

https://dspace.org/

Extensions of Solovays system S without independent sets of axioms

Igor Gorbunov; Dmitry Shkatov

ELSEVIER SCIENCE BV
https://hdl.handle.net/10539/41124
WOS

Downloaded from a DSpace repository. https://dspace.org/

Annals of Pure and Applied Logic 175 (2024) 103360

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Extensions of Solovay's system **S** without independent sets of axioms

Igor Gorbunov [a], Dmitry Shkatov [b,a,*]

[a] *Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow 127051, Russia*
[b] *School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Private Bag 3, Wits 2050, South Africa*

ARTICLE INFO

Article history:
Received 23 June 2022
Received in revised form 23 August 2023
Accepted 26 August 2023
Available online 1 September 2023

MSC:
03B45
03B22
03F45
03F07

Keywords:
Quasi-normal modal logic
Independent axiomatizability
Lattices of logics

ABSTRACT

Chagrov and Zakharyaschev posed the problem of existence of extensions of Solovay's system **S**, which is a non-normalizable quasi-normal modal logic, that do not admit deductively independent sets of axioms. This paper gives a solution by exhibiting countably many extensions of **S** without deductively independent sets of axioms.

© 2023 Elsevier B.V. All rights reserved.

1. Introduction

A logic is said to be *independently axiomatizable* if it can be represented by an axiomatic system with a deductively independent set of axioms. Independent axiomatizability is of importance to the study of lattices of logics: if, in such a lattice, every logic is independently axiomatizable, then all the elements of the lattice have a modular representation—in particular, by removing a formula from a logic's axiomatization we assuredly obtain a different logic.

Chagrov and Zakharyaschev [3] had shown that the lattice of intermediate logics—which are consistent extensions of the intuitionistic propositional logic—and the lattice of normal modal logics—which are extensions of the modal logic **K** closed under Substitution (Sub), Modus Ponens (MP), and Necessitation (N),

* Corresponding author.
E-mail addresses: i_gorbunov@mail.ru (I. Gorbunov), shkatov@gmail.com (D. Shkatov).

https://doi.org/10.1016/j.apal.2023.103360
0168-0072/© 2023 Elsevier B.V. All rights reserved.

the rule stipulating that $\Box\varphi$ is a theorem whenever φ is a theorem—both contain logics without independent axiomatizations. The normal modal logics obtained by Chagrov and Zakharyaschev do, however, have independent axiomatizations if the rule (N) is not required: the deductive weakening through the removal of (N) makes it possible to obtain deductively independent, in a new sense, sets of axioms. Therefore, Chagrov and Zakharyaschev pose [3, p. 301] the problem of whether the lattice of quasi-normal modal logics, i.e., extensions of **K** closed under (Sub), (MP), but not necessarily under (N)—and, more specifically, the lattice of extensions of Solovay's system **S**, which is the most prominent quasi-normal modal logic—contain logics not admitting independent sets of axioms. The authors [5] have earlier given a partial solution by obtaining quasi-normal modal logics, but not extensions of **S**, without independent sets of axioms. Here, we obtain not independently axiomatizable extensions of **S**.

We note that **S** plays an important role among quasi-normal modal logics: the study of quasi-normal, as opposed to normal,[1] modal logics has been motivated by interest in **S**, which was proposed by Solovay [8] as the propositional modal logic of arithmetically true principles of provability in Peano arithmetic PA (replacing 'arithmetically true' by 'provable in PA', gives us, as Solovay had shown in the same paper, the normal logic **GL** [8,2]). System **S** can be obtained axiomatically by adding to **GL** the axiom $\Box p \to p$, which can be interpreted as saying that, if p is provable in PA, then p is true. Observe that **S** cannot be consistently normalized: its closure under (Sub), (MP), and (N) is inconsistent due to the presence of the Löb axiom $\Box(\Box p \to p) \to \Box p$: necessitate $\Box p \to p$ and apply (MP) twice to obtain p. To avoid inconsistency, axiomatic systems for extensions of **S** lack (N) as a rule of inference.

Our proof of existence of extensions of **S** without independet sets of axioms essentially involves semantics. To explain our choice of semantic apparatus, we briefly outline our proof. To obtain the main result, stated as Theorem 7.9, we use an observation originally made by Kleiman [6] in the context of axiomatizations of varieties of groups and adapted to logics[2] by Chagrov and Zakharyaschev [3, Lemma 1.1]:

Lemma 1.1. *If a logic L_1 is independently axiomatizable, then, for every finitely axiomatizable logic L_2 such that $L_2 \subset L_1$, the interval $[L_2, L_1] = \{L : L_2 \subseteq L \subseteq L_1\}$ of logics contains an immediate predecessor of L_1, i.e., a logic L_0 such that $L_2 \subseteq L_0 \subset L_1$ and no logic L satisfies the condition $L_0 \subset L \subset L_1$.*

Proof. Suppose first that L_1 has a finite axiomatization. Let φ be the conjunction of the axioms of L_1. Then every chain $\mathcal{C}$ of logics in the set $\{L : L_2 \subseteq L \subseteq L_1 \;\&\; \varphi \notin L\}$ is bounded above by $\bigcup\mathcal{C}$. Hence, by Zorn's lemma, this set contains a maximal logic; this logic is an immediate predecessor of L_1 in $[L_2, L_1]$.

Suppose next that L_1 has an infinite independent set $\{\varphi_n : n \in \mathbb{N}\}$ of axioms. Since L_2 is a finitely axiomatizable sublogic of L_1, there exists $m \in \mathbb{N}$ such that L_2 is contained in the logic with axioms $\varphi_0, \ldots, \varphi_m$. Let L_3 be the logic with axioms $\varphi_0, \ldots, \varphi_m, \varphi_{m+2}, \varphi_{m+3}, \ldots$. Then $L_2 \subseteq L_3 \subseteq L_1$ and, since the set $\{\varphi_n : n \in \mathbb{N}\}$ is independent, $\varphi_{m+1} \notin L_3$. By Zorn's lemma, the set $\{L : L_3 \subseteq L \subseteq L_1 \;\&\; \varphi_{m+1} \notin L\}$ contains a maximal logic; this logic is an immediate predecessor of L_1 in $[L_3, L_1]$ and, hence, in $[L_2, L_1]$. □

In view of Lemma 1.1, to show that a logic L_1 is not independently axiomatizable, it suffices to exhibit a finitely axiomatizable proper sublogic L_2 of L_1 such that the interval $[L_2, L_1]$ does not contain an immediate predecessor of L_1.

In the present paper, the role of L_1 from the statement of Lemma 1.1 is played by each of the logics from the countable set $\{\mathbf{L}^s_\omega : s \geqslant 2\}$: each logic $\mathbf{L}^s_\omega$, with $s \geqslant 2$, is shown not to allow an independent set of axioms. For each logic $\mathbf{L}^s_\omega$, the role of L_2 from the statement of Lemma 1.1 is played by the logic $\mathbf{L}^s$ obtained by adding a finite set of axioms to the system **S**. Logics $\mathbf{L}^s$ are finitely axiomatizable with (Sub) and (MP), and hence meet the requirement of Lemma 1.1.

[1] The terminology is due to Segerberg [7].

[2] Formally, Chagrov and Zakharyaschev speak of normal modal logics; it is, however, easy to see from their proof, given here as Lemma 1.1, that normality is not essential to their argument.

To prove that, for each $s \geqslant 2$, the interval $[\mathbf{L}^s, \mathbf{L}^s_\omega]$ does not contain an immediate predecessor of $\mathbf{L}^s_\omega$, we show that every frame validating $\mathbf{L}^s$ (for short, $\mathbf{L}^s$-frame) belongs to one of the special classes $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_\omega$, and, moreover, that $\mathbf{L}^s$ has frames in each of these classes; we also show that, if a logic L has a frame in $\mathscr{F}_n$, for some $n \in \mathbb{N}^+$ (we denote by $\mathbb{N}^+$ the set of positive integers), then L is bound to have a frame in $\mathscr{F}_{n+1}$. The logic $\mathbf{L}^s_\omega$ is then defined as the logic of $\mathbf{L}^s$-frames not belonging to $\bigcup_{n\in\mathbb{N}^+} \mathscr{F}_n$. We then use the following argument to prove that $\mathbf{L}^s_\omega$ does not have immediate predecessors in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$. Suppose that L_0 is an immediate predecessor of $\mathbf{L}^s_\omega$ in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$, i.e., $L_0 \subset \mathbf{L}^s_\omega$ and there exists no logic L such that $L_0 \subset L \subset \mathbf{L}^s_\omega$. Provided (the assumption we return to shortly) L_0 can be separated from $\mathbf{L}^s_\omega$ by frames, L_0 has a frame in $\mathscr{F}_n$, for some $n \in \mathbb{N}^+$. Let k be the smallest number such that L_0 has a frame in $\mathscr{F}_k$, and let L'_0 be the logic of the L_0-frames not belonging to $\mathscr{F}_k$. Since L_0 is bound to have a frame in $\mathscr{F}_{k+1}$, it follows that L'_0 has a frame in $\mathscr{F}_{k+1}$; we also show that L'_0 has no frames in $\mathscr{F}_k$; hence, contrary to the assumption, $L_0 \subset L'_0 \subset \mathbf{L}^s_\omega$. Thus, no logic is an immediate predecessor of $\mathbf{L}^s_\omega$ in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$. Since $\mathbf{L}^s$ is finitely axiomatizable, it follows, by Lemma 1.1, that $\mathbf{L}^s_\omega$ is not independently axiomatizable.

This outline of the structure of our argument shows that, when choosing our semantic apparatus, we need to pick a class $\mathscr{C}$ of frames that is complete for the lattice $\mathcal{E}\,\mathbf{S}$ of extensions of $\mathbf{S}$, in the sense that every consistent logic in $\mathcal{E}\,\mathbf{S}$ is determined by a non-empty subclass of $\mathscr{C}$; then, we can ensure that every proper sublogic of $\mathbf{L}^s_\omega$ can be separated from $\mathbf{L}^s_\omega$ by frames from $\mathscr{C}$. On the other hand, we need to prove the existence of $\mathbf{L}^s$-frames in every class $\mathscr{F}_\kappa$, with $\kappa \in \mathbb{N}^+$ or $\kappa = \omega$. To simultaneously satisfy these requirements, we have chosen $\mathscr{C}$ to be the class of refined general frames.[3]

The paper is structured as follows. Section 2 contains preliminaries on—in particular, quasi-normal—modal logics. In Section 3, we establish semantic conditions for some formulas over refined general frames. In Section 4, we consider sets of points in refined frames that play an important role in describing frames of the logics $\mathbf{L}^s$. In Section 5, we define the logics $\mathbf{L}^s$, prove their finite axiomatizability, and describe their frames. In Section 6, we consider classes $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_\omega$ of frames playing a pivotal role in our argument. We present our main result in Section 7. We conclude, in Section 8, with questions for future study.

2. Preliminaries

In this section, we present background on propositional modal logic, mostly to fix the terminology and notation used in the paper; for more extensive background, we refer the reader to standard textbooks [4,1].

2.1. Syntax

We work with the standard propositional monomodal language containing a countable supply of proposition letters, the Boolean constant $\bot$, the binary Boolean connective $\to$, the unary modal connective $\Box$, and parentheses. Formulas are defined in the usual way. We use lower-case Latin letters like p and q for proposition letters, lower-case Greek letters like φ and ψ for formulas, and upper-case Greek letters like Γ and Δ for sets of formulas. We use the standard abbreviations $\neg\varphi = (\varphi \to \bot)$, $\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$, $\varphi \vee \psi = (\neg\varphi \to \psi)$, $\Diamond\varphi = \neg\Box\neg\varphi$, $\Box^+\varphi = \Box\varphi \wedge \varphi$, and $\Diamond^+\varphi = \Diamond\varphi \vee \varphi$, as well as the following ones: $\Box^0\varphi = \varphi$, $\Box^{n+1}\varphi = \Box\Box^n\varphi$, and $\Diamond^n\varphi = \neg\Box^n\neg\varphi$, for every $n \in \mathbb{N}$. When parentheses are omitted, $\neg$ and $\Box$ are assumed to bind tighter than $\wedge$ and $\vee$, which are assumed to bind tighter than $\to$. The following formulas shall be referred to often (here, p and q are fixed proposition letters):

[3] We have chosen to avoid dealing with descriptive general frames, due to the difficulty of establishing compactness of specific frames, which would lengthen and complicate the proofs. This deprives us of the use of the Sahlquist correspondence theorem [4, Theorem 10.30]; this, however, creates only a minor inconvenience since we deal with only two, very simple, Sahlquist formulas. Our earlier result on existence of quasi-normal modal logics without independent axioms [5, Theorem 1.1] relied substantially on the use of Kripke frames. This approach does not work for extensions of $\mathbf{S}$, since $\mathbf{S}$, as is well known, does not admit Kripke frames: over Kripke frames, the Löb axiom requires irreflexivity, while $\Box p \to p$ corresponds to reflexivity.

$$\begin{aligned} \boldsymbol{k} &= \Box(p \to q) \to (\Box p \to \Box q);\\ \boldsymbol{ref} &= \Box p \to p;\\ \boldsymbol{tra} &= \Box p \to \Box\Box p;\\ \boldsymbol{la} &= \Box(\Box p \to p) \to \Box p. \end{aligned}$$

Following standard practice in modal logic, we identify logics with sets of formulas, conceived of as their theorems. We distinguish between normal and quasi-normal modal logics.

