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**2016**

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# RESOURCING LEARNER ERRORS AND MISCONCEPTIONS ON GRADE 10 FRACTIONAL EQUATIONS AT A MATHEMATICS CLINIC


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A research report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in partial fulfilment of the requirements for the degree of MSc.

## Abstract

The purpose of this study, conducted at a mathematics clinic, was to investigate the misconceptions that learners display through errors they make when solving algebraic equations involving fractions. A teaching intervention to address those errors and misconceptions was done at a mathematics clinic. A mathematics clinic is a remedial facility where low-attaining students attend sessions, by choice or by referrals. In this study teaching intervention was used to address learners' errors and misconceptions. The assumption of the study was that learners are knowledge constructors that use previously-learned knowledge as the basis of new knowledge. Since their previous knowledge contains errors and misconceptions, the construction of new knowledge results in errors.

This research was mainly qualitative. Data were collected, using a sample of 17 grade 10 learners, though the work of only 13 of them was analysed. Two participants wrote the pre-test, but did not participate in the subsequent data collection, and the other two did not solve some of the equations in the pre- and post-tests. There were three stages of data collection; pre-test, teaching intervention and post-test.

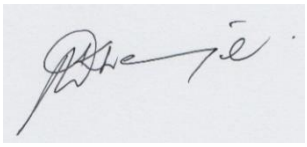
Pre- and post-tests were analysed for errors committed by learners, and the teaching intervention sessions were analysed for opportunities of learning provided. Transcripts were produced from the teaching intervention sessions. They were also analysed to check how students participated in constructing mathematical meanings, and also how effectively their attention was focused on the object of learning. The errors found in learners' equation-solving were like-term errors, lowest common denominator errors, careless errors, sign errors and restriction errors. The comparison of the number of learners who committed these errors in the pre- and the post-test was insightful. Of 13 learners, 4 committed like-term errors in the pre-test and just 1 in the post-test; 4 committed LCD errors both in the pre- and post-tests; 9 committed careless errors (other errors) in the pre-test, and 6 learners in the post-test; 7 committed sign errors in the pre-test and 1 in the post-test; and 12 committed restriction errors in the pre-test, and 9 in the post-test. These findings suggest that teaching intervention is a necessary pedagogical technique, and needs to be employed when addressing learners' errors and misconceptions in mathematics. Reduction in learners' errors and misconceptions was evident after the teaching intervention suggesting that the mathematics clinic provided learning opportunities for participants.

## Acknowledgement

I would like to extend my gratitude to the University of the Witwatersrand, Johannesburg and all my lecturers that contributed to my academic knowledge, opportunities and experiences in my struggle to fulfil the requirements of my MSc Degree in Mathematics Education. I would like to particularly thank my supervisor, Dr Judah Makonye for his passion for what he does and for all the guidance, advice, support and encouragement he provided me with when doing this research report. It would not have been possible to complete this degree without him. I would also like to extend my appreciation to my school and the headmistress for allowing me to conduct the research. This research would not have been possible without the grade 10 mathematics learners and their mathematics clinic's teacher who voluntarily agreed to take part. All your support was invaluable.

## Declaration

I declare that this research report is my own unaided work. It is submitted for the degree of a Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.



**Duduzile Winnie Khanyile**

25 August 2016

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| LCD | Lowest Common Denominator |
| OTL | Opportunity To Learn |
| PCK | Pedagogical Content Knowledge |
| TIMSS | Third International Mathematics and Science Study |
| ZPD | Zone of Proximal Development |
| NCTM | National Council of Teachers of Mathematics |
| MTPCK | Mathematics Teacher Pedagogical Content Knowledge |

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# CHAPTER 1: BACKGROUND TO THE STUDY

## 1.1 Introduction

In South Africa mathematics results at the end of grade 12 are poor and have not improved in, at least, the past decade (Taylor & Taylor, 2013). The Department of Basic Education reported that just above 50% of students passed mathematics by at least 30% in 2014 (Taylor & Taylor, 2013). These figures call for a continued investigation into why students perform so badly in mathematics. In the Third International Mathematics and Science Study (TIMSS) (1995), South African grade 8 learners came last out of 41 countries which completed the study. These reports are according to Reddy (2004). Again, in TIMSS-R (1999) South African grade 8 learners performed poorly (Reddy, 2004). They obtained a mean score of 275, significantly lower than international benchmark mean score of 487. When TIMSS-R (2003) was again conducted, South African grade 8 learners' performance showed no improvement (Reddy, 2004). Learners' poor performance in mathematics indicates a serious problem in the teaching and learning of mathematics in South Africa. Brodie (2010), and Luneta and Makonye (2010) suggest that learner-centred teaching approaches need to be adopted if teaching is to address the problem. It can be argued that if mathematics teachers are unqualified, and not innovative in the classroom, there are great chances of learners' poor performance in mathematics continuing. Learners' poor performance can be caused by teacher-centred approaches, where teachers care more about the subject they teach than the learners who learn it, or by unknowledgeable teachers. Stols (2013) supports this when he points out that the high rate of poor performance can be attributed to, among other things, unequal opportunities to learn due to unqualified teachers.

Mji and Makgato (2006) also attribute learners' poor performance to teachers' lack of pedagogical content knowledge (PCK), a result of teachers being under-qualified or unqualified to teach mathematics. Shulman (1986) suggests that pedagogical content knowledge goes beyond subject matter knowledge. According to her, pedagogical content knowledge involves teaching for understanding, and knowing what makes some topics easy and others difficult for learners. It also involves being able to handle conceptions and preconceptions which are often misconceptions that learners bring to mathematical classrooms (Shulman, 1986). Arguably these misconceptions arise from teachers' poor pedagogical content knowledge, and therefore lead to learners' poor performance in mathematics. It is therefore not surprising that many learners find mathematics difficult and therefore are unmotivated (Hall, 2002; Mji & Makgato, 2006; Brodie, 2010). One topic in particular, that learners find difficult, is fractions.

Kerslake (1986); Figueras, Males and Otten (2008) and Mhakure, Jacobs & Julie (2014) confirm that fractions is one of the problematic areas of mathematics in the Senior Phase (i.e. grades 7 to 9) and Further Education and Training Phase (i.e. grades 10 to 12). Hence the need to investigate role of errors and misconceptions in teaching and learning fractions. In addition, the fractional equations section is one of the areas of mathematics that requires investigation.

It is common to hear people saying that "mathematics is not for everyone". The subject is perceived to be for certain individuals, as is believed to be difficult (Brodie, 2010). Ball (2003) confirms this, pointing out that there is a "cultural belief that only some people have what it takes to learn mathematics" (p.34). It would make sense then to suggest that learners lose marks in mathematics assessments because they lack confidence in what they do emanating from thinking that it is impossible for them to be good at mathematics. This causes them to make errors when solving given equations or problems. These errors are also a result of overgeneralization of their past experiences (Olivier, 1989) in new mathematical situations. Learners' errors and misconceptions may emanate from their pre-knowledge, how the teacher explains a particular concept in relation to what they already know, the curriculum, the environment or an interaction between these variables (Moru, Qhobela, Poka & Nchejane, 2014).

White (2005) suggests that when a learner answers a mathematical question, she or he is confronted with challenges, including reading or decoding, comprehension, transformation, process skills and encoding; and along the way it is always possible for the learners to make careless errors (White, 2005). It therefore becomes important for the teacher to be able to identify and diagnose learners' errors in order to be able to classify them correctly (White, 2005; Chinnappan & Forrester, 2014). It is also important to acknowledge that learners' errors create an opportunity for the teacher to reflect on his or her teaching instruction. It is for this reason that Nesher (1987) suggests that learners' errors are their valuable contributions to teaching and learning, while, on the other hand, White (2005) suggests that learners' errors are the sources of their thinking. In other words, Nesher (1987) and White (2005) confirm that the learners' contributions to learning mathematics is to reveal their erroneous mathematical thinking. Teachers then need to use this understood erroneous thinking as a teaching resource.

Learners' errors and misconceptions are common in algebra. Usiskin (2004) argues that algebra is the basis of all mathematics in high school and beyond. It is therefore important that the progression from arithmetical language to algebraic language is addressed as learners progress from arithmetic to algebra (Usiskin, 2004). Herscovics and Linchevski (1994) study confirmed the cognitive gap between arithmetic and algebra when learners solve first-degree equations. The

topic of equations is one of the areas in mathematics in which learners display misconceptions through mathematical errors (Hall, 2002; Mhakure et al., 2014). In addition, fractional equations have been identified by this researcher, the literature and the mathematics clinic's teacher as one of the areas in which many learners display misconceptions. Adi (1978) and Robson, Abell, and Boustead (2012) point out that many high school learners and early university students struggle to solve equations that involve fractions. It is, therefore, important that learners' errors and misconceptions in equations, and in fractional equations in particular, are addressed in order for learners to cope with equations in other areas of mathematics such as Geometry and Trigonometry (Hall, 2002) and in university.

This study attempts to contribute in this area of mathematics, investigating errors and misconceptions in fractional equations as it is one of the sections in which learners have problems (Kerslake, 1986; Robson et al., 2012; Mhakure et al., 2014).

## 1.2 Research problem

Algebra is considered to be one of the most abstract branches of mathematics (Egodawatte, 2011). As a combination of algebra and equations, it can be argued that learners find solving fractional equations even more challenging. Figueras et al. (2008) and Mhakure et al. (2014) suggest that learners face a significant challenge when they have to simplify rational expressions, while Kerslake (1986) found that learners of the ages 12 to 15 find working with equivalent fractions difficult. It may be that learners struggle to solve fractional equations because the nature of fractional equations requires the knowledge and understanding of equivalent fractions. Mhakure et al. (2014) reiterate this, pointing out that the multifaceted nature of fractions is a "major contributing factor to the core difficulties experienced by teachers and students during the teaching and learning of fractions" (p.1). Therefore, it would be expected for learners to have difficulty in solving fractional equations. Analyzing learners' errors in equations and trying to improve the teaching and learning of this topic is therefore important (Hall, 2002).

Continued investigation into the reasons for making errors in solving fractional equations is necessary to build PCK on the topic. Adi (1978) argues that solutions of many mathematical problems require students to be fluent in solving equations, including fractional equations. Also in other science subjects, such as Physics, equations are used frequently in formulae to express relationships between variables (Hall, 2002). It can be argued that most of the formulae in Physics involve fractions; for example: voltage (V), current (I) and resistance (R) formula:

$I = \frac{V}{R}$, yet learners find it difficult to manipulate such equations. This points to the fact that PCK that eventually benefit learners, should be built on the topic of fractional equations.

Olivier (1989) points out that when a learner's conceptual structure interacts with new concepts, misconceptions influence new learning negatively, as they are the source of errors. Learners' mathematical errors present the teacher with opportunities to find ways to improve the quality of their teaching (Nesher, 1987; Hall, 2002; Moru et al., 2014). Teachers are able to achieve this if they are aware of mathematical errors learners tend to make (Nesher, 1987; Makonye, 2012). This includes the solving of fractional equations as well. The ability of identifying, diagnosing, analyzing and interpreting learners' mathematical errors and misconceptions is therefore important (Chinnappan & Forrester, 2014; Moru et al., 2014; Shalem, Sapire & Sorto, 2014). It helps to build mathematics teacher pedagogical content knowledge (MTPCK).

While the common core standards of mathematics stress "conceptual understanding as a key component of mathematical expertise" (Wiggins, 2014, p.1), I have, as a mathematics teacher, noticed that many learners prefer to 'learn' mathematics by memorizing the rules. Kilpatrick, Swafford & Findell (2001) define conceptual understanding as "an integrated and functional grasp of mathematical ideas" (p.118). It can be argued that memorizing the rules is related to procedures without connection to mathematical concepts or contexts (Stein, Grover & Henningsen, 1996). Memorizing allows learners to retrieve mathematical rules when doing mathematics assessments, and thereby get good marks but memorizing rules has many limitations. Kilpatrick et al. (2001) define procedural fluency as "knowing when and how to use procedures appropriately, skill in performing them flexibly, accurately and efficiently" (p.121). They regard conceptual understanding and procedural fluency as competing in mathematics. The teacher's role is therefore crucial in making learners realize that knowing the rules and being able to use them correctly does not necessarily help them understand the central mathematical concepts.

Learners need to be aware that understanding mathematics is to be able to '*do*' mathematics (Stein et al.,1996). They suggest that '*doing mathematics'* is the ability to solve a mathematical problem without using procedures, but instead by using multiple mathematical methods to approach a mathematical problem. Without identifying, diagnosing, analyzing and addressing learners' mathematical errors and misconceptions it is difficult to develop learners into *doing mathematics.*

Solving equations involving fractions is a mathematical skill not easily understood by learners. Chinnappan and Forrester's study (2014) suggests this when pre-service teachers tried to explain how they had simplified fractions.

Mhakure et al. (2014) note that it is difficult for students to conceptualize algebraic fractions. They maintain that it is critically important for fractions to be taught to senior primary school learners. Among other reasons, they point out that fractions can help learners access higher mathematics. Thus, if learners have problems with algebraic fractions, they are likely to find topics like number theory and calculus difficult (Mhakure et al., 2014). I therefore argue that learners will also find solving fractional equations difficult if they do not master the simplification of algebraic expressions, particularly equivalent fractions.

## 1.3 Aim of the study and research questions

The purpose of this study is to investigate what learning opportunities are created if learners' errors and misconceptions on grade 10 algebraic equations involving fractions are used as resources in teaching interventions at a mathematics clinic. The research also aims to explain learning gains, if any, of such intervention.

The following research questions are therefore posed:

- What errors and misconceptions do students display when solving equations involving fractions?
- What opportunities to learn avail themselves to learners with errors and misconceptions in solving equations involving fractions at a mathematics clinic.
- What gains in learning, if any, occur through the mathematics clinic interventions?

## 1.4 The rationale of the study

In this study I attempt to investigate learners' mathematical errors when solving fractional equations, and further investigate whether the teacher's intervention reduces these errors which result from misconceptions that the learners have. When teachers mark their students' work it is important to diagnose learners' errors and analyse their source in order to be able to teach in a way that will elicit these errors in future (Nesher, 1987; White, 2005; Luneta & Makonye, 2010). For example, Luneta and Makonye (2010) argue that teaching must be directed at responding to learners' mathematical difficulties, as the learner-centred approach suggests, while Nesher (1987) argues that a good teaching strategy must purposely allow for errors in the process of learning. Addressing learners' mathematical errors through teaching interventions should

therefore focus learners' attention on the errors they make when solving fractional equations, and reduce or eliminate them in future.

With the ongoing concern about the grade 12 mathematics results in South Africa, it is vital to keep the investigation going on the cause of the poor performance in mathematics; and seek ways to ameliorate the situation. The demands of the curriculum require teachers to review their teaching approaches and adopt new teaching approaches to teaching and learning mathematics (Brodie, 2010). It is therefore important to conduct a study that will give insight into some of the causes of the high mathematics failure rate. I believe that it is worthwhile to investigate the types of errors learners make in fractional equations. In addition, I investigate whether a teaching intervention specifically aimed at addressing these errors and misconceptions in fractional equations helps to reduce them and provides opportunities to learn.

As a researcher in the field of education, this type of research is important for me to develop understanding of the source of learners' errors and misconceptions. Gaining this knowledge will equip both other teachers and me with a deeper understanding of why students have misconceptions. The research will also assist in terms of how we (teachers) can try to eliminate the possibilities of contributing to the learners' errors and misconceptions in our teaching. Such a study helps to build mathematics teacher pedagogical knowledge on the topic of algebraic equations, and also allied topics.

Since algebra is a key topic in learning mathematics (Usiskin, 2004; Egodawatte, 2011), it is vital that misconceptions in algebra, including fractional equations, are addressed as early as possible in order to afford learners the opportunities to study mathematics with understanding at a higher level.

Hall (2002) points out that "mathematical errors provide valuable insight for the teacher into pupils' thinking as well as for the students themselves" (p.8). When learners make errors they reveal their incomplete knowledge, and afford the teacher an opportunity to contribute additional knowledge (Nesher, 1987).

It is important that learners' errors and misconceptions in equations are addressed adequately, in order to diminish the chances of errors which learners carry from one section of mathematics to the next. Hall (2002) alludes to this fact when pointing out in his study that "equations are a central part of any mathematics course" (p. 4). Fractional equations are even more critical in the teaching and learning of mathematics as its problems arise from arithmetic fractions. As Kerslake (1986) points out, learners find simplifying fractions difficult, and therefore avoid using fractions

in their simplifications. They would rather revert to other methods that they had learnt before fractions were introduced to them. They therefore need to be assisted in achieving conceptual understanding of solving equations that involve fractions in order to be able to apply fractional equations knowledge in other areas of mathematics. They have to be 'fluent' in solving fractional equations, since they are a sub-section of equations. Their fluency will only be possible if their incomplete knowledge of solving fractional equations is revealed in order to allow for teacher's additional knowledge through teaching intervention. Steinbring (2001) and Watson (2003) argue that effectiveness in teaching intervention can be achieved when the learner is provided with the opportunity to actively construct his or her own mathematical knowledge. Teaching intervention also allows for learners to interact with one another, with the mathematical learning material and with the teacher. Leikin and Zaslavsky (1997) refer to this as "significant activities for effective learning" (p.331). It is therefore important for learners to realize their mathematical misconceptions in fractions and fractional equations in order to benefit from teaching intervention.

The TIMSS study report in 2012 found that South Africa was amongst those countries that were ranked low in mathematics and science performance. Amongst other reasons for this was teachers' inadequate qualifications for teaching mathematics and/or science. This study, therefore, will certainly help mathematics teachers to reflect on how they teach fractional equations. What misconceptions do they, as teachers, have that are possibly transferred to the learners? How can these misconceptions be addressed? An understanding of the use of the theory of variation in mathematics lessons will certainly prove to be of great benefit to the mathematics teachers, as shown in the study by Tong (2012).

This study will not only help teachers gain insight into teaching approaches that may contribute to learners' mathematical misconceptions, but will also help them realize that they can use learners' errors and misconceptions to inform their teaching strategies (Nesher, 1987).

Finally, this research project is important in developing my own expertise in mathematics education research.

## 1.5 Conclusion

In this chapter I introduced the study and briefly discussed general problems in the teaching and learning of mathematics. I also discussed problems in teaching and learning algebra in general, fractions, equations and fractional equations in particular. I articulated my research purpose, as well as my research questions. In the next chapters I discuss the theoretical framework that

underpins my study; the literature review; methodology and research design; analysis and discussion of my results.

# CHAPTER 2: THEORETICAL FRAMEWORK AND LITERATURE REVIEW

## 2.1 Introduction

This chapter discusses the theoretical framework and the literature review that guides this research. Miles and Huberman (1994) point out that the "conceptual framework explains either graphically or in narrative form, the main things to be studied" (p.18). It also includes the assumed theory/theories used to inform the research. While it is important to relate one's study with some existing theory or concepts, Maxwell (2005) argues that a conceptual framework should be *constructed* and *built* by the researcher himself or herself, and incorporate concepts borrowed from other studies. These concepts are found in literature that was reviewed.

Literature review guides the researcher with regard to what has been studied and found about the topic. Reviewing literature helps in locating a gap in the field of interest that new research may fill (Creswell, 2012). Literature review also helps to inform the researcher about the thinking, ideas, research questions, methodologies, analysis and findings of similar past research. It makes the researcher aware that his/her research does not occur in a vacuum, and so their research must add to the knowledge base in an existing research field. It is therefore important to analyse the literature review relating to this area of this study – learners' errors and misconceptions in mathematics in general, and in fractional equations in particular, to establish an informed base of this study.

In this study I use the theory of variation (Marton & Booth, 1997) as one of my conceptual frameworks. According to the theory of variation, learners' attention should be focused on the 'object of learning' (Tong, 2012). I also use the conception of knowledge in Hatano's (1996) terms. Among other forms of constructing knowledge, Hatano (1996) points out that knowledge construction involves restructuring, and that it is "acquired domain by domain" (p.199). Lastly, I use Skemp's (1976) notions of understanding – instrumental and relational understanding. He defines instrumental understanding as knowing the "rules without reasons" (p.46), and relational understanding as "knowing both what to do and why" (p.46). I now discuss below, in greater detail, the theoretical framework pertaining to this study.

## 2.2 Theoretical Framework

### 2.2.1 The Theory of Variation

The purpose of teaching mathematics is to develop learners' capabilities and competencies in mathematics. Learners need to acquire knowledge of what is being learned. It is therefore important to be clear about what is being learned (Marton, Runesson & Tsui, 2004). In the context of teaching and learning mathematics, the theory of variation offers possibilities to view how students may discern the mathematics objects they are taught. Drawing from the theory of variation, Tong (2012) posits that "there is no single way to understand, experience or think about a particular phenomenon" (p.4). This explains that different people understand or interpret what they learn differently. In a way, the theory of variation helps to explain how students' viewing of the mathematics being taught can result in errors and misconceptions. As Hatano (1996) and Olivier (1989) argue, learners do not always learn what they are taught, but instead, construct their own meanings. Teachers and learners have a classroom discussion together in order for the creation of mathematical knowledge to be facilitated. These constructions that students make of the mathematics concepts learnt are regarded as concept images (Tall & Vinner, 1981), and can be quite different from the concept definitions (*ibid*) targeted in teaching. According to Tall and Vinner (*ibid*), the concept image is the schema, or cognitive picture a learner has about an object of learning (Sfard, 1991; Marton & Booth, 1997). On the other hand, the concept definition is the actual mathematical conceptions about that mathematical object which is commonly held by mathematicians (Tall & Vinner, 1981).

The theory of variation presents the forms of learning that help the teacher at least to try to make sure that it is possible that what is intended for the learners to learn is learned. Tong (2012) points out that the theory of variation presents four forms in which an 'object of learning' (Sfard, 1991) can be manipulated. He regards the 'object of learning' as a targeted mathematical concept to be learned - for example, solving a fractional equation. He points out that the object of learning could be drawn from the curriculum or "a teacher's assessment of students' needs" (p.3). This assessment could be from, for example, diagnosis of learner's errors and misconceptions, as being pursued in this research. Manipulation of the object of learning takes place in order to focus students' attention on a particular element of that object of learning that needs to be understood (Marton & Booth, 1997; Tong, 2012).

The four manipulations of the variation theory are: contrast, separation, generalization and fusion (Tong, 2012). These manipulations are important, as they focus learners' attention on what is the main component to be learned. Nesher (1987) points out that when we hold beliefs that clash

with counter-evidence, those beliefs "become the focus of attention and inquiry" (p.34). It can be argued that Marton and Booth's (1997) and Tong's (2012) manipulations, therefore, are meant to highlight learners' beliefs against counter-evidence.

The theory of variation concerns directing students' attention to the peculiar features or aspects of the object of learning (Marton et al., 2004; Ling Lo, 2012; Tong, 2012). Peculiar features can be focused by contrasting examples. When highlighting the peculiar features of the object of learning, the teacher shows the learners what the object is and what it is not (Marton, et al., 2004; Ling Lo, 2012; Tong, 2012). *Contrast* is used to focus the students' attention on the features that the object has and does not have. Analogously, a learner cannot understand what the darkness is if she or he has not experienced light. This contrasting can be applied in the context of teaching fractional equations with the intention of focusing learners' attention on its features with which learners struggle to identify. For instance, learners need to be able to discern different fractional equations structures through identifying how each structure differs from the others. By contrasting examples such as: $\frac{2x}{3}+\frac{x}{5}=\frac{3x}{5}$ and $\frac{2}{3x}+\frac{1}{5x}=\frac{3}{5x}$, as well as $\frac{1}{2x-4}-\frac{2}{x+2}=\frac{1}{x-2}$ and $\frac{4}{x^2-x-2}-\frac{2}{x+1}=\frac{1}{x-2}$, learners should realize that the features of these four fractional equations differ if their attention is drawn to the nature of the denominators. In this case the contrast highlights the aspects of the fractional equations structure. It becomes vital to give different examples to learners in order for them to experience variation. So, contrast enables them to see or understand the object of learning in a new way – by comparing it to other similar objects that have different features (Marton et al., 2004; Tong, 2012).

*Separation* serves to separate particular aspects of an object from other objects. In the case of the equations mentioned above, through contrast, one can see that all the equations are fractional, but that in the first two, the denominators are monomials with numerical denominators in the first, and algebraic denominators in the latter. In the second set of equations the first contains binomial denominators only, whereas in the second one, one term has a quadratic trinomial expression. When comparing these types of equations with other algebraic equations, such as $2x+5=3x$ one can easily separate the fractional equation from equations that do not contain division or denominators. After learners' attention has been directed to the aspect of the structure of $2x+5=3x$ and $\frac{2x}{3}+\frac{x}{5}=\frac{3x}{5}$ through contrast, they should be able to separate the form of equation where they have to get rid of the denominators first, from the ones in which they simply begin by grouping like terms. When learners solve fractional equations $\frac{2x}{3}+\frac{x}{5}=\frac{3x}{5}$ they should be able to recognize that they need to determine the lowest common denominator first. This

particular feature of determining the LCD first separates it from an equation such as $2x + 5 = 3x$ in which grouping like terms would be the first step of working out the solution (Marton et al., 2004; Tong, 2012).

