On members of Lucas sequences which are products of factorials

by

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A thesis submitted to the Faculty of Science

University of the Witwatersrand

Johannesburg

in fulfillment of the requirements

for the degree of Doctor of Philosophy (Mathematics)

July 2021

ABSTRACT

We show that if $\{U_n\}_{n\geq 0}$ is a Lucas sequence, then the largest n such that $|U_n| = m_1!m_2!\cdots m_k!$ with $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ satisfies $n < 62000$. When the roots of the Lucas sequence are real, we have $n \in \{1, 2, 3, 4, 6, 12\}$. As a consequence, we show that if $\{X_n\}_{n\geq 1}$ is the sequence of X coordinates of a Pell equation $X^2 - dY^2 = \pm 1$ with a nonsquare integer $d > 1$, then $X_n = m!$ implies $n = 1$.

Moreover, we show that the largest n such that $|U_n| = C_{m_1}C_{m_2}\cdots C_{m_k}$ with $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, where C_m is the mth Catalan number satisfies $n < 6500$. In case the roots of the Lucas sequence are real, we have $n \in \{1, 2, 3, 4, 6, 8, 12\}$. As a consequence, we show that if $\{X_n\}_{n\geq 1}$ is the sequence of the X coordinates of a Pell equation $X^2 - dY^2 = \pm 1$ with a nonsquare integer $d > 1$, then $X_n = C_m$ implies $n = 1$.

PREFACE

Certain chapters and sections of this PhD thesis have been publised in two accredited journals. The two articles in which these appear are listed in accordance with the numbering in the bibliography.

[11] S. Laishram, F. Luca and M. Sias, *On members of Lucas sequences which are products of factorials*, Monatshefte für Mathematik **193** (2020), 329–359.

[12] S. Laishram, F. Luca and M. Sias, *On members of Lucas sequences which are products of Catalan numbers*, International Journal of Number Theory (2021), http://doi.org/10.1142/S1793042121500457.

The sections and chapters relevant to the two articles are as follows.

Certain parts of Section 1.2 appear in [11] and [12].

A part of Section 2.3 and the whole of Sections 2.5 to 2.7 appear in [11] and [12].

Chapters 3, 4 and 5 appear in [11].

Chapters 6 and 7 appear in [12].

A portion of Chapter 8 appears in [11] and [12].

All the computations in this manuscript were carried out in SageMath.

ACKNOWLEDGEMENTS

I dedicate this thesis to the memory of my grandmother, Mary Sias. Her labour of love and humble sagacity have left an indelible impression. I would like to convey my sincere gratitude to Professor Florian Luca for his unwavering support and guidance throughout the study. The financial assistance of the National Research Foundation (NRF) and The Carnegie Foundation of New York towards this research is hereby acknowledged as well. Opinions expressed and conclusions arrived at, are those of the author and are not to be attributed necessarily to the NRF or The Carnegie Foundation of New York.

DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg.

Mark Anthony Sias

Signed on the 13th day of July 2021, at Johannesburg, South Africa.

Contents

Chapter 1

Introduction

1.1 Outline of thesis

The primary purpose of this thesis is to determine upper bounds on n when the nth term of a Lucas sequence is expressible as a product of factorials. Such determination is subject to the nature of the roots of the characteristic polynomial of the Lucas sequence. Since we use an adaptation of the proof of the Primitive Divisor Theorem as our point of departure, the condition of coprimality of the coefficients of the characteristic polynomial is of vital importance for this study.

We start off by providing an overview of the thesis in this chapter. As a rationale for this thesis, Section 1.2, exhibits the ubiquitous nature of recurrence sequences. Brief definitions of Lucas sequences, companion Lucas sequences and the Pell equation are given. The section also provides a succinct survey of various authors' work which are closely related to this thesis.

Chapter 2 presents some of the background mathematical machinery and notions connected with the various approaches adopted to assist with the resolution of lemmas and proofs in the thesis. These are, in no order of preference, the concepts of primitive divisors of Lucas sequences, Cyclotomic polynomials and Linear forms in

logarithms, among others.

In Chapter 3 a series of lemmas are proven en route to establishing an upper bound on n when the nth member of a Lucas sequence is expressible as a product of factorials. Some of these proofs exploit sieve methods, the properties of Cyclotomic polynomials, and results related to Linear forms in logartihms, inter alia. The question of an upper bound on n is also settled for the members of the companion Lucas sequences expressible as products of factorials.

Chapter 4 deals with the same question in case the roots are real. In such instances we find that for specific values of n an infinite number of solutions exist where the nth members of certain Lucas sequences are expressible as products of factorials. Furthermore, under the *abc* conjecture, such products of Lucas sequences reveal only finitely many solutions for certain values of n. We also determine that there are infinitely many solutions (for particular values of n) where the companion Lucas sequences may be written as products of factorials.

In Chapter 5 we formulate and prove a corollary regarding the X- coordinates of Pell equations which are products of factorials.

As captured by the heading of Chapter 6, settling the problem in case the Lucas sequences and companion Lucas sequences are expressible as products of Catalan numbers and middle binomial coefficients, is the overriding theme of this chapter.

Chapter 7 addresses the question of bounding n when the nth member of the sequence of X- coordinates of a Pell equation equate to a Catalan number or a middle binomial coefficient.

Chapter 8 provides the concluding remarks to the thesis as well as an outline of

potential directions for further study.

1.2 Motivation

As Lucas sequences belong to the broader family of linear recurrence sequences, which permeate many different spheres within mathematics and related fields it is therefore only apt to afford - at least - a cursory glance at recurrence sequences in general, in this section. Definitions of Lucas and companion Lucas sequences and the Pell equation are given. We also provide a brief description of some articles which are closely related to the published work contained in Chapters 3 to 7. Some of these serve, to some extent, as impetus for the work contained in the aforementioned chapters.

The importance of linear recurrence sequences hardly needs explanation since its study is clearly of intrinsic interest and has been a pivotal aspect of number theory for some years, see [8]. Furthermore, linear recurrence sequences arise in many fields of mathematics and computer science, for example, the theoretical aspects of power series which represent rational functions, pseudo-random number generators, k-regular and automatic sequences and cellular automata, see [8]. Linear recurrences are also formed by sequences of solutions of certain Diophantine equations, see [8]. In many interesting instances a diverse range of power series, such as the zeta-functions of algebraic varieties of finite fields, generating functions that arise from group theory, dynamical zeta functions of many dynamical systems, Hilbert series and commutative algebra, Poincare series, and closely related series, are all known to be rational, see [8]. The coefficients of the series representing such functions are linear recurrence sequences, see [8].

In addition, recurrence sequences enter many spheres of the mathematical sciences in a broader sense (which includes applied mathematics and applied computer science), see [8]. For instance, we have that quite a large number of systems of orthogonal polynomials, including the Tchebychev polynomials and the Dickson polynomials, which are their finite field counterparts, satisfy linear recurrence relations, see [8]. Linear recurrence sequences also play a key role in approximation theory and cryptography and have found application in time series analysis and computer graphics, see [8].

Let

$$\mathcal{P}F := \{\pm \prod_{j=1}^{k} m_j! : k \geq 1 \text{ and } 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k\}$$

be the set of integers which are the product of factorials. Members of $\mathcal{P}F$ are sometimes called *Jordan-Polya* numbers and several recent papers investigate their arithmetic properties. For example, the counting function of $\mathcal{P}F$ (that is, the cardinality of the set $\mathcal{P}F \cap [1, x]$ for positive real numbers x) was recently studied in [7], while the occurrence of perfect powers in $\mathcal{P}F$ was studied in [3].

At this stage, it is appropriate to briefly define the Lucas sequence $\{U_n\}_{n\geq 0}$ and the companion Lucas sequence $\{V_n\}_{n\geq 0}$.

Definition 1.1 (see [23]) *A* **Lucas sequence** $\{U_n\}_{n\geq 0}$ *is a linear recurrent sequence of order 2 (or binary recurrence) defined by*

$$U_{n+2} = rU_{n+1} + sU_n \qquad n \geq 0,$$

for r and s positive integers such that $\gcd(r, s) = 1$, $U_0 = 0$, $U_1 = 1$ *and the ratio α/β of the roots α, β of $x^2 - rx - s = 0$ is not a root of* 1. *The* **companion Lucas**

sequence $\{V_n\}_{n\geq 0}$ *of parameters* (r, s) *is given by*

$$V_n = \alpha^n + \beta^n \qquad \text{for all} \qquad n \geq 0.$$

Alternatively, it can be defined recursively as

$$V_{n+2} = rV_{n+1} + sV_n \qquad \text{for all} \quad n \geq 0$$

with initial conditions $V_0 = 2,\ V_1 = r$.

We will have a bit more to say about Lucas sequences in the following chapter.

In [14], it was shown that if $t \geq 1$ is any fixed integer, then the Diophantine equation

$$\prod_{i=1}^{t} U_{n_i} \in \mathcal{P}F \tag{1.1}$$

has only finitely many positive integer solutions $1 \leq n_1 \leq n_2 \leq \cdots \leq n_t$ and they are all effectively computable. When $(r, s) = (1, 1)$ then $U_n = F_n$ is the nth Fibonacci number. For this particular case, it was shown in [18] that the largest solution of equation (1.1) in which t is also an indeterminate but with the condition on the indices $1 \leq n_1 < n_2 < \cdots < n_t$ is

$$F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$$

Similar results can be proved when in (1.1) all U_{n_i}'s are replaced by V_{n_i}'s although we have not seen this being explicitly done in the literature.

The Diophantine equation $x^2 - dy^2 = 1$, usually known as Pell's equation, is of considerable importance in number theory. We provide a succinct definition of this equation.

Definition 1.2 *The* **Pell equation** *is the Diophantine equation with unknowns* $(x, y) \in \mathbb{Z}^2$ *defined by*

$$x^2 - dy^2 = 1,$$

where d *is a positive integer which is not a perfect square.*

For such d, let (x_1, y_1) be the minimal solution in positive integers of the equation

$$x^2 - dy^2 = \pm 1. \tag{1.2}$$

It is known that (x_1, y_1) exists.

From the theory of Pell equations, all positive integer solutions (x, y) of Equation (1.2) are of the form $(x, y) = (x_m, y_m)$ for some positive integer m.

We refer to x and y as the coordinates of the Pell equation. In the next chapter we will show the connection between Pell equations and binary recurrences in general and with Lucas sequences in particular.

Arithmetic properties of the coordinates x or y of Pell equations have been studied before. For example, values of n such that x_n is a square have been studied by Ljunggren [13]. He proved that there are at most two such values of n. This was improved later in [26] where it was shown that in fact there is at most one such n except for $d = 1785$, for which both x_1 and x_2 are squares.

Inspired by the following problems, we study here the Diophantine equation obtained by imposing that a member of the Lucas sequences $\{U_n\}_{n\geq 0}$ or $\{V_n\}_{n\geq 0}$ is a product of middle binomial coefficients and Catalan numbers.

Let

$$B_m := \binom{2m}{m} \quad \text{and} \quad C_m := \frac{1}{m+1}\binom{2m}{m} \quad \text{for} \quad m \geq 0,$$

be the middle binomial coefficient and Catalan number, respectively. For each m, we write D_m for one of the numbers B_m, C_m. Let

$$\mathcal{P}BC := \{\pm \prod_{j=1}^{k} D_{m_j} : D_{m_j} \in \{B_{m_j}, C_{m_j}\},\ k \geq 1,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k\}$$

be the set of integers which are products of middle binomial coefficients and Catalan numbers. Diophantine equations with members of $\mathcal{P}BC$ have been studied before. For example, in [15], the authors characterised all nontrivial solutions of the system of two equations

$$\sum_{i=1}^{n} ip_i = \sum_{j=1}^{r} jq_j \quad \text{and} \quad \prod_{i=1}^{n} B_i^{p_i} = \prod_{j=1}^{r} B_j^{q_j}.$$

This system of equations arose naturally from a question in topology concerning n-dimensional complexes which do not embed in $\mathbb{R}^{2n}$ and characterising non-homotopic pairs of such with the same homology. In [16], it was shown that the largest positive integer solution (n, m) of the Diophantine equation

$$F_n = C_m$$

is $(n, m) = (5, 3)$. In [17], it is shown that if $\{u_n\}_{n\geq 0}$ is any nondegenerate binary recurrence of integers, then the equation $u_n = B_m$ has only finitely many positive integer solutions (n, m).

Chapter 2

Preliminaries

To comprehend some of the aspects of the thesis requires some degree of familiarity with the concepts captured as the headings of the ensuing six sections together with the associated concepts linked to Section 2.1. As mentioned previously, the Primitive Divisor Theorem is a key ingredient in the resolution of the questions posed in Section 1.1 - which the thesis answers - so that the various manifestations of this theorem, by virtue of necessity, take centre stage in this chapter.

2.1 Binary recurrence sequences

Definition 2.1 (see [23]) *Let $k \geq 1$ be an integer. A sequence $\{w_n\}_{n\geq 0} \subseteq \mathbb{C}$ is called* ***linearly recurrent of order*** *k if the recurrence*

$$w_{n+k} = a_1 w_{n+k-1} + a_2 w_{n+k-2} + \ldots + a_k w_n \tag{2.1}$$

holds for all $n \geq 0$ with $a_1, \ldots, a_k \in \mathbb{C}$.

Definition 2.2 (see [8]) *Suppose that $a_k \neq 0$. If $a_1, \ldots, a_k \in \mathbb{Z}$ and $w_0, \ldots, w_{k-1} \in \mathbb{Z}$, then, by induction on n, we find that w_n is an integer for all $n \geq 0$.*

The polynomial

$$f(X) = X^k - a_1X^{k-1} - \cdots - a_k \in \mathbb{C}[X]$$

is called the ***characteristic polynomial*** *of* $\{w_n\}_{n\geq 0}$.

Suppose now that $f(X) = \prod_{i=1}^{s}(X-\alpha_i)^{\sigma_i}$, where $\alpha_1, \ldots, \alpha_s$ are the distinct roots of $f(X)$ with multiplicities $\sigma_1, \ldots, \sigma_s$, respectively.

Proposition 2.3 (see [20]) *Suppose that* $f(X) \in \mathbb{Z}[X]$ *has* k *distinct roots. Then there exist constants* $c_1, \ldots, c_k \in \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ *such that the formula*

$$w_n = \sum_{i=1}^{k} c_i\alpha_i^n \tag{2.2}$$

holds for all $n \geq 0$.

For $k = 2$, the sequence $\{w_n\}_{n\geq 0}$ is called **binary recurrent**. Here we observe that its characteristic polynomial is of the form

$$f(X) = X^2 - a_1X - a_2 = (X-\alpha_1)(X-\alpha_2).$$

Definition 2.4 (see [9]) *A binary recurrent sequence* $\{w_n\}_{n\geq 0}$ *whose general term is given by equation* (2.2)*, with* $k = 2$*, is referred to as* ***nondegenerate*** *if* $c_1c_2\alpha_1\alpha_2 \neq 0$ *and* α_1/α_2 *is not a root of unity.*

2.1.1 Binary recurrences associated to the Pell equation, $x^2 - dy^2 = \pm 1$.

Taking Definition 1.2 and the discussion connected thereto into account, we let

$$\chi = x_1 + \sqrt{d}y_1 \quad \text{and} \quad \gamma = x_1 - \sqrt{d}y_1,$$

where

$$x_m + \sqrt{d}y_m = (x_1 + \sqrt{d}y_1)^m = \chi^m.$$

By conjugating the above relation, we have

$$x_m - \sqrt{d}y_m = (x_1 - \sqrt{d}y_1)^m = \gamma^m.$$

Hence,

$$x_m = \frac{\chi^m + \gamma^m}{2} \quad \text{and} \quad y_m = \frac{\chi^m - \gamma^m}{2\sqrt{d}} \quad \text{for all } m \geq 1. \tag{2.3}$$

Putting $(x_0, y_0) = (1, 0)$ so that equation (2.3) holds with $m = 0$ as well, turns out to be quite useful. It is evident that $\{x_m\}_{m\geq 0}$ and $\{y_m\}_{m\geq 0}$ are binary recurrent sequences with characteristic polynomial

$$\begin{aligned} f(X) &= X^2 - a_1 X - a_2 \\ &= (X - \chi)(X - \gamma) \\ &= X^2 - (\chi + \gamma)X + \chi\gamma \\ &= X^2 - 2x_1 X \pm 1. \end{aligned}$$

2.1.2 Lucas sequences

Latching on to Definition 1.1, we observe that $(\alpha, \beta) = \left(\frac{r+\sqrt{r^2+4s}}{2}, \frac{r-\sqrt{r^2+4s}}{2}\right)$. We let $\Delta := r^2+4s$. Also $r = \alpha+\beta$ and $s = -\alpha\beta$. The initial conditions lead to $U_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ with $n = 0, 1, \ldots$ Note that we refer to the pair (α, β) as a Lucas pair, see [4].

The following example serves to illustrate the connection between the Pell equations and Lucas sequences.

Example 2.5 *Suppose (x_m, y_m) are all positive integer solutions to the Pell equations $x^2 - dy^2 = \pm 1$ where $d > 0$ is not a perfect square. If (x_1, y_1) is the smallest such solution, then*

$$x_m + y_m\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^m.$$

Let $\chi = x_1 + \sqrt{d}y_1$ and $\gamma = x_1 - \sqrt{d}y_1$. Then we have that $\chi+\gamma = 2x_1$ and $\chi\gamma = \pm 1$. We also have

$$x_m = \frac{\chi^m + \gamma^m}{2} \quad \text{and} \quad y_m = \frac{\chi^m - \gamma^m}{2\sqrt{d}}.$$

Clearly, $\left\{\frac{y_m}{y_1}\right\}_{m\geq 0}$ is the Lucas sequence with roots (χ, γ) and characteristic polynomial $x^2 - rx - s$, where $r = 2x_1$ and $s = \pm 1$.

2.2 Cyclotomic polynomials

The family of polynomials which has the form $X^n - 1$ have the key property that their roots are exactly the nth roots of unity. It is, however, imperative to bear in mind that many of these polynomials have common factors. Consider, for example, that for any positive integers k and n, the $\gcd(X^k - 1, X^n - 1) = X^{\gcd(k,n)} - 1$. These greatest common factors show that there exist kth roots of unity which are also nth roots of unity. Defining a polynomial $\Phi_n(X)$ whose roots are all nth roots of unity, but not kth roots of unity for all $k < n$, enables the removal of all common factors among these polynomials.

Definition 2.6 *The polynomial*

$$\Phi_n(X) = \prod_{\substack{(d,n)=1 \\ d \leq n}} \left(X - e^{\frac{2\pi i d}{n}} \right)$$

is called the nth ***Cyclotomic polynomial****. This polynomial has degree* $\varphi(n)$.

Example 2.7 *The following equation*

$$X^n - 1 = \prod_{d|n} \Phi_d(X) \tag{2.4}$$

is very important. To illustrate this, we consider the list

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$$

and write every fraction $\frac{m}{n}$ *as a fraction in its simplest form* $\frac{s}{d}$ *with* $\gcd(s, d) = 1$ *and some divisor* d *of* n*. It is easily discernable that all divisors* d *of* n *arise in this manner. Furthermore, for fixed* d*, all* s *which have* $1 \leq s \leq d$ *and are relatively*

prime to d also appear. As $e^{\frac{2\pi im}{n}} = e^{\frac{2\pi is}{d}}$ is a root of $\Phi_d(X)$, we get Equation 2.4. To see how this works, we observe that

$$\left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right).$$

The last number 1 leads to the only root of $\Phi_1(X) = X - 1$. The number $\frac{1}{2}$ corresponds to the only root of unity of order 2, which is -1. Thus $\Phi_2(X) = X + 1$. The numbers $\frac{1}{3}$ and $\frac{2}{3}$ correspond to cubic roots of unity, therefore $\Phi_3(X) = X^2 + X + 1$. Lastly, $\frac{1}{6}$ and $\frac{5}{6}$ correspond to the two roots of unity of order 6, thus $\Phi_6(X) = X^2 - X + 1$.

Proposition 2.8 (see [10]) *For all $m \in \mathbb{N}$, $\Phi_m(X) \in \mathbb{Z}[X]$.*

Remark 2.9 It is a well known fact that for a positive integer m,

$$\Phi_m(\alpha, \beta) = \prod_{d|m} (\alpha^{\frac{m}{d}} - \beta^{\frac{m}{d}})^{\mu(d)},$$

where $\mu(d)$ is Möbius function of d.

2.3 The Primitive Divisor Theorem

Primitive divisors invariably arise en route to establishing a reasonable upper bound on n considering the nature of the roots of the characteristic polynomial of the Lucas sequence. In reading this section, a slight familiarity with the notion of a 'primitive divisor' will facilitate understanding of the various guises or forms in which the theorem is enunciated and of certain aspects of the proofs employed to resolve the questions alluded to in Chapter 1.

Definition 2.10 *We say that a prime $p \mid U_n$ is a* ***primitive divisor*** *of U_n if $p \nmid U_t$ for $t < n$ and $p \nmid r^2 + 4s$.*

There exists a very useful relationship between primitive divisors of Lucas sequences and Cyclotomic polynomials, which is proved in [4, Theorem 2.4]. We state this as lemmas.

We note that here, α_p and ν_p indicate the exponent at which p appears in the factorisation of U_k^1, while $P(n_0)$ denotes the largest prime factor of n_0.
Let $m \mid n$. Denote by $U_k^1 := \left(\frac{\alpha_1^k - \beta_1^k}{\alpha_1 - \beta_1}\right)$, where $\alpha_1 = \alpha^{n/m}$ and $\beta_1 = \beta^{n/m}$.

Lemma 2.11 *For $n > 4, n \notin \{6, 12\}$,*

$$\prod_{\substack{p^{\alpha_p} \| U_n^1 \\ p \text{ primitive}}} p^{\alpha_p} = \frac{\Phi_n(\alpha, \beta)}{\delta},$$

where $\delta \in \{2, P(n_0)\}$ and $\Phi_n(\alpha, \beta)$ is the specialisation of the homogenization $\Phi_n(X, Y)$ of the nth cyclotomic polynomial $\Phi_n(X)$ in the pair (α, β).

Furthermore, from [4, Theorem 2.4] we have

Lemma 2.12 *For $\ell > 4, \ell \notin \{6, 12\}$ and putting*

$$U_\ell^1 = \frac{\alpha_1^\ell - \beta_1^\ell}{\alpha_1 - \beta_1},$$

we have

$$\prod_{\substack{p^{\nu_p} \| U_\ell^1 \\ p \text{ primitive}}} p^{\nu_p} = \frac{\Phi_\ell(\alpha_1, \beta_1)}{\delta_\ell},$$

where $\delta_\ell \in \{1, 2, P(\ell)\}$ and $\Phi_\ell(\alpha_1, \beta_1)$ is the specialisation of the homogenization $\Phi_\ell(X, Y)$ of the ℓth cyclotomic polynomial $\Phi_\ell(X)$ in the pair (α_1, β_1).

Now consider the version of the Primitive Divisor Theorem attributed to Zsigmondy, see [28].

Theorem 2.13 *If the roots of the characteristic polynomial of* $\{U_n\}_{n\geq 0}$ *are coprime positive integers and if* $n > 6$, *then* U_n *has a primitive divisor.*

The following example serves to illustrate Zsigmondy's Theorem.

