Group invariant solutions for contaminant transport in saturated soils under radial uniform water flow background

Moshe Moses Potsane

School of Computational and Applied Mathematics,
University of the Witwatersrand,
Johannesburg, South Africa.

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Declaration

I declare that the contents of this dissertation are original except where due references have been made. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree to any other institution.

Moshe Moses Potsane

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Abstract

The transport of chemicals through soils to the groundwater or precipitation at the soils surfaces leads to degradation of the resources such as soil fertility, drinking water and so on. Serious consequences may be suffered in the long run. In this dissertation, we consider macroscopic deterministic models describing contaminant transport in saturated soils under uniform radial water flow backgrounds. The arising convection-dispersion equation given in terms of the stream functions is analyzed using classical Lie point symmetries. A number of exotic Lie point symmetries are admitted. Group invariant solutions are classified according to the elements of the one-dimensional optimal systems. We analyze the group invariant solutions which satisfy some physical boundary conditions.

The governing equation describing movements of contaminants under radial water flow background may be given in conserved form. As such, the conserved form of the governing equation may be written as a system of first order partial differential equation referred to as an auxiliary system, by an introduction of the nonlocal variable. The resulting system of equations admits a number of (local) point symmetries which induce the nonlocal symmetries for the original governing equation. We construct classes of solutions using the admitted genuine nonlocal symmetries, which include the invariant solutions obtained via corresponding point symmetries of the governing equation.

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Chapter 1

Introduction

1.1 Background on contaminant transport in soils

Describing and predicting water and contaminant movements in soils is important to the theory of soil salinity and underground water pollution. Soil salinity is the accumulation of salts such as sodium, magnesium and calcium through contamination of the soil. Among others, the sources of contamination are movements of agricultural and industrial contaminants such as fertilizers and pesticides. The effects of soil salinity include the reduction in productivity of land and the management of agriculture as plants become unable to draw enough water from the soil. As such, many authors have worked extensively on the effects and management implementations of saline soils and considerable progress has been made. In particular, Bresler *et al.* [1] made a major contribution to the theory and literature of saline soils. They have considered principles and models describing saline soils and classified the causes of salinity. It was worth noting that groundwater is polluted and soil is salinated over a large time period. As such, it is almost impossible for environmentalists to manage agriculture. In that regard, computational and mathematical models have been implemented to predict solute transport. Such methods require a quantitative understanding of the mechanisms controlling solute transport. The scope of most mathematical models pertains the fundamental advection-dispersion equation. This equation also known as the convection-dispersion equation (CDE) is a partial differential equation (PDE) and is most often used to model solute transport in porous media by dispersion and convection. Dispersion and convection are considered to be the most commanding mechanisms in solute transport [2, 3]. Dispersion is a process which results from the microscopic non-uniformity of the flow velocity in the soil's conduction pores while convection illustrates changes in solute concentrations due to water moving and carrying solutes with it [4].

The CDE can be developed both microscopically and macroscopically by Brownian motion and Fick's law respectively. Some solutions are constructed for constraint assumptions, as such, investigation and study of these problems is extremely difficult and challenging [5], even when the problem is given in terms of linear PDEs [6]. For this reason, few exact solutions for mathematical models describing solute transport exist. A compendium of exact solutions is given in [7]. Researchers have used different techniques to obtain exact solutions (see e.g. [8, 9, 10]). In particular, Chen *et al.* [11] used the power series method to constructed analytical solutions for two-dimensional advection-dispersion equation with transverse dispersivities depending linearly on the spatial variable. Yadav *et al.* [12] constructed analytical solutions for solute transport in a semi-infinite porous domain. Numerical models are able to simulate complex reactive transport phenomena but can be time consuming to construct and subject to numerical discretization errors [9]. Also, available packages have significant disagreement in their prediction of solute transport [13]. Therefore, exact solutions are very important because they are needed both as validation tests for numerical schemes and also to provide insight into the water and solute transport processes.

1.2 Aims and objectives of dissertation

In this dissertation we consider models describing contaminant transport in saturated soils. The objectives are two fold:

Firstly, to derive the Lie point symmetries and classify the group invariant solutions according to the one-dimensional optimal systems. Wherever possible we analyze the group invariant solutions.

Secondly, to derive nonlocal symmetries and construct classes of exact solutions using the derived nonlocal symmetry techniques. These solutions may not be constructed by any other symmetry methods.

1.3 Outline of dissertation

In Chapter 2, mathematical modeling of contaminant transport processes in soils is provided.

In Chapter 3, we provide a brief account on the symmetry techniques. In particular, Lie point (local) symmetries and potential (nonlocal) symmetries.

Chapter 4 deals with the classification of group invariant solutions for contaminant transport in saturated soils under radial uniform water flows. The results presented in Chapter 4 have been submitted for consideration to be published by in the journal Applied Mathematics and Computation and is under review (see [14]).

Chapter 5 deals with nonlocal symmetries and classes of exact solutions for contaminant transport in saturated soils under radial water flow background. The results in this Chapter have been submitted for possible publication in the journal Quaestiones Mathematicae (see [15]).

The conclusions are provided in Chapter 6.

Chapter 2

Derivation of the contaminant transport equation

2.1 Introduction

Transport of contaminants in soil results from convection, molecular diffusion, and hydrodynamic dispersion. The equation we consider in this dissertation stems from the macroscopic deterministic models based on local conservation laws [16]. The equation is derived by considering the convection, diffusion, and hydrodynamic dispersion fluxes of solutes in soil. Solute transport occurs in solid, liquid, and gaseous phases [17], but our main focus in this dissertation is the transport of solutes in liquid phase. The theoretical background in this Chapter is obtained from Hillel [2].

2.2 Convective transport of solutes

We first consider the convective transport of solutes, which is the mass flow of water. This concept has a convective flux $\mathbf{J_c}$ given by

$$\mathbf{J_c} = c\mathbf{V}, \tag{2.1}$$

where $\mathbf{V}$ is the volumetric Darcian water flux. This flux expresses the mass of the component passing through a unit area of porous medium normal to $\mathbf{V}$ per unit time [17].

2.3 Diffusion of solutes

Diffusion is a movement of solutes from zones where their concentration is high to where it is lower by means of random molecular motion. Diffusion processes are of great importance in the study of soil. Hillel [2] showed that diffusion in gaseous phase such as oxygen, carbon dioxide and nitrogen can have a significant influence on the soil's chemical and biological processes. Diffusion can also be associated with solutes in the soil's liquid phase. The assumption that concentration gradients exist in a solute results in a diffusion of solutes by Fick's law. Sumner [3] considered the one-dimensional mass flux due to molecular diffusion.
We consider the two-dimensional mass flux relationship $\mathbf{J_d}$ given by

$$\mathbf{J_d} = -\theta D_0 \nabla c, \tag{2.2}$$

where θ is the volumetric water content, D_0 is the diffusion coefficient for a particular solute diffusing in a bulk water, c is the solute concentration and ∇c is that solute's effective concentration gradient. Here ∇ is given in Cartesian coordinates (x, y).

2.4 Hydrodynamic dispersion of fluids

The dispersive flux is a consequence of the fact that both the velocity and the concentration vary from point to point within a fluid phase that occupies the entire void space, or part of it. The dispersive flux $\mathbf{J_h}$ is given by [18, 19] as

$$\mathbf{J_h} = -\theta D_e(v) \nabla c, \tag{2.3}$$

where D_e is the dispersion coefficient. The dispersion coefficient is found to be an increasing function of water speed given by $v = |\mathbf{V}|/\theta$. This function can be modeled by the power law $D_e = D_1 v^m$ with experiments showing that D_1, the proportionality constant is positive and $1 \leq m \leq 2$ [19].

2.5 Combined transport of solutes

We now combine the three mechanisms of solute transport, i.e. convection, diffusion, and dispersion by adding their fluxes to get the macroscopic flux. This is given by

$$\begin{aligned} \mathbf{J} &= \mathbf{J_d} + \mathbf{J_c} + \mathbf{J_h} \\ &= -\theta D_0 \nabla c + c\mathbf{V} - \theta D_e(v) \nabla c. \end{aligned} \tag{2.4}$$

Combining equation (2.4) with the continuity equation describing mass conservation

$$\frac{\partial(\theta c)}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{2.5}$$

we obtain

$$\frac{\partial(\theta c)}{\partial t} = \nabla \cdot (\theta D_0 \nabla c + \theta D_e(v) \nabla c) - \nabla(c\mathbf{V}), \tag{2.6}$$

where t is time. This equation reduces to

$$\frac{\partial(\theta c)}{\partial t} = \nabla \cdot (\theta D(v) \nabla c) - \nabla(c\mathbf{V}), \tag{2.7}$$

where $D(v) = D_0 + D_e(v)$ denotes the coefficient of hydrodynamic dispersion [17]. This is because dispersion predominates molecular diffusion ($D_e \gg D_0$) [2]. Experimental and theoretical observations show that the dispersion coefficient can take the power law form $D(v) = \lambda v^p$ with λ being a proportionality constant and $1 \leq p \leq 2$ (see e.g. [17]). Note that $\lambda \neq 0$.

For flow in saturated soils, the continuity equation is $\nabla \cdot \mathbf{V}$=0. For two-dimensional steady saturated water flows, $\theta = \theta_s$, where θ_s is the water content at saturation. By Darcy's law, $\mathbf{V} = -k_s \nabla \Phi$, where Φ is the total hydraulic pressure head, and k_s is the hydraulic conductivity at saturation [18], we get Laplace's equation $\nabla^2 \Phi = 0$. Equation (2.7) then becomes

$$\frac{\partial c}{\partial t} = \nabla \cdot (D(v) \nabla c) + k \nabla \Phi \cdot \nabla c, \tag{2.8}$$

where $k = k_s/\theta_s$ and $v = |k \nabla \Phi|$. Note that this problem becomes extremely difficult to solve exactly when v must be the modulus of the potential flow velocity field for an incompressible fluid (see e.g. [6]). The key to analyzing equation (2.8) is to view points in the plane as complex numbers, $z = x + iy$. This enables a transformation of the independent spatial variables from Cartesian coordinates (x, y) to the streamline coordinates (ϕ, ψ). If the analytic complex potential is given by, $\Psi = \phi + i\psi$, then it is known that the velocity potential, ϕ, is the real part of Ψ, where the harmonic conjugate, ψ, is the stream function (see e.g. [20]). The resulting solute transport equation is given by (see [19])

$$\frac{\partial c}{\partial t} = v^2 \overline{\nabla} \cdot [D(v) \overline{\nabla} c] + v^2 \frac{\partial c}{\partial \phi}, \tag{2.9}$$

where $\overline{\nabla} = \left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \psi} \right)$. The velocity potential $\phi(x, y)$ and the conjugate harmonic stream function $\psi(x, y)$ satisfy the Cauchy-Riemann equations $\phi_x = \psi_y$ and $\phi_y = -\psi_x$ as well as the Laplace equations $\nabla^2 \phi = 0$ and $\nabla^2 \psi = 0$. In radial water flows, the velocity potential is given by $\phi = -(Q/\theta_s) \log r$ and the stream function $\psi = -(Q/\theta_s) \arctan(y/x)$, where Q is the source strength (pumping rate). Introducing the normalized concentration, velocity potential and time given by $C = c/c_s$, $\phi = -\log R$ and $\tau = t/t_s$ respectively with c_s, t_s and R being the concentration, time at soil saturation and the distance or radius from the point source, we may write equation (2.9) as

$$\frac{\partial C}{\partial \tau} = v^2 \overline{\nabla} \cdot [D(v) \overline{\nabla} C] + v^2 \frac{\partial C}{\partial \phi}. \tag{2.10}$$

In this case, the relevant normalized point source, the water velocity and the dispersion coefficient are given by $\phi = -\log R$, $v = e^{\phi}$ and $D(v) = \lambda e^{p\phi}$, respectively (see e.g. [21]). One may simply omit the dependence of contaminant concentration on the clockwise polar angle coordinate. As such, a three dimensional equation is reduced to a two dimensional equation.

2.6 Concluding remarks

In this chapter we have provided models which describe contaminant transport in saturated soils. Equation (2.10) is the governing equation and will be analyzed given a radial water flow background.

Chapter 3

Symmetries of differential equations

3.1 Introduction

This chapter presents the underlying theory and notation used throughout the dissertation. We discuss the key features of the Lie symmetry approach for differential equations. In brief, a symmetry of a differential equation is an invertible transformation of the dependent and independent variables that does not change the original differential equation. Symmetries depend continuously on a parameter and form a group; the *one-parameter group of transformations*[1].

3.2 Partial differential equations

Definition 3.1. A scalar kth-order ($k \geq 2$) system F of s partial differential equations is defined by

$$F^{\sigma}(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \sigma = 1, \ldots, s, \tag{3.1}$$

[1]A discussion on one-parameter group of transformations is given in [22, 23, 24, 25, 26]

where $u = (u^1, u^2, \ldots, u^m)$, called the dependent variable is a function of the independent variable $x = (x^1, x^2, \ldots, x^n)$ and $u_{(1)}, \ldots, u_{(k)}$ are the first up to the kth-order partial derivatives, i.e. $u_{(1)} = \{u_i^\mu\}$, $u_{(2)} = \{u_{ij}^\mu\}$, $u_{(k)} = \{u_{i_1 \ldots i_k}^\mu\}$ for $\mu = 1, \ldots, m$ and $i, j, i_1, \ldots, i_k = 1, \ldots, n$.

Partial derivatives can be represented by the following notation

$$u_i^\mu = \frac{\partial u^\mu}{\partial x^i}, \quad u_{ij}^\mu = \frac{\partial^2 u^\mu}{\partial x^i \partial x^j}, \quad u_{ijk}^\mu = \frac{\partial^3 u^\mu}{\partial x^i \partial x^j \partial x^k}, \ldots.$$

Solving equations of the form (3.1) implies obtaining a function $u = (u^1, u^2, \ldots, u^m)$ verifying (3.1). System (3.1) is said to be form invariant if it can be written in the form

$$F^\sigma(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(k)}) = 0, \quad \sigma = 1, \ldots, \bar{s}, \tag{3.2}$$

where $\bar{x}$ and $\bar{u}$ are the new transformed variables[2].

3.3 Lie point symmetries and prolongation formulas

Definition 3.2. The total derivative operator with respect to the independent variables x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_i^\mu} + \ldots, \quad i = 1, \ldots, n. \tag{3.3}$$

We seek the one-parameter Lie group of transformations

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon \xi^i(x, u) + \mathcal{O}(\epsilon^2), \\ \bar{u}^\mu &= u^\mu + \epsilon \eta^\mu(x, u) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3.4}$$

which leave the system (3.1) invariant. These transformations are generated by the generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\mu(x, u) \frac{\partial}{\partial u^\mu}. \tag{3.5}$$

[2]Transformed variables are given by $\bar{x}^i = f^i(x, u)$ and $\bar{u}^i = g^\alpha(x, u)$, $i = 1, \ldots, n$; $\alpha = 1, \ldots, m$. See [23] for more on transformed variables.

The kth-extended transformations of (3.4) are given by

$$\left.\begin{array}{rcl}\bar{u}^{\mu}_{i} & = & u^{\mu}_{i} + \epsilon\zeta^{\mu}_{i}(x,u,u_{(1)}) + \mathcal{O}(\epsilon^{2}),\\ \bar{u}^{\mu}_{ij} & = & u^{\mu}_{ij} + \epsilon\zeta^{\mu}_{ij}(x,u,u_{(1)},u_{(2)}) + \mathcal{O}(\epsilon^{2}),\\ & \vdots & \\ \bar{u}^{\mu}_{i_1,i_2,\ldots,i_k} & = & u^{\mu}_{i_1,i_2,\ldots,i_k} + \epsilon\zeta^{\mu}_{i_1,i_2,\ldots,i_k}(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}) + \mathcal{O}(\epsilon^{2}).\end{array}\right\} \tag{3.6}$$

Theorem 3.3. *The extended infinitesimals satisfy the recursion relations*

$$\left.\begin{array}{rcl}\zeta^{\mu}_{i} & = & D_i(\eta^{\mu}) - u^{\mu}_{j}D_i(\xi^{j}),\\ \zeta^{\mu}_{ij} & = & D_j(\zeta^{\mu}_{i}) - u^{\mu}_{il}D_j(\xi^{l}),\\ & \vdots & \\ \zeta^{\mu}_{i_1,i_2,\ldots,i_k} & = & D_{i_k}(\zeta^{\mu}_{i_1,i_2,\ldots,i_{k-1}}) - u^{\mu}_{i_1,i_2,\ldots,i_{k-1}l}D_{i_k}(\xi^{l}).\end{array}\right\} \tag{3.7}$$

These recursion relations are known as the prolongation formulas.

Introducing the *Lie characteristic function* given by

$$W^{\mu} = \eta^{\mu} - \xi^{j}u^{\mu}_{j}, \tag{3.8}$$

then we can equivalently write (3.7) as

$$\left.\begin{array}{rcl}\zeta^{\mu}_{i} & = & D_i(W^{\mu}) + \xi^{j}u^{\mu}_{ji},\\ \zeta^{\mu}_{ij} & = & D_iD_j(W^{\mu}) + \xi^{k}u^{\mu}_{kj},\\ & \vdots & \\ \zeta^{\mu}_{i_1,i_2,\ldots,i_k} & = & D_{i_1}\ldots D_{i_k}(W^{\mu}) + \xi^{j}u^{\mu}_{ji_1\ldots i_k}.\end{array}\right\} \tag{3.9}$$

The kth-extended infinitesimal generator (prolonged generator) is given by

$$\begin{aligned} X^{[k]} &= \xi^{i}(x,u)\frac{\partial}{\partial x^{i}} + \eta^{\mu}(x,u)\frac{\partial}{\partial u^{\mu}} + \zeta^{\mu}_{i}(x,u,u_{(1)})\frac{\partial}{\partial u^{\mu}_{i}} + \ldots \\ &\quad + \zeta^{\mu}_{i_1i_2\ldots i_k}(x,u,u_{(1),\ldots,u_{(k)}})\frac{\partial}{\partial u^{\mu}_{i_1i_2\ldots i_k}}, \quad k \geq 1. \end{aligned} \tag{3.10}$$

3.4 Determining equations and Lie algebras

Given a system of the form (3.1), the determining equations are given by

$$X^{[k]}F^{\sigma}(x,u,u_{(1)},\ldots,u_{(k)})\big|_{F^{\sigma}(x,u,u_{(1)},\ldots,u_{(k)})=0}=0, \quad \sigma=1,2,\ldots,s. \tag{3.11}$$

This is a set of over-determined linear homogeneous partial differential equations for the unknown functions $\xi^i(x,u)$ and $\eta^{\mu}(x,u)$. An important property of the determining equations is the determination of symmetries which span the Lie algebra. A Lie algebra is a vector space $\mathcal{L}$ over some field $\mathcal{F}$ together with a binary operation called the Lie bracket. If we consider the generators

$$X_1=\xi_1^i(x,u)\frac{\partial}{\partial x^i}+\eta_1^{\mu}(x,u)\frac{\partial}{\partial u^{\mu}}, \tag{3.12}$$

and

$$X_2=\xi_2^i(x,u)\frac{\partial}{\partial x^i}+\eta_2^{\mu}(x,u)\frac{\partial}{\partial u^{\mu}}, \tag{3.13}$$

we can write the commutator, $[X_1,X_2]$ as

$$\begin{aligned}[X_1,X_2] &= \big(X_1(\xi_2^i)-X_2(\xi_1^i)\big)\frac{\partial}{\partial x^i}+\left(X_1(\eta_2^{\mu})-X_2(\eta_1^{\mu})\right)\frac{\partial}{\partial u^{\mu}}\\ &= X_1X_2-X_2X_1. \end{aligned} \tag{3.14}$$

We say that X_1 and X_2 span the vector space $\mathcal{L}$ if the following axioms hold

(a) **Bilinearity**. If $X_1,X_2,X_3\in\mathcal{L}$ then

$$[aX_1+bX_2,X_3]=a\,[X_1,X_3]+b\,[X_2,X_3]\,,\quad a,b\in\mathbb{R},$$

$$[X_1,aX_2+bX_3]=a\,[X_1,X_2]+b\,[X_1,X_3]\,,\quad a,b\in\mathbb{R}.$$

(b) **Skew-symmetry**. If $X_1,X_2\in\mathcal{L}$ then

$$[X_1,X_2]=-\,[X_2,X_1]\,.$$

(c) **Jacobi identity**. If $X_1,X_2,X_3\in\mathcal{L}$ then

$$[[X_1,X_2]\,,X_3]+[[X_2,X_3]\,,X_1]+[[X_3,X_1]\,,X_2]=0.$$

(d) **Linear combination**. If $X_1, X_2 \in \mathcal{L}$ then

$$aX_1 + bX_2 \in \mathcal{L}, \quad a, b \in \mathbb{R}.$$

(e) **Associativity**. If $X_1, X_2, X_3 \in \mathcal{L}$ then

$$X_1 + (X_2 + X_3) = (X_1 + X_2) + X_3.$$

(f) **Commutativity**. If $X_1, X_2 \in \mathcal{L}$ then

$$X_1 + X_2 = X_2 + X_1.$$

3.5 Nonlocal symmetries

A symmetry generator of the form

$$X = \xi^i(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})\frac{\partial}{\partial x^i} + \eta^\mu(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})\frac{\partial}{\partial u^\mu},$$

is known as a *local* symmetry generator if ξ^i and η^μ only depend on the independent variables x, dependent variables u and and its derivatives evaluated at x. On the other hand, a *nonlocal* symmetry generator also known as a *potential* symmetry has ξ^i and η^μ depending not only on the independent and dependent variables but also on integrals of the dependent variables. Bluman *et al.* [27] introduced a method for obtaining new classes of symmetries for a partial differential equation written in conservative form. The method is as follows; Given a scalar kth-order partial differential equation $R\{x, u\}$ with independent variable $x = (x^1, x^2, \ldots, x^n)$ and dependent variables $u = (u^1, u^2, \ldots, u^m)$. If $R\{x, u\}$ can be written in conserved form, it is possible to write the corresponding auxiliary system of PDEs $S\{x, u, v\}$ by introducing the *potential* variable $v = (v^1, v^2, \ldots, v^{n-1})$. A partial differential equation in conserved form $R\{x, u\}$ is written as

$$D_i f^i(x, u, u_{(1)}, \ldots, u_{(k-1)}) = 0, \tag{3.15}$$

where D_i is the total derivative operator with respect to the independent variables x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_i} + \ldots, \quad i = 1, \ldots, n. \tag{3.16}$$

The corresponding auxiliary system of PDEs $S\{x, u, v\}$ is given by [28]

$$\begin{aligned}
f^1(x, u, u_{(1)}, \ldots, u_{(k-1)}) &= \frac{\partial v^1}{\partial x_2}, \\
f^j(x, u, u_{(1)}, \ldots, u_{(k-1)}) &= (-1)^{j-1} \left(\frac{\partial v^j}{\partial x_{j+1}} + \frac{\partial v^{j-1}}{\partial x_{j-1}} \right), \quad 1 < k < n, \\
f^n(x, u, u_{(1)}, \ldots, u_{(k-1)}) &= (-1)^{n-1} \frac{\partial v^{n-1}}{\partial x_{n-1}}.
\end{aligned} \tag{3.17}$$

Local symmetries admitted by (3.17) may induce nonlocal (potential) symmetries of $R\{x, u\}$. Pucci and Saccomandi [29] presented necessary conditions for a PDE written in conserved form to admit potential symmetries. A link between nonlocal symmetries and reduction methods of order two is provided in [30]. Nonlocal symmetry techniques have attracted interest from many symmetry analysts (see e.g. [31, 32, 33, 34, 35]). Bluman and Kumei [28] have shown that it is possible to obtain a new class of solutions using nonlocal symmetries. These solutions may not be obtained using local symmetries. Any solution $(u(x), v(x))$ of $S\{x, u, v\}$ will define a solution $u(x)$ of $R\{x, u\}$ since $R\{x, u\}$ is enclosed in $S\{x, u, v\}$.

3.6 Invariant solutions

Definition 3.4. [28] A function $u = U(x)$ is an invariant solution of (3.1) corresponding to the generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u)^\mu \frac{\partial}{\partial u^\mu}, \tag{3.18}$$

if and only if

- $u = U(x)$ is an invariant surface of (3.18), i.e.

$$\xi^i\left(x, U(x)\right)\frac{\partial U}{\partial x^i} = \eta^\mu\left(x, U(x)\right),$$

- $u = U(x)$ solves (3.1).

The method for obtaining invariant solutions involves solving the characteristic equations[3] given by

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}. \tag{3.19}$$

3.7 Optimal systems of subalgebras

Definition 3.5. [22] An optimal system of $l-$parameter group-invariant solutions to a differential equation (or system of differential equations) is a collections of solutions with the properties

(i) Each solution in the list is invariant under some $l-$parameter symmetry group of the differential equation (or system of differential equations).