*Generalization* requires learners to know and recognize the common characteristics of an equation. In this study, if learners were able to differentiate between an algebraic expression and an algebraic equation, they would have been able to generalize. Furthermore, this would enable them to decide whether to simplify or to find the solution of a variable. Before they solve the equation they need to know what makes it an equation. For instance, $2x + x$ is an algebraic expression, and $2x + x = 3$ is an algebraic equation. (Marton et al., 2004; Tong, 2012)

From my own teaching experience, learning is even more effective when learners are able to contrast, separate and generalize at the same time. This is what Marton, et al. (2004) refer to as *fusion*. In fusion learners are expected to handle different critical aspects of an object simultaneously. When learners are given the equation: $\frac{4}{x^2-x-2} - \frac{2}{x+1} = \frac{1}{x-2}$ to solve they are required to notice that this is an algebraic equation. It consists of an equal sign, which separates one algebraic expression on the left from another on the right. It is a fractional equation with polynomial and algebraic denominators. Learners also need to recognize that this is not an algebraic expression. Recognition of all these characteristics, and being able to handle them simultaneously, would enable a learner to approach the equation's solution, with due care, successfully. The importance of fusion in teaching mathematics can be seen in examinations. Often, learners who are low-achievers perform better when they write assessments at the end of each mathematics topic than at the end of the mathematics curriculum. One of the reasons for this is their inability to handle integrated concepts in one mathematical problem. They find it difficult to handle the features of the object of learning that are integrated to form a linking and holistic conception of that object of learning (Marton & Booth, 1997; Tong, 2012).

According to Mhakure et al. (2014) learners carry misconceptions over from the simplification of algebraic fractions to fractional equations. Fusion is important in the teaching of fractional equations, as students need to be familiar with different structures of fractional equations; and be able to identify and relate these features to one another to form the schema for solving fractional equations. These different features range from relating solutions of common fractions such as addition, subtraction, multiplication and division of arithmetical fractions to the handling of algebraic fractions and algebraic fractional equations.

Variation Theory can also be seen in the lens of constructivism, in that when learners regard particular features of an object they begin to construct ideas about that object. I now discuss the conception of knowledge.

### 2.2.2 Conception of knowledge acquisition

We may assume that if a teacher understands how children construct mathematical knowledge they will understand better how learners come to have misconceptions. According to Olivier (1989) and Hatano (1996), it is impossible for knowledge to be transferred intact from one person to another, but a child constructs his or her own knowledge. In the case of solving equations, for instance, a teacher may explain that when solving $2x + 5 = 3x$, one needs to subtract $3x$ and $5$ from both sides of the equation. This will result in $2x - 3x = -5$. In his or her observation the learner, however, sees $3x$ moved from the right hand side to the left with the sign changed and $5$ from left hand side to the right with the sign changed. The learner then considers that she 'understands' how to solve the equation. When she has to solve a similar equation on her own, she will simply move the terms from left to right and vice versa, and make a sign error in the process. This is because she constructed her own knowledge, and did not consider what the teacher was telling her. This is an example of the fact that a teacher might be teaching a particular concept, but the learner might be learning something different from the teacher's objective. In other words, a learner might have a concept image which differs from the concept definition (Tall & Vinner, 1981) as discussed earlier. Teaching and learning fractional equations is no different, as is often seen in the errors they make in their solutions. For instance, to solve the equation $\frac{2x}{3} + \frac{x}{5} = \frac{3x}{5}$, having 'understood' that the denominators need to be gotten rid of, a learner might solve the equation as follows: $\frac{2x}{3} \times 3 + \frac{x}{5} \times 5 = \frac{3x}{5} \times 5$, resulting in $2x + x = 3x$, which will lead to the solution $0 = 0$, suggesting that the equation is true for all values of $x$. However, this will not be the correct solution of the equation. A misconception would have played a role in producing this erroneous solution because a learner constructed his or her own knowledge.

Hatano (1996) sums up knowledge acquisition in five interrelated characterizations. Table 1 shows how knowledge is acquired, according to him.

## Table 1: Characterizations of knowledge acquisition

| **Characterization** | **Actions leading to construction of knowledge** |
|---|---|
| Knowledge is constructed. | Learners' invention of knowledge is a by-product of their problem-solving or comprehension activity;<br>Teacher's verbalization of target knowledge;<br>Transmission of knowledge. |
| Knowledge is restructured. | Acquired pieces of knowledge cause conceptual change through reorganization of this knowledge. |
| The process of knowledge acquisition is constrained. | The acquired knowledge is often similar between the individuals, and these take place under constraints. These constraints make it difficult for new knowledge to come to mind. |
| Knowledge is acquired domain by domain. | The body of knowledge is divided into a number of domains. Knowledge acquired through problem solving or comprehension activity is stored within the domain. |
| Knowledge acquisition is situated. | Knowledge acquisition occurs in contexts, and therefore situated within these contexts. |

In these terms errors and misconceptions in fractional equations, among other things, can be attributed to the fact that learners construct and reorganize their own knowledge. This new knowledge may resist being comprehended because it is constrained. Simplifying an arithmetic fraction $\frac{2}{3}+\frac{1}{4}$ may be understood differently from simplifying an algebraic expression $\frac{2}{3x}+\frac{1}{4x}$, even though in both cases the concept of equivalent fractions applies. This is because learners may think that the method used to simplify arithmetic fractions is constrained to arithmetical fractions only, and may not be used in algebraic fractions. Hence they find it difficult to transit from arithmetic to algebra (Usiskin, 2004). From such misconceptions learners find solving equations involving fractions difficult: they are unable to assimilate and accommodate new knowledge because they consider it a completely separate item of knowledge.

Mathematics classrooms, together with other learning resources, should be utilized effectively, because that is where knowledge acquisition occurs and is situated, including learning to solve

fractional equations. The other resources include teachers themselves, textbooks, the classroom environment, etc. It is therefore vital that the teacher possesses adequate knowledge in the mathematical topic she or he is to teach to learners. This is where teachers' pedagogical content knowledge (PCK) becomes important, as they need to understand that a learner will need to restructure the knowledge acquired and to be there to guide them in doing so. By guiding learners in constructing and restructuring their acquired knowledge they will be providing them with the opportunities to attempt to *understand relationally* (Skemp, 1976).

### 2.2.3 Relational and Instrumental understanding

Skemp (1976) categorizes understanding into two types; understanding *relationally* and understanding *instrumentally*. He defines relational understanding as "knowing both what to do and why" (p.46), and argues that he could not regard instrumental understanding as understanding at all, as he describes it as "rules without reasons" (p.46). It can be argued that when mathematical conceptual understanding does not occur according to the teacher's intention, a learner resorts to mathematical procedures (Kilpatrick et al., 2001). Skemp (1976) would refer to this situation as a *mis-match*.

Skemp (1976) refers to two situations that can occur in mathematics classrooms as *mis-matches*. The first occurs when a teacher wants learners to understand relationally but the learners' goal is to understand instrumentally. The second occurs when learners want to understand relationally but the teacher's goal is for them to understand instrumentally. In the first situation learners do not care about understanding concepts; all they want is the rules to get to the answer. What frustrates the teacher about the first situation is that when he or she asks a question that does not seem to fit in with the rule, learners get it wrong. The teacher's PCK is therefore important in addressing such situations, as it empowers them to teach mathematical concepts for relational understanding, and can make the learners realize that simply using the rules to solve a mathematical problem is not enough.

In the second situation, Skemp (1976) argues that it is more damaging when a teacher teaches instrumentally while learners want a relational understanding of mathematical concepts. This situation could be one of the sources of mathematical misconceptions. Unfortunately, this happens even if the new mathematics textbooks or other resources are aimed at relational understanding (Skemp, 1976). When a teacher teaches mainly according to the textbook and its examples, without aiming at learners' conceptual understanding, learners' are 'forced' to learn instrumentally. In linear and fractional equations alike, an example of this situation would be a

learner trying to understand why when 'moving' terms across the equal sign, the term's sign changes. When the teacher's explanation directs the learner to given examples in the textbook, without the conceptual focus on the additive inverses in order to get rid of a term from one side to the other, that teacher forces that learner to understand instrumentally. The learner will then resort to memorization of the rules, and then develop erroneous guiding principles (Nesher, 1987) about solving equations.

Since learners' knowledge is constructed and restructured (Hatano, 1996) in the learners' minds through the interaction with his or her experience, misconceptions are bound to happen; and even more so when the teaching is aimed at instrumental understanding. In the studies conducted on learners' errors and misconceptions in mathematics, equations and fractional equations have been evidence of instrumental understanding in that learning. These errors and misconceptions have been discussed in the literature review that follows.

## 2.3 Literature review

In this discussion I draw from literature that I have used, starting with discussing the difference between slips, errors and misconceptions.

### 2.3.1 Slips, errors and misconceptions

It is important to note that a mathematical error could be caused by different actions (Hansen, 2011). They could be the result of carelessness, misinterpretation of symbols or text, not checking the solution carefully at the end of a mathematical problem, and so on (Hansen, 2011). Olivier (1989) distinguishes *errors* from *slips* and *misconceptions*. He refers to *slips* as unsystematic mistakes that produce wrong answers because of faulty processing. They are easy to detect and can be corrected there and then (Olivier, 1989). An example of a slip would be a careless error when adding like terms, as in the following example: $2x - 3 + 4x = 3$, resulting in $-2x = 6$. In this example $4x$ was subtracted from $2x$ resulting in $-2x$ instead of adding it. This careless error will lead to an incorrect solution. That type of error gives no indication of conceptual misunderstanding. For this reason, White (2005) refers to *slips* as *careless errors*, and defines them as those which occur even if the student knows how to get the correct answer at the time the incorrect answer is given.

*Misconceptions* are "underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors", and give rise to systematic wrong answers, referred to as *errors* (Nesher, 1987; Olivier, 1989, p.3). Nesher (1987) agrees with Olivier (1989), and points out that

a misconception "denotes a line of thinking that causes a series of errors, all resulting from an incorrect underlying premise" (p.35). An example of an error caused by an underlying erroneous guiding principle, or misconception, can be seen in the solution of this equation: $\frac{2x}{3} \times 3 + \frac{x}{5} \times 5 = \frac{3x}{5} \times 5 \Rightarrow 2x + x = 3x$. In this example a learner understands that in order to solve this equation she or he needs to get rid of the denominators. However, he or she has the misconception that each term must be multiplied by its denominator, instead of determining the LCD and multiply each term by that.

Errors in this study should be understood according to Olivier (1989) and Nesher's (1987) terms.

### 2.3.2 Learners' errors and misconceptions in common fractions

Mhakure et al. (2014) point out that the multifaceted construct of fractions is the source of learners' difficulties in fractions. They name these constructs as: part-whole, ratio, operator, measure and quotient. If learners had a complete understanding of the relationship between these constructs, perhaps their misconceptions in fractions could be reduced. This is also mentioned in Kerslake's (1986) study where learners only knew that a fraction is a part of whole, but could not extend this to understanding other aspects of fractions. When solving mathematical problems that involve fractions in the FET, fractional errors and misconceptions that learners bring from senior primary school to FET phase become obstacles. These misconceptions include relating decimals to fractions. Many learners in high school cannot recognize $0{,}21$ as $\frac{21}{100}$, neither can they understand a ratio $21:100$ as $\frac{21}{100}$, or its meaning as $21$ in $100$ (Kerslake, 1986; Mhakure et al., 2014). Lack of understanding of such constructs, including equivalent fractions, is reflected in their errors when attempting to solve fractional equations of the form: $0{,}3x + 0{,}2 = 15$ in the FET phase.

In Kerslake's (1986) study, the findings indicated that learners of the ages 12 to 15 lack conceptual understanding of arithmetic fractions. Learners in her study did not understand fractions as numbers, even though they knew that a fraction is a part of a whole. Learners also found it difficult to recognize that there is a connection between $\frac{x}{y}$ and $x \div y$. They also could not link the algorithm for finding equivalent fractions with a diagrammatic illustration of equivalent fractions. If learners do not understand the constructs pointed out by Kerslake (1986) and Mhakure et al. (2014), they will find learning fractions difficult, because learning of fractions requires a "deep understanding of all" (p. 2) these sub-constructs (Mhakure et al., 2014). With

the congested FET phase curriculum, learners' chances of learning fractional skills in the FET phase are reduced (Mhakure et al., 2014).

Mhakure et al. (2014) bring up the point of the irrelevance of fractions in learners' daily lives as the historical reason for learners not engaging with the learning of fractions. However, Usiskin (2007) seems to disagree with the notion of the irrelevance of fractions, and he relates fractions to algebra. This speaks to situations such as relating fractions to numbers, which Kerslake (1986) found to be a problem for learners of 12 to 15 years old. Learners need to understand that in talking about 21% of people, they are talking about 21 in 100 people, which is $\frac{21}{100}$ people. Usiskin (2007) points out that competence in fractions is important to those who take algebra skills as important. Algebra skills are indeed important in mathematics (Hall, 2002) as it is the basis of all mathematics (Usiskin, 2004). Thus it stands to reason that the learning of fractions is important. Learners need to be assisted in reducing errors and misconceptions identified in arithmetic in order to understand fractions in algebra. Usiskin (2004) suggests that arithmetic is seen everywhere in our daily lives, whereas algebra is hidden. The link between arithmetical and algebraic fractions needs to be created in order to eliminate or reduce learners' misconceptions in fractions.

It is therefore important to address learners' errors and misconceptions in fractions as early as primary school, and to find ways to make fractions more meaningful in mathematically solving everyday problems.

### 2.3.3 Learners' errors and misconceptions when solving fractional equations

Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) suggest that mathematical errors may be procedural or conceptual. Procedural errors are produced when learners incorrectly apply computational skills, procedures, algorithms and definitions when learning a particular mathematical concept. On the other hand, conceptual errors are produced when learners have incorrect knowledge of the underlying structure of mathematics, that is, they lack understanding of "the relationships and interconnections of ideas that explain and give meaning to mathematical procedures" (Eisenhart et al., 1993). When solving fractional equations, it becomes evident in learners' errors that they lack procedural skills such as the steps involved in solving fractional equations and the rules governing those steps (algorithms). They lack relational understanding. This was evident in the studies of Figueras et al. (2008) and Mhakure et al. (2014), as discussed above. However, Thomas and Tall (1988) argue that combining procedural and conceptual understanding in solving mathematical problems results in versatile thinking. This suggests that

procedural understanding has its place, but needs to be combined with conceptual understanding. Thomas and Tall (1988) define versatile thinking as being able to switch between the procedural and the conceptual view in order to be able to fit procedures into the whole conceptual structure. They add that if learners lack conceptual understanding, they are not able to think with versatility. For instance, when solving a fractional equation $\frac{2x}{3} + \frac{x}{5} = \frac{3x}{5}$ , a learner who does not have the conceptual understanding of the fact that the equal sign represents equivalence; she or he would not multiply both sides of the equation by the same LCD to maintain equivalence.

Mathematical errors include the application of wrong procedures, incorrect definition of a mathematical concept, and generalizing a rule merely after seeing it working in a few particular instances (Moru et al., 2014). From my experience, errors due to application of incorrect procedures is common among learners. According to Figueras et al. (2008) and Mhakure et al. (2014), learners find simplification of algebraic expressions difficult. This leads to errors and misconceptions in fractional equations.

Legutko (2008) classifies errors in mathematics as mathematical and didactical. He points out that a mathematical error is committed by a student or teacher who considers an untrue mathematical statement as true, whereas didactical error is considered a situation "when teachers' behaviour is contradictory to the didactics, methodological and common sense guidelines" (p.149). In relation to this study, fractional equations section is one of the sections where students 'confidently' make errors because they are convinced that the rule they use is correct. It can be argued that this is caused by misconceptions they bring from the simplification of algebraic fractions (Mhakure et al., 2014) and other areas of mathematics such as arithmetic. Nesher (1987) suggests that misconceptions are "derived from previous instruction" (p.35). For instance, in Kerslake's (1986) study the findings suggested that learners have misconceptions in arithmetic fractions; arguably they bring these across to algebra and continue with the same misconceptions in algebraic fractional equations.

In their analysis of errors grade 10 learners made in the simplification of algebraic and fractional equations, Mhakure et al. (2014) found four categories of learners' errors, namely: (1) *cancellation error*. This term is also used by Figueras, Males and Otten (2008) in their analysis of learners' errors when simplifying rational expressions. For example, learners would simplify the following algebraic fraction as shown: $\frac{5\cancel{a}+a^2}{\cancel{a}} = 5 + a^2$ or $\frac{5a+\cancel{a^2}}{\cancel{a}} = 5a + a = 6a$, cancelling 'like' variables, instead of factorizing the numerator before simplifying by cancelling. This type of error arises from the overgeneralization of simplifying expressions such as: $\frac{ab}{b} = b$.

(2) *Performing a mistaken operation and 'grouping like terms' error*; as in the following example: $2x + y = 2xy$, where learners 'added' unlike terms as if there was a multiplication operation sign between the terms. This is also referred to as *conjoining* (Falle, 2007). Learners bring this type of error from arithmetic. They understand that the equal sign suggests that one must always have an answer on the right hand side of the equation. They believe that if they leave their answer as two terms, there is no closure in their solution.

(3) *Factorization errors:* Learners factorized the sum between two squares as the difference between two squares because they wanted the same brackets that could cancel each other in the numerator and in the denominator. An example of the error they made is: $\frac{a}{a+b} - \frac{a^2+b^2}{b^2-a^2} = \frac{a}{a+b} - \frac{\cancel{(a+b)}(a-b)}{\cancel{(a+b)}(a-b)}$. In this example learners disregarded the signs of the two variables '$a$' and '$b$' in the denominator.

(4) Lastly, Mhakure et al. (2014) found errors where learners converted an algebraic fraction into a fractional equation. They termed this the *equation error*.

Of 10 learners whose errors were analysed, 5 made the cancellation error, 4 made mistaken operation and grouping like terms, 4 made the equation error and 2 made factorization errors. I have observed similar errors when learners solve fractional equations. I also found these errors when conducting a similar study focusing on errors and misconceptions in the simplification of algebraic fractions for my Honours research project.

Figueras et al. (2008) investigated the understanding in the simplification of rational expressions by algebra students. They found that 79% of 750 items were incorrectly simplified, while 74% of the attempted items contained at least one error. They found seven categories of errors committed by algebra students in their study. One of the categories they found was the *cancellation error* (Figueras et al., 2008); for example: $\frac{2x^2+\cancel{x}}{\cancel{x}} = 2x^2$ or $\frac{2x^{\cancel{2}}+x}{\cancel{x}} = 2x + x = 3x$, which is the same as what Mhakure et al. (2014) found in their study. The cancellation error is due to the use of *visual cues* (Figueras et al., 2008). By visual cues they mean that learners confuse $\frac{pq}{q}$ with $\frac{p+q}{q}$, because they recall a cancellation rule that only applies to the former.

Conceptual understanding of the simplification of algebraic expressions is essential in order to be able to solve fractional equations. Sometimes learners are required to factorize the denominator before they determine the lowest common denominator. For example in the equation $\frac{2}{x^2+2x} = 5$, learners would need to factorize the $x^2 + 2x$ first, in order to determine the

correct lowest common denominator. Learners with misconceptions from simplification of algebraic expressions are likely to make a *cancellation error* before determining the lowest common denominator. They are likely to simplify the left hand side of the equation as follows: $\frac{2}{x^2+2x} = 5 \Rightarrow \frac{1}{x^2+x} = 5$, before they determine the lowest common denominator. Such misconceptions relate to the learners' inability to factorize expressions in the numerator or denominator before doing anything else.

Hall (2002) points out that learners tend to use procedural approach when solving equations by using a trial-and-error method or arithmetical approach, such as using the reversal process. Some errors found in Hall's (2002) study on linear equations include the transposing error, the switching addends error and the division error, which was classified as structural error. Hall (2002) suggests that the transposing error is caused by oversimplification of the process of transposing. He also points out that the switching addends error appears more frequently in algebra than arithmetic.

These findings indicate that when learners learn fractional equations they bring misconceptions from simplification of algebraic fractions and apply arithmetic principles. These errors need to be identified again in fractional equations, diagnosed, and a continuing attempt must be made to address them.

### 2.3.4 Teaching intervention

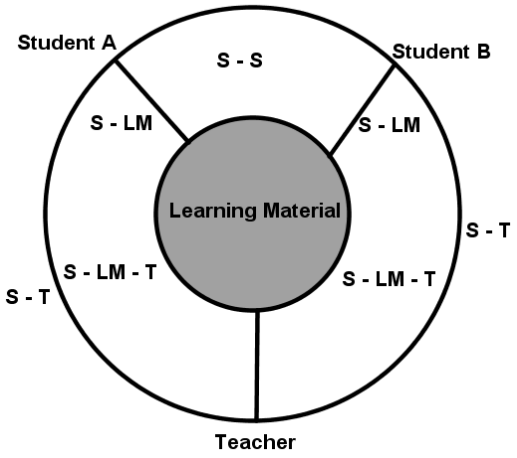
Baker, Gersten and Lee (2002) argue that, as part of teaching intervention, learners' errors and misconceptions can be used as an effective instructional method. They suggest that teachers need to be able to predict learners' errors and misconceptions in order to prepare in advance to use them "to help learners understand correct solutions" (p.53) to mathematical problems. Steinbring (2005) describes teaching-intervention as a '*didactic triangle*' whose three vertices are mathematical knowledge, the student and the teacher. These vertices have to interact by putting a learner at the centre and to help him or her to communicate and construct mathematics object of learning with the mediation of the teacher. Nesher (1987) agrees with Steinbring (2005) when she points out that for the subject matter to be learned an expert teacher is needed in order to successfully bring the learner to know the subject matter by using different pedagogical strategies. Over and above these three interacting components in the learning environment, Kaldrimidou, Sakonidis and Tzekaki (2003), added interaction as one of the components in a learning environment. Therefore, teaching intervention should provide opportunities for learners to explicitly express their thinking through communication with the expertise of the teacher.

Teaching intervention is a form of feedback which Robson et. al (2012) suggest is recognized as an important contribution to learning.

Steinbring (2001) argues that mathematical knowledge is about the relationship between mathematics and its concepts. Learners actively construct these relationships in social processes of teaching and learning. When they construct these relationships they get an opportunity to think independently. Duval (2000) points out that learning mathematics is not only to gain a practice of particular concepts and to apply algorithms, but that "it is also to take over the thought processes which enable a student to understand concepts and their application" (p.2). For this reason, the teacher should mediate between mathematical concepts and learners' thought processes as learners try to construct mathematical concepts (Steinbring, 2001).

Teacher interaction in learning environments has to be analysed using learners' and teachers' mathematical examples. It is also important to interpret learners' intentions or meaning in their articulations as they construct "relevant mathematical relations in the present exemplary learning environment with their own mathematical conceptions" (Steinbring, 2001, p.212).
Kaldrimidou et al. (2003) use three categories of teaching intervention, namely; (a) "re-setting the problem", (b) "providing clues and help for the solution", and (c) "imposition of the solution". It is common for learners to misunderstand a mathematical problem and end up giving an incorrect solution. In discussion of that problem later, the teacher needs to 'kick-start' the problem in order to allow for learners' contributions. Providing clues and help to learners becomes inevitable in such situations. However, providing a solution to mathematical problems without learners' contributions does not provide any learning opportunities. Discussing learners' mathematical errors through teaching intervention should provide learners with learning opportunities. Leikin and Zaslavsky (1997) support this suggestion when they say "task-related verbal interactions are closely related to learning outcomes" (p.2). So, if there is no teacher-learner interaction when addressing learners' errors and misconceptions, there is no learning taking place.

Leikin and Zaslavsky (1997) emphasise that "students' interactions with one another, with learning material, or with the teacher – are significant activities for effective learning" (p. 331). They identify five interactions that may occur in the teaching and learning of mathematics: student-student (S-S), student-learning material (S-LM), student-teacher (S-T), student-learning material-student (S-LM-S) and student-learning material-teacher (S-LM-T). In this lesson all these interactions occurred. The following types of student interactions during learning are taken from Leikin and Zaslavsky (1997):




Types of student interactions during learning

Leikin and Zaslavsky's notion of student interaction is supported by Vygotsky (1978), who suggests that social interaction is a vehicle for learning. Learners learn better to solve problems when they work cooperatively before they can solve problems on their own (Vygotsky, 1978).

The research discussed above shows how important it is to analyse teacher-learner interaction while observing whether learners are being able to overcome their misconceptions. By addressing errors and misconceptions through teaching intervention, a teacher is providing learners with opportunities to learn (Stols, 2013).

### 2.3.5 Opportunity to learn (OTL)

In his South African study, Stols (2013) points out opportunity to learn as an attribute to increasing learner performance. He further points out that OTL includes a number of factors, such as teacher qualification, curriculum materials, teacher's professional development and safety and security of the learning environment. It can be argued that the interaction of all these variables is what provides learners with opportunities to learn mathematics. An adequately qualified mathematics teacher who constantly gets professional development is able to assess and ensure the safety of the learning environment for the learners. He or she then delivers the subject matter that is in line with the intended curriculum, and ensures that opportunities to learn in the classroom are provided for the learners.

Watson (2003) argues that the most important factor that influences learning is the nature of mathematical tasks. She argues that variation within a task is important. Teaching intervention is therefore critical in ensuring that these variations provide learners with opportunities to learn.

