murray: The minimum temperature? What did they say, the minimum temperature, what did they say? The minimum temperature – that is the lowest temperature that you going to get today.
*Several answers are mumbled. Some of them include minus temperatures.*
Mrs Murray: So, nobody looked at the TV. Nobody watches the news. [*Learners shouting*] I want to know the minimum. What does minimum mean to you? [*Learners shouting*] The lowest. So, who said minus three? Okay, let's take minus three. Now, what is minus three degrees – that is in Centigrade – what does it mean to you? Okay, just hang on.
*Lesson is interrupted by a teacher bringing a message.*
Mrs Murray: Alright, sorry about that. Okay
Learner *Inaudible*
Mrs Murray: That means it is very cold. Right. Okay, what does negative three degrees mean to you, besides very cold?
Learner: Below zero
Mrs Murray: Below zero. Very good. Is zero a number?
*Learners mumble Yes and No*
Mrs Murray: Is it a number? [*more mumbles*] It is a number. Yes, it is a very important number. If you don't count zero in as a number, you are going to make mistakes along the way. Is zero positive or negative?
Learner: Neither
Mrs Murray: Neither. Good. Well done. Okay. …

**Excerpt 4: Lesson 1, 08:38 – 11:32**

In this episode the focus of the enacted object of learning is *giving meaning to negative numbers and calculations that include negative numbers*. The minimum temperature and maximum temperature illustrate order of integers to the learners and that 0 has a place on the number line, but is neither positive nor negative. In order to make it possible for the learners to discern the critical feature of *the structure of the number system* Mrs Murray "generalises" the structure of the number system (Marton et al., 2004, p. 16). After introducing negative numbers as cold temperatures, she

opens up varying appearances of number system by the *ruler*. The little *ruler* is used to assist learners to order integers out of context.

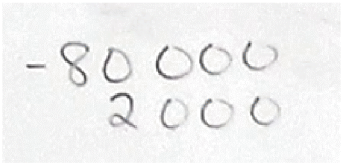
### Addition as increasing and subtraction as decreasing

Mrs Murray uses the *ruler* to describe addition as increasing (moving right on the number line) and subtraction as decreasing (moving left of the number system). In this explanation, learners always add and subtract by a positive number. The case of adding and subtracting a negative numbers is not discussed. Refer to Excerpt 1 on page 24 and 2 on page 25. This episode provides the opportunity for conceptual understanding. The critical feature that is made possible to discern in this episode is *addition as increasing and subtraction as decreasing*. This critical feature is made possible to discern by "contrast" (Marton et al., 2004, p. 16). Mrs Murray makes it possible for the learners to discern addition as increasing by contrasting it to subtraction as decreasing. She also summarises this variation with the following illustration:




Mrs Murray uses the context of debt to show adding to negative number reduces the balance, i.e. the number gets bigger.

Mrs Murray: You owing money. You have a negative balance in the bank. Okay, so say for instance, you go and buy yourself a car, your first car, you've got a job so the bank is going to give you a loan, okay, and you go and buy your first car. And you earn, just say you earn a nice round number, R20 000 a month, okay, and um, you are paying off the loan of a car, and um, when you get your statement, okay, say you bought a car for R80 000, you are going to have a negative balance of R80 000 [*Writes –80 000 on board*] Now, they charge you interest and all sorts of things like that, but lets not go into the interest and simply do percentage and things like that. But, say for instance that the bank tells you that you have got to pay R2 000 a month, okay, to pay off that loan. Nice, easy numbers. Okay, and if I pay R2 000 off of that, what am I owing the bank now?



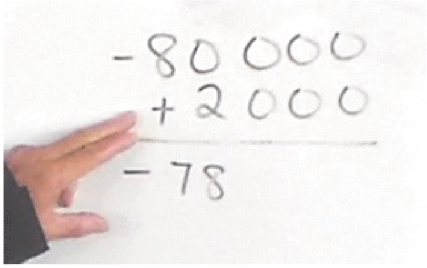

Ssh, Andrew is going to tell me.

Andrew: Umm. [*inaudible*]

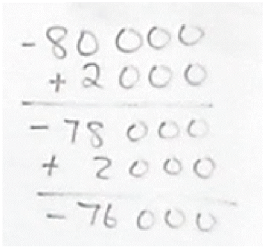
Mrs Murray: Okay, if I am paying R2 000 off that loan, what am I going to owe the bank now?

*A learner calls out seventy-eight*

Mrs Murray: Seventy eight, good. So, now I have a negative balance of seventy-eight. So, I am actually putting a positive amount in there, aren't I? Okay, so I am really putting a positive amount in there, because I am putting real money in there. [*Points at the plus sign*]




And I have now got, a negative balance of seventy-eight thousand. [*Writes in zeroes for thousands*]. Okay, do you understand Andrew? Alright? If, I the following month pay another two-thousand rand, I put another positive amount in, okay?




My balance is then going to be seventy-six thousand and negative seventy-six thousand … until I have paid it off. Okay, so there is another way of explaining negative amounts. Does it make a little bit more sense now to you? Okay, just remember that this is a new thing and if you are not sure about something, you've got to stop and ask because otherwise, I am going to push ahead and you might not know the difference between and you might not understand what you doing. Okay,

Learner: So, negative numbers could also be an outstanding amount?

Mrs Millar: An outstanding amount, okay…

**Excerpt 5: Lesson 1, 16:24 – 18:43**

### 5.2.2 Critical features for 'Mastering the rules that can be used to do calculations that include negative numbers'

The critical features identified for *mastering the rules that can be used to do calculations that include negative numbers* were the specific rules that were rehearsed in the lesson. The way that Mrs Murray makes these rules possible for the learners to discern is described below.

In episode 12 of lesson 1, Mrs Murray intentionally shifts the enacted object of learning to *mastering the rules that can be used to do calculations that include negative numbers.* She begins by suggesting that the *ruler* is just a back-up to check answers.

> Mrs Murray: Now, those of you who got a number of them incorrect, it's actually not a bad idea to keep the ruler and to work with the ruler as a back-up, in fact that is not a bad idea for all of you to keep the ruler as a back up okay to check your answer on the second method …
>
> **Excerpt 6: Lesson 1, 34:51 – 35:13**

Excerpt 3 on page 25 describes how Mrs Murray introduces the first rules to the learners. Again she encourages learners to put away their rulers and so shows preference for the rules.

> Mrs Murray: Okay, let's see, don't look at your ruler, let's see how you cope with that rule. Let's take minus um, five plus eight. Without looking at your ruler, who can give me that answer? … using the rule that I have just given you. Okay.
>
> **Excerpt 7: Lesson 1, 36:56 – 17:14**

The rules that Mrs Murray referred to in Excerpt 3 on page 25 and that she repeats numerous times throughout the three lessons are:

- When the signs are the same, keep the sign and add the numbers.
- When the signs are different, use the sign of the bigger number and subtract the numbers.

These are the rules that are used for questions in the forms +a + b, –a – b, +a – b and –b + a.

Mrs Murray emphasises rules over *ruler* method again in the second lesson.

> Mrs Murray: Alright, I am just going to recap on what we learned yesterday and I want you to try and not to use the ruler. The ruler was to introduce the topic to you and for you to check and if you have difficulty um, with the addition and subtraction of integers, okay. So, I want you to have a look now at what I am going to do on the board and I want you to try and understand the second method without the ruler and we are going to take it one thing at a time.
>
> **Excerpt 8: Lesson 2, 01:46 – 02:17**

> Mrs Murray: Do you see that you've really got to keep a Maths hat on here and you've got to concentrate and you've got to remember the rule.
>
> **Excerpt 9: Lesson 2, 36:33 – 36:47**

> Mrs Murray: … we need more practice with, especially with the second rule and, not using the ruler.
>
> **Excerpt 10: Lesson 2, 44:07 – 44:55**

In lesson 1, learners do an activity on their worksheet where they are instructed to use the rules to do the following calculations:

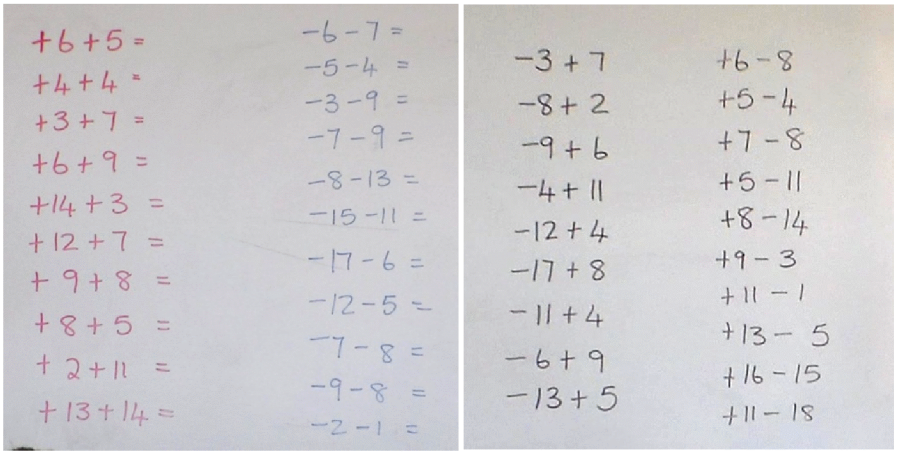
The questions on the worksheet were as follows:

| | |
|---|---|
| a) 9 + 6 | g) 7 – 3 |

| | |
|---|---|
| b) –13 + 6 | h) 2 + 6 |
| c) 8 – 3 | i) 7 – 14 |
| d) –3 + 8 | j) –2 – 6 |
| e) –8 + 2 | k) 7 – 12 |
| f) 4 – 8 | l) –9 – 3 |

There is no systematic pattern of variation in these questions, but they do cover all versions of the two rules that have been made possible to discern so far.