(ii) If there exists another solution which is invariant under an $l-$parameter symmetry group, then there is a further symmetry generator admitted by the equation (or system) which maps this old solution to the new one.

Clearly, an optimal system is a set of elements which lead to invariant solutions that are not equivalent by any transformation. There are different techniques used to determine optimal systems. In particular, Ovsiannikov [36] presented a technique based on finding the matrix of inner automorphism corresponding to the operators of the adjoint group of a given Lie algebra. Olver [22] presented a technique which involves representing the adjoint action as follows

$$\mathrm{Ad}\left(e^{\epsilon X_i}\right)X_j = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}(\mathrm{Ad}X_i)^n X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i,[X_i,X_j]] - \cdots, \tag{3.20}$$

[3]For more on characteristic equations see e.g. [26, 28].

where $[X_i, X_j] = X_iX_j - X_jX_i$ is the commutator of X_i and X_j. Equation (3.20) is used to identify equivalent symmetry generators, i.e. those that give rise to equivalent invariant solutions in the sense that one solution can be transformed into another by a point transformation.

3.8 Concluding remarks

In this chapter we have given a brief account on the Lie symmetry methods which will be utilized in this dissertation. First, we discussed the determination of Lie point symmetries and the construction of the one-dimensional optimal system. Furthermore, nonlocal symmetries are briefly discussed.

Chapter 4

Classification of the group invariant solutions

4.1 Introduction

In this chapter we consider a contaminant transport equation in streamline coordinates. We consider contaminant transport under steady radial water flow in saturated soils. In this case, the relevant normalized point source, the water velocity and the dispersion coefficient are given by $\phi = -\log R$, $v = e^{\phi}$ and $D(v) = \lambda e^{p\phi}$, respectively (see e.g. [21]). One may simply omit the dependence of contaminant concentration on the clockwise polar angle coordinate ψ and seek axisymmetric solutions. We employ Lie point symmetries to reduce the governing equation into ordinary differential equations (ODEs). This is primarily done by employing optimal systems, hence, classification of group invariant solutions. Various cases of p and λ are considered as the number of symmetries vary with p and λ. We note that equation (2.10) only admits extra symmetries when $p = 0$ or $p = 2$. These play an important role in determining the forms of the dispersion coefficient. We consider the constant dispersion coefficient and the velocity dependent dispersion for symmetry analysis. The derivation of the Lie point symmetries is given and invariant solutions are presented.

4.2 Lie point symmetries for a constant dispersion coefficient

Given $p = 0$, then the dispersion coefficient becomes a nonzero proportionality constant, λ. The equation in question (2.10) is given by

$$\frac{\partial C}{\partial \tau} = \lambda e^{2\phi}\frac{\partial^2 C}{\partial \phi^2} + e^{2\phi}\frac{\partial C}{\partial \phi}. \tag{4.1}$$

The Lie point symmetry generator of equation (4.1) is of the form

$$X = \xi^1(\tau,\phi,C)\frac{\partial}{\partial \tau} + \xi^2(\tau,\phi,C)\frac{\partial}{\partial \phi} + \eta(\tau,\phi,C)\frac{\partial}{\partial C}. \tag{4.2}$$

We seek to solve the determining equations given by

$$X^{[2]}\left(\frac{\partial C}{\partial \tau} - \lambda e^{2\phi}\frac{\partial^2 C}{\partial \phi^2} - e^{2\phi}\frac{\partial C}{\partial \phi}\right)\bigg|_{(4.1)} = 0. \tag{4.3}$$

Here $X^{[2]}$ is the operator:

$$X^{[2]} = X + \zeta_\tau\frac{\partial}{\partial C_\tau} + \zeta_\phi\frac{\partial}{\partial C_\phi} + \zeta_{\tau\tau}\frac{\partial}{\partial C_{\phi\phi}} + \zeta_{\tau\phi}\frac{\partial}{\partial C_{\tau\phi}} + \zeta_{\phi\phi}\frac{\partial}{\partial C_{\phi\phi}}, \tag{4.4}$$

where

$$\begin{aligned}\zeta_\tau &= D_\tau(\eta) - C_\tau D_\tau(\xi^1) - C_\phi D_\tau(\xi^2)\\ &= \eta_\tau + C_\tau(\eta_C - \xi^1_\tau) - C_\tau^2\xi^1_C - C_\phi\xi^2_\tau - C_\phi C_\tau\xi^2_C,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\zeta_\phi &= D_\phi(\eta) - C_\tau D_\phi(\xi^1) - C_\phi D_\phi(\xi^2)\\ &= \eta_\phi + C_\phi(\eta_C - \xi^2_\phi) - C_\phi^2\xi^2_C - C_\tau\xi^1_\phi - C_\phi C_\tau\xi^1_C,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\zeta_{\tau\tau} &= D_\tau(\zeta_\tau) - C_{\tau\tau}D_\tau(\xi^1) - C_{\tau\phi}D_\tau(\xi^2)\\ &= \eta_{\tau\tau} + 2C_\tau\eta_{\tau C} + C_{\tau\tau}\eta_C + C_\tau^2\eta_{CC} - 2C_{\tau\phi}\xi^2_\tau - C_\phi\xi^2_{\tau\tau} - 2C_\phi C_\tau\xi^2_{\tau C}\\ &\quad - \xi^2_C(C_\phi C_{\tau\tau} + 2C_\tau C_{\phi\tau}) - C_\phi C_\tau^2\xi^2_{CC} - 2C_{\tau\tau}\xi^1_\tau - C_\tau\xi^1_{\tau\tau} - 2C_\tau^2\xi^1_{\tau C}\\ &\quad - 3C_\tau C_{\tau\tau}\xi^1_C - C_\tau^3\xi^1_{CC},\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\zeta_{\tau\phi} = \zeta_{\phi\tau} &= D_\phi(\zeta_\tau) - C_{\tau\tau}D_\phi(\xi^1) - C_{\tau\phi}D_\phi(\xi^2) \\
&= \eta_{\tau\phi} + 2C_\tau\eta_{\phi C} + C_{\phi\tau}\eta_C + C_\tau C_\phi\eta_{CC} - C_{\tau\tau}\xi^1_\phi - C_{\tau\phi}\xi^1_\tau \\
&\quad - \xi^1_C(C_\phi C_{\tau\tau} + 2C_\tau C_{\tau\phi}) - C_\tau(\xi^1_{\phi\tau} + C_\tau\xi^1_{\phi C} + C_\phi\xi^1_{\tau C} + C_\tau C_\phi\xi^1_{CC}) \\
&\quad - \xi^2_C(C_\tau C_{\phi\phi} + 2C_\phi C_{\phi\tau}) - C_\phi(\xi^2_{\tau\phi} + C_\tau\xi^2_{\phi C} + C_\phi\xi^2_{\tau C} + C_\tau C_\phi\xi^2_{CC}), \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\zeta_{\phi\phi} &= D_\phi(\zeta_\phi) - C_{\tau\phi}D_\phi(\xi^1) - C_{\phi\phi}D_\phi(\xi^2) \\
&= \eta_{\phi\phi} + 2C_\phi\eta_{\phi C} + C_{\phi\phi}\eta_C + C_\phi^2\eta_{CC} - 2C_{\phi\phi}\xi^2_\phi - C_\phi\xi^2_{\phi\phi} - 2C_\phi^2\xi^2_{\phi C} \\
&\quad - \xi^1_C(C_\tau C_{\phi\phi} + 2C_\phi C_{\phi\tau}) - C_\phi^3\xi^2_{CC} - 2C_{\phi\tau}\xi^1_\phi - C_\tau\xi^1_{\phi\phi} - 2C_\phi C_\tau\xi^1_{\phi C} \\
&\quad - C_\phi^2 C_\tau\xi^1_{CC} - 3C_\phi C_{\phi\phi}\xi^2_C. \qquad (4.9)
\end{aligned}$$

From the determining equation (4.3), we obtain

$$\left(\zeta_\tau - e^{2\phi}\zeta_\phi - 2e^{2\phi}C_\phi\xi^2 - \lambda e^{2\phi}\zeta_{\phi\phi} - 2\lambda e^{2\phi}C_{\phi\phi}\xi^2\right)\Big|_{(4.1)} = 0. \qquad (4.10)$$

The expressions for ζ_τ, ζ_ϕ and $\zeta_{\phi\phi}$ are substituted into the determining equation (4.10). We also take into account the following equations

$$C_\tau = e^{2\phi}C_\phi + \lambda e^{2\phi}C_{\phi\phi},$$

$$C_{\tau\phi} = C_{\phi\tau} = 2e^{2\phi}C_\phi + (e^{2\phi} + 2\lambda e^{2\phi})C_{\phi\phi} + \lambda e^{2\phi}C_{\phi\phi\phi},$$

to obtain

$$\begin{aligned}
&2\lambda e^{2\phi}\xi^2 C_{\phi\phi} + 2e^{2\phi}\xi^2 C_\phi + \lambda e^{2\phi}\eta_{CC}C_\phi^2 + 2\lambda e^{2\phi}\eta_{\phi C}C_\phi \\
&-\lambda^2 e^{4\phi}\xi^1{}_{CC}C_{\phi\phi}{C_\phi}^2 - 4\lambda^2 e^{4\phi}\xi^1{}_C C_{\phi\phi}C_\phi - 2\lambda^2 e^{4\phi}\xi^1_C C_{\phi\phi\phi}C_\phi \\
&-2\lambda^2 e^{4\phi}\xi^1_{\phi C}C_{\phi\phi}C_\phi - 4\lambda^2 e^{4\phi}\xi^1_\phi C_{\phi\phi} - 2\lambda^2 e^{4\phi}\xi^1_\phi C_{\phi\phi\phi} \\
&-\lambda^2 e^{4\phi}\xi^1_{\phi\phi}C_{\phi\phi} - \lambda e^{2\phi}\xi^2_{CC}C_\phi^3 - 2\lambda e^{2\phi}\xi^2_{\phi C}C_\phi^2 - 2\lambda e^{2\phi}\xi^2_C C_{\phi\phi}C_\phi \\
&-\lambda e^{2\phi}\xi^2_{\phi\phi}C_\phi - 2\lambda e^{2\phi}\xi^2_\phi C_{\phi\phi} + \lambda e^{2\phi}\xi^1_\tau C_{\phi\phi} - \lambda e^{4\phi}\xi^1_{CC}C_\phi^3 \\
&-4\lambda e^{4\phi}\xi^1_C C_\phi^2 - 2\lambda e^{4\phi}\xi^1_{\phi C}C_\phi^2 - 2\lambda e^{4\phi}\xi^1_C C_{\phi\phi}C_\phi - 4\lambda e^{4\phi}\xi^1_\phi C_\phi \\
&-\lambda e^{4\phi}\xi^1_{\phi\phi}C_\phi - 3\lambda e^{4\phi}\xi^1_\phi C_{\phi\phi} + \xi^2_\tau C_\phi - e^{2\phi}\xi^2_\phi C_\phi + e^{2\phi}\xi^1_\tau C_\phi \\
&-e^{4\phi}\xi^1_\phi C_\phi + \lambda e^{2\phi}\eta_{\phi\phi} - \eta_\tau + e^{2\phi}\eta_\phi = 0. \qquad (4.11)
\end{aligned}$$

Since the functions to be determined $\xi^1(\tau,\phi,C)$, $\xi^2(\tau,\phi,C)$ and $\eta(\tau,\phi,C)$ are independent of the derivatives of C, we can equate coefficients of separate powers of the derivatives of C to zero.

$$
\begin{aligned}
C_\phi C_{\phi\phi\phi} &: \lambda\xi^1_C = 0,\\
C_{\phi\phi\phi} &: \lambda\xi^1_\phi = 0,\\
C_\phi^2 C_{\phi\phi} &: \lambda\xi^1_{CC} = 0,\\
C_\phi C_{\phi\phi} &: \lambda\left((2\lambda+1)e^{2\phi}\xi^1_C + \lambda e^{2\phi}\xi^1_{\phi C} + \xi^2_C\right) = 0,\\
C_{\phi\phi} &: \lambda\left(-2\xi^2 + (4\lambda+3)e^{2\phi}\xi^1_\phi + \lambda e^{2\phi}\xi^1_{\phi\phi} + 2\xi^2_\phi - \xi^1_\tau\right) = 0,\\
C_\phi^3 &: \lambda\left(\xi^2_{CC} + e^{2\phi}\xi^1_{CC}\right) = 0,\\
C_\phi^2 &: \lambda\left(-\eta_{CC} + 4e^{2\phi}\xi^1_C + 2\left(\xi^2_{\phi C} + e^{2\phi}\xi^1_{\phi C}\right)\right) = 0,\\
C_\phi &: 2e^{2\phi}\xi^2 + 2\lambda e^{2\phi}\eta_{\phi C} - \lambda e^{2\phi}\xi^2_{\phi\phi} - (4\lambda+1)e^{4\phi}\xi^1_\phi\\
&\quad -\lambda e^{4\phi}\xi^1_{\phi\phi} + \xi^2_\tau - e^{2\phi}\xi^2_\phi + e^{2\phi}\xi^1_\tau = 0,\\
\text{remainder} &: \lambda e^{2\phi}\eta_{\phi\phi} - \eta_\tau + e^{2\phi}\eta_\phi = 0.
\end{aligned}
\tag{4.12}
$$

Since $\lambda \neq 0$, then $\xi^1_\phi = 0$, $\xi^1_C = 0$ and $\xi^1_{CC} = 0$. Also, from the fourth equation in (4.12) $\xi^2_C = 0$. Equations in (4.12) simplify to the following

$$
\begin{aligned}
\xi^1_\phi &= 0\\
\xi^1_C &= 0,\\
\xi^2_C &= 0,\\
\eta_{CC} &= 0,\\
2\xi^2_\phi - 2\xi^2 - \xi^1_\tau &= 0,\\
2e^{2\phi}\xi^2 + 2\lambda e^{2\phi}\eta_{\phi C} - \lambda e^{2\phi}\xi^2_{\phi\phi} + \xi^2_\tau - e^{2\phi}\xi^2_\phi + e^{2\phi}\xi^1_\tau &= 0,\\
\lambda e^{2\phi}\eta_{\phi\phi} - \eta_\tau + e^{2\phi}\eta_\phi &= 0.
\end{aligned}
\tag{4.13}
$$

From the governing equations (4.13) it can be observed that $\eta = A(\tau,\phi) + B(\tau,\phi)C$, $\xi^1 = D(\tau)$ and $\xi^2 = E(\tau,\phi)$. Here A, B, D and E are arbitrary functions of the specified variables. The remaining determining equations are given below with the prime denoting differentiation with respect to τ.

$$2E + D' - 2E_\phi = 0, \tag{4.14}$$

$$2e^{2\phi}E + e^{2\phi}D' - e^{2\phi}E_\phi + 2\lambda e^{2\phi}B_\phi - \lambda e^{2\phi}E_{\phi\phi} + E_\tau = 0, \tag{4.15}$$

$$e^{2\phi}\left(A_\phi + CB_\phi\right) + \lambda e^{2\phi}\left(A_{\phi\phi} + CB_{\phi\phi}\right) - \left(A_{\tau\tau} + CB_{\tau\tau}\right) = 0. \tag{4.16}$$

Equation (4.16) may be separated into two equation by considering powers of C. These are given by

$$e^{2\phi}A_\phi + \lambda e^{2\phi}A_{\phi\phi} - A_\tau = 0, \tag{4.17}$$

and

$$e^{2\phi}B_\phi + \lambda e^{2\phi}B_{\phi\phi} - B_\tau = 0. \tag{4.18}$$

The solution of (4.14) given by $D(\tau,\phi) = e^\phi F(\tau) - D'/2$ substituted into (4.15) yields an expression for $B(\tau,\phi)$ given by

$$B(\tau,\phi) = \frac{1}{4\lambda}\left(2(\lambda-1)e^\phi F + 2e^{-\phi}F' - \frac{e^{-2\phi}}{2}D''\right) + G(\tau), \tag{4.19}$$

where F and G are arbitrary functions of τ. This expression is substituted into (4.18) to give

$$4(\lambda^2-1)e^{5\phi}F - 8\lambda e^{2\phi}G' + 2e^{2\phi}D'' - 4\lambda e^{2\phi}D'' - 4e^\phi F'' + D'' = 0. \tag{4.20}$$

Since the functions D, F and G are independent of ϕ we can separate (4.20) to obtain

$$(\lambda^2-1)F = 0, \tag{4.21}$$

$$2(1-2\lambda)D'' - 8\lambda G' = 0, \tag{4.22}$$

$$D''' = 0, \tag{4.23}$$

$$F'' = 0. \tag{4.24}$$

From (4.23) and (4.24) we obtain $D(\tau) = k_1 + \tau k_2 + \tau^2 k_3$ and $F(\tau) = k_4 + \tau k_5$ respectively. We note that from (4.21) three cases for λ arise since $F \neq 0$. When $\lambda = 1$ then $G(\tau) = -(k_3/2)\tau + k_6$. When $\lambda = -1$ then $G(\tau) = -(3k_3/2)\tau + k_6$. When λ is neither 1, -1 nor 0, then the $G(\tau) = (1/2\lambda)\left(\tau(k_3 - 2\lambda k_3)\right) + k_4$. The expressions for ξ^1, ξ^2 and η are given below

Case 1: $\lambda = 1$.

$$\xi^1 = k_1 + \tau(k_2 + \tau k_3), \tag{4.25}$$

$$\xi^2 = -\frac{k_2}{2} - \tau k_3 + e^{\phi} k_4 + \tau e^{\phi} k_5, \tag{4.26}$$

$$\eta = C\left(-\frac{\tau k_3}{2} + \frac{e^{-2\phi}}{4}(-k_3 + 2e^{\phi}k_5) + k_6\right) + A(\tau, \phi), \tag{4.27}$$

where $A(\tau, \phi)$ is an arbitrary function satisfying equation (4.1) and $k_1, \ldots, k_6$ are arbitrary constants. The resulting one-parameter Lie point symmetries besides the infinite symmetry generator, $X_\omega = \omega(\tau, \phi)\dfrac{\partial}{\partial C}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau}, \\
X_2 &= C\frac{\partial}{\partial C}, \\
X_3 &= e^{\phi}\frac{\partial}{\partial \phi}, \\
X_4 &= 2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi}, \\
X_5 &= \tau\frac{\partial}{\partial \phi} + \left(\frac{e^{-2\phi}}{2}\right)C\frac{\partial}{\partial C}, \\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} - \left(\frac{\tau}{2} + \frac{e^{-2\phi}}{4}\right)C\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{4.28}$$

Case 2: $\lambda = -1$.

$$\xi^1 = k_1 + \tau(k_2 + \tau k_3), \tag{4.29}$$

$$\xi^2 = -\frac{k_2}{2} - \tau k_3 + e^{\phi} k_4 + \tau e^{\phi} k_5, \tag{4.30}$$

$$\eta = \frac{C}{4}\left(e^{-2\phi} k_3 - 6\tau k_3 - 2e^{-\phi} k_5 + 4e^{\phi}(k_4 + \tau k_5) + 4k_6\right) + \omega(\tau, \phi), \tag{4.31}$$

where $\omega(\tau, \phi)$ is an arbitrary function satisfying equation (4.1) and $k_1, \ldots, k_6$ are arbitrary constants. The resulting one-parameter Lie point symmetries besides the infinite symmetry generator, $X_\omega = \omega(\tau, \phi)\dfrac{\partial}{\partial C}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C},\\
X_3 &= e^{\phi}\frac{\partial}{\partial \phi} - e^{\phi} C\frac{\partial}{\partial C},\\
X_4 &= 2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi},\\
X_5 &= \tau\frac{\partial}{\partial \phi} + \left(\tau - \frac{e^{-2\phi}}{2}\right) C\frac{\partial}{\partial C},\\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} - \left(\frac{3\tau}{2} - \frac{e^{-2\phi}}{4}\right) C\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{4.32}$$

Invariant solutions for the negative dispersion coefficient, i.e. $\lambda = -1$ are explored in the next Chapter.

Case 3: $\lambda \neq 1, -1, 0$. Arbitrary λ.

$$\xi^1 = k_1 + \tau(k_2 + \tau k_3), \tag{4.33}$$

$$\xi^2 = -\frac{k_2}{2} - \tau k_3, \tag{4.34}$$

$$\eta = -\frac{C}{4\lambda}\left(e^{-2\phi}k_3 + 2(2\lambda - 1)\tau k_3 - 4\lambda k_4\right) + \omega(\tau, \phi), \tag{4.35}$$

where $\omega(\tau, \phi)$ is an arbitrary function satisfying equation (4.1) and $k_1, \ldots, k_4$ are arbitrary constants. The resulting one-parameter Lie point symmetries besides the infinite symmetry generator, $X_\omega = \omega(\tau, \phi)\dfrac{\partial}{\partial C}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C},\\
X_3 &= 2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi},\\
X_4 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left(\lambda\tau - \frac{\tau}{2} + \frac{e^{-2\phi}}{4}\right)C\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{4.36}$$

4.3 Group invariant solutions for a constant dispersion coefficient with $\lambda = 1$

4.3.1 Optimal system for symmetry generators in (4.28)

We adopt the method in [22] to construct the one-dimensional system of sub-algebras of the algebra spanned by the base vectors. To construct the optimal system we first need to determine the commutators of the admitted symmetries. We consider symmetry generators (4.28). Given a general operator

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6, \tag{4.37}$$

where $a_1, \ldots, a_6$ are arbitrary constants. We aim to simplify as many of the coefficients $a_1, \ldots, a_6$ as possible using the adjoint action. We first construct the commutators of the admitted symmetries in (4.28). For illustration we show calculations in the following examples:

$$\begin{aligned}[X_1, X_2] &= X_1X_2 - X_2X_1 \\ &= \frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial C}\right) - \frac{\partial}{\partial C}\left(\frac{\partial}{\partial \tau}\right) \\ &= 0, \end{aligned} \tag{4.38}$$

$$\begin{aligned}[X_1, X_4] &= X_1X_4 - X_4X_1 \\ &= \frac{\partial}{\partial \tau}\left(2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi}\right) - \left(2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi}\right)\left(\frac{\partial}{\partial \tau}\right) \\ &= 2\frac{\partial}{\partial \tau} \\ &= 2X_1, \end{aligned} \tag{4.39}$$

and so on.

All the calculations are summarized in Table 4.1.

Table 4.1: Commutators of the admitted symmetries (4.28).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$2X_1$	$2X_3$	$4X_4 - 2X_2$
X_2	0	0	0	0	0	0
X_3	0	0	0	X_3	$-X_2$	$2X_5$
X_4	$-2X_1$	0	$-X_3$	0	X_5	$2X_6$
X_5	$-2X_3$	0	X_2	$-X_5$	0	0
X_6	$2X_2 - 4X_4$	0	$-2X_5$	$-2X_6$	0	0

Furthermore we construct a set of one-dimensional subalgebras which are equivalent to a unique element of the set under some element of the adjoint representation given by

$$\mathrm{Ad}\left(e^{\epsilon X_i}\right) X_j = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}(\mathrm{Ad}X_i)^n X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots, \tag{4.40}$$

where the commutator of X_i and X_j is defined above. For illustration we show calculations in the following examples:

$$\begin{aligned} \mathrm{Ad}\left(e^{\epsilon X_4}\right) X_1 &= X_1 - \epsilon\,[X_4, X_1] + \frac{\epsilon^2}{2!}\,[X_4, [X_4, X_1]] - \cdots \\ &= X_1 + 2\epsilon X_1 + 2\epsilon^2 X_1 + \cdots \\ &= \exp{(2\epsilon)} X_1, \end{aligned} \tag{4.41}$$

$$\begin{aligned} \mathrm{Ad}\left(e^{\epsilon X_4}\right) X_3 &= X_3 - \epsilon\,[X_5, X_3] + \frac{\epsilon^2}{2!}\,[X_5, [X_5, X_3]] - \cdots \\ &= X_3 - \epsilon X_2 + \frac{1}{2!}\epsilon^2\,[X_5, X_2] \\ &= -\epsilon X_2 + X_3, \end{aligned} \tag{4.42}$$

and so on.

All the calculations are summarized in Table 4.2.

Table 4.2: Adjoint representation for the base vectors (4.28).

