

https://dspace.org/

Group theoretical evaluation of the action of the dilatation operator in the large N limit

Tribelhorn, Laila

2016

http://hdl.handle.net/10539/21667

Tribelhorn, Laila (2016) Group theoretical evaluation of the action of the dilatation operator in the large N limit, University of Witwatersrand, Johannesburg, <http://wiredspace.wits.ac.za/handle/10539/21667>

Downloaded from a DSpace repository. https://dspace.org/

Group Theoretical Evaluation of the Action of the Dilatation Operator in the Large N Limit

Laila Tribelhorn* 567662
Faculty of Science, School of Physics
University of the Witwatersrand

Supervised by Professor R. de Mello Koch†

A dissertation submitted to the University of the Witwatersrand in fulfilment of the requirements for candidacy for the degree of Master of Science.
Johannesburg, 2016



Abstract

Restricted Schur polynomials can be used to describe large N, non-planar limits of $\mathcal{N} = 4$ super Yang-Mills theory. The $\mathcal{R}$-symmetry generators commute with the dilatation operator. For small deformations of $\frac{1}{2}$-BPS operators, the matrix elements of these generators have been computed and a set of recursion relations for the matrix elements of the dilatation operator are obtained from this commutation relation. Together with the knowledge that the smallest eigenvalues of the dilatation operator (corresponding to BPS operators) vanish, these recursion relations can be used to determine the matrix elements of the dilatation operator. Studies up to now have computed the matrix elements of the $su(2)$ generators in the displaced corners approximation. Our first novel result is the computation of the exact $su(2)$ generators. We obtain the matrix elements for the $su(3)$ generators in the displaced corners approximation and exactly, for the first time. This is the first step to computing exact matrix elements of the dilatation operator.

* laila.tribelhorn@students.wits.ac.za
† robert.demellokoch@wits.ac.za

Declaration

I, Laila Tribelhorn, declare that the novel work presented in this dissertation is my own unless otherwise stated. This work has not been submitted before for examination in any other university.

March, 2016

Acknowledgements

I would like to thank my supervisor, Robert de Mello Koch, for his tireless dedication and encouragement. His vast and detailed knowledge and willingness to share it has been invaluable in the course of writing this dissertation. I would like to thank him for sharing his passion for physics, a trait which is perhaps the most motivating of all.

Secondly I thank the National Research Foundation and the University of the Witwatersrand for the financial support I have received.

Contents

0 Introduction **1**
- 0.1 Overview . . . 1
- 0.2 Background . . . 1
- 0.3 Motivating the Duality . . . 4
- 0.4 Warming up to AdS Space . . . 5
- 0.5 Giant Gravitons in AdS . . . 10
  - 0.5.1 Dipole Analogy . . . 10
  - 0.5.2 Embedding in S^4: $AdS_7 \times S^4$ . . . 14
- 0.6 Matrix Models . . . 17
  - 0.6.1 Ribbon Graphs . . . 17
  - 0.6.2 The N Dependence of a Diagram . . . 23
  - 0.6.3 Complex Matrix Model . . . 23
- 0.7 Large N and the Duality . . . 25
  - 0.7.1 Factorisation . . . 25
  - 0.7.2 Triangulating a Surface . . . 26
- 0.8 Connection with Young Diagrams . . . 28

1 Tools and Objectives **29**
- 1.1 Restricted Schur Polynomials . . . 29
- 1.2 $su(2)$ and $su(3)$ Algebra . . . 31
- 1.3 Calculating a Projection Operator . . . 33

2 $SU(2)$ **36**
- 2.1 Numerical Calculation of Projectors . . . 36
- 2.2 Displaced Corners Approximation . . . 40
- 2.3 Exact Generators . . . 41
  - 2.3.1 Traces and Identities Needed . . . 41
  - 2.3.2 Exact Traces . . . 43
  - 2.3.3 Exact Matrix Elements of J_- . . . 44
  - 2.3.4 First Steps to Calculating the Dilatation Operator . . . 46

3 $SU(3)$ **49**
- 3.1 Numerical Calculation of Projectors . . . 49
- 3.2 Displaced Corners Approximation . . . 52
  - 3.2.1 Calculating the $SU(2)$ Clebsch-Gordon coefficients . . . 53
  - 3.2.2 Calculating the Traces . . . 54
  - 3.2.3 The Displaced Corners Generators . . . 55
- 3.3 Exact Generators . . . 56
  - 3.3.1 The Structure of the Problem . . . 57
  - 3.3.2 Traces and Identities Needed . . . 57
  - 3.3.3 Exact Traces . . . 59
  - 3.3.4 Exact Matrix Elements of $\mathrm{Tr}\left(Y \frac{d}{dX}\right)$ . . . 59

4 Discussion **61**
- 4.1 Summary . . . 61
- 4.2 Outlook . . . 61

A A Proof of Stoke's Theorem **62**

B Inequivalent, Irreducible Representations **64**

C Symmetric group and Representation Theory **65**

D Schur Polynomials **68**

0 Introduction

0.1 Overview

The AdS/CFT correspondence as proposed by Maldacena [1, 2, 3] conjectures a duality between $\mathcal{N} = 4$ super Yang-Mills theory and type IIB string theory on an asymptotically $AdS_5 \times S^5$ background. This means that insights in the field theory have an interpretation in the string theory and vice versa, as motivated in Section 0.3.

Studies of the anomalous dimensions of restricted Schur polynomials, which provide a basis for the multimatrix sector in the field theory, have shown giant graviton brane states in the field theory to be emergent [4, 5, 6, 7, 8, 9]. The operators dual to these giant gravitons have a bare dimension that grows like N in the large N limit [10, 11, 12]. To capture the large N limit for correlators of this type we must sum non-planar Feynman diagrams, as illustrated in Section 0.6. This is a non-perturbative problem in the string theory.

Thus far results are limited to $\frac{1}{2}$-BPS and small deformations of those operators. This can be accomplished by using the displaced corners approximation as shown in Sections 2.2 and 3.2. When the operators have dimensions that grow faster than N, this approximation is no longer valid.

The aim of this dissertation is to compute matrix elements of certain symmetry generators exactly. The $su(2)$ generators were computed in [13] in the displaced corners approximation. We compute the exact $su(2)$ generators in Section 2.3.

Novel results can be found in Section 3 where the matrix elements of the $su(3)$ generators are computed both exactly and in the displaced corners approximation.

In Section 2.3.4 we present a starting point for the computation of the dilatation operator which will utilise the generators we have calculated. This is the object we are interested in since it maps into the Hamiltonian operator in the dual string theory. We begin by noting the requirement that these generators must commute with the dilatation operator which yield a set of (exact) recursion relations (when the generators are exact).

0.2 Background

Modern physics can be divided into two broad subgroups. These subgroups consist of physics on large scales and on small scales, respectively. Specifically, one problem faced is trying to find a theory of quantum gravity. That is a theory of gravity, which is well described by Einstein's theory of general relativity on large scales, on the much smaller scales where physics is well described by quantum theories of mechanics and fields.

Quantum field theories are written in terms of bare parameters. These are not physical. We don't measure these in the lab. In a perturbative study of quantum field theory, divergences appear in diagrams that contain loops. Virtual particles with any momentum allowed by momentum conservation run in these loops, so that sums over all allowed virtual particle states becomes integrals over particle 4-momenta. These integrals are often divergent. In renormalised perturbation theory, we split our parameters into a sum of a physical parameter (which is measured) and a bare parameter (which isn't). The bare parameters appear in our counter terms and these cancel the divergences in our theory so that the physical parameters are finite. The BPHZ theorem (for Bogoliubov, Parasiuk, Hepp and Zimmermann) states that all divergences are removed by counter terms to all orders in the perturbation, so long as the theory is renormalisable [14],[15],[16]. These high energy (or short-distance) modes thus have little effect on the low energy theory. It is the low energy modes that yield quantitative physical predictions.

Equivalently, we can introduce a momentum cut-off, Λ, such that integrating over momentum when calculating Feynman diagrams does not give us arbitrarily large values. This seems like an unphysical thing to do. It was Wilson's insight, however, that any quantum field theory (QFT) is intrinsically defined with a cutoff Λ that is

physically significant. His insight was that QFT is not a fundamental theory. It is an effective theory with energy scale defined by the length-scale of what we are observing. For example, friction is not understood by considering all the interactions of the atoms of two objects that are moving against each other. We need only understand it macroscopically to make sense of it. The length scales of atoms is simply too small and we instead imagine a collection of atoms and study the effective interaction of that collection. Similarly, QFT breaks down at length scales that are very small:

$$\Lambda = \frac{\hbar}{l}$$

where l is the shortest length scale. New physics is needed to describe what happens beyond this scale.

So QFTs that tell us about nature need to be renormalisable. We are interested in quantum gravity and so we can ask whether we can write down a renormalisable QFT that couples to gravity. To this end, consider the Einstein-Hilbert action.

$$\mathcal{S}_{EH} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} R$$

where G_N is our coupling to gravity, R is the Ricci scalar and g is the metric determinant. Metric components are unitless. We can see this by considering

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

The left hand side of the equality has dimensions of L^2 and each dx^μ has dimension L. Thus we see that $g_{\mu\nu}$ has dimension L^0 so it is dimensionless. The Ricci scalar is built out of derivatives and products of Christoffel symbols. For the purpose of dimensional analysis, we only note that these are proportional to the derivative, ∂_μ, squared. Thus, R has dimensions L^{-2}. We integrate over $d^d x$ which has dimension L^d. The action is dimensionless giving us that

$$[G_N] = 2 - d$$

If $d > 2$ then $[G_N] < 0$. This tells us that G_N is irrelevant. Relevant and marginal operators have dimension greater than or equal to zero. They grow as we integrate out high energy modes (Wilson's idea). Irrelevant operators do not do this. They flow to zero at low energy. Thus, Wilson's renormalization group flows explains why all low energy theories describing nature are renormalizable theories.

The problem with non-renormalisable field theories is that they cannot be continued to high energies without encountering some difficulties. Unlike with renormalisable theories, the couplings do not remain small and so we cannot trust perturbative calculations. As we study our theory at higher and higher energies, so do our couplings grow. Eventually these couplings become infinite. Quantum gravity is a non-renormalisable theory. By its nature, we need to understand this theory on small scales, and therefore high energies, in order to understand it. The fact that we have these divergences at high energies are indicative that there are some physical effects missing in our theory. Remember that it was Wilson's insight that told us that the physical effects in a theory are finite so that the non-physical effects are removed with the divergences. The fact that our theory is missing some physical effects is part of why we say we have lost predictive power. With a renormalisable theory we can trust our couplings will remain small at high energies. In a non-renormalisable theory, we realise we are missing information at high energies. Thus QFT has failed to give us physical description of quantum gravity.

General relativity is a statement about the relation between matter and spacetime. To quote John Wheeler, spacetime tells matter how to move and matter tells spacetime how to curve. We can relate the mass of an object in a spacetime to the force of gravity (in a Newtonian mindset) one would feel around that object. This framework breaks down on smaller scales but theories of QFT do not provide an alternative description. Where quantum field theory fails, however, string theory succeeds and these are theories of quantum gravity.

The AdS/CFT correspondence [1] proposes a duality between a theory with quantum gravity and one without. This duality claims an exact equivalence between $\mathcal{N} = 4$ Super Yang-Mills (SYM) and string theory on an $AdS_5 \times S^5$ background. The former is a quantum field theory and the latter is spacetime with negative curvature. The duality implies that we can study $\mathcal{N} = 4$ SYM and learn something about the string theory.

In the $\mathcal{N} = 4$ SYM theory, quantum fields are represented by $N \times N$ matrices. Observables are represented as traces of products of these matrices. The number of matrices used to construct an operator is related to the scaling dimension of the operator. If we use N^α matrices to construct our observable, then we are describing

different objects in the string theory for different α. For example, consider the table below.

Observables given by various scalings of fields in $\mathcal{N}=4$ SYM (N^α)				
N^α	Object	$N=9$	$N=9^2$	$N=9^3$
$\mathcal{O}(1)$	Graviton	$\mathrm{tr}(Z^3)$	$\mathrm{tr}(Z^3)$	$\mathrm{tr}(Z^3)$
$\mathcal{O}(\sqrt{N})$	String	$\mathrm{tr}(Z^3)$	$\mathrm{tr}(Z^9)$	$\mathrm{tr}(Z^{27})$
$\mathcal{O}(N)$	Giant Graviton	$\mathrm{tr}(Z^3)+\ldots$	$\mathrm{tr}(Z^{27})+\ldots$	$\mathrm{tr}(Z^{243})+\ldots$
$\mathcal{O}(N^2)$	New Geometry	$\mathrm{tr}(Z^3)+\ldots$	$\mathrm{tr}(Z^{243})+\ldots$	$\mathrm{tr}(Z^{19683})+\ldots$

We should see interesting physics in the limit where we construct our observables out of N^α matrices and let $N\to\infty$ (for $\alpha>\frac{1}{2}$ this is a non-planar limit). Already at $\mathcal{O}(N^2)$, our states are so heavy that they curve the space time around them and we start to see new geometries emerging. However, we do not have the tools to study the dynamics in this limit. This means we need to build tools and construct methods if we want to study the physics in this limit.

We will be working in the field theory side (as this is comparatively much better understood) but we identify operators over here with states in the dual quantum gravity. That is, operators in $\mathcal{N}=4$ SYM correspond to states in the dual quantum gravity on an asymptotically $AdS_5\times S^5$ background, with the dimensions of the operators mapping into energies in the quantum gravity [2]. This means the physics we work on has an interpretation in the dual quantum gravity. Thus our question develops into the problem of identifying what operators correspond to which states.

We can look at conserved quantities and label states using their eigenvalues. This is a familiar concept: for example, we might look at eigenkets of the Hamiltonian which are labelled by specific energy eigenvalues. Operators that correspond to conserved quantities, like the Hamiltonian, generate symmetries (by Noether's theorem). This association between observables and symmetries becomes very natural at the quantum level. In the quantum theory observables are represented by Hermitian operators and symmetries by unitary transformations. Consider a unitary transformation U which implements a symmetry transformation. Then

$$U^\dagger = e^{-iO^\dagger} = U^{-1},$$

which follows since U is unitary. We see that this implies that $O=O^\dagger$ i.e.: O is Hermitian. So for each symmetry we can find a Hermitian operator.

We can match operators by matching the symmetries in the field theory with those in the quantum gravity. Eigenvalues of these generators will label states (in the quantum gravity) and operators (in the field theory) and we can match states and operators if they carry the same quantum numbers. However this matching is not one to one so we may not get a unique statement about the matching. For example, we might match energy eigenvalues but in a case where we have degeneracies this does not uniquely determine what should be matched. The spectrum of the dilatation operator in $\mathcal{N}=4$ SYM maps into energies in the quantum gravity. Thus, computing the action of the dilatation operator is of especial interest to us.

The dilatation operator is usually found by methods which employ summing Feynman diagrams. We want to study operators whose dimensions grow with N – these are giant gravitons. This means we will need to include many $\frac{1}{N}$ corrections to capture the leading large N limit. These corrections take the form of non-planar Feynman diagrams. Developing methods to sum this huge class of diagrams, in order that the spectrum of the dilatation operator can be computed, is one of the main goals of this MSc dissertation.

The reason to study these operators is that in the dual quantum gravity we would like to study states that are heavy enough to deform the spacetime. These correspond to operators whose dimensions grow like N^2. We already struggle to study the non-planar large N limit of the theory for the giant graviton case, so studying new geometries seems like an impossible task. By studying the comparatively simpler problem of giant gravitons, we start to build up a working knowledge and useful techniques which extend the gauge/gravity duality into new large N but non-planar settings.

The problem we run into, with the above goal in mind, is that we need to sum many Feynman diagrams to

determine the dilatation operator. In the non-planar case, the diagrams we need to sum explode so that we are summing huge numbers of Feynman diagrams even at one loop (the first quantum correction). The number of diagrams to sum increases at each loop and we need to evaluate the dilatation operator at all loops if we want a set of anomalous dimensions that can be compared to string theory. This point deserves further explanation.

When we study a theory with strong coupling perturbatively, the higher order terms are not negligible and we must include the full set of quantum corrections for the theory to make physical predictions. We want to study $\mathcal{N} = 4$ SYM at strong coupling and large N because this simplifies the string theory. The reason is twofold. Firstly, AdS/CFT has a strong-weak coupling nature. λ is the 't Hooft coupling (which we will see in more detail in Section 0.6) and is related to the curvature of AdS_5 by $\lambda = \frac{R^4}{l_s^4}$ where l_s is the string length. Thus large λ means $l_s << R$ so that the spacetime looks flat which simplifies the theory. Secondly, the string coupling g_s is equal to $\frac{1}{N^2}$. Thus, at large N the string coupling is small and we can ignore the higher order terms in the string theory loop expansion.

0.3 Motivating the Duality

In string theory, closed and open strings are the fundamental objects. Open strings have endpoints on objects called D-branes[1], while closed strings exist apart from the branes. A D-brane generalises the concept of a point particle to higher dimensions. A D0 brane is a point particle, a D1 brane looks like a line (i.e.: a string), a D2 brane[2] looks like a 2 dimensional object (i.e.: a membrane) and so forth.

We cannot break strings up into more fundamental pieces pieces. If we think about holding a piece of string (putting tension on the string), we know we can wiggle it to produce a wavelike motion. If we wiggle the string to produce a certain wavelength, then we can produce standing waves of different modes. Each mode has its own energy. What we see is that the various energy excitations of the string correspond to different particles.

We have gaps between the energies of different excitations. Specifically, there must be a gap between the mode of an unexcited string and the first mode. The existence of this gap allows us to define the low energy effective theory. The string tension is related to the size of the string and so the size of the string provides a natural cut off for our theory. That is, our theory holds only on distances larger than this cut-off scale which is why we can construct a low energy effective theory. If we integrate out the high energy modes (modes smaller than the cut-off), then we obtain the low energy effective theory.

The low energy effective action of a closed string describes gravitons. For the open string, the action describes a spin 1 gauge theory. If we think about how this relates to our D-branes, then we see that the theory that describes the physics on the brane is a gauge theory. We can turn questions we have about these branes into questions about strings. For example, if we want to know how two branes interact then we can consider a string exchanged between the two branes. We might wonder if the brane has mass. Then we can think about a string interacting with the brane. This must be a gravitational interaction – the strength of which we can relate to mass. In this way, any question we have about the D-brane can be phrased as a question about open strings and answered using string theory. In order to do brane physics, we do not need to add new degrees of freedom – the open strings and their endpoints are enough.

In the supergravity, we have objects called p-branes, where p is the dimension of the object. They are like black holes but they are infinitely long cylinders or planes or hyperplanes with a horizon. The event horizon in a spacetime is where the metric component g_{00} is zero. A 3-brane is a 3 dimensional object surrounded by a horizon.

In the low energy theory (supergravity), we have small momenta and therefore long wavelengths. Rayleigh's criterion places a limit on the resolution ability of any imaging process to be on the order of the wavelength of the wave used to measure it. This is best understood in the case of single slit diffraction. Diffraction imposes a limit

[1] D for Dirichlet boundary condition, which fixes the endpoint of the string on the brane in some direction.
[2] D3 branes feature in Type IIB string theory and will play an important role in what follows.

on our ability to resolve images. The defining equation for the first minimum in a diffraction pattern is given by

$$\sin\theta_m = \frac{\lambda}{a_0}$$

where λ is the wavelength of the source and a_0 is the size of the aperture. This equation tells us that we will see a diffraction pattern when $\lambda < a_0$ and that when $\lambda = a_0$ the first minimum is at $\frac{\pi}{2}$. This tells us that, for long wavelength waves, the wave is spread out so much that we cannot resolve maxima and minima and so we cannot say where the source of the wave is.

Thus the long wavelengths in the supergravity mean that the supergravity modes don't see the modes on the D-brane and so the branes do not interact with the closed strings. The low energy Lagrangian density is thus split into a gravitational part on $\mathcal{M}_{10}$ and a gauge theory part (Supersymmetric Yang-Mills), with no interactions mixing them.

The same can be said about interactions in the p-brane description at low-energy. However, the geometry in this description is not flat. Far away from the p-branes, the space looks flat and we have supergravity on $\mathcal{M}_{10}$. Near to the p-branes, the gravitational potential is a deep well (it looks like an attractive force) and so the energies of our string modes are red-shifted. This means that all the modes are at a low energy, so we have to retain a complete description of the string – not just the supergravity modes. The Lagrangian density is thus split into gravity on $\mathcal{M}_{10}$ and into type IIB strings on $AdS_5 \times S^5$.

If D-branes are the same as p-branes then the above two descriptions must agree and

$$\mathcal{L}_{\mathcal{N}=4\text{SYM}} \equiv \mathcal{L}^{\text{IIB strings}}_{AdS_5\times S^5} \tag{0.3.1}$$

This claims an equivalence between $\mathcal{N} = 4$ SYM and a string theory on $AdS_5 \times S^5$. The left hand side is a flat space theory of particles. It has the gauge group $U(N)$. The right hand side is a string theory in higher dimensions, so it has many more degrees of freedom arising from these extra dimensions. We can try to build a dictionary that maps quantities in either theory to each other. For example, we might ask what the anomalous dimensions in the gauge theory map into in the string theory. We will motivate later how these anomalous dimensions map into energies in the quantum gravity.

0.4 Warming up to AdS Space

Let's start by looking at the metric for Schwarzschild spacetime.

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{0.4.1}$$

This describes a black hole at the origin. The event horizon is at $r = 2GM$, which we can see because $g_{00} = 0$. The action looks like

$$\mathcal{S} = m\int ds = m\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau = \int L dt \tag{0.4.2}$$

Here we have used an Affine parametrisation

$$-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$$

We work out the equations of motion using the Euler-Lagrange equation.

$$\begin{aligned}
&\frac{\partial L}{\partial \dot{x}^\alpha} = -\frac{m}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}g_{\alpha\mu}\dot{x}^\mu \\
&\implies \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = -\frac{m}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\beta}\dot{x}^\beta\dot{x}^\mu + g_{\alpha\mu}\ddot{x}^\mu\right) \\
&\frac{dL}{dx^\alpha} = \frac{m}{2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\left(-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu\right) \\
&\implies \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu = \frac{\partial g_{\alpha\mu}}{\partial x^\beta}\dot{x}^\beta\dot{x}^\alpha + g_{\alpha\mu}\ddot{x}^\mu
\end{aligned} \tag{0.4.3}$$

where we used the Affine parametrisation to get the last equality. In nearly flat space and in the non-relativistic limit,

$$-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = (\dot{x}^0)^2 - \vec{x}\cdot\vec{x} = 1 \tag{0.4.4}$$

If we set the speed of light, c, to 1 then $\dot{x}^0 \approx 1$ and $\dot{x}^i \approx 0$. Consider now $\alpha = i$ in (0.4.3).

$$\begin{aligned}
&\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^i}\dot{x}^\mu\dot{x}^\nu = \frac{\partial g_{ii}}{\partial x^\beta}\dot{x}^\beta\dot{x}^i + g_{ii}\ddot{x}^i \\
&\implies g_{ii}\ddot{x}^i = \frac{1}{2}\frac{\partial g_{00}}{\partial x^i}\dot{x}^0\dot{x}^0 \\
&\implies \ddot{x}^i = \frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = -\frac{\partial}{\partial x^i}\Phi_N
\end{aligned} \tag{0.4.5}$$

So the Newtonian potential, Φ_N, is $-\frac{1}{2}g_{00}$. In terms of our Schwarzschild metric, $\Phi_N = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)$.

$$\vec{F} = -\nabla\Phi_N = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\Phi_N = -\frac{GM}{r^2}\hat{r} \tag{0.4.6}$$

This tells us that M in the metric is indeed the mass of the object. To summarise, we have shown that the g_{00} component of the metric becomes the gravitational potential in the non-relativistic limit and the limit of an almost flat spacetime. We will use this insight to build up intuition for other spacetime metrics we will consider.

Consider a metric on a 10 dimensional spacetime of the form

$$ds^2 = -\frac{dt^2 - d\vec{x}\cdot d\vec{x}}{\sqrt{1+\frac{C}{r^4}}} + \sqrt{1+\frac{C}{r^4}}(dr^2 + r^2 d\Omega_5^2) \tag{0.4.7}$$

C is some constant and $d\vec{x}\cdot d\vec{x} = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. We see as r gets large, the space begins to look flat and we expect to recover Newtonian physics (so as $r \to \infty$ our space starts to look like $\mathcal{M}_{10}$). Thus, in this limit, we can calculate what the Newtonian potential for this space should look like.

$$\Phi_N = -\frac{1}{2}g_{00} = \frac{1}{2}\left(1+\frac{C}{r^4}\right)^{-\frac{1}{2}} \approx \frac{1}{2}\left(1 - \frac{C}{2r^4}\right) = \frac{1}{2} - \frac{C}{4r^4} \tag{0.4.8}$$

Our approximation holds since we are at large r so $\frac{1}{r^4}$ is small. For the 4-dimensional Schwarzschild metric, the gravitational potential fell off like $\frac{1}{r^2}$ or $\frac{1}{r^{d-2}}$. The potential for this 10-dimensional spacetime does not fall of like $\frac{1}{r^8}$ as expected. Instead it looks like we are in 6 dimensions. We can interpret this as the metric telling us we are filling $3+1$ dimensions (from the $(dt)^2 - d\vec{x}\cdot d\vec{x}$ terms) and that we have 6 transverse dimensions. We have S^5 rotational invariance. Think of an infinite rod in 3 spatial dimensions. It completely occupies one of the dimensions, so there are only two directions the field can spread in. As a result the electric field sourced by the rod falls off as $\frac{1}{r}$ and not as $\frac{1}{r^2}$. This is analogous to what we have here. It only makes sense to talk about the gravitational potential in the space that isn't already filled.

Looking at our metric, we see that $g_{00} = 0$ at $r = 0$, so there is a horizon at the origin. We have a black 3-brane here. We can probe the metric more by making a few convenient adjustments. For the physics close to the brane we are concerned with small r so we can drop the $+1$ term under the square roots since the term that goes like $\frac{1}{r^4}$ dominates the behaviour. We rename our constant C to C^2. The latter adjustment is perfectly reasonable since C was arbitrary to begin with. Now we have

$$ds^2 = -\frac{r^2}{C}(dt^2 - d\vec{x}\cdot d\vec{x}) + \frac{C}{r^2}dr^2 + Cd\Omega_5^2 \tag{0.4.9}$$

This is the metric of $AdS_5 \times S^5$. We can study the potential after this change. We use the relation we derived earlier

$$\Phi \equiv -\frac{1}{2}g_{00} = \frac{r^2}{2C} \tag{0.4.10}$$

This is quadratic, just like the potential for the harmonic oscillator. What this means is that, for small r, any object launched radially outward from the brane (at $r = 0$) will ultimately return to the brane. So geodesics on $AdS_5 \times S^5$ will go away from the origin and then come back, just like with the harmonic oscillator.

If we make the coordinate transformation

$$t \to \alpha t, \quad \vec{x} \to \alpha\vec{x}, \quad r \to \frac{r}{\alpha}$$

then our metric, ds^2, is unchanged. In order to leave the metric invariant, we had to have r scaling inversely with $\vec{x}$, so it looks like r is scaling like an energy.

We are interested in studying $AdS_5 \times S^5$ as that is the space in which type the type IIB string theory we are interested in lives. A metric on this space is (0.4.9), with $C = 1$ and the coordinate transformation $z = \frac{1}{r}$.

$$ds^2 = \frac{1}{z^2}(-dt^2 + d\vec{x} \cdot d\vec{x} + dz^2) + d\Omega_5^2 \tag{0.4.11}$$

This is the metric in the Poincaré patch of AdS_5. We have a horizon as $z \to \infty$ and the boundary of the spacetime is at $z = 0$. This metric is defined to cover a patch of AdS_5 only and not the whole spacetime. We have a choice which patch we want to cover, each with its own metric. Our discussion will only consider the above metric and we will see later what portion of the space it covers.

We will explore how this metric can be obtained by embedding into flat space using Poincaré coordinates and then we will see how to obtain a metric which describes the whole of the space (AdS_{p+2} for generality). We are embedding our curved manifold into a flat space. In curved spaces, the geometry we are used to using in flat space (Euclidean geometry) no longer holds. In particular, the way we measure distances between points is different. We can figure out what the metric in this curved space is if we can figure out how to measure distances between points. That is, we are figuring out how to measure distances between points on the curved space by using the rule (essentially the Pythagoras theorem) that tells us how to measure the distance between points on the flat space.

The AdS_{p+2} geometry has constant negative curvature and its metric describes a hyperbolic geometry. We want to induce the metric on our manifold by embedding into flat space. Our manifold is a surface in embedding space. Just like we have the equation of a sphere in $(p+2)$ dimensions, $\sum_{i=0}^{p+1}(X^i)^2 = R^2$, we have a negative curvature analogue which describes a hyperboloid

$$(X^0)^2 + (X^{p+2})^2 - \sum_{i=1}^{p+1}(X^i)^2 = R^2 \tag{0.4.12}$$

where R is the radius of curvature. Note that when we embed a hyperbolic space into a flat space we appear to gain a timelike dimension: this is required to represent the isometry of the hyperbolic space with the flat embedding space. That is, our embedding space is a $(p+3)$ flat spacetime with two timelike coordinates. Our metric is

$$ds^2 = -(dX^0)^2 - (dX^{p+2})^2 + \sum_{i=1}^{p+1}(dX^i)^2 \tag{0.4.13}$$

Note that we are ignoring the sphere part and looking only at AdS_{p+2} spacetime. To get the Poincaré patch, we choose the following coordinates, which satisfy our defining equation for our embedding space (0.4.12).

$$\begin{aligned}
X^0 &= \frac{R^2}{2r}\left(1 + \frac{r^2}{R^4}(R^2 + \vec{x}\cdot\vec{x} - t^2)\right) \\
X^i &= \frac{r}{R}x^i \qquad\qquad i = 1, \ldots, p \\
X^{p+2} &= \frac{r}{R}t \\
X^{p+1} &= \frac{R^2}{2r}\left(1 - \frac{r^2}{R^4}(R^2 - \vec{x}\cdot\vec{x} + t^2)\right)
\end{aligned} \tag{0.4.14}$$

We can work out the differentials and plug this into (0.4.13) to obtain

$$ds^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}d\vec{x}^2 \tag{0.4.15}$$

We set $R^2 = 1$ and $z = \frac{1}{r}$ to obtain the AdS part of (0.4.11). Our coordinates now look like

$$\begin{aligned}
X^0 &= \frac{z}{2}\left(1 + \frac{1 + \vec{x}\cdot\vec{x} - t^2}{z^2}\right) \\
X^i &= \frac{x^i}{z} \qquad\qquad i = 1, \ldots, p \\
X^{p+2} &= \frac{t}{z} \\
X^{p+1} &= \frac{z}{2}\left(1 - \frac{1 - \vec{x}\cdot\vec{x} + t^2}{z^2}\right)
\end{aligned} \tag{0.4.16}$$

Then

$$\frac{1}{z} = \frac{X^0 - X^{p+2}}{2} \tag{0.4.17}$$

The left hand side describes a hyperbola with asymptote along the line $z = 0$. This splits our space up into two regions: $X^0 > X^{p+2}$ and $X^0 < X^{p+2}$. We have to choose which region we are in by specifying the range of $r = \frac{1}{z}$. This is what we mean by having a patch on the space and this makes it clear that we are choosing which patch to cover. It is most natural to choose $r = \frac{1}{z} \geq 0$ as this looks like a radial coordinate.[3]

We can identify global coordinates, also satisfying (0.4.12), that describe the entire AdS_{p+2} space. We use

$$\begin{aligned} X^0 &= R\cosh\rho\cos\tau \\ X^i &= R\sinh\rho\,\eta^i \\ X^{p+2} &= R\cosh\rho\sin\tau \end{aligned} \tag{0.4.18}$$

It is convenient to make a coordinate choice where the η^i parametrise a unit sphere. In our case, this sphere is a p-sphere. For example, the unit sphere in 3 dimensions is given by the coordinate transformation $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$, $z = \cos\phi$. This gives the metric $ds^2 = -dt^2 + d\phi^2 + \sin\phi d\theta^2$. Similarly in AdS_5 where our metric has two timelike coordinates, we make a coordinate transformation to spherical coordinates on the X^i for $i = 1 \ldots p+1$. Now we see that the X^i describe a p-sphere with radius $R\sinh\rho$. The components η^i look like

$$\begin{aligned} \eta^1 &= \cos(\theta_1) \\ \eta^2 &= \sin(\theta_1)\cos(\theta_2) \\ \eta^3 &= \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ &\vdots \\ \eta^p &= \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{p-1})\cos(\theta_p) \\ \eta^{p+1} &= \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{p-1})\sin(\theta_p) \end{aligned} \tag{0.4.19}$$

Here $\theta_1, \ldots, \theta_{p-1} \in [0, \pi]$ and $\theta_p \in [0, 2\pi]$. Note that $\eta^i\eta_i = 1 \implies d\eta^i\eta_i = \eta^i d\eta_i = 0$.

Now we have everything we need to calculate the induced metric for these coordinates.

$$ds^2_{\text{global}} = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2) \tag{0.4.20}$$

Let's examine whether we really cover the whole space. First, consider that X^0 and X^{p+2} together describe a circle of fixed radius, for ρ constant. As we vary τ we move around the circle and varying ρ changes the radius of the circle with no restriction on how big this radius can be. Now consider X^i. The vector component ensures that we can point in any direction, no matter which value we fix ρ to be. So we truly can get to any point in the space using these coordinates.