At the beginning of the second lesson Mrs Murray has the following questions written on her board:



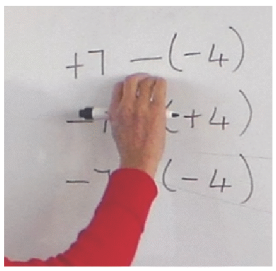

In this lesson, Mrs Murray uses variation to make it possible for learners to discern the rules. In the first column, learners are adding to a positive number. All questions are in the form +a + b, where a and b vary. In the second column, learners are subtracting from a negative number. All questions are in the form –a – b, and again a and b vary. These two columns of questions make it possible for the learners to discern the rule "*when the signs are the same, keep the sign and add the numbers*."

In the third column, learners are adding to a negative number. The questions are in the form +a – b, where a and b vary. In the fourth column, learners are subtracting from a positive number. All the questions are in the form –a + b and a and b vary. The third and fourth columns of questions make it possible for the learners to discern the rule "*when the signs are different, use the sign of the bigger number and subtract the numbers*."

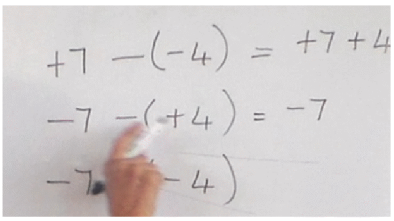
At the end of the second lesson, Mrs Murray returns to a question that a learner had asked earlier in the lesson, "What happens when you minus two numbers together?" Mrs Murray explains how this is done as follows:

Mrs Murray: Um, okay, now who asked a question earlier that is an

important question because that was where I was going to go to next. What happens when you minus two numbers together? [*writes – on board*] uh two numbers from each other and they could be negative or positive? So what happens, if I go like this: I've got positive seven minus negative four [*writes +7 – (–4)*] Or negative seven minus positive four [*writes –7 – (+4)*] Or I can even have negative seven minus negative four [*writes –7 – (–4)*], okay. Alright, let's have a look at those, I think all combinations there.

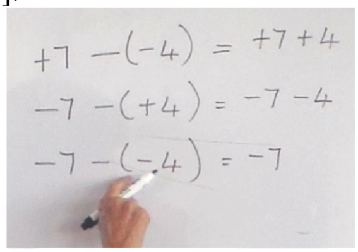



Now what happens here is, if I have two negatives together, okay it's almost like multiplying negative together, it becomes positive. So, if you ever see two negative right next to each other like that, they become positive. So this minus minus four becomes plus four. So, in other words, this means seven plus four. [*writes = +7 + 4 next to +7 – (–4)*]. Okay, if you put it into, what you have been doing now.




Okay, this one here is minus seven, there is a negative and a positive together okay, so that's positive. We going to do an exercise in a minute. Okay, and that then becomes minus four. [*writes – 4 behind ... =–7*].

This one is negative seven. Two negatives together makes a …?



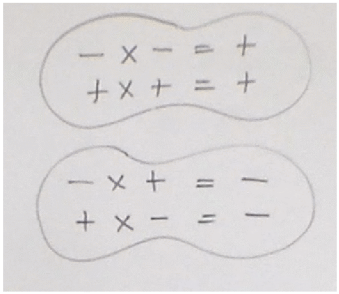

Learners: Positive

Mrs Murray: Positive. Plus four [*writes + 4 behind ... = –7*]. So, we going to do an exercise first, okay, in your books. Negative times a negative is equal to a positive. A positive times a positive is equal to a positive. So, if the signs are the same, the answer will be positive when you multiply them together. So, in other words, if you have two negatives together or two positives together the answer will be positive.

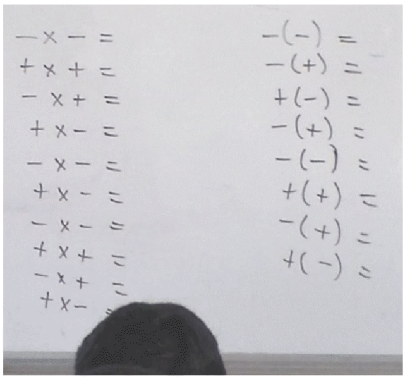
**Excerpt 11: Lesson 2, 45:57 – 48:01**

She has used this question to introduce learners to calculations that have brackets. She explains that what happens is that when two signs are together it becomes "almost like multiplying". Mrs Murray has answered the question, "What happens when you minus two numbers together?" by introducing another rule. No reason has been given as to why, when you subtract a negative, you add. Neither does Mrs Murray link this method to the number line used previously. Mrs Murray

summarises the rules that she has just explained to the learners by writing the following on the board.




Mrs Murray continues to reinforce these rules by asking the learners to "quickly practice to see if they get it right". The questions written on the board for the learners to practice are:




Neither the summary nor the activity on the board have any context and are therefore stripped of any meaning.

Thus we see that when the object of learning is *mastering the rules that can be used to do calculations that include negative numbers* the actual rules themselves and rehearsal of the various forms of the rules becomes the sole focus of the classroom. For this reason I suggest the rules themselves (listed below) become the critical features in the classroom when that object of learning is in focus.

- When the signs are the same, keep the sign and add the numbers.
- When the signs are different, use the sign of the bigger number and subtract the numbers.
- Positive × Positive = Positive
- Negative × Negative = Positive
- Positive × Negative = Negative, and
- Negative × Positive = Negative.

These last four rules were sometimes worded as:

- When the signs are the same, the answer is positive
- When the signs are different the answer is negative

The analysis of the lessons raised the questions:

Where both of these enacted objects of learning also intended objects of learning?

And which critical features did Mrs Murray intend to make possible for her learners to discern?

What follows is an analysis of the interviews with Mrs Murray in order to identify the intended object (or objects) of learning and the critical features that Mrs Murray intended to make possible for learners to discern.

## 5.3 The intended object of learning

The intended object of learning is the object of learning as seen from the teacher's perspective. It is the object of learning that the teacher strives for (Marton et al., 2004).

Mrs Murray's intended object of learning appears to focus on *giving meaning to negative numbers and calculations that include negative number* and *mastering the rules that can be used to do calculations that include negative numbers*. However, there is more emphasis on making it possible for learners to discern the rules. There is also evidence of Mrs Murray intending to merge these two objects of learning.

### 5.3.1 Giving meaning to negative numbers and calculations that include negative numbers

The only evidence that Mrs Murray gives that she intends to focus on *giving meaning to negative numbers and calculations that include negative numbers* is in the first interview. Mrs Murray indicates that she wants to "introduce negative numbers" and that to do this she gives the learners a little *ruler*. This could be interpreted as intending to make it possible for learners to discern the *structure of the number system*.

| | |
|---|---|
| Interviewer: | Thank you very much. So tomorrow you are teaching the Grade 7's adding and subtracting negative numbers? |
| Mrs Murray: | I am going to teach them adding and subtracting, but also introducing the negative numbers. They don't, they haven't really been introduced to negative numbers at all yet. |
| Interviewer: | Okay |
| Mrs Murray: | So, incidentally they come across one or two |
| Interviewer: | And what factors did you take into account when you were planning your lesson? |
| Mrs Murray: | The fact that it is brand new to them and the fact that a lot of children at high school do battle with the negatives and positives. So, I do teach two methods and they are going to have a little ruler with negative and positive numbers that I always give them and they start |

off with that and I find that a lot better when they have something concrete in front of them. And I do like the concrete a lot.

**Excerpt 12: Teacher Interview 1, 0:11 – 1:06**

Although this is the only evidence of a focus purely on *giving meaning to negative numbers and calculations that include negative numbers*, Mrs Murray does express an intention to use *the meaning of negative numbers and calculations that include negative numbers* to explain the rules that can be used to do the calculations. She indicates her intention to teach two methods. The first method that she has described is to use a *ruler*. Learners start off with the *ruler* because it is concrete. This suggests that the second method (the rules) is more abstract and sophisticated.

### 5.3.2 A merge between Meaning and Rules

Mrs Murray expresses how she intends to make it possible for the learners to discern why the rules for adding and subtracting negative numbers work – the link between the meaning and the rules.

Interviewer: What do the learners need to understand in order to follow those rules of the second method that you taught them?

Mrs Murray: Well, I think they've got to have more of an inner understanding I think. I wouldn't, normally I wouldn't have gone as quickly. We would have mastered that *ruler* a lot more and I would have made them discover another method and it would have taken a little bit more time. So then I would have lead them into the fact that, if they, what is happening to the numbers if they both negative and if they are both positive what is happening and then the numbers are different signs what happens. And they would have discovered that for themselves, but it would have taken a little bit longer. In that instance, I normally do pair work or in groups of three and they discuss it, so they break a while and we discuss things a little bit, but I fast tracked things so I think it shows you that self-discovery is far better because, and group-work and discussing it so I like it to be … but I changed my method a little bit and I thought – let's see what happens.

Interviewer: You asked the learners, "Who is still a little bit insecure with method 2?", and there were a few of them that did. What do you think they need to still understand in order to make them feel more secure?