Example 2.14 *We consider the sequence of integers of the form*

$$w_n = 2^n - 1,$$

where $1 \leq n \in \mathbb{N}$. *This generates the following sample of elements of the sequence:*

$$1, 3, 7, 15, 31, 63, \cdots .$$

We note that $w_1 = 1, w_4 = 15 = 3 \cdot 5$ (*and* $3 \mid w_2$), $w_6 = 63 = 3^2 \cdot 7$. *It is evident that no primitive divisors exist for 63.*

Zsigmondy's theorem was rediscovered independently by Birkhoff and Vandiver more than a decade later in the instance when the roots of the characteristic polynomial are integers.

It is worth noting that Birkhoff and Vandiver independently furnished an elementary proof stating that for n a fixed positive integer, the set of primes $p \equiv 1 \pmod{n}$, is infinite, see [5].

For the case when the roots of the characteristic polynomial of $\{U_n\}_{n\geq 0}$ are real, we have the following.

Lemma 2.15 *The primitive divisors of U_n are all congruent to* $\pm 1 \pmod{n}$.

In 1913, Carmichael established that for α and β real and n not equal to 1, 2,3,4 and 6, then the Lucas numbers, $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ have one or more primitive divisors excluding $n = 12$, $r = 1$ and $s = -1$ (for the characteristic polynomial

$$f(X) = X^2 - rX - s = (X - \alpha)(X - \beta)),$$

see [6]. Carmichael's result is in essence an extension of Zsigmondy's theorem.

What follows is the statement of the Primitive Divisor Theorem which is an extension of Zsigmondy's theorem.

Theorem 2.16 *If $n \notin \{1, 2, 3, 4, 6\}$ then U_n has a primitive divisor except when $((\alpha, \beta), n)$ is of the form $\left(\pm\left(\frac{r+\sqrt{\Delta}}{2}, \frac{r-\sqrt{\Delta}}{2}\right), n\right)$*

Table 2.1: The triples

n	(r, Δ)
5	$(1,5),(1,-7),(2,-40),(1,-11),(1,-15),(12,-76),(12,-1364)$
7	$(1,-7),(1,-19)$
8	$(2,-24),(1,-7)$
10	$(2,-8),(5,-3)$
12	$(1,5)(1,-7)(1,-11)(2,-56)(1,-15)(1,-19)$
13	$(1,-7)$
18	$(1,-7)$
30	$(1,-7)$

Schinzel established that when α and β are complex nonreal and if n is sufficiently large, the nth term in the companion Lucas sequence has a primitive divisor, see [24]. In 1976, it was proven by Stewart that if $n = 5$ or $n > 6$ then there are a finite number of Lucas sequences that do not exhibit a primitive divisor and that such sequences may be determined, see [25].

Building on the work of Schinzel and Stewart, Bilu, Hanrot and Voutier settled the problem in the case where the roots are complex nonreal by proving that for $n > 30$, every Lucas number has a primitive divisor, see [4].

2.4 Linear forms in logarithms

It was in 1966 when A. Baker [1] provided an effective lower bound on the absolute value of a nonzero linear form in logarithms of algebraic numbers; that is, for a nonzero expression of the form

$$\sum_{i=1}^{n} b_i \log \alpha_i,$$

for $\alpha_1, \ldots, \alpha_n$ algebraic numbers and $b_1, \ldots, b_n$ are integers. His findings ushered in the dawn of a new era in the effective resolution of Diophantine equations of certain types. Such Diophantine equations can be reduced to exponential ones; i.e., where the unknown variables are in the exponents.
We furnish the reader with some preliminaries regarding algebraic numbers.

Proposition 2.17 (see [10]) *If α is an algebraic number then α is the root of a unique monic irreducible $f(x)$ in $\mathbb{Q}[x]$. Furthermore, if $g(x) \in \mathbb{Q}[x]$, $g(\alpha) = 0$, then $f(x) \mid g(x)$ in $\mathbb{Q}[x]$.*

Remark 2.18 The polynomial of interest in the foregoing proposition depends therefore only upon α and is called the **minimal polynomial** of α. If the degree of the minimal polynomial is d, then α is called an algebraic number of degree d. Since $f(x)$ is irreducible of degree d, by invoking the fundamental theorem of algebra in conjunction with the fact that $f(x)$ does not have repeated roots, we see that $f(x)$ is the minimal polynomial for each of its roots. If α and β are roots of $f(x)$ then α and β are said to be **conjugates**, see [10].

We now define the concepts of 'height' and the 'logarithmic height' of an algebraic number, see [1].

Definition 2.19 *Let α be an algebraic number of degree d. We let*

$$f(x) = \sum_{i=0}^{d} a_i x^{d-i} \in \mathbb{Z}[X].$$

*be the minimal polynomial of α with $a_0 > 0$ and $(a_0, \ldots, a_d) = 1$. Putting $H(\alpha) := \max\{|a_i| : i = 0, \ldots, d\}$, we let it denote the **height** of α. We now write*

$$f(X) = a_0 \prod_{i=1}^{d} (X - \alpha^{(i)}),$$

*where $\alpha = \alpha^{(1)}$. The **logarithmic height** of α is*

$$h(\alpha) = \frac{1}{d} \left(\log |a_0| + \sum_{i=1}^{d} \log \max\{|\alpha^{(i)}|, 1\} \right).$$

Remark 2.20 The numbers $\alpha^{(i)}$ are called the conjugates of α.

Example 2.21 *If $\alpha = \frac{a}{b}$ is a rational number, where a and b are relatively prime integers with $b > 0$, then $H(\alpha) = \max\{|a|, b\}$ and $h(\alpha) = \log \max\{|a|, b\}$.*

2.5 Sieves

A sieve is basically an inclusion-exclusion argument. Many results in the theory of primes can be proved using sieves.

The next lemma follows easily from the Brun-Titchmarsh inequality given by Montgomery and Vaughan [21, Theorem 2] since $\pi(1;q,l)=0$ and $\pi(y;q,l)\le\pi(y+1;q,l)-\pi(1;q,l)$. Recall that $\pi(y;q,l)$ stands for the number of primes $p\le y$ and $p\equiv l \pmod q$.

Lemma 2.22 *Let q be a positive integer, l be coprime to q and $y>q$. Then*

$$\pi(y;q,l)\le\frac{2y}{\varphi(q)\log(y/q)} \quad\text{and}\quad \pi(2y;q,l)-\pi(y;q,l)\le\frac{2y}{\varphi(q)\log(y/q)}.$$

As usual, let

$$\psi(y;q,l):=\sum_{\substack{p^t\le y\\ p\equiv l \pmod m}}\log p \quad\text{and}\quad \theta(y;q,l):=\sum_{\substack{p\le y\\ p\equiv l \pmod m}}\log p.$$

The following estimates are from [22, Table 2]. We have taken into account the estimates for $\theta\#$ defined in [22, Table 2] for $q\in\{8,16,24\}$.

Lemma 2.23 *Let $q\in\{8,9,12,16,24\}$ or $5\le q\le 23$ be a prime and ℓ_0 be an integer coprime to q with $\ell_0\not\equiv 1 \pmod q$. Then for $y\ge q$, we have*

$$\psi(y;q,1)+\psi(y;q,\ell_0)\le\frac{2y}{\varphi(q)}\left(1+\frac{\epsilon_\psi\varphi(q)}{\sqrt{y}}\right) \tag{2.5}$$

and

$$\theta(y;q,1)+\theta(y;q,\ell_0)\ge\frac{2y}{\varphi(q)}\left(1-\frac{\epsilon_\theta\varphi(q)}{\sqrt{y}}\right), \tag{2.6}$$

where ϵ_ψ and ϵ_θ are given by

q	5	7	8	9	12	16	24	$11 \le q \le 23$
ϵ_ψ	.807	.78	.927	.789	.863	.774	.745	.912
ϵ_θ	1.413	1.106	1.5	1.11	1.5	1.03	1.5	1.1

Further,

$$\frac{y}{\varphi(24)}\left(1-\frac{\varphi(24)}{\sqrt{y}}\right) \le \theta(y;24,5) \le \psi(y;24,5) \le \frac{y}{\varphi(q)}\left(1+\frac{0.745\varphi(24)}{\sqrt{y}}\right).$$

As a consequence, we have the following result.

Lemma 2.24 *Let $q \in \{8,9,12,16,24\}$ or $5 \le q \le 23$ be a prime and ℓ_0 be an integer coprime to q with $\ell_0 \not\equiv 1 \pmod q$. Then for $y \ge 1500$, we have*

$$\begin{aligned}&\sum_{l=1,\ell_0}\left(\psi(2y;q,l)-\theta(y;q,l)+\theta\left(\frac{2y}{3};q,l\right)-\theta\left(\frac{y}{2};q,l\right)+\theta\left(\frac{2y}{5};q,l\right)\right)\\ \le&\frac{y}{\varphi(q)}\left\{\frac{47}{15}+\frac{2\sqrt{2}\epsilon_\psi}{\sqrt{y}}\left(1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}\right)+\frac{2\epsilon_\theta}{\sqrt{y}}\left(1+\frac{1}{\sqrt{2}}\right)\right\},\end{aligned} \tag{2.7}$$

where ϵ_ψ and ϵ_θ are given in Lemma 2.23. Also for each $y \ge 15$, there is a prime $p \equiv 5 \pmod{24}$ with $y+1 < p \le 2y$. Further, for $y \ge 6$, there is a prime $p \equiv \pm 5 \pmod 8$ with $y+1 < p \le 2y$. And for $y \ge 9$, there is a prime $p \equiv 5 \pmod{12}$ with $y+1 < p \le 2y$.

Proof. The assertion (2.7) is immediate from Lemma 2.23 and using the inequality $\theta(y;q,l) \le \psi(y;q,l)$ valid for all y. For primes $p \equiv 5 \pmod{24}$, again from Lemma 2.23, we have

$$\theta(2y;24,5) - \theta(y+1;24,5) \geq \frac{2y}{\varphi(24)}\left(1 - \frac{\varphi(24)}{\sqrt{2y}}\right)$$
$$- \frac{y+1}{\varphi(24)}\left(1 + \frac{0.745\varphi(24)}{\sqrt{y+1}}\right)$$
$$= \frac{y}{\varphi(24)}\left\{1 - \frac{2\varphi(24)}{\sqrt{2y}} - \frac{1}{y} - \frac{0.745\varphi(q)}{\sqrt{y+1}} - \frac{0.745\varphi(q)}{y\sqrt{y+1}}\right\} > 0,$$

for $y \geq 400$. Thus, there is a prime $p \equiv 5 \pmod{24}$ with $y+1 < p \leq 2y$ for $y \geq 400$. This is also true for $15 \leq y < 400$ by checking at integer values of y. Since a prime congruent to 5 (mod 24) is also congruent to 5 (mod 8) and 5 (mod 12), the last two assertions can be obtained by checking it in the range $6 \leq y < 15$.

□

2.6 The *abc* conjecture and associated notation

This very important conjecture has several possible statements, however the following lemma will suffice. Firstly, a definition.

Definition 2.25 *For a nonzero natural integer n we define the **radical** $\gamma(n)$ of n as the product of the prime numbers dividing n, i.e., $\gamma(n) = \prod_{p|n} p$.*

Lemma 2.26 *Let $\epsilon > 0$. There exists a constant κ_ε depending only on ε such that given pairwise coprime positive integers a, b, c with $a + b = c$, we have*

$$c < \kappa_\varepsilon \cdot \gamma(abc)^{1+\epsilon}.$$

Henceforth, for two expressions A and B we shall use the Landau symbols $A = O(B)$

and $A = o(B)$ as well the Vinogradov symbol $A \ll B$ instead of writing constants explicitly. Recall that $A = O(B)$ and $A \ll B$ are equivalent to the fact that $|A| \leq \kappa B$ for some constant κ, while $A = o(B)$ means $A/B \to 0$. Further, $A \asymp B$ means both $A \ll B$ and $B \ll A$ hold. If the implied constants depend on some parameter ε, for example, then we write $A = O_\varepsilon(B)$, $A \ll_\varepsilon B$ and $A \asymp_\varepsilon B$ to indicate this dependence. With this notation, the *abc* conjecture then is the statement that if $\varepsilon > 0$ is given then for all positive integers a, b, c with $a + b = c$ and $\gcd(a, b) = 1$, we have $c \ll_\varepsilon \gamma(abc)^{1+\varepsilon}$.

2.7 Primes in Arithmetic progressions

In this thesis the resolution of a number of lemmas, and hence the resolution of certain theorems hinges, to a large degree, on the results of primes in arithmetic progressions.

The next lemma follows from [2, Corollary 1.6].

Lemma 2.27 *Let m be either a prime ≤ 30 or $m \in \{8, 9, 12, 16, 24\}$. Then*

$$\pi(y; m, 1) + \pi(y; m, -1) \leq \frac{y}{\varphi(m) \log y} \left(2 + \frac{5}{\log y}\right) \quad \text{for } y > 1. \tag{2.8}$$

Also

$$\pi(y; 12, 1) + \pi(y; 12, 7) \leq \frac{y}{\varphi(12) \log y} \left(2 + \frac{5}{\log y}\right) \quad \text{for } y > 1. \tag{2.9}$$

Proof. From [2, Corollary 1.6], we have for $m \leq 1200$ and $\gcd(\ell, m) = 1$ that

$$\pi(y; m, \ell) \leq \frac{y}{\varphi(m) \log y} \left(1 + \frac{5}{2 \log y}\right) \quad \text{for} \quad y \geq 50m^2.$$

This implies the assertions (2.8) and (2.9) for m, y in the statement of the lemma with $y \geq 50m^2$. For m, y in the statement of the lemma with $y < 50m^2$, we check that

$$\max_{1<y<50m^2} \left\{ \frac{\varphi(m)\log y}{y} \left(\pi(y; m, 1) + \pi(y; m, -1) - 2\right) \log y \right\} < 5.$$

In fact the largest value of the above quantity found over the ranges of m and y was 4.2635 for $m = 29$ and $y = 523$. Hence assertion (2.8) follows. Similarly, we check that the assertion (2.9) holds for $1 < y < 50 \cdot 12^2$.

□

Let n_0 be a positive integer. For an integer ℓ, define

$$M_{n_0}(\ell) := \log \left(\prod_{\substack{p^{\nu_p} \| \ell \\ p \equiv \pm 1 \pmod{n_0}}} p^{\nu_p} \right) = \sum_{\substack{p^{\nu_p} \| \ell \\ p \equiv \pm 1 \pmod{n_0}}} \nu_p \log p. \tag{2.10}$$

We prove a number of results to estimate lower and upper bounds for $M_{n_0}(U_n)$ and $M_{n_0}(V_n)$ for some divisors n_0 of n.

We assume without loss of generality that $|\alpha| \geq |\beta|$ and that α is positive if α and β are real.

We begin by proving a lower bound for $M_{n_0}(U_n)$ and $M_{n_0}(V_n)$ for some divisors n_0 of n. Throughout this chapter, we use $x := \beta/\alpha$.

Lemma 2.28 *Let n be a positive integer and $p < p_1$ be distinct primes and $t \geq 0, h > 0, h_1 > 0$ be integers. Let $n_0 \in \{p^h, p^h p_1^{h_1}\}$, $n_0 > 4, n_0 \notin \{6, 12\}$ be such*

that $n_0 p^t \mid n$. *Then*

$$M_{n_0}(U_n) \geq$$
$$\begin{cases} n\left(1-\frac{1}{p^{t+1}}\right)\log|\alpha| + \log\left(\frac{1-x^n}{1-x^{n/p^{t+1}}}\right) - \log(p^{t+1}), & n_0 = p^h; \\ n\left(1-\frac{1}{p^{t+1}}\right)\left(1-\frac{1}{p_1}\right)\log|\alpha| + \log\left(\frac{1-x^n}{1-x^{\frac{n}{p^{t+1}}}}\right) - \log(pp_1)^{t+1}, & n_0 = p^h p_1^{h_1}, \end{cases} \tag{2.11}$$

and for $n_0 = p^h, p > 2$,

$$M_{n_0}(V_n) \geq n\left(1-\frac{1}{p^{t+1}}\right)\log|\alpha| + \log\left(\frac{1+x^n}{1+x^{n/p^{t+1}}}\right) - \log(p^{t+1}). \tag{2.12}$$

Proof. Let n_0 be the divisor of n given in the statement of the lemma. Let $m = n_0 p^t$. Write

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \left(\frac{(\alpha^{n/m})^m - (\beta^{n/m})^m}{\alpha^{n/m} - \beta^{n/m}}\right)\left(\frac{\alpha^{n/m} - \beta^{n/m}}{\alpha - \beta}\right).$$

We define α_1, β_1 and U_ℓ^1 as in Section 2.3. Also $V_\ell^1 = \alpha_1^\ell + \beta_1^\ell$ for $\ell \geq 0$.

Then $\{U_\ell^1\}_{\ell\geq 0}$ and$\{V_\ell^1\}_{\ell\geq 0}$ are the Lucas sequences with parameters (r_1, s_1), where $(r_1, s_1) = (\alpha_1+\beta_1, -\alpha_1\beta_1) = (V_{n/m}, (-1)^{n/m-1}s^{n/m})$. Further, we have $U_n = U_m^1 U_{n/m}$ and $V_n = V_m^1$ implying

$$M_{n_0}(U_n) \geq M_{n_0}(U_m^1) \qquad \text{and} \qquad M_{n_0}(V_n) = M_{n_0}(V_m^1).$$

Observe that $U_m^1 = U_{n_0 p^t}^1$ is divisible by each $U_{n_0 p^i}^1, 0 \leq i \leq t$. From Lemma 2.15, we have that the primitive divisors of $U_{n_0 p^i}^1$ for $0 \leq i \leq t$ are all congruent to one of ± 1 modulo $n_0 p^i$. We now look at the primitive part of U_ℓ^1. This is the part of U_ℓ^1 built up only with powers of primitive prime divisors of U_ℓ^1. Thus, the primitive

parts of $U^1_{n_0p^i}$ for $0 \leq i \leq t$ divide U^1_m. Hence,

$$M_{n_0}(U_n) \geq M_{n_0}(U^1_m) \geq M_{n_0}\left(\prod_{i=0}^{t} U^1_{n_0p^i}\right).$$

For a positive integer m, let

$$\Phi_m(\alpha,\beta) = \prod_{\substack{1\leq k\leq n\\(k,m)=1}} (\alpha - e^{2\pi ik/m}\beta) \tag{2.13}$$

be the specialisation of the homogenization $\Phi_m(X,Y)$ of the mth cyclotomic polynomial $\Phi_m(X)$ in the pair (α,β).

We bear in mind the preceding definitions in this section and Lemma 2.12. Since primitive divisors of U^1_ℓ are congruent to one of ± 1 modulo ℓ, we obtain by taking $\ell = n_0p^i$ for $0 \leq i \leq t$ that

$$M_{n_0}(U_n) \geq M_{n_0}\left(\prod_{i=0}^{t} U^1_{n_0p^i}\right) \geq \left(\prod_{i=0}^{t} |\Phi_{n_0p^i}(\alpha_1,\beta_1)|\right)(P(n_0))^{-t-1}. \tag{2.14}$$

Also from the fact that $V_n = V^1_{n_0p^t}$ is divisible by each $V_{n_0p^i}, 0 \leq i \leq t$ (here n_0, p are both odd) and the primitive part of $V_{n_0p^i}$ is exactly the primitive part of $U^1_{2n_0p^i}$, we obtain similarly

$$M_{n_0}(V_n) \geq M_{2n_0}\left(\prod_{i=0}^{t} U^1_{2n_0p^i}\right) \geq \left(\prod_{i=0}^{t} |\Phi_{2n_0p^i}(\alpha_1,\beta_1)|\right)(P(n_0))^{-t-1}. \tag{2.15}$$

Therefore, it remains to estimate the right–hand sides of Inequalities (2.14) and (2.15).

Remark 2.9 which exhibits the connection between Cyclotomic polynomials and the

Möbius function of n now comes into effect.

Hence, we have, by using $\alpha_1^{n_0 p^t} = \alpha^n$,

$$\begin{aligned}\prod_{i=0}^{t} \Phi_{n_0 p^i}(\alpha_1, \beta_1) =& \prod_{i=0}^{t} \frac{\alpha_1^{p^{h+i}} - \beta_1^{p^{h+i}}}{\alpha_1^{p^{h+i-1}} - \beta_1^{p^{h+i-1}}} = \frac{\alpha_1^{p^{h+t}} - \beta_1^{p^{h+t}}}{\alpha_1^{p^{h-1}} - \beta_1^{p^{h-1}}} \\ =& \frac{\alpha^n - \beta^n}{\alpha^{n/p^{t+1}} - \beta^{n/p^{t+1}}}, \quad n_0 = p^h;\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\prod_{i=0}^{t} \Phi_{n_0 p^i}(\alpha_1, \beta_1) =& \prod_{i=0}^{t} \frac{(\alpha_1^{p^{h+i} p_1^{h_1}} - \beta_1^{p^{h+i} p_1^{h_1}})(\alpha_1^{p^{h+i-1} p_1^{h_1-1}} - \beta_1^{p^{h+i-1} p_1^{h_1-1}})}{(\alpha_1^{p^{h+i-1} p_1^{h_1}} - \beta_1^{p^{h+i-1} p_1^{h_1}})(\alpha_1^{p^{h+i} p_1^{h_1-1}} - \beta_1^{p^{h+i} p_1^{h_1-1}})} \\ =& \frac{(\alpha_1^{p^{h+t} p_1^{h_1}} - \beta_1^{p^{h+t} p_1^{h_1}})(\alpha_1^{p^{h-1} p_1^{h_1-1}} - \beta_1^{p^{h-1} p_1^{h_1-1}})}{(\alpha_1^{p^{h-1} p_1^{h_1}} - \beta_1^{p^{h-1} p_1^{h_1}})(\alpha_1^{p^{h+t} p_1^{h_1-1}} - \beta_1^{p^{h+t} p_1^{h_1-1}})} \\ =& \left(\frac{\alpha^n - \beta^n}{\alpha^{\frac{n}{p^{t+1}}} - \beta^{\frac{n}{p^{t+1}}}} \right) \left(\frac{\alpha^{\frac{n}{p_1 p^{t+1}}} - \beta^{\frac{n}{p_1 p^{t+1}}}}{\alpha^{n/p_1} - \beta^{n/p_1}} \right), \quad n_0 = p^h p_1^{h_1}.\end{aligned} \tag{2.17}$$

Also,

$$\prod_{i=0}^{t} \Phi_{2n_0 p^i}(\alpha_1, \beta_1) = \frac{\alpha_1^{p^{h+t}} + \beta_1^{p^{h+t}}}{\alpha_1^{p^{h-1}} + \beta_1^{p^{h-1}}} = \frac{\alpha^n + \beta^n}{\alpha^{\frac{n}{p^{t+1}}} + \beta^{\frac{n}{p^{t+1}}}}, \quad n_0 = p^h. \tag{2.18}$$

From $|\alpha| \geq |\beta|$, we have $|x| \leq 1$. Taking out the powers of α in (2.16)–(2.18) and further using in (2.17) the inequality

$$\left| \frac{1-y}{1-y^{p^{t+1}}} \right| \geq \frac{1}{p^{t+1}} \quad \text{valid for all} \quad p, \quad \text{where} \quad y := x^{\frac{n}{p_1 p^{t+1}}} \quad \text{has} \quad |y| \leq 1,$$

we get the assertions (2.11) and (2.12) from (2.14) and (2.15), respectively.

□

Chapter 3

Products of factorials : real and non-real roots

Having collected some of the the basic results in Chapter 2, in the present chapter we embark on the proof of the main theorem as stated in Section 3.1. In this direction we make the following additional assumptions pertaining to the Lucas sequences. We assume that $r^2 + 4s \neq 0$ and again let $\alpha,\ \beta$ be the roots of the quadratic equation $\lambda^2 - r\lambda - s = 0$ and assume without loss of generality that $|\alpha| \geq |\beta|$.