A *normal modal logic* is a set L of formulas containing the classical propositional tautologies and the formula $\boldsymbol{k}$ and closed under (Sub), (MP), and (N). The smallest normal modal logic is called $\mathbf{K}$. Every normal logic can be represented as $\mathbf{K} \oplus \Gamma$, where $\Delta \oplus \Gamma$ is the closure of the set $\Delta \cup \Gamma$ under (Sub), (MP), and (N); we write $\Delta \oplus \varphi$ instead of $\Delta \oplus \{\varphi\}$. We recall the definitions of the standard normal modal logics we shall be working with:

$$\begin{aligned} \mathbf{K4} &= \mathbf{K} \oplus \boldsymbol{tra};\\ \mathbf{GL} &= \mathbf{K4} \oplus \boldsymbol{la}. \end{aligned}$$

A *quasi-normal modal logic* is a set L of formulas including $\mathbf{K}$ and closed under (Sub) and (MP), but not necessarily under (N). Every quasi-normal logic can be represented as $\mathbf{K} + \Gamma$, where $\Delta + \Gamma$ is the closure of the set $\Delta \cup \Gamma$ under (Sub) and (MP). We shall be concerned with the quasi-normal logic $\mathbf{S}$ defined by

$$\mathbf{S} = \mathbf{GL} + \boldsymbol{ref}.$$

Logic simpliciter refers, in this paper, to both normal and quasi-normal logics.

A quasi-normal logic L' is a *quasi-normal extension* of a logic L if $L' = L + \Gamma$, for some set Γ of formulas. The set of quasi-normal extensions of L is denoted by $\mathcal{E}\,L$. For every logic L, the set $\mathcal{E}\,L$ forms a lattice with respect to set inclusion. For every $L_1, L_2 \in \mathcal{E}\,\mathbf{K}$, the *interval* $[L_1, L_2]$ is the set $\{L \in \mathcal{E}\,\mathbf{K} : L_1 \subseteq L \subseteq L_2\}$.

By definition, every normal modal logic is quasi-normal. For normal logics, either normality or quasi-normality can be underscored through a representation by an axiomatic system. We distinguish two types of modal axiomatic systems; for both, we stipulate a (not necessarily recursively enumerable) set of axioms; the difference lies in the stipulated rules of inference.

We say that a formula φ is *N-derivable* from a set Γ of formulas, and write $\Gamma \vdash^N \varphi$, if φ can be obtained from Γ by finitely many applications of (Sub), (MP), and (N). A logic L is *independently N-axiomatizable* if there exists a set Γ of formulas such that

$$L = \{\varphi : \Gamma \vdash^N \varphi\}, \tag{1}$$

but $\Gamma - \{\varphi\} \nvdash^N \varphi$, for any $\varphi \in \Gamma$. If there exists a finite set Γ satisfying (1), then L is *finitely N-axiomatizable*.

We say that a formula φ is *QN-derivable* from a set Γ of formulas, and write $\Gamma \vdash \varphi$, if φ can be obtained from Γ by finitely many applications of (Sub) and (MP). A logic L is *independently QN-axiomatizable* if there exists a set Γ of formulas such that

$$L = \{\varphi : \Gamma \vdash \varphi\}, \tag{2}$$

but $\Gamma - \{\varphi\} \nvdash \varphi$, for any $\varphi \in \Gamma$. If there exists a finite set Γ satisfying (2), then L is *finitely QN-axiomatizable*. If there exists a recursive set Γ satisfying (2), then L is *recursively QN-axiomatizable*.

Since deductively dependent formulas can always be removed from a finite set of axioms, every finitely QN-axiomatizable logic is independently QN-axiomatizable; similarly for N-axiomatizability. As the example by Chagrov and Zakharyaschev [3, p. 301] shows, a logic that is not independently N-axiomatizable can be independently QN-axiomatizable: dependency of axioms may arise due to the additional rule of inference.

2.2. Semantics

We use the standard notation for relations and operations on sets: if $W \neq \varnothing$, $R \subseteq W \times W$, $x \in W$, and $X \subseteq W$, then

- R^0 is identity on W, and $R^{n+1} = R \circ R^n$ whenever $n \in \mathbb{N}$;
- $R^* = \bigcup_{n=0}^{\infty} R_n$; this is the reflexive transitive closure of R;
- R^{-1} is the converse of R;
- $R(x) = \{y \in W : xRy\}$;
- $\Box X = \{x \in W : R(x) \subseteq X\}$;
- $\Diamond X = \{x \in W : R(x) \cap X \neq \varnothing\}$;
- $|X|$ is the cardinality of X;
- $R \restriction X = R \cap (X \times X)$.

For reasons given in the introduction, our semantic apparatus is refined (general) frames with a distinguished point; a distinguished point is a standard way to model logics not closed under (N). A *frame with a distinguished point* is a structure $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ where

- W is a non-empty set of *points*;
- R is a binary *accessibility* relation on W;
- $\Pi \subseteq 2^W$ is a set of *admissible subsets of* W satisfying the following closure conditions:

$$\varnothing \in \Pi; \qquad X \in \Pi \Longrightarrow \overline{X}, \Box X \in \Pi; \qquad X, Y \in \Pi \Longrightarrow X \cup Y \in \Pi; \tag{3}$$

- $d \in W$ is a *distinguished point*.

Thus, Π is a modal algebra of subsets of W. In this paper, all frames have a distinguished point; hence, we usually omit the modifier 'with a distinguished point'. Since $\Diamond X = \overline{\Box\overline{X}}$, for every $X \subseteq W$, the set Π is closed under $\Diamond$; clearly, Π is also closed under $\cap$ and $-$ (set difference). If $y \in R(x)$, we say that y is *accessible* from x and that x *sees* y. We say that a point y is *accessible from a point* x *in exactly one step*, and write $y \in R^{=1}(x)$, if $y \in R(x)$, but $y \notin R^n(x)$, for any $n \geqslant 2$.

Given a frame $\langle W, R, \Pi, d\rangle$ and a set $X \subseteq W$, we recursively define $\Box^0 X = X$; $\Box^{n+1} X = \Box\Box^n X$, for every $n \in \mathbb{N}$. We also define $\Diamond^n X = \overline{\Box^n\overline{X}}$, for every $n \in \mathbb{N}$.

A frame $\langle W, R, \Pi, d\rangle$ is

- *differentiated* if, for every $x, y \in W$,

$$x \neq y \iff (\exists X \in \Pi)\,[x \in X \,\&\, y \notin X]; \tag{4}$$

- *tight* if, for every $x, y \in W$,

$$xRy \iff (\forall X \in \Pi)\,[y \in X \Rightarrow x \in \Diamond X]; \tag{5}$$

- *refined* if it is differentiated and tight;
- *compact* if each $\Sigma \subseteq \Pi$ has a non-empty intersection whenever every finite subset of Σ has a non-empty intersection;
- *descriptive* if it is refined and compact;
- *a Kripke frame* if $\Pi = 2^W$.

A *valuation* on a frame $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ is a map v assigning to a proposition letter p a set $v(p) \in \Pi$. The satisfaction relation between frames $\mathfrak{F}$, points x, valuations v, and formulas φ is defined recursively:

- $\mathfrak{F}, x \models^v p \leftrightharpoons x \in v(p)$ if p is a proposition letter;
- $\mathfrak{F}, x \not\models^v \bot$;
- $\mathfrak{F}, x \models^v \varphi_1 \to \varphi_2 \leftrightharpoons \mathfrak{F}, x \not\models^v \varphi_1$ or $\mathfrak{F}, x \models^v \varphi_2$;
- $\mathfrak{F}, x \models^v \Box\varphi_1 \leftrightharpoons \mathfrak{F}, x' \models^v \varphi_1$ whenever $x' \in R(x)$.

Let $\mathfrak{F}$ be a frame with a distinguished point d, let φ be a formula, and let Γ be a set of formulas; then

- we say that φ is *valid at x in* $\mathfrak{F}$, and write $\mathfrak{F}, x \models \varphi$, if $\mathfrak{F}, x \models^v \varphi$, for every valuation v on $\mathfrak{F}$;
- we say that φ is *valid* on $\mathfrak{F}$, and write $\mathfrak{F} \models \varphi$, if $\mathfrak{F}, d \models \varphi$;
- we say that Γ is *valid* on $\mathfrak{F}$, and write $\mathfrak{F} \models \Gamma$, if $\mathfrak{F}, d \models \varphi$, for every $\varphi \in \Gamma$.

When $\mathfrak{F}$ is clear from the context, we write $x \models^v \varphi$ instead of $\mathfrak{F}, x \models^v \varphi$, and $x \models \varphi$ instead of $\mathfrak{F}, x \models \varphi$. The following is well known and straightforward to check:

Fact 2.1. The formula $\Box^+\boldsymbol{k}$ is valid on every frame with a distinguished point.

A frame with a distinguished point $\langle W, R, \Pi, d\rangle$ is *rooted* if $W = R^*(d)$. Since we are interested in formulas valid only at the distinguished point of a frame, we may assume, without a loss of generality, that all frames are rooted. Hence, in the rest of the paper, frames are always assumed to be rooted.

If L is a logic, a frame $\mathfrak{F}$ is said to be a *frame for* L, or simply an *L-frame*, if $\mathfrak{F} \models L$; the class of L-frames is denoted by $\mathtt{Fr}\, L$. If $\mathscr{C}$ is a class of frames, $\mathtt{L}\mathscr{C}$ denotes the set of formulas valid on every frame in $\mathscr{C}$; we write $\mathtt{L}\mathfrak{F}$ instead of $\mathtt{L}\{\mathfrak{F}\}$. The following is well known [4, Proposition 5.57] and easy to check:

Proposition 2.2 *(Soundness of frame semantics). For every class $\mathscr{C}$ of frames with a distinguished point, the set $\mathtt{L}\mathscr{C}$ is a quasi-normal modal logic.*

A logic L is *determined* by a class $\mathscr{C}$ of frames if $L = \mathtt{L}\mathscr{C}$. A class $\mathscr{C}_0$ of frames is *complete* for a class $\mathscr{L}$ of logics if, for every consistent logic $L \in \mathscr{L}$, there exists a non-empty subclass $\mathscr{C}$ of $\mathscr{C}_0$ such that $L = \mathtt{L}\mathscr{C}$. It is well known [4, Theorems 8.36 and 8.61] that the class of rooted descriptive frames with a distinguished point is complete for $\mathcal{E}\,\mathbf{K}$. Since descriptive frames are refined, the following is also true:

Proposition 2.3 *(Completeness of rooted refined frame semantics). The class of rooted refined frames with a distinguished point is complete for $\mathcal{E}\,\mathbf{K}$.*

Since $\mathcal{E}\,\mathbf{S} \subseteq \mathcal{E}\,\mathbf{K}$, the following is also true:

Corollary 2.4. *The class of rooted refined frames with a distinguished point is complete for $\mathcal{E}\,\mathbf{S}$.*

Remark 2.5. In view of Corollary 2.4, in the rest of the paper, a frame simpliciter, unless indicated otherwise, means a rooted refined frame with a distinguished point.

3. Semantic conditions for formulas on refined frames

In this section, we establish semantic conditions over refined frames with a distinguished point for formulas that shall be used in (finite) *QN*-axiomatizations of the logics $\mathbf{L}^s$. We also introduce here some frame-related terminology used in subsequent sections.

Definition 3.1. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame. Then

- the point $x \in W$ is *transitive* if $R^2(x) \subseteq R(x)$;
- $\mathfrak{F}$ is *transitive* if every point in W is transitive;
- $\mathfrak{F}$ is *point-transitive* if d is transitive.

We shall often use the following fact (recall that all the frames are assumed to be rooted):

Fact 3.2. If a frame $\langle W, R, \Pi, d\rangle$ is point-transitive, then $W = \{d\} \cup R(d)$.

Proof. Suppose that $x \in W - \{d\}$. Since the frame is rooted, there exists $n \geqslant 1$ such that $x \in R^n(d)$. Since d is transitive, it follows that $x \in R(d)$. □

Lemma 3.3. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame and let $x \in W$. Then $\mathfrak{F}, x \models \boldsymbol{tra}$ if, and only if, x is transitive.*

Proof. Assume that x is not transitive. Then there exists $y \in W$ such that $y \in R^2(x) - R(x)$. By (5), there exists $X \in \Pi$ such that $y \in X$ and $x \notin \Diamond X$. Let v be a valuation with $v(p) = \overline{X}$. Then $x \not\models^v \boldsymbol{tra}$. Conversely, assume that there exists a valuation v such that $x \not\models^v \boldsymbol{tra}$. Since $x \not\models^v \Box\Box p$, there exists $y \in R^2(x) - v(p)$. Since $x \models^v \Box p$, it follows that $y \notin R(x)$. Hence, $y \in R^2(x) - R(x)$. Thus, x is not transitive. □

Lemma 3.4. *Let $\mathfrak{F}$ be a frame. Then $\mathfrak{F} \models \Box^+\boldsymbol{tra}$ if, and only if, $\mathfrak{F}$ is transitive.*

Proof. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$. Suppose that $\mathfrak{F}$ is transitive. Then every point in $\{d\} \cup R(d)$ is transitive. Hence, by Lemma 3.3, $d \models \Box^+\boldsymbol{tra}$, which means that $\mathfrak{F} \models \Box^+\boldsymbol{tra}$. Conversely, suppose that $d \models \Box^+\boldsymbol{tra}$. Then $d \models \boldsymbol{tra}$, and so, by Lemma 3.3, d is transitive. Hence, by Fact 3.2, $W = \{d\} \cup R(d)$. Since $d \models \Box^+\boldsymbol{tra}$, it follows that $\mathfrak{F}, x \models \boldsymbol{tra}$, for every $x \in W$. Hence, by Lemma 3.3, $\mathfrak{F}$ is transitive. □

We next define a number of properties of point-transitive frames.