We are interested in what happens at the boundary of our space[4]. Looking at our metric, we can examine what happens as ρ gets large. We could do the same for τ but we know $\cos\tau$ and $\sin\tau$ are periodic, so we won't learn anything new.

For large ρ,

$$\cosh\rho = \frac{e^\rho + \mathrm{e}^{-\rho}}{2} \approx e^\rho \approx \frac{e^\rho - e^{-\rho}}{2} = \sinh\rho$$

and so our metric becomes

$$ds^2 = \left(\frac{Re^\rho}{2}\right)^2\left(-d\tau^2 + d\Omega_p^2 + 4e^{-2\rho}d\rho^2\right) \tag{0.4.21}$$

The boundary of our manifold is at large ρ. Here any coordinate difference Δx^μ leads to a vanishingly small contribution to the proper distance from the ρ coordinate. This can be seen by noticing that Δs^2 is some number multiplied by $(-\Delta\tau^2 + \Delta\Omega_p^2 + 4e^{-2\rho}\Delta\rho^2)$. The coefficient of $\Delta\rho^2$ is exponentially decreasing with increasing ρ and scales this term to be very small at large ρ compared with the other terms. It is thus negligible and our metric at the boundary simplifies.

$$ds^2_{\text{boundary}} = \left(\frac{Re^\rho}{2}\right)^2\left(-d\tau^2 + d\Omega_p^2\right) \tag{0.4.22}$$

[3] Recall that the horizon is at $r = 0$ and we reach the boundary of the spacetime as $r \to \infty$

[4] The AdS/ CFT duality conjectures that the CFT lives on the boundary of our space. So we can think of AdS as living inside a box (like the harmonic oscillator): that is where our string theory is. The CFT lives on the surface of that box.

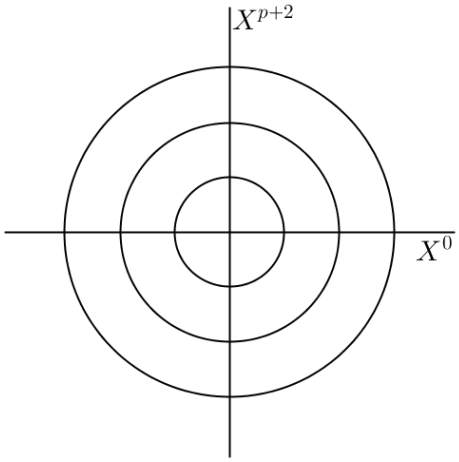


Figure 1: Circles of radius fixed by ρ

We see that this metric has the form $\mathbb{R} \times S^p$. So the boundary of our space, AdS_{p+2}, is $\partial AdS_{p+2} = \mathbb{R} \times S^p$.

$\mathcal{N} = 4$ SYM lives on the surface of our AdS_5 box. We usually think to define the field theory on Minkowski space, $\mathbb{R}^{1,3}$, and not $\mathbb{R} \times S^3$. However, $\mathbb{R}^{1,3}$ and $\mathbb{R} \times S^3$ are related by a conformal transformation (detailed below). $\mathcal{N} = 4$ SYM is a conformal field theory and any CFT is invariant under conformal transformations. This means that we can define $\mathcal{N} = 4$ SYM on both $\mathbb{R}^{1,3}$ and $\mathbb{R} \times S^3$.

To show that Minkowski spacetime and $\mathbb{R} \times S^3$ are related by a conformal transformation, we can Wick rotate to get the Euclidean signature for the CFT metric.

$$ds^2 = -dt^2 + d\vec{x} \cdot d\vec{x} \to dt_E^2 + d\vec{x} \cdot d\vec{x} = ds_E^2 \tag{0.4.23}$$

By a change of coordinates we can obtain

$$ds_E^2 = dr^2 + r^2 d\Omega_3^2 \tag{0.4.24}$$

We change variables so that $r = e^\tau \implies dr = e^\tau d\tau$ and then we perform a scale transformation to absorb the e^τ. $\mathcal{N} = 4$ SYM is a CFT and so the above conformal transformation will leave all physical predictions invariant. The metric we obtain

$$ds_E^2 = d\tau^2 + d\Omega_3^2 \tag{0.4.25}$$

is $\mathbb{R} \times S^3$. We see that our time coordinate corresponds to $\mathbb{R}$. There is a symmetry for translating $\mathbb{R}$ in $\mathbb{R} \times S^p$. We can try and match this symmetry in the string theory with one in the field theory. We know that energy is conserved when we have a symmetry in time translations. So it is natural to look inside the box and determine what the conserved quantity in the field theory is.

Consider the time translation $\tau \to \tau + a$. We defined $r = e^\tau$. Under this translation, $r \to e^a e^\tau = e^a r$. This is a scale transformation so our conserved quantity is a dimension. If we scale our coordinate x by λ, then our operator scales by a factor λ^{-D} i.e.: $\mathcal{O}(\lambda x) = \lambda^{-D}\mathcal{O}(x)$. The classical scaling dimension is 1 when the two point function, $\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle$, depends on the inverse of the distance squared, $|x_1 - x_2|^{-2}$. This is when $\mathcal{O}$ looks something like the scalar field and has dimensions of L^{-1}. Later, in Section 0.6, we will see that physical observables are traces

of complex fields, Z and $Z^\dagger$. If we set $c = \hbar = 1$, then the action is dimensionless and the kinetic term gives us

$$\mathcal{S} = \int d^4x \, \mathrm{Tr}(\partial_\mu Z \partial^\mu Z^\dagger)$$

Dimensional analysis tells us $[Z] = [Z^\dagger] = L^{-1}$. However, observables may take the form of something like $\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^3)$. This will have dimension $D = 5$. When we start looking at quantum corrections, D will include the anomalous dimensions.

So time translations in the string theory really correspond to scaling in the field theory (dilatations). By exploring the scaling dimensions in the field theory, we can learn about the energies in the string theory.

In $\mathcal{N} = 4$ SYM we have $\mathcal{R}$-symmetry which is $SO(6)$. We have 6 scalar fields, $\phi_1, \ldots, \phi_6$. We build our complex fields by taking $Z = \phi_1 + i\phi_2$, for example. We have conservation of $\mathcal{R}$-charge in the QFT. The space S^5 enjoys an $SO(6)$ isometry given by the group of rotations. We know the conserved quantity associated with rotations is angular momentum. We match these conserved quantities since they both come from the $SO(6)$ symmetry of each theory. Thus, angular momentum in the string theory can be identified with $\mathcal{R}$-charge in the field theory.

In general, we expect to be able to match symmetries (or conserved quantities) in each theory. In quantum theory, operators describe observables. We can study different operators in the field theory and the values of conserved quantities that label the operator, and see how it matches up with the string theory. The AdS/ CFT duality predicts we should be able to match operators in the field theory to states in the string theory [17]. We are especially interested in looking at the dilatation operator as this allows us to learn about the energies of states in the string theory.

0.5 Giant Gravitons in AdS

A graviton is a point particle that would mediate the gravitational force on the quantum scale. The graviton corresponds to operators in the gauge theory obtained by taking the trace of a product of the complex adjoint scalar fields. The gravitons that we get when we take order N fields have the topology of S^3 when we study the $AdS_5 \times S^5$ background and are macroscopic in size[5]. In fact, they have a size of the order of one unit of the spacetime radius. Hence, we call them giant gravitons.

These giant gravitons were discovered in [17] and those findings are reviewed here.

0.5.1 Dipole Analogy

We are given the Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^i \dot{x}_{1i} + \dot{x}_2^j \dot{x}_{2j}) + \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{k}{2}(x_1 - x_2)^2 \tag{0.5.1.1}$$

This describes two point particles in a magnetic field, $\vec{B}$. They are connected by a spring with spring constant, k. They have the same mass and equal but opposite charge which we've scaled to 1 for simplicity. Suppose that the magnetic field points in the $\hat{z}$ direction. We can then consider the motion of the particles in the x-y plane. Classically, we expect the particles to feel a force (Lorentz force) proportional to the magnetic field and their speed (as well as the size of the charge). Let's forget the spring term for the moment and consider the resultant equations of motion. They are:

[5] When we studied the global coordinates for the induced metric on AdS_{p+2} we had a component for the metric on S^p.

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1^k}\right) = \frac{\partial \mathcal{L}}{\partial x_1^k} \\
&\implies \frac{d}{dt}\left(m\dot{x}_{1k} + \frac{B}{2}\epsilon_{kj}x_1^j\right) = \frac{B}{2}\epsilon_{ik}\dot{x}_1^i \\
&\implies m\ddot{x}_{1k} = B\epsilon_{ik}\dot{x_1}^i
\end{aligned} \tag{0.5.1.2}$$

Similarly,

$$m\ddot{x}_{2k} = -B\epsilon_{ik}\dot{x}_2^i \tag{0.5.1.3}$$

This confirms that these charges, when put in a magnetic field and given some momentum, will feel an equal Lorentz force but in opposite directions (positive and negative charges). The Lorentz force is perpendicular to the velocity of the charge and perpendicular to the magnetic field. The more momentum the dipole has, the stronger the Lorentz force that each charge feels. This results in a movement of the two charges away from each other (but perpendicular to the momentum of dipole) so that the dipole is stretched. We know that the particles are coupled by the spring. The spring will have an effect on the ability of the dipole to stretch due to the Lorentz force. The magnitude of the spring constant will constrain how wide the dipole can stretch.

Now we can send the mass to zero and focus on the magnetic field and spring terms in the Lagrangian. We find that the magnitude of the centre of mass momentum, $|P|$ is proportional to the magnitude of the separation, $\Delta = \frac{x_1 - x_2}{2}$ [6]. Allowing the dipole to move along the surface of a sphere of radius, R, and magnetic flux, N, we see that when the dipole is the size of the sphere ($R = \Delta$), the momentum of the dipole will be at $2BR$ and the angular momentum, $L = PR$, will be at a maximum. It is at a maximum because we are assuming a fixed magnetic field and the largest chord we can draw in a sphere is the diameter. Of course, since Gauss' law tells us the magnetic flux through a closed surface is zero, and we clearly have identified the sphere with having a non-zero magnetic flux, then we must conclude there is a magnetic monopole at the centre of the sphere with strength $2\pi N = \Omega_2 BR^2$ [7]. So the maximum angular momentum is of the order N. The fact that we have a magnetic monopole at the centre of the sphere is necessary to ensure we have a constant magnetic field going through the sphere at all points on a given radius.

We can make the same argument much more precisely and the steps that follow are based largely on [17].

Since we are working on a sphere, it is natural to work in spherical coordinates. We are on the surface of a 2-sphere so we need to specify two angles (2 coordinates) to specify our position. We need an azimuthal angle, θ, and we have ϕ measuring angular distance from the equator ($\theta \in [0, 2\pi]$ and $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$). We choose to have only the A_θ component of the vector potential nonzero. We want to couple the velocity of the dipole to the magnetic field. Consider the state that has each charge a pole of the sphere. Then they have the same coordinates but with opposite sign for ϕ. This applies whenever the charges are at antipodal points.

Consider $\vec{B} = B\hat{r}$[8]. We know that

$$\begin{aligned}
&\nabla \times \vec{A} = \vec{B} \\
\implies &\nabla \times (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = B\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \\
&= B\cos\theta\sin\phi\hat{x} + B\sin\theta\sin\phi\hat{y} + B\cos\phi\hat{z}
\end{aligned} \tag{0.5.1.4}$$

We also have that

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \implies \nabla\phi + \vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{0.5.1.5}$$

The electromagnetic portion of the Lagrangian is given by

$$\mathcal{L}_B = -q(\vec{A}\cdot\dot{\vec{x}} - \phi) = -q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z} - \phi) \tag{0.5.1.6}$$

[6] Make a change of coordinates to $X = \frac{x_1 + x_2}{2}$ and Δ as above. We find that $\dot{X}^i\Delta^j = \frac{1}{4}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j)$. Using this, the term in the Lagrangian coupling to the magnetic field becomes $2B\epsilon_{ij}\dot{X}^i\Delta^j$. The canonical momentum is $\frac{\partial \mathcal{L}}{\partial \dot{X}^i}$ which we can see is proportional to Δ.

[7] We want a uniform magnetic field. This is achieved by a monopole located at the centre.

[8] We have a magnetic monopole at the centre of the sphere so the field lines point radially outwards

which comes from the action

$$\begin{aligned}\mathcal{S}_B &= q\int A_\mu dx^\mu = q\int(\phi dt - \vec{A}d\vec{x}) = q\int\left(\phi dt - \vec{A}\frac{d\vec{x}}{dt}dt\right)\\ &= q\int(\phi - \vec{A}\cdot\dot{\vec{x}})dt\end{aligned} \tag{0.5.1.7}$$

Here we have chosen the convention that q describes a negative test charge. This sign choice should be reflected in our resulting equations of motion.

We are considering charges of mass m with a kinetic energy of $\frac{1}{2}m\dot{\vec{x}}^2$ so we need to add another term to the Lagrangian: the kinetic term. The Lagrangian describing the system is

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_T = -q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z} - \phi) + \frac{1}{2}m\dot{\vec{x}}^2 \tag{0.5.1.8}$$

We can check this is indeed the correct Lagrangian by checking if we get the correct equations of motion.

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = -q\dot{A}_x + m\ddot{x} = \frac{\partial\mathcal{L}}{\partial x} = q\left(\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial x}\dot{x} - \frac{\partial A_y}{\partial x}y - \frac{\partial}{\partial x}\dot{A}_z\dot{z}\right) \tag{0.5.1.9}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = -q\dot{A}_y + m\ddot{y} = \frac{\partial\mathcal{L}}{\partial y} = q\left(\frac{\partial\phi}{\partial y} - \frac{\partial A_x}{\partial y}\dot{x} - \frac{\partial A_y}{\partial y}\dot{y} - \frac{\partial A_z}{\partial y}\dot{z}\right) \tag{0.5.1.10}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} = -q\dot{A}_z + m\ddot{z} = \frac{\partial\mathcal{L}}{\partial z} = q\left(\frac{\partial\phi}{\partial z} - \frac{\partial A_x}{\partial z}\dot{x} - \frac{\partial A_y}{\partial z}\dot{y} - \frac{\partial A_z}{\partial z}\dot{z}\right) \tag{0.5.1.11}$$

$$\begin{aligned}&\implies \sum_{i=1}^{3}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} = -q\dot{\vec{A}} + m\ddot{\vec{x}}\\ &= q\nabla\phi - q\dot{x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}\right) - q\dot{y}\left(\frac{\partial A_y}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_y}{\partial z}\right) - q\dot{z}\left(\frac{\partial A_z}{\partial x} + \frac{\partial A_z}{\partial y} + \frac{\partial A_z}{\partial z}\right)\end{aligned} \tag{0.5.1.12}$$

Now we make note of the following:

$$\begin{aligned}\dot{\vec{x}}\times(\nabla\times\vec{A}) &= \nabla(\dot{\vec{x}}\cdot\vec{A}) - \vec{A}(\nabla\dot{\vec{x}})\\ &= \dot{x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}\right) + \dot{y}\left(\frac{\partial A_y}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_y}{\partial z}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} + \frac{\partial A_z}{\partial y} + \frac{\partial A_z}{\partial z}\right)\end{aligned} \tag{0.5.1.13}$$

Using this, (0.5.1.12) and (0.5.1.5) we obtain

$$\begin{aligned}&-q\dot{\vec{A}} + m\ddot{\vec{x}} = q\nabla\phi + q\vec{E} + m\ddot{\vec{x}} = q\nabla\phi - q\nabla(\dot{\vec{x}}\cdot\vec{A})\\ &\implies m\ddot{\vec{x}} = -q(\dot{\vec{x}}\times\nabla\times\vec{A}) - q\vec{E} = -q(\dot{\vec{x}}\times\vec{B}) - q\vec{E}\end{aligned} \tag{0.5.1.14}$$

which are our equations of motion. They describe the motion of a negative test charge in a magnetic field. Note that the Lorentz force is conventionally written in terms of a positive test charge: the result is simply a sign change.

We repeat the argument for spherical coordinates. We have that

$$\left(\frac{1}{r}\frac{\partial A_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}\right)\hat{\theta} + \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\phi} = B\hat{r} \tag{0.5.1.15}$$

Here $\vec{B} = B\hat{r}$ as before. We'll denote the scalar potential as $\tilde{\phi}$ so as not to confuse this with our coordinate ϕ. In this choice of coordinates our position and velocity vectors are

$$\begin{aligned}\vec{r} &= r\hat{r}\\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}\end{aligned}$$

We get the second line as follows. First, we note that the transformation from Cartesian to spherical coordinates is

$$\begin{aligned}x &= r\cos\phi\sin\theta\\ y &= r\sin\phi\cos\theta\\ z &= r\cos\phi\end{aligned}$$

In spherical coordinates, $\vec{r}$ is our position vector; θ and ϕ are angles. Only the coordinate $r = |\vec{r}|$ has dimensions of length.

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = r\cos\phi\sin\theta\hat{x} + r\sin\phi\sin\theta\hat{y} + r\cos\theta\hat{z} \tag{0.5.1.16}$$

Our unit vectors are a little more complicated then before. We look at how a small change in one of our coordinates will affect our position vector, $\vec{r}$, in terms of our Cartesian coordinates and then we divide by that magnitude to get our unit vector. That is

$$\begin{aligned}
\hat{r} &= \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z} \\
\hat{\theta} &= \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|} = \frac{r\cos\phi\cos\theta\hat{x} + r\sin\phi\cos\theta\hat{y} - r\sin\theta\hat{z}}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}} = \cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z} \\
\hat{\phi} &= \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|} = \frac{-r\sin\phi\sin\theta\hat{x} + r\cos\phi\sin\theta\hat{y}}{\sqrt{r^2\sin^2\theta}} = -\sin\phi\hat{x} + \cos\phi\hat{y}
\end{aligned} \tag{0.5.1.17}$$

This notation is useful to calculate the velocity vector, $\dot{\vec{r}}$, because we know how to treat Cartesian unit vectors under operations like differentiation.

$$\begin{aligned}
\dot{\vec{r}} &= (\dot{r}\cos\phi\sin\theta - r\sin\phi\sin\theta\dot{\phi} + r\cos\phi\cos\theta\dot{\theta})\hat{x} + (\dot{r}\sin\phi\sin\theta + r\cos\phi\sin\theta\dot{\phi} + r\sin\phi\cos\theta\dot{\theta})\hat{y} + (\dot{r}\cos\theta - r\sin\theta\dot{\theta})\hat{z} \\
&= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}
\end{aligned} \tag{0.5.1.18}$$

We will make use of

$$\begin{aligned}
\dot{\vec{r}}\times(\nabla\times\vec{A}) = \nabla(\dot{\vec{r}}\cdot\vec{A}) &= \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\left(\dot{r}A_r + r\dot{\theta}A_\theta + r\dot{\phi}\sin\theta A_\phi\right) \\
&= \left(\dot{r}\frac{\partial A_r}{\partial r} + r\frac{\partial A_\theta}{\partial r}\dot{\theta} + A_\theta\dot{\theta} + \sin\theta A_\phi\dot{\phi} + r\sin\theta\frac{\partial A_\phi}{\partial r}\dot{\phi}\right)\hat{r} \\
&+ \left(\frac{\dot{r}}{r}\frac{\partial A_r}{\partial\theta} + \frac{\partial A_\theta}{\partial\theta}\dot{\theta} + \sin\theta\frac{\partial A_\phi}{\partial\theta}\dot{\phi} + \cos\theta A_\phi\dot{\phi}\right)\hat{\theta} \\
&+ \left(\frac{\dot{r}}{r\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}\dot{\theta} + \frac{\partial A_\phi}{\partial\phi}\dot{\phi}\right)\hat{\phi}
\end{aligned} \tag{0.5.1.19}$$

Now equations of motion can be calculated as before:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}}\right) = \frac{\partial\mathcal{L}}{\partial\vec{r}} \tag{0.5.1.20}$$

Our kinetic term in the Lagrangian is $\mathcal{L}_T = \frac{1}{2}m\dot{\vec{r}}^2$.

Using the Euler-Lagrange equation we get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = q\dot{A}_r + m\ddot{r} = \frac{\partial\mathcal{L}}{\partial r} &= q\left(A_\theta\dot{\theta} + \frac{\partial A_\theta}{\partial r}\dot{\theta} + \sin\theta A_\phi\dot{\phi} + r\sin\theta\frac{\partial A_\phi}{\partial r}\dot{\phi} + \frac{\partial A_r}{\partial r}\dot{r} - \frac{\partial\tilde{\phi}}{\partial r}\right) \\
\frac{d}{dt}\left(\frac{1}{r}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = q\dot{A}_\theta = \frac{1}{r}\frac{\partial\mathcal{L}}{\partial\theta} &= q\left(\frac{\partial A_\theta}{\partial\theta}\dot{\theta} + \cos\theta A_\phi\dot{\phi} + \sin\theta\frac{\partial A_\phi}{\partial\theta}\dot{\phi} + \frac{1}{r}\frac{\partial A_r}{\partial\theta}\dot{r} - \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial\theta}\right) \\
\frac{d}{dt}\left(\frac{1}{r\sin\theta}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = q\dot{A}_\phi = \frac{1}{r\sin\theta}\frac{\partial\mathcal{L}}{\partial\phi} &= q\left(\frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}\dot{\theta} + \frac{\partial A_\phi}{\partial\phi}\dot{\phi} + \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}\dot{r} - \frac{1}{r\sin\theta}\frac{\partial\tilde{\phi}}{\partial\phi}\right) \\
&\implies q\dot{\vec{A}} + m\ddot{\vec{r}} = q(\dot{\vec{r}}\times(\nabla\times\vec{A})) - q\nabla\tilde{\phi} = q(\dot{\vec{r}}\times\vec{B}) - q\nabla\tilde{\phi} \\
&\implies q\dot{\vec{r}}\times\vec{B} + q\vec{E} = m\ddot{\vec{r}}
\end{aligned} \tag{0.5.1.21}$$

Here we have used the convention that we are working with a positive test charge, contrary to before, and so our equations of motion describe a positive test charge moving in a magnetic field.

We can now say that, assuming only A_θ is non-zero[9], our Lagrangian is

$$\mathcal{L} = A_\theta r\dot{\theta} = A_\theta R\cos\phi\dot{\theta} \tag{0.5.1.22}$$

[9]We fix our gauge so that A_θ is the only non-zero component of the vector potential. This is possible since our field is uniform on the sphere.

We have dropped the kinetic term since it is small compared to the coupling to the magnetic field i.e.: we can send the mass to zero.

We choose

$$A_\theta = N\frac{1-\sin\phi}{2R\cos\phi} \tag{0.5.1.23}$$

We can check that this is reasonable by making note that $r = R\cos\phi$ (see Figure[2]) and is a maximum, i.e.: $r = R$, when $\phi = 0 + 2k\pi, k \in \mathbb{Z}$. Using this, we can take the curl of our vector potential and check that we recover the expression for the flux, which we do.

The electromagnetic term in the Lagrangian looks like

$$\mathcal{L}_B = A_\theta R\cos\phi\dot{\theta} + A_\theta R\cos(-\phi)\dot{\theta} = -N\sin\phi\dot{\theta} \tag{0.5.1.24}$$

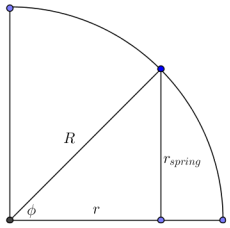


Figure 2: An arc of a great circle cutting the poles of the sphere

The spring coupling term is

$$\mathcal{L}_S = -\frac{k}{2}R^2\sin^2\phi \tag{0.5.1.25}$$

Here we have used the chord length instead of arc length for the purpose of simplification. Derivative of the total Lagrangian then with respect to $\dot{\theta}$ is just $\frac{\partial\mathcal{L}_B}{\partial\dot{\theta}}$ which gives us the angular momentum. This leads us to

$$|L_{max}| = N\sin(\pi/2) = N \tag{0.5.1.26}$$

0.5.2 Embedding in S^4: $AdS_7 \times S^4$

Consider a relativistic spherical membrane[10] moving in S^4. The membrane has no net charge but it does have a dipole moment. We have a background field strength which it couples to. We can parametrise S^4 such that

$$\begin{aligned}
X_1 &= R\cos\theta_1 \\
X_2 &= R\sin\theta_1\cos\theta_2 \\
X_3 &= R\sin\theta_1\sin\theta_2\cos\theta_3 \\
X_4 &= R\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\
X_5 &= R\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4
\end{aligned} \tag{0.5.2.1}$$

Here θ_4 is the azimuthal angle and goes from 0 to 2π while the other angles go from 0 to π. We see that the above formulae parametrise a sphere S^4. That is, $\sum_{i=1}^{5} X_i^2 = R^2$. Now we embed a spherical membrane into this S^4. We can parametrise this membrane using the azimuthal angle, θ_4, and another angle, choose θ_3. By (0.5.2.1) we

[10]This membrane is the giant graviton.

see that this means the membrane (or brane) can only move in the $X_1 - X_2$ plane (the only coordinates left that are not involved in our parametrisation of the sphere). It is natural to define $r = R\sin\theta_1 \sin\theta_2$ which leaves us with the 2-sphere

$$\begin{aligned} X_3 &= r\cos\theta_3 \\ X_4 &= r\sin\theta_3\cos\theta_4 \\ X_5 &= r\sin\theta_3\sin\theta_4 \end{aligned} \tag{0.5.2.2}$$

The size of the brane (with radius r) depends on its location in the $X_1 - X_2$ plane (since r depends on θ_1 and θ_2, as does X_1 and X_2. We also have $X_3^2 + X_4^2 + X_5^2 = r^2 \implies X_1^2 + X_2^2 = R^2 - r^2$. That is, r is radius of the brane whilst $\sqrt{R^2 - r^2}$ is the radius of the circle on which the brane orbits. So circles in this plane describe branes of fixed size. In terms of our embedding coordinates then,

$$\begin{aligned} X_1 &= \sqrt{R^2 - r^2}\cos\phi \\ X_2 &= \sqrt{R^2 - r^2}\sin\phi \end{aligned} \tag{0.5.2.3}$$

The metric on the 4-sphere (embedded into 5 dimensional Euclidean space) is

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2 \tag{0.5.2.4}$$

where (assuming only ϕ is time-dependent)

$$\begin{aligned} dX_1 &= \frac{-r}{\sqrt{R^2 - r^2}}\cos\phi dr - \sqrt{R^2 - r^2}\sin\phi d\phi - \sqrt{R^2 - r^2}\sin\phi\dot{\phi}dt \\ dX_2 &= \frac{-r}{\sqrt{R^2 - r^2}}\sin\phi dr + \sqrt{R^2 - r^2}\cos\phi d\phi + \sqrt{R^2 - r^2}\cos\phi\dot{\phi}dt \\ dX_3 &= \cos\theta_3 dr - r\sin\theta_3 d\theta_3 \\ dX_4 &= \sin\theta_3\cos\theta_4 dr + r\cos\theta_3\cos\theta_4 d\theta_3 - r\sin\theta_3\sin\theta_4 d\theta_4 \\ dX_5 &= \sin\theta_3\sin\theta_4 dr + r\cos\theta_3\sin\theta_4 d\theta_3 + r\sin\theta_3\cos\theta_4 d\theta_4 \end{aligned} \tag{0.5.2.5}$$

This gives us almost all the terms we need to compute the induced metric. That is, we want the metric on the worldvolume from the metric on the spacetime. We compute the elements by

$$g_{ij} = \frac{\partial X^M}{\partial\xi^i}\frac{\partial X^N}{\partial\xi^j}G_{MN} \tag{0.5.2.6}$$

where G_{MN} is the metric on the spacetime and g_{ij} is the metric induced on the worldvolume. We have, already, the transformations for r, θ_3, θ_4 and ϕ.

$$g_{tt} = \left(\frac{\partial X_1}{\partial t}\right)^2 G_{X_1X_1} + \left(\frac{\partial X_2}{\partial t}\right)^2 G_{X_2X_2} + \cdots + \left(\frac{\partial X_5}{\partial t}\right)^2 G_{X_5X_5} + \left(\frac{\partial\tau}{\partial t}\right)^2 G_{\tau\tau} \tag{0.5.2.7}$$

Our brane must be positioned in the full 11 dimensional space. We choose to put it at the origin of AdS_7 which leads to the simplification $G_{\tau\tau} = -1$. Thus $g_{tt} = -1 + (R^2 - r^2)\sin^2\phi\dot{\phi}^2 + (R^2 - r^2)\cos^2\phi\dot{\phi}^2$. Following this process, we find that our metric becomes

$$ds^2 = \frac{R^2}{R^2 - r^2}dr^2 + (R^2 - r^2)d\phi^2 + r^2 d\theta_3^2 + r^2\sin^2\theta_3 d\theta_4^2 + (-1 + (R^2 - r^2)\dot{\phi}^2)dt^2 \tag{0.5.2.8}$$

$$\implies \sqrt{-g} = Rr^2\sin\theta_3\sqrt{1 - (R^2 - r^2)\dot{\phi}^2} \tag{0.5.2.9}$$

We can now compute the kinetic-like term of the action, which is given by the Dirac-Born-Infeld action.

$$\begin{aligned} S_{DBI} &= -T\int r^2\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}dtd\theta_3 d\theta_4 \\ &= -T\Omega_2\int r^2\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}dt \end{aligned} \tag{0.5.2.10}$$

where T is the tension of the membrane and is given by $T = \frac{1}{4\pi^2 l_p^3}$[11].

The term that couples us to the background field is called the Chern-Simons coupling. Each orbit, the brane

[11] It looks like we have dropped a factor of R but this follows from the definition of our tension to give us a dimensionally correct action

sweeps out a 3-dimensional surface in S^4 (the worldvolume of the brane after 1 orbit). This surface forms a boundary of a 4-manifold, call it Σ. If we integrate the flux over this surface, then we must obtain the contribution of the 4-form field strength to the action of the brane per orbit. The background flux, $F \equiv dC$, is the constant flux density, B, multiplied by the infinitesimal volume element on S^4. This gives us

$$S_B = \oint_{\text{wv}} C = \int_\Sigma F = B\text{vol}(\Sigma) \tag{0.5.2.11}$$

This volume is given by

$$\text{vol}(\Sigma) = R\Omega_2 \int_0^{2\pi} d\phi \int_0^r r'^2 dr' = \frac{8\pi^2}{3} R r^3 \tag{0.5.2.12}$$

A short proof for the four dimensional Stoke's theorem can be found in Appendix A.

We make the ansatz that $\phi = \omega_0 t$ so that $\omega_0 = \dot{\phi}$. If T is the period of the orbit of the brane, then $\omega_0 T = 2\pi$.

$$\mathcal{L}_B = \frac{\dot{\phi}}{2\pi} B\Omega_4 R r^3 \tag{0.5.2.13}$$

We can see this is true by checking if we get the correct action back.

$$\begin{aligned}
\mathcal{S}_B &= B\frac{8\pi^2}{3} R r^3 = B\Omega_4 R r^3 \\
&= \int_{t_{\text{initial}}}^{t_{\text{final}}} dt \mathcal{L} \\
&= \int_{t_{\text{initial}}}^{t_{\text{final}}} \frac{\dot{\phi}}{2\pi} B\Omega_4 R r^3 = \int_0^T \frac{d(\omega_0 t)}{dt} \frac{B\Omega_4 R r^2}{2\pi} dt \\
&= \frac{\omega_0 T}{2\pi} \mathcal{S}_B = \mathcal{S}_B
\end{aligned} \tag{0.5.2.14}$$

Here we have integrated over one period of the orbit of the brane. We can do this because we calculated the Chern-Simons term in the action by using Stoke's theorem. Stoke's theorem required us to integrate over a closed path on the boundary of some surface. In our case, this closed path was precisely the orbit of the brane.

Quantisation of the flux requires that

$$\Omega_4 B R^4 = 2\pi N \implies B = \frac{2\pi N}{R^4 \Omega_4} \implies \mathcal{L}_B = \dot{\phi} N \frac{r^3}{R^3} \tag{0.5.2.15}$$

Defining $m = \Omega_2 T r^2$ we have the full Lagrangian

$$\mathcal{L} = -m\sqrt{1 - \dot{\phi}^2(R^2 - r^2)} + N\frac{r^3}{R^3}\dot{\phi} \tag{0.5.2.16}$$

The angular momentum is

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{m\dot{\phi}(R^2 - r^2)}{\sqrt{1 - \dot{\phi}^2(R^2 - r^2)}} + mr \tag{0.5.2.17}$$

where the definition of the membrane tension and $R = l_p(\pi N)^{\frac{1}{3}}$ give $\frac{N}{R^3} = T\Omega_2$. This function is strictly increasing for $r > 0$ so it is clear that the angular momentum has a minimum at $r = 0$ (the smallest the membrane can be) and a maximum at $r = R$ (the maximum size the membrane can be). This gives us that $L_{max} = N$, just like in the dipole case. This fact, and the behaviour of $L(r)$ in general, implies that the angular momentum of the membrane is greater at larger r or, rather, that the membrane size increases with increasing angular momentum. The membrane must fit in S^4 and so it cannot be bigger than R. When $r > R$ the first term, which looks like $\frac{mv\Delta R}{\sqrt{1-v^2}}$ where $v = \dot{\phi}R$ is the velocity of the centre of mass of the brane, suggests the centre of mass is going faster than the speed of light because the denominator now looks like $\sqrt{1+v^2}$. Our relativistic particle can do no such thing, which enforces the upper bound on r.