Mrs Murray: Well, I need to, they actually need to see a lot of examples with their *ruler* where both are negative and have a look at the answer being bigger and you are adding them together and they need to see a lot more examples of both positive and then they need to see a lot of examples where there two different signs and what happens to the answer compared to the others. So they actually do need to see that a lot more and I think that is where the discovery and the groups come in quite a lot more. And I think tomorrow my first point of departure is to consolidate and to get that more out of them before I push on.

**Excerpt 13: Teacher Interview 2, 02:50 – 5:00**

In order to make it possible for the learners to discern the connection between the meaning of addition and subtraction of negative numbers and the rules, Mrs Murray intends to use lots of examples. She does express some intention to use variation. She will do this by first giving examples where both are negative (and the numbers will change) and then where both are negative and then when the signs are different.

Although Mrs Murray describes her intention to make a connection between *giving meaning to negative numbers and calculations that include negative numbers* and *mastering the rules that can be used to do calculations that include negative numbers*, we see in the enacted object of learning a preference for the second method, i.e. the rules and recourse to meaning as "for the weaker learners".

### 5.3.3 Mastering the rules that can be used to do calculations that include negative numbers

In the second interview Mrs Murray expresses her preference on mastering the rules as opposed to using the *ruler* method:

| | |
|---|---|
| Interviewer: | So in tomorrow's lesson, what aspects of the content in tomorrow's lesson do you think are critical that the learners understand? |
| Mrs Murray: | That we go over the second rule, and, before I even get onto the brackets story because … I think I've got to consolidate that second rule. [*Bell rings, interview pauses*] So definitely to consolidate and, um, to reinforce that second rule[11] because at the end of the day so even the weaker children can go back to ruler and I don't take it away from them even during exams and they can take it with them when they go to high school. Somewhere along the line they have got to understand that second rule and they've got to apply it because they can't keep …, although, I've seen grade 10's and 11's that are literally that battle with that rule and they use that ruler because, then, if their understanding is poor, they resort to calculator and I want to try and get them comfortable enough where they don't feel the need to use a calculator. |

**Excerpt 14: Teacher Interview 2, 5:06 – 6:18**

Mrs Murray feels that even though weaker learners may use the *ruler* – they have to reach the point where they master the second rule.

[11]The second rule being referred to in this interview is, "When the signs are different, you use the sign of the bigger number and subtract the numbers."

This can be illustrated further. When Mrs Murray was asked what she thought a learner did to get an incorrect answer, she responded that it was the rule that she had not remembered.

| | |
|---|---|
| Interviewer: | Another question was negative six minus eight and Anna answered negative two. How do you think she got negative two? |
| Mrs Murray: | I think instead of adding the two negatives together, she subtracted the numbers, because remember that was when I taught the second rule and so when the signs were the same, they had to add them and I think she, you know, its quite a lot all at once and um to remember that when the signs are the same you've got to add them and when the signs are different. So, what we were doing was that when the signs were different we subtracted, so she subtracted, but kept the sign of the bigger number, and I think that is what happened there. |

**Excerpt 15: Teacher interview 2, 2:05 – 2:50**

In the third interview, Mrs Murray was asked what she intends to focus on in the next lesson to make adding and subtracting with brackets possible to learn. Again, her focus was on reinforcing the rules.

| | |
|---|---|
| Interviewer: | Okay, so from this, what have you focussed on in the next lesson to make adding and subtracting with brackets possible to learn? |
| Mrs Murray: | Okay, what I am going to do is as you saw yesterday I did negative and positives together multiplying them and using the brackets that when they see two negatives together, they are going to make it into a positive. |

**Excerpt 16: Teacher interview 3, 5:49 – 6:09**

The discussion above indicates that Mrs Murray's focus was more on *mastering the rules that can be used to do calculations that include negative numbers* and she intended to make it possible for learners to discern the rules. For Mrs Murray, the rules were the critical feature in the intended object of learning.

The discussion, so far, illustrates a major focus on *mastering the rules that can be used to do calculations that include negative numbers* in the intended object of learning and in the enacted object of learning. So, finally I consider the focus of the lived object of learning and which critical features did the learners discern?

## 5.4 The lived object of learning

The lived object of learning describes what the critical features the learner may or may not have discerned (Marton et al., 2004). The learner test showed improvement in learner performance. The table below shows the number of learners who got questions 1, 2, 5, 7 and 8 correct.

**Table 4: Summary of results of learner pre- and post-test**

| Ques | 1 | 2a | 2b | 2c | 2d | 2e | 2f | 2g |
|---|---|---|---|---|---|---|---|---|
| **Pre** | 17 | 28 | 21 | 14 | 10 | 17 | 9 | 14 |
| **Post** | 24 | 27 | 21 | 16 | 18 | 19 | 18 | 17 |

| Ques | 5a | 5b | 5c | 5d | 5e | 5f | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **Pre** | 14 | 5 | 10 | 7 | 8 | 19 | 1 | 7 |
| **Post** | 19 | 15 | 12 | 14 | 8 | 19 | 8 | 18 |

Questions 3, 4, 6 and 9 were analysed according to the reasons given for the answers and not the answer, therefore these questions are not included in the table. The explanations written accompanying these numerical answers in the test were analysed to explore the nature of the lived object of learning. This was supplemented by interviews with three students, to whom I've given the pseudonyms Janine, Richard and Refilwe. The results of this analysis are reported below

### 5.4.1 The structure of the number system

In the learner test, the first question asked learners to order numbers in descending order. Seventeen (17) out of the 28 learners ordered the numbers correctly in descending order in the pre-test. From the eight (8) that did not answer correctly, seven (7) of them ordered the positive numbers correctly, and then ordered the absolute values of the negatives in descending order, i.e. 15;4; –18; –5; –3. One learner ignored the negative signs (−18; 15; −5; 4; −3)

Twenty-six (26) out of the 28 learners ordered the numbers correctly in descending order in the post-test. Of the two, that did not, one left the question blank and one ordered correctly in ascending order.

This provides evidence that the *structure of the numbers system* was discerned in the lived object of learning *giving meaning to negative numbers and calculations that include negative numbers*.

In the learner interviews, Janine made a statement that indicated that she understood *the structure of the number system* with negative numbers being below zero.

| | |
|---|---|
| Interviewer: | When Mrs Murray started teaching you about negative numbers she asked you who watched the weather today and what is the minimum temperature? And if I remember correctly, the minimum temperature was negative or minus three degrees Celsius. What does it mean if a numbers got a minus sign in front of it? |
| Janine: | It means it is obviously negative instead of positive and that it is below zero. |

**Excerpt 17: Janine's interview, 0:51 – 01:19**

Refilwe did not respond to questions in the interview by describing the structure of the number system. Richard however had a very good sense of the number system which is discussed in the next section.

### 5.4.2 Addition as increasing and subtraction as decreasing

In question 3 of the test, learners were asked to write one positive and one negative number to make the statements ___ + ___ = 1 and ___ + ___ = –1 true and to motivate their answer. In question 4 of the test, learners were asked to write two negative numbers to make the statements ___ – ___ = 1 and ___ – ___ = –1 true and to motivate their answer. Out of the twenty-eight learners, only one motivated their answer by describing movement along a number line. They explained that –4 + 5 = 1 because, "your answer moves forward 5 times and your answer is 1" and –3 + 2 = –1 because "you move 2 forward and land on –1." Although evidence from the written test does not allow us to conclude that the others did not or could not see addition as increasing and subtraction as decreasing it is clear from this that for almost all the learners this seeing addition and subtraction in that was not at the view that was elicited here. Instead, as I report in 5.4.3 below, the majority of motivations made recourse to the rules.

Richard was interviewed because of his interesting reference to 0. He was the only learner interviewed that showed evidence that he had discerned *the structure of the number system* and *addition as increasing*.

| | |
|---|---|
| Interviewer: | Question number three asks you to write one negative and one positive number to make each of these number sentences correct. So for a) you've said negative one plus negative two is equal to one [–1 + (–2) = 1]. Which one of those is the negative number and which one is the positive number? |
| Richard: | Well, if you look at it, two uh, negative, wait the higher you go the less the value the number is so negative two plus negative one equals which, I know I got that wrong. |
| Interviewer: | Why do you think it is wrong now? |
| Richard: | Well because negative one plus negative two will be negative three. Because, ja, so, you kind of have to say negative one plus positive two to get your answer. Because, my same concept, zero is still a number. So negative one plus one equals zero and you've still got another one which equals one. |

**Excerpt 18: Richard's interview, 1:48 – 2:57**

"The higher you go the less the value the number is" suggests that Richard has a clear concept of the order of integers. Richard expresses this feature again later in the interview:

> Richard: Cause the higher you go from zero downwards the number might still get bigger, but the value gets less.
>
> **Excerpt 19: Richard's interview, 04:26 – 04:33**

Richard also uses 0 as a reference point on the number line. As Richard tries to explain his reasoning that $-1 + (-1) = 1$, he realises that he has made a mistake and changes his answer to $-1 + (+2) = 1$. To get +2, he adds 1 to –1 to get 0 and then adds another 1 to get to 1. So he has found his answer by working out how much –1 has to increase to get to 1.

Evidence from Janine's interview suggests that she did not pay attention to *addition as increasing and subtraction as decreasing*. Janine was asked how she calculated 4 – 2.

> Janine: You look at the positive because that is a positive and the sign of it is positive and because they are different you subtract.
>
> **Excerpt 20: Janine's interview, 02:56 – 03:04**

Despite answering a question that Janine would have done since Grade 1 by taking away or decreasing, she resorts to explaining her method by referring to a rule that has no link to conceptual understanding.