We also let

$$\mathcal{P}F := \{\pm \prod_{j=1}^{k} m_j! : k \geq 1 \text{ and } 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k\}$$

be the set of integers which are the product of factorials.

Here we also prove a series of lemmas which exploit certain properties of cyclotomic polynomials, linear forms in logarithms and the logarithmic height function, among others. Sieves also come to the fore with the resolution of a lemma in Section 3.3 which is a result of a refinement of the Brun - Titchmarsh inequality due to Montgomery and Vaughan. Throughout the proofs of the lemmas, leading up to the main result in Section 3.4, we use $\omega(n), P(n), \mu(n), \varphi(n)$ and $\gamma(n)$ with the regular

meaning as being the number of distinct prime factors of n, the largest prime factor of n, the Möbius function of n, the Euler phi function of n and the radical of n, respectively.

3.1 Main Result

The resolution of the following theorem is the subject of this chapter.

Theorem 3.1 *The equation*

$$U_n = \pm m_1!m_2!\cdots m_k! \quad \textit{where } k \geq 1 \quad \textit{and} \quad 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k \tag{3.1}$$

implies $n < 62000$. *When* α, β *are real, then* $n \in \{1, 2, 3, 4, 6, 12\}$.

The equation

$$V_n = \pm m_1!m_2!\cdots m_k, \qquad 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k. \tag{3.2}$$

implies $n = 2$ *or* n *is odd and* $n < 31000$. *Further,* $n \in \{1, 2, 3\}$ *when* α, β *are real.*

Note that $U_1 = 1 \in \mathcal{P}F$ and $V_1 = U_2 = r$ which is in $\mathcal{P}F$ if $r \in \mathcal{P}F$. So, when studying Equations (3.1) and (3.2), we may assume that $n \geq 2$.

3.2 Preliminaries

For a positive integer m, let $q_0 := q_0(m)$ be the least odd prime divisor of m. We make the convention that $q_0 := 1$ if m is a power of 2. Let $\eta := \eta(m) = 0$ if m is odd and 1 if m is even. Note that $m \equiv \gamma \pmod 2 \equiv 2^\eta \pmod 2$. Put $\gamma_0 := \gamma/(2^\eta q_0)$. We shall assume that $m \geq 4$ whenever m is a power of 2.

Lemma 3.2 *Let m be a positive integer. We have*

$$\log|\Phi_m(\alpha,\beta)| = \varphi(m)\log|\alpha| + \sum_{d|\gamma}\mu(d)\log\left|1-\left(\frac{\beta}{\alpha}\right)^{\frac{m}{d}}\right|. \tag{3.3}$$

Putting $x = \beta/\alpha$, we have

$$\sum_{d|\gamma}\mu(d)\log\left|1-x^{\frac{m}{d}}\right| = \begin{cases} \log\left|1+x^{\frac{m}{2}}\right|, & q_0 = 1; \\ \displaystyle\sum_{\substack{\mu(d)=1\\ d|\gamma_0}}\log\left|\frac{1+x^{\frac{m}{2d}}}{1+x^{\frac{m}{2q_0d}}}\right| - \sum_{\substack{\mu(d)=-1\\ d|\gamma_0}}\log\left|\frac{1+x^{\frac{m}{2d}}}{1+x^{\frac{m}{2q_0d}}}\right|, & \eta = 1,\ q_1 > 1; \\ \displaystyle\sum_{\substack{\mu(d)=1\\ d|\gamma_0}}\log\left|\frac{1-x^{\frac{m}{d}}}{1-x^{\frac{m}{q_0d}}}\right| - \sum_{\substack{\mu(d)=-1\\ d|\gamma_0}}\log\left|\frac{1-x^{\frac{m}{d}}}{1-x^{\frac{m}{q_0d}}}\right|, & \eta = 0,\ q_1 > 1. \end{cases} \tag{3.4}$$

Proof. We take cognisance of Remark 2.9. The assertion (3.3) follows by taking logarithms and using

$$\sum_{d|m}\mu(d)m/d = \varphi(m).$$

Next we write $m = \gamma\delta$, where $\gamma = \gamma(m)$. Every divisor d of m with $\mu(d)^2 = 1$ is also a divisor of γ. For m even, each divisor d of $\gamma/2$ gives two divisors d and $2d$ of γ. From

$$\log|1 - x^{m/d}| - \log|1 - x^{m/(2d)}| = \log|1 + x^{m/(2d)}|,$$

we see that

$$\sum_{d|\gamma} \mu(d) \log\left|1 - x^{\frac{m}{d}}\right| = \sum_{d|\gamma/2} \mu(d) \log\left|1 + x^{\frac{m}{2d}}\right|.$$

Now the assertion (3.4) follows when $q_0 = 1$; i.e., when $m = 2^{e_0}$ is a power of 2 for which $\gamma = 2$. Hence, we now assume that $\omega(m) \geq 2$ when m is even.

Recall that we have put $\gamma_0 = \gamma/(2^\eta q_0)$, where $\eta = 0, 1$ according to whether m is odd or even, respectively. Then $\omega(\gamma_0) = \omega(m) - 1 - \eta$ and every divisor d of γ_0 gives exactly two divisors d and $q_0 d$ of $\gamma/2^\eta$ which are both divisors of $m/2^\eta$ whose Möbius function is nonzero. Further, $\mu(d) = 1$ gives $\mu(q_0 d) = -1$ and $\mu(d) = -1$ gives $\mu(q_0 d) = 1$. Hence, for m odd, we have

$$\sum_{d|\gamma} \mu(d) \log\left|1 - x^{\frac{m}{d}}\right| = \sum_{\substack{\mu(d)=1 \\ d|\gamma_0}} \log\left|\frac{1 - x^{\frac{m}{d}}}{1 - x^{\frac{m}{q_0 d}}}\right| + \sum_{\substack{\mu(d)=-1 \\ d|\gamma_0}} \log\left|\frac{1 - x^{\frac{m}{q_0 d}}}{1 - x^{\frac{m}{d}}}\right|,$$

which gives (3.4) for m odd, and for m even

$$\begin{aligned}\sum_{d|\gamma} \mu(d) \log\left|1 - x^{\frac{m}{d}}\right| &= \sum_{d|\gamma/2} \mu(d) \log\left|1 + x^{\frac{m}{2d}}\right| \\ &= \sum_{\substack{\mu(d)=1 \\ d|\gamma_0}} \log\left|\frac{1 + x^{\frac{m}{2d}}}{1 + x^{\frac{m}{2q_0 d}}}\right| + \sum_{\substack{\mu(d)=-1 \\ d|\gamma_0}} \log\left|\frac{1 + x^{\frac{m}{2q_0 d}}}{1 + x^{\frac{m}{2d}}}\right|,\end{aligned}$$

which gives (3.4) for m even.

□

Remark 3.3 The reader will notice that the connection between the Möbius function of n and the Euler phi function of n, together with their properties, have come into play to expedite the proof of the lemma above. We also observe that the assumption of $\omega(m) \geq 2$ when m is even together with the definition of other relevant constructs have of consequence $d^2 \nmid m/2^\eta$, and hence $\mu(m/2^\eta) \neq 0$. So we can only have $\mu(d) = \pm 1$.

Lemma 3.4 *Assume that α and β are real and $|\alpha| \geq |\beta|$. Let $m \geq 4$. Then*

$$\log|\Phi_m(\alpha,\beta)| \geq \varphi(m)\log|\alpha| - \begin{cases} \log 2, & \text{for } \omega(m) \leq 1+\eta, \\ \frac{2^{\omega(m)}}{2^{2+\eta}}\left(\log 2q_0\right), & \text{for } \omega(m) \geq 2+\eta. \end{cases} \tag{3.5}$$

Proof. Let α and β be real and put $x = \beta/\alpha$ as before. When $m = 2^{e_0}$ with $m \geq 4$, we get that $m/2$ even and so $1 + x^{m/2} > 0$ implying the assertion (3.5) from (3.4). Thus, we assume that $\omega(m) \geq 2$ when m is even. We keep the notations γ, q_0, γ_0 used before. We will use (3.4). For $0 < |y| < 1$, we have

$$\frac{1-y^{q_0}}{1-y} = 1 + \sum_{i=1}^{q_0-1} y^i > 1 \quad \text{if} \quad y > 0,$$

and

$$\frac{1-y^{q_0}}{1-y} = \frac{1-y(y^{(q_0-1)/2})^2}{1-y} \geq \frac{1}{1-y} > \frac{1}{2} \quad \text{if} \quad y < 0.$$

Hence, putting $y := x^{m/dq_0}$ if m is odd and $y = -x^{m/2dq_0}$ if m is even, we get

$$\left|\frac{1-x^{\frac{m}{d}}}{1-x^{\frac{m}{q_0 d}}}\right| \geq \frac{1}{2} \quad \text{and} \quad \left|\frac{1+x^{\frac{m}{2d}}}{1+x^{\frac{m}{2q_0 d}}}\right| \geq \frac{1}{2}.$$

This together with relation (3.4) already gives (3.5) if $\gamma_0 = 1$. For $\gamma_0 > 1$, the above relations together with

$$\left|\frac{1-x^{\frac{m}{d}}}{1-x^{\frac{m}{q_0 d}}}\right| \leq q_0 \quad \text{and} \quad \left|\frac{1+x^{\frac{m}{2d}}}{1+x^{\frac{m}{2q_0 d}}}\right| \leq q_0,$$

relation (3.4), the fact that $\omega(\gamma_0) = \omega(m) - 1 - \eta$ and observing that there are exactly $2^{\omega(\gamma_0)-1}$ divisors d of γ_0 with $\mu(d) = 1$ and exactly $2^{\omega(\gamma_0)-1}$ divisors d of γ_0 with $\mu(d) = -1$, give (3.5).

□

We need the following result stated in Voutier [27, Lemma 5] for getting a lower bound for $|\Phi_n(\alpha, \beta)|$ when α and β are complex conjugates. Let us recall Voutier's notation from Theorem 2 on page 408 in [27]. That involves a parameter d which is the degree of β/α. Since we will use it only when α and β are complex conjugates, we take $d = 2$. Then for a real number $B \geq 2$ we put

$$c_1(B) := \min\left\{\frac{\log 2}{32\log^2 B} + \frac{B}{8\log^2 B}, 22.36\left(\frac{1}{2} + \frac{0.28}{\log B} + \frac{1.8}{2\log B}\right)^2\right\}.$$

Below is Voutier's lemma for the case when α and β are complex conjugated (case (ii) of Lemma 5). Voutier has a height condition namely that $dh(\beta/\alpha) > 8$, where h is the height function. For us, $2\alpha = r + i\sqrt{\Delta}$ and $2\beta = r - \sqrt{\Delta}$ where $\Delta = r^2 + 4s < 0$, so $|\alpha| = |\beta|$. The minimal polynomial of β/α is $sx^2 - (r^2 + 2s)x + s$ and β/α and its complex conjugate α/β have absolute values 1. Thus,

$$h\left(\frac{\beta}{\alpha}\right) = \frac{\log|s|}{2} = \frac{\log|\alpha| + \log|\beta|}{2} = \log|\alpha|.$$

Hence, Voutier's condition is satisfied in our case when $\log|\alpha| > 4$.

Remark 3.5 It is important that the logarithmic height $h(\frac{\beta}{\alpha})$ should not be constrained to lie within a certain range of values. If it is, our Theorem 3.1 would not hold in general.

Lemma 3.6 *Let α and β be complex conjugates with $\log|\alpha| > 4$. For $t \geq 3$, putting*

$t_* := t/\gcd(t,2)$, *we have*

$$\log|\alpha^t - \beta^t| \geq \log|\alpha| \left(t - 8(c_1(t_*) + 0.02)\log^2 t_*\right).$$

Furthermore,

$$\log|\alpha^t + \beta^t| \geq \log|\alpha| \left(t - 8(c_1(t) + 0.02)\log^2 t\right).$$

Remark 3.7 We make one comment about the above lemma. Voutier only states an equivalent inequality to the first one from Lemma 3.6 for the case of $\alpha^t - \beta^t$. However, at the end of his proof he states that by the same reasoning, also the second inequality of Lemma 3.6 holds.

We will use the following consequence of Lemma 3.6.

Lemma 3.8 *Let α and β be complex conjugates with $\log|\alpha| > 4$. Then*

$$\log\left|1 \pm \left(\frac{\beta}{\alpha}\right)^t\right| \geq -\log|\alpha| \quad \textit{for} \quad t = 1, 2. \tag{3.6}$$

For $t \geq 3$, putting $t_* := t/\gcd(t,2)$, we have

$$\log|\alpha^t - \beta^t| \geq \log|\alpha| \left(t - 44.72\left(\log t_* + 2.36\right)^2 - 0.16(\log t_*)^2\right), \tag{3.7}$$

and

$$\log|\alpha^t + \beta^t| \geq \log|\alpha| \left(t - 44.72\left(\log t + 2.36\right)^2 - 0.16(\log t)^2\right). \tag{3.8}$$

Proof. Inequalities (3.6) are clear using that $|\alpha^t \pm \beta^t| \geq 1$ for $t = 1, 2$. For Inequalities (3.7) and (3.8), we use Lemma 3.6, and keep only the second expression inside the minimum which defines $c_1(B)$, namely

$$c_1(B) \leq \frac{22.36(\log B + 2.36)^2}{4\log^2 B}$$

with $B = t_*$ and $B = t$, respectively.

□

Lemma 3.9 *Let α and β be complex conjugates with $\log|\alpha| > 4$. Let*

$$f(t) := 44.88 \log^2 t + 211.08 \log t + 249.08.$$

for $t > 1$. Then

$$\log|\Phi_m(\alpha,\beta)| \geq \varphi(m)\log|\alpha| - \max\left\{1, \frac{2^{\omega(m)}}{2^{2+\eta}}\right\}\left(f(m/2^\eta)\log\alpha + \log 2q_0\right). \tag{3.9}$$

Further, for m with $1 \leq \omega(m) \leq 6+\eta$, we have

$$\begin{aligned}\log|\Phi_m(\alpha,\beta)| \geq{}& \left(\varphi(m) - f(m/2^\eta) - (m/2^\eta)g_{\omega(m)-1-\eta}\right)\log|\alpha| \\ &- \begin{cases} \log 2, & \text{if} \quad \omega(m) \leq 1+\eta; \\ \frac{2^{\omega(m)}}{2^{2+\eta}}\log 2q_0, & \text{if} \quad \omega(m) \geq 2+\eta, \end{cases}\end{aligned} \tag{3.10}$$

where $g_{\omega(m)}$ is given by the following table:

$\omega(m)-1-\eta$	0	1	2	3	4	5
$g_{\omega(m)-\eta}$	0	0	0.0286	0.0598	0.0934	0.1238

Proof. Let α and β be complex conjugates with $\log|\alpha| > 4$. We use Lemmas 3.2 and 3.8, in particular, (3.3), (3.4) and (3.7) for m odd and (3.3), (3.4) and (3.8) for m even. Let $x = \beta/\alpha$ be as before. If $\omega(m) \geq 2+\eta$, then there are $2^{\omega(m)-2-\eta}$ divisors d of γ_0 with $\mu(d) = 1$ and an equal number of divisors d with $\mu(d) = -1$. As usual, here $\eta = 0, 1$ according as m is odd or even, respectively. Hence, from (3.4) and the inequalities

$$\left|1 \pm x^{\frac{m}{2^\eta q_0 d}}\right| \leq 2, \quad \left|\frac{1 - x^{\frac{m}{d}}}{1 - x^{\frac{m}{q_0 d}}}\right| \leq q_0 \quad \text{and} \quad \left|\frac{1 + x^{\frac{m}{2d}}}{1 + x^{\frac{m}{2q_0 d}}}\right| \leq q_0,$$

we get

$$\sum_{d|\gamma}\mu(d)\log\left|1-x^{\frac{m}{d}}\right| \geq \begin{cases} \log\left|1+x^{\frac{m}{2}}\right|-\log 2, & \omega(m)\leq 2,\\ & \eta=1;\\ \displaystyle\sum_{\substack{\mu(d)=1\\ d|\gamma_0}}\log\left|1+x^{\frac{m}{2d}}\right|-2^{\omega(m)-3}\log 2q_0, & \omega(m)\geq 3,\\ & \eta=1;\\ \log|1-x^m|-\log 2, & \omega(m)=1,\\ & \eta=0;\\ \displaystyle\sum_{\substack{\mu(d)=1\\ d|\gamma_0}}\log\left|1-x^{\frac{m}{d}}\right|-2^{\omega(m)-2}\log 2q_0, & \omega(m)\geq 2,\\ & \eta=0. \end{cases} \tag{3.11}$$

We have

$$44.72(\log t+2.36)^2+0.16\log^2 t\leq 44.88\log^2 t+211.08\log t+249.08=f(t).$$

Putting $t:=m/d$ in (3.7) when m is odd and $t:=m/(2d)$ in (3.8) when m is even, we get

$$\log\left|1-x^{\frac{m}{d}}\right|\geq -f(m/d)\log|\alpha| \quad \text{and} \quad \log\left|1+x^{\frac{m}{2d}}\right|\geq -f(m/(2d))\log|\alpha|. \tag{3.12}$$

Observe that

$$f\left(\frac{m}{2^\eta d}\right)\leq f\left(\frac{m}{2^\eta}\right) \qquad \text{for} \qquad d\geq 1.$$

Now the Inequalities (3.9) as well as (3.10) for $\omega(m)\leq 1+\eta$ follow easily from (3.3) and (3.11). Thus, it remains to look at (3.10) for the case when $\omega(m)\geq 2+\eta$. So, we take $\omega(m)\geq 2+\eta$. There are $2^{\omega(m)-2-\eta}$ divisors d of γ_0 with $\mu(d)=1$ and an equal number of divisors d with $\mu(d)=-1$. If $\omega(m)=2+\eta$, the only such d is

$d = 1$. For $\omega(d) \geq 3+\eta$, since $q_0 \geq 3$ and $q_0 \nmid \gamma_0$, each $d > 1$ with $d \mid \gamma_0$ and $\mu(d) = 1$ has its least prime factor at least 5. For such $d > 1$, we apply the trivial inequality

$$\log \left|1 \pm x^{m/2^\eta d}\right| \geq -\frac{(m/2^\eta)\log|\alpha|}{d},$$

Summing up the above inequalities, we get that $g_{\omega(m)-1-\eta}$ can be taken to be an upper bound for the numbers

$$\sum_{\substack{d\mid\gamma_0\\ d>1\\ \mu(d)=1}} \frac{1}{d} = \frac{1}{2}\left(\prod_{p\mid\gamma_0}\left(1+\frac{1}{p}\right)+\prod_{p\mid\gamma_0}\left(1-\frac{1}{p}\right)\right)-1, \tag{3.13}$$

where the upper bound is taken over all squarefree numbers γ_0 whose smallest prime factor is at least 5. The values for $g_{\omega(m)-1-\eta}$ appearing in the table for $\omega(m)-1-\eta \in \{2,3,4,5\}$ are obtained by computing the right–hand sides of (3.13) above for $\gamma_0 = 5\cdot 7,\ 5\cdot 7\cdot 11,\ 5\cdot 7\cdot 11\cdot 13,\ 5\cdot 7\cdot 11\cdot 13\cdot 17$, respectively.

□

3.3 Upper bound for prime powers dividing a product of factorials

In the present section, we use Lemmas 2.22 and 2.27 to get an upper bound for prime powers dividing a product of factorials. The allusion to an adaptation of the Primitive Divisor Theorem is evident in the hypothesis of Lemma 3.10 below. A subtle interplay of various tools utilised, inclusive of Stirling's formula, the Abel summation formula, the fact that the primitive divisors of U_n are congruent to one of ± 1 modulo n, in conjunction with Lemmas 2.22 and 2.27, culminate in the establishment of an upper bound for prime powers dividing a product of factorials under the given hypotheses

of Lemma 3.10.

We assume $\mathcal{F} \in \mathcal{P}F$ is such that $\mathcal{F} > 1$. Thus,

$$\mathcal{F} := \mathcal{F}(m_1, m_2, \ldots, m_k) := m_1!m_2! \cdots m_k! \tag{3.14}$$

for some positive integers $1 < m_1 \leq \cdots \leq m_k$. Let n_0 be a positive integer and let $a_0 = -1$ if $n_0 \neq 12$ and $a_0 \in \{-1, 7\}$ if $n_0 = 12$. Define (this is similar to Equation (2.10))

$$M_{n_0}(\mathcal{F}) := \log \left(\prod_{\substack{p^{\alpha_p} \| \mathcal{F} \\ p \equiv 1, a_0 \pmod{n_0}}} p^{\alpha_p} \right) = \sum_{\substack{p^{\alpha_p} \| \mathcal{F} \\ p \equiv 1, a_0 \pmod{n_0}}} \alpha_p \log p. \tag{3.15}$$

We use analytic methods to get an upper bound of $M_{n_0}(\mathcal{F})$ and prove the following lemma.