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	$-2\epsilon X_1 + X_4$	$-2\epsilon X_3 + X_5$	$4\epsilon^2 X_1 + 2\epsilon X_2 - 4\epsilon X_4 + X_6$
X_2	X_1	X_2	X_3	X_4	X_5	X_6
X_3	X_1	X_2	X_3	$-\epsilon X_3 + X_4$	$\epsilon X_2 + X_5$	$-\epsilon^2 X_2 + X_6 - 2\epsilon X_5$
X_4	$e^{2\epsilon} X_1$	X_2	$e^{\epsilon} X_3$	X_4	$e^{-\epsilon} X_5$	$e^{-2\epsilon} X_6$
X_5	$X_1 + 2\epsilon X_3 - \epsilon^2 X_2$	X_2	$-\epsilon X_2 + X_3$	$X_4 + \epsilon X_5$	X_5	X_6
X_6	$X_1 - 2\epsilon X_2 + 4\epsilon X_4 + 4\epsilon^2 X_6$	X_2	$X_3 + 2\epsilon X_5$	$X_4 + 2\epsilon X_6$	X_5	X_6

It remains to use the adjoint table to simplify as much as possible the constants in equation (4.37). If X is given as in (4.37), then

$$\widetilde{X} = \sum_{i=1}^{6} \widetilde{a_i} X_i = \mathrm{Ad}(\exp(\alpha X_1)) \circ \mathrm{Ad}(\exp(\beta X_6))X. \tag{4.43}$$

This means referring to Table 4.2, we firstly act on (4.37) by $\mathrm{Ad}(\exp(\alpha X_1))$ to obtain

$$\begin{aligned} X' &= (a_1 - 2\alpha a_4 + 4\alpha^2 a_6)X_1 + (a_2 + 2\alpha a_6))X_2 \\ &\quad + (a_3 - 2\alpha a_5)X_3 + (a_4 - 4\alpha a_6)X_4 + a_5 X_5 + a_6 X_6, \end{aligned} \tag{4.44}$$

and acting on (4.44) by $\mathrm{Ad}(\exp(\beta X_6))$ we obtain

$$\begin{aligned} \widetilde{X} &= (a_1 - 2\alpha a_4 + 4\alpha^2 a_6)X_1 + \left[a_2 + 2\alpha a_6 - 2\beta(a_1 - 2\alpha a_4 + 4\alpha^2 a_6)\right] X_2 \\ &\quad + (a_3 - 2\alpha a_5)X_3 + \left[a_4 - 4\alpha a_6 + 4\beta(a_1 - 2\alpha a_4 + 4\alpha^2 a_6)\right] X_4 \\ &\quad + [2\beta(a_3 - 2\alpha a_5) + a_5] \\ &\quad + \left[a_6 + 2\beta(a_4 - 4\alpha a_6) + 4\beta^2(a_1 - 2\alpha a_4 + 4\alpha^2 a_6)\right] X_6. \end{aligned} \tag{4.45}$$

This has the coefficients

$$\left.\begin{aligned} \widetilde{a_1} &= a_1 - 2\alpha a_4 + 4\alpha^2 a_6, \\ \widetilde{a_4} &= a_4 - 4\alpha a_6 + 4\beta(a_1 - 2\alpha a_4 + 4\alpha^2 a_6), \\ \widetilde{a_6} &= a_6 + 2\beta(a_4 - 4\alpha a_6) + 4\beta^2(a_1 - 2\alpha a_4 + 4\alpha^2 a_6). \end{aligned}\right\} \tag{4.46}$$

The key here, is the recognition of the function $\eta(X) = a_4^2 - 4a_1 a_6$, which is invariant under the full adjoint action [22]. To begin the simplification we concentrate on the constants a_1, a_4, a_6.

Three cases arise.

Case 1: If $\eta(X) > 0$, then we choose α to be a real root of $a_1 - 2\alpha a_4 + 4\alpha^2 a_6 = 0$ and $\beta = a_6/(8\alpha a_6 - 2a_4)$. This implies $\widetilde{a_1} = \widetilde{a_6} = 0$ and $\widetilde{a_4} = \sqrt{\eta(X)} \neq 0$. Thus, X is equivalent to the vector $\widetilde{X} = X_4 + \widetilde{a_2} X_2 + \widetilde{a_3} X_3 + \widetilde{a_5} X_5$. Acting on $\widetilde{X}$ by adjoint maps generated by X_5 and X_3, namely $\mathrm{Ad}(\exp(-\widetilde{a_5} X_5))$ and $\mathrm{Ad}(\exp(\widetilde{a_3} X_3))$, as such X_5 and X_3 in $\widetilde{X}$ vanish. No further simplifications are possible, therefore X is equivalent to a multiple of $X_4 + aX_2$ for some $a \in \mathbb{R}$ provided $\eta(X) > 0$.

Case 2: If $\eta(X) < 0$, we set $\alpha = 0$ and $\beta = -a_4/4a_1$ so that $\widetilde{a_4} = 0$. For the simplification process, one may then assume both the coefficients of X_1 and X_6 to be unity. Thus, X is equivalent to the vector $\widetilde{X} = X_1 + X_6 + \widetilde{a_2}X_2 + \widetilde{a_3}X_3 + \widetilde{a_5}X_5$. Acting on $\widetilde{X}$ by adjoint maps generated by X_5 and X_3, namely $\mathrm{Ad}\left(\exp\left(\left(-\widetilde{a_3}/2\right)X_5\right)\right)$ and $\mathrm{Ad}\left(\exp\left(\left(\widetilde{a_5}/2\right)X_3\right)\right)$, as such X_5 and X_3 in $\widetilde{X}$ vanish. No further simplifications are possible, as such X is equivalent to $X_1 + X_6 + aX_2$, $a \in \mathbb{R}$ given $\eta(X) < 0$.

Case 3: $\eta(X) = 0$. Two subcases arise.

(i) If not all the coefficients a_1, a_4, a_6 are zero, then we are free to choose α and β such that $\widetilde{a_1} \neq 0$ and $\widetilde{a_4} = \widetilde{a_6} = 0$. In this case X is equivalent to a multiple of $\widetilde{X} = X_1 + \widetilde{a_2}X_2 + \widetilde{a_3}X_3 + \widetilde{a_5}X_5$. Acting on $\widetilde{X}$ by adjoint maps generated by X_1 and X_3, namely $\mathrm{Ad}\left(\exp\left(\left(\widetilde{a_3}/2\widetilde{a_5}\right)X_5\right)\right)$ and $\mathrm{Ad}\left(\exp\left(\left(-\widetilde{a_2}/\widetilde{a_5}\right)X_3\right)\right)$, $\widetilde{a_5} \neq 0$, as such X_2 and X_3 in $\widetilde{X}$ vanish. This means we now have $\widetilde{X'} = X_1 + \widetilde{a_5}$. If we further act on $\widetilde{X'}$ by any group generated by X_4, namely, $\mathrm{Ad}\left(\exp\left(\epsilon X_4\right)\right)$ then we obtain $\widetilde{X'} = X_1 + \widetilde{a_5}e^{2\epsilon}X_5$. Therefore, depending on the sign of $\widetilde{a_5}$ find that X is equivalent to a multiple of $X_1 \pm X_5$. If on the other hand $\widetilde{a_5} = 0$, then acting on $\widetilde{X}$ by $\mathrm{Ad}\left(\exp\left(-\left(\widetilde{a_3}/2\right)X_5\right)\right)$ we find that X is equivalent to a multiple of $X_1 + aX_2, \quad a \in \mathbb{R}$.

(ii) If all the coefficients a_1, a_4, a_6 are zero, and assuming $a_3 \neq 0$ (say $a_3 = 1$), then acting on X by $\mathrm{Ad}\left(\exp\left(\widetilde{a_2}X_5\right)\right)$ and $\mathrm{Ad}\left(\exp\left(\left(-\widetilde{a_5}/2\right)X_5\right)\right)$ yield X_3, a multiple of X. If $a_3 = 0$ but $a_5 \neq 0$, acting by any group generated by X_1, namely, $\mathrm{Ad}\left(\exp\left(\epsilon X_1\right)\right)$ gives a nonzero coefficient in front X_3 implying that this case is similar to the case when $a_3 \neq 0$. Thus, the only remaining vectors are multiples of X_2.

The set of one-dimensional optimal system is given by

$$\{X_4 + aX_2, X_1 + X_6 + aX_2, X_1 \pm X_5, X_1 + aX_2, X_2, X_3\}, \quad a \in \mathbb{R}. \tag{4.47}$$

If we admit a discrete symmetry so that $X_1 + X_5$ is mapped to $X_1 - X_5$, then the number of the elements in the optimal system is reduced by one.

4.3.2 Symmetry reductions and group invariant solutions

We construct group invariant solutions for equation (4.1) with $\lambda = 1$. This is done by the use of symmetries by reducing a PDE to an ODE. We use members of the optimal system to perform the reductions and solve the equation, hence we classify the group invariant solutions by the optimal system. Wherever they appear k_1 and k_2 are arbitrary constants, ${}_1F_1(b, c; z)$ and $U(b, c; z)$ are the confluent hypergeometric functions, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are the Airy functions, while $J_n(z)$ and $\Gamma(z)$ represent the Bessel function of first kind and Euler gamma function respectively and $\mathrm{erf}(z)$ represents an error function. A well documented review of such functions is presented by Abramowitz and Stegun [37]. Some of the solutions are given in terms of complex numbers, i.e. we have an imaginary unit satisfying $i = \sqrt{-1}$.

(i) Invariance under $X_4 + aX_2$:

$$X_4 + aX_2 = 2\tau\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi} + aC\frac{\partial}{\partial C}.$$

The corresponding characteristic equation is given by

$$\frac{d\tau}{2\tau} = -\frac{d\phi}{1} = \frac{dC}{aC}. \tag{4.48}$$

Solving the characteristic equation (4.48) gives the following integrals

$$I_1 = \tau e^{2\phi}, \tag{4.49}$$

$$I_2 = \frac{C}{\tau^{a/2}}. \tag{4.50}$$

Equation (4.49) is obtained by solving the first two terms in (4.48) while (4.50) is obtained by solving the first and the last term in (4.48). This gives rise to the group invariant solution

$$C = \tau^{\frac{a}{2}} G(\gamma), \tag{4.51}$$

where $\gamma = \tau e^{2\phi}$ denotes the similarity variable. Upon substitution of (4.51) into (4.1) we obtain the following second order ordinary differential equation

$$\gamma^3 G'' + \frac{1}{4}\gamma(6\gamma - 1)G' - \frac{1}{8}aG = 0, \tag{4.52}$$

where a prime denotes differentiation with respect to the similarity variable γ. The solution to (4.52) varies depending on the sign of a. This gives rise to three cases, i.e. $a < 0$, $a = 0$, $a > 0$.

(a) $a < 0$. The solution of (4.52) is given by

$$G(\gamma) = \frac{1}{2}\sqrt{\frac{1}{\gamma}}\ k_2\ {}_1F_1\left(\frac{a+1}{2}; \frac{3}{2}; -\frac{1}{4\gamma}\right) + k_1\ {}_1F_1\left(\frac{a}{2}; \frac{1}{2}; -\frac{1}{4\gamma}\right), \tag{4.53}$$

and consequently the group invariant solution is given by

$$C(\tau, \phi) = \tau^{a/2}\left[\frac{1}{2}\sqrt{\frac{1}{\tau e^{2\phi}}}\ k_2\ {}_1F_1\left(\frac{a+1}{2}; \frac{3}{2}; -\frac{1}{4\tau e^{2\phi}}\right) + k_1\ {}_1F_1\left(\frac{a}{2}; \frac{1}{2}; -\frac{1}{4\tau e^{2\phi}}\right)\right]. \tag{4.54}$$

(b) $a = 0$. The solution of (4.52) is given by

$$G(\gamma) = k_2 - 2\sqrt{\pi}\ k_1\ \mathrm{erf}\left(\frac{1}{2\sqrt{\gamma}}\right), \tag{4.55}$$

and consequently the group invariant solution is given by

$$C(\tau, \phi) = k_2 - 2\sqrt{\pi}\ k_1\ \mathrm{erf}\left(\frac{1}{2\sqrt{\tau e^{2\phi}}}\right). \tag{4.56}$$

(c) $a > 0$. The solution of (4.52) is given by

$$G(\gamma) = \frac{1}{2}\sqrt{\frac{1}{\gamma}}\ k_2\ {}_1F_1\left(\frac{1-a}{2}; \frac{3}{2}; -\frac{1}{4\gamma}\right) + k_1\ {}_1F_1\left(-\frac{a}{2}; \frac{1}{2}; -\frac{1}{4\gamma}\right), \tag{4.57}$$

and consequently the group invariant solution is given by

$$C(\tau,\phi) = \tau^{a/2}\left[\frac{1}{2}\sqrt{\frac{1}{\tau e^{2\phi}}}\; k_2 \; {}_1F_1\left(\frac{1-a}{2};\frac{3}{2};-\frac{1}{4\tau e^{2\phi}}\right) + k_1 \; {}_1F_1\left(-\frac{a}{2};\frac{1}{2};-\frac{1}{4\tau e^{2\phi}}\right)\right]. \tag{4.58}$$

(ii) Invariance under $X_1 + aX_2$:

$$X_1 + aX_2 = \frac{\partial}{\partial \tau} + aC\frac{\partial}{\partial C}.$$

The corresponding characteristic equation is

$$\frac{d\tau}{1} = \frac{d\phi}{0} = \frac{dC}{aC}. \tag{4.59}$$

Solving the characteristic equation (4.59) gives the following integrals

$$I_1 = \phi, \tag{4.60}$$

$$I_2 = \frac{C}{e^{a\tau}}. \tag{4.61}$$

This gives rise to the group invariant solution

$$C = e^{a\tau}G(\gamma), \tag{4.62}$$

where $\gamma = \phi$ is the similarity variable. Substituting (4.62) into (4.1) gives the ODE

$$G'' + G' - ae^{-2\phi}G = 0, \tag{4.63}$$

where prime again denotes differentiation with respect to the similarity variable γ. The solution for (4.63) will also depend on the sign of a.

(a) $a < 0$. The solution for (4.63) is given by

$$G(\gamma) = k_1 \cos(\sqrt{a}e^{-\gamma}) - k_2 \sin(\sqrt{a}e^{-\gamma}), \tag{4.64}$$

and the group invariant solution is given by

$$C(\tau,\phi) = e^{a\tau}\left(k_1 \cos(\sqrt{a}e^{-\phi}) - k_2 \sin(\sqrt{a}e^{-\phi})\right). \tag{4.65}$$

(b) $a = 0$. The solution for (4.63) is given by

$$G(\gamma) = -e^{-\gamma}k_1 + k_2, \tag{4.66}$$

and the group invariant solution is given by

$$C(\tau, \phi) = -e^{-\phi}k_1 + k_2. \tag{4.67}$$

(c) $a > 0$. The solution for (4.63) is given by

$$G(\gamma) = k_1 \cosh(\sqrt{a}e^{-\gamma}) - i\ k_2 \sinh(\sqrt{a}e^{-\gamma}), \tag{4.68}$$

and the group invariant solution is given by

$$C(\tau, \phi) = e^{a\tau}\left(k_1 \cosh(\sqrt{a}e^{-\phi}) - i\ k_2 \sinh(\sqrt{a}e^{-\phi})\right). \tag{4.69}$$

(iii) Invariance under $X_1 + X_5$ has the characteristic equation given by

$$\frac{d\tau}{1} = \frac{d\phi}{2e^{\phi}\tau} = \frac{dC}{e^{-\phi}C}. \tag{4.70}$$

Solving (4.70) yields the group invariant solution

$$C = \exp\left(\tau e^{-\phi} + \frac{2}{3}\tau^3\right)G(\gamma), \tag{4.71}$$

where $\gamma = \tau^2 + e^{-\phi}$ is the similarity variable. Substituting (4.71) into (4.1) gives the ODE

$$G'' - \gamma G = 0, \tag{4.72}$$

where prime again denotes differentiation with respect to the similarity variable γ. The solution to (4.72) is given by

$$G(\gamma) = \mathrm{Ai}(\gamma)k_1 + \mathrm{Bi}(\gamma)k_2, \tag{4.73}$$

and the group invariant solution is given by

$$C(\tau, \phi) = \exp\left(\tau e^{-\phi} + \frac{2}{3}\tau^3\right)\left(\mathrm{Ai}(\tau^2 + e^{-\phi})k_1 + \mathrm{Bi}(\tau^2 + e^{-\phi})k_2\right). \tag{4.74}$$

(iv) Invariance under $X_1 + X_6 + aX_2$ has the characteristic equation

$$\frac{d\tau}{1+4\tau^2} = -\frac{d\phi}{4\tau} = \frac{dC}{(a - 2\tau - e^{-2\phi})C}, \tag{4.75}$$

and this gives the group invariant solution

$$C(\tau,\phi) = \exp\left(-\frac{\tau}{e^{2\phi(1+4\tau^2)}} - \frac{1}{4}\ln(1+4\tau^2) + \frac{a}{2}\arctan(2\tau)\right)G(\gamma), \tag{4.76}$$

where $\gamma = e^{\phi}\sqrt{1+4\tau^2}$ is again the similarity variable. Upon substituting the group invariant solution (4.76) into (4.1) we obtain the ODE

$$\gamma^6 G'' + 2\gamma^5 G' + (1 - a\gamma^2)G = 0, \tag{4.77}$$

where prime denotes differentiation with respect to the similarity variable γ. The solution for (4.77) will also depend on the sign of a.

(a) $a < 0$ The solution for (4.77) is given by

$$\begin{aligned} G(\gamma) = {} & e^{-i/2\gamma^2}\sqrt{\frac{1}{2\gamma^2}}\; k_2\; {}_1F_1\left(\frac{3+ia}{4}; \frac{3}{2}; \frac{i}{\gamma^2}\right) \\ & + e^{-i/2\gamma^2}\sqrt{\frac{1}{2\gamma^2}}\; k_1\; U\left(\frac{3+ia}{4}; \frac{3}{2}; \frac{i}{\gamma^2}\right), \end{aligned} \tag{4.78}$$

and the group invariant solution is given by

$$\begin{aligned} C(\tau,\phi) = {} & \exp\left(-\frac{\tau}{e^{2\phi}(1+4\tau^2)} - \frac{1}{4}\ln(1+4\tau^2) + \frac{a}{2}\arctan(2\tau)\right)\times \\ & \left[e^{-i/(2e^{2\phi}(1+4\tau^2))}\sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; k_2\; {}_1F_1\left(\frac{3+ia}{4}; \frac{3}{2}; \frac{i}{e^{2\phi}(1+4\tau^2)}\right)\right. \\ & \left. + e^{-i/(2e^{2\phi}(1+4\tau^2))}\sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; k_1\; U\left(\frac{3+ia}{4}; \frac{3}{2}; \frac{i}{e^{2\phi}(1+4\tau^2)}\right)\right]. \end{aligned} \tag{4.79}$$

(b) $a = 0$. The solution for (4.77) is given by

$$G(\gamma) = \sqrt{\frac{1}{2\gamma}}\; J_{-\frac{1}{4}}\left(\frac{1}{2\gamma^2}\right) k_1\Gamma\left(\frac{3}{4}\right) + \sqrt{\frac{1}{2\gamma}}\; J_{\frac{1}{4}}\left(\frac{1}{2\gamma^2}\right) k_2\Gamma\left(\frac{5}{4}\right), \tag{4.80}$$

and the group invariant solution is given by

$$C(\tau,\phi) = \exp\left(-\frac{\tau}{e^{2\phi}(1+4\tau^2)} - \frac{1}{4}\ln(1+4\tau^2)\right)\times \left[\sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; J_{-\frac{1}{4}}\left(\frac{1}{2e^{2\phi}(1+4\tau^2)}\right) k_1\Gamma\left(\frac{3}{4}\right) + \sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; J_{\frac{1}{4}}\left(\frac{1}{2e^{2\phi}(1+4\tau^2)}\right) k_2\Gamma\left(\frac{5}{4}\right)\right]. \tag{4.81}$$

(c) $a > 0$. The solution for (4.77) is given by

$$G(\gamma) = e^{-i/2\gamma^2}\sqrt{\frac{1}{2\gamma^2}}\; k_2\; {}_1F_1\left(\frac{3-ia}{4};\frac{3}{2};\frac{i}{\gamma^2}\right) + e^{-i/2\gamma^2}\sqrt{\frac{1}{2\gamma^2}}\; k_1\; U\left(\frac{3-ia}{4};\frac{3}{2};\frac{i}{\gamma^2}\right), \tag{4.82}$$

and the group invariant solution is given by

$$C(\tau,\phi) = \exp\left(-\frac{\tau}{e^{2\phi}(1+4\tau^2)} - \frac{1}{4}\ln(1+4\tau^2) + \frac{a}{2}\arctan(2\tau)\right)\times \left[e^{-i/(2e^{2\phi}(1+4\tau^2))}\sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; k_2\; {}_1F_1\left(\frac{3-ia}{4};\frac{3}{2};\frac{i}{e^{2\phi}(1+4\tau^2)}\right) + e^{-i/(2e^{2\phi}(1+4\tau^2))}\sqrt{\frac{1}{2e^{2\phi}(1+4\tau^2)}}\; k_1\; U\left(\frac{3-ia}{4};\frac{3}{2};\frac{i}{e^{2\phi}(1+4\tau^2)}\right)\right]. \tag{4.83}$$

4.4 Invariant solutions for a constant dispersion coefficient with $\lambda \neq 0, 1, -1$

4.4.1 Optimal system for the generators in (4.36)

Given a general operator

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{4.84}$$

where $a_1, \ldots, a_4$ are arbitrary constants. We aim to simplify as many of the coefficients $a_1, \ldots, a_4$ as possible using the adjoint action. We construct the commutators and the adjoint representation of the admitted symmetries in (4.36). These are given in Table 4.3 and Table 4.4.

Table 4.3: Commutators of the admitted symmetries (4.36).

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$2X_1$	$4\lambda X_3 + (2-4\lambda)X_2$
X_2	0	0	0	0
X_3	$-2X_1$	0	0	$2X_4$
X_4	$-4\lambda X_3 + (4\lambda - 2)X_2$	0	$-2X_4$	0

Table 4.4: Adjoint representation for the base vectors (4.36).

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$-2\epsilon X_1 + X_3$	$4\lambda\epsilon^2 X_1 - (2-4\lambda)\epsilon X_2 - 4\lambda\epsilon X_3 + X_4$
X_2	X_1	X_2	X_3	X_4
X_3	$e^{2\epsilon} X_1$	X_2	X_3	$e^{-2\epsilon} X_4$
X_4	$X_1 + (2-4\lambda)\epsilon X_2 + 4\lambda\epsilon X_3 + 4\lambda\epsilon^2 X_4$	X_2	$X_3 + 2\epsilon X_4$	X_4

We can simplify the constants $a_1, \ldots, a_4$ in (4.84) by applying the adjoint maps to it. If we first suppose that $a_4 \neq 0$, we can take $a_4 = 1$. From (4.84) we have

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + X_4. \tag{4.85}$$

Acting on (4.85) by $\mathrm{Ad}\left(\exp\left((a_2/(2-4\lambda))X_1\right)\right)$, we obtain

$$\widetilde{X} = \widetilde{a_1} X_1 + \widetilde{a_3} X_3 + X_4, \tag{4.86}$$

where $\widetilde{a_1}$ and $\widetilde{a_3}$ are the constants given by $\widetilde{a_1} = a_1 + 4\lambda\left(a_2/(2-4\lambda)\right)^2 - \left(2a_2a_3/\left(2-4\lambda\right)\right)$ and $\widetilde{a_3} = a_3 - 4\lambda\left(a_2/\left(2-4\lambda\right)\right)$. If we further act on $\widetilde{X}$ by the group generated by X_3, i.e. $\mathrm{Ad}\left(\exp(\epsilon X_3)\right)$ we obtain $\widetilde{X'} = \widetilde{a_1} e^{4\epsilon} X_1 + \widetilde{a_3} e^{2\epsilon} X_3 + X_4$. Depending on the signs of $\widetilde{a_1}$ and $\widetilde{a_3}$ we can make the coefficients of X_1 and X_3 to be either $+1, -1$ or 0. No further simplifications are possible from the entries in the adjoint table, therefore X is equivalent to a scalar multiple of $\alpha X_1 + \beta X_2 + X_4$, with α and β being either $+1, -1$ or 0. If we now suppose $a_4 = 0$, we have

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3. \tag{4.87}$$

We can now assume $a_3 \neq 0$ and suppose $a_3 = 1$. This gives

$$X = a_1 X_1 + a_2 X_2 + X_3. \tag{4.88}$$

Acting on X in (4.88) by $\mathrm{Ad}\left(\exp\left((a_1/2)X_1\right)\right)$ gives $\widetilde{X} = \widetilde{a_2} X_2 + X_3$. No further simplifications are possible, so X is equivalent to a scalar multiple of $aX_2 + X_3$, $a \in \mathbb{R}$. If we now take $a_3 = 0$, we have

$$X = a_1 X_1 + a_2 X_2. \tag{4.89}$$

We can now assume $a_2 \neq 0$ and suppose $a_2 = 1$. This gives

$$X = a_1 X_1 + X_2. \tag{4.90}$$

Acting on X in (4.90) by the group generated by X_3, i.e. $\mathrm{Ad}\left(\exp(\epsilon X_3)\right)$ we obtain $\widetilde{X} = a_1 e^{2\epsilon} X_1 + X_2$. Depending on the sign of a_1, we can make the coefficient of X_1 to be either $+1, -1$ or 0. Therefore X is equivalent to $X_2 + X_1$, $X_2 - X_1$ or X_2. The only remaining case is if $a_2 = 0$. This means X is equivalent to X_1. The set of one-dimensional optimal system is given by

$$\left\{\alpha X_1 + \beta X_3 + X_4, aX_2 + X_3, X_2 \pm X_1, X_2, X_1, \right\}, \quad a \in \mathbb{R}, \tag{4.91}$$

where α and β are either $+1, -1$ or 0.