This cut-off on the angular momentum is the stringy exclusion principle. Some states that we would expect are missing due to this upper bound placed on the momentum. Since there is a largest momentum, there is also a smallest distance such that we can no longer resolve points on the sphere.

For the energy, we get

$$E = \dot{\phi}L - \mathcal{L} = \sqrt{\left(\frac{Nr^2}{R^3}\right)^2 + \frac{(L - Nr^3/R^3)^2}{R^2 - r^2}} \tag{0.5.2.18}$$

We can examine what happens to the energy as $r \to R$ and at $L = N$. The second term inside the square root is an indeterminate form in this limit but we see the r^3 term in the numerator increases to R more rapidly then the quadratic r^2 in the denominator. So the numerator tends to zero faster than the denominator and that whole term is zero in the above limit. This leaves us with

$$E|_{r=R} = \frac{N}{R} \tag{0.5.2.19}$$

This is in agreement with the energy of a Kaluza-Klein graviton with angular momentum L. We also note that the Kaluza-Klein graviton has a maximum angular momentum in accordance with the stringy exclusion principle.[12]

The extension to $AdS_5 \times S^5$ follows quite similarly and the angular momentum in this case is given by

$$L = \frac{m\dot{\phi}(R^2 - r^2)}{\sqrt{1 - \dot{\phi}^2(R^2 - r^2)}} + N\frac{r^4}{R^4} \tag{0.5.2.20}$$

where $m = T\Omega_3 r^3$. The behaviour of this function is in complete agreement with the previous case, with angular momentum increasing the size of our graviton to the maximum value R. This extension is important to us since we plan to explore the physics of $AdS_5 \times S^5$, where Maldacena conjectured the duality between string theory and $\mathcal{N} = 4$ SYM on the boundary of this space.

0.6 Matrix Models

Matrix models in 0 dimensions are a good toy model for non-Abelian gauge theories like $\mathcal{N} = 4$ SYM. They allow us to learn enough about the theory so that we can start to look at more realistic problems. The quantum fields of $\mathcal{N} = 4$ SYM are $N \times N$ matrices. By taking the trace of products of these fields we get observables. In the planar limit, where the number of fields are fixed or grow at most like $\sqrt{N}$ and $N \to \infty$, the theory simplifies. 't Hooft [18] proposed that gauge theories in the large N limit are equivalent to string theories. By studying matrix models at large N, we learn something about the dynamics of strings and gravitons. We can explore matrix models in detail to get a feel for the machinery used to describe the planar limit. Later, we will adopt a new method which will allow us to explore physics outside of the planar limit.

0.6.1 Ribbon Graphs

We will be using the path integral formalism. The simplification of working in 0 dimensions is that the universe has only a single event and so our field assigns a single value to this one event. The generating functional of a correlation function for a scalar field theory in 0 dimensions looks like

$$Z[j] = \mathcal{N}\int_{-\infty}^{\infty} d\phi\, e^{-\mathcal{S}+j\phi} \tag{0.6.1.1}$$

where $\mathcal{N}$ is the normalisation and $\mathcal{S}$ is the action for the theory. The reason for introducing this generating functional is that it allows us to calculate correlation functions by taking derivatives of $Z[j]$. The correlation function looks like

$$\langle \phi^n \rangle = \int_{-\infty}^{\infty} d\phi\, e^{-\mathcal{S}}\phi^n \tag{0.6.1.2}$$

In a $3+1$ dimensional QFT, our generating functional looks like

$$Z[J] = \int [D\phi]\, e^{i\mathcal{S}+\int d^4x\, \phi(x)J(x)} \tag{0.6.1.3}$$

[12] We have compactified onto the sphere, reducing the number of dimensions we work with. Our single field in the full spacetime has reduced to many fields in the reduced spacetime, without losing dependence on the quantum numbers of the full space. This is Kaluza-Klein reduction.

and the correlation functions are

$$\begin{aligned}\langle\phi(x_1)\dots\phi(x_n)\rangle &= \int[D\phi]\ \phi(x_1)\dots\phi(x_n)e^{i\mathcal{S}} \\ &= \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0}\end{aligned} \tag{0.6.1.4}$$

We can analytically continue $t \to it$ taking us to Euclidean space. This results in $i\mathcal{S} \to -\mathcal{S}$ so that the path integral measure starts to resemble the 0 dimensional calculation. We will also replace our scalar fields with matrix valued fields so that we study fields, M, living in the Lie algebra of $U(N)$. Finally, moving to 0 dimensions gives us

$$Z[J] = \int[dM]\ e^{-\frac{1}{2}Tr(M^2)+Tr(JM)} \tag{0.6.1.5}$$

Let's study M in more detail. It lives in the Lie algebra of $U(N)$ so it is an $N \times N$ hermitian matrix. This tells us the diagonal elements, M_{ii}, are real. The elements in the upper triangle are the complex conjugate of the elements in the lower triangle: $M_{kl} = M^*_{lk};\ \ k < l$. Then the measure, $[dM]$, requires us to integrate over the N diagonal elements, plus the $\frac{1}{2}N(N-1)$ real parts of the elements above the diagonal (M^r_{kl}) plus the $\frac{1}{2}N(N-1)$ imaginary parts of the elements above the diagonal (M^i_{kl}). This is a total of N^2 real integrals. Our correlation function looks like

$$\begin{aligned}\langle\dots\rangle &\equiv \mathcal{N}\int[dM]\ e^{-\frac{\omega}{2}Tr(M^2)}\dots \\ &= \mathcal{N}\prod_{i=1}^{N}\int_{-\infty}^{\infty}dM_{ii}\prod_{\substack{k,l=1\\k>l}}^{N}\int_{-\infty}^{\infty}dM^r_{kl}\int_{-\infty}^{\infty}dM^i_{kl}\dots\end{aligned} \tag{0.6.1.6}$$

We normalise the expectation value of 1 to be 1 so that

$$\mathcal{N}\int[dM]\ e^{-\frac{1}{2}\omega Tr(M^2)} = 1 \tag{0.6.1.7}$$

For arbitrary N,

$$\mathcal{N} = \left(\frac{1}{\sqrt{2}}\right)^N\left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2}$$

Now we are in a position to check whether we can use our generating functional to determine correlators. Consider

$$\langle M_{ij}M_{kl}\rangle = \int[dM]\ e^{-\frac{\omega}{2}}M_{ij}M_{kl} \tag{0.6.1.8}$$

Taking derivatives of Z with respect to J should replace the $M_{ij}M_{kl}$ in our correlator.

$$\begin{aligned}\frac{d}{dJ_{ji}}e^{Tr(JM)} &= e^{Tr(JM)}M_{kl}\delta_{jl}\delta_{ik} = e^{Tr(JM)}M_{ij} \\ \implies \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z[J]\Big|_{J=0} &= \langle M_{ij}M_{kl}\rangle\end{aligned} \tag{0.6.1.9}$$

Now that we know we can use our Z to compute correlation functions, we can turn to evaluating $Z[J]$. Consider

$$\begin{aligned}-\frac{\omega}{2}Tr(M^2)+Tr(JM) &= -\frac{\omega}{2}Tr\left(M^2-2\frac{JM}{\omega}\right) \\ &= -\frac{\omega}{2}Tr\left[\left(M-\frac{J}{\omega}\right)^2-\left(\frac{J}{\omega}\right)^2\right] \\ &= -\frac{\omega}{2}Tr\left(M-\frac{J}{\omega}\right)^2+\frac{1}{2\omega}Tr(J^2)\end{aligned} \tag{0.6.1.10}$$

where we have completed the square in the second line. We can change variables now so that $M' = M - J$ and use the fact that J is not dependent on M (we fix J to be some matrix with elements of fixed value as we integrate over M). Then $[dM] = [dM']$. This leaves us with

$$Z[J] = e^{\frac{1}{2}Tr(J^2)} \tag{0.6.1.11}$$

Now let's calculate the correlator (0.6.1.8).

$$\begin{aligned}\langle M_{ij}M_{kl}\rangle &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{lk}}Z[J]\Big|_{J=0}\\ &= \frac{1}{2\omega}\frac{d}{dJ_{ij}}(2J_{lk})Z[J]\Big|_{J=0}\\ &= \frac{1}{\omega}\delta_{il}\delta_{jk}\end{aligned} \tag{0.6.1.12}$$

These calculations can get cumbersome when we want to consider more complicated correlators. Feynman diagrams provide a convenient language for discussing perturbative quantum field theories. This language is common when talking about scalar fields. It turns out there is a Feynman diagram language for matrix model theories and we can draw diagrams for these correlators. In this case, they are called ribbon graphs. Our rules for drawing Feynman diagrams in this theory are first stated and then detailed:

Each ribbon comes with a factor $\frac{1}{\omega}$.
Each edge has a Kronecker delta.

Ribbons are lines that join indices. Ribbon lines of a single ribbon do not cross. We join indices that are the same. Edges refer to the edge of the ribbon. We count edges going from point to point. In this case, points correspond to the indices of our Kronecker deltas.

We can try reproduce our result for $\langle M_{ij}M_{kl}\rangle$ using the ribbon graph method. We start by drawing a horizontal line. On the line we have four points, one for each index. We label them as they appear in our correlator (i then j then k then l). We start at the leftmost point, i. We cannot join it to j as ij appear as a pair in the correlation function. If tried to join them, then we would not be able to make a ribbon as a ribbon is made up of two lines. We can only join i to l and j to k if we want uncrossed ribbon lines. Our graph looks like

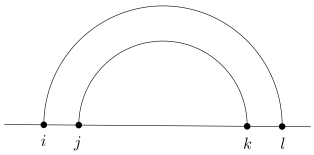


Figure 3: The ribbon graph for $\langle M_{ij}M_{kl}\rangle$

There is only one ribbon, so we acquire a factor $\frac{1}{\omega}$. The outer edge joins i and l so we acquire δ_{il} and a δ_{jk} for the inner edge. Thus

$$\langle M_{ij}M_{kl}\rangle = \frac{1}{\omega}\delta_{il}\delta_{jk}$$

as before.

The physical observables in a non-Abelian gauge theory are invariant under the local gauge symmetry. In our 0 dimensional toy model, we call

$$M \to UMU^\dagger \tag{0.6.1.13}$$

a gauge symmetry. If this transformation leaves our theory invariant, then $M' \equiv UMU^\dagger$ and M generate equivalent $U(N)$ representations and share the same eigenvalues (see Appendices B and C for more on representations). If M is diagonalisable, then $\mathrm{Tr}(M^N)$ can be written as the sum of λ_i^N where λ_i are the eigenvalues of M. Our gauge invariant observables are traces of products of M.

For example, consider

$$\langle \mathrm{Tr}(M^2)\rangle = \langle M_{ij}M_{ji}\rangle = \frac{1}{\omega}\delta_{ii}\delta_{jj} = \frac{N^2}{\omega} \tag{0.6.1.14}$$

Our indices run from $1 \ldots N$ hence the N^2. Notice that the answer for our correlator is just a polynomial. This suggests that we can modify our ribbon graph notation to exclude indices. Here are our new rules with this insight:

Link points that are labelled by the same index.
Replace index pairs by a solid line, indicating which indices are summed.
Acquire an N for each closed loop.
Acquire a $\frac{1}{\omega}$ for each ribbon.

For $\langle M_{ij}M_{ji}\rangle$, we start by drawing four points. Our convention will be to connect points that have the same indices with lines on the bottom of our graph. We must connect the outer points and the inner points. This replaces labelling the points with indices which will be cumbersome for larger calculations. We join lines at the top as usual: lines of the same ribbon mustn't cross, otherwise we join however we can until all ribbons in all possible combinations are drawn.

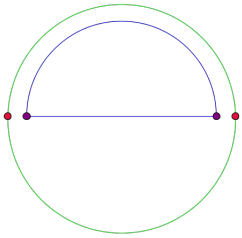


Figure 4: The ribbon graph for $\langle \mathrm{Tr}(M^2)\rangle$

The green line in Figure 4 shows one closed loop and the blue line shows another closed loop. These give us a factor N^2. The points that are coloured in the same are just to reinforce that they represent repeated indices. At the top we have a ribbon, so we gain $\frac{1}{\omega}$.

Consider $\langle \mathrm{Tr}(M^4)\rangle$. Now we will have four pairs of points that need joining. We have only one way to indicate summed indices but we have multiple ways of connecting ribbons. For example, our first ribbon (on the leftmost side) could be connected to the next pair of dots, or the third pair or the fourth. Once that is connected, the second ribbon can only connect with the remaining unconnected point pair. So we can connect in $3 \times 1 = 3!!$ ways. In general, for $\langle \mathrm{Tr}(M^{2n})\rangle$ we will have $(2n-1)!!$ diagrams.

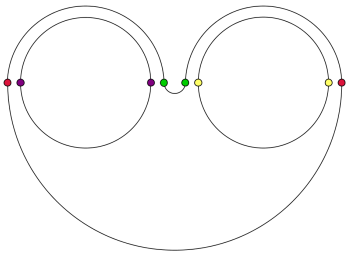


Figure 5: A ribbon graph for $\langle \mathrm{Tr}(M^4)\rangle$

Here are some calculated examples:

$$\begin{aligned}
\langle \mathrm{Tr}(M^2)\rangle &= \frac{N^2}{\omega} \\
\langle \mathrm{Tr}(M^4)\rangle &= \frac{1}{\omega^2}(2N^3+N) \\
\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2)\rangle &= \frac{1}{\omega^2}(N^4+2N^2) \\
\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle &= \frac{1}{\omega^3}(2N^5+9N^3+4N)
\end{aligned} \tag{0.6.1.15}$$

We see something interesting in the limit where we take $N \to \infty$.

$$\begin{aligned}
\langle \mathrm{Tr}(M^2)\rangle &= \frac{N^2}{\omega} \\
\langle \mathrm{Tr}(M^4)\rangle &= \frac{2N^3}{\omega^2}\left(1+\frac{1}{2N^2}\right) \to \frac{2N^3}{\omega^2} \\
\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2)\rangle &= \frac{N^4}{\omega^2}\left(1+\frac{2}{N^2}\right) \to \frac{N^4}{\omega^2} \\
\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle &= \frac{2N^5}{\omega^3}\left(1+\frac{9}{2N^2}+\frac{4}{N^4}\right) \to \frac{2N^5}{\omega^3}
\end{aligned} \tag{0.6.1.16}$$

Remarkably, in this limit, the expectation value of products of our gauge invariant observables are products of their expectation values. That is

$$\begin{aligned}
\langle \mathrm{Tr}(M^2)\rangle\langle \mathrm{Tr}(M^2)\rangle &= \langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2)\rangle \\
\langle \mathrm{Tr}(M^2)\rangle\langle \mathrm{Tr}(M^4)\rangle &= \langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle
\end{aligned} \tag{0.6.1.17}$$

This is called factorisation. The fact that it holds tells us that taking $N \to \infty$, which was 't Hooft's idea [18], is taking a classical limit of the theory. See Section 0.7.1 for a more detailed explanation of factorisation and its implications for our theory.

Up till this point we have only considered the free field theory. We will now consider adding an interaction term. The concepts are the same as the scalar field theory (which we are more familiar with), and we add a coupling term of strength g to the exponential. Now correlators look like

$$\langle \dots \rangle = \mathcal{N}\int [dM] e^{-\frac{\omega}{2}\mathrm{Tr}(M^2)-g\mathrm{Tr}(M^4)} \dots \tag{0.6.1.18}$$

We can complete the square in the exponent, as before, to obtain a Gaussian integral which we know how to do. We keep the same normalisation as before. The difference now comes in with the interaction term which we expand using a Taylor series. Our generating functional is

$$\begin{aligned}
Z[J] &= \sum_{q=0}^{\infty}\left(-g\frac{d}{dJ_{ba}}\frac{d}{dJ_{cb}}\frac{d}{dJ_{dc}}\frac{d}{dJ_{ad}}\right)^q \frac{\mathcal{N}}{q!}\int [dM] e^{\frac{\omega}{2}\mathrm{Tr}(M^2)+\frac{1}{2\omega}\mathrm{Tr}(J^2)} \\
&= \sum_{q=0}^{\infty}\frac{1}{q!}\left(-g\frac{d}{dJ_{ba}}\frac{d}{dJ_{cb}}\frac{d}{dJ_{dc}}\frac{d}{dJ_{ad}}\right)^q e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}
\end{aligned} \tag{0.6.1.19}$$

where we have used the earlier result that

$$\frac{d}{dJ_{ij}} \leftrightarrow M_{ji}$$

We can see that we will have many more terms than we did for the free theory. In fact, due to the interaction vertex, we have 15 diagrams for $\langle \mathrm{Tr}(M^2)\rangle$ whereas we only had 1 diagram for the same correlator in the free theory. In the scalar field theory, we know that adding loops comes with factors of $\hbar$. The more loops we have, the further we move away from the classical limit. As we add interaction vertices, i.e.: increase the allowed orders of g, we form an increasing number of loops. Usually it is sufficient to say the coupling is weak and neglect these higher order terms. We will see why this is not the case here with the following example.

Consider $\langle \mathrm{Tr}(M^2)\rangle$. The method outlined above gives us

$$\langle \mathrm{Tr}(M^2)\rangle = \left[1-g\left(\frac{2N^3}{\omega^2}+\frac{N}{\omega^2}\right)\right]\frac{N^2}{\omega} - g\frac{8N^3}{\omega^3} - g\frac{4N}{\omega^3} \tag{0.6.1.20}$$

The vacuum diagrams are the diagrams which do not connect the pair of dots representing our matrix elements with the interaction vertex and are written in parenthesis above. Renormalising our partition function for the interacting theory will naturally remove these vacuum graphs.

Before we had normalised $Z'[J=0]=1$. We will now use $Z[J] = (Z'[J=0])^{-1}Z'[J]$. So if we want to work out correlators in the interacting theory that are normalised to exclude vacuum graphs, then we work out the non-normalised correlators and divide by the partition function of the non-normalised theory at $J=0$. For

example

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}}Z(J)\Big|_{J=0} \\ &= \frac{1}{Z'(J=0)}\frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}}Z'(J)\Big|_{J=0} \\ &= \frac{1}{\omega^3}\frac{N^2[\omega^2 - g(2N^3+N)] - 8gN^3 - 4gN}{\frac{\omega^2-g(2N^3+N)}{\omega^2}}\end{aligned} \tag{0.6.1.21}$$

Here we have only gone up to first order in g. To be more explicit for the denominator, recall that

$$Z'[J] = \sum_{q=0}^{\infty}\frac{1}{q!}\left(-g\frac{d}{dJ_{ba}}\frac{d}{dJ_{cb}}\frac{d}{dJ_{dc}}\frac{d}{dJ_{ad}}\right)^q \mathcal{N}\int [dM]e^{\frac{\omega}{2}\mathrm{Tr}(M^2)} \tag{0.6.1.22}$$

When $q=0$ the only term we have is

$$\mathcal{N}\int [dM]e^{\frac{\omega}{2}\mathrm{Tr}(M^2)} = 1 \tag{0.6.1.23}$$

from our original definition of $\mathcal{N}$. When $q=1$ we have the above term plus

$$-g\langle M_{ab}M_{bc}M_{cd}M_{da}\rangle = -g\langle \mathrm{Tr}(M^4)\rangle = -g(2N^3+N)\frac{1}{\omega^2} \tag{0.6.1.24}$$

Thus, our normalised correlator, (0.6.1.21), is

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle &= \frac{1}{\omega^3}\left\{N^2[\omega^2 - g(2N^3+N)] - 8gN^3 - 4gN\right\}\left\{1 + \frac{g(2N^3+N)}{\omega^2} - \mathcal{O}(g^2)\right\} \\ &= \frac{1}{\omega^3}\left\{[\omega^2 + g(2N^3+N) - g(2N^3+N)] - 8gN^3 - 4gN\right\} + \mathcal{O}(g^2) \\ &= \frac{N^2}{\omega} - \frac{g}{\omega^3}(8N^3+4N)\end{aligned} \tag{0.6.1.25}$$

For the free theory, we saw that we could take $N\to\infty$ and treat $\frac{1}{N}$ as a small parameter to get the classical limit: factorisation held. In scalar field theories, we usually consider the coupling to be weak and this is why higher order terms are less relevant. In light of this idea, we choose to assume g is small and take $N\to\infty$. To this end, we consider $\langle \mathrm{Tr}(M^2)\rangle$ again, but this time up to order g^2.

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle &= \frac{N^2}{\omega} - \frac{g}{\omega^3}(8N^3+4N) + \frac{g^2}{\omega^5}(144N^4+224N^2) \\ &\approx \frac{N^2}{\omega} - \frac{gN^3}{\omega^3}\left(8+\frac{4}{N^2}\right) + \frac{g^2N^4}{\omega^5}\left(144+\frac{224}{N^2}\right)\end{aligned} \tag{0.6.1.26}$$

There does not appear to be a good and consistent way of taking $N\to\infty$ since the term with the highest power in N is a term with the highest power in g. Even if we consider the coupling to be weak, we can't seem to make sense of the large powers of N. We don't have a good concept of what it means to be weakly coupled here. That is, it is not a well defined question to ask whether $\frac{1}{g}$ or N dominates.

We modify the way we scale by introducing a double scaling limit. We take $N\to\infty$ and $g\to 0$. We know that "$0\cdot\infty$" is an indeterminate form so we define our scaling to go such that $\lambda = gN$ is fixed. Further, we choose λ to be small. Now we have

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle &= \frac{N^2}{\omega} - \frac{N^2}{\omega^3}\left(8\lambda + \frac{4\lambda}{N^2}\right) + \frac{N^2}{\omega^5}\left(144\lambda^2 + \frac{224\lambda^2}{N^2}\right) \\ &\approx \frac{N^2}{\omega} - \frac{8\lambda N^2}{\omega^3} + \frac{144\lambda^2N^2}{\omega^5}\end{aligned} \tag{0.6.1.27}$$

In the first line we can see that the leading term is of order N^2 and that holds for each term in the perturbation series (for each power of λ).

λ is called the 't Hooft coupling. We can do an expansion in this coupling since each term looks like some function of λ. In general, for an observable O,

$$\langle O\rangle = \sum_{n=0}^{\infty} f_n(\lambda)N^{2-2n} \tag{0.6.1.28}$$

We can see from the correlator we calculated that $f_n(\lambda)$ is not a trivial function, especially at higher orders of λ. This choice of coupling has not reduced the complexity of our calculation. We also have preserved the structure of terms so, like the free theory, we have terms that go like $\frac{1}{N^2}$ that will fall away at large N. This is interesting because it introduces $\frac{1}{N^2}$ as a new source of uncertainty. See Section 0.7.1 for more detail.

0.6.2 The N Dependence of a Diagram

There is a neat way to determine the N dependence of each diagram. We start by rescaling our matrix fields and determining the rescaled partition function.

$$\begin{aligned} & M = \sqrt{N}M' \\ & \implies \frac{\omega}{2}\mathrm{Tr}(M^2) = \frac{N\omega}{2}\mathrm{Tr}(M'^2) \\ & \implies g\mathrm{Tr}(M^4) = gN^2\mathrm{Tr}(M'^4) = N\lambda\mathrm{Tr}(M'^4) && \because \lambda = gN \\ & \implies \int [dM']e^{-\frac{N\omega}{2}\mathrm{Tr}(M'^2)} \sum_{q=0}^{\infty} \left(-N\lambda\mathrm{Tr}(M'^4)\right)^q \frac{1}{q!} = Z[0] && (0.6.2.1) \end{aligned}$$

It looks like we have made the transformation $\omega \to N\omega$. Correlators transform as

$$\langle M_{ij}M_{kl}\rangle = \frac{\delta_{ij}\delta_{kl}}{\omega} \to \frac{\delta_{ij}\delta_{kl}}{N\omega}$$

Usually we would assign a factor $-g$ for each vertex. Now we assign a $-\lambda N$ so that we incorporate the 't Hooft coupling. We can expand the sum in the partition function, $Z[0]$, to get

$$Z[0] = \int [dM']e^{-\frac{N\omega}{2}\mathrm{Tr}(M'^2)} \left(1 - \lambda N\mathrm{Tr}(M'^4) + \frac{\lambda^2 N^2}{2}(\mathrm{Tr}(M'^4))^2 + \mathcal{O}(\lambda^3)\right) \quad (0.6.2.2)$$

The first order in λ gives us three diagrams. Two of them have three closed loops, one vertex and two ribbons. So these two diagrams each contribute a factor

$$\left(\frac{1}{N\omega}\right)^2 (-\lambda N)N^3$$

according to our Feynman rules. We introduce some new terminology here. Each loop encloses/ borders a surface. We call these surfaces faces. We count ribbons by starting and ending on a vertex (we know how to count ribbons in the free theory already). We call ribbons the edges of the faces. Vertices do not get a name change. In the new terminology we get an N for each face, a $\frac{1}{N\omega}$ for each edge and a $-\lambda N$ for each vertex. In general, a diagram will contribute a factor

$$\left(\frac{1}{N\omega}\right)^E (-\lambda N)^V (N)^F \quad (0.6.2.3)$$

to the sum. We are now in a position to state the N dependence of a graph with E edges, F faces and V vertices. It is N^{F+V-E}. The quantity $F + V - E$ is a topological invariant called the Euler characteristic. To better understand this topological interpretation, see Section 0.7.2.

0.6.3 Complex Matrix Model

Up to now we have studied matrix models in 0 dimensions where our operators were Hermitian. This was a toy model for a non-Abelian gauge theory like $\mathcal{N} = 4$ SYM. Whilst this taught us most of the intuition we need, we need to modify the model in order to account for the fact that operators in $\mathcal{N} = 4$ SYM are built from complex fields. In particular, the half BPS operators[13] are built from a single complex matrix. In this discussion, we will call this matrix Z. The methodology is very similar to what we have had before and so this discussion is less

[13]To be half BPS means the operators are invariant under half the supersymmetries of the theory. In this theory, the four types of supercharges (hence $\mathcal{N} = 4$) generate the superymmetry. Half BPS means that half of these supercharges commute with the fields. A consequence of this is that two and three point correlation functions are given exactly by their free field limits.

detailed.

Consider the Hermitian matrices M_1 and M_2. Define

$$Z = \frac{M_1 + iM_2}{\sqrt{2}} \qquad\qquad Z^\dagger = \frac{M_1 - iM_2}{\sqrt{2}} \tag{0.6.3.1}$$

Our new correlation function is

$$\begin{aligned}\langle \dots \rangle &= \int [dZdZ^\dagger]\, e^{-\omega \mathrm{Tr}(ZZ^\dagger)} \dots \\ &= \int [dM_1][dM_2]\, e^{-\frac{\omega}{2}[\mathrm{Tr}(M_1^2)+\mathrm{Tr}(M_2^2)]} \dots \end{aligned} \tag{0.6.3.2}$$

where $\langle 1 \rangle = 1$. We introduce the generating function

$$Z[J_1, J_2] = \int [dM_1][dM_2]\, e^{-\frac{\omega}{2}[\mathrm{Tr}(M_1^2)+\mathrm{Tr}(M_2^2)]+\mathrm{Tr}(J_1 M_1)+\mathrm{Tr}(J_2 M_2)} \tag{0.6.3.3}$$

We know from previous sections that

$$\langle M_{ij} M_{kl} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} Z[J] \Big|_{J=0}$$

Like before, we complete the square in the numerator of our partition function but for both M_1 and M_2 separately. We then shift the measure of our integral and use our normalisation condition to obtain

$$Z[J_1, J_2] = e^{\frac{1}{2\omega}[\mathrm{Tr}(J_1^2)+\mathrm{Tr}(J_2^2)]} \tag{0.6.3.4}$$

We are now in a position to calculate the two point function. Using $Z_{ij} = (M_1)_{ij} + i(M_2)_{ij}$ and $Z^\dagger_{kl} = (M_1)_{kl} - i(M_2)_{kl}$ we get the following results:

$$\begin{aligned}\langle Z_{ij} Z^\dagger_{kl} \rangle &= \frac{1}{\omega} \delta_{ij} \delta_{kl} \\ \langle Z_{ij} Z_{kl} \rangle &= 0 = \langle Z^\dagger_{ij} Z^\dagger_{kl} \rangle \end{aligned} \tag{0.6.3.5}$$

This tells us that we must add a new rule when drawing our Feynman diagrams. We must differentiate between which pairs of dots represent the indices of Z and which pair of dots represent the indices of $Z^\dagger$. Ribbons that start on dots for the Zs must end on dots for the $Z^\dagger$s. This reduces the allowed combinations we can have and so the number of graphs we have for this model is less than when we had in a Hermitian matrix model. Recall that physical observables are given by expectation values of traces of operators. In general,

$$\langle \mathrm{Tr}(Z^J) \mathrm{Tr}(Z^{\dagger J}) \rangle = \text{a sum of } J! \text{ graphs}$$

The leading term plus first two correction terms for the above correlator is

$$\langle \mathrm{Tr}(Z^J) \mathrm{Tr}(Z^{\dagger J}) \rangle = JN^J + \frac{J^4 N^{J-2}}{6} + \frac{J^5 N^{J-2}}{24} \tag{0.6.3.6}$$

This is the genus 0 (sphere) and genus 1 (torus) contribution to the sum. If we hold the scale dimension of our operators fixed at $\mathcal{O}(1)$, then we can neglect the higher genus contributions as we take $N \to \infty$. If the number of fields in each trace grows as we take $N \to \infty$ then we can no longer ignore terms corresponding to the higher genus ribbon graphs. We make this transition precisely when we let J grow like $\sqrt{N}$, i.e.: this is the transition point between the planar and non-planar limit of the theory. We can see this by looking at (0.6.3.6) and noticing that when $J = \sqrt{N}$, the genus 1 and genus 0 contributions are of the same order in N.

Ultimately, we want to study the non-planar limit of the theory as this is where new physics is emergent. We learn from the above discussion that the non-planar limit is a much more difficult problem to study than the planar limit. In fact, when we study the non-planar limit, different trace structures mix. To see this, first recall that in section 0.4 we discussed how the operators $Z, Z^\dagger$ scale (they have scale dimension 1). Now we can define the normalised operator as follows:

$$O_n \equiv \frac{1}{\sqrt{nN^n}} \mathrm{Tr}(Z^n) \tag{0.6.3.7}$$

Correlators then look like

$$\langle O_n O^\dagger_m \rangle = \delta_{nm} \left(1 + \mathcal{O}\left(\frac{1}{N^2} \right) \right) \tag{0.6.3.8}$$

A multitrace correlator would look as follows in this notation:

$$\langle O_n O_m O^\dagger_{n+m}\rangle = \frac{\sqrt{nm(n+m)}}{N}\left(1+O\left(\frac{1}{N^2}\right)\right) \tag{0.6.3.9}$$

If the dimensions of our operators scale like $\mathcal{O}(1)$, then the above correlator goes to zero as $N\to\infty$. This tells us that different multitrace structures do not mix in the planar limit and so we only have to study single trace operators. When our operators have scale dimension of $\mathcal{O}(N)$ then the above correlator cannot be neglected so we must consider all possible multitrace structures.

This is not an easy task to study. In Section 0.8 and Appendix D we show how an alternative method to summing ribbon graphs can simplify our task. This method involves finding a basis of operators that diagonalise the two point function, the Schur polynomial basis, and we see that this is equivalent to finding a set of projectors on the vector space $V_N^{\otimes n}$ (where $Z^{\otimes n}$ lives). This allows us to replace a question in quantum field theory by one in group theory. These same tools are applied to solve the problem that this dissertation is based on.

0.7 Large N and the Duality

0.7.1 Factorisation

Consider some quantum system. This system may be in a number of different states, labelled by i. Suppose we want to perform a measurement. The value of an observable O_I in the state i is $O_I(i)$. The probability to be in this state is μ_i with $\sum_i \mu_i = 1$ (normalisation) and $\mu_i \geq 0\ \forall i$. Then the expected value of O_I (or the average value) is $\sum_i \mu_i O_I(i)$. Suppose factorisation was true of this system, that is

$$\langle O_{I_1},\dots,O_{I_n}\rangle = \langle O_{I_1}\rangle\dots\langle O_{I_n}\rangle \tag{0.7.1.1}$$

for any observables O_I. We know what the probability to be in the state i is so we can rewrite the left hand side as

$$\langle O_{I_1},\dots,O_{I_n}\rangle = \sum_i \mu_i O_{I_1}(i)O_{I_2}(i)\dots O_{I_n}(i) \tag{0.7.1.2}$$

We can rewrite the right hand side of (0.7.1.1) as

$$\langle O_{I_1}\rangle\dots\langle O_{I_n}\rangle = \sum_{i_1}\mu_{i_1}O_{I_1}(i_1)\cdot\sum_{i_2}\mu_{i_2}O_{I_2}(i_2)\cdots\sum_{i_n}\mu_{i_n}O_{I_n}(i_n) \tag{0.7.1.3}$$

We see that the left hand side of (0.7.1.1) involves a sum over i, which is the number of states we have. The right hand side, however, involves a sum over i for each expectation value so we end up summing the number of states to the power of n. The only way to achieve equality is if we only have one state participating in the sum. That is, $\mu_i = 1$ for $i = i^*$ and $\mu_i = 0$ for $i \neq i^*$. Now both sides give

$$O_{I_1}(i^*)O_{I_2}(i^*)\dots O_{I_n}(i^*) \tag{0.7.1.4}$$

We can only be in one state: this is the classical limit. That is, for factorisation to hold our system must be in a classical limit.