Further evidence that Janine was not focusing on *addition as increasing and subtraction as decreasing* is described below:

> Interviewer: In question number five you had to fill in a missing digit, so for number five a) [–10 + ____ = –7], you said negative ten plus three is equal to negative seven. Can you maybe show me on your number line how you found the three?
>
> Janine: I don't know how to do it with a number line, but I knew like how to do it in my head because I thought that if you have the negative and it equals a negative because that number is smaller than that number, the seven is smaller than the ten …
>
> Interviewer: Okay
>
> Janine: … then um. the other number that was missing obviously has to be a positive. So the difference between so it would be a positive and a negative so you would have to subtract so then you think, okay, so what would I subtract from ten to get seven, and then I got three
>
> **Excerpt 21: Janine's interview, 05:02 – 05:55**

Janine was unable to show that –10 increased by 3 to give –7, but she could manipulate the rules correctly to find 3.

### 5.4.3 The Rules

Rules appeared to dominate the intended and enacted objects of learning. The following discussion showed that the rules also dominated the lived object of learning.

Where learners were asked to motivate their answers in the post-test, thirteen (13) out of twenty-eight (28) motivations explicitly referred to the rules used to add and subtract the numbers. There were only two learners that gave a clear indication that they were using "Motion along an Arithmetic Path". The other learners gave very vague motivations like, "I think it is correct because if you add 7 to −6 you'll get 1". This indicated that many of the learners focused on *mastering the rules that can be used to do calculations that include negative numbers.*

From the learners' interviews three different relationships to the rules emerged. Janine was able to use the rules correctly and fluently. Refilwe could use the rules, but her explanations showed les confidence. She used the phrase "plus is sort of like subtracting" three times in her test and again in her interview. Richard differed from most of the class in that he did not refer to the rules in his explanations in the test or in the interview. He focused on the fact that zero was a number on the number line and used 0 as a reference point for all his calculations.

Janine had answered every one of the questions correctly in her post-test. She also motivated answers exclusively in terms of the rules. In Excerpt 21 above, Janine had described how she has manipulated the rules to find a missing value. In question 9 on the post-test, learners were asked to write down how they would explain to a friend how to work out –2 – (–3). Janine's answer to this question was:

> When two negatives are next to each other they become positive. (–2 + 3 = ?) Because the two signs are different, you take the sign of the higher number and then subtract the two numbers. (The 3 is bigger so your answer should be +)

**Excerpt 22: Janine's post test, answer to question 9**

Refilwe was the third learner interviewed. She also showed evidence of focusing on *mastering the rules that can be used to do calculations that include negative numbers*. She particularly articulated that she had discerned the rules for eliminating brackets. For example, in question 4b) in her post test she explained that –6 – (–5) = –1 because, "a negative times a negative makes a positive". Further evidence that Refilwe had discerned the rule *negative times negative is positive* was given in her interview.

| | |
|---|---|
| Interviewer: | Here [question 7 and 8], you were asked to circle the correct answers, twelve minus negative fifteen and negative twelve minus fifteen. They have different answers. Please explain to me the difference between those two questions. |
| Refilwe: | Well the two negatives in 7), well they would make a positive |
| Interviewer | Okay |
| Refilwe: | So you would say twelve plus fifteen is twenty seven. |

**Excerpt 23: Refilwe's interview, 05:01 – 05:30**

Unusually, there was no evidence in Richard's test or interview that he had used rules. Instead, he used increasing and decreasing along the number line, emphasising that he need to include 0 (See excerpt 18 on page 42).

No evidence was found in either the learner tests or their interviews to suggest that learners had discerned a merge between meaning and rules. This does not mean that the link was not discerned by any learners – just that no evidence was found.

## 5.5 Summary

Two objects of learning were in focus in this sequence of lessons on adding and subtracting negative numbers. These were:

- *Giving meaning to negative numbers and calculations that include negative numbers.*
- *Mastering the rules that can be used to do calculations that include negative numbers.*

Critical features that were made possible to discern when *giving meaning to negative numbers and calculations including negative numbers* was in focus were:

- *The structure of the number system*
- *Addition as increasing and subtraction as decreasing*

Critical features that were made possible to discern when *mastering the rules that can be used to do calculations including negative numbers* were the following rules:

- *When the signs are the same, keep the sign and add the numbers*
- *When the signs are different, keep the sign of the bigger number and subtract the numbers.*
- *Positive × Positive = Positive*
- *Negative × Negative = Positive*
- *Positive × Negative = Negative*

- *Negative × Positive = Negative*

In the intended object of learning there was a focus making the rules possible to discern.
In the lived object of learning, most learners showed evidence that they had discerned the rules. Very few learners showed evidence of discerning *addition as increasing and subtraction as decreasing*.
In lesson 1 the enacted object of learning focussed on making it possible to discern t*he structure of the number system and addition as increasing and subtraction as decreasing*. In lesson 2 and 3 the focus shifted to making it possible for learners to discern the *rules*.

This movement away from meaning to rules and the strong focus on rules in the last two lessons created some contradictions which will be discussed in the following chapter.

# CHAPTER 6 - Discussion

## 6.1 Introduction

The analysis of the sequence of lessons on addition and subtraction of negative numbers showed a dwindling focus on the object of learning from *giving meaning to negative numbers and calculations that include negative numbers* to *mastering the rules that can be used to do calculations that include negative numbers*. This movement away from meaning to rules created some contradictions. Contradictions that became obvious in two areas during the analysis were:

- *The structure of the number system or order of integers*
- *The meaning of arithmetic operations and*
- *The dual meaning of the minus sign (and the plus sign).*

One of the consequences of these contradictions was confusion around the commutative property and it appeared to suggest that subtraction was commutative. The shift in focus also raises an issue about the relationship between *conceptual understanding* and *procedural fluency* (Kilpatrick et al., 2001). Mrs Murray indicated her intention to make it possible for learners to discern a link between adding and subtracting on a number line (meaning) and the rules (Refer to Excerpt 13 on page 39). How this was enacted will be discussed in this chapter.

And, finally, this research was inspired by the 2011 study by Runesson, et al. where they identified four features that were critical for learners in a Swedish classroom to discern in order to learn addition and subtraction of negative numbers. I will compare and discuss the critical features that I identified in this study to the critical features that were identified in their study

## 6.2 Contradictions that arise between the objects of learning

### 6.2.1 The structure of the number line or order of integers

In the analysis of the data, the critical feature of *the structure of the number system* which emerged during the focus on *giving meaning to negative numbers and calculations that include negative numbers* became distorted when the focus shifted to *mastering rules that can be used to do calculations that include negative numbers*. We saw that when Mrs Murray's focus was on the first object of learning (the meaning of negative numbers) she introduced negative numbers to the

learners using the context of temperature (See Excerpt 4 on page 31) and emphasised the order of integers using this context. She then also introduced the "little *ruler"*(or number line) that learners could use as a reference to order a given set of integers.

During these activities learners were focused on the structure of number system and how numbers are ordered. The number line gave them a visual tool to make this possible to discern. However, we see that when Mrs Murray's focus shifts to the other object of learning, (*mastering the rules that can be used to do calculations that include negative numbers*) she focuses on a different kind of ordering. Her rules are phrased in a way that says "Take the sign of the bigger number" when she actually means "take the sign of the number whose absolute value is bigger". Davis (2010) would argue that she has decoupled the signifier from the integer which produces a "whole number".

Mrs Murray is aware of this contradiction. When talking to learners about how to calculate $-6 - 7$ using rules, she says:

> Mrs Murray: Minus thirteen, so the value, it almost seems as though it increases, even though when I told you earlier that negative thirteen was actually a smaller value than negative seven or negative six. But, the actual number, seems a little confusing, seems to increase. Okay, so I am just reinforcing this rule. So, turn your ruler upside down so that you are not looking at it and I want you to do these. Okay, so if the sign is the same you keeping the sign, so in other words, you are going to put your sign there [writes – 11 in on the board] and you are going to add them together. Okay, this one [points to –6 – 7 =] they both negative, you adding them together you keep the sign and you add them together [writes –13 on the board] Got it? Anybody confused? Anybody not happy? You sure? Okay, we will do the others just now, so just write those two columns down and do those so that you are practising that skill really quickly.
>
> **Excerpt24: Lesson 2, 04:33 – 05:33**

In this explanation Mrs Murray tells the learners that it gets confusing because it "seems to increase" even though she had told the learners earlier that "negative thirteen is actually smaller than negative seven". So, when learners are using a number line, the order of integers is clear, but when using the rules, smaller numbers are now bigger numbers.

It is interesting to note that her instructions to "turn over the ruler" and "practise" prioritises the rule method and suggests leaving the number line picture behind. This is perhaps indicative of the difference between the meaning of "bigger number" depending on what object of learning is in

focus. Mrs Murray's actions seem to suggest that it is better to leave behind the image of the number line when the focus switches to the rules for adding and subtracting integers.

The order of integers is also problematic because the plus and minus signs are seen exclusively as *structural signifiers* so a sign that would indicate an operation is used to indicate the magnitude of a number. This will be discussed further in the next section.

Mrs Murray uses the *ruler* or number line to give meaning to the order of integers as well as a model for "Arithmetic Motion Along a Path" (Kilhamn, 2009). In other words, her focus is on subtraction as taking away. Runesson et al. (2011) use the number line to focus on subtraction as difference as well. They consider *subtraction could be both seen as 'a take away' and as a 'difference' between numbers* as a critical feature for teaching and learning negative numbers.