4.4.2 Symmetry reductions and group invariant solutions

We construct group invariant solutions for equation (4.1) with arbitrary λ. Invariance under X_1 and X_2 is trivial and hence not considered. However, a linear combination of these form part of the one-dimensional optimal system and will be used to seek a group invariant solution. Wherever they appear k_1 and k_2 are arbitrary constants, ${}_1F_1(b, c; z)$ is the confluent hypergeometric function, $\Gamma(z)$ and $\Gamma(b, z)$ represent the Euler gamma function and an incomplete gamma function respectively. $I_n(z)$ is the modified Bessel function of first kind. A well documented review of such functions is presented by Abramowitz and Stegun [37].

1. Invariance under $X_2 + X_1$:

$$X_2 + X_1 = \frac{\partial}{\partial \tau} + C\frac{\partial}{\partial C}.$$

The corresponding characteristic equation is given by

$$\frac{d\tau}{1} = \frac{d\phi}{0} = \frac{dC}{C}. \tag{4.92}$$

Solving (4.92) we find the group invariant solution

$$C = e^{\tau} G(\gamma), \tag{4.93}$$

where $\gamma = \phi$ is the similarity variable. Upon substitution of (4.93) into (4.1) we obtain the following second order ordinary differential equation

$$\lambda G'' + G' - e^{-2\gamma} G = 0, \tag{4.94}$$

where a prime denotes differentiation with respect to the similarity variable γ. The solution of (4.94) is given by

$$\begin{aligned} G(\gamma) =& 2^{-1/2\lambda} \lambda^{-1/4\lambda} (e^{-2\gamma})^{1/4\lambda} \left[I_{-\frac{1}{2\lambda}} \left(\sqrt{\frac{e^{-2\gamma}}{\lambda}} \right) k_1 \Gamma \left(1 - \frac{1}{2\lambda} \right) \right. \\ & \left. + i \; I_{\frac{1}{2\lambda}} \left(\sqrt{\frac{e^{-2\gamma}}{\lambda}} \right) k_2 \Gamma \left(1 + \frac{1}{2\lambda} \right) \right], \end{aligned} \tag{4.95}$$

and the group invariant solution is given by

$$\begin{aligned} C(\tau, \phi) =& e^{\tau} \left\{ 2^{-1/2\lambda} \lambda^{-1/4\lambda} (e^{-2\phi})^{1/4\lambda} \left[I_{-\frac{1}{2\lambda}} \left(\sqrt{\frac{e^{-2\phi}}{\lambda}} \right) k_1 \Gamma \left(1 - \frac{1}{2\lambda} \right) \right. \right. \\ & \left. \left. + i \; I_{\frac{1}{2\lambda}} \left(\sqrt{\frac{e^{-2\phi}}{\lambda}} \right) k_2 \Gamma \left(1 + \frac{1}{2\lambda} \right) \right] \right\}. \end{aligned} \tag{4.96}$$

We mention here that invariance under $X_2 - X_1$ is similar to that of $X_2 + X_1$ by means of the discrete transformations.

2. Invariance under $aX_2 + X_3$:

$$aX_2 + X_3 = 2\tau \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \phi} + aC \frac{\partial}{\partial C}.$$

The corresponding characteristic equation is given by

$$\frac{d\tau}{2\tau} = -\frac{d\phi}{1} = \frac{dC}{aC}. \tag{4.97}$$

Solving (4.97) we find the group invariant solution

$$C = \tau^{a/2} G(\gamma), \tag{4.98}$$

where $\gamma = \tau e^{2\phi}$ is the similarity variable. Upon substituting (4.98) into (4.1) we obtain the following second order ordinary differential equation

$$\lambda \gamma^3 G'' + \frac{1}{4} \gamma \left(2(1 + 2\lambda)\gamma - 1 \right) G' - \frac{a}{8} G = 0, \tag{4.99}$$

where a prime denotes differentiation with respect to the similarity variable γ. The solution for (4.99) is considered for three cases, i.e. $a < 0$, $a = 0$ and $a > 0$.

(a) $a < 0$ he solution for (4.99) is given by

$$\begin{aligned} G(\gamma) =& k_1 \; {}_1F_1\left(\frac{a}{2}; 1 + \frac{1}{2\lambda}; -\frac{1}{4\lambda\gamma}\right) \\ &+ 2^{-1/\lambda} \left(\frac{1}{\gamma}\right)^{\frac{1}{2\lambda}} \lambda^{-1/2\lambda} k_2 \; {}_1F_1\left(\frac{a}{2} + \frac{1}{2\lambda}; 1 + \frac{1}{2\lambda}; -\frac{1}{4\lambda\gamma}\right), \end{aligned} \tag{4.100}$$

and the group invariant solution is given by

$$\begin{aligned} C(\tau, \phi) = \tau^{a/2} \Bigg[& k_1 \; {}_1F_1\left(\frac{a}{2}; 1 + \frac{1}{2\lambda}; -\frac{e^{-2\phi}}{4\lambda\tau}\right) \\ &+ 2^{-1/\lambda} \left(\frac{e^{-2\phi}}{\tau}\right)^{\frac{1}{2\lambda}} \lambda^{-1/2\lambda} k_2 \; {}_1F_1\left(\frac{a}{2} + \frac{1}{2\lambda}; 1 + \frac{1}{2\lambda}; -\frac{e^{-2\phi}}{4\lambda\tau}\right) \Bigg]. \end{aligned} \tag{4.101}$$

(b) $a = 0$. The solution for (4.99) is given by

$$G(\gamma) = k_2 + 2^{1/\lambda} \gamma^{-1/2\lambda} \left(\frac{1}{\lambda\gamma}\right)^{-\frac{1}{2\lambda}} k_1 \; \Gamma\left(\frac{1}{2\lambda}, \frac{1}{4\lambda\gamma}\right), \tag{4.102}$$

and the group invariant solution is given by

$$C(\tau, \phi) = k_2 + 2^{1/\lambda} (\tau e^{2\phi})^{-1/2\lambda} \left(\frac{e^{-2\phi}}{\lambda\tau}\right)^{-\frac{1}{2\lambda}} k_1 \; \Gamma\left(\frac{1}{2\lambda}, \frac{e^{-2\phi}}{4\lambda\tau}\right). \tag{4.103}$$

(c) $a > 0$. The solution for (4.99) is given by

$$\begin{aligned} G(\gamma) =& k_1 \; {}_1F_1\left(-\frac{a}{2}; 1 - \frac{1}{2\lambda}; -\frac{1}{4\lambda\gamma}\right) \\ &+ 2^{-1/\lambda} \left(\frac{1}{\gamma}\right)^{\frac{1}{2\lambda}} \lambda^{-1/2\lambda} k_2 \; {}_1F_1\left(-\frac{a}{2} + \frac{1}{2\lambda}; 1 + \frac{1}{2\lambda}; -\frac{1}{4\lambda\gamma}\right), \end{aligned} \tag{4.104}$$

and the group invariant solution is given by

$$C(\tau,\phi) = \tau^{a/2}\Bigg[k_1\ {}_1F_1\left(-\frac{a}{2};1-\frac{1}{2\lambda};-\frac{e^{-2\phi}}{4\lambda\tau}\right) + 2^{-1/\lambda}\left(\frac{e^{-2\phi}}{\tau}\right)^{\frac{1}{2\lambda}}\lambda^{-1/2\lambda}k_2\ {}_1F_1\left(-\frac{a}{2}+\frac{1}{2\lambda};1+\frac{1}{2\lambda};-\frac{e^{-2\phi}}{4\lambda\tau}\right)\Bigg]. \tag{4.105}$$

3. Invariance under $aX_1 + bX_3 + X_4$:

In this case, a and b are either $0, -1, 1$. Considering the characteristic equation corresponding to $aX_1 + bX_3 + X_4$ given by

$$\frac{d\tau}{a+2b\tau+4\tau^2} = -\frac{d\phi}{b+4\tau} = -\frac{\lambda dC}{(e^{-2\phi}-2\tau+4\lambda\tau)C}. \tag{4.106}$$

We obtain the group invariant solution

$$C = \exp\left[-\frac{1}{\lambda}\left(\frac{\tau}{e^{2\phi}(a+2b\tau+4\tau^2)} - \frac{b(2\lambda-1)\arctan\left(\frac{b+4\tau}{\sqrt{4a-b^2}}\right)}{2\sqrt{4a-b^2}} + \frac{2\lambda-1}{4}\ln\left(a+2b\tau+4\tau^2\right)\right)\right]G(\gamma), \tag{4.107}$$

where $\gamma = \sqrt{a+2\tau(b+2\tau)}\ e^{\phi}$ is the similarity variable. Upon substitution of (4.107) into (4.1) we obtain the following second order ordinary differential equation

$$\lambda^2\gamma^6 G'' + \lambda\gamma^3\left((1+\lambda)\gamma^2 - b\right)G' + aG = 0, \tag{4.108}$$

where a prime denotes differentiation with respect to the similarity variable γ. From (4.108) we can let a and b to be either 0, -1 and 1. As an example, if we make both a and b to be 0, then we obtain an ODE

$$\lambda\gamma G'' + (1+\lambda)G' = 0, \tag{4.109}$$

with a solution given by

$$G(\gamma) = -\lambda\gamma^{-1/\lambda}k_1 + k_2. \tag{4.110}$$

The group invariant solutions is therefore given by

$$C(\tau,\phi) = \left(-\lambda(\tau e^{\phi})^{-1/\lambda}k_1 + k_2\right)\exp\left[\frac{1}{2}\left(\frac{1}{\lambda}-2\right)\ln(\tau) - \frac{1}{4\lambda\tau e^{2\phi}}\right]. \tag{4.111}$$

The other cases follow in the same way from (4.108) by considering all possible values of a and b.

4.5 Lie point symmetries for a velocity dependent dispersion coefficient

The case $p = 2$, is in agreement with solute transport theory. This implies that the dispersion coefficient is now given in terms of the water pore velocity. In this case, the equation in question with arbitrary proportionality constant λ, is given by

$$\frac{\partial C}{\partial \tau} = \lambda e^{4\phi}\frac{\partial^2 C}{\partial \phi^2} + e^{2\phi}(1+2\lambda e^{2\phi})\frac{\partial C}{\partial \phi}. \tag{4.112}$$

We seek to solve the determining equations given by

$$X^{[2]}\left(\frac{\partial C}{\partial \tau} - \lambda e^{4\phi}\frac{\partial^2 C}{\partial \phi^2} - e^{2\phi}(1+2\lambda e^{2\phi})\frac{\partial C}{\partial \phi}\right)\Bigg|_{(4.112)} = 0. \tag{4.113}$$

The operator $X^{[2]}$ is as given in (4.4). From the determining equation (4.113), we obtain

$$\begin{aligned}&\big(\zeta_\tau - e^{2\phi}\zeta_\phi - 2e^{2\phi}C_\phi\xi^2 - 2\lambda e^{4\phi}\zeta_\phi - 8\lambda e^{4\phi}C_\phi\xi^2 - \lambda e^{4\phi}\zeta_{\phi\phi}\\ &-4\lambda e^{4\phi}C_{\phi\phi}\xi^2\big)\Big|_{(4.112)} = 0.\end{aligned} \tag{4.114}$$

Upon substituting expressions for ζ_τ, ζ_ϕ and $\zeta_{\phi\phi}$ given by (4.5), (4.6) and (4.9) respectively we then obtain the following expression

$$\begin{aligned}
&2\lambda^2 e^{8\phi}\xi^1_{CC}C_\phi^3 + \lambda e^{6\phi}\xi^1_{CC}C_\phi^3 + \lambda e^{4\phi}\xi^2_{CC}C_\phi^3 + 16\lambda^2 e^{8\phi}\xi^1_C C_\phi^2 \\
&+4\lambda e^{6\phi}\xi^1_C C_\phi^2 - \lambda e^{4\phi}\eta_{CC}C_\phi^2 + \lambda^2 e^{8\phi}\xi^1_{CC}C_\phi^2 C_{\phi\phi} + 4\lambda^2 e^{8\phi}\xi^1_{\phi C}C_\phi^2 \\
&+2\lambda e^{6\phi}\xi^1_{\phi C}C_\phi^2 + 2\lambda e^{4\phi}\xi^2_{\phi C}C_\phi^2 - 2e^{2\phi}\xi^2 C_\phi - 8\lambda e^{4\phi}\xi^2 C_\phi \\
&+12\lambda^2 e^{8\phi}\xi^1_C C_\phi C_{\phi\phi} + 2\lambda e^{6\phi}\xi^1_C C_\phi C_{\phi\phi} + 2\lambda^2 e^{8\phi}\xi^1_C C_\phi C_{\phi\phi\phi} \\
&+2\lambda e^{4\phi}\xi^2_C C_\phi C_{\phi\phi} + e^{4\phi}\xi^1_\phi C_\phi + 20\lambda^2 e^{8\phi}\xi^1_\phi C_\phi + 8\lambda e^{6\phi}\xi^1_\phi C_\phi \\
&+e^{2\phi}\xi^2_\phi C_\phi + 2\lambda e^{4\phi}\xi^2_\phi C_\phi - 2\lambda e^{4\phi}\eta_{\phi C}C_\phi + 2\lambda^2 e^{8\phi}\xi^1_{\phi C}C_\phi C_{\phi\phi} \\
&+2\lambda^2 e^{8\phi}\xi^1_{\phi\phi}C_\phi + \lambda e^{6\phi}\xi^1_{\phi\phi}C_\phi + \lambda e^{4\phi}\xi^2_{\phi\phi}C_\phi - e^{2\phi}\xi^1_\tau C_\phi - 2\lambda e^{4\phi}\xi^1_\tau C_\phi \\
&-\xi^2_\tau C_\phi - 4\lambda e^{4\phi}\xi^2 C_{\phi\phi} - e^{2\phi}\eta_\phi - 2\lambda e^{4\phi}\eta_\phi + 14\lambda^2 e^{8\phi}\xi^1_\phi C_{\phi\phi} \\
&+3\lambda e^{6\phi}\xi^1_\phi C_{\phi\phi} + 2\lambda^2 e^{8\phi}\xi^1_\phi C_{\phi\phi\phi} + 2\lambda e^{4\phi}\xi^2_\phi C_{\phi\phi} - \lambda e^{4\phi}\eta_{\phi\phi} \\
&+\lambda^2 e^{8\phi}\xi^1_{\phi\phi}C_{\phi\phi} + \eta_\tau - \lambda e^{4\phi}\xi^1_\tau C_{\phi\phi} = 0.
\end{aligned} \tag{4.115}$$

The separation of the monomials gives rise to

$$\begin{aligned}
C_\phi C_{\phi\phi\phi} &: \lambda\xi^1_C = 0, \\
C_{\phi\phi\phi} &: \lambda\xi^1_\phi = 0, \\
C_\phi^2 C_{\phi\phi} &: \lambda\xi^1_{CC} = 0, \\
C_\phi C_{\phi\phi} &: \lambda\left(\left(6\lambda e^{4\phi} + e^{2\phi}\right)\xi^1_C + \lambda e^{4\phi}\xi^1_{\phi C} + \xi^2_C\right) = 0, \\
C_{\phi\phi} &: \lambda\Big(-4\xi^2 + e^{2\phi}\left(14\lambda e^{2\phi} + 3\right)\xi^1_\phi + \lambda e^{4\phi}\xi^1_{\phi\phi} + 2\xi^2_\phi - \xi^1_\tau\Big) = 0, \\
C_\phi^3 &: \lambda\left(\left(2\lambda e^{4\phi} + e^{2\phi}\right)\xi^1_{CC} + \xi^2_{CC}\right) = 0, \qquad (4.116) \\
C_\phi^2 &: \lambda\Big(-\eta_{CC} + 4e^{2\phi}\left(4\lambda e^{2\phi} + 1\right)\xi^1_C + 2\Big(\left(2\lambda e^{4\phi} + e^{2\phi}\right)\xi^1_{\phi C} \\
&\quad +\xi^2_{\phi C}\Big)\Big) = 0, \\
C_\phi &: 2e^{2\phi}\left(4\lambda e^{2\phi} + 1\right)\xi^2 + 2\lambda e^{4\phi}\eta_{\phi C} - 2\lambda^2 e^{8\phi}\xi^1_{\phi\phi} \\
&\quad -e^{4\phi}\left(20\lambda^2 e^{4\phi} + 8\lambda e^{2\phi} + 1\right)\xi^1_\phi - 2\lambda e^{4\phi}\xi^2_\phi - \lambda e^{4\phi}\xi^2_{\phi\phi} \\
&\quad +2\lambda e^{4\phi}\xi^1_\tau - \lambda e^{6\phi}\xi^1_{\phi\phi} + \xi^2_\tau - e^{2\phi}\xi^2_\phi + e^{2\phi}\xi^1_\tau = 0, \\
\text{remainder} &: \left(2\lambda e^{4\phi} + e^{2\phi}\right)\eta_\phi + \lambda e^{4\phi}\eta_{\phi\phi} - \eta_\tau = 0.
\end{aligned}$$

Since $\lambda \neq 0$, the over-determined system simplifies to

$$\begin{aligned} \xi^1_\phi &= 0, \\ \xi^1_C &= 0, \\ \xi^2_C &= 0, \\ \eta_{CC} &= 0, \\ 4\xi^2 - 2\xi^2_\phi + \xi^1_\tau &= 0, \\ 2e^{2\phi}\left(4\lambda e^{2\phi}+1\right)\xi^2 + 2\lambda e^{4\phi}\eta_{\phi C} - 2\lambda e^{4\phi}\xi^2_\phi - \lambda e^{4\phi}\xi^2_{\phi\phi} + 2\lambda e^{4\phi}\xi^1_\tau + \xi^2_\tau & \\ -e^{2\phi}\xi^2_\phi + e^{2\phi}\xi^1_\tau &= 0, \\ \left(2\lambda e^{4\phi} + e^{2\phi}\right)\eta_\phi + \lambda e^{4\phi}\eta_{\phi\phi} - \eta_\tau &= 0. \end{aligned} \tag{4.117}$$

From the governing equations (4.117), it can be observed that $\eta = A(\tau, \phi) + B(\tau, \phi)C$, $\xi^1 = D(\tau)$ and $\xi^2 = E(\tau, \phi)$. Here A, B, D and E are arbitrary functions of the specified variables. The remaining determining equations are given below with the prime denoting differentiation with respect to τ

$$4E + D' - 2E_\phi = 0, \tag{4.118}$$

$$\begin{aligned} 2e^{2\phi}(1+4\lambda e^{2\phi})E + e^{2\phi}(1+2\lambda e^{2\phi})D' - e^{2\phi}E_\phi - 2\lambda e^{4\phi}E_\phi + 2\lambda e^{4\phi}B_\phi & \\ -\lambda e^{4\phi}E_{\phi\phi} + E_\tau &= 0, \end{aligned} \tag{4.119}$$

$$e^{2\phi}(1+2\lambda e^{2\phi})\left(A_\phi + CB_\phi\right) + \lambda e^{4\phi}\left(A_{\phi\phi} + CB_{\phi\phi}\right) - \left(A_{\tau\tau} + CB_{\tau\tau}\right) = 0. \tag{4.120}$$

Equation (4.120) may be separated into two equation by considering powers of C. These are given by

$$e^{2\phi}\left(1+2\lambda e^{2\phi}\right)A_\phi + \lambda e^{4\phi}A_{\phi\phi} - A_{\tau\tau} = 0, \tag{4.121}$$

and

$$e^{2\phi}\left(1+2\lambda e^{2\phi}\right)B_\phi + \lambda e^{4\phi}B_{\phi\phi} - B_{\tau\tau} = 0. \tag{4.122}$$

The solution of (4.118) given by $D(\tau, \phi) = e^{2\phi}F(\tau) - D'/4$ substituted into (4.119) yields an expression for $B(\tau, \phi)$ given by

$$B(\tau, \phi) = \frac{1}{8\lambda}\left(e^{-2\phi}(D' + 2F') - \frac{e^{-4\phi}}{4}D''\right) + G(\tau), \tag{4.123}$$

where F and G are arbitrary functions of τ. This expression is substituted into (4.122) to give

$$8e^{4\phi}D' + 16e^{4\phi}F' + 32\lambda e^{4\phi}G' + 8\lambda e^{4\phi}D'' + 8e^{2\phi}F'' - D'' = 0. \tag{4.124}$$

Since the functions D, F and G are independent of ϕ, we can separate (4.124) to obtain

$$D' + 2F' + \lambda(D'' + 4G') = 0, \tag{4.125}$$

$$D''' = 0, \tag{4.126}$$

$$F'' = 0. \tag{4.127}$$

The solution of (4.125) to (4.127) is

$$D(\tau) = k_1 + \tau k_2 + \tau^2 k_3, \tag{4.128}$$

$$F(\tau) = k_4 + \tau k_5, \tag{4.129}$$

$$G(\tau) = -\frac{k_2}{4\lambda}\tau - \frac{k_3}{2}\tau\left(1 + \frac{\tau}{2\lambda}\right) - \frac{k_5}{2\lambda}\tau + k_6. \tag{4.130}$$

The expressions for ξ^1, ξ^2 and η are therefore given by

$$\xi^1 = k_1 + \tau(k_2 + \tau k_3), \tag{4.131}$$

$$\xi^2 = -\frac{1}{4}(k_2 + 2k_3\tau) + e^{2\phi}(k_4 + k_5\tau), \tag{4.132}$$

$$\eta = -\frac{e^{-4\phi}C}{16\lambda}\big[k_3 - 2e^{2\phi}\left(k_2 + (k_3\tau + k_5)\right) + 4e^{4\phi}\big(k_3\tau^2 + \tau(k_2 + 2\lambda k_3 + 2k_5) - 4\lambda k_6\big)\big] + A(\tau, \phi), \tag{4.133}$$

where $A(\tau, \phi)$ is an arbitrary function satisfying equation (4.1) and $k_1, \ldots, k_6$ are arbitrary constants. The resulting Lie point symmetries including the infinite symmetry generator, $X_A = A(\tau, \phi)\dfrac{\partial}{\partial C}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C},\\
X_3 &= e^{2\phi}\frac{\partial}{\partial \phi},\\
X_4 &= \tau\frac{\partial}{\partial \phi} + \frac{1}{\lambda}\left(\frac{e^{-4\phi}}{4} - \frac{\tau e^{-2\phi}}{2}\right)C\frac{\partial}{\partial C},\\
X_5 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{4}\frac{\partial}{\partial \phi} + \frac{1}{\lambda}\left(\frac{e^{-2\phi}}{8} - \frac{\tau}{4}\right)C\frac{\partial}{\partial C},\\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \frac{\tau}{2}\frac{\partial}{\partial \phi} + \frac{1}{\lambda}\left(\frac{\tau e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\lambda\tau}{2} - \frac{\tau^2}{4}\right)C\frac{\partial}{\partial C}.
\end{aligned}\right\} \quad (4.134)$$

Note that the admitted symmetry structure and number is not affected by the constant λ, that is we obtain the same symmetries up to the specified λ value.

4.6 Invariant solutions for a velocity-dependent dispersion coefficient

4.6.1 Optimal system for generators in (4.134)

Repeating the calculations as in section (4.3.1), we construct the one-dimensional optimal system for the Lie point symmetries in equation (4.134) and obtain

$$\left\{X_5 + aX_2, X_1 + X_6 + aX_2, X_1 \pm X_4, X_1 + aX_2, X_2, X_3\right\}, \quad a \in \mathbb{R}. \quad (4.135)$$

If we admit a discrete symmetry so that $X_1 + X_5$ is mapped to $X_1 - X_5$, then the number of the elements in the optimal system is reduced by one.

