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First integrals, conserved vectors of nonlinear partial difference equations

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Abstract
We conduct a symmetry analysis of the discrete Fitzhugh-Nagumo (FN) and the Kolmogorov Petrovskii Piskunov (KPP) equations for which the discrete versions are produced using a formal discretization method. The primary aspect of the procedure is to generate a symmetry algebra using a technique developed by Hydon. Then, we derive a method for the construction of conserved vectors or first integral vectors for the discrete equations.

Keywords Symmetry algebra · Partial difference equations · Conserved vectors · First integral vectors · Fitzhugh-Nagumo equation

Mathematics Subject Classification 22E40 · 35Bxx · 70S10 · 22E05

1 Introduction

Lie groups were developed in 1870 by Sophus Lie to study the symmetry of differential equations that led to analytical solutions. The study of nonlinear differential equations has widely utilized Lie groups to find exact solutions. The details of Lie's approach can be found in [1, 2]. Since then differential equations have been successfully solved, and their classification and solution have been characterized by using Lie's theory. Much more recently, the Lie group theory has been used for discrete equations such as differential-difference equations, difference equations, or q-difference equations [3]. The literature offers significant works and applications, [4, 5]. Although it is feasible to determine symmetry groups automatically, doing so is typically laborious and susceptible to errors. Several packages use symbolic manipulations in this matter. Here are a few of the works that we have highlighted: Schwartz [6], Vu and Carminati [7], Herod [8], Baumann [9], and Rocha [10].

The Fitzhugh-Nagumo (FN) and the Kolmogorov Petrovskii Piskunov (KPP) equations represent the heat equations with cubic and quadratic nonlinearities representing emanating from some 'external forces'. In this study, we focus on applying the theory of Lie groups to the equivalent difference/discrete versions of the equations. In general, difference equations that are used to approximate the solutions of a differential equation and do not take into

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account the symmetries of the original equation may lead to incorrect mathematical findings. There are several techniques that have been investigated in the literature that enable one to develop a scheme while preserving the symmetry of the original differential equation. The direct examination of the symmetries of difference equations forms the basis of this method. The direct technique we follow here was introduced by Hydon [11]. We found this technique to be more useful and adequate for the difference equations. Hydon [12, 13] investigated the conservation laws for the nonlinear partial difference (PDEs) with two independent variables. Folly-Gbetoula also followed the technique for some difference equations [14–17]. Some other techniques exist in the literature that preserve the symmetry algebra [18].

Principally, our approach is motivated by the first integral technique followed by Hydon [19]. For this investigation, we mainly deal with the shift operator that was introduced in [11]. This operator plays a leading role in the derivation of the symmetry algebra and conserved vectors as well. If we are dealing with a function $g(n)$, then the *forward shift* of the operator is defined as $S_n : n \mapsto n+1$. In general

$$S_n^r\{g(n)\} = g(n+r), \quad S_n^r w_i = w_{i+r}.$$

In contrast, the operator for a *reverse shift* of the operator is defined as $S_n : n \mapsto n-1$. In a general way,

$$S_n^r\{g(n)\} = g(n-r), \quad S_n^r w_i = w_{i-r}.$$

This article is organized as follows: The theory of difference equations is introduced in Sect. 1 together with the Lie symmetry work. Section 2 provides the general strategy for the symmetry generators and first integral vectors or conserved vectors of the nonlinear PDEs. The adopted method is used for the discrete illustrations of the considered equations in Sect. 3. There is a conclusion offered at the end.

2 Symmetries and conserved vectors

Let k and l be two integers and w_k^l be a function that depends on the independent variables k, l. If ω is a function of the dependent and independent variables then the partial difference equation (Type-I) with the above variables is given by

$$w_{k+1}^{l+1} = \omega(k, l, w_k^l, w_k^{l+1}, w_{k+1}^l). \tag{1}$$

In this study, we consider a point transformation [11] given by

$$\Gamma_1 : (k, l, w_k^l, w_k^{l+1}, w_{k+1}^l, w_{k+1}^{l+1}) \mapsto (k, l, \hat{w}_k^l, \hat{w}_k^{l+1}, \hat{w}_{k+1}^l, \hat{w}_{k+1}^{l+1}). \tag{2}$$

If we consider Γ_1 to be a symmetry for Eq. (1), then we obtain

$$\hat{w}_{k+1}^{l+1} = \omega(k, l, \hat{w}_k^l, \hat{w}_k^{l+1}, \hat{w}_{k+1}^l), \tag{3}$$

whenever (1) holds. We can obtain Lie symmetries by linearizing the symmetry condition about the identity. We look for symmetry with one-parameter (local) Lie groups of the type

$$\hat{w}_k^l = w_k^l + \epsilon Q(k, l, w_k^l) + O(\epsilon^2). \tag{4}$$

In the above equation, Q is the characteristic function of the one-parameter group. We obtain the one-parameter (local) group transformation as

$$\begin{aligned}\hat{w}_{k+1}^{l} &= w_{k+1}^{l} + \epsilon Q(k+1, l, w_{k+1}^{l}) + O(\epsilon^2),\\ \hat{w}_{k}^{l+1} &= w_{k}^{l+1} + \epsilon Q(k, l+1, w_{k}^{l+1}) + O(\epsilon^2),\\ \hat{w}_{k+1}^{l+1} &= w_{k+1}^{l+1} + \epsilon Q(k+1, l+1, w_{k+1}^{l+1}) + O(\epsilon^2).\end{aligned} \tag{5}$$

We obtain the linearized symmetry condition by expanding Eqs. (4)–(5) to first order in ϵ as

$$S_k S_l Q - V\omega = 0. \tag{6}$$

In Eq. (6) the differential function V is a vector field and is given by

$$V = Q\frac{\partial}{\partial w_k^l} + (S_k Q)\frac{\partial}{\partial w_{k+1}^l} + (S_l Q)\frac{\partial}{\partial w_k^{l+1}}\cdot \tag{7}$$

A similar situation arises when we have a Type-II partial difference equation is given by

$$w_{k+1}^{l+1} = \omega(k, l, w_k^l, w_{k-1}^{l+1}, w_k^{l+1}). \tag{8}$$

The transformation [11] in this case is as follows

$$\Gamma_2 : (k, l, w_k^l, w_{k-1}^{l+1}, w_k^{l+1}, w_{k+1}^{l+1}) \mapsto (k, l, \hat{w}_k^l, \hat{v}_{k-1}^{l+1}, \hat{w}_k^{l+1}, \hat{w}_{k+1}^{l+1}). \tag{9}$$