Stein et al. (1996) define a mathematical task as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). The nature and structure of mathematical tasks should create opportunities for learners' mathematical thinking, and allow for learners' engagement. Teachers should design mathematical tasks that demand learners' cognition, and allow for learners' cognitive processing. These types of tasks create learners' opportunities to learn (Stein et al., 1996).

Watson (2003) argues that for a learner to be able to use appropriate mathematical language, she or he must have been afforded the opportunity and must have been encouraged to do so. Learners need the teachers' input in recognizing and validating the mathematical rules they use in the classroom (*ibid*). This, in turn, provides them with opportunities to learn.

As Stols (2013) would agree, a teacher must possess pedagogical content knowledge (PCK) in order to provide learners with adequate opportunities to learn. I argue that these can be seen in the varied nature of examples as Watson (2003) suggests, and mathematical explanations that the teacher provides (Steinbring, 2001; Kaldrimidou et al., 2003; Watson, 2003).

## 2.4 Conclusion

In this chapter I have discussed the conceptual framework that guides this study and the literature that focuses on the simplification of algebraic fractions, equations and fractional equations. As elaborated above, these mathematical topics are closely related. I have discussed the theory of variation according to Marton and Booth (1997), Marton et al. (2004) and Tong (2012), acquisition of knowledge according to Hatano (1996) and instrumental and relational understanding (Skemp, 1976).

As is evident in the discussion above, the manipulations that are presented by the theory of variation allowing learning through contrasting examples, separation, generalization and fusion provide learners with the opportunities to learn. The teacher should be there to allow for these manipulations to occur as the object of learning is scrutinized for learners' understanding of its features. The teacher should also guide learners by allowing them to construct their own knowledge (Hatano, 1996), intervening when necessary. This intervention should close the learners' *zone of proximal development (ZPD)* (Vygotsky, 1978) allowing for learners' relational understanding. The zone of proximal development is defined as "the distance between the actual development level as determined *by* independent problem solving and the level of potential development as determined *through* problem solving under adult guidance or in collaboration

with more capable peers" (*ibid*, p. 86). The next chapter discusses the research design and methodology that was used in an attempt to answer the research questions.

# CHAPTER 3: THE RESEARCH METHODOLOGY AND RESEARCH DESIGN

## 3.1 Introduction

This chapter discusses the research methodology and research design used in collecting data for the research questions. In this research the types of errors learners exhibit when solving fractional equations were investigated. Learning opportunities were also investigated in teaching intervention targeting those errors. The research also looked at whether the learners' errors and misconceptions could be reduced by that teaching intervention for re-learning fractional equations at a mathematics clinic.

The research design of the study was qualitative, with text as the main form of data collected through transcripts of video-recorded lessons. Data was also collected from learners' scripts. The intention of collecting data from learners' scripts was to determine the types of errors they exhibit in algebraic tasks. Opie (2004) suggests that qualitative data are all in the form of words. Qualitative research falls under the interpretive paradigm. Guba and Lincoln (1994) refer to a paradigm as "an approach to thinking about and doing research" (p.31). Hatch (2002) points out that "naturalistic qualitative research methods are the data collection and analytic tools of the constructivist paradigm" (p. 15). Hatch (2002) views the constructivist / interpretive paradigm as one in which knowledge is constructed by the individuals and, therefore, is subjective. Investigating the types of errors learners make when solving fractional equations and hearing learners' voices as they construct their own knowledge as individuals in the learning environment further qualifies this study as qualitative.

This study is qualitative in the sense that the data collected is empirical and context-based, and collected from real classroom interaction between learners, learning material and the teacher. Furthermore, it was collected under natural conditions. Data collected in this study include video recordings and transcripts produced from these recordings (Wilson, 2009).

As I explore learners' errors and also understand what students can learn in a mathematics clinic about their errors, qualitative data was a suitable type for use in this study. This is a case study because it gives an opportunity for learners' fractional equations errors and misconceptions to be studied in depth in a limited amount of time as Opie (2004) suggests. In this study the researcher focuses on the use of errors and misconceptions as a teaching resource at a mathematics clinic. The Mathematics clinic is attended by a small group of learners. This same group was chosen as participants in this study. McMillan and Schumacher (2006) argue that in a case study one phenomenon is chosen by the researcher in order to understand in depth. In a

case study the phenomenon is studied in its real-life context (Wilson, 2009). The case in my study is learners' errors and misconceptions in fractional equations displayed at a mathematics clinic, and addressing those misconceptions through teaching intervention. Wilson (2009) defines a case study as "a traditional, systematic approach to looking at events, collecting data, analyzing information and reporting the results with the end goal of describing the case under investigation as fully and accurately as possible" (p. 204). This definition confirms the process that the research took.

## 3.2 Sampling

Hatch (2002) points out that a sample is a part of statistical population that is studied in order to gain information about the whole. Whereas positivistic research findings on a sample are used to prove a theory and to generalize to a population, in qualitative studies the results are inductive, used to generate a theory. That means that results in a qualitative study cannot, in general, be extrapolated to a population. They only pertain to a certain case. As such, samples in qualitative study are purposive and non-representative (Creswell, 2012).

Purposeful and convenience sampling were used in this study. Creswell (2009) argues that the idea behind qualitative research is to "purposefully select participants or sites that will best help the researcher understand the problem and the research question" (p.178). The sample in this study consists of a group of learners that I have taught previously and some that I was still teaching. I decided to conduct research on these participants because I identified the mathematics clinic at my school as the rich source of data for this study. The school and the mathematics clinic class as research sites were intentionally selected for rich data, hence purposeful sampling. Creswell (2012) refers to purposeful sampling as intentional selection of participants or sites for the purpose of understanding the central problem. The researcher chooses participants or sites according to whether the data collection from those participants or site will be 'rich'. The researcher searches for information-rich participants, groups or sites to study (McMillan & Schumacher, 2006). The learners in this sample are the ones who most need help, and therefore provided rich data of errors and misconceptions in fractional equations. It was convenient to approach the principal, the head of mathematics, the mathematics teacher, learners and their parents about conducting the study. Learners and their teacher were also willing and available to participate in this research. They understood the educational benefit of participating in the study better (Creswell, 2012). Silverman (1997) adds that convenience sampling is a statistical method

of drawing representative data by selecting people because of their availability or easy access, as is the case in this study.

The learners in this study were just a small sample – 17 out of 109 grade 10 learners who take mathematics as a subject at my school. Some of these learners acknowledge that they generally find mathematics challenging while others are referred to mathematics clinic by their mathematics teachers. Learners then regularly attend this remedial facility provided by the school, called mathematics clinic. The clinic offers extra mathematics lessons to high school learners who find certain mathematics topics difficult, and are generally among the low achievers in mathematics. They attend mathematics clinic lessons two afternoons a week, after school normal hours. These lessons are each, one-hour long. However, data for this study was collected during normal mathematics lessons due to the unavailability of some participants after school hours.

This research took place at a girls' private school in Johannesburg, South Africa. It was conducted with grade 10 learners of average age 16, generally low achievers in mathematics and attend the clinic regularly. They were chosen according to their grade 9 average end-of-year results, their formal and informal assessments in the first term and mid-year examination of the current year. Also, most of them were in the bottom set, and a few came from the second – bottom set. In this school mathematics classes are streamed, that is, learners are allocated to classes according to ability, determined from their previous year's results. The bottom set consists of learners with an average of 40% to 50%, and the second bottom an average of 51% to 60%. Learners in the sample had slightly varying mathematical abilities.

Besides the mathematics clinic teacher, of the initial 17 participants, the number reduced to 13, due to the fact that two participants only wrote the pre-test, and did not participate in subsequent data collection sessions. Of three pre-test items, the other two participants only answered the first two.

The learners in this study write Independent Examinations Board (IEB) examinations at the end of their grade 12 year. The IEB serves independent, or private schools in South Africa. It conducts its own summative assessments, as distinct from those of the public schools.

### 3.3 Data collection methods and research instruments

The research was conducted on the topic of fractional equations, introduced in grade 9, and extended in grade 10. The learners therefore had had an opportunity to learn how to solve fractional equations presented in different forms. Data was collected in a numbers of forms,

namely; pre-test, video-recording of teaching intervention sessions, teacher's interview and post-test.

### 3.3.1 Pre-test

This helped to answer the first research question: "What errors and misconceptions do students display when solving equations involving fractions?" Pre-test in this study refers to items tasks that were testing learners' pre-knowledge of fractional equations concept. It consisted of three items of fractional equations. The learners had to solve the equations and give explanations on how they had solved them. The items were selected according to their structure and the degree of difficulty, hence only three. The learners were given notice well in advance to prepare for the pre-test. By the time the data was collected they had been led in revising the topic as preparation for the mid-year examination. The tasks were written under controlled, or test conditions, and learners were given 40 minutes to complete the task (Appendix A). Their solutions to fractional equations were analysed for the errors committed. On the answer sheet was a section alongside their working in which they were asked to explain how they solved their equation. The reasons for those errors from their explanations were analysed in terms of learners' procedural and/or conceptual understanding, and recorded. Errors were then discussed with the mathematics clinic's teacher, and given to her to use for her teaching intervention.

### 3.3.2 Video-recording of teaching intervention sessions

This instrument helped in answering the second research question: "What opportunities to learn are made available to learners with misconceptions on solving equations involving fractions at a mathematics clinic?" The teacher's intervention sessions, two one-hour lessons, were video-recorded by the researcher. The researcher was looking at different aspects of the lesson, such as discussions of errors made through contrasting examples, in order to focus learners' attention on the strategies of solving fractional equations, including discussion of restricted solutions. When these aspects were not focused on in the teaching intervention, another lesson video-recording was done.

The researcher did not participate in the lessons, that is, she did not pose any questions or manipulate the situation in any way; she was just an observer (Cohen, Manion & Morrison, 2011). The video recording was transcribed, and transcripts were produced. The teachers' and learners' explanations, and the questions posed by both teacher and learners, as well as the responses to those questions, were analysed for the learning opportunities they availed to learners.

### 3.3.3 Interview with the teacher

After the second session of teaching intervention, I interviewed the teacher about the session. I had to conduct the interview in order to understand the teacher's experience of the intervention session. In addition to that I wanted to find out about the questions the learners asked in their groups during the session. It was not easy to hear what learners were discussing and asking the teacher since all groups were discussing the task at the same time.

### 3.3.4 Post-test

The other form of the data collection instrument was the post-test. Post-test in this study refers to items tasks that were testing learners' gained knowledge from the teaching intervention. This again consisted of three fractional equations to be solved. Again, there was a column requiring learners to explain how they had solved the equations. This type of data collection instrument helped to answer the third research question of this study, which was: "What gains in learning are met through the intervention?"

The items in the post-test were similar, but slightly different to the pre-test ones (Appendix B). These items were designed in a way that allowed the researcher to examine if the type of errors made in the pre-testing stage were reduced by teacher's intervention. This was further clarified in the learners' explanations column in the task.

This method of data collection was necessary, as it would help to check whether the learners benefited from teaching intervention. This was determined by comparing the types of errors that were made in the pre-testing task with those of the post-testing task. Learners' explanations of how they had solved the equations were the evidence of whether gains had been made through the intervention. The learners were given notice of the post-test. The task was written under controlled conditions, and the learners were given 40 minutes to complete the task.

## 3.4 Analysis of data

In order to analyse the data in this research the researcher used the pre-designed conceptual framework, as well as her experiential knowledge of learners' errors in fractional equations. The categories of errors were determined by those found in literature of similar studies such as Figueras et al. (2008) and Mhakure et al. (2014) studies, as well as some that emerged that may not have been evident in the literature.

The researcher's experiential knowledge together with the method of analysis used by Figueras et al. (2008) and Makonye and Khanyile (2015) were adopted in the analysis of data in this

research. The errors from the pre-test and post-test tasks were recorded and categorized. Errors' categories adopted from Figueras et al. (2008) and Makonye and Khanyile (2015) were combined with the new categories of errors that had not been mentioned in the previous studies, but that had been experienced by the researcher in her own teaching.

The transcript produced from observing the video was analysed according to the theory of variation (Marton et al., 2004; Ling Lo, 2012; Tong, 2012). The Theory of Variation uses four forms of manipulation of the 'object of learning' in order to provoke and focus learners' thinking. According to Marton & Runesson (2003), in Tong (2012), these manipulations are contrast, separation, generalization and fusion. I analysed the conversations according to Steinbring's (2001) classification of analyses of teacher's intervention. These classifications are (a) mathematical examples used, (b) teacher's questions and/or responses to learners' questions, (c) learner's articulation of their mathematical explanations in relation to examples given, and (d) interpretation of learners' articulations. The nature of mathematical tasks on fractional equations to be solved was also analysed according to Watson (2003). It must be noted that Watson's (2003) analysis of mathematical tasks involves the theory of variation.

The post-test task was analysed using the same categories as the pre-test task.

## 3.5 Rigour

Rigour pertains to the issue of reliability and validity of the research. Seale and Silverman (1997) argue that authenticity is more of the issue in qualitative research than reliability. Validity focuses on how the authentic understanding of people's experiences is ensured in the research. On the other hand, reliability concerns with consistency of the research results if the research is conducted later using the same data collection instruments. Validity and reliability in this research are discussed in detail in the next paragraphs.

### 3.5.1 Validity

To address the issue of validity I submitted my first data collection instrument (a pre-test task) to the head of the mathematics department and to the Mathematics Clinic's teacher, both from my school, to moderate. I also discussed with them in detail my other data collection instruments. Maxwell (1992) points out that validity can be descriptive, interpretive or theoretical. Descriptive validity is about factual accuracy (Maxwell, 1992). According to Maxwell's (1992) definition of descriptive validity, this study provides facts, as all data were collected by the researcher. Data collected and the learners' responses to the pre- and post-tests provided evidence of accuracy.

The video-recording of lessons was done to address the issue of accuracy when transcribing conversations. Teacher's intervention involved interpretive validity, as I was required to interpret the actions and words of the participants in the lesson. Maxwell (1992) argues that the "account of participants' meanings are never a matter of direct access" (p. 290), but they are interpreted and "constructed by the researcher on the basis of the participants' accounts and other evidence" (p. 290). These interpretations were done using the conceptual framework discussed earlier; this qualifies the validity to be a theoretical one.

The process of data collection went from pre-tests to teaching intervention. At the teaching intervention stage participants were given the copies of their pre-tests in order to see the errors they had made, while the teacher was addressing those errors at the same time. This allowed the participants to check the accuracy of the data collected by the researcher from their pre-test. According to McMillan and Schumacher (2006), one of the ways of increasing validity is for the data collected to be informally checked by the participants. Another aspect that increases the validity of this study is the fact that the language used in all forms of data collection was that of the participants – English. Even though not all participants had English as their home language, all of them were fluent in it. They had all attended English-medium schools from pre-school level, and they all do English as home language level: they were comfortable with English.

As is clear from this discussion, the data collection instruments used in this study measured what they intended to measure (Wilson, 2009) in order to answer the research questions of the study. In order to strengthen the validity of this research a pilot study was conducted by collecting data in a form of a test in fractional equations.

### 3.5.2 Reliability

According to Maxwell (1992), reliability addresses a particular threat to validity. It is a precondition for validity, according to Wilson (2009). Creswell (2009) suggests that reliability in qualitative research can be achieved by following certain reliability procedures. The data collection tools used in this study: in particular, pre- and post-tests, video-recording, interview as well as transcripts, have been proved in other studies to collect reliable data. In order to address the issue of reliability, data was collected from the participants without any interference of the researcher. The pre- and post-tests had unambiguous instructions. The procedures of test administration were consistent in terms of time allocation, the time at which the tests were taken and the level of supervision. The fact that the participants were made aware that their performance in the tests would not affect their individual or class mathematics averages helped

in relieving them of stress and nerves. Therefore, the participants were more relaxed, and that maximized their performance, especially because they were preparing for their mid-year examinations at the same time. Creswell (2012) argues that in order to ensure the reliability of the research, the questions on instruments must be free of ambiguity, the procedures of test administration must not vary, and the participants must be in a good state of mind in order to interpret the questions correctly.

In the collection of data through video recording the researcher was an observer only, not a participant in any way (Miles & Huberman, 1994). The transcripts produced from the videos were recorded accurately in terms of the participants' utterances (McMillan & Schumacher, 2006; Wilson, 2009; Creswell, 2012).

The coding of data used in the study is consistent with that of other researchers, and defined according to their findings. Data coding created by the researcher was clearly defined, and its origin explained. Of five data codes used in this study three have been used by other researchers in similar studies, such as Figueras et al. (2008) and Mhakure et al. (2014), and two were made up by the researcher from the emerging learners' errors in the data of this research.

The discussion above attest to the reliability of this study.

### 3.5.3 Ethical considerations

The ethics clearance was applied for, and was granted by the Ethics Committee of the University of the Witwatersrand. The researcher's protocol number is 2015ECE010M.
Evidence of the permission and consent from everyone who took part in this study is provided (See Appendices H to L for the ethics letters sent to the research participants). Permission was granted by the Headmistress, the Head of Mathematics department, the Mathematics Clinic's teacher, the learners and their parents, all of the school where the study was conducted.

## 3.6 Conclusion

This chapter has given a comprehensive description of the research methodology and design. Sampling, research instruments, procedures used to collect data, validity, reliability and ethical considerations were discussed in detail. The next chapter focuses on data analysis and discussion of results for this study.

# CHAPTER 4: ANALYSIS AND DISCUSSION OF RESULTS

## 4.1 Introduction

This chapter presents the collected data, the report on its analysis and discusses results in light of the theoretical framework. Data analysis concerns making sense of the data in relation to the research aims and questions. It requires critical thinking. As Hatch (2002) points out, data analysis is a systematic way of searching for meaning in the data. Analyzing data requires the researcher to organize, present and examine data with the intention of detecting patterns and relationships between variables. Hatch (2002) argues that data analysis could be from typology and deduction; or inductive and grounded (Glasser & Strauss, 1967). With typology the data obtained is mapped into pre-determined categories as a preliminary way of sorting data. Typologies should be derived from theories, literature or research objectives such as research questions, and should be consistent with these objectives. On the other hand, in the inductive (grounded) approach (Glaser & Strauss, 1967) refers to analysis using categories that emerge from the data itself, without any pre-determined ideas. This type of analysis moves from specific to general. The researcher searches for patterns of meaning, so that general statements or theories about the phenomena being studied can be made (Hatch, 2002).

In this study both models of data analysis are used, in accordance with Hatch (2002), who suggests, that, in practice, both the deductive and inductive data analysis methods be used.

In this research I analyse and discuss data collected when attempting to answer the research questions. The study attempted to investigate the errors and misconceptions that learners display when solving fractional equations. It also investigated opportunities to learn created when learners' errors and misconceptions were used as pedagogical cognitive resources in a learning intervention at a mathematics clinic.

I begin my data analysis with categorizing learners' mathematical errors in the pre-test, then I move to teaching intervention tasks and then the post-test.

## 4.2 Some error codes used in analysing learners' errors in the study

In order to gain insight into the learners' errors and misconceptions when solving equations involving fractions, raw data were analysed, using five categories. Three of these were pre-determined from the literature, while the two of them were made up by the researcher from the emerging findings.

*Like-Term Error* was coded as (T). Among the errors that Figueras et al. (2008) found and coded in their study on the simplification of algebraic fractions, they identified two types of like-term errors. An error where learners committed an error other than division, (usually simplifying like terms in the numerator with the ones in the denominator where there was a subtraction sign between the terms) they termed '*Like-term error 1* (*T1*)". Where learners performed an incorrect operation in the numerator or denominator; (for instance, incorrect addition of terms) they termed "*Like-term error 2 (T2)*". In this study only the like-term error 2 was observed, and therefore was adopted as simply a like-term error (T). Like-term error was characterized by the incorrect answer from adding like terms. For example: $3y + 6 - 6y = 6 + 3y$.

*Sign error* was coded as SE. This was characterized as making a sign error when simplifying either the left hand side or right hand side of the equation by the distribution law. This was an error where a learner multiplied a negative term outside the bracket by a negative term inside the bracket or vice versa and obtained an answer with the incorrect sign. For instance; $-4(y - 6) \Rightarrow -4y - 24$.

*Restriction error* was coded as RE. These were characterized as not stating the restricted solutions to the given equation, or by stating incorrect restrictions. For instance; in $\frac{2}{6x} + \frac{1}{3} = 4$ it should be stated that $zero$ is a restricted value of $x$, as $x$ cannot be equal to $zero$.

*Lowest common denominator error* was coded as LCDE (Makonye & Khanyile, 2015).
The LCDE error was characterized as determining the incorrect lowest common denominator or by determining the LCD before factorizing the denominator, where it was necessary to factorize the denominator first. An example of this error would be: $\frac{2}{6x} + \frac{1}{3} = 4 \Rightarrow \frac{2+2}{6} = 24$.

*Other error* was coded as OE (Figueras et al., 2008). OE error was characterized as errors that could not be categorized under the above categories. These errors could not be linked to any conceptual misunderstanding, but could only be seen as careless errors such as: $-4(y - 6) = -4 + 24$.

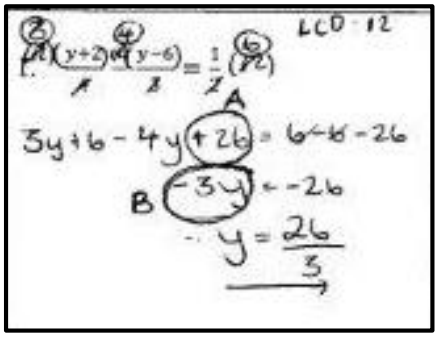
### 4.3 Learners' errors in the pre-test

Item 1:

The first item required learners to solve the equation: $\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$. The following example shows one of the learners' solutions to this equation.

**(a) Like-Term Error**

Learner C:




Learner C was able to determine 12, which is the correct LCD of 4, 3 and 2. She multiplied both sides of the equation by 12 in order to maintain the equivalence, simplified the left side of the equation by the distributive law and obtained: $3y + 6 - 4y + 26 = 6$. However, in this step she erroneously wrote 26 (see A) for $-4 \times -6$ instead of 24. Also, when adding unknown terms $(3y - 4y)$, Learner C obtained $-3y$ $(see\ B)$ instead of $-y$, making like-term error. When she continued with procedures, she managed to maintain the equivalence as she divided both sides of the equation by $-1$ in order to isolate $y$. Her incorrect solution of $y$ is due to the like-term error she had made in the second-last step.

Learner C showed a good understanding of solving equations involving fractions. The like-term error she made is a slip which could have been caused by the lack of concentration.

One learner made like-term error in this item.

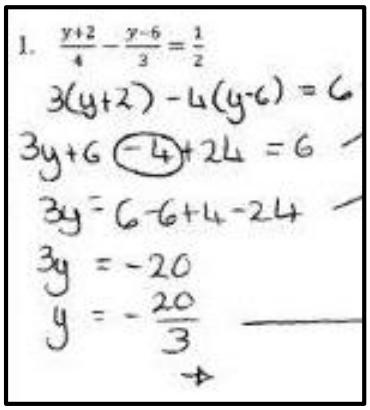
**(b) LCD Error**

None of the learners committed the LCD error in this item. This shows that they understood how the arithmetic LCD is derived.

**(c) Other error**

The following example shows the learner who made errors termed ‘other error’. Most of the errors in this category were careless errors.

Learner A



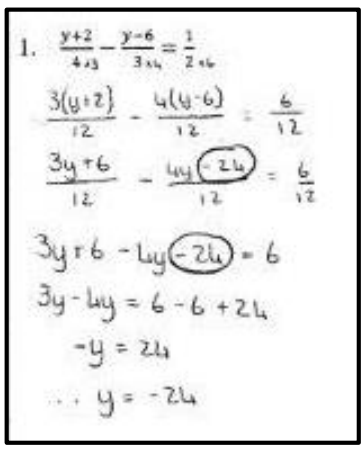

When distributing $-4$ into $(y-6)$, learner A obtained $-4$ instead of $-4y$. This is merely carelessness, as this error cannot be connected to any misconception. The learner simply forgot to write the variable $y$ after multiplying $-4$ by $y$, and all other subsequent steps were executed correctly. Grouping and addition of like terms were done correctly. The incorrect solution of $y$, which is $-\frac{20}{3}$,was due to the careless error she made.

Three learners made the other error in this item:

**(d) Sign error**

Let us consider the following learner's solution in the example below:

Learner G




From the learner's solution to the equation, having done everything correctly, the learner made a sign error ($-4\times-6=-24$) in step 3. The solution to $y$ was incorrect only because of that sign error. Every concept and algorithm related to solving fractional equations and simple equations was well understood and executed correctly. She did not check her solution carefully.

I would call such an error ‘sign error’, relating to failure to work with operations of directed numbers.

Five learners made the sign error in this item.

Item 2:

The second item required learners to solve the equation: $1 - \frac{2}{p} = 2 - \frac{1}{p}$. In this equation there is a variable $p$ in the denominator of the second term on the left and on the right. This requires learners to give extra information about the solution of $p$, either before or after solving the equation.

**(a) Like-Term Error**

None of the learners made like-term error in this equation.