4.6.2 Symmetry reductions and group invariant solutions

(i) A solution invariant under $X_5 + aX_2$ assumes the form

$$C = \exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda} + \frac{a}{8\lambda}\ln(\tau)\right)G(\gamma), \tag{4.136}$$

where $\gamma = e^{\phi}\sqrt[4]{\tau}$ is the similarity variable and G is an arbitrary function satisfying the ODE

$$\lambda^2\gamma^6 G'' - \frac{1}{4}\lambda\gamma(1 - 12\lambda\gamma^4)G' - \frac{a}{8}G = 0. \tag{4.137}$$

Upon solving (4.137) and taking into account (4.136), we obtain the group invariant solutions

(a) $a < 0$.

$$\begin{aligned} C =& \exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda} + \frac{a}{8\lambda}\ln(\tau)\right) \\ &\times \left[\frac{k_2 e^{-2\phi}}{4\sqrt{\lambda\tau}}\,{}_1F_1\left(\frac{4\lambda + a}{8\lambda}, \frac{3}{2}; -\frac{e^{-4\phi}}{16\lambda\tau}\right) + k_1\,{}_1F_1\left(\frac{a}{8\lambda}, \frac{1}{2}; -\frac{e^{-4\phi}}{16\lambda\tau}\right)\right]. \end{aligned} \tag{4.138}$$

(b) $a = 0$.

$$C = \exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda}\right)\left[k_2 - \sqrt{\lambda\pi}\; k_1 \,\mathrm{erf}\left(\frac{e^{-2\phi}}{4\sqrt{\lambda\tau}}\right)\right]. \tag{4.139}$$

(c) $a > 0$

$$\begin{aligned} C =& \exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda} + \frac{a}{8\lambda}\ln(\tau)\right)\left[\frac{k_2 e^{-2\phi}}{4\sqrt{\lambda\tau}}\,{}_1F_1\left(\frac{4\lambda - a}{8\lambda}, \frac{3}{2}; -\frac{e^{-4\phi}}{16\lambda\tau}\right)\right. \\ &\left. + k_1\,{}_1F_1\left(-\frac{a}{8\lambda}, \frac{1}{2}; -\frac{e^{-4\phi}}{16\lambda\tau}\right)\right]. \end{aligned} \tag{4.140}$$

(ii) A solution invariant under $X_1 + X_4$ assumes the form

$$C = \exp\left(\frac{8\lambda\tau^3}{3} + \tau e^{-2\phi} - \tau^2\right)G(\gamma), \tag{4.141}$$

where $\gamma = 4\tau^2 + e^{-2\phi}$ is the similarity variable and G satisfies

$$G'' - \frac{1}{2\lambda}G' - \frac{\gamma}{4\lambda}G = 0. \tag{4.142}$$

The group invariant solution is given by

$$C = \exp\left(\frac{8\lambda\tau^3}{3} + \tau e^{-2\phi} - \tau^2\right) \times \left\{ e^{\frac{4\tau^2+e^{-2\phi}}{4\lambda}} \left[k_1 \text{ Ai}\left(\frac{1+4\lambda(4\tau^2+e^{-2\phi})}{(4\lambda)^{4/3}}\right) + k_2 \text{ Bi}\left(\frac{1+\lambda 4(4\tau^2+e^{-2\phi})}{(4\lambda)^{4/3}}\right)\right]\right\}. \tag{4.143}$$

(iii) The invariant solution under $X_1 + aX_2$ is given by

$$C = e^{a\tau}G(\gamma), \tag{4.144}$$

where $\gamma = \phi$ and G satisfies

$$G'' + \left(2 + \frac{e^{-2\phi}}{\lambda}\right)G' - \frac{a}{\lambda}e^{-4\phi}G = 0. \tag{4.145}$$

Upon solving (4.137) and taking into account (4.144), we obtain the group invariant solutions

(a) $a < 0$.

$$C = e^{a\tau}\left[\exp\left(-\frac{1}{4\lambda}\left(-1-\sqrt{1-4a\lambda}\right)e^{-2\phi}\right)k_1 + \exp\left(-\frac{1}{4\lambda}\left(-1+\sqrt{1-4a\lambda}\right)e^{-2\phi}\right)k_2\right]. \tag{4.146}$$

(b) $a = 0$

$$C = k_2 - \lambda\exp\left(\frac{e^{-2\phi}}{2\lambda}\right)k_1. \tag{4.147}$$

(c) $a > 0$

$$C = e^{a\tau}\exp\left[\left(\frac{1}{4\lambda}\left(1-\sqrt{1+4a\lambda}\right)e^{-2\phi}\right)k_1 + \exp\left(\frac{1}{4\lambda}\left(1+\sqrt{1+4a\lambda}\right)e^{-2\phi}\right)k_2\right]. \tag{4.148}$$

(iv) A solution invariant under $X_1 + X_6 + aX_2$ assumes the form

$$C = \exp\left(-\frac{\tau e^{-4\phi}}{1+16\tau^2} - \frac{1}{4}\ln(1+16\tau^2) + \frac{1}{4}e^{-2\phi} - \frac{\tau}{4} + \frac{1}{16}(1+4a)\arctan(4\tau)\right) G(\gamma), \tag{4.149}$$

where $\gamma = e^{\phi}\sqrt[4]{1+16\tau}$ is the group invariant and G satisfies

$$\gamma^{10}G'' + 3\gamma^9 G' + \left(1 - \frac{1}{4}(1+4a)\gamma^4\right) G = 0. \tag{4.150}$$

Equation (4.150) is difficult to solve exactly. However, one may solve (4.150) by specifying a. For example, if $a = -1/4$ then (4.150) reduces to a simpler form. We omit such a solution in this dissertation.

4.7 Some physical examples

4.7.1 Given constant dispersion coefficient

Example (a)

Suppose a concentration C_0 of a solute is supplied to a single point, in an instant of time. We require to determine the subsequent concentration of the pollution at various distances from where it was released (see e.g. [38]). We would expect concentration to vanish at large distance, that is

$$C(\phi, \tau) \to 0, \quad \text{as} \quad \phi \to -\infty. \tag{4.151}$$

The generator X_4 in equation (4.28) (also, an element of the one dimensional optimal systems given $a = 0$) leads to an invariant solution in functional form given by

$$C = G(\gamma), \tag{4.152}$$

where $\gamma = \sqrt{\tau}e^{\phi}$ and G satisfies the equation

$$2\gamma^3 G'' + (4\gamma^2 - 1)G' = 0,$$

and hence

$$G = c_1 + c_2 \mathrm{erf}\left(\frac{1}{2\gamma}\right).$$

$\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-\nu^2} d\nu$ is the error function [37]. In terms of the original variable and subject to the boundary conditions, we obtain

$$C = C_0 \mathrm{erfc}\left(\frac{e^{-\phi}}{2\sqrt{\tau}}\right). \tag{4.153}$$

Here, the erfc(.) is the complement error function defined by (1-erf(.)). Solution (4.153) is depicted in Figure 4.1.

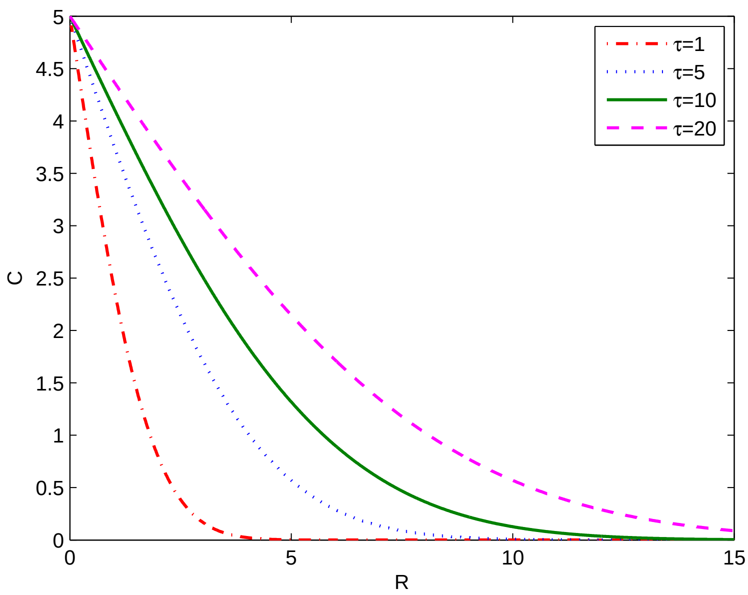

Figure 4.1: Contaminant concentration profile along the radius. Here $C_0 = 5$.

Total flux across $R = R_a$ is given by

$$C + \frac{1}{R}\frac{\partial C}{\partial R}\bigg|_{R=R_a} = C_0 \mathrm{erfc}\left(\frac{R_a}{2\sqrt{\tau}}\right) - \frac{C_0 e^{-R_a^2/4\tau}}{\sqrt{\pi\tau}R_a}. \tag{4.154}$$

The contaminant flux (4.154) is depicted in Figure 4.2. Total flux across $R_a = 4$, increases and flattens at large time.

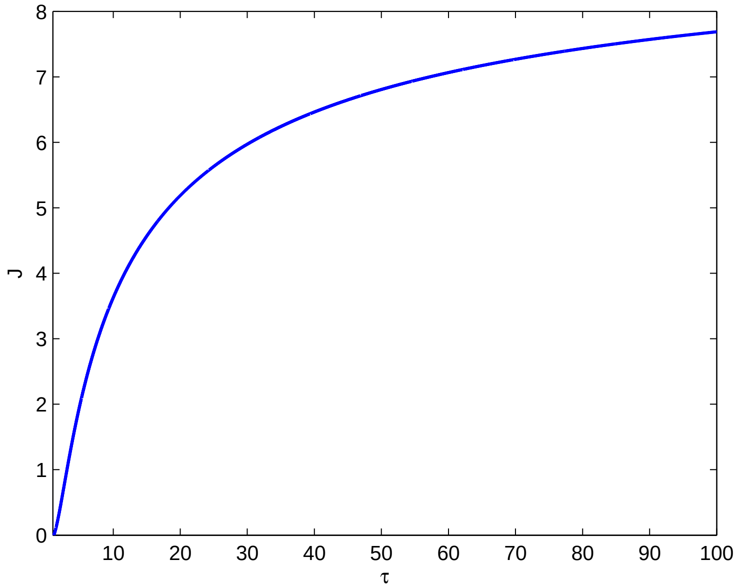

Figure 4.2: Solute flux across a fixed radius. Here $R_a = 4$ and $C_0 = 10$.

Example (b)

The X_6-invariant solution is given in functional form as

$$C = \frac{1}{\sqrt{\tau}} \exp\left(-\frac{e^{-2\phi}}{4\tau}\right) G(\gamma), \tag{4.155}$$

where

$$\gamma = \tau e^{\phi} \quad \text{and} \quad G \quad \text{satisfies the ODE} \quad \gamma G'' + G' = 0.$$

We impose the boundary conditions

$$C \to 0, \quad \phi \to -\infty \quad \text{and} \quad C = \omega(\tau), \quad \phi \to \infty.$$

Infinite concentration at the origin implies that there is a high supply of contaminants at this point. Furthermore, contaminant concentration vanishes when time evolves. In terms of the original variables we obtain the exact (group-invariant) solution given by

$$C = \frac{1}{\sqrt{\tau}} \exp\left(-\frac{e^{-2\phi}}{4\tau}\right). \tag{4.156}$$

Solution (4.156) is depicted in Figures 4.3 and 4.4. In Figure 4.3, a sharp peak of concentration is observed shortly after $\tau = 0$ and decreases at later stage. This may be interpreted as an injection of contaminants at a single point, that is, the concentration at a single point increases but due to diffusion at larger time it smoothed out. Note that here we have restricted our analysis using symmetry generator X_6. This symmetry generator leads to simpler and realistic exact solution. In Figure 4.4, we observe that concentration at the origin decreases with time. Furthermore, this concentration vanishes at large distances.

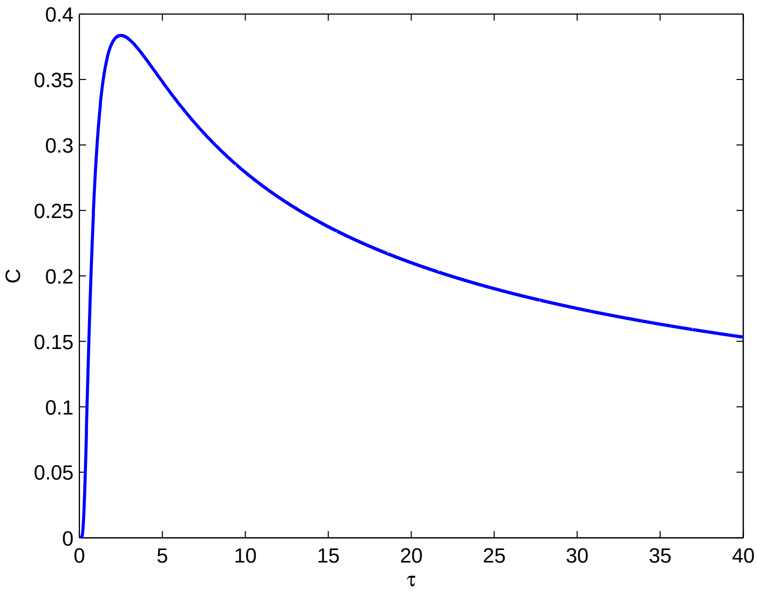

Figure 4.3: Contaminant concentration profile as time evolves.

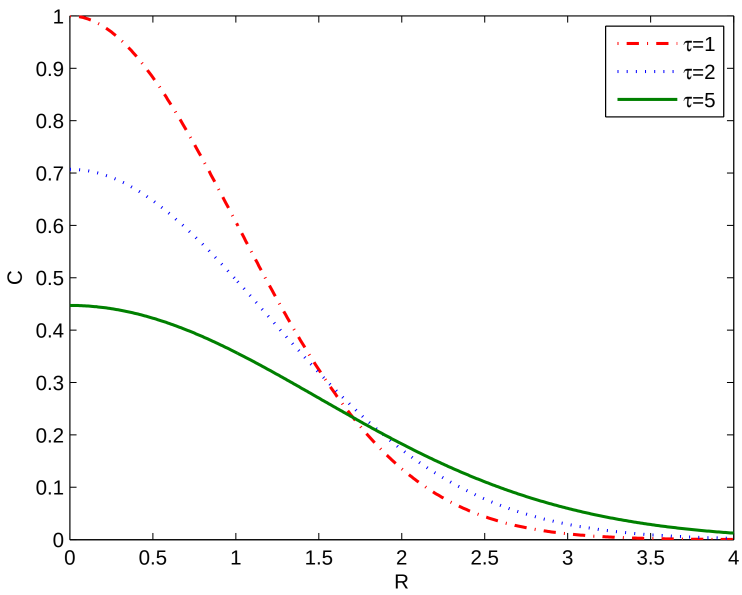

Figure 4.4: Contaminant concentration profile at various fixed times.

4.7.2 Given velocity-dependent dispersion coefficient

Example (a) Steady state solution

The time translation leads to the analysis of the steady state contaminant transport. Steady state solutions may be constructed subject to the following imposed boundary conditions;

$$C = C_0, \qquad R = 0 \tag{4.157}$$

and

$$C + \frac{1}{R}\frac{dC}{dR} = 0, \qquad R = R_a. \tag{4.158}$$

The boundary condition (4.157) implies that pollutants are supplied at the origin, and boundary condition (4.158) correspond to the assumption that pollutants are not carried though at some distance R_a, rather it accumulates here. We obtain the exact solution

$$C = C_0\left[\left(1 - \frac{1}{\Delta}\right) + \frac{1}{\Delta}\exp\left(\frac{R^p}{\lambda p}\right)\right], \tag{4.159}$$

where Δ is given by

$$\Delta = 1 - \left(1 + \frac{R_a^{p-2}}{\lambda}\right) \exp\left(\frac{R_a^p}{\lambda p}\right).$$

The solution (4.159) is depicted in Figures 4.5, 4.6 and 4.7. We observe in Figure 4.5 that concentration starts decreasing and converge to some value at large distance for $p = 2$ than for lower values of p, whereas λ has an opposite effect as shown in 4.6.

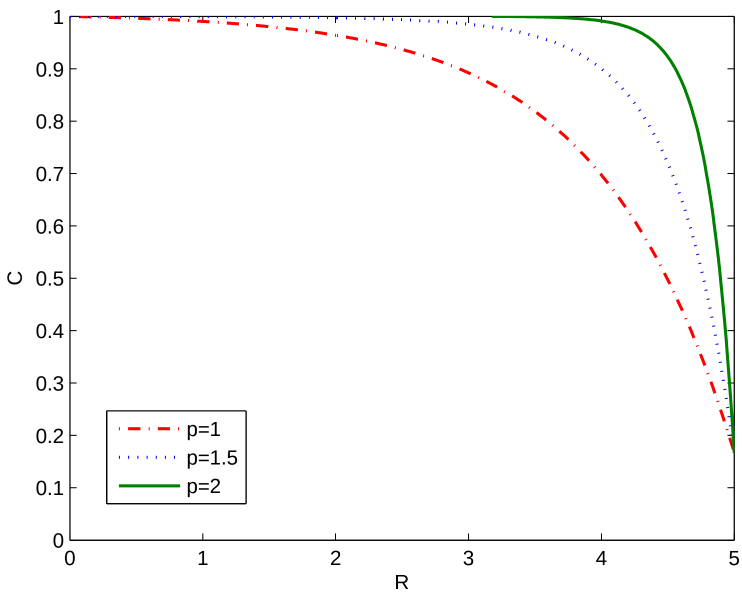

Figure 4.5: Steady contaminant profile for concentration given in equation (4.159). Here $\lambda = 1$.

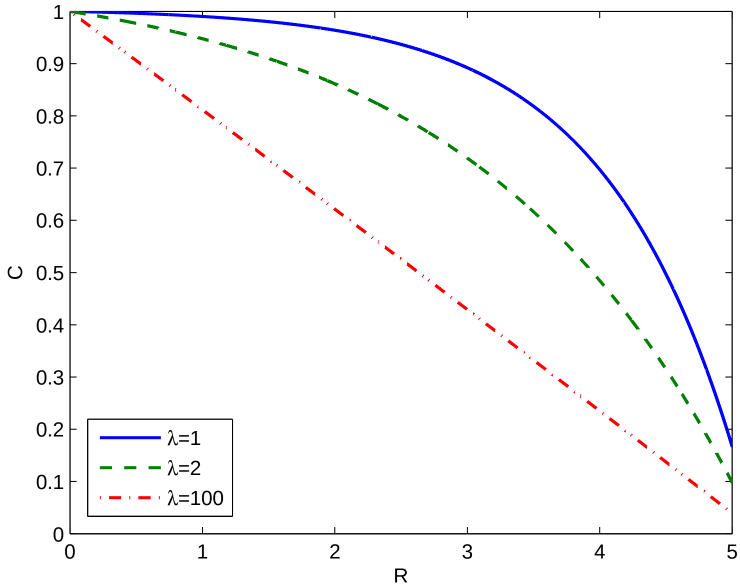

Figure 4.6: Steady contaminant profile for concentration given in equation (4.159). Here $R = R_a = 2$.

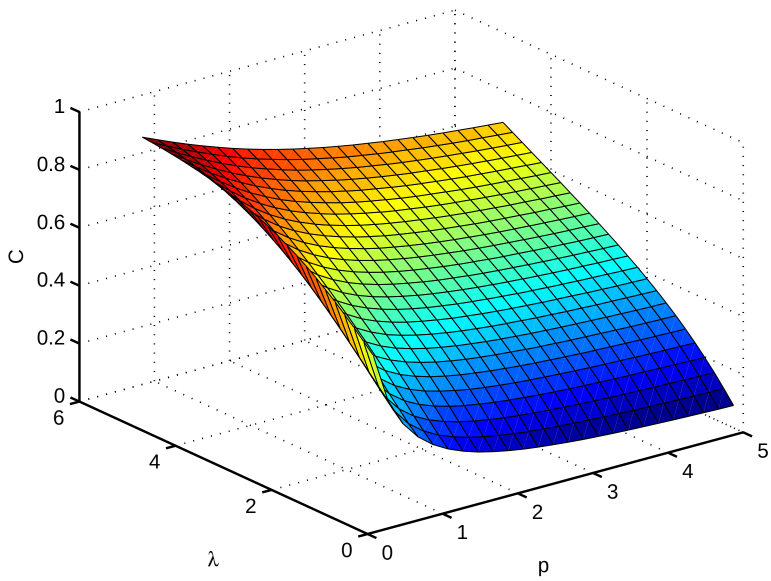

Figure 4.7: Effects of λ and p on concentration profile for solution given in equation (4.159). Here $R = R_a = 2$.

Example (b) Transient state solution

It is quite difficult to construct exact solutions for transient contaminant transport subject to these boundary conditions (4.157) and (4.158). However, if one assumes that at an initial time, say $\tau = 1$, the concentration at the point source is given by a constant and that this concentration vanishes at large distances and prolonged periods, then using the symmetry combination $X_1 + aX_2$ from Table 4, the group invariant solution is given by

$$C = C_0 e^{a\tau} \exp\left[\left(\frac{1-\sqrt{1-4a\lambda}}{4\lambda}\right) R^2\right], \quad \forall \quad a < 0 \quad \text{and} \quad \lambda > 0. \tag{4.160}$$

Solution (4.160) is depicted in Figures 4.8 and 4.9.

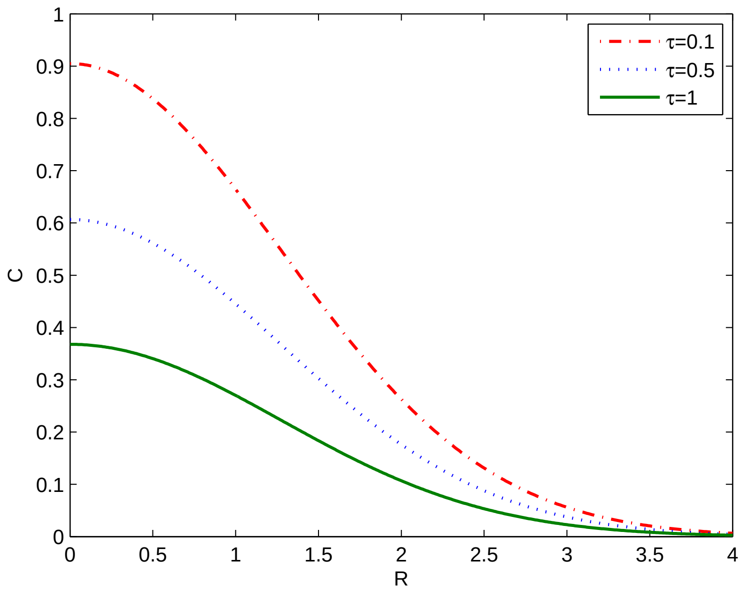

Figure 4.8: Concentration profile for solution given in equation (4.160). Here $R = R_a = 2,\ C_0 = 1$ and $\lambda = 1$

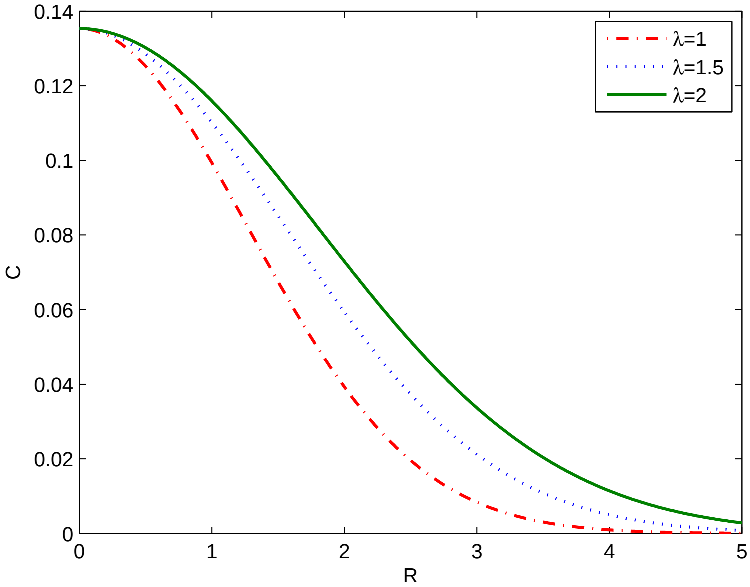

Figure 4.9: Effects of λ on the concentration profile for solution given in equation (4.160). Here $R = R_a = 2,\ C_0 = 1$ and $\tau = 2$.

In his work, Philip [39] considered the instantaneous point source for contaminant dispersion during radial water flow in porous media. Exact solutions were constructed for the two- and three- dimensional models with dispersion coefficient depending on Péclet number defined by $Pe = vL/D(v)$, L is the length of the soil column. Here, we consider models in stream functions coordinates. Exact close-form (similarity) solutions are constructed using the elements of the one dimensional optimal systems. These new solutions may be viewed as representing the continual supply of contaminant at a point (source) which are dispersed radially.

4.8 Concluding remarks

We have focused only on the two dimensional solute concentration field within water from a single injection well. The considered problem is a significant improvement in the study of solute transport under radial water background since we analyze the convection dispersion equation in stream functions. We have observed that extra Lie point symmetries are admitted when the dispersion coefficient is a constant or when it is given as a power law function of velocity, with exponent being given by two. We have classified the group invariant solutions by the elements of the optimal systems. In fact, new exact solutions are constructed. The symmetry (invariant) solutions are obtainable when dispersion coefficient is constant or is dependent on the velocity.