Definition 3.5. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a point-transitive frame. We define the following properties of $\mathfrak{F}$:

(S1) $X - \Diamond X \neq \varnothing$ whenever $X \in \Pi - \{\varnothing\}$.
(S2) $x \in \Diamond X \Longrightarrow x \in \Diamond(X - \Diamond X)$ whenever $x \in R(d)$ and $X \in \Pi$.
(S3) $d \in X \Longrightarrow d \in \Diamond X$ whenever $X \in \Pi$;
(S4) there do not exist $x_1, x_2 \in W$ and $X_1, X_2 \in \Pi$ such that

$$X_1 \cap X_2 = \varnothing, \quad x_1 \in X_1 - \Diamond(X_1 \cup X_2), \quad x_2 \in X_2 - \Diamond(X_1 \cup X_2),$$

and either $x_1 \in R^{=1}(d)$ (in this case, d is irreflexive) or there exists $x \in R(d)$ such that $x_1 \in R^{=1}(x)$ and $x_2 \in R(x)$ (in this case, d can be either reflexive or irreflexive).

Lemma 3.6. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a point-transitive frame. Then $\mathfrak{F} \models \boldsymbol{la}$ if, and only if, $\mathfrak{F}$ satisfies property* (S1) *of Definition 3.5.*

Proof. For the 'only if', let $d \models \boldsymbol{la}$ and let $X \in \Pi - \{\varnothing\}$. We distinguish two cases.

Case $R(d) \cap X \neq \varnothing$: Let v be a valuation with $v(p) = \overline{X}$; hence, $X = \overline{v(p)}$. Then $d \not\models^v \Box p$. Since, by assumption, $d \models^v \boldsymbol{la}$, it follows that $d \not\models^v \Box(\Box p \to p)$. Hence, there exists $x \in R(d)$ such that $x \models^v \Box p$ but $x \not\models^v p$. Then $x \in X$, but $x \notin \Diamond X$; hence, $x \in X - \Diamond X$. Thus, in this case, $X - \Diamond X \neq \varnothing$.

Case $R(d) \cap X = \varnothing$: Since $\mathfrak{F}$ is point-transitive, by Fact 3.2, $W = \{d\} \cup R(d)$. Since $X \neq \varnothing$, it follows that $X = \{d\}$ and d is irreflexive. Hence, $d \in X - \Diamond X$. Thus, in this case, too, $X - \Diamond X \neq \varnothing$.

For the 'if', assume that $X - \Diamond X \neq \varnothing$ whenever $X \in \overline{\Pi} - \{\varnothing\}$. Let v be a valuation such that $d \models^v \Box(\Box p \to p)$ but $d \not\models^v \Box p$. Then $d \in \Diamond\overline{v(p)}$, and so $\overline{v(p)} \neq \varnothing$. Since $\overline{v(p)} \in \overline{\Pi}$, by assumption, $\overline{v(p)} - \Diamond\overline{v(p)} \neq \varnothing$. Choose $x \in \overline{v(p)} - \Diamond\overline{v(p)}$. Then $x \models^v \Box p$, but $x \not\models^v p$. Since $d \not\models^v \Box p$, it follows that $x \neq d$; hence, by Fact 3.2, $x \in R(d)$. Since $d \models^v \Box(\Box p \to p)$, it follows that $x \models^v \Box p \to p$, contrary to the choice of x. $\Box$

Lemma 3.7. *Let $\mathfrak{F} = \langle W, R, \overline{\Pi}, d\rangle$ be a frame. Then $\mathfrak{F} \models \Box\boldsymbol{la}$ if, and only if, $\mathfrak{F}$ satisfies property* (S2) *of Definition 3.5.*

Proof. For the 'only if', assume that $d \models \Box\,\boldsymbol{la}$, $X \in \overline{\Pi}$, and $x \in R(d) \cap \Diamond X$. Let v be a valuation with $v(p) = \overline{X}$; hence, $X = \overline{v(p)}$. Then $x \not\models^v \Box p$. Since $d \models \Box\,\boldsymbol{la}$, and so $x \models \boldsymbol{la}$, it follows that $x \not\models^v \Box(\Box p \to p)$. Thus, $x \in \Diamond(X - \Diamond X)$.

For the 'if', assume that $x \in R(d) \cap \Diamond X$ implies that $x \in \Diamond(X - \Diamond X)$ whenever $X \in \overline{\Pi}$. Let v be a valuation such that $d \not\models^v \Box\,\boldsymbol{la}$. Then there exists $y \in R(d)$ such that $y \models^v \Box(\Box p \to p)$ but $y \not\models^v \Box p$. Thus, $y \in \Diamond\overline{v(p)}$. Hence, by assumption, $y \in \Diamond(\overline{v(p)} - \Diamond\overline{v(p)})$, which is impossible since $y \models^v \Box(\Box p \to p)$. $\Box$

Lemma 3.8. *Let $\mathfrak{F} = \langle W, R, \overline{\Pi}, d\rangle$ be a refined frame. Then $\mathfrak{F} \models \boldsymbol{ref}$ if, and only if, $\mathfrak{F}$ satisfies property* (S3) *of Definition 3.5.*

Proof. For the 'only if', assume that there exists $X \in \overline{\Pi}$ such that $d \in X - \Diamond X$. Let v be a valuation with $v(p) = \overline{X}$. Then $d \not\models^v \boldsymbol{ref}$. For the 'if', let v be a valuation such that $d \not\models^v \boldsymbol{ref}$. Then $d \models^v \Box p$ but $d \not\models^v p$. Set $X = \overline{v(p)}$. Then $X \in \overline{\Pi}$ and $d \in X - \Diamond X$. $\Box$

We next consider frames characterized by properties of chains and antichains of their points.

Definition 3.9. Let $\mathfrak{F} = \langle W, R, \overline{\Pi}, d\rangle$ be a frame. A non-empty subset B of W is a *chain* in $\mathfrak{F}$ if it is linearly ordered by R, i.e., for every pair of distinct points $x, y \in B$, either $y \in R(x)$ or $x \in R(y)$. A non-empty subset B of W is an *antichain* in $\mathfrak{F}$ if, for every pair of distinct points $x, y \in B$, neither $y \in R(x)$ nor $x \in R(y)$; in particular, every singleton subset of W is an antichain. If an antichain is a singleton, it is *trivial*; otherwise, it is *non-trivial*.

Definition 3.10. Let $\mathfrak{F}$ be a frame. The *breadth of* $\mathfrak{F}$, denoted by $bw\,\mathfrak{F}$, is the supremum of cardinalities of antichains in $\mathfrak{F}$. If there exists $n \in \mathbb{N}^+$ such that $bw\,\mathfrak{F} = n$, then $\mathfrak{F}$ is a *frame of finite width*.

Define, for every $n \in \mathbb{N}^+$,

$$\boldsymbol{bw}_n = \bigwedge_{0 \leqslant i \leqslant n} \Diamond p_i \to \bigvee_{0 \leqslant i < j \leqslant n} \big(\Diamond(p_i \wedge \Diamond^+ p_j) \vee \Diamond(p_j \wedge \Diamond^+ p_i)\big).$$

The following is a straightforward consequence of a well-known fact [4, Proposition 10.32] about validity on (rooted) refined transitive frames without a distinguished point:

Lemma 3.11. *Let $\mathfrak{F}$ be a point-transitive frame and let $n \in \mathbb{N}^+$. Then $\mathfrak{F} \models \boldsymbol{bw}_n$ if, and only if, $bw\,\mathfrak{F} \leqslant n$.*

Definition 3.12. Let $\mathfrak{F} = \langle W, R, \overline{\Pi}, d\rangle$ be a frame and V a finite, non-empty subset of W. A system of sets $B \subseteq \overline{\Pi}$ is a *differentiating system for* V if

- for every $x \in V$, there exists $X \in B$ such that $x \in X$;

- $X' \cap X'' = \varnothing$ whenever $X', X'' \in B$ and $X' \neq X''$.

Lemma 3.13. *Every differentiated (and, hence, every refined) frame $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ contains a differentiating system for every finite, non-empty subset of W.*

Proof. Let V be a finite, non-empty subset of W. We show, by induction on $|V|$, that $\mathfrak{F}$ contains a differentiating system for V.

If $|V| = 1$, then $\{W\}$ is a differentiating system for V. Assume the claim holds for every subset of W of cardinality n, with $n \geqslant 1$. Suppose that $V = \{x_1, \ldots, x_n, x\}$. By inductive hypothesis, $\mathfrak{F}$ contains a differentiating system for $\{x_1, \ldots, x_n\}$. Let $\{X_1, \ldots, X_n\}$ be such a system, with $x_i \in X_i$. By (4), for each $i \in \{1, \ldots, n\}$, there exists $Y_i \in \Pi$ such that $x \in Y_i$ but $x_i \notin Y_i$. Set $Y = \bigcap\limits_{1 \leqslant i \leqslant n} Y_i$ and $Z_i = X_i - Y$, for each $i \in \{1, \ldots, n\}$. Then $\{Z_1, \ldots, Z_n, Y\}$ is a differentiating system for V. □

Lemma 3.14. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a differentiated frame of finite width, $X \in \Pi$ and $x \in X - \Diamond X$. Then $\{x\} \in \Pi$.*

Proof. By assumption, $X - \Diamond X \neq \varnothing$. Clearly, $X - \Diamond X$ is an antichain. Since, by assumption, $\mathfrak{F}$ is a frame of finite width, the set $X - \Diamond X$ is finite, so let $X - \Diamond X = \{x_1, \ldots, x_n\}$. By Lemma 3.13, $\mathfrak{F}$ contains a differentiating system for $X - \Diamond X$. Let $X_1, \ldots, X_n$ be such a system, with $x_i \in X_i$. Then $\{x_i\} = X_i \cap (X - \Diamond X)$, for each $i \in \{1, \ldots, n\}$. By Definition 3.12, $X_i \in \Pi$; hence, $\{x_i\} \in \Pi$. □

We next consider frame conditions for the formulas $\Box^+\boldsymbol{inf}$ defined by

$$\begin{aligned} \alpha(r_1, r_2) &= r_1 \wedge \Box\neg r_1 \wedge \Box^+\neg r_2; \\ \boldsymbol{inf} &= \Diamond\alpha(p, q) \wedge \Diamond\alpha(q, p) \to \Diamond^2\alpha(p, q). \end{aligned}$$

Lemma 3.15. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a transitive frame of finite width and let $x \in W$. Then $x \not\models \boldsymbol{inf}$ if, and only if, there exist $x_1, x_2 \in R(x)$ and $X_1, X_2 \in \Pi$ such that $X_1 \cap X_2 = \varnothing$, $x_1 \in X_1 - \Diamond(X_1 \cup X_2)$, $x_2 \in X_2 - \Diamond(X_1 \cup X_2)$, and $x_1 \in R^{=1}(x)$.*

Proof. Let $x_1, x_2 \in R(x)$ be the points satisfying the statement of the lemma. By Lemma 3.14, $\{x_1\}, \{x_2\} \in \Pi$. Let v be a valuation with $v(p) = \{x_1\}$ and $v(q) = \{x_2\}$. Since $x_1, x_2 \in R(x)$, it follows that $x \models^v \Diamond\alpha(p, q) \wedge \Diamond\alpha(q, p)$. Since $x_1 \in R^{=1}(x)$, it follows that $x \not\models^v \Diamond^2\alpha(p, q)$. Hence, $x \not\models \boldsymbol{inf}$.

Conversely, let $x \not\models^v \boldsymbol{inf}$, for some valuation v. Then there exist $x_1, x_2 \in R(x)$ such that $x_1 \models^v \alpha(p, q)$ and $x_2 \models^v \alpha(q, p)$. Hence, $x_1 \in v(p) - \Diamond v(p)$, and so, by Lemma 3.14, $\{x_1\} \in \Pi$. It follows similarly that $\{x_2\} \in \Pi$. Set $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$. Then $x_1 \in X_1$ and, since $x_1 \notin \Diamond v(p)$, also $x_1 \notin \Diamond X_1$; similarly, $x_2 \in X_2$ and $x_2 \notin \Diamond X_2$. Since $x_1 \models^v p$ but $x_2 \not\models^v p$, surely $x_1 \neq x_2$. Hence, $X_1 \cap X_2 = \varnothing$. Since $x_1 \models^v \Box\neg q$ and $x_2 \models^v q$, it follows that $x_1 \notin \Diamond X_2$. Since $x_2 \models^v \Box\neg p$ and $x_1 \models^v p$, it follows that $x_2 \notin \Diamond X_1$. Hence,

$$x_1 \in X_1 - \Diamond(X_1 \cup X_2) \quad \text{and} \quad x_2 \in X_2 - \Diamond(X_1 \cup X_2).$$

Since $d \not\models \Diamond^2\alpha(p, q)$, surely $x_1 \notin R^2(x)$, and so, since R is transitive, $x_1 \notin R^n(x)$, for any $n \geqslant 2$. Hence, $x_1 \in R^{=1}(x)$. Thus, x_1 and x_2 are the points satisfying the statement of the lemma. □

Lemma 3.16. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a transitive frame of finite width. Then $\mathfrak{F} \models \Box^+\boldsymbol{inf}$ if, and only if, $\mathfrak{F}$ satisfies property* (S4) *of Definition 3.5.*

Proof. Immediate from Lemma 3.15. □

4. Slices of points in frames

In this section, which mostly consists of a series of simple lemmas, we consider special sets of points in frames and discuss their properties captured by modal formulas.