The limit as $N\to\infty$ is the classical limit because factorisation holds. In particular, the large N limit of $\mathcal{N}=4$ SYM is given by the classical limit of type IIB string theory on $AdS_5\times S^5$.

It is natural to ask what the other terms are that fall away when we take N to be large. If we look back at (0.6.1.16), then we see that the first order correction to this stringy classical limit goes like $\frac{1}{N^2}$. In quantum field theory, the first order quantum correction comes with an $\hbar$. If we relate these ideas then $\frac{1}{N^2}\equiv\hbar_{\text{string}}$. So with strings, just like with field theory, we have quantum corrections or string corrections. It turns out that this is not the only uncertainty we have in string theory.

We are used to having the coupling constant as being a source of fundamental uncertainty and it is related to $\hbar$. For every order in the perturbation (or each vertex in our Feynman diagrams) we introduce an $\hbar$. This is

related to an uncertainty in the string theory – the string tension. The fact that we have strings replacing points introduces these new uncertainty because we cannot resolve the string into points. The string tension is inversely related to the size of the string and this gives us new uncertainty.

Our theory has gauge group $U(N)$ so our observables are built out of $N \times N$ Hermitian matrices. We saw that $\frac{1}{N^2}$ appeared as a new source of fundamental uncertainty (in the free and interacting theory), and so the size of our matrices in the field theory controls this uncertainty.

0.7.2 Triangulating a Surface

In Section 0.6.2, we saw that the N dependence of a diagram was given by a number called the Euler characteristic which we called a topological invariant. We will now motivate that statement and discuss its implications for the duality between $\mathcal{N} = 4$ SYM and string theory on an asymptotically $AdS_5 \times S^5$ background.

Imagine two loops of ribbon tied to each other. The knot is the vertex. We can arrange these loops onto a sphere so that edges of the ribbon do not cross. This is one of the terms we would get from order λ in the above correlator. It triangulates a 2 dimensional surface, as can be seen in Figure 6. The ribbon divides the surface into patches. These patches are the faces we have introduced above.

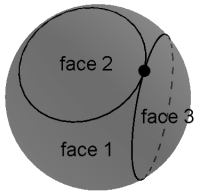


Figure 6: A representation of an $\mathcal{O}(\lambda)$ ribbon graph triangulating a sphere. The ribbons are shown as black lines.

We can imagine more intricate triangulations. Consider some such triangulation. It has four faces when viewed from one side. We can stretch this object horizontally as shown in Figure 7, creating an extra face in the middle.

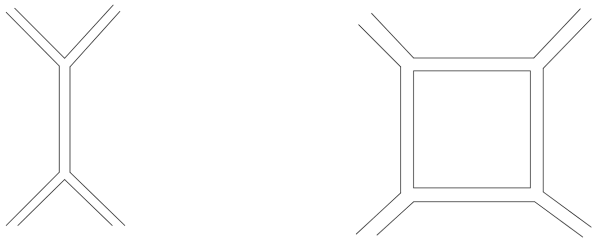


Figure 7: Stretching a piece of the ribbon configuration on one side of a sphere

Notice that, for this section of the triangulation, we started with four faces, five edges and two vertices. After the stretch we have five faces, eight edges and four vertices. Relating quantities before and after (indicated

with the primed variables) the stretch we have

$$F' = F + 1$$
$$E' = E + 3$$
$$V' = V + 2$$

Thus,

$$F' - E' + V' = F - E + V$$

We could also shrink our original picture along the vertical direction. In this case we again find

$$F'' - E'' + V'' = F - E + V$$

and the Euler characteristic is unchanged by the shrinking.

These deformations have all been homeomorphisms. They preserve all the topological properties of a space. This is why the Euler characteristic is a topological invariant: it does not change under a homeomorphism. This is not only true for the sphere example we have considered above. Some ribbon graphs might triangulate a torus or a pretzel. Intuitively, the more ribbons we have, the harder it becomes to triangulate a surface like a sphere (remember that our ribbons cannot cross on the surface). We see that we need surfaces like a torus, which is a sphere with a handle on it, to give us new ways to place ribbons so that they do not cross.

We do not need the triangulation to calculate what the Euler characteristic is; it can be computed directly from the topology of the surface. Every two dimensional oriented surface is topologically equivalent to a sphere with some handles stuck onto it or some holes cut out of it. So we can think of the torus as being a sphere with a handle on it. By stretching and shrinking our surface in various places, we can imagine moulding this sphere with a handle into a torus shape. The Euler characteristic for a surface has a definition in terms of the number of handles, H, and holes, h: $\chi = 2 - 2H - h$. For a sphere, there are no holes or handles. This gives an Euler characteristic of two. We can check this against the triangulation in Figure 6: $\chi = F - E + V = 3 - 2 + 1 = 2$. A torus has one handle, so it has an Euler characteristic of zero. A pretzel is a sphere with two handles, so it has an Euler characteristic of negative four. Each time we add a handle, our Euler characteristic decreases by two.

Recall that Euler characteristic gives us the N dependence of each ribbon graph. For the ribbon graph triangulating the sphere in Figure 6, the N dependence is two. The ribbon graph that triangulates a torus is shown in Figure 8. There is only one face (or one closed loop) in this diagram and the N dependence is zero. The surface that has an Euler characteristic of zero is the torus, so this diagram must triangulate the torus. It is remarkable that the topology of the surfaces these ribbon graphs triangulate determine the graphs' N dependence.

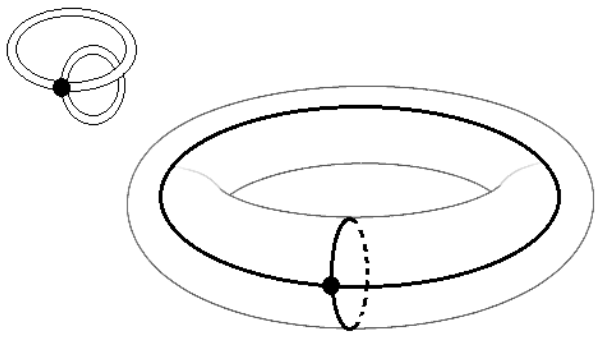


Figure 8: An $\mathcal{O}(\lambda)$ ribbon graph that triangulates a torus

When we study a perturbative quantum field theory, we use Feynman diagrams as a tool to sum terms in the perturbation series. The fact that the N dependence is related to the topology of a surface suggests that summing ribbon graphs has something to do with summing over surfaces. For a non-matrix model theory, like the scalar

particle, we perform a path integral quantisation of the theory by summing over all possible worldlines. Similarly for a string theory, we must sum over all possible worldsheets the string can follow in spacetime [14]. The fact that a string traces out a surface as it moves through spacetime draws a link between a matrix model theory, where summing over ribbon graphs is related to summing over surfaces, and a string theory. In particular, $\mathcal{N} = 4$ SYM is a matrix model theory. So we see here further motivation in claiming a duality between $\mathcal{N} = 4$ SYM and a string theory which is realised by the AdS/CFT correspondence. This correspondence is a precise guess for a type of string theory that is dual to a particular matrix model (type IIB strings on an asymptotically $AdS_5 \times S^5$ background geometry dual to $\mathcal{N} = 4$ SYM).

Looking back at 0.6.1.16, we see that the first correction to the leading order term in our correlation functions went like $\frac{1}{N^2}$. This was a quantum correction to the classical limit (obtained by taking $N \to \infty$) which we related to $\hbar_{\text{string}}$ in Section 0.7.1. The Euler characteristic showed us that when the N dependence of our graphs decreased by two, we needed to a glue a handle onto our sphere. This implies that for each $\hbar_{\text{string}}$ we have for the string, we need a higher genus correction. So the genus of our surface tells us the order of the quantum correction. This goes to motivate that we are summing over surfaces and that there is a duality between a matrix model theory and a string theory.

0.8 Connection with Young Diagrams

Young diagrams that characterise representations of $U(N)$ have a restriction on the length of their columns (no more than N boxes). The fully antisymmetric representation corresponds to a single column of N rows and corresponds to a giant graviton embedded in the S^n part of the $AdS_m \times S^n$ background. So we can identify boxes with discrete lumps of angular momenta. This means we have a maximum angular momentum allowed which we know corresponds to a maximum size of our graviton. This means that the Young diagram prescription encodes the stringy exclusion principle as we saw it in Section 0.5.

We can consider also a Young diagram consisting of a single single row of N boxes. This corresponds to a giant graviton expanded in the AdS_m part of the $AdS_m \times S^n$ background. This is the fully symmetric representation. This time, a limit is placed on the number of AdS giants we can have and not the size of the giants. This is because there is no restriction placed on the length of rows by working in a representation of $U(N)$. We can have at most N rows since these representations can have at most N boxes in a column. So there is at most N AdS giants. The size and number of rows and columns tells us whether we are in a fully symmetric, mostly symmetric, mostly antisymmetric or fully antisymmetric representation or something in between.

In [11], Schur polynomials composed of $\mathcal{O}(N)$ fields were proposed as the natural gauge theory duals to $\frac{1}{2}$-BPS giant gravitons. See Appendix D for details on Schur polynomials. These Schur polynomial operators allow for a natural identification of operators in the gauge theory corresponding to giant gravitons and new background geometries in the string theory side of the AdS/CFT correspondence.

Young diagrams label representations of the symmetric group (see Appendix C). Since they are composed from a discrete number of boxes, they discretise the geometry of the membrane. This discretisation removes many modes and therefore many of the infinities from the worldvolume theory.

[14] As a particle moves through spacetime, it traces out a line (worldline). A string is one-dimensional, and so it traces out a surface as it moves through spacetime (worldsheet).

1 Tools and Objectives

Schur polynomials (see Appendix D) with only Z fields are identified with $\frac{1}{2}$-BPS operators. A $\frac{1}{2}$-BPS state is a supersymmetric state. It has the most symmetry possible after the vacuum. This makes it simple to work with. We can learn new physics by moving away from this these highly symmetric states. In particular, we can consider a small deformation of $\frac{1}{2}$-BPS operators by adding a small number of Y fields such that the number of Z fields, n, is much larger than the number of Y fields, m. We will see in the following section that this is called the distant corners approximation, and how it applies to the algebra of the group of global $\mathcal{R}$-symmetry transformations. In particular, we can calculate the generators of the algebra in this approximation. Later, we will compute these generators exactly. This exact calculation will be the tool that allows us to go beyond small deformations of $\frac{1}{2}$-BPS. By adding these extra fields, we are studying the multimatrix model which allow for excitations in the string theory. Our operators here are the restricted Schur polynomials. In the following sections we will be building on these ideas and learning how to use the tools that will eventually allow us to calculate the generators of the $su(2)$ and $su(3)$ algebra. The reader is encouraged to study Appendices B, C and D at this point since these appendices contain background that will be used in the discussion which follows.

1.1 Restricted Schur Polynomials

For the matrix theory, we learn that finding an orthogonal basis of operators is reduced to the problem of constructing a suitable set of projection operators. This means that we have replaced a question in quantum field theory by a question in representation theory. This leads us to the Schur polynomials (see Appendix D) which provide a basis for the single matrix sector of $\mathcal{N} = 4$ SYM. They are conveniently labelled by Young diagrams and are the characters of irreducible representations of $SU(N)$. We will start by studying the algebra of $SU(2)$ built on a restricted Schur polynomial constructed out of two complex fields.

To form our basis, we need to look at the construction of our projection operators. Before we can build these, we have to understand some notation. In particular, we introduce the Young diagram. These are essentially a collection of n boxes arranged in rows such that row lengths don't increase as one moves from the top of the page to the bottom of the page. Much like English writing, we begin all rows on the left. For a given n, all Young diagrams with n boxes label a complete set of inequivalent, irreducible representations of S_n. They are useful structures, and it is not surprising they appear in our notation given our inclination toward group theory. We can label a given diagram by R, and this is partitioned by n boxes. This simply means that each row has a specific number of boxes. Filling the boxes with integers $1, ..., n$ such that, if the boxes are dropped in this order one has a valid Young diagram at each step, gives us an object called a Young-Yamanouchi symbol, which label states in the carrier space of representation R.

Each of these symbols label a basis vector in the S_n representation, R. For each Young diagram, R, the number of distinct Young-Yamanouchi symbols it allows is equal to the dimension of that representation. Each Young diagram labels an irreducible matrix representation in the vector space. For two matrix representations to be equivalent, $\Gamma_R(g)$ and $\widetilde{\Gamma}_R(g)$; $\forall g \in \mathcal{G}$, their traces need to be equal (see Appendix B). This trace is called the character of the group element g and it is denoted by $\chi_R(g)$.

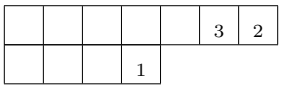
Consider the vector space $V_N^{\otimes n}$. We can simultaneously diagonalise the actions of the symmetric and unitary group on this space, leading to collections of states that are irreducible representations of both. The Young diagram that labels a particular subspace is then both an irreducible representation of the symmetric group and an irreducible representation of the unitary group. This allows us to see how $V_N^{\otimes n}$ decomposes. This naturally leads to the introduction of a projection operator, with which we can project onto one of the subspaces (labelled by a Young diagram). These operators are given by

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma$$

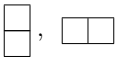
In this dissertation, we wish to study the algebra of $su(2)$ and later $su(3)$. The restricted Schur polynomials

in $su(2)$ are constructed out of two complex fields. This means we need to modify our projection operator to take this new field into account. To this end, consider S_{n+m}. Here n tells us how many of the first type of field (Z field) and m tells us how many of the second type of field (Y field) we have. We say that r is a symmetric group representation that organises the Z fields and s organises the Y fields. Our Young diagram $R \vdash (n+m)$ can be thought of as being built out of $r \vdash n$ and $s \vdash m$. Operators that are labelled by Young diagrams of g long rows are dual to a system of g giant gravitons (the Z fields) dressed by open strings (the Y fields)[15]. This corresponds to exciting the giant graviton.

Now (r,s) labels an irreducible representation of $S_n \times S_m \subset S_{n+m}$. If we pull m boxes off R to form s, then r is what remains with $(n+m)-m=n$ boxes. The way in which we pull boxes off R will determine what subspaces are possible ($S_{n+m} \to S_n \times S_m$). For example, consider the partially decorated Young diagram R



If we pull one box off row 1 and one box off row 2 then we can form the following Young diagrams for s:



Whereas, if we pulled both boxes of the first row we could only form the latter diagram. Our projection operator now takes the form

$$P_{R,(r,s)} = \frac{1}{n!}\frac{1}{m!}\sum_{\sigma_1 \in S_n}\sum_{\sigma_2 \in S_m} \chi_r(\sigma_1)\chi_s(\sigma_2)\Gamma_R(\sigma_1\sigma_2)$$

One might ask how feasible this process of construction would be in the limit where we take $n+m \to \infty$. It turns out, and this is where the strength of our new formalism really shows, that the above projector can be written as

$$P_{R,(r,s)} = \frac{1}{m!}\sum_{\sigma \in S_m} \chi_s(\sigma)\Gamma_R(\sigma)$$

This projector acts only on the boxes removed from R to leave r. Notice that the form of subspace r does not feature. So we could have as many boxes as we like in r. We can take that number to infinity. So long as m is finite, we are able to build the projectors we need.

Thinking about $su(2)$ in general again, there are three generators that close the algebra. They are J_-, J_+ and J_3. The third follows from the first two (by the commutation relation $[J_+, J_-] = J_3$), so it will be sufficient to concern ourselves with the construction of only $J_\pm$. Recall that n and m partition the subspaces, (r,s), of the entire irrep. The action of these $J_\pm$ on our operators, built from the restricted Schur basis where R is a Young diagram that labels an irrep of S_{n+m} and (r,s) are a pair of Young diagrams that label an irreducible representation of $S_n \times S_m \subset S_{n+m}$,

$$\chi_{R,(r,s)\vec{\mu}}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}(\Gamma_R(\sigma))\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}), \tag{1.1.1}$$

yield an expression for elements of $J_\pm$ in terms of the trace of the product of two projection operators, i.e.:

$$\mathrm{Tr}(P_{R,(r,s)}P_{R,(r^+,s^-)})$$

We know how to calculate the first of these projectors. The second is obtained as the first but by first removing a box from s and adding it to r.

Let's study the form of (1.1.1) in more detail. This is the restricted Schur polynomial. We call it the restricted Schur polynomial because we replace the usual character of our irreps with an object called the restricted character. This encodes the way we have partitioned the Young diagram R into $n+m$, where $r \vdash n$ and $s \vdash m$ are subspaces of R. In each of these subspaces we must consider S_n and S_m separately such that we study permutations in $S_n \times S_m$. This comes from the following logic.

We are working $V_N^{\otimes n+m}$. In the single matrix model, the following action of the permutations $\sigma \in S_n$ left $Z^{\otimes n}$ invariant.

$$(\sigma)^I_J Z^{\otimes n}(\sigma^{-1})^K_L = (Z^{\otimes n})^I_L \tag{1.1.2}$$

[15]For this to hold n must be $\mathcal{O}(N)$ and m must be $\mathcal{O}(1)$.

When we consider two complex matrix models, we have the same invariance holding for the product $Z^{\otimes n}Y^{\otimes m}$, but now $\sigma \in S_n \times S_m$. This is because we cannot swap Zs and Ys without changing the structure. Our permutations must act separately on each type of field so that we restrict to the subgroup $S_n \times S_m$. From (1.1.2) and using the cyclicity of the trace we obtain the relation

$$\mathrm{Tr}(\rho Z^{\otimes n}Y^{\otimes m}) = \mathrm{Tr}(\sigma^{-1}\rho\sigma Z^{\otimes n}Y^{\otimes m}) \tag{1.1.3}$$

This allows us to define a notion of restricted conjugate. Two elements g_1 and g_2 in S_{n+m} are restricted conjugate to one another if they satisfy the relation

$$g_1 = \sigma^{-1}g_2\sigma \tag{1.1.4}$$

for $\sigma \in S_n \times S_m$. The is like the equivalence relation, conjugate to, which we used when classifying irreps in Appendix B but with the added restriction that σ is in the subgroup. When studying single matrix models, we saw that the number of conjugacy classes was equal to the number of irreps and that this was equal to the number of physical observables. Now, in our multi matrix model, we have that the number of restricted conjugacy classes is equal to the number of physical observables.

A complete set of functions on the restricted conjugacy class are given by taking a restricted trace of the matrix irrep of the group element. The restricted trace is a trace over the subspace. The restricted character is defined as the restricted trace of a group element.

$$\chi_{R,(r,s)\vec{\mu}}(\sigma) = \mathrm{Tr}_{R,(r,s)\vec{\mu}}(\Gamma_R(\sigma)) \tag{1.1.5}$$

We label multiplicities with $\vec{\mu}$. Row and column indices in the restricted trace are traced over different subspaces (S_n and S_m).

We are able to write the restricted Schur polynomials in terms of projectors, like with the Schur polynomials in Appendix D.

$$\chi_{R,(r,s)\vec{\mu}}(Z^{\otimes n}Y^{\otimes m}) = \mathrm{Tr}(P_{R,(r,s)\vec{\mu}}Z^{\otimes n}Y^{\otimes m}) \tag{1.1.6}$$

Here $\sigma \in S_{n+m}$. The projector is defined as

$$P_{R,(r,s)\vec{\mu}} \equiv \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\vec{\mu}}(\sigma)\sigma \tag{1.1.7}$$

which acts on S_{n+m}. This allows us to write the two point function of restricted Schur polynomials as

$$\langle \chi_{R,(r,s)\vec{\mu}}(X,Y)\chi_{T,(t,u)\vec{\nu}^\dagger}(Z,Y)\rangle = n!m!\mathrm{Tr}(P_{R,(r,s)\vec{\mu}}P_{T,(t,u)\vec{\nu}}) = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\mu_2\nu_1}\delta_{\mu_1\nu_2}\frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r\mathrm{hooks}_s} \tag{1.1.8}$$

which features prominently in subsequent sections. We note that these projectors are not projection operators. In general, $\vec{\mu} \neq \vec{\nu}$ and we have that

$$P_{R,(r,s)\mu_1\mu_2}P_{T,(t,u)\nu_1\nu_2} = \mathcal{A}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\mu_2\nu_1}P_{R,(r,s)\mu_1\nu_2} \tag{1.1.9}$$

where $\mathcal{A}$ is a number. The product of two projection operators is either zero, or one of the projectors, $P_RP_T = \delta_{RT}P_R$, by definition of a projection operator. Multiplicities label different copies of a representation. The projector on the right hand side of (1.1.9) has multiplicity labels that are different to the projectors on the left hand side, hence these objects are all different. However, if the multiplicity labels are the same, then these objects are projection operators. In general, they are not and we call them intertwining maps as they map us between different copies of the irrep. To illustrate:

$$\Gamma_{(r,s)\mu_1}(\sigma)P_{R,(r,s)\vec{\mu}} = P_{R,(r,s)\vec{\mu}}\Gamma_{(r,s)\mu_2}(\sigma) \tag{1.1.10}$$

1.2 $su(2)$ and $su(3)$ Algebra

The elementary fields of $\mathcal{N} = 4$ SYM are represented by $N \times N$ Hermitian matrices. These are six scalar fields, ϕ_i; $i = 1, \ldots, 6$, with matrices in the $u(N)$ Lie algebra. The theory enjoys an $SO(6)$ symmetry which rotates the scalars in the vector representation. We form the complex fields from these scalar fields as follows.

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6 \tag{1.2.1}$$

Operators, restricted Schur polynomials, that are built out of only Y and Z fields belong to the $SU(2)$ sector of the theory and operators that are built out of all three complex fields belong to the $SU(3)$ sector. We want to construct the generators of the global symmetries which are subgroups of the $SO(6)$ which rotates these complex fields.

Angular momentum generates rotations so it is no surprise that we can realise the $su(2)$ algebra by defining

$$J_+ = \mathrm{Tr}\left(Y\frac{d}{dZ}\right), \qquad J_- = \mathrm{Tr}\left(Z\frac{d}{dY}\right), \qquad J_3 = \mathrm{Tr}\left(Y\frac{d}{dY} - Z\frac{d}{dZ}\right) \tag{1.2.2}$$

which close the usual algebra

$$[J_+, J_-] = J_3, \qquad [J_3, J_\pm] = \pm 2 J_\pm \tag{1.2.3}$$

The non-trivial generators we want to study for $su(3)$ include

$$\mathrm{Tr}\left(Z\frac{d}{dY}\right), \quad \mathrm{Tr}\left(Z\frac{d}{dX}\right), \quad \mathrm{Tr}\left(Y\frac{d}{dZ}\right), \quad \mathrm{Tr}\left(Y\frac{d}{dX}\right), \quad \mathrm{Tr}\left(X\frac{d}{dY}\right), \quad \mathrm{Tr}\left(X\frac{d}{dZ}\right) \tag{1.2.4}$$

In section 0.8 we talked about how the number of rows in our Young diagram is related to the number of giant gravitons. We saw in the previous section that the construction of the restricted Schur polynomial in a representation that has more than two rows was complicated by the need to keep track of multiplicity labels. Later we will consider only two rows (two giant gravitons) which means we can drop multiplicity labels but for now we will keep the discussion general. Our operators are built out of n Z fields and m Y fields such that $r \vdash n$ and $s \vdash m$. The restricted Schur polynomial is

$$\chi_{R,(r,s)\vec{\mu}}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}(\Gamma_R(\sigma))\mathrm{Tr}(\sigma Y^{\otimes m}\otimes Z^{\otimes n}) \tag{1.2.5}$$

We want to work out what the action of the generator J_- on our operator is.

$$\begin{aligned}
J_-\chi_{R,(r,s)\vec{\mu}}(Z,Y) &= \mathrm{Tr}\left(Z\frac{d}{dY}\right)\chi_{R,(r,s)\vec{\mu}}(Z,Y) \\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}(\sigma Y^{\otimes\, m-1}\otimes Z^{\otimes\, n+1}) \\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}_{(r,s)\vec{\mu}}(\Gamma_R(\sigma))\sum_{T,(t^+,u^-)\vec{\nu}} \frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\chi_{T,(t^+,u^-)\vec{\nu}^*}(\sigma^{-1})\chi_{T,(t^+,u^-)\vec{\nu}}(Z,Y) \\
&= \sum_{T,(t^+,u^-)\vec{\nu}} \frac{d_T(n+1)}{d_{t^+}d_{u^-}(n+m)!}\frac{(n+m)!}{d_T}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s)\vec{\mu}}P_{T,(t^+,u^-)\vec{\nu}^*})\chi_{T,(t^+,u^-)\vec{\nu}}(Z,Y) \\
&= \sum_{(t^+,u^-)\vec{\nu}} \frac{n+1}{d_{t^+}d_{u^-}}\mathrm{Tr}_R(P_{R,(r,s)\vec{\mu}}P_{R,(t^+,u^-)\vec{\nu}^*})\chi_{R,(t^+,u^-)\vec{\nu}}(Z,Y)
\end{aligned} \tag{1.2.6}$$

where we have used the inverse Fourier transform

$$\mathrm{Tr}(\sigma Y^{\otimes\, m-1}\otimes Z^{\otimes\, n+1}) = \sum_{T,(t^+,u^-)\vec{\nu}} \frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\chi_{T,(t^+,u^-)\vec{\nu}^*}(\sigma^{-1})\chi_{T,(t^+,u^-)\vec{\nu}}(Z,Y) \tag{1.2.7}$$

and the fundamental orthogonality relation

$$\sum_{g\in\mathcal{G}}\Gamma_R(g)_{ab}\Gamma_T(g^{-1})_{cd} = \frac{|\mathcal{G}|}{d_R}\delta_{RT}\delta_{bc}\delta_{ad} \tag{1.2.8}$$

and $t^+ \vdash n+1$, $u^- \vdash m-1$, $T \vdash m+n$ and d_T is the dimension of the representation T. Note that R and T label the irreps (see appendix B). Our operators obey

$$\langle\chi_{R,(r,s)\vec{\mu}}(Z,Y)\chi^\dagger_{T,(t,u)\vec{\nu}}(Z,Y)\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\vec{\mu}\vec{\nu}}\frac{f_R\mathrm{hooks}_R}{\mathrm{hooks}_r\ \mathrm{hooks}_s} \tag{1.2.9}$$

By rescaling

$$\chi_{R,(r,s)\vec{\mu}}(Z,Y) = \sqrt{\frac{f_R\mathrm{hooks}_R}{\mathrm{hooks}_r\mathrm{hooks}_s}}O_{R,(r,s)\vec{\mu}}(Z,Y)$$

we can get operators, $O_{R,(r,s)\vec{\mu}}(Z,Y)$, with two point function equal to 1. That is

$$\langle O_{R,(r,s)\vec{\mu}}(Z,Y)O^\dagger_{T,(t,u)\vec{\nu}}(Z,Y)\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\vec{\mu}\vec{\nu}} \tag{1.2.10}$$

Now we can look at the action of J_- on our normalised operators.

$$J_- O_{R,(r,s)\vec{\mu}}(Z,Y) = \sum_{T,(t^+,u^-)\vec{\nu}} (J_-)_{T,(t^+,u^-)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} O_{T,(t^+,u^-)\vec{\nu}}(Z,Y) \tag{1.2.11}$$

This allows us to identify the matrix elements of J_- with aspects that relate to our representation and our choice of basis:

$$(J_-)_{T,(t^+,u^-)\vec{\nu}\,,\,R,(r,s)\vec{\mu}} = \sum_i \delta_{RT}\delta_{t_i^+ r}\frac{(n+1)}{d_{t^+}d_{u^-}}\sqrt{\frac{\text{hooks}_r\text{hooks}_s}{\text{hooks}_{t^+}\text{hooks}_{u^-}}}\text{Tr}_R(P_{R,(r,s)\vec{\mu}}P_{R,(t^+,u^-)\vec{\nu}^*}) \tag{1.2.12}$$

We will learn in the next section that these matrix elements depend only on our representations. In particular, when we consider two rows, the operators inside the trace are projection operators and we will learn how to calculate them.

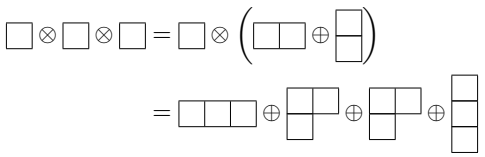
1.3 Calculating a Projection Operator

In (1.2.12), the projectors that appear

$$P_{R,(r,s)\vec{\mu}} \equiv \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\vec{\mu}}(\sigma)\Gamma_R(\sigma) \tag{1.3.1}$$

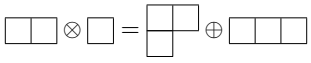
are not actually projection operators. However, when we consider labelling our representations by Young diagrams that have only two rows, we find that we no longer have multiplicities and these projectors behave like usual projection operators. That is, they satisfy $P^2 = P$, where P is shorthand for a general projection operator.

The reason why we have no multiplicities when we restrict to two rows is as follows. First, imagine a Young diagram with $B = b_1 + b_2 + 1$ boxes with $b_1 > b_2 > 1$. Suppose it has three rows, with only one box in the third row, b_2 boxes in the second row and b_1 boxes in the first row. Suppose we remove a box from each row to construct the subspace (r,s) of R. Then we are left with a Young diagram that has $b_1 - 1$ boxes in the first row, $b_2 - 1$ boxes in the second row and zero boxes in all other rows plus the three boxes we have pulled off. From these pulled off boxes, we can construct the following representations of S_3:



(1.3.2)

So there are two ways of constructing the 2-dimensional representation $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$. This means we have two copies of the same subspace (r,s). This is why we need multiplicity labels when we consider Young diagrams of more than two rows. Two rows are special because, no matter how many boxes we remove from R, we will never have a situation like the above where we construct s out the tensor product of more than two Young diagrams. For example, we might have:



(1.3.3)

This illustrates that we can only end up with two distinct Young diagrams.

The projection operator is defined as

$$P_{R,(r,s)} = \frac{d_r}{n!}\frac{d_s}{m!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m}\chi_r(\sigma_1)\chi_s(\sigma_2)\Gamma_R(\sigma_1\sigma_2) \tag{1.3.4}$$

and it acts on the vector space $V^{\otimes n+m}$. We can understand this projector as follows. We have some representation R which we decompose into the subspaces r and s. We form the subspace s by removing boxes from R. For example, we might remove 2 boxes from the first row of R and 1 box from the second row of R. The possible Young diagrams are as in (1.3.3). The representation r is in the shape of R less the boxes removed to form s. The number of boxes in s is m such that $s \vdash m$, and there are n boxes in r such that $r \vdash n$. The dimension of the subspace labelled by r is denoted by d_r. Similarly for d_s. The characters of each of these subspaces is given

by χ_r and χ_s, respectively. The irrep of the space is labelled by R and denoted by Γ_R. The projection operator above projects us onto a particular representation for the subspaces r and s. The way we construct the Young diagram for s determines the shape of the Young diagram for r. So there is only one representation of r that we can project to for a given way of removing boxes from R to produce s. We can construct a new projection operator for each possible representation of s. But there is only representation of r so the sum over all the $P_{R,(r,s)}$ is really a sum over the representations of s multiplied by some constant. We have

$$\sum_{(r,s)} P_{R,(r,s)} = \frac{d_r}{n!} \sum_{\sigma_1 \in S_n} \chi_r(\sigma_1)\Gamma(\sigma_1) \left(\sum_s \frac{d_s}{m!} \sum_{\sigma_2 \in S_m} \chi_s(\sigma_2)\Gamma(\sigma_2) \right) \tag{1.3.5}$$

Since we are working with projection operators, the above equation must sum to the identity. Looking at the term in brackets we see that it looks like a sum over a projection operator P_s. This should separately sum to the identity. This tells us that

$$\frac{d_r}{n!} \sum_{\sigma_1 \in S_n} \chi_r(\sigma_1)\Gamma(\sigma_1) = \mathbb{1}_{n \times n}$$

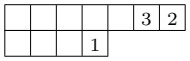
We can thus rewrite (1.3.4) as

$$P_{R,(r,s)} = \frac{d_s}{m!} \sum_{\sigma \in S_m} \chi_s(\sigma)\Gamma_R(\sigma) \tag{1.3.6}$$

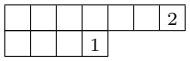
Notice that this formula has no dependence on n. So the size of our Young diagram R can be completely arbitrary. No matter the size, we are equally capable of calculating the projection operator $P_{r,(r,s)}$, even as $n \to \infty$. The only changes that will complicate the calculation is by increasing the number of boxes in s. This is precisely the reason why the restricted Schur polynomial basis is a natural choice for our computation of the matrix elements of J_- and J_+. If we can calculate these projection operators, then we can compute these matrix elements. The above formula is convincing to show that we will be able to calculate the projection operators we need without specifying r.

A numerical study of these projectors will give us a check for when we later compute general formulae for the matrix elements. The numerical study is outlined below and the results are included in subsequent sections.

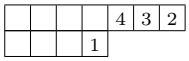
We need to specify what m is and how we pull boxes off R to form a Young diagram in representation s. Each Young diagram we are able to form for s represents a conjugacy class in S_m and thus has a restricted character associated with it. There is an irrep representing each $\sigma \in S_m$. Consider the following partially labelled Young diagram, R.