If we consider Γ_2 to be a symmetry for Eq. (8), then we obtain

$$\hat{w}_{k+1}^{l+1} = \omega(k, l, \hat{w}_k^l, \hat{v}_{k-1}^{l+1}, \hat{w}_k^{l+1}), \tag{10}$$

whenever (8) holds. We obtain the Lie symmetries by linearizing the symmetry condition about the identity. We look for symmetry with one-parameter (local) Lie groups of the type

$$\hat{w}_k^l = w_k^l + \epsilon Q(k, l, w_k^l) + O(\epsilon^2). \tag{11}$$

The characteristic of the one-parameter group is the function Q. Additionally, we have

$$\begin{aligned}\hat{v}_{k-1}^{l+1} &= w_{k-1}^{l+1} + \epsilon Q(k-1, l+1, w_{k-1}^{l+1}) + O(\epsilon^2),\\ \hat{w}_{k}^{l+1} &= w_{k}^{l+1} + \epsilon Q(k, l+1, w_{k}^{l+1}) + O(\epsilon^2),\\ \hat{v}_{k+1}^{l+1} &= w_{k+1}^{l+1} + \epsilon Q(k+1, l+1, w_{k+1}^{l+1}) + O(\epsilon^2).\end{aligned} \tag{12}$$

The linearized symmetry condition is obtained by expanding Eqs. (11)–(12) to first order in ϵ as

$$S_k S_l Q - V\omega = 0. \tag{13}$$

In Eq. (13) the differential function V is a vector field and defined by

$$V = Q\frac{\partial}{\partial w_k^l} + (S_{-k} S_l Q)\frac{\partial}{\partial w_{k-1}^{l+1}} + (S_l Q)\frac{\partial}{\partial w_k^{l+1}}\cdot \tag{14}$$

2.1 Conserved vectors for nonlinear PDEs

We consider a PDE of Type-I variables given by (1). Let (ϕ, ψ) be our required conserved vector for Eq. (1). Then we define components ϕ and ψ as

$$\phi = \phi(k, l, w_k^l, w_{k+1}^l), \qquad \psi = \psi(k, l, w_k^l, w_k^{l+1}). \tag{15}$$

The conserved vector condition that can be imposed is given by

$$S_l\phi + S_k\psi = \phi + \psi, \qquad \frac{\partial \phi}{\partial w_k^l} \neq 0, \qquad \frac{\partial \psi}{\partial w_k^l} \neq 0. \tag{16}$$

Expanded form of the Eq. (16) is

$$S_l\phi(k, l, w_k^l, w_{k+1}^l) + S_k\psi(k, l, w_k^l, w_k^{l+1}) = \phi(k, l, w_k^l, w_{k+1}^l) + \psi(k, l, w_k^l, w_k^{l+1}), \tag{17}$$

or

$$\phi(k, l+1, w_k^{l+1}, \omega) + \psi(k+1, l, w_{k+1}^l, \omega) = \phi(k, l, w_k^l, w_{k+1}^l) + \psi(k, l, w_k^l, w_k^{l+1}). \tag{18}$$

Here we define some functions based on components ϕ and ψ as

$$\begin{aligned} P_1(k, l, w_k^l) &= \frac{\partial \phi}{\partial w_k^l}, & R_1(k, l, w_k^l) &= \frac{\partial \psi}{\partial w_k^l}, \\ P_2 &= \frac{\partial \phi}{\partial w_{k+1}^l}, & R_2 &= \frac{\partial \psi}{\partial w_k^{l+1}}\cdot \end{aligned} \tag{19}$$

We differentiate Eq. (18) w.r.t. variable w_k^l

$$\frac{\partial \phi}{\partial w_k^l}(k, l+1, w_k^{l+1}, \omega) + \frac{\partial \psi}{\partial w_k^l}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial w_k^l} + \frac{\partial \psi}{\partial w_k^l} \tag{20}$$

Using the chain rule we get

$$(S_l P_2 + S_k R_2)\frac{\partial \omega}{\partial w_k^l} = P_1 + R_1. \tag{21}$$

Using the linear property of differential operators we have

$$P_1 = (S_l P_2)\frac{\partial \omega}{\partial w_k^l}, \qquad R_1 = (S_k R_2)\frac{\partial \omega}{\partial w_k^l}\cdot \tag{22}$$

Differentiating Eq. (18) w.r.t. variable w_{k+1}^l, we get

$$\frac{\partial \phi}{\partial w_{k+1}^l}(k, l+1, w_k^{l+1}, \omega) + \frac{\partial \psi}{\partial w_{k+1}^l}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial w_{k+1}^l} + \frac{\partial \psi}{\partial w_{k+1}^l}\cdot \tag{23}$$

By writing the expressions in the form of shift operator, we obtain

$$P_2 = S_l P_2 \frac{\partial \omega}{\partial w_{k+1}^l} + S_k R_2 \frac{\partial \omega}{\partial w_{k+1}^l}\cdot \tag{24}$$

Now we differentiate Eq. (18) w.r.t. second variable w_k^{l+1}

$$\frac{\partial \phi}{\partial w_k^{l+1}}(k, l+1, w_k^{l+1}, \omega) + \frac{\partial \psi}{\partial w_k^{l+1}}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial w_k^{l+1}} + \frac{\partial \psi}{\partial w_k^{l+1}}\cdot \tag{25}$$

We simplify the above relations as

$$R_2 = S_l P_2 \frac{\partial \omega}{\partial w_k^{l+1}} + S_k R_2 \frac{\partial \omega}{\partial w_k^{l+1}}\cdot \tag{26}$$

Inserting Eq. (24) in Eq. (26) and then repeating the converse process, we get

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial w^l_{k+1}}[S_l P_2 + S_l P_2 \frac{\partial \omega}{\partial w^{l+1}_k} + S^2_k R_2 \frac{\partial \omega}{\partial w^{l+1}_k}] \\ R_2 &= \frac{\partial \omega}{\partial w^{l+1}_k}[S_k R_2 + S_k R_2 \frac{\partial \omega}{\partial w^l_{k+1}} + S^2_l P_2 \frac{\partial \omega}{\partial w^l_{k+1}}]. \end{aligned} \tag{27}$$

The Eq. (27) leads to a way of defining characteristic functions P_2 and R_2 given by