Definition 4.1. Let $\mathfrak{F}$ be a frame. Then

$$S_0^{\mathfrak{F}} = \varnothing; \qquad S_n^{\mathfrak{F}} = \Box^n\varnothing - \Box^{n-1}\varnothing, \text{ for every } n \in \mathbb{N}^+.$$

When $\mathfrak{F}$ is clear from the context, we write S_n instead of $S_n^{\mathfrak{F}}$.

Lemma 4.2. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame. Then $S_n^{\mathfrak{F}} \in \Pi$, for every $n \in \mathbb{N}$.*

Proof. By (3), $\varnothing \in \Pi$ and Π is closed under $-$ and $\Box$. □

Lemma 4.3. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame and $x \in W$. Then, for every $n \in \mathbb{N}^+$,*

$$x \in S_n^{\mathfrak{F}} \iff x \models \Box^n\bot \text{ and } x \not\models \Box^{n-1}\bot.$$

Proof. We show, by induction on n, that, for every $n \in \mathbb{N}$,

$$x \in \Box^n\varnothing \iff x \models \Box^n\bot.$$

The statement of the lemma then follows by definition of $S_n^{\mathfrak{F}}$. If $n = 0$, the equivalence holds trivially. For the inductive step, observe that

$$x \in \Box^{n+1}\varnothing \iff R(x) \subseteq \Box^n\varnothing \iff \forall y\,(xRy \Rightarrow y \models \Box^n\bot) \iff x \models \Box^{n+1}\bot. \quad \Box$$

Thus, if R is transitive, then $x \in S_n^{\mathfrak{F}}$ if, and only if, the cardinality of the longest R-chain in $\mathfrak{F}$ beginning with x equals n.

Lemma 4.4. *For every frame $\mathfrak{F}$ and every $n \in \mathbb{N}^+$, the set $S_n^{\mathfrak{F}}$ is an antichain containing only irreflexive points.*

Proof. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$. Suppose that $x, y \in S_n$. By Lemma 4.3, $x \models \Box^n\bot$ and $y \not\models \Box^{n-1}\bot$; hence, $y \notin R(x)$. If we take x to be y, we see that every point in S_n is irreflexive. □

Lemma 4.5. *Let $\mathfrak{F}$ be a frame of finite width and $x \in S_n^{\mathfrak{F}}$, for some $n \in \mathbb{N}^+$. Then $\{x\} \in \Pi$.*

Proof. By Lemma 4.4, $x \in S_n - \Diamond S_n$. By Lemma 4.2, $S_n \in \Pi$. Hence, by Lemma 3.14, $\{x\} \in \Pi$. □

Lemma 4.6. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame, let $n \geqslant 2$, and let $x \in S_n^{\mathfrak{F}}$. Then $R(x) \cap S_{n-1}^{\mathfrak{F}} \neq \varnothing$.*

Proof. By Lemma 4.3, $x \models \Box^n\bot$ and $x \not\models \Box^{n-1}\bot$. Since $x \not\models \Box^{n-1}\bot$, there exists $y \in R(x)$ such that $y \not\models \Box^{n-2}\bot$. Since $x \models \Box^n\bot$ and $y \in R(x)$, it follows that $y \models \Box^{n-1}\bot$. Hence, by Lemma 4.3, $y \in S_{n-1}$. □

Lemma 4.7. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a transitive frame, let $n \geqslant 2$, and let $x \in S_n^{\mathfrak{F}}$. Then $R(x) \cap S_k^{\mathfrak{F}} \neq \varnothing$, for every $k \in \{1, \ldots, n-1\}$.*

Proof. By repeated applications of Lemma 4.6 and transitivity of R. □

Note that Lemma 4.7 implies that, if $n \geqslant 2$ and $S^{\mathfrak{F}}_n \neq \varnothing$, then $S^{\mathfrak{F}}_k \neq \varnothing$ whenever $1 \leqslant k \leqslant n-1$.

Lemma 4.8. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a transitive frame, let $n \in \mathbb{N}^+$, and let $x \in S^{\mathfrak{F}}_n$. Then $R(x) \subseteq \bigcup_{m<n} S^{\mathfrak{F}}_m$.*

Proof. Suppose that $y \in R(x)$. By Lemma 4.3, $x \models \Box^n\bot$; hence, $y \models \Box^{n-1}\bot$. Thus, $\{k : y \models \Box^k\bot\} \neq \varnothing$. Set $m = \min\{k : y \models \Box^k\bot\}$. Then, by Lemma 4.3, $y \in S_m$. Since $y \models \Box^{n-1}\bot$, surely $m < n$. □

Lemma 4.9. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a point-transitive frame such that $\mathfrak{F} \models \boldsymbol{la}$ and $d \notin \bigcup_{i\in\mathbb{N}} S^{\mathfrak{F}}_i$. Then $S^{\mathfrak{F}}_n \neq \varnothing$, for every $n \in \mathbb{N}^+$.*

Proof. Let $n \in \mathbb{N}^+$. Set $\Sigma_n = \bigcup_{1\leqslant i<n} S_i$. By assumption, $d \notin \bigcup_{i\in\mathbb{N}} S_i$; hence, $d \notin \Sigma_n$; thus, $\overline{\Sigma_n} \neq \varnothing$. By Lemma 4.2, $\overline{\Sigma_n} \in \Pi$. Hence, by Lemma 3.6, $\overline{\Sigma_n} - \Diamond\overline{\Sigma_n} \neq \varnothing$. Let $x \in \overline{\Sigma_n} - \Diamond\overline{\Sigma_n}$. Then $x \notin \Diamond\overline{\Sigma_n}$, and so $R(x) \subseteq \Sigma_n (= \bigcup_{1\leqslant i<n} S_i)$. Hence, $x \models \Box^n\bot$, and so $\{k : x \models \Box^k\bot\} \neq \varnothing$. Let $m = \min\{k : x \models \Box^k\bot\}$. Then, by Lemma 4.3, $x \in S_m$. Since $x \models \Box^n\bot$, surely, $m \leqslant n$. Since $x \notin \Sigma_n$, it follows that $m = n$. Hence, $x \in S_n$, and so $S_n \neq \varnothing$. □

Lemma 4.10. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a frame such that $\mathfrak{F} \models \mathbf{S}$. Then $d \notin \bigcup_{n\in\mathbb{N}} S^{\mathfrak{F}}_n$.*

Proof. Suppose that $d \in S^{\mathfrak{F}}_n$, for some $n \in \mathbb{N}^+$. By Lemma 4.2, $S^{\mathfrak{F}}_n \in \Pi$. Hence, by Lemma 3.8, $d \in \Diamond S^{\mathfrak{F}}_n$, in contradiction with Lemma 4.4. □

Lemma 4.11. *If $\mathfrak{F}$ is a frame such that $\mathfrak{F} \models \mathbf{S}$, then $S^{\mathfrak{F}}_n \neq \varnothing$, for every $n \in \mathbb{N}^+$.*

Proof. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$. By Lemma 4.10, $d \notin \bigcup_{n\in\mathbb{N}} S^{\mathfrak{F}}_n$. Since $\mathfrak{F} \models \boldsymbol{la}$, by Lemma 4.9, $S^{\mathfrak{F}}_n \neq \varnothing$, for every $n \in \mathbb{N}^+$. □

Lemma 4.12. *Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ be a transitive frame such that $\mathfrak{F} \models \Box^+\boldsymbol{la}$ and let $x \in W - \bigcup_{i\in\mathbb{N}} S^{\mathfrak{F}}_i$. Then $R(x) \cap S^{\mathfrak{F}}_n \neq \varnothing$, for every $n \in \mathbb{N}^+$.*

Proof. We proceed by induction on n.

Since $x \notin S_1$, surely $x \in \Diamond W$. Hence, by Lemma 3.7, $x \in \Diamond(W - \Diamond W)$, i.e., $x \in \Diamond S_1$. Assuming that the statement holds for n, we prove it for $n+1$.

Suppose that, on the contrary, $x \notin \Diamond S_{n+1}$. By inductive hypothesis, $x \in \Diamond S_n$. Set

$$X = \Diamond S_n \cap \overline{\Diamond S_{n+1}} \cap \overline{S_{n+1}}.$$

Then $x \in X$, and so $X \neq \varnothing$. By Lemma 4.2, $X \in \Pi$. Hence, by Lemma 3.6, $X - \Diamond X \neq \varnothing$. Let $y \in X - \Diamond X$. Since $y \in X$ and, by definition, $X \subseteq \Diamond S_n$, surely $R(y) \neq \varnothing$. Let $z \in R(y)$. Since $y \in X \subseteq \overline{\Diamond S_{n+1}}$, surely $z \in \overline{S_{n+1}}$; hence, by transitivity, $z \in \overline{\Diamond S_{n+1}}$. Since $y \notin \Diamond X$, surely $z \notin X$. Thus, $z \in \overline{\Diamond S_{n+1}} \cap \overline{S_{n+1}}$, but $z \notin X$; hence, $z \notin \Diamond S_n$, by definition of X. Then, by inductive hypothesis, $z \in \bigcup_{i\in\mathbb{N}} S_i$. Since $z \notin \Diamond S_n$, by Lemma 4.7, $z \in \bigcup_{1\leqslant i\leqslant n} S_i$. Since z is an arbitrary element of $R(y)$, it follows that $R(y) \subseteq \bigcup_{1\leqslant i\leqslant n} S_i$, and hence $y \in \bigcup_{1\leqslant i\leqslant n+1} S_i$. Since $y \in X \subseteq \Diamond S_n$, it follows that $y \in S_{n+1}$. This is, however, impossible since $y \in X \subseteq \overline{S_{n+1}}$. □

5. Logics $\mathbf{L}^s$ and their frames

We now define extensions of **S** that shall be shown not to admit deductively independent sets of axioms.

Definition 5.1. For every $s \geqslant 2$, let

$$\mathbf{L}^s = \mathbf{S} + \boldsymbol{bw}_s + \Box^+\boldsymbol{inf}.$$

We next show that logics $\mathbf{L}^s$ are finitely *QN*-axiomatizable. For every formula φ, denote by $sub\,\varphi$ the set of substitution instances of φ. For every set Δ of formulas, let $\Box^+\Delta = \{\Box^+\varphi : \varphi \in \Delta\}$. Lastly, denote by **CL** the set of Boolean, or classical, tautologies.

Lemma 5.2. *Let Δ be a set of formulas such that $\Delta \vdash \psi$ whenever*

$$\psi \in \mathbf{CL} \cup sub\,\boldsymbol{k} \cup sub\,\boldsymbol{tra},$$

and let φ be a formula such that $\Delta \vdash^N \varphi$. Then $\Box^+\Delta \vdash \varphi$ and $\Box^+\Delta \vdash \Box\varphi$.

Proof. We proceed by induction on an N-derivation of φ from Δ.

If $\varphi \in \Delta$, then $\Box^+\varphi \in \Box^+\Delta$; hence, by classical logic, $\Box^+\Delta \vdash \varphi$ and $\Box^+\Delta \vdash \Box\varphi$.

Suppose that φ has been obtained from ψ by (Sub). By inductive hypothesis, $\Box^+\Delta \vdash \psi$ and $\Box^+\Delta \vdash \Box\psi$. Hence, by (Sub), $\Box^+\Delta \vdash \varphi$ and $\Box^+\Delta \vdash \Box\varphi$.

Suppose that φ has been obtained from ψ and $\psi \to \varphi$ by (MP). By inductive hypothesis, $\Box^+\Delta \vdash \psi$ and $\Box^+\Delta \vdash \psi \to \varphi$; hence, by (MP), $\Box^+\Delta \vdash \varphi$. Also by inductive hypothesis, $\Box^+\Delta \vdash \Box\psi$ and $\Box^+\Delta \vdash \Box(\psi \to \varphi)$. By assumption, $\Delta \vdash sub\,\boldsymbol{k}$; hence, $\Delta \vdash \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$. Thus, by (MP), $\Box^+\Delta \vdash \Box\varphi$.

Suppose that φ has been obtained from ψ by (N), and so $\varphi = \Box\psi$. By inductive hypothesis, $\Box^+\Delta \vdash \Box\psi$, i.e., $\Box^+\Delta \vdash \varphi$. By assumption, $\Box^+\Delta \vdash sub\,\boldsymbol{tra}$; hence, $\Box^+\Delta \vdash \Box\psi \to \Box\Box\psi$. Thus, by (MP), $\Box^+\Delta \vdash \Box\Box\psi$, i.e., $\Box^+\Delta \vdash \Box\varphi$. □

Recall that a set Θ of Boolean tautologies is *deductively complete* if every Boolean tautology can be obtained from Θ by finitely many applications of (Sub) and (MP). The next proposition establishes the finite *QN*-axiomatizability of the modal logic **GL**:

Proposition 5.3. *Let Θ be a finite deductively complete set of Boolean tautologies. Then*

$$\mathbf{GL} = \{\varphi : \Box^+\Theta \cup \{\Box^+\boldsymbol{k}, \Box^+\boldsymbol{tra}, \Box^+\boldsymbol{la}\} \vdash \varphi\}.$$

Proof. The definition of **GL** implies that

$$\mathbf{GL} = \{\varphi : \Theta \cup \{\boldsymbol{k}, \boldsymbol{tra}, \boldsymbol{la}\} \vdash^{\mathsf{N}} \varphi\}.$$

Taking **GL** for Δ in the statement of Lemma 5.2, we see that

$$\mathbf{GL} \subseteq \{\varphi : \Box^+\Theta \cup \{\Box^+\boldsymbol{k}, \Box^+\boldsymbol{tra}, \Box^+\boldsymbol{la}\} \vdash \varphi\}.$$

The converse inclusion follows from the definition of **GL**. □

Since $\mathbf{L}^s = \mathbf{GL} + \boldsymbol{ref} + \boldsymbol{bw}_s + \Box^+\boldsymbol{inf}$, by Proposition 5.3 we obtain the following:

Proposition 5.4. *Every logic $\mathbf{L}^s$ is finitely QN-axiomatizable.*

We next characterize $\mathbf{L}^s$-frames.

