Some topological aspects of modular quasi-metric spaces

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Declaration

The work described in this PhD thesis was carried out under the supervision of Professor Olivier Olela Otafudu, University of the Witwatersrand, Johannesburg, South Africa, from February 2017 to November 2019.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others, it is duly acknowledged in the text.

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Declaration 1 - Publications

1. O. Olela Otafudu, K. Sebogodi, Topologies on modular quasi-pseudometric spaces, Afr. Mat. (under review).
2. O. Olela Otafudu, K. Sebogodi, On w-Isbell-convexity, Adv. Oper. Theory (under review).

Abstract

In this PhD thesis, we present the modular quasi-pseudometric on a nonempty set, a concept that generalises modular pseudometrics to the framework of quasi-metric spaces. We investigate the topological aspects of a set equipped with a modular quasi-pseudometric. It turns out that a set equipped with a modular quasi-pseudometric is a bitopological space in the sense of Kelly. Moreover, we introduce the theory of Isbell-convexity in the setting of modular quasi-pseudometrics which we shall call w-Isbell-convexity. Furthermore, we prove that given a modular set, w-Isbell-convexity is equivalent to Isbell-convexity whenever the modular quasi-pseudometric is continuous from the right on the set of positive numbers. We also study the boundedness of a set endowed with a modular quasi-pseudometric and we shall call it w-boundedness. Consequently, we show that a self w-nonexpansive map in a w-Isbell-convex modular set has a fixed point and the set of fixed points is w-Isbell-convex whenever the modular quasi-pseudometric is continuous from the right on the set of positive numbers.

Dedication

To my family.

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Contents

Introduction

In 2008, Chistyakov in [19] introduced the concept of a modular metric space induced by F-modulars. He developed the theory of a modular on an arbitrary set and studied the notion of a metric induced by a modular that he called the modular metric. For more details about the theory of F-modulars we refer the reader to [17, 19, 44, 54, 68]. He defined a modular metric as follows:

Definition 0.0.1. *[17] Given a nonempty set S, a function $m : (0,\infty) \times S \times S \to [0,\infty]$ is said to be a modular metric if it satisfies the following conditions:*

1. $m(\lambda, a, b) = 0$ *if and only if* $a = b$ *for all* $a, b \in S$ *and for any* $\lambda > 0$,
2. $m(\lambda, a, b) = m(\lambda, b, a)$ *for all* $a, b \in S$ *and for any* $\lambda > 0$,
3. $m(\lambda + \mu, a, b) \leq m(\lambda, a, c) + m(\mu, c, b)$ *for all* $a, b, c \in S$ *and for any* $\lambda, \mu > 0$.
   If in this definition the function m satisfies the weaker condition
4. $m(\lambda, a, a) = 0$ *for any* $\lambda > 0$ *and* $a \in S$
   instead of condition 1, then m is called a modular pseudometric.

For any $a_0 \in S$, the set $S_m(a_0) = \{a \in S : \lim_{\lambda \to \infty} m(\lambda, a, a_0) = 0\}$ is called a modular set. For simplicity, we shall write S_m for $S_m(a_0)$. Furthermore, Chistyakov defined the metric d_m on the modular set S_m by $d_m(a, b) = \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\}$ for all $a, b \in S_m$ and he called the pair (S_m, d_m) a modular metric space. In recent years, there has been a growing interest in the studies of electrorheological fluids (other authors prefer to call them "smart fluids" e.g lithium polymethacrylate) because using sufficient accuracy, these fluids can be modelled by using classical Lesbesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a constant. One of the most interesting problems in this setting is the famous Dirichlet energy problem (see [28, 29]). The classical technique used so far in studying this problem is to convert the energy function, naturally defined by a modular, to a convoluted and complicated problem which involves a norm (the Luxemburg norm). Afrah in [2] suggested that the modular metric approach is more natural and has not been used extensively.

Modular metric

The main idea behind this new concept of modular metric is its physical interpretation. Informally speaking, in the same way that a metric on a set represents the nonnegative finite distance between any two points of a set, a modular metric on a set is generally the average velocity $m(\lambda, a, b)$ it takes to cover a distance between points a and point b over time $\lambda > 0$.

So a priori, one cannot make any "symmetry assumption" on such distance and by implication no "symmetry assumption" should be made on the velocity. For instance in economics, the velocity is the rate at which money changes hand within an economy. This is a typical natural asymmetric setting. This is a great motivation for us to introduce the concept of modular metric that does not satisfy the symmetry axiom of a modular metric. In this thesis, we focus on some topological aspects of a set equipped with a modular quasi-pseudometric. We show that a set equipped with a modular quasi-pseudometric has three topologies, one induced by the modular quasi-pseudometric, another one induced by the transpose modular quasi-pseudometric and the third one induced by the symmetrized modular quasi-pseudometric.

Based on the studies on Hausdorff distance, Chistyakov in [16] introduced the concept of pseudo-modular on the power set of a nonempty set equipped with a modular metric in the following sense:

Definition 0.0.2. *[16] Let m be a modular metric on a nonempty set S. Given $A, B \in \mathcal{P}(S)$ and $\lambda > 0$, then the function*

$W_m : (0, \infty) \times \mathcal{P}(S) \times \mathcal{P}(S) \to [0, \infty]$ *defined by*

$$W_m(\lambda, A, B) = \sup_{a \in A} \inf_{b \in B} m(\lambda, a, b)$$

such that

1. $W_m(\lambda, A, A) = 0$ *for all* $A \in \mathcal{P}(S)$ *and* $\lambda > 0$,
2. $W_m(\lambda + \mu, A, B) \leq W_m(\lambda, A, C) + W_m(\mu, C, B)$ *for all* $A, B, C \in \mathcal{P}(S)$ *and* $\mu, \lambda > 0$

is called Hausdorff pseudomodular.

It turns out that $W_m(\lambda, A, B) \neq W_m(\lambda, B, A)$, in general, for all $A, B \subseteq S$ and $\lambda > 0$.

Moreover, the function W defined by

$$W(\lambda, A, B) = \max\{W_m(\lambda, A, B), W_m(\lambda, B, A)\} \tag{1}$$

for all $A, B \in \mathcal{P}(S)$ and $\lambda > 0$ is a modular pseudometric on $\mathcal{P}(S)$.

One observes that the functions

$$W_m(\lambda, A, B) = \sup_{a \in A} \inf_{b \in B} m(\lambda, a, b)$$

and

$$W_m^t(\lambda, A, B) = W_m(\lambda, B, A) = \sup_{b \in B} \inf_{a \in A} m(\lambda, b, a)$$

satisfy all the axioms of a modular metric on the power set $\mathcal{P}(S)$ except the symmetry axiom. So W_m and W_m^t are modular pseudometrics on $\mathcal{P}(S)$ which are not symmetric. The symmetrized modular pseudometric W in (1) is called the Hausdorff pseudomodular induced by the modular metric m.

We study the Hausdorff modular quasi-pseudometric induced by the modular quasi-pseudometric and its transpose and we show that the modular quasi-pseudometric generated by the Hausdorff pseudometric induced by the sum of modular quasi-pseudometric and its transpose modular quasi-pseudometric has similar behaviour like the Pompéu quasi-pseudometric. It turns out that this Hausdorff modular pseudometric is just a symmetrisation of a Hausdorff modular pseudometric which does not satisfy the symmetry axiom of a modular pseudometric. Moreover, in [16] Chistyakov introduced a topology $\tau(m)$ on the modular set S_m in the following sense:

Definition 0.0.3. *[16] Let S_m be a modular set and $\tau(m)$ be the family of all w-open subsets of S_m. Then a subset O of S_m is $\tau(m)$-open if and only if for any $\lambda > 0$ and $a \in O$, there exists $\mu > 0$ such that the entourage set*

$$B_{\lambda,\mu}(a) = \{b \in S_m : m(\lambda, a, b) \leq \mu\}$$

is inside the subset O.

We use this definition to study the topologies induced by a modular quasi-pseudometric on a nonempty set S.

Isbell-convexity

A metric space (S, d) is said to be hyperconvex (see [38, Definition 4.1]) if for each family $(x_i)_{i\in I}$ of points in S and each family of positive real numbers $(r_i)_{i\in I}$, the conditions $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$ imply that $\emptyset \neq \bigcap_{i\in I} C_d(x_i, r_i)$. In this case, $C_d(x, r)$ denotes the closed ball of (non-negative) radius r at x. Dress [23] and Isbell [31] independently constructed the hyperconvex hull of a metric space.

Though using different terminology, Salbany in [65, Proposition 9] observed that in a quasi-pseudometric space the concept of hyperconvexity can be redefined in the following sense:

Definition 0.0.4. *A quasi-pseudometric space (S, d) is called Isbell-convex if for each family $(x_i)_{i\in I}$ of points in S and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the conditions $d(x_i, x_j) \leq r_i + s_j$ for all $i, j \in I$ imply that*

$$\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^t}(x_i, s_i)) \neq \emptyset.$$

We find it important to point out that the set of the reals equipped with the usual metric is not Isbell-convex (see [34, Example 2]), although it yields the paradigmatic example of a hyperconvex metric space [38, Proposition

4.1]. In [34], Olela Otafudu et al. investigated the concept of an Isbell-convex quasi-pseudometric space. Compared to investigations by Jawhari, Pouzet and Misane [32], their investigations developed quite naturally in parallel with the well-known metric theory of hyperconvexity (see e.g. [24] and [38]). Though many classical ideas about hyperconvexity can be generalized properly from the modular metric setting to the modular quasi-pseudometric setting, these generalizations are not always trivial and sometimes the asymmetric setting requires interesting new variations of old arguments. Naturally this has led to the belief that Isbell-convexity on a modular set equipped with a modular quasi-pseudometric may also be investigated. In this thesis, we establish that given a modular set, w-Isbell-convexity is equivalent to Isbell-convexity whenever the modular quasi-pseudometric is continuous from the right on the set of positive numbers.

More recently in [40], Khamsi and Kozlowski presented a fixed point result for pointwise nonexpansive and asymptotic pointwise nonexpansive mappings acting on modular functions spaces. The theory of nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s (see e.g. Belluce and Kirk [8], Browder [12], Bruck [13] and Lim [50]), and generalized to other metric spaces (see e.g. [26, 27, 42]), and modular function spaces (see e.g. [39]).

Afrah in [2] studied the concept of one-local retract in the more general setting of modular metric spaces. He proved the existence of common fixed points for a family of modular nonexpansive mappings defined on nonempty w-closed and w-bounded subsets of a modular metric space.

In this thesis we also study the boundedness (w-boundedness) of a set endowed with a modular quasi-pseudometric and show that a self-nonexpansive map (w-nonxpansive map) on a w-Isbell-convex modular set has a fixed point. Moreover, we show that the set of fixed points is w-Isbell convex whenever its modular quasi-pseudometric is continuous from the right on the set of positive numbers.

Organization of the thesis

Chapter 1

This chapter is devoted to some useful background concepts and definitions that will be used throughout the thesis. In the first section, we present a summary of the theory of quasi-pseudometric spaces and give interesting examples. We then proceed to revisit the well-known concept of Hausdorff quasi-pseudometric in the second section. The third section is devoted to the concept of an Isbell-convex quasi-pseudometric space.

Chapter 2

In this chapter we summarise the results of Chistyakov on modular metrics (see [16, 17, 18]). The first section of this chapter reviews the definition of a modular metric and considers some examples. In the second section, we discuss the concept of a modular set. We show that a modular set equipped with a pseudometric induced by a modular pseudometric turns out to be a metric space. In the third section, we review convergence in the modular pseudometric setting. The fourth section explores the usual metric space terminology in the modular metric setting that characterizes modular open and modular closed sets in the modular set S_m^*. In the fifth section and the sixth section we look into the notion of modular convergence in a set and discuss topology on a modular set respectively. In the last section of this chapter, we discuss the concept of Hausdorff pseudomodular on the power set equipped with a modular metric.

Chapter 3

In this chapter we start our own investigations on a modular pseudometric which does not satisfy the symmetric axiom of a modular pseudometric and we call it modular quasi-pseudometric. In the first section, we define and present the concept of modular quasi-pseudometric and show that a modular quasi-pseudometric is a nonincreasing function. We define a modular set in the modular quasi-pseudometric setting, and we show that whenever a modular set is equipped with a T_0-quasi-metric enduced by a modular quasi-pseudometric, it becomes a T_0-quasi-pseudometric space. Since for nonsymmetric distance functions there are several completeness concepts, the second section is devoted to discussing Cauchy sequences and left-(right)-Cauchy sequences. In the third section of this chapter, we study some topological aspects on a set equipped with a modular quasi-pseudometric. We show that a set equipped with a modular quasi-pseudometric has three topologies. In the fourth section, we generalise the Hausdorff modular quasi-pseudometric induced by modular quasi-pseudometric and its transpose. We show that the modular quasi-pseudometric generated by Hausdorff pseudomodular induced by the sum of modular quasi-pseudometric and its transpose modular quasi-pseudometric has similar behavior to the Pompéiu quasi-pseudomeric.

Chapter 4

In this chapter we present the concept of Isbell-convexity in the modular quasi-pseudometric setting. The first section is devoted to the introduction and presentation of the theory of Isbell-convexity in the setting of a modular quasi-pseudometric. We show that on a modular set, w-Isbell-convexity is equivalent to Isbell-convexity whenever the modular quasi-pseudometric is continuous from the right on the set of positive numbers. In the second part of this chapter, we discuss the w-boundedness of a set endowed with a modular quasi-pseudometric. We eventually show that a self w-nonexpansive map on a w-Isbell-convex modular set has a fixed point. We consequently prove that if a modular quasi-pseudometric is continuous from the right on the set of positive numbers and given a w-nonexpansive map, then the set of fixed points is nonempty and w-Isbell-convex.

Chapter 5

In this chapter we give a summary of the results we have achieved and give some open problems for future studies.

Chapter 1

Preliminaries

In this chapter, we begin by considering some basic definitions, properties, notations and some results that will be useful in the thesis. We will first present the theory of quasi-metric spaces and later look into and summarise facts about Hausdorff quasi-pseudometrics defined on a power set. We refer the reader for instance to [49] or [22] for further details on quasi-pseudometric spaces and on Hausdorff quasi-pseudometrics. We shall also recall some concepts of Isbell-convexity in the latter section of this chapter.

1.1 Quasi-pseudometric

Let us start by recalling some basic concepts from the theory of quasi-pseudometric spaces as they form a basis for the thesis.

Definition 1.1.1. *[49, Definition 2.1] Let S be a nonempty set and let $q : S \times S \to [0, \infty]$ be a function such that*

1. *$q(a, a) = 0$ for all $a \in S$;*
2. *$q(a, b) \leq q(a, c) + q(c, b)$ for all $a, b, c \in S$ (where q attains infinite values the triangle inequality is interpreted in the obvious way).*

Then q is called an extended quasi-pseudometric on S. If q is an extended quasi-pseudometric on S, then we shall call the pair (S, q) an extended quasi-pseudometric space.

We shall call q a quasi-pseudometric provided that q is a mapping into $S \times S \to [0, \infty)$ (meaning the function q does not attain infinite values). A(n extended) quasi-pseudometric q on S is called a quasi-metric (or T_0-quasi metric) if for all $a, b \in S$, $q(a, b) = 0$ implies $a = b$.
We next give a practical T_0-quasi-metric example.

Example 1.1.1. *[66, Example 1.1.1] Let $q(a, b)$ be a "distance above sea level" function such that a is a point at sea level and b is some point vertical to a. Now, if $q(a, b) = 10$, it means b is 10 units above a. But if $q(a, b) = 0$, it does not necessarily mean that both a and b are at sea level. It could mean that b is below sea level. So $q(a, b) = 0$ because the function is defined for above sea level. Hence $q(a, b)$ is a T_0-quasi-metric.*

Most importantly, we are mainly interested in quasi-pseudometrics in the following.

Remark 1.1.1. *When q is a(n extended) quasi-pseudometric on S, then $q^{-1} : S \times S \to [0, \infty]$ defined by $q^{-1}(a, b) = q(b, a)$ for all $a, b \in S$ is also a(n extended) quasi-pseudometric of S, called the transpose (extended) quasi-pseudometric of q. Lastly, q is called a(n extended) pseudometric on S if $q = q^{-1}$. As usual, for any T_0-quasi-pseudometric q, the function $q^s = \max\{q, q^{-1}\} = q \bigvee q^{-1}$ is a pseudometric (see [49]).*

We give the following examples for clarification.

Example 1.1.2. *[66, Example 1.1.2] Let $\mathbb{R}$ be the set of real numbers. The function*

$$u(a, b) = a \dot{-} b = \max\{a - b, 0\}$$

for all $a, b \in \mathbb{R}$ is a T_0-quasi-metric on $\mathbb{R}$. Moreover, the function $u^s = |a - b|$ for all $a, b \in \mathbb{R}$ is the usual metric on $\mathbb{R}$.

Example 1.1.3. *[66, Example 1.1.3] Let $\mathbb{R}$ be the set of real numbers and the function q defined by*

$$q(a, b) = \begin{cases} e^a - e^b & \text{if } a \geq b \\ e^b - e^a & \text{if } a < b \end{cases}$$

for all $a, b \in \mathbb{R}$ is a T_0-quasi-metric on $\mathbb{R}$ and the pair $(\mathbb{R}, q)$ is a T_0-quasi-metric space.

Remark 1.1.2. *Let (S, q) be a(n extended) quasi-pseudometric space.*

1. *For each $a \in S$ and $\epsilon > 0$, the set $B_q(a, \epsilon) = \{b \in S : q(a, b) < \epsilon\}$ denotes an open ball with centre a and radius ϵ.*

2. *The collection of all open balls forms a base for a topology $\tau(q)$ and such a topology is said to be a topology induced by q on S.*

3. *We also have the set $C_q(a, \epsilon) = \{b \in S : q(a, b) \leq \epsilon\}$ for all $a \in S$ and $\epsilon > 0$, which is called a closed ϵ-ball with centre a and radius ϵ.*

Note that $C_q(a, \epsilon)$ is $\tau(q^{-1})$ closed but not $\tau(q)$-closed in general.

Remark 1.1.3. *The topology induced by q^s (i.e $\tau(q^s)$), is finer than other topologies (i.e. $\tau(q)$ and $\tau(q^{-1})$).*

A map $f : (S, q) \to (X, p)$ between two quasi-pseudometric spaces (S, q) and (X, p) is called an isometry (or an isometric map) if $p(f(a), f(b)) = q(a, b)$ for all $a, b \in S$. Moreover, if (S, q) is a T_0-quasi-metric space, then f is said to be injective if $f : (S, q) \to (X, p)$ is an isometric map between two quasi-pseudometric spaces (S, q) and (X, p) (see [45, Lemma 4]).

Furthermore, if there exists a bijective isometry $f : (S, q) \to (X, p)$ between two quasi-pseudometric spaces (S, q) and (X, p), then these quasi-pseudometric spaces are said to be isometric.
Moreover, given two quasi-pseudometric spaces (S, q) and (X, p), such that there is a map $f : (S, q) \to (X, p)$ between them, then such a map is said to be non-expansive provided that $p(f(a), f(b)) \leq q(a, b)$ for all $a, b \in S$.

Example 1.1.4. *[56, Example 3.3] Let (S, q) be a T_0-quasi-metric space and let $\mathcal{P}(S) \equiv 2^S$ denote the family of subsets of S and denote the set of the nonempty subsets of S by $\mathcal{P}_0(S)$. Then for any $A \in \mathcal{P}_0(S)$, set $(f_A)_1(a) := dist(A, a)$ and $(f_A)_2(a) := dist(a, A)$ for all $a \in S$. Then for the function pair $f_A = ((f_A)_1, (f_A)_2)$, we have $(f_A)_1 : (S, q) \to (\mathbb{R}, u)$ is a nonexpansive map and $(f_A)_2 : (S, q) \to (\mathbb{R}, u)$ is a nonexpansive map where u is a T_0-quasi-metric defined in Example 1.1.2.*

1.2 Hausdorff quasi-pseudometric

In this section, we recall the well known concept of Hausdorff quasi-pseudometric. Let (S, q) be a quasi-pseudometric space and let $\mathcal{P}_0(S)$ be the set of all nonempty subsets of S. Given $B \in \mathcal{P}_0(S)$, we shall set the distance function between a point and a set as $dist(a, B) = \inf\{q(a, b) : b \in B\}$ and $dist(B, a) = \inf\{q(b, a) : b \in B\}$ where $a \in S$.
For any $A, B \in \mathcal{P}_0(S)$, we set the distance function between sets as

$$q_H(A, B) = \max\left\{\sup_{b \in B} dist(A, b), \sup_{a \in A} dist(a, B)\right\}$$

(see [49] and compare [10]). Then q_H is called an extended Hausdorff (-Bourbarki) quasi-pseudometric on $\mathcal{P}_0(S)$. When q_H is restricted to the set of all the nonempty subsets B of S that satisfy $B = cl_{\tau(q)}B \cap cl_{\tau(q^{-1})}B$, where $cl_\tau B$ is the closure of B with respect to topology τ, then q_H is an extended T_0-quasi-metric on $\mathcal{P}_0(S)$ (compare [47, page 164]).

Remark 1.2.1. It is maybe important to point out that the Hausdorff distance function on $\mathcal{C}_0(S)$ (where $\mathcal{C}_0(S)$ is the set of all nonempty, closed and bounded subsets of a metric space (S, d)) can be "divided" into two non-symmetric distance functions. Furthermore, if (S, q) is a quasi-pseudometric space, then the mapping $q : \mathcal{P}_0(S) \times \mathcal{P}_0(S) \to [0, \infty]$, is called the upper Hausdorff extended quasi-pseudometric on $\mathcal{P}_0(S)$ (see [59]) defined by

$$q_q(A, B) = \sup_{b \in B} q(A, b)$$

for each $A, B \in \mathcal{P}_0(S)$. One can effortlessly prove that $qq(A, B)$ is a quasi-pseudometric on $\mathcal{P}_0(S)$. Similarly the transpose quasi-pseudometric q^{-1} is called the lower Hausdorff extended quasi-pseudometric on $\mathcal{P}_0(S)$ (see [59]) defined by

$$q_q^{-1}(A, B) = \sup_{a \in A} q(a, B)$$

for each $A, B \in \mathcal{P}(S)$.
Moreover, the join $q_q^s = q_q \bigvee q_q^{-1} = \max\{q_q, q_q^{-1}\}$ is the Hausdorff extended quasi-pseudometric on $\mathcal{P}_0(S)$ (see [10, 48]) and Hausdorff pseudometric if q is a pseudometric itself (see [10]). It is known that that q_q, q_q^{-1} and q_q^s are quasi-pseudometric on $\mathcal{P}_0(S)$ whenever q is bounded.

Clearly $q_q(\{a\},\{b\}) = q(a,b)$, hence the upper Hausdorff quasi-pseudometric q_q is an extension of the ordinary quasi-pseudometric q. Now if q is unbounded, then q_q attains infinite value because whenever $a \in S$, we have $q_q(\{a\}, S) = \infty$. Furthermore, the upper Hausdorff quasi-pseudometric is invariant when sets are replaced by their closures. Particularly, the upper Hausdorff quasi-pseudometric between a set (say A) and its closure ($\bar{A}$) must be zero. Lastly, if we restrict q_q to act only on nonempty closed subsets of $\mathcal{C}_0(S)$, then q_q would satisfy the usual axioms of q. Hence q_q defines an extended real valued quasi-pseudometric on $\mathcal{C}_0(S)$.

Example 1.2.1. *(compare [34, remark 4]) Let (S,q) be a T_0-quasi-metric space and let $\mathcal{P}(S) \equiv 2^S$ denote the family of subsets of S (called power set) and symbolise the set of nonempty subsets of S by $\mathcal{P}_0(S)$. Then for any $A \in \mathcal{P}_0(S)$, set $(f_A)_1(a) := dist(A,a)$ and $(f_A)_2(a) := dist(a,A)$ for all $a \in S$. Then for the function pair $f_A = ((f_A)_1, (f_A)_2)$, we have $(f_A)_1 : (S,q) \to (\mathbb{R}, u)$ is a nonexpansive map and $(f_A)_2 : (S,q) \to (\mathbb{R}, u)$ is a nonexpansive map, where $u(a,b) = \max\{a-b, 0\}$ for all $a,b \in \mathbb{R}$. Furthermore, $q_H(A,B) = Q_q(f_A, f_B)$ whenever $A, B \in \mathcal{P}_0(S)$ and $q_H(A,B)$ is the extended Hausdorff quasi-pseudometric on $\mathcal{P}_0(S)$ and Q_q is the extended T_0-quasi-metric defined by*

$$Q_q(f,g) = \sup_{a\in S}(f_1(a) \dot{-} g_1(a)) \vee \sup_{a\in S}(g_2(a) \dot{-} f_2(a))$$

on $\mathcal{C}(S,q)$ (set of continuous real valued functions on S) whenever $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{C}(S,q)$.