### 6.2.2 The meaning of arithmetic operations

Gallardo and Rojano (1994) describe three uses in subtraction, *taking away*, *completing* and the *difference between two numbers*. Galbraith (1974) blames the difficulty with negative numbers on this strong association with take away. Mrs Murray used the *ruler* to model subtraction as take-away, but in one episode she described subtraction as *difference,* but used it as *completing*. Mrs Murray had not indicated that this was a critical feature in the intended object of learning. However, it did feature in the first lesson when she was discussing temperature. Mrs Murray asks the learners what the difference is between −3 and 16.

Mrs Murray: Now, I want to know who is clever enough. What is the difference between these two temperatures? What is the difference? *Learners are mumbling* No, not the English difference, I don't want an explanation on the difference between a girl and a boy and a negative and positive temperature I want to know the difference. What does difference in Maths mean to you? What are we going to do?

*Learners mumble responses*

Mrs Murray: We are going to subtract. So what is the difference between those two numbers?

Learner: Twenty degrees

Mrs Murray: Twenty.

Learner: Could be.

*Other learners call out nineteen degrees*

Mrs Murray: Okay, what is the difference between those?

Janine: Nineteen

Mrs Millar: Nineteen degrees. Well done. So, negative three, okay, the temperature climbs all the way to sixteen degrees so it got warmer by nineteen degrees. Let's look at our ruler now. Okay, have a look at the ruler and put your finger on minus three. Okay, and I want you to count the degrees as it got warmer. So, at five o'clock this morning it was negative three degrees and then it got warmer, it went to negative two, negative one, is zero a degree?

Learners: Yes

Mrs Millar: Yes. It is a number. You got to count it in. And go hop from number to number until you get to sixteen and tell me how many times you hopped.

*Mumbles of eighteen and nineteen*

Mrs Millar: Don't guess. Do the hopping. Maths is not a guessing game.

Learner: Nineteen

Mrs Millar: Nineteen. Well done. Did everyone hop and see? Okay, alright. That is just one area where you can see negative numbers as well as positive numbers because negative numbers are actually very difficult to do something in a concrete way. You know whenever I introduce something new to you you get Diene's blocks and you get things to play with and build with and do things physically so that you can actually visualise what you are doing, but negative numbers are not as easy – so it is very easy for me to use temperature, especially now that it's cold to introduce it to you. Okay, now we are going to hold the thought on saying negative three minus sixteen or getting an answer that we actually get the answer at the end of the day.

**Excerpt 25: Lesson 1, 12:22 – 14:36**

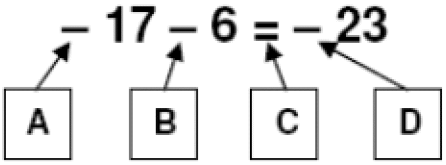
Mrs Murray explains that difference in Maths means subtract. The learners find that the answer is nineteen. The method used to explain this answer is by checking on the number line. Learners start and −3 and then count up nineteen to check that they get to 16. There is no reference to the distance between −3 and 16. Mrs Murray does not formalise this meaning by discussing whether the difference between −3 and 16 was −3 – 16 or 16 – (−3). Later, she tells the learners to "hold the thought on saying negative three minus sixteen". −3 – 16 is negative nineteen, not nineteen as the learners answered. What has not been made possible to discern here is that *subtraction is always from the position of perspective*. This was a critical feature identified by the Swedish teachers.

### 6.2.3 The dual meaning of the minus sign

The research literature suggests that it is important to recognise the different functions of the minus sign. When the focus is on *giving meaning to negative number and calculations that include negative number*, the learners are working with a number line. In this method the structural and operational signifiers are distinguished, but when the focus shifts to *mastering rules that can be*

*used to do calculations that include negative number* all minus signs are seen as structural signifiers. The following discussion will explain this contradiction.

Consider this question from the learner test.




When the objective of learning in focus is *meaning of negative numbers and calculations that include negative numbers* learners would use a number line to work this question out. 'A' indicates that 17 is negative (*structural*) and 'B' indicates that learners should move left (*operational*) 6 places on the number line.

However, when the object of learning in focus is *mastering rules that can be used to do calculations that include negative numbers* the rule that learners would use to answer this question is: "When the signs are the same, keep the sign and add the numbers". From this explanation, 'B' is given the same meaning as 'A' i.e. they are both *structural signifiers*. The operation that the rule prescribes for this type of calculation (–a – b) is addition, but there is no *plus* sign in this question. This scenario indicates a breakdown in the mathematical meaning of this calculation. Similarly for calculations of the type –a + b, the plus sign is an *operational signifier*, but if the learners use the rule "when the signs are different, keep the sign and subtract the numbers" the plus sign is used an a *structural signifier*.

When the rules are in focus, all signs are given a unary function or behave as *structural signifiers* and the operation disappears. The examples I have given illustrate how one rule tells you to add, when the operation tells you to subtract and another rule tells you to subtract, when the operation tells you to add.

The learners bring into this space of learning questions about the function of the minus sign and whether a sign is a *structural* or *operational signifier*. In the second lesson, a learner asked Mrs Murray, "How do you know if this is a minus or a negative?" Mrs Murray reflected on this question more in her third interview.

| | |
|---|---|
| Interviewer: | What do you think she was trying to understand when she asked that question? |
| Mrs Murray: | I think she was trying to understand where the sign belonged |

> because in the past they always had a number minus another number and now what we are doing is taking that minus sign and we changing their thinking to a negative sign, although it can still be a minus sign and so, she wanted to know whether that minus sign or negative sign belongs to that number and that is what she was trying to get at. You know, when you are talking about negatives, what belongs to the number and where do you pick up the negative sign? What is that meaning of that negative sign because there can be a dual meaning depending on what you know from their past experiences to where they going.
>
> **Excerpt 26: Teacher Interview 3, 00:36 – 01:31**

Mrs Murray describes "taking that minus sign and changing their thinking to a negative sign, although it can still be a minus sign …" Her explanation indicates that she is aware that the minus sign has other meanings, but quite purposefully talks about changing the learners thinking. Changing learners' thinking to recognise all signs as *structural signifiers* is necessary in order for the rules to work.

The teacher described in the research by Davis (2010) used these same rules. Davis argues that the teacher and learners have not operated on integers, but on the more familiar whole numbers". In a similar fashion, Mrs Murray uses numbers as integers when she is focused on *giving meaning to negative numbers and calculations that include negative numbers*, but when she focuses on *mastering the rules that can be used to do calculations that include negative numbers* the signs are detached from the integers leaving "whole numbers" and signs that only have a unary function.

These rules were not a critical feature to the teachers in Runesson et al's study. From the students' pre-test there was evidence that the students could answer problems using the rule 'two minus signs make a plus' and no understanding of addition and subtraction with negative numbers. The Swedish teachers "wanted to teach for understanding, not just to get the students to come up with the correct answer." (Runesson et al., 2011, p. 267)

By analysing a pre-test, the Swedish teachers realised that the different meanings of the operational sign for subtraction and the 'minus' sign for a negative number was probably confusing for their students (Runesson et al., 2001). They chose to vary examples like '$8 - 3 = 5$ and $3 - 8 = -5$ to make it possible for their students to discern the different meanings of the sign for positive or a negative, but still keep the sign for the operation the same.

## 6.3 Confusions around the commutative property

One of the implications of the unary function dominating when the focus is on *mastering the rules that can be used to do calculations that include negative numbers* is that the commutative property of operations becomes blurred. Addition is commutative, whereas subtraction is not commutative. However, if the rules are used and signs are seen as *structural* signifiers, then all calculations could be turned into addition sums, for example −3 – 5 = (−3) + (−5) and +3 – 5 = (+3) + (−5). This makes it appear as if subtraction is also commutative when dealing with negative numbers.

Mrs Murray explained that the subtraction calculations could be done in any order and she encouraged learners to take the 'smaller number' away from the 'bigger number'. This feature was introduced when a learner asked, "Doesn't the bigger number have to be in front?" The class had just calculated –5 – 3 on their *rulers* by hopping left of −5. Mrs Murray responds that it does not matter and then uses the ruler to show learners that −3 + 5 = +5 − 3 and −5 − 3 = − 3 – 5.

In the case of −3 + 5 = +5 − 3, the binary function of the plus sign is changed to a unary function when the order of operations is reversed, i.e. *operational signifier* becomes a *structural signifier*. This is not as obvious in the second case (−5 – 3 = −3 – 5). The equality in each pair of calculations can best be explained by using the commutative property of addition. In other words, by maintaining the binary function, this could be explained as: (−3) + (+5) = (+5) + (−3) and (−5) + (−3) = (−3) + (−5).

A problem arises when a minus sign maintains its binary function. Take, for example, −5 − 3. If the minus sign is used as subtraction and the order of numbers changes, then (−5) – (+3) changes to (+3) – (−5). But, (−5) – (+3) ≠ (+3) – (−5), which is true since subtraction in not commutative. Clearly, there is a mathematical contradiction depending on whether signs are used with a unary function or with a binary function.