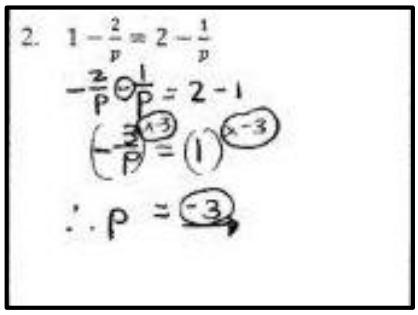
**(b) LCD Error**

None of the learners made the LCD error in this equation. This shows that they understood how to formulate an algebraic LCD where a monomial expression is involved in the denominator.

**(c) Other error**

The following example shows some of the learners’ other errors which could not be classified as either procedural or conceptual.

Learner M



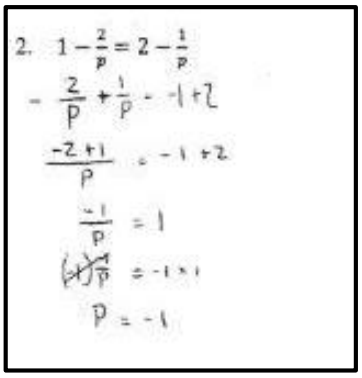

In this solution Learner M’s first error was a sign error. When transposing $-\frac{1}{p}$ from the right to the left side of the equation, the sign remained the same instead of changing to positive. She was, however, able to change the sign of 1 to negative when she transposed it from the left side to the right side of the equation: $-\frac{2}{p} - \frac{1}{p} = 2 - 1$. This error is caused by a misconception regarding grouping of like terms to the same side of the equation: she ‘moved’ them instead of ‘adding’ or ‘subtracting’ them from one side and also should have done the same on the other side of the

equation in order to maintain the equivalence of the equation. If a learner says to herself: I add $-\frac{1}{p}$ on the right and add it on the left, the chances of making a sign error due to transposing a term are slim. The learner correctly added like terms after the error, obtaining: $-\frac{3}{p} = 1$. However, instead of multiplying both sides of the equation by $p$ to eliminate the denominator, learner M incorrectly multiplied both sides of the equation by the numerator $-3$. Furthermore, instead of getting $\frac{9}{p} = -3$ from that, she obtained $p = -3$. This answer is coincidentally correct, because the right hand side of the equation is 1; had it been any other number, greater or less than 1, the answer would have been different. If she had multiplied both sides of the equation by $p$, like this: $p \times -\frac{3}{p} = p \times 1$ she would have got the same solution: $p = -3$ which, would have been incorrect anyway, but would have been the result of the first transposing error.

The learner consciously knew that she was supposed to isolate $p$, but she also had to eliminate the denominator $p$. She thought if she eliminated the denominator $p$ on the left side of the equation, she might end up losing $p$, for which she needed the solution. She did not realise that when multiplying both sides by $p$, $p$ will be eliminated on the left side but will reappear on the right side of the equation.

The following solution is for Learner G, who used the same approach as M, but successfully added $-\frac{1}{p}$ to the right side of the equation in order to eliminate it on the right; she also added it on the left side of the equation in order to maintain equivalence.

Learner G




She correctly simplified the two-term fractional expression on the left side of the equation to obtain $\frac{-2+1}{p} = -1 + 2$, and also correctly added the constant terms in the numerator and on the right side respectively to obtain $\frac{-1}{p} = 1$. However, like M, from this step, G multiplied both sides

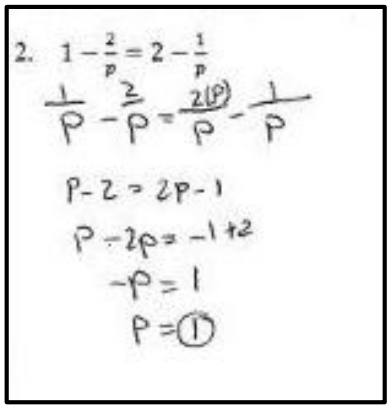
of the equation by the numerator $-1$, when she was supposed to multiply by the denominator $p$, in order to eliminate the denominator. In her explanation she wrote: "multiply everything by $-1$ to get the denominator to the top". She coincidentally ends up with the correct solution for $p$, which is $-1$. Again, as explained earlier, this solution is coincidentally correct because the right hand side of the equation is 1. This confirms Nesher's (1987) and Luneta's and Makonye's (2010) arguments that learners sometimes obtain correct answers despite using incorrect mathematical procedures.

Two learners committed other errors in this equation.

**(d) Sign Error**

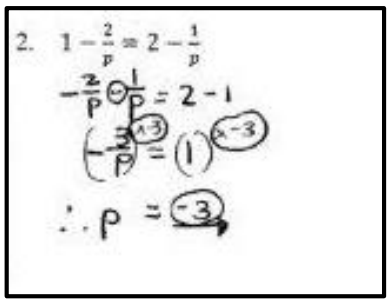
Consider the learners' solutions in the following example:

Learner F




Having correctly executed all the necessary steps to solve a fractional equation, Leaner F did not divide both sides of the equation by $-1$ in the last step in order to maintain equivalence in the equation. From the step $-p = 1$, only $-p$ was divided by $-1$ to obtain $p$, but 1 was not divided by the same $-1$ to obtain $-1$. Thus equivalence was not maintained, which led to the incorrect solution $p = 1$. This error led to a sign error as the solution of $p$ should be $-1$ and not 1.

Learner M



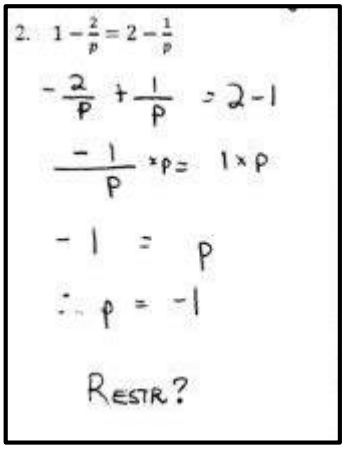
Learner M committed a sign error when transposing $-\frac{1}{p}$ from the right to the left side of the equation. Instead of adding $-\frac{1}{p}$ to both sides with the intention of eliminating it on the right side, she 'moved' it to the left, and hence forgot to change the sign. The learner in this case does not understand how a term is eliminated on one side of the equation, and also did not maintain equivalence in the equation.

Two learners made the sign error in this equation.

**(e) Restriction Error**

Consider the learner's solution in the following example.

Learner T




In this solution, Learner T did not indicate that there is a restricted solution of $p$, since $p$ is in the denominator. If she had found $p$ to be zero, she would have possibly thought that was the correct solution of $p$, which would not have been the case.

Failure to state the restriction indicates the lack of understanding the meaning of restricted solutions when solving fractional equations.

None of the learners stated incorrect restrictions, but neither did those who committed a restriction error in this question state the restriction at all. Ten learners made the restriction error in this question.

Item 3:

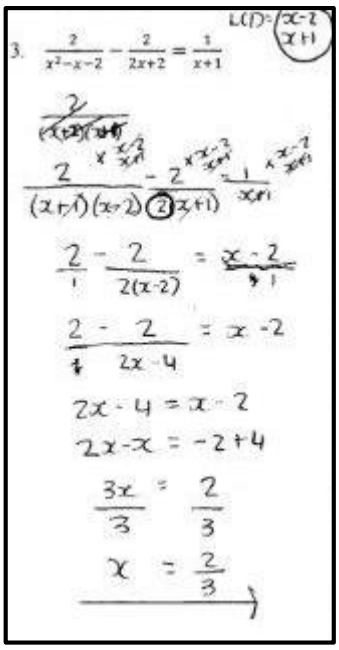
The third item required learners to solve the equation: $\frac{2}{x^2-x-2}-\frac{2}{2x+2}=\frac{1}{x+1}$. In this equation there is a quadratic trinomial in the denominator that needs to be factorised first, and two

binomials, of which one also needs to be factorised first. Factorising the denominator would help learners identify the restricted values of $x$, since $x$ is in all the denominators.

**(a) Like-Term Error**

The example below shows the solution of one of the learners.

Learner P



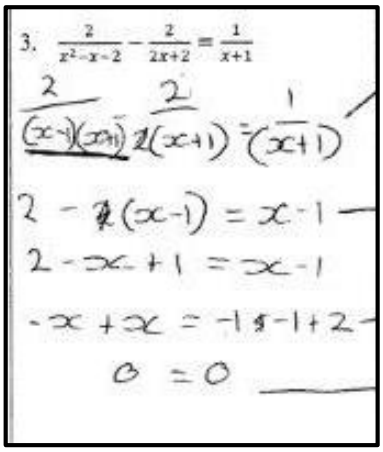

Learner P's third-last step is: $2x - x = -2 + 4$. The learner grouped like terms with the terms containing unknowns on the left and the constant terms on the right. While the right hand side of the equation was simplified correctly; i.e. $-2 + 4 = 2$, when simplifying $2x - x$ on the left side, the learner obtained $3x$, and this led to $3x = 2$. Like terms were added incorrectly, hence making a like-term error.

Three learners of the 13 committed this error in this equation.

**(b) LCD Error**

This equation required learners to factorise the denominators $x^2 - x - 2$ and $2x + 2$ first in order to easily determine the LCD. Factorising these denominators would further help the learners to identify the restricted values of $x$. I analyse the following example for the incorrect LCD obtained due to the incorrect factors of the denominators, the incorrect formulation of the LCD, or for not factorising the denominators that needed to be factorised before determining the denominators. The example below shows one of the learners' LCD errors when attempting to solve this equation.

Learner A



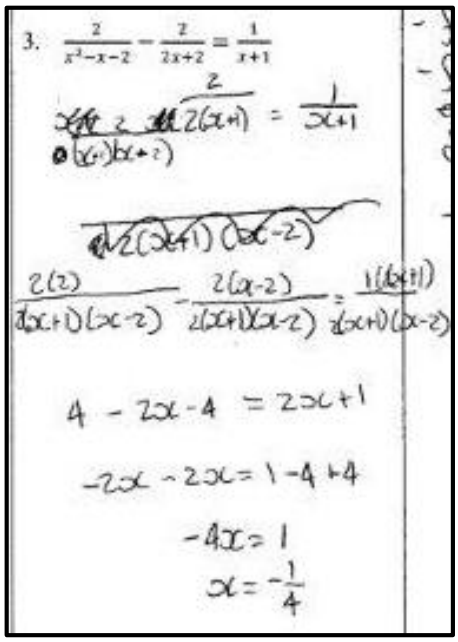

Learner A committed an LCD error which was due to the incorrect factors of $x^2 - x - 2$. She found the factors $(x-1)(x+1)$. When these two factors are simplified or multiplied out, they give $x^2 - 1$, which is the difference between two squares, and not $x^2 - x - 2$, which is a quadratic trinomial. This shows that the learner does not have a concept of factorizing a quadratic trinomial. Learner A does not understand that the factors of $-2$ (the third term of the expression) should give the sum of $-1$, the coefficient of $x$, the middle term. She needs to also understand that this method of factorising a quadratic trinomial applies when the value of $a$ is 1, in the general quadratic expression $ax^2 + bx + c$. Without knowing how to factorise quadratic trinomials it is impossible to solve a fractional equation with a quadratic trinomial in the denominator. It is important, however, to note that although Learner A did not find the correct factors of $x^2 - x - 2$, she was able to 'build' the LCD from the factors she obtained. Had she not had a problem with the quadratic trinomial factors and some sign errors (that will be discussed later), she would have been able to solve the equation correctly.

Four of 13 learners committed the LCD error in this equation.

**(c) Other Error**

The following example shows learners' solutions that contain the errors termed 'other errors' - mostly careless errors.

Learner F



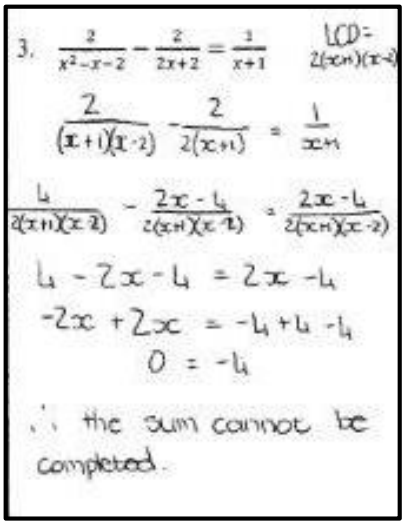

F, having made a sign error, also committed a careless error on the right hand side of the equation. Although she did not show how she obtained $2x + 1$ on the right hand side (step 3), when she multiplied $\frac{1}{x+1}$ on the right by the LCD $2(x + 1)(x - 2)$, she should have obtained $2x - 4$, but she obtained $2x + 1$. Together with the sign error made on the left side in the same step 3, the incorrect solution $x = -\frac{1}{4}$ was obtained. This error shows a lack of concentration.

Six of 13 learners committed the 'other error' in this equation.

**(d) Sign error**

Consider the following learner's solution in the example below.

Learner G



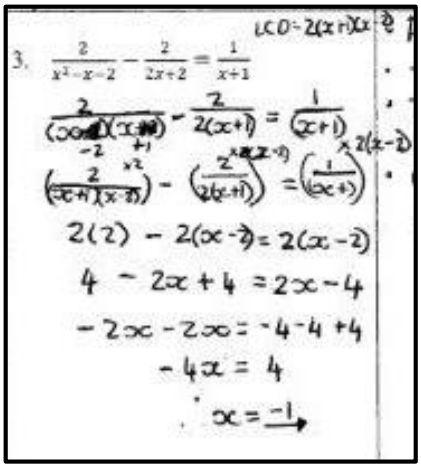

The learner made two sign errors. The first one was in step 1: $-2(x-2)$, giving the product of $-2x-4$ in step 2. Although this step is not clearly shown, this error occurred when the learner was multiplying $-\frac{2}{2(x+1)}$ by the LCD $2(x+1)(x-2)$ and was caused by ignoring the existence of brackets around $x-2$. The second sign error occurred when transposing $2x$ from the right to the left side. Instead of subtracting it from both the left and the right, G just 'moved' it and therefore forgot to change the sign. Again, a lack of understanding of the reasons for moving the terms' sides results in this kind of error.

Four of 13 learners committed sign error in this equation.

**(e) Restriction Error**

Consider the learner's solution in the example below.

Learner M




M did not state the restriction at all. One of the denominators in the equation is $x + 1$, but she found the solution $x = -1$. According to her solution, $-1$ is the correct solution of $x$. However, $-1$ is not applicable, as it makes the equation undefined; but because she did not first state the restricted values of $x$ initially, she takes this solution as valid. This error shows that M does not understand the importance of restrictions, or of why restrictions have to be stated.

Ten learners did not state the restrictions, and two stated incomplete or incorrect restrictions.

Table 2 shows the number of learners who committed errors in each category per equation.

**Table 2: Error frequencies of learners in each error category per equation in pre-test:**

| | $\frac{y+2}{4}-\frac{y-6}{3}=\frac{1}{2}$ | | | | $1-\frac{2}{p}=2-\frac{1}{p}$ | | | | | $\frac{2}{x^2-x-2}-\frac{2}{2x+2}=\frac{1}{x+1}$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Learner | T | LCDE | OE | SE | T | LCDE | OE | SE | RE | T | LCDE | OE | SE | RE |
| A | | | ✓ | | | | | | ✓ | | ✓ | | ✓ | ✓ |
| B | | | | ✓ | | | | | ✓ | | ✓ | | | ✓ |
| C | ✓ | | ✓ | | | | | | | | | | | |
| D | | | | | | | | | | | | ✓ | | ✓ |
| F | | | | | | | | ✓ | ✓ | | | ✓ | ✓ | ✓ |
| G | | | | ✓ | | | ✓ | | ✓ | | | | ✓ | ✓ |
| H | | | | ✓ | | | | | ✓ | ✓ | | ✓ | | ✓ |
| J | | | ✓ | | | | | | | | | | | ✓ |
| M | | | | | | | ✓ | ✓ | ✓ | ✓ | | ✓ | | ✓ |
| P | | | | | | | | | ✓ | ✓ | | ✓ | | ✓ |
| Q | | | | ✓ | | | | | ✓ | | | | ✓ | ✓ |
| R | | | | | | | | | ✓ | | ✓ | ✓ | | ✓ |
| T | | | | ✓ | | | | | ✓ | | ✓ | | | ✓ |
| Total | 1 | 0 | 3 | 5 | 0 | 0 | 2 | 2 | 10 | 3 | 4 | 6 | 4 | 12 |

### 4.4 Analysis of opportunities to learn in teaching intervention

The teaching intervention was analysed with respect to the opportunities to learn created for the learners. Opportunities to learn in this study were classified as: (i) cognitive opportunity to learn (Stein et al., 1996; Watson, 2003) (ii) opportunity to learn from the learners' explanations regarding the solutions to equations (this is provided in the right column of the learners pre- and post- tests (Appendices A and B) and (iii) opportunities to learn provided in the (a) lesson to

address learners' errors and misconceptions and (b) through the teaching intervention worksheet (Appendix D).

### 4.4.1 Cognitive opportunity to learn

In both the pre- and the post-test equations, items to be solved ranged from simple to more complex equations. The first item consisted of numerical denominators only. In these equations, for example: $\frac{y+2}{4}-\frac{y-6}{3}=\frac{1}{2}$ (pre-test) and $\frac{2x-3}{2}+\frac{x+1}{3}=\frac{3x-1}{3}$ (post-test); learners had to transform the equations into equivalent equations that did not contain fractions. They were given the opportunity to apply known basic facts of equivalent fractions to eliminate denominators. They were also given the opportunity to use previous knowledge of writing a number as a product of its prime factors in order to work out the lowest common denominator; for instance, in the equation above, the denominators $4=2\times2=2^2$, $3=3^1$, and $2=2^1$. From these prime factors the lowest common denominator will be $2^2\times3^1=12$. After this they would have the opportunity to apply their knowledge of the distributive law. For example, the left side of the equation: $\frac{12}{1}\times\frac{(y+2)}{4}-\frac{12}{1}\times\frac{(y-6)}{3}$ $\Rightarrow$ $3(y+2)-4(y-6)\Rightarrow3y+6-4y+24$. The application of the distributive law also gives the learners the opportunity to display their understanding when distributing a negative number into the terms inside the brackets in a pre-test item. The learners' accuracy in grouping and simplifying like terms to obtain the value of $x$ and $y$ respectively, was assessed in these equations. The structure of these equations provided the learners with the opportunity to learn meaningful mathematical procedures by engaging with the equations at hand.

In the second equations: $1-\frac{2}{p}=2-\frac{1}{p}$ (pre-test) and $\frac{2}{3x}-\frac{1}{6}=\frac{6}{x}$ (post-test), the opportunity to learn was provided for learners to display their understanding of restricted solutions as $p$ and $x$ could not be zero. They were expected to show that they knew that if $x=0$ or $p=0$ the equation would be meaningless, since division by zero is not acceptable in mathematics. They were expected to state restrictions before or after solving the equations. The nature or structure of these equations provided the learners with an opportunity to contrast (Marton & Booth, 1997) them with the first equations to see what was the same and what was different with regard to the denominators.

In the third equations: $\frac{2}{x^2-x-2}-\frac{2}{2x+2}=\frac{1}{x+1}$ (pre-test) and $\frac{1}{x^2-2x-3}+\frac{4}{x+1}=\frac{3}{x-3}$ (post-test), deciding on restrictions required more thinking than in the second equations. Firstly, these equations are more complex, since the denominators are quadratic trinomials and a binomial

(pre-test) that needed to be written as a product of their factors first. For example; $\frac{2}{x^2-x-2} - \frac{2}{2x+2} = \frac{1}{x+1} \Rightarrow \frac{2}{(x+1)(x-2)} - \frac{2}{2(x+1)} = \frac{1}{x+1}$. One of the denominators in the post-test equation needed to be written as a product of its factors as well, for instance; $\frac{1}{(x-3)(x+1)} + \frac{4}{x+1} = \frac{3}{x-3}$. This step was important before determining the lowest common denominator which would make all fractions equivalent while 'dropping' (simplifying) the denominators at the same time. For example; the LCD in the pre-test equation is $2(x+1)(x-2)$. Simplifying all terms with the denominators by the LCD:

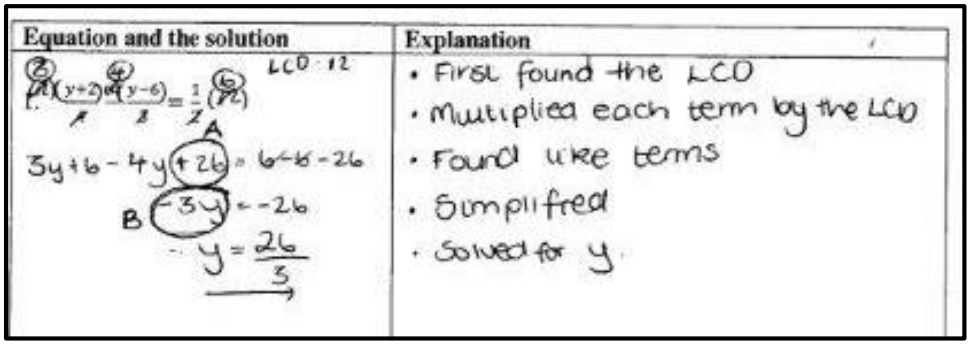
$\frac{2(x+1)(x-2)}{1} \times \frac{2}{(x+1)(x-2)} - \frac{2(x+1)(x-2)}{1} \times \frac{2}{2(x+1)} = \frac{2(x+1)(x-2)}{1} \times \frac{1}{x+1} \Rightarrow 4 - 2(x-2) = 2(x-2) \Rightarrow 4 - 2x + 4 = 2x - 4$. Stating restrictions in this case required more of the learners' cognitive engagement, because they had to think of the values of $x$ in both instances that would make the denominators zero. The learners had to use their pre-knowledge of additive inverses, and the fact that they give the sum of zero, for their restrictions. The participants' cognitive opportunity was provided by the complexity of the equation, which required more engagement than the previous ones. These equations also gave them the opportunity to contrast the equations' structure to the first and the second ones before attempting to solve them.

### 4.4.2 Opportunity to learn from learners' own explanations

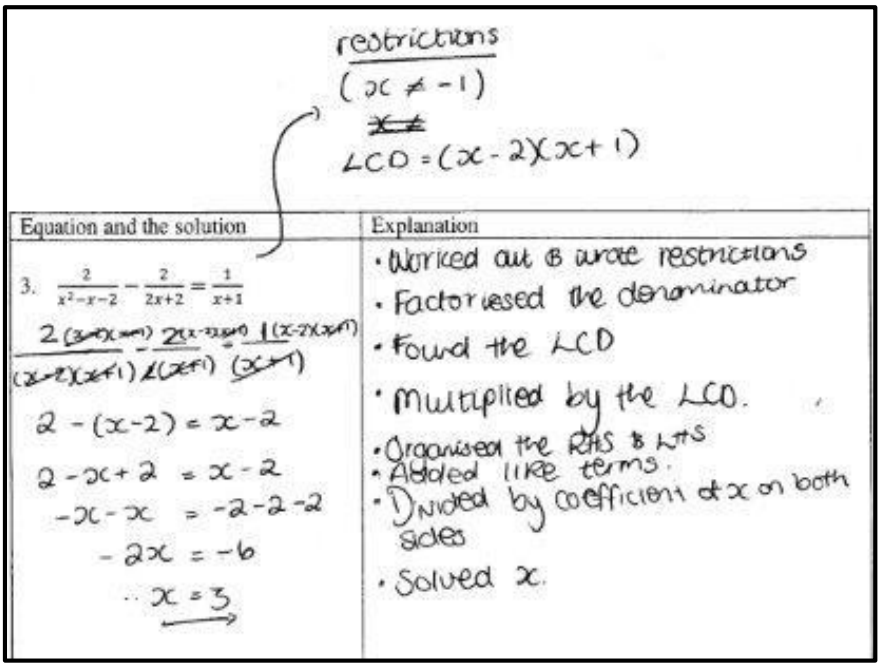
While solving the equations, learners were instructed to explain how they solved them step-by-step. This was done in order for them to make their thinking visible. These steps were intended to describe the actions to be taken which led to procedures to use in solving the equations. The participants were therefore given an opportunity to decide strategically on the goals, actions and procedures (Robson et al., 2012) applicable to solve each equation. Some of the learners' actions and procedures are shown in the examples below:

Learner C

(a)



(b)




In example (a) with the *goal* of getting rid of the denominators in the equation, Learner C took actions of finding the LCD, and multiplied each term by the LCD. She also had a *goal* of combining like terms in order to group the terms with a variable for which she was solving, and her *actions* were to add and simplify them. This learner's final point of explanation was 'solved for $y$'. In that explanation there is a *goal* (to solve for $y$) but the *action* is not explained, *how* she would solve for $y$ (Robson et al., 2012). The learner's explanation shows that she knows the procedures of solving a fractional equation. However, she does not elaborate on each step in terms of how to execute it. For instance, she does not explain how she is going to find the LCD in order to convince the reader that she knows exactly what finding the LCD involves. When the learner says that she multiplied each term by the LCD, she needed to include the reason why she did that, for instance, 'multiplied each term by the LCD in order to eliminate the denominators'. In her third step, the she said 'found like terms' - *and then?* Again, she was supposed to go further, and mention what she did with those like terms: something like: 'I grouped them'. Her last step is 'solved for y'. She should have mentioned how she did that, for instance: "I divided both sides of the equation by $-3$ to solve for $y$".