1.3 Isbell-convexity

In this section we recall some concepts on Isbell-convexity and Isbell-completeness on a quasi-pseudometric space.

Definition 1.3.1. *[34] A quasi-pseudometric space (S,q) said to be Isbell-convex if for each family of $(x_i)_{i\in I}$ of points in S and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$, the following condition holds:*

If $\quad q(x_i,x_j) \leq r_i + s_j \quad$ *for all* $\quad i,j \in I, \quad$ *then* $\quad \bigcap_{i\in I} C_q(x_i,r_i) \cap C_{q^{-1}}(x_i,s_i) \neq \emptyset.$

Remark 1.3.1. *We are mainly interested in T_0-quasi-metric space, and for that we do not only require r_i or s_i (for all $i \in I$) to be positive in Definition 1.3.1 but also allow them to be zero.*

Example 1.3.1. *[34] Let the set $\mathbb{R}$ of real numbers be equipped with the T_0-quasi-metric $u(a,b) = \max\{a-b, 0\}$ for all $a, b \in \mathbb{R}$. Then $(\mathbb{R}, u)$ is Isbell-convex.*

Corollary 1.3.1. *[34] Let $(\mathbb{R}, u)$ be the Isbell-convex T_0-quasi-metric space defined in Example 1.3.1. Then the subspace $[0, \infty)$ of $(\mathbb{R}, u)$ is also Isbell-convex.*

Example 1.3.2. *[34] Let the set $\mathbb{R}$ of real numbers be equipped with the usual metric $u^s(a,b) = |a-b|$ for all $a, b \in \mathbb{R}$. Then $(\mathbb{R}, u^s)$ is not Isbell-convex.*

Definition 1.3.2. *[34] A quasi-pseudometric space (S, q) is said to be metrically-convex if for any points $a, b \in S$ and nonnegative numbers r and s such that $q(a,b) \leq r + s$, then there exists $c \in S$ such that $q(a,c) \leq r$ and $q(c,b) \leq s$.*

Example 1.3.3. *[34] Let $\mathbb{R}$ be the set of real numbers and q be the Sorgenfrey quasi-pseudometric on $\mathbb{R}$ which is defined as*

$$q(a,b) = \begin{cases} a - b & \text{if } a \geq b \\ 1 & \text{otherwise} \end{cases}$$

for all $a, b \in \mathbb{R}$. Then q is not metrically convex. Indeed, if we let $a = \frac{1}{2}$ and $b = 1$, then $q(a,b) = 1 \leq \frac{1}{2} + \frac{1}{2}$. But there is no $c \in \mathbb{R}$ such that $q(\frac{1}{2}, c) \leq \frac{1}{2}$ and $q(c, 1) \leq \frac{1}{2}$ because such c would have to less than $\frac{1}{2}$ and greater that 1.

Definition 1.3.3. *[34] Let (S, q) be a quasi-pseudometric space. A family of balls $\Bigg(C_q(x_i, r_i), C_{q^{-1}}(x_i, s_i)\Bigg)_{i \in I}$ with $r_i, s_i \geq 0$ and $x_i \in S$ is said to have the mixed binary intersection property if for all $i, j \in I$, we have*
$C_q(x_i, r_i) \cap C_{q^{-1}}(x_j, s_j) \neq \emptyset$.

Definition 1.3.4. *[34] A quasi-pseudometric space (S, q) is said to be Isbell-complete if for every family $\Bigg(C_q(x_i, r_i), C_{q^{-1}}(x_i, s_i)\Bigg)_{i \in I}$ of balls with the mixed binary intersection property has $\bigcap_{i \in I} \Bigg(C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)\Bigg) \neq \emptyset$.*

Lemma 1.3.2. *[34] A quasi-pseudometric space (S, q) is Isbell-convex if and only if it is metrically convex and Isbell-complete.*

Chapter 2

Modular metric

In this chapter, we summarise the results of Chistyakov on modular metric spaces (see [17, 18]).

2.1 Modular metric

Definition 2.1.1. *[17, Definition 2.1.] Let S be a nonempty set. A function $m : (0,\infty) \times S \times S \to [0,\infty]$ is said to be a modular metric on S if it satisfies the following axioms:*

1. *given $a, b \in S$, $a = b$ if and only if $m(\lambda, a, b) = 0$ for all $\lambda > 0$;*
2. *$m(\lambda, a, b) = m(\lambda, b, a)$ for all $\lambda > 0$ and $a, b \in S$;*
3. *$m(\lambda + \mu, a, b) \leq m(\lambda, a, c) + m(\mu, c, b)$ for all $\lambda, \mu > 0$ and $a, b, c \in S$.*

In the following remark we give a few observations from Definition 2.1.1.

Remark 2.1.1. *[2]*

1. *If instead of axiom (1) in the above definition, the function m satisfies a weaker condition (1') $m(\lambda, a, a) = 0$ for all $\lambda > 0$ and $a \in S$, then m would be called a modular pseudometric on S.*
2. *If instead of axiom (1), the function m satisfies (1') and a stronger condition (1") for any $a, b \in S$ and $\lambda > 0$ with $a \neq b$, then $m(\lambda, a, b) \neq 0$ and such m is said to be a strict modular metric on S.*

3. *A modular metric m on S is said to be regular if the following weaker version of (1) is satisfied (1"') given $a, b \in S$, $a = b$ if and only if $m(\lambda, a, b) = 0$ for some $\lambda > 0$.*

Remark 2.1.2. *[16]*

1. *If we assume $m : (0, \infty) \times S \times S \to (-\infty, \infty]$ in the definition of a modular (pseudo)metric, then the assumption does not lead to any greater generality. Actually, setting $a = b$ and $\mu = \lambda > 0$ in the triangle inequality property axiom of m and also taking into account (1'), we get*
$$0 = m(2\lambda, a, a) \leq m(\lambda, a, c) + m(\lambda, c, a) = 2m(\lambda, a, c)$$
and so, $m(\lambda, a, c) \geq 0$ or $m(\lambda, a, c) = \infty$ for all $\lambda > 0$ and $a, c \in S$ hence m is well defined in $m : (0, \infty) \times S \times S \to [0, \infty]$.

2. *If $m(\lambda, a, b) = m(a, b)$ for all $a, b \in S$, meaning m being independent of $\lambda > 0$, then m would just be an extended metric on S and if m was not defined for infinite values, then m would be just a usual metric on S.*

3. *If $m(\lambda, a, b) = m(\lambda)$ for all $a, b \in S$ and $\lambda > 0$, meaning m being independent of $a, b \in S$, then $m \equiv 0$ by virtue of (1) and (1') and m would be a (pseudo) modular on S (see [57] for more details on modular on a real linear space).*

We now explore a few simple examples of modular (pseudo)metrics on a nonempty set.

Example 2.1.1. *[17, Example 2.4 (a)] Let S be a nonempty set. The function $m : (0, \infty) \times S \times S \to [0, \infty]$ defined by*

$$m(\lambda, a, b) = \begin{cases} \infty & \text{if } a \neq b, \\ 0 & \text{if } a = b \end{cases}$$

for all $a, b \in S$ and $\lambda > 0$ is a modular metric on S.

Example 2.1.2. *[17, Example 2.4 (b)] Given a nonempty set S with $a, b \in S$, the function $m : (0, \infty) \times S \times S \to [0, \infty]$ defined by*

$$m(\lambda, a, b) = \frac{d(a, b)}{\phi(\lambda)},$$

where $\phi : (0, \infty) \to (0, \infty)$ is a nondecreasing function and d is a (pseudo)metric on S, is a modular (pseudo)metric on S.

Remark 2.1.3. *In Example 2.1.2 above, it is important for the function $\phi : (0, \infty) \to (0, \infty)$ to be nondecreasing because if it is not nondecreasing then m in Example 2.1.2 would not be nonincreasing (see Lemma 2.1.1).*

Example 2.1.3. *[17, Example 2.4 (c)] Let (S, d) be a (pseudo)metric space. The function defined by*

$$m(\lambda, a, b) = \begin{cases} \infty & \text{if } 0 < \lambda \leq d(a, b), \\ 0 & \text{if } \lambda > d(a, b) \end{cases}$$

for all $a, b \in S$ and $\lambda > 0$ is a modular (pseudo)metric on S and the pair (S, m) is a modular (pseudo)metric space.

Remark 2.1.4. *In Example 2.1.3, if we replace the condition $\lambda \leq d(a, b)$ with $\lambda < d(a, b)$ and condition $\lambda > d(a, b)$ with $\lambda \geq d(a, b)$, it will not lead to any significant results.*

One of the essential properties of a modular (pseudo)metric m on S is its monotonicity.

Lemma 2.1.1. *[17, page 3] Let S be a nonempty set. Given $a, b \in S$, the function $m : (0, \infty) \times S \times S \to [0, \infty]$ is nonincreasing on $(0, \infty)$.*

Proof. Let $\lambda, \mu \in (0, \infty)$ such that $\lambda > \mu$. Setting $c = a$ in conjunction to property (3) of Definition 2.1.1, we have

$$m(\lambda, a, b) = m(\lambda - \mu + \mu, a, b) \leq m(\lambda - \mu, a, a) + m(\mu, a, b). \tag{2.1}$$

Meaning

$$m(\lambda, a, b) \leq m(\mu, a, b)$$

since $m(\lambda - \mu, a, a) = 0$ by property (1'). This shows that though $\lambda > \mu$, $m(\lambda, a, b) \leq m(\mu, a, b)$ for all $a, b \in S$. Hence m is nonincreasing.

□

Consequently, given $a, b \in S$ and $\lambda > 0$, the limit from the right exists in $[0, \infty]$ and is given by

$$m(\lambda + 0, a, b) = \lim_{\mu \to \lambda + 0} m(\mu, a, b) = \sup\{m(\mu, a, b) : \mu > \lambda\} \tag{2.2}$$

and the limit from the left also exists in $[0, \infty]$ and is defined as

$$m(\lambda - 0, a, b) = \lim_{\mu \to \lambda - 0} m(\mu, a, b) = \inf\{m(\mu, a, b) : 0 < \mu < \lambda\}. \tag{2.3}$$

Now, combining equation (2.3) and equation (2.2), we have the following inequalities that hold for all $0 < \mu < \lambda$

$$m(\lambda+0, a, b) \leq m(\lambda, a, b) \leq m(\lambda-0, a, b) \leq m(\mu+0, a, b) \leq m(\mu, a, b) \leq m(\mu-0, a, b). \tag{2.4}$$

Using the monotonicity property of m, for any $0 < \mu < \mu_1 < \lambda_1 < \lambda$, we have:

$$m(\lambda, a, b) \leq m(\lambda_1, a, b) \leq m(\mu_1, a, b) \leq m(\mu, a, b)$$

and m remain to pass the limits as $\lambda_1 \to \lambda - 0$ and $\mu_1 \to \mu + 0$.

2.2 Modular sets

In the previous section, we have explored modular (pseudo)metrics. In this section, we show that a modular (pseudo)metric induces an equivalence relation on the set S, and we shall call the quotient set a *modular set*. Moreover, we show that if a modular set is equipped with a metric induced by a modular metric, then the modular set is a metric space.

Definition 2.2.1. *[17, page 3] Let S be a nonempty set and m be a modular (pseudo)metric on S. We define the binary relation $\overset{m}{\sim}$ on S by*

$$a \overset{m}{\sim} b \qquad \text{if and only if} \qquad \lim_{\lambda \to \infty} m(\lambda, a, b) = 0 \tag{2.5}$$

for all $a, b \in S$ and $\lambda > 0$.

Lemma 2.2.1. *[66, Lemma 2.2.2] If S is a nonempty set, then the binary relation defined on S in Definition 2.2.1 above is an equivalence relation on S.*

Definition 2.2.2. *[66, Definition 2.2.3] Let m be a modular metric on a nonempty set S with an equivalence relation $\overset{m}{\sim}$ defined on S. We denote the quotient set of S with respect to $\overset{m}{\sim}$ as $S\backslash \overset{m}{\sim}$ and define the set $S_m(a)$ as*

$$S_m(a) = \{b \in S : b \overset{m}{\sim} a\} = \left\{b \in S : \lim_{\lambda\to\infty} m(\lambda, a, b) = 0\right\}$$

of which it is the equivalence class of an element $a \in S$ in the quotient set $S\backslash \overset{m}{\sim}$. If we fix $a \in S$, and set $S_m(a) = S_m$, then we call the set S_m a modular set.

We define another type of modular set. For a fixed $a \in S$, we have

$$S_m^* = S_m^*(a) = \{b \in S : \exists\lambda > 0 \text{ such that } m(\lambda, a, b) < \infty\}.$$

In the following remark, we give few observations about a modular set.

Remark 2.2.1. *[16]*

1. *Given an equivalence class $S_m(a)$, any other elements that belongs to equivalence class should be equivalent to a (i.e. $y \overset{m}{\sim} z$ if and only if $y, z \in S_m(a)$).*
2. *It is clear $S_m \subset S_m^*$ but this inclusion may be proper in general.*
3. *It turns out that the function $\tilde{d} : (S\backslash \overset{m}{\sim}) \times (S\backslash \overset{m}{\sim}) \to [0, \infty]$ defined on the quotient set $S\backslash \overset{m}{\sim}$ by*

$$\tilde{d}(S_m(a), S_m(b)) = \lim_{\lambda\to\infty} m(\lambda, a, b)$$

for all $a, b \in S$ is well defined and satisfies the axioms of an extended metric as the function may take infinite values.

Next we give some examples of modular sets.

Example 2.2.1. *[17] Let (S, d) be a metric space and m be a modular metric on S defined by*

$$m(\lambda, a, b) = \begin{cases} \infty & \text{if } 0 < \lambda \leq d(a, b), \\ 0 & \text{if } \lambda > d(a, b) \end{cases}$$

for all $a, b \in S$ and $\lambda > 0$.

In this case, we see that as $\lambda \to \infty$ we get $m \equiv 0$, and for that we have $S_m(a) = S$ for all $a \in S$ and $\tilde{d} \equiv 0$. Therefore, the quotient pair $(S \backslash \overset{m}{\sim}, \tilde{d})$ degenerates.

Example 2.2.2. *[66, Example 2.2.1] Let S be a nonempty set and m be a modular metric defined on S by*

$$m(\lambda, a, b) = \begin{cases} \infty & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

for all $a, b \in S$ and $\lambda > 0$. Then the modular set in this case is a singleton (i.e. $S_m(a) = \{a\}$).

Example 2.2.3. *[66, Example 2.2.2.] Consider a modular metric m defined in Example 2.1.2. For any $a \in S$, a modular set is given by*

$$S_m(a) = \left\{ b \in S : \lim_{\lambda \to \infty} \frac{d(a,b)}{\phi(\lambda)} = 0 \right\}$$

for all $\lambda > 0$. Since ϕ is nondecreasing, then it is either bounded above or not bounded above. If it is not bounded above, then $S_m(a) = S$. But if ϕ is bounded above, then that would imply $d(a,b) = 0$. Then $S_m(a) = \{a\}$.

Remark 2.2.2. *It is worth noting that $S_m(a) \subseteq S$.*

The following theorem is one of the main results of the Chistyakov's paper [17]. The theorem shows that whenever a modular set is equipped with a metric induced by its modular metric and then the modular set is a metric space.

Theorem 2.2.2. *[17, Theorem 2.6.] Let S be a nonempty set and m be a modular (pseudo) metric on S. If d_m is a function defined on S by*

$$d_m(a,b) = \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\}$$

for all $a, b \in S$, then d_m is an extended (pseudo) metric on S. Furthermore, the restriction of d_m to S_m is a (pseudo) metric on S_m.

Proof. For us to show that d_m is a metric on S_m, we have to show that the function d_m satisfies all the properties of a metric. But we start by checking if d_m is well defined.

Suppose $a, b \in S_m$. Then $a \overset{m}{\sim} b$ (i.e $\lim\limits_{\lambda \to \infty} m(\lambda, a, b) = 0$). Hence the smallest value of d_m is $d_m(a, b) = 0$.
Furthermore for any $\lambda > 0$ we have some $\lambda_0 > 0$ such that when $\lambda > \lambda_0$, we have $m(\lambda, a, b) \leq 1$.
If we let $\lambda_1 = \max\{1, \lambda_0\}$, then $m(\lambda, a, b) \leq 1 \leq \lambda_1$. So we have

$$d_m(a, b) = \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\} \leq \lambda_1 < \infty.$$

This implies that $d_m(a, b)$ is finite. Therefore $d_m(a, b) \in [0, \infty)$ for all $a, b \in S_m$. Thus the function d_m is well defined. We now prove that d_m is a metric.

1. If $a = b$ where $a, b \in S_m$, then $m(\lambda, a, b) = 0$ for all $\lambda > 0$ since m is a modular metric on S. Then we have

$$\begin{aligned} d_m(a, b) &= \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\} \\ &= 0. \end{aligned}$$

   Therefore $d_m(a, b) = 0$. If $d_m(a, b) = 0$, then

$$d_m(a, b) = \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\}$$

$$0 = \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\}.$$

   This implies that $m(\lambda, a, b) = 0$ since $\lambda \neq 0$. This means that $a = b$ because m is a modular metric on S.

2. We prove the symmetry condition. Since m is a modular metric on S, then m possesses the symmetry property. Thus we have

$$\begin{aligned} d_m(a, b) &= \inf\{\lambda > 0 : m(\lambda, a, b) \leq \lambda\} \\ &= \inf\{\lambda > 0 : m(\lambda, b, a) \leq \lambda\} \\ &= d_m(b, a). \end{aligned}$$

   Therefore $d_m(a, b) = d_m(b, a)$.

3. Now we prove the triangle inequality (i.e. $d_m(a, b) \leq d_m(a, c) + d_m(c, b)$ for all $a, b, c \in S_m$).

By the definition of d_m, for any $\lambda, \mu > 0$ we have

$$d_m(a,b) < \lambda \qquad and \qquad d_m(c,b) < \mu$$

and furthermore we have

$$m(\lambda, a, b) \leq \lambda \qquad and \qquad m(\mu, c, b) \leq \mu.$$

Since m is a modular metric on S, then

$$\begin{aligned} m(\lambda + \mu, a, b) &\leq m(\lambda, a, c) + m(\mu, c, b) \\ &\leq \lambda + \mu. \end{aligned}$$

Moreover by the definition of $d_m(a,b)$ it follows that

$$d_m(a,b) \leq \lambda + \mu.$$

Furthermore

$$\begin{aligned} d_m(a,b) &= \inf\{\lambda, \mu > 0 : m(\lambda + \mu, a, b) \leq \lambda + \mu\} \\ &\leq \inf\{\lambda > 0 : m(\lambda, a, c) \leq \lambda\} + \inf\{\mu > 0 : m(\mu, c, b) \leq \mu\} \\ &= d_m(a,c) + d_m(c,b). \end{aligned}$$

Therefore we have

$$d_m(a,b) \leq d_m(a,c) + d_m(c,b) \quad for\, all \quad a,b,c \in S_m \quad and \quad \lambda, \mu > 0.$$

Thus d_m is a metric on the set on S_m. $\square$

We next provide examples of a metric induced by a modular metric.

Example 2.2.4. *[17, Example 2.7. (a)] Let m be a modular metric defined in Example 2.2.2. If we set $S_m = \{a\}$, then the corresponding metric d_m induced by modular metric is $d_m(a,b) = 0$ for all $a, b \in S_m$.*

Example 2.2.5. *[17, Example 2.7. (b)] Suppose S is a nonempty set and let m be a modular metric defined in Example 2.1.2 and d_m be a metric defined in Theorem 2.2.2.*
If ϕ is bounded from above, then $S_m = \{a\}$ and then $d_m(a,b) = 0$ for all

$a, b \in S_m$. And if ϕ is not bounded from above (i.e. $\phi(\lambda) \to \infty$ as $\lambda \to \infty$), then

$$S_m = S \qquad \text{and} \qquad d_m = \psi^{-1}[d(a,b)] \tag{2.6}$$

where ψ is a strictly increasing function defined on $\psi : (0, \infty) \to (0, \infty)$ by $\psi(\lambda) = \lambda\phi(\lambda)$ for all $\lambda > 0$ and ψ^{-1} is the inverse function of ψ.
For assertion (2.6) to be well-defined, we need ψ to be one-to-one (for the inverse ψ^{-1} to exist) such that $\psi(\lambda) \to \infty$ as $\lambda \to \infty$ and $\psi(\lambda) \to 0$ as $\lambda \to 0^+$.

Example 2.2.6. *[17, Example 2.7 (b)] Let S be a nonempty set and m be the modular metric defined in Example 2.1.2. If $\phi(\lambda) = \lambda^p$ where p is a nonnegative constant , then we get that (S_m, d_m) is a metric space and $d_m(a, b) = [d(a, b)]^{\frac{1}{p+1}}$.*

Example 2.2.7. *[17, Example 2.7. (c)] Suppose S is a nonempty set and m be the modular metric defined in Example 2.1.3. If $S_m = S$, then $d_m(a, b) = d(a, b)$.*

Definition 2.2.3. *[63, page 182] A function $\psi : [0, \infty) \to \mathbb{R}$ is said to be convex if*

$$\psi(\lambda a + (1 - \lambda)b) \leq \lambda\psi(a) + (1 - \lambda)\psi(b)$$

for all $a, b \in [0, \infty)$ and $\lambda \in [0, 1]$.

Example 2.2.8. *[17, Example 2.9 (c)] Given a (pseudo)metric space (S, d) and a convex function $\psi : [0, \infty) \to (0, \infty)$ which has a maximum only at zero (this means ψ is a strictly increasing, continuous and admits inverse function ψ^{-1}), we set*

$$m(\lambda, a, b) = \lambda\psi\left(\frac{d(a,b)}{\lambda}\right)$$

for all $\lambda > 0$ and $a, b \in S$. Then $m(\lambda, a, b)$ is a modular (pseudo)metric on S.

Proof. We omit the proof for property (1) and (2) of Definition 2.1.1 since they follow from non degeneracy and symmetry properties of a metric and modular metric. We show the triangle inequality.

$$\begin{aligned} m(\lambda+\mu,a,b) &\leq (\lambda+\mu)\psi\left(\frac{\lambda}{\lambda+\mu}.\frac{d(a,c)}{\lambda}+\frac{\mu}{\lambda+\mu}.\frac{d(c,b)}{\mu}\right) \\ &\leq (\lambda+\mu)\left[\frac{\lambda}{\lambda+\mu}\psi\left(\frac{d(a,c)}{\lambda}\right)+\frac{\mu}{\lambda+\mu}\psi\left(\frac{d(c,b)}{\mu}\right)\right] \\ &= \lambda\psi\left(\frac{d(a,c)}{\lambda}\right)+\mu\psi\left(\frac{d(c,b)}{\mu}\right) \\ &= m(\lambda,a,c)+m(\mu,c,b) \end{aligned}$$

for all $a,b \in S$ and $\lambda,\mu > 0$. Moreover, if the function ψ is such that $\psi'(0) = \lim_{\lambda\to 0^+} \frac{\psi(\lambda)}{\lambda} = 0$, then for all $a \in S$, we get that

$$\lim_{\lambda\to\infty} m(\lambda,a,a_0) = \lim_{\lambda\to\infty} \lambda\psi\left[\frac{d(a,a_0)}{\lambda}\right] = \lim_{\lambda\to 0^+} \frac{d(a,a_0)}{\mu}\psi(\mu) = 0$$

and so $S_m = S$. □

Remark 2.2.3. *[17] The type of modular metric defined in Example 2.2.8 is not allowed in classical theory of modulars on linear spaces.*

The following theorem shows the relations between the parameter λ, the modular metric $m(\lambda,a,b)$ and the metric d_m induced by the modular metric m.

Theorem 2.2.3. *[17, Theorem 2.2.11.] Let S be a nonempty set and let d_m be the metric induced by a modular metric m on S. On a modular set S_m, given $a,b \in S_m$ and $\lambda > 0$, we have*

(a) if $d_m(a,b) < \lambda$, then $m(\lambda-0,a,b) \leq d_m(a,b) < \lambda$;

(b) if $m(\lambda,a,b) = \lambda$, then $d_m(a,b) = \lambda$;

(c) if $\lambda = d_m(a,b) > 0$, then $m(\lambda+0,a,b) \leq \lambda \leq m(\lambda-0,a,b)$.