Chapter 5

Nonlocal symmetries and classes of exact solutions

5.1 Introduction

In this Chapter, we consider a form of a convection dispersion equation given in terms of the stream functions. The governing equation $R\{\tau, \phi, C\}$ describing movements of contaminants under radial water flow background is given in conserved form. As such, the conserved form of the governing equation may be written as a system of first order partial differential equations referred to as an auxiliary system, say $S\{\tau, \phi, C, \vartheta\}$ by an introduction of the nonlocal (or potential variable). The resulting system of equations admits a number of (local) point symmetries which induce the nonlocal symmetries of the original governing equation. We construct a class of solutions using admitted genuine nonlocal symmetries which includes the invariant solutions obtained via corresponding point symmetries of the governing equation. We will consider a method presented in [29] for symmetry reductions using nonlocal symmetries. The method finds families of solutions $\mathbb{S}_{\mathbb{F}}$ and $\mathbb{S}^*_{\mathbb{F}}$. We point out here that the solutions $\mathbb{S}_{\mathbb{F}}$ are enclosed in the solutions $\mathbb{S}^*_{\mathbb{F}}$. As such, we refer to $\mathbb{S}^*_{\mathbb{F}}$ as a wider class of solutions obtained by nonlocal symmetries.

In brief, a point symmetry of $S\{\tau, \phi, C, \vartheta\}$ has the characteristic system

$$\frac{d\tau}{\xi^1} = \frac{d\phi}{\xi^2} = \frac{dC}{\eta^1} = \frac{d\vartheta}{\eta^2}.$$

and the invariant surface conditions [28]

$$\xi^1(\tau, \phi, C, \vartheta)C_\tau + \xi^2(\tau, \phi, C, \vartheta)C_\phi - \eta^1(\tau, \phi, C, \vartheta) = 0, \tag{5.1}$$

$$\xi^1(\tau, \phi, C, \vartheta)\vartheta_\tau + \xi^2(\tau, \phi, C, \vartheta)\vartheta_\phi - \eta^2(\tau, \phi, C, \vartheta) = 0. \tag{5.2}$$

We mention here that the general solution of the invariant surface conditions is obtained by solving the characteristic system. The general solution of the characteristic system assumes the form

$$I_1 = \varphi_1(\tau, \phi, C, \vartheta), \tag{5.3}$$

$$I_2 = \varphi_2(\tau, \phi, C, \vartheta), \tag{5.4}$$

$$I_3 = \varphi_3(\tau, \phi, C, \vartheta), \tag{5.5}$$

where I_1, I_2 and I_3 are the constants of integration. If we define the similarity variable as $I_1 = z$ and the similarity functions as $I_2 = h_1(z)$ and $I_3 = h_2(z)$, then we can rewrite (5.3), (5.4) and (5.5) as

$$C = F(\tau, \phi, C, \vartheta, h_1(z), h_2(z)), \tag{5.6}$$

$$\vartheta = G(\tau, \phi, C, \vartheta, h_1(z), h_2(z)), \tag{5.7}$$

$$H(\tau, \phi, C, \vartheta, h_1(z), h_2(z)) = 0, \tag{5.8}$$

where F, G and H are arbitrary functions. The functions $h_1(z)$ and $h_2(z)$ are determined by substituting (5.6), (5.7) and (5.8) into the auxiliary system $S\{\tau, \phi, C, \vartheta\}$. These lead to a class of solutions $\mathbb{S}_{\mathbb{F}}$. To find the solutions $\mathbb{S}^*_{\mathbb{F}}$, we impose (5.6) into $R\{\tau, \phi, C\}$.

The model $R\{\tau, \phi, C\}$ to be investigated is given by

$$e^{-2\phi}\frac{\partial C}{\partial \tau} = \frac{\partial}{\partial \phi}\left(\lambda e^{p\phi}\frac{\partial C}{\partial \phi}\right) + \frac{\partial C}{\partial \phi}, \quad \lambda \neq 0. \tag{5.9}$$

$R\{\tau, \phi, C\}$ can be written in conserved form as

$$D_\tau \left(e^{-2\phi} C\right) = D_\phi \left(\lambda e^{p\phi} \frac{\partial C}{\partial \phi} + C\right). \tag{5.10}$$

where $D_\tau = \dfrac{\partial}{\partial \tau}$ and $D_\phi = \dfrac{\partial}{\partial \phi}$. By introducing an auxiliary potential variable $\vartheta(\tau, \phi)$ as a further unknown function, we obtain the auxiliary system $S\{\tau, \phi, C, \vartheta\}$ given by

$$\left.\begin{array}{rcl} \vartheta_\phi &=& e^{-2\phi} C, \\ \vartheta_\tau &=& \lambda e^{p\phi} \dfrac{\partial C}{\partial \phi} + C. \end{array}\right\} \tag{5.11}$$

Point symmetries of the system $S\{\tau, \phi, C, \vartheta\}$ are the nonlocal symmetries of $R\{\tau, \phi, C\}$. Two cases of p are considered. When $p = 0$ the equation has a constant dispersion coefficient and when $p = 2$ the equation has a velocity-dependent dispersion coefficient. The outline of the derivation of the Lie point symmetries of (5.11) is presented in Appendix A and Appendix B.

5.2 Nonlocal symmetries for a constant dispersion coefficient

Given $p = 0$ in (5.9) and (5.11), the dispersion coefficient $D(v)$ becomes a nonzero proportionality constant λ. It turns out that λ must satisfy three cases for the system (5.11) to admit local symmetries which induce nonlocal symmetries of (5.9). These cases are $\lambda = -1$, $\lambda = -1/3$ and the general case with $\lambda \neq 0, -1, -1/3$. The local symmetries of (5.11) including the infinite symmetry generator, $X_\omega = e^{2\phi}\omega(\tau, \phi)\dfrac{\partial}{\partial C} + \omega_\phi \dfrac{\partial}{\partial \vartheta}$, are presented below

Case 1: $\lambda = -1$.

$$
\left.
\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau}, \\
X_2 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta}, \\
X_3 &= e^{-2\phi}\frac{\partial}{\partial \phi} + C\frac{\partial}{\partial C}, \\
X_4 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C}, \\
X_5 &= \tau\frac{\partial}{\partial \phi} - \left[\left(\frac{e^{-2\phi}}{2} - \tau\right)C - \frac{\vartheta}{2}\right]\frac{\partial}{\partial C} - \frac{e^{-2\phi}}{2}\vartheta\frac{\partial}{\partial \vartheta}, \\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} + \left[\left(\frac{e^{-2\phi}}{4} - \frac{5}{2}\tau\right)C - \frac{\vartheta}{2}\right]\frac{\partial}{\partial C} \\
&\quad + \left(\frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)\vartheta\frac{\partial}{\partial \vartheta},
\end{aligned}
\right\} \tag{5.12}
$$

In all the symmetries above, X_5 and X_6 are the only genuine nonlocal symmetries.

Case 2: $\lambda = -1/3$.

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau}, \\
X_2 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta}, \\
X_3 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C}, \\
X_4 &= \frac{\partial}{\partial \phi} + \left(2C + e^{2\phi}\vartheta\right)\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta}, \\
X_5 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} + \left[\left(\frac{3e^{-2\phi}}{4} - \frac{7\tau}{2}\right)C - \frac{3}{2}\vartheta\right]\frac{\partial}{\partial C}, \\
X_6 &= \tau\frac{\partial}{\partial \phi} - \left[\left(\frac{3e^{-2\phi}}{2} - 2\tau\right)C - \left(\frac{3}{2} + e^{2\phi}\tau\right)\vartheta\right]\frac{\partial}{\partial C} \\
&\quad - \left(\frac{3e^{-2\phi}}{2} - \tau\right)\vartheta\frac{\partial}{\partial \vartheta},
\end{aligned}\right\} \tag{5.13}$$

The only genuine nonlocal symmetries of (5.9) with $p = 0$ are X_4, X_5 and X_6.

Case 3: $\lambda \neq -1, -1/3, 0$. Arbitrary λ.

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C},\\
X_3 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C},\\
X_4 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left[\left(2\lambda\tau + \frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)C + \frac{1}{2}\vartheta\right]\frac{\partial}{\partial C}\\
&\quad -\frac{1}{\lambda}\left(\frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)\vartheta\frac{\partial}{\partial \vartheta}.
\end{aligned}\right\} \tag{5.14}$$

The only genuine nonlocal symmetry is X_4. Here $\omega(\tau, \phi)$ is an arbitrary function satisfying equation (5.9) with $p = 0$.

5.3 Invariant solutions for a constant dispersion coefficient

We apply nonlocal symmetries to find solution the family of solutions. We point out that the solutions $\mathbb{S}_{\mathbb{F}}$ are contained in solutions $\mathbb{S}^*_{\mathbb{F}}$.

5.3.1 *Case 1:* $\lambda = -1$

We consider the nonlocal symmetries X_5 and X_6.

(a) For the nonlocal symmetry X_5, the corresponding characteristic system related to the invariant surface conditions:

$$2\tau C_\phi + (e^{-2\phi} - 2\tau)C - \vartheta = 0, \tag{5.15}$$

$$2\tau\vartheta_\phi - e^{-2\phi}\vartheta = 0, \tag{5.16}$$

is given by

$$\frac{d\tau}{0} = \frac{d\phi}{2\tau e^{\phi}} = \frac{d\vartheta}{-e^{-\phi}\vartheta} = \frac{dC}{-e^{-\phi}C + 2e^{\phi}\tau C + e^{\phi}\vartheta}. \tag{5.17}$$

Solving the characteristic system (5.17) gives the following three integrals

$$I_1 = \tau, \tag{5.18}$$

$$I_2 = \vartheta \exp\left(-\frac{e^{-2\phi}}{4\tau}\right), \tag{5.19}$$

$$I_3 = \left(C + \frac{\vartheta}{2\tau}\right) \exp\left(-\phi - \frac{e^{-2\phi}}{4\tau}\right). \tag{5.20}$$

If we define the similarity variable as $I_1 = z$ and the similarity functions as $I_2 = h_1(z)$ and $I_3 = h_2(z)$, then the solutions of the invariant surface conditions are

$$C = h_2(z) \exp\left(\phi + \frac{e^{-2\phi}}{4\tau}\right) - \frac{1}{2\tau} h_1(z) \exp\left(\frac{e^{-2\phi}}{4\tau},\right) \tag{5.21}$$

$$\vartheta = h_1(z) \exp\left(\frac{e^{-2\phi}}{4\tau}\right), \tag{5.22}$$

$$z = \tau. \tag{5.23}$$

The elimination of ϑ in the invariant surface conditions leads to the PDE

$$\Upsilon^* \quad : \quad 4\tau^2 e^{2\phi} C_{\phi\phi} - 4\tau^2 e^{2\phi} C_{\phi} - \left(2\tau + e^{-2\phi}\right) C = 0. \tag{5.24}$$

The family of solutions for (5.24) is given by (5.21). In order to find the family of solutions $\mathbb{S}^*_{\mathbb{F}}$, we substitute (5.21) into the PDE (5.9) to reduce it to an ODE. This is given by

$$4z^2 e^{\phi} h_2' - 2z h_1' + 2z e^{\phi} h_2 - h_1 = 0. \tag{5.25}$$

This gives rise to two ODEs

$$2z h_1' + h_1 = 0, \tag{5.26}$$

$$2z h_2' + h_2 = 0, \tag{5.27}$$

which have the solutions

$$h_1(z) = \frac{k_1}{\sqrt{z}}, \tag{5.28}$$

$$h_2(z) = \frac{k_2}{\sqrt{z}}, \tag{5.29}$$

where k_1 and k_2 are arbitrary constants. The solutions $\mathbb{S}^*_{\mathbb{F}}$ are therefore given by

$$C = \frac{1}{2}\tau^{-\frac{3}{2}}\left(2\tau e^{\phi}k_1 - k_2\right)\exp\left(\frac{e^{-2\phi}}{4\tau}\right). \tag{5.30}$$

The family of solutions $\mathbb{S}_{\mathbb{F}}$ are obtained by substituting (5.21), (5.22) and (5.23) into the auxiliary system (5.11) with $p = 0$ to obtain the system

$$h_2 = 0, \tag{5.31}$$

$$2zh_1' + h_1 = 0, \tag{5.32}$$

with solutions

$$h_2 = 0, \tag{5.33}$$

$$h_1 = \frac{k_1}{\sqrt{z}}, \tag{5.34}$$

where k_1 is an arbitrary constant. The solution $\mathbb{S}_{\mathbb{F}}$ is therefore given as.

$$C = -\frac{1}{2}\tau^{-\frac{3}{2}}k_1\exp\left(\frac{e^{-2\phi}}{4\tau}\right). \tag{5.35}$$

These are also solutions of the PDE

$$\bar{\Upsilon} \quad : \quad 2\tau C_\phi + e^{-2\phi}C = 0. \tag{5.36}$$

(b) For the nonlocal symmetry X_6, the corresponding characteristic system related to the invariant surface conditions:

$$4\tau C_\phi - 4\tau^2 C_\tau + (e^{-2\phi} - 10\tau)C - 2\vartheta = 0 \tag{5.37}$$

$$4\tau\vartheta_\phi - 4\tau^2\vartheta_\phi + (e^{-2\phi} - 2\tau)\vartheta = 0 \tag{5.38}$$

Has three integrals given by

$$I_1 = \tau e^{\phi}, \tag{5.39}$$

$$I_2 = \vartheta \exp\left(-\frac{\phi}{2} - \frac{e^{-2\phi}}{4\tau}\right), \tag{5.40}$$

$$I_3 = \left(e^{-\phi}C + \frac{e^{-\phi}}{2\tau}\vartheta\right)\exp\left(-\frac{3}{2}\phi - \frac{e^{-2\phi}}{4\tau}\right). \tag{5.41}$$

If we again make the assumptions that $I_1 = z$, $I_2 = h_1(z)$ and $I_3 = h_2(z)$, then the solutions of the invariant surface conditions are

$$C = h_2(z)\exp\left(\frac{5}{2}\phi + \frac{e^{-2\phi}}{4\tau}\right) - \frac{h_1(z)}{2\tau}\exp\left(\frac{\phi}{2} + \frac{e^{-2\phi}}{4\tau}\right), \tag{5.42}$$

$$\vartheta = h_1(z)\exp\left(\frac{\phi}{2} + \frac{e^{-2\phi}}{4\tau}\right), \tag{5.43}$$

$$z = \tau e^{\phi}. \tag{5.44}$$

The elimination of ϑ in the invariant surface conditions leads to the PDE

$$\begin{aligned}\Upsilon^* \quad : \quad & 16\tau^2 e^{2\phi}C_{\phi\phi} + 16\tau^4 e^{2\phi}C_{\tau\tau} - 32\tau^3 e^{2\phi}C_{\phi\tau} + 8\tau(1 - 8\tau e^{2\phi})C_\phi \\ & - 8\tau(\tau - 10\tau^2 e^{2\phi})C_\tau + (e^{-2\phi} - 20\tau + 60\tau^2 e^{2\phi})C = 0. \end{aligned} \tag{5.45}$$

The family of solutions for (5.45) is given by (5.42). In order to find the family of solutions $\mathbb{S}^*_{\mathbb{F}}$, we substitute (5.42) into the PDE (5.9) to reduce it to an ODE. This is gives rise to the system

$$4z^2 h_1'' + 4zh_1' + 8h_2 - h_1 = 0, \tag{5.46}$$

$$4z^2 h_2'' + 20zh_2' + 15h_2 = 0, \tag{5.47}$$

which upon solving gives

$$h_2(z) = \frac{k_1}{z^{\frac{3}{2}}} + \frac{k_2}{z^{\frac{5}{2}}}, \tag{5.48}$$

$$h_1(z) = \frac{-2k_2 + 3z[-2k_1 + z(1+z)k_3 + i\ z(z-1)k_4]}{6z^{\frac{5}{2}}}. \tag{5.49}$$

The family of solutions $\mathbb{S}^*_{\mathbb{F}}$ are therefore given by

$$C = \frac{\tau^{-3}}{12\sqrt{\tau e^{\phi}}}\exp\left(\frac{e^{-2\phi}}{4\tau} - \frac{3\phi}{2}\right)\rho, \tag{5.50}$$

where

$$\rho = 2k_2 + 3\tau e^{\phi}\left[(2+4\tau e^{2\phi})k_1 + e^{\phi}\left(4k_2 - \tau(k_3 + \tau e^{\phi}(k_3 + i\ k_4) - i\ k_4)\right)\right],$$

and $k_1, \ldots, k_4$ are arbitrary constants. The family of solutions

The family of solutions $\mathbb{S}_{\mathbb{F}}$ are obtained by substituting (5.42), (5.43) and (5.44) into the auxiliary system (5.11) with $p = 0$ to obtain the system

$$2zh_1' - 2h_2 + h_1 = 0, \tag{5.51}$$

$$2zh_2' + 3h_2 = 0, \tag{5.52}$$

with solutions

$$h_1 = -\frac{k_1}{z^{\frac{3}{2}}} + \frac{k_2}{\sqrt{z}} \tag{5.53}$$

$$h_2 = \frac{k_1}{z^{\frac{3}{2}}} \tag{5.54}$$

where k_1 and k_2 are arbitrary constants. The solution $\mathbb{S}_{\mathbb{F}}$ is therefore given as.

$$C = \frac{1}{2}\tau^{-3}\sqrt{\tau e^{\phi}}\left(k_1 + 2\tau e^{2\phi}k_1 - \tau e^{\phi}k_2\right)\exp\left(\frac{e^{-2\phi} - 6\tau\phi}{4\tau}\right). \tag{5.55}$$

These are also solutions of the PDE

$$\bar{\Upsilon} \quad : \quad 4\tau C_\phi + 4\tau^2(2e^{2\phi}\tau - 1)C_\tau + (e^{-2\phi} - 4\tau + 12e^{2\phi}\tau^2)C = 0. \tag{5.56}$$

5.3.2 *Case 2:* $\lambda = -1/3$

Considering the nonlocal symmetries X_4, X_5 and X_6.

(a) The nonlocal symmetry X_4 has the invariant surface conditions

$$C_\phi - 2C - e^{2\phi}\vartheta = 0, \tag{5.57}$$

$$\vartheta_\phi - \vartheta = 0. \tag{5.58}$$

The characteristic system related to X_4 has three integrals, i.e.

$$I_1 = \tau, \tag{5.59}$$

$$I_2 = \vartheta e^{-\phi}, \tag{5.60}$$

$$I_3 = (C - \vartheta)e^{-2\phi}. \tag{5.61}$$

If we define the similarity variable as $I_1 = z$ and the similarity functions as $I_2 = h_1(z)$ ans $I_3 = h_2(z)$ we then obtain the following

$$C = h_2(z)e^{2\phi} + h_1(z)e^{3\phi}, \tag{5.62}$$

$$\vartheta = h_1(z)e^{\phi}, \tag{5.63}$$

$$z = \tau. \tag{5.64}$$

In this case, (5.62) is the family of solutions of the PDE

$$\Upsilon^* \quad : \quad C_{\phi\phi} - 5C_\phi + 6C = 0. \tag{5.65}$$

The substitution of (5.62) into the PDE (5.9) gives the system

$$h_2(z) = 0, \tag{5.66}$$

$$h_1'(z) = 0, \tag{5.67}$$

with solution

$$h_2(z) = 0, \tag{5.68}$$

$$h_1'(z) = k_1. \tag{5.69}$$

The family of solutions $\mathbb{S}^*_{\mathbb{F}}$ are consequently

$$C = k_1 e^{3\phi}. \tag{5.70}$$

This is similar to the family $\mathbb{S}_{\mathbb{F}}$.

(b) The nonlocal symmetry X_5 has the invariant surface conditions

$$4\tau C_\phi - 4\tau^2 C_\tau + (3e^{-2\phi} - 14\tau)C - 6\vartheta = 0. \tag{5.71}$$

$$4\tau\vartheta_\phi - 4\tau^2\vartheta_\tau + (3e^{-2\phi} - 6\tau)\vartheta = 0. \tag{5.72}$$

The characteristic system related to X_5 has three integrals, i.e.

$$I_1 = \tau e^{\phi}, \tag{5.73}$$

$$I_2 = \vartheta \exp\left[\frac{3}{4}\left(-2\phi - \frac{e^{-2\phi}}{\tau}\right)\right], \tag{5.74}$$

$$I_3 = \left(C + \frac{3\vartheta}{2\tau}\right)\exp\left(-\frac{7\phi}{2} - \frac{3e^{-2\phi}}{4\tau}\right). \tag{5.75}$$

These can be rewritten in terms of the similarity variable z and similarity functions $h_1(z)$ and $h_2(z)$ as

$$C = h_2(z)\exp\left(\frac{7\phi}{2} + \frac{3e^{-2\phi}}{4\tau}\right) - \frac{3e^{-\phi}}{2\tau}h_1(z)\exp\left(\frac{5\phi}{2} + \frac{3e^{-2\phi}}{4\tau}\right), \tag{5.76}$$

$$\vartheta = h_1(z)\exp\left[\frac{3}{4}\left(2\phi + \frac{e^{-2\phi}}{\tau}\right)\right], \tag{5.77}$$

$$z = \tau e^{\phi}. \tag{5.78}$$

The substitution of (5.76) gives the system

$$4z^2h_2'' + 20zh_2' + 7h_2 = 0, \tag{5.79}$$

$$4z^2h_1'' + 4zh_1' + 8h_2 - 9h_1 = 0, \tag{5.80}$$

with solution

$$h_2(z) = \frac{k_1}{\sqrt{z}} + \frac{k_2}{z^{\frac{7}{2}}}, \tag{5.81}$$

$$h_1(z) = \frac{-2k_2 + 5z^2[2zk_1 + k_3 + z^3k_3 + i(z^3 - 1)k_4]}{10z^{\frac{7}{2}}}. \tag{5.82}$$

The family of solutions $\mathbb{S}_{\mathbb{F}}^*$ are consequently

$$C = \frac{\tau^{-4}}{20\sqrt{\tau e^{\phi}}}\exp\left[\frac{3}{4}\left(\frac{e^{-2\phi}}{\tau} - 2\phi\right)\right]\rho, \tag{5.83}$$

where

$$\rho = 6k_2 + 5\tau e^{2\phi}(4k_2 + \tau(-3k_3 + \tau e^{\phi}(-6k_1 \tag{5.84}$$

$$+ \tau e^{2\phi}(4k_1 - 3\tau(k_3 + ik_4))) + 3ik_4)). \tag{5.85}$$

The solutions $\mathbb{S}_{\mathbb{F}}$ are

$$C = \frac{\tau^{-2}}{2\sqrt{\tau e^{\phi}}}[\tau e^{\phi}(2\tau e^{2\phi} - 3)k_1 - 3k_2]\exp\left(\frac{\phi}{2} + \frac{3e^{-2\phi}}{4\tau}\right) \tag{5.86}$$

(c) For X_6 the solutions $\mathbb{S}_{\mathbb{F}}$ and $\mathbb{S}^*_{\mathbb{F}}$ are given by

$$C = \frac{1}{2}\tau^{-\frac{3}{2}}(2\tau e^{2\phi} - 3)\exp\left(\phi + \frac{3e^{-2\phi}}{4\tau}\right). \tag{5.87}$$

5.3.3 *Case 3: General case* $\lambda \neq -1, -1/3, 0$

Considering the nonlocal symmetry X_4. We obtain the solutions

$$\begin{aligned}
\mathbb{S}^*_{\mathbb{F}} \quad : \quad C = \frac{1}{2}\tau^{-3}\exp\left(-\frac{\phi}{2\lambda} - \frac{e^{-2\phi}}{4\lambda\tau}\right) \times \Bigg[&2\tau(\tau e^{\phi})^{-\frac{1}{2\lambda}}(k_1 + (\tau e^{\phi})^{\frac{1}{\lambda}}k_2) \\
&+ \frac{e^{2\phi}}{\lambda}\left(-\frac{1}{4\lambda^2 - 1}(2(\tau e^{\phi})^{-\frac{1}{2\lambda}}\lambda^2((1-2\lambda)k_1 + (\tau e^{\phi})^{\frac{1}{\lambda}}(1+2\lambda)k_2)\right) \\
&+ (\lambda(k_2 - k_1) + \tau^2 e^{2\phi}k_3)\cosh\left(\frac{\ln(\tau e^{\phi})}{2\lambda}\right) \\
&+ (\lambda(k_2 + k_1) + i\ \tau^2 e^{2\phi}k_4)\sinh\left(\frac{\ln(\tau e^{\phi})}{2\lambda}\right)\Bigg]
\end{aligned} \tag{5.88}$$

and

$$\begin{aligned}
\mathbb{S}_{\mathbb{F}} \quad : \quad C = \frac{2\tau^{-3}(\tau e)^{-\frac{1}{2\lambda}}}{\lambda(1+2\lambda)}\exp\left(-\frac{e^{-2\phi}}{4\lambda\tau} + \frac{\phi}{2\lambda} + 2\phi\right) \times \\
[\lambda(2(1+2\lambda)\tau e^{2\phi} - 1)k_1 + (1+2\lambda)(\tau e)^{2+\frac{1}{\lambda}}k_2]
\end{aligned} \tag{5.89}$$

5.4 Nonlocal symmetries for a velocity dependent dispersion coefficient

Given $p = 2$ in (5.9) and (5.11), the dispersion coefficient is now given in terms of the water pore velocity v. In this case (5.11) with arbitrary proportionality constant λ admits the point symmetries given by

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= e^{2\phi}\frac{\partial}{\partial \phi},\\
X_3 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta},\\
X_4 &= \tau\frac{\partial}{\partial \phi} + \frac{1}{\lambda}\left[\left(\frac{e^{-4\phi}}{4} - \frac{\tau e^{-2\phi}}{2}\right)C - \frac{e^{-2\phi}}{2}\vartheta\right]\frac{\partial}{\partial C}\\
&\quad +\frac{1}{\lambda}\left(\frac{e^{-4\phi}}{4} - \frac{\tau e^{-2\phi}}{2}\right)\vartheta\frac{\partial}{\partial \vartheta},\\
X_5 &= \tau\frac{\partial}{\partial \phi} - \frac{1}{4}\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left[\left(\frac{\lambda}{2} - \frac{e^{-2\phi}}{8} + \frac{\tau}{4}\right)C + \frac{1}{4}\vartheta\right]\frac{\partial}{\partial C},\\
&\quad +\frac{1}{\lambda}\left(\frac{e^{-2\phi}}{8} - \frac{\tau}{4}\right)\vartheta\frac{\partial}{\partial \vartheta},\\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \frac{1}{2}\tau\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left(\frac{\lambda\tau}{2} + \frac{e^{-4\phi}}{16} - \frac{\tau e^{-2\phi}}{4} + \frac{\tau^2}{4}\right)\vartheta\frac{\partial}{\partial \vartheta}\\
&\quad -\frac{1}{\lambda}\left[\left(\frac{3\lambda\tau}{2} + \frac{e^{-4\phi}}{16} - \frac{\tau e^{-2\phi}}{4} + \frac{\tau^2}{4}\right)C - \left(\frac{e^{-2\phi}}{4} + \frac{\tau}{2}\right)\vartheta\right]\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{5.90}$$

The only genuine nonlocal symmetries are X_4, X_5 and X_6.