Mrs Murray encourages learners to change the perspective of the calculation and put the bigger number first, "ignoring the signs". For example, in Lesson 1 a learner asked for help with calculating –3 + 8.

| | |
|---|---|
| Richard: | Mumbles |
| Mrs Murray: | Is it positive? Ja |
| Richard | And then its minus three |
| Mrs Murray: | So, it almost becomes like saying eight minus three. Remember earlier, I said it does not matter which way you put the numbers. So, it is almost like saying positive eight minus three. Okay, which will |

| | |
|---|---|
| | not be negative five, would it? |
| Richard: | No |
| Mrs Murray: | Okay, what would it be? |
| Richard: | A normal five |
| Mrs Murray: | Yes normal five or positive five. Put the positive sign in front. Okay. |

**Excerpt 27: Lesson 1, 01:57 – 02:34**

In this explanation Mrs Murray has explained that –3 + 8 is "almost like" 8 – 3. She has reinforced that order doesn't matter. In this case it works because (−3) + (+8) = (+8) + (−3) and addition is commutative.

Following this Mrs Murray describes how to calculate –13 + 6.

| | |
|---|---|
| Mrs Murray: | Minus thirteen plus six. The signs are different. The larger number is thirteen; the sign is negative so I put my negative sign down. Thirteen minus six is seven. Minus seven. |

**Excerpt 28: Lesson 1, 06:08 – 06:18**

Although the calculation is −13 + 6, Mrs Murray says that "thirteen minus six is seven", but there is no equality between −13 + 6 and 13 – 6. This change in perspective makes no mathematical sense. Mrs Murray accommodates this by immediately saying "minus seven". This is because 13 is the 'larger number', so the sign is negative. Again, there is confusion about the order of integers. By ignoring signs 13 is bigger than 6, but in actual fact –13 is smaller than 6.

The reasoning behind why the order could be changed did not come through in these lessons. In the learner test, they were asked to find one negative number and one positive number to make this number sentence true: ____ + ____ = 1. In the test, Refilwe answered that −5 + 6 = 1 and she motivated her answer by explaining "When you plus it is sort of like subtracting with positive answers". She was asked to explain what she meant in her interview.

| | |
|---|---|
| Refilwe: | Well, like I was saying six minus five is equal to one. |
| Interviewer: | Okay so instead of negative five plus six, you see it as six minus five? |
| Refilwe | Ja |
| Interviewer: | The negative five plus six sorry, negative five plus six is the same as six minus five? |
| Refilwe | Yes. |
| Interviewer: | Why? |
| Refilwe: | … I'm not sure |

**Excerpt 29: Refilwe's interview, 01:13 – 01:47**

Refilwe's response suggests that for her the order did not matter and $-5 + 6$ was the same as $6 - 5$. However the fact she could not explain why this was so suggests she was following a procedure, but without understanding.

This discussion has focused on the contradictions that resulted from a shift in focus from *giving meaning to negative numbers and calculations that include negative numbers* to *mastering the rules that can be used to do calculations that include negative numbers*. The structure of the number system and the meaning of the number system were considered within both of these objects of learning and evidence of how these critical features were discerned in the intended, enacted and lived objects of learning was discussed. The exclusive use of the unary function of the minus (and plus) sign when *mastering the rules that can be used to do calculations that include negative numbers* lead to a breakdown in the absence of the commutative property in subtraction.

Mrs Murray did attempt to link these two objects of learning as indicated in her interview. How this intended object of learning was enacted will now be discussed.

## 6.4 Finding a link between the objects of learning

Mrs Murray expressed her intention to make it possible for her learners to discern the relationship between using the ruler and using the rules to calculate addition and subtraction of negative numbers. Mrs Murray says that, "they actually need to see a lot of examples with their ruler where both are negative and have a look at the answer being bigger and you are adding them together".

This intention suggest that Mrs Murray intends to focus on *giving meaning to negative numbers and calculations including negative numbers.* Learners need to see that when subtracting (a positive number) from a negative number, the answer decreases or moves left on the number line. The result is a bigger *absolute value* (or as Mrs Murray says, "ignoring signs"), and hence the rule is to "add". See Excerpt 13 on page 39

The contradiction between the order of integers seen on the number line and the order of integers when using these rules has already been discussed. A possible reason could be that it would be true to say that when you subtract (a positive) number from a negative number, the *absolute value* increases, but Grade 7's are not exposed to absolute value so Mrs Murray cannot use this language in her lesson.

There was no evidence in either learner tests or interviews that the learners had made the link between *meaning of negative numbers and calculations that include negative numbers* and *mastering rules that can be used to do calculations that include negative numbers*. I am not suggesting that no learner made this link, only that there was no evidence to support it. The link between the two objects of learning can be interrogated further using Kilpatrick et al.s' (2001) five strands of mathematical proficiency.

## 6.5 Conceptual understanding vs. Procedural fluency

The two objects of learning align with conceptual understanding and procedural fluency. *Mastering the rules that can be used to do calculations that include negative numbers* improves learners' procedural fluency. When the rules are used correctly, they do allow learners to find an answer to a calculation quickly and accurately. *Giving meaning to negative numbers and calculations that include negative numbers* allows learners to gain conceptual understanding.

The previous discussion explains that Mrs Murray was aware of this and she intended to make it possible for her learners to discern the link by seeing more and more examples. However despite the intention, we see in the enacted object of learning that the emphasis in procedural fluency made it very difficult for the learners to make this the link. In fact, it produced ideas that contradicted what was conceptual understanding.

The strong focus on *procedural fluency* almost dissolved the other strands. Kilpatrick et al. (2001) warn that in school mathematics *procedural fluency* and *conceptual understanding* often compete for attention. Venkat & Adler (2012) also express concern that the magic of rules indicates inattention to the representations that are being operated on and frequently produce incorrect answers.

## 6.6 Comparison with Davis' study

The description of the teacher's interaction with his learners in Davis (2010) provides evidence that this teacher focussed on *mastering the rules that can be used to do calculations that include negative numbers*. There is no evidence given that there was any focus on *giving meaning to*

*negative numbers and calculations that include negative numbers.* It is then possible, that in Davis' (2010) study, the contradictions that arose from focusing on two objects of learning did not arise and procedural fluency was the priority in this teacher and learner exchange. The findings of my study and Davis' (2010) raise an interesting question in relation to the link between procedural fluency and conceptual understanding. The "rules without reasons" approach to negative numbers (which one could argue Davis' teacher proposes) is internally consistent albeit limited. In contrast the attempt by the teacher in my study to provide a conceptual basis for negative numbers alongside the set of procedural rules produces contradictions.

## 6.7 Summary

The focus on two objects of learning during the teaching and learning of negative numbers in this sequence of lessons caused some contradictions and tensions which have been discussed in this chapter. Although the teacher paid some attention to merging these two objects of learning, however the emphasis on rules made this merge difficult to conceptualise.

The two objects of learning align themselves with a history of previous research and discussion on conceptual understanding and procedural fluency.

# CHAPTER 7 - Conclusion

The intention of this research report was to analyse teaching and learning addition and subtraction with negative numbers by providing a rich description of the experiences of a Grade 7 Mathematics teacher and her learners during a sequence of lessons on addition and subtraction of negative numbers. Critical features of teaching addition and subtraction in the intended, enacted and lived object of learning in a South African classroom were explored. Also, two objects of learning were working alongside each other which created real difficulties and contradictions.

The critical features identified in the intended and enacted object of learning in this case study of a teacher and a group of Grade 7 learners in South Africa were:

1. *The structure of the number system*
2. *Addition (of positive numbers) as increasing and subtraction (of positive numbers) as decreasing.*
3. When the signs are the same, keep the sign and add the numbers.
4. When the signs are different, use the sign of the bigger number and subtract the numbers.
5. Positive × Positive = Positive
6. Negative × Negative = Positive
7. Positive × Negative = Negative, and
8. Negative × Positive = Negative.

These critical features were very different from the ones identified by the teachers in the Learning Study conducted in Sweden by Runesson, Kullberg and Maunula (2011). Davis' (2010) study provides evidence that the use of rules is not unique to Mrs Murray. It is possible, but not researched, that many South African teachers use rules similar to these to teach learners how to do calculations that include negative numbers. Unfortunately, time and scope restrictions in this research meant that I could not engage in a Learning Study, but this analysis has thrown up fascinating information about the tensions of having two objects of learning.

The structure of the number system changed depending on what object of learning was in focus. When *meaning of negative numbers and calculations that include negative numbers* was in focus

then −6 was less than 2, but when *mastering rules that are used to do calculations that include negative numbers* was in focus, then 2 was less than −6. This happened because the sign was stripped away from the number so that "counting numbers" were in focus as opposed to integers. When the focus was on *mastering rules that are used to do calculations that include negative numbers* the binary function of the minus sign was ignored and subtraction appeared to be commutative.

The literature reviewed shows that models and metaphors for operating on negative numbers have limitations and this proves a challenge for teaching. It is not surprising then that Mrs Murray abandoned the *ruler* (*Motion Along an Arithmetic Path)* for rules and struggled to connect them in a meaningful way. Some of the challenge then in teaching negative numbers is to think of how a transition from the more concrete representations could work.

The evidence of this analysis would be a powerful starting point for a discussion about what one might do differently if one really did want to weave together the strands of procedural fluency and conceptual understanding. The literature on models, the critical features identified in this study and the discussion on conceptual understanding and procedural fluency provide an excellent foundation on which to create a new Learning Study in a South African classroom. A researcher, working with a teacher, could share some of the previous research as a starting point to identify a critical feature for teaching negative numbers. They could then draw on some of the metaphors and models to consider how discerning this critical feature may be possible to discern. By reflecting on this process together, researcher and teacher may identify other critical features for teaching negative numbers. By sharing these possible critical features with other teachers and the ways of making these critical features possible to discern may make teaching and learning negative numbers a POSITIVE experience.