Proposition 5.5. *For every $s \geqslant 2$, the logic $\mathbf{L}^s$ is determined by the class of transitive frames of breadth at most s satisfying conditions* (S1) *through* (S4).

Proof. Immediate from Proposition 5.3, Fact 2.1, and Lemmas 3.4, 3.6, 3.7, 3.8, 3.11, and 3.16. □

Lemma 5.6. *Let $s \geqslant 2$ and $\mathfrak{F} \in \mathtt{Fr}\,\mathbf{L}^s$, and let $n \in \mathbb{N}^+$. Then every point in $S^{\mathfrak{F}}_n$ is accessible from some point in $S^{\mathfrak{F}}_{n+1}$.*

Proof. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$.

Assume that $x_1 \in S_n$ and that $R^{-1}(x_1) \cap S_{n+1} = \varnothing$. By Lemma 4.10, $x_1 \neq d$; hence, by Fact 3.2, $x_1 \in R(d)$, and so $\Diamond\{x_1\} \neq \varnothing$. By Lemma 4.5, $\{x_1\} \in \Pi$. Hence, $\Diamond\{x_1\} \in \Pi$. Thus, by Lemma 3.6, $\Diamond\{x_1\} - \Diamond^2\{x_1\} \neq \varnothing$.

Let $y \in \Diamond\{x_1\} - \Diamond^2\{x_1\}$; thus, $x_1 \in R^{=1}(y)$, by transitivity of R. By assumption, $R^{-1}(x_1) \cap S_{n+1} = \varnothing$; hence, $y \notin S_{n+1}$. Thus, by definition of S_{n+1}, there exists $z \in R(y)$ such that $z \notin \bigcup\limits_{1 \leqslant i \leqslant n} S_i$. Observe that this implies that $z \neq x_1$. Since $y \notin \Diamond^2\{x_1\}$, it follows that $x_1 \notin R(z)$; on the other hand, by Lemma 4.8, $z \notin R(x_1)$. Thus, $\{x_1, z\}$ is a non-trivial antichain. We next show that $z \notin \bigcup\limits_{i \in \mathbb{N}} S_i$. Suppose not. Then, by Lemma 4.5, $\{x_1\}, \{z\} \in \Pi$. As we have seen, $x_1 \in R^{=1}(y)$. Hence, $\mathfrak{F}$ violates (S4): to obtain counterexample, take points $x_1 \in R^{=1}(y)$ and $z \in R(y)$ and sets $X_1 = \{x_1\}$ and $X_2 = \{z\}$. This contradicts Proposition 5.5. Hence, $z \notin \bigcup\limits_{i \in \mathbb{N}} S_i$.

Then, by Lemma 4.12, $R(z) \cap S_n \neq \varnothing$. Let $x_2 \in R(z) \cap S_n$, and so $x_2 \in R(y)$, by transitivity of R. But then $\mathfrak{F}$ violates (S4): to obtain a counterexample, take points $x_1 \in R^{=1}(y)$ and $x_2 \in R(y)$, and sets $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$. This, again, contradicts Proposition 5.5. □

Lemma 5.7. *Let $s \geqslant 2$ and $\mathfrak{F} \in \mathtt{Fr}\,\mathbf{L}^s$, and let $x \in \bigcup\limits_{i \in \mathbb{N}} S^{\mathfrak{F}}_i$. Then x does not see any non-trivial antichains.*

Proof. Let $x \in S_n$ and let $\{w', w''\}$ to be a non-trivial antichain such that $\{w', w''\} \subseteq R(x)$. By Lemma 4.8, $R(x) \subseteq \bigcup\limits_{m < n} S_m$. Hence, every R-path from x to both w' and w'' is finite and acyclic. Then there exists $u \in R(x) \cup \{x\} \subseteq \bigcup\limits_{m \leqslant n} S^{\mathfrak{F}}_m$ such that $w', w'' \in R(u)$, but no $u' \in R(u)$ satisfies $w', w'' \in R(u')$—to see this, go backwards along $R^{=1}$ from w' to x and choose u to be the first point that sees w''.

Now, let

$$uR^{=1}y_1R^{=1}y_2R^{=1}\ldots R^{=1}w' \quad \text{and} \quad uR^{=1}z_1R^{=1}z_1R^{=1}\ldots R^{=1}w''$$

be $R^{=1}$-paths from u to, respectively, w' and w''. Set $Y = \{y_1, z_1\}$. By choice of u, it follows that $y_1 \neq z_1$. Moreover, $\{y_1, z_1\}$ is an antichain—otherwise, either $w' \in R(z_1)$ or $w'' \in R(y_1)$, contrary to the choice of u. Since $y_1, z_1 \in R^{=1}(u) \subseteq \bigcup\limits_{m < n} S^{\mathfrak{F}}_m$, by Lemma 4.5, $Y \in \Pi$. Since Y is an antichain and, by Lemma 4.4, both y_1 and z_1 are irreflexive, $y_1, z_1 \in Y - \Diamond Y$. By Lemma 3.14, $\{y_1\}, \{z_1\} \in \Pi$. By Lemma 4.11, $u \neq d$, and hence, by Fact 3.2, $u \in R(d)$. By Lemma 4.4, u is irreflexive. Hence, $\mathfrak{F}$ violates (S4): to obtain counterexample, take points $y_1 \in R^{=1}(u)$ and $z_1 \in R^{=1}(u)$, and sets $X_1 = \{y_1\}$ and $X_2 = \{z_1\}$. This, however, contradicts Proposition 5.5. □

6. Special classes of frames

In this section, we consider classes of frames that shall be used to define logics $\mathbf{L}^s_\omega$ without independent QN-axiomatizations.

Definition 6.1. For each $n \in \mathbb{N}^+$, let $\mathscr{F}_n$ be the class of frames such that $\mathfrak{F} \in \mathscr{F}_n$ if, and only if,

- $\mathfrak{F}$ is transitive;
- $|S^{\mathfrak{F}}_i| = 1$, for every $i \in \{1, \ldots, n-1\}$;
- $|S^{\mathfrak{F}}_i| > 1$, for every $i \geqslant n$.

Let also $\mathscr{F}_\omega$ be the class frames such that $\mathfrak{F} \in \mathscr{F}_\omega$ if, and only if,

- $\mathfrak{F}$ is transitive;
- $|S^{\mathfrak{F}}_i| = 1$, for every $i \geqslant 1$.

Examples of frames belonging to classes $\mathscr{F}_n$ and $\mathscr{F}_\omega$ are depicted, respectively, on the left and on the right of Fig. 1 (in the figure, arrows that can be inferred by transitivity are omitted).

Lemma 6.2. *For every $s \geqslant 2$, $\mathtt{Fr}\,\mathbf{L}^s \subseteq \big(\bigcup_{n \in \mathbb{N}^+} \mathscr{F}_n\big) \cup \mathscr{F}_\omega$.*

Proof. Suppose that $\mathfrak{F} \in \mathtt{Fr}\,\mathbf{L}^s$. By Lemma 4.11, $|S^{\mathfrak{F}}_n| \geqslant 1$, for every $n \in \mathbb{N}^+$.

Fix $n \in \mathbb{N}^+$. Due to Lemmas 4.6 and 5.7, we can define a map $f\colon S^{\mathfrak{F}}_{n+1} \to S^{\mathfrak{F}}_n$ by

$$f(x) = y \quad \Longleftrightarrow \quad y \in R(x).$$

By Lemma 5.6, f is a surjection. Hence, $|S^{\mathfrak{F}}_{n+1}| \geqslant |S^{\mathfrak{F}}_n|$, for every $n \in \mathbb{N}^+$.

Now, if $|S^{\mathfrak{F}}_n| = 1$, for every $n \in \mathbb{N}^+$, then $\mathfrak{F} \in \mathscr{F}_\omega$. Otherwise, there exists $n \in \mathbb{N}^+$ such that $|S^{\mathfrak{F}}_n| > 1$. Let $k = \min\{k : |S^{\mathfrak{F}}_k| > 1\}$. Then $|S^{\mathfrak{F}}_i| > 1$, for every $i \geqslant k$. Hence, in this case, $\mathfrak{F} \in \mathscr{F}_k$. □

We next show that each logic $\mathbf{L}^s$ has frames in every class $\mathscr{F}_\kappa$, where $\kappa \in \mathbb{N}^+$ or $\kappa = \omega$.

Lemma 6.3. *For every $s \geqslant 2$ and every $n \in \mathbb{N}^+$, $\mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_n \neq \varnothing$.*

Proof. For every $s \geqslant 2$ and every $n \in \mathbb{N}^+$, we exhibit a frame that we show to be in $\mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_n$. Let (see Fig. 1, left, for the case $s = 2$)

- $W = \{d\} \cup \{a_1, \ldots, a_{n-1}\} \cup \{b^j_m : m \geqslant n, 1 \leqslant j \leqslant s\} \cup \{c^j_m : m \in \mathbb{N}^+, 1 \leqslant j \leqslant s\} \cup \{e_m : m \in \mathbb{N}^+\}$;
- R be a binary relation on W such that
  - $R(a_k) = \{a_1, \ldots, a_{k-1}\}$, for every $k \in \{1, \ldots, n-1\}$;
  - $R(b^j_k) = \{a_1, \ldots, a_{n-1}\} \cup \{b^j_m : n \leqslant m < k\}$, for every $j \in \{1, \ldots, s\}$ and every $k \geqslant n$;
  - $R(c^j_k) = \{a_1, \ldots, a_{n-1}\} \cup \{b^j_m : m \geqslant n\} \cup \{c^j_m : 1 \leqslant m < k\}$, for every $j \in \{1, \ldots, s\}$ and every $k \in \mathbb{N}^+$;
  - $R(e_k) = \{a_1, \ldots, a_{n-1}\} \cup \{b^j_m : m \geqslant n, 1 \leqslant j \leqslant s\} \cup \{c^j_m : m \in \mathbb{N}^+, 1 \leqslant j \leqslant s\} \cup \{e_1, \ldots, e_{k-1}\}$, for every $k \in \mathbb{N}^+$;
  - $R(d) = W - \{d\}$;
- Π is the set of elements of the modal algebra generated by singletons from 2^W other than $\{d\}$, i.e., Π is the subset of 2^W such that

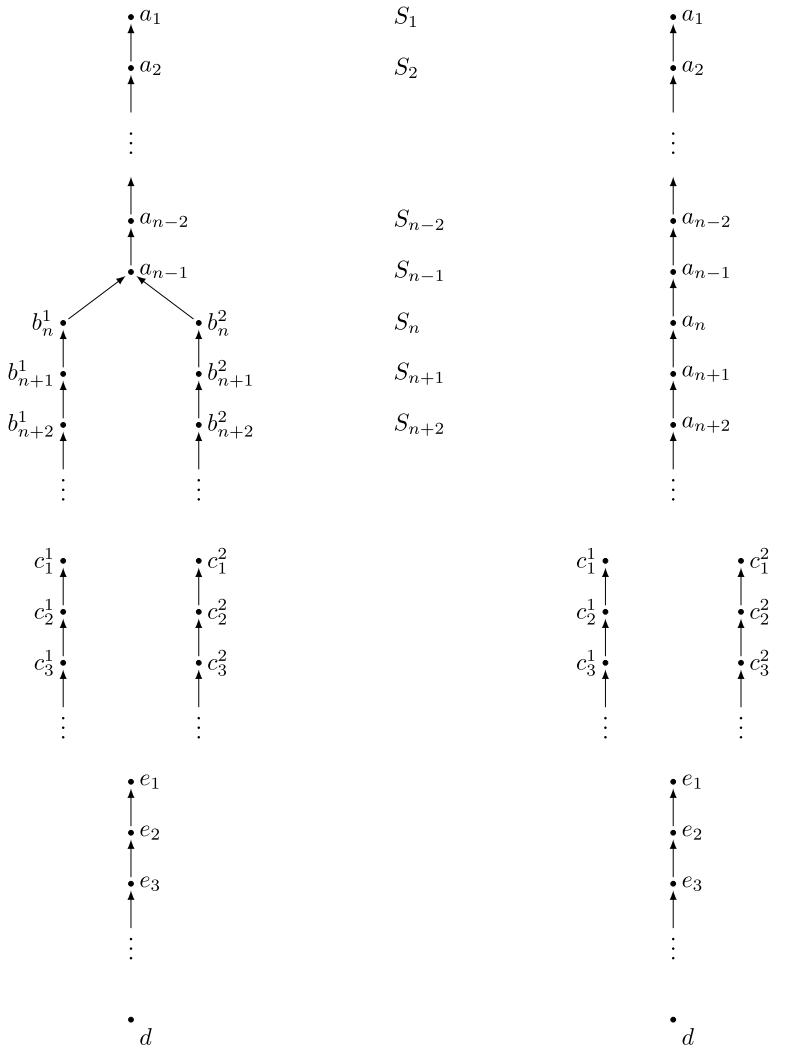


Fig. 1. Frames $\mathfrak{F}_n^2$ (left) and $\mathfrak{F}_\omega^2$ (right).