In example (b) Learner C's first *goal* was to determine the $x$ values that would make the denominators zero. Her actions were to write the denominators as the product of their factors (i. e., factorise), and then find the additive inverses of the constant terms in the denominators and make them restrictions. The next *goal* was to get rid of the denominators in the equation; the

learner took *actions* of finding the LCD and multiplied each term by the LCD. She then had a *goal* of combining like terms in order to group them with the variable for which she was solving. In order to achieve this, her *actions* were to add and simplify them. In this example, she had a goal of isolating the variable, as well as the action on that, which is to 'divide by coefficient of $x$ on both sides', in order to solve for $x$ (Robson et al., 2012). The learner's first step makes it clear that she understands that she has to 'work out' the values of $x$ that will make the denominator zero. This shows that she understands that when the denominator is not a monomial variable, some thinking goes into working out the restricted values of the variable. In this equation her step 5 is better than her step 3 in figure 14 (a): she mentions that she organized the RHS and LHS, however, using the phrase 'grouped like terms' would have made it even better understood. The last two steps are much clearer as she explains how she would isolate $x$.

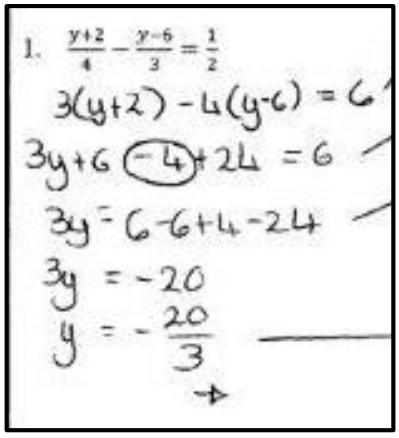
### 4.4.3 Opportunity to learn provided in the first lesson to address learners' errors and misconceptions

In the feedback lesson the teacher discussed the pre-test items by explaining how learners were supposed to solve equations. She started by stating how equations are solved.

**Opportunity to learn through teacher-learners' conversations**

(a) Below is one of the learners' incorrect solutions to the equations $\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$ and the extracts taken from the video lesson 1 transcript (see Appendix D).

Learner A




*1. Teacher: When solving a mathematical equation, whatever you do on the left of the equal sign, you have to do it on the right.*

$$\frac{3(y+2)}{12} - \frac{4(y-6)}{12} = \frac{1}{2}$$

$$\frac{3y+6-4y+24}{12} = \frac{1}{2}$$

$$\frac{-y+30}{12} = \frac{1}{2}$$

$$-y + 30 = 6$$

$$-y = -24$$

$$y = 24$$

2. *Teacher: How to get rid of the denominator? Multiply each term by the LCD on both sides of the equal sign.*

*Inverse operations (undoing what is done to the variable we are solving for).*

In the lesson introduction learners are reminded about the equation solving concept – "*whatever you do on the left of the equal sign, you have to do it on the right*" and strategic decisions – "*multiply each term by the LCD on both sides of the equal sign*" (action) (Robson et al., 2012).

Learners A's solution in figure 15 above indicates that she is aware of the procedure that 'whatever you do on the left of the equal sign, you have to do it on the right'. In her solution we see this in the step: $3y + 6 - 4 + 24 = 6$ and $3y = 6 - 6 + 4 - 24$. Although she did not show this action explicitly on the left in her step, she consciously did the following: $3y + 6 - 6 - 4 + 4 + 24 - 24 = 6 - 6 + 4 - 24$.

(b) The extracts below show the conversation between the learner and the teacher. The learner needs some clarification.

*3. Learner: In the previous example, how come for the* $\frac{1}{2}$ *on the right hand side, you did not multiply it by the LCD?*

*4. Teacher: We did, in the next step:* $\frac{-y+30}{\cancel{12}} \times \cancel{12} = \frac{1}{2} \times \cancel{12}\, 6$*, although it is not shown but the fact that we got* 6 *on the right means that we multiplied* $\frac{1}{2}$ *by* $12$.

$$-y + 30 = 6$$

$$-y + 30 - 30 = 6 - 30$$

$$-y = -24$$

$$y = 24$$

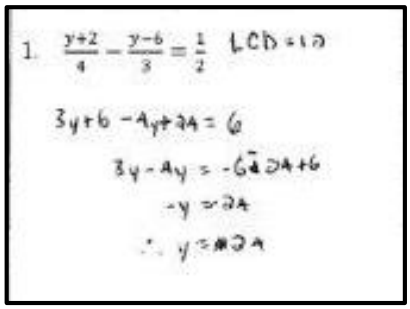
In these extracts, even though the step $'-y+30=6'$ is shown, the learner did not understand that the right hand side was also multiplied by the LCD (12) because the step was not shown when this simplification happened. The teacher had to show this step explicitly so that the learner could understand how 6 on the right hand side was obtained. The opportunity to learn was created when the learner asked for clarity.

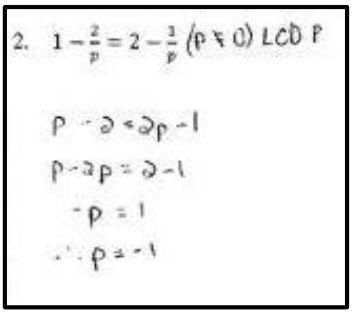
(c) In the following example the learners' solutions are shown, and the extracts show that they are being focused on the similarities and the differences of the first and the second equation.

First equation: $\frac{y+2}{4}-\frac{y-6}{3}=\frac{1}{2}$ Second equation: $1-\frac{2}{p}=2-\frac{1}{p}$

Learner D

(i) (ii)




*5. Teacher [writes the second equation on the board]:* $1-\frac{2}{p}=2-\frac{1}{p}$

*What is the same between this and the previous equation?*

*6. Learner 1: They are both fractions.*

*7. Learner 2: It is also an equation.*

*8. Teacher: What's different?*

*9. Learner 3: There is a variable in the denominator.*

*10. Teacher: The principle of how we solve is still the same. We want p all by itself and the value of p that will make this true. We want the value of p that will make this mathematical statement true.*

…

*11. Teacher: Solving* $1-\frac{2}{p}=2-\frac{1}{p}$*, the LCD is p, if we multiply each term by the LCD as follows:* $\frac{p}{1}\times 1-\frac{p}{1}\times\frac{2}{p}=p\times 2-\frac{p}{1}\times\frac{1}{p}$*, we get* $p-2=2p-1$*, we now need to group like terms, terms with p on the left and constants on the right:*

$p - 2p = -1 + 2$, *simplify the left and right hand sides of the equation* $\Rightarrow -p = 1$, *if we divide the left and the right hand sides by* $-1 \Rightarrow p = -1$.

*We are saying* $p = -1$*; if I have the variable in the denominator, I have to check that my solution is not creating any mathematical issues. Anyone has an idea what that issue might be?*

12. *Learner D: We have to assume that our solution is correct.*
13. *Teacher: You are right, we have to assume that our solution is perfect and that* $'-1'$ *is that perfect solution.*
14. *Learner B: The denominator must not be equal to zero.*
15. *Teacher: We have to check that the solution to this equation does not break any mathematical rules we know. If, for argument sake, we have got a value of zero, a value of zero for p will break a mathematical rule and that is a division by zero. So, if there is a variable in the denominator, you always have to state the restriction upfront. So,* $p \neq 0$.

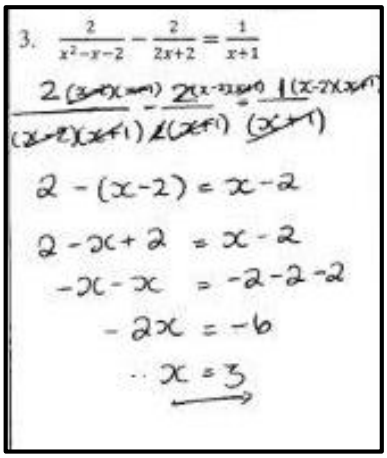
In the first part of the conversation the teacher introduces the concept of restricted values of a variable. She focuses attention on the fact that the denominators in this equation differ from the first equation's denominators in the sense that some denominators in the second equation have a variable. Learners are now focused on the *object of learning* (Sfard, 1991; Marton & Booth, 1997; Marton, Runesson & Tsui, 2004; Tong, 2012; Ling Lo, 2012) – the denominator that requires them to determine the restriction of a variable. The opportunity to learn created for learners in this conversation is that they were made to realise that the value of $p$ must not break a mathematical division by zero rule.

In the second part of the conversation the learners are seen participating actively in the lesson trying to construct mathematical meaning. That meaning is negotiated by the individual learners (Watson, 2003). This discussion provides learners with the opportunity to learn as their thinking is provoked.

(d) The following extracts show the opportunity to learn in seeing learners' attention focused on the specific aspect of the object of learning when solving $\frac{2}{x^2-x-2} - \frac{2}{2x-2} = \frac{1}{x+1}$. The teacher starts by comparing this equation with $\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$ and $1 - \frac{2}{p} = 2 - \frac{1}{p}$.

The example below shows one of the learners' solutions for the equation $\frac{2}{x^2-x-2} - \frac{2}{2x-2} = \frac{1}{x+1}$.

Learner C




*16. Teacher [writes on the board and then asks]:* $\frac{2}{x^2-x-2} - \frac{2}{2x-2} = \frac{1}{x+1}$ *; What is the same about this equation as to the previous ones?*

*[Learners start to answer without raising their hands]*

*17. Teacher: I may have to take hands on this one.*

*18. Learner K: There are variables in the denominator.*

*19. Teacher: Which tells us …?*

*20. Learner C: We have to have restrictions.*

*21. Teacher: What else is the same?*

*22. Learner B: Fractions.*

*23. Teacher [repeating the learner's answer]: Fractions.*

*24. Learner D: Equation.*

….

*25. Teacher: There is something different about this equation. Anyone spotted the difference?*

*26. Learner C: There is more than one term in the denominator.*

*27. Teacher: There is one other thing that you have to do if you have denominators that look like these.*

*28. Learner C: We have to factorize.*

*29. Teacher: You do need to factorize.*

*30. Teacher: So, this is a quadratic trinomial [underlining the denominator of the first term of the expression on the left hand side of the equation]. So, we need to factorize it before we take this any further, before we even do our restrictions.*

*31. Teacher: Because I am hoping that when this is factorized, we might get something similar to these [pointing at the denominators of the other terms of the equation].*

Again, in the above exchanges attention is focused on the differences between the expressions in the denominators of $\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$, $1 - \frac{2}{p} = 2 - \frac{1}{p}$, and $\frac{2}{x^2-x-2} - \frac{2}{2x-2} = \frac{1}{x+1}$. The teacher focuses their attention on the difference between numerical, monomial variables and polynomial variables in the denominators. In line 20 the learner realizes that she needs to have restrictions before solving the equation. The same learner (line 26) mentions another important point, that there is more than one term in the expressions in the denominator, and therefore there is a need to factorize it (line 28).

### 4.4.4 Opportunity to learn provided by the teaching intervention worksheet (second lesson)

The participants in this session worked in 6 groups of 2 or 3. The worksheet consisted of three tasks:

task A – an arithmetic task, where learners' pre-knowledge of writing a number as a product of its prime factors, equivalent fractions and determining the lowest common denominator were assessed;

task B – an algebraic task aimed at determining learners' pre-knowledge of writing an algebraic expression as a product of its factors, equivalent fractions and determining the lowest common denominator;

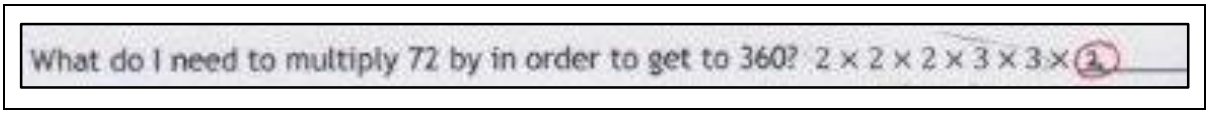
task C – aimed at giving the participants an opportunity to display their understanding of restricted solutions when solving a fractional equation with variables in the denominator. Also given on the worksheet were the reflection questions on what learners had suggested in the previous task of restrictions and a cumulative task where they were given algorithms of solving an equation involving fractions. In the cumulative task participants were given the following equation to solve: $\frac{1}{x^2-x} + \frac{2}{x-1} = \frac{1}{x}$ and also given the solution $x = -2$ for the equation.

Attention was focused on a number of aspects chosen as the *objects of learning* (Sfard, 1991; Marton et al., 2004; Ling Lo, 2012) in the teaching intervention worksheet. These objects of learning included writing 72 and 20 as a product of their prime factors, connection between factors of numbers and their LCD, identifying common factors, writing algebraic expressions as the product of their factors, and 'building' the LCD from the factors of algebraic expressions.

In the following examples, the opportunity to learn was given to the participants: however, misconceptions of various concepts were revealed by their errors as they were working on the tasks provided in the worksheet.

(a) The example below shows an error when responding to the arithmetical question: What do I need to multiply 72 by in order to get 360? Only one of the six groups of learners made this error.

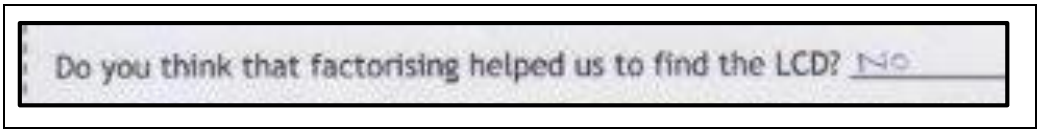
Learners A and B:




This question was intended to provide learners with the opportunity to learn to write numbers as a product of their prime factors: 72 as the product of its factors was an object of learning in this case (Sfard, 1991; Marton & Booth, 1997; Marton, Runesson & Tsui, 2004; Ling Lo, 2012). While the learners were working on this question, the circled '2' was the error committed. This group could not determine the prime factor that was supposed to complete the arithmetic sentence. This error shows that the learners in this group did not understand the concept of writing a number as a product of its prime factors, which is purely arithmetic. Also, they could not link this section with the section learned in grade 8 – using the 'fish bone' or ladder method to find the prime factors of a number.

(b) The example below shows an erroneous answer to the question: Do you think factorizing helped us to find the LCD? Two groups of participants committed this error.

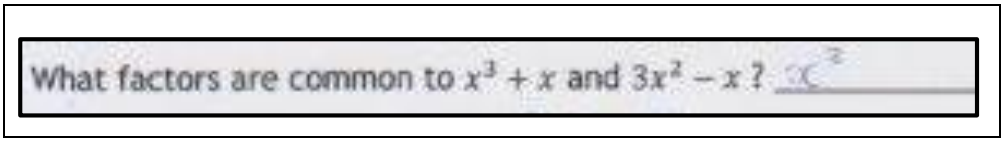
Learners M, P and Q




The learners were provided with an opportunity to realise that there is a connection between the factors of numbers and their LCD. They did not seem to have noticed that connection.

(c) The example below shows what they thought about the question: What factors are common to $x^3 + x$ and $3x^2 - x$? Two groups committed this error.

Learners F, K and R



This question provided the learners with the opportunity to factorise algebraic expressions. They just wrote down the product of the two common factors, and considered that as their common factor.

(d) The example below shows the erroneous response to the question: What factors are not common? The participants gave the following erroneous response:

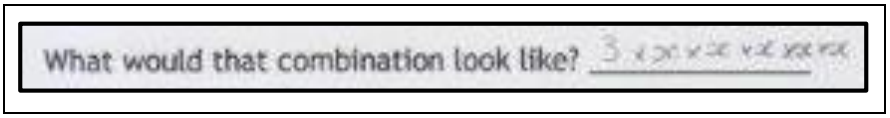
Learners F, K and R




In that error it seems that they had no concept of what common and non-common factors are.

(e) The example below shows the learners experiencing difficulty in 'building' the LCD from the algebraic factors. They committed the following error:

Learners D, H and S:

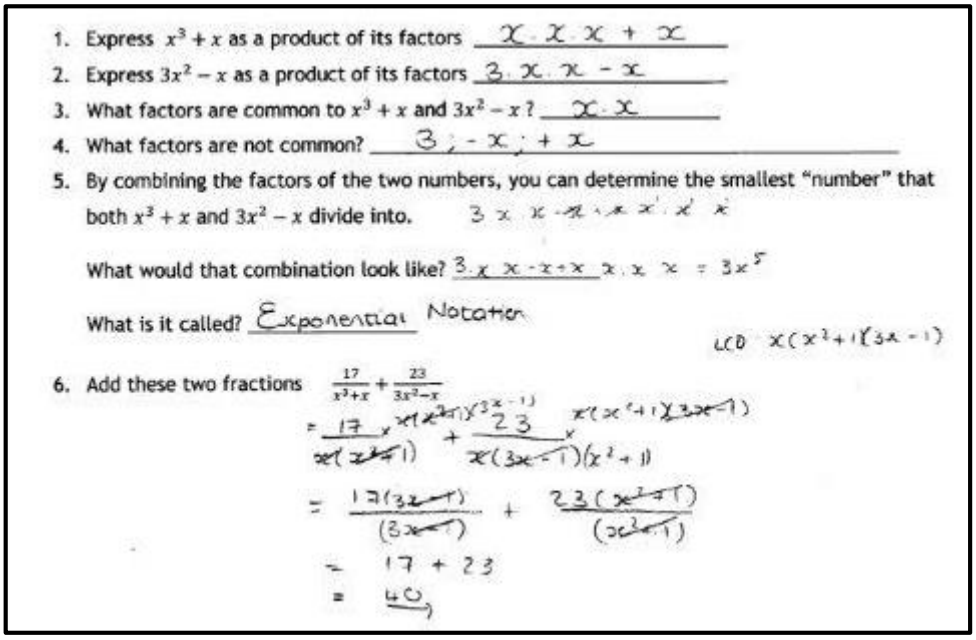



In this error they did not understand how to 'build' the algebraic LCD from the factors of algebraic expressions. The opportunity to learn was provided, it revealed that the learners have a misconception with regard to building an algebraic LCD.

In task B the participants were given an opportunity to consolidate all the concepts covered in task A by simplifying the following fractional algebraic expression: $\frac{17}{x^3+x} + \frac{23}{3x^2-x}$. Some groups - as expected, given how they answered the previous questions - were not able to solve the equation correctly. However, what was interesting was that some groups who could not find the correct combination of LCD of $x^3 + x$ and $3x^2 - x$ managed to determine the correct LCD in this expression.

(f) Let us look at the following group's response to the questions asked in task B:

Learners C and T


1. Express $x^3 + x$ as a product of its factors $x \cdot x \cdot x + x$
2. Express $3x^2 - x$ as a product of its factors $3 \cdot x \cdot x - x$
3. What factors are common to $x^3 + x$ and $3x^2 - x$? $x \cdot x$
4. What factors are not common? $3 ; -x ; +x$
5. By combining the factors of the two numbers, you can determine the smallest "number" that both $x^3 + x$ and $3x^2 - x$ divide into. $3 \times x \cdot x \cdot x \cdot x \cdot x$

   What would that combination look like? $3 \cdot x \cdot x \cdot x + x \cdot x \cdot x = 3x^5$

   What is it called? Exponential Notation

   LCD $x(x^2+1)(3x-1)$
6. Add these two fractions $\frac{17}{x^3+x} + \frac{23}{3x^2-x}$

$= \frac{17}{x(x^2+1)} \times \frac{x(x^2+1)(3x-1)}{1} + \frac{23}{x(3x-1)(x^2+1)} \times \frac{x(x^2+1)(3x-1)}{1}$

$= \frac{17(3x-1)}{(3x-1)} + \frac{23(x^2+1)}{(x^2+1)}$

$= 17 + 23$

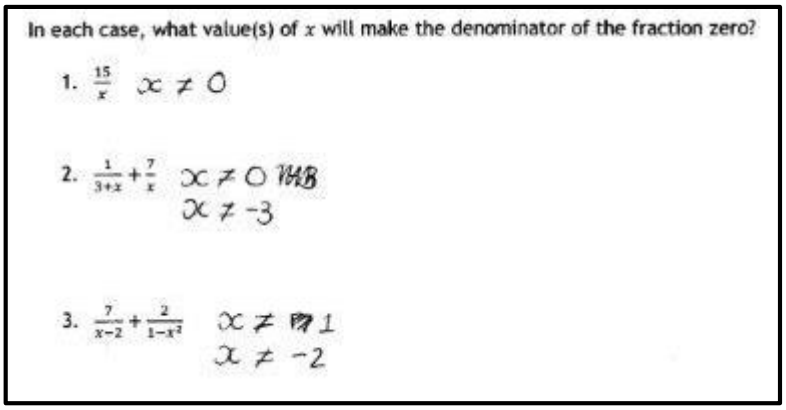
$= 40$

Even though the group was unable to respond to questions 1 to 5 correctly, they were able to determine the correct LCD.

In task C the participants were given the opportunity to learn about the values of the variable that would make the denominator zero (restricted values of the variable). This made restrictions the object of learning. Three groups showed an incomplete understanding of the concept in the sense that they provided some corrected restricted values but left out others.

(g) The following is an example of this:

Learners M, P and Q


In each case, what value(s) of $x$ will make the denominator of the fraction zero?

1. $\frac{15}{x}$ $x \neq 0$
2. $\frac{1}{3+x} + \frac{7}{x}$ $x \neq 0$ IMB
   $x \neq -3$
3. $\frac{7}{x-2} + \frac{2}{1-x^2}$ $x \neq$ [illegible] 1
   $x \neq -2$

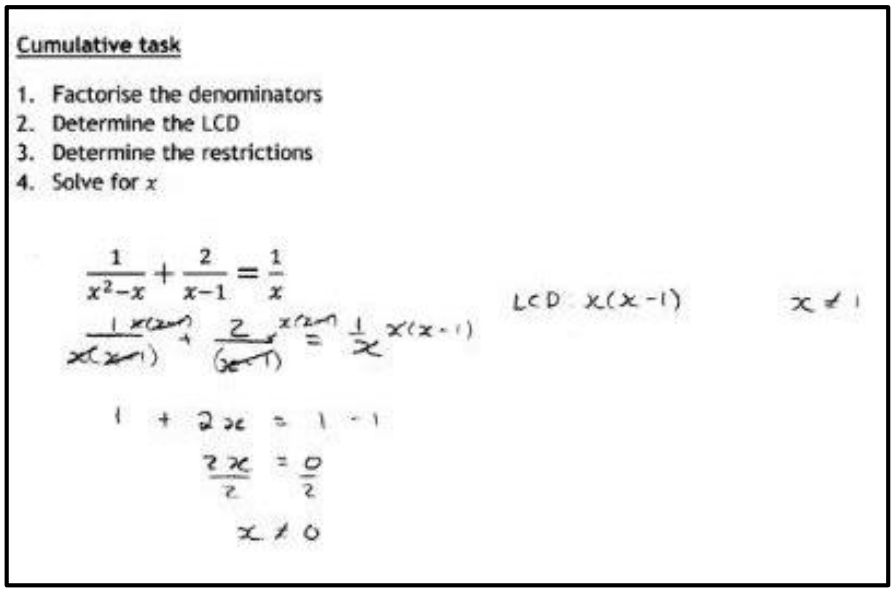
Here the restricted values of $x$ were correctly determined for the first two expressions. However in the third expression the participants stated that $x \neq -2$. This shows that either the learners

had made a random error or that they did not understand the concept altogether. Also for $x \neq 1$, the restriction is correct, but they do not realize that $(-1)^2 = 1$, which makes $-1$ a restricted value as well.

In the cumulative task the learners were given the equation $\frac{1}{x^2-x} + \frac{2}{x-1} = \frac{1}{x}$ to solve. They were given the opportunity to solve the fractional equation in order to apply all the knowledge from the previous tasks A and B. All the groups managed to solve the equation correctly, with the exception of one group.

(h) The example below shows that groups' incorrect solutions to the equation.

Learners C and T




In this equation the participants showed that they understand how to write algebraic expressions as a product of their factors, and how to build the LCD from those factors. However, they stated an incomplete set of restricted values, as they left out 0 and 1 as restricted values. They also committed a careless error on the right hand side of the equation when simplifying $\frac{1}{x}.x\ (x-1)$: instead of obtaining $x-1$, they obtained 1. This led to the solution $x = 0$. They then realised that $x \neq 0$, but did not explain what this solution meant about the equation.

### 4.4.5 Interview with the teacher

After the teaching intervention worksheet, the interview with the teacher was conducted. The following extracts from the interview show what the teacher thought about the learners' pre-knowledge.