Moreover, given the continuity assumptions of m on $(0,\infty)$, we have:

(d) if m is continuous from the right on $(0,\infty)$, then

$$d_m(a,b) \leq \lambda \quad \text{if and only if} \quad m(\lambda,a,b) \leq \lambda;$$

(e) if m is continuous from the left on $(0, \infty)$, then

$$d_m(a, b) < \lambda \qquad \textit{if and only if} \qquad m(\lambda, a, b) < \lambda;$$

(f) if m is continuous $(0, \infty)$, then

$$d_m(a, b) = \lambda \qquad \textit{if and only if} \qquad m(\lambda, a, b) = \lambda.$$

We give examples to demonstrate Theorem 2.2.3.

Example 2.2.9. *[16, Example 2.2.12.] Let (S, d) be a metric space and $h : (0, \infty) \to (0, \infty)$ be a nondecreasing function. We define the modular metric m on S by*

$$m(\lambda, a, b) = \frac{d(a, b)}{h(\lambda) + d(a, b)}, \tag{2.7}$$

where $\lambda > 0$ and $a, b \in S$. If we set $h(\lambda) = \lambda^p$ with $p > 0$ and given that m is continuous on $(0, \infty)$, then by Theorem 2.2.3 (f) above, the value $\lambda = d_m(a, b)$ with $a \neq b$ satisfies the equation $m(\lambda, a, b) = \lambda$, that is,

$$\lambda^{p+1} + d(a, b)\lambda - d(a, b) \quad = \quad 0. \tag{2.8}$$

If $p = 1$, the resulting quadratic equation solution is given by

$$d_m(a, b) = \frac{\sqrt{(d(a, b))^2 + 4d(a, b)} - d(a, b)}{2}.$$

If $p = 2$, the solution for λ for the cubic equation resulting from equation (2.8) is given by Cardano's formula:

$$d_m(a, b) = \sqrt[3]{\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3}} - \sqrt[3]{-\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3}}$$

where x is the value $d(a, b)$.

Example 2.2.10. *[16, Example 1.3.3] Let S be a nonempty set and d_m be the metric induced by a modular metric m on S. In general, for any $a, b \in S$, there exists $\mu > 0$ such that $m(\lambda - 0, a, b) < \mu$ for all $\lambda > 0$, then $d_m(a, b) < \mu$ for all $a, b \in S$.*

Since $m(\lambda, a, b) \leq m(\lambda - 0, a, b) < \mu$, if $\lambda = \mu$ then we get $d_m(a, b) \leq \mu$.

If we assume $d_m(a,b) = \mu$ for some $a,b \in S$ with $a \neq b$. Then by Theorem 2.2.3 (c), we have $m(\lambda,a,b) \geq m(\lambda+0,a,b) > \lambda$ for all $0 < \lambda < d_m(a,b) = \mu$. This implies that

$$m(\mu-0,a,b) = \lim_{\lambda\to\mu-0} m(\lambda,a,b) \geq \mu.$$

This is a contradiction because we assumed $d_m(a,b) = \mu$.

Hence $m(\lambda-0,a,b) < 1 = \mu$ for modular metric m in equation (2.7) to hold.

2.3 The metric convergence

In this section, we study the metric convergence of sequences in S induced by a metric in the extended (pseudo)metric space (S,d), (pseudo)metric spaces (S_m^*, d_m) and (S_m^*, d_m^*).

Definition 2.3.1. *[16, page 65] Let m be a modular metric on a nonempty set S. A sequence (x_n) in S is said to converge to an element $x \in S$ if $d_m(x_n,x) \to 0$ as $n \to \infty$ where d_m is the metric induced by m.*

We shall call this kind of convergence induced by the metric d_m, a metric convergence. Without loss of meaning, we shall denote that a sequence (x_n) converges to an element x in S as $x_n \to x$ and omit as "$n \to \infty$".

We can also define closedness of a set using metric convergence.

Theorem 2.3.1. *[16, Theorem 4.1.1] Let m be a modular metric on a nonempty set S. Suppose (x_n) is a sequence in S with $x \in S$, then we have*

$$x_n \to x \quad \text{if and only if} \quad m(\lambda, x_n, x) \to 0 \quad \text{for all} \quad \lambda > 0. \tag{2.9}$$

Proof. $\Rightarrow$ Let $x_n \to x$. Given $\epsilon > 0$, $\exists n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_m(x_n,x) < \epsilon$ for all $n \geq n_0(\epsilon)$ where d_m is the metric induced by m on S.

By inequalities (2.4) and Theorem 2.2.3(a), we have

$$m(\epsilon, x_n, x) \leq m(\epsilon-0, x_n, x) < \epsilon \quad \text{for all} \quad n \geq n_0(\epsilon). \tag{2.10}$$

Now, if $\lambda > 0$, for any $n \geq n_0(\min\{\epsilon, \lambda\})$, by inequality (2.1) we get

$$m(\lambda, x_n, x) \leq m(\min\{\epsilon, \lambda\}, x_n, x) < \min\{\epsilon, \lambda\} \leq \epsilon \tag{2.11}$$

and this implies $m(\lambda, x_n, x) \to 0$.

$\Leftarrow$ If $\epsilon > 0$, $m(\epsilon, x_n, x) \to 0$ means $\exists n_1(\epsilon) \in \mathbb{N}$ such that $m(\epsilon, x_n, x) \leq \epsilon$ for all $n \geq n_1(\epsilon)$. By definition of d_m in Theorem 2.2.2, we have $d_m(x_n, x) \leq \epsilon$ for $n \geq n_1(\epsilon)$, which implies $d_m(x_n, x) \to 0$. □

Theorem 2.3.2. *[17, Theorem 2.13 (b)] Let m be a modular metric on a nonempty set S and (x_n) be a sequence in S with $x \in S$. The assertion (2.9) holds for Cauchy sequences also.*

Proof. The assertion for Cauchy sequences (x_n) is of the form:

$$\lim_{n,m\to\infty} d_m(x_n, x_m) = 0 \quad \text{if and only if} \quad \lim_{n,m\to\infty} m(\lambda, x_n, x_m) = 0 \quad \text{for all} \quad \lambda > 0.$$

The proof is similar to that of Theorem 2.3.1 above. □

Theorem 2.3.3. *[16] Let m be a modular metric on a nonempty set S and (x_n) be a sequence in S with $x \in S$. Then modular sets $S_m^*(a)$ and $S_m(a)$ are closed with respect to the metric convergence.*

Proof. Since a sequence (x_n) in S and $x \in S$, suppose $x_n \to x$. Given $\epsilon > 0$, if $\lambda_0 > 0$, then for $n_0 = n_0(\epsilon) \in \mathbb{N}$, we have

$$m(\epsilon + \lambda_0, x, a) \leq m(\epsilon, x, x_{n_0}) + m(\lambda_0, x_{n_0}, a)$$

by triangle property of modular metric m. Now, as we have seen from inequality 2.10 that $m(\epsilon, x_n, x) < \epsilon$ then we have

$$m(\epsilon + \lambda_0, x, a) \leq \epsilon + m(\lambda_0, x_{n_0}, a). \tag{2.12}$$

If (x_n) in S_m^*, then $x_{n_0} \in S_m^*$, and $\lambda_0 > 0$ such that $m(\lambda_0, x_{n_0}, a) < \infty$. Therefore, we have $m(\epsilon + \lambda_0, x, a) < \infty$, which implies $x \in S_m^*$ since a is already an element of S_m^* as it is the centre. Therefore, by Definition 2.6.2, S_m^* is closed.

Suppose (x_n) in S_m. Then $x_{n_0} \in S_m$ implies $m(\lambda, x_{n_0}, a) \to 0$ as $\lambda \to \infty$. Therefore, there exists some $\lambda_0 = \lambda_0(\epsilon) > 0$ such that $m(\lambda_0, x_{n_0}, a) < \epsilon$.

Now from (2.12) and monotonicity of modular metric m, we have

$$m(\lambda, x, a) \leq m(\epsilon + \lambda_0, x, a) < 2\epsilon$$

for all $\lambda \geq \epsilon + \lambda_0$. This means $m(\lambda, x, a) \to 0$ as $\lambda \to \infty$. Hence $x \in S_m$. Then by Definition 2.6.2, S_m is closed.

□

2.4 The metric topology

In Section 2.2, we have learnt that whenever a modular set is equipped with a metric induced by a modular metric, then the modular is a metric space. We labelled d_m to be the metric induced by a modular metric. In this section, we study topologies induced by this metric d_m and we shall call such topology a metric topology and denote it as $\tau(d_m)$.

Definition 2.4.1. *[16, page 66] Let S be a nonempty set and d_m be the metric induced by modular metric m on S. Given $a \in S$ and $r > 0$, we now define and denote an open ball of radius r and centre a in modular metric sense as the set*

$$B(a, r) \equiv B_{d_m}(a, r) = \{b \in S : d_m(a, b) < r\}.$$

Definition 2.4.2. *[16, page 67] Let S be a nonempty set and $B_{d_m}(a, r)$ be an open ball of radius r and centre a. A nonempty set $O \subset S$ is said to be open if for every $a \in O$, $\exists r = r(a) > 0$ such that $B_{d_m}(a, r) \subset O$.*

Remark 2.4.1. *Considering Theorem 2.2.2, we note the following:*

1. *If $a \in S_m^*(a)$, then $B_{d_m}(a, r) \subset S_m^*(a)$ for all $r > 0$ (i.e. $S_m^*(a)$ is open).*
2. *A set $O \subset S$ is open if and only if the intersection $O \cap S_m^*(a)$ is open for all $a \in S$.*

We give examples of open sets.

Example 2.4.1. *[16]*

1. *An open ball $B_{d_m}(a,r) \subset S_m^*(a)$ is open.*

2. *An open set $\emptyset \subset S_m^*(a)$ is open.*

3. *The modular set $S_m^* \subseteq S_m^*(a)$ is open.*

Definition 2.4.3. *[16, page 67] Let S be a nonempty set and S_m^* be a modular set. We define the interior of a set $A \subset S_m^*$, which we shall denote as A^o, as the largest open set contained in A.*

One can look at A^o as the union of all open sets in A.

Clearly

$$A^o = \{a \in S : B_{d_m}(a,r) \subset A \quad \text{for some} \quad r > 0\}$$

and $A^o \subset A$. Lastly, the set A is open if and only if $A^o = A$.

Definition 2.4.4. *[16, page 67] Let S_m^* be a modular set and d_m be the metric induced by modular metric m on S_m^*. A collection or family of all open sets in S_m^* form a topology, which we shall call metric topology on S_m^* and denote it by $\tau(d_m)$ since it is induced by d_m on S_m^*.*

Remark 2.4.2. *[16] If O be an arbitrary set with no assumed structure and $\tau(d_m)$ is a metric topology defined in Definition 2.4.4, then it is obvious that $O \in \tau(d_m)$ (i.e. O is open) if and only if its complement $O^c = S_m^* \backslash O$ is closed with respect to metric convergence.*

We give examples of closed sets.

Example 2.4.2. *[16] Let S_m^* be a modular set. Then in modular sets S_m^*.*

1. *$\emptyset$ is closed.*

2. *A closed ball of radius $r > 0$ and centre at $a \in S_m^*$ is closed:*

$$\bar{B}_{d_m}(a,r) = \{b \in S_m^* : d_m(a,b) \leq r\} = \{b \in S_m^* : m(r+0,a,b) \leq r\}$$

3. *Modular sets S_m^* and S_m are closed.*

For more clarification, Theorem 2.2.3 can be consulted.

Definition 2.4.5. *[16, page 68] Let A be a subset of a modular set S_m^*. The closure of A, which we shall denote by $\bar{A}$, is the smallest closed set containing A.*

This implies that $\bar{A}$ is the intersection of all closed sets $F \supset A$ containing A.

In the following remark we give a few observations.

Remark 2.4.3. *[16] Let A be a subset of a modular set S_m^*. Then it is obvious that*

1. *$\bar{A} = ((A^c)^o)^c$ is closed.*
2. *$A \subset \bar{A}$.*
3. *A is closed if and only if $\bar{A} = A$.*

We can now define the closure of A. Let $B_{\mu,\mu}$ be a modular entourage defined in Definition 2.6.1. Then we define the closure of A with respect to modular entourage as a set

$$\bar{A} = \{a \in S_m^* : A \cap B_{\mu,\mu}(a) \neq \emptyset \quad \text{for all} \quad \mu > 0\} \tag{2.13}$$

which make sense since $\bar{A}$ is the set of points in S_m^* which are not in the set $(A^c)^o$.

Remark 2.4.4. *[16] In general, we have*

1. *$\overline{B_{d_m}(a,r)} \subset \bar{B}_{d_m}(a,r)$*
2. *$B_{d_m}(a,r) \subset (\bar{B}_{d_m}(a,r))^o$*

We give nontrivial examples by the following lemma.

Lemma 2.4.1. *[16] Let S be a nonempty set and m be a modular metric defined on S. If A to be a subset of a modular set S_m^* with $x \in S_m^*$, then the following statements are equivalent:*

1. *$x \in \bar{A}$;*

2. *there exists a sequence* (x_n) *in* A *such that* $m(\lambda, x_n, x) \to 0$ *for all* $\lambda > 0$*;*

3. *for every* $\lambda > 0$ *there is sequence* $(x_n(\lambda))$ *in* A *such that* $m(\lambda, x_n(\lambda), x) \to 0$.

Proof. $1 \Rightarrow 2$ We have seen from Theorem 2.3.1 that $x_n \to x$ if and only if $m(\lambda, x_n, x) \to 0$ for all $\lambda > 0$. Therefore if there exists a sequence (x_n) in A such that $d_m(x_n, x) \to 0$, then $x \in \bar{A}$.

$2 \Leftarrow 3$ It is obvious that (2) implies (3) (i.e. $x_n(\lambda) = x_n$ for all $\lambda > 0$ and $n \in \mathbb{N}$).

Therefore, we prove that (3) implies (2).

Given $k \in \mathbb{N}$, there exists x_n in A such that

$$m\left(\frac{1}{k}, x_n\left(\frac{1}{k}\right), x\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, we choose $n_1 \in \mathbb{N}$ such that

$$m(1, x_{n_1}(1), x) < 1.$$

We pick $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and

$$m\left(\frac{1}{2}, x_{n_2}\left(\frac{1}{2}\right), x\right) < \frac{1}{2}.$$

By induction, if $k \geq 3$ and $n_{k-1} \in \mathbb{N}$ is already chosen, pick $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $m(\frac{1}{k}, x_{n_k}(\frac{1}{k}), x) < \frac{1}{k}$. Let $y_k = x_{n_k}(\frac{1}{k})$ for all $k \in \mathbb{N}$, so that (y_k) is in A.

We claim that $m(\lambda, y_k, x) \to 0$ as $k \to \infty$ for all $\lambda > 0$. Actually given $\lambda > 0$, let $k_0 = k_0(\lambda) \in \mathbb{N}$ such that $k_0 > \frac{1}{\lambda}$. Then, for all $k \geq k_0$, we find $\frac{1}{k} < \lambda$, and so, by monotonicity of m,

$$m(\lambda, y_k, x) \leq m\left(\frac{1}{k}, y_k, x\right) < \frac{1}{k}$$

is equivalent to metric convergence in modular set S_m^*.

□

2.5 Modular Convergence

In this section we look into the notion of modular convergence in a set.

Definition 2.5.1. *[16] Let S be a nonempty set on S. A sequence (x_n) in S is said to be modular convergent (m-convergent) to an element $x \in S$ (denoted as $x_n \stackrel{m}{\to} x$) if there is $\lambda_0 = \lambda_0(x_n, x) > 0$ such that $m(\lambda_0, x_n, x) \to 0$, where m is a modular metric defined on S.*

From Definition 2.5.1 above, if such element x exists, it is called a modular limit (m-limit) of (x_n). Since m is nonincreasing on $(0, \infty)$ as we have seen Lemma 2.1.1 and by inequality (2.1), we get that $m(\lambda, x_n, x) \to 0$ for all $\lambda > \lambda_0$.

It is clear that metric convergence induced by metric d_m implies modular convergence. We show this by the following examples.

Example 2.5.1. *[16] Let (S, d) be a metric space. For $p \geq 0$, let $m(\lambda, a, b) = \dfrac{d(a, b)}{\lambda^p}$ be a modular metric defined in Example 2.2.6. We have seen that $S^*_m = S$ and $d_m(a, b) = \sqrt[p+1]{d(a, b)}$, then metric convergence (d_m-convergence) and modular convergence (m-convergence) in S^*_m are equivalent to the usual convergence (d-convergence) in S.*

Example 2.5.2. *[16] Let (S, d) be a metric space. Let*

$$m(\lambda, a, b) = \begin{cases} \infty & \text{if } 0 < \lambda < d(a, b), \\ 0 & \text{if } \lambda \geq d(a, b) \end{cases}$$

be the modular metric defined in Example 2.1.3. We have $S^*_m = S_m = S$ and $d_m = d$.

Remark 2.5.1. *[16] Let S be a nonempty set and m be the modular metric defined in Example 2.5.2. A d_m-convergent sequence (x_n) is bounded in the metric space (S, d) but not vice versa.*

Corollary 2.5.1. *[16] Let S be a nonempty set and m be the modular metric defined in Example 2.5.2. The m-convergence of (x_n) in S^*_m is equivalent to the boundedness of (x_n) in (S, d).*

Proof. ($\Rightarrow$) Let (S, d) be a metric space. Now if $x_n \stackrel{m}{\to} x$ with $x \in S$, then $m(\lambda_0, x_n, x) \to 0$ for some $\lambda_0 > 0$ according to the Definition 2.5.1 of modular convergence. Therefore, $\exists n_0 \in \mathbb{N}$ such that if $n > n_0$, then $m(\lambda_0, x_n, x) < 1$ since $m(\lambda, a, b) = 0$ if $\lambda \geq d(a, b)$. Now $m(\lambda_0, x_n, x) < 1$ implies $\lambda_0 \geq d(x_n, x)$. Hence (x_n) is in $\bar{B}(a, r_0)$ with $r_0 = \max\{d(x_1, a), ..., d(x_{n_0}, a), \lambda_0\}$. Therefore, modular convergence implies metric convergence.

($\Leftarrow$) If (x_n) is in $\bar{B}(a, r)$ for some $a \in S$ and $r > 0$, then $x_n \stackrel{m}{\to} b$ for every $b \in S$. By triangular inequality of a metric, we have $d(x_n, b) \leq d(x_n, a) + d(a, b) \leq \lambda_0$ for $n \in \mathbb{N}$ and $\lambda_0 = r + d(a, b)$. Then $m(\lambda_0, x_n, b) = 0$ for all $n \in \mathbb{N}$. Therefore, metric convergence implies modular convergence. $\square$

Now we define some topological aspects of m-convergence.

Definition 2.5.2. *[16] Let S be a nonempty set. Elements $a, b \in S$ are said to be modular equal (m-equal) if there is $\lambda > 0$ such that $m(\lambda, a, b) = 0$ and we shall symbolise this by $a \stackrel{m}{=} b$.*

Remark 2.5.2. *[16] When we consider (1") in Definition 2.1.1, we then define m-equality as: For any $a, b \in S$,*

$$a \stackrel{m}{=} b \quad \textit{is equivalent to} \quad a = b \quad \textit{if and only if } m \textit{ is strict.}$$

Example 2.5.3. *[16] Let S be a nonempty set and let m be a modular metric defined in Example 2.5.2. Considering modular m in Example 2.5.2, then $a \stackrel{m}{=} b$ for all $a, b \in S$.*

Definition 2.5.3. *[16] Let S be a nonempty set and m be the modular metric defined on S. A set $A \subset S$ is said to be closed with respect to m-convergence if for any sequence (x_n) in A such that $x_n \stackrel{m}{\to} x$ with $x \in S$, then $x \in A$.*

Theorem 2.5.2. *[16] Let S be a nonempty set and m be a modular metric defined on S. Modular spaces $S_m^*(x_0)$ and $S_m(x_0)$ are closed with respect to m-convergence.*

Proof. Suppose a sequence (x_n) is in S such that $x_n \stackrel{m}{\to} x$ with $x \in S$. Then we have $m(\lambda_0, x_n, x) \to 0$ for some $\lambda_0 > 0$ by definition of modular convergence. Given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $m(\lambda_0, x_n, x) \leq \epsilon$ for all $n \geq n_0$. By the triangle inequality of a modular metric, if $\lambda_1 > 0$, then we have

$$m(\lambda_0 + \lambda_1, x, a) \leq m(\lambda_0, x, x_{n_0}) + m(\lambda_1, x_{n_0}, a) \leq \epsilon + m(\lambda_1, x_{n_0}, a) \quad (2.14)$$

Let a sequence (x_n) be in S_m^*. Since $x_{n_0} \in S_m^*$, then $m(\lambda_1, x_{n_0}, a) < \infty$ for some $\lambda_1 > 0$, therefore inequality (2.14) implies $m(\lambda_0 + \lambda_1, x, a) < \infty$, which means $x \in S_m^*$. Hence S_m^* is closed with respect to modular convergence.

If a sequence (x_n) is in S_m, then $x_{n_0} \in S_m$, therefore $m(\lambda_1, x_{n_0}, a) \to 0$ as $\lambda \to \infty$. Let $\lambda_1 = \lambda_1(\epsilon) > 0$ such that $m(\lambda_1, x_{n_0}, a) \leq \epsilon$. Therefore, if $\lambda \geq \lambda_0 + \lambda_1$, then inequality (2.14) means $m(\lambda, x, a) \leq m(\lambda_0 + \lambda_1, x, a) \leq 2\epsilon$ which implies $m(\lambda, x, a) \to 0$ as $\lambda \to \infty$. Therefore $x \in S_m$. Hence S_m is closed with respect to modular convergence.

□

Theorem 2.5.3. *[16] Let S be a nonempty set and m be a modular metric defined on S. An m-convergent sequence in S_m^* is bounded (in metric d_m sense) and all its m-limits are equal. Furthermore, if m is a strict modular metric on S, then its limit point is unique.*

Proof. We shall first show that an m-convergent sequence in S_m^* is bounded then later show that its m-limits are equal. Let (x_n) be a sequence in S_m^* such that $x_n \overset{m}{\to} x$ and $x \in S$. From the above Theorem 2.5.2, we have seen that $x \in S_m^*$ and also that, and there is $n_0 \in \mathbb{N}$ such that $m(\lambda_0, x_n, x) \leq \lambda_0$ for all $n > n_0$. This implies $d_m(x_n, x) \leq \lambda_0$. By Theorem 2.2.2, d_m is an extended metric if its points are elements of S_m^*, therefore a sequence (x_n) is in $\bar{B}_{d_m}(x, r)$ where

$$r = \max\{d_m(x_1, x), ..., d_m(x_{n_0}, x), \lambda_0\}.$$

Therefore an m-convergent sequence in S_m^* is bounded.

We now show that m-limits of m-convergent sequences are m-equal. If $x_n \overset{m}{\to} a$ and $x_n \overset{m}{\to} b$, then $m(\lambda, x_n, a) \to 0$ and $m(\mu, x_n, b) \to 0$ for some $\lambda, \mu > 0$. Using the triangle inequality property of a modular metric, we have

$$m(\lambda + \mu, a, b) \leq m(\lambda, a, x_n) + m(\mu, x_n, b) \to 0$$

hence $m(\lambda + \mu, a, b) = 0$. Thus $a \overset{m}{=} b$.
Furthermore, if m is strict, we have $a = b$.

□

2.6 Modular topology

In this section we discuss a topology induced by a modular metric on a modular set. We shall call such a topology a modular topology.

Definition 2.6.1. *[16, page 67] Let S_m^* be a modular set and a be an element in S_m^*. Given $\lambda, \mu > 0$, we define a modular entourage (m-entourage) with centre a and relative to λ and μ as the set*

$$B_{\lambda,\mu}(a) \equiv B_{\lambda,\mu}^m(a) = \{b \in S_m^* : m(\lambda, a, b) < \mu\}.$$

In the following remark we give a few observations from Definition 2.6.1.

Remark 2.6.1. *[16]*

1. *It is obvious that $a \in B_{\lambda,\mu}(a)$.*
2. *If $0 < \mu_1 < \mu_2$, then $B_{\lambda,\mu_1}(a) \subset B_{\lambda,\mu_2}(a)$*
3. *If $0 < \lambda_1 < \lambda < \lambda_2$ and $\mu > 0$, considering inequality (2.4), we have*

$$B_{\lambda_1+0,\mu}(a) \subset B_{\lambda-0,\mu}(a) \subset B_{\lambda,\mu}(a) \subset B_{\lambda+0,\mu}(a) \subset B_{\lambda_2-0,\mu}(a).$$

4. *The mappings $\lambda \mapsto B_{\lambda,\mu}(a)$ and $\mu \mapsto B_{\lambda,\mu}(a)$ are non-decreasing.*
5. *For $\lambda > 0$, using Theorem 2.2.3 we have $B_{d_m}(a, \lambda) = B_{\lambda-0,\lambda}(a)$.*
6. *Consequently, if $0 < \mu < \lambda$ we have*

$$B_{\mu,\mu}(a) \subset B_{\mu,\lambda}(a) \subset B_{d_m}(a, \lambda) = B_{\lambda-0,\lambda}(a) \subset B_{\lambda,\lambda}(a).$$

In the next remark, we show how observations in Remark 2.6.1 give characterization of open sets induced by a modular metric m on a modular set S_m^*.