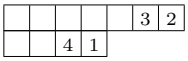


The numbered boxes are the boxes we pull off to construct s. The numbers label states according to how they transform under a chain of subgroups. For example, this Young diagram is in S_{n+m} and the three numbered boxes form a Young diagram in S_3 such that the action of S_3 on this state will permute the boxes 1,2 and 3. Suppose we want to consider s^- which is a Young diagram with $m-1=2$ boxes. The numbers tell us in which order we pull boxes off R to construct s: first 1, then 2, then 3 and so on. So now we have



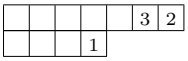
The numbered boxes here form a Young diagram in S_2. The leftover boxes from r such that the number of boxes in the first row of r has increased by one box. We could also consider s^+ which is a Young diagram in S_4. We could take three boxes off the first row and one box off the second row or two boxes off the first and second rows. In this case R looks like either of the following.

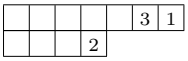


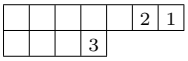


The first adds a box to the first two of r and the second diagram adds a box to the second row of r.

Going back to R we see we can have three possible states. These are

$|1\rangle =$ 

$|2\rangle =$ 

$|3\rangle =$ 

We got them by filling in the numbers 1,2 and 3 into the rightmost empty boxes of R (two boxes in the first row and one in the second row) in all possible ways that produce a valid Young diagram. This tells us that the dimension of the subspace s is 3. Thus the irreps $\Gamma_R(\sigma)$ will be 3×3 matrices for $\sigma \in S_3$. These permutations, σ, are partitioned into conjugacy classes. Each conjugacy class has a different character, χ_s. In order to calculate the projection operator, $P_{R,(r,s)}$, then we need to specify:

The number of boxes, m.
The shape of the Young diagram s.
The way boxes are removed off R to form s.

We need to calculate:

The irrep $\Gamma_R(\sigma)$ for each possible $\sigma \in S_m$.
The dimension of s.

And finally we can look up the character $\chi_s(\sigma)$ in a character table for each $\sigma \in S_m$, since these are specified generally for each conjugacy class.

We are interested in calculating

$$\mathrm{Tr}(P_{R,(r,s)} P_{R,(t,u)})$$

for $t = r^+$ and $u = s^-$ or $t = r^-$ and $u = s^+$. The $+$ superscript means we have added a box to the subspace representation and the $-$ superscript means we have removed a box. The total number of boxes partitioned by R $(n+m)$ is fixed so that $r^+ \vdash n+1$ and $s^- \vdash m-1$. We calculate the projection operator $P_{R,(t,u)}$ with this in mind using the same process outlined above.

For $su(3)$ the process is almost exactly the same as $su(2)$. In what follows we note only the differences with the $su(2)$ computation. The identity (1.2.7) must be upgraded. For the case of three complex matrices the statement of the completeness of the restricted Schur polynomials is [19]

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m} \otimes X^{\otimes p}) = \sum_{R,(r,s,t)\vec{\alpha}\vec{\beta}} \frac{d_R n!m!p!}{d_r d_s d_t (n+m+p)!} \chi_{R,(r,s,t)\vec{\alpha}\vec{\beta}}(\sigma^{-1}) \chi_{R,(r,s,t)\vec{\beta}\vec{\alpha}}(Z,Y,X)\,. \tag{1.3.7}$$

Consider the generator

$$\begin{aligned}\mathrm{Tr}\left(X\frac{d}{dY}\right)\chi_{R,(r,s,t)\vec{\alpha}\vec{\beta}}(Z,Y,X) = \frac{m}{n!m!p!} \sum_{\sigma\in S_{n+m+p}} \mathrm{Tr}(P_{R,(r,s,t)\vec{\alpha}\vec{\beta}}\Gamma^R(\sigma))\times \\ \sum_{T,(t_1,t_2,t_3)\vec{\delta}\vec{\gamma}} \frac{d_T n!(m-1)!(p+1)!}{d_{t_1}d_{t_2}d_{t_3}(n+m+p)!} \chi_{T,(t_1,t_2,t_3)\vec{\delta}\vec{\gamma}}(\sigma^{-1})\chi_{T,(t_1,t_2,t_3)\vec{\gamma}\vec{\delta}}(Z,Y,X)\,.\end{aligned} \tag{1.3.8}$$

In the above, $T \vdash n+m+p$, $t_1 \vdash n$, $t_2 \vdash m-1$ and $t_3 \vdash p+1$. Carrying out the sum over σ in (1.3.8) using the completeness relation, we find

$$\begin{aligned}\mathrm{Tr}\left(X\frac{d}{dY}\right)\chi_{R,(r_1,r_2,r_3)\vec{\alpha}\vec{\beta}}(Z,Y,X) = \frac{m}{n!m!p!} \sum_{(t_1,t_2,t_3)\vec{\delta}\vec{\gamma}} \frac{(m-1)!n!(p+1)!}{d_{t_1}d_{t_2}d_{t_3}} \\ \times \mathrm{Tr}\left(P_{R,(r_1,r_2,r_3)\vec{\alpha}\vec{\beta}} P_{R,(t_1,t_2,t_3)\vec{\delta}\vec{\gamma}}\right)\chi_{R,(t_1,t_2,t_3)\vec{\gamma}\vec{\delta}}(Z,Y,X)\,.\end{aligned} \tag{1.3.9}$$

To evaluate this generator, we need to evaluate the trace

$$\mathrm{Tr}\left(P_{R,(\vec{r})\vec{\alpha}\vec{\beta}} P_{R,(\vec{t})\vec{\delta}\vec{\gamma}}\right)\,. \tag{1.3.10}$$

Without multiplicity labels, the projection operator we are interested in is

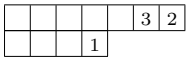
$$\begin{aligned}P_{R,(r,s,t)} &= \frac{d_r d_s d_t}{n!m!p!} \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m}\sum_{\sigma_3\in S_p} \chi_r(\sigma_1)\chi_s(\sigma_2)\chi_t(\sigma_3)\Gamma_R(\sigma_1\sigma_2\sigma_3) \\ &= \frac{d_s d_t}{m!p!} \sum_{\sigma_1\in S_m}\sum_{\sigma_2\in S_p} \chi_s(\sigma_1)\chi_t(\sigma_2)\Gamma_R(\sigma_1\sigma_2)\end{aligned} \tag{1.3.11}$$

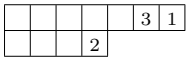
where $t \vdash p$ and the total number of boxes in the irrep R is $n+m+p$. We label states in R by numbering $n+m$ boxes. The total number of states gives the size of our matrix Γ. We first remove boxes to form the Young diagram t and then we remove boxes to form s. The construction of these projectors follows in more detail in Section 2.1 and Section 3.1.

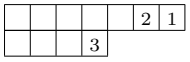
2 $SU(2)$

2.1 Numerical Calculation of Projectors

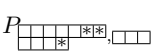
Consider the subspace $s \vdash m = 3$. Pull two boxes off row 1 of R and one box off row 2. We have $\frac{3!}{2!1!} = 3$ states. Label the states as follows

$|1\rangle =$ 

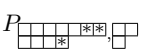
$|2\rangle =$ 

$|3\rangle =$ 

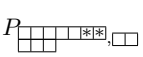
Note that the size of R is arbitrary and we could have chosen to have any number of empty boxes above. Consider also the subspaces s_1^-, formed by dropping a single box from row 1 of s, and s_2^-, formed by dropping a single box off row 2 of s. The following projectors are possible:

$$P^{(2,1)}_{\square\square\square} = P_{\text{}} \tag{2.1.1}$$

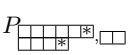
(2.1.2)

$$P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = P_{\text{}} \tag{2.1.3}$$

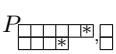
(2.1.4)

$$P^{(2)}_{\square\square} = P_{\text{}} \tag{2.1.5}$$

(2.1.6)

$$P^{(1,1)}_{\square\square} = P_{\text{}} \tag{2.1.7}$$

(2.1.8)

$$P^{(1,1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = P_{\text{}} \tag{2.1.9}$$

The notation is such that $P^{(2,1)}_{\square\square\square}$ is a projector onto (r, s). The Young diagram $\square\square\square$ is s (recall that R is arbitrary so we do not need to specify r as well). The superscript $(2,1)$ tells us how many boxes were removed off each row of R.

We want to take the trace of the product of projectors for subspace $s \vdash m$ and $s^- \vdash m-1$.

Call the number of blocks in row 1 and row 2 of r by r_1 and r_2 respectively (since the number of boxes in r, nn is arbitrary). The numerical results are then:

$$\mathrm{Tr}\left(P^{(2,1)}_{\square\square\square} P^{(1,1)}_{\square\square}\right) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) \tag{2.1.10}$$

$$\mathrm{Tr}\left(P^{(2,1)}_{\square\square\square} P^{(1,1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) = 0 \tag{2.1.11}$$

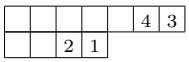
$$\mathrm{Tr}\left(P^{(2,1)}_{\square\square\square} P^{(2)}_{\square\square}\right) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right) \tag{2.1.12}$$

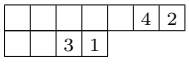
$$\mathrm{Tr}\left(P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} P^{(1,1)}_{\square\square}\right) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right) \tag{2.1.13}$$

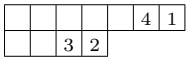
$$\mathrm{Tr}\left(P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} P^{(1,1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) = 1 \tag{2.1.14}$$

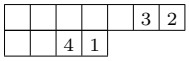
$$\mathrm{Tr}\left(P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} P^{(2)}_{\square\square}\right) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) \tag{2.1.15}$$

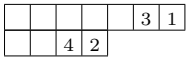
Consider the case with $m = 4$. States are produced by removing two boxes from row 1 and two boxes from row 2. We have $\frac{4!}{2!2!} = 6$ states. We can label states as follows:

$|1\rangle =$ 

$|2\rangle =$ 

$|3\rangle =$ 

$|4\rangle =$ 

$|5\rangle =$ 

$|6\rangle =$ 

The following projectors are possible:

$$P^{(2,2)}_{\square\square\square\square} = P_{\square\square\square\square\square\square\ast\ast,\square\square\square\square} \tag{2.1.16}$$

(2.1.17)

$$P^{(2,2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} = P_{\square\square\square\square\square\square\ast\ast,\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} \tag{2.1.18}$$

(2.1.19)

$$P^{(2,2)}_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} = P_{\square\square\square\square\square\square\ast\ast,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} \tag{2.1.20}$$

(2.1.21)

$$P^{(2,1)}_{\square\square\square} = P_{\square\square\square\square\square\ast\ast,\square\square\square} \tag{2.1.22}$$

(2.1.23)

$$P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = P_{\square\square\square\square\square\ast\ast,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \tag{2.1.24}$$

(2.1.25)

$$P^{(1,2)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = P_{\square\square\square\square\square\ast,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \tag{2.1.26}$$

(2.1.27)

$$P^{(1,2)}_{\square\square\square} = P_{\square\square\square\square\square\ast,\square\square\square} \tag{2.1.28}$$

We compute the traces in a similar fashion to the $m = 3$ case. We obtain:

$$\mathrm{Tr}\left(P^{(2,2)}_{\square\square\square\square}P^{(1,2)}_{\square\square\square}\right) = \frac{1}{2} - \frac{1}{r_1 - r_2 + 1} \tag{2.1.29}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\square\square\square\square}P^{(1,2)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 0 \tag{2.1.30}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\square\square\square\square}P^{(2,1)}_{\square\square\square}\right) = \frac{1}{2} + \frac{1}{r_1 - r_2 + 1} \tag{2.1.31}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\square\square\square\square}P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 0 \tag{2.1.32}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}P^{(1,2)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 1 + \frac{1}{r_1 - r_2 + 1} \tag{2.1.33}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}P^{(1,2)}_{\square\square\square}\right) = 0 \tag{2.1.34}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}P^{(2,1)}_{\square\square\square}\right) = 0 \tag{2.1.35}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 1 - \frac{1}{r_1 - r_2 + 1} \tag{2.1.36}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}P^{(1,2)}_{\square\square\square}\right) = \frac{1}{2} + \frac{1}{r_1 - r_2 + 1} \tag{2.1.37}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}P^{(1,2)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 1 - \frac{1}{r_1 - r_2 + 1} \tag{2.1.38}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}P^{(2,1)}_{\square\square\square}\right) = \frac{1}{2} - \frac{1}{r_1 - r_2 + 1} \tag{2.1.39}$$

$$\mathrm{Tr}\left(P^{(2,2)}_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}P^{(2,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\right) = 1 + \frac{1}{r_1 - r_2 + 1} \tag{2.1.40}$$

Consider $m = 5$. We construct s by pulling 3 boxes off the first row and 2 boxes off the second row. We have $\frac{5!}{3!2!} = 10$ states. We can label states as follows:

$$|1\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 4 & 3 \\ \hline \ & \ & 2 & 1 \\ \cline{1-4}\end{array}$$

$$|2\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 4 & 2 \\ \hline \ & \ & 3 & 1 \\ \cline{1-4}\end{array}$$

$$|3\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 4 & 1 \\ \hline \ & \ & 3 & 2 \\ \cline{1-4}\end{array}$$

$$|4\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 3 & 2 \\ \hline \ & \ & 4 & 1 \\ \cline{1-4}\end{array}$$

$$|5\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 3 & 1 \\ \hline \ & \ & 4 & 2 \\ \cline{1-4}\end{array}$$

$$|6\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 5 & 2 & 1 \\ \hline \ & \ & 4 & 3 \\ \cline{1-4}\end{array}$$

$$|7\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 4 & 3 & 2 \\ \hline \ & \ & 5 & 1 \\ \cline{1-4}\end{array}$$

$$|8\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 4 & 3 & 1 \\ \hline \ & \ & 5 & 2 \\ \cline{1-4}\end{array}$$

$$|9\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 4 & 2 & 1 \\ \hline \ & \ & 5 & 3 \\ \cline{1-4}\end{array}$$

$$|10\rangle = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & 3 & 2 & 1 \\ \hline \ & \ & 5 & 3 \\ \cline{1-4}\end{array}$$

The following projectors are possible:

$$P^{(3,2)}_{\square\square\square\square\square} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&*&*&*\\\square&\square&*&*\end{smallmatrix},\square\square\square\square\square} \tag{2.1.41}$$

(2.1.42)

$$P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&*&*&*\\\square&\square&*&*\end{smallmatrix},\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}} \tag{2.1.43}$$

(2.1.44)

$$P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square\\\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&*&*&*\\\square&\square&*&*\end{smallmatrix},\begin{smallmatrix}\square&\square&\square&\square\\\square\end{smallmatrix}} \tag{2.1.45}$$

(2.1.46)

$$P^{(2,2)}_{\square\square\square\square} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&*&*\\\square&\square&*&*\end{smallmatrix},\square\square\square\square} \tag{2.1.47}$$

(2.1.48)

$$P^{(2,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&*&*\\\square&\square&*&*\end{smallmatrix},\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}} \tag{2.1.49}$$

(2.1.50)

$$P^{(2,2)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&*&*\\\square&\square&*&*\end{smallmatrix},\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}} \tag{2.1.51}$$

(2.1.52)

$$P^{(3,1)}_{\square\square\square\square} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&*&*&*\\\square&\square&\square&*\end{smallmatrix},\square\square\square\square} \tag{2.1.53}$$

(2.1.54)

$$P^{(3,1)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square&\square&\square&\square&\square&*&*&*\\\square&\square&\square&*\end{smallmatrix},\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}} \tag{2.1.55}$$

Similarly to the previous examples, we compute the traces. We obtain:

$$\mathrm{Tr}\left(P^{(3,2)}_{\square\square\square\square\square}P^{(2,2)}_{\square\square\square\square}\right) = \frac{3}{5}\left(1-\frac{2}{r_1-r_2+1}\right) \tag{2.1.56}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\square\square\square\square\square}P^{(2,2)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 0 \tag{2.1.57}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\square\square\square\square\square}P^{(2,2)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}\right) = 0 \tag{2.1.58}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\square\square\square\square\square}P^{(3,1)}_{\square\square\square\square}\right) = \frac{2}{5}\left(1+\frac{3}{r_1-r_2+1}\right) \tag{2.1.59}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\square\square\square\square\square}P^{(3,1)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 0 \tag{2.1.60}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(2,2)}_{\square\square\square\square}\right) = 0 \tag{2.1.61}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(2,2)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 1+\frac{2}{r_1-r_2+1} \tag{2.1.62}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(2,2)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}\right) = 2 \tag{2.1.63}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(3,1)}_{\square\square\square\square}\right) = 0 \tag{2.1.64}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(3,1)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 2\left(1-\frac{1}{r_1-r_2+1}\right) \tag{2.1.65}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square&\square\\\square\end{smallmatrix}}P^{(2,2)}_{\square\square\square\square}\right) = \frac{2}{5}\left(1+\frac{3}{r_1-r_2+1}\right) \tag{2.1.66}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(2,2)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 2\left(1-\frac{1}{r_1-r_2+1}\right) \tag{2.1.67}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square\\\square&\square\end{smallmatrix}}P^{(2,2)}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}}\right) = 0 \tag{2.1.68}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square&\square\\\square\end{smallmatrix}}P^{(3,1)}_{\square\square\square\square}\right) = \frac{3}{5}\left(1-\frac{2}{r_1-r_2+1}\right) \tag{2.1.69}$$

$$\mathrm{Tr}\left(P^{(3,2)}_{\begin{smallmatrix}\square&\square&\square&\square&\square\\\square\end{smallmatrix}}P^{(3,1)}_{\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}}\right) = 1+\frac{2}{r_1-r_2+1} \tag{2.1.70}$$

In the displaced corners approximation (see the next section) only the leading term survives. When we calculate the exact traces, we will see we are able to reproduce these examples.

2.2 Displaced Corners Approximation

In the displaced corners approximation we take $r_1 >> r_2$ and $m << n$. In this approximation, the right hand corners of our Young diagrams R are well separated. When we act with $S_m \subset S_{n+m}$ on the Young-Yamanouchi states, we permute the labels of the Y boxes. We only swap boxes whose labels differ by 1 such that we consider swaps of the form $(k, k+1)$ where the integer $k \leq m-1$. This allows us to work out the matrices $\Gamma_R(\sigma)$ for $\sigma \in S_m$ (we can build any σ by taking a product of 2-cycles). We use the formula

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle$$

where $|R\rangle$ is some Young-Yamanouchi state in the irrep R and $|R_{(k,k+1)}\rangle$ is the same state but with the labels on boxes k and $k+1$ swapped. The c_k are the contents of each box labelled by k (see Appendix C). If we want to swap boxes in the same row, then we have to swap boxes that are next to each other and this is not a valid Young-Yamanouchi symbol. The above formula takes care of this problem since, in this case, $|c_k - c_{k+1}| = 1$ and the coefficient of $|R_{(k,k+1)}\rangle$ is zero leaving us with only the state $|R\rangle$. When we swap boxes between the first and second row, $c_k - c_{k+1} \propto r_1 - r_2$ and so we are only left with the state $|R_{(k,k+1)}\rangle$.

The $su(2)$ generators in the displaced corners approximation have already been calculated in [13]. The results are summarised here.

Since we are considering restricted Schur polynomials labelled by a Young diagram R with at most 2 rows, there is no multiplicity index and we can use the $SU(2)$ state labels j, j_3 with $-j \leq j_3 \leq j$ as usual (we are labelling states, $|j, j^3\rangle$, in representation s by their angular momenta which has a natural interpretation in terms of giant gravitons in the string theory). Denote the length of row i in s by s_i (respectively R_i, r_i in R, r). The translation between the two labels is (see [7])

$$\begin{aligned} s_1 &= \frac{m}{2} + j \qquad s_2 = \frac{m}{2} - j\,, \\ R_1 &= r_1 + \frac{m}{2} + j_3 \qquad R_2 = r_2 + \frac{m}{2} - j_3\,. \end{aligned} \tag{2.2.1}$$

The computation of the relevant traces now reduces to the computation of $SU(2)$ Clebsch-Gordan coefficients. For detailed examples of the computations required see [7, 20] and especially [13] which computes precisely the traces that are used here. The result is

$$\begin{aligned} J_- O^{(n,m)}(r_1, j, j_3) = A_- O^{(n+1,m-1)}(r_1+1, j+\frac{1}{2}, j_3-\frac{1}{2}) + B_- O^{(n+1,m-1)}(r_1+1, j-\frac{1}{2}, j_3-\frac{1}{2}) \\ + C_- O^{(n+1,m-1)}(r_1, j+\frac{1}{2}, j_3+\frac{1}{2}) + D_- O^{(n+1,m-1)}(r_1, j-\frac{1}{2}, j_3+\frac{1}{2}) \end{aligned} \tag{2.2.2}$$

where

$$A_- = \sqrt{r_1}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}\frac{j-j_3+1}{2j+2}}\,, \tag{2.2.3}$$

$$B_- = \sqrt{r_1}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}\frac{j+j_3}{2j}}\,, \tag{2.2.4}$$

$$C_- = \sqrt{r_2}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}\frac{j+j_3+1}{2j+2}}\,, \tag{2.2.5}$$

$$D_- = \sqrt{r_2}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}\frac{j-j_3}{2j}} \tag{2.2.6}$$

These generators do not close the correct $su(2)$ algebra, although it is correct to the leading order in $\frac{m}{n}$, as expected [13].

In the next section we will work out the exact $su(2)$ generators. We will check that the J_- matrix elements reduce to the above displaced corners elements.

2.3 Exact Generators

2.3.1 Traces and Identities Needed

Our computation involves a restricted character that is easily evaluated using the methods developed in [21]. See also [22] for a closely related recent discussion. Introduce indices with the following ranges

$$I, J = 1, 2, ..., m+n, \qquad \alpha, \beta = m+1, m+2, ..., m+n, \qquad a, b = 1, 2, ..., m.$$

We will compute the restricted character

$$\chi_{R,(r,s)}(\,(a,\alpha)\,) = \mathrm{Tr}(P_{R,(r,s)}(a,\alpha)\,)\,. \tag{2.3.1.1}$$

Denote the row lengths of these Young diagrams by R_1, R_2, r_1, r_2 and s_1, s_2. By $\sum_{IJ}(I,J)$ we mean the sum of all distinct two cycles. For S_n this sum runs over $n(n-1)/2$ terms. It is easy to establish the identity

$$\sum_{a,\alpha}(a,\alpha) = \sum_{IJ}(I,J) - \sum_{ab}(a,b) - \sum_{\alpha\beta}(\alpha,\beta)\,. \tag{2.3.1.2}$$

The sum over all two cycles is a Casimir with eigenvalue equal to the number of row pairs minus the number of column pairs [23]. Since our Young diagrams have two rows, it follows that

$$\sum_{IJ}\mathrm{Tr}(P_{R,(r,s)}(I,J)\,) = \frac{R_1(R_1-1)}{2} + \frac{R_2(R_2-1)}{2} - R_2\,, \tag{2.3.1.3}$$

$$\sum_{\alpha\beta}\mathrm{Tr}(P_{R,(r,s)}(\alpha,\beta)\,) = \frac{r_1(r_1-1)}{2} + \frac{r_2(r_2-1)}{2} - r_2\,, \tag{2.3.1.4}$$

$$\sum_{ab}\mathrm{Tr}(P_{R,(r,s)}(a,b)\,) = \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} - s_2\,. \tag{2.3.1.5}$$

Taking the trace of (2.3.1.2) it follows that

$$\begin{aligned}\sum_{a,\alpha}\chi_{R,(r,s)}(\,(a,\alpha)\,) &= nm\chi_{R,(r,s)}(\,(m,m+1)\,)\\ &= \frac{2j_3^2 - 2j(j+1) + 2j_3(r_1-r_2+1) + m(r_1+r_2)}{2}d_r d_s\end{aligned} \tag{2.3.1.6}$$

which is the formula we were after.

We now derive identities which will later give the answer for the trace over a sum of projectors. First we will compute a trace that is simple enough to evaluate exactly. This trace, together with the identities we obtain, allow us to determine the rest of the traces.
We want to compute

$$T = \mathrm{Tr}(P_{R,(r,s)}P_{R,(r^+,s^-)}) = \sum_i \langle R,(r,s);i|P_{R,(r^+,s^-)}|R,(r,s);i\rangle \tag{2.3.1.7}$$

with $R \vdash n+m$, $r \vdash n$, $r^+ \vdash n+1$, $s \vdash m$ and $s^- \vdash m-1$. One way of embedding the S_n irrep r within the S_{n+m} irrep R is to remove boxes from R to obtain r. Assume that we remove m_1 boxes from the first row of R and m_2 boxes from the second row of R. The first basic trace we will compute assumes that r^+ is given by adding

one box to the second row of r. The Young diagram labelling s is a single row of m boxes. The Young diagram labelling s^- is a single row of $m-1$ boxes. It is clear that the only states that participate on the right hand side of the above equation have box m (the last box of s which is the first box of r^+) in the second row of r^+. Further, only the subspace of s corresponding to s^- contributes. Thus, we need to do the sum over i making sure that we arrange two things: box m must sit in the second row and we must project to s^-. Projecting to s^- is easy: we know how to construct a projector that will accomplish this. To fix box m, note that the last box of s can be in the first row or the second row. By fixing the content of this box we can ensure its in the second row. This content is measured by the Jucys-Murphy element which lives in the S_m group algebra [23]. Using this element we can construct the operator

$$\left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \tag{2.3.1.8}$$

which gives 1 when acting on states for which the last box of r^+ is in the second row and zero when the last box of r^+ is in the first row. Clearly then

$$\begin{aligned}
T &= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \chi_{s^-}(\sigma)\Gamma_R(\sigma)\right] |R,(r,s);i\rangle \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \Gamma_R(\sigma)\right] |R,(r,s);i\rangle \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \Gamma_s(\sigma)\right] |R,(r,s);i\rangle \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] |R,(r,s);i\rangle \\
&= \frac{r_1 d_r d_s}{r_1 - r_2 + 1} - n\frac{\chi_{R,(r,s)}(\,(m,m+1)\,)}{r_1 - r_2 + 1} \\
&= \frac{m-2j_3}{4m}\left[2 + \frac{m+2j_3}{r_1 - r_2 + 1}\right] d_r d_s
\end{aligned} \tag{2.3.1.9}$$

where to obtain the second line we use $\chi_{s^-}(\sigma) = 1$ since s is the 1-dimensional representation with a single row of boxes; to obtain the third line we use the fact that state $|R,(r,s);i\rangle$ belongs to an irrep of $S_n \times S_m$, to obtain the fourth line we use the fact that in irrep s we have $\Gamma_s(\sigma) = 1$ and to obtain the last line use the formula for the restricted character derived in the previous section.

The second basic trace that we will need arises when r^+ is given by adding one box to row 1 of r. The second basic trace is easily computed, in exactly the same way, to be

$$T = \left[\frac{m+2j_3}{2m} - \frac{(m^2 - 4(j_3)^2)}{4m(r_1 - r_2 + 1)}\right] d_r d_s\,. \tag{2.3.1.10}$$

Note that d_s appears, but we have $d_s = 1$ for both these basic traces.

Now we derive two identities. To obtain the first identity, assume r^+ is given by adding a box to row 2 of r. It is trivial to obtain the formula for r^+ given by adding a box to row 1. In general s subduces two possible s^-: s_1^- and s_2^-, where s_1^- is obtained by dropping a box from row 1 of s and s_2^- is obtained by dropping a box from row 2 of s. It is now rather straightforward to compute (we use $s = s_1^- \oplus s_2^-$ below)

$$\begin{aligned}
T &= \sum_{K=1}^{2} \mathrm{Tr}\left(P_{R,(r,s)} P_{R,(r^+,s_K^-)}\right) \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \sum_K \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \chi_{s_K^-}(\sigma)\Gamma_R(\sigma)\right] |R,(r,s);i\rangle \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{m!}\sum_{\sigma\in S_m} \chi_s(\sigma)\Gamma_R(\sigma)\right] |R,(r,s);i\rangle \\
&= \sum_i \langle R,(r,s);i| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] |R,(r,s);i\rangle \\
&= \left[\frac{m-2j_3}{2m} + \frac{2j(j+1) - 2j_3^2 - m}{2m(r_1 - r_2 + 1)}\right] d_r d_s\,.
\end{aligned} \tag{2.3.1.11}$$

This is the first sum identity we wanted to prove.

For the second sum identity, we start by noting that the generic $s^- \vdash m-1$ can be subduced by two possible representations, s_1 and s_2. If you drop a box from row 1 of s_1 you get s^- and if you drop a box from row 2 of s_2 you get s^-. We want to compute

$$T = \sum_{J=1}^{2} \mathrm{Tr}(P_{R,(r,s_J)} P_{R,(r^+,s^-)}) \,. \tag{2.3.1.12}$$

We can easily extend the sum above to a sum over all the possible s: since the only irreps that can subduce s^- are s_1 and s_2, the additional terms all vanish. Now,

$$\sum_s P_{R,(r,s)} \tag{2.3.1.13}$$

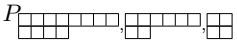
projects us from R to r. Thus, the product $P_{R,(r,s_J)} P_{R,(r^+,s^-)}$ projects us to $r \oplus \square \oplus s^-$ and hence

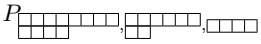
$$T = \sum_{J=1}^{2} \mathrm{Tr}(P_{R,(r,s_J)} P_{R,(r^+,s^-)}) = d_r d_{s^-} \tag{2.3.1.14}$$

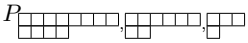
Using the two basic traces and the two basic sum identities, we can compute any trace we want. This is easily illustrated with an example.

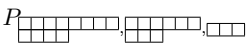
Consider the numerical example given in Section 2.1 for $m = 4$. We constructed the subspace s by removing 2 boxes from row 1 and 2 from row 2 such that we had $j_3 = 2 - \frac{4}{2} = 0$ and $j = 2$ (for $s_1 = 4$), $j = 0$ (for $s_1 = 2$) and $j = 1$ (for $s_1 = 3$).

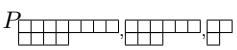
Consider the following projectors for some Young diagram R. Note that the number of boxes in R is arbitrary, we are concerned with the shape of s and the way in which we remove boxes from R to form r.

$P_1 = P$  (2.3.1.15)

$P_2 = P$  (2.3.1.16)

$P_3 = P$  (2.3.1.17)

$P_A = P$  (2.3.1.18)

$P_B = P$  (2.3.1.19)

We see that r^+ is formed by adding a box to the second row of r. We could also construct projectors onto $R, (r^+, s^-)$ where r^+ is formed by adding a box the first row of r. Notice that $s = (3,1) \to s^- = (3)$ or $s^- = (2,1)$ implies we can use the first sum identity, $s = (3,1)$ or $s = (2,2) \to s^- = (2,1)$ implies we can use the second sum identity, $s = (4) \to s^- = (3)$ implies we can use the basic trace, $s = (4)$ or $s = (3,1) \to s^- = (3)$ implies we can use the second sum identity. So we calculate $\mathrm{Tr}(P_2 P_A)$ using the basic trace. Then we can calculate $\mathrm{Tr}(P_3 P_A)$ using the second sum identity. Given this, $\mathrm{Tr}(P_3 P_B)$ must be calculated using the first sum identity. Then we can use the second sum identity to calculate $\mathrm{Tr}(P_1 P_B)$.

2.3.2 Exact Traces

The results of the previous section are completely summarised in the following for exact traces. We will shift from the $R, (r,s)$ notation to the r_1, j, j_3, n, m notation.
The exact results for the traces we need are

$$\begin{aligned} &\mathrm{Tr}\left(P(r_1, j, j_3, n, m) P(r_1, j - \frac{1}{2}, j_3 + \frac{1}{2}, n+1, m-1) \right) \\ &\qquad = \left[\frac{j - j_3}{2j} + \frac{j^2 - j_3^2}{2j(r_1 - r_2 + 1)} \right] d_r d_{s^-} \,, \end{aligned} \tag{2.3.2.1}$$

$$\mathrm{Tr}\left(P(r_1, j, j_3, n, m) P(r_1, j + \frac{1}{2}, j_3 + \frac{1}{2}, n+1, m-1) \right)$$

$$= \left[\frac{j+j_3+1}{2j+2} - \frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right] d_r d_{s^-} , \tag{2.3.2.2}$$

$$\mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1+1,j-\frac{1}{2},j_3-\frac{1}{2},n+1,m-1)\right) = \left[\frac{j+j_3}{2j} - \frac{j^2-j_3^2}{2j(r_1-r_2+1)}\right] d_r d_{s^-} , \tag{2.3.2.3}$$

$$\mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1+1,j+\frac{1}{2},j_3-\frac{1}{2},n+1,m-1)\right) = \left[\frac{j-j_3+1}{2j+2} + \frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right] d_r d_{s^-} . \tag{2.3.2.4}$$

For the same example ($m=4$) in Section 2.1, we check these exact traces reproduce the numerical results. We get:

$$\mathrm{Tr}(P_2 P_A) = \mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1,j-\frac{1}{2},j_3+\frac{1}{2},n+1,m-1)\right)\Big|_{j=2,j_3=0} = \left[\frac{1}{2}+\frac{1}{r_1-r_2+1}\right] d_r \tag{2.3.2.5}$$

which is in agreement with our numerical result (2.1.31), modulo d_r (note here that $d_{s^-}=1$).

$$\mathrm{Tr}(P_1 P_B) = \mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1,j+\frac{1}{2},j_3+\frac{1}{2},n+1,m-1)\right)\Big|_{j=0,j_3=0} = \left[1-\frac{1}{r1-r2+1}\right] d_r \tag{2.3.2.6}$$

where $d_{s^-}=2$. This corresponds to (2.1.36), modulo d_r.

$$\mathrm{Tr}(P_3 P_B) = \mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1,j-\frac{1}{2},j_3+\frac{1}{2},n+1,m-1)\right)\Big|_{j=1,j_3=0} = \left[1+\frac{1}{r1-r2+1}\right] d_r \tag{2.3.2.7}$$

which corresponds to (2.1.40).

$$\mathrm{Tr}(P_3 P_A) = \mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1,j+\frac{1}{2},j_3+\frac{1}{2},n+1,m-1)\right)\Big|_{j=1,j_3=0} = \left[\frac{1}{2}-\frac{1}{r1-r2+1}\right] d_r \tag{2.3.2.8}$$

which corresponds to (2.1.39).