*Researcher: Firstly, in the task you set for the learners, the focus was on factors and equivalent fractions; did you think this was the source of their errors in the fractional equations pre-test?*

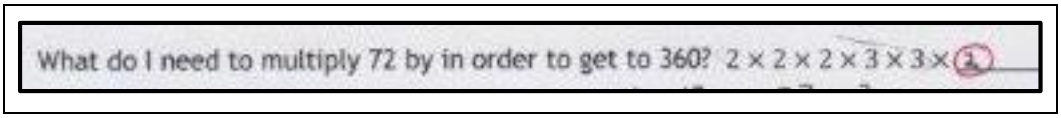
*Teacher: I think the factors thing, especially in algebra, the girls are not able to see the variables as factors and I see that from the Form Is (grade 8s) and Form IIs (grade 9s) as well. So trying to get them to see, especially with more than one term in the denominator, for example* $x^3 + x$ *that, that can be written as a product of its factors; that* $x$ *is a factor and* $x^2 + 1$ *is another factor. It's interesting that the idea of writing a number as a product of its factors seems to be …hmm…, even the terminology is something that the girls struggle to engage with. They don't really understand what it means to write a number as a product of its factors. I think for me that was the starting point. So, in terms of the equivalent fractions in task A that was really just the numerical examples where I was just trying to get them to write the denominator* 20 *and* 72 *as products of their factors, so that we could then find equivalent fractions for* $\frac{15}{20}$ *and* $\frac{23}{72}$ *that have the same denominators. We teach them this technique, even the ladder method in Form I (grade 8) to write a number as a product of its prime factors, so that we can then identify the lowest common multiple. I think even the connection between the lowest common multiple (LCM) and the lowest common denominator (LCD) is a link that other girls struggle with. So, in the task, I wanted to show them how that tool that we taught them in grade 8 is so useful in terms of actually factorizing a number and then finding an LCD and then keep going from that numerical example whereby writing a number as a product of its factors makes it easier for us to find an LCD. We apply that same principle in algebra where we have variables. By writing the denominator as a product of its factors we are then able to identify what factors are common and what factors are not common in order to build this algebraic expression that represents our LCD and then what the remaining factors are.*

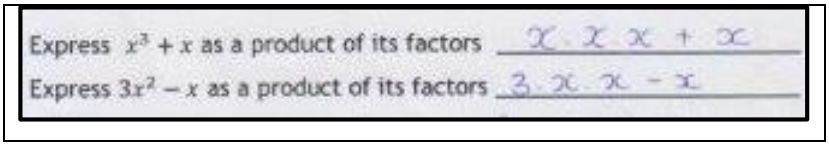
In the teacher's response, as we have already seen in the errors that the learners committed in the teaching intervention task, there is an emphasis on the gaps that the learners still have, despite having learned these concepts in previous grades and also in grade 10 this year. The following extracts clarify what learners found difficult as they were discussing the questions on the worksheets in groups.

*Researcher :* *When the lesson was in progress I noticed that the learners were mostly working in pairs and you went from group to group responding to the questions they had. What were part of the worksheet did they find challenging?*

*Teacher :* *That was a very interesting thing for me and I was quite surprised. They really struggled with task A; this whole idea of writing a number as a product of its prime factors. In retrospect, I think I would have done that as a guided activity on the board, pull all of that together so that they really got the gist of what I was wanting them to see and writing the denominator as a product of its factors. Oh! And then also in task B when in question one, it says express $x^3 + x$ as a product of its factors, learners did not realise that, that means to factorise, and that was a real surprise to me, that, writing an expression as a product of its factors did not mean to factorise, to them. They worked it out in expanded form instead of factorizing it.*

What the teacher is saying here is evident in the learners' responses, as shown in the following examples some of which were discussed earlier:



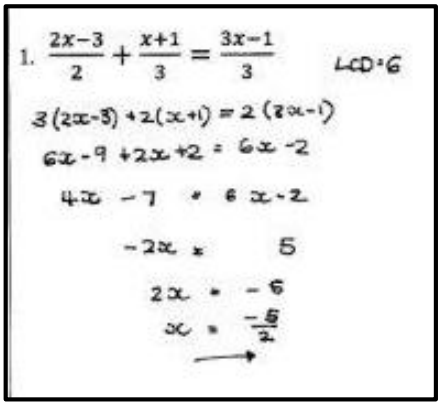

In these examples it is evident that some of the learners struggled with writing a number or an algebraic expression as a product of its factors. As the teacher pointed out, they did not realize that writing a number as a product of its factors means 'factorize'. This type of error leads to LCD errors. This section of the curriculum was covered in grade 8.

## 4.5 Learners' errors in the post-test

In the post-test the first item required learners to solve the equation: $\frac{2x-3}{2} + \frac{x+1}{3} = \frac{3x-1}{3}$. The following example shows one of their solutions to this equation.

**(a) Like-Term Error**

Learner B




Learner B committed a like-term error in step 3: $4x - 7 = 6x - 2$. In her previous step: $6x - 9 + 2x + 2 = 6x - 2$, she added $6x$ and $2x$ and erroneously obtained $4x$ instead of $8x$. She appears to have subtracted $2x$ from $6x$ instead of adding it. This error led to the incorrect solution $x = -\frac{5}{2}$.

One learner of the 13 committed the like-term error in this equation.

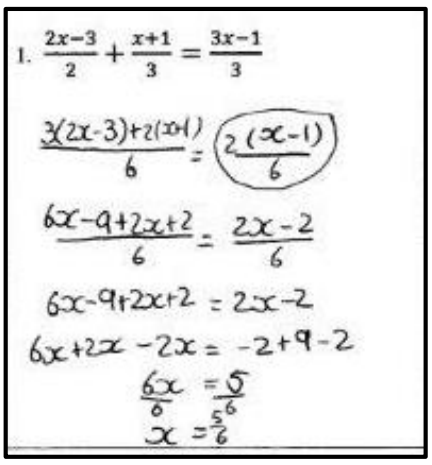
**(a) LCD Error**

None of the learners committed the LCD error. All of them obtained the correct LCD of 2 and 3, which is 6.

**(c) Other Error**

The following example shows one of the learner's 'other errors' which are mostly due to carelessness.

Learner R:



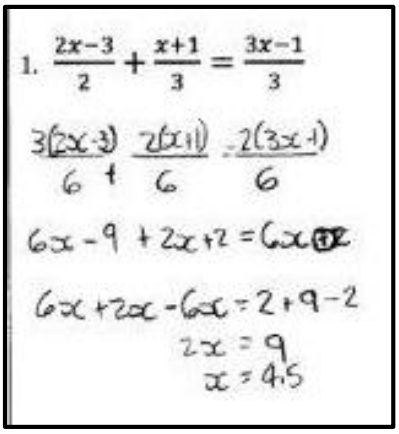
Learner R committed an error when writing $x-1$ instead of $3x-1$ after simplifying 3 with 6. Instead of writing $2(3x-1)$, she wrote $2(x-1)$ leaving out 3, the coefficient of $x$. This was due to carelessness. Had she checked her solution before submitting, she would have corrected the error.

Four of the learners committed the 'other error'.

**(d) Sign Error**

Consider the learner's solution:

Learner F




When distributing 2 into $'-1'$ when simplifying $2(3x-1)$, she obtained the incorrect answer 2 instead of $'-2'$, committing a sign error.

Only one of the learners committed a sign error in this equation.

Item 2

The second item required learners to solve the equation: $\frac{2}{3x}-\frac{1}{6}=\frac{6}{x}$. Here there is a variable $x$ in the denominator. Stating the restricted solution to this equation is necessary, since $x$ is in the denominators in the first term on the left and the right sides. In the analysis of the solutions below I will focus only on the like-term error.

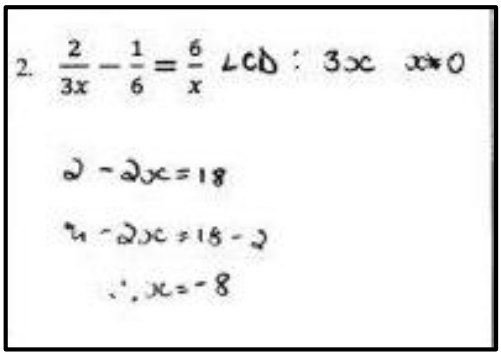
**(a) Like-Term Error**

There was no like-term error committed in this equation.

**(b) LCD Error**

The following figure shows one of the LCD errors committed by learners when solving this equation.

Learner D

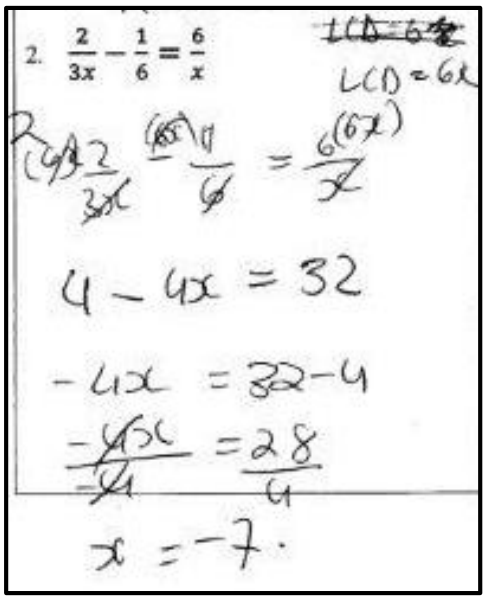



Learner D determined the LCD as $3x$ instead of $6x$. When the second term $'-\frac{1}{6}'$ was multiplied by $3x$, she obtained $2x$ instead of $\frac{x}{2}$. This incorrect term led to the incorrect solution $x = -8$.

**(c) Other Error**

Consider the following learner's solution in the example below:

Learner Q

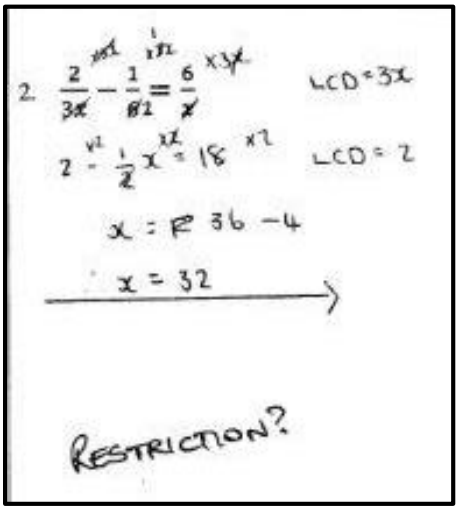



Learner Q committed a careless error in the second term on the left side of the equation. She multiplied $-\frac{1}{6}$ by the LCD: $6x$, but obtained an erroneous answer - $-4x$ instead of $-x$. This error seemed to have been caused by incorrect copying of the second term from the original equation. The learner copied $-\frac{4}{6}$ instead of $-\frac{1}{6}$.

**(d) Sign Error**

The example below shows one of the learners' solutions.

Learner P

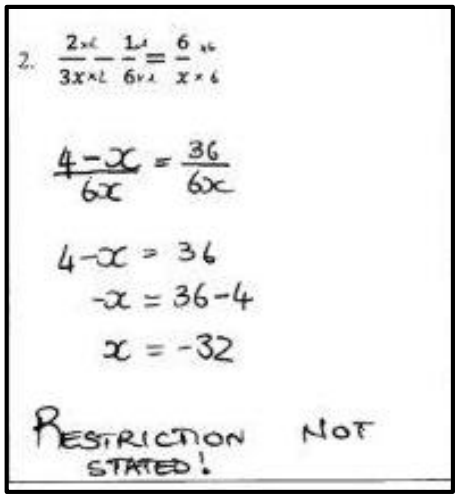



Here, Learner P committed a sign error in her final solution. This was caused by the careless error of copying the product of $-\frac{1}{2}x \times 2$ as $x$ instead of $-x$. This resulted in the solution $x = 32$ instead of $x = -32$. Only one of the learners committed a sign error in this equation.

**(e) Restriction error**

Consider the following learner's solutions.

Learner R




Of 13 learners, 7 did not state that the restriction $x \neq 0$, which is the restricted solution of $x$. Learner R is just one example of learners who committed a restriction error.

Item 3

The third item required the learners to solve the equation: $\frac{1}{x^2-2x-3}+\frac{4}{x+1}=\frac{3}{x-3}$. In this equation there is a variable $x$ in the denominator, and an expression that needs to be factorized before determining the LCD. Stating the restricted solutions to this equation was necessary, since $x$ is in the denominators of all the terms.

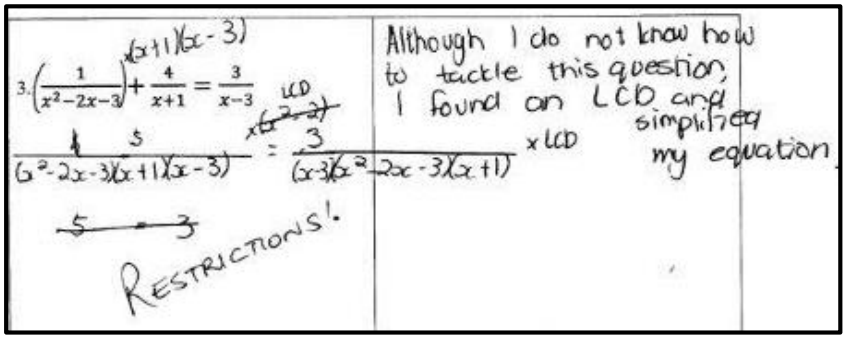
**(a) Like-Term Error, Other Error and Sign Error.**

Of the 13 learners, 12 were able to solve this equation correctly. There was no like-term error, other error and sign error committed by learners when solving this equation.

**(b) LCD Error**

The following learner's solution shows how she committed the LCD error.

Learner T

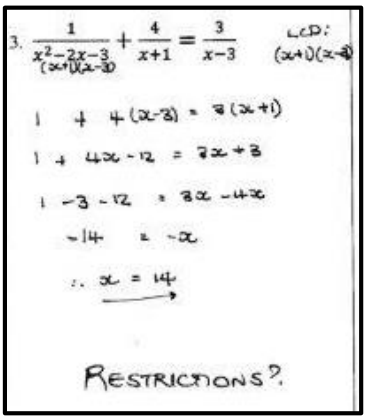



Learner T was not able to tackle the equation. Firstly, she was not able to determine the correct LCD, due to not factorizing the denominator first. She just used the quadratic trinomial as part of the LCD without factoring it; thus her LCD is $(x^2-2x-3)(x+1)(x-3)$. After this she appears to have added the numerators on the left hand side and written them over the LCD. She seems to have had a problem in understanding how the LCD is built in an expression that needs to be factorized.

**(c) Restriction Error**

Consider the following learner's solution in the example below.

Learner B




Here, Learner B did not state the restrictions at all. For the denominators $x - 3$ and $x + 1$ the values of $x$ that would make the denominator zero are 3 and $-1$ respectively, since $3 - 3 = 0$ and $-1 + 1 = 0$. Seven learners did not state the restrictions in this equation, while two gave incorrect restricted values of $x$.

Table 3 shows the number of learners who committed errors in each category per equation.

**Table 3: Error frequencies of learners in each error category per equation in post-test:**

| | $\frac{2x-3}{2}+\frac{x+1}{3}=\frac{3x-1}{3}$ | | | | $\frac{2}{3x}-\frac{1}{6}=\frac{6}{x}$ | | | | | $\frac{1}{x^2-2x-3}+\frac{4}{x+1}=\frac{3}{x-3}$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Learner | T | LCDE | OE | SE | T | LCDE | OE | SE | RE | T | LCDE | OE | SE | RE |
| A | | | | | | | | | ✓ | | | | | ✓ |
| B | ✓ | | | | | | | | ✓ | | | | | ✓ |
| C | | | ✓ | | | | | | | | | | | ✓ |
| D | | | | | | ✓ | | | | | | | | |
| F | | | | ✓ | | | | | | | | | | |
| G | | | | | | ✓ | | | | | | | | |
| H | | | | | | | | | ✓ | | | | | ✓ |
| J | | | | | | | | | | | | | | |
| M | | | | | | | ✓ | | | | | | | ✓ |
| P | | | ✓ | | | ✓ | ✓ | ✓ | ✓ | | | | | ✓ |
| Q | | | ✓ | | | | ✓ | | ✓ | | | | | ✓ |
| R | | | ✓ | | | | | | ✓ | | | | | ✓ |
| T | | | | | | | ✓ | | ✓ | | ✓ | | | ✓ |
| Total | 1 | 0 | 4 | 1 | 0 | 3 | 4 | 1 | 7 | 0 | 1 | 0 | 0 | 9 |

Besides the errors mentioned in Figueras et al. (2008), Hall (2002) and Mhakure et al. (2014) in the simplification of rational expressions and equation solving, such as like term errors, LCD errors and careless errors as dealt with above, two further errors emerged – sign error and restriction error.

### 4.6 Discussion

The study revealed that learners' errors in algebraic fractions arise from arithmetical misconceptions, such as simplification of numerical fractions, equivalence and writing numbers

as the product of their prime factors. This was seen in errors in the pre-test and in the teaching intervention task. Learners bring these misconceptions to algebra, hence they find algebra challenging as Hall (2002) and Mhakure et. al (2014) argue. Fractional equations require skills in executing arithmetical fractions. Chinnappan and Forrester (2014) state that conceptual understanding of fractions present difficulties to primary school learners. They are challenged by transferring whole numbers to fractions. They suggest that fractions are a problematic area of learning for both learners and teachers. The errors that the learners committed when solving fractional equations in this study support this argument.

Learners seem unable to restructure their fractional knowledge that already exists in their schema into solving fractional equations. The reason is that they do not have a strong conceptual understanding of arithmetic fractions. Their knowledge of fractions simplification, learned prior to fractional equations, constrains new knowledge acquisition (Hatano, 1996).

In the pre-test and post-test alike, the learners needed to execute their procedures well in order to get to conceptual understanding of solving fractional equations. Basically they needed to demonstrate their instrumental understanding in order to reach relational understanding (Skemp, 1976). Chinnappan and Forrester (2014) argue that procedural knowledge supports instrumental understanding, while conceptual knowledge supports relational understanding. They further maintain that as the learners' procedural understanding develops, their conceptual knowledge can be influenced. In this study the hope was that learners would start by showing procedural understanding: at least that would be a starting point on their way to conceptual understanding.

Furthermore, this study showed that underperformance of learners in mathematics is also caused by carelessness. Learners, in the pre- and post-test, committed careless errors. These suggest that they fail to check their solutions after solving equations.

In order to address the errors and misconceptions displayed in the pre-test, a teaching intervention worksheet was designed to give the participants the same opportunity to learn.
In this study the exchanges in the classroom interaction as shown in 4.4.3 above show a very important role of the teacher. It shows that the teacher needs to help learners move from their comfortable knowledge zone in order to reach their potential development (Vygotsky, 1978). Vygotsky (1978) refers to this as a *zone of proximal development*, which he defines as "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable learners" (p.86). In this study the learners had an

opportunity to reveal their actual development level when solving equations that involve fractions. They made errors that revealed their misconceptions, and the teacher used their errors as a teaching and learning tool, while addressing them at the same time. In the pre-test four learners committed a like-term error, while only one committed this error in the post-test; nine learners committed careless errors, while six committed this error in the post-test; seven committed a sign error while only one committed this error in the post test; and twelve committed a restriction error while nine committed this error in the post-test. Thus, after the teaching intervention, there was evidence of the shift in learning that occurred. This shift occurred through the theory of variation.

In the classroom conversation between the teacher and the learners, the teacher focused attention on specific aspects of equations. An example of this could be seen in exchanges 5 to 9, discussed above: *Teacher [writes the second equation on the board]:* $1 - \frac{2}{p} = 2 - \frac{1}{p}$

*What is the same between this and the previous equation?*

*Learner 1: They are both fractions.*

*Learner 2: It is also an equation.*

*Teacher: What's different?*

*Learner 3: There is a variable in the denominator.*

In the theory of variation, the learner cannot understand what something is "without knowing what it is not" (Ling Lo, 2012; p. 5). In the extract above the teacher focuses attention on a special feature of the equation in question. That special feature is having a variable in the denominator. The teacher wanted the learners to notice the invariants and variants in the equation. She wanted to introduce the meaning of a variable in the denominator and the way to deal with an equation of this nature. By this contrast a shared meaning occurred. With the exchanges and the example of the equation it is evident that meanings occur when the difference between objects is identified. The sameness in objects provides the point of departure, but the difference provides meaning (Ling Lo, 2012). While dealing with fractional equations, learners need to learn other structures of fractional equations. The teacher should be there to ensure that the critical aspects of the object are made available to learners. We saw in this study how effective the learning is when the variation theory is applied. Constructivist theory advocates the principle of a teacher giving a minimum of direction and guidance to the learner. An example of the principle is the scaffolding strategy, which aims at providing guidance to a learner only when it is necessary.

The theory of variation therefore provides learners with such an opportunity to learn; by being exposed to two contrasting situations or examples in order for them to have a conceptual understanding of the mathematical concept at hand.

The theory of variation helps in dealing with learners' concept images (Tall & Vinner, 1981) about solving fractional equations. In this study the concept images were revealed in the pre-test's learners' errors. This revelation gave the teacher the opportunity to draw their attention to the object of learning (Sfard, 1991; Marton & Booth, 1997) in order to acquire a concept definition. In addition, the teaching intervention provided the teacher with the opportunity to view how learners discern the concept of solving fractional equations while providing learners with more opportunities to learn.

The errors committed in the pre-test were helpful in designing the teaching intervention worksheet, in order to try to address them adequately. The teaching intervention where the teacher addressed the errors committed in the pre-test, in a feedback lesson in class, was necessary, as it allowed all the participants the opportunity to learn, even though they had learned the topic in class from other teachers. The improvement in making errors in the post-test suggest a shift in learning. The learners eventually had a relational understanding (Skemp, 1976) of the fractional equation's concept.

When the interaction took place among the learners themselves the questions they asked also showed instrumental understanding (Skemp, 1976) of algebraic concepts. For instance, some of the learners could not formulate the LCD from the algebraic expressions when given them in isolation. However, when these algebraic expressions were in the equation they managed to 'build' the correct LCD. Again, assimilation of the knowledge of formulating an LCD became a problem for them, thus confirming that "what the teacher teaches is not necessarily what the learner learns". Learners construct and restructure their own knowledge "domain by domain" (Hatano, 1996). Sometimes new knowledge is difficult to be added to pre-knowledge, as the latter is constrained (Hatano, 1996). In this study some learners found it difficult to assimilate the new knowledge of solving fractional equations to previously learned arithmetical knowledge. It was therefore important for the teacher to focus their attention on the object of learning.

The object of learning in this study was the given equations to be solved. Marton et al., (2004) refer to the object of learning as a *capability*, while Ling Lo (2012) refers to it as a specific aspect of the object on which the learners are focused. The opportunity to learn for the learners starts with what Sfard (1991) refers to as interiorisation and condensation, which focus on the

processes to be carried out. The teacher is helping to 'kick start' the learning. As Ling Lo (2012) suggests, the teacher is there to help make the critical aspects of the object available to students. *Revoicing* is seen in lines 23 and 29 of the classroom interaction set out above. Revoicing happens when a teacher repeats a learner's contribution to a class discussion (O'Connor & Michaels, 1996). Revoicing helps a learner to rethink his or her claim and reflect as the teacher might be requiring further explanation, or she might be emphasizing the correctness of the answer.

The learners' performance in post-test equation solving improved. This is evident when comparing the number of learners that committed various errors in the pre-test to the number in the post test (see Table 4.3 above). With the exception of LCD errors, of which the number remained the same, in all the other error categories the numbers decreased. However, the restriction and careless errors still need to be worked on, as the number of these errors in the post-test decreased only slightly.

## 4.7 Conclusion

In this chapter I analysed the data for this study, and discussed and interpreted results. Errors and misconceptions that the participants made in solving fractional equations were evidence to the fact that Hatano (1996) and Olivier (1989) pointed out, that knowledge cannot be transferred from the teacher to the learner, but that a learner constructs his or her own knowledge. A learner does not necessarily learn what the teacher teaches him, but he or she might be learning something completely different from the teacher's objectives. The variation theory applied in the teaching intervention proved to be a better option in making the learning effective. This was evident in seeing the decrease in a number of errors from pre-test to post-test.

Other researchers have found similar errors in the area of solving equations; however, in all the literature that I reviewed, there was no mention of the importance of stating restrictions when solving equations that involve fractions and variables in the denominator. In this study it became evident that the learners need to understand the importance and the meaning of restrictions when solving fractional equations. Teaching intervention lessons and worksheets in this study played an important role, as we have seen in Table 4.3, where the number of errors in the post-test was significantly lower in some types of errors than in the pre-test. However, analysis of other error categories (OE) showed that careless errors contribute significantly in the learners' poor performance in mathematics. Teaching intervention worksheets proved to have given the learners an opportunity to learn by constructing their own knowledge (Hatano, 1996; Olivier,

1989) through interaction between themselves, and with the learning material (Leikin & Zaslavsky, 1997). These interactions were intended to give them an opportunity to learn and understand relationally (Skemp, 1976). Skemp emphasizes that there is no adequate understanding for a learner other than relational understanding. The next chapter concludes my report, reflects on the whole research, and implications and recommendations for South African Education will be discussed.

# CHAPTER 5: CONCLUSION

## 5.1 Introduction

This research investigated the types of errors that learners make when solving equations that involve fractions. It also investigated learning opportunities that are provided by the teaching intervention targeting those errors. The research focused on the errors and misconceptions of low-performing grade 10 learners in solving fractional equations. It then determined whether those learners' attendance at a mathematics clinic where a teaching intervention occurred in order to address these errors, helps them to develop their competence in solving fractional algebraic equations. The research addressed the following questions:

(i) What errors and misconceptions do students display when solving equations involving fractions?

(ii) What opportunities to learn avail themselves to learners with errors and misconceptions in solving equations involving fractions at a mathematics clinic.