Lemma 3.10 *(a) For a positive integer $n_0 \geq 30$ and $m_k \geq 3$, we have*

$$M_{n_0}(\mathcal{F}) < \frac{4}{\varphi(n_0)}(1 + \log\log n_0)\left(\log \mathcal{F} - 1.4 \log m_k\right). \tag{3.16}$$

(b) Let $\mathcal{D} := \{8, 9, 12, 16, 24\} \cup \{5 \leq p \leq 30 : p \text{ prime}\}$. For $n_0 \in \mathcal{D}$, we have

$$M_{n_0}(\mathcal{F}) \leq \frac{2.2}{\varphi(n_0)}\left(\log \mathcal{F} - 1.4 \log m_k\right). \tag{3.17}$$

Further for $n_0 \in \{8, 12\}$, we have

$$M_{n_0}(\mathcal{F}) \leq \begin{cases} 0.23 \log \mathcal{F} & \text{when } n_0 = 8 \text{ and } 7 \leq m_k < 17; \\ 0.3 \log \mathcal{F} & \text{when } n_0 = 8 \text{ and } 17 \leq m_k \leq 47; \\ 0.28 \log \mathcal{F} & \text{when } n_0 = 12 \text{ and } 7 \leq m_k \leq 16. \end{cases} \tag{3.18}$$

Proof. Using $t! \geq \sqrt{2\pi t}(t/e)^t = \sqrt{2\pi e}(t/e)^{t+1/2}$ for $t > 1$, we get

$$\begin{aligned}\log \mathcal{F} &\geq \sum_{i=1}^{k} \left(m_i + \frac{1}{2}\right)(\log m_i - 1) + \frac{1 + \log 2\pi}{2} \\ &\geq \sum_{m_i} (m_i - 1)(\log m_i - 1) + 1.5 \log m_k - 1 + \frac{\log 2\pi}{2}.\end{aligned}$$

Since $m_k \geq 3$, we have $0.1 \log m_k - 1 + \frac{\log 2\pi}{2} > 0$ and hence

$$\sum_{m_i} (m_i - 1)(\log m_i - 1) \leq \log \mathcal{F} - 1.4 \log m_k. \tag{3.19}$$

For a prime number p and a positive integer t we write $\nu_p(t)$ for the exact exponent of p in the prime factorization of t. For a prime p, we have

$$\alpha_p = \nu_p\left(\prod_{i=1}^{k} m_i!\right) \leq \sum_{m_i \geq p} \frac{m_i - 1}{p - 1},$$

by using the fact that

$$\nu_p(t!) = \frac{t - \sigma_p(t)}{p - 1} \leq \frac{t - 1}{p - 1},$$

where $\sigma_p(t)$ is the sum of digits of t in base p. Recall that $a_0 = -1$ if $n_0 \neq 12$ and $a_0 \in \{-1, 7\}$ if $n_0 = 12$. Let Q_1 be the least prime congruent to one of $1, a_0 \pmod{n_0}$. Hence,

$$M_{n_0}(\mathcal{F}) \leq \sum_{p \equiv 1, a_0 \pmod{n_0}} \alpha_p \log p \leq \sum_{m_i \geq Q_1} (m_i - 1) \sum_{\substack{p \leq m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p - 1}. \tag{3.20}$$

We next give upper bounds on the inner sums in the right most side above. For that, we assume from now on that $m_i \geq Q_1$ since the sum is 0 for $m_i < Q_1$. We first consider $n_0 \geq 30$. We have $a_0 = -1$. Assume first that $m_k \leq 4n_0 - 2$ and we shall consider the case $m_k > 4n_0 - 2$ later. Note that $Q_1 \geq n_0 - 1$ in this case. For $m_i \geq Q_1$ and primes $p \leq m_i$ with $p \equiv \pm 1 \pmod{n_0}$, we have in fact $p \leq 3n_0 + 1$

because $p \le 4n_0 - 2$. We obtain

$$\sum_{\substack{p \le m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} \le \frac{\log m_i}{3n_0+1} \left(\sum_{t=1}^{3} \left(\frac{3n_0+1}{tn_0-2} + \frac{3n_0+1}{tn_0} \right) \right) < \frac{5.5(\log m_i - 1)}{n_0}.$$

In the above, we used that $m_i \ge p \ge Q_1 \ge n_0 - 1$. Indeed, the function $(\log m_i)/(\log m_i - 1)$ is increasing for $m_i \ge n_0 \ge 30$, and the maximum of the function

$$\left(\frac{\log n_0}{\log n_0 - 1} \right) \left(\frac{n_0}{3n_0+1} \right) \left(\sum_{t=1}^{3} \frac{3n_0+1}{tn_0-2} + \frac{3n_0+1}{tn_0} \right)$$

for $n_0 \ge 30$ is at $n_0 = 30$ where its value is smaller than 5.34. Since $n_0 > \varphi(n_0)$, we get

$$\sum_{\substack{p \le m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} \le \frac{5.5(\log m_i - 1)}{\varphi(n_0)}.$$

This together with (3.20) and (3.19) imply (3.16) since

$$5.5 \le 4 + 4 \log\log 30 \le 4 + 4 \log\log n_0.$$

From now on, we assume $m_k > 4n_0 - 2$. We use a variation of an argument from [19]. We split the argument as in [19] into two parts, namely according to $p \le 4n_0 - 2$ or $p > 4n_0 - 2$. Since $p \equiv \pm 1 \pmod{n_0}$, the conditions $p > 4n_0 - 2$ and $p > 3n_0 + 1$ (or $p \le 4n_0 - 2$ and $p \le 3n_0 + 1$) are the same. Now, for $p \le 4n_0 - 2$, we have $p \le 3n_0 + 1$, so $\log p \le \log(3n_0 + 1)$, and hence

$$\sum_{\substack{p \le 4n_0 - 2 \\ p \equiv \pm 1 \pmod{n_0}}} \frac{\log p}{p-1} - (\pi_1(3n_0+1) + \pi_{-1}(3n_0+1) - 1) \frac{\log(3n_0+1)}{3n_0+1}$$

$$\le \begin{cases} \left(1 + \sum_{t=1}^{3} \left(\frac{3n_0+1}{tn_0-1-1} - 1 + \frac{3n_0+1}{tn_0+1-1} - 1 \right) \right) \frac{\log(3n_0+1)}{3n_0+1}, & \text{if } n_0 \text{ is even;} \\ \left(1 + \left(\frac{3n_0+1}{2n_0-2} - 1 + \frac{3n_0+1}{2n_0} - 1 \right) \right) \frac{\log(3n_0+1)}{3n_0+1}, & \text{if } n_0 \text{ is odd,} \end{cases}$$

$$\leq \begin{cases} \left(1 + \sum_{t=1}^{3} \left(\frac{91}{30t-2} + \frac{3}{t} + \frac{1}{30t} - 2\right)\right) \frac{\log(3n_0+1)}{3n_0} & \text{if } n_0 \text{ is even;} \\ \left(\frac{91}{58} + \frac{91}{60} - 1\right) \frac{\log(3n_0+1)}{3n_0} & \text{if } n_0 \text{ is odd,} \end{cases}$$

$$\leq \begin{cases} \frac{6.42 \log(3n_0)}{3n_0}, & \text{if } n_0 \text{ is even;} \\ \frac{2.09 \log(3n_0)}{3n_0}, & \text{if } n_0 \text{ is odd,} \end{cases}$$

since $n_0 \geq 30$. We obtain, from the fact that $\varphi(n_0) \leq n_0/2$ for n_0 even, that

$$\frac{6.42 \log(3n_0)}{3n_0} \leq \frac{6.42 \log(3n_0)}{6\varphi(n_0)} = \frac{1.07 \log(3n_0)}{\varphi(n_0)}$$

for n_0 even. The inequality

$$\frac{2.09 \log(3n_0)}{3n_0} \leq \frac{1.07 \log(3n_0)}{\varphi(n_0)}$$

also holds for n_0 odd since $\varphi(n_0) < n_0$. For $p > 4n_0 - 2$, we have

$$\begin{aligned} \sum_{\substack{p \equiv \pm 1 \pmod{n} \\ p > 4n_0 - 2}} \frac{\log p}{p-1} &= \sum_{\substack{p \equiv \pm 1 \pmod{n} \\ p > 3n_0 + 1}} \frac{\log p}{p-1} \\ &\leq \sum_{\substack{p \equiv \pm 1 \pmod{n} \\ p > 3n_0 + 1}} \frac{\log p}{p} + \sum_{t \geq 3n_0 + 2} \left(\frac{\log(t-1)}{t-1} - \frac{\log t}{t}\right) \\ &\leq \sum_{\substack{p \equiv \pm 1 \pmod{n_0} \\ p > 3n_0 + 1}} \frac{\log p}{p} + \frac{\log(3n_0+1)}{3n_0+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{\substack{p \equiv \pm 1 \pmod{n} \\ p \leq m_i}} \frac{\log p}{p-1} - (\pi_1(3n_0+1) + \pi_{-1}(3n_0+1)) \frac{\log(3n_0+1)}{3n_0+1} \\ &\leq \sum_{\substack{p \equiv \pm 1 \pmod{n_0} \\ p > 3n_0 + 1}} \frac{\log p}{p} + \frac{1.07 \log(3n_0)}{\varphi(n_0)}. \end{aligned} \tag{3.21}$$

We now get an upper bound for the sum of $(\log p)/p$ over primes p congruent to one of ± 1 modulo n_0. Recall that $\pi(x; n_0, a)$ stands for the number of primes $p \leq x$

satisfying $p \equiv a \pmod{n_0}$. For simplicity, we put

$$\pi_{\pm 1}(x) := \pi(x; n_0, 1) + \pi(x; n_0, -1).$$

By Lemma 2.22, we have

$$\pi_{\pm 1}(x) \le \frac{4x}{\varphi(n_0)\log(x/n_0)} \quad \text{for} \quad x > n_0.$$

Since the conditions $p > 4n_0 - 2$ and $p > 3n_0 + 1$ are the same, by Abel's summation formula, we have

$$\begin{aligned}
&\sum_{\substack{4n_0-2<p\le m_i \\ p\equiv\pm 1 \pmod{n_0}}} \frac{\log p}{p} + \pi_{\pm 1}(3n_0+1)\frac{\log(3n_0+1)}{3n_0+1} \\
=&\sum_{\substack{3n_0+1<p\le m_i \\ p\equiv\pm 1 \pmod{n_0}}} \frac{\log p}{p} + \pi_{\pm 1}(3n_0+1)\frac{\log(3n_0+1)}{3n_0+1} \\
\le&\pi_{\pm 1}(m_i)\frac{\log m_i}{m_i} - \pi_{\pm 1}(3n_0+1)\frac{\log(3n_0+1)}{3n_0+1} \\
&- \int_{3n_0+1}^{m_1} \pi_{\pm 1}(t)\left(\frac{1-\log t}{t^2}\right)dt + \pi_{\pm 1}(3n_0+1)\frac{\log(3n_0+1)}{3n_0+1} \\
\le&\frac{4}{\varphi(n_0)}\frac{m_i}{\log(m_i/n_0)}\frac{\log m_i}{m_i} + \frac{4}{\varphi(n_0)}\int_{3n_0+1}^{m_i}\frac{\log(t/n_0)+\log n_0 - 1}{t\log(t/n_0)}dt \\
\le&\frac{4}{\varphi(n_0)}\frac{\log m_i}{\log(m_i/n_0)} + \frac{4}{\varphi(n_0)}\int_{3n_0+1}^{m_i}\left(\frac{1}{t} + \frac{\log n_0 - 1}{t\log(t/n_0)}\right)dt \\
<&\frac{4}{\varphi(n_0)}\left(\frac{\log m_i}{\log(m_i/n_0)} + \log m_i - \log(3n_0) + (\log n_0 - 1)(\log\log(m_i/n_0))\right).
\end{aligned}$$

This together with (3.21) imply that

$$\begin{aligned}
&\sum_{\substack{p\equiv\pm 1 \pmod{n_0} \\ p\le m_i}} \frac{\log p}{p-1} \\
\le\ & \frac{4}{\varphi(n_0)}\left(\frac{\log m_i}{\log(m_i/n_0)} + \log\left(\frac{m_i}{3n_0}\right) + (\log n_0 - 1)(\log\log(m_i/n_0))\right) \\
+\ & \frac{1.07\log(3n_0)}{\varphi(n_0)}
\end{aligned}$$

$$\leq \frac{4}{\varphi(n_0)}\left(\frac{\log m_i}{\log(m_i/n_0)} + \log m_i + (\log n_0 - 1)(\log\log(m_i/n_0))\right.$$
$$- \left.\frac{2.93}{4}\log(3n_0)\right).$$

We put $m_i := n_0^{1+c_i}$. Since m_i is fixed in the subsequent argument, we write $c := c_i$ in order not to clutter the notation (that is, we omit the dependence on i of c). Then $m_i \geq 4n_0 - 1 \geq 3.93n_0$ for $n_0 \geq 30$ giving $c \geq \log 3.93/\log n_0$. Hence,

$$\sum_{\substack{p\leq m_i \\ p\equiv\pm 1 \pmod{n_0}}} \frac{\log p}{p-1}$$
$$\leq \frac{4}{\varphi(n_0)}\left(\frac{c+1}{c} + (c+1)\log n_0 + (\log n_0 - 1)(\log\log n_0 + \log c)\right.$$
$$- \left.\frac{2.93\log(3n_0)}{4}\right)$$
$$\leq \frac{4((c+1)\log n_0 - 1)}{\varphi(n_0)}\left(1 + \frac{(\log n_0 - 1)}{(c+1)\log n_0 - 1}(\log\log n_0 + \log c)\right.$$
$$+ \left.\frac{2 + \frac{1}{c} - \frac{2.93}{4}\log(3n_0)}{(c+1)\log n_0 - 1}\right)$$
$$\leq \frac{4(\log m_i - 1)}{\varphi(n_0)}\left(1 + \frac{\log\log n_0}{c+1} + \frac{\log c}{c+1} + \frac{2 + \frac{1}{c} - \frac{2.93}{4}\log(3n_0)}{(c+1)\log n_0 - 1}\right).$$

If $c \geq 1$; that is, if $m_i \geq n_0^2$, then the expression inside the bracket is at most

$$1 + \frac{\log\log n_0}{2} + \frac{1}{e+1} < 1 + \log\log n_0, \quad \text{since} \quad n_0 \geq 30.$$

If $c < 1$, but

$$2.93(\log 3n_0)/4 - 2 - 1/c \geq 0,$$

the expression inside the bracket is at most $1 + \log\log n$. Assume now that

$$2.93(\log 3n_0)/4 < 2 + 1/c.$$

Since $c \geq \log 3.93/\log n_0$, we have $1/c \leq (\log n_0)/\log 3.93$ and therefore

$$\begin{aligned}\frac{2+\frac{1}{c}-\frac{2.93}{4}\log(3n_0)}{(c+1)\log n_0 - 1} &\leq \frac{2+\frac{\log n_0}{\log 3.93}-\frac{2.93\log(3n_0)}{4}}{\log n_0 + \log 3.93 - 1} \\ &< \frac{2-\frac{2.93\log 3}{4}}{\log n_0 + \log 3.93 - 1} \\ &\leq 0.32.\end{aligned}$$

Also

$$2.93(\log(3n_0))/4 < 2 + 1/c \quad \text{implies} \quad 1/c \geq 2.93(\log 3n_0)/4 - 2 > (\log n_0)/3,$$

since $n_0 \geq 30$. Therefore,

$$\frac{\log c}{c+1} \leq \frac{-\log(1/c)}{2} \leq -\frac{\log((\log n_0)/3)}{2} = \frac{\log 3}{2} - \frac{\log\log n_0}{2}.$$

Putting these estimates together, we get

$$\begin{aligned}\sum_{\substack{p\leq m_i \\ p\equiv\pm 1 \pmod{n_0}}} \frac{\log p}{p-1} &\leq \frac{4(\log m_i - 1)}{\varphi(n_0)}\left(1 + \frac{\log\log n}{2} + \frac{\log 3}{2} + 0.32\right) \\ &\leq \frac{4(\log m_i - 1)}{\varphi(n_0)}\left(1 + \log\log n_0\right),\end{aligned}$$

since $n_0 \geq 30$. Finally, inserting this estimate into (3.20), we get

$$\log M_{n_0} \leq \frac{4(1+\log\log n_0)}{\varphi(n_0)}\left(\sum_{m_i \geq n_0 - 1}(m_i - 1)(\log m_i - 1)\right),$$

which together with (3.19) give the assertion (3.16).

We now consider $n_0 \in \mathcal{D}$. Recall that $a_0 = -1$ if $n_0 \neq 12$ and that $a_0 \in \{-1, 7\}$ if $n_0 = 12$. If the inequality

$$\sum_{\substack{p\leq m_i \\ p\equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} < \frac{2.2(\log m_i - 1)}{\varphi(n_0)} \tag{3.22}$$

is valid for all m_i, then, together with (3.20) and (3.19), we get the assertion (3.17). Therefore, we now show that (3.22) is valid for all m_i. First we check that (3.22) is valid at all values of $m_i \le 400$ which are primes congruent to one of $1, a_0$ modulo n_0 and hence (3.22) is valid for all $m_i \le 400$. Thus, we consider $m_i > 400$.

Let $\pi_{1,a_0}(x) := \pi(x; n_0, 1) + \pi(x; n_0, a_0)$. We have, from Lemma 2.27, that

$$\pi_{1,a_0}(x) \le \frac{x}{\varphi(n_0)\log x}\left(2 + \frac{5}{\log x}\right).$$

Put $n_1 = 70$. By Abel's summation formula again, we have, for $m_i > 400$,

$$\begin{aligned}
&\sum_{\substack{n_1 < p \le m_i \\ p \equiv \pm 1 \pmod{n_0}}} \frac{\log p}{p} + \pi_{1,a_0}(n_1)\frac{\log n_1}{n_1} \\
\le &\pi_{1,a_0}(m_i)\frac{\log m_i}{m_i} - \int_{n_1}^{m_i} \pi_{1,a_0}(t)\frac{1-\log t}{t^2}dt \\
\le &\frac{m_i}{\varphi(n_0)\log m_i}\left(2 + \frac{5}{\log m_i}\right)\frac{\log m_i}{m_i} + \frac{1}{\varphi(n_0)}\int_{n_1}^{m_i} \frac{t}{\log t}\left(2 + \frac{5}{\log t}\right)\frac{\log t - 1}{t\log t}dt \\
\le &\frac{1}{\varphi(n_0)}\left(2 + \frac{5}{\log m_i}\right) + \frac{1}{\varphi(n_0)}\int_{n_1}^{m_i}\left(2 + \frac{5}{\log t}\right)\left(1 - \frac{1}{\log t}\right)\frac{dt}{t} \\
< &\frac{1}{\varphi(n_0)}\left(2 + \frac{5}{\log m_i}\right) + \frac{1}{\varphi(n_0)}\int_{n_1}^{m_i}\left(2 + \frac{3}{\log t} - \frac{5}{\log^2 t}\right)\frac{dt}{t} \\
\le &\frac{1}{\varphi(n_0)}\left(2 + \frac{5}{\log m_i} + 2(\log m_i - \log n_1) + 3(\log\log m_i - \log\log n_1)\right. \\
&\left. + \frac{5}{\log m_i} - \frac{5}{\log n_1}\right) \\
\le &\frac{1}{\varphi(n_0)}\left(2\log m_i - 2 + \delta_1 + (\log m_i - 1 + 1)\delta_2\right),
\end{aligned}$$

where

$$\delta_2 := \frac{3\log\log m_i - 1.23\log n_1}{\log m_i} \quad \text{and} \quad \delta_1 := 4 - 0.65\log n_1 - 3\log\log n_1 < -3.1,$$

since $n_1 = 70$, and where we have used that

$$\frac{10}{\log m_i} - \frac{5}{\log n_1} - 0.12\log n_1 \le \frac{10}{\log 400} - \frac{5}{\log 70} - 0.12\log 70 < 0,$$

since $m_i > 400$. We have

$$\delta_2 = \frac{3}{n_1^{0.41}} \frac{\log \frac{\log m_i}{n_1^{0.41}}}{\frac{\log m_i}{n_1^{0.41}}} \leq \frac{3}{n_1^{0.41}} \cdot \frac{1}{e} = \frac{3}{e \cdot 70^{0.41}} < 0.2.$$

Hence,

$$\sum_{\substack{n_1 < p \leq m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p} + \pi_{1,a_0}(n_1) \frac{\log n_1}{n_1} \leq \frac{1}{\varphi(n_0)} \left(2.2(\log m_i - 1) - 2.9\right).$$

Therefore, we obtain

$$\begin{aligned}
\sum_{\substack{p \leq m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} &\leq \sum_{\substack{p \leq n_1 \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} + \sum_{\substack{n_1 < p \leq m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p} + \frac{\log n_1}{n_1} \\
&\leq \sum_{\substack{p \leq n_1 \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} - \left(\pi_{1,a_0}(n_1) - 1\right) \frac{\log n_1}{n_1} \\
&\quad + \sum_{\substack{n_1 < p \leq m_i \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p} + \pi_{1,a_0}(n_1) \frac{\log n_1}{n_1} \\
&\leq \sum_{\substack{p \leq n_1 \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} - \left(\pi_{1,a_0}(n_1) - 1\right) \frac{\log n_1}{n_1} - \frac{2.9}{\varphi(n_0)} \\
&\quad + \frac{2.2(\log m_i - 1)}{\varphi(n_0)}.
\end{aligned}$$

For each $n_0 \in \mathcal{D}$, putting $n_1 := 70$, we checked that

$$\sum_{\substack{p \leq n_1 \\ p \equiv 1, a_0 \pmod{n_0}}} \frac{\log p}{p-1} - \left(\pi_{1,a_0}(n_1) - 1\right) \frac{\log n_1}{n_1} - \frac{2.9}{\varphi(n_0)} < 0.$$

This implies (3.22) and hence the proof of (3.17) is complete.

Finally, we prove (3.18). For that, we take $n_0 \in \{8, 12\}$ and we assume that $m_k \leq n_2$, where $n_2 = 47$ if $n_0 = 8$ and $n_2 = 16$ if $n_0 = 12$. Recall that Q_1 is the least prime congruent to one of $1, a_0$ modulo n_0. Let $Q_1 < Q_2 < \cdots < Q_{\ell_2}$ be all the primes

congruent to one of $1, a_0 \pmod{n_0}$ which are at most n_2. In fact we have

$$\ell_2 = 6: \ Q_1 = 7, Q_2 = 17, Q_3 = 23, Q_4 = 31, Q_5 = 41, Q_6 = 47,$$

if $n_0 = 8$ and

$$\ell_2 = 2: \ Q_1 = 11, Q_2 = 13 \qquad \text{if} \quad a_0 = -1;$$
$$\ell_2 = 2: \ Q_1 = 7, Q_2 = 13 \qquad \text{if} \quad a_0 = 7,$$

if $n_0 = 12$. Let t_j be the number of m_is with $m_i = j$ in (3.14). Then

$$\log \mathcal{F} = \sum_{j \le m_k} t_j \log j!.$$

Since $m_i \le m_k \le 47$, we see that $Q_h^2 > 47 \ge m_k$ for $h \ge 1$. Hence, for $p = Q_h$,

$$\alpha_p = \nu_p \left(\prod_{i=1}^{k} m_i! \right) = \sum_j t_j \nu_p(j!) \le \sum_{j \ge p} t_j \frac{j}{p} = \frac{1}{p} \sum_{j \ge p} t_j j.$$

Therefore,

$$\begin{aligned} M_{n_0}(\mathcal{F}) &= \sum_{Q_\ell \le m_k} \alpha_{Q_\ell} \log Q_\ell \\ &\le \sum_{Q_\ell \le m_k} \left(\frac{\log Q_\ell}{Q_\ell} \sum_{j \ge Q_\ell} t_j j \right) = \sum_{j \le m_k} \left(\sum_{Q_\ell \le j} \frac{\log Q_\ell}{Q_\ell} \right) t_j j. \end{aligned}$$

Let ℓ_0 be the maximum such that $Q_{\ell_0} \le m_k$ so that

$$\log \mathcal{F} = \sum_{h=1}^{\ell_0} \left(\sum_{Q_h \le j < Q_{h+1}} t_j \log j! \right).$$

Since $j \mapsto j/\log j!$ is a decreasing function of j, we have

$$\begin{aligned} M_{n_0}(\mathcal{F}) &\le \sum_{h=1}^{\ell_0} \sum_{Q_h \le j < Q_{h+1}} t_j \log j! \frac{j}{\log j!} \left(\sum_{i=1}^{h} \frac{\log Q_i}{Q_i} \right) \\ &\le \sum_{h=1}^{\ell_0} \sum_{Q_h \le j < Q_{h+1}} t_j \log j! \frac{Q_h}{\log Q_h!} \left(\sum_{i=1}^{h} \frac{\log Q_i}{Q_i} \right). \end{aligned}$$

Let

$$\delta_h := \frac{Q_h}{\log Q_h!}\left(\sum_{i=1}^{h}\frac{\log Q_i}{Q_i}\right) \qquad \text{for} \quad 1 \le h \le \ell_2,$$

and put $\delta_0 := \max_{1\le h\le \ell_2}\delta_h$. Then

$$M_{n_0}(\mathcal{F}) \le \begin{cases} \delta_1 \displaystyle\sum_{Q_1\le j<Q_2} t_j \log j! = \delta_1 \log \mathcal{F}, & \text{if } Q_1 \le m_k < Q_2; \\ \delta_0 \displaystyle\sum_{h=1}^{\ell_0}\sum_{Q_h\le j<Q_{h+1}} t_j \log j! = \delta_0 \log \mathcal{F}, & \text{if } m_k \ge Q_2. \end{cases}$$

We computed the values of δ_0 and δ_1 for $n_0 \in \{8, 12\}$. We found numerically that $\delta_1 \le 0.23, \delta_0 \le 0.3$ if $n_0 = 8$ and $\delta_1 \le \delta_0 \le 0.28$ if $n_0 = 12$. This gives the assertion (3.18) and the proof is complete.