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial w^l_{k+1}}[S_l P_2 + S_l P_2 \frac{\partial \omega}{\partial w^{l+1}_k} + S^2_k P_2 \frac{\partial \omega}{\partial w^{l+1}_k}] \\ R_2 &= \frac{\partial \omega}{\partial w^{l+1}_k}[S_k R_2 + S_k R_2 \frac{\partial \omega}{\partial w^l_{k+1}} + S^2_l R_2 \frac{\partial \omega}{\partial w^l_{k+1}}], \end{aligned} \tag{28}$$

such that $S^2_l P_2 = S^2_k R_2$ holds. By critically examining Eq. (19), we can define the integrability condition for the conserved vectors as

$$\frac{\partial P_1}{\partial w^l_{k+1}} = \frac{\partial P_2}{\partial w^l_k}, \qquad \frac{\partial R_1}{\partial w^{l+1}_k} = \frac{\partial R_2}{\partial w^l_k}. \tag{29}$$

Then, by following the above integrability condition one can define the conserved vector (ϕ, ψ) in the following way

$$\begin{aligned} \phi &= \int P_1 dw^l_k + P_2 w^l_{k+1} + G(k, l) \\ \psi &= \int R_1 dw^l_k + R_2 w^{l+1}_k + H(k, l), \end{aligned} \tag{30}$$

where the functions $G(k, l)$ and $H(k, l)$ can be determined by integrability condition (29). Now, we consider for the PDE of Type-II given by (8). On similar lines, we consider here the conserved vector (ϕ, ψ) for Eq. (8). Then we define the components ϕ, ψ as

$$\phi = \phi(k, l, w^l_k, w^{l+1}_{k-1}), \qquad \psi = \psi(k, l, w^l_k, w^{l+1}_k). \tag{31}$$

The first integral condition that can be imposed is given by

$$S^2_k \phi + S_k \psi = \phi + \psi, \qquad \frac{\partial \phi}{\partial w^l_k} \neq 0, \qquad \frac{\partial \psi}{\partial w^l_k} \neq 0. \tag{32}$$

Expand form of the Eq. (32) is

$$S^2_k \phi(k, l, w^l_k, w^{l+1}_{k-1}) + S_k \psi(k, l, w^l_k, w^{l+1}_k) = \phi(k, l, w^l_k, w^{l+1}_{k-1}) + \psi(k, l, w^l_k, w^{l+1}_k), \tag{33}$$

or

$$\phi(k+2, l, v^l_{k+2}, \omega) + \psi(k+1, l, w^l_{k+1}, \omega) = \phi(k, l, w^l_k, w^{l+1}_{k-1}) + \psi(k, l, w^l_k, w^{l+1}_k). \tag{34}$$

We specify a few functions like previously

$$\begin{aligned} P_1(k, l, w^l_k) &= \frac{\partial \phi}{\partial w^l_k}, \qquad & R_1(k, l, w^l_k) &= \frac{\partial \psi}{\partial w^l_k}, \\ P_2 &= \frac{\partial \phi}{\partial w^{l+1}_{k-1}}, \qquad & R_2 &= \frac{\partial \psi}{\partial w^{l+1}_k}. \end{aligned} \tag{35}$$

We differentiate Eq. (34) w.r.t. variable w_k^l

$$\frac{\partial \phi}{\partial w_k^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial w_k^l}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial w_k^l} + \frac{\partial \psi}{\partial w_k^l} \cdot \tag{36}$$

Using the chain rule we get

$$(S_k^2 P_2 + S_k R_2)\frac{\partial \omega}{\partial w_k^l} = P_1 + R_1. \tag{37}$$

Using the linear property of the shift and differential operators we get

$$P_1 = S_k^2(P_2)\frac{\partial \omega}{\partial w_k^l}, \qquad R_1 = S_k(R_2)\frac{\partial \omega}{\partial w_k^l} \cdot \tag{38}$$

Differentiating (34) w.r.t. second variable w_{k+1}^l, we get

$$\frac{\partial \phi}{\partial w_{k+1}^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial w_{k+1}^l}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial w_{k+1}^l} + \frac{\partial \psi}{\partial w_{k+1}^l} \cdot \tag{39}$$

Now writing the expressions in the form shift operator, we obtain

$$P_2 = [S_l(P_2) + S_k(R_2)]\frac{\partial \omega}{\partial w_{k+1}^l} \cdot \tag{40}$$

Now we differentiate Eq. (34) w.r.t. w_{k+2}^l to obtain

$$\frac{\partial \phi}{\partial v_{k+2}^l}(k+2, l, v_{k+2}^l, \omega) + \frac{\partial \psi}{\partial v_{k+2}^l}(k+1, l, w_{k+1}^l, \omega) = \frac{\partial \phi}{\partial v_{k+2}^l} + \frac{\partial \psi}{\partial v_{k+2}^l} \cdot \tag{41}$$

We simplify the above relations as

$$R_2 = S_k^2(P_2) + S_{-k}S_l(R_2)\frac{\partial \omega}{\partial v_{k+2}^l} \cdot \tag{42}$$

Inserting Eq. (40) in Eq. (42) and then repeating the process in converse, we get

$$\begin{aligned} P_2 &= \frac{\partial \omega}{\partial w_{k+1}^l}\left[S_l(P_2) + \frac{\partial \omega}{\partial v_{k+2}^l}\left\{S_k^2(P_2) + S_l(R_2)\right\}\right], \\ R_2 &= \frac{\partial \omega}{\partial v_{k+2}^l}\left[S_{-k}S_l(R_2) + \frac{\partial \omega}{\partial w_{k+1}^l}\left\{S_k^2 S_l(P_2) + S_k(R_2)\right\}\right]. \end{aligned} \tag{43}$$

This Eq. (43) leads to a way of defining a characteristic function p_2 and R_2 as

$$\begin{aligned} P_2 &= S_k\left(\frac{\partial \omega}{\partial w_k^l}\right)\left[S_l(P_2) + S_k^2\left(\frac{\partial \omega}{\partial w_k^l}\right)S_k^2(P_2) + S_k^2\left(\frac{\partial \omega}{\partial w_k^l}\right)S_l(P_2)\right\}\right] \\ R_2 &= S_k^2\left(\frac{\partial \omega}{\partial w_k^l}\right)\left[S_{-k}S_l(R_2) + S_k\left(\frac{\partial \omega}{\partial w_k^l}\right)S_k(R_2) + S_k\left(\frac{\partial \omega}{\partial w_k^l}\right)S_k^2 S_l(R_2)\right\}\right], \end{aligned} \tag{44}$$

such that $S_l(R_2) = S_k^2 S_l(P_2)$ holds. By critically working over Eq. (35), we can define the integrability condition for the conserved vectors in this case as

$$\frac{\partial P_1}{\partial w_{k-1}^{l+1}} = \frac{\partial P_2}{\partial w_k^l}, \qquad \frac{\partial R_1}{\partial w_k^{l+1}} = \frac{\partial R_2}{\partial w_k^l} \cdot \tag{45}$$

Then, by following the above integrability condition one can define the conserved vector (ϕ, ψ) in the following way

$$\begin{aligned}\phi &= \int P_1 dw_k^l + P_2 w_{k-1}^{l+1} + G(k,l),\\ \psi &= \int R_1 dw_k^l + R_2 w_k^{l+1} + H(k,l),\end{aligned} \tag{46}$$

where the functions $G(k,l)$ and $H(k,l)$ can be determined by integrability condition (45). Now we follow the method adopted here to some nonlinear discrete PDEs, particularly to discrete FN equation and the KPP equation.