Remark 2.6.2. *[16] Let O be an arbitrary set with no assumed structure and let $\tau(d_m)$ be the metric topology. Then*

$$O \in \tau(d_m) \text{ if and only if } \forall a \in O, \exists \mu > 0 \text{ such that } B_{\mu,\mu}(a) \subset O \quad (2.15)$$

and the interior of a set $A \subset S_m^*$ is

$$A^o = \{a \in A : B_{\mu,\mu}(a) \subset A\} \quad \textit{for some} \quad \mu > 0.$$

We define closed sets as follows.

Definition 2.6.2. *[16] Let A be a subset of the modular set S_m^*. A is said to be closed if its complement $A^c = S_m^* \backslash A$ is open.*

This implies that the interior of a complement is the same as the complement itself (i.e. $(A^c)^o = A^c$), meaning A is closed with respect to metric convergence.

Definition 2.6.3. *[16] A nonempty set $O \subset S$ is said to be modular open (m-open) if for every $a \in O$ and $\lambda > 0$ there is $\mu = \mu(x, \lambda) > 0$ such that $B_{\lambda,\mu}(a) \subset O$.*

We can compare the following remarks with Remark 2.4.1.

Remark 2.6.3. *[16]*

1. *If $a \in S_m^*(a)$, then $B_{\lambda,\mu}(a) \subset S_m^*(a)$ for all $\lambda, \mu > 0$.*
2. *Similarly to Remark 2.4.1(2), $O \subset S$ is m-open if and only if the intersection $O \cap S_m^*(a)$ is m-open for all $a \in S$.*

As we have seen in Section 2.4, where we discussed a topology induced by metric d_m. Now we will be considering modular open subsets of S_m^*. We denote the family of all m-open sets of S_m^* by $\tau(m)$. Obviously $\tau(m)$ is a topology on S_m^*. This implies that $\tau(m)$ contains the empty set $\emptyset$ and S_m^*, and this family is closed under arbitrary unions and finite intersections. We shall call this family $\tau(m)$ the modular topology and abbreviate it with m-topology on S_m^*.

Lemma 2.6.1. *[16] Let $O_\psi(a) = \bigcup\limits_{\lambda>0} B_{\lambda,\psi(\lambda)}(a)$, where $a \in S_m^*$ and $\psi : (0, \infty) \to (0, \infty)$ is a nonincreasing function on the set of positive numbers. Then $O_\psi(a) \in \tau(m)$ (i.e. $O_\psi(a)$ is open).*

Proof. We want to show that $O_\psi(a)$ is open.

Meaning we want to show that given $b \in O_\psi(a)$, there is a positive number $\mu = \mu(b,\lambda) > 0$ which depends on b and λ such that $B_{\lambda,\mu}(b) \subset O_\psi(a)$ for any $\lambda > 0$.

Now since $b \in O_\psi(a)$, there exists $\lambda_0 = \lambda_0(b) > 0$ such that $b \in B_{\lambda_0,\psi(\lambda_0)}(a)$, which implies $m(\lambda_0, a, b) < \psi(\lambda_0)$.

Let $\mu = \mu(b,\lambda) = \psi(\lambda_0+\lambda) - m(\lambda_0, a, b)$. Since $\mu \geq \psi(\lambda_0) - m(\lambda_0, a, b) > 0$, then ψ, μ are well defined.

Let $c \in B_{\lambda,\mu}(b)$, then we have $m(\lambda, b, c) < \mu$. By the triangle inequality axiom of a modular metric, we have

$$m(\lambda_0 + \lambda, a, c) \leq m(\lambda_0, a, b) + m(\lambda, b, c) < m(\lambda_0, a, b) + \mu = \psi(\lambda_0 + \lambda),$$

so we have $c \in B_{\lambda_0+\lambda,\psi(\lambda_0+\lambda)}(a) \subset O_\psi(a)$. Therefore $O_\psi(a)$ is open. Which implies $O_\psi(a) \in \tau(m)$ since $B_{\mu,\mu}(b) \subset O_\psi(a)$. $\square$

Remark 2.6.4. *[16] We can see that from Lemma 2.6.1 above that* $O_{a,\epsilon} = \bigcup_{\lambda>0} B_{\lambda,\epsilon}(a) \in \tau(m)$ *for all* $\epsilon > 0$.

Comparing Remark 2.6.4 with Remark 2.6.2, we see that Remark 2.6.4 is more general.

Remark 2.6.5. *[16] The family* $\{O_{a,\epsilon} : \epsilon > 0\}$ *may not form a neighbourhood base for* $\tau(m)$ *at the point* $a \in S_m^*$.

Let us now look at some examples.

Example 2.6.1. *[16] We will denote* $B^d(a,r)$ *as the usual open ball in a metric space* (S,d) *and* $\bar{B}^d(a,r)$ *the usual closed ball in a metric space* (S,d).

We recall that from Example 2.5.1, for $r > 0$ *and* $\lambda, \mu > 0$ *we have*

$$\begin{aligned} B_{d_m}(a,r) &= \{b \in S : d_m(a,b) < r\} \\ &= \{b \in S : [d(a,b)]^{\frac{1}{p+1}} < r\} \\ &= \{b \in S : d(a,b) < r^{p+1}\} \\ &= B^d(a, r^{p+1}) \end{aligned}$$

and we also have

$$\begin{aligned} B_{\lambda,\mu}(a) &= \{b \in S : m(\lambda, a, b) < \mu\} \\ &= \{b \in S : \frac{d(a,b)}{\lambda^p} < \mu\} \\ &= \{b \in S : d(a,b) < \mu\lambda^p\} \\ &= B^d(a, \lambda^p \mu). \end{aligned}$$

Since $B_{\lambda,\mu}(a) = B_{d_m}(a,r)$ *with* $\mu = \frac{r^{p+1}}{\lambda^p}$*,we get that* $\tau(m) = \tau(d_m)$*.*

In Example 2.5.2, for $r > 0$ and $\lambda, \mu > 0$, we have $B_{d_m}(a,r) = B^d(a,r)$ and $B_{\lambda,\mu}(a) = \bar{B}^d(a,\mu)$.

Hence,

$$\emptyset \neq O \in \tau(m) \quad \text{if and only if} \quad \bar{B}^d(a,\lambda) \subset O \quad \text{for all } a \in O \quad \text{and} \quad \lambda > 0.$$

This equivalent to $O = S$. Therefore, we have that the m-topology $\tau(m) = \{\emptyset, S\}$ is a nonmetrizable (antidiscrete) topology on $S^*_m = S$, which is a proper subfamily of the metric topology $\tau(d_m) = \tau(d)$.

Remark 2.6.6. *[16]*

1. *The* m*-convergence* $x_n \overset{m}{\to} x$ *implies the convergence* $x_n \overset{\tau(m)}{\to} x$ *in the topology* $\tau(m)$ *but not vice versa.*
2. *Considering Example 2.5.2, only bounded sequences in metric space* (S,d) *are* m*-convergent whereas all sequences in* (S,d) *are convergent in the nonmetrizable topology* $\tau(m)$*.*

Example 2.6.2. *[16] Suppose* $m(\lambda, a, b) = \psi(\frac{d(a,b)}{\lambda})$ *where the function* $\psi : [0,\infty) \to [0,\infty]$ *is nondecreasing and such that* $\psi(0) = 0, \psi \not\equiv 0$ *and* $\psi \not\equiv \infty$*, meaning the function can attain zero value and infinite values, but not be constant at zero nor at infinity. Let* (S,d) *be a metric space and we have* $S^*_m = S$ *for all* $a, b \in S^*_m$ *and* $\lambda > 0$*, and the modular entourages for* m *are given by*

$$B_{\lambda,\mu}(a) = \begin{cases} \{a\} & \text{if } \lambda > 0 \text{ and } 0 < \mu \leq 1, \\ B^d(a, \lambda\mu) & \text{if } \lambda > 0 \text{ and } \mu > 1. \end{cases}$$

Since

$$d_m(a,b) = \begin{cases} \max\{1, \sqrt{d(a,b)}\} & \textit{if } a \neq b, \\ 0 & \textit{if } a = b, \end{cases}$$

then the open balls induced by modular metric m in (S_m^*, d_m) are of the form

$$B_{d_m}(a,r) = \{b \in S : d_m(a,b) < r\} = \begin{cases} \{a\} & 0 < r \leq 1, \\ B^d(a, r^2) & \textit{if } r > 1. \end{cases}$$

Hence $\tau(m) = \tau(d_m) = \mathcal{P}(S)$ is the discrete topology on S.

Theorem 2.6.2. *[16] Let S be a nonempty set and $\tau(m)$ be a modular topology induced by a metric modular m on S and $\tau(d_m)$ be a metric topology induced a metric d_m. Then we have $\tau(m) \subset \tau(d_m)$.*

Proof. We want to show that every m-open set is open.

Let $O \in \tau(m)$. This means for any $a \in O$ and $\lambda = 1$, there is $\mu_0 = \mu_0(a,1) > 0$ such that $B_{1,\mu_0}(a) \subset O$. If $\mu = \min\{1, \mu_0\}$ then $B_{\mu,\mu}(a) \subset B_{1,\mu_0}(a) \subset O$.

By characterization of open sets in S_m^* in terms of m in assertion (2.15), we get $O \in \tau(d_m)$. Hence $\tau(m) \subset \tau(d_m)$. □

Theorem 2.6.3. *[16] Let S be a nonempty set and S_m^* be a modular set. The modular topology $\tau(m)$ is the finest topology among all topologies τ on S_m^* such that, whenever given a sequence (x_n) in S_m^* and $x \in S_m^*$, $x_n \overset{m}{\to} x$ implies $x_n \overset{\tau(m)}{\to} x$.*

Proof. We start proving first that $x_n \overset{m}{\to} x$ implies $x_n \overset{\tau(m)}{\to} x$ and then later we prove that $\tau(m)$ is the finest topology among the topologies τ.

Let $O \in \tau(m)$ be such that $a \in O$. Since $x_n \overset{m}{\to} x$, then $m(\lambda_0, x_n, x) \to 0$ for some $\lambda_0 > 0$. Now since $O \in \tau(m)$, then O is m-open, which implies there exists $\mu_0 = \mu_0(x, \lambda_0) > 0$ such that $B_{\lambda_0,\mu_0}(a) \subset O$.

Let $n_0 = n_0(\mu_0) \in \mathbb{N}$ such that $m(\lambda_0, x_n, x) < \mu_0$ for all $n \geq n_0$. Then we have (x_n) is in $B_{\lambda_0,\mu_0}(a) \subset O$ for $n \geq n_0$, which implies $x_n \overset{\tau(m)}{\to} x$.

Now we show that $\tau(m)$ is the finest topology among the topologies τ. Which means that we want to show that if τ is a topology on S_m^* such that $x_n \overset{m}{\to} x$ implies $x_n \overset{\tau}{\to} x$ then $\tau \subset \tau(m)$. We shall prove this by contradiction.

Assume that $O \notin \tau(m)$ for some $O \in \tau$. Then there is $a \in O$ and $\lambda_0 > 0$ such that for every $n \in \mathbb{N}$ there exists (x_n) in $B_{\lambda_0, \frac{1}{n}}(a_0)$ with (x_n) not in O (i.e. (x_n) is in O^c). This means we have $m(\lambda_0, x_n, a_0) < \frac{1}{n}$. We get that $x_n \xrightarrow{m} a_0$, and so by the assumption $x_n \xrightarrow{\tau} a_0$. Since $O \in \tau$ and $a_0 \in O$, there exits $n_0 \in \mathbb{N}$ such that (x_n) in O for all $n \geq n_0$, which is a contradiction with the condition (x_n) is in $O^c = S_m^* \backslash O$. Hence $\tau(m)$ is the finest topology among all topologies τ on S_m^*. $\square$

2.7 Hausdorff pseudomodular

In this section, we discuss the concept of the Hausdorff pseudomodular on the power set of a nonempty set S equipped with a modular metric m. We start by first recalling the construction of Hausdorff distance then later introduce a modular pseudometric on the power set induced by the modular pseudometric m.

Definition 2.7.1. *[6] Let (S, d) be a metric space. We define Hausdorff distance between any two nonempty subsets A and B of S as*

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

We give a few observations of the above formula in the following remark.

Remark 2.7.1. *[6] Let H_d be Hausdorff distance defined above.*

1. *It is clear that $H_d(\{a\}, \{b\}) = d(a, b)$, so the Hausdorff distance is an extension of the usual metric d.*

2. *When the metric d is unbounded, then one can obtain infinite Hausdorff distance. For example, given $a \in S$, then we have $H_d(\{a\}, S) = \infty$.*

3. *We notice that Hausdorff distance is invariant under replacing sets by their closures induced by d. In fact, the Hausdorff distance between any nonempty set and its closure must be zero.*

4. *If we restrict the sets to be nonempty closed sets, then H_d would satisfy the usual properties of a metric and since Hausdorff distances assumes infinite values then it would define an extended real-valued metric on a set of nonempty closed subsets of S.*

The Hausdorff distance has been studied in [7, 9, 43, 53].

We adopt the convention that $\sup\emptyset = 0$ and $\inf\emptyset = \infty$.

Definition 2.7.2. *[16, page 15] Let m be a modular metric on a nonempty set S. We define a function $E : (0,\infty)\times\mathcal{P}(S)\times\mathcal{P}(S)\to[0,\infty]$ as*

$$E(\lambda, A, B) = \sup_{a\in A}\inf_{b\in B} m(\lambda, a, b) \tag{2.16}$$

for all $\lambda > 0$ and for any set $A, B \in \mathcal{P}(S)$.
Moreover, we set

$$E(\lambda, \emptyset, B) = 0 \quad \textit{for all } \lambda > 0 \textit{ and for any} \qquad B \in \mathcal{P}(S)$$

and

$$E(\lambda, A, \emptyset) = \infty \qquad \textit{for all } \lambda > 0 \textit{ and} \qquad \textit{for any nonempty set} \quad A \in \mathcal{P}(S).$$

Proposition 2.7.1. *[16, Proposition 1.3.11.] Let S be a nonempty set and m be a modular metric defined on S. The function $E : (0,\infty)\times\mathcal{P}(S)\times\mathcal{P}(S)\to[0,\infty]$ is well-defined and satisfies the following properties:*

1. *$E(\lambda, A, B) = 0$ for all $\lambda > 0$ and $A, B \in \mathcal{P}(S)$;*
2. *$E(\lambda+\mu, A, B) \le E(\lambda, A, C) + E(\mu, C, B)$ for all $\mu, \lambda > 0$ and $A, B \in \mathcal{P}(S)$.*

Proof. We will provide the proof in more general settings. See Proposition 3.4.1. □

Definition 2.7.3. *Let m be a modular metric on a nonempty set S. Suppose E is a function defined in Definition 2.7.2, then we define a Hausdorff pseudomodular M induced by m as a function $M : (0,\infty)\times\mathcal{P}(S)\times\mathcal{P}(S)\to[0,\infty]$ defined by*

$$M(\lambda, A, B) = \max\{E(\lambda, A, B), E(\lambda, B, A)\} \tag{2.17}$$

that satisfies the following properties:-

1. *$M(\lambda, A, A) = 0$ for all $\lambda > 0$ and $A \in \mathcal{P}(S)$;*

2. $M(\lambda, A, B) = M(\lambda, B, A)$ *for all* $\lambda > 0$ *and* $A, B \in \mathcal{P}(S)$.

3. $M(\lambda+\mu, A, B) \leq M(\lambda, A, C)+M(\mu, C, B)$ *for all* $\mu, \lambda > 0$ *and* $A, B \in \mathcal{P}(S)$.

Remark 2.7.2. *[16] From the convention we adopted, we get* $M(\lambda, \emptyset, \emptyset) = 0$ *and* $M(\lambda, A, \emptyset) = 0$ *for all* $\lambda > 0$ *and for any nonemptyset* $A \in \mathcal{P}(S)$.

It turns out that there are two ways of obtaining a distance function (induced by a modular metric m) on the power set $\mathcal{P}(S)$ which we show in the following definitions.

Definition 2.7.4. *[16] Let* M *be a Hausdorff pseudomodular on* $\mathcal{P}(S)$. *We define a function* $d_m : \mathcal{P}(S) \times \mathcal{P}(S) \to [0, \infty]$ *as*

$$d_M(A, B) = \inf\{\lambda > 0 : M(\lambda, A, B) \leq \lambda\} \tag{2.18}$$

for all $\lambda > 0$ *and for any* $A, B \in \mathcal{P}(S)$.

Definition 2.7.5. *[16] Consider the metric* d_m *defined in Theorem 2.2.2. Suppose* $\mathcal{P}(S)$ *is the power set of a nonempty set* S, *then we define a function* $H_{d_m} : \mathcal{P}(S) \times \mathcal{P}(S) \to [0, \infty]$ *as*

$$H_{d_m}(A, B) = \max\{\sup_{a \in A} d_m(a, B), \sup_{b \in B} d_m(b, A)\} \tag{2.19}$$

for all $A, B \in \mathcal{P}(S)$.

Note that the distance functions d_M and H_{d_m} which are defined in the above definitions, coincide on the power set.

Chapter 3

Topologies on modular quasi-pseudometric spaces

In this chapter, we start our own investigations on a modular metric that does not satisfy the symmetric axiom of modular pseudometric and we shall call it modular quasi-pseudometric. Then, we study topologies induced by a modular quasi-pseudometric. We later study Hausdorff distances induced by modular quasi-pseudometrics.

3.1 Modular quasi-pseudometric

The following definition can be compared to Definition 2.1.1.

Definition 3.1.1. *Consider a nonempty set X. A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a modular quasi-pseudometric on X if it satisfies the following conditions:*

1. $w(\lambda, x, x) = 0$ *whenever* $x \in X$ *and* $\lambda \in (0, \infty)$,
2. $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y)$ *whenever* $x, y, z \in X$ *and* $\lambda, \mu \in (0, \infty)$.

   We shall say that w is a modular quasi-metric provided that w also satisfies the following condition: whenever $x, y \in X$ and $\lambda \in (0, \infty)$,

3. $w(\lambda, x, y) = 0$ *and* $w(\lambda, y, x) = 0$ *imply* $x = y$.

Remark 3.1.1. *Let w be a modular quasi-(pseudo)metric on a set X, then the function $w^t : (0,\infty) \times X \times X \to [0,\infty]$ defined by $w^t(\lambda, x, y) = w(\lambda, y, x)$ whenever $x, y \in X$ and $\lambda \in (0,\infty)$ is also a modular quasi-pseudometric on X that we call the transpose modular quasi-pseudometric of w.*
Moreover, note that for any modular quasi-pseudometric w on X, the function $w^s(\lambda, x, y) = \max\{w(\lambda, x, y), w^t(\lambda, x, y)\}$ whenever $x, y \in X$ and $\lambda \in (0,\infty)$ is a modular (pseudo)metric on X (see [16, 18, 19]). In addition, we observe that the function
$w^m : (0,\infty) \times X \times X \to [0,\infty]$ defined by

$$w^m(\lambda, x, y) = w(\lambda, x, y) + w^t(\lambda, x, y) \quad \textit{whenever} \quad x, y \in X \quad \textit{and} \quad \lambda > 0$$

is also a modular (pseudo)metric on X.

For any modular quasi-pseudometric w on a set X, if $w = w^t$, then w is a modular pseudometric on X.

We give a few remarks concerning Definition 3.1.1.

Remark 3.1.2.

1. *The assumption $w : (0,\infty) \times X \times X \to (-\infty,\infty]$ (that w assumes negative values) does not lead to any great results. Actually, setting $y = x$ and $\mu = \lambda > 0$ in (b), we get*

   $$0 = w(2\lambda, x, x) \leq w(\lambda, x, z) + w(\lambda, z, x) = w^m(\lambda, x, z)$$

   which implies $w^m(\lambda, x, z) \geq 0$ or $w^m(\lambda, x, z) = \infty$ for all $\lambda > 0$ and $x, z \in X$. Which just implies w is non-negative.

2. *If $w(\lambda, x, y) = w(\lambda)$ does not depend on $x, y \in X$, then by (a), we get $w \equiv 0$, which is just a pseudomodular on X.*

3. *If $w(\lambda, x, y) = w(x, y)$ is independent of $\lambda > 0$, then w is an extended quasi-pseudo metric on X and w would be quasi-pseudometric on X if it does not assume infinite values.*

It is obvious that on a nonempty set X endowed with a modular quasi-pseudometric w we have

$$w(\lambda, x, y) \leq w^s(\lambda, x, y) \tag{3.1}$$

and

$$w^t(\lambda, x, y) \leq w^s(\lambda, x, y) \tag{3.2}$$

whenever $\lambda > 0$ and $x, y \in X$.

Lemma 3.1.1. *(compare Lemma 2.1.1) Let w be a modular quasi-pseudometric on a set X and let $x, y \in X$. Define a function $\psi : (0, \infty) \to [0, \infty]$ by $\psi(\lambda) = w(\lambda, x, y)$. Then ψ is noninincreasing.*

Proof. Let $x, y \in X$ and $\lambda, \mu \in (0, \infty)$ with $\mu < \lambda$. Since $\lambda - \mu > 0$, we have

$$\psi(\lambda) = w(\lambda, x, y) = w(\lambda - \mu + \mu, x, y) \leq w(\lambda - \mu, x, x) + w(\mu, x, y)$$

by Definition 2(b). It follows that

$$\psi(\lambda) = w(\lambda, x, y) \leq w(\mu, x, y) = \psi(\mu)$$

since $w(\lambda - \mu, x, x) = 0$ as w is a modular quasi-pseudometric. Therefore ψ is a nonincreasing function. □

The following remark is just the consequence of Lemma 3.1.1 above and it can be compared to the essential property of a modular pseudometric (see [16, page 6]).

Remark 3.1.3. *If w is a modular quasi-pseudometric on a nonempty set X. Then for any $x, y \in X$ and $\lambda > 0$, the following inequalities hold:*

$$\sup\{w(\mu, x, y) : \mu > \lambda\} \leq w(\lambda, x, y) \leq \inf\{w(\mu, x, y) : 0 < \mu < \lambda\}, \tag{3.3}$$

$$\inf\{w(\mu, x, y) : 0 < \mu < \lambda\} \leq \sup\{w(\gamma, x, y) : \gamma > \mu\} \leq w(\mu, x, y) \tag{3.4}$$

and

$$w(\mu, x, y) \leq \inf\{w(\gamma, x, y) : 0 < \gamma < \mu\}. \tag{3.5}$$

Let w be a modular quasi-pseudometric on a nonempty set X. Then for any $x \in X$, we define the set $X_w^0(x)$ by

$$X_w^0(x) = \{y \in X : \lim_{\lambda \to \infty} w(\lambda, x, y) = 0 = \lim_{\lambda \to \infty} w^t(\lambda, x, y)\}.$$

Consider an element $x_0 \in X$. We set $X_w = X_w^0(x_0)$ and call the set X_m a w-modular set.
Let us fix an element $x_0 \in X$. The set

$$X_w^* \equiv X_w^*(x_0) = \{x \in X : w(\lambda, x, x_0) < \infty \text{ and } w(\lambda, x_0, x) < \infty \text{ for some } \lambda > 0\}$$

is also a *w-modular set* (around x_0) and x_0 is the *center* of X_w^*. It is easy to see that $X_w \subseteq X_w^*$. This is not new since it has been observed in the symmetric case (see [17, Section 3.1]).

Remark 3.1.4. *If w is a modular quasi-pseudometric on a nonempty set X and $x_0 \in X$, then we observe that $X_w^*(x_0) = X_{w^s}^*(x_0)$.*

Proof. ($\Rightarrow$) If $x \in X_{w^s}^*$, then $w^s(\lambda, x, x_0) < \infty$ for some $\lambda > 0$. From inequalities (3.1) and (3.2) we have $w(\lambda, x, x_0) < \infty$ and $w(\lambda, x_0, x) < \infty$ for some $\lambda > 0$. Thus $x \in X_w^*(x_0)$.
($\Leftarrow$) If $y \in X_w^*(x_0)$, then we have $w(\lambda, y, x_0) < \infty$ and $w(\lambda, x_0, y) < \infty$ for some $\lambda > 0$. Since

$$w^s(\lambda, y, x_0) = \max\{w(\lambda, y, x_0), w(\lambda, x_0, y)\},$$

we have $w^s(\lambda, y, x_0) < \infty$ for some $\lambda > 0$. Hence $y \in X_{w^s}^*(x_0)$. □

Theorem 3.1.2. *(compare Theorem 2.2.2) Consider a nonempty set X and a modular quasi-metric w on X. Then the function q_w on X defined by*

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \le \lambda\}$$

whenever $x, y \in X$, is an extended T_0-quasi-metric on X. Moreover, the restriction of q_w to X_w (or X_w^*) is a T_0-quasi-metric on X_w (or X_w^*).