3.4 Proof of Theorem 3.1

Proof. We recall that $|\alpha| \ge |\beta|$. Further, we may replace (α, β) by $(-\alpha, -\beta)$. This replacement changes the pair (r, s) to $(-r, s)$, while $|U_n|$ and $|V_n|$ are not affected. Thus, we may assume that $r > 0$. When α, β are real, these conventions imply that α is positive so $\alpha > |\beta|$.

Note that $U_1 = 1 \in \mathcal{P}F$. In fact, if $U_n = \pm 1$ (or $V_n = \pm 1$) then U_n (or V_n) are also in $\mathcal{P}F$. The equations $U_n = \pm 1$ and $V_n = \pm 1$ are important from the Diophantine point of view. However, such equations have been solved completely and we refer to [4] for more details. For this reason, whenever we study Equations (3.1) and (3.2), we omit the cases $n = 1$, $U_n = \pm 1$ and $V_n = \pm 1$. Thus, we also assume that $m_1 > 1$.

We first treat the case of the sequence $\{U_n\}_{n\ge 0}$. Assume that Equation (3.1) has a

solution. Then $|U_n| = \mathcal{F} = m_1!m_2!\cdots m_k!$. For a divisor n_0 of n, we will compare the upper bound of $M_{n_0}(\mathcal{F})$ given by Lemma 3.10 with a lower bound on it obtained by using Lemmas 3.4 and 3.9. We will choose a suitable divisor n_0 of n such that these bounds contradict each other and hence for n with such divisors n_0, $|U_n|$ cannot be a product of factorials. It is known that primitive divisors of U_n are congruent to one of ± 1 modulo n. From the well known result from [4], we know that a primitive divisor for U_n exists for all $n > 30$, see Theorem 2.16. Further for $5 \leq n \leq 30$, $n \neq 6$, the pairs (r, s) for which a primitive divisor for U_n does not exist are given by

n	(r, s)
5	$(1,1), (1,-2), (1,-3), (1,-4), (2,-11), (12,-55), (12,-377)$
7	$(1,-2), (1,-5)$
8	$(1,-2), (2,-7)$
10	$(2,-3), (5,-7), (5,-18)$
12	$(1,1), (1,-2), (1,-3), (1,-4), (1,-5), (2,-15)$
$13, 18, 30$	$(1,-2)$

Compare the above table with the one in Theorem 2.16. We checked that for (r, s) given above with $n \geq 5, n \neq 6$, Equation (3.1) does not hold except when

$$(r, s, n) = (1, 1, 12), \quad \text{where} \quad U_{12} = 144 = 3!4! = 2!2!3!3!,$$

or when (r, s, n) belongs to

$$\{(1,-2,5), (1,-3,5), (12,-55,5), (12,-377,5), (1,-5,7), (1,-2,13)\},$$

for which $U_n = \pm 1$. Hence, we assume now that U_n has a primitive prime divisor p and so $p \equiv \pm 1 \pmod{n}$. From (3.1), we have that $p \mid m_k!$ and so $m_k \geq p$. Let $P_n := P(U_n)$ be the largest primitive divisor of U_n. Then $m_k \geq P_n$,

$P_n \equiv \pm 1 \pmod{n}$, and therefore

$$2|\alpha|^n \geq \left|\frac{\alpha^n - \beta^n}{\alpha - \beta}\right| = |U_n| \geq m_k! \geq P_n!. \tag{3.23}$$

Let n_0 be a divisor of n. Write

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{(\alpha^{n/n_0})^{n_0} - (\beta^{n/n_0})^{n_0}}{\alpha^{n/n_0} - \beta^{n/n_0}} \times \frac{\alpha^{n/n_0} - \beta^{n/n_0}}{\alpha - \beta}.$$

Let $\alpha_1 := \alpha^{n/n_0}$ and $\beta_1 := \beta^{n/n_0}$ and put

$$U^1_{n_0} := \frac{\alpha_1^{n_0} - \beta_1^{n_0}}{\alpha_1 - \beta_1}.$$

Then $U_n = U^1_{n_0} U_{n/n_0}$ and primitive prime divisors of U_n divide $U^1_{n_0}$. Let Q_1 be the least prime congruent to one of ± 1 modulo n_0 as before. Then $m_k \geq P_n \geq Q_1$. Hence, from (3.23), we have

$$n_0 \log |\alpha_1| = \frac{n}{n_0} \log |\alpha| \geq \log(\max\{m_k, Q_1\}!) - \log 2. \tag{3.24}$$

Since $Q_1 \geq n_0 - 1$, we obtain by using $t! \geq \sqrt{2\pi t}(t/e)^t > 2(t/e)^{t+1/2}$ for $t > 1$ that

$$\begin{aligned} n_0 \log |\alpha_1| &\geq \log(n_0 - 1)! - \log 2 \geq \left(n_0 - \frac{1}{2}\right)(\log(n_0 - 1) - 1) \\ &= n_0 \log n_0 \left(1 - \frac{1}{2n_0}\right)\left(1 - \frac{1}{\log n_0} - \frac{\log(1 + \frac{1}{n_0 - 1})}{\log n_0}\right) \\ &\geq 0.68 n_0 \log n_0 \quad \text{for} \quad n_0 \geq 30. \end{aligned} \tag{3.25}$$

We now look at the primitive part of $U^1_{n_0}$. This is the part of $U^1_{n_0}$ built up only with powers of primitive prime divisors of $U^1_{n_0}$. These primes are all congruent to one of ± 1 modulo n_0. We recall Lemma 2.11. Thus, from (3.1), we have

$\mathcal{F} = |U_n| = |U_{n_0}^1 U_{n/n_0}|$ giving

$$M_{n_0}(\mathcal{F}) = \log \left(\prod_{\substack{p^{\nu_p} \| \mathcal{F} \\ p \equiv \pm 1 \pmod{n_0}}} p^{\nu_p} \right) \geq \log \left(\prod_{\substack{p^{\alpha_p} \| U_{n_0}^1 \\ p \text{ primitive}}} p^{\alpha_p} \right) \tag{3.26}$$

$$\geq \log |\Phi_{n_0}(\alpha_1, \beta_1)| - \log P(n_0).$$

Now we complete the proof by choosing suitable n_0 as follows. We consider different cases according to whether α and β are complex conjugated or α and β are real. The case when α and β are complex conjugated gives the worst (highest) bounds so we start with it. Thus, assume that α and β are complex conjugates. For the proof of Theorem 3.1 for the case of U_n, we may assume that $n \geq 62000$. Let $\mathcal{P}_1 < \mathcal{P}_2 < \cdots < \mathcal{P}_{\omega(n)}$ be prime powers dividing n such that

$$n = \mathcal{P}_1 \mathcal{P}_2 \cdots \mathcal{P}_{\omega(n)}.$$

We choose

$$n_0 := P_{\omega(n)} P_{\omega(n)-1} \cdots P_{\omega(n)-t},$$

where $t \geq 0$ be the least integer such that $n_0 \geq 62000$. Let $\omega(n_0) = \omega$ and write

$$n_0 = 2^{e_0} q_0^{e_0} q_1^{e_1} q_2^{e_2} \cdots q_{\omega-1-\eta_0}^{e_{\omega-1-\eta_0}},$$

where q_i's are odd primes with $q_0 < q_1 < q_2 < \cdots < q_{\omega-1-\eta_0}$ and $\eta_0 = 0$ if n_0 is odd and $\eta_0 = 1$ if n_0 is even. From

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 > 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 > 62000,$$

we see that $\omega = \omega(n_0) \leq 6 + \eta_0$. Furthermore, from (3.25) and the fact that $n_0 \geq 62000$, we obtain

$$\log |\alpha_1| \geq 0.68 \log n_0 \geq 0.68 \log 62000 > 4.$$

We use (3.10) in Lemma 3.9 with α_1, β_1, n_0 in place of α, β, m, respectively along with (3.26). We obtain

$$M_{n_0} \geq (\varphi(n_0) - f(n_0/(2^{\eta_0})) - (n_0/2^{\eta_0}) g_{\omega-\eta_0-1}) \log|\alpha_1| - \begin{cases} \log 2 + \log 2, & \text{if } n_0 = 2^{e_0}; \\ \log 2 + \log q_{\omega-1-\eta_0}, & \text{if } \omega = \omega(n_0) \leq 1 + \eta_0, n \neq 2^{e_0}; \\ \frac{2^\omega}{2^{2+\eta_0}} \log 2q_0 + \log q_{\omega-1-\eta_0}, & \text{if } \omega = \omega(n_0) \geq 2 + \eta_0, \end{cases}$$

where

$$f(t) = 44.88 \log^2 t + 211.08 \log t + 249.08$$

and $g_{\omega-\eta_0-1}$ is given in the statement of Lemma 3.9. Since $\omega = \omega(n_0) \leq 6 + \eta_0$ and $n_0 \geq 2^{\eta_0} q_1 q_2 \cdots q_{\omega-\eta_0}$, we get

$$\frac{M_{n_0}}{n_0 \log|\alpha_1|} \geq \begin{cases} \frac{\varphi(n_0)}{n_0} - \frac{1}{n_0} f(\frac{n_0}{2}) - \frac{0.1238}{2} - \frac{16 \log n_0}{n_0 \log|\alpha_1|}, & \text{for even } n_0; \\ \frac{\varphi(n_0)}{n_0} - \frac{1}{n_0} f(n_0) - 0.1238 - \frac{16 \log n_0}{n_0 \log|\alpha_1|}, & \text{for odd } n_0. \end{cases} \tag{3.27}$$

Again $\omega \leq 6 + \eta_0$ gives

$$\frac{\varphi(n_0)}{n_0} = \prod_{p|n_0} \left(1 - \frac{1}{p}\right) \geq \prod_{k=2-\eta_0}^{7} \left(1 - \frac{1}{p_k}\right) = \begin{cases} \frac{3072}{17017}, & \text{for even } n_0; \\ \frac{6144}{17017}, & \text{for odd } n_0, \end{cases}$$

where p is a prime and p_k is the kth prime. Now we compare (3.27) with the upper bound of M_{n_0} given by (3.16). We have from $m_k \geq n_0 - 1$,

$$\log \mathcal{F} - 1.4 \log m_k \leq \log|U_n| - \log 2 \leq n_0 \log|\alpha_1|,$$

which together with (3.16) and $\log|\alpha_1| \geq 0.68 \log n_0$ (from (3.25)) give

$$4 + 4\log\log n_0 \geq \begin{cases} \frac{3072}{17017}\left(\frac{3072 n_0}{17017} - f(\frac{n_0}{2}) - 0.0619 n_0 - \frac{16}{0.68}\right) & \text{for even } n_0; \\ \frac{6144}{17017}\left(\frac{6144 n_0}{17017} - f(n_0) - 0.1238 n_0 - \frac{16}{0.68}\right) & \text{for odd } n_0. \end{cases} \tag{3.28}$$

This is a contradiction for $n_0 \geq 61762$ (or $n_0 \geq 30691$) according as whether n_0 is even (or odd), respectively. Hence, the assertion of Theorem 3.1 for Equation (3.1) follows when α and β are complex conjugates.

Now let α and β be real and assume that $n \notin \{1, 2, 3, 4, 6, 12\}$. We choose n_0 as follows:

$$n_0 := \begin{cases} q, & \text{if} \quad P(n) = q \geq 5; \\ 9, & \text{if} \quad P(n) = 3 \text{ and } 9 \mid n; \\ 16, & \text{if} \quad P(n) = 3, 9 \nmid n \text{ and } 16 \mid n; \\ 24, & \text{if} \quad n = 24; \\ 8, & \text{if} \quad n = 8. \end{cases}$$

Note that $\omega(n_0) = 1$ except when $n = 24$. Now we use (3.26) along with (3.5) in Lemma 3.4 with $m = n_0$ and α_1, β_1 in place of α, β to obtain

$$M_{n_0} \geq \varphi(n_0) \log|\alpha_1| - \log 2 - \log P(n_0). \tag{3.29}$$

Let $n_0 \neq 8$. We now compare the lower bound of M_{n_0} with the upper bound of M_{n_0} given by (3.16) and using

$$\log \mathcal{F} - 1.4 \log m_k \leq n_0 \log|\alpha_1|,$$

and $\varphi(n_0) = \varphi(q) = q - 1$ for $n_0 = q$ to obtain

$$\frac{4(1 + \log\log q)}{q-1} \geq 1 - \frac{1}{q} - \frac{\log 2q}{n_0 \log|\alpha_1|} \qquad \text{if } n_0 = q > 30,$$

and

$$\frac{2.2}{\varphi(n_0)} \geq \begin{cases} \frac{1}{3} - \frac{\log 6}{n_0 \log|\alpha_1|} & \text{if } n_0 = 24; \\ \frac{\varphi(n_0)}{n_0} - \frac{\log 2P(n_0)}{n_0 \log|\alpha_1|} & \text{if } n_0 < 30, \quad n_0 \neq 24. \end{cases}$$

We use the inequalities $n_0 \log|\alpha_0| \geq 0.68q \log q$ given by (3.25) for the case when $n_0 = q > 30$ and $n_0 \log|\alpha_1| \geq \log Q_1! - \log 2$ given by (3.24) for the case when $n_0 < 30$. Further, we use the fact that $4(1+\log\log q)/(q-1)$ is a decreasing function of q and $1 - 1/q \geq 1 - 1/31$ for $q > 30$. We obtain,

$$\frac{4}{31-1}(1 + \log\log 31) \geq 1 - \frac{1}{31} - \frac{1}{0.68 \cdot 31}\left(1 + \frac{\log 2}{\log 31}\right) \quad \text{if } n_0 = q > 30 \tag{3.30}$$

and

$$\frac{2.2}{\varphi(n_0)} \geq \begin{cases} \frac{1}{3} - \frac{\log 4}{\log(23!/2)} & \text{if } n_0 = 24; \\ \frac{\varphi(n_0)}{n_0} - \frac{\log 2P(n_0)}{\log(Q_1!/2)} & \text{if } n_0 < 30, \quad n_0 \neq 24. \end{cases} \tag{3.31}$$

We checked that the above inequalities are not valid. Thus, Equation (3.1) has no solution for all $n \geq 5, n \notin \{6, 8, 12\}$.

Let $n_0 = 8$. Then we have $n_0 = n = 8$ so that $\alpha_1 = \alpha$ and $\beta_1 = \beta$. We use the inequality

$$|U_n| = \left|\frac{\alpha^n - \beta^n}{\alpha - \beta}\right| \leq n|\alpha|^{n-1},$$

to obtain

$$\log \mathcal{F} = \log|U_8| \leq 7\log|\alpha| + \log 8.$$

Then, we have

$$\frac{\log \mathcal{F} - 1.4 \log m_k}{8 \log |\alpha|} < \frac{\log \mathcal{F}}{8 \log |\alpha|} \leq \frac{7}{8} + \frac{\log 8}{8 \log |\alpha|}.$$

Using this, we compare (3.29) with the upper bound of M_{n_0} given by (3.17) and (3.18). We obtain

$$\frac{1}{2} - \frac{\log 4}{8 \log |\alpha|} \leq \delta_8 \left(\frac{7}{8} + \frac{\log 8}{8 \log |\alpha|} \right) \quad \text{where} \quad \delta_8 := \begin{cases} 0.23, & \text{if } 7 \leq m_k < 17; \\ 0.3, & \text{if } 17 \leq m_k \leq 47; \\ \frac{2.2}{4}, & \text{if } m_k > 47, \end{cases}$$

implying

$$8 \log |\alpha| \leq \frac{\log 4 + \delta_8 \log 8}{1/2 - 7\delta_8/8}.$$

Also from (3.24), we get

$$8 \log |\alpha| \geq \log m_k! - \log 2 \geq \begin{cases} \log 7! - \log 2, & \text{if} \quad 7 \leq m_k < 17; \\ \log 17! - \log 2, & \text{if} \quad 17 \leq m_k \leq 47; \\ \log 47! - \log 2, & \text{if} \quad m_k > 47. \end{cases}$$

Comparing the above upper and lower bounds for $8 \log |\alpha|$ over all the cases $7 \leq m_k < 17, 17 \leq m_k \leq 47$ and $m_k > 47$ above, we arrive at a contradiction. Thus the assertion of Theorem 3.1 for Equation (3.1) holds for real α, β.

Finally, we consider Equation (3.2). The first ingredient is that the upper bound $|V_n| \leq 2|\alpha|^n$ holds as in the case of U_n. Since $V_n = U_{2n}/U_n$, we see that primitive divisors of V_n are the primitive divisors of U_{2n}. From the table listed in this section we find that the values of $n \geq 4$ for which V_n does not have a primitive divisor are the same as the ones for which U_{2n} has no primitive divisors. For these n all belonging to the set $\{4, 5, 6, 9\}$ and corresponding pairs (r, s), we find that (3.2) has

no solution. Hence, we now assume that $n \geq 4$ has a primitive divisor. Let $n = 2m$ be even. Then

$$V_{2m} = \alpha^{2m} + \beta^{2m} = (\alpha^m + \beta^m)^2 - 2(\alpha\beta)^m = V_m^2 - 2(-s)^m.$$

Since $\gcd(V_m, s) = 1$ for all $m \geq 1$, we may assume that $2 \nmid s$. Then $\nu_2(V_m^2 - 2(-s)^m) \leq 1$ implies $\nu_2(V_{2m}) \leq 1$. From Equation (3.2), we obtain $m_k \leq 3$. This contradicts the fact that V_{2m} has a primitive divisor which is a primitive divisor of U_{4m} congruent to one of ± 1 modulo $4m$. Thus, Equation (3.2) does not have a solution for n even with $n \geq 4$.

From now on, we take n odd with $n \geq 5$. For a divisor $n_0 \mid n$, we have

$$V_n = \alpha^n + \beta^n = (\alpha^{n/n_0})^{n_0} + (\beta^{n/n_0})^{n_0} =: V_{n_0}^1 = \frac{U_{2n_0}^1}{U_{n_0}^1},$$

which is the n_0th term of another Lucas sequence having r replaced by $\alpha^{n/n_0} + \beta^{n/n_0} = V_{n/n_0}$ and s replaced by $-(\alpha\beta)^{n/n_0} = -s^{n/n_0}$. Since primitive divisors of $V_{n_0}^1$ are the primitive divisors of $U_{2n_0}^1$ which are in one of residue classes ± 1 modulo $2n_0$, we consider

$$\begin{aligned} M_{2n_0} := M_{2n_0}(\mathcal{F}) &= \log \left(\prod_{\substack{p^{\nu_p} \| \mathcal{F} \\ p \equiv \pm 1 \pmod{2n_0}}} p^{\nu_p} \right) \geq \log \left(\prod_{\substack{p^{\alpha_p} \| U_{2n_0} \\ p \text{ primitive}}} p^{\alpha_p} \right) \\ &\geq \log |\Phi_{2n_0}(\alpha, \beta)| - \log \max\{3, P(n_0)\}. \end{aligned} \tag{3.32}$$

We choose n_0 as in the proof for Equation (3.1) above. Since $n_0 \mid n$ and n is odd, we have n_0 is odd. Hence, primes congruent to one of ± 1 modulo $2n_0$ are the same as primes congruent to one of ± 1 modulo n_0, which gives that $M_{2n_0}(\mathcal{F}) = M_{n_0}(\mathcal{F})$. Also $\varphi(2n_0) = \varphi(n_0)$. In case of real α, β, we have that n_0 is a prime power and we see that the lower bounds of $\log \Phi_{n_0}(\alpha_1, \beta_1))$ and $\log \Phi_{2n_0}(\alpha_1, \beta_1))$ given in the statement of Lemma 3.4 are exactly the same. In case of complex conjugates α, β,

we have that n_0 is an odd number with $\omega(n_0) \leq 6$ and we see from the statement of Lemma 3.9 that the lower bounds on $\log \Phi_{n_0}(\alpha_1, \beta_1))$ and $\log \Phi_{2n_0}(\alpha_1, \beta_1))$ given in (3.10) are exactly the same because of the equality $n_0 g_{\omega(n_0)-1} = (2n_0/2) g_{\omega(2n_0)-2}$ for n_0 odd.

Also

$$\log \mathcal{F} = \log |V_{n_0}^1| \leq n_0 \log |\alpha_1| + \log 2$$

gives the inequality (3.28) when α and β are complex conjugates which is a contradiction for $n_0 \geq 31000$ as stated after (3.28). Thus, Equation (3.2) has no solution for $n \geq 31000$ when α, β are complex conjugates. Let α, β be real. Then we obtain the Inequality (3.30) when $n_0 \geq 30$ and (3.31) when $n_0 < 30$. We get a contradiction as in Equation (3.1) above. Hence, Equation (3.2) has no solution for all $n \geq 4$ when α, β are real. Now the assertion of Theorem 3.1 for Equation (3.2) follows.

□

Chapter 4

Small values of n and infinitely many solutions

Here, we let $n \in \{2, 3, 4, 6, 12\}$. Let α, β be real which is the case when $r^2 + 4s > 0$. Thus, $1 \leq r < 2\alpha$. We have

$$\begin{aligned}
U_2 &= \alpha + \beta = r; \\
U_3 &= \alpha^2 + \alpha\beta + \beta^2 = r^2 + s; \\
U_4 &= (\alpha + \beta)(\alpha^2 + \beta^2) = r(r^2 + 2s); \\
U_6 &= U_3(\alpha^3 + \beta^3) = (r^2 + s)(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = r(r^2 + s)(r^2 + 3s); \\
U_{12} &= U_6(\alpha^6 + \beta^6) = r(r^2 + s)(r^2 + 3s)(\alpha^2 + \beta^2)(\alpha^4 - \alpha^2\beta^2 + \beta^4) \\
&= r(r^2 + s)(r^2 + 2s)(r^2 + 3s)((r^2 + 2s)^2 - 3s^2)
\end{aligned}$$

and

$$\begin{aligned}
V_1 &= \alpha + \beta = r; \\
V_2 &= \alpha^2 + \beta^2 = r^2 + 2s; \\
V_3 &= (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) = r(r^2 + 3s).
\end{aligned}$$

It is easy to see that there are infinitely many examples when U_2, U_3 and U_4 are a product of factorials. Indeed, take any N which is a product of factorials. If $r = N$, then $U_2 = N$. For $U_3 = N$, take any prime number p which does not divide N such that $p < \sqrt{N}$. Take $r = p$ and put $s := N - p^2 > 0$. Then $s > 0$ is coprime to r. Thus, r and s are coprime and $\Delta = r^2 + 4s > 0$. To see that p exists if N is sufficiently large, note that for all $m > 1$, the interval $(m, 2m]$ contains a prime number p by Bertrand's postulate. Thus, $p \leq (2m)^2 < m!$ for $m \geq 5$. Thus, if the largest factorial $m_k!$ from the representation of N as a product of factorials satisfies $m_k \geq 5$, then such p exists. It remains to look at numbers N of the form $2^a 3^b$ with $a \leq b$, and for these $p = 5$ satisfies the condition that $p^2 < N$ provided $N \geq 26$. Hence, whenever $N \geq 26$, we can find r, s coprime with $\Delta = r^2 + 4s > 0$ such that $U_3 = N$. For $U_4 = N$, assume $4 \mid N$ and choose $r := 2^\ell$, where $\ell := \nu_2(N/2)$. Then let $s := (N/r - r^2)/2$. Since $2 \| N/r$ and $4 \mid r^2$, it follows that s is odd therefore coprime to r. If $r^3 < N$, then s is also positive. There are infinitely many N's for which this condition fails (like $N = 2^k \cdot 3 = 2!^{k-1}3!$ with $k \geq 4$) but for example if $N = m!$ and $m > 24$, then this condition holds. Indeed, in this case, $\nu_2(N) < m$, so

$$r^3 < 2^{3m} < (m/e)^m < m!.$$

Thus, there are infinitely many instances for which $U_4 = N$. Similarly there are infinitely many n for which $V_n = N$ for $n \in \{1, 2, 3\}$. Perhaps there are only finitely many instances for $U_n = N$ for $n \in \{6, 12\}$. We don't have an unconditional proof of this but we present herein our conditional proof under the *abc* conjecture. Before proceeding with our proof conditioned upon the *abc* conjecture, we prove a lemma in Section 4.2.