- $\varnothing \in \Pi$;
- $\{x\} \in \Pi$ whenever $x \in W - \{d\}$;
- if $X_1, X_2 \in \Pi$, then $X_1 \cup X_2 \in \Pi$;
- if $X \in \Pi$, then $\overline{X}, \Diamond X \in \Pi$.

Lastly, let $\mathfrak{F}_n^s = \langle W, R, \Pi, d\rangle$. By definition, Π satisfies (3); hence, $\mathfrak{F}_n^s = \langle W, R, \Pi, d\rangle$ is a frame. We shall prove that $\mathfrak{F}_n^s \in \mathtt{Fr} \cap \mathscr{F}_n\mathbf{L}^s$. First, we prove the following:

Sublemma 6.4. *In the frame $\mathfrak{F}_n^s$, the set Π does not contain $\{d\}$.*

Proof. For every $x \in W$, define

$$[d, x] = \{d, x\} \cup \{y \in W \mid dRyRx\}.$$

We prove, by induction on Π, that, for every $X \in \Pi$,

$$\begin{aligned} d \in X &\implies (\exists k \in \mathbb{N}^+)\, [d, e_k] \subseteq X; \\ d \notin X &\implies (\exists k \in \mathbb{N}^+)\, [d, e_k] \cap X = \varnothing. \end{aligned} \tag{6}$$

Clearly, the statement of the sublemma follows from (6).

Surely, (6) holds if $X = \varnothing$ or $X = \{x\}$, for some $x \in W - \{d\}$. Assuming that X satisfies (6), we show that so do $\overline{X}$ and $\Diamond X$.

Suppose that $d \in \overline{X}$. Then $d \notin X$. By inductive hypothesis, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \cap X = \varnothing$. Hence, $[d, e_k] \subseteq \overline{X}$. Suppose that $d \notin \overline{X}$. Then $d \in X$. By inductive hypothesis, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \subseteq X$. Hence, $[d, e_k] \cap \overline{X} = \varnothing$.

Suppose that $d \in \Diamond X$. Then $\Diamond X \neq \varnothing$, and so $X \neq \varnothing$. We distinguish two cases. Case $d \in X$: In this case, by inductive hypothesis, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \subseteq X$. Then the definition of R implies that $[d, e_{k-1}] \subseteq \Diamond X$. Case $d \notin X$: In this case, by inductive hypothesis, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \cap X = \varnothing$. Then the definition of R implies that $[d, e_k] \subseteq \Diamond X$.

Suppose that $d \notin \Diamond X$. Observe that $R(d) = W - \{d\}$. By inductive hypothesis, $X \neq \{d\}$. Thus, $X = \varnothing$, and so $\Diamond X = \varnothing$. Hence, $[d, e_k] \cap \Diamond X = \varnothing$, for every $k \in \mathbb{N}^+$.

Assume that X_1 and X_2 satisfy (6) and let $X = X_1 \cup X_2$. Suppose that $d \in X$. Then $d \in X_1$ or $d \in X_2$. If $d \in X_1$, then, by inductive hypothesis, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \subseteq X_1 \subseteq X$. If $d \in X_2$, then, for the same reason, there exists $k \in \mathbb{N}^+$ such that $[d, e_k] \subseteq X_2 \subseteq X$. Suppose that $d \notin X$. Then $d \notin X_1$ and $d \notin X_2$. By inductive hypothesis, there exist $k_1, k_2 \in \mathbb{N}^+$ such that $[d, e_{k_1}] \cap X_1 = \varnothing$ and $[d, e_{k_2}] \cap X_2 = \varnothing$. Choose $k = \max\{k_1, k_2\}$. Then $[d, e_k] \cap X = \varnothing$. □

We next show that the frame $\mathfrak{F}^s_n$ is refined. By definition of Π, every singleton subset of W except $\{d\}$ is in Π; hence, $\mathfrak{F}^s_n$ satisfies (4). For (5), observe that the only point of W not accessible from any point of W is d. Thus, by Sublemma 6.4, if a point $x \in W$ is accessible from some point of W, then $\{x\} \in \Pi$. Hence, $\mathfrak{F}^s_n$ satisfies (5). Thus, $\mathfrak{F}^s_n$ is refined.

It should be clear that $\mathfrak{F}^s_n \in \mathscr{F}_n$. It, thus, remains to show that $\mathfrak{F}^s_n \in \mathtt{Fr}\,\mathbf{L}^s$. Evidently, $\mathfrak{F}^s_n$ is transitive, and $bw\,\mathfrak{F}^s_n = s$. We next show that $\mathfrak{F}^s_n$ satisfies (S1) through (S4).

(S1) Assume that $X \in \Pi - \{\varnothing\}$. Observe that R^{-1} is a well-founded strict partial order on W. Hence, X, being non-empty, contains a maximal, with respect to R, point; this point, clearly, belongs to $X - \Diamond X$; hence, $X - \Diamond X \neq \varnothing$.

(S2) Assume that $x \in \Diamond X$ and $X \in \Pi$; the assumption that $x \in R(d)$ is not needed here. Since $\Diamond X \neq \varnothing$, surely $X \neq \varnothing$. Hence, as we have just seen, X contains a maximal, with respect to R, point; since R is transitive, this point is accessible from x; hence, $x \in \Diamond(X - \Diamond X)$.

(S3) Assume that $X \in \Pi$ and $d \in X$. Then $X \neq \varnothing$. By Sublemma 6.4, $X \neq \{d\}$; hence, $X \cap \big(W - \{d\}\big) \neq \varnothing$. Since, by definition of R, we have $R(d) = W - \{d\}$, it follows that $X \cap R(d) \neq \varnothing$, i.e., $d \in \Diamond X$.

(S4) Let $x_1, x_2 \in W$ and $X_1, X_2 \in \Pi$ be such that

$$X_1 \cap X_2 = \varnothing, \quad x_1 \in X_1 - \Diamond(X_1 \cup X_2), \quad \text{and} \quad x_2 \in X_2 - \Diamond(X_1 \cup X_2).$$

We show that $x_1 \notin R^{=1}(x)$, for every $x \in W$. By assumption, $\{x_1, x_2\}$ is a non-trivial antichain. Hence, by definition of R, if $x_1, x_2 \in R(x)$, then $x \in \{d\} \cup \{e_k \mid k \in \mathbb{N}^+\}$ and $\{x_1, x_2\} \cap \{e_k \mid k \in \mathbb{N}^+\} = \varnothing$. But then $x_1 \notin R^{=1}(x)$.

Thus, by Proposition 5.5, $\mathfrak{F}^s_n \in \mathtt{Fr}\,\mathbf{L}^s$. Hence, $\mathfrak{F}^s_n \in \mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_n$, and so $\mathscr{F}_n \cap \mathtt{Fr}\,\mathbf{L}^s \neq \varnothing$. □

Lemma 6.5. *For every* $s \geqslant 2$, $\mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_\omega \neq \varnothing$.

Proof. The proof is similar to the proof of Lemma 6.3: for every $s \geqslant 2$, we exhibit a frame that we show to be in $\mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_\omega$. Let (see Fig. 1, right, for the case $s = 2$)

- $W = \{d\} \cup \{a_m : m \in \mathbb{N}^+\} \cup \{c^j_m : m \in \mathbb{N}^+, 1 \leqslant j \leqslant s\} \cup \{e_m : m \in \mathbb{N}^+\}$;
- R be a binary relation on W such that
  - $R(a_k) = \{a_1, \ldots, a_{k-1}\}$, for every $k \in \mathbb{N}^+$;
  - $R(c^j_k) = \{a_m : m \in \mathbb{N}^+\} \cup \{c^j_m : 1 \leqslant m < k\}$, for every $j \in \{1, \ldots, s\}$ and every $k \in \mathbb{N}^+$;
  - $R(e_k) = \{a_m : m \in \mathbb{N}^+\} \cup \{c^j_m : m \in \mathbb{N}^+, 1 \leqslant j \leqslant s\} \cup \{e_1, \ldots, e_{k-1}\}$, for every $k \in \mathbb{N}^+$;
  - $R(d) = W - \{d\}$;
- Π is the set of elements of the modal algebra generated by singletons from 2^W other than $\{d\}$, i.e., Π is the subset of 2^W such that
  - $\varnothing \in \Pi$;
  - $\{x\} \in \Pi$ whenever $x \in W - \{d\}$;
  - if $X_1, X_2 \in \Pi$, then $X_1 \cup X_2 \in \Pi$;
  - if $X \in \Pi$, then $\overline{X}, \Diamond X \in \Pi$.

Lastly, let $\mathfrak{F}^s_\omega = \langle W, R, \Pi, d\rangle$. By definition, Π satisfies (3); hence, $\mathfrak{F}^s_\omega = \langle W, R, \Pi, d\rangle$ is a frame. The proof that $\mathfrak{F}^s_\omega \in \mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_\omega$ is analogous to the argument from the proof of Lemma 6.3 showing that $\mathfrak{F}^s_n \in \mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_n$. □

We next show that, for every $\mathbf{S}$-frame $\mathfrak{F} \in \mathscr{F}_n$, there exists $\mathfrak{F}' \in \mathscr{F}_{n+1}$ such that $\mathfrak{F}' \models \mathtt{L}\,\mathfrak{F}$. To that end, we adapt to our setting the notion of p-morphism between general frames without distinguished points. We use the standard notation $f[X]$ and $f^{-1}[X]$ for, respectively, the image and the pre-image of the set X under map f.

Definition 6.6. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$ and $\mathfrak{F}' = \langle W', R', \Pi', d'\rangle$ be frames. A map $f\colon W \to W'$ is a *p-morphism* from $\mathfrak{F}$ onto $\mathfrak{F}'$ if

(P1) $f[W] = W'$,
(P2) $f(d) = d'$,

and, for every $x, y \in W$ and every $Y \in \Pi'$,

(P3) $xRy \Rightarrow f(x)R'f(y)$,
(P4) $f(x)R'f(y) \Rightarrow (\exists z \in R(x))\, f(z) = f(y)$,
(P5) $f^{-1}[Y] \in \Pi$.

If f is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, we write $f\colon \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. If there exists a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, we say that $\mathfrak{F}'$ is a *p-morphic image* of $\mathfrak{F}$ and write $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$.

Since, as is straightforward to check, p-morphisms preserve validity at a point of a frame, we obtain the following analogue of the well-known result about modal logics of frames without distinguished points:

Proposition 6.7. *If* $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, *then* $\mathtt{L}\,\mathfrak{F} \subseteq \mathtt{L}\,\mathfrak{F}'$.

Lemma 6.8. *Let* $n \in \mathbb{N}^+$ *and* $\mathfrak{F} \in \mathscr{F}_n \cap \mathtt{Fr}\,\mathbf{S}$. *Then* $\mathscr{F}_{n+1} \cap \mathtt{Fr}\,\mathtt{L}\,\mathfrak{F} \neq \varnothing$.

Proof. We obtain a frame $\mathfrak{F}' \in \mathscr{F}_{n+1}$ and a p-morphism $f\colon \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ by identifying all points in $S_n^{\mathfrak{F}}$ and mapping each point in $S_n^{\mathfrak{F}}$ onto the new point, i.e., by taking the standard quotient of $S_n^{\mathfrak{F}}$. Then, by Proposition 6.7, $\mathfrak{F}' \in \mathtt{Fr}\,\mathtt{L}\,\mathfrak{F}$.

Suppose that $\mathfrak{F} = \langle W, R, \Pi, d\rangle$. Assuming that $a \notin W$, let

- $V = W - S_n^{\mathfrak{F}}$;
- $W' = V \cup \{a\}$;
- R' be a binary relation on W' such that
  - $R' \restriction V = R \restriction V$;
  - $a \notin R'(a)$;
  - $(\forall y \in V)\,[aR'y \Leftrightarrow (\exists x \in S_n^{\mathfrak{F}})\,xRy]$;
  - $(\forall x \in V)\,[xR'a \Leftrightarrow (\exists y \in S_n^{\mathfrak{F}})\,xRy]$;
- $f : W \to W'$ be a map defined by

$$f(x) = \begin{cases} x \text{ if } x \in V; \\ a \text{ if } x \in S_n^{\mathfrak{F}}; \end{cases}$$

- $\Pi' = \{f[X] \,:\, X \in \Pi\}$.

Set $\mathfrak{F}' = \langle W', R', \Pi', d\rangle$; observe that, by Lemma 4.10, $d \notin S_n^{\mathfrak{F}}$; hence, $d \in V \subseteq W'$.

We first show that Π' satisfies (3), and so $\mathfrak{F}'$ is a frame, and that $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. We begin by proving that (P1) through (P5) from Definition 6.6 are satisfied for the frame $\mathfrak{F}$, the structure $\mathfrak{F}'$, and the map f.

It should be clear that (P1) is satisfied. Condition (P2) is satisfied since f is an identity map on V and, as we have seen, $d \in V$. That (P3) is satisfied follows from the definitions of R' and f together with Lemma 4.4, which ensures that $R(x) \cap S_n^{\mathfrak{F}} = \varnothing$ whenever $x \in S_n^{\mathfrak{F}}$.