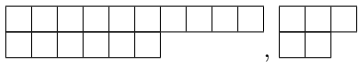
These results are enough to illustrate that we can compute any trace we like using only the four formulae for the exact traces. These will allow us to construct the exact matrix elements of $J_\pm$.

2.3.3 Exact Matrix Elements of J_-

Consider (1.2.12). We have calculated the traces analytically in the previous section. We have some terms that can be simplified further: this is the coefficient with dimensions and hooks of the subspace irreps. We have

$$\sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_{t^+}\mathrm{hooks}_{u^-}}}\frac{(n+1)}{d_{t^+}d_{u^-}} d_r d_{u^-} = (n+1)\sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_{t^+}\mathrm{hooks}_{u^-}}}\frac{\mathrm{hooks}_{t^+}}{(n+1)!}\frac{n!}{\mathrm{hooks}_r} = \sqrt{\frac{\mathrm{hooks}_{t^+}\mathrm{hooks}_s}{\mathrm{hooks}_r \mathrm{hooks}_{u^-}}} \tag{2.3.3.1}$$

Consider the 2 row Young diagrams representing a possible arrangement for subspaces r and s of R.



We can work out the hooks as follows. For r_2 it is easy: $(r_2)!$. From the end of r_2 to the end of r_1 it is also straightforward: $(r_1 - r_2)!$. Since we only have two rows, the contribution to the hooks from the first row goes like $(r_1 + 1)!$ but we overcount since $r_1 > r_2$ so we divide by $(r_1 - r_2 + 1)!$. Thus we have

$$\text{hooks}_r = \frac{(r_1 - r_2)!(r_1 + 1)!(r_2)!}{(r_1 - r_2 + 1)!} = \frac{(r_1 + 1)!(r_2)!}{r_1 - r_2 + 1} \tag{2.3.3.2}$$

Similarly

$$\text{hooks}_s = \frac{(s_1 + 1)!(s_2)!}{s_1 - s_2 + 1} \tag{2.3.3.3}$$

We change notation such that $s_1 = \frac{m}{2} + j$ and $s_2 = \frac{m}{2} + j$. Thus

$$\text{hooks}_s = \frac{(j + \frac{m}{2} + 1)!(\frac{m}{2} - j)!}{2j + 2} \tag{2.3.3.4}$$

We have four cases to study for u^- and t^+ together. We will illustrate how to calculate the hooks for one such case. Consider $r_1 + 1$, $j + \frac{1}{2}$ and $j_2 - \frac{1}{2}$. Then

$$\text{hooks}_{t^+} = \frac{(r_1 + 2)!r_2!}{r_1 - r_2 + 1} \tag{2.3.3.5}$$

and

$$\text{hooks}_{u^-} = \frac{(j + \frac{m}{2} + 1)!(\frac{m}{2} - j - 1)!}{2j + 2} \tag{2.3.3.6}$$

This gives

$$\sqrt{\frac{\text{hooks}_{t^+}\text{hooks}_s}{\text{hooks}_r\text{hooks}_{u^-}}} = \sqrt{\frac{(r_1 + 2)(r_1 - r_2 + 1)}{(r_1 - r_2 + 2)}}\sqrt{\frac{(m - 2j)}{2}\frac{2j + 2}{2j + 1}} \tag{2.3.3.7}$$

The action of J_- on the normalised operator is

$$\begin{aligned} J_- O^{(n,m)}(r_1, j, j_3) &= A_- O^{(n+1,m-1)}(r_1 + 1, j + \frac{1}{2}, j_3 - \frac{1}{2}) + B_- O^{(n+1,m-1)}(r_1 + 1, j - \frac{1}{2}, j_3 - \frac{1}{2}) \\ &+ C_- O^{(n+1,m-1)}(r_1, j + \frac{1}{2}, j_3 + \frac{1}{2}) + D_- O^{(n+1,m-1)}(r_1, j - \frac{1}{2}, j_3 + \frac{1}{2}) \end{aligned} \tag{2.3.3.8}$$

where the A_-, B_-, C_- and D_- are the matrix elements. Using the exact traces and computing the remaining coefficients using the same procedure used previously we get

$$A_- = \sqrt{\frac{(r_1 + 2)(r_1 - r_2 + 1)}{(r_1 - r_2 + 2)}}\sqrt{\frac{m - 2j}{2}\frac{2j + 2}{2j + 1}}\left[\frac{j - j_3 + 1}{2j + 2} + \frac{(j + 1)^2 - j_3^2}{(2j + 2)(r_1 - r_2 + 1)}\right] \tag{2.3.3.9}$$

$$B_- = \sqrt{\frac{(r_1 + 2)(r_1 - r_2 + 1)}{(r_1 - r_2 + 2)}}\sqrt{\frac{m + 2j + 2}{2}\frac{2j}{2j + 1}}\left[\frac{j + j_3}{2j} - \frac{j^2 - j_3^2}{2j(r_1 - r_2 + 1)}\right] \tag{2.3.3.10}$$

$$C_- = \sqrt{\frac{(r_2 + 1)(r_1 - r_2 + 1)}{(r_1 - r_2)}}\sqrt{\frac{m - 2j}{2}\frac{2j + 2}{2j + 1}}\left[\frac{j + j_3 + 1}{2j + 2} - \frac{(j + 1)^2 - j_3^2}{(2j + 2)(r_1 - r_2 + 1)}\right] \tag{2.3.3.11}$$

$$D_- = \sqrt{\frac{(r_2 + 1)(r_1 - r_2 + 1)}{(r_1 - r_2)}}\sqrt{\frac{m + 2j + 2}{2}\frac{2j}{2j + 1}}\left[\frac{j - j_3}{2j} + \frac{j^2 - j_3^2}{2j(r_1 - r_2 + 1)}\right] \tag{2.3.3.12}$$

The action of J_3 is particularly simple

$$J_3\, O^{(n,m)}(r_1, j, j_3) = (m - n)\, O^{(n,m)}(r_1, j, j_3) \tag{2.3.3.13}$$

To get the matrix elements of J_+ we can dagger the above expressions. In a bra-ket notation the relevant formula is

$$\langle n + 1, r^+, m - 1, j', j_3'|J_+|n, r, m, j, j_3\rangle = \langle n, r, m, j, j_3|J_-|n + 1, r^+, m - 1, j', j_3', p + 1\rangle \tag{2.3.3.14}$$

The above generators close the correct $su(2)$ algebra, which is a good check. Further, in the displaced corners approximation we have $r_1 - r_2 \gg 1$. In this limit we see that the above generators reduce to the formulas we obtained in the displaced corners limit.

For example, consider A_-. In the displaced corners approximation $r_1 - r_2 + 1 \approx r_1 - r_2$ and $r_1 + 2 \approx r_1$. The term with denominator proportional to $r_1 - r_2$ goes to zero so that we are left with precisely (2.2.3).

Finally, for $n = 3$ and $m = 2$ we have checked by hand, using explicit examples, that the above action is indeed correct. This completes the derivation of the exact $su(2)$ generators.

2.3.4 First Steps to Calculating the Dilatation Operator

One motivation behind attempting to solve for the matrix elements of the dilatation operator in a similar form to our generators is that the one loop and two loop structure of the dilatation operator factorises into a piece that acts only on the r label of Young diagram representation and another piece that acts only on the s label [24]. A good guess for the one loop action of the dilatation operator on our normalised operators [13] is

$$D_2 O^{(n,m)}(r_1, j, j_3) = \sum_{c=-1}^{1} \sum_{d=-1}^{1} \beta^{(n,m)}_{r_1,j,j_3}(c,d) O^{(n,m)}(r_1 + c, j + d, j_3) \tag{2.3.4.1}$$

At one loop, the dilatation operator shifts a single box. So row 1 might gain or lose a box ($r_1 \to r_1 \pm 1$) or we can have that s_1 might gain or lose a box ($s_1 \to s_1 \pm 1$) meaning that $j \to j \pm 1$. We should also consider the effect on j_3, if any. All three variables, r_1, j, j_3, determine how a box moves between representations. If j_3 does not change it means we are fixing the number of boxes we pull off each row, despite the representation. If we do that, then the matrix element, β, depends on the change in r_1 and j only. The matrix coefficients of the dilatation operator map us to $O^{(n,m)}(r_1, j, j_3)$.

The dilatation operator is

$$D_2 = -g^2_{YM} \mathrm{Tr}[Y, Z][\partial_Y, \partial_Z] \tag{2.3.4.2}$$

J_- is $\mathrm{Tr}\left(Z \frac{d}{dY}\right)$. Previously we saw that the action of J_- on our operators is

$$J_- O_{R,(r,s)}(Z, Y) = \sum_{T,(t^+,u^-)} (J_-)_{T,(t^+,u^-),R,(r,s)} O_{T,(t^+,u^-)}(Z, Y) \tag{2.3.4.3}$$

Comparing (2.3.4.1) with (2.3.4.3), we see that the action changes the subspace the operator is defined on. We also have a sum appearing with some matrix element coefficient. So, the form (2.3.4.1) is familiar to something we have seen before.

The fact that we are using the displaced corners approximation is what fixes j_3. Consider that we know we can work out the effect of permuting two adjacent (differ by 1) numbers in a Young diagram. The result is the same diagram, with some coefficient, and a diagram where those numbers are swapped, again with some coefficient. These coefficients go like $\frac{1}{r_1 - r_2 + 1}$ and $\sqrt{1 - \frac{1}{(r_1 - r_2 + 1)^2}}$ respectively. In the displaced corners approximation, these coefficients go like 0 and 1, respectively. So only one diagram contributes and the sum over j_3 falls away. (Recall the j_3 tells us how m boxes are distributed in a Young diagram R. The displaced corners approximation fixes the diagrams possible, thereby fixing j_3).

We will be looking at J_+ for this discussion. The matrix elements of this generator is easily related to to J_- by Hermitian conjugation. We introduce the notation

$$J_+ O^{(n,m)}_{r_1,j,j_3} = \sum_{a=-1}^{0} \sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1,j,j_3}(a,b) O^{(n-1,m+1)}_{r_1+a, j+b, j_2-\frac{1}{2}-a} \tag{2.3.4.4}$$

We start off with following equation implied by commutation relation $[D_2, J_+] = 0$.

$$\sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}}\sum_{c=-1}^{1}\sum_{d=-1}^{1}\big(\beta^{(n,m)}_{r_1,j,j_3}(c,d)\alpha^{(n,m)}_{r_1+c,j+d,j_3}(a,b) - \alpha^{(n,m)}_{r_1,j,j_3}(a,b)\beta^{(m-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a}(c,d)\big)O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a} = 0 \tag{2.3.4.5}$$

We have that our α coefficient terms are proportional to either $\sqrt{r_1}$ or $\sqrt{r_2}$, depending on the value that a takes. At large N, α terms with different values of a must separately vanish since our operators are linearly independent.

At large N, both n and m are infinite. In this limit we can replace the j, j_3 and r_1, r_2 with continuous variables. Under the transformations

$$2j = 2\sqrt{m}x_j \qquad r_1 - r_2 = 2l = 2\sqrt{n}x$$
$$R_1 = r_1 + m_1 \qquad R_2 = r_2 + m_2$$
$$2j_3 = 2\sqrt{m}x_{j_3}$$

and with $a = 0, b = \frac{1}{2}, c = 1, d = 1$, we note the following:

$$r_1 + r_2 = n$$
$$\therefore r_1 \to r_1 + c \implies r_1' = r_1 + 1 \implies r_2' = r_2 - 1$$
$$\implies r_1' - r_2' = 2\sqrt{n}x - 2 = 2\sqrt{n}x' \implies x' = x - \frac{1}{\sqrt{n}}$$
$$j \to j + d \implies x_j \to x_j + \frac{1}{\sqrt{m}}$$
$$r_1 + r_2 = n - 1$$
$$\therefore r_1' = r_1 + a \implies r_1' = r_1 \implies r_1' - r_2' = r_1 - r_2 + 1 = 2\sqrt{n-1}x'$$
$$\implies 2\sqrt{n}x + 1 = 2\sqrt{n-1}x' \implies x' = \left(x + \frac{1}{2\sqrt{n}}\right)\sqrt{\frac{n}{n-1}}$$
$$j + b \implies 2\sqrt{m}x_j + 1 = 2\sqrt{m+1}x_j' \implies x_j' = \left(x_j + \frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}}$$
$$j_3 - b \implies \sqrt{m}x_{j_3} - \frac{1}{2} = \sqrt{m+1}x_{j_3} \implies x_{j_3}' = \left(x_{j_3} - \frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}}$$

We obtain

$$0 = \beta_{1,1}(x, x_j, x_{j_3}, n, m)\alpha_{0,\frac{1}{2}}(x + \frac{1}{\sqrt{n}}, x_j + \frac{1}{\sqrt{m}}, x_{j_3}, n, m) - \alpha_{0,\frac{1}{2}}(x, x_j, x_{j_3}, n, m)\beta_{1,1}(\left(x + \frac{1}{2\sqrt{n}}\right)\sqrt{\frac{n}{n-1}}, \left(x_j + \frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}}, \left(x_{j_3} - \frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}}, n-1, m+1) \tag{2.3.4.6}$$

We want to determine which matrix coeefficients of $J_\pm$ correspond to the α coefficients we see above. In order to do so, we consider the action of J_+ on our observables.

$$J_+ O_{R,(r,s)}(Z,Y) = \sum_{T,(t^-,u^+)}(J_+)_{T,(t^-,u^+),R,(r,s)}O_{T,(t^-,u^+)}(Z,Y)$$

where

$$(J_+)_{T,(t^-,u^+),R,(r,s)} = \sqrt{\frac{\text{hooks}_r\text{hooks}_s}{\text{hooks}_{t^-}\text{hooks}_{u^+}}}\sum_i \delta_{RT}\delta_{t^-r_i'}\frac{m+1}{d_{t^-}d_{u^+}}\text{Tr}_R(P_{R,(r,s)}P_{R,(t^-,u^+)})$$

We are interested in determining $\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})$. To do so, we must first expand the sum

$$\begin{aligned}J_+O^{(n,m)}_{r_1,j,j_3} &= \sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}}\alpha^{(n,m)}_{r_1,j,j_3}(a,b)O^{(n-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a}\\ &= \alpha^{(n,m)}_{r_1,j,j_3}(-1,-\frac{1}{2})O^{(n-1,m+1)}_{r_1-1,j-\frac{1}{2},j_3+\frac{1}{2}} + \alpha^{(n,m)}_{r_1,j,j_3}(-1,\frac{1}{2})O^{(n-1,m+1)}_{r_1-1,j+\frac{1}{2},j_3+\frac{1}{2}}\\ &+ \alpha^{(n,m)}_{r_1,j,j_3}(0,-\frac{1}{2})O^{(n-1,m+1)}_{r_1,j-\frac{1}{2},j_3-\frac{1}{2}} + \alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})O^{(m-1,m+1)}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2}}\end{aligned} \tag{2.3.4.7}$$

for $a=0$, $b=\frac{1}{2}$, $c=1$, $d=1$. For $\alpha(0,\frac{1}{2})$, we see that $j\to j+\frac{1}{2}$, $j_3\to j_3-\frac{1}{2}$ and $r_1\to r_1$. So the trace we need is precisely

$$\mathrm{Tr}\left(P(r_1,j+\frac{1}{2},j_3-\frac{1}{2},n-1,m+1)P(r_1,j,j_3,n,m)\right)=\left[\frac{j-j_3+1}{2j+1}+\frac{j^2+j-j_3^2-j_3}{(2j+1)(r_1-r_2+2)}\right]d_r \tag{2.3.4.8}$$

That is

$$\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})=\sqrt{\frac{\mathrm{hooks}_r\mathrm{hooks}_s}{\mathrm{hooks}_{t^-}\mathrm{hooks}_{u^+}}}\sum_i\delta_{RT}\delta_{t^-r_i'}\frac{(m+1)d_r}{d_{t^-}d_{u^+}}\left[\frac{j-j_3+1}{2j+1}+\frac{j^2+j-j_3^2-j_3}{(2j+1)(r_1-r_2+2)}\right] \tag{2.3.4.9}$$

In the large N limit, we can treat our variables as being continuous. We make the trade as before and we obtain (we will drop everything but the trace part for now)

$$\begin{aligned}\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})&=\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m)\\&=\frac{\sqrt{m}x_j-\sqrt{m}x_{j_3}+1}{2\sqrt{m}x_j+1}+\frac{\sqrt{m}x_j(\sqrt{m}x_j+1)-\sqrt{m}x_{j_3}(\sqrt{m}x_{j_3}+1)}{(2\sqrt{m}x_j+1)(r_1-r_2+2)}\end{aligned} \tag{2.3.4.10}$$

$$\begin{aligned}\alpha^{(n,m)}_{r_1+1,j+1,j_3}(0,\frac{1}{2})&=\alpha_{0,\frac{1}{2}}\left(x+\frac{1}{\sqrt{n}},x_j+\frac{1}{\sqrt{m}},x_{j_3},n,m\right)\\&=\frac{\sqrt{m}x_j-\sqrt{m}x_{j_3}+2}{2\sqrt{m}x_j+3}+\frac{(\sqrt{m}x_j+1)(\sqrt{m}x_j+2)-\sqrt{m}x_{j_3}(\sqrt{m}x_{j_3}+1)}{(2\sqrt{m}x_j+3)(r_1-r_2+2)}\end{aligned} \tag{2.3.4.11}$$

We can also make the change $r_1-r_2=2\sqrt{n}x$ in this limit.

Putting it together with the prefactor, we have that

$$\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})=\frac{1}{\sqrt{r_2}}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\frac{n}{d_s}\left[\frac{j-j_3+1}{2j+1}+\frac{j^2+j-j_3^2-j_3}{(2j+1)(r_1-r_2+2)}\right] \tag{2.3.4.12}$$

and we can make the change of variables, going to continuous variables in the large N limit, as before.

Next we make the ansatz

$$\beta=nmf^{(0)}(x_j,x_{j_3})+n\sqrt{m}f^{(1)}(x_j,x_{j_3})+nf^{(2)}(x_j,x_{j_3})+\frac{nf^{(3)}(x_j,x_{j_3})}{\sqrt{m}}+\mathcal{O}\left(\frac{n}{m}\right) \tag{2.3.4.13}$$

We Taylor expand these coefficients along with the J_+ coefficients to leading order in n to obtain

$$\begin{aligned}&\beta_{1,1}(x,x_j,x_{j_3},n,m)\alpha_{0,\frac{1}{2}}(x+\frac{1}{\sqrt{n}},x_j+\frac{1}{\sqrt{m}},x_{j_3},n,m)\\&\approx n^2\left[\left(\frac{(x_j-x_{j_3})}{2\sqrt{2}x_j}\right)m^{\frac{3}{2}}+\left(\frac{x_j+4x_j^3+7x_{j_3}-4x_j^2x_{j_3}^2}{8\sqrt{2}x_j^2}\right)m+\mathcal{O}(m^{\frac{1}{2}})\right]f^0_{1,1}(x_j,x_{j_3})\end{aligned} \tag{2.3.4.14}$$

$$\begin{aligned}&\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m)\beta_{1,1}(\left(x+\frac{1}{2\sqrt{n}}\right)\sqrt{\frac{n}{n-1}},\left(x_j+\frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}},\left(x_{j_3}-\frac{1}{2\sqrt{m}}\right)\sqrt{\frac{m}{m+1}},n-1,m+1)\\&\approx\left[\left(\frac{(x_j-x_{j_3})}{2\sqrt{2}x_j}\right)m^{\frac{3}{2}}+\left(\frac{x_j+4x_j^3+3x_{j_3}-4x_j^2x_{j_3}^2}{8\sqrt{2}x_j^2}\right)m+\mathcal{O}(m^{\frac{1}{2}})\right]\\&\times\left[f^0_{1,1}(x_j,x_{j_3})+\frac{1}{2\sqrt{m}}\left(-\frac{\partial f^0_{1,1}}{\partial x_{j_3}}(x_j,x_{j_3})+\frac{\partial f^0_{1,1}}{\partial x_j}(x_j,x_{j_3})\right)+\mathcal{O}\left(\frac{1}{m}\right)\right]\end{aligned} \tag{2.3.4.15}$$

Dividing through by n^2m we have that

$$0=\frac{x_{j_3}}{2\sqrt{2}x_j^2}f^0_{1,1}(x_j,x_{j_3})-\frac{1}{2}\left[-\frac{\partial f}{\partial x_{j_3}}(x_j,x_{j_3})+\frac{\partial f}{\partial x_j}(x_j,x_{j_3})\right]\frac{x_j-x_{j_3}}{2\sqrt{2}x_j}+\mathcal{O}\left(\frac{1}{m}\right)^{\frac{1}{2}} \tag{2.3.4.16}$$

In the large N limit, terms that go like $\frac{1}{\sqrt{m}}$ are vanishing, so we can drop them. Multiplying through by $4\sqrt{2}x_j^2$, we are left with

$$0=2x_{j_3}f^0_{1,1}(x_j,x_{j_3})+x_j(x_{j_3}-x_j)\left(\frac{\partial f^0_{1,1}}{\partial x_j}(x_j,x_{j_3})-\frac{\partial f^0_{1,1}}{\partial x_{j_3}}(x_j,x_{j_2})\right) \tag{2.3.4.17}$$

This is a differential equation which allows for the computation of the one loop matrix elements of the dilatation operator in the displaced corners approximation at large N. The solution to the differential equation agrees with the explicit computation of the dilatation operator.

3 $SU(3)$

3.1 Numerical Calculation of Projectors

In this section, we will consider three fields: X, Y and Z. We will have n Z fields, as before, where $r \vdash n$, m Y fields where $s \vdash m$ and p X fields, where $t \vdash p$ so that our restricted Schur polynomials are labelled by Young diagrams $R, (r, s, t)$. Like with the $SU(2)$ case, our first step is to evaluate the trace of the products of two types of projectors. The first will be some representation $R, (r, s, t)$ and the second will be some representation $R, (r \pm 1, s \mp 1, t)$ or $R, (r \pm 1, s, t \mp 1)$ or $R, (r, s \pm 1, t \mp 1)$. This comes from the fact the we know that our generators act like
$\mathrm{Tr}\left(X\frac{d}{dY}\right)$, $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$, $\mathrm{Tr}\left(Z\frac{d}{dY}\right)$, $\mathrm{Tr}\left(Y\frac{d}{dZ}\right)$, $\mathrm{Tr}\left(X\frac{d}{dZ}\right)$ and $\mathrm{Tr}\left(Z\frac{d}{dX}\right)$.
We need 8 generators since we are working in $SU(3)$ which implies $3^2 - 1$ generators. The last two are
$\mathrm{Tr}\left(Y\frac{d}{dY} - Z\frac{d}{dZ}\right)$ and $\mathrm{Tr}\left(Z\frac{d}{dZ} + Y\frac{d}{dY} - 2X\frac{d}{dX}\right)$.

Consider $m + p = 5$. We pull three boxes off row 1 of our Young diagram and two boxes off row 2. We can construct $\frac{5!}{3!2!} = 10$ distinct states labelled by the following Young-Yamanouchi diagrams:

$|1\rangle =$
					5	4	3
		2	1				

$|2\rangle =$
					5	4	2
		3	1				

$|3\rangle =$
					5	4	1
		3	2				

$|4\rangle =$
					5	3	2
		4	1				

$|5\rangle =$
					5	3	1
		4	2				

$|6\rangle =$
					5	2	1
		4	3				

$|7\rangle =$
					4	3	2
		5	1				

$|8\rangle =$
					4	3	1
		5	2				

$|9\rangle =$
					4	2	1
		5	3				

$|10\rangle =$
					3	2	1
		5	3				

Note that these are the total number of distinct states. The dimension of the carrier space is $2^{m+p} = 32$ since we have the choice of taking each box from either row 1 or row 2. Now imagine we build t by first removing 2 boxes from row 1 and 1 box from row 2. We then pull s boxes off by removing 1 box from each row. This restricts us to 6 states. We can keep track of these states by assigning the numbers 4,5 to boxes in s and the numbers 1,2,3 to boxes in t. This leaves us with states $|4\rangle$ through $|9\rangle$.

Let's start off with the generator $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$. We can form t^- by dropping a t box from either the first or

second row. The following projectors, $P_{s,t}$, are possible:

$$P^{(1,1),(2,1)}_{\ydiagram{2},\ydiagram{2,1}} = P_{\ydiagram{5,3,1},\ydiagram{2},\ydiagram{2,1}} \tag{3.1.1}$$

$$P^{(1,1),(2,1)}_{\ydiagram{2},\ydiagram{3,1}} = P_{\ydiagram{5,3,1},\ydiagram{2},\ydiagram{3,1}} \tag{3.1.2}$$

$$P^{(1,1),(2,1)}_{\ydiagram{1,1},\ydiagram{2,1}} = P_{\ydiagram{5,3,1},\ydiagram{1,1},\ydiagram{2,1}} \tag{3.1.3}$$

$$P^{(1,1),(2,1)}_{\ydiagram{1,1},\ydiagram{3,1}} = P_{\ydiagram{5,3,1},\ydiagram{1,1},\ydiagram{3,1}} \tag{3.1.4}$$

$$P^{(2,1),(1,1)}_{\ydiagram{2,1},\ydiagram{2}} = P_{\ydiagram{5,3,1},\ydiagram{2,1},\ydiagram{2}} \tag{3.1.5}$$

$$P^{(2,1),(1,1)}_{\ydiagram{2,1},\ydiagram{1,1}} = P_{\ydiagram{5,3,1},\ydiagram{2,1},\ydiagram{1,1}} \tag{3.1.6}$$

$$P^{(2,1),(1,1)}_{\ydiagram{3,1},\ydiagram{2}} = P_{\ydiagram{5,3,1},\ydiagram{3,1},\ydiagram{2}} \tag{3.1.7}$$

$$P^{(2,1),(1,1)}_{\ydiagram{3,1},\ydiagram{1,1}} = P_{\ydiagram{5,3,1},\ydiagram{3,1},\ydiagram{1,1}} \tag{3.1.8}$$

$$P^{(1,2),(2,0)}_{\ydiagram{2,1},\ydiagram{2}} = P_{\ydiagram{5,3,1},\ydiagram{2,1},\ydiagram{2}} \tag{3.1.9}$$

$$P^{(1,2),(2,0)}_{\ydiagram{3,1},\ydiagram{2}} = P_{\ydiagram{5,3,1},\ydiagram{3,1},\ydiagram{2}} \tag{3.1.10}$$

where the last 6 are obtained by dropping an s box in favour of a t box.
We can compute the following traces numerically and the results we get are:

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{1}{9}\left(1-\frac{2}{r_1-r_2+1}+\frac{2}{r_1-r_2+2}\right) \tag{3.1.11}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}\right)=\frac{1}{3}\left(1-\frac{2}{r_1-r_2+1}\right) \tag{3.1.12}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\square\square\square,\square\square}\right)=\frac{2}{9}\left(1+\frac{2}{r_1-r_2+1}\right)\left(1+\frac{1}{r_1-r_2+2}\right) \tag{3.1.13}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\square\square\square,\square}\right)=\frac{2}{3}\left(1+\frac{1}{r_1-r_2+2}\right) \tag{3.1.14}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(1,2),(2,0)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{2}{9}\left(1+\frac{1}{r_1-r_2+1}\right) \tag{3.1.15}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(1,2),(2,0)}_{\square\square\square,\square\square}\right)=\frac{4}{9}\left(1-\frac{2}{r_1-r_2+1}\right) \tag{3.1.16}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{2}{9}\left(1-\frac{1}{r_1-r_2+1}\right)\left(1-\frac{2}{r_1-r_2+2}\right) \tag{3.1.17}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}\right)=0 \tag{3.1.18}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(2,1),(1,1)}_{\square\square\square,\square\square}\right)=\frac{4}{9}\left(1-\frac{2}{r_1-r_2+1}+\frac{2}{r_1-r_2+1}\right) \tag{3.1.19}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(2,1),(1,1)}_{\square\square\square,\square}\right)=0 \tag{3.1.20}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(1,2),(2,0)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{1}{9}\left(1+\frac{6}{r_1-r_2}-\frac{2}{r_1-r_2+1}\right) \tag{3.1.21}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square}P^{(1,2),(2,0)}_{\square\square\square,\square\square}\right)=\frac{2}{9}\left(1+\frac{4}{r_1-r_2+1}-\frac{3}{r_1-r_2}\right) \tag{3.1.22}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{1}{3}\left(1+\frac{2}{r_1-r_2+1}\right) \tag{3.1.23}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}\right)=1 \tag{3.1.24}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\square\square\square,\square\square}\right)=0 \tag{3.1.25}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(2,1),(1,1)}_{\square\square\square,\square}\right)=0 \tag{3.1.26}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(1,2),(2,0)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{2}{3}\left(1-\frac{1}{r_1-r_2+1}\right) \tag{3.1.27}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}P^{(1,2),(2,0)}_{\square\square\square,\square\square}\right)=0 \tag{3.1.28}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{2}{3}\left(1-\frac{1}{r_1-r_2+1}\right) \tag{3.1.29}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(2,1),(1,1)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}\right)=0=\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(2,1),(1,1)}_{\square\square\square,\square\square}\right)=\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(2,1),(1,1)}_{\square\square\square,\square}\right) \tag{3.1.30}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(1,2),(2,0)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square}\right)=\frac{1}{3}\left(1+\frac{2}{r_1-r_2+1}\right) \tag{3.1.31}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square,\square\square\square}P^{(1,2),(2,0)}_{\square\square\square,\square\square}\right)=0 \tag{3.1.32}$$

For the next few generators, we will include only a few of the possible projectors we can produce and therefore not all possible trace computations will feature (starting from the same $R,(r,s,t)$ as above.

Consider $\mathrm{Tr}\left(X\frac{d}{dY}\right)$. We will construct t^+ by removing three boxes of the first row amd one box off the sec-

ond. We will construct s^- by then taking one box from the second row. Some projectors are:

$$P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square\square\square\square\square\\\square\square\end{smallmatrix},\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \tag{3.1.33}$$

$$P^{(1,1),(2,1)}_{\square\square,\square\square\square} = P_{\begin{smallmatrix}\square\square\square\square\square\\\square\square\end{smallmatrix},\square\square,\square\square\square} \tag{3.1.34}$$

$$P^{(0,1),(3,1)}_{\square,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} = P_{\begin{smallmatrix}\square\square\square\square\\\square\square\end{smallmatrix},\square,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} \tag{3.1.35}$$

$$P^{(0,1),(3,1)}_{\square,\square\square\square\square} = P_{\begin{smallmatrix}\square\square\square\square\\\square\square\end{smallmatrix},\square,\square\square\square\square} \tag{3.1.36}$$

The states that feature for the first two projectors are states $|4\rangle$ through $|9\rangle$, as outlined earlier. The states that feature for the latter two projectors are $|7\rangle$ through $|10\rangle$. The traces we compute are

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} P^{(0,1),(3,1)}_{\square,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}\right) = 1 + \frac{1}{r_1 - r_2 + 1} \tag{3.1.37}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} P^{(0,1),(3,1)}_{\square,\square\square\square\square}\right) = 0 \tag{3.1.38}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square} P^{(0,1),(3,1)}_{\square,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}}\right) = \frac{1}{8}\left(1 + \frac{6}{r_1 - r_2} - \frac{2}{r_1 - r_2 + 1}\right) \tag{3.1.39}$$

$$\mathrm{Tr}\left(P^{(1,1),(2,1)}_{\square\square,\square\square\square} P^{(0,1),(3,1)}_{\square,\square\square\square\square}\right) = \frac{3}{8}\left(1 - \frac{2}{r_1 - r_2} + \frac{2}{r_1 - r_2 + 1}\right) \tag{3.1.40}$$

3.2 Displaced Corners Approximation

The computation is almost exactly the same as for the $su(2)$ generators, so we will only point out what the differences are and quote the final results. Note that the result of this section and the following one have been put together in the paper [25].