(iii) What gains in learning, if any, occur through the mathematics clinic interventions?

In an attempt to answer these research questions, data was collected from seventeen grade 10 learners who attend a mathematics clinic at a private high school in Johannesburg where this researcher teaches. The data collected was analysed both deductively and inductively with the intention of ensuring that the research questions of this study were fully answered.

The teacher is in the classroom to ensure that learners' attention is focused on the object of learning, with minimal guidance. For this reason, data in the teaching intervention lessons of this research was also analysed through the theory of variation. The theory of variation's main thesis is that one "cannot know what something is without knowing what it is not" (Ling Lo, 2012, p.5). In this study data was analysed mainly through contrasting examples. In contrasting examples, learners' focus was directed on the variants and invariants of the object of learning, and the different features of equations involving fractions. The contrasting led to learners' ability to distinguishing different types of denominators' expressions in fractional equations from those with numerical denominators, monomial algebraic expressions and with polynomial algebraic expressions. Contrast allowed them to distinguish one type of equation from the others, and to handle each type accordingly. Furthermore, it enabled the learners to identify common features of an equation, and the difference between algebraic equations and algebraic expressions (generalization). In the end they were expected to contrast, separate and generalize; that is, they

were expected to handle all aspects of the object of learning simultaneously (fusion) (Marton et al., 2004).

### 5.2 Responding to the research questions of the study

The findings of this study are presented according to the three stages of data collection.

*Errors and misconceptions in the learners' pre- and post-test solutions to equations*

With respect to the first research question: "What errors and misconceptions do students display when solving equations involving fractions", the findings showed that learners in grade 10 are able to determine the lowest common denominator of numerical denominators in fractions. However, they have errors and misconceptions in solving equations that involve algebraic denominators. The errors included like-term errors, LCD errors, sign errors, careless errors and restriction errors. The following table shows the types of errors committed in the pre- and post-tests. The number of learners who committed each type of error is given as a portion of the total of 13 learners.

Table 4 shows errors committed in pre- and post-tests as discussed below.

**Table 4: Error frequencies of learners in each category of errors in the pre- and post-tests**

| **Error category** | **Pre-test** | **Post-test** |
|---|---|---|
| Like-Term Error 2 (T2) | 4 | 1 |
| LCD Error (LCDE) | 4 | 4 |
| Other Error (OE) | 9 | 6 |
| Sign Error (SE) | 7 | 1 |
| Restriction Error (RE) | 12 | 9 |

Like-term errors were committed when learners added like terms and then gave an incorrect sum. The type of error seemed to be caused by carelessness. Like-term error was committed by four out of 13 learners in the pre-test, and only one in the post-test.

LCD errors were due to the learners' inability to factorize quadratic trinomials. The concept of formulating an LCD from the factors of the denominators' expressions proved to have been well understood. Of the 13 learners, four committed an LCD error both in the pre- and post-tests.

The Sign error was committed by seven learners in the pre-test and by only one in the post-test. Sign errors indicated carelessness rather than conceptual misunderstanding. This was seen where learners had correctly added like-terms and obtain the correct sums or differences, but would randomly make a sign error.

Other error or careless error seemed to have contributed most significantly: nine of them in the pre-test and seven in the post-test. This error was one that could not be categorized as any of the other error categories in this study, and, moreover, could not be identified as conceptual in nature. They all appeared to be careless errors.

The restriction error also contributed significantly to errors in this study. This suggests that the learners were still not sure about the meaning of a variable and its solutions in the denominator. Almost all the learners in the pre-test either gave incorrect restricted solutions or did not state them at all, with the sole exception of one, who successfully stated restrictions correctly in all items in the post-test, where it was required. Nine learners committed the same restriction error in the post-test. Whether the learners who did not state the restriction error simply forgot or did not know how to state them was not clear in the study. One learner stated that $x \neq 0$ when she had a denominator: $x + 1$. This suggests that she had no conceptual understanding at all of restricted solutions in fractional equations.

*Opportunities to learn created in the teaching intervention*

With respect to the second research question: "What opportunities to learn avail themselves to learners with errors and misconceptions in solving equations involving fractions at a mathematics clinic", the findings indicated that the range in the structure of the equations in this study created opportunities to learn (Watson, 2003). Opportunities to engage with more complex equations involving fractions were created. Fractional equations in both pre- and post-tests ranged from those with numerical denominator, binomial algebraic expressions to those with quadratic expressions. The learners got an opportunity to compare and contrast the different structures of fractional equations. With the variation theory that was used to focus their attention on different features of fractional equations, they learned to focus on the denominator first. In the teaching intervention attention was focused on writing the denominator as a product of its factors, the formulation of LCD and on how the denominators are eliminated in order to solve a

fractional equation. All these opportunities were created by the different structures of equations presented to the learners to solve.

*Gains in learning that occurred*

With respect to the third research question: "What gains in learning, if any, occur through the mathematics clinic interventions?, the different tasks in the teaching intervention gave learners experience of engaging with various tasks in their working groups. These tasks required different skills, ranging from writing a number or algebraic expression as a product of its factors, simplifying algebraic expressions, formulating the LCD and deciding on restricted solutions to fractional equations. The intervention lesson and the intervention task gave enough information on different aspects of solving fractional equations, and the learners were thus given an opportunity to "rethink and restructure existing assumptions and understandings" (Watson, 2003, p. 3). Table 4.3 shows that the number of learners who committed errors in the pre-test decreased in the post test with the exception of the LCD error. This is evidence that a learning shift occurred in the teaching intervention.

## 5.3 Reflections

Conducting this research was challenging in terms of collecting data that would produce meaningful results. The literature reviewed yielded no study specifically on fractional equations, yet dealing with fractions is the critical skill in negotiating mathematics. It is of concern that learners find dealing with algebraic fractions particularly difficult (Kerslake, 1986; Hall, 2002; Mhakure et al., 2014). The findings of this study therefore are new, and can hardly be compared to any other previous findings. However, several aspects of this study are similar to previous studies, for example, the difficulty experienced by learners when solving equations that involve division.

## 5.4 Limitations

Research that involves another teacher has its hazards, especially if that teacher is new in the profession. It is challenging to convince her that you are not checking up on her teaching or pedagogical skills. I found this particularly challenging, and resolved it by discussing the aim of the research and research questions with her. Another difficult aspect of this study was collecting all data. Many of the learners who took part were involved in different sports and other extra-curricular activities after school, and it became a problem to have a lesson with all of them present, since the clinic takes place after school on Mondays and Wednesdays. It was difficult to find a day when all participants were available. In order to address this problem permission was

granted to use some of the mathematics lessons in the normal timetable to collect data. I owe a debt of gratitude to the Head of Mathematics for this. The participants came from two classes – the bottom and second-last sets. The researcher and teacher identified periods at which both classes would be having mathematics.

With hindsight, I would encourage the participants to answer all the questions in a pre- or post-test, to make it possible to analyse all data collected from every participant. When all participants do what they are supposed to do, it makes it easier for the researcher, even though it is of no great consequence in a qualitative study. I would also collect data from interviews directly after a pre-test in order to understand the origin of their mathematical misconceptions better.

In similar hindsight I would choose a grade 10 class. I view this grade as very important, as it is the entry level into a Further Education and Training (FET) phase. It is important that learners master basic mathematical knowledge before grade 10 in order to cope with the senior school mathematics and beyond. To my mind grade 10 is critical to a solid foundation in mathematics. Thus checking mathematical skills in terms of errors and misconceptions learners bring with to FET phase from the previous phase is essential.

## 5.5 Implications and recommendations

Low-performing students need to be given more time when taught; and the difficulties they face in learning mathematics constitute pedagogical content knowledge for the teacher. This means that once the teacher is aware of learners' errors and misconceptions, in particular in mathematics, they must anticipate and prepare for them in order to help learners correct them. Teachers must always probe and elicit learner difficulties in all mathematics topics, and use them as resources for teaching (National Council of Teachers of Mathematics, 2015). NCTM (2015) argues that teacher eliciting learners' thinking on particular mathematics topics or concepts, and using that information in teaching, is a hallmark of mathematics teaching excellence.

An aspect that became clear in this study is that these students needed to be given an opportunity to interact amongst themselves, with the learning material, and with the teacher in order to deal with their mathematical misconceptions. This kind of interaction provides them with a more meaningful opportunity to construct their own knowledge in the social realm. Those interactions create the opportunity to learn mathematics. It appeared that more teaching intervention sessions with low-attaining students would help them gain confidence, particularly if their errors and misconceptions are kept in focus in the interventions. When teaching fractional equations, the

learners' attention needs to be focused on working accurately and always stating restrictions, as this is very important for future complex topics such as functions.

Based on this study, it was evident that teaching intervention that includes previously learned topics, particularly arithmetic expressions, helps to improve learners' understanding of algebraic expressions in mathematics. This suggests that it is vital to teach from known to unknown.

This study revealed that errors and misconceptions that students have in fractional equations can be diminished by a teaching intervention directed at them. Through teaching interventions and practice, learners can learn to work accurately in order to avoid making like-term errors, careless errors; they can formulate algebraic LCD and can focus on and understand restricted solutions to equations.

It is therefore important that teaching intervention be conducted when addressing learners' errors and misconceptions as it provides a safe space for their reconstruction of mathematics knowledge. Mathematics teachers need to approach fractional equations topic with great care, especially when teaching low-performing students. As was evident in this study, learners have many misconceptions that emanate from their prior knowledge, particularly from arithmetic fractions. Employing teaching intervention when addressing learners' errors and misconceptions proved successful in this study. In light of the above, it is therefore recommended that:

Teaching interventions used in addressing learners' errors and misconceptions as this exposes them to a number of opportunities to learn. Teaching intervention also allows for effective interactions between learners and the teacher, learners and the teaching material and learners and their peers.

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**Appendix A**

**PRE-TEST ITEMS** **Date:_____**

**Learner Code: _____**

(a) Solve the following equations, stating the restrictions where necessary.

(b) Explain in words using mathematical language, in **bullet points,** how you have solved the equations.

| **Equation and the solution** | **Explanation** |
|---|---|
| 1. $\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$ | |
| 2. $1 - \frac{2}{p} = 2 - \frac{1}{p}$ | |
| 3. $\frac{2}{x^2-x-2} - \frac{2}{2x+2} = \frac{1}{x+1}$ | |

**Appendix B**

**POST-TEST ITEMS** **Date:_____**

**Learner Code: _____**

(a) Solve the following equations, stating the restrictions where necessary.

(b) Explain in words using mathematical language, in **bullet points,** how you have solved the equations.

| **Equation and the solution** | **Explanation** |
|---|---|
| 1. $\frac{2x-3}{2}+\frac{x+1}{3}=\frac{3x-1}{3}$ | |
| 2. $\frac{2}{3x}-\frac{1}{6}=\frac{6}{x}$ | |
| 3. $\frac{1}{2x^2-x-3}+\frac{5}{x-4}=\frac{3}{2+x}$ | |

**Appendix C**

**The transcript of a lesson video 1: Teaching intervention: Date: 04 August 2015**

***The teacher addressed the errors committed by learners in their pre-test task of fractional equations.***

[The teacher explains by pointing at the equation written on the board, the meaning of solving an equation]

Teacher : When solving a mathematical equation, whatever you do on the left of the equal sign, you have to do it on the right.

$$\frac{3(y+2)}{12} - \frac{4(y-6)}{12} = \frac{1}{2}$$

$$\frac{3y+6-4y+24}{12} = \frac{1}{2}$$

$$\frac{-y+30}{12} = \frac{1}{2}$$

$$-y + 30 = 6$$

$$-y = -24$$

$$y = 24$$

Teacher : How to get rid of the denominator? Multiply each term by the LCD on both sides of the equal sign.

Inverse operations (undoing what is done to a variable we are solving for).

Learner C: In the previous example, how come for the half on the right hand side, you did not multiply it by the LCD?

Teacher : We did, in the next term: $\frac{-y+30}{\cancel{12}} \times \cancel{12} = \frac{1}{\cancel{2}} \times \cancel{12}\, 6$

$$-y + 30 = 6$$

$$-y + 30 - 30 = 6 - 30$$

$$-y = -24$$

$$y = 24$$

Teacher : So, let's go back to our original question again: $\frac{(y+2)}{4} - \frac{(y-6)}{3} = \frac{1}{2}$

In terms of what you described on your piece of paper, who has got the short cut, or a quicker method?

Learner D: Multiply each term by the LCD.

Teacher *(writing on the board, repeating what the learner said]*: Multiply each term by the LCD.

$$\frac{(y+2)}{4} \times \frac{\cancel{12}\ 3}{1} - \frac{(y-6)}{3} \times \frac{\cancel{12}\ 4}{1} = \frac{1}{\cancel{2}} \times \frac{\cancel{12}\ 6}{1}$$

Multiply 3 into the bracket and multiply 4 into the bracket:

$$\frac{(y+2)}{4} \times \frac{\cancel{12}\ 3}{1} - \frac{(y-6)}{3} \times \frac{\cancel{12}\ 4}{1} = \frac{1}{\cancel{2}} \times \frac{\cancel{12}\ 6}{1}$$

$3y + 6$, subtract 4 times $y$, subtract negative 6; we get $3y + 6 - 4y + 24 = 6$

$$-y + 30 = 6$$

$$-y = -24$$

$$y = 24$$

Teacher *[writes the second equation on the board]*: $1 - \frac{2}{p} = 2 - \frac{1}{p}$

What is the same between this and the previous equation?

L1 : They are both fractions.

L2 : It is also an equation.

Teacher : What's different?

L : There is a variable in the denominator.

Teacher : The principle of how we solve, is still the same. We want $p$ all by itself, and the value of $p$ that will make this true. We want the value of $p$ that will make this mathematical statement true.

Teacher : We will have to multiply every single term by the LCD. How many terms have we got?

Learners : Four.

Teacher : Four in total, two on the left, two on the right.

What is our $LCD$?

Learners : $p$

Teacher *[Writing on the board and talking]*: Multiply this term by $p\ over\ one$, this one by $p\ over\ one$, this one by $p\ over\ one$, and this one by $p\ over\ one$.

So, I have $p$, these two cancel, so, I have $p - 2 = 2p - 1$.

Teacher : Solving $1 - \frac{2}{p} = 2 - \frac{1}{p}$, we are saying $p = -1$, if I have the variable in the denominator, I have to check that my solution is not creating any mathematical issues. Anyone has an idea what that issue might be?

Learner D: We have to assume that our solution is correct.

Teacher : You are right, we have to assume that our solution is perfect and that $'-1'$ is that perfect solution.

Learner B: The denominator must not be equal to zero.

Teacher : We have to check that the solution to this equation does not break any mathematical rules we know. If, for argument sake, we have got a value of zero, a value of zero for $p$ will break a mathematical rule and that is a division by zero. So, if there is a variable in the denominator, you always have to state the restriction upfront. So, $p \neq 0$.

Teacher : Fortunately, we did not get zero as our solution, so we are safe. If, for argument sake, this value of $p$ has been zero, I would have to say we have to reject this solution because the value of $p$, actually the value of $p$ as zero is not allowed. Any questions?

Teacher *[writes on the board and then asked]*: $\frac{2}{x^2-x-2} - \frac{2}{2x-2} = \frac{1}{x+1}$; What is the same about this equation as to the previous ones?

*[Learners start to answer without raising their hands]*

Teacher : I may have to take hands on this one.

Learner K: There are variables in the denominator.

Teacher : Which tells us …?

Learner C: We have to have restrictions.

Teacher : What else is the same?

Learner D: Fractions.

Teacher *[repeating the learner's answer]*: Fractions.

Learner D: Equation.

Teacher *[repeating the learner's answer]*: Equation. So, ultimately we want to get $x$ equal to some value. That value will make this mathematical statement true.

Teacher : There is something different about this equation. Anyone spotted the difference?

Learner C: There are more than one term in the denominator.

Teacher : There is one other thing that you have to do if you have denominators that look like these.

Learner C: We have to factorize.

Teacher : You do need to factorize.

Teacher : So, this is a quadratic trinomial *[underlining the denominator of the first term of the expression on the left hand side of the equation]*. So, we need to factorize it before we take this any further, before we even do our restrictions.

Teacher : Because I am hoping that when this is factorized, we might get something similar to these *[pointing at the denominators of other terms of the equation]*.
Okay, let's do that, so what are the factors of this quadratic expression?
$\frac{2}{(x+1)(x-2)} - \frac{2}{2x+2} = \frac{1}{x+1}$; we have two brackets; my signs will be the same or different? *(writing two brackets in the denominator of the first term):*
$\frac{2}{(\ )(\ )}$

Learners : Different.

Teacher : How do you know they will be different?

Learners *[Starting to answer without raising their hands]:*

Teacher : I need to take hands on this one. *[Calling Learner D, to give the answer]*.

Learner D: Because the coefficient of 2 is negative.

Teacher : Right, therefore the signs will be different. Okay, one is gonna be plus and one will be minus. Okay, we will have $\frac{2}{(x+)(x-)}$, 2 and 1 are the factors of 2, which factor of 2 will be positive and which one will be negative?

Learner C: 1 will be positive and 2 will be negative.

Teacher : The 1 will be positive and 2 will be negative: $\frac{2}{(x+1)(x-2)}$. Everyone happy with why that is?

Learners : Yes.

Teacher : If we foil this out *[pointing at the two brackets of factors in the denominator]*, we will get $-2x + x$, that will give us negative $x$. Fantastic! Are you happy with that?

Teacher : In the next line, you need to take out the highest common factor, don't forget to do that, it still applies here. This side can't be factorized, just put it in brackets, so that it looks neater. $\frac{2}{(x+1)(x-)} - \frac{2}{2(x+1)} = \frac{1}{(x+1)}$

Teacher : Now we are looking for what's the same, what is different in building out LCD. And remember at this point we are counting these as factors *[pointing at all the denominators, and then wrote 'LCD' on the board]:*

Teacher : An LCD is the smallest number that each one of these *[pointing at all denominators]* acts as a factor of. So, we need to find some number that each of these fits into, as a factor. Does that make sense? So, let's find that.

Learners : $(x + 1)$ *and* $(x - 2)$.

Teacher : $(x + 1)$ is definitely a factor, it appears here, here, and here *[pointing at all the denominators that contain* $(x + 1)$.

Teacher : What else is the denominator?

Learner C: $(x - 2)$.

Teacher : $(x - 2)$, so both $(x + 1)$ and $(x - 2)$ have been accounted for, another one that we still need to build into in number …?

Learners : 2.

Teacher : It is 2 and we need to put that in the front: $2(x + 1)(x - 2)$. Can you see how every part here has been represented in our LCD that we built?

Teacher : At this point we need to work on our restrictions. We are looking at what values of $x$, we are allowed to have and what we are not allowed to have.

Any values of $x$ that we are not allowed to have?

Learners : negative 1.

Teacher : Okay, $x$ cannot equal negative 1; so, $x \neq 1$, what else?

Learners : 2

Teacher : Can we have $x$ equal zero?

Learners : Yes.

Teacher : Why would it be okay if $x$ is equal to zero in this specific equation?

Learner C: Because we don't break maths rules.

Teacher : Why is it not breaking the maths rules?

Learner C: It won't be undefined.

Teacher : You getting there, you very close.

*[Asking learner B, who has offered to give the answer, to give the answer]*.

Learner B: Because if $x = 0$, we will have to subtract 2 from 0, in the denominator and we do not get zero.

Teacher : Right, if I just look at this one *[pointing at* $(x - 2)$*]*, if $x = 0$, I will have $0 - 2$, which is not equal to zero, I will get negative 2 and not zero and if I have $0 + 1$, I will not get zero. So, $x \neq 0$ is not a valid restriction in this specific equation because $x$ is allowed to equal zero. $x$ being equal zero will not cause us any problems in any of these fractions. Great!

Teacher : Remember each step from previous question, the more efficient method to get rid of the denominator. What would we have to do?

Learners : Multiply each term by the LCD.

Teacher : Multiply each term by the LCD, let's do that! This will look a little bit more complicated.

*[Writing on the board]:*

$$\frac{2(x+1)(x-2)}{1} \times \frac{2}{(x+1)(x-2)} - \frac{2}{2(x+1)} \times \frac{2(x+1)(x-2)}{1} = \frac{1}{(x+1)} \times \frac{2(x+1)(x-2)}{1}$$

What is the next thing that I have to do?

Learners : Cancel.

Teacher : Cancel, right, this is gonna cancel with that and this with that *[cancelling in the first term on the LHS of the equation]*. I will sort out each term as we cancel. This 2 multiplied by this is 4. Do I have any denominator left?

$$\frac{2(x+1)(x-2)}{1} \times \frac{2}{(x+1)(x-2)}$$

Learners : No.

Teacher : At this point I have this 2 cancelling with this one and this cancelling with this. $\frac{2}{\cancel{2(x+1)}} \times \frac{\cancel{2(x+1)}(x-2)}{1}$ and this side I have this cancelling with this, cancel those out we have nothing left in the denominator.

$$\frac{1}{\cancel{(x+1)}} \times \frac{2\cancel{(x+1)}(x-2)}{1}$$

So, now I have $4 - 2(x-2) = 2(x-2)$.

Teacher : So, we have $4 - 2(x-2) = 2(x-2)$ *giving us* $4 - 2x + 4 = 2x - 4$. At this point collect like terms, variables to the left constants to the right.

*[the bell for the end of the lesson went off]*.

Teacher *[continues]:* So we gonna minus $2x$ this side *[on the right]*:

$4 - 2x + 4 = -2x - 2x - 4$ and then minus $2x$ this side *[on the left]*. These on the right will give zero, they will cancel each other out. We are almost done. On the left I will have $-2x - 2x = -4x$ and I also have got 4 $plus$ 4. I need to get those to the other side, so I must subtract 8 this side *[on the left]* and subtract 8 on the right. So, $'-8'$ will cancel out $4 + 4$ on the left, therefore: $-4x = -12$.

Teacher : What do we need to do to get $x$ by itself?

Learners : Divide by $'-4'$ both sides.

Teacher : Divide both sides by $'-4'$. These cancel: $\frac{\cancel{-4}x}{\cancel{-4}}$ and $\frac{-12}{-4}$ is …?

Learners : 3.

Teacher : Therefore $x = 3$. Again I would like you to take a look at your solutions and the errors you made to get those solutions.

***THE END***.

**Appendix D**

**Teaching intervention task: 29 September 2015**

**Form III Working with fractions**

**Task A**

Add these two fractions $\frac{17}{20} + \frac{23}{72}$

***You need to make the denominators the same. Before you begin, consider:***

20 expressed as the product of its prime factors is: $2 \times 2 \times 5$

72 expressed as the product of its prime factors is: $2 \times 2 \times 2 \times 3 \times 3$

If I combine those two sets of factors, I create a larger number that both 20 and 72 "fit into"

$2 \times 2 \times 5 \times 2 \times 2 \times 2 \times 3 \times 3 = 1440$ (a common multiple)

That is the same as saying $20 \times 72 = 1440$

This is not the SMALLEST number that they both fit into because we have a repeat:
$2 \times 2$ are factors common to both 20 and 72.

If I take the common factor $(2 \times 2)$ and then add in the other factors that are not common, I get

$(2 \times 2) \times (5) \times (2 \times 3 \times 3) = 360$ (the LCM)

What do I need to multiply 20 by in order to get to 360? $2 \times 2 \times 5 \times$ _____

What do I need to multiply 72 by in order to get to 360? $2 \times 2 \times 2 \times 3 \times 3 \times$ _____

Now use this information to add the fractions.

Do you think that factorising helped us to find the LCD? _____

**Task B**

1. Express $x^3 + x$ as a product of its factors _____
2. Express $3x^2 - x$ as a product of its factors _____
3. What factors are common to $x^3 + x$ and $3x^2 - x$ ? _____
4. What factors are not common? _____
5. By combining the factors of the two numbers, you can determine the smallest "number" that both $x^3 + x$ and $3x^2 - x$ divide into.

   What would that combination look like? _____

   What is it called? _____

6. Add these two fractions $\frac{17}{x^3+x} + \frac{23}{3x^2-x}$

**Task C**

In each case, what value(s) of $x$ will make the denominator of the fraction zero?

1. $\frac{15}{x}$

2. $\frac{1}{3+x}+\frac{7}{x}$

3. $\frac{7}{x-2}+\frac{2}{1-x^2}$

*Reflect:*

a) Why does it cause a problem, mathematically, for the denominator of a fraction to be zero?
b) Consider question 3. Does factorising help us to determine the "disallowed" values of the variable?

**Cumulative task**

1. Factorise the denominators
2. Determine the LCD
3. Determine the restrictions
4. Solve for $x$

$$\frac{1}{x^2-x}+\frac{2}{x-1}=\frac{1}{x} \qquad (x=-2)$$

*Reflect:*

a) Briefly describe the type of question above.
b) What was the first step needed in the solution?
c) Was there an extra step you had to do first and why was this necessary?
d) Are there any values of $x$ that we would have to reject?