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## *APPENDIX 1 – Learner Test*

### Test: Adding and Subtracting Negative Numbers

Name: _____

1. Write the numbers in order of size. Start with the largest number.

   – 3 15 – 5 4 – 18

   _____

2. Calculate the answers:

   a) 4 – 2 = _____ b) 2 – 4 = _____

   c) 4 + (– 2) = _____ d) – 4 + 2 = _____

   e) – 4 + (– 2) = _____ f) – 4 – 2 = _____

   g) – 4 – (– 2) = _____

3. Write **one negative** and **one positive** number to make each of these number sentences correct.

   a) ____ + ____ = 1

   Motivate your answer: _____

   b) ____ + ____ = – 1

   Motivate your answer: _____

4. Write **two negative** numbers to make these number sentences correct.

   a) ____ – ____ = 1

   Motivate your answer: _____

   _____

   b) ____ – ____ = – 1

   Motivate your answer: _____

   _____

5. Fill in the missing value that makes each of the following number sentences true.

a) – 10 + _____ = – 7

b) _____ – 1 = – 5

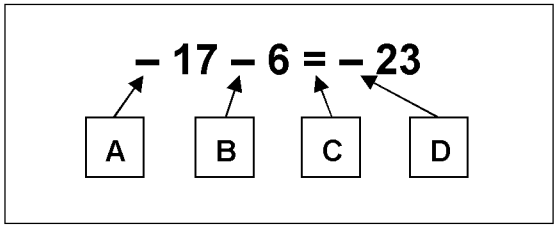
c) _____ + (– 5) = 9

d) _____ – (– 2) = – 4

e) 1 – _____ = 2

f) – 6+ _____= – 8

6. Explain what each of the signs in this number sentence mean.




**A:** _____

**B:** _____

**C:** _____

**D:** _____

7. Circle the correct answer, or the correct answers:
12 – (– 15) =_____

A) 3 B) – 3 C) 27 D) – 27

8. Circle the correct answer, or the correct answers:
– 12 – 15 = _____

A) 3 B) – 3 C) 27 D) – 27

9. Write down how you would explain to a friend how to work out: – 2 – (–3)?

_____

_____

_____

_____

## *APPENDIX 2 – Letter to Teacher*

June 2012

Dear Teacher,

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand and am doing research on mathematics education.

My research topicis: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

I am interested in hearing about your and your Grade 7's experiences in a sequence of lessons about adding and subtracting negative numbers.

I was wondering if you would mind if I interviewed you prior to the first lesson on addition and subtraction of negative numbers about your expectations for the lesson. Following this interview, I would like to sit as an observer in your lessons with a Grade 7 class whileyou are teaching addition and subtraction of negative numbers. This means that I will watch your lessons, but not interfere in your teaching in any way. I would appreciate observing one class' lessons over the period in which you teach them addition and subtraction of negative numbers (possibly two or three days). During the observation of lessons, I will be looking at how your learners' needs influence the expectations you had of the lesson. In order to collect all this information, I would appreciate using a video camera to record the lessons.

I would also like to interview you each day of this sequence of lessons to reflect upon the previous lesson and hear your expectations of the lessons to follow. There would be three or four interviews in total and each interview should take no more than 20 minutes. I would like to audio-record these interviews.

Before and after this sequence of lessons, I would like these Grade 7 learners to write a 30 minute pre- and post-test. The pre-test will be used to create a description of learners' previous experience of adding and subtracting negative numbers. A summary of the results of this pre-test will be made available to you before the sequence of lessons commences. The post-test will be used to analyse the learners' understanding of the features of adding and subtracting negative numbers and the methods they use to solve these problems. The test is attached at the end of this information sheet. A summary of the results of the post-test will be made available to you. Permission to conduct these tests will be requested from the learners and their parents

Lastly, I would like to interview three of these learners after the sequence of lessons in order to collect more information about their thinking in answering the post-test questions. Permission to conduct these interviews will be requested from the learners and their parents. Each interview should take no more than 30 minutes and will take place at a time that does not inconvenience their normal school day. I would like to audio-record these interviews.

The video of the observed lessons, the audio-tapes of interviews and the learners' pre- and post-tests will only be observed by me and possibly my supervising lecturer. They will not be shared with colleagues, learners or principals. Once this research is complete, all of this data will be destroyed.

When writing my research report, I will not use your name, the school's name or any of your learners' names. This is so that no one who reads the project knows your identity. I do not expect any risks to you as a result of your participation in the study. You will not get paid for participating. Any information relating to the study will have no impact on your personal job reviews or any other assessment.

I would like to invite you to participate in this study. If you are willing to participate in this study, please would you fill in and sign the consent form on the following page. You are free to withdraw permission for data to be collected or used for research at any stage along the way without any impact on you.

I am willing to provide you and your colleagues with feedback from this research. Please do not hesitate to contact me if you require further detail or clarification.

Best wishes,



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## TEACHER CONSENT FORM 1

Please complete, sign and return the form below indicating whether you do or do not agree to participate.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7**

Researcher**: Kerryn Vollmer**
Date:2012
Unversity of the Witwatersrand

Please circle appropriate selection below:

I **give / do not give** permission for **data** (non video-based) of myself collected within Arbor Primary School for the study listed above to be used for research purposes. This will include audio recorded data collected during the classroom sessions.

I **give / do not give** permission for myself to be interviewed prior to observed lessons and at the conclusion of each observed lesson and for my responses to be audio recorded for research purposes.

Teacher name: _____

Teacher signature: _____

Date: _____

## TEACHER CONSENT FORM 2

Please complete, sign and return the form below indicating whether you do or do not agree to participate.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Date: 2012

Unversity of the Witwatersrand

Please circle appropriate selection below:

I **give / do not give** permission for **video data** of myself conducting classroom lessons collected within Arbor Primary School for the study listed above to be used for research purposes.

Teacher name: _____

Teacher signature: _____

Date: _____

## *APPENDIX 3 – Letter to learners re: test and observation*

June 2012

Dear learner,

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand.

I am doing research on: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

I am trying to find out more about learners' experiences in a sequence of lessons about adding and subtracting negative numbers.

I was wondering whether you would mind if I sit in your class, just to watch for two or three days. I would like to video record these lessons on addition and subtraction of negative numbers.

I need your help with seeing how well you understand adding and subtracting negative numbers. To do this, I would like to ask you to write two short tests. The first one will be before your teacher teaches you about adding and subtracting negative numbers. Thesecond one will be after you have been taught about adding and subtracting negative numbers. These tests will be written in class time. Your tests will not be used by your teacher for marks.

All the information that I collect about you (the video of your lessons and your tests) will only be seen by me and my supervising lecturer. I will store this information safely and will destroy it 3 years after I have completed my studies. I will not be using your name, your teacher's name or your school's name. So no one can identify you, and all information about you will be kept confidential in all my writing about the study.

If you agree to help me, you will be free to change your mind at any stage and withdraw from the study. This will not have any impact on you, your teacher or your school.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join me in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you



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**LEARNER CONSENT FORM 1**

Please complete, sign and return the form below indicating whether you do or do not agree to participate. Also please state if you are 18 years or older.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date:2012

Unversity of the Witwatersrand

Please circle appropriate selection below

I **give / do not give** permission for **data** (non video-based) of myself collected within Arbor Primary School for the study listed above to be used for research purposes. This will include audio-recorded data collected during the classroom lessons.

I **give / do not give** permission to be tested in order for **data** to be used for research purposes.

I am 18 years or older [ ] Yes
[ ] No

Learner name: _____

Learner signature: _____

Date: _____

## LEARNER CONSENT FORM 2

Please complete, sign and return the form below indicating whether you do or do not agree to participate. Also please state if you are 18 years or older.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date:2012

Unversity of the Witwatersrand

Please circle appropriate selection below

I **give / do not give** permission for **video data** of myself collected within Arbor Primary School for the study listed above to be used for research purposes.

I am 18 years or older [ ] Yes
[ ] No

Learner name: _____

Learner signature: _____

Date: _____

## *APPENDIX 4 - Letter to parents re: test and observation*

June 2012

Dear Parent / Guardian

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand and doing research on mathematics education.

My research topicis: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

I am interested in hearing about learners' and teacher's experiences of lessons on addition and subtraction of negative numbers. I have chosen a Grade 7 class because this is the grade where negative numbers are first introduced to learners and is therefore critical to their further understanding of this topic.

I was wondering if you would mind me sitting in your child's Maths class, to observe these lessons. It is likely to take two or three days for the teacher to teach this topic. I would like to video-record these lessons.

I would like to request that your child writes two tests, one before this topic is taught and one after the topic has been taught. These tests will be written in class time. These tests will not be used by the teacher to calculate any school marks.

All the information that I collect about your child (the video of their lessons and their tests) will only be seen by me and my supervising lecturer. I will store this information safely and will destroy it 3 years after I have completed my studies. Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

Your child will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in their participating and your child will not be paid for this study.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

KLVollme

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(Office) 011 717 3464
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## PARENT/GAURDIAN CONSENT FORM 1

Please complete, sign and return the form below indicating whether you do or do not agree to allow your child to participate.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date:2012

Unversity of the Witwatersrand

Please circle appropriate selection below:

I **give / do not give** permission for **data** (non video-based) of my child collected within Arbor Primary School for the study listed above to be used for research purposes. This will include audio recorded data collected during the classroom lessons.

I **give / do not give** permission for my child to be tested in order to obtain **data** to be used for research purposes.