Proof. The proof is similar to that of Theorem 2.2.2. The only difference here is the symmetry axiom of a metric is not satisfied by q_w because $w(\lambda, x, y) \neq w(\lambda, y, x)$ whenever $x, y \in X_w$. □

Remark 3.1.5. *If w is a modular quasi-metric on a set X, then we observe that for any $x, y \in X_w$ we have*

$$q_{w^t}(x, y) = \inf\{\lambda > 0 : w^t(\lambda, x, y) \le \lambda\}$$

$$= \inf\{\lambda > 0 : w(\lambda, y, x) \leq \lambda\}$$

$$= q_w(y, x) = (q_w)^t(x, y).$$

Furthermore, if $x, y \in X$, then $q_w(x, y) \in [0, \infty]$ and in this case q_w is an extended T_0-quasi-metric.

Example 3.1.1. *Let (X, q) be a quasi-pseudometric space. If $p \geq 0$, then the function $w(\lambda, x, y) = \dfrac{q(x, y)}{\lambda^p}$ whenever $\lambda > 0$ and $x, y \in X$ is modular quasi-pseudometric on X. It is readily checked that $X_w = X_w^* = X$ and*

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$$

$$= \inf\{\lambda > 0 : \frac{q(x, y)}{\lambda^p} \leq \lambda\}$$

$$= \inf\{\lambda > 0 : q(x, y) \leq \lambda^{p+1}\}.$$

Now taking the infimum we get $q_w(x, y) = [q(x, y)]^{\frac{1}{p+1}}$.

The following lemma is a nonsymmetric version of Theorem 2.2.3 and it does not need to be proved again in our context, because the symmetry axiom of a modular metric does not have an effect on the proof of [16, Theorem 2.2.11]. Therefore, this result still holds in our context.

Lemma 3.1.3. *(compare Theorem 2.2.3) Consider a nonempty set X and w a modular quasi-metric on X. If $\lambda > 0$ and $x, y \in X_w$, then we have:*

1. *if $q_w(x, y) < \lambda$, then $\inf\{w(\mu, x, y) : 0 < \mu < \lambda\} \leq q_w(x, y) < \lambda$ and if $\inf\{w(\mu, x, y) : 0 < \mu < \lambda\} < \lambda$, then $q_w(x, y) < \lambda$;*

2. *if $q_w(x, y) > \lambda$, then $\sup\{w(\mu, x, y) : \mu > \lambda\} \geq q_w(x, y) > \lambda$, and if $\sup\{w(\mu x, y) : \mu > \lambda\} > \lambda$, then $q_w(x, y) > \lambda$;*

3. *$q_w(x, y) = \lambda$ if and only if $\sup\{w(\mu, x, y) : \mu > \lambda\} \leq \lambda \leq \inf\{w(\mu, x, y) : 0 < \mu < \lambda\} < \lambda$.*

3.2 Convergence in modular quasi-pseudometric spaces

In this section we study convergence in modular quasi-pseudometric spaces. We study convergence induced by a quasi-metric that is induced by a mod-

ular quasi-pseudometric and also study convergence induced by a modular quasi-pseudometric.
The following definition can be compared to Definition 2.3.1.

Definition 3.2.1. *Let w be a modular quasi-pseudometric on a set X. We say that a sequence (a_n) in X_w is q_w-convergent to $a \in X_w$ if*

$$\lim_{n\to\infty} q_w(a_n, a) = 0.$$

In the same way as in modular metric settings (see for instance [16, Theorem 4.1.1]), one can show by analogy that

$$\lim_{n\to\infty} q_w(a_n, a) = 0 \quad \text{if and only if} \quad w(\lambda, a_n, a) = 0 \quad \text{whenever} \quad \lambda > 0.$$

It is well-known that for nonsymmetric distance functions there are several completeness concepts. Here, we are going to discuss the case of $(q_w)^s$-Cauchy sequences and left (q_w)-Cauchy sequences and right (q_w)-Cauchy sequences.

From [16, Theorem 4.1.1 (b)], it is easy to see that if (a_n) is a sequence in X_w, then for all $\lambda > 0$, we have:

$$(a_n) \text{ is } (q_w)^s\text{–Cauchy if and only if } \lim_{n,m\to\infty} w^s(\lambda, a_n, a_m) = 0 \text{ whenever } \lambda > 0.$$

If (X, q) is a quasi-pseudometric space, then a sequence (x_n) in X is called left q-Cauchy if for every $\epsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $q(x, x_n) < \epsilon$ and similarly a sequence (x_n) in X is called right q-Cauchy if for every $\epsilon > 0$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $q(x_n, x) < \epsilon$ (see for instance [22]).

Proposition 3.2.1. *Let w be a modular quasi-pseudometric on a set X and (x_n) be a sequence in X_w. Then (x_n) is left q_w-Cauchy if and only if for any $\epsilon > 0$, there exist $x \in X_w$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $\lim_{n\to\infty} w(\lambda, x, x_n) = 0$ whenever $\lambda > 0$.*

Proof. $\Rightarrow$ Let $\lambda > 0$ and for any $\epsilon > 0$. Suppose that (x_n) is left q_w-Cauchy. Then there exist $x \in X_w$ and $n_0 \in \mathbb{N}$ such that $q_w(x, x_n) < \epsilon$. Since $q_w(x, x_n) = \inf\{\lambda > 0 : w(\lambda, x, x_n) \leq \lambda\}$, we have two cases $(0 < \epsilon < \lambda$ or $\epsilon \geq \lambda)$.

Case 1. If $0 < \epsilon < \lambda$, then $w(\lambda, x, x_n) \leq \epsilon$ whenever $n \geq n_0$.
Moreover,
$w(\lambda, x, x_n) \leq w(\epsilon, x, x_n) < \epsilon < \lambda$ whenever $n \geq n_0$. Hence

$$\lim_{n\to\infty} w(\lambda, x, x_n) = 0 \quad \text{whenever} \quad \lambda > 0.$$

Thus, for arbitrary $\epsilon > 0$, there exist $x \in X_w$ and $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$,

$$\lim_{n\to\infty} w(\lambda, x, x_n) = 0.$$

Case 2. If $\epsilon \geq \lambda$. Suppose $\lambda = \frac{\epsilon}{2}$. It follows that

$$w(\lambda, x, x_n) \leq w\left(\frac{\lambda}{2}, x, x_n\right) \leq \frac{\lambda}{2} < \lambda \leq \epsilon$$

whenever $\lambda > 0$. Hence, for an arbitrary $\epsilon > 0$, there exist $x \in X_w$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ such that

$$\lim_{n\to\infty} w(\lambda, x, x_n) = 0.$$

$\Leftarrow$ Let ϵ >0 and $\lim_{n\to\infty} w(\epsilon, x, x_n) = 0$ for some $x \in X_w$. Then there exists $n_0 \in \mathbb{N}$ such that $w(\epsilon, x, x_n) \leq \epsilon$ whenever $n \geq n_0$. Thus $q_w(x, x_n) < \epsilon$ whenever $n \geq n_0$. Hence (x_n) is left q_w-Cauchy.

□

Corollary 3.2.2. *Let w be a modular quasi-pseudometric on a set X and (x_n) be a sequence in X_w. Then (x_n) is right q_w-Cauchy if and only if for any $\epsilon > 0$, there exist $x \in X_w$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $\lim_{n\to\infty} w(\lambda, x_n, x) = 0$ whenever $\lambda > 0$.*

Proof. The proof follows by duality from the proof of Proposition 3.2.1. □

Definition 3.2.2. *(compare Definition 2.5.1) Let w be a modular quasi-pseudometric on a nonempty set X and (x_n) be a sequence in X_w^*. We say that the sequence (x_n) is w-convergent to $x \in X_w^*$ (denoted by $(x_n) \xrightarrow{w} x$) if*

$$\lim_{n\to\infty} w(\lambda, x_n, x) = 0 \quad \textit{for some} \quad \lambda > 0.$$

Using the transpose modular quasi-pseudometric w^t of the modular quasi-pseudometric w, one defines the w^t-convergence in the similar way.

Proposition 3.2.3. *Let w be a modular quasi-pseudometric on a set X and (x_n) a sequence in X_w^*.*

1. *If (x_n) is w-convergent to $x \in X_w^*$ and (x_n) is w^t-convergent to $y \in X_w^*$, then $w(\lambda, y, x) = 0$ for some $\lambda > 0$.*

2. *If (x_n) is w-convergent to $x \in X_w^*$ and $w(\lambda, x, y) = 0$ whenever $\lambda > 0$, then (x_n) is also w-convergent to $y \in X_w^*$.*

Proof. 1. Suppose that (x_n) is w-convergent to x, then for some $\mu > 0$, we have
$w(\mu, x_n, x) \to 0$ as $n \to \infty$. Moreover, for some $\nu > 0$, we have $w^t(\nu, x_n, y) = w(\nu, y, x_n) \to 0$ as $n \to \infty$ since (x_n) is w^t-convergent to y. Then we have

$$w(\nu + \mu, y, x) \leq w(\nu, y, x_n) + w(\mu, x_n, x).$$

For some $\lambda > 0$ such that $\lambda > \mu + \nu$. It follows that

$$w(\lambda, y, x) \leq w(\nu + \mu, y, x) \leq w(\nu, y, x_n) + w(\mu, x_n, x). \tag{3.6}$$

Thus $w(\lambda, y, x) = 0$ by letting $n \to \infty$ in the inequality (3.6).

2. Suppose (x_n) is w-convergent to $x \in X_w^*$ and $w(\lambda, x, y) = 0$ whenever $\lambda > 0$. Then $w(\mu, x_n, x) \to 0$ as $n \to \infty$. For some ν such that $\nu > \lambda + \mu$, we have

$$w(\nu, x_n, y) \leq w(\mu, x_n, x) + w(\lambda, x, y) = w(\mu, x_n, x) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Therefore, the sequence (x_n) is w-convergent to y. □

Corollary 3.2.4. *Let w be a modular quasi-pseudometric on a set X and (x_n) a sequence in X_w^*. If (x_n) is w^t-convergent to $x \in X_w^*$ and $w(\lambda, y, x) = 0$ whenever $\lambda > 0$, then (x_n) is also w^t-convergent to $y \in X_w^*$.*

3.3 Topologies

In this section, present our investigations on topologies induced by a modular quasi-pseudometric w on a nonempty set X. We extend ideas from Section 2.6 to our general context.
Consider a modular quasi-pseudometric w on a set X . Let $\lambda, \mu > 0$ and $x \in X_w^*$. Then the set $B^w_{\lambda,\mu}(x)$ defined by

$$B^w_{\lambda,\mu}(x) = \{y \in X_w^* : w(\lambda, x, y) < \mu\}$$

is called the $w_<$-entourage about x relative to λ and μ and

$$C^w_{\lambda,\mu}(x) = \{y \in X_w^* : w(\lambda, x, y) \leq \mu\}$$

is called the $w_\leq$-*entourage*. Moreover, $w^t_<$-entourage and $w^t_\leq$-entourage are defined dually.

Example 3.3.1. *Let $\mathbb{R}$ be equipped with modular quasi-metric w defined by*

$$w(\lambda, x, y) = \begin{cases} \infty & \text{if } x > y, \\ 0 & \text{if } x \leq y \end{cases}$$

for all $x, y \in \mathbb{R}$ and $\lambda > 0$.
It is readily checked that $X_w = X_w(x_0) = \{x_0\}$, where $x_0 \in \mathbb{R}$ and $q_w(x, y) = 0$. Moreover, for any $r > 0$, we have

$$B_{q_w}(x_0, r) = \{y \in X_w : 0 = q_w(x, y) < r\} = \{x_0\}$$

and

$$B^w_{r,r}(x_0) = \{y \in X_w : w(r, x_0, y) \leq r\} = \{x_0\} = B^{w^s}_{r,r}(x_0) = C^w_{r,r}(x_0).$$

Lemma 3.3.1. *Let w be a modular quasi-pseudometric on X. Then for any $x \in X_w^*$ and $\lambda, \mu > 0$, we have*

$$B^{w^s}_{\lambda,\mu}(x) \subset B^w_{\lambda,\mu}(x) \tag{3.8}$$

and

$$B^{w^s}_{\lambda,\mu}(x) \subset B^{w^t}_{\lambda,\mu}(x), \tag{3.9}$$

with similar inclusions for the $w_{\leq}$-entourages.

Proof. Let $y \in B^{w^s}_{\lambda,\mu}(x)$. Then $w(\lambda, x, y) \leq w^s(\lambda, x, y) < \mu$ by the inequality (3.1). Hence $y \in B^w_{\lambda,\mu}(x)$. The other inclusion is obtained by duality and inequality (3.2). $\square$

Remark 3.3.1. *Let w be a modular quasi-pseudometric on X and let $x \in X$.*

1. *For all $\lambda, \mu > 0$, $x \in B^w_{\lambda,\mu}(x)$ and*

$$B^{w^s}_{\lambda,\mu}(x) = B^w_{\lambda,\mu}(x) \cap B^{w^t}_{\lambda,\mu}(x). \tag{3.10}$$

2. *If $0 < \lambda_1 < \lambda < \lambda_2$, then we have*

$$B^w_{\lambda_1^+,\mu}(x) \subset B^w_{\lambda^-,\mu}(x) \subset B^w_{\lambda,\mu}(x) \subset B^w_{\lambda^+,\mu}(x) \subset B^w_{\lambda_2^-,\mu}(x)$$

where,

$$B^w_{\lambda^+,\mu} := \{y \in X_w : \sup\{w(\nu, x, y) : \nu > \lambda\} < \mu\}$$

and

$$B^w_{\lambda^-,\mu} := \{y \in X_w : \inf\{w(\nu, x, y) : 0 < \nu < \lambda\} < \mu\}.$$

3. *If $0 < \mu < \lambda$, then*

$$B^w_{\mu,\mu}(x) \subset B^w_{\mu,\lambda}(x) \subset B_{q_w}(x, \lambda) = B^w_{\lambda^-,\lambda}(x) \subset B^w_{\lambda,\lambda}(x) \tag{3.11}$$

where $B_{q_w}(x, \lambda)$ denotes an open ball (with centre x and radius λ) induced by quasi-pseudometric defined on a modular set X^*_w.

The following properties of $w_<$-entourages are not new since they have been already observed in [16] for modular metrics. Therefore, we omit to prove them and because their proofs do not depend on the symmetry axiom of a modular pseudometric.

Lemma 3.3.2. *(compare Remark 2.6.1) Let w be a modular quasi-pseudometric on X and x be in X^*_w. Then we have:*

1. *If $0 < \mu_1 \leq \mu_2$, then $B^w_{\lambda,\mu_1}(x) \subset B^w_{\lambda,\mu_2}(x)$ whenever $\lambda > 0$.*

2. If $0 < \mu < \lambda$, then $B^w_{\mu,\mu}(x) \subset B^w_{\mu,\lambda}(x) \subseteq B_{q^o_w}(x,\lambda) \subset B^w_{\lambda,\lambda}(x)$.

Example 3.3.2. *Let us consider the quasi-pseudometric* $w(\lambda,x,y) = \dfrac{q(x,y)}{\lambda^p}$ *whenever* $\lambda > 0$ *and* $x, y \in X$ *in Example 3.1.1 where* (X,q) *is quasi-pseudometric space. If* $x \in X$ *and* $\lambda, \mu > 0$, *then let us compute* $B^w_{\mu,\lambda}(x)$ *and* $B_{q_w}(x,r)$ *with* $r > 0$.
Indeed,

$$\begin{aligned} B_{q_w}(x,r) &= \{y \in X^*_w : q_w(x,y) < r\} = \{y \in X : [q(x,y)]^{\frac{1}{p+1}} < r\} \\ &= \{y \in X^*_w : q(x,y) < r^{p+1}\} \\ &= B_q(x, r^{p+1}) \end{aligned}$$

and

$$\begin{aligned} B^w_{\mu,\lambda}(x) &= \{y \in X : w(\lambda,x,y) < \mu\} = \left\{ y \in X : \frac{q(x,y)}{\lambda^p} < \mu \right\} \\ &= \{y \in X : q(x,y) < \mu\lambda^p\} \\ &= B_q(x, \mu\lambda^p). \end{aligned}$$

If $\mu = \dfrac{r^{p+1}}{\lambda^p}$, *then* $B^w_{\mu,\lambda}(x) = B_{q_w}(x,r)$.

Example 3.3.3. *Consider a quasi-pseudometric space* (X,q). *If we equipped* X *with the modular quasi-pseudometric* $w(\lambda,x,y) = \dfrac{q(x,y)}{\lambda^p}$ *for all* $x, y \in X$ *and* $\lambda > 0$, *where* p *is strictly positive constant. Then it follows that* $X_w = X$ *and* $q_w(x,y) = [q(x,y)]^{\frac{1}{p+1}}$.
Furthermore, for any $\lambda > 0$, *we have*

$$\begin{aligned} B_{q_w}(x,\lambda) &= \left\{ y \in X_w : [q(x,y)]^{\frac{1}{p+1}} < \lambda \right\} \\ &= \left\{ y \in X_w : \frac{q(x,y)}{\lambda^p} < \lambda \right\} \\ &= \{y \in X_w : w(\lambda,x,y) < \lambda\} = B^w_{\lambda,\lambda}(x). \end{aligned}$$

Since $w^s(\lambda,x,y) = \frac{q^s(x,y)}{\lambda^p}$, *it follows that*

$$B^{w^s}_{\lambda,\lambda} = B_{(q_w)^s}(x,y) \subseteq B_{q_w}(x,\lambda) = B^w_{\lambda,\lambda}(x).$$

Lemma 3.3.3. *Let* w *be a modular quasi-pseudometric on* X *and* $x \in X_w$.

Then, for all $\lambda > 0$ we have

1. $B_{q_w}(x, \lambda) \subseteq B^w_{\lambda,\lambda}(x)$,
2. $C_{q_w}(x, \lambda) \subseteq C^w_{\lambda,\lambda}(x)$.

Proof. Let $z \in B_{q_w}(x, \lambda)$, then $q_w(x, z) < \lambda$. By definition of the quasi-pseudometric q_w, it follows that $w(\mu, x, z) \leq \mu < \lambda$ for all $\mu > 0$. Since $\lambda - \mu > 0$, then

$$\begin{aligned} w(\lambda, x, z) &= w(\lambda - \mu + \mu, x, z) \\ &\leq w(\lambda - \mu, x, x) + w(\mu, x, z) \\ &= w(\mu, x, z) \leq \mu < \lambda. \end{aligned}$$

Hence $z \in B^w_{\lambda,\lambda}(x)$.
The other inclusion follows by a similar argument. □

Note that in Lemma 3.3.3, $B_{q_w}(x, \lambda)$ denotes a q_w-open ball with centre x and radius λ and $C_{q_w}(x, \lambda)$ denotes a q_w-closed ball with centre x and radius λ .
Every modular quasi-pseudometric induces a topology denoted by $\tau(w)$ on X^*_w and it is defined exactly as in modular metric (see and compare Section 2.6).

Definition 3.3.1. *(compare Definition 2.6.3) Let w be a modular quasi-pseudometric on X. We say that a nonempty subset $V \subseteq X$ belongs to $\tau(w)$ (or V is $\tau(w)$-open) if there exists $\mu(x, \lambda) > 0$ such that $B^w_{\lambda,\mu}(x) \subset V$ whenever $x \in V$ and $\lambda > 0$.*

Remark 3.3.2. *For $x \in X$ and $\lambda > 0$, we point out that the $w_<$-entourage $B^w_{\lambda,\mu}(x)$ is not $\tau(w)$-open in general. This fact has been observed in the case of modular pseudometrics by Chistyakov.*

Lemma 3.3.4. *Let w be a modular quasi-pseudometric on X. Then $V \in \tau(w)$ if and only if for any $x \in V$, there exists $\mu = \mu(x) > 0$ such that $B^w_{\mu,\mu}(x) \in V$.*

Proof. This follows from (3.11). □

Theorem 3.3.5. *Let w be a modular quasi-pseudometric on X. Then we have*

1. $\tau(w) \subset \tau(q_w)$,

2. $\tau(w) \subset \tau(q_w^t)$.

Moreover, we have $\tau(w) \subset \tau(q_w^s)$.

Proof. We only prove (1). The proof of (2) will follow by similar arguments. Let $V \in \tau(w)$ and $\lambda = 1$. Then for any $x \in X$, there exists $\mu_0 = \mu_0(x, 1) > 0$ such that $B^w_{1,\mu_0}(x) \subset V$. Let $\mu = \min\{1, \mu_0\}$. Then by (3.11) we have $B^w_{\mu,\mu} \subset B^w_{1,\mu_0} \subset V$ and by Lemma 3.3.4, it follows that $V \in \tau(q_w)$.
Furthermore, since the topology induced by q_w^s is finer than the topologies induced by $\tau(q_w)$ and $\tau(q_w^t)$, we then have $\tau(w) \subset \tau(q_w^s)$. $\square$

Using the transpose modular quasi-pseudometric w^t, one obtains another topology $\tau(w^t)$ generated by the transpose modular quasi-pseudometric w^t. The third one is the topology $\tau(w^s)$ generated by the modular pseudometric w^s and it is called *modular topology* in X_w^* in the sense of Definition 2.6.3.

Remark 3.3.3. *The w-modular set X_w^* is a space with two topology $\tau(w)$ and $\tau(w^t)$. Then X_w^* can be seen a bitopological space in the sense of Kelly [33]. Therefore, any result related to a bitopological space can be applied to $(X_w^*, \tau(w), \tau(w^t))$.*

Proposition 3.3.6. *Let w be a modular quasi-pseudometric on X. Then the topology $\tau(w^s)$ is finer than the topologies $\tau(w)$ and $\tau(w^t)$. This means that:*

1. *any $\tau(w)$-open set is $\tau(w^s)$-open, similarly any $\tau(w^t)$-open set is $\tau(w^s)$-open,*

2. *the identity map $(X_w^*, \tau(w^s))$ to $(X_w^*, \tau(w))$ and to $(X_w^*, \tau(w^t))$ are continuous,*

3. *a sequence (x_n) in X_w^* is $\tau(w^s)$-convergent to $x \in X_w^*$ if and only if (x_n) is $\tau(w)$-convergent to x and $\tau(w^t)$-convergent to x.*

Proof. We prove assertions 1 and 3 and we leave the assertion 2 since it follows from 1.

1. Let U be a $\tau(w)$-open. Then for any $x \in U$ and $\lambda > 0$, there exists $\mu > 0$ such that $B^{w}_{\lambda,\mu}(x) \subseteq U$. By inclusion (3.8), we have $B^{w^s}_{\lambda,\mu}(x) \subseteq U$. Hence U is $\tau(w^s)$-open. Analogously, one shows that any $\tau(w^t)$-open is $\tau(w^s)$-open.

3. Suppose (x_n) is $\tau(w^s)$-convergent to x. Let $x \in U \in \tau(w)$. From 1 it follows that $U \in \tau(w^s)$. Then there exists $n_0 \in \mathbb{N}$ such that (x_n) in U for all $n \geq n_0$ since (x_n) is $\tau(w^s)$-convergent to x. Thus (x_n) is $\tau(w)$-convergent to x. By similar arguments (x_n) is $\tau(w^t)$-convergent to x.
   Conversely, let $x \in U = A \cap B \in \tau(w^s)$ with $A \in \tau(w)$ and $B \in \tau(w^t)$ since A and B belong to $\tau(w^s)$. Then there is $n_0^{'} \in \mathbb{N}$ such that (x_n) in A for all $n \geq n_0$ since (x_n) is $\tau(w)$-convergent to x.
   Similarly, we have that there is $n_0'' \in \mathbb{N}$ such that (x_n) B for all $n \geq n_0''$ since (x_n) is $\tau(w^t)$-convergent to x. Let $n_0 = \max\{n_0', n_0''\}$. Then for all $n \geq n_0$ with have that (x_n) in $A \cap B = U \in \tau(w^s)$. Hence (x_n) is $\tau(w^s)$-convergent x.