4.1 Main Result

Theorem 4.1 *Let α, β be real. Then the equation (3.1) with $n \in \{2,3,4\}$ has infinitely many solutions. Further under the abc conjecture, equation (3.1) with $n \in \{6,12\}$ has only finitely many solutions. There are infinitely many solutions of (3.2) for real α, β and $n \in \{1,2,3\}$.*

4.2 Preliminaries

Lemma 4.2 *Let $N = m_1! \cdots m_k!$. Then $\gamma(N) = N^{o(1)}$ as $N \to \infty$.*

Proof. Since $m! = m^{m(1+o(1))}$ and $\gamma(m!) = \prod_{p \le m} p = e^{m(1+o(1))}$, it follows that $\gamma(m!) = m!^{(1+o(1))/\log\log m}$ as $m \to \infty$. Thus, for all $\varepsilon > 0$, there exists m_0 such that if $m > m_0(\varepsilon)$, the inequality $\gamma(m!) < (m!)^{\varepsilon}$ holds. Now assume $m_1 \le m_2 \le \cdots \le m_k$. Write

$$\begin{aligned} N &= \prod_{i=1}^{k} m_i! := \prod_{m \le m_k} m!^{\alpha(m)} \\ &= \prod_{\substack{m \le m_0 \\ \alpha(m) < 1/\varepsilon}} m!^{\alpha(m)} \prod_{\substack{m \le m_0 \\ \alpha(m) \ge 1/\varepsilon}} m!^{\alpha(m)} \prod_{m > m_0} m!^{\alpha(m)} \\ &:= M_1 M_2 M_3. \end{aligned}$$

Clearly, $M_1 \le m_0!^{m_0/\varepsilon} = O_\varepsilon(1)$. Furthermore,

$$\gamma(M_2) \le \prod_{\substack{m \le m_0 \\ \alpha(m) \ge 1/\varepsilon}} m!^{\alpha(m)} \le M_2^{\varepsilon},$$

and also $\gamma(M_3) \leq M_3^{\varepsilon}$ by the choice of m_0. Thus,

$$\gamma(N) \ll_{\varepsilon} N^{\varepsilon},$$

showing that $\gamma(N) < N^{2\varepsilon}$ for any sufficiently large N (with respect to ε), which is what we wanted to prove.

4.3 Proof of Theorem 4.1

Proof. Since $n = 1, 2, 3, 4$, were treated already, it remains to discuss the cases $n \in \{6, 12\}$. For $n = 6$, let

$$d_1 := r \leq 2\alpha, \ d_2 := r^2 + s \asymp \alpha^2, \ d_3 := r^2 + 3s \asymp \alpha^2.$$

Since $d_1 d_2 d_3 = N$, it follows by Lemma 4.2 that the estimates $\gamma(d_i) = d_i^{o(1)}$ hold for $i = 2, 3$ as $N \to \infty$. Since $d_3 - 3d_2 = -2d_1^2$, and $\gcd(d_i, d_j) = O(1)$ for $1 \leq i < j \leq 3$, we may apply the *abc* conjecture, namely Lemma 2.26, to this last equation $3d_2 = d_3 + 2d_1^2$ getting that

$$\alpha^2 \ll d_2 \ll \gamma(d_1 d_2 d_3)^{1+\varepsilon} = (d_1 d_2 d_3)^{o(1)} = \alpha^{o(1)},$$

so α is bounded. Since $U_6 \mid U_{12}$ and $U_6 \gg U_{12}^{1/3}$, this argument works for $n = 12$ as well. This completes the proof.

□

Chapter 5

Products of factorials : the X- coordinates of the Pell equation

We present a corollary regarding X-coordinates of Pell equations which are in $\mathcal{P}F$. For a positive integer d which is square-free, let (X_n, Y_n) be the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$ in positive integers (X, Y) (namely, the positive integers X, Y satisfy either $X^2 - dY^2 = 1$ or $X^2 - dY^2 = -1$). We supplement this with the following result on values of X_n which are in $\mathcal{P}F$.

5.1 Main Result

Theorem 5.1 *Let (X_n, Y_n) be the nth solution in positive integers of the equation $X^2 - dY^2 = \pm 1$ for some squarefree integer d. Then $X_n \in \mathcal{P}F$ implies $n = 1$. Similarly, let (W_n, Z_n) be the nth solution in positive integers of the equation $W^2 - dZ^2 = \pm 4$ for some squarefree integer d. Then $W_n \in \mathcal{P}F$ implies $n = 1$ except for the cases $(n, d) = (2, 2),\ (3, 5),\ (3, 13)$ for which $W_2 = 3!$, $W_3 = 2!2!$ and $W_3 = 3!3!$, respectively, with solutions*

$$3!^2 - 2 \cdot 4^2 = 4, \quad (2!2!)^2 - 5 \cdot 2^2 = -4 \quad \textit{and} \quad (3!3!)^2 - 13 \cdot 10^2 = -4.$$

5.2 Preliminaries

First we prove the following result for $s = \pm 1$.

Lemma 5.2 *Let $s \in \{-1, 1\}$ and $r \geq 1$. Then $V_n \in \mathcal{P}F$ with $n > 1$ implies*

$$\begin{aligned} &n = 2 : (r, s) = (2, 1), \quad V_2 = 3!; \\ &n = 3 : (r, s) = (1, 1), \quad V_3 = 2!2!; \\ &n = 3 : (r, s) = (3, 1), \quad V_3 = 3!3!. \end{aligned} \tag{5.1}$$

Proof. By Theorem 3.1, $V_n \in \mathcal{P}F$ implies $n \leq 3$. Hence, we take $n \in \{2, 3\}$. Let $V_2 = r^2 + 2s \in \mathcal{P}F$. Since $\gcd(r, s) = 1$, we have $\nu_2(r^2 + 2s) \leq 1$ giving $V_2 = r^2 + 2s = \pm 2!$ or $V_2 = r^2 + 2s = \pm 3!$. We check that this is possible only when $(r, s) \in \{(1, -1), (2, 1), (2, -1)\}$, which are the exceptions mentioned in the statement of this lemma except that the cases $(r, s) = (1, -1),\ (2, -1)$ give degenerate solutions, namely solutions for which α/β is a root of unity.

Let $V_3 = r(r^2 + 3s) \in \mathcal{P}F$. For $r \leq 3$, we check that indeed $V_3 \in \mathcal{P}F$ only for those pairs $(r, s) \in \{(1, 1), (1, -1), (2, -1), (3, 1)\}$ and again the pairs $(r, s) \in \{(1, -1), (2, -1)\}$ are not convenient. We now take $r \geq 4$. Since $\gcd(r, s) = 1$, we observe that $\nu_2(r^2 + 3s) \leq 2$ and $\nu_3(r^2 + 3s) \leq 1$. Further, we observe that primes $p > 3$ dividing $r^2 + 3s$ satisfy $\left(\dfrac{-3s}{p}\right) = 1$, where $(\frac{\cdot}{\cdot})$ is the Legendre symbol. We get $p \equiv 1, 7 \pmod{12}$ if $p > 3$ when $s = 1$ and $p \equiv \pm 1 \pmod{12}$ if $p > 3$ when $s = -1$. We take $n_0 = 12$ and $a_0 = 7, -1$, according to whether $s = 1, -1$, respectively. From the equation

$$V_3 = r(r^2 + 3s) = \mathcal{F} = m_1!m_2! \cdots m_k!,$$

we consider upper and lower bounds of $M_{12}(\mathcal{F})$. Since $r \geq 4$, we have

$$M_{12}(\mathcal{F}) \geq \log(r^2+3s) - \log 12 \geq \log r^2 + \log\left(1 - \frac{3}{4^2}\right) - \log 12 > \log r^2 - \log 16,$$

and

$$\log m_k! \leq \log \mathcal{F} \leq \log r(r^2+3s) \leq \log r^3 \left(1 + \frac{3}{r^2}\right) \leq 3\log r + \log 19 - \log 16,$$

implying

$$\log r \geq (\log m_k! - \log 19/16)/3.$$

Note that $m_k \geq 7$ since $r \geq 4$. Now we use Lemma 3.10 (Inequalities (3.18) and (3.17)) to obtain $M_{12}(\mathcal{F}) \leq \delta_3 \log \mathcal{F}$, where

$$\delta_3 := \begin{cases} 0.28 & \text{if } 7 \leq m_k \leq 16; \\ \frac{2.2}{4} & \text{if } m_k > 16. \end{cases}$$

Then

$$\delta_3(3\log r + \log 19/16) \geq \delta_3 \log \mathcal{F} \geq M_{12}(\mathcal{F}) \geq 2\log r - \log 16,$$

implying

$$3\delta_3 \log 19/16 + 3\log 16 \geq (2 - 3\delta_3)(3\log r) = (2-3\delta_3)(\log m_k! - \log 19/16).$$

This gives

$$\frac{2\log 19 + \log 16}{2 - 3\delta_3} \geq \log m_k! \geq \begin{cases} \log 7!, & \text{if } 7 \leq m_k \leq 16; \\ \log 17!, & \text{if } m_k > 16, \end{cases}$$

and we find that this a contradiction for the above choices of δ_3. Hence, $V_3 \notin \mathcal{P}F$ for $r \geq 4$ and the assertion of Lemma 5.2 follows.

5.3 Proof of Theorem 5.1

Proof. Let d be a squarefree positive integer and assume that $\varepsilon \in \{-1, 1\}$. Let (X_n, Y_n) be the nth solution in positive integers of the Pell equation $X^2 - dY^2 = \varepsilon$. Then $X_n = (\alpha^n + \beta^n)/2$ where (α, β) are the two roots of the quadratic $x^2 - (2X_1)x + \varepsilon = 0$. Thus, $X_n \in \mathcal{P}F$ gives $V_n \in \mathcal{P}F$ where $V_n = \alpha^n + \beta^n$ and $s = -\alpha\beta = \pm 1$. Then for $n > 1$, V_n is given by Lemma 5.2, namely $V_n \in \{3!, 2!2!, 3!3!\}$. Among these solutions, $X_n = V_n/2 \notin \mathcal{P}F$ except for $X_n = 2!2!/2 = 2!$. Thus, the only possibility is $X_n = 2$. Since $n > 1$, we have $2 = X_n \geq X_2 = 2X_1^2 \pm 1$, giving $X_1 < 1$, a contradiction. The assertion of Theorem 5.1 for X_n follows.

We now consider solutions (W_n, Z_n) of $W^2 - dZ^2 = 4\varepsilon$ with $\varepsilon \in \{-1, 1\}$. Assume that $W_n \in \mathcal{P}F$. Note that by putting

$$\alpha := (W_1 + \sqrt{d}Z_1)/2 \quad \beta := (W_1 - \sqrt{d}Z_1)/2,$$

we have that $W_n = \alpha^n + \beta^n = V_n$ and $s = -\alpha\beta = \pm 1$. By Lemma 5.2, we get that either $n = 2$ with $V_2 = 3!$ or $n = 3$ with $V_3 = 2!^2$, or $V_3 = 3!^2$. All three are valid choices for $d = 2$, $(W_2, Z_2) = (3!, 4)$ for which $3!^2 - 2 \cdot 4^2 = 4$ (and $(W_1, Z_1) = (2, 2)$), $d = 5$, $(W_3, Z_3) = (2!^2, 2)$ for which $(2!^2)^2 - 5 \cdot 2^2 = -4$ (and $(W_1, Z_1) = (1, 1)$), and $d = 13$, $(W_3, Z_3) = (3!^2, 10)$ for which $(3!^2)^2 - 13 \cdot 10^2 = -4$ (and $(W_1, Z_1) = (3, 1)$), respectively.

This finishes the proof of Theorem 5.1.

□

Chapter 6

Products of Catalan numbers and middle binomial coefficients : real and nonreal roots

This chapter introduces the case when members of Lucas sequences and companion Lucas sequences are products of Catalan numbers and middle binomial coeffients.

We let

$$B_m := \binom{2m}{m} \quad \text{and} \quad C_m := \frac{1}{m+1}\binom{2m}{m} \quad \text{for} \quad m \geq 0,$$

be the middle binomial coefficient and Catalan number, respectively. For each m, we write D_m for one of the numbers B_m, C_m. Let

$$\mathcal{P}BC := \{\pm \prod_{j=1}^{k} D_{m_j} : D_{m_j} \in \{B_{m_j}, C_{m_j}\},\ k \geq 1,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_k\}$$

be the set of integers which are products of middle binomial coefficients and Catalan numbers.

As in Chapter 3 we also prove a series of lemmas which exploit certain characteristics of cyclotomic polynomials and linear forms in logarithms, among others. Sieves are

utilised again with the resolution of a lemma which is a result of a refinement of the Brun - Titchmarsh inequality due to Montgomery and Vaughan. Various subcases are considered when the roots of the characteristic polynomial are complex conjugates or real, given certain values of n.

6.1 Main Result

Our theorem is the following.

Theorem 6.1 *For each m, let $D_m \in \{B_m, C_m\}$. The equation*

$$U_n = \pm D_{m_1} D_{m_2} \cdots D_{m_k}, \quad \text{where} \quad k \geq 1 \quad \text{and} \quad 1 \leq m_1 \leq \cdots \leq m_k, \tag{6.1}$$

implies $n < 6500$ if n is odd and $n \leq 720$ if n is even. Further when α, β are real, then $n \in \{1, 2, 3, 4, 6, 8, 12\}$. Additionally,

$$V_n = \pm D_{m_1} D_{m_2} \cdots D_{m_k}, \quad \text{where} \quad k \geq 1 \quad \text{and} \quad 1 \leq m_1 \leq \cdots \leq m_k, \tag{6.2}$$

implies $n < 6500$ and $4 \nmid n$. Further, when α, β are real, then $n \in \{1, 2, 3, 6\}$.

The proof of this theorem is provided in Section 6.4.

Note that $U_1 = 1 \in \mathcal{P}BC$. For this reason, whenever we look at Equation (6.1), we omit $n = 1$ and assume $n \geq 2$.

6.2 Preliminaries

The following result follows from Lemma 3.8, which is a consequence of Voutier [27, Lemma 5].

Lemma 6.2 *Let α and β be complex conjugates with $\log|\alpha| > 4$. Let*

$$f(\ell) := 44.88\log^2\ell + 211.08\log\ell + 249.08 \quad \text{for} \quad \ell > 1. \tag{6.3}$$

Then for integer $\ell \geq 3$, we have

$$\log|\alpha^\ell - \beta^\ell| \geq \log|\alpha| \left(\ell - f\left(\frac{\ell}{\gcd(\ell,2)}\right)\right) \tag{6.4}$$

and

$$\log|\alpha^\ell + \beta^\ell| \geq \log|\alpha| \left(\ell - f(\ell)\right). \tag{6.5}$$

The following lemma gives us a range for the parameters (r,s) in case when α is real, positive and lies in an interval $[c_1, c_2]$.

Lemma 6.3 *Let $c_1 \leq \alpha \leq c_2$ where c_1, c_2 are positive reals and $r^2 + 4s > 0$. For $s > 0$, we have $r < c_2$ and*

$$\max\left\{c_1(c_1 - r), \frac{c_1^2 - r^2}{4}\right\} \leq s \leq c_2(c_2 - r).$$

For $s < 0$, we have $c_1 \leq r \leq 2c_2$ and

$$c_2(r - c_2) \leq |s| < \frac{r^2}{4}.$$

Further, if $r < 2c_1$, then $|s| < c_1(r - c_1)$.

Proof. We have $2c_1 \leq 2\alpha = r + \sqrt{r^2 + 4s} \leq 2c_2$. This gives the inequality

$r^2+4s \le (2c_2-r)^2$ implying $s \le c_2(c_2-r)$. If $2c_1 > r$, we then have $r^2+4s \ge (2c_1-r)^2$ giving $s \ge c_1(c_1-r)$.

Let $s > 0$. Then $r < \alpha \le c_2$ giving $r < c_2$ and $s \le c_2(c_2-r)$. If $c_1 > r$, then $2c_1 > r$ and therefore $s \ge c_1(c_1-r)$. Also

$$2c_1 \le r + \sqrt{r^2+4s} \le 2\sqrt{r^2+4s}$$

gives $s \ge \dfrac{c_1^2 - r^2}{4}$ implying

$$s \ge \max\left\{c_1(c_1-r), \frac{c_1^2-r^2}{4}\right\}.$$

Let $s < 0$. Then $c_1 \le \alpha < r < r + \sqrt{r^2+4s} \le 2c_2$ giving $c_1 < r < 2c_2$. Also $r^2 + 4s > 0$ gives $|s| = -s < r^2/4$. From $s \le c_2(c_2-r)$, we get

$$|s| = -s \ge c_2(r-c_2).$$

If $r < 2c_1$, then $s \ge c_1(c_1-r)$ implying $|s| = -s \le c_1(r-c_1)$.

The following lemma is proved using Stirling's formula.

Lemma 6.4 *The function $m \mapsto \log(C_m/2)/m$ is increasing for $m \ge 7$. Hence,*

$$\log\left(\frac{B_m}{2}\right) > \log\left(\frac{C_m}{2}\right) > \begin{cases} m & \text{for} \quad m \ge 14; \\ 1.36m & \text{for} \quad m \ge 400; \\ 1.38m & \text{for} \quad m \ge 2100. \end{cases} \tag{6.6}$$

Further, given $M \ge 7$ and $m \le M$, we have

$$\frac{m \log 2m}{\log(C_m/2)} \frac{\log(C_M/2)}{M} \le 1.0001 \log 2M. \tag{6.7}$$

Proof. We recall Stirling's formula. For a positive integer ν, we have

$$\nu! = \sqrt{2\pi\nu}\; e^{-\nu}\nu^{\nu}e^{\theta_\nu}, \qquad \text{where} \qquad \frac{1}{12\nu+1} < \theta_\nu < \frac{1}{12\nu}.$$

From $C_m = \frac{(2m)!}{(m+1)(m!)^2}$, we have

$$m\log 4 - \sigma_m < \log(C_m/2) < m\log 4 - \tau_m, \tag{6.8}$$

where

$$\sigma_m := \log 2 + \log(m+1) + \log\sqrt{\pi m} + \frac{1}{6m} - \frac{1}{24m+1}$$
$$\text{and} \quad \tau_m := \log 2 + \log(m+1) + \log\sqrt{\pi m} + \frac{2}{12m+1} - \frac{1}{24m}.$$

We have $C_m < 4^m/\sqrt{\pi m}$ and

$$\frac{4^m}{\sqrt{\pi m}}\left(\frac{m+2}{4m+2}\right)^m = \frac{(1+3/(2m+1))^m}{\sqrt{\pi m}} \le \frac{e^{\frac{3m}{2m+1}}}{\sqrt{\pi m}} < \frac{e^{3/2}}{\sqrt{\pi m}} < 1 \quad \text{for} \quad m \ge 7.$$

Hence, from $C_{m+1}/C_m = (4m+2)/(m+2)$, we get

$$m\log\left(\frac{C_{m+1}}{2}\right) - (m+1)\log\left(\frac{C_m}{2}\right) \ge m\log\left(\frac{C_{m+1}}{C_m}\right) - \log C_m > 0$$

for $m \ge 7$. This shows that $\log(C_m/2)/m$ is an increasing function for $m \ge 7$. Hence, the assertion (6.6) follows by calculating $\log(C_m/2)/m$ at $m = 14, 400, 2100$, respectively.

From (6.8), we have

$$\frac{m\log 2m}{\log(C_m/2)} \le \frac{\log 2m}{\log 4 - \sigma_m/m}$$

and the right–hand side is an increasing function of m. Therefore, from $m \le M$ and Inequality (6.8) again, we get

$$\left(\frac{m\log 2m}{\log(\frac{C_m}{2})}\right)\left(\frac{\log(C_M/2)}{M}\right) \le \frac{\log 2M}{\log 4 - \sigma_M/M}(\log 4 - \tau_M/M)$$

$$
\begin{aligned}
&= (\log 2M)\left(1+\frac{\sigma_M-\tau_M}{M\log 4-\sigma_M}\right)\\
&\le (\log 2M)\left(1+\frac{\frac{1}{24M+1}+\frac{4}{12M+1}}{24M(M\log 4-\sigma_M)}\right)\\
&\le 1.0001\log 2M,
\end{aligned}
$$

since $M \ge 7$, implying the assertion (6.7).

□

6.3 Upper bound for prime powers dividing a product of Catalan numbers and middle binomial coefficients

In the present section, we use Lemmas 6.4, 2.22 and 2.24 to get an upper bound for prime powers dividing a product of Catalan numbers and middle binomial coefficients. To this end, analytic methods are exploited.

For positive integers $1 \le m_1 \le m_2 \le \cdots \le m_k$, let

$$\mathcal{D} := \mathcal{D}(m_1, m_2, \ldots, m_k) := \prod_{i=1}^{k} D_{m_i}, \quad D_{m_i} \in \{B_{m_i}, C_{m_i}\}.$$

In practice, since $B_1 = 2 = C_2$ and $C_1 = 1$, it follows that unless $|U_n| = 1$ in Equation (6.1), we may always assume that $m_1 > 1$. The same goes when $|V_n| > 1$ in (6.2). Let n_0 be a positive integer. Recall the definition of $M_{n_0}(\ell)$ given in (2.10). We use analytic methods to find an upper bound for $M_{n_0}(\mathcal{D})$. This is the content of the following lemma.

Lemma 6.5 *For $n_0 \geq 25$, we have*

$$M_{n_0}(\mathcal{D}) \leq \begin{cases} \left(\frac{3.9}{\varphi(n_0)} + \frac{2.92\log(3n_0)}{n_0}\right)(\log \mathcal{D} - \log 2), & \text{if } n_0 \text{ is even;} \\ \left(\frac{3.9}{\varphi(n_0)} + \frac{1.46\log(3n_0)}{n_0}\right)(\log \mathcal{D} - \log 2), & \text{if } n_0 \text{ is odd.} \end{cases} \tag{6.9}$$

Let $n_0 \in \{9, 16, 24\}$ or $5 \leq n_0 \leq 23$ be a prime. We have

$$M_{n_0}(\mathcal{D}) \leq \begin{cases} \frac{\delta_0}{\varphi(n_0)} \log \mathcal{D}, & \text{if} \quad m_k < 1500; \\ \frac{\delta_0}{\varphi(n_0)} (\log \mathcal{D} - \log 2), & \text{if} \quad m_k \geq 1500; \end{cases} \tag{6.10}$$

where δ_0 is given by

n_0	5	7	9	16	24	$11 \leq n_0 \leq 23$
δ_0	2.61	3.19	3.57	2.89	2.746	3.3

Proof. Let t_{1j} and t_{2j} be the number of i's such that $D_{m_i} = C_j$ and $D_{m_i} = B_j$, respectively. Put $t_j = t_{1j} + t_{2j}$. Then

$$\log \mathcal{D} = \sum_{1 \leq i \leq k} \log D_{m_i} = \sum_{1 < j \leq m_k} (t_{1j} \log C_j + t_{2j} \log B_j) \geq \sum_{1 < j \leq m_k} t_j \log C_j$$

since $B_m > C_m$. Let $7 \leq M \leq m_k$ be an integer which we will choose later on. Using Lemma 6.4, we get

$$\begin{aligned} \log \mathcal{D} &\geq \sum_{j \leq M} t_j \log C_j + \sum_{j > M} t_j \log C_j \\ &\geq \log 2 + \sum_{j \leq M} t_j \log(C_j/2) + \frac{\log(C_M/2)}{M} \sum_{j > M} t_j j, \end{aligned}$$

so that

$$\sum_{j > M} t_j j \leq \frac{M}{\log(C_M/2)} \left(\log \mathcal{D} - \log 2 - \sum_{j \leq M} t_j \log(C_j/2) \right). \tag{6.11}$$

Here, as usual, the empty sum is taken to be 0. For a prime number p and a positive integer t, we write $\nu_p(t)$ for the exact exponent of p in the prime factorization of t.