Learner name: _____

Parent Name: _____

Parent signature: _____

Date: _____

## PARENT/GAURDIAN CONSENT FORM 2

Please complete, sign and return the form below indicating whether you do or do not agree to allow your child to participate.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date: 2012

Unversity of the Witwatersrand

Please circle appropriate selection below:

I **give / do not give** permission for **video data** of my child collected within Arbor Primary School for the study listed above to be used for research purposes.

Learner name: _____

Parent Name: _____

Parent signature: _____

Date: _____

## *APPENDIX 5 – Letter to learners re: interview*

August 2012

Dear learner,

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand.

I am doing research on: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

I am trying to find out more about learners' experiences in a sequence of lessons about adding and subtracting negative numbers.

Thank you for your helping me with my research.

I am hoping I could ask you for some more help. I would like to invite you to participate in an interview to explain more to me *how* you worked out some of your answers in your test. This interview would take place during the week of 13 August at a time convenient to you in Mrs Millar's classroom. Please see the attached reply slip to choose a time that best suits you. I would like to audio-record this interview.

Only my supervising lecturer and I will listen to this recording. As with the other information I have collected, I will also store this safely and will destroy it 3 years after I have completed by studies. I will still not be using your name, your teacher's name or your school's name. So no one can identify you, and all information about you will be kept confidential in all my writing about the study.

If you agree to help me again, you will be free to change your mind at any stage and withdraw from the study. This will not have any impact on you, your teacher or your school.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you

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(Office) 011 717 3464
(Cell) 084 926 9812

Please complete, sign and return the form below indicating whether you do or do not agree to participate. Also please state if you are 18 years or older.

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date:2012

Unversity of the Witwatersrand

Please circle appropriate selection below

I **give / do not give** permission for **data** (non video-based) of myself collected within Arbor Primary School for the study listed above to be used for research purposes. This will include audio-recorded data collected during the classroom lessons.

I **give / do not give** permission for myself to be interviewed for no longer than 30 minutes after school.

I am 18 years or older [ ] Yes
[ ] No

Please tick a day and time that would be most suitable to you:

| | | |
|---|---|---|
| Monday 13 August | 2:00 – 2:30 | |
| Monday 13 August | 2:30 – 3:00 | |
| Tuesday 14 August | 2:00 – 2:30 | |
| Tuesday 14 August | 2:30 – 3:00 | |
| Wednesday 15 August | 2:00 – 2:30 | |
| Wednesday 15 August | 2:30 – 3:00 | |
| Thursday 16 August | 2:00 – 2:30 | |
| Thursday 16 August | 2:30 – 3:00 | |
| Friday 17 August | 2:00 – 2:30 | |

Please feel free to suggest another time: _____

Learner name: _____

Learner signature: _____

Date: _____

## *APPENDIX 6 – Letter to parents re: interview*

August 2012

Dear Parent / Guardian

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand and doing research on mathematics education.

My research topicis: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

I am interested in hearing about learners' and teacher's experiences of lessons on addition and subtraction of negative numbers.

Thank you for allowing me to observe your child's lessons on adding and subtracting negative numbers and for allowing them to write a test to assist me with my research.

Please could I interview your child to get a deeper understanding of their thinking in the test that they wrote.This interview would take place during the week of 13 August at a time convenient to you and your child in Mrs Millar's classroom. Please see the attached reply slip to choose a time that best suits you. I would like to audio-record this interview.

Only my supervising lecturer and I will listen to this recording. As with the other information, I will store this safely and will destroy it 3 years after I have completed by studies. Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

Your child will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in their participating and your child will not be paid for this study.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

Kerryn Vollmer
RADMASTE
Wits School of Education
emailkerryn.vollmer@wits.ac.za
(Home) 011 849 8641
(Office) 011 717 3464
(Cell) 084 926 9812

Please complete, sign and return the form below to Mrs Millar before Wednesday 8 August

Reasearch project: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.**

Researcher**: Kerryn Vollmer**

Date:2012

Unversity of the Witwatersrand

Please circle appropriate selection below:

I **give / do not give** permission for **data** (non video-based) of my child collected within Arbor Primary School for the study listed above to be used for research purposes. This will include audio recorded data collected during the classroom lessons.

I **give / do not give** permission for my child to be interviewed for no longer than an 30 minutes after school.

Please tick a day and time that would be most suitable to you:

| | | |
|---|---|---|
| Monday 13 August | 2:00 – 2:30 | |
| Monday 13 August | 2:30 – 3:00 | |
| Tuesday 14 August | 2:00 – 2:30 | |
| Tuesday 14 August | 2:30 – 3:00 | |
| Wednesday 15 August | 2:00 – 2:30 | |
| Wednesday 15 August | 2:30 – 3:00 | |
| Thursday 16 August | 2:00 – 2:30 | |
| Thursday 16 August | 2:30 – 3:00 | |
| Friday 17 August | 2:00 – 2:30 | |

Please feel free to suggest another time: _____

Learner name: _____

Parent Name: _____

Parent signature: _____

Date: _____

## *APPENDIX 7 – Letter to school principal*

June 2012

Dear Principal

This letter serves to confirm our previous discussion regarding my invitation for your school's participation in my proposed research study.

My name is Kerryn Vollmer and I am currently studying for a master's degree at the University of Witwatersrand and am doing research on mathematics education.

My research topicis: **Why so negative about negatives? The intended, enacted and lived object of learning negative numbers in Grade 7.** I am interested in hearing about a Grade 7 Mathematics teacher and her Grade 7 learners' experiences on a sequence of lessons about adding and subtracting negative numbers.

I would like to observe a Grade 7 Mathematics teacher for the period in which they teach addition and subtraction of negative numbers (two to three days). I will in no way interfere with or assist them during these lessons. I do not anticipate any disruption to the normal school practices.

I would like to collect the following data in your school.

- observation of a Grade 7 Mathematics teacher teaching addition and subtraction of negative numbers during normal school hours
- video-recording a Grade 7 teacher teaching addition and subtraction of negative numbers during normal school hours
- three or four 20 minute interviews with the teacher at the end of the school day.
- results of a pre-test and post-test written by the Grade 7 learners prior to and following these lessons
- three 30 minute interviews with three selected learners from this class after school at the completion of the observations and testing.

I undertake to maintain anonymity and confidentiality of the participants and the school in all academic writing about the study. After three to five years all data related to the study will be destroyed.

There are no foreseeable risks in participating in the study. The participants will not get paid for participating. Any information relating to the study will have no impact on learners' school marks or other assessments and/or the teachers' job reviews or other assessments.

You are free to withdraw permission for data to be collected or used for research at any stage along the way without any impact. I will be willing to share interesting findings with you and your Mathematics Department.

Please do not hesitate to contact me if you require further details or clarification.

Best wishes,



Kerryn Vollmer
RADMASTE
Wits School of Education
kerryn.vollmer@wits.ac.za
(Home) 011 849 8641
(Office) 011 717 3464
(Cell) 084 926 9812

## *APPENDIX 8 – Ethics Clearance from GDE*

education
Department: Education
GAUTENG PROVINCE

For administrative use:
Reference no. D2012/193

# GDE RESEARCH APPROVAL LETTER

| | |
|---|---|
| Date: | 12 December 2011 |
| Validity of research Approval: | 6 February 2012 to 30 September 2012 |
| Name of Researcher: | Vollmer K.L. |
| Address of Researcher: | P.O. Box 15655 |
| | Farramere |
| | 1518 |
| Telephone Number: | 011 849 8641 / 084 926 9812 |
| Email address: | kerrynleighv@yahoo.co.uk |
| Research Topic: | A study of teacher's selection and use of examples to encourage Grade 8 learners' understanding of operating on integers |
| Number and type of schools: | ONE Secondary School |
| District/s/HO | Johannesburg South |

***Re: Approval in Respect of Request to Conduct Research***

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

**The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:**

1. *The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.*
2. *The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.*
3. *A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.* 1

*Making education a societal priority*

**Office of the Director: Knowledge Management and Research**
9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

4. *A letter / document that outlines the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.*
5. *The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.*
6. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
7. *Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year.*
8. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
9. *It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.*
10. *The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.*
11. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.*
12. *On completion of the study the researcher must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.*
13. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.*
14. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

Dr David Makhado

Director: Knowledge Management and Research

2011 / 12 / 12



*Making education a societal priority*

**Office of the Director: Knowledge Management and Research**

9th Floor, 111 Commissioner Street, Johannesburg, 2001
P.O. Box 7710, Johannesburg, 2000 Tel: (011) 355 0506
Email: David.Makhado@gauteng.gov.za
Website: www.education.gpg.gov.za

## *APPENDIX 9 – Ethics Clearance from Wits School of Education*

Wits School of Education

27 St Andrews Road, Parktown, Johannesburg, 2193 • Private Bag 3, Wits 2050, South Africa
Tel: +27 11 717-3064 • Fax: +27 11 717-3100 • E-mail: enquiries@educ.wits.ac.za • Website: www.wits

Student number: 9507621K

25 January 2012

Ms. Kerryn Leigh Vollmer
Kerryn.vollmer@wits.ac.za

Dear Ms. Vollmer

**Re: Application for Ethics: Master of Science**

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

**A study of teacher's selection and use of examples to encourage grade 8 learners' understanding of operating on integers**

The committee recently met and I am pleased to inform you that clearance was granted. The committee was delighted about the ways in which you have taken care of and given consideration to the ethical dimensions of your research project. Congratulations to you and your supervisor!

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

**The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.**

All the best with your research project.

Yours sincerely

M Mabeta

Matsie Mabeta
Wits School of Education

**(011) 717 3416**