□

Proposition 3.3.7. *If w is a modular quasi-metric on a set X, then the topologies $\tau(w)$ and $\tau(w^t)$ are T_0 on X^*_w.*

Proof. Let $x, y \in X^*_w$ with $x \neq y$. Since w is a modular quasi-metric, we have $\max\{w(\lambda, x, y), w(\lambda, y, x)\} > 0$ whenever $\lambda > 0$.
If $w(\lambda, x, y) > 0$. Then suppose that $x \in U \in \tau(w)$. It follows that $B^w_{\lambda,\mu}(x) \in U$ for some $\mu > 0$ and whenever $\lambda > 0$. Moreover, $0 < \mu' < w(\lambda, x, y) < \mu$, where $\mu' = \dfrac{1}{2}w(\lambda, x, y)$.
Thus $w(\lambda, x, y) > \mu'$ whenever $0 < \mu' < \mu$ and $\lambda > 0$. Hence by Lemma 3.3.2, we have

$$y \notin B^w_{\lambda,\mu'}(x) \subset B^w_{\lambda,\mu}(x) \subset U.$$

If $w(\lambda, y, x) > 0$. Let $y \in V \in \tau(w)$. Then $B^w_{\lambda,\mu}(y) \in V$ for some $\mu > 0$ and whenever $\lambda > 0$. By duality we have

$$x \notin B^w_{\lambda,\mu''}(y) \subset B^w_{\lambda,\mu}(x) \subset V$$

whenever $0 < \mu'' < \mu$ and $\lambda > 0$ where, $\mu'' = \dfrac{1}{2}w(\lambda, y, x)$. Therefore, the topology $\tau(w)$ is T_0. Using similar arguments, one can show that the topology $\tau(w^t)$ is T_0. $\square$

3.4 Hausdorff quasi-pseudometric modular

Let w be modular quasi-pseudometric on a nonempty set X. Let $\mathcal{P}(X)$ be the power set of X. We assume that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.
If $A, B \in \mathcal{P}(X)$ and $\lambda > 0$, then we set

$$\Phi^w_\lambda(A, B) =: \sup_{x \in A} \inf_{y \in B} w(\lambda, x, y).$$

It follows that $\Phi^w_\lambda \in [0, \infty]$.
In general $\Phi^w_\lambda(A, B) \neq \Phi^w_\lambda(B, A)$ and since $w(\lambda, x, y) \neq w^t(\lambda, x, y)$ whenever $x, y \in X$ and $\lambda > 0$. These are a motivation to define the function $\Phi^{w^t}_\lambda$ in the following way

$$\Phi^{w^t}_\lambda(A, B) =: \Phi^w_\lambda(B, A)$$

whenever $A, B \in \mathcal{P}(X)$ and $\lambda > 0$.
Moreover, we set

$$\Phi^w_\lambda(\emptyset, B) = 0 \quad \text{whenever} \quad \lambda > 0 \quad \text{and} \quad B \subset X$$

and

$$\Phi^w_\lambda(A, \emptyset) = \infty \quad \text{whenever} \quad \lambda > 0 \quad \text{and} \quad \emptyset \neq A \subset X.$$

Remark 3.4.1. *We observe that $\Phi^{w^t}_\lambda(\emptyset, B) = \infty$ whenever $\emptyset \neq B \subset X$ and $\lambda > 0$ and $\Phi^{w^t}_\lambda(A, \emptyset) = 0$ whenever $A \subset X$ and $\lambda > 0$. Note for any $\lambda > 0$, we have $\Phi^w_\lambda(\emptyset, \emptyset) = 0 = \Phi^{w^t}_\lambda(\emptyset, \emptyset)$.*

Proposition 3.4.1. *(compare Proposition 2.7.1) Let w be a modular quasi-pseudometric on a set X. Then the function $\Phi^w : (0, \infty) \times \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ defined by $\Phi^w(\lambda, A, B) = \Phi^w_\lambda(A, B)$ whenever $\lambda > 0$ and $A, B \in \mathcal{P}(X)$ is well defined and satisfies the following properties:*

1. *$\Phi^w(\lambda, A, B) = 0$ whenever $\lambda > 0$ and $A \subseteq B \subset X$;*
2. *$\Phi^w(\lambda + \mu, A, B) \leq \Phi^w(\lambda, A, C) + \Phi^w(\mu, C, B)$ whenever $\lambda, \mu > 0$ and A, B and C are subsets of X.*

Proof. 1. Case 1. If $A=\emptyset$, then from the assumption that $\sup\emptyset=0$, we have $\Phi^w(\lambda,\emptyset,B)=0$ for all $\lambda>0$ and $B\subset X$.

Case 2. Suppose that $A\neq\emptyset$. Given $x\in A$, then $x\in B$ since $A\subseteq B\subset X$. Then we have $0\leq \inf_{y\in B} w(\lambda,x,y)\leq w(\lambda,x,x)=0$. Now since $x\in A$ is arbitrary, then $\Phi^w(\lambda,A,B)=\sup_{x\in A}\inf_{y\in B} w(\lambda,x,y)=0$ for all $\lambda>0$ and $A\subseteq B\subset X$.

2. Case 1. If atleast one of the sets A,B or C is empty, we have the following:

(a) if $A=\emptyset$, then we have

$$0\leq 0+\Phi^w(\lambda,C,B).$$

(b) if $B=\emptyset$, then we have

$$\infty\leq \Phi^w(\lambda,A,C)+\infty.$$

(c) if $C=\emptyset$, then we have

$$\Phi^w(\lambda+\mu,A,B)\leq \infty+0.$$

Case 2. We assume that none of the sets A,B and C is empty. Given $x\in A, z\in C$ and $\lambda,\mu>0$, then by triangle inequality we get that

$$\inf_{y\in B} w(\lambda+\mu,x,y)\leq w(\lambda+\mu,x,y_1)\leq w(\lambda,x,z)+w(\mu,z,y_1)\qquad \text{for all } y_1\in B.$$

Taking the infimum over all $y_1\in B$, we get

$$\begin{aligned}\inf_{y\in B} w(\lambda+\mu,x,y)&\leq w(\lambda,x,z)+\inf_{y_1\in B} w(\mu,z,y_1)\\ &\leq w(\lambda,x,z)+\Phi^w(\mu,C,B).\end{aligned}$$

Now taking the infimum over all $z\in C$, we get

$$\inf_{y\in B} w(\lambda+\mu,x,y)\leq \inf_{z\in C} w(\lambda,x,z)+\Phi^w(\mu,C,B).$$

Now taking the supremum over all $x \in A$, we get

$$\sup_{x\in A}\inf_{y\in B} w(\lambda+\mu, x, y) \leq \sup_{x\in A}\inf_{z\in C} w(\lambda, x, z) + \Phi^w(\mu, C, B)$$

hence

$$\Phi^w(\lambda+\mu, A, B) \leq \Phi^w(\lambda, A, C) + \Phi^w(\mu, C, B)$$

whenever $\lambda, \mu > 0$ and A, B and C are subsets of X.

$\square$

Definition 3.4.1. *Let w be a modular quasi-pseudometric on a set X. Then we define the function $W_w : (0, \infty) \times \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ by*

$$W_w(\lambda, A, B) = \Phi^w_\lambda(A, B) \quad \textit{whenever} \quad \lambda > 0 \quad \textit{and} \quad A, B \in \mathcal{P}(X).$$

Corollary 3.4.2. *If w is a modular quasi-pseudometric on a set X, then the function W_w in Definition 3.4.1 satisfies the following properties:*

1. *$W_w(\lambda, A, A) = 0$ whenever $\lambda > 0$ and $A \subset X$;*
2. *$W_w(\lambda+\mu, A, B) \leq W_w(\lambda, A, C) + W_w(\mu, C, B)$ whenever $\lambda, \mu > 0$ and A, B and C are subsets of X.*

Proof. Follows from Proposition 3.4.1. $\square$

Definition 3.4.2. *If w is a modular quasi-pseudometric on a set X. Then with respect to Lemma 3.4.2 the function W_w is a modular quasi-pseudometric on $\mathcal{P}(X)$. Thus we call W_w the w-Hausdorff modular quasi-pseudometric induced by w.*

Let W_w be the w-Hausdorff modular quasi-pseudometric induced by the modular quasi-pseudometric w on a set X. Then the function $W^t_w : (0, \infty) \times \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty]$ defined by

$$W^t_w(\lambda, A, B) = \Phi^{w^t}_\lambda(A, B) \quad \text{whenever} \quad \lambda > 0 \quad \text{and} \quad A, B \subset X$$

is also a modular quasi-pseudometric on $\mathcal{P}(X)$ that we call w^t-Hausdorff modular quasi-pseudometric induced by w^t. Furthermore, we note that the function W_w^s defined by

$$W_w^s(\lambda, A, B) = \max\{W_w(\lambda, A, B), W_w^t(\lambda, A, B)\}$$

whenever $A, B \subset X$ and $\lambda > 0$ is a modular pseudometric on $\mathcal{P}(X)$. Note that the function W_w^s coincides with the Hausdorff modular pseudometric in the sense of Definition 2.7.3.

Remark 3.4.2. *Let w be a modular quasi-pseudometric on a set X. Then it is easy to see that for any $\lambda > 0$ and any $A, B \subset X$, we have*

$$W_w^t(\lambda, A, B) = W_{w^t}(\lambda, A, B).$$

Furthermore, $W_w(\lambda, A, \emptyset) = 0$ whenever $\emptyset \neq A \subset X$ and $W_w(\lambda, \emptyset, B) = \infty$ whenever $B \subset X$.

If w is a modular quasi-pseudometric on a set X and W_w is the w-Hausdorff modular quasi-pseudometric induced by w, then we denote by $Q_{q_w}^-$ the lower Hausdorff quasi-pseudometric of q_w on $\mathcal{P}_0(X)$ and $Q_{q_w}^+$ the upper Hausdorff quasi-pseudometric of q_w on $\mathcal{P}_0(X)$.

Theorem 3.4.3. *Let w be a modular quasi-pseudometric on a set X and W_w be the w-Hausdorff modular quasi-pseudometric induced by w on X. Then we have that*

$$Q_{q_w}^-(A, B) = q_{W_w}(A, B) \quad \textit{whenever} \quad A, B \in \mathcal{P}_0(X).$$

Proof. $\Leftarrow$ Suppose that

$$q_{W_w}(A, B) = \inf\{\lambda > 0 : W_w(\lambda, A, B) \leq \lambda\} < \infty \quad \text{and} \quad q_{W_w}(A, B) < \lambda$$

for any $A, B \in \mathcal{P}_0(X)$. Then by Lemma 3.1.3(1), we have

$$\begin{aligned} W_w(\lambda, A, B) &= \sup_{a \in A} \inf_{b \in B} w(\lambda, a, b) \\ &\leq \inf\{W_w(\mu, A, B) : \mu < \lambda\} < \lambda. \end{aligned}$$

It follows that whenever $a \in A$, there exists $b_a \in B$ such that $w(\lambda, a, b_a) < \lambda$. Hence $q_w(a, b_a) \leq \lambda$ by Theorem 3.1.2.

Moreover, $\inf\limits_{b \in B} q_w(a,b) \leq q_w(a,b_a) \leq \lambda$ whenever $a \in A$ and then

$$Q^-_{q_w}(A,B) = \sup_{a \in A} \inf_{b \in B} q_w(a,b) \leq \lambda, \text{whenever} \lambda > 0.$$

Thus $Q^-_{q_w}(A,B) \leq \lambda$ for all $\lambda > q_{W_w}(A,B)$. So

$$Q^-_{q_w}(A,B) \leq q_{W_w}(A,B). \tag{3.12}$$

$\Rightarrow$ Let $\lambda > 0$ and suppose that $Q^-_{q_w}(A,B) < \infty$ and $Q^-_{q_w}(A,B) < \lambda$. Then we have that

$$\sup_{a \in A} \inf_{b \in B} q_w(a,b) < \lambda$$

It follows that whenever $a \in A$, $\inf\limits_{b \in B} q_w(a,b) < \lambda$. Moreover, whenever $a \in A$, there exists $b_a \in B$ such that $q_w(a,b_a) < \lambda$.
Since by definition of q_w, we have $w(\lambda,a,b_a) \leq \lambda$ and $\inf\limits_{b \in B} w(\lambda,a,b) \leq \lambda$ whenever $a \in A$, then

$$W_w(\lambda,A,B) = \sup_{a \in A} \inf_{b \in B} w(\lambda,a,b) \leq \lambda.$$

Thus by definition of q_{W_w} we have

$$q_{W_w}(A,B) \leq \lambda \quad \text{for all} \quad \lambda > Q^-_{q_w}(A,B).$$

So

$$q_{W_w}(A,B) \leq Q^-_{q_w}(A,B). \tag{3.13}$$

Therefore, from (3.12) and (3.13) we have

$$Q^-_{q_w}(A,B) = q_{W_w}(A,B) \quad \text{whenever} \quad A, B \in \mathcal{P}_0(X).$$

$\square$

The following result follows by similar arguments of the above proof and Remark 3.1.5.

Corollary 3.4.4. *Let w be a modular quasi-pseudometric on a set X and W_w be the w-Hausdorff modular quasi-pseudometric induced by w on X.*

Then we have that

$$Q^+_{q_{w^t}}(A,B) = q_{W_{w^t}}(A,B) = (q_{W_w})^t(B,A) \quad \textit{whenever} \quad A,B \in \mathcal{P}_0(X).$$

Theorem 3.4.5. *Let X be a nonempty set. If we equip X with a modular quasi-pseudometric w, then for any $A,B \in \mathcal{P}_0(X)$ we have*

$$Q^-_{q_w}(A,B) + Q^-_{q_{w^t}}(A,B) = q_{W_{w^m}}(A,B)$$

where w^m is the modular pseudometric in Remark 3.1.1.

Proof. ($\Leftarrow$) Suppose that $q_{W_{w^m}}(A,B)$ is finite and $q_{W_{w^m}}(A,B) < \lambda$ for any $\lambda > 0$. Then $W_{w^m}(\lambda,A,B) < \lambda$ by definition of $q_{W_{w^m}}(A,B)$.
Hence whenever $a \in A$, there exists $b_a \in B$ such that

$$w^m(\lambda,a,b_a) = w(\lambda,a,b_a) + w^t(\lambda,a,b_a) < \lambda.$$

Furthermore, whenever $a \in A$,

$$\inf_{b\in B}\{q_w(a,b)\} + \inf_{b\in B}\{q_{w^t}(a,b)\} \leq q_w(a,b_a) + q_{w^t}(a,b_a) \leq \lambda.$$

So

$$\sup_{a\in A}\inf_{b\in B}\{q_w(a,b)\} + \sup_{a\in A}\inf_{b\in B}\{q_{w^t}(a,b)\} \leq \lambda \quad \text{for all} \quad \lambda > q_{W_{w^m}}(A,B).$$

Moreover,

$$Q^-_{q_w}(A,B) + Q^-_{q_{w^t}}(A,B) \leq \lambda \quad \text{for all} \quad \lambda > q_{W_{w^m}}(A,B).$$

Thus

$$Q^-_{q_w}(A,B) + Q^-_{q_{w^t}}(A,B) \leq q_{W_{w^m}}(A,B). \tag{3.14}$$

($\Rightarrow$) By similar arguments as in the proof of Theorem 3.4.3 we have

$$q_{W_{w^m}}(A,B) \leq Q^-_{q_w}(A,B) + Q^-_{q_{w^t}}(A,B). \tag{3.15}$$

$\square$

Chapter 4

On w-Isbell-convexity

In this chapter, we continue our investigations on a modular quasi-pseudometric spaces. We introduce the theory of Isbell-convexity in the setting of modular quasi-pseudometric spaces and we call it w-Isbell-convexity. We then study some fixed point theorems on self nonexpansive map on a modular set equipped with the structure of Isbell-convexity that are w-Isbell-convexity.

4.1 Isbell-convexity on modular quasi-pseudometric spaces

Definition 4.1.1. *Let w be a modular quasi-pseudometric on a nonempty set X. We say that X_w is w-Isbell-convex if for any family of points $(x_i)_{i\in I}$ in X_w and families of points $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ in $(0,\infty)$ such that*

$$w(\lambda_i+\mu_j, x_i, x_j) \leq \lambda_i + \mu_j, \quad \textit{for all} \quad i,j\in I,$$

then

$$\bigcap_{i\in I}[C^{w}_{\lambda_i,\lambda_i}(x_i)\cap C^{w^t}_{\mu_i,\mu_i}(x_i)] \neq \emptyset.$$

Remark 4.1.1. *Let (X,q) be an Isbell-convex T_0-quasi-metric space. If we equip X with the modular quasi-metric w_q defined by $w_q(\lambda,x,y) = q(x,y)$ for all $x,y\in X$, then its modular set $X_w(x_0)=\{x_0\}$ is w-Isbell-convex.*

Example 4.1.1. *For any T_0-quasi-metric space (X,q). If X is equipped with the modular quasi-metric $w(\lambda,x,y) = \dfrac{q(x,y)}{\psi(\lambda)}$ for all $x,y\in X$ and $\lambda>0$,* *where $\psi : (0, \infty) \to (0, \infty)$ is a bounded nondecreasing function. It follows that the modular set $X_w(x) = \{x\}$ for all $x \in X$ is w-Isbell-convex. Indeed for any $\lambda, \mu > 0$ such that $0 = w(\lambda + \mu, x, x) \leq \lambda + \mu$. It follows that $x \in B^w_{\lambda,\lambda}(x) \cap B^w_{\mu,\mu}(x) \subset C^w_{\lambda,\lambda}(x) \cap C^w_{\mu,\mu}(x)$.*

Definition 4.1.2. *Let w be a modular quasi-pseudometric on a nonempty set X. We say that X_w is w-metrically convex if for any points $x, y \in X_w$ and positive numbers λ and μ such that*

$$w(\lambda + \mu, x, y) \leq \lambda + \mu,$$

there exists $z \in X_w$ such that $w(\lambda, x, z) \leq \lambda$ and $w(\mu, z, y) \leq \mu$.

Example 4.1.2. *Let the Sorgenfrey T_0-quasi-metric u on $\mathbb{R}$ defined by*

$$u(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ 1 & \text{if } x < y \end{cases}$$

for all $x, y \in \mathbb{R}$.
If we equip $\mathbb{R}$ with the modular quasi-metric $w(\lambda, x, y) = \dfrac{u(x, y)}{\lambda}$ for any $x, y \in \mathbb{R}$ and $\lambda > 0$, then $X_w = \mathbb{R}$ is not w-metrically convex. Indeed, we have $w(1, \dfrac{1}{2}, 1) = \dfrac{u(\frac{1}{2}, 1)}{1} = 1 \leq \dfrac{1}{2} + \dfrac{1}{2}$ but there is no $z \in \mathbb{R}$ such that $w(\dfrac{1}{2}, \dfrac{1}{2}, z) = \dfrac{u(\frac{1}{2}, z)}{\frac{1}{2}} \leq \dfrac{1}{2}$ and $w(\dfrac{1}{2}, z, 1) = \dfrac{u(z, 1)}{\frac{1}{2}} \leq \dfrac{1}{2}$, if such z exists it would satisfy $z \leq \frac{1}{2}$ and $z \geq 1$.

Remark 4.1.2. *It is easy to see that if (X_w, q_w) is metrically convex, then X_w is w-metrically convex too.*

Proof. Let $x, y \in X_w$ and $\lambda, \mu > 0$ such that $w(\lambda+\mu, x, y) \leq \lambda+\mu$. It follows that $q_w(x, y) \leq \lambda+\mu$. By metrical convexity of (X_w, q_w), there exists $z \in X_w$ such that $z \in C_{q_w}(x, \lambda) \bigcap C_{(q_w)^t}(y, \mu) \subseteq C^w_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(y)$.
Hence there exists $z \in X_w$ such that $w(\lambda, x, z) \leq \lambda$ and $w(\mu, z, y) \leq \mu$. Therefore, X_w is w-metrically convex. □

Definition 4.1.3. *Let w be a modular quasi-pseudometric on a nonempty set X. A family $(C^w_{\lambda_i,\lambda_i}(x_i), C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I}$ of double $w_{\leq}$-entourages with $\lambda_i, \mu_i > 0$ and $x_i \in X_w$ for all $i \in I$ is said to have the mixed binary intersection property provided that $C^w_{\lambda_i,\lambda_i}(x_i) \bigcap C^{w^t}_{\mu_j,\mu_j}(x_j) \neq \emptyset$ for all $i, j \in I$.*

Definition 4.1.4. *Let w be a modular quasi-pseudometric on a nonempty set X. The modular set X_w is called w-Isbell-complete if every family $(C^w_{\lambda_i,\lambda_i}(x_i), C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I}$ of double $w_{\leq}$-entourages, where $\lambda_i, \mu_i > 0$ and $x_i \in X_w$ for all $i, j \in I$, having the mixed binary intersection property satisfies*

$$\bigcap_{i\in I}\left[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)\right] \neq \emptyset.$$

Proposition 4.1.1. *If w is a modular quasi-pseudometric on X, then X_w is w-Isbell-convex if and only if X_w is w-metrically-convex and w-Isbell-complete.*

Proof. ($\Longrightarrow$) Suppose that X_w is w-Isbell-convex. Let $x_1, x_2 \in X_w$ and $\lambda_1, \mu_2 > 0$ such that $w(\lambda_1 + \mu_2, x_1, x_2) \leq \lambda_1 + \mu_2$. Then, we set $\lambda_2 = \mu_1 = w(\lambda_2 + \mu_1, x_2, x_1)$. By w-Isbell-convexity of X_w, there exists

$$a \in C^w_{\lambda_1,\lambda_1}(x_1) \cap C^{w^t}_{\mu_2,\mu_2}(x_2).$$

It follows that $w(\lambda_1, x_1, a) \leq \lambda_1$ and $w(\mu_2, a, x_2) \leq \mu_2$. Hence X_w is w-metrically convex.
Consider the family $(C^w_{\lambda_i,\lambda_i}(x_i), C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I}$ of $w_{\leq}$-entourages having mixed binary intersection property.
For all $i, j \in I$, there exits

$$z \in C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_j,\mu_j}(x_j).$$

In this case we have

$$w(\lambda_i + \mu_j, x_i, x_j) \leq w(\lambda_i, x_i, z) + w(\mu_j, z, x_j) \leq \lambda_i + \mu_j.$$

Thus $\bigcap_{i\in I}[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)] \neq \emptyset$ by the w-Isbell-convexity of X_w. Therefore, X_w is w-Isbell-complete.
($\Longleftarrow$) Let X_w be w-metrically convex and w-Isbell-complete. If $(x_i)_{i\in I}$ is a family of points in X_w and $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ are families of points in $(0, \infty)$ such that

$$w(\lambda_i + \mu_j, x_i, x_j) \leq \lambda_i + \mu_j \quad \text{for all} \quad i, j \in I.$$

By w-metrical convexity of X_w there exists $z \in X_w$ such that $w(\lambda_i, x_i, z) \leq \lambda_i$

and $w(\mu_j, z, x_j) \leq \mu_j$ for all $i, j \in I$.
Then the family $(C^w_{\lambda_i,\lambda_i}(x_i), C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I}$ has the mixed binary intersection property. Since X_w is w-Isbell-complete, we have

$$\bigcap_{i\in I}\Big(C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)\Big) \neq \emptyset.$$

$\square$

Proposition 4.1.2. *Let w be a modular quasi-pseudometric on a nonempty set X. If (X_w, q_w) is an Isbell-convex quasi-pseudometric space, then X_w is w-Isbell-convex.*

Proof. Suppose (X_w, q_w) is an Isbell-convex quasi-pseudometric space. Let $(x_i)_{i\in I}$ be a family of points in X_w and $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ be families of points in $(0, \infty)$ such that

$$w(\lambda_i + \mu_j, x_i, x_j) \leq \lambda_i + \mu_j \quad \text{for all} \quad i, j \in I.$$

Then

$$q_w(x_i, x_j) = \inf\{\lambda > 0 : w(\lambda, x_i, x_j) \leq \lambda\} \leq \lambda_i + \mu_j \quad \text{for all} \quad i, j \in I.$$

By Isbell-convexity of (X_w, q_w) we have

$$\bigcap_{i\in I}\Big[C_{q_w}(x_i, \lambda_i) \cap C_{(q_w)^t}(x_i, \mu_i)\Big] \neq \emptyset.$$

Since by Lemma 3.3.3, $C_{q_w}(x_i, \lambda_i) \subseteq C^w_{\lambda_i,\lambda_i}(x_i)$ and $C_{(q_w)^t}(x_i, \mu_i) \subseteq C^{w^t}_{\mu_i,\mu_i}(x_i)$, It follows that

$$\emptyset \neq \bigcap_{i\in I}\Big[C_{q_w}(x_i, \lambda_i) \cap C_{(q_w)^t}(x_i, \mu_i)\Big] \subseteq \bigcap_{i\in I}\Big[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)\Big].$$

Therefore, X_w is w-Isbell-convex. $\square$

Example 4.1.3. *Let (X, q) be an Isbell-convex quasi-metric space. If we equip X with the modular quasi-metric $w_q(\lambda, x, y) = \dfrac{q(x, y)}{\lambda^p}$, where $p \geq 1$. Then X_w is w-Isbell-convex but (X_w, q_w) is not Isbell-convex quasi-metric space.*

Proof. Let a family $(x_i)_{i\in I}$ of point in X_w and two family of positive real numbers $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ such that

$$w(\lambda_i + \mu_j, x_i, x_j) \leq \lambda_i + \mu_j \quad \text{whenever} \quad i, j \in I.$$

Then $\dfrac{q(x_i, x_j)}{K} \leq \lambda_i + \mu_j$, where $K = \lambda_i + \mu_j$ whenever $i, j \in I$. Furthermore, whenever $i, j \in I$, we have $q(x_i, x_j) \leq K\lambda_i + K\mu_j$. Since (X, q) is Isbell-convex, then it follows that there exists $z \in X_w$ such that $q(x_i, z) \leq K\lambda_i$ and $q(z, x_j) \leq K\mu_j$ whenever $i \in I$. Hence

$$w(\lambda_i, x_i, z) = \frac{q(x_i, z)}{K} \leq \lambda_i$$

and

$$w(\mu_i, z, x_j) = \frac{q(z, x_j)}{K} \leq \mu_j$$

whenever $i, j \in I$. Thus X_w is w-metrically convex. Moreover, one shows that X_w is w-Isbell-complete. Therefore, X_w is w-Isbell-complete. But the quasi-metric $q_w(x, y) = (q(x, y))^{\frac{1}{p+1}}$ is not Isbell-convex. □

Definition 4.1.5. *Let X be a nonemptyset and w be a modular quasi-pseudometric on X. Given $x, y \in X$, we say that w is continuous from the right at each point $\lambda > 0$ whenever*

$$\lim_{\mu \to \lambda + 0} w(\mu, x, y) = \sup\{w(\mu, x, y) : \mu > \lambda\}.$$

The following has been observed in [16, Theorem 2.2.11] and its proof does not depend on the symmetry axiom of a modular pseudometric.