5.5 Invariant solutions for a velocity dependent dispersion coefficient

By applying the same procedure as in section (5.3) we obtain the solutions for a velocity dependent dispersion. We note again here that the solutions $\mathbb{S}_{\mathbb{F}}$ are contained in $\mathbb{S}^*_{\mathbb{F}}$.

(a) The nonlocal symmetry X_4 gives the family of solutions

$$\mathbb{S}_{\mathbb{F}}^{*} \quad : \quad C = \frac{\tau^{-\frac{3}{2}}}{4\lambda}(k_1 + 4\lambda\tau e^{2\phi}k_2)\exp\left(-\frac{e^{-4\phi}}{16\lambda\tau} + \frac{e^{-2\phi}}{4\lambda} + \frac{\tau}{4\lambda} - 2\phi\right) \quad (5.91)$$

and

$$\mathbb{S}_{\mathbb{F}} \quad : \quad C = \frac{\tau^{-\frac{3}{2}}}{4\lambda}(1 - 2\tau e^{-2\phi})k_1\exp\left(-\frac{e^{-4\phi}}{16\lambda\tau} + \frac{e^{-2\phi}}{4\lambda} + \frac{\tau}{4\lambda} - 2\phi\right). \quad (5.92)$$

(b) The nonlocal symmetry X_5 gives the family of solutions

$$\begin{aligned}\mathbb{S}_{\mathbb{F}}^{*} \quad : \quad C = \frac{1}{2}\exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda}\right) \times \Bigg[\frac{1}{\lambda}\left(4\sqrt{\lambda\pi}\ k_3 \ \mathrm{erf}\left(\frac{1}{4\sqrt{\lambda\tau e^{4\phi}}}\right) - k_4\right) \\ + \frac{2}{\sqrt{\tau e^{4\phi}}}\exp\left(2\phi - \frac{e^{-4\phi}}{16\lambda\tau}\right)\left(k_1 - 4\sqrt{\lambda\pi}\ k_2 \ \mathrm{erfi}\left(\frac{1}{4\sqrt{\lambda\tau e^{4\phi}}}\right)\right)\Bigg]\end{aligned} \quad (5.93)$$

and

$$\begin{aligned}\mathbb{S}_{\mathbb{F}} \quad : \quad C = \frac{1}{2}\exp\left(\frac{e^{-2\phi}}{4\lambda} - \frac{\tau}{4\lambda}\right) \times \Bigg[\frac{2\ k_1}{\sqrt{\tau e^{4\phi}}}\exp\left(2\phi - \frac{e^{-4\phi}}{16\lambda\tau}\right) \\ - \frac{k_2}{\lambda} + \frac{\sqrt{\pi}\ k_1}{\sqrt{\lambda}}\ \mathrm{erf}\left(\frac{1}{4\sqrt{\lambda\tau e^{4\phi}}}\right)\Bigg].\end{aligned} \quad (5.94)$$

Here, erf and erfi are the error function and the complex error function, respectively [37].

(c) The nonlocal symmetry X_6 gives the family of solutions

$$\mathbb{S}_{\mathbb{F}}^{*} \quad : \quad C = \frac{\tau^{-4}}{24\lambda}\sqrt{\tau e^{2\phi}}\exp\left[-7\phi - \frac{e^{-4\phi}}{16\lambda\tau}\left(1 - 2\tau e^{2\phi}\right)^2\right]\rho, \quad (5.95)$$

where

$$\rho = [-3\tau e^{2\phi}k_1 - k_2 + 6\tau e^{4\phi}(4\lambda k_2 + \tau k_3) + 6\tau^2 e^{6\phi}(4\lambda k_1 + \tau k_4)],$$

and

$$\mathbb{S}_{\mathbb{F}} \quad : \quad C = \frac{\tau^{-\frac{5}{2}}}{8\lambda e^{5\phi}}\exp\left[\phi - \frac{e^{-4\phi}}{16\lambda\tau}\left(1 - 2\tau e^{2\phi}\right)^2\right]\rho, \quad (5.96)$$

where

$$\rho = [2\tau e^{2\phi}(k_1 + k_2) - 4\tau e^{4\phi}(\tau k_2 - 2\lambda k_1) - k_1].$$

5.6 Concluding remarks

We have successfully applied the nonlocal symmetry techniques to a model describing contaminant transport in saturated soils under radial water flow background. The nonlocal symmetry analysis resulted in a number of exotic and rich array of symmetries being admitted. As far as we know, nonlocal symmetry techniques have not been used for these models and as such, we have constructed new exact solutions. These solution are not obtainable by any other symmetry techniques and are not yet recorded in literature.

Chapter 6

Conclusion

This dissertation is a body of new exact (group invariant) solutions for a model arising in contaminant transport theory.

Firstly, models describing contaminant transport under radial water flow background are analyzed using Lie point symmetry techniques. We have observed that extra Lie point symmetries are admitted when the dispersion coefficient is a constant λ or when it is given as a power law function of velocity λv^p, with the exponent being given by two. In our case $v = e^{\phi}$. When the dispersion coefficient is constant, the Lie point symmetries of (2.10) vary according to three λ cases. In particular, we have $\lambda = 1$, $\lambda = -1$ and a general case where λ is just arbitrary. When the dispersion coefficient is velocity dependent, we observe that the Lie point symmetries of (2.10) depend on λ. We note that the admitted symmetry structure and number is not affected by the constant λ, that is we obtain the same symmetries up to the specified λ value.

The optimal system of one-dimensional subalgebras of the admitted symmetry Lie algebra is constructed. Furthermore, group-invariant solutions are classified according to the elements of the optimal systems (see e.g. [22]).

On the other hand, in chapter 5 we analyzed the governing equation using nonlocal symmetry techniques. It is possible to construct exact solutions using nonlocal symmetries. These solutions may not be obtained using any other symmetry techniques. We have constructed families of exact solutions by the method of reduction of order (see e.g. [29]). The method finds the exact solutions $\mathbb{S}_{\mathbb{F}}$ which are always contained in $\mathbb{S}^*_{\mathbb{F}}$. As such, we refer to $\mathbb{S}^*_{\mathbb{F}}$ as a wider class or family of exact solutions.

Appendix A

Nonlocal symmetries for a constant dispersion coefficient

The derivation of nonlocal symmetries of the contaminant transport equation with $p = 0$ will be outlined. The model $R\{\tau, \phi, C\}$ to be investigated is given by

$$e^{-2\phi}\frac{\partial C}{\partial \tau} = \lambda\frac{\partial^2 C}{\partial \phi^2} + \frac{\partial C}{\partial \phi}, \quad \lambda \neq 0. \tag{A.1}$$

Equation (A.1) can be written in conserved form as

$$D_\tau\left(e^{-2\phi}C\right) = D_\phi\left(\lambda\frac{\partial C}{\partial \phi} + C\right), \tag{A.2}$$

where $D_\tau = \dfrac{\partial}{\partial \tau}$ and $D_\phi = \dfrac{\partial}{\partial \phi}$. By introducing an auxiliary potential variable $\vartheta(\tau, \phi)$ as a further unknown function, we obtain the auxiliary system $S\{\tau, \phi, C, \vartheta\}$ given by

$$\left.\begin{array}{rcl} \vartheta_\phi &=& e^{-2\phi}C, \\ \vartheta_\tau &=& \lambda\dfrac{\partial C}{\partial \phi} + C. \end{array}\right\} \tag{A.3}$$

The Lie point symmetry generator of the system (A.3) is of the form

$$X = \xi^1(\tau, \phi, C, \vartheta)\frac{\partial}{\partial \tau} + \xi^2(\tau, \phi, C, \vartheta)\frac{\partial}{\partial \phi} + \eta^1(\tau, \phi, C, \vartheta)\frac{\partial}{\partial C} + \eta^2(\tau, \phi, C, \vartheta)\frac{\partial}{\partial \vartheta}. \tag{A.4}$$

We seek to solve the determining equations given by

$$X^{[1]}\left(\vartheta_\phi - e^{-2\phi}C\right)\Big|_{(A.3)} = 0, \quad X^{[1]}\left(\vartheta_\tau - \lambda\frac{\partial C}{\partial \phi} - C\right)\Big|_{(A.3)} = 0. \tag{A.5}$$

Here $X^{[1]}$ is the operator:

$$X^{[1]} = \xi^1\frac{\partial}{\partial \tau} + \xi^2\frac{\partial}{\partial \phi} + \eta^1\frac{\partial}{\partial C} + \eta^2\frac{\partial}{\partial \vartheta} + \eta^1_a\frac{\partial}{\partial C_\tau} + \eta^1_b\frac{\partial}{\partial C_\phi} + \eta^2_a\frac{\partial}{\partial \vartheta_\tau} + \eta^2_b\frac{\partial}{\partial \vartheta_\phi}, \tag{A.6}$$

where

$$\eta^1_a = \eta^1_\tau + \left(\eta^1_C + \xi^1_\tau\right)C_\tau - \xi^1_C C^2_\tau - \xi^2_\tau C_\phi - \xi^2_C C_\tau C_\phi + \eta^1_\vartheta \vartheta_\tau - \xi^1_\vartheta C_\tau \vartheta_\tau - \xi^2_\vartheta C_\phi \vartheta_\tau, \tag{A.7}$$

$$\eta^1_a = \eta^1_\phi + \left(\eta^1_C - \xi^2_\phi\right)C_\tau - \xi^1_\phi C_\tau - \xi^1_C C_\phi C_\tau - \xi^2_C C^2_\phi + \eta^1_\vartheta \vartheta_\phi - \xi^1_\vartheta C_\tau \vartheta_\phi - \xi^2_\vartheta C_\phi \vartheta_\phi, \tag{A.8}$$

$$\eta^1_a = \eta^2_\tau - \left(\eta^2_\vartheta + \xi^1_\tau\right)\vartheta_\tau - \xi^1_\vartheta \vartheta^2_\tau + \eta^2_C C_\tau - \xi^1_C C_\tau \vartheta_\tau - \xi^2_\tau \vartheta_\phi - \xi^2_C C_\tau \vartheta_\phi - \xi^2_\vartheta \vartheta_\tau \vartheta_\phi, \tag{A.9}$$

$$\eta^1_a = \eta^2_\phi + \left(\eta^2_\vartheta - \xi^2_\phi\right)\vartheta_\phi - \xi^2_\vartheta \vartheta^2_\phi + \eta^2_C C_\phi - \xi^1_\phi \vartheta_\tau - \xi^1_C C_\phi \vartheta_\tau - \xi^1_\vartheta \vartheta_\phi \vartheta_\tau - \xi^2_C C_\phi \vartheta_\phi. \tag{A.10}$$

From the determining equations (A.5) we obtain

$$e^{-2\phi}\eta^1 - 2e^{-2\phi}\xi^2 C + e^{-4\phi}{\xi^2}_\vartheta C^2 + e^{-2\phi}{\xi^1}_\vartheta C^2 - e^{-2\phi}\eta^2_\vartheta C - \eta^2_C C_\phi + \lambda {\xi^1}_C {C_\phi}^2 + \lambda {\xi^1}_\phi C_\phi + e^{-2\phi}{\xi^2}_C C C_\phi + e^{-2\phi}{\xi^2}_\phi C + {\xi^1}_C C C_\phi + {\xi^1}_\phi C + \lambda e^{-2\phi}{\xi^1}_\vartheta C C_\phi - \eta^2_\phi = 0 \tag{A.11}$$

and

$$\eta_1 + e^{-2\phi}{\xi^2}_\vartheta C^2 + {\xi^1}_\vartheta C^2 + \lambda\eta^1_C C_\phi - \eta^2_C C_\tau + \lambda e^{-2\phi}\eta^1_\vartheta C - \lambda\eta^2_\vartheta C_\phi - \eta^2_\vartheta C - \lambda {\xi^2}_C {C_\phi}^2 - \lambda {\xi^2}_\phi C_\phi - \lambda {\xi^1}_\phi C_\tau + \lambda {\xi^1}_\tau C_\phi + e^{-2\phi}{\xi^2}_C C C_\tau + e^{-2\phi}{\xi^2}_\tau C + {\xi^1}_C C C_\tau + {\xi^1}_\tau C + \lambda^2 {\xi^1}_\vartheta {C_\phi}^2 - \lambda e^{-2\phi}{\xi^1}_\vartheta C C_\tau + 2\lambda {\xi^1}_\vartheta C C_\phi + \lambda\eta^1_\phi - \eta^2_\tau = 0. \tag{A.12}$$

The separation of the monomials, viz., the coefficients of separate powers of the derivatives of C gives the following over determined system

$$\begin{aligned}
&\lambda\xi^1_C = 0,\\
&\lambda(\xi^2_C - \lambda\xi^1_\vartheta) = 0,\\
&\lambda(\xi^1_\vartheta C + e^{2\phi}\xi^1_\phi) = 0,\\
&\lambda(\eta^1_C + 2\xi^1_\vartheta C - \eta^2_\vartheta - \xi^2_\phi + \xi^1_\tau) = 0,\\
&\xi^2_C - \lambda\xi^1_\vartheta C - e^{2\phi}\eta^2_C + e^{2\phi}\xi^1_C C - \lambda e^{2\phi}\xi^1_\phi = 0\\
&2e^{2\phi}\eta^1 - 2e^{2\phi}\xi^2 C + \xi^2_\vartheta C^2 + e^{2\phi}\xi^1_\vartheta C^2 - e^{2\phi}\eta^2_\vartheta C + e^{2\phi}\xi^2_\phi C + e^{4\phi}\xi^1_\phi C - e^{4\phi}\eta^2_\phi = 0,\\
&2e^{2\phi}\xi^2 C + \lambda\eta^1_\vartheta C + \xi^2_\tau C - e^{2\phi}\xi^2_\phi C + e^{2\phi}\xi^1_\tau C - e^{4\phi}\xi^1_\phi C + \lambda e^{2\phi}\eta^1_\phi - e^{2\phi}\eta^2_\tau + e^{4\phi}\eta^2_\phi = 0.
\end{aligned} \tag{A.13}$$

Since $\lambda \neq 0$, the first two equations in (A.13) give $\xi^1 = A(\tau, \phi, \vartheta)$ and $\xi^2 = B(\tau, \phi, \vartheta) + \lambda A_\vartheta C$. Here A and B are simply arbitrary functions of τ, ϕ and ϑ. The fifth equation in (A.13) gives $\eta^2 = D(\tau, \phi, \vartheta)$. Here D is an arbitrary function of τ, ϕ and ϑ. If we now consider the third equation in (A.13), we note that A is simply a function of τ since $A_\vartheta = 0$ and $A_\phi = 0$, as such, we have $\xi^1 = A(\tau)$. The overdetermined system simplifies into the following equations with prime denoting differentiation with respect to τ

$$A' - D_\vartheta - B_\phi + \eta^1_C = 0, \tag{A.14}$$

$$e^{2\phi}\eta^1 - 2e^{2\phi}B\ C + B_\vartheta C^2 - e^{2\phi}D_\vartheta C + e^{2\phi}B_\phi C - e^{4\phi}D_\phi = 0, \tag{A.15}$$

$$\begin{aligned}
2e^{2\phi}B\ C + e^{2\phi}A'C - e^{2\phi}B_\phi C + e^{4\phi}D_\phi + B_\tau C - e^{2\phi}D_\tau&\\
+\lambda\eta^1_\vartheta C + \lambda e^{2\phi}\eta^1_\phi = 0.&
\end{aligned} \tag{A.16}$$

Equation (A.14) gives

$$\eta^1 = E(\tau, \phi, \vartheta) - (A' - B_\phi - D_\vartheta)C, \tag{A.17}$$

where E is an arbitrary function of τ, ϕ and ϑ. Substituting (A.17) into (A.15) gives

$$2e^{2\phi}B\ C - e^{2\phi}E + e^{2\phi}A'C - B_{\vartheta}C^2 - 2e^{2\phi}B_{\phi}C + e^{4\phi}D_{\phi} = 0. \quad \text{(A.18)}$$

The functions A, B, D and E are all independent of C. Hence equating the coefficients of powers of C to zero gives

$$C^2 : B_{\vartheta} = 0, \quad \text{(A.19)}$$

$$C : 2B(\tau, \phi, \vartheta) + A' - 2B_{\phi} = 0, \quad \text{(A.20)}$$

$$C^0 : E(\tau, \phi, \vartheta) - e^{2\phi}D_{\phi} = 0. \quad \text{(A.21)}$$

Substituting (A.17) into (A.16) gives

$$2e^{2\phi}B\ C + e^{2\phi}A'C + \lambda E_{\vartheta}C + \lambda D_{\vartheta\vartheta}C^2 - e^{2\phi}B_{\phi}C + e^{4\phi}D_{\phi} + \lambda e^{2\phi}E_{\phi}$$
$$+\lambda B_{\phi\vartheta}C^2 + \lambda e^{2\phi}D_{\phi\vartheta}C + \lambda e^{2\phi}B_{\phi\phi}C + B_{\tau}C - e^{2\phi}D_{\tau} = 0. \quad \text{(A.22)}$$

Equation (A.22) separates as follows

$$C^2 : B_{\phi\vartheta} + D_{\vartheta\vartheta} = 0, \quad \text{(A.23)}$$

$$C : 2e^{2\phi}B + e^{2\phi}A' + \lambda E_{\vartheta} - e^{2\phi}B_{\phi} + \lambda e^{2\phi}D_{\phi\vartheta} + \lambda e^{2\phi}B_{\phi\phi} + B_{\tau} = 0, \quad \text{(A.24)}$$

$$C^0 : e^{2\phi}D_{\phi} + \lambda E_{\phi} - D_{\tau} = 0. \quad \text{(A.25)}$$

Equation (A.19) shows that B is an arbitrary function independent of ϑ, therefore from now on we will take B as a function of τ and ϕ. If we realize that $D = G(\tau, \phi) + H(\tau, \phi)\vartheta$ with G and H being arbitrary functions of τ and ϕ, it remains for us to solve the following system of equations

$$2B + A' - 2B_{\phi} = 0, \quad \text{(A.26)}$$

$$E - e^{2\phi}\left(G_{\phi} + \vartheta H_{\phi}\right) = 0, \quad \text{(A.27)}$$

$$e^{2\phi}G_{\phi} + e^{2\phi}H_{\phi}\vartheta - G_{\tau} - H_{\tau}\vartheta + \lambda E_{\phi} = 0, \quad \text{(A.28)}$$

$$2e^{2\phi}B + e^{2\phi}A' - e^{2\phi}B_{\phi} + \lambda e^{2\phi}H_{\phi} + \lambda e^{2\phi}B_{\phi\phi} + B_{\tau} + \lambda E_{\vartheta} = 0. \quad \text{(A.29)}$$

Equation (A.26) gives

$$B(\tau,\phi) = e^{\phi}K(\tau) - \frac{A'}{2}, \tag{A.30}$$

where K is an arbitrary function of τ. Substituting (A.30) into (A.29) leads to the following equations

$$2(1+\lambda)e^{3\phi}K(\tau) + 2e^{\phi}K' - A'' + 2\lambda e^{2\phi}H_{\phi} + 2\lambda E_{\vartheta} = 0, \tag{A.31}$$

and

$$e^{2\phi}G_{\phi} + e^{2\phi}H_{\phi}\vartheta - G_{\tau} - H_{\tau}\vartheta + \lambda E_{\phi} = 0. \tag{A.32}$$

If we also substitute an expression for E found in (A.27) the we are only left with the following equations to solve

$$(1+2\lambda)G_{\phi} + \lambda e^{2\phi}G_{\phi\phi} - G_{\tau} = 0, \tag{A.33}$$

$$(1+2\lambda)H_{\phi} + \lambda e^{2\phi}H_{\phi\phi} - H_{\tau} = 0, \tag{A.34}$$

$$2(1+\lambda)e^{3\phi}K(\tau) + 2e^{\phi}K' - A'' + 4\lambda e^{2\phi}H_{\phi} = 0. \tag{A.35}$$

From equation (A.35) we obtain

$$H(\tau,\phi) = \frac{1}{4\lambda}\left(-2(1+\lambda)e^{\phi}K(\tau) + 2e^{-\phi}K' - \frac{1}{2}e^{-2\phi}A''\right) + L(\tau). \tag{A.36}$$

Substitution (A.36) into (A.34) gives

$$4(3\lambda^2+4\lambda+1)e^{5\phi}K(\tau) + 8\lambda e^{2\phi}L' - 2e^{2\phi}A'' + 4e^{\phi}K'' - A'' = 0. \tag{A.37}$$

This separates as follows

$$e^{5\phi} : (1+\lambda)(1+3\lambda)K(\tau) = 0, \tag{A.38}$$

$$e^{2\phi} : 8\lambda L' - 2A'' = 0, \tag{A.39}$$

$$e^{\phi} : K'' = 0, \tag{A.40}$$

$$\text{remainder} : A''' = 0. \tag{A.41}$$

Therefore we have

$$A(\tau) = k_1 + \tau(k_2 + \tau k_3), \tag{A.42}$$

$$K(\tau) = k_4 + \tau k_5, \tag{A.43}$$

$$L(\tau) = \frac{1}{2\lambda}\tau k_3 + k_6. \tag{A.44}$$

From (A.38), we note that since $K(\tau) \neq 0$, then we have three conditions arising, i.e. $\lambda = -1$, $\lambda = -1/3$ and $\lambda \neq 0, -1, -1/3$. As such, We have the following cases for symmetries of (A.1).