It is again simplest to focus on the two row example and to use the distant corners approximation. The translation of the restricted Schur polynomial $\chi_{R,(r,s,t)}(Z,Y,X)$ to $SU(2)$ state labels is as follows

$$\begin{aligned} t_1 &= \frac{p}{2} + k \qquad t_2 = \frac{p}{2} - k\,, \\ s_1 &= \frac{m}{2} + j \qquad s_2 = \frac{m}{2} - j\,, \\ R_1 &= r_1 + \frac{m+p}{2} + j_3 + k_3 \qquad R_2 = r_2 + \frac{m+p}{2} - j_3 - k_3\,. \end{aligned} \tag{3.2.1}$$

It appears as though the above labels may appear to be over complete. That is, given n, m, p as well as $r, k, j, k_3 + j_3$ we can reconstruct the Young diagram labels R, r, s and t. It seems that we need only the sum $k_3 + j_3$ and not the individual values j_3, k_3. This is certainly enough to construct r. The point is that, even when R has two rows, when we restrict S_{p+m+n} to $S_p \times S_m \times S_n$ we need a multiplicity label. Specifying k_3 and j_3 independently resolves the multiplicity. The simplest way to see this is to note that we can first restrict S_{p+m+n} to $S_p \times S_{m+n}$ without multiplicity, and then restrict S_{m+n} to $S_m \times S_n$, again without multiplicity. The first restriction introduces (k, k_3) and the second (j, j_3). To illustrate this, consider removing $m + p = 5$ boxes from R such that $j_3 + k_3 = \frac{1}{2}$. We remove the boxes that will build t first. We have

$$\begin{smallmatrix}\square\square\square\square\square\\\square\square\end{smallmatrix} \tag{3.2.2}$$

For t we can have only the Young diagram $\square\square$ and for s we can have either $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ or $\square\square\square$. But $j_3 + k_3 = \frac{1}{2}$ could also imply we removed boxes like

$$\begin{smallmatrix}\square\square\square\square\square\\\square\square\end{smallmatrix} \tag{3.2.3}$$

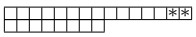
Now we can construct either $\square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ for t and either $\square\square\square$ or $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ for s. There is clearly some ambiguity and it is resolved by specifying both j_3 and k_3 separately[16].

[16] This is equivalent to specifying the representation R/p which is the Young diagram R after p boxes have been removed. We will use this notation when we calculate the exact generators in Section 3.3.

In the distant corners approximation, the computation of the traces needed to compute the generators is reduced to the computation of $SU(2)$ Clebsch-Gordan coefficients. Since the boxes associated to the X fields (organized by Young diagram $t \vdash p$) are removed first, the generators $\mathrm{Tr}\left(Z\frac{d}{dY}\right)$ and $\mathrm{Tr}\left(Y\frac{d}{dZ}\right)$ are unchanged from the formulas we obtained above. If we compute the action of $\mathrm{Tr}\left(X\frac{d}{dY}\right)$, the action of any other generator can be computed by taking the hermitian conjugate or by using the $su(3)$ algebra. Consequently, we only need (and quote) the action of $\mathrm{Tr}\left(X\frac{d}{dY}\right)$.

3.2.1 Calculating the $SU(2)$ Clebsch-Gordon coefficients

We start by illustrating the notation we will be using in the following sections.
Consider the Young diagram R.



We build the Young diagram s, $\square\square$. Each block carries angular momentum $j = \frac{1}{2}$. Blocks pulled of row 1 of R have spin $\frac{1}{2}$ and blocks pulled off row 2 of R have spin $-\frac{1}{2}$. The states we build are labelled by $|j, j_3\rangle$. For s we have

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = |1,1\rangle \tag{3.2.1.1}$$

This gives us that $j = j_3 = 1$, which is indeed what we have[17]. If we instead pull a box off each row, we can construct two representations: $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ for s. We have

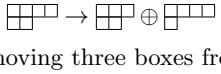
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = \#_1\,|0,0\rangle + \#_2\,|1,0\rangle \tag{3.2.1.2}$$

where $\#$ is some number. The coefficient of $\#_1$ are states with $j = j_3 = 0$, which are represented by all Young Yamanouchi states in $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. The coefficient of $\#_2$ are states with $j = 1$, $j_3 = 0$ which are represented by all Young Yamanouchi states in $\square\square$. Lastly we consider pulling two boxes off the second row of R. We have

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle = |1,-1\rangle \tag{3.2.1.3}$$

which again reproduces what we have: $j = 1$ and $j_3 = -1$. So we really are working in angular momentum representations of $SU(2)$.

Consider some Young-Yamanouchi state, $|j, j_3\rangle$ labelled by s. We can decompose a Young diagram into the direct product of two subgroup representations by removing a box from either the first or second row. For example



(3.2.1.4)

Suppose we pick $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$. We can form this by removing three boxes from row 1 of R and two boxes from row 2 of R or the other way around. Then

$$|j, j_3\rangle = \#_1 \left|j-\frac{1}{2}, j_3-\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \#_2 \left|j-\frac{1}{2}, j_3+\frac{1}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle \tag{3.2.1.5}$$

For the other diagram,

$$|j, j_3\rangle = \#_3 \left|j+\frac{1}{2}, j_3-\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \#_4 \left|j+\frac{1}{2}, j_3+\frac{1}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle \tag{3.2.1.6}$$

In this way we can decouple the original state $|j, j_3\rangle$ into one either the $j - \frac{1}{2}$ representation or the $j + \frac{1}{2}$ representation. Since our Young diagrams represent a collection of Young-Yamanouchi states, we include the index i to keep track of these states and to introduce a summation over them. We need to assign limits to these sums. We know the number of states in a representation is equal to the dimension of the representation so we sum from 1 to $d_{s^{m-1}_{j\pm\frac{1}{2}}}$. This notation is a bit cumbersome so we will use $s^{m-1}_{j\pm\frac{1}{2}}$ instead, noting that we mean the dimension of this space. By summing over all Young-Yamanouchi states we are building the full representation.

[17] We are using $|l_1, m_1\rangle \otimes |l_2, m_2\rangle = \sum_{l=|l_1-l_2|}^{l_1+l_2} |l, m_1 + m_2\rangle$ and $m = -l, \ldots, l$. Here m corresponds to the j_3 label and l to the j label.

Now we need to determine the coefficients $\#_i \ \forall i = 1, \ldots, 4$. Consider

$$\left\langle \frac{1}{2}, \frac{1}{2} \middle| \left\langle j - \frac{1}{2}, j_3 - \frac{1}{2} \middle| j, j_3 \right\rangle = \#_1 \left\langle \frac{1}{2}, \frac{1}{2} \middle| \left\langle j - \frac{1}{2}, j_3 - \frac{1}{2} \middle| j - \frac{1}{2}, j_3 - \frac{1}{2} \right\rangle \middle| \frac{1}{2}, \frac{1}{2} \right\rangle = \#_1 \tag{3.2.1.7}$$

Notice that the inner products in the coefficients of $\#_2$, $\#_3$ and $\#_4$ are zero. We can do something similar to obtain each $\#_i$. The term

$$\left\langle \frac{1}{2}, \frac{1}{2} \middle| \left\langle j - \frac{1}{2}, j_3 - \frac{1}{2} \middle| j, j_3 \right\rangle$$

is the Clebsch-Gordon coefficient. These are calculated with the formula

$$\begin{aligned} \langle j_1, m_1; j_2, m_2 | j, m \rangle = \delta_{m, m_1 + m_2} & \sqrt{\frac{(2j+1)(j + j_1 - j_2)!(j - j_1 + j_2)!(j_1 + j_2 - j)!}{(j_1 + j_2 + j + 1)!}} \\ & \times \sqrt{(j+m)!(j-m)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!} \\ & \times \sum_k \frac{(-1)^k}{k!(j_1 + j_2 - j - k)!(j_1 - m_1 - k)!(j_2 + m_2 - k)!(j - j_2 + m_1 + k)!(j - j_1 - m_2 + k)!} \end{aligned} \tag{3.2.1.8}$$

where we have replaced the j_3 with m for notational convenience. Our Clebsch-Gordon coefficients are

$$\left\langle \frac{1}{2}, \frac{1}{2}; j - \frac{1}{2}, j_3 - \frac{1}{2} \middle| j, j_3 \right\rangle = \sqrt{\frac{j + j_3}{2j}} \tag{3.2.1.9}$$

$$\left\langle \frac{1}{2}, \frac{1}{2}; j + \frac{1}{2}, j_3 - \frac{1}{2} \middle| j, j_3 \right\rangle = \sqrt{\frac{j - j_3 + 1}{2)j + 1)}} \tag{3.2.1.10}$$

$$\left\langle \frac{1}{2}, -\frac{1}{2}; j - \frac{1}{2}, j_3 + \frac{1}{2} \middle| j, j_3 \right\rangle = \sqrt{\frac{j - j_3}{2j}} \tag{3.2.1.11}$$

$$\left\langle \frac{1}{2}, -\frac{1}{2}; j + \frac{1}{2}, j_3 + \frac{1}{2} \middle| j, j_3 \right\rangle = -\sqrt{\frac{j + j_3 + 1}{2(j+1)}} \tag{3.2.1.12}$$

3.2.2 Calculating the Traces

Our projection operators are constructed as follows

$$\begin{aligned} P^{m,p} &= P^m(j, j_3) \otimes P^p(k, k_3) \\ &= \sum_{i=1}^{s_j^m} |j, j_3; i\rangle\langle j, j_3; i| \otimes \sum_{I=1}^{t_k^p} |k, k_3; I\rangle\langle k, k_3; I| \end{aligned} \tag{3.2.2.1}$$

We saw how to decouple states in $s \vdash m$ in terms of $s^- \vdash m - 1$ in the previous section. The idea is similar for $t \vdash p$, which we want to decouple in terms of $t^+ \vdash p + 1$. We adapt the ideas by starting with t^+ and working out the decomposition in terms of representations of t. For example, we will have

$$\begin{aligned} \left| k + \frac{1}{2}, k_3 - \frac{1}{2} \right\rangle &= \sqrt{\frac{(k + \frac{1}{2}) - (k_3 - \frac{1}{2})}{2(k + \frac{1}{2})}} |k, k_3\rangle + \sqrt{\frac{(k + \frac{1}{2}) + (k_3 - \frac{1}{2})}{2(k + \frac{1}{2})}} |k, k_3 - 1\rangle \\ &= \sqrt{\frac{k - k_3 + 1}{2k + 1}} |k, k_3\rangle + \sqrt{\frac{k + k_3}{2k + 1}} |k, k_3 - 1\rangle \end{aligned} \tag{3.2.2.2}$$

These Clebsch-Gordon coefficients were obtained by translating $k \to k + \frac{1}{2}$ and $k_3 \to k_3 - \frac{1}{2}$ in the formulae calculated in the previous section. In this notation, we do not decompose states in $t \vdash p$ labelled by $|k, k_3\rangle$ but rather states in $t^+ \vdash p + 1$. We will see this explicitly when we calculate

$$\mathrm{Tr}(P^{m,p} \cdot P^{m-1,p+1})$$

This trace is only nonzero if the m^{th} box is the $(p+1)^{th}$ box. That is, the $j_3 \pm \frac{1}{2}$ and $k_3 \pm \frac{1}{2}$ labels must match up so that when j_3 increases, k_3 decreases. This is because j_3 tells us which row in R we are removing an s box

from. This box becomes a t box, and k_3 tells us from which row in R we are getting this box. Thus, there are only eight possible combinations of $j \pm \frac{1}{1}$, $k \pm \frac{1}{2}$, $j_3 \pm \frac{1}{2}$ and $k_3 \pm \frac{1}{2}$ in the above trace and not sixteen. We calculate one of these traces for illustration, thereafter we quote the answers.

$$\begin{aligned}
&\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j+\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})\right) = \mathrm{Tr}\Big[\sum_i \Big(\sqrt{\frac{j-j_3+1}{2(j+1)}}\Big|j+\frac{1}{2},j_3-\frac{1}{2};i\Big\rangle\Big|\frac{1}{2},\frac{1}{2}\Big\rangle \\
&-\sqrt{\frac{j+j_3+1}{2(j+1)}}\Big|j+\frac{1}{2},j_3+\frac{1}{2};i\Big\rangle\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\Big)\Big(\sqrt{\frac{j-j_3+1}{2(j+1)}}\Big|j+\frac{1}{2},j_3-\frac{1}{2};i\Big\rangle\Big|\frac{1}{2},\frac{1}{2}\Big\rangle \\
&-\sqrt{\frac{j+j_3+1}{2(j+1)}}\Big|j+\frac{1}{2},j_3+\frac{1}{2};i\Big\rangle\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\Big)^T \otimes \sum_I |k,k_3;I\rangle\langle k,k_3;I| \cdot \sum_i \Big|j+\frac{1}{2},j_3-\frac{1}{2};i\Big\rangle\Big\langle j+\frac{1}{2},j_3-\frac{1}{2};i\Big| \\
&\otimes \sum_I \Big(-\sqrt{\frac{k+k_3+1}{2k+1}}|k,k_3+1;I\rangle\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle + \sqrt{\frac{k-k_3}{2k+1}}|k,k_3;I\rangle\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\Big) \\
&\times\Big(-\sqrt{\frac{k+k_3+1}{2k+1}}|k,k_3+1;L\rangle\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle + \sqrt{\frac{k-k_3}{2k+1}}|k,k_3;L\rangle\Big|\frac{1}{2},\frac{1}{2}\Big\rangle\Big)^T\Big] \\
&= \left(\frac{j-j_3+1}{2(j+1)}\right)\left(\frac{k-k_3}{2k+1}\right)\sum_{i,I}\delta_{ii}\delta_{II} \\
&= \left(\frac{j-j_3+1}{2(j+1)}\right)\left(\frac{k-k_3}{2k+1}\right) d_{s^{m-1}_{j+\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.3)
\end{aligned}$$

The rest of the traces are:

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j+\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})\right) = \left(\frac{j-j_3+1}{2(j+1)}\right)\left(\frac{k+k_3+1}{2k+1}\right)) d_{s^{m-1}_{j+\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.4)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j+\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})\right) = \left(\frac{j+j_3+1}{2(j+1)}\right)\left(\frac{k-k_3+1}{2k+1}\right) d_{s^{m-1}_{j+\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.5)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j+\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})\right) = \left(\frac{j+j_3+1}{2(j+1)}\right)\left(\frac{k+k_3}{2k+1}\right) d_{s^{m-1}_{j+\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.6)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j-\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})\right) = \left(\frac{j-j_3}{2j}\right)\left(\frac{k-k_3+1}{2k+1}\right) d_{s^{m-1}_{j-\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.7)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j-\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})\right) = \left(\frac{j-j_3}{2j}\right)\left(\frac{k+k_3}{2k+1}\right) d_{s^{m-1}_{j-\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.8)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j-\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})\right) = \left(\frac{j+j_3}{2j}\right)\left(\frac{k+k_3+1}{2k+1}\right) d_{s^{m-1}_{j-\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.9)$$

$$\mathrm{Tr}\left(P^{m,p}(j,j_3,k,k_3)\cdot P^{m-1,p+1}(j-\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})\right) = \left(\frac{j+j_3}{2j}\right)\left(\frac{k-k_3}{2k+1}\right) d_{s^{m-1}_{j-\frac{1}{2}}} d_{t^p_k} \qquad (3.2.2.10)$$

3.2.3 The Displaced Corners Generators

The matrix elements of the generator, $\mathrm{Tr}\left(X\frac{d}{dY}\right)$, have the form

$$\sqrt{\frac{\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_t}{\mathrm{hooks}_r\mathrm{hooks}_{s^-}\mathrm{hooks}_{t^+}}}\frac{(p+1)}{d_{s^-}d_{t^+}}\delta_{RT}\delta_{ru}\mathrm{Tr}_R\left(P^{m,p}\cdot P^{m-1,p+1}\right) \qquad (3.2.3.1)$$

We have already calculated the traces in the previous section. We need to calculate the coefficients of the traces now. We can rewrite the above as

$$\sqrt{\frac{\mathrm{hooks}_s\mathrm{hooks}_{t^+}}{\mathrm{hooks}_{s^-}\mathrm{hooks}_t}}\frac{\delta_{RT}\delta_{ru}}{d_{s^-}d_t}\mathrm{Tr}_R\left(P^{m,p}\cdot P^{m-1,p+1}\right) \qquad (3.2.3.2)$$

Notice that the dimensions of the representations s^- and t will cancel out with the same dimensions which came out of calculating the traces. The calculation of the hooks is the same as with $SU(2)$ and we get

$$\begin{aligned}
j \to j+\frac{1}{2} &\implies \frac{\text{hooks}_s}{\text{hooks}_{s^-}} = \frac{(\frac{m}{2}-j)(2j+2)}{2j+1} \\
j \to j-\frac{1}{2} &\implies \frac{\text{hooks}_s}{\text{hooks}_{s^-}} = \frac{(\frac{m}{2}+j+1)(2j)}{2j+1} \\
k \to k+\frac{1}{2} &\implies \frac{\text{hooks}_{t^+}}{\text{hooks}_t} = \frac{(\frac{p}{2}+k+2)(2k+1)}{2k+2} \\
k \to k-\frac{1}{2} &\implies \frac{\text{hooks}_{t^+}}{\text{hooks}_t} = \frac{(\frac{p}{2}-k+1)(2k+1)}{2k}
\end{aligned}$$

We find

$$\begin{aligned}
&\text{Tr}\left(X\frac{d}{dY}\right)O^{(n,m,p)}_{R,r,j,j_3,k,k_3} \\
&= \frac{j+j_3}{2j}\frac{k+k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}} \\
&\quad+\frac{j+j_3}{2j}\frac{k-k_3}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}} \\
&+\frac{j-j_3+1}{2j+2}\frac{k+k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}} \\
&\quad+\frac{j-j_3+1}{2j+2}\frac{k-k_3}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}} \\
&+\frac{j-j_3}{2j}\frac{k-k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}} \\
&\quad+\frac{j-j_3}{2j}\frac{k+k_3}{2k+1}\sqrt{\left(\frac{m}{2}+j+1\right)\frac{2j}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j-\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}} \\
&+\frac{j+j_3+1}{2j+2}\frac{k-k_3+1}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}+k+2\right)\frac{2k+1}{2k+2}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}} \\
&\quad+\frac{j+j_3+1}{2j+2}\frac{k+k_3}{2k+1}\sqrt{\left(\frac{m}{2}-j\right)\frac{2j+2}{2j+1}}\sqrt{\left(\frac{p}{2}-k+1\right)\frac{2k+1}{2k}}O^{(n,m-1,p+1)}_{R,r,j+\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}}\,.
\end{aligned} \tag{3.2.3.3}$$

Once again, the $su(3)$ algebra is not obeyed exactly. It is obeyed to leading order in $\frac{m}{n}$ and $\frac{p}{n}$, which is exactly what we'd expect.

3.3 Exact Generators

In this section we compute the matrix elements of the $su(3)$ generators exactly. For the $su(2)$ algebra, we only needed to compute J_-. We obtain J_+ by taking the hermitian conjugate of J_- and the action of J_3 follows from the algebra. For the $su(3)$ algebra we again only need to compute a single generator. First, it follows that since the boxes associated to the X fields (organized by Young diagram $t \vdash p$) are removed first, the generators $\text{Tr}\left(Z\frac{d}{dY}\right)$ and $\text{Tr}\left(Y\frac{d}{dZ}\right)$ are unchanged from the formulas we obtained for $su(2)$. We will compute the action of $\text{Tr}\left(X\frac{d}{dY}\right)$. The action of $\text{Tr}\left(Y\frac{d}{dX}\right)$ follows from this by taking the hermitian conjugate and the action of any other generator can then be computed using the $su(3)$ algebra. Consequently, we only need the action of $\text{Tr}\left(X\frac{d}{dY}\right)$.

The approach that worked for the $su(2)$ algebra can be used to compute the $su(3)$ generators. In particular, there are again identities we can derive for traces over sums of projectors, and specific matrix elements that we are able to compute exactly. These identities then completely determine the traces and hence the generator in general. In the end however, the final results have been verified by checking that the generators reduce to the displaced corners result, close the correct algebra and give the correct action for small values of m, n and p where the action can be computed explicitly.

In what follows we freely move between the[18] $R, R/p, (r,s,t)$ and R, r, j, j_3, k, k_3 labelling of our operators. Henceforth we refine our notation by writing the number of Zs (n), the number of Ys (m) and the number of Xs (p)

[18]Recall that the labelling R/p is equivalent to specifying j_3 and k_3 separately. See the comment after (3.2.1).

using a superscript.

3.3.1 The Structure of the Problem

The states of irrep (r,s,t) of $S_n \times S_m \times S_p$ are obtained in the carrier space of irrep R of S_{n+m+p}, by associating boxes inside R with r (Z boxes), s (Y boxes) or t (X boxes). These three species of boxes are filled independently as follows: X boxes are filled with the labels $1,2,...,p$, Y boxes with $p+1,p+2,...,p+m$ and Z boxes with $p+m+1,p+m+2,...,p+m+n$. Each species is filled independently of the others. It is clear that each vector belonging to this basis is the tensor product of a Z vector, a Y vector and an X vector. The Z boxes are already organized according to r. Thus, we can write our projector as

$$\begin{aligned} P_{R,(r,s,t)} &= \frac{1}{m!}\sum_{\sigma\in S_m}\chi_s(\sigma)\Gamma_R(\sigma)\,\frac{1}{p!}\sum_{\tau\in S_p}\chi_t(\tau)\Gamma_R(\tau) \\ &\equiv \mathbf{1}_r^R \otimes P_s^R \otimes P_t^R \end{aligned} \tag{3.3.1.1}$$

where we have restricted this operator to the subspace spanned by states filled as we just described and S_m acts on $p+1,p+2,...,p+m$ while S_p acts on $1,2,...,p$. The tensor product structure is also evident in this projector. To compute the trace

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,s,t)}\,P^{n,m-1,p+1}_{R,(r,s^-,t^+)}) = \mathrm{Tr}(\mathbf{1}_r^R \otimes P_s^R \otimes P_t^R\,\mathbf{1}_r^R \otimes P_{s^-}^R \otimes P_{t^+}^R) \tag{3.3.1.2}$$

we can decompose P_s^R into a sum of 2 projectors onto S_{m-1} irreps

$$P_s^R = \alpha_1 P_{s_1'}^R + \alpha_2 P_{s_2'}^R \tag{3.3.1.3}$$

and $P_{t^+}^R$ into a sum of 2 projectors onto S_p irreps

$$P_{t^+}^R = \beta_1 P_{t_1^{+\prime}}^R + \beta_2 P_{t_2^{+\prime}}^R\,. \tag{3.3.1.4}$$

Our notation uses T_i' to denote the Young diagram obtained by dropping a box from row i of T. Assuming, for example, that $s_1' = s^-$ and $t_2^{+\prime} = t$ we find

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,s,t)}\,P^{n,m-1,p+1}_{R,(r,s^-,t^+)}) = d_r d_{s^-} d_t \alpha_1\beta_2\,. \tag{3.3.1.5}$$

Clearly then, our traces are determined once we have understood how to decompose the projectors. The problems (3.3.1.3) and (3.3.1.4) are independent of each other. The problem of determining the decomposition (3.3.1.4), can be understood by studying the $S_{n+m+p} \to S_{n+m} \times S_p$ problem. This shows that the results of the decomposition (3.3.1.4) can be read straight from the $su(2)$ trace results of the previous section (formulas (2.3.2.1), (2.3.2.2), (2.3.2.3) and (2.3.2.4)), after replacing $r_1 - r_2 + 1 \to r_1 - r_2 + 2j_3 + 1$. This can be understood by noting that, for the $su(2)$ case, the displaced corners result for the traces and the exact traces differed by a factor proportional to $\frac{1}{r_1-r_2+1}$. The denominator is the content of the last box in row 1 of r less the content of the last box of row 2 in r. If we swap these boxes in some Young-Yamanouchi state of r, we get the state back with some prefactor and the state where the boxes are swapped with a prefactor proportional to $\frac{1}{r_1-r_2+1}$. When we consider the representation R/p, the content of the last box in row 1 is r_1+m_1-1 (where m_1 is the number of boxes pulled off row 1 of R/p to construct s) and the content of the last box in row 2 is r_1+m_2-2 (where m_2 is the number of boxes pulled off row 2 of R/p to construct s). The difference between these contents is $r_1-r_2+2j_3+1$.

The above argument motivates that we only need to study the decomposition (3.3.1.3).

3.3.2 Traces and Identities Needed

Since the two decompositions (3.3.1.4) and (3.3.1.3) are independent of each other, when studying the problem (3.3.1.3) we can simplify things enormously by setting $p=0$. Consequently, we need only study the problem of computing

$$\mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\,P^{n,m-1,1}_{R,(r,s^-,\square)}) \tag{3.3.2.1}$$

This collection of traces can again be computed by deriving two trace identities as we did for the $su(2)$ problem. The simplest identity, derived exactly as for the $su(2)$ case, is

$$\sum_{s} \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,s^-,\square)}) = d_r d_{s^-} \tag{3.3.2.2}$$

The second identity we need follows from considering

$$\sum_{s^-} \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,s^-,\square)}) = \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, \hat{C}) \tag{3.3.2.3}$$

Assume for concreteness that $\square$ in $P^{n,m-1,1}_{R,(r,s^-,\square)}$ sits in the first row of R. On the LHS of the above equation, since we sum over all s^-, the sum $\sum_{s^-} P^{n,m-1,1}_{R,(r,s^-,\square)}$ projects the Z boxes to r and the first box (corresponding to $\square$) to the right most box of the first row of R. Since $P^{n,m,0}_{R,(r,s)}$ already projects the Z boxes to r, the net affect is that $\sum_{s^-} P^{n,m-1,1}_{R,(r,s^-,\square)}$ projects $\square$ to the right most box of the first row of R. Arguing exactly as we did for the $su(2)$ problem, we can accomplish this using a Jucys-Murphy element. Denote the number of boxes removed from row i of R to obtain r by m_i. A simple argument now shows that we can write $\hat{C}$ as follows

$$\hat{C} = \frac{\sum_{i=2}^{m+n}(i,1) - (r_2 + m_2 - 2)}{r_1 - r_2 + m_1 - m_2 + 1} \tag{3.3.2.4}$$

With this form we have

$$\mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, \hat{C}) = \frac{d_r \chi_s((m,m-1))(m-1) + n\chi_{R,(r,s)}(m,m+1) - (r_2+m_2-2)d_r d_s}{r_1 - r_2 + 2j_3 + 1} \tag{3.3.2.5}$$

This is very similar to what we had before, and we see that the same restricted character appears. A new feature is the appearance of an S_m character, which is easily evaluated using the Murnaghan-Nakayama rule[26] (recall that $s = 2j$)

$$\chi_s((m,m-1)) = d_s \frac{2j + 2j^2 + \frac{m}{2}(m-4)}{m(m-1)} \tag{3.3.2.6}$$

This is everything we need to evaluate (3.3.2.5). In the end we find

$$\sum_{s^-} \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,s^-,\square)}) = d_r d_s \left[\frac{m + 2j_3}{2m} + \frac{2j(j+1) - 2j_3^2 - m}{2m(r_1 - r_2 + 2j_3 + 1)}\right]. \tag{3.3.2.7}$$

Remarkably, these are exactly the same identities we had from the $su(2)$ problem, except that $r_1 - r_2$ is replaced with $r_1 - r_2 + 2j_3$. There are only two non-zero terms on the LHS of the identities (3.3.2.2) and (3.3.2.7).

There are a number of traces that are simple enough that they can be computed immediately. The traces

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,j,k,k)} P^{n,m-1,p+1}_{R,(r,j-1,j-1,k+1,k+1)}) = d_r \tag{3.3.2.8}$$

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,-j,k,-k)} P^{n,m-1,p+1}_{R,(r,j-1,-j+1,k+1,-k-1)}) = d_r \tag{3.3.2.9}$$

follow because the subspaces we project to are the same for both projectors appearing in the trace. For the special case that $r_1 = r_2$, the Y boxes are already organized in an irrep (the Y boxes in row 2 of R sit beneath the Y boxes in row 1, which fixes the irrep) so that we reduce to the formulae we obtained for the $su(2)$ case. The traces we obtain in this way are as follows

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,j,k,k_3)} P^{n,m+1,p-1}_{R,(r,j-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})}) = d_r d_s d_{t^-} \left[\frac{k-k_3}{2k} + \frac{k^2-k_3^2}{2k(2j+1)}\right] \tag{3.3.2.10}$$

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,j,k,k_3)} P^{n,m+1,p-1}_{R,(r,j-\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})}) = d_r d_s d_{t^-} \left[\frac{k+k_3+1}{2k+2} - \frac{(k+1)^2-k_3^2}{(2k+2)(2j+1)}\right] \tag{3.3.2.11}$$

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,j,k,k_3)} P^{n,m+1,p-1}_{R,(r,j+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})}) = d_r d_s d_{t^-} \left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(2j+1)}\right] \tag{3.3.2.12}$$

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,j,j,k,k_3)} P^{n,m+1,p-1}_{R,(r,j+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})}) = d_r d_s d_{t^-} \left[\frac{k-k_3+1}{2k+2} + \frac{(k+1)^2-k_3^2}{(2k+2)(2j+1)}\right] \tag{3.3.2.13}$$

We now have enough to write the general results.

3.3.3 Exact Traces

We have understood now why the traces we will have will look like the $su(2)$ traces. To summarise, the differences are that we take a product of the trace in relevant matrix elements of J_- multiplied with the trace in corresponding relevant matrix elements of J_+, the contents of last boxes change such that $r_1 - r_2 \to r_1 - r_2 + 2j_3$, and r_1 and r_2 are fixed under the action of $\mathrm{Tr}\left(X\frac{d}{dY}\right)$ and $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$.

The exact results for the traces we need are

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j+\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k+k_3}{2k} - \tfrac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j+j_3+1}{2j+1} + \tfrac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \quad (3.3.3.1)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j-\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k+k_3}{2k} - \tfrac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j-j_3}{2j+1} - \tfrac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \quad (3.3.3.2)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j+\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k-k_3+1}{2k+2} + \tfrac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j+j_3+1}{2j+1} + \tfrac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \quad (3.3.3.3)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j-\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k-k_3+1}{2k+2} + \tfrac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j-j_3}{2j+1} - \tfrac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \quad (3.3.3.4)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k-k_3}{2k} + \tfrac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j-j_3+1}{2j+1} - \tfrac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \quad (3.3.3.5)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j-\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k-k_3}{2k} + \tfrac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j+j_3}{2j+1} + \tfrac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \quad (3.3.3.6)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k+k_3+1}{2k+2} - \tfrac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j-j_3+1}{2j+1} - \tfrac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \quad (3.3.3.7)$$

$$\mathrm{Tr}(P^{n,m,p}_{r_1,j,j_3,k,k_3}P^{n,m+1,p-1}_{r_1,j-\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2}}) = d_r d_s d_{t^-}\left[\tfrac{k+k_3+1}{2k+2} - \tfrac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\left[\tfrac{j+j_3}{2j+1} + \tfrac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \quad (3.3.3.8)$$

These traces exactly reproduce the numerical examples given in Section 3.1. In the displaced corners approximation (and by daggering), terms with denominator that go like $r_1 - r_2 + 2j_3$ go to zero so that we reproduce the traces we calculated in Section 3.2.2.

3.3.4 Exact Matrix Elements of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$

The action on our normalised operator is

$$\mathrm{Tr}\left(Y\frac{d}{dX}\right)O^{n,m,p}_{r_1,j,j_3,k,k_3} = \sum_{a,b,c,d=-\frac{1}{2}}^{\frac{1}{2}} \alpha(a,b,c,d)\, O^{n,m+1,p-1}_{r_1,j+a,j_3+b,k+c,k_3+d} \quad (3.3.4.1)$$

where

$$\alpha(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j+j_3+1}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \quad (3.3.4.2)$$

$$\alpha(-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right]$$

$$\times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j-j_3}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{3.3.4.3}$$

$$\alpha(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})=\sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k-k_3+1}{2k+2}+\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j+j_3+1}{2j+1}+\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{3.3.4.4}$$

$$\alpha(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})=\sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k-k_3+1}{2k+2}+\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j-j_3}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{3.3.4.5}$$

$$\alpha(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})=\sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k-k_3}{2k}+\frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j-j_3+1}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{3.3.4.6}$$

$$\alpha(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})=\sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k-k_3}{2k}+\frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j+j_3}{2j+1}+\frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{3.3.4.7}$$

$$\alpha(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})=\sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k+k_3+1}{2k+2}-\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j-j_3+1}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{3.3.4.8}$$

$$\alpha(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})=\sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k+k_3+1}{2k+2}-\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right]$$
$$\times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j+j_3}{2j+1}+\frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{3.3.4.9}$$

Notice that we can obtain $\mathrm{Tr}\left(X\frac{d}{dY}\right)$ by swapping the variables $j\leftrightarrow k$, $j_3\leftrightarrow k_3$ and $m\leftrightarrow p$. This is due to the symmetry of our generators. We can see this because in both cases we remove $m+p$ boxes off R. We have chosen the convention that we pull p boxes off first but we could have chosen to pull m boxes off first at the cost of the variable changes listed above. By doing this we see that we reproduce the results of Section 3.2.3 in the displaced corners approximation.

The complete set of generators can now easily be obtained by hermitian conjugation and by using the $su(3)$ algebra. It is straightforward to check that they close the correct algebra and check their action for small n, m and p like we did for the $su(2)$ case.