3 Derivation of partial difference equations

3.1 Discrete Fitzhugh-Nagumo equation

For the derivation of the PDEs, we first consider the discrete FN equation given by

$$w_t = w_{xx} - w(1-w)(\rho - w), \tag{47}$$

where $w(x,t)$ is an unknown function depending upon the temporal t and spatial x variables and $0 < \rho < 1$. Eq. (47) is a crucial nonlinear reaction-diffusion equation used to simulate the transmission of nerve impulses [20, 21] and is also employed in biology, population genetics, circuit theory [22], and other fields. For $\rho = -1$ this Eq. (47) becomes the Newell-Whitehead equation that represents the dynamical features of the Rayleigh Benard convection of binary fluid mixtures near the bifurcation point [23]. On the right hand of this Eq. (47), the term $w(1-w)$ indicates the reaction or growth term. We begin with an implicit finite difference scheme, which is a backward difference in time and a central second-order difference in space, for the discretization of a single-domain diffusion equation. When implicit discretization is used, the approximations for the derivatives are defined as

$$w_t \approx \frac{w_k^{l+1} - w_k^l}{\triangle t}, \tag{48}$$

$$w_{xx} \approx \frac{w_{k+1}^{l+1} - 2w_k^{l+1} + w_{k-1}^{l+1}}{2(\triangle x)^2}. \tag{49}$$

The Eq. (47) in the form of first and second-order differences becomes

$$\frac{w_k^{l+1} - w_k^l}{\triangle t} \approx \frac{w_{k+1}^{l+1} - 2w_k^{l+1} + w_{k-1}^{l+1}}{2(\triangle x)^2} - w_k^l(1-w_k^l)(\rho - w_k^l). \tag{50}$$

Truncating the error terms and simplifying, we get

$$w_k^{l+1} - w_k^l = \nu(w_{k+1}^{l+1} - 2w_k^{l+1} + w_{k-1}^{l+1}) - (\triangle t)w_k^l(1-w_k^l)(\rho - w_k^l), \tag{51}$$

where $\nu = \frac{(\triangle t)}{2(\triangle x)^2}\cdot$ Arranging the expressions in terms of indices we get an implicit formulation

$$w_{k+1}^{l+1} = Aw_k^{l+1} + Bw_k^l - w_{k-1}^{l+1} + C(w_k^l)^2 - D(w_k^l)^3.$$

The parameters appearing in above equation follow as $A = 2 + \frac{1}{\nu},\ B = \frac{1}{\nu}(\rho\triangle t),\ C = \frac{\triangle t}{\nu}(1-\rho),\ D = \frac{\triangle t}{\nu}\cdot$ Here, we will calculate the symmetries of the discrete FN equation, which is the equation mentioned above.

3.1.1 Symmetries

For this study, we consider the discrete FN equation

$$w_{k+1}^{l+1} = Aw_k^{l+1} + Bw_k^l - w_{k-1}^{l+1} + C(w_k^l)^2 - D(w_k^l)^3. \tag{52}$$

In our classification, the above equation is of Type-II given by (8)

$$w_{k+1}^{l+1} = \omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1}). \tag{53}$$

Note that in the above equation the $\omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1})$ is a smooth function. We consider $Q = Q(k, l, w_k^l)$ a lower order possible characteristic for the Eq. (52) and then the symmetry condition (13) simplifies to

$$Q(k+1, l+1, \omega) - V\omega = 0. \tag{54}$$

The symmetry generator in the above equation is given by (14)

$$V = Q\frac{\partial}{\partial w_k^l} + (S_k Q)\frac{\partial}{\partial w_k^{l+1}} + (S_l Q)\frac{\partial}{\partial w_{k-1}^{l+1}}.$$

We further simplify the symmetry condition (54) to have the following form

$$\begin{aligned} &Q(k+1, l+1, \omega) - (B + 2Cw_k^l - 3D(w_k^l)^2)Q(k, l, w_k^l) - AQ(k, l \\ &\quad + 1, w_k^{l+1}) + Q(k-1, l+1, w_{k-1}^{l+1}) = 0. \end{aligned} \tag{55}$$

We define the differential operator for the above equation as

$$L_0 = \frac{\partial}{\partial w_k^l} + \frac{\partial w_k^{l+1}}{\partial w_k^l}\frac{\partial}{\partial w_k^{l+1}}. \tag{56}$$

By applying the differential operator to the above equation, we get

$$\begin{aligned} &(B + 2Cw_k^l - 3D(w_k^l)^2)Q'(k, l, w_k^l) + (2C - 6Dw_k^l)Q(k, l, w_k^l) \\ &\quad - (B + 2Cw_k^l - 3D(w_k^l)^2)Q'(k, l+1, w_k^{l+1}) = 0. \end{aligned} \tag{57}$$

We solve the above equation for the required characteristic, and we obtain

$$Q(k, l, w_k^l) = c_1(k, l)\frac{(Bw_k^l + C(w_k^l)^2 - D(w_k^l)^3)}{(B + 2Cw_k^l - 3D(w_k^l)^2)} + \frac{c_2(k, l)}{(B + 2Cw_k^l - 3D(w_k^l)^2)}. \tag{58}$$

For simplicity we take $c_1(x, t) = 1 = c_2(x, t)$. Thus we get

$$Q(k, l, w_k^l) = \frac{1}{3}w_k^l - \frac{C}{9D} + \frac{(5Cw_k^l + BC)}{9D(B + 2Cw_k^l - 3D(w_k^l)^2)} + \frac{1}{(B + 2Cw_k^l - 3D(w_k^l)^2)}. \tag{59}$$