Remark 4.1.3. *Let w be a modular quasi-pseudometric on X. If w is continous from the right on $(0, \infty)$, then $q_w(x, y) \leq \lambda$ if and only if $w(\lambda, x, y) \leq \lambda$ for any $x, y \in X_w$ and positive number λ.*

It is natural to wonder about the converse of Proposition 4.1.2 above.

Lemma 4.1.3. *Let w be a modular quasi-pseudometric on X. If w is continuous from the right on $(0, \infty)$, then X_w is w-metrically convex if and only if (X_w, q_w) is metrically convex quasi-pseudometric space.*

Proof. ($\Longrightarrow$) Suppose that X_w is w-metrically convex. Let $x, y \in X_w$ and $\lambda, \mu > 0$ such that $q_w(x, y) \leq \lambda + \mu$.

Since w is continuous from the right on $(0,\infty)$ we have $w(\lambda+\mu,x,y)\leq\lambda+\mu$. It follows that there exists $a\in X_w$ such that $w(\lambda,x,a)\leq\lambda$ and $w(\mu,a,z)\leq\mu$. Then $q_w(x,a)\leq\lambda$ and $q_w(a,y)\leq\mu$ by the right continuity of w. So (X_w,q_w) is metrically convex.
($\Leftarrow$) Follows from Remark 4.1.2.

$\square$

Lemma 4.1.4. *Let w be a modular quasi-pseudometric on X. If w is continuous from the right on $(0,\infty)$, then X_w is w-Isbell-complete if and only if (X_w,q_w) is an Isbell-complete quasi-pseudometric space.*

Theorem 4.1.5. *Let w be a modular quasi-pseudometric on X. If w is continuous from the right on $(0,\infty)$, then X_w is w-Isbell-convex if and only if (X_w,q_w) is an Isbell-convex quasi-pseudometric space.*

Proof. Suppose that X_w is w-Isbell-convex. Then, by Proposition 4.1.1 X_w is w-Isbell-complete and w-metrically convex. Thus (X_w,q_w) is Isbell-complete and metrically convex quasi-pseudometric space by Lemma 4.1.3 and Lemma 4.1.4. Therefore, (X_w,q_w) is an Isbell-convex quasi-pseudometric space by [34, Proposition 1]. The converse follows by similar arguments. $\square$

4.2 Nonexpansive maps

In this section we study boundedness of a subset of a modular set. This concept was introduced in [2], where the authors were studying the existence of fixed points of modular contractive maps in a modular metric space. In our studies we generalise this concept to our context.

Let w be a modular quasi-pseudometric on X. We say that a nonempty subset A of X_w is *w-bounded* if there exists $x\in X_w$ such that

$$A\subseteq C^{w}_{\lambda,\lambda}(x)\cap C^{w^t}_{\mu,\mu}(x)\quad\text{for some}\quad\lambda,\mu>0.$$

Remark 4.2.1. *If w is a modular quasi-pseudometric on a set X, then boundedness on (X_w,q_w) implies w-boundedness. This observation follows from the fact that $C_{q_w}(x,\lambda)\subseteq C^{w}_{\lambda,\lambda}(x)$ and $C_{(q_w)^t}(x,\lambda)\subseteq C^{w^t}_{\lambda,\lambda}(x)$ for all $\lambda>0$ and $x\in X_w$.*

Definition 4.2.1. *Let A be a w-bounded subset of X_w. Then we define the w-diameter of A denoted by $diam_w(A)$ by*

$$diam_w(A) = \sup\{w(\lambda, x, y) : x, y \in A\}$$

for some $\lambda > 0$.

Lemma 4.2.1. *Let w be a modular quasi-pseudometric on X. If A is a w-bounded subset of X_w, then $diam_w(A) < \infty$.*

Proof. Suppose that A is w-bounded. Then for some $x \in X_w$, we have $A \subseteq C^{w}_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(x)$ for some $\lambda, \mu > 0$.
If $z, y \in A$ then

$$w(\lambda, x, z) \leq \lambda \quad \text{and} \quad w(\mu, z, x) \leq \mu$$

and

$$w(\lambda, x, y) \leq \lambda \quad \text{and} \quad w(\mu, y, x) \leq \mu.$$

It follows that

$$w(\mu + \lambda, z, y) \leq w(\mu, z, x) + w(\lambda, x, y) \leq \mu + \lambda$$

and

$$w(\mu + \lambda, y, z) \leq w(\mu, y, x) + w(\lambda, x, z) \leq \mu + \lambda.$$

Hence for $\lambda' = \mu + \lambda > 0$, we have

$$\sup\{w(\lambda', z, y) : z, y \in A\} \leq \lambda'.$$

Therefore, $diam_w(A) < \infty$. $\square$

Lemma 4.2.2. *Let w be a modular quasi-pseudometric on X. If w is continuous from the right on $(0, \infty)$, then boundedness on (X_w, q_w) is equivalent to w-boundedness.*

Proof. We only prove the sufficient condition since the necessary condition follows from Remark 4.2.1. Suppose that A is a w-bounded subset of X_w.

Then there exists $x \in X_w$ such that $A \subseteq C^w_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(x)$ for some $\lambda, \mu > 0$. Let $y \in A$. Then $w(\lambda, x, y) \leq \lambda$ and $w(\mu, y, x) \leq \mu$. By the right continuity of w on $(0, \infty)$, we have

$$q_w(x, y) \leq \lambda \quad \text{and} \quad q_w(y, x) \leq \mu$$

for some $x \in X_w$ and $\lambda, \mu > 0$.
Thus $A \subseteq C_{q_w}(x, \lambda) \cap C_{(q_w)^t}(x, \mu)$. Therefore, A is bounded on (X_w, q_w). □

Proposition 4.2.3. *Let w be a modular quasi-pseudometric on X. If X_w is a w-Isbell-convex and $(x_i)_{i\in I}$ is a family of points in X_w and $(\lambda_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ are families of positive real numbers such that $w(\lambda_i + \mu_j, x_i, x_j) \leq \lambda_i + \mu_j$ for all $i, j \in I$. Then the set $A := \bigcap_{i\in I}[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)]$ is nonempty and w-Isbell-convex.*

Proof. Observe that the set A is nonempty by w-Isbell-convexity of X_w. Now we show that the set A is w-Isbell-convex. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a family of points in A and $(\lambda_\alpha)_{\alpha\in\Gamma}$ and $(\mu_\alpha)_{\alpha\in\Gamma}$ be families of positive real numbers such that $w(\lambda_\alpha + \mu_\beta, x_\alpha, x_\beta) \leq \lambda_\alpha + \mu_\beta$ for all $\alpha, \beta \in \Gamma$. We have to show that the family of $w_{\leq}$-entourages

$$((C^w_{\lambda_\alpha,\lambda_\alpha}(x_\alpha)_{\alpha\in\Gamma}, (C^w_{\lambda_i,\lambda_i}(x_i))_{i\in I}; C^{w^t}_{\mu_\alpha,\mu_\alpha}(x_\alpha)_{\alpha\in\Gamma}, (C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I})$$

satisfies the hypothesis of w-Isbell-convexity.
Then, in particular, for all $\alpha \in \Gamma$ and $i \in I$, we have that

$$w(\mu_i, x_\alpha, x_i) \leq \mu_i < \lambda_\alpha + \mu_i$$

and

$$w(\lambda_i, x_i, x_\alpha) \leq \lambda_i < \lambda_i + \mu_\alpha$$

since $x_\alpha \in A$.
By w-Isbell-convexity of X_w it follows that

$$\begin{aligned}\emptyset \neq \bigcap_{\alpha\in\Gamma}[C^w_{\lambda_\alpha,\lambda_\alpha}(x_\alpha) \cap C^{w^t}_{\mu_\alpha,\mu_\alpha}(x_\alpha)] \cap \bigcap_{i\in I}[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)] \\ = A \cap \bigcap_{\alpha\in\Gamma}[C^w_{\lambda_\alpha,\lambda_\alpha}(x_\alpha) \cap C^{w^t}_{\mu_\alpha,\mu_\alpha}(x_\alpha)].\end{aligned}$$

Hence A is w-Isbell-convex. $\square$

Definition 4.2.2. *Let w be a modular quasi-pseudometric on X. A nonempty and w-bounded subset A of X_w is called w-admissible if $A = bicov_w(A)$.*

In the sequel, we will denote by $\mathcal{A}_w(X_w)$, the set of all w-admissible subsets of X_w.

Remark 4.2.2. *Observe that a w-admissible subset of X_w can be written as the intersection of a family of the form $C^w_{\lambda,\lambda}(x) \cap C^{w^t}_{\mu,\mu}(x)$, where $x \in X_w$ and $\lambda, \mu > 0$.*

Lemma 4.2.4. *Let w be a modular quasi-pseudometric on X which is continuous from the right on $(0, \infty)$. Then*

$$C_{q_w}(x, \lambda) = C^w_{\lambda,\lambda}(x)$$

and

$$C_{(q_w)^t}(x, \lambda) = C^{w^t}_{\lambda,\lambda}(x)$$

for all $\lambda > 0$ and $x \in X_w$.

Proof. We only prove the equality $C_{q_w}(x, \lambda) = C^w_{\lambda,\lambda}(x)$ and the equality $C_{(q_w)^t}(x, \lambda) = C^{w^t}_{\lambda,\lambda}(x)$ will follow by duality.
Since we know that $C_{q_w}(x, \lambda) \subseteq C^w_{\lambda,\lambda}(x)$, then we only prove the reverse inclusion.

If $a \in C^w_{\lambda,\lambda}(x)$, then $w(\lambda, x, a) \leq \lambda$ which is equivalent to $q_w(x, a) \leq \lambda$ by the right continuity of w. Thus $a \in C_{q_w}(x, \lambda)$. $\square$

Corollary 4.2.5. *Let w be a modular quasi-pseudometric on X which is continuous from the right on $(0, \infty)$ and $A \subseteq X_w$. Then A is w-admissible if and only if A is q_w-admissible.*

For a w-bounded subset A of X_α, we set

$$cov(A)_w := \bigcap \{C^w_{\lambda,\lambda}(x) : A \subseteq C^w_{\lambda,\lambda}(x), x \in X_w, \lambda > 0\} \tag{4.1}$$

and

$$cov(A)_{w^t} := \bigcap \{C^{w^t}_{\mu,\mu}(x) : A \subseteq C^{w^t}_{\mu,\mu}(x), x \in X_w, \mu > 0\}. \tag{4.2}$$

Moreover, we define the $w-$*bicover* of A by $bicov_w(A) := cov(A)_w \cap cov(A)_{w^t}$. Moreover, we set:

$$r^w_{x,\lambda}(A) := \sup_{y\in A}\{w(\lambda, x, y) : \quad \text{for some} \quad \lambda > 0\}, \qquad \text{where} \quad x \in X_w$$

and

$$r^{w^t}_{x,\mu}(A) := \sup_{y\in A}\{w(\mu, y, x) : \quad \text{for some} \quad \mu > 0\}, \qquad \text{where} \quad x \in X_w.$$

Then, let

$$r_x(A) := r^w_{x,\lambda}(A) \vee r^{w^t}_{x,\lambda}(A), \qquad \text{where} \quad x \in X_w.$$

Proposition 4.2.6. *Let w be a modular quasi-pseudometric on X and A be a w-bounded subset of X_w. Then we have:*

1. $bicov_w(A) = \bigcap_{x\in X}\left[C^w_{r^w_{x,\lambda}(A), r^w_{x,\lambda}(A)}(x) \cap C^{w^t}_{r^{w^t}_{x,\lambda}(A), r^{w^t}_{x,\lambda}(A)}(x)\right].$
2. $r_x(bicov_w(A)) = r_x(A)$.

Proof. 1. We first prove that

$$bicov_w(A) \subseteq \bigcap_{x\in X}\left[C^w_{r^w_{x,\lambda}(A), r^w_{x,\lambda}(A)}(x) \cap C^{w^t}_{r^{w^t}_{x,\lambda}(A), r^{w^t}_{x,\lambda}(A)}(x)\right].$$

Let $x \in X_w$. For any $y \in A$ with $x \neq y$ and some $\lambda > 0$, we have

$$w(r^w_{x,\lambda}(A), x, y) \leq \sup\{w(r^w_{x,\lambda}(A), x, y) \quad \text{for some} \quad r^w_{x,\lambda}(A) > 0\}.$$

Then $w(r^w_{x,\lambda}(A), x, y) \leq r^w_{x,\lambda}(A)$ which implies that $y \in C^w_{r^w_{x,\lambda}(A), r^w_{x,\lambda}(A)}(x)$ for all $x \in X_w$. So $A \subseteq C^w_{r^w_{x,\lambda}(A), r^w_{x,\lambda}(A)}(x)$ for all $x \in X_w$.
Hence

$$cov(A)_w \subseteq C^w_{r^w_{x,\lambda}(A), r^w_{x,\lambda}(A)}(x) \quad \text{for all} \quad x \in X_w. \tag{4.3}$$

By analogue arguments one has $A \subseteq C^{w^t}_{r^{w^t}_{x,\lambda}(A), r^{w^t}_{x,\lambda}(A)}(x)$ for all $x \in X_w$. Moreover,

$$cov(A)_{w^t} \subseteq C^{w^t}_{r^{w^t}_{x,\lambda}(A), r^{w^t}_{x,\lambda}(A)}(x) \quad \text{for all} \quad x \in X_w. \tag{4.4}$$

Combining (4.3) and (4.4) we have

$$bicov_w(A) \subseteq \bigcap_{x\in X}\left[C^{w}_{r^{w}_{x,\lambda}(A),r^{w}_{x,\lambda}(A)}(x)\cap C^{w^t}_{r^{w^t}_{x,\lambda}(A),r^{w^t}_{x,\lambda}(A)}(x)\right].$$

For the reverse inclusion, let $A\subseteq C^{w}_{\lambda,\lambda}(x)$ and A$\subseteq C^{w}_{\mu,\mu}(x)$ with $x\in X_w$ and $\lambda,\mu>0$.
Then $w(\lambda,x,y)\leq\lambda$ and $w(\mu,y,x)\leq\mu$ for all $y\in A$. It follows that

$$r^{w}_{x,r}(A)=\sup_{y\in A}\{w(r,x,y)\quad\text{for some}\quad r>0\}\leq\lambda.$$

Furthermore,

$$C^{w}_{r^{w}_{x,r}(A),r^{w}_{x,r}(A)}(x)\subseteq C^{w}_{\lambda,\lambda}(x)\quad\text{for all}\quad x\in X_w.$$

Hence

$$C^{w}_{r^{w}_{x,r}(A),r^{w}_{x,r}(A)}(x)\subseteq cov(A)_w\quad\text{for all}\quad x\in X_w. \tag{4.5}$$

By duality one has

$$C^{w^t}_{r^{w^t}_{x,r}(A),r^{w^t}_{x,r}(A)}(x)\subseteq cov(A)_{w^t}\quad\text{for all}\quad x\in X_w. \tag{4.6}$$

From inclusions (4.5) and (4.6) we have

$$\bigcap_{x\in X}\left[C^{w}_{r^{w}_{x,r}(A),r^{w}_{x,r}(A)}(x)\cap C^{w^t}_{r^{w^t}_{x,r}(A),r^{w^t}_{x,r}(A)}(x)\right]\subseteq bicov_w(A).$$

2. Indeed,

$$r^{w}_{x,\lambda}(bicov_w(A))=\sup\left\{w(\lambda,x,y):y\in\bigcap_{x\in X}\left[C^{w}_{r^{w}_{x,\lambda}(A),r^{w}_{x,\lambda}(A)}(x)\cap C^{w^t}_{r^{w^t}_{x,\lambda}(A),r^{w^t}_{x,\lambda}(A)}(x)\right]\right.$$

$\left.\text{for some } \lambda>0\right\}.$
In particular $y\in bicov_w(A)$ implies that

$$w(r^{w}_{x,\lambda}(A),x,y)\leq r^{w}_{x,\lambda}(A)\leq r_x(A)$$

and

$$w(r^{w^t}_{x,\lambda}(A),y,x)\leq r^{w^t}_{x,\lambda}(A)\leq r_x(A).$$

It follows that

$$r^{w}_{x,\lambda}(bicov_w(A)) \le r_x(A)$$

and

$$r^{w^t}_{x,\lambda}(bicov_w(A)) \le r_x(A).$$

Hence

$$r_x(bicov_w(A)) \le r_x(A).$$

We obtain the reverse inequality since $A \subseteq bicov_w(A)$.

□

Definition 4.2.3. *Let w be a modular quasi-pseudometric on X. Given a subset A of X_w, we define for $\lambda, \mu > 0$ the λ, μ-parallel set of A as*

$$P_{\lambda,\mu}(A) = \bigcup_{a \in A} [C^{w}_{\lambda,\lambda}(a) \cap C^{w^t}_{\mu,\mu}(a)].$$

Observe that for any $\lambda > 0$, we have that

$$P_{\lambda,\lambda}(A) = \bigcup_{a \in A} C^{w^s}_{\lambda,\lambda}(a).$$

Proposition 4.2.7. *Let w be a modular quasi-pseudometric on X. If X_w is w-Isbell-convex and A is a w-admissible subset of X_w, that is $A = \bigcap_{i \in I} C^{w}_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)$ with $x_i \in X_w$ and $\lambda_i, \mu_i > 0$ for each $i \in I \neq \emptyset$, then*

$$P_{\lambda,\mu}(A) = \bigcap_{i \in I} [C^{w}_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i) \cap C^{w^t}_{\mu_i+\mu,\lambda_i+\mu}(x_i)] \tag{4.7}$$

for all $\lambda, \mu > 0$.

Proof. Let $y \in P_{\lambda,\mu}(A)$. Then, for some $a \in A$ we have $w(\lambda, a, y) \le \lambda$ and $w(\mu, y, a) \le \mu$. Furthermore, for each $i \in I$,

$$w(\lambda_i + \lambda, x_i, y) \le w(\lambda_i, x_i, a) + w(\lambda, a, y) \le \lambda_i + \lambda,$$

and

$$w(\mu_i + \mu, y, x_i) \le w(\mu, y, a) + w(\mu_i, a, x_i) \le \mu_i + \mu.$$

It follows that $y \in C^w_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i) \cap C^{w^t}_{\mu_i+\mu,\lambda_i+\mu}(x_i)$ for all $i \in I$. Hence

$$P_{\lambda,\mu}(A) \subseteq \bigcap_{i\in I}[C^w_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i) \cap C^{w^t}_{\mu_i+\mu,\lambda_i+\mu}(x_i)].$$

We now suppose that $y \in \bigcap_{i\in I}[C^w_{\lambda_i+\lambda,\lambda_i+\lambda}(x_i) \cap C^{w^t}_{\mu_i+\mu,\lambda_i+\mu}(x_i)]$. Hence, for any $i \in I$,

$$w(\lambda_i + \lambda, x_i, y) \leq \lambda_i + \lambda,$$

and

$$w(\mu_i + \mu, y, x_i) \leq \mu_i + \mu.$$

For any $a \in A$ and $i, j \in I$ we have

$$w(\lambda_i + \mu_j, x_i, x_j) \leq w(\lambda_i, x_i, a) + w(\mu_i, a, x_j) \leq \lambda_i + \mu_j$$

by definition of A and the triangle inequality. Thus the family of $w_{\leq}$-entourages $[(C^w_{\lambda_i,\lambda_i}(x_i))_{i\in I}, C^w_{\lambda,\lambda}(y); (C^{w^t}_{\mu_i,\mu_i}(x_i))_{i\in I}, C^{w^t}_{\mu,\mu}(y)]$ satisfies the hypothesis of w-Isbell-convexity of X_w. Then

$$\begin{aligned}
\emptyset &\neq \left(\bigcap_{i\in I} C^w_{\lambda_i,\lambda_i}(x_i)\right) \cap C^w_{\lambda,\lambda}(y) \cap \left(\bigcap_{i\in I} C^{w^t}_{\mu_i,\mu_i}(x_i)\right) \cap C^{w^t}_{\mu,\mu}(y) \\
&= \bigcap_{i\in I}[C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)] \cap (C^w_{\lambda,\lambda}(y) \cap C^{w^t}_{\mu,\mu}(y)) \\
&= A \cap (C^w_{\lambda,\lambda}(y) \cap C^{w^t}_{\mu,\mu}(y)).
\end{aligned}$$

It follows that $w(\lambda, y, a) \leq \lambda$ and $w(\mu, a, y) \leq \mu$ for some $a \in A$.
Therefore, $y \in P_{\lambda,\mu}(A)$. □

The following definition is motivated from [10, Theorem 5.1] and [10, Theorem 5.2] since their proofs do not involve the symmetry axiom of a modular metric. Therefore, we keep the same definition in the asymmetric setting.

Definition 4.2.4. *Let w be a modular metric on a set X. Then we say that a map $T : X_w \to X_w$ is a w-Lipschitz if there exists a $k > 0$ such that*

$$w(k\lambda, T(x), T(y)) \leq w(\lambda, x, y) \quad \textit{for all} \quad \lambda > 0 \quad \textit{and} \quad x, y \in X_w.$$

If $k = 1$, then the map T is called w-nonexpansive map.

Remark 4.2.3. *In light of [10, Theorem 5.1] and [10, Theorem 5.2], one can easily prove that for a w-Lipschitz map $T : X_w \to X_w$, where w is modular quasi-pseudometric on X. We have that $w(k\lambda, T(x), T(y)) \leq w(\lambda, x, y)$ implies $q_w(T(x), T(y)) \leq kq_w(x, y)$ for some $k > 0$ and for all $\lambda > 0$ and $x, y \in X_w$.*

Theorem 4.2.8. *(compare [55, Lemma 5.3]) Let w be a modular quasi-pseudometric on X which is continuous from the right on $(0, \infty)$ and X_w be w-Isbell-convex. If A is a w-admissible subset of X_w, then there is a w-nonexpansive retraction R of $P_{\lambda_1,\lambda_2}(A)$ onto A such that $w(\lambda_2, x, R(x)) \leq \lambda_2$ and $w(\lambda_1, R(x), x) \leq \lambda_1$ for all $x \in P_{\lambda_1,\lambda_2}(A)$ and $\lambda_1, \lambda_2 > 0$.*

Proof. Suppose that A is w-admissible. Then

$$A = \bigcap_{i \in I} C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i) \quad \text{with} \quad I \neq \emptyset.$$

Since $P_{\lambda_2,\lambda_1}(A)$ is an intersection of $w_{\leq}$-entourages from Proposition 4.2.7, then $P_{\lambda_1,\lambda_2}(A)$ is w-admissible in X_w. Moreover, from Proposition 4.2.3 we have that $P_{\lambda_1,\lambda_2}(A)$ is w-Isbell-convex. Let us consider the family Ω defined by $\Omega = \{(D, R_D) : A \subseteq D \subseteq P_{\lambda_1,\lambda_2}(A)$ and $R_D : D \to A$ is a nonexpansive retraction such that $w(\lambda_2, x, R_D(x)) \leq \lambda_2$ and $w(\lambda_1, R_D(x), x) \leq \lambda_1$ for all $x \in D\}$.