Since f is an identity map on V, it suffices to check (P4) for the points in $S_n^{\mathfrak{F}} \subseteq W$ and for $a \in W'$. Assume first that $f(x) = a$ and $aR'f(y)$. Then $f(y) \neq a$, by definition of R'. Hence, $f(y) \in V$, and, thus, $f(y) = y$. By definition of R', there exists $z \in S_n^{\mathfrak{F}}$ such that zRy. By Lemma 4.8, $y \in S_m^{\mathfrak{F}}$, for some $m < n$. Since $\mathfrak{F} \in \mathscr{F}_n$ and $m < n$, surely $|S_m^{\mathfrak{F}}| = 1$; hence, $S_m^{\mathfrak{F}} = \{y\}$. As $f(x) = a$, the definition of f implies that $x \in S_n^{\mathfrak{F}}$. Hence, by Lemma 4.7, xRy. Assume next that $f(x)R'a$. Then $f(x) \neq a$, by definition of R'. Hence, $f(x) = x$, by definition of f. Thus, by definition of R', there exists $y \in S_n^{\mathfrak{F}}$, i.e., $y \in W$ with $f(y) = a$, such that xRy. Hence, (P4) is satisfied.

To prove that (P5) is satisfied, we show that, for every $Y \subseteq W'$,

$$Y \in \Pi' \iff f^{-1}[Y] \in \Pi. \tag{7}$$

Assume that $f^{-1}[Y] \in \Pi$. Then $f[f^{-1}[Y]] \in \Pi'$, by definition of Π'. Since $f[f^{-1}[Y]] = Y$, it follows that $Y \in \Pi'$.

Conversely, assume that $Y \in \Pi'$, i.e., $Y = f[X]$, for some $X \in \Pi$. We distinguish two cases. Case $Y \subseteq V$: Since f is an identity map on V, it follows that $f^{-1}[Y] = Y = X$. Hence, in this case, $f^{-1}[Y] \in \Pi$. Case $Y \nsubseteq V$: Then $a \in Y$. Set $Y' = Y - \{a\}$. Then $Y' \subseteq V$ and, hence,

$$f^{-1}[Y] = f^{-1}[Y' \cup \{a\}] = f^{-1}[Y'] \cup f^{-1}(a) = Y' \cup S_n^{\mathfrak{F}}.$$

Since $Y = f[X]$, it follows that $Y' \subseteq X \subseteq Y' \cup S_n^{\mathfrak{F}}$, and so $Y' \cup S_n^{\mathfrak{F}} = X \cup S_n^{\mathfrak{F}}$. By Lemma 4.2, $X \cup S_n^{\mathfrak{F}} \in \Pi$. Since $f^{-1}[Y] = Y' \cup S_n^{\mathfrak{F}} = X \cup S_n^{\mathfrak{F}}$, it follows that, in this case, too, $f^{-1}[Y] \in \Pi$.

This proves (7), which implies that (P5) is satisfied.

Thus, (P1) through (P5) are satisfied for $\mathfrak{F}$, $\mathfrak{F}'$, and f. We next prove—relying on $\mathfrak{F}$, $\mathfrak{F}'$, and f satisfying (P1) through (P4), as well as (7)—that Π' satisfies (3), and so $\mathfrak{F}'$ is a frame.

Since $\varnothing \in \Pi$ and $f[\varnothing] = \varnothing$, the definition of Π' implies that $\varnothing \in \Pi'$.

To prove the closure of Π' under $\cup$, assume that $Y_1, Y_2 \in \Pi'$. Then, by (7), $f^{-1}[Y_1] \in \Pi$ and $f^{-1}[Y_2] \in \Pi$. Hence, $f^{-1}[Y_1] \cup f^{-1}[Y_2] \in \Pi$. Since

$$f^{-1}[Y_1 \cup Y_2] = f^{-1}[Y_1] \cup f^{-1}[Y_2],$$

it follows that $f^{-1}[Y_1 \cup Y_2] \in \Pi$. Thus, by (7), $Y_1 \cup Y_2 \in \Pi'$.

To prove the closure of Π' under $\overline{\ }$ and $\Box$, assume that $Y \in \Pi'$.

By (7), $f^{-1}[Y] \in \Pi$; hence, $\overline{f^{-1}[Y]} \in \Pi$. Since $\overline{f^{-1}[Y]} = f^{-1}[\overline{Y}]$, it follows that $f^{-1}[\overline{Y}] \in \Pi$. Thus, by (7), $\overline{Y} \in \Pi'$.

We next show that $\Box Y \in \Pi'$. To that end, we first prove that, for every $Z \subseteq W'$,

$$f^{-1}[\Diamond Z] = \Diamond f^{-1}[Z]. \tag{8}$$

Assume that $x \in \Diamond f^{-1}[Z]$. Then there exists $y \in f^{-1}[Z]$ such that xRy. Thus, $f(y) \in Z$, and, by (P3), $f(x)R'f(y)$. Then $f(x) \in \Diamond Z$; hence, $x \in f^{-1}[\Diamond Z]$.

Conversely, assume that $x \in f^{-1}[\Diamond Z]$. Then $f(x) \in \Diamond Z$, i.e., there exists $w \in Z$ such that $f(x)R'w$. By (P1), there exists $y \in W$ with $w = f(y)$. Hence, by (P4), there exists $z \in R(x)$ such that $f(z) = f(y) = w$. But then $z \in f^{-1}[Z]$. Hence, $x \in \Diamond f^{-1}[Z]$. This proves (8).

Now, by assumption, $Y \in \Pi'$; hence, by (7), $f^{-1}[Y] \in \Pi$, and so $\Box f^{-1}[Y] \in \Pi$. By (8),

$$f^{-1}[\Box Y] = f^{-1}[\overline{\Diamond \overline{Y}}] = \overline{\Diamond \overline{f^{-1}[Y]}} = \Box f^{-1}[Y].$$

Hence, $f^{-1}[\Box Y] \in \Pi$. Thus, by (7), $\Box Y \in \Pi'$.

Thus, $\mathfrak{F}'$ is a frame, and $f \colon \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$.

We next show that $\mathfrak{F}'$ is refined, i.e. that it satisfies (4) and (5).

For (4), assume that $x, y \in W'$ and $x \neq y$. We distinguish two cases.

Case $x = a$ or $y = a$: Observe that $S^{\mathfrak{F}'}_n = \{a\}$. Thus, by Lemma 4.2, either $\{x\} \in \Pi'$ or $\{y\} \in \Pi'$. Hence, in this case, (4) is satisfied.

Case $x \neq a$ and $y \neq a$: Then $x, y \in V$. Since $\mathfrak{F}$ is refined, by (4), there exists $X \in \Pi$ such that $x \in X$ and $y \notin X$. Let $Z = X - S^{\mathfrak{F}}_n$. Then $x \in Z$ and $y \notin Z$. By Lemma 4.2, $Z \in \Pi$; hence, $f[Z] \in \Pi'$, by definition of Π'. Since $Z \subseteq V$, it follows that $f[Z] = Z$; hence, $x \in f[Z]$ and $y \notin f[Z]$. Thus, in this case, too, (4) is satisfied.

For (5), assume that $y \notin R'(x)$. If $y = a$, then, as we have seen, $\{y\} \in \Pi'$; hence, in this case, (5) is satisfied. So, assume that $y \neq a$. Then $y \in V$. Let $w \in f^{-1}(x)$. By (P3), $y \notin R(w)$. Since $\mathfrak{F}$ is refined, by (5), there exists $X \in \Pi$ such that $y \in X$ and $w \notin \Diamond X$. Let $Y = X - S^{\mathfrak{F}}_n$. Then $y \in Y$ and $f[Y] = Y$; hence, $y \in f[Y]$. By Lemma 4.2, $Y \in \Pi$; hence, $f[Y] \in \Pi'$. We next show that $x \notin \Diamond f[Y]$.

First, suppose that $w \notin S^{\mathfrak{F}}_n$. Then $w = x$ and $x \in V$. Since $f[Y] \subseteq V$ and $R' \restriction V = R \restriction V$, in this case, $x \notin \Diamond f[Y]$. Second, suppose that $w \in S^{\mathfrak{F}}_n$, and so $x = a$. Since $\mathfrak{F} \in \mathscr{F}_n$, clearly, $|S^{\mathfrak{F}}_{n-1}| = 1$; hence, by Lemma 4.8, $R(z) = R(z')$, for every $z, z' \in S^{\mathfrak{F}}_n$. Since $w \notin \Diamond X$ and $Y \subseteq X$, surely $w \notin \Diamond Y$. Since, by assumption, $w \in S^{\mathfrak{F}}_n$, it follows that $S^{\mathfrak{F}}_n \cap \Diamond Y = \varnothing$. Since $Y \subseteq V$, and so $f[Y] = Y$, the definition of R' implies that $a \notin \Diamond f[Y]$, i.e., that $x \notin \Diamond f[Y]$.

Thus, $\mathfrak{F}'$ is refined. It is straightforward to check that $\mathfrak{F}'$ satisfies all the other conditions for membership in $\mathscr{F}_{n+1}$, so we leave this to the reader.

Thus, $\mathfrak{F}' \in \mathscr{F}_{n+1}$ and, by Proposition 6.7, $\mathfrak{F}' \in \mathtt{Fr}\,\mathtt{L}\,\mathfrak{F}$. Hence, $\mathscr{F}_{n+1} \cap \mathtt{Fr}\,\mathtt{L}\,\mathfrak{F} \neq \varnothing$. □

From Lemma 6.8 we immediately obtain the following:

Proposition 6.9. *Let $L \in \mathcal{E}\,\mathbf{S}$. If $n \in \mathbb{N}^+$ and $\mathtt{Fr}\,L \cap \mathscr{F}_n \neq \varnothing$, then $\mathtt{Fr}\,L \cap \mathscr{F}_m \neq \varnothing$ whenever $m \geqslant n$.*

7. Main result

In this section, we present our main result, the existence of a countable family of extensions of system **S** without independent QN-axiomatizations. We start by defining the logics that shall be shown not to be independently QN-axiomatizable:

Definition 7.1. Let $s \geqslant 2$. For every $n \in \mathbb{N}^+$, define

$$\mathscr{C}^s_n = \mathtt{Fr}\,\mathbf{L}^s - \bigcup_{1 \leqslant i < n} \mathscr{F}_i \quad \text{and} \quad \mathbf{L}^s_n = \mathtt{L}\,\mathscr{C}^s_n.$$

Also, define

$$\mathscr{C}^s_\omega = \mathtt{Fr}\,\mathbf{L}^s - \bigcup_{i \in \mathbb{N}^+} \mathscr{F}_i \quad \text{and} \quad \mathbf{L}^s_\omega = \mathtt{L}\,\mathscr{C}^s_\omega.$$

Observe that $\mathbf{L}^s_1 = \mathbf{L}^s$. We first prove three auxiliary lemmas. For every $n \in \mathbb{N}^+$, define

$$\sigma_n = \Diamond(p \wedge \Box^n\bot \wedge \neg\Box^{n-1}\bot) \to \Box(\Box^n\bot \wedge \neg\Box^{n-1}\bot \to p).$$

Lemma 7.2. *Let $\mathfrak{F}$ be a point-transitive differentiated frame and let $n \in \mathbb{N}^+$. Then $\mathfrak{F} \models \sigma_n$ if, and only if, $|S^{\mathfrak{F}}_n| \leqslant 1$.*

Proof. Let $\mathfrak{F} = \langle W, R, \Pi, d\rangle$.

Assume that $|S_n| > 1$. Let $x_1, x_2 \in S_n$ and $x_1 \neq x_2$. By Fact 3.2, $W = R(d) \cup \{d\}$. By Lemma 4.4, $x_2 \notin R(x_1)$ and $x_1 \notin R(x_2)$. Hence, $d \notin \{x_1, x_2\}$, and so $\{x_1, x_2\} \subseteq R(d)$. By Lemma 3.13, $\mathfrak{F}$ contains a differentiating system for $\{x_1, x_2\}$; so, let $\{X_1, X_2\}$ be such a system with $x_1 \in X_1$ and $x_2 \in X_2$. Let v be a valuation with $v(p) = X_1$. Then, by Lemma 4.3, $x_1 \models^v p \wedge \Box^n\bot \wedge \neg\Box^{n-1}\bot$ and $x_2 \models^v \neg p \wedge \Box^n\bot \wedge \neg\Box^{n-1}\bot$. But then $d \not\models^v \sigma_n$, and so $\mathfrak{F} \not\models \sigma_n$.

Conversely, let v be a valuation such that $d \not\models^v \sigma_n$. Then there exists $x \in R(d)$ such that $x \models^v p$ and $x \models^v \Box^n\bot \wedge \neg\Box^{n-1}\bot$, and there exists $y \in R(d)$ such that $y \not\models^v p$ and $y \models^v \Box^n\bot \wedge \neg\Box^{n-1}\bot$. Then $x \neq y$ and, by Lemma 4.3, $x, y \in S_n$. Hence, $|S_n| > 1$. □

Lemma 7.3. *Let $s \geqslant 2$ and $n \in \mathbb{N}^+$. Then $\sigma_n \notin \mathbf{L}^s_n$.*

Proof. By Lemma 6.3, $\mathscr{F}_n \cap \mathtt{Fr}\,\mathbf{L}^s \neq \varnothing$. Hence, $\mathscr{F}_n \cap \mathscr{C}^s_n \neq \varnothing$. Let $\mathfrak{F} \in \mathscr{F}_n \cap \mathscr{C}^s_n$. By Lemma 7.2, $\mathfrak{F} \not\models \sigma_n$. Since $\mathfrak{F} \in \mathscr{C}^s_n$ and $\mathbf{L}^s_n = \mathtt{L}\,\mathscr{C}^s_n$, it follows that $\sigma_n \notin \mathbf{L}^s_n$. □

Lemma 7.4. *Let $s \geqslant 2$. Then $\sigma_n \in \mathbf{L}^s_\omega$, for every $n \in \mathbb{N}^+$.*

Proof. Let $n \in \mathbb{N}^+$. By Lemma 6.2, $\mathscr{C}^s_\omega \subseteq \mathscr{F}_\omega$. By Lemma 7.2, every frame in $\mathscr{F}_\omega$, and so every frame in $\mathscr{C}^s_\omega$, validates σ_n. Since $\mathbf{L}^s_\omega = \mathtt{L}\,\mathscr{C}^s_\omega$, it follows that $\sigma_n \in \mathbf{L}^s_\omega$. □

We next prove our two main technical lemmas.