The symmetry algebra for the discrete FN Eq (52) then follow as

$$
\begin{aligned}
V_1 &= \frac{\partial}{\partial w_k^l} + \frac{\partial}{\partial w_k^{l+1}} + \frac{\partial}{\partial w_{k-1}^{l+1}}, \qquad V_2 = w_k^l \frac{\partial}{\partial w_k^l} + w_k^{l+1} \frac{\partial}{\partial w_k^{l+1}} + w_{k-1}^{l+1} \frac{\partial}{\partial w_{k-1}^{l+1}},\\
V_3 &= \frac{C(5w_k^l + B)}{9D(B + 2Cw_k^l} \frac{\partial}{\partial w_k^l} + \frac{C(5w_k^{l+1} + B)}{9D(B + 2Cw_k^{l+1}} \frac{\partial}{\partial w_k^{l+1}} + \frac{C(5w_{k-1}^{l+1} + B)}{9D(B + 2Cw_{k-1}^{l+1}} \frac{\partial}{\partial w_{k-1}^{l+1}},\\
V_4 &= \frac{1}{(B + 2Cw_k^l - 3D(w_k^l)^2)} \frac{\partial}{\partial w_k^l} + \frac{1}{(B + 2Cw_k^{l+1} - 3D(w_k^{l+1})^2)} \frac{\partial}{\partial w_k^{l+1}}\\
&\quad + \frac{1}{(B + 2Cw_{k-1}^{l+1} - 3D(w_{k-1}^{l+1})^2)} \frac{\partial}{\partial w_{k-1}^{l+1}}.
\end{aligned} \tag{60}
$$

3.1.2 First integral vectors

This section deals with the results related to the first integral vectors derived in the above subsection. We first consider the discrete FN equation given by

$$w_{k+1}^{l+1} = Aw_k^{l+1} + Bw_k^l - w_{k-1}^{l+1} + C(w_k^l)^2 - D(w_k^l)^3.$$

This equation is of type-II given in Eq (8)

$$w_{k+1}^{l+1} = \omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1})$$

Note that in the above equation the $\omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1})$ is a smooth function and it indicates the right side of Eq (8). For this smooth function, the characteristic function is defined by $P_2 = P_2(k, l, w_k^l)$. Then we follow the first integral vector condition (28) for P_2 as

$$
\begin{aligned}
&- P_2(k, l, w_k^l) + (B + 2Cw_{k+1}^l - 3D(w_{k+1}^l)^2)[P_2(k, l+1, w_k^{l+1})\\
&\quad + (B + 2Cv_{k+2}^l - 3D(v_{k+2}^l)^2)\{P_2(k+2, l, v_{k+2}^l) + P_2(k, l+1, w_k^{l+1})\}] = 0.
\end{aligned} \tag{61}
$$

We apply the operator (56) to simplify above in different arguments

$$
\begin{aligned}
&- P_2'(k, l, w_k^l) - \frac{1}{A}(B + 2Cw_k^l - 3D(w_k^l)^2)(B + 2Cw_{k+1}^l - 3D(w_{k+1}^l)^2)[1 + (B\\
&\quad + 2Cv_{k+2}^l - 3D(v_{k+2}^l)^2)]P_2'(k, l+1, w_k^{l+1}) = 0.
\end{aligned} \tag{62}
$$

When we differentiate this equation with respect to w_k^l and simplify it, we obtain

$$P_2(k, l, w_k^l) = c_1(k, l)(Bw_k^l + C(w_k^l)^2 - D(w_k^l)^3) + c_2(k, l). \tag{63}$$

Inserting the characteristic through Eq (61) and (62), we get

$$P_2(k, l, w_k^l) = (-1)^l(Bw_k^l + C(w_k^l)^2 - D(w_k^l)^3) + (-1)^k. \tag{64}$$

We define P_1 by Eq (22) as

$$P_1(k-1, l+1, w_{k-1}^{l+1}) = (B+2Cv_{k-3}^{l+1}-3D(v_{k-3}^{l+1})^2)[(-1)^{l+1}\{B+2Cw_{k-1}^{l+1})-3D(w_{k-1}^{l+1})^2)\}]. \tag{65}$$

The relations for the integrability condition (28) follow as

$$\begin{aligned}\frac{\partial P_1}{\partial w^l_{k+1}} &= (B+2Cv^{l+1}_{k-3}-3D(v^{l+1}_{k-3})^2)[(-1)^{l+1}\{Bw^{l+1}_{k-1} \\ &\quad + C(w^{l+1}_{k-1})^2 - D(w^{l+1}_{k-1})^3)\} + (-1)^k], \\ \frac{\partial P_2}{\partial w^l_k} &= (-1)^l(B+2Cw^l_k-3D(w^l_k)^2).\end{aligned} \tag{66}$$

The component ϕ of the conserved vector is given by

$$\begin{aligned}\phi &= B(-1)^l v^l_{k-1}w^l_k[1-(B+2Cv^{l+1}_{k-3}-3D(v^{l+1}_{k-3})^2)] \\ &\quad + C(-1)^l w^l_k w^{l+1}_{k-1}[w^l_k-(B+2Cv^{l+1}_{k-3}-3D(v^{l+1}_{k-3})^2)w^{l+1}_{k-1}] \\ &\quad + (-1)^l w^l_k w^{l+1}_{k-1}[(w^l_k)^2-(B+2Cv^{l+1}_{k-3}-3D(v^{l+1}_{k-3})^2)(w^{l+1}_{k-1})^2] \\ &\quad + (-1)^k[w^{l+1}_{k-1})+(B+2Cv^{l+1}_{k-3}-3D(v^{l+1}_{k-3})^2)w^l_k]+G(k,l).\end{aligned} \tag{67}$$

To obtain the second component ψ of the conserved vector we proceed now. We take the characteristic $R_2 = R_2(k,l,w^l_k)$ and by using (28) we get

$$\begin{aligned}&-R_2(k,l,w^l_k)+(B+2Cv^l_{k+2}-3D(v^l_{k+2})^2)[R_2(k-1,l+1,w^{l+1}_{k-1}) \\ &\quad + (B+2Cw^l_{k+1}-3D(w^l_{k+1})^2)\{R_2(k+1,l,w^l_{k+1})+R_2(k+2,l+1,v^{l+1}_{k+2})\}]=0.\end{aligned} \tag{68}$$