If the map I_A is the identity on A, then $(A, I_A) \in \Omega$. Hence $\Omega \neq \emptyset$. Furthermore, one can partially order the family Ω by $(D, R_D) \leq (E, R_E)$ if and only if $D \subseteq E$ and the nonexpansive map R_D is an extension of the nonexpansive map R_E.

It follows that every chain in $(\Omega, \leq)$ is bounded above. Thus by Zorn's lemma, Ω has a maximal element. Suppose that (D, R_D) is the maximal element of $(\Omega, \leq)$. We now prove that $D = P_{\lambda_1,\lambda_2}(A)$. Suppose that there exists an element $x \in P_{\lambda_1,\lambda_2}(A)$ with $x \notin D$ and we consider the set

$$\mathcal{C} = \left[\bigcap_{d \in D} C^w_{\lambda',\lambda'}(R_D(d)) \cap C^{w^t}_{\mu',\mu'}(R_D(d))\right] \cap \left[\bigcap_{i \in I} C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)\right]$$

$$\cap [C^w_{\lambda_1,\lambda_1}(x) \cap C^{w^t}_{\lambda_2,\lambda_2}(x)],$$

where $\lambda' = w(\lambda', d, x)$ and $\mu' = w(\mu', x, d)$ for all $d \in D$.

We first show that the set $\mathcal{C} \neq \emptyset$. To achieve this we only need to show that the family

$$[(C^w_{\mu_i,\mu_i}(R_D(d)))_{d\in D}, (C^w_{\lambda_i,\lambda_i}(x_i))_{i\in I}, (C^w_{\lambda_1,\lambda_1}(x)); (C^w_{\lambda',\lambda'}(R_D(d)))_{d\in D},$$

$$(C^{w^t}_{\mu',\mu'}(R_D(d)))_{i\in I}, (C^{w^t}_{\lambda_2,\lambda_2}(x))]$$

of double $w_\leq$-entourages has the mixed binary intersection property in light of Proposition 4.1.1.
If $d_1, d_2 \in D$, then

$$\begin{aligned} w(\lambda'+\mu', R_D(d_1), R_D(d_2)) &\leq w(\lambda'+\mu', d_1, d_2) \\ &\leq w(\lambda', d_1, x) + w(\mu', x, d_2) \\ &= \lambda' + \mu'. \end{aligned}$$

Thus by the w-admissibility of A, we have
$C^w_{\lambda',\lambda'}(R_D(d)) \cap C^{w^t}_{\mu',\mu'}(R_D(d)) \neq \emptyset$. From the definition of A as w-admissible set in X_w, for any $i, j \in I$ it is easy to see that $C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i) \neq \emptyset$.

Moreover, for each $d \in D, R_D(d) \in A = \bigcap_{i\in I} C^w_{\lambda_i,\lambda_i}(x_i) \cap C^{w^t}_{\mu_i,\mu_i}(x_i)$. It follows that

$$C^w_{\lambda',\lambda'}(R_D(d)) \cap C^{w^t}_{\mu_i,\mu_i}(x_i) \neq \emptyset$$

and

$$C^w_{\mu',\mu'}(R_D(d)) \cap C^{w^t}_{\lambda_i,\lambda_i}(R_D(d)) \neq \emptyset,$$

for any $d \in D$ and $i \in I$.
Furthermore, since

$$x \in P_{\lambda_1,\lambda_2}(A) = \bigcap_{i\in I} [C^w_{\lambda_i+\lambda_1,\lambda_i+\lambda_1}(x_i) \cap C^{w^t}_{\mu_i+\lambda_2,\lambda_i+\lambda_2}(x_i)]$$

by Proposition 4.2.7, there exists $y \in A$ such that $x \in C^w_{\lambda_1,\lambda_1}(y) \cap C^{w^t}_{\lambda_2,\lambda_2}(y)$ and, therefore, we have

$$(C^w_{\lambda_1,\lambda_1}(x) \cap C^{w^t}_{\lambda_2,\lambda_2}(x)) \cap (C_{\lambda_i,\lambda_i}(x_i) \cap C_{\mu_i,\mu_i}(x_i)) \neq \emptyset$$

for all $i \in I$.

However, if $d \in D$, then from the assumption on R_D we have

$$w(\lambda_1 + \lambda', R_D(d), x) \leq w(\lambda_1, R_D(d), d) + w(\lambda', d, x) \leq \lambda' + \lambda_1$$

and

$$w(\mu' + \lambda_2, x, R_D(d)) \leq w(\mu', x, d) + w(\lambda_2, d, R_D(d)) \leq \mu' + \lambda_2.$$

Thus by w-metric convexity of X_w, we have

$$C^w_{\lambda',\lambda'}(R_D(d)) \cap C^{w^t}_{\lambda_2,\lambda_2}(x) \neq \emptyset$$

as well as

$$C^{w^t}_{\mu',\mu'}(R_D(d)) \cap C^w_{\lambda_1,\lambda_1}(x) \neq \emptyset.$$

Obviously, we have $C^w_{\lambda_1,\lambda_1}(x) \cap C^{w^t}_{\lambda_2,\lambda_2}(x) \neq \emptyset$. Therefore, we just proved that the family

$$[(C^w_{\lambda',\lambda'}(R_D(d)))_{d\in D}, (C^w_{\lambda_i,\lambda_i}(x_i))_{i\in I}, (C^w_{\lambda_1,\lambda_1}(x)), (C^w_{\lambda',\lambda'}(R_D(d)))_{d\in D},$$

$$(C^{w^t}_{\mu',\mu'}(R_D(d)))_{i\in I}, (C^{w^t}_{\lambda_2,\lambda_2}(x))]$$

of double $w_\leq$ -entourages has the mixed binary intersection property. Therefore, $\emptyset \neq \mathcal{C} \subseteq A$. We now suppose that $z \in \mathcal{C}$ and we define map $R' : D \cup \{x\} \to A$ by $R'(d) = R_D(d)$ if $d \in D$ and $R'(x) = z$. Then, we have for any $d \in D$

$$w(\mu', R'(x), R'(d)) = w(\mu', z, R_D(d)) \leq \mu' = w(\mu', x, d)$$

and

$$w(\lambda', R'(d), R'(x)) = w(\lambda', R_D(d), z) \leq \lambda' = w(\lambda', d, x).$$

So R' is nonexpansive.

Moreover,

$$w(\lambda_1, R'(x), x) = w(\lambda_1, z, x) \leq \lambda_1$$

and

$$w(\lambda_2, x, R'(x)) = w(\lambda_2, x, z) \leq \lambda_2$$

since $z \in \mathcal{C}$.

Hence $(D \cup \{x\}, R') \in \Omega$ which contradicts the maximality of D. Therefore, $D = P_{\lambda_1,\lambda_2}(A)$. $\square$

Theorem 4.2.9. *Let w be a modular quasi-pseudometric on X and X_w be w-Isbell-convex. If w is continuous from the right on $(0,\infty)$ and $T : X_w \to X_w$ is a w-nonexpansive map, then the fixed point set $Fix(T)$ is nonempty and w-Isbell-convex.*

Proof. Since $T : X_w \to X_w$ is a w-nonexpansive map, then $T : (X_w, q_w) \to (X_w, q_w)$ is a nonexpansive map.
Hence for any $\lambda > 0$, we have $w(\lambda, T(x), T(y)) \leq w(\lambda, x, y)$ for all $x, y \in X_w$.
Then from Remark 4.2.3 when $k = 1$, we have $q_w(T(x), T(y)) \leq q_w(x, y)$ for all $x, y \in X_w$.
Observe that (X_w, q_w) is an Isbell-convex quasi-pseudometric space by Theorem 4.1.5. Moreover, we have $Fix(T)$ is nonempty and Isbell-convex by [49, Theorem 3.3]. Therefore, $Fix(T)$ is w-Isbell-convex by Theorem 4.1.5. $\square$

Definition 4.2.5. *Let w be a modular quasi-pseudometric on X and $T : X_w \to X_w$ map. For $\lambda_1, \lambda_2 > 0$, we define the set $F_{\lambda_1,\lambda_2}(T)$ by*

$$F_{\lambda_1,\lambda_2}(T) = \{x \in X_w : w(\lambda_2, x, T(x)) \leq \lambda_2 \quad \textit{and} \quad w(\lambda_1, T(x), x) \leq \lambda_1\}.$$

Corollary 4.2.10. *Let w be a modular quasi-pseudometric on X and X_w be w-Isbell-convex. If w is continuous from the right on $(0,\infty)$ and $T : X_w \to X_w$ is a w-nonexpansive map, then the set $F_{\lambda_1,\lambda_2}(T)$ is w-Isbell-convex whenever $F_{\lambda_1,\lambda_2}(T)$ is nonempty.*

Chapter 5

Conclusion and open problems

5.1 Conclusion

In this thesis, many aspects of a modular metric that does not satisfy the symmetry condition have been defined. In this chapter, we discuss the conclusion of our investigations and point out some open problems encountered throughout this present work. These open problems may comprise the topics of future research.

In the first part of our investigations, we introduced a notion of modular metric that does not satisfy the symmetric axiom of a modular metric. We called this a modular quasi-pseudometric. We proved that a modular quasi-pseudometric is a nonincreasing function. We defined a modular set around a point which in essence is an equivalence class. In Theorem 3.1.2, we showed that a modular set equipped with T_0-quasi-metric induced by a modular quasi-pseudometric is a quasi-metric space. It is well-known that for a nonsymmetric distance functions there are several completeness concepts. Hence we explored convergence in a set in the modular quasi-pseudometric setting. We studied topologies induced by a modular quasi-pseudometric on a nonempty set and it turns out that a set equipped with a modular quasi-pseudometric has three topologies, one induced by the modular quasi-pseudometric, one by transpose modular quasi-pseudometric and the third one induced by the symmetrized modular quasi-pseudometric. We showed that both topologies are T_0 on the modular set and the third topology, which is called the modular topology generated by the modular pseudometric is finer than the topologies induced by the modular quasi-pseudometric and its transpose. We also studied the Hausdorff modular quasi-pseudometric induced by the modular quasi-pseudometric and its transpose. We showed that the modular quasi-pseudometric generated by the Hausdorff pseudometric induced by the sum of modular quasi-pseudometric and its transpose modular quasi-pseudometric, has similar behaviour to the Poméiu quasi-pseudometric.

In the second part of this work we introduced the theory of Isbell-convexity in the setting of modular quasi-pseudometric. We called this concept w-Isbell-convexity. We studied the connections between Isbell-convexity and w-Isbell-convexity on a modular set and realised that they are actually equivalent whenever the modular quasi-pseudometric is continuous from the right on the set of positive numbers. We studied the boundedness of a set endowed with a modular quasi-pseudometric and proved that w-Isbell-convexity implies w-metrical-convexity and w-Isbell-completeness and that the converse is also true. We consequently studied a self-nonexpansive map on a w-Isbell-convex modular set and show that it has a fixed point. We showed that the fixed point sets of self-nonexpansive map is nonempty and is w-Isbell-convex.

5.2 Open problems

Our conclusion leads us to list some open problems that we came across throughout our present investigations. We would like to study these problems in future work.

One-local retract in modular quasi-pseudometric spaces.

In [36], Khamsi introduced and studied one-local retract on metric spaces. His paper was inspired by Baillon's results [5] on hyperconvex metric spaces. He defined *one-local retract* as: a subset A of M is said to be one-local retract of M if for every family $\{B_i; i \in I\}$ of closed balls centered in A with non-empty intersections (i.e. $A \cap (\bigcap B_i) \neq \emptyset$). It is immediate that each nonexpansive retract of M is a one-local retract (but not conversely). He also remarked that $A \subset M$ is a nonexpansive retract of M if there exists a nonexpansive map $r : M \to A$ such that $r(a) = a$ for every $a \in A$.

Not long ago, in [3], the authors introduced the concept of one-local in modular function spaces and proved the existence of common fixed points for commutative mappings.

About six years back, Abdou in [2] introduced the concept of one local retract in a more general setting in modular metric spaces and proved the existence of common fixed points for a family of nonexpansive modular mappings defined on nonempty w-closed w-bounded subsets in metric modular spaces.

Problem 5.2.1. How can we introduce the theory of one-local retract in modular quasi-metric spaces to extend the results from [55] to the context of modular quasi-metric spaces.

Problem 5.2.2. Let w be a modular quasi-pseudometric on X. If a subset A of X is one local retract in X, is A one local retract in (X_w, q_w)?

Problem 5.2.3. If w is not continuous from the right on $(0, \infty)$, under what conditions will the Theorem 4.2.8 hold?

Bornology of modular quasi-pseudometric spaces.

The well known concept of bornology has been investigated by many authors. For instance in [6], a bornology on a set X is defined as:

Definition 5.2.1. *Let X be a non-empty set without any assumed structure. A family $\mathcal{B}$ of subsets of X is called a bornology on X provided the following conditions are satisfied :*

1. *$\mathcal{B}$ form a cover of X, (i.e. $X = \bigcup \mathcal{B}$),*
2. *whenever $B \in \mathcal{B}$ and $A \subseteq B$, then $A \in \mathcal{B}$ and*
3. *$\mathcal{B}$ is stable under finite unions (i.e. if $B_1, ..., B_n \in \mathcal{B}$, then $\bigcup_{i=1}^{n} B_i \in \mathcal{B}$).*

Condition (2) implies that a bornology on a set is a hereditary family, meaning a bornology must contain the empty set. The smallest bornology on a set is the family of its finite subsets and the largest is the power set $\mathcal{P}(X)$. If (X, d) is a metric space, then its bornology of bounded subsets is denoted $\mathcal{B}_d(X)$ and it is called *metric bornology* determined by d. So we would like to investigate bornologies of modular quasi-metric spaces. This is inspired by [56] as they generalised Beer's results in [6] on bornologies of extended metric spaces to the framework of quasi-metric spaces. Hu in [30] investigated bornologies defined on a metric space that correspond to the bornology of bounded sets, for example, compact metric spaces. Over the years as the concept continued to be investigated, Borwein [11] used bornologies to develop a unified theory of differentiability for functions defined on general normed spaces, characterizing various geometric properties of spaces in terms of the relationships between different bornological derivatives on various classes of functions.

Problem 5.2.4. How one can define a bornology induced by a modular quasi-pseudometric spaces?

Problem 5.2.5. What are the connections between the bornology induced by a modular quasi-pseudometric w on a set X and the quasi-metric Bornology induced by q_w?

Common fixed points of nonexpansive mappings on modular quasi-pseudometric spaces.

In many cases, particularly in applications of integral operators, approximation, and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. In recent years, there was a strong interest in studying the fixed point property in modular function spaces after the first paper [39] was published in 1990.

In [58], Penot presented an abstract version of Kirk's fixed point theorem [41] for nonexpansive mappings. Many results of fixed point in metric spaces were developed after Penot's formulation. Using Penot's work, the author in [37] proved some results in metric spaces with uniform normal structure similar to the ones known in Banach spaces.

Problem 5.2.6. It is natural to wonder if does there exist a common fixed point of family of commutative w-nonexpansive mappings on a modular quasi-pseudometric space.

Bibliography

[1] T. Abreu, E. Corbacho and V. Tarieladze, Uniform type hyperspaces, Math. Pannon. 19 (2008), 155-170.

[2] Afrah A.N. Abdou, One-local retract and common fixed point in modular metric spaces. Abstr. Appl. Anal. 2013, Art. ID 672069, 8 pp.

[3] S.A. Al-Mezel, A. Al-Roqi, and M.A. Khamsi, One-local retract and common fixed point in modular function spaces, Fixed Point Theory and Appl. vol. 2012, article 109, 13 pp, 2012.

[4] A. Andrikopoulos, A solution to the completion problem for quasi-pseudometric spaces, Int. J. Math. Math. Sci. 2013 (2013), Article ID 278381, 12 pp.

[5] J.B. Baillon, Nonexpansive mappings and hyperconvex space, Fixed Point Theory and its Applications (R.R. Brown, ed.), Amer. Math. Soc., Providence, RI, Contemporary Math. 72 (1988), 11-19.

[6] G. Beer, The structure of extended real-valued metric spaces, Set-valued Var. Anal. 21 (2013), 591-602.

[7] G. Beer, Topologies on Closed and Closed Convex Sets. Kluwer, Dordrecht (1993).

[8] L.P. Belluce and W. A. Kirk, Nonexpansive mappings and fixed-points in Banach spaces, Illinois J. Math. 11 (1967), 474-479.

[9] C. Berge, Topological Spaces. Macmillan, New York (1963).

[10] G. Berthiaume, On quasi-uniformities in hyperspaces, Proc. Amer. Math. Soc. 66 (1977), 335-343.

[11] J. Borwein, M. Fabiana and J. Vanderwerff, Locally Lipschitz functions and bornological derivatives, CECM Report No. 93:012.

[12] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. of Sci. U.S.A. 54 (1965), 1041-1044.

[13] R.E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappings, Pacifc J. Math. 53 (1964), 59-71.

[14] V.V. Chistyakov, Lipschitzian Nemytskii operators in the cones of mappings of bounded Wiener ϕ-variation, Folia Math. 11 (2004), 15-39.

[15] V.V. Chistyakov, Metric modulars and their application, Dokl. Math. 73 (2006), 32-35.

[16] V.V. Chistyakov, Metric modular spaces. Theory and applications, SpringerBriefs in Mathematics, Springer, Cham, 2015.

[17] V.V. Chistyakov, Modular metric spaces, I: Basic concepts, Nonlinear Anal. 72 (2010), 1-14.

[18] V.V. Chistyakov, Modular metric spaces, II. Application to superposition operators, Nonlinear Anal. 72 (2010), 15-30.

[19] V.V. Chistyakov, Modular metric spaces generated by F-modulars, Folia Math. 15 (2008), 3-24.

[20] V.V. Chistyakov, Selections of bounded variation, J. Appl. Anal. 10 (2004), 1-82.

[21] S. Cobzas, Completeness in quasi-metric spaces and Ekeland Variational Principle, Topology Appl. 158 (2011), 1073-1084.

[22] S. Cobzas, Functional analysis in asymmetric normed spaces, Frontiers in Mathematics, Springer, Basel, 2013.

[23] A.W.M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, Adv. Math. 53 (1984), 321-402.

[24] R. Espínola, M.A. Khamsi, Introduction to hyperconvex spaces, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001.

[25] P. Fletcher and W.F. Lindgren, Quasi-uniform Spaces, Dekker, Lecture notes in Pure and Applied Mathematics, 77. Marcel Dekker, New York, 1982.

[26] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Monographs and Textbooks in Pure and Applied Mathematics, 83. Marcel Dekker, New York, 1984.

[27] K. Goebel, T. Sekowski, and A. Stachura, Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball, Nonlinear Anal. 4 (1980), 1011-1021.

[28] P. Harjulehto, P. H*ästö*, M. Koskenoja, and S. Varonen, The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, Potential Anal. 25 (2006), 205-222.

[29] J. Heinonen, T. Kilpel*ä*inen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, New York, 1993.

[30] S.-T. Hu, Boundedness in a topological space, J. Math. Pures Appl. 28 (1949), 287-320.

[31] J.R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1964), 65-76.

[32] E.M. Jawhari, M. Pouzet, D. Misane, Retracts: graphs and ordered sets from the metric point of view, Contemp. Math. 57 (1986), 175-226.

[33] J.C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963), 71-89.

[34] E. Kemajou, H.-P. Künzi and O. Olela Otafudu, The Isbell-hull of a di-space, Topol. Appl. 159 (2012), 2463-2475.

[35] M.A. Khamsi, On asymptotically nonexpansive mappings in hyperconvex metric spaces, Proc. Amer. Math. Soc. 132 (2004), 365-373.

[36] M.A. Khamsi, One-local retract and common fixed point for commuting mappings in metric spaces, Nonlinear Anal. 27 (1996), 1307-1313.

[37] M.A. Khamsi, On metric spaces with uniform normal structure, Proc. Amer. Math. Soc. 106 (1989), 723-726.

[38] M.A. Khamsi, W. A. Kirk, An introduction to metric spaces and Fixed point theory, Wiley-Interscience, New York, 2001.

[39] M.A. Khamsi, W. M. Kozlowski, and S. Reich, Fixed point theory in modular function spaces, Nonlinear Anal. 14 (1990), 935-953.

[40] M.A. Khamsi and W. M. Kozlowski, On asymptotic pointwise nonexpansive mappings in modular function spaces, J. Math. Appl. 380 (2011), 697-708.

[41] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly, 72 (1965), 1004-1006.

[42] W.A. Kirk, Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees, Fixed Point Theory and Appl. (2004) 309-316.

[43] E. Klein, A. Thompson, Theory of Correspondences, Wiley-Interscience, New York, 1984.

[44] S. Koshi, T. Shimogaki, On F-norms of quasi-modular spaces, J. Fac. Sci. Hokkaido Univ. Ser. I. 15 (1961), 202-218.

[45] H.-P. Künzi and C.M. Kivuvu, A double completion for an arbitrary T_0-quasi-metric space, J. Log. Algebr. Program. 76 (2008), 251-269.

[46] H.-P. Künzi, J. Rodríguez-López and S. Romaguera, Hyperspaces of a weightable quasi-metric space: application to models in the theory of computation. Math. Comput. Modelling 52 (2010), 674-682.

[47] H.-P. Künzi and C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. 20 (1995), 161-183.

[48] H.-P. Künzi and S. Romaguera, Quasi-metric spaces, quasi-metric hyperspaces and uniform local compactness, Rend. Instit. Mat. Univ. Trieste 30 (1999), Suppl., 141 - 152.

[49] H.-P. Künzi and O. Olela Otafudu, q-hyperconvexity in quasi-pseudometric spaces and fixed point theorems, J. funct. Spaces Appl. (2012) Art. ID 765903, 18 pp.

[50] T.C. Lim, A fixed point theorem for families on nonexpansive mappings, Pacifc J. Math. 53 (1974), 487-493.

[51] L. Maligranda, Orlicz Spaces and Interpolation, Seminars in Math. 5, Univ. of Campinas, Brazil, 1989.

[52] J. Musielak and W. Orlicz, On modular spaces, Studia Math. 18 (1959), 49-65.

[53] S. Nadler, Hyperspaces of Sets. Dekker, New York, 1978.

[54] H. Nakano, Modulared semi-ordered linear spaces, Maruzen Co., Tokyo, 1950.

[55] O. Olela Otafudu, On one-local retract in quasi-metric spaces, Topology Proc. 45 (2015), 271-281.

[56] O. Olela Otafudu, W.B. Toko and D. Mukonda, On bornology of extended quasi-metric spaces, Hacet. J. Math. Stat. (to appear).

[57] W. Orlicz, A note on modular spaces, I, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 (1961), 157-162.

[58] J.-P. Penot, Fixed point theorems without convexity, Bull. Soc. Math. France. Mém. 60 (1961), 129-152.

[59] J. Rodríguez-López and S. Romaguera, Wijsman and hit-and-miss topologies of quasi-metric spaces, Set-Valued Anal. 11 (2003), 323-344.

[60] J. Rodríguez-López and M.A. Sánchez-Granero, Some properties of bornological convergences, Topology Appl. 158 (2011), 101-117.

[61] J. Rodríguez-López, M.P. Schellekens and O. Valero, An extension of the dual complexity space and an application to computer science, Topology Appl. 156 (2009), 3052-3061.

[62] S. Romaguera and O. Valero, Domain theoretic characterisations of quasi-metric completeness in terms of formal balls, Math. Structures Comput. Sci. 20 (2010), 453-472.

[63] K.A. Ross, Elementary Analysis, second edition, Springer.

[64] S. Salbany, Bitopological spaces, compactifications and completions, Math. Monographs, University of Cape Town, Depart. Math., 1974.

[65] S. Salbany, Injective objects and morphisms, Categorical topology and its relation to analysis, algebra and combinatorics, Prague, 1988, World Sci. Publ., Teaneck, NJ, 1989, 394 - 409.

[66] K. Sebogodi, On Modular Quasi-Pseudometric Spaces, North West University, 2016.

[67] Y.U. Gaba, Startpoints and (α, γ)-contractions in quasi-pseudometric spaces, J. Math. 2014, Art. ID 709253, 8 pp.

[68] S. Yamamuro, On conjugate space of Nakano space, Trans. Amer. Math. Soc. 90 (1959), 291-311.