Lemma 7.5. *Let $s \geqslant 2$. Then $\mathbf{L}^s \subset \mathbf{L}^s_\omega$.*

Proof. By definition of the classes $\mathscr{C}^s_n$ and $\mathscr{C}^s_\omega$,

$$\mathscr{C}^s_\omega \subseteq \mathscr{C}^s_1 \quad \text{and} \quad \mathscr{C}^s_1 = \mathtt{Fr}\,\mathbf{L}^s;$$

hence, $\mathbf{L}^s \subseteq \mathbf{L}^s_\omega$. By Lemma 6.3, $\mathscr{F}_1 \cap \mathtt{Fr}\,\mathbf{L}^s \neq \varnothing$. Hence, by Lemma 7.2, $\sigma_1 \notin \mathbf{L}^s$. On the other hand, by Lemma 7.4, $\sigma_1 \in \mathbf{L}^s_\omega$. Hence, $\mathbf{L}^s \subset \mathbf{L}^s_\omega$. □

Lemma 7.6. *For every $s \geqslant 2$, the logic $\mathbf{L}^s_\omega$ does not have an immediate predecessor in the interval $[\mathbf{L}^s, \mathbf{L}^s_\omega]$.*

Proof. Let L_0 be an immediate predecessor of $\mathbf{L}^s_\omega$ in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$, i.e., let $\mathbf{L}^s \subseteq L_0 \subset \mathbf{L}^s_\omega$ and let there be no logic L such that $L_0 \subset L \subset \mathbf{L}^s_\omega$.

By Lemma 6.5, $\mathbf{L}^s_\omega$ is consistent; hence, so is L_0. Thus, by Proposition 2.3, there exists a non-empty class $\mathscr{C}_0$ of refined frames such that $L_0 = \mathtt{L}\,\mathscr{C}_0$. Since, by assumption, $L_0 \subset \mathbf{L}^s_\omega$, it follows that $\mathscr{C}^s_\omega \subset \mathscr{C}_0$. Since, by assumption, $\mathbf{L}^s \subseteq L_0$, and so $\mathscr{C}_0 \subseteq \mathtt{Fr}\,\mathbf{L}^s$, it follows, by Lemma 6.2, that $\mathscr{C}_0 - \mathscr{C}^s_\omega \subseteq \bigcup\limits_{i \in \mathbb{N}^+} \mathscr{F}_i$.

Now, let

$$k = \min\{n \in \mathbb{N}^+ : \mathscr{C}_0 \cap \mathscr{F}_n \neq \varnothing\}; \qquad \mathscr{C}'_0 = \mathscr{C}_0 - \mathscr{F}_k; \qquad L'_0 = \mathtt{L}\,\mathscr{C}'_0.$$

Since $\mathscr{C}^s_\omega \subseteq \mathscr{C}'_0 \subseteq \mathscr{C}_0$, surely $L_0 \subseteq L'_0 \subseteq \mathbf{L}^s_\omega$. We next show that these inclusions are, in fact, strict.

By choice of k, surely $\mathscr{C}_0 \cap \mathscr{F}_k \neq \varnothing$; hence, by Lemma 7.2, $\sigma_k \notin L_0$. On the other hand, since

$$\mathscr{C}'_0 \subseteq \mathtt{Fr}\,\mathbf{L}^s - \bigcup_{1 \leqslant i \leqslant k} \mathscr{F}_i,$$

by Lemmas 6.2 and 7.2, $\sigma_k \in L'_0$. Thus, $L_0 \subset L'_0$.

By Lemma 7.4, $\sigma_{k+1} \in \mathbf{L}^s_\omega$. We next show that $\sigma_{k+1} \notin L'_0$ and, hence, $L'_0 \subset \mathbf{L}^s_\omega$.

As we have seen, $\mathscr{C}_0 \cap \mathscr{F}_k \neq \varnothing$; so, let $\mathfrak{F} \in \mathscr{C}_0 \cap \mathscr{F}_k$. Since $\mathscr{C}_0 = \mathtt{Fr}\,L_0$, by Proposition 6.9, there exists $\mathfrak{F}' \in \mathscr{C}_0 \cap \mathscr{F}_{k+1}$. Since $\mathscr{C}'_0 = \mathscr{C}_0 - \mathscr{F}_k$ and $\mathscr{F}_k \cap \mathscr{F}_{k+1} = \varnothing$, it follows that $\mathfrak{F}' \in \mathscr{C}'_0$. Since $\mathfrak{F}' \in \mathscr{F}_{k+1}$ and $L'_0 = \mathtt{L}\,\mathscr{C}'_0$, by Lemma 7.2, $\sigma_{k+1} \notin L'_0$.

Thus, $L_0 \subset L'_0 \subset \mathbf{L}^s_\omega$, contrary to the assumption. □

Lemma 7.7. *For every $s \geqslant 2$, the logic $\mathbf{L}^s_\omega$ is not independently QN-axiomatizable.*

Proof. By Proposition 5.4, $\mathbf{L}^s$ is finitely *QN*-axiomatizable. By Lemma 7.5, $\mathbf{L}^s \subset \mathbf{L}^s_\omega$. By Lemma 7.6, $\mathbf{L}^s_\omega$ does not have an immediate predecessor in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$. Hence, by Lemma 1.1, $\mathbf{L}^s_\omega$ is not independently *QN*-axiomatizable. □

To prove our main result, Theorem 7.9, it remains to establish the following:

Lemma 7.8. $\mathbf{L}^s_\omega \neq \mathbf{L}^r_\omega$ *whenever* $s \neq r$.

Proof. We may assume that $r < s$. We show that $\boldsymbol{bw}_r \in \mathbf{L}^r_\omega$, but $\boldsymbol{bw}_r \notin \mathbf{L}^s_\omega$; hence, $\mathbf{L}^s_\omega \neq \mathbf{L}^r_\omega$.

Since $\mathtt{Fr}\,\mathbf{L}^r_\omega \subseteq \mathtt{Fr}\,\mathbf{L}^r$, by Proposition 5.5, every $\mathbf{L}^r_\omega$-frame has width at most r. Hence, by Lemma 3.11, $\boldsymbol{bw}_r \in \mathbf{L}^r_\omega$. Now, consider the frame $\mathfrak{F}^s_\omega$ defined in the proof of Lemma 6.5. As we have seen, $\mathfrak{F}^s_\omega \in \mathtt{Fr}\,\mathbf{L}^s \cap \mathscr{F}_\omega$. Hence $\mathfrak{F}^s_\omega \in \mathscr{C}^s_\omega (= \mathtt{Fr}\,\mathbf{L}^s_\omega)$. Since $s > r$, by Lemma 3.11, $\mathfrak{F}^s_\omega \not\models \boldsymbol{bw}_r$. Hence, $\boldsymbol{bw}_r \notin \mathbf{L}^s_\omega$. □

We are in a position to prove our main result:

Theorem 7.9. *There exist countably many extensions of the quasi-normal modal logic* **S** *not admitting deductively independent sets of axioms.*

Proof. Immediate from Lemmas 7.7 and 7.8. □

Remark 7.10. Since logics $\mathbf{L}^s_\omega$ are extensions of **S**, and therefore contain formulas $\Box(\Box p \to p) \to \Box p$ and $\Box p \to p$, their closures under (N) are inconsistent. Thus, logics $\mathbf{L}^s_\omega$ cannot be consistently normalized.

Since logics $\mathbf{L}^s_\omega$ are not independently *QN*-axiomatizable, they are not finitely *QN*-axiomatizable. They are, however, as we next show, recursively *QN*-axiomatizable.

Proposition 7.11. *For every $s \geqslant 2$, the logic $\mathbf{L}^s_\omega$ is recursively QN-axiomatizable.*

Proof. We show that $\mathbf{L}^s_\omega = \mathbf{L}^s + \{\sigma_n : n \in \mathbb{N}^+\}$.

By Lemma 6.2, $\mathtt{Fr}\,\mathbf{L}^s \subseteq \big(\bigcup_{n\in\mathbb{N}^+} \mathscr{F}_n\big) \cup \mathscr{F}_\omega$. Hence, by Lemma 7.2, for every $\mathfrak{F} \in \mathtt{Fr}\,\mathbf{L}^s$,

$$\mathfrak{F} \not\models \sigma_n \iff \mathfrak{F} \in \{\mathscr{F}_1, \ldots, \mathscr{F}_n\}.$$

Thus,

$$\mathbf{L}^s_\omega = \mathsf{L}(\mathtt{Fr}\,\mathbf{L}^s - \bigcup_{n\in\mathbb{N}^+} \mathscr{F}_n) = \mathsf{L}(\{\mathfrak{F} \in \mathtt{Fr}\,\mathbf{L}^s : \mathfrak{F} \models \sigma_n, n \in \mathbb{N}^+\}) =$$

$$\mathbf{L}^s + \{\sigma_n : n \in \mathbb{N}^+\}. \quad \Box$$

8. Discussion

Propositional logics not admitting deductively independent sets of axioms exists in the lattices of intermediate logics [3], normal modal logics [3], and, as has been shown here, quasi-normal modal logics, in particular among extensions of **S**.

It is well known that intermediate logics have counterparts, obtained through the Gödel–Tarski translation, in the lattice of normal extensions of the modal logic **S4**. This connection was drawn upon by Chagrov and Zakharyaschev [3] in simultaneously obtaining both intermediate and normal modal logics without independent sets of axioms; consequently, the normal modal logics they obtained extend **S4**. There does exist a similar connection between normal extensions of the modal logic **K4** and extensions of Visser's basic logic **BL** [9], which is a logic in the intuitionistic language with Kripke semantics allowing irreflexive points. Even though the lattice of extensions of **K4** contains logics not admitting independent sets of axioms [3], the properties of frames essentially used in defining those logics are not invariant under the Gödel–Tarski translation. It, therefore, remains to be seen if there exist not independently axiomatizable logics among the extensions of **BL**.

Another question for future study is whether continua of logics without independent sets of axioms can be obtained in various lattices of logics. All the known examples of logics without independent axiomatizations, including those presented here, involve countable families of logics. It remains unclear whether a continuum of such logics can be obtained. The standard method for obtaining continua of extensions of a logic L with a certain property—which involves identifying a countable set Γ of formulas independent over L and defining, for every countable $\Gamma' \subseteq \Gamma$, the logic $L{+}\Gamma'$—is not compatible with a technique for obtaining logics without independent sets of axioms used here. Recall that, for every $s \geqslant 2$, we obtain a finitely axiomatizable logic $\mathbf{L}^s$ and its recursively, but not independently, axiomatizable proper extension $\mathbf{L}^s_\omega$ such that $\mathbf{L}^s_\omega$ has no immediate predecessor in $[\mathbf{L}^s, \mathbf{L}^s_\omega]$. Moreover, as we have seen in the proof of Proposition 7.11, $\mathbf{L}^s_\omega$ is recursively axiomatizable over $\mathbf{L}^s_\omega$, to wit $\mathbf{L}^s_\omega = \mathbf{L}^s + \{\sigma_n : n \in \mathbb{N}^+\}$. Let $A' = \{\alpha_1, \alpha_2, \ldots,\}$ be a countable set of formulas independent over $\mathbf{L}^s_\omega$; applying the standard method of obtaining a continuum of logics with a certain property, i.e., adding A' to $\mathbf{L}^s_\omega$, we obtain a logic with an independent set $\{\sigma_1 \wedge \alpha_1, \sigma_2 \wedge \alpha_2, \ldots\}$ of axioms.

Declaration of competing interest

We have no conflict of interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are grateful to the anonymous reviewer for comments and suggestions that helped to improve the paper. The work on this paper has been supported by Russian Science Foundation, project 21-18-00195.

References

[1] Patrick Blackburn, Maarten de Rijke, Yde Venema, Modal Logic, Cambridge Tracts in Theoretical Computer Science, vol. 53, Cambridge University Press, 2001.
[2] George S. Boolos, The Logic of Provability, Cambridge University Press, 1993.
[3] Alexander Chagrov, Michael Zakharyaschev, On the independent axiomatizability of modal and intermediate logics, J. Log. Comput. 5 (3) (1995) 287–302.
[4] Alexander Chagrov, Michael Zakharyaschev, Modal Logic, Oxford University Press, 1997.
[5] Igor Gorbunov, Dmitry Shkatov, On independent axiomatizability of quasi-normal modal logics, Stud. Log. 110 (5) (2022) 1189–1217.
[6] Yury Kleiman, Some questions in the theory of varieties of groups, Math. USSR, Izv. 22 (1) (1984) 33–66.
[7] Krister Segerberg, An Essay in Classical Modal Logic, Upsalla, 1971.
[8] Robert M. Solovay, Provability interpretations of modal logic, Isr. J. Math. 25 (1976) 287–304.
[9] Albert Visser, A propositional logic with explicit fixed points, Stud. Log. 40 (1981) 155–175.