By applying the operator (56) we get

$$\begin{aligned}&\frac{1}{A}(B+2Cv^l_{k+2}-3D(v^l_{k+2})^2)(B+2Cw^l_k \\ &\quad -3D(w^l_k)^2)R_2'(k-1,l+1,w^{l+1}_{k-1}) \\ &\quad -R_2'(k,l,w^l_k)\}]=0.\end{aligned} \tag{69}$$

We simplify the above equation to get

$$R_2(k,l,w^l_k) = c_1(k,l)\{Bw^l_k+C(w^l_k)^2-D(v^l_{k+2})^3\}+c_2(k,l). \tag{70}$$

Further, using (68) and (69), we have

$$R_2(k,l,w^l_k) = C_1\left(\frac{A}{B^2}\right)^{k-1}\{Bw^l_k+C(w^l_k)^2-D(v^l_{k+2})^3\}+C_2\left(\frac{1}{B-1}\right)^{l-1}. \tag{71}$$

On the other hand the relation for R_1 (22) follows as

$$\begin{aligned}R_1(k,l+1,w^{l+1}_k) &= (B+2Cw^{l+1}_{k-1}-3D(w^{l+1}_{k-1})^2)[C_1\left(\frac{A}{B^2}\right)^{k-1}\{Bw^l_k \\ &\quad + C(w^l_k)^2-D(v^l_{k+2})^3\}+C_2\left(\frac{1}{B-1}\right)^l].\end{aligned} \tag{72}$$

The relation for condition (29) becomes

$$\begin{aligned}\frac{\partial R_1}{\partial w^{l+1}_k} &= (B+2Cw^{l+1}_{k-1}-3D(w^{l+1}_{k-1})^2)[C_1\left(\frac{A}{B^2}\right)^{k-1}\{B+2C(w^{l+1}_k-3D(w^{l+1}_k)^2\}], \\ \frac{\partial R_2}{\partial w^l_k} &= C_1\left(\frac{A}{B^2}\right)^{k-1}\{B+2Cw^l_k-3D(v^l_{k+2})^2\}.\end{aligned} \tag{73}$$

From Eq. (30), we follow the second component ψ of the conserved vector as

$$\begin{aligned}\psi =& C_1\left(\frac{A}{B^2}\right)^{k-1} B w_k^l w_k^{l+1}[1+(B+2Cv_{k-3}^{l+1}-3D(w_{k-1}^{l+1})^2)]\\&+C_1\left(\frac{A}{B^2}\right)^{k-1} C w_k^l w_k^{l+1}[w_k^l+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)w_k^{l+1}]\\&+C_1\left(\frac{A}{B^2}\right)^{k-1} D w_k^l w_k^{l+1}[(w_k^l)^2+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)(w_k^{l+1})^2]\\&+C_2\left(\frac{1}{B}-1\right)^{l-1}[w_k^{l+1})+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)w_k^l]+H(k,l).\end{aligned} \tag{74}$$

By using condition (32) we get the following conserved vector (ϕ, ψ) as

$$\begin{aligned}\phi =& B(-1)^l v_{k-1}^l w_k^l[1-(B+2Cv_{k-3}^{l+1}-3D(v_{k-3}^{l+1})^2)]\\&+C(-1)^l w_k^l w_{k-1}^{l+1}[w_k^l-(B+2Cv_{k-3}^{l+1}-3D(v_{k-3}^{l+1})^2)w_{k-1}^{l+1}]\\&+(-1)^l w_k^l w_{k-1}^{l+1}[(w_k^l)^2-(B+2Cv_{k-3}^{l+1}-3D(v_{k-3}^{l+1})^2)(w_{k-1}^{l+1})^2]\\&+(-1)^k[w_{k-1}^{l+1})+(B+2Cv_{k-3}^{l+1}-3D(v_{k-3}^{l+1})^2)w_k^l]+(k-1)^k,\\ \psi =& C_1\left(\frac{A}{B^2}\right)^{k-1} B w_k^l w_k^{l+1}[1+(B+2Cv_{k-3}^{l+1}-3D(w_{k-1}^{l+1})^2)]\\&+C_1\left(\frac{A}{B^2}\right)^{k-1} C w_k^l w_k^{l+1}[w_k^l+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)w_k^{l+1}]\\&+C_1\left(\frac{A}{B^2}\right)^{k-1} D w_k^l w_k^{l+1}[(w_k^l)^2+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)(w_k^{l+1})^2]\\&+C_2\left(\frac{1}{B}-1\right)^{l-1}[w_k^{l+1})+(B+2Cw_{k-1}^{l+1}-3D(w_{k-1}^{l+1})^2)w_k^l]+(-1)^l.\end{aligned} \tag{75}$$

Consequently, we have the following theorem.

Theorem 3.1 *The conserve vector for the discrete FN Eq (52) has components given by (75).*

3.2 Discrete Kolmogorov Petrovskii Piskunov equation

Consider the one-dimensional KPP equation [24]

$$w_t = w_{xx} + aw + bw^2, \tag{76}$$

where $w(x,t)$ represents the propagation of the dominant genes and $x,\ t$ are independent variables. To characterize the evolutionary model for the propagation of dominant genes across populations, Kolmogorov, Petrovsky, and Piskunov proposed a nonlinear PDE known as KPP equation. Later, the KPP equation has been used in a variety of scientific disciplines, including physics as it relates to combustion, biology as it relates to nerve impulse propagation, chemical kinetics as it relates to concentration wave propagation and plasma as it relates to development of a group of duffing oscillators. We begin with an implicit finite difference scheme, which is a backward difference in time and a central second-order difference in space, for the discretization of a single-domain diffusion equation. When implicit discretization is used, the approximations for the derivatives are defined in Eqs (48) and (49). The Eq

(76) in the form of first and second-order differences becomes

$$\frac{w_k^{l+1}-w_k^l}{\triangle t}=\frac{w_{k+1}^{l+1}-2w_k^{l+1}+w_{k-1}^{l+1}}{2(\triangle x)^2}+aw_k^l+b(w_k^l)^2. \tag{77}$$

Truncating the error terms and simplifying, we get

$$w_k^{l+1}-w_k^l=\nu(w_{k+1}^{l+1}-2w_k^{l+1}+w_{k-1}^{l+1})+(\triangle t)(aw_k^l+b(w_k^l)^2), \tag{78}$$

where $\nu=\frac{(\triangle t)}{2(\triangle x)^2}\cdot$ Arranging the expressions in terms of indices we get an implicit difference equation

$$w_{k+1}^{l+1}=Aw_k^{l+1}-Bw_k^l-w_{k-1}^{l+1}-C(w_k^l)^2.$$

The parameters appearing in above Eq (78) follow as $A=\frac{1}{\nu}+2,\ B=\frac{1+\triangle t}{\nu},\ C=\frac{\triangle t}{\nu}\cdot$ Here, we will calculate the symmetries of the discrete KPP equation, which is the equation mentioned above.