Given a positive integer j, let

$$\xi_1(j) := \sum_{p\equiv\pm1 \pmod{n_0}} \nu_p(C_j)\log p \quad \text{and} \quad \xi_2(j) := \sum_{p\equiv\pm1 \pmod{n_0}} \nu_p(B_j)\log p.$$

Then $\xi_1(j) \le \xi_2(j)$ and hence $M_{n_0}(\mathcal{D}) \le \sum_j t_j\xi_2(j)$. For a prime p, we have

$$\nu_p(B_j) = \sum_{\ell\ge1}\left(\left\lfloor\frac{2j}{p^\ell}\right\rfloor - 2\left\lfloor\frac{j}{p^\ell}\right\rfloor\right) \le \begin{cases} 1, & \text{if } \frac{2j}{2i} < p \le \frac{2j}{2i-1},\ i\in\{1,2\};\\ 0, & \text{if } \dfrac{2j}{2i+1} < p \le \frac{2j}{2i},\ i\in\{1,2\};\\ \left\lfloor\frac{\log(2j)}{\log p}\right\rfloor, & \text{if } p \le \dfrac{2j}{5}. \end{cases}$$

Therefore,

$$\begin{aligned}\xi_2(j) &\le \sum_{\substack{(2j)^{1/2}<p\le2j\\ p\equiv\pm1 \pmod{n_0}}}\left(\left\lfloor\frac{2j}{p}\right\rfloor - 2\left\lfloor\frac{j}{p}\right\rfloor\right)\log p + \sum_{\substack{p\le(2j)^{1/2}\\ p\equiv\pm1 \pmod{n_0}}}\left\lfloor\frac{\log(2j)}{\log p}\right\rfloor\log p\\ &\le \sum_{\substack{p\le2j\\ p\equiv\pm1 \pmod{n_0}}}\left\lfloor\frac{\log(2j)}{\log p}\right\rfloor\log p - \sum_{\substack{1\le i\le2\\ \frac{2j}{2i-1}<p\le\frac{2j}{2i}\\ p\equiv\pm1 \pmod{n_0}}}\log p\\ &\le \sum_{\ell\in\{1,-1\}}\left\{\psi(2j;n_0,\ell) + \sum_{t=2}^{3}\theta(2j/t;n_0,\ell) - \sum_{t=1}^{2}\theta(j/t;n_0,\ell)\right\}.\end{aligned} \tag{6.12}$$

Recall that $\pi(x;n_0,\ell)$ stands for the number of primes $p\le x$ satisfying the congruence $p\equiv\ell \pmod{n_0}$. We put $\pi_{\pm1}(x) := \pi(x;n_0,1) + \pi(x;n_0,-1)$. Then

$$\xi_2(j) \le (\pi_{\pm1}(2j) - \pi_{\pm1}(j) + \pi_{\pm1}(2j/3))\log(2j), \tag{6.13}$$

by (6.12). Let us assume that $n_0 \ge 25$. Let $t>0$. We split the analysis in two cases according to whether $2j \le (3n_0)^{1+1/t}$ or $2j > (3n_0)^{1+1/t}$.

Assume first that $2j \ge (3n_0)^{1+1/t}$. Then $2j/3n_0 \ge (2j)^{1/(1+t)}$ and therefore

$$\log(j/n_0) \ge \log(2j/3n_0) \ge (\log(2j))/(1+t).$$

From (6.13) and Lemma 2.22, we get

$$\xi_2(j) \le \frac{4j\log 2j}{\varphi(n_0)\log(j/n_0)} + \frac{(4j/3)\log 2j}{\varphi(n_0)\log(2j/3n_0)} \le \frac{16(1+t)j}{3\varphi(n_0)}.$$

In the smaller range $2j \le (3n_0)^{1+1/t}$, using the trivial estimates and the fact that primes congruent to one of ± 1 modulo n_0 are of the form $2ln_0 \pm 1$ when n_0 is odd, we get

$$\begin{aligned}
&\pi_{1,-1}(2j) - \pi_{1,-1}(j) + \pi_{1,-1}(2j/3) \le \pi_{1,-1}(2j) \\
&\le \begin{cases} \frac{2j-1}{n_0} + \frac{2j+1}{n_0} = \frac{4j}{n_0}, & \text{if } n_0 \text{ is even;} \\ \frac{2j-1}{2n_0} + \frac{2j+1}{2n_0} = \frac{2j}{n_0}, & \text{if } n_0 \text{ is odd.} \end{cases}
\end{aligned}$$

Let $\eta := 1, 2$ according to whether n_0 is even or odd, respectively. From (6.13), we get

$$\xi_2(j) \le \left(\frac{4}{\eta}\right)\frac{j\log 2j}{n_0}.$$

We choose

$$t := \frac{3.0003}{4\eta}\frac{\varphi(n_0)\log 3n_0}{n_0} \qquad \text{and} \qquad M := \left\lfloor \frac{1}{2}(3n_0)^{1+\frac{1}{t}} \right\rfloor.$$

Since $n_0/\varphi(n_0) \ge 2/\eta$, we observe that

$$\frac{1}{2}(3n_0)^{1+\frac{1}{t}} \ge 1.5n_0 \exp\left(\frac{8}{3.0003}\right) > 686 \quad \text{for} \quad n_0 \ge 32,$$

which together with $M \ge 686$ for each $25 \le n_0 < 32$ imply $M \ge 686$ for all $n_0 \ge 25$. From (6.11), we have

$$\begin{aligned}
&M_{n_0}(\mathcal{D}) \le \sum_j t_j \xi_2(j) \\
&\le \frac{M}{\log(\frac{C_M}{2})}\frac{16(1+t)}{3\varphi(n_0)}\left(\log\frac{\mathcal{D}}{2} - \sum_{j\le M} t_j \log\left(\frac{C_j}{2}\right)\right) + \sum_{j\le M}\frac{4t_j j\log 2j}{\eta n_0}
\end{aligned}$$

$$\leq \frac{M}{\log(\frac{C_M}{2})}\frac{16(1+t)}{3\varphi(n_0)}\log\frac{\mathcal{D}}{2} - \sum_{j\leq M} t_j\left(\frac{M\log(C_j/2)}{\log(C_M/2)}\frac{16(1+t)}{3\varphi(n_0)} - \frac{4j\log 2j}{\eta n_0}\right).$$

Since $M \geq 7$, we get from (6.7) and $\log 2M \leq (1+1/t)\log 3n_0$ that

$$\begin{aligned}
&\frac{M\log(C_j/2)}{\log(C_M/2)}\frac{16(1+t)}{3\varphi(n_0)} - \frac{4j\log 2j}{\eta n_0} \\
=&\frac{4M\log(C_j/2)}{n_0\log(C_M/2)}\left(\frac{4(1+t)n_0}{3\varphi(n_0)} - \frac{j\log 2j}{\eta\log(C_j/2)}\frac{\log(C_M/2)}{M}\right) \\
\geq&\frac{16M\log(C_j/2)}{3n_0\log(C_M/2)}\frac{n_0}{\varphi(n_0)}\left(\frac{(1+t)n_0}{\varphi(n_0)} - \frac{3\varphi(n_0)}{4\eta n_0}\frac{1.0001(1+t)\log 3n_0}{t}\right) \geq 0,
\end{aligned}$$

for

$$t = \frac{3.0003}{4\eta}\frac{\varphi(n_0)\log 3n_0}{n_0}.$$

Therefore, we have from $M \geq 686$ and Lemma 6.4, that

$$\begin{aligned}
M_{n_0}(\mathcal{D}) &\leq \frac{M}{\log(C_M/2)}\frac{16(1+t)}{3\varphi(n_0)}\log\frac{\mathcal{D}}{2} \\
&\leq \frac{16}{3}\frac{686}{\log(C_{686}/2)}\left(\frac{1}{\varphi(n_0)} + 3.0003\left(\frac{\log 3n_0}{4\eta n_0}\right)\right)(\log\mathcal{D} - \log 2),
\end{aligned}$$

which gives the assertion (6.9).

We now consider $n_0 \leq 24$ as given in the statement of the lemma. Then either $n_0 \in \{9, 16, 24\}$, or n_0 is a prime with $5 \leq n_0 \leq 23$. We check with exact computations that for $j \leq 1500$,

$$\xi_1(j) \leq \frac{\delta_0\log C_j}{\varphi(n_0)} \quad \text{and} \quad \xi_2(j) \leq \frac{\delta_0\log B_j}{\varphi(n_0)},$$

where δ_0 are given in the statement of the lemma. Hence, we have

$$M_{n_0}(\mathcal{D}) = \sum_{1<j\leq m_k}(t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) \leq \frac{\delta_0}{\varphi(n_0)}\sum_{1<j\leq m_k}(t_{1j}\log C_j + t_{2j}\log B_j),$$

which gives the assertion (6.10) for $m_k < 1500$.

We now take $m_k \geq 1500$. From (6.12) and Lemma 2.24, we get

$$\xi_1(j) \leq \xi_2(j) \leq \frac{\delta_1 j}{\varphi(n_0)} \leq \frac{\delta_1 j}{\log(C_j/2)} \frac{\log(D_j/2)}{\varphi(n_0)} \quad \text{for} \quad j \geq 1500,$$

where

$$\delta_1 = \frac{47}{15} + \frac{2\sqrt{2}\epsilon_\psi}{\sqrt{1500}} \left(1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}\right) + \frac{2\epsilon_\theta}{\sqrt{1500}} \left(1 + \frac{1}{\sqrt{2}}\right),$$

and ϵ_ψ and ϵ_θ are given in Lemma 2.23. By Lemma 6.4, we have

$$\frac{\delta_1 j}{\log(C_j/2)} \leq \frac{\delta_1}{1.37} \quad \text{for each} \quad j \geq 1500,$$

and we find that $\delta_1/1.37 \leq \delta_0$. Thus,

$$\xi_1(j) \leq \xi_2(j) \leq \frac{\delta_0 \log(D_j/2)}{\varphi(n_0)} \quad \text{for each} \quad j \geq 1500,$$

and therefore

$$\begin{aligned} M_{n_0}(\mathcal{D}) &= \sum_{j<1500} (t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) + \sum_{j\geq 1500} (t_{1j}\xi_1(j) + t_{2j}\xi_2(j)) \\ &\leq \frac{\delta_0}{\varphi(n_0)} \sum_{j<1500} (t_{1j} \log C_j + t_{2j} \log B_j) + \frac{\delta_0}{\varphi(n_0)} \sum_{j\geq 1500} \left(t_{1j} \log \frac{C_j}{2} + t_{2j} \log \frac{B_j}{2}\right) \\ &\leq \frac{\delta_0}{\varphi(n_0)} (\log \mathcal{D} - \log 2)\,. \end{aligned}$$

Hence, the assertion (6.10) follows and the proof is complete.

6.4 Proof of Theorem 6.1

Proof. We first treat the case of the sequence $\{U_n\}_{n\geq 0}$. Assume that Equation (6.1) has a solution. Then

$$|U_n| = \mathcal{D} = D_{m_1} \cdots D_{m_k}, \quad D_{m_i} \in \{B_{m_i}, C_{m_i}\}.$$

For a divisor n_0 of n, we will compare the upper bound of $M_{n_0}(\mathcal{D})$ given by Lemma 6.5 with a lower bound on it obtained by using Lemma 2.28. We will choose a suitable divisor n_0 of n such that these bounds contradict each other and hence for n with such divisors n_0, $|U_n|$ cannot be a product of Catalan numbers and middle binomial coefficients.

In Table 2.1, we checked that for (r, s) with $n \geq 5, n \neq 6$, Equation (6.1) holds in several instances. The roots (α, β) are real only when $(r, s) = (1, 1)$ and then

$$(r, s, n) = (1, 1, 5),\ (1, 1, 12), \quad U_5 = C_3,\ U_{12} = B_1^6 C_2^2 = B_1^2 B_2^2.$$

Hence, we assume now that U_n has a primitive prime divisor p and so $p \equiv \pm 1 \pmod{n}$. Let P_n be the largest primitive divisor of U_n. From (6.1), we have that $P_n \mid B_{m_i}$ for some $i = 1, \ldots, k$, and so $2m_k \geq P_n + 1$ since P_n is odd. Let Q_n be the least prime congruent to one of ± 1 modulo n. Then $2m_k \geq P_n + 1 \geq Q_n + 1$ and therefore

$$2|\alpha|^n \geq \left|\frac{\alpha^n - \beta^n}{\alpha - \beta}\right| = |U_n| \geq C_{m_k}. \tag{6.14}$$

From Lemma 6.4, we have

$$n \log |\alpha| \geq \log(C_{m_k}/2) \geq \begin{cases} 1.36 m_k \geq 0.68(Q_n + 1) \geq 0.68n, & n \geq 400; \\ 1.38 m_k \geq 0.69(Q_n + 1) \geq 0.69n, & n \geq 4200, \end{cases} \tag{6.15}$$

since $Q_n \geq n - 1$.

We have

$$\log \mathcal{D} \leq \log |U_n| \leq n \log |\alpha| + \log |1 - x^n|.$$

Now we complete the proof by choosing suitable n_0 and comparing upper and lower

bounds of $M_{n_0}(U_n) = \mathcal{M}_{n_0}(\mathcal{D})$. For $n_0 \in \{9, 16, 24\}$, or n_0 an odd prime power, we define

$$g(n_0) := \begin{cases} \frac{\delta_0}{\varphi(n_0)}, & \text{if } n_0 \in \{9, 16, 24\}, \text{ or } n = p \leq 23; \\ \frac{3.9}{\varphi(n_0)} + \frac{1.46 \log 3n_0}{n_0}, & \text{if } n_0 \geq 25 \text{ is odd}, \end{cases} \tag{6.16}$$

where δ_0 is stated in Lemma 6.5. By Lemma 6.5, we have

$$M_{n_0}(\mathcal{D}) \leq g(n_0) \log |U_n| \leq g(n_0) \left(n \log |\alpha| + \log |1 - x^n|\right). \tag{6.17}$$

Let $p^{h+t} \mid n$, where p is a prime and $h > 0, t \geq 0$ are integers such that $p^h > 4$. Taking $n_0 = p^h$ and using (2.11) in Lemma 2.28, we get a lower bound for $M_{n_0}(U_n) = M_{n_0}(\mathcal{D})$ which we compare with (6.17). We obtain

$$\begin{aligned} & g(n_0) \left(n \log |\alpha| + \log |1 - x^n|\right) \\ & \geq \left(1 - \frac{1}{p^{t+1}}\right) n \log |\alpha| + \log |1 - x^n| - \log |1 - x^{n/p^{t+1}}| - \log(p^{t+1}), \end{aligned}$$

implying

$$\left(1 - \frac{1}{p^{t+1}} - g(n_0)\right) \leq \frac{(g(n_0) - 1) \log |1 - x^n| + \log |1 - x^{n/p^{t+1}}| + \log(p^{t+1})}{n \log |\alpha|}. \tag{6.18}$$

We consider different cases.

6.4.1 The case when n is even

We assume that $n > 720$. We choose $n_0 = p^h$ and t as follows:

$$(n_0, t) \in \{(2^4, 1), (3^2, 1), (5, 1)\} \cup \{(p, 0) : p > 5\}. \tag{6.19}$$

Since $2^4 \cdot 3^2 \cdot 5 = 720$, we find that for each even $n > 720$, there is some (n_0, t) in (6.19) with $n_0 p^t \mid n$. From the triangle inequality

$$2|\alpha^{n/2}| \leq |\alpha^{n/2} - \beta^{n/2}| + |\alpha^{n/2} + \beta^{n/2}|,$$

we have either $|\alpha^{n/2} - \beta^{n/2}| \geq |\alpha|^{\frac{n}{2}}$, or $|\alpha^{n/2} + \beta^{n/2}| \geq |\alpha|^{\frac{n}{2}}$. Therefore,

$$\begin{aligned} |\alpha^n - \beta^n| &= |\alpha^{n/2} - \beta^{n/2}||\alpha^{n/2} + \beta^{n/2}| \\ &\geq \begin{cases} |\alpha|^{\frac{n}{2}}|\alpha^{\frac{n}{2}} + \beta^{\frac{n}{2}}| = |\alpha|^{\frac{n}{2}}|V_{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}, & |\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}; \\ |\alpha - \beta||\frac{\alpha^{\frac{n}{2}} - \beta^{\frac{n}{2}}}{\alpha - \beta}||\alpha|^{\frac{n}{2}} \geq |U_{\frac{n}{2}}||\alpha|^{\frac{n}{2}}, \geq |\alpha|^{\frac{n}{2}} & |\alpha^{\frac{n}{2}} + \beta^{\frac{n}{2}}| \geq |\alpha|^{\frac{n}{2}}, \end{cases} \end{aligned}$$

since $V_{n/2}, U_{n/2}$ are integers and $|\alpha - \beta| \geq 1$. Hence,

$$|1 - x^n| = |\alpha|^{-n}|\alpha^n - \beta^n| \geq |\alpha|^{-\frac{n}{2}}.$$

Using the above inequality together with the inequality $|1 - x^{n/p^{t+1}}| \leq 2$ (since $|x| \leq 1$) in (6.18), we get

$$\frac{\log(2p^{t+1})}{n \log|\alpha|} \geq 1 - \frac{1}{p^{t+1}} - g(n_0) + \frac{g(n_0) - 1}{2} = \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2}.$$

From (6.15), we have

$$0 \geq \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2} - \frac{\log(2p^{t+1})}{0.68n}.$$

For a fixed choice of $n_0 = p^h$ and t, the right–hand side of the above inequality is an increasing function of n. We check that for (n_0, t) in (6.19) with $n_0 < 29$, the above inequality is not valid at $n = 720$ and hence it is not valid for any $n \geq 720$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime, which together with the observation that $g(p)$ is a decreasing function of p, we obtain

$$0 \geq \frac{1}{2} - \frac{1}{p^{t+1}} - \frac{g(n_0)}{2} - \frac{\log(2p^{t+1})}{0.65n} \geq \frac{1}{2} - \frac{1}{29} - \frac{g(29)}{2} - \frac{\log(2 \times 29)}{0.68n}.$$

We check that the right–most side is positive for $n \geq 720$ and hence we get a contradiction for all $n \geq 720$. Thus, Equation (6.1) has no even solution $n > 720$.

6.4.2 The case when α, β are complex conjugates

From the previous section, we may assume that either $n > 720$ is odd or n is an even number ≤ 720. Since we are shooting for the inequality $n < 6500$, we may assume that $n \geq 6500$ is odd. Also, we have $Q_n \geq 2n-1$ which together with $2m_k \geq Q_n+1$, Inequality (6.14) and Lemma 6.4 give

$$\log |U_n| \geq 1.38n \qquad \text{and} \qquad n \log |\alpha| \geq 1.38n.$$

We choose n_0 of the form p^h and t given by

$$(n_0, t) \in \{(3^2, 2), (5, 1), (7, 1)\} \cup \{(p, 0) : p \geq 11\}. \tag{6.20}$$

Since $3^3 \cdot 5^2 \cdot 7 < 6500$, we find that for each odd $n \geq 6500$, there is some (n_0, t) in (6.20) with $n_0 p^t \mid n$.

First we consider the case when $\log |\alpha| \leq 4$. We use

$$|1 - x^n| = \frac{|\alpha - \beta||U_n|}{|\alpha|^n} \geq \frac{|U_n|}{|\alpha|^n}, \quad |1 - x^{n/p^{t+1}}| \leq 2 \quad \text{and} \quad \log |\alpha| \leq 4$$

in (2.11) and compare it with (6.17) to get

$$g(n_0) \log |U_n| \geq M_{n_0}(U_n) \geq \log |U_n| - \frac{4n}{p^{t+1}} - \log(2p^{t+1}).$$

Since $\log |U_n| \geq 1.38n$, we obtain

$$0 \geq 1.38(1 - g(n_0)) - \frac{4}{p^{t+1}} - \frac{\log(2p^{t+1})}{n}. \tag{6.21}$$

For a fixed choice of $n_0 = p^h$ and t, the right–hand side of the above inequality is

an increasing function of n. We check that for (n_0, t) in (6.20) with $n_0 < 29$, the above inequality is not valid at $n = 6500$ and hence it is not valid for any $n \geq 6500$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime and $t = 0$, which together with the observation that $g(p)$ is a decreasing function of p, we obtain

$$\begin{aligned} 0 &\geq 1.38(1 - g(n_0)) - \frac{4}{p^{t+1}} - \frac{\log(2p^{t+1})}{n} \\ &\geq 1.38(1 - g(29)) - \frac{4}{29} - \frac{\log(2 \cdot 29)}{n}. \end{aligned}$$

We check that the right–most side is positive for $n \geq 6500$ and hence we get a contradiction for any $n \geq 6500$. Thus, Equation (6.1) does not have an odd solution $n \geq 6500$ in case $\log |\alpha| < 4$.

Assume now that $\log |\alpha| > 4$. By Lemma 6.2, we get

$$\log |1 - x^n| \geq -f(n) \log |\alpha|,$$

where $f(n)$ is given by formula (6.3). Using this inequality along with

$$|1 - x^n| \leq 2 \quad \text{and} \quad n \log |\alpha| \geq 4n$$

in (6.18), we obtain

$$0 \geq 1 - \frac{1}{p^{t+1}} - g(n_0) + \frac{(1 - g(n_0))f(n)}{n} - \frac{\log(2p^{t+1})}{4n}. \tag{6.22}$$

For a fixed $n_0 = p^h$ and t, the right–hand side of the above inequality is an increasing function of n. We check that for (n_0, t) in (6.20) with $n_0 < 29$, the above inequality is not valid at $n = 6500$ and hence it is not valid for any $n \geq 6500$. Further, for $n_0 \geq 29$, we have $n_0 = p$ is prime and $t = 0$ and hence the right–hand side of the above inequality is at least

$$1 - \frac{1}{29} - g(29) + \frac{(1 - g(29))f(n)}{n} - \frac{\log(2 \cdot 29)}{4n}.$$

We check that the above quantity is positive for $n \geq 6500$ and hence we get a contradiction for any $n \geq 6500$. Thus, Equation (6.1) has no odd solution $n \geq 6500$ in case $\log|\alpha| > 4$.

6.4.3 The case when α, β are real and $n \geq 5, n \notin \{6, 8, 12, 24\}$

We now consider the case when α and β are real. Recall that in this case $\alpha > 0$ and $U_n > 0$. For the proof of Theorem 6.1, we may assume that $n \geq 5$, $n \notin \{6, 8, 12, 24\}$. We will consider the case $n = 24$ separately in the next section. We choose $n_0 = p^h$ with $t = 0$ as

$$n_0 \in \{2^4, 3^2\} \cup \{p : p \geq 5\}. \tag{6.23}$$

Note that each $n \geq 5, n \notin \{6, 8, 12, 24\}$ is divisible by some n_0 in (6.23).