4 Discussion

4.1 Summary

Motivation was provided for claiming a duality between $\mathcal{N} = 4$ super Yang-Mills theory and string theory on an $AdS_5 \times S^5$ background. Giant gravitons were studied in the string theory which had an interpretation in the Young diagram representation used to label inequivalent, irreducible representations in the matrix sector of the field theory. We went on to motivate matrix model field theories at large N as a tool for learning about the dynamics of strings and gravitons. We studied a toy model for this field theory and saw that summing ribbon graphs was related to summing over surfaces. This computation is simplified by using the Schur polynomial basis which diagonalises the two point function. The problem of building the basis of Schur polynomials is equivalent to finding a set of projectors on $V_N^{\otimes n}$. This allowed us to replace a question in field theory with one in group theory.

The restricted Schur polynomials provided a basis for the multi-matrix field theory. We saw how to construct the projection operators for the field theory with two complex matrices ($SU(2)$) and three complex matrices ($SU(3)$). We constructed the global symmetry generators for $su(2)$ and $su(3)$ which rotates these complex fields. We did this in both the displaced corners approximation and exactly. We were able to do this by considering the action of the generators on restricted Schur polynomials and noticing that this was proportional to the trace of the product of projection operators. Thus, our computation centred around computing these traces analytically. We checked our results with some numerical examples and were able to recover the displaced corners' results by taking the appropriate limits of the exact generators.

Lastly, we exploited the form of the action of our generators and we used the relation $[D_2, J_\pm] = 0$ to show how one would find a recursion relation to solve for matrix elements of the dilatation operator in the displaced corners approximation. The key insight here was that the dilatation operator conserved j_3 in the displaced corners approximation at large N.

4.2 Outlook

The computations of this dissertation have been motivated by work done in [13], which suggest a method for exploring the action of the dilatation operator. Here the generators for the $su(2)$ sector were computed in the displaced corners approximation. We saw, using a well motivated guess for the form of the action of the dilatation operator given in [13], that we could obtain a set of recursion relations by enforcing the commutation relation between the dilatation operator and the $su(2)$ generators. The results of that paper reproduce the known one and two loop results and provide definite predictions for the higher loop structure of the dilatation operator which is supported by work done in [27, 28].

Since the results of [13] are in the displaced corners approximation, they are restricted to small deformations of $\frac{1}{2}$-BPS. The fact that the exact generators have been computed in this dissertation means that imposing that the generators must commute with the dilatation operator will lead to a new set of recursion relations. These recursion relations will be valid for all operators belonging to the relevant sector.

A A Proof of Stoke's Theorem

The statement of the 3 dimensional Stoke's theorem is

$$\oint_{\partial\Sigma} \vec{F} \cdot d\vec{x} = \int_{\Sigma} (\nabla \times \vec{F}) \cdot d\vec{a} \tag{A.1}$$

where Σ is a surface with boundary $\partial\Sigma$, $\vec{F}$ is some vector field, $d\vec{x}$ is tangent to the boundary and $d\vec{a}$ is normal to the surface. We can see a problem start to form for higher dimensions, because the idea of a normal to a plane is no longer intuitive. In fact, there is no normal component in a single direction. In the 4 dimensional case, we have planes that are normal to other planes. The notion of the curl then makes less sense. With the right hand rule, the curl usually tells us how a vector field is rotating about a given axis. It doesn't make sense to talk about the rotation about an axis in 4 dimensions. Instead, we use what is called the wedge product. The wedge product is antisymmetric, like the cross product but it maps vectors to bivectors. The case is special in 3 dimensions and we can treat these two things as the same but in higher dimensions the wedge product of 2 vectors gives us something that is distinct from a vector.

An important thing to note about the Chern-Simons term is that the vector appears contracted with another vector (so an inner product) so that it looks like we have a Lorentz scalar, or a 1-form. Forms, in differential geometry, are independent of coordinate system.

$$A \equiv A_\mu dx^\mu \tag{A.2}$$

A quick demonstration of why this is a scalar is as follows. First we look at how dx^μ transforms.

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{A.3}$$

Next, we look at how our vector field, A_μ, transforms.

$$A'_\mu = \frac{\partial x^\rho}{\partial x'^\mu} A_\rho \tag{A.4}$$

Then

$$A'_\mu dx'^\mu = A_\rho \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = A_\rho \frac{\partial x^\rho}{\partial x^\nu} dx^\nu = A_\rho \delta^\rho_\nu dx^\nu = A_\nu dx^\nu \tag{A.5}$$

where the second equality follows from application of the chain rule.

We also have that

$$dA = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu \tag{A.6}$$

An alternative way of writing Stoke's theorem is

$$\oint_{\partial\Sigma} A = \int_{\Sigma} dA \tag{A.7}$$

which we can see is Stoke's theorem in 3 dimensions by plugging in our definitions of A and dA. The advantage of this notation is that it is general and so we can use it in higher dimensions.

Now comes the task of showing that this holds in 4 dimensions. The following is a guide that builds on the intuition of the 3-dimensional Stoke's theorem. First, we prove the 4-dimensional Stoke's theorem for a very small cube in four dimensions. Our logic is that we build any shape by stacking cubes.

We are looking at a D2 brane. We can think of a D0 brane as being a point particle in space which has a corresponding worldline. We integrate along that line when we apply Stoke's theorem. For a D1 brane, we have a worldsheet to integrate over. The D2 brane has a worldvolume that looks 3 dimensional. We associate the vector field, A_μ, with a point particle. We have a rank 2 tensor, $A_{\mu\nu}$, for the D1 brane and a rank 3 tensor, $A_{\mu\alpha\rho}$, for the D2 brane.

We imagine taking an infinitesimal, cubic slice of our volume. We first examine the surface of this object. A surface in 4 dimensional space is 3 dimensional. In this simpler study we choose to work in a Euclidean space with X_1, X_2, X_3, X_4 coordinates. We will orient our surface so that one side of it lies in the X_2-X_3-X_4 plane. We choose to study a small section of the volume so that we can imagine this surface is something like a cube in

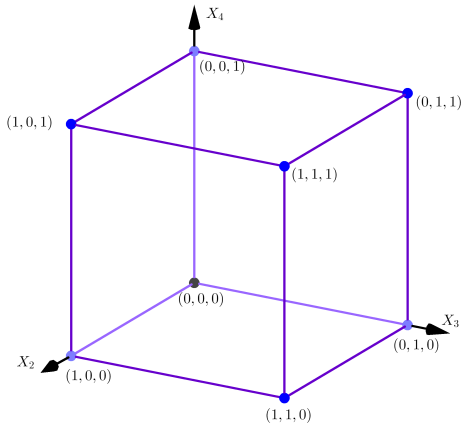


Figure 9: A surface in the X_2-X_3-X_4 plane

3 dimensions. We will have eight different faces (each of which are 3 dimensional cubes) to consider: two faces in each of the four planes, X_1-X_2-X_3, X_1-X_2-X_4, X_1-X_3-X_4 and X_2-X_3-X_4.
We imagine this surface to have sides of length ϵ, so Σ is composed of infinitesimal cubes in $\mathbb{R}^3$

$$\Sigma = \{(X_1, X_2, X_3, X_4) : 0 \leq X_i \leq \epsilon \ \ \forall i = 1, \ldots, 4\} \tag{A.8}$$

We can write our boundary components in the X_2-X_3-X_4 plane as

$$\begin{aligned} S_1 &= \{(0, X_2, X_3, X_4) \in \Sigma\} \\ S_2 &= \{(\epsilon, X_2, X_3, X_4) \in \Sigma\} \end{aligned} \tag{A.9}$$

We have that

$$\mathbf{A} = A_{\mu\alpha\rho}(X_1, X_2, X_3, X_4) dX_2 \wedge dX_3 \wedge dX_4 \tag{A.10}$$

and so

$$\begin{aligned} d\mathbf{A}_1 &= \sum \frac{\partial \mathbf{A}}{\partial X_i} dX_i \wedge dX_2 \wedge dX_3 \wedge dX_4 \\ &= \frac{\partial \mathbf{A}}{\partial X_1} dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4 \end{aligned} \tag{A.11}$$

where the second equality follow from the antisymmetry of the wedge product (which implies $dX_i \wedge dX_i = 0 \ \ \forall i$). The subscript $d\mathbf{A}_1$ is to reinforce we are in the X_2-X_3-X_4 plane.

We choose to parametrise these boundary components so that they face in opposite directions. This is easier to visualise in the 3 dimensional case, where we consider the surface of a cube with squares being the opposite faces (in analogy with S_1 and S_2), pictured below. The idea here is that we break our integral over the entire surface up into an integral of the sum of infinitesimal pieces of the integral. With our parametrisation of the boundary, we see that sides of different faces that lie next to one another cancel out. This is because our parametrisation has opposite sides facing opposite directions. When we add up all the pieces, only the the outer pieces will remain: the boundary of the whole surface. Generalising this concept to our 4 dimensions, we want our boundary parametrised so that opposite components have opposite orientations.

For the plane we are in, the only non-zero components are S_1 and S_2. We said they need to be parametrised to have opposite orientations. Call the boundary in this plane $\partial\Sigma_1$. Then

$$\oint_{\partial\Sigma_1} \mathbf{A}_1 = \oint_{C_1} A_{\mu\alpha\rho}(X_2, X_3, X_4) dX_2 \wedge dX_3 \wedge dX_4 + \oint_{C_2} A_{\mu\alpha\rho}(X_2, X_3, X_4) dX_2 \wedge dX_3 \wedge dX_4 \tag{A.12}$$

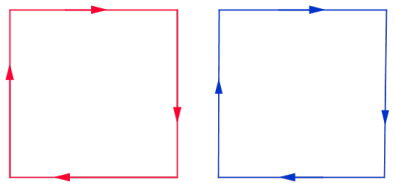


Figure 10: Parametrisation of 2 side-by-side faces of a cube

where C_1 is the boundary of S_1 and C_2 is the boundary of S_2.

$$\oint_{\partial\Sigma_1} \mathbf{A}_1 = \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon A_{\mu\alpha\rho}(\epsilon, X_2, X_3, X_4) dX_2 \wedge dX_3 \wedge dX_4 - \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon A_{\mu\alpha\rho}(0, X_2, X_3, X_4) dX_2 \wedge dX_3 \wedge dX_4 \tag{A.13}$$

Now the fundamental theorem of calculus tells us that, for some function $F(t)$ that is differentiable and where $\frac{dF}{dt} = f$,

$$\int_a^b f dt = F(b) - F(a)$$

This tells us that

$$\begin{aligned} \oint_{\partial\Sigma_1} \mathbf{A}_1 &= \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon \int_0^\epsilon \frac{\partial A_{\mu\alpha\rho}}{\partial X_1}(X_1, X_2, X_3, X_4) dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4 \\ &= \int_{\Sigma_1} d\mathbf{A}_1 \end{aligned} \tag{A.14}$$

By stacking these cubes together we can obtain any finite volume. Applying Stoke's theorem to each small cube, with a little effort we find the same result for the full volume. We can do the same thing for the remaining six faces, pairing them like we did here. This proves Stoke's theorem.

B Inequivalent, Irreducible Representations

For any group, there are a finite number of inequivalent irreducible representations (irreps). These representations are the building blocks of the representation theory of the group and any representation in the group can be built out of some combination of irreps. This is why they are useful to study. In general, there are an infinite number of representations of any group, $\mathcal{G}$, and we cannot hope to list all of them.

A matrix representation of $\mathcal{G}$ is a map, $\Gamma_R(\cdot)$, from $\mathcal{G}$ to the group of matrices $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ such that, for $\forall g_1, g_2 \in \mathcal{G}$, the following relation is satisfied.

$$\Gamma_R(g_1) \cdot \Gamma_R(g_2) = \Gamma_R(g_1 \cdot g_2) \tag{B.1}$$

This means that group composition is realised as matrix multiplication of the representations. This ensures that multiplication of two or more group elements yields another element of the group (by the group composition axiom, $g_1 \cdot g_2 \in \mathcal{G}$). When two representations are related by

$$\widetilde{\Gamma}_R(g) = M^{-1} \Gamma_R(g) M \tag{B.2}$$

we say that they are equivalent representations, for all $g \in \mathcal{G}$ and M is any invertible matrix. One can see that $\widetilde{\Gamma}_R(g)$ is indeed a representation of the group by checking it obeys (B.1), which it does. Since there are an infinite number of invertible matrices M, there are an infinite number of representations equivalent to the representation $\Gamma_R(g)$. We can shorten this list by considering only the inequivalent representations of a group. However, there are still an infinite number of these.

Similar matrices have the same eigenvalues. We could use this to check to see whether two representations are equivalent. However, finding the eigenvalues of a matrix is not a simple problem, especially for large matrices. Computing the trace of a matrix is a much simpler problem. However, two matrices having equal traces does not imply they have the same eigenvalues. What we can consider instead is $\Gamma^n_R(g)$. If this representation is a d dimensional matrix then

$$\mathrm{Tr}(\Gamma^n_R(g)) = \sum_{i=1}^{d} \lambda_i^n$$

where the λ_i are the eigenvalues of $\Gamma_R(g)$. Now if $\mathrm{Tr}(\Gamma^n_R(g)) = \mathrm{Tr}(\widetilde{\Gamma}^n_R(g))$ for all n, then $\Gamma_R(g)$ and $\widetilde{\Gamma}_R$ have the same eigenvalues. What we have shown is that if the set of matrices N_i (i can be any number larger than 1) are equivalently related to $\widetilde{N}_i$ by $\widetilde{N}_i = M^{-1}N_iM$, then they satisfy the condition $\mathrm{Tr}(N_i^n) = \mathrm{Tr}(\widetilde{N}_i^n)\ \ \forall n$. However, the converse is not necessarily true. This is because the invertible matrix M is arbitrary in this case (i.e.: can be a different invertible matrix for each N_i). If we want the converse statement to be true then we need to strengthen our condition for testing for equivalent matrices. Now we consider an arbitrary product of matrices. For example

$$\mathrm{Tr}([N_1N_2\dots N_8]^n) = \mathrm{Tr}([\widetilde{N}_1\widetilde{N}_2\dots \widetilde{N}_8]^n)$$

This condition needs M to be the same for each of the N_i equivalent to some $\widetilde{N}_i$. If these matrices are representations of a group, then we can use the defining equation of a representation (B.1) to rewrite it in the more compact form

$$\mathrm{Tr}(\Gamma_R(g)) = \mathrm{Tr}(\widetilde{\Gamma}_R(g)) \tag{B.3}$$

This is the condition that tells us when two representations, $\Gamma_R(g)$ and $\widetilde{\Gamma}_R(g)$, are equivalent. The trace of the representation of a group is denoted by

$$\mathrm{Tr}(\Gamma_R(g)) \equiv \chi_R(g)$$

and is called the character of group element g in representation R. Finally, we can say that two representations are equivalent if they have the same character.

Two group elements, g_1 and g_2, have the same character when they obey the equivalence relation

$$g_1 = g^{-1}g_2g$$

We say these group elements are conjugate. This equivalence relation partitions the group into conjugacy classes. All elements in the same conjugacy class then will have the same character.

We understand what it means for representations to be equivalent. Now we need to learn what it means for a representation to be reducible. Recall that we are looking for the inequivalent, irreducible representations as this list is finite. If a representation can be written as the direct sum of two other representations, then it is reducible. Any representation that is equivalent to a block diagonal representation is the direct sum of two other representations. Thus any representation that is equivalent to a block diagonal representation is called a reducible representation. This block diagonal representation has at least two invariant subspaces. Each block in the diagonal will act in a different subspace such that the subspaces are not mixed by the action of the group (so subspaces are invariant). An irreducible representation is one which has no invariant subspaces under the action of the group. The number of inequivalent, irreducible representations is equal to the number of conjugacy classes. This number is finite for finite groups.

C Symmetric group and Representation Theory

The symmetric group, S_n is a group of operations

$$S_n = \{g_1, g_2, \dots g_p\}$$

such that each g_i; $i = 1, \dots p$ is a permutation of any number of objects between 1 and n. The order of this group is $p = n!$. For example, the group S_2 has $2! = 2$ elements. There are two objects we can permute: we say these objects are 1 and 2. This is standard notation so whenever we talk about the symmetric group we are talking about permuting the numbers $1, \dots n$. The kinds of permutations we can write down for S_2 is the swap (12) (which is also its inverse) and the identity, $\mathbb{1} = (1)(2)$ – no swap. This notation is called cycle notation and it is spelled out for (12) as follows:

$$1 \to 2 \quad 2 \to 1$$

In cycle notation each object is followed by its image and the last object's image in a cycle is the first object. In S_3 we have the group element (123). This permutes all three objects such that

$$1 \to 2 \quad 2 \to 3 \quad 3 \to 1$$

This a cycle of length three. We can decompose this into a product of 2 cycles, $(\cdot\cdot)$, as follows

$$(123) = (12)(23)$$

We consider the right action of the group so that we read the right hand side of the above equation as (starting with 1)

$$1 \to 2 \quad 2 \to 3 \quad 3 \to 2 \to 1$$

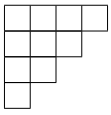
Any k-cycle (a cycle of length k) can be decomposed into a product of 2-cycles (called transpositions) by noticing that

$$(g_1 g_2 \dots g_{n-1} g_n) = (g_1 g_2)(g_2 \dots g_n g_{n-1}) \tag{C.1}$$

and repeatedly applying it. Further, we can decompose any cycle into the the product of adjacent 2-cycles. This is a product of 2-cycles of the form $(p-1, p)$ where $p \leq n$ in S_n. For example, we can write the identity in S_2 as the product of an adjacent 2 cycle and its inverse $(12)(12) = \mathbb{1}$. We can see statement holds by using (C.1) and noting that any 2-cycle can be written as the product of adjacent transpositions. For example $(13) = (12)(23)(12)$. Later we will use this decomposition to form matrix representations of the action of group elements. These adjacent transpositions are sufficient to build the representation of the whole group because we can form any element in the group by taking the relevant product of a number of these adjacent transpositions.

Now that we have understood what is meant by cycle structure, we can introduce some new notation that will represent the cycle structure of group elements in S_n. These representations are partitions of n. That is, a set of positive integers $[n_1, n_2, \dots, n_k]$ such that $n_1 \geq n_2 \geq \dots \geq n_k$ and $n_1 + n_2 + \dots + n_k = n$. By representing group elements in this way (by their cycle structure), we are partitioning the group and each partition has a finite number of group elements in it. Partitions are often visualised by Young diagrams.

Young diagrams of shape R consist of n boxes arranged in rows and columns. We say $R \vdash n$. The Young diagrams are left justified so that the number of boxes in each column is always greater than the number of boxes in the column to the right. Similarly, the number of boxes in each row is always greater than or equal to the number of boxes in the row beneath it. For example



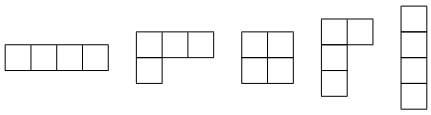
The first row has n_i boxes. Since $n_1 + n_2 + n_3 + n_4 = n$ we see that this Young diagram is representing a partition of a group element in S_{10}. So the number of boxes in the Young diagram tells us to which group the particular element represented belongs to. The structure of the Young diagram has a deeper interpretations as well. Each row represents a cycle in the group. The above Young diagram represents the cycle $(\cdot\cdot\cdot\cdot)(\cdot\cdot\cdot)(\cdot\cdot)(\cdot)$. So Young diagrams partition group elements according to their cycle structure. These partitions are called conjugacy classes.

Elements in the same conjugacy class (g_1, g_2) obey the equivalence relation

$$g_1 = g^{-1} g_2 g$$

for any g in the same group. In the previous section, we saw that two group elements that are conjugate have the same character ($\chi_R(g) = \mathrm{Tr}(\Gamma_R(g))$. Young diagrams label representations of the symmetric group that are given by a set of matrices $\Gamma_R(g)$ that act on the vector space $V_R^{S_n}$. These representations are the inequivalent,

irreducible representations of the group which we can see since they partition the group elements into conjugacy classes. For example, S_4 has the following possible Young diagrams:



So S_4 has five irreps or, equivalently, five conjugacy classes.

We can fill in the numbers $1, \ldots, n$ into the empty boxes of a Young diagram. The convention we use is that numbers in each row must decrease rightwards and numbers in each column must decrease downwards. By filling in the numbers $1, \ldots n$ into the empty boxes of a Young diagram, we form what is called a Young tableaux. When these label elements of a (complete) basis of $V_R^{S_n}$, they are called Young-Yamanouchi symbols. The number of Young-Yamanouchi states, $|R\rangle$, a representation, R, has is equal to the dimension of the Young diagram. In order to calculate the dimension of a Young diagram, we need to know two things. The first is the the number of boxes and the second is the hook lengths of each box. The hook length of a box x is calculated by drawing a horizontal line going rightwards from the box and a vertical line going downwards till the end of the Young diagram. The number of boxes these lines cross, including the starting box is the hook length. We denote the product of the hook lengths in Young diagram R as $\prod_{x\in R} \text{hook}(x) \equiv \text{hooks}_R$. The dimension of a Young diagram is defined as

$$d_R = \frac{n!}{\text{hooks}_R} \tag{C.2}$$

For the Young diagram



the dimension is $d_R = \frac{4!}{4\cdot 2\cdot 1\cdot 1} = 3$. This means this representation has three associated Young-Yamanouchi states i.e.: we can fill the numbers 1, 2, 3 and 4 into the empty boxes in three ways:

$$|1\rangle = \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, \quad |2\rangle = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, \quad |3\rangle = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \tag{C.3}$$

We are almost ready to see how to calculate the matrix representations of elements in a group in a particular irrep. We now introduce the content of a box in Young diagram R. Each box x in row i and column j has content $c_x = j - i$. For example, consider the following Young diagram with contents filled in.

0	1	2	3
−1	0	1	
−2	−1		
−3			

Earlier we said we wanted to consider permutations which were a product of adjacent 2-cycles. We will consider the action of adjacent 2-cycles on our Young-Yamanouchi states. This sufficient because representations satisfy $\Gamma_R(g_1 g_2) = \Gamma_R(g_1)\Gamma_R(g_2)$ where g_1 and g_2 are elements of the group. We will consider the action of $\Gamma_R(12)$ on the Young-Yamanouchi states given in (C.3). We denote the Young diagram after a swap, $(k, k+1)$, by $R_{(k,k+1)}$. Matrix elements of adjacent transpositions are specified by

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle \tag{C.4}$$

Then

$$\Gamma_{\begin{array}{|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}}((12))|1\rangle = -|1\rangle$$

$$\Gamma_{\begin{array}{|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}}((12))|2\rangle = -\frac{1}{3}|2\rangle + \frac{\sqrt{8}}{3}|3\rangle$$

$$\Gamma_{\begin{array}{|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}}((12))|3\rangle = |3\rangle + \frac{\sqrt{8}}{3}|2\rangle$$

which yields the matrix representation

$$\Gamma_R((12)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3} \\ 0 & \frac{\sqrt{8}}{3} & -\frac{1}{3} \end{pmatrix}$$

D Schur Polynomials

In [11], the exact two point function of Schur polynomials in the free field limit was calculated to be

$$\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle = \delta_{RS} f_R \tag{D.1}$$

Since this correlator is diagonal in the Young diagram labels, it is only nonzero when $R = S$. The f_R denoted the products of weights in a Young diagram R. The weight of a box x is $N + j - i$. This is the content of a box x plus N. The result in [11] was obtained by exploiting the link between the symmetric group and the unitary group. The quantity f_R appears in the definition of the dimension of irreps of the unitary group:

$$Dim_R = \frac{f_R}{\text{hooks}_R}$$

This insight reduces our computations of Section 0.6.3 drastically. For example

$$\langle \chi_{\square\square}(Z)\chi^\dagger_{\square\square}(Z)\rangle = N(N+1)(N+2) \tag{D.2}$$

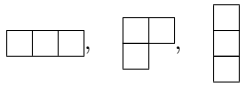
where

$$\chi_{\square\square}(Z) = \frac{1}{6}\left((\text{Tr}(Z))^3 + 3(\text{Tr}(z))(\text{Tr}(Z^2)) + 2\text{Tr}(Z^3)\right)$$

This demonstrates the effectiveness of this language to describe quantum gravity: it hints that we can study non-perturbative physics by summing all the ribbon graphs. In this section we define Schur polynomial labelled by Young diagram R and use it to reproduce (D.1).

We start off by making some comments about the unitary group, $U(N)$ since the action of our complex matrix model is invariant under the $U(N)$ symmetry. The irreps of $U(N)$ are labelled by Young diagrams. Unlike the symmetric group, S_n, where the number of boxes was restricted to n, the number of boxes in an irrep of $U(N)$ can have any number of boxes but the number of rows must be $\leq N$. There are many connections between $U(N)$ and S_n and these form what is known as the Schur-Weyl or Frobenius-Schur duality. States of $U(N)$ are labelled by Gelfan-Tsetlin patterns. This labelling chooses basis states that are simultaneous eigenstates of the matrix J_z. Thus, this basis is a natural choice for studying angular momentum. We will not go into details about the construction of these states but we note that they are related to Young diagrams in that they label states according to how they transform under a chain of subgroups.

Consider the vector space $V_N^{\otimes 3}$. The dimension of this vector space is N^3. There are three possible Young diagrams we can draw:



The dimensions of each of these in the S_n representation is 1, 2, 1 respectively and $\frac{N(N+1)(N+2)}{6}$, $\frac{N(N+1)(N-1)}{3}$, $\frac{N(N-1)(N-2)}{6}$ respectively. States are labelled by both the Young-Yamanouchi symbols and the Gelfand-Tsetlin patterns such that the total number of states is

$$1 \times \frac{N(N+1)(N+2)}{6} + 2 \times \frac{N(N+1)(N-1)}{3} + 1 \times \frac{N(N-1)(N-2)}{6} = N^3$$

as required. So the multiplicity of the $U(N)$ representations we obtain is organised by irreps of the symmetric group.
There are three operators we can construct for $n = 3$ and they are

$$\text{Tr}(Z^3) \quad \text{Tr}(Z^2)\text{Tr}(Z) \quad \text{Tr}(Z)^3$$

These operators are related to the shape of the Young diagrams. That is, they are related to the cycle structure of elements in S_n. For example, the 3-cycle is represented by $\square\square$ and corresponds to $\text{Tr}(Z^3)$. The 2-cycle is represented by $\begin{smallmatrix}\square\\\square\end{smallmatrix}\square$ and corresponds to $\text{Tr}(Z^2)\text{Tr}(Z)$.

The Schur polynomial is defined as follows.

$$\chi_r(Z) \equiv \frac{1}{n!}\sum_{\sigma\in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_n}_{i_{\sigma(n)}} \tag{D.3}$$

Here R is a Young diagram of n boxes (labels an irrep pf the symmetric group), and $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in irrep R. The Schur polynomial $\chi_R(U)$ is the character of an element $U \in SU(N)$ in irrep R.

We have seen how to construct a projection operator onto an irrep R. This projector is defined as

$$P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \tag{D.4}$$

and $\mathrm{Tr}(P_R) = Dim_R$. We can rewrite our Schur polynomial in terms of this projection operator.

$$\begin{aligned}\chi_R(Z) &= \frac{1}{d_R}\left(\frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_n}_{i_{\sigma(n)}}\right) \\ &= \frac{1}{d_R}\mathrm{Tr}\left(P_R Z^{\otimes n}\right)\end{aligned} \tag{D.5}$$

One last insight we need is that summing over the Wick contractions in a correlation function can be understood as the problem of summing over permutations. We have that

$$\langle Z^I_J (Z^\dagger)^K_L \rangle = \sum_{\sigma\in S_n} \sigma^I_L (\sigma^{-1})^K_J \tag{D.6}$$

This shorthand notation keeps track of the indices under matrix multiplication and can be expanded as $Z^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \dots Z^{i_N}_{j_n}$ and similarly for the other matrices.

Now we are ready to show (D.1) holds.

$$\begin{aligned}\langle \chi_R(Z)\chi^\dagger_Z(Z)\rangle &= \frac{1}{d_R d_S}(P_R)^I_J (P_S)^K_L \langle (Z^{\otimes n})^J_I (Z^{\dagger\otimes n})^L_K\rangle \\ &= \frac{1}{d_R d_S}(P_R)^I_J (P_S)^K_L \sum_{\sigma\in S_n}(\sigma^{-1})^J_K (\sigma)^L_I \\ &= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(P_R\sigma^{-1}P_S\sigma) \\ &= \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(P_R P_S) \qquad\qquad \because P_S\sigma = \sigma P_S \\ &= \frac{\delta_{RS} n!}{d_R d_S}\mathrm{Tr}(P_R) = \frac{\delta_{RS} n!}{d_R d_S} Dim_R \\ &= \delta_{RS} f_R\end{aligned} \tag{D.7}$$

where we have used the fact that the order of the group S_n is $n!$ in the second last line and the definitions of Dim_R and d_R to get the last line. The above steps could be carried out for any projection operator so that the field theory problem of computing a correlation function has been replaced with the group theory problem of constructing a set of projection operators.

References

[1] J.Maldacena. The Large N Limit of Superconformal Field Theories and Supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998. 1, 2

[2] E. Witten. Anti-de Sitter Space and Holography. *Adv.Theor.Math.Phys*, 2:253–291, 1998. 1, 3

[3] S. Gubser, I. Klebanov, and A. Polyakov. Gauge Theory Correlators from non-Critical String Theory. *Phys. Lett. B*, 428(105), 1998. 1

[4] R. de Mello Koch, J. Smolic, and M. Smolic. Giant Gravitons – with Strings Attached (I). *JHEP*, 0706(074), 2007. 1

[5] R. de Mello Koch, J. Smolic, and M. Smolic. Giant Gravitons – with Strings Attached (II). *JHEP*, 0709(049), 2007. 1

[6] D. Bekker, R. de Mello Koch, and M. Stephanou. Giant Gravitons – with Strings Attached (III). *JHEP*, 0802(029), 2008. 1

[7] R. de Mello Koch, M. Dessein, and D. Giataganas et al. Giant Graviton Oscillators. *JHEP*, 2011:1–62, 2011. 1, 40

[8] R. de Mello Koch and S. Ramgoolam. A double coset ansatz for integrability in AdS/CFT. *JHEP*, 1206(083), 20012. 1

[9] R. de Mello Koch, N. Tahiridimbiosa, and C. Mathwin. Anomalous Dimensions of Heavy Operators from Magnon Energies. 2015. arXiv:1506.05224 [hep-th]. 1

[10] V. Balasubramanian et al. Giant Gravitons in Conformal Field Theory. *JHEP*, 0204(034), 2002. 1

[11] S. Corley, A. Jevicki, and S. Ramgoolam. Exact Correlators of Giant Gravitons from Dual N=4 SYM Theory. *Adv. Theor. Math. Phys*, 5(4):809–839, 2001. 1, 28, 68

[12] D. Berenstein. A Toy Model for the AdS/CFT Correspondence. *JHEP*, 0407(018), 2004. 1

[13] R. de Mello Koch, S. Graham, and I. Messamah. Higher Loop Nonplanar Anomalous Dimensions from Symmetry. *JHEP*, 1402(125), 2014. 1, 40, 41, 46, 61

[14] N. N. Bogoliubov and O.S. Parasiuk. On the Multiplication of Propagators in Quantum Field Theory. *Acta Math*, 97:227–326, 1957. 1

[15] K. Hepp. Proof of the Bogoliubov-Parasiuk Theorem on Renormalisation. *Comm. Math. Phys.*, 2:301–326, 1966. 1

[16] W. Zimmermann. Local Operator Products and Renormalisation in Quantum Field Theory. In *S. Deser, et al.*, Proceedings of the 1970 Brandeis Summer Institute in Theoretical Physics, pages 399–589, 1970. 1

[17] J. McGreevy, L. Susskind, and N. Toumbas. Invasion of the Giant Gravitons from Anti de Sitter Space. *JHEP*, 0006:008, 2000. 10, 11

[18] G. 't Hooft. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B*, 72:461, 1974. 17, 21

[19] R. de Mello Koch, P. Diaz, and N. Nokwara. Restricted Schur Poynomials for Fermions and Integrability in the su(2-3) Sector. *JHEP*, 1303(173), 2013. 35

[20] W. Carlson, R. de Mello Koch, and H. Lin. Nonplanar Integrability. *JHEP*, 1103(105), 2011. 40

[21] R. Bhattacharyya, de Mello Koch R, and M. Stephanou. Exact Multi-Restricted Schur Polynomial Correlators. *JHEP*, 0806(101), 2008. 41

[22] P. Mattioli and S. Ramgoolam. Permutation Centralizer Algebras and Multi-Matrix Invariants. 2016. arXiv:1601.06086 [hep-th]. 41

[23] J. Chen, J. Ping, and F. Wang. *Group Representation Theory for Physicists*. World Scientific, 2002. 41, 42

[24] R. de Mello Koch et al. Nonplanar Integrability at Two Loops. *JHEP*, 1210(144), 2012. 46

[25] N. Bornman, R. de Mello Koch, and L. Tribelhorn. Rotating Restricted Schur Polynomials. 2016. arXiv:1602.05675 [hep-th]. 52

[26] M. Hamermesh. *Group Theory and its Application to Physical Problems*. Addison-Wesley, 1962. 58

[27] N. Beisert. The SU(2–2) dynamic S-matrix. *Adv. Theor. Math. Phys.*, 12(945), 2008. 61

[28] N. Beisert. The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2–2) Symmetry. *J. Stat. Mech.*, 0701(P01017), 2007. 61