3.2.1 Symmetries

For this study, we consider the discrete KPP equation derived above

$$w_{k+1}^{l+1}=Aw_k^{l+1}-Bw_k^l-w_{k-1}^{l+1}-C(w_k^l)^2. \tag{79}$$

In our classification, this above equation is of Type-II given by

$$w_{k+1}^{l+1}=\omega(k,l,w_k^l,w_k^{l+1},w_{k-1}^{l+1}). \tag{80}$$

Note that in above equation the $\omega(k,l,w_k^l,w_k^{l+1},w_{k-1}^{l+1})$ is a smooth function. We consider $Q=Q(k,l,w_k^l)$ a lower order possible characteristic for the Eq (79) and then the symmetry condition (13) simplifies to

$$Q(k+1,l+1,\omega)-V\omega=0. \tag{81}$$

The symmetry generator in the above equation is given by (14)

$$V=Q\frac{\partial}{\partial w_k^l}+(S_kQ)\frac{\partial}{\partial w_k^{l+1}}+(S_lQ)\frac{\partial}{\partial w_{k-1}^{l+1}}\cdot$$

We further simplify the symmetry condition (81) to have the following form

$$Q(k+1,l+1,\omega)+Q(k-1,l+1,w_{k-1}^{l+1})-AQ(k,l+1,w_k^{l+1})+(B+2Cw_k^l)Q(k,l,w_k^l)=0. \tag{82}$$

We apply the differential operator (56) to the above equation to get

$$-(B+2Cw_k^l)Q'(k,l+1,w_k^{l+1})+(B+2Cw_k^l)Q'(k,l,w_k^l)+2CQ(k,l,w_k^l)=0. \tag{83}$$

By solving the above equation for the required characteristic, we obtain

$$Q(k,l,w_k^l)=\frac{1}{4C}c_1(k,l)(B+2Cw_k^l)+\frac{\lambda_2^2}{4C(B+2Cw_k^l)}c_1(k,l)+\frac{c_2(k,l)}{B+2Cw_k^l}\cdot \tag{84}$$

To determine the values of constants, we use Eq (82) and Eq (83) to have the final form of the characteristic.

$$Q(k,l,w_k^l)=\frac{(-1)^{l+1}}{4C}B+\frac{(-1)^{l+2}}{2}w_k^l+\frac{(-1)^l\lambda_2^2+4Cc_1\big(-\frac{1}{A}\big)^{k-1}}{(B-2Cw_k^l)}\cdot \tag{85}$$

The symmetry algebra for the discrete KPP Eq (79) is given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial w_k^l} - \frac{\partial}{\partial w_k^{l+1}} + \frac{\partial}{\partial w_{k-1}^{l+1}}, \\ V_2 &= w_k^l \frac{\partial}{\partial w_k^l} - w_k^{l+1} \frac{\partial}{\partial w_k^{l+1}} + w_{k-1}^{l+1} \frac{\partial}{\partial w_{k-1}^{l+1}}, \\ V_3 &= \frac{1}{(B + 2Cw_k^l)} \frac{\partial}{\partial w_k^l} - \frac{1}{(B + 2Cw_k^{l+1})} \frac{\partial}{\partial w_k^{l+1}} + \frac{1}{(B + 2Cw_{k-1}^{l+1})} \frac{\partial}{\partial w_{k-1}^{l+1}}. \end{aligned} \tag{86}$$

3.2.2 First integral vectors

This section deals with the conserved vectors of the discrete KPP equation (79) given by

$$w_{k+1}^{l+1} = Aw_k^{l+1} - Bw_k^l - w_{k-1}^{l+1} - C(w_k^l)^2.$$

This equation is of type-II given in Eq (8)

$$w_{k+1}^{l+1} = \omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1}).$$

Note that in the above equation the $\omega(k, l, w_k^l, w_k^{l+1}, w_{k-1}^{l+1})$ is a smooth function and it indicates the right side of Eq (79). For this smooth function, the characteristic function is defined by $P_2 = P_2(k, l, w_k^l)$. Then we follow the first integral vector condition (44) for P_2 as

$$\begin{aligned} &P_2(k, l, w_k^l) + (B - 2Cw_{k+1}^l)\big[P_2(k, l+1, w_k^{l+1}) \\ &- (B - 2Cv_{k+2}^l)P_2(k+2, l, v_{k+2}^l) - (B - 2Cv_{k+2}^l)P_2(k, l+1, w_k^{l+1})\big] = 0. \end{aligned} \tag{87}$$

We apply the operator (56) to simplify the above equation in different arguments

$$P_2'(k, l, w_k^l) + \frac{1}{A}(B - 2Cw_k^l)(B - 2Cw_{k+1}^l)(1 - B + 2Cv_{k+2}^l)P_2'(k, l+1, w_k^{l+1}) = 0. \tag{88}$$

When we differentiate this equation with respect to w_l^k and simplify it, we obtain

$$P_2(k, l, w_k^l) = Bc_1(k, l)w_k^l - Cc_1(k, l)(w_k^l)^2 + c_2(k, l). \tag{89}$$

Inserting the characteristic through Eq (87) and (88), we get

$$P_2(k, l, w_k^l) = c_1 B\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_k^l + c_1 C\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} (w_k^l)^2 + c_2\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(B+1)}\right)^{l-1}. \tag{90}$$

We define P_1 by Eq (38) as

$$\begin{aligned} P_1(k-1, l+1, w_{k-1}^{l+1}) &= (-B + 2Cv_{k-3}^{l+1})\left\{c_1 B\left(\frac{A}{\lambda_2^2(B-1)}\right)^l w_{k-1}^{l+1}\right. \\ &\left. - c_1 C\left(\frac{A}{\lambda_2^2(B-1)}\right)^l (w_{k-1}^{l+1})^2 + c_2\left(\frac{\lambda_2^2 - 1}{\lambda_2^2(B+1)}\right)^l\right\}. \end{aligned} \tag{91}$$

The relations for the integrability condition (45) is given by

$$\begin{aligned}\frac{\partial P_1}{\partial w_{k-1}^{l+1}} &= (-B+2Cv_{k-3}^{l+1})\left\{c_1 B\left(\frac{A}{\lambda_2^2(B-1)}\right)^l - 2c_1 C\left(\frac{A}{\lambda_2^2(B-1)}\right)^l w_{k-1}^{l+1}\right\},\\ \frac{\partial P_2}{\partial w_k^l} &= c_1 B\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} - c_1 C\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_k^l.\end{aligned} \tag{92}$$