**Appendix E**

## The transcript of a lesson video 2: Date:29 September 2015

*The researcher discussed the second lesson with the teacher, in terms of what she (the researcher) wanted the teacher to discuss in the second teaching intervention with learners. The teacher then prepared tasks on fractions and factors and these tasks were discussed with the researcher before the lesson. The task focused on factors, HCF and LCM and they also linked these with the addition or subtraction of algebraic fractions. Task A was on factors, Task B was on factorizing algebraic expressions and Task C was on simplifying algebraic fractions that contain a variable in the denominator, with some denominators to be factorised first before obtaining the LCD. Some questions for the learners to reflect on Task C were set after Task C.*

Teacher : Thanks for coming again. I hope you will find this exercise helpful. Remember, last time we were looking at equations with fractions in them. You wrote a pre-test, you answered a couple of questions and we went over those questions together and now I am trying to address some of common mistakes that you made in that task.

This is giving you the opportunity to think a little bit more deeply about some of the aspects of what is required in solving an equation. Mostly we will be working in pairs, but I will be walking around and chatting to you. So, if there is something you are not sure about, just shout.

There are three tasks, and we have to try and finish these tasks in this session. The first one has quite a lot of information. Look through it, try to make sense of it, see how you gonna use its understanding to answer task B and task C and then the final task which will help you to draw everything together.

*[Learners then started to work on the tasks assigned to them.]*

All learners in groups were discussing and engaging with the tasks. The teacher walked around during discussions which the teacher had to address. Since the learners were discussing in their groups, the class was noisy and therefore was not possible to hear the

questions asked and the teacher's responses to those questions. I therefore interviewed the teacher after watching the lesson video.

**Appendix F**

**The interview with the teacher after the second session of teaching intervention.**

**Date: 6 October 2015**

Researcher : Firstly, in the task for the learners, the focus was on factors and equivalent fractions; in your opinion, did you think this was the source of their errors in the fractional equations pre-test?

Teacher : I think the factors thing, especially in Algebra, the girls are not able to see the variables as factors and I see that from the Form Is (Grade 8) and Form IIs (Grade 9). So trying to get them to see, especially with more than one term in the denominator, for example $x^3 + x$ that, that can be written as a product of its factors; that $x$ is a factor and $x^2 + 1$ is another factor. It's interesting that the idea of writing a number as a product of its factors seems to be …hmm…, even the terminology is something that the girls struggle to engage with. They don't really understand what it means to write a number as a product of its factors. I think for me that was the starting point. So, in terms of the equivalent fractions in task A that was really just the numerical examples where I was just trying to get them to write the denominator $20$ and $72$ as products of their factors, so that we could then find equivalent fractions for $\frac{15}{20}$ and $\frac{23}{72}$ that have the same denominators. We teach them this technique, even the ladder method in Form I (Grade 8) to write a number as a product of its prime factors, so that we can then identify the lowest common multiple. I think even the connection between the lowest common multiple (LCM) and the lowest common denominator (LCD) is a link that other girls struggle with. So, in the task, I wanted to show them how that tool that we taught them in Grade 8 is so useful in terms of actually factorising a number and then finding an LCD and then keep going from that numerical example whereby writing a number as a product of its factors makes it easier for us to find an LCD. We apply that same principle in

algebra where we have variables. By writing the denominator as a product of its factors we are then able to identify what factors are common and what factors are not common in order to build this algebraic expression that represents our LCD and then what the remaining factors are.

Researcher : When the lesson was in progress I noticed that the learners were mostly working in pairs and you went from group to group responding to the questions they had in each group. What were part of the worksheet did they find challenging?

Teacher : That was a very interesting thing for me and I was quite surprised. They really struggled with task A; this whole idea of writing a number as a product of its prime factors. In retrospect, I think I would have done that as a guided activity on the board, pull all of that together so that they really got the gist of what I was wanting them to see and writing the denominator as a product of its factors. Oh! And then also in task B when in question one, it says express $x^3 + x$ as a product of its factors, learners did not realise that, that means to factorise, and that was a real surprise to me, that, writing an expression as a product of its factors did not mean to factorise to them. They worked it out in expanded form instead of factorizing it.

Researcher : Thank you very much, I think my second last question has been answered and it was 'What did you find surprising regarding learners pre-knowledge?' One more question, how did you address the questions that they posed and what kind of questions they posed?

Teacher : Generally, in my teaching, when a learner asks a question I take it back and ask them guiding questions to try and get them to identify the key structure of the question. So, rather than giving them the answer straight, so I ask questions like; 'what about this?'; 'what do you think about the following?'; and really try to push them and guide them into thinking about the right thing. It is something I have been thinking a lot about in terms of my teaching. I am not sure how you see that as a strategy because I think quite a few girls find that frustrating when you are trying to get them to answer, they have a question in

their minds and they want the answer to that question. I sometimes find that by answering a question with a question, they get frustrated because I am not answering their question which means they don't get feedback in terms of their question. I think sometimes, perhaps that is not helpful, I think I need to learn when to give the answer and when to be asking a question in response to a question and that is definitely an area that I need to work on.

Researcher： Thank you very much for your time.

## Appendix G: A letter of ethics clearance

**Wits School of Education**

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

29 April 2015

Student Number: 448872

Protocol Number: 2015ECE010M

Dear Winnie Khanyile

**Application for Ethics Clearance: Master of Science**

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

**Learners' errors and misconceptions as a teaching resource in re-learning fractional linear equations at a Mathematics Clinic.**

The committee recently met and I am pleased to inform you that **clearance was granted**. However, there were a few small issues which the committee would appreciate you attending to before embarking on your research.

**The following comments were made:**

- The process to video recording needs to be clarified, How will students who do not want to be video recorded going to be dealt with? How will those who are being video recorded have their identities protected? These need to be clarified both in methodological terms and in terms of anonymity and confidentiality
- **Interviews**. The candidate ticked the box related to interviews but failed to provide the interview protocol. Also, in terms of the description of the research it is not clear exactly when and where the interviews will be conducted.
- **Location of research**. The candidate indicates that the research will be conducted at a Maths Clinic but it is unclear where the Maths Clinic is located and one can only infer that is located at the school from which permission is being requested. Please make this clearer.
- **Clarity of language**. The information letters (and the proposal) needs to be edited for language clarity.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

**The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.**

All the best with your research project.

Yours sincerely,

M Maseko

Wits School of Education

011 717-3416

Cc Supervisor: Dr Judah Makonye

**Appendix H**

06 May 2015

**A request for permission to do research at St Mary's School, Waverley**

Dear Ms King

As you are aware, I am currently an MSc (in Science and Mathematics Education) student at the University of Witwatersrand. I am conducting research on the topic of grade 10 equations involving fractions. The purpose of the research is to investigate the effect of the teacher's intervention in addressing learners' misconceptions that are displayed in their errors when solving equations that involve fractions. I have chosen to do this research on grade 10 learners as I have taught some of these learners before and currently teach some of them. The other reason for choosing a grade 10 class is that it is the class that seem to have more learners who attend Mathematics Clinic in the afternoon twice a week. Since Mathematics classes are in the afternoon, conducting this research will not interfere with the teaching time; instead it will add more value to the Mathematics Clinic lessons. The research will take place between Monday 18th May 2015 and Friday 18th September 2015.

I request your permission to carry out this research at your school. The research will involve:

- Girls doing a written task of 40 minutes consisting of three equations involving fractions to be solved.
- Their task will be assessed but no marks will be allocated. Instead their errors will be the focus of the research.
- Discussion of learners' solutions will take place in one of the afternoon Mathematics sessions, where their Mathematics Clinic teacher will be helping them understand the errors they would have made. This session will be video-recorded in order to accurately capture the discussion.
- They will be re-assessed after the discussion session to check if the teacher's intervention was effective.

The data collected will be used only for a research report, purely for academic purposes.

The girls' participation in the research will be **confidential** and **anonymity** is guaranteed. Confidentiality and anonymity is assured by using letters of alphabets on the learners' answer scripts (e.g. Learner A) as well as on data analysis. I will save data collected on a USB with a password so that if it gets lost no one else could access it. All data video-recordings will be destroyed between 3-5 years after the research is done.

Girls' participation in this research is voluntary and refusal to participation will bear neither prejudice nor penalty of any kind. Girls may withdraw at any time if they wish to discontinue with participation.

Should you have any queries regarding this research, please do not hesitate to contact my supervisor on the following contact details:

Name: Dr J Makonye

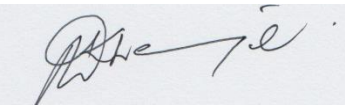
Office M 87 Marang Block, Wits School of Education, University of Witwatersrand

Phone: 011 717 3206

Email: Judah.Makonye@wits.ac.za

I would like to request you to fill in the consent form accompanying this letter and return it to me by **Friday 15th May 2015.**

Yours sincerely



Winnie Khanyile

PROTOCOL NUMBER: 2015ECE010M

**A permission from the Headmistress:**

St Mary's School
Waverley
Founded 1888

8 May 2015

**Permission from the school principal**

I, DEANNE KING, am the principal of ST MARY'S SCHOOL. I have read and understood the content of the letter about learners' participation in the research project on learners' errors on linear equations involving fractions in the Mathematics clinic Grade 10 class. I allow / do not allow Mrs Duduzile Winnie Khanyile (Protocol Number: 2015ECE010M) to do the research described above at this school.

Signature: _____

Signed at WAVERLEY on this day of 11 MAY 2015.

MS D KING
HEADMISTRESS
ST MARY'S SCHOOL,
WAVERLEY
P O BOX 981
HIGHLANDS NORTH, 2037

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Principal - Ms D King BA HDE (Wits)
Reg No 1925/008073/08

## Appendix I: A request for permission to do research in the Mathematics classes

St Mary's School
Waverley
Founded 1888

**A request to the Head of Mathematics Department for permission to do research at some of the Mathematics Clinic's lessons at Mary's School, Waverley**

55 Athol Street
Waverley
2090

8 May 2015

Dear Mrs Zlobinsky-Roux

As you are aware, I am currently an MSc (in Science and Mathematics Education) student at the University of Witwatersrand. I am conducting research on the topic of Grade 10 linear equations involving fractions. The purpose of the research is to investigate the effect of the teacher's intervention in addressing leaners' misconceptions that are displayed in their errors when solving linear equations that involve fractions. I have chosen to do this research on Grade 10 learners as I have taught some of these learners before and currently do teach some of them.

My other reason for choosing a Grade 10 class is that it is the class that seems to have more learners who attend Mathematics clinic in the afternoons, twice a week. Since Mathematics classes are in the afternoon, conducting this research will not interfere with the teaching time; instead it will add more value to the Mathematics clinic lessons. The research will take place between Monday 18 May and Friday 5 June 2015.

I request your permission to carry out this research at your school. The research will involve:

- Girls doing a written task of 40 minutes consisting of three linear equations involving fractions to be solved
- Their task will be assessed but no marks will be allocated. Instead their errors will be the focus of the research
- Discussion of learners' solutions will take place in one of the afternoon Mathematics sessions, where their Mathematics clinic teacher will be helping them understand the errors they would have made. This session will be video-recorded in order to accurately capture the discussion
- They will be re-tested after the discussion session to check if the teacher's intervention was effective

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Principal - Ms D King BA HDE (Wits)
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Founded 1888

The data collected will be used only for a research report, purely for academic purposes.

The girls' participation in the research will be **confidential** and **anonymity** is guaranteed. Confidentiality and anonymity is assured by using letters of alphabets on the learners' answer scripts (e.g. Learner A) as well as on data analysis. I will save data collected on a USB with a password so that if it gets lost no one else could access it. All data video-recordings will be destroyed within three to five years of completion of the study.

Girls' participation in this research is voluntary and refusal to participate will bear neither prejudice nor penalty of any kind. Girls may withdraw at any time if they wish to discontinue with participation.

Should you have any queries regarding this research, please do not hesitate to contact my supervisor on the following contact details:

Name: Dr J Makonye
Office M 87 Marang Block, Wits School of Education, University of Witwatersrand
Phone: 011 717 3206
Email: Judah.Makonye@wits.ac.za

I should like to request that you to fill in the consent form accompanying this letter and return it to me by **Friday 15 May 2015.**

Yours sincerely
***Winnie Khanyile***
PROTOCOL NUMBER: 2015ECE010M

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Dr A Jammine Mrs A Mackay Mr F Maraschin Mr A Mazwai Mrs D Mokhobo Prof L Nongxa Mr A Pullinger
Principal - Ms D King BA HDE (Wits)
Reg No 1925/008073/08

**A permission from the HoD of Mathematics**

St Mary's School
Waverley
Founded 1888

8 May 2015

**Permission from the Head of Mathematics Department**

I, Ingrid Zlobinsky-Roux, am the Head of Mathematics Department at St Mary's School, Waverley. I have read and understood the content of the letter about learners' participation in the research project on learners' errors on linear equations involving fractions in the Mathematics clinic Grade 10 class. I allow / ~~do not allow~~ Mrs Duduzile Winnie Khanyile (Protocol Number: 2015ECE010M) to do the research described above at the Mathematics clinic's lessons.

Signature: I Z Roux

Signed at Johannesburg on this day of 8 May 2015.

55 Athol Street, Waverley, Johannesburg, Gauteng, South Africa P O Box 981, Highlands North 2037
www.stmarysschool.co.za
SENIOR SCHOOL: Tel: 011 531 1800 Fax: 086 504 1122 Email: smsenior@stmary.co.za
JUNIOR SCHOOL: Tel: 011 531 1880 Fax: 011 440 1226 Email: smjunior@stmary.co.za

Directors: Mr ND Carman (Chairman) The Rt Revd S Moreo (Alternate: Revd J Pretorius) Mrs T Chaane Dr G Codrington Mr R Dow Dr A Jammine Mrs A Mackay Mr F Maraschin Mr A Mazwai Mrs D Mokhobo Prof L Nongxa Mr A Pullinger
Principal - Ms D King BA HDE (Wits)
Reg No 1925/008073/08

**Appendix J: Teacher's information and invitation letter**

06 May 2015

Dear Mrs McKechnie

I am currently an MSc (in Science and Mathematics Education) student at the University of Witwatersrand. I am conducting research on the topic of grade 10 fractional equations. The purpose of the research is to investigate the effect of using learners' errors and misconceptions as a resource in learning equations involving fractions through teacher's intervention at a Mathematics Clinic. I have chosen to do this research on grade 10 learners as I have taught some of these learners before and currently teach some of them. The other reason for choosing a grade 10 class is that it is the class that seem to have more learners who attend Mathematics Clinic in the afternoon twice a week. The research will take place between Monday 18th May 2015 and Friday 18th

September 2015. I would, therefore, like to invite you to participate in this research.

The research will involve:

- Doing a written task of 40 minutes consisting of three equations involving fractions to be solved.
- The task will be assessed but no marks will be allocated. Instead your errors will be the focus of the research.
- Discussion of the learners' solutions will take place in one of the afternoon Mathematics Clinic sessions, where you will be helping them to understand the errors they would have made. This session will be video-recorded in order to accurately capture the discussion.
- They will be re-assessed after the discussion session to check if the teacher's intervention was effective.

The data collected from this research will be used only for a research report, purely for academic purposes.

Your participation in the research will be **confidential** and **anonymity** is guaranteed. Confidentiality and anonymity will be assured by using letters of alphabets on the learners' answer scripts (e.g. Learner A) as well as on data analysis. I will save data collected on a USB

with a password so that if it gets lost no one else could access it. All data and video-recordings will be destroyed between 3-5 years after the research is done.

Participation in this research is voluntary and refusal to participation will bear neither prejudice nor penalty of any kind. You may withdraw at any time if you wish to discontinue with participation.

Should you have any queries please do not hesitate to contact me on:

Mrs Winnie Khanyile:

Telephone number: 011 531 1800 (w)/ 011 531 1864 (a/h)

Email: winnie.khanyile@stmary.co.za

My supervisor is Dr J Makonye

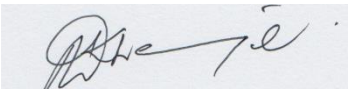
Office M87 Marang Block, Wits School of Education, University of Witwatersrand

Phone: 011 717 3206

Email: Judah.Makonye@wits.ac.za

If you are interested in participating in this research, please fill in the consent form attached and return it to me by **Friday 15th May 2015**.

Yours sincerely



Mrs W Khanyile

PROTOCOL NUMBER: 2015ECE010M

**Teacher's consent form for participation in research and for video-recording of the lesson, on the grade 10 topic of equations involving fractions.**

I, _____(name and surname), I have read and understood the content of the letter about participation in the research project on learners' errors and misconceptions on grade 10 equations involving fractions.

(1) I agree / do not agree (delete inapplicable) to participate in the research on grade 10 equations involving fractions.

(2) I agree / do not agree (delete inapplicable) for the class discussion of solutions to the equations to be video-recorded.

Signature:_____

Signed at _____ on this day of _____2015.

Researcher: W Khanyile

PROTOCOL NUMBER: 2015ECE010M

**Acceptance of invitation to take part in the research, from the teacher**

St Mary's School
Waverley
Founded 1888

**Teacher's consent form for participation in research on the Grade 10 topic of linear equations involving fractions**

I, Philippa McKechnie (name and surname), have read and understood the content of the letter about participation in the research
project on learners' errors and misconceptions on Grade 10 linear equations involving fractions.

I agree ~~I do not agree~~ (delete inapplicable) to participate in the research on Grade 10 linear equations involving fractions.

Signature: P McKechnie

Signed at Johannesburg on this day of 15 May 2015.

Contact details:
Telephone number: /
Cellphone number: 082 502 6711

Researcher: W Khanyile
PROTOCOL NUMBER: 2015ECE010M

55 Athol Street, Waverley, Johannesburg, Gauteng, South Africa P O Box 981, Highlands North 2037
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Directors: Mr ND Carman (Chairman) The Rt Revd S Moreo (Alternate: Revd J Pretorius) Mrs T Chaane Dr G Codrington Mr R Dow Dr A Jammine Mrs A Mackay Mr F Maraschin Mr A Mazwai Mrs D Mokhobo Prof L Nongxa Mr A Pullinger
Principal - Ms D King BA HDE (Wits)
Reg No 1925/008073/08

St Mary's School
Waverley
Founded 1888

**Teacher's consent form for video-recording in research on the grade 10 topic of equations involving fractions**

I, Philippa McKechnie (name and surname), I have read and understood the content of the letter about participation in the research project on learners' errors and misconceptions on grade 10 linear equations involving fractions.

I agree ~~/ do not agree~~ (delete inapplicable) to video-recorded in the research on grade 10 linear equations involving fractions.

Signature: PMcK

Signed at Johannesburg on this day of 15 May 2015.

Contact details:

Telephone number: —

Cellphone number: 082 502 6711

Researcher: W Khanyile

PROTOCOL NUMBER: 2015ECE010M

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Principal - Ms D King BA HDE (Wits)
Reg No 1925/008073/08

**Appendix K: Learner's information and invitation letter:**

06 May 2015

Dear Learner

I am currently an MSc (in Science and Mathematics Education) student at the University of Witwatersrand. I am conducting research on the topic of grade 10 equations involving fractions. The purpose of the research is to investigate the effect of using learners' errors and misconceptions as a resource in learning equations involving fractions through teacher's intervention at a Mathematics Clinic. I have chosen to do this research on grade 10 learners as I have taught some of these learners before and currently teach some of them. The other reason for choosing a grade 10 class is that it is the class that seem to have more learners who attend Mathematics Clinic in the afternoon twice a week. The research will take place between Monday 18th May 2015 and Friday 18th September 2015. I would, therefore, like to invite you to participate in this research.

The research will involve:

- Doing a written task of 40 minutes consisting of three equations involving fractions to be solved.
- The task will be assessed but no marks will be allocated. Instead your errors will be the focus of the research.
- Discussion of your solutions will take place in one of the afternoon Mathematics sessions, where your Mathematics Clinic teacher will be helping you understand the errors you would have made. This session will be video-recorded in order to accurately capture the discussion.
- You will be re-assessed after the discussion session to check if the teacher's intervention was effective.

The data collected from this research will be used for a research report and purely for academic purposes.

Your participation in the research will be **confidential** and **anonymity** is guaranteed. Confidentiality and anonymity will be assured by using letters of alphabets on the learners'

answer scripts (e.g. Learner A) as well as on data analysis. I will save data collected on a USB with a password so that if it gets lost no one else could access it. All data and video-recordings will be destroyed between 3-5 years after the research is done.

Participation in this research is voluntary and refusal to participation will bear neither prejudice nor penalty of any kind. You may withdraw at any time if you wish to discontinue with participation.

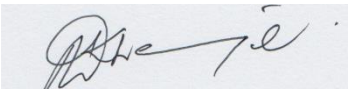
Should you have any queries please do not hesitate to contact me on:

Telephone: 011 531 1864

Email: winnie.khanyile@stmary.co.za

If you are interested in participating in this research, please fill in the consent form attached and return it to me by **Friday 15th May 2015**.

Yours sincerely



Mrs W Khanyile

PROTOCOL NUMBER: 2015ECE010M

**Learner's consent form for video-recording in research on the grade 10 topic of equations involving fractions:**

I, _____(name and surname), I have read and understood the content of the letter about participation in the research project on learners' errors and misconceptions on grade 10 equations involving fractions.

(1) I agree / do not agree (delete inapplicable) to participate in the research on grade 10 equations involving fractions.

(2) I agree / do not agree (delete inapplicable) for the class discussion of solutions to the equations to be video-recorded.

Signature:_____

Signed at _____ on this day of _____2015.

Researcher: W Khanyile

PROTOCOL NUMBER: 2015ECE010M

**Learner's consent form for participation in research on the grade 10 topic of equations involving fractions:**

I, _____(name and surname), I have read and understood the content of the letter about participation in the research project on learners' errors and misconceptions on grade 10 equations involving fractions.

(3) I agree / do not agree (delete inapplicable) to participate in the research on grade 10 equations involving fractions.

(4) I agree / do not agree (delete inapplicable) for the class discussion of solutions to the equations to be video-recorded.

Signature:_____

Signed at _____ on this day of _____2015.

Researcher: W Khanyile

PROTOCOL NUMBER: 2015ECE010M

## Appendix L: Learners' parents' information letter and request and request for their daughters to take part in the research.

06 May 2015

Dear Parents

**Request for permission for your daughter to participate in research on the grade 10 topic of equations involving fractions**

I am Mrs Winnie Khanyile, a Mathematics teacher at St Mary's School, Waverley. I am currently an MSc (in Science and Mathematics Education) student at the University of Witwatersrand. I am conducting research on the learning of grade 10 equations involving fractions. The purpose of the research is to investigate the effect of the teacher's intervention in addressing leaners' misconceptions that are displayed in their errors when solving equations that involve fractions. I have chosen to do this research on grade 10 learners as I have taught some of these learners before and currently teach some of them. The other reason for choosing a grade 10 class is that it is the class that seem to have more learners who attend Mathematics Clinic in the afternoon twice a week.

The research will take place between Monday 18th May 2015 and Friday 18th September 2015.

I would, therefore, like to request your permission for your daughter to participate in this research which will be conducted through:

- Girls doing a written task of 40 minutes consisting of three equations involving fractions to be solved.
- Their task will be assessed but no marks will be allocated. Instead their errors will be the focus of the research.
- Discussion of learners' solutions will take place in one of the afternoon Mathematics sessions, where their Mathematics Clinic teacher will be helping them understand the errors they would have made. This session will be video-recorded in order to accurately capture the discussion.
- They will be re-assessed after the discussion session to check if the teacher's intervention was effective.

The data collected from this research will be used only for a research report, purely for academic purposes.

Your daughter's participation in the research will be **confidential** and her **anonymity** is guaranteed. Confidentiality and anonymity is assured by using letters of alphabets on the learners' answer scripts (e.g. Learner A) as well as on data analysis. I will save data collected on a USB with a password so that if it gets lost no one else could access it. All data and video-recordings will be destroyed between 3-5 years after the research is done.

Participation in this research is voluntary and refusal to participation will bear neither prejudice nor penalty of any kind. Your daughter may withdraw at any time if you or she wishes to discontinue with participation.

Should you have any queries please contact me or my supervisor using contact details below.

My contact details are:

Mrs Winnie Khanyile

Telephone number: 011 531 1800 (w)/ 011 531 1864 (a/h)

Email: winnie.khanyile@stmary.co.za

My supervisor is Dr J Makonye

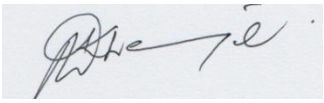
Office M87 Marang Block, Wits School of Education, University of Witwatersrand

Phone: 011 717 3206

Email: Judah.Makonye@wits.ac.za

I would like to request you to fill in the consent form accompanying this letter and return it to me by **Friday 15th May 2015**.

Yours sincerely



W Khanyile

PROTOCOL NUMBER: 2015ECE010M

**Parent/guardian consent form for learner participation in research on the grade 10 topic of equations involving fractions:**

I, _____(name and surname), is the parent / guardian (delete inapplicable) of _____ (your daughter's name and surname). I have read and understood the content of the letter about my daughter's participation in the research project on learners' errors on grade 10 equations involving fractions.

(1) I agree / do not agree (delete inapplicable) for my daughter to participate in the research on grade 10 equations involving fractions.

(2) I agree / do not agree (delete inapplicable) for the class discussion of solutions to the equations to be video-recorded.

Signature:_____

Signed at _____ on this day of _____2015.

Researcher: W Khanyile

PROTOCOL NUMBER: 2015ECE010M