Case 1: $\lambda = -1$.

$$\begin{aligned}
\xi^1 &= k_1 + \tau(k_2 + \tau k_3), \\
\xi^2 &= -\frac{k_2}{2} - \tau k_3 + e^{\phi}k_4 + \tau e^{\phi}k_5, \\
\eta^1 &= -\frac{1}{2}\vartheta(k_3 - e^{\phi}k_5) + \frac{C}{4}\bigg(- 4k_2 + e^{-2\phi}k_3 - 10\tau k_3 - 2e^{-\phi}k_5 \\
&\qquad + 4e^{\phi}(k_4 + \tau k_5) + 4k_6\bigg) + e^{2\phi}\omega(\tau, \phi), \\
\eta^2 &= \vartheta\left(-\frac{\tau}{2}k_3 + \frac{e^{-2\phi}}{4}(k_3 - 2e^{\phi}k_5) + k_6\right) + \omega(\tau, \phi),
\end{aligned}$$

where $\omega(\tau, \phi)$ is an arbitrary function satisfying equation (A.1) and $k_1, \ldots, k_6$ are arbitrary constants. The resulting Lie point symmetries including the

infinite symmetry generator, $X_\omega = e^{2\phi}\omega(\tau,\phi)\dfrac{\partial}{\partial C} + \omega_\phi \dfrac{\partial}{\partial \vartheta}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta},\\
X_3 &= e^{-2\phi}\frac{\partial}{\partial \phi} + C\frac{\partial}{\partial C},\\
X_4 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C},\\
X_5 &= \tau\frac{\partial}{\partial \phi} - \left[\left(\frac{e^{-2\phi}}{2} - \tau\right)C - \frac{\vartheta}{2}\right]\frac{\partial}{\partial C} - \frac{e^{-2\phi}}{2}\vartheta\frac{\partial}{\partial \vartheta},\\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} + \left[\left(\frac{e^{-2\phi}}{4} - \frac{5}{2}\tau\right)C - \frac{\vartheta}{2}\right]\frac{\partial}{\partial C}\\
&\quad + \left(\frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)\vartheta\frac{\partial}{\partial \vartheta}.
\end{aligned}\right\} \tag{A.45}$$

In all the symmetries above, X_5 and X_6 are the only genuine nonlocal symmetries.

Case 2: $\lambda = -1/3$.

$$\begin{aligned}
\xi^1 &= k_1 + \tau(k_2 + \tau k_3),\\
\xi^2 &= -\frac{k_2}{2} - \tau k_3 + e^{\phi}k_4 + \tau e^{\phi}k_5,\\
\eta^1 &= \frac{e^{-2\phi}}{4}\Big[2e^{2\phi}\vartheta\left(-3k_3 + 3e^{\phi}k_5 + 2e^{3\phi}(k_4 + \tau k_5)\right)\\
&\qquad + \left(3k_3 - 6e^{\phi}k_5 + 8e^{3\phi}(k_4 + \tau k_5) - 2e^{2\phi}(2k_2 + 7\tau k_3 - 2k_6)\right)C\Big] + e^{4\phi}\omega_\phi,\\
\eta^2 &= \frac{\vartheta}{4}\left(3e^{-2\phi}k_3 - 6\tau k_3 - 6e^{-\phi}k_5 + 4e^{\phi}(k_4 + \tau k_5) + 4k_6\right) + \omega(\tau,\phi),
\end{aligned}$$

where $\omega(\tau,\phi)$ is an arbitrary function satisfying equation (A.1) and $k_1,\ldots,k_6$ are arbitrary constants. The resulting Lie point symmetries including the

infinite symmetry generator, $X_\omega = e^{2\phi}\omega(\tau,\phi)\dfrac{\partial}{\partial C} + \omega_\phi \dfrac{\partial}{\partial \vartheta}$, are

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta},\\
X_3 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C},\\
X_4 &= \frac{\partial}{\partial \phi} + \left(2C + e^{2\phi}\vartheta\right)\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta},\\
X_5 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} + \left[\left(\frac{3e^{-2\phi}}{4} - \frac{7\tau}{2}\right)C - \frac{3}{2}\vartheta\right]\frac{\partial}{\partial C},\\
X_6 &= \tau\frac{\partial}{\partial \phi} - \left[\left(\frac{3e^{-2\phi}}{2} - 2\tau\right)C - \left(\frac{3}{2} + e^{2\phi}\tau\right)\vartheta\right]\frac{\partial}{\partial C}\\
&\quad - \left(\frac{3e^{-2\phi}}{2} - \tau\right)\vartheta\frac{\partial}{\partial \vartheta}.
\end{aligned}\right\} \quad \text{(A.46)}$$

The only genuine nonlocal symmetries of (A.1) are X_4, X_5 and X_6.

Case 3: $\lambda \neq -1, -1/3, 0$. Arbitrary λ.

$$\begin{aligned}
\xi^1 &= k_1 + \tau(k_2 + \tau k_3),\\
\xi^2 &= -\frac{k_2}{2} - \tau k_3,\\
\eta^1 &= -\frac{1}{4\lambda}\left(-2\vartheta k_3 + \left((e^{-2\phi} - 2\tau)k_3 + 4\lambda(k_2 + 2\tau k_3 - k_4)\right)C\right) + e^{2\phi}\omega_\phi,\\
\eta^2 &= -\frac{1}{4\lambda}\left(e^{-2\phi}k_3 - 2\tau k_3 - 4\lambda k_4\right)\vartheta + \omega(\tau,\phi),
\end{aligned}$$

where $\omega(\tau,\phi)$ is an arbitrary function satisfying equation (A.1) and $k_1,\ldots,k_4$ are arbitrary constants. The resulting Lie point symmetries including the

infinite symmetry generator, $X_\omega = e^{2\phi}\omega(\tau,\phi)\dfrac{\partial}{\partial C} + \omega_\phi \dfrac{\partial}{\partial \vartheta}$, are

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial \tau}, \\ X_2 &= C\frac{\partial}{\partial C}, \\ X_3 &= \tau\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial}{\partial \phi} - C\frac{\partial}{\partial C}, \\ X_4 &= \tau^2\frac{\partial}{\partial \tau} - \tau\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left[\left(2\lambda\tau + \frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)C + \frac{1}{2}\vartheta\right]\frac{\partial}{\partial C} \\ &\quad -\frac{1}{\lambda}\left(\frac{e^{-2\phi}}{4} - \frac{\tau}{2}\right)\vartheta\frac{\partial}{\partial \vartheta}. \end{aligned}\right\} \tag{A.47}$$

The only genuine nonlocal symmetry of (A.1) is X_4.

Appendix B

Nonlocal symmetries for a velocity dependent dispersion coefficient

The derivation of nonlocal symmetries of the contaminant transport equation with $p = 2$ will be outlined. The model $R\{\tau, \phi, C\}$ to be investigated is given by

$$e^{-2\phi}\frac{\partial C}{\partial \tau} = \left(\lambda e^{2\phi}\frac{\partial C}{\partial \phi}\right) + \frac{\partial C}{\partial \phi}, \quad \lambda \neq 0. \tag{B.1}$$

Equation (B.1) can be written in conserved form as

$$D_\tau\left(e^{-2\phi}C\right) = D_\phi\left(\lambda e^{2\phi}\frac{\partial C}{\partial \phi} + C\right), \tag{B.2}$$

where $D_\tau = \dfrac{\partial}{\partial \tau}$ and $D_\phi = \dfrac{\partial}{\partial \phi}$. By introducing an auxiliary potential variable $\vartheta(\tau, \phi)$ as a further unknown function, we obtain the auxiliary system $S\{\tau, \phi, C, \vartheta\}$ given by

$$\left.\begin{aligned} \vartheta_\phi &= e^{-2\phi}C, \\ \vartheta_\tau &= \lambda e^{2\phi}\frac{\partial C}{\partial \phi} + C. \end{aligned}\right\} \tag{B.3}$$

The Lie point symmetry generator of the system (B.3) is of the form

$$X = \xi^1(\tau,\phi,C,\vartheta)\frac{\partial}{\partial\tau} + \xi^2(\tau,\phi,C,\vartheta)\frac{\partial}{\partial\phi} + \eta^1(\tau,\phi,C,\vartheta)\frac{\partial}{\partial C} + \eta^2(\tau,\phi,C,\vartheta)\frac{\partial}{\partial\vartheta}. \tag{B.4}$$

We seek to solve the determining equations given by

$$X^{[1]}\left(\vartheta_\phi - e^{-2\phi}C\right)\Big|_{(B.3)} = 0, \quad X^{[1]}\left(\vartheta_\tau - \lambda e^{2\phi}\frac{\partial C}{\partial\phi} - C\right)\Big|_{(B.3)} = 0. \tag{B.5}$$

Here $X^{[1]}$ is the operator:

$$X^{[1]} = \xi^1\frac{\partial}{\partial\tau} + \xi^2\frac{\partial}{\partial\phi} + \eta^1\frac{\partial}{\partial C} + \eta^2\frac{\partial}{\partial\vartheta} + \eta^1_a\frac{\partial}{\partial C_\tau} + \eta^1_b\frac{\partial}{\partial C_\phi} + \eta^2_a\frac{\partial}{\partial\vartheta_\tau} + \eta^2_b\frac{\partial}{\partial\vartheta_\phi}, \tag{B.6}$$

where

$$\eta^1_a = \eta^1_\tau + \left(\eta^1_C + \xi^1_\tau\right)C_\tau - \xi^1_C C^2_\tau - \xi^2_\tau C_\phi - \xi^2_C C_\tau C_\phi + \eta^1_\vartheta\vartheta_\tau - \xi^1_\vartheta C_\tau\vartheta_\tau - \xi^2_\vartheta C_\phi\vartheta_\tau, \tag{B.7}$$

$$\eta^1_a = \eta^1_\phi + \left(\eta^1_C - \xi^2_\phi\right)C_\tau - \xi^1_\phi C_\tau - \xi^1_C C_\phi C_\tau - \xi^2_C C^2_\phi + \eta^1_\vartheta\vartheta_\phi - \xi^1_\vartheta C_\tau\vartheta_\phi - \xi^2_\vartheta C_\phi\vartheta_\phi, \tag{B.8}$$

$$\eta^1_a = \eta^2_\tau - \left(\eta^2_\vartheta + \xi^1_\tau\right)\vartheta_\tau - \xi^1_\vartheta\vartheta^2_\tau + \eta^2_C C_\tau - \xi^1_C C_\tau\vartheta_\tau - \xi^2_\tau\vartheta_\phi - \xi^2_C C_\tau\vartheta_\phi - \xi^2_\vartheta\vartheta_\tau\vartheta_\phi, \tag{B.9}$$

$$\eta^1_a = \eta^2_\phi + \left(\eta^2_\vartheta - \xi^2_\phi\right)\vartheta_\phi - \xi^2_\vartheta\vartheta^2_\phi + \eta^2_C C_\phi - \xi^1_\phi\vartheta_\tau - \xi^1_C C_\phi\vartheta_\tau - \xi^1_\vartheta\vartheta_\phi\vartheta_\tau - \xi^2_C C_\phi\vartheta_\phi. \tag{B.10}$$

From the determining equations (B.5) we obtain

$$e^{-2\phi}\eta^1 - 2e^{-2\phi}\xi^2 C + e^{-4\phi}\xi^2_\vartheta C^2 + e^{-2\phi}\xi^1_\vartheta C^2 - e^{-2\phi}\eta^2_\vartheta C + \lambda\xi^1_\vartheta C C_\phi - \eta^2_C C_\phi + \lambda e^{2\phi}\xi^1_C C^2_\phi + \lambda e^{2\phi}\xi^1_\phi C_\phi + e^{-2\phi}\xi^2_C C C_\phi + e^{-2\phi}\xi^2_\phi C + \xi^1_C C C_\phi + \xi^1_\phi C - \eta^2_\phi = 0, \tag{B.11}$$

and

$$
\begin{aligned}
&2\lambda e^{2\phi}\xi^2 C_\phi + \eta^1 - \eta^2_\tau + \eta^2_C C_\tau + \xi^1_C C C_\tau + \xi^1_\tau C - \lambda \xi^1_\vartheta C C_\tau - \lambda e^{2\phi}\xi^1_\phi C_\tau \\
&+ \lambda e^{2\phi}\xi^1_\tau C_\phi + e^{-2\phi}\xi^2_C C C_\tau + e^{-2\phi}\xi^2_\tau C + \xi^1_\vartheta C^2 + e^{-2\phi}\xi^2_\vartheta C^2 + \lambda \eta^1_\vartheta C - \eta^2_\vartheta C \\
&- \lambda e^{2\phi}\eta^2_\vartheta C_\phi + \lambda^2 e^{4\phi}\xi^1_\vartheta C^2_\phi + 2\lambda e^{2\phi}\xi^1_\vartheta C C_\phi + \lambda e^{2\phi}\eta^1_C C_\phi - \lambda e^{2\phi}\xi^2_C C^2_\phi \\
&- \lambda e^{2\phi}\xi^2_\phi C_\phi + \lambda e^{2\phi}\eta^1_\phi = 0. \qquad \text{(B.12)}
\end{aligned}
$$

The separation of the monomials leads to

$$
\begin{aligned}
&\lambda \xi^1_C = 0, \\
&\lambda\left(\xi^1_\vartheta C + e^{2\phi}\xi^1_\phi\right) = 0, \\
&\lambda\left(\xi^2_C - \lambda e^{2\phi}\xi^1_\vartheta\right) = 0, \\
&\lambda\left(2\xi^2 + \xi^1_\tau + \eta^1_C + 2\xi^1_\vartheta C - \eta^2_\vartheta - \xi^2_\phi\right) = 0, \qquad \text{(B.13)} \\
&\xi^2_C C - \lambda e^{2\phi}\xi^1_\vartheta C - e^{2\phi}\eta^2_C + e^{2\phi}\xi^1_C C - \lambda e^{4\phi}\xi^1_\phi = 0, \\
&e^{2\phi}\eta^1 - 2e^{2\phi}\xi^2 C + \xi^2_\vartheta C^2 + e^{2\phi}\xi^1_\vartheta C^2 - e^{2\phi}\eta^2_\vartheta C + e^{2\phi}\xi^2_\phi C + e^{4\phi}\xi^1_\phi C - e^{4\phi}\eta^2_\phi = 0, \\
&2e^{2\phi}\xi^2 C + \xi^2_\tau C + e^{2\phi}\xi^1_\tau C - e^{2\phi}\eta^2_\tau + \lambda e^{2\phi}\eta^1_\vartheta C - e^{2\phi}\xi^2_\phi C - e^{4\phi}\xi^1_\phi C + \lambda e^{4\phi}\eta^1_\phi + e^{4\phi}\eta^2_\phi = 0.
\end{aligned}
$$

Since $\lambda \neq 0$, the first two equations in (B.13) give $\xi^1 = A(\tau)$. Here A is an arbitrary function of τ. The third and fourth equations give $\xi^2 = B(\tau, \phi, \vartheta)$ and $\eta^2 = D(\tau, \phi, \vartheta)$. Here B and D are arbitrary functions of τ, ϕ and ϑ. The overdetermined system simplifies into the following equations with prime denoting differentiation with respect to τ

$$
2B + A' - D_\vartheta - B_\phi + \eta^1_C = 0, \qquad \text{(B.14)}
$$

$$
e^{2\phi}\eta^1 - 2e^{2\phi}B\,C + B_\vartheta C^2 - e^{2\phi}D_\vartheta C + e^{2\phi}B_\phi C - e^{4\phi}D_\phi = 0, \qquad \text{(B.15)}
$$

$$
\begin{aligned}
&2e^{2\phi}B\,C + e^{2\phi}A'C - e^{2\phi}B_\phi C + e^{4\phi}D_\phi + B_\tau C - e^{2\phi}D_\tau \\
&+\lambda \eta^1_\vartheta C + \lambda e^{2\phi}\eta^1_\phi = 0. \qquad \text{(B.16)}
\end{aligned}
$$

Equation (B.14) gives

$$\eta^1 = E(\tau, \phi, \vartheta) - (2B + A' - D_\vartheta - B_\phi)C, \tag{B.17}$$

where E is an arbitrary function of τ, ϕ and ϑ. Substituting (B.17) into (B.15) gives

$$4e^{2\phi}B\ C - e^{2\phi}E + e^{2\phi}A'C - B_\vartheta C^2 - 2e^{2\phi}B_\phi C + e^{4\phi}D_\phi = 0. \tag{B.18}$$

Since the functions A, B, D and E are all independent of C, we can equate the coefficients of powers of C to zero. This gives

$$C^2 : B_\vartheta = 0, \tag{B.19}$$

$$C : 2B + A' - 2B_\phi = 0, \tag{B.20}$$

$$C^0 : E - e^{2\phi}D_\phi = 0. \tag{B.21}$$

Substituting (B.17) into (B.16) gives

$$\begin{aligned}2e^{2\phi}B\ C + e^{2\phi}A'C - 2\lambda e^{2\phi}B_\vartheta C^2 + \lambda e^{2\phi}E_\vartheta C + \lambda e^{2\phi}D_{\vartheta\vartheta}C^2 - e^{2\phi}B_\phi C \\ -2\lambda e^{4\phi}B_\phi C + e^{4\phi}D_\phi + \lambda e^{4\phi}E_\phi + \lambda e^{2\phi}B_{\phi\vartheta}C^2 + \lambda e^{4\phi}D_{\phi\vartheta}C + \lambda e^{4\phi}B_{\phi\phi}C \\ +B_\tau C - e^{2\phi}D_\tau = 0.\end{aligned} \tag{B.22}$$

Equation (B.22) separates as follows

$$C^2 : 2B_\vartheta - B_{\phi\vartheta} - D_{\vartheta\vartheta} = 0, \tag{B.23}$$

$$\begin{aligned}C : 2e^{2\phi}B + e^{2\phi}A' + \lambda e^{2\phi}E_\vartheta - e^{2\phi}B_\phi - 2\lambda e^{4\phi}B_\phi + \lambda e^{4\phi}D_{\phi\vartheta} + \lambda e^{4\phi}B_{\phi\phi} \\ + B_\tau = 0,\end{aligned} \tag{B.24}$$

$$C^0 : e^{2\phi}D_\phi + \lambda e^{2\phi}E_\phi - D_\tau = 0. \tag{B.25}$$

Equation (B.19) shows that B is an arbitrary function independent of ϑ, therefore from now on we will take B as a function of τ and ϕ. From (B.23) we note that $D = G(\tau, \phi) + H(\tau, \phi)\vartheta$ with G and H being arbitrary functions of τ and ϕ. We also note that from (B.20) that

$$B = e^{2\phi}H(\tau) - \frac{A'}{4}. \tag{B.26}$$

Equation (B.21) leads to

$$E = e^{2\phi} \left(F_\phi + G_\phi \vartheta \right). \tag{B.27}$$

From equation (B.24), we now have

$$\begin{gathered} 2e^{2\phi} B\ C + e^{2\phi} A' C - e^{2\phi} B_\phi C - 2\lambda e^{4\phi} B_\phi C + e^{4\phi} F_\phi + e^{4\phi} G_\phi \vartheta \\ + \lambda e^{4\phi} G_\phi C + \lambda e^{4\phi} B_{\phi\phi} C + B_\tau C - e^{2\phi} F_\tau - e^{2\phi} G_\tau \vartheta + \lambda e^{2\phi} E_\vartheta C + \lambda e^{4\phi} E_\phi = 0. \end{gathered} \tag{B.28}$$

This separates as follows

$$\begin{aligned} C &: 2e^{2\phi} B + e^{2\phi} A' - e^{2\phi} B_\phi - 2\lambda e^{4\phi} B_\phi + \lambda e^{4\phi} G_\phi + \lambda e^{4\phi} B_{\phi\phi} \\ &\qquad + B_\tau + \lambda e^{2\phi} E_\vartheta = 0, \qquad \text{(B.29)} \\ C^0 &: e^{2\phi} F_\phi + e^{2\phi} G_\phi \vartheta - F_\tau - G_\tau \vartheta + \lambda e^{2\phi} E_\phi = 0. \qquad \text{(B.30)} \end{aligned}$$

Substituting (B.26) and (B.27) into (B.29) gives

$$2e^{2\phi} A' + 4e^{2\phi} H' - A'' + 8\lambda e^{4\phi} G_\phi = 0. \tag{B.31}$$

This suggests that

$$G(\tau, \phi) = \frac{1}{8\lambda} \left(e^{-2\phi} \left(A'' + 2H' \right) - \frac{1}{4} e^{-4\phi} A'' \right) + K(\tau), \tag{B.32}$$

where K is an arbitrary function of τ. Substituting (B.27) into (B.30) leads to the equations

$$e^{2\phi} \left(1 + 2\lambda e^{2\phi} \right) G_\phi + \lambda e^{4\phi} G_{\phi\phi} - G_\tau = 0, \tag{B.33}$$

and

$$e^{2\phi} \left(1 + 2\lambda e^{2\phi} \right) F_\phi + \lambda e^{4\phi} F_{\phi\phi} - F_\tau = 0. \tag{B.34}$$

Substituting (B.32) into (B.33) gives

$$8e^{4\phi} A' + 16e^{4\phi} H' + 32\lambda e^{4\phi} K' + 8\lambda e^{4\phi} A'' + 8e^{2\phi} H'' - A'' = 0. \tag{B.35}$$

This separates as follows

$$e^{4\phi} : A' + 2H' + \lambda\left(4K' + A''\right) = 0, \tag{B.36}$$

$$e^{2\phi} : H'' = 0, \tag{B.37}$$

$$\text{remainder} : A''' = 0. \tag{B.38}$$

Therefore we have

$$A(\tau) = k_1 + \tau(k_2 + \tau k_3), \tag{B.39}$$

$$H(\tau) = k_4 + \tau k_5, \tag{B.40}$$

$$K(\tau) = -\frac{\tau}{4\lambda}k_2 - \frac{\tau}{4\lambda}\left(2\lambda + \tau\right)k_3 - \frac{\tau}{4\lambda}k_5 + k_6. \tag{B.41}$$

Therefore we have the following coefficients

$$\begin{aligned}
\xi^1 &= k_1 + \tau(k_2 + \tau k_3),\\
\xi^2 &= -\frac{1}{4}\left(k_2 + 2\tau k_3\right) + e^{2\phi}\left(k_4 + k_5\tau\right),\\
\eta^1 &= \frac{e^{-4\phi}}{16\lambda}\Big[2e^{2\phi}(2\vartheta k_3 + (k_2 + 2(\tau k_3 + k_5))C) - k_3 C\\
&\quad - 4e^{4\phi}(\tau^2 k_3 C + (k_2 + 2k_5)\vartheta + (2\vartheta k_3 + (k_2 + 6\lambda k_3 + 2k_5)C)\tau\\
&\quad + 2\lambda(k_2 - 2k_6)C\Big] + e^{2\phi}\omega_\phi,\\
\eta^2 &= -\frac{e^{-4\phi}}{16\lambda}\Big[(k_3 - 2e^{2\phi}(k_2 + 2(\tau k_3 + k_5))\\
&\quad + 4e^{4\phi}(\tau^2 k_3 + (k_2 + 2\lambda k_3 + 2k_5)\tau - 4\lambda k_6))\vartheta\Big] + \omega(\tau, \phi),
\end{aligned}$$

where $\omega(\tau, \phi)$ is an arbitrary function satisfying equation (B.1) and $k_1, \ldots, k_6$ are arbitrary constants. The resulting Lie point symmetries including the infinite symmetry generator, $X_\omega = e^{2\phi}\omega(\tau,\phi)\frac{\partial}{\partial C} + \omega_\phi \frac{\partial}{\partial \vartheta}$, are given by

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial \tau},\\
X_2 &= e^{2\phi}\frac{\partial}{\partial \phi},\\
X_3 &= C\frac{\partial}{\partial C} + \vartheta\frac{\partial}{\partial \vartheta},\\
X_4 &= \tau\frac{\partial}{\partial \phi} + \frac{1}{\lambda}\left[\left(\frac{e^{-4\phi}}{4} - \frac{\tau e^{-2\phi}}{2}\right)C - \frac{e^{-2\phi}}{2}\vartheta\right]\frac{\partial}{\partial C}\\
&\quad + \frac{1}{\lambda}\left(\frac{e^{-4\phi}}{4} - \frac{\tau e^{-2\phi}}{2}\right)\vartheta\frac{\partial}{\partial \vartheta},\\
X_5 &= \tau\frac{\partial}{\partial \phi} - \frac{1}{4}\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left[\left(\frac{\lambda}{2} - \frac{e^{-2\phi}}{8} + \frac{\tau}{4}\right)C + \frac{1}{4}\vartheta\right]\frac{\partial}{\partial C},\\
&\quad + \frac{1}{\lambda}\left(\frac{e^{-2\phi}}{8} - \frac{\tau}{4}\right)\vartheta\frac{\partial}{\partial \vartheta},\\
X_6 &= \tau^2\frac{\partial}{\partial \tau} - \frac{1}{2}\tau\frac{\partial}{\partial \phi} - \frac{1}{\lambda}\left(\frac{\lambda\tau}{2} + \frac{e^{-4\phi}}{16} - \frac{\tau e^{-2\phi}}{4} + \frac{\tau^2}{4}\right)\vartheta\frac{\partial}{\partial \vartheta}\\
&\quad - \frac{1}{\lambda}\left[\left(\frac{3\lambda\tau}{2} + \frac{e^{-4\phi}}{16} - \frac{\tau e^{-2\phi}}{4} + \frac{\tau^2}{4}\right)C - \left(\frac{e^{-2\phi}}{4} + \frac{\tau}{2}\right)\vartheta\right]\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{B.42}$$

The only genuine nonlocal symmetries are X_4, X_5 and X_6.

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