The component ϕ of the conserved vector we obtain using Eq (46)

$$\begin{aligned}\phi =& c_1 B\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_k^l w_{k-1}^{l+1}\left\{1-(B-2Cv_{k-3}^{l+1})\left(\frac{A}{\lambda_2^2(B-1)}\right)\right\}\\ &- c_1 C\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_{k-1}^{l+1} w_k^l\left\{w_k^l - (B-2Cv_{k-3}^{l+1})\left(\frac{A}{\lambda_2^2(B-1)}\right)w_{k-1}^{l+1}\right\}\\ &+ c_2\left(\frac{\lambda_2^2-1}{\lambda_2^2(B+1)}\right)^{l-1}\left\{w_{k-1}^{l+1} - (B-2Cv_{k-3}^{l+1})\left(\frac{\lambda_2^2-1}{\lambda_2^2(B-1)}\right)w_k^l\right\} + G(k,l).\end{aligned} \tag{93}$$

To obtain the second component ψ of the conserved vector we proceed now. We take the characteristic $R_2 = R_2(k, l, w_k^l)$ and by using (44) we get

$$\begin{aligned}&R_2(k,l,w_k^l) + (B-2Cv_{k+2}^l)\big[R_2(k-1,l+1,w_{k-1}^{l+1})\\ &\quad - (B-2Cw_{k+1}^l)R_2(k+1,l,w_{k+1}^l) - (B-2Cw_{k+1}^l)R_2(k+2,l+1,v_{k+2}^{l+1})\big] = 0.\end{aligned} \tag{94}$$

By applying the operator (56) we get

$$R_2'(k,l,w_k^l) - (B-2Cw_k^l)(B-2Cv_{k+2}^l)R_2'(k-1,l+1,w_{k-1}^{l+1}) = 0. \tag{95}$$

We simplify the above equation to get

$$R_2(k,l,w_k^l) = Bc_1(k,l)w_k^l - Cc_1(k,l)(w_k^l)^2 + c_2(k,l). \tag{96}$$

Further using (94) and (95), we have

$$R_2(k,l,w_k^l) = c_1(B)^{2l-1}w_k^l - c_1 C(B)^{2l-2}(w_k^l)^2 + c_2\left(\frac{B+1}{B}\right)^{l-1}. \tag{97}$$

On the other hand, the relation for R_1 is as follows

$$R_1(k,l+1,w_k^{l+1}) = -(B-2Cv_{k-2}^{l+2})\left\{c_1(B)^{2l+1}w_k^{l+1} - c_1 C(B)^{2l}(w_k^{l+1})^2 + c_2\left(\frac{B+1}{B}\right)^l\right\}. \tag{98}$$

The relation for condition (45) becomes

$$\begin{aligned}\frac{\partial R_1}{\partial w_k^{l+1}} &= -(B-2Cv_{k-2}^{l+2})\{c_1(B)^{2l+1} - 2c_1 C(B)^{2l}w_k^{l+1}\},\\ \frac{\partial R_2}{\partial w_k^l} &= c_1(B)^{2l-1} - 2c_1 C(B)^{2l-2}w_k^l.\end{aligned} \tag{99}$$

From Eq (46), we obtain the second component ψ of the conserved vector as

$$\begin{aligned}\psi =& c_1(B)^{2l-1}w_k^l w_k^{l+1}\{1-(B-2Cv_{k-2}^{l+2})\lambda_2^2\}\\ &- c_1C(B)^{2l-2}w_k^l w_k^{l+1}\{w_k^l-\lambda_2^2(B-2Cv_{k-2}^{l+2})w_k^{l+1}\}\\ &+ c_2\left(\frac{B+1}{B}\right)^{l-1}\{w_k^{l+1}-(B-2Cv_{k-2}^{l+2})w_k^l\}+H(k,l).\end{aligned} \tag{100}$$

Consequently, we arrive at the following theorem.

Theorem 3.2 *The conserved vector for the discrete KPP equation (79) has components*

$$\begin{aligned}\phi =& c_1B\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_k^l w_{k-1}^{l+1}\left\{1-(B-2Cv_{k-3}^{l+1})\left(\frac{A}{\lambda_2^2(B-1)}\right)\right\}\\ &- c_1C\left(\frac{A}{\lambda_2^2(B-1)}\right)^{l-1} w_{k-1}^{l+1}w_k^l\left\{w_k^l-(B-2Cv_{k-3}^{l+1})\left(\frac{A}{\lambda_2^2(B-1)}\right)w_{k-1}^{l+1}\right\}\\ &+ c_2\left(\frac{\lambda_2^2-1}{\lambda_2^2(B+1)}\right)^{l-1}\left\{w_{k-1}^{l+1}-(B-2Cv_{k-3}^{l+1})\left(\frac{\lambda_2^2-1}{\lambda_2^2(B-1)}\right)w_k^l\right\}+(-1)k \\ \psi =& c_1(B)^{2l-1}w_k^l w_k^{l+1}\{1-(B-2Cv_{k-2}^{l+2})\lambda_2^2\}\\ &- c_1C(B)^{2l-2}w_k^l w_k^{l+1}\{w_k^l-\lambda_2^2(B-2Cv_{k-2}^{l+2})w_k^{l+1}\}\\ &+ c_2\left(\frac{B+1}{B}\right)^{l-1}\{w_k^{l+1}-(B-2Cv_{k-2}^{l+2})w_k^l\}+(-1)^l.\end{aligned} \tag{101}$$

4 Conclusions

The symmetry analysis of the PDEs related to the FN and KPP equations was done in this work. Independent of the underlying symmetry characteristics, we constructed the first integrals. The interplay between symmetries and first integrals usually lead to double reduction of the system as far as differential equations are concerned but for discrete equations, this study is its infancy and it is highly likely that this exists for discrete equations. Furthermore, the notion of a discrete manifold has not, to the best of our knowledge, been pursued but the methods of the study here may contribute to a better understanding of this novel concept. Hydon has, in papers cited here, discussed the ideas of a discrete Lie algebra-this is important as the symmetry generators of symmetry form a closed Lie algebra.

Data Availibility Statement For the current study, no data sharing is applicable because no data were investigated or developed for this article.

Declarations

Conflicts of Interest The authors professed no conflicts of interest.

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