Let $n_0 = 2^4 = 16$. Then $p = 2$, $4 \mid n$ and hence

$$g(16)\log|1 - x^n| < 0 \quad \text{and} \quad \frac{1 - x^n}{1 - x^{n/p}} = 1 + \sum_{i=1}^{p-1} x^{in/p} > 1.$$

Using this in (6.18) together with $n \geq 16$ and $\alpha \geq \dfrac{1+\sqrt{5}}{2}$, we get

$$0 \geq 1 - \frac{1}{2} - g(16) - \frac{\log 2}{n \log \alpha} \geq \frac{1}{2} - g(16) - \frac{\log 2}{16 \log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

We find that the right–most quantity is positive, which is a contradiction. Thus, Equation (6.1) has no solution when $\alpha,\ \beta$ are real with $16 \mid n$.

Let $n_0 \neq 2^4$. Then $p > 2$. Writing

$$1 - x^n = (1 - x^{n/p})\left(\frac{1 - x^n}{1 - x^{n/p}}\right),$$

we have

$$|1-x^{n/p}| \le 2 \quad \text{and} \quad \frac{1-x^n}{1-x^{n/p}} = \begin{cases} 1+\sum_{i=1}^{p-1} y^i > 1, & y = x^{\frac{n}{p}} > 0; \\ \frac{1-y(y^{(p-1)/2})^2}{1-y} \ge \frac{1}{1-y} > \frac{1}{2}, & y = x^{\frac{n}{p}} < 0. \end{cases}$$

Using this in (6.18), we obtain

$$\begin{aligned} \log \alpha &\le \frac{(g(n_0)-1)\log\left(\frac{1-x^n}{1-x^{n/p}}\right) + g(n_0)\log(1-x^{n/p}) + \log p}{n(1-1/p-g(n_0)))} \\ &\le \frac{(1-g(n_0))\log 2 + g(n_0)\log 2 + \log p}{n(1-1/p-g(n_0))} \\ &= \frac{\log(2p)}{n(1-1/p-g(n_0))}. \end{aligned}$$

This together with $n \ge n_0$ and $\alpha \ge \dfrac{1+\sqrt{5}}{2}$ give

$$\begin{aligned} \log\left(\frac{1+\sqrt{5}}{2}\right) \le \log\alpha &\le \frac{\log(2p)}{n(1-1/p-g(n_0))} \\ &\le \begin{cases} \frac{\log 2p}{n_0(1-1/p-g(n_0))}, & n_0 < 29; \\ \frac{\log(2\cdot 29)}{29(1-1/29-g(29))}, & n_0 = p \ge 29. \end{cases} \end{aligned} \tag{6.24}$$

We check that the right–most quantity exceeds $\log\left(\dfrac{1+\sqrt{5}}{2}\right)$ except when $n_0 = p \in \{5,7\}$. Further, for $n_0 = p \in \{5,7\}$, putting $n = p\ell$, we obtain by using (6.15),

$$\log(C_{m_k}/2) \le n\log\alpha = p\ell\log\alpha \le \frac{\log 2p}{1-1/p-g(p)} \le \begin{cases} 15.62, & \text{if } p=5; \\ 8.11, & \text{if } p=7. \end{cases}$$

This gives $m_k \le 15,9$ according to whether $n_0 = p = 5,7$, respectively. Further $1 \le \ell \le 6,2$, according as $p = 5,7$, respectively since $\alpha \ge \dfrac{1+\sqrt{5}}{2}$. This together

with $P_n \le P(B_{m_i})$ for some $i = 1, \dots, k$, yield

$$\log\left(\frac{1+\sqrt{5}}{2}\right) \le \log\alpha \le \begin{cases} \frac{15.62}{5\ell}, & \text{if } p = 5; \\ \frac{8.11}{7\ell}, & \text{if } p = 7 \end{cases} \quad \text{and} \quad P_{p\ell} \le \begin{cases} 29, & \text{if } p = 5; \\ 19, & \text{if } p = 7. \end{cases}$$

For the pairs (r, s) given by Lemma 6.3 with the conditions above, we check that Equation (6.1) has no solution with $n = p\ell$. Therefore, Equation (6.1) has no solution for α, β real and $n \ge 5, n \notin \{6, 8, 12, 24\}$.

6.4.4 The case when α, β are real and $n = 24$

Let α, β be real and $n = 24$. Then $|x| < 1$. We have

$$\log \mathcal{D} = \log U_{24} = \log\left(\frac{\alpha^{24} - \beta^{24}}{\alpha - \beta}\right) = 23\log\alpha + \log\left(\frac{1-x^{12}}{1-x}\right) + \log(1+x^{12}). \tag{6.25}$$

We take $n_0 = n = 24$. Let $g > 0$ and $\lambda \ge 0$ be such that

$$M_{24}(\mathcal{D}) \le g\left(\log\mathcal{D} - \lambda\right) \le g(23\log\alpha + \log\left|\frac{1-x^{12}}{1-x}\right| + \log|1+x^{12}| - \lambda\right). \tag{6.26}$$

In particular,

$$g \le g_0(24) = \frac{2.746}{8} \quad \text{and} \quad \lambda = 0,$$

by (6.17). We now take $n_0 = 24, t = 0$ in (2.11) to get a lower bound for $M_{24}(U_{24}) = M_{24}(\mathcal{D})$ and compare it with (6.26) to obtain

$$8\log|\alpha| + \log|1+x^{12}| - \log 6 \le g\left(23\log\alpha + \log\left|\frac{1-x^{12}}{1-x}\right| + \log|1+x^{12}| - \lambda\right).$$

This gives

$$(8-23g)\log\alpha \le g\left(\log\left|\frac{1-x^{12}}{1-x}\right| + \left(1-\frac{1}{g}\right)\log|1+x^{12}| + \frac{\log 6}{g} - \lambda\right).$$

Assume that $x<0$. Then

$$\frac{1-x^{12}}{1-x} = 1 + x\left(\frac{1-x^{11}}{1-x}\right) < 1 \quad \text{and} \quad 1+x^{12} > 1,$$

which together with $g<1$ imply the right hand side of the above inequality is strictly less than $\log 6$.

Assume next that $x>0$. Then

$$\frac{1-x^{12}}{1-x} = 1 + x + x^2 + \cdots x^{11} < 12,$$

since $x<1$. For any x_0 with $0<x_0<1$, we have

$$\log\left|\frac{1-x^{12}}{1-x}\right| + \left(1-\frac{1}{g}\right)\log|1+x^{12}| \le \begin{cases} \log 12 + \left(1-\frac{1}{g}\right)\log(1+x_0), & x > x_0^{\frac{1}{12}}, \\ \log\left|\frac{1-x_0}{1-x_0^{\frac{1}{12}}}\right|, & x \le x_0^{\frac{1}{12}}. \end{cases}$$

Putting

$$y_0 := y_0(g,x_0) = \frac{\log 6}{g} - \lambda + \max\left(\log 12 + \left(1-\frac{1}{g}\right)\log(1+x_0), \log\left|\frac{1-x_0}{1-x_0^{\frac{1}{12}}}\right|\right),$$

we get

$$\log\alpha < \frac{y_0 g}{8-23g} \qquad \text{or} \qquad \alpha < \exp\left(\frac{y_0 g}{8-23g}\right). \tag{6.27}$$

As stated before, we have

$$g \le g_0(24) = \frac{2.746}{8} < 0.3433 \quad \text{and} \quad \lambda = 0,$$

by (6.17). Taking $g = 0.3433, \lambda = 0$ and $x_0 = 0.298$, we get $y_0 \leq 7.21$ and hence $\log \alpha < 23.78$ by (6.27). However, for $m_k \geq 420$, we have

$$\log \alpha \geq \frac{\log(C_{m_k}/2)}{24} \geq \frac{\log(C_{240}/2)}{24} > 24.82,$$

by (6.15). Thus, $m_k < 420$.

For each $j < 420$, let

$$\epsilon_{1j} := \frac{M_{24}(C_j)}{\log C_j} \qquad \text{and} \qquad \epsilon_{2j} := \frac{M_{24}(B_j)}{\log B_j}.$$

Then $\epsilon_{1j} = \epsilon_{2j} = 0$ for $j < 12$ since 23 is the least prime congruent to one of ± 1 modulo 24. We check that $\max(\epsilon_{1j}, \epsilon_{2j}) \leq 0.3433$ for $j < 420$.
Write $U_{24} = \mathcal{D} = \prod_{i=1}^{k} D_{m_i}$ as

$$\log \mathcal{D} = \sum_j t_{1j} \log C_j + \sum_j t_{2j} \log B_j,$$

where

$$t_{1j} := \#\{i : D_{m_i} = C_j\} \qquad \text{and} \qquad t_{2j} := \#\{i : D_{m_i} = B_j\}.$$

Let $\epsilon_0 \geq \max_j\{\epsilon_{1j}, \epsilon_{2j}\}$ for j such that $t_{1j} + t_{2j} > 0$. Then

$$\begin{aligned} M_{24}(U_{24}) &= \sum_j (\epsilon_{1j} t_{1j} \log C_j + \epsilon_{2j} t_{2j} \log B_j) \\ &= \epsilon_0 \sum_j \left(\left(1 - \left(1 - \frac{\epsilon_{1j}}{\epsilon_0}\right)\right) t_{1j} \log C_j \right. \\ &\quad \left. + \left(1 - \left(1 - \frac{\epsilon_{2j}}{\epsilon_0}\right)\right) t_{2j} \log B_j \right) \\ &\leq \epsilon_0 \left(\log \mathcal{D} - \sum_j t_{1j} \lambda_{1j} - \sum_j t_{2j} \lambda_{2j} \right), \end{aligned}$$

where

$$\lambda_{1j} := \left(1 - \frac{\epsilon_{1j}}{\epsilon_0}\right) \log C_j \quad \text{and} \quad \lambda_{2j} := \left(1 - \frac{\epsilon_{2j}}{\epsilon b_0}\right) \log B_j.$$

It is clear that $\lambda_{1j} \geq 0$ and $\lambda_{2j} \geq 0$.

Suppose that $\alpha \leq 100$. Then $t_{1j} + t_{2j} > 0$ implies $j \leq m_k \leq 85$ by (6.15) since $\log(C_{86}/2) > 24 \log 100$. For $j \leq 85$ and $j \notin \{37, 38, 39, 40, 41, 42, 43\}$, we find that $\epsilon_{1j}, \epsilon_{2j} \leq 0.29$. Taking $g = 0.29$ and $\lambda = 0$ in (6.26) and taking $x_0 = 0.25$, we get $y_0 \leq 8.12$ and $\alpha < 5.88$ so $\log \alpha \leq 1.771$. By (6.15) again, we have $j \leq m_k \leq 32$ since $\log(C_{33}/2) > 24 \log 5.88$ and we furthermore have $P_{24} \leq 61$. We check that Equation (6.1) with $P_{24} \leq 61$ and $\alpha \leq 5.88$ is not possible. Here, we use Lemma 6.3 to find all possible pairs (r, s) with $\alpha \leq 5.88$. Thus, we assume $t_{1j} + t_{2j} > 0$ for some $j \in \{37, 38, 39, 40, 41, 42, 43\}$. Also

$$\log \alpha \geq \frac{\log(C_{37}/2)}{24} > 2.0379,$$

by (6.15). Again $t_{1j} + t_{2j} > 0$ implies $j \leq m_k \leq 46$ since

$$\log C_{37} + \log C_{47} > 24 \log 100 + \log 2 \geq \log U_{24}.$$

Hence, $P_{24} \leq 89$. Further $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid U_{24}$ also since $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid C_j \mid B_j$ for $j \in \{37, 38, 39, 40, 41, 42, 43\}$. For the pairs (r, s) with $2.0379 < \log \alpha \leq \log 100$ given by Lemma 6.3, we check that $P_{24} \leq 89$ and $47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \mid U_{24}$ is not possible. Therefore, Equation (6.1) has no solution when $\alpha \leq 100$.

From now on, we assume that $\alpha > 100$. Suppose that $t_{1j} = 0$ for $j \in \{37, 38\}$. Then we find that $\max(\epsilon_{1j}, \epsilon_{2j}) \leq \epsilon_0 = 0.324$ for $j < 420$ with $j \neq 37, 38$, and also $\epsilon_{2j} < 0.324$ for $j = 37, 38$. By taking $g = 0.324$ and $\lambda = 0$ in (6.26) and further $x_0 = 0.28$, we get $y_0 \leq 7.5$ and $\alpha < 84.3$. This is not possible. Therefore, we

have $t_{1j} > 0$ for $j = 37$ or $j = 38$. Then $\max(\epsilon_{1j}, \epsilon_{1j}) \le \epsilon_0 = 0.3433$ for $j < 420$. Taking $g = \epsilon_0 = 0.3433$ and $\lambda = \sum_j t_{1j}\lambda_{1j} + \sum_j t_{2j}\lambda_{2j}$ in (6.26) and further taking $x_0 = 0.298$, we obtain $y_0 \le 7.21 - \lambda$ and

$$\log \alpha < \frac{0.3433\left(7.21 - \sum_j t_{1j}\lambda_{1j} - \sum_j t_{2j}\lambda_{2j} j\right)}{8 - 23 \times 0.3433}.$$

Together with $\alpha > 100$, this gives

$$\sum_j t_{1j}\lambda_{1j} + \sum_j t_{2j}\lambda_{2j} \le 7.21 - \left(\frac{8}{0.3433} - 23\right)\log 100 \le 5.8136. \tag{6.28}$$

We compute the values of λ_{1j} and λ_{2j} for $j < 420$ and find that

$$\lambda_{1j} \le 5.8136 \quad \text{for} \quad j \in T_1 := \{j : j \le 6\} \cup \{12, 13, 14, 37, 38, 39, 40, 41\},$$

and

$$\lambda_{2j} \le 5.8136 \quad \text{for} \quad j \in T_2 := \{j : j \le 5\} \cup \{12, 37, 38\}.$$

Thus, by (6.28), we may suppose that $t_{1j} > 0$ implies $j \in T_1$ and $t_{2j} > 0$ implies $j \in T_2$. Recall that we have $t_{1j} > 0$ for $j = 37$ or $j = 38$. Write t_4, t_5, t_{12} for t_{1j} according to whether $j = 4, 5, 12$, respectively. We find that $\lambda_{1j} \ge 2.639, 3.737, 3.111$ according to whether $j = 4, 5, 12$, respectively. Hence, from (6.28), we have $t_4 \le 2, t_5 \le 1$ and $t_{12} \le 1$. Put

$$\log \mathcal{D}_1 := \sum_{j \in T_1, j \ne 4,5,12} t_{1j} \log C_j + \sum_{j \in T_2} t_{2j} \log B_j,$$

so that

$$\log \mathcal{D} = \log \mathcal{D}_1 + t_4 \log C_4 + t_5 \log C_5 + t_{12} \log C_{12}. \tag{6.29}$$

We now consider $M_8(\mathcal{D})$ given by (2.10). We find that $M_8(C_j) < 0.46 \log C_j$ for all

$j \in T_1$ except when $j \in \{4, 5, 12\}$ and $M_8(B_j) < 0.46 \log B_j$ for $j \in T_2$ and further

$$M_8(C_4) \leq 0.74, \quad M_8(C_5) \leq 0.53 \quad \text{and} \quad M_8(C_{12}) \leq 0.65.$$

Hence, from (6.29), (6.25) and the fact that $\dfrac{1-x^{12}}{1-x} < 12$, we get

$$\begin{aligned} M_8(\mathcal{D}) <& 0.46 \log \mathcal{D}_1 + 0.74 t_4 \log C_4 + 0.53 t_5 \log C_5 + 0.65 t_{12} \log C_{12} \\ <& 0.46 \log \mathcal{D} + 0.28 t_4 \log C_4 + 0.07 t_5 \log C_5 + 0.19 t_{12} \log C_{12} \\ <& 0.46(23 \log \alpha + \log 12 + \log(1 + x^{12})) \\ & + 0.56 \log C_4 + 0.07 \log C_5 + 0.19 \log C_{12}, \end{aligned}$$

since $t_4 \leq 2, t_5 \leq 1$ and $t_{12} \leq 1$. Comparing the above inequality with the lower bound of $M_8(\mathcal{D}) = M_8(U_{24})$ given by (2.11) with $n_0 = 2^3$ and $t = 0$, we obtain

$$\begin{aligned} 4.07 >& 0.56 \log C_4 + 0.07 \log C_5 + 0.19 \log C_{12} \\ >& (12 - 0.46 \times 23) \log \alpha + (1 - 0.46) \log(1 + x^{12}) - 0.46 \log 12 - \log 2 \\ >& (12 - 0.46 \times 23) \log 100 - 0.46 \log 12 - \log 2 > 4.7 \end{aligned}$$

since $1 + x^{12} > 0$ and $\alpha > 100$. This is a contradiction. Therefore, Equation (6.1) has no solution with $n = 24$ when α and β are real.

6.4.5 The case of Equation (6.2)

We now consider Equation (6.2). Since $V_n = U_{2n}/U_n$, we see that primitive divisors of V_n are the primitive divisors of U_{2n}. From Table 2.1 in Theorem 2.16, we find that the values of $n \geq 4$ for which V_n does not have a primitive divisor which are given by the instances for which U_{2n} has no primitive divisors, belong to the set $\{4, 5, 6, 9\}$. For $n \in \{4, 5, 6, 9\}$ and corresponding pairs (r, s) (which are given by pairs (r, s) corresponding to $2n$ in the table), we check that Equation (6.2) has no solution.

Hence, for the proof of Theorem 6.1, we now assume that $n \geq 4$ and further V_n has a primitive divisor which is congruent to one of ± 1 modulo $2n$. Since this is a primitive divisor of U_{2n}, we keep the notation P_{2n} for the largest such.

Let $n = 4t$ be even. Then

$$V_{4t} = \alpha^{4t} + \beta^{4t} = (\alpha^{2t} + \beta^{2t})^2 - 2(\alpha\beta)^{2t} = V_{2t}^2 - 2(-s)^{2t}.$$

For an odd prime $p \mid V_{4t}$, we see that 2 is a quadratic residue modulo p and hence $p \equiv \pm 1 \pmod 8$. We observe that both B_m and C_m are divisible by each prime $m + 1 < p \leq 2m$. By Lemma 2.24, there is a prime $p \equiv \pm 5 \pmod 8$ with $m + 1 < p \leq 2m$ for each $m \geq 6$. Thus, Equation (6.2) implies $m_k \leq 5$ which together with the fact that V_n has a primitive prime divisor give $n = 4t = 4$. Further, $\gcd(V_t, s) = 1$ for all $t \geq 1$ gives $\nu_2(V_{2t}^2 - 2(s)^{2t}) \leq 1$ implying $\nu_2(V_4) \leq 1$. Considering $\nu_2(B_m), \nu_2(C_m)$ for $2 \leq m \leq 5$ and using the fact that V_4 has a primitive prime divisor which is congruent to $\pm 1 \pmod 8$, we get $V_4 = C_4 = 14$. Now $14 = V_4 = \alpha^4 + \beta^4 = r^2(r^2 + 4s) + 2s^2$ and $\gcd(V_4, s) = 1$ give s odd and r even. Reducing the above relation modulo 8, we get $14 \equiv 2 \pmod 8$ which is a contradiction. Thus, Equation (6.2) does not have a solution for n even with $4 \mid n$.

From now on, we take n odd with $n \geq 5$ or $2||n$ with $n \geq 6$. We have

$$\log \mathcal{D} = \sum_{i=1}^{k} \log D_{m_i} = \log |V_n| = \log |\alpha^n + \beta^n| = n \log |\alpha| + \log |1 + x^n|.$$

Since V_n has a primitive divisor which is congruent to one of ± 1 modulo $2n$, we have $2m_k - 1 \geq 2n - 1$ or $m_k \geq n$. By Lemma 6.4 and since $|1 + x^n| \leq 2$, we have

$$\log |\alpha| + \log 2 \geq \log |V_n| \geq \log 2 + \log C_{m_k/2} \geq \log 2 + 1.38n \text{ for } n \geq 2100. \tag{6.30}$$

Let p be an odd prime and $h > 0, t \geq 0$ be such that $p^{h+t} \mid n$. Taking $n_0 = p^h$,

we use the estimate (2.12) of Lemma 2.28 to get a lower bound for the quantity $M_{n_0}(\mathcal{D}) = M_{n_0}(V_n)$ and compare it with the upper bound given by Lemma 6.5 to obtain

$$g(n_0)(n\log|\alpha| + \log|1+x^n|) \geq \left(1 - \frac{1}{p^{t+1}}\right) n\log|\alpha| + \log\left|\frac{1+x^n}{1+x^{n/p}}\right| - \log p^{t+1},$$

implying

$$\left(1 - \frac{1}{p^{t+1}} - g(n_0)\right) \leq \frac{(g(n_0)-1)\log|1+x^n| + \log|1+x^{n/p^{t+1}}| + \log(p^{t+1})}{n\log|\alpha|}, \tag{6.31}$$

where $g(n_0)$ is given by (6.16). We consider different cases as in the analysis for U_n.

Let α and β be complex conjugates. We may assume that $n \geq 6500$. We choose $n_0 = p^h$ and t given by (6.20). Assume that $\log|\alpha| \leq 4$. Then using

$$|1+x^n| = \frac{|V_n|}{|\alpha|^n}, \quad |1+x^{n/p^{t+1}}| \leq 2 \quad \text{and} \quad \log|\alpha| \leq 4,$$

along with (6.30) in (6.31), we obtain

$$\begin{aligned} 0 \geq & \frac{(1-g(n_0))\log|V_n| + \log(2p^{t+1})}{n\log|\alpha|} - \frac{1}{p^{t+1}} \\ \geq & \frac{1.38(1-g(n_0)) + \log(2p^{t+1})}{4n} - \frac{1}{p^{t+1}}, \end{aligned}$$

which is the inequality (6.21). As in the case of U_n in Section 6.4.2 we a get a contradiction. Assume now that $\log|\alpha| > 4$. By Lemma 6.2, we get

$$\log|1+x^n| \geq -f(n)\log|\alpha|,$$

where $f(n)$ be given by (6.3). Using this along with $|1+x^{n/p^{t+1}}| \leq 2$ and $n\log|\alpha| > 4n$ in (6.31), we obtain the inequality (6.22). As in the case of U_n in Section 6.4.2, we a get a contradiction. Therefore, Equation (6.2) has no solution $n \geq 6500$.

Let α and β be real. Then $\alpha > 0$. We take $n \geq 5, n \neq 6$. We choose $n_0 = p^h$ and t given by (6.23), $n_0 \neq 2^4$. Since p is odd, writing

$$1 + x^n = (1 + x^{n/p}) \left(\frac{1 + x^n}{1 + x^{n/p}} \right),$$

we have

$$|1 + x^{n/p}| \leq 2 \quad \text{and} \quad \frac{1 + x^n}{1 + x^{n/p}} = \begin{cases} 1 + \sum_{i=1}^{p-1} (-y)^i > 1, & y = x^{\frac{n}{p}} < 0; \\ \frac{1+y^p}{1+y} \geq \frac{1}{1+y} > \frac{1}{2}, & y = x^{\frac{n}{p}} > 0. \end{cases}$$

Using this in (6.31), we obtain

$$\begin{aligned} \log \alpha &\leq \frac{(g(n_0) - 1) \log \left(\frac{1+x^n}{1+x^{n/p}} \right) + g(n_0) \log(1 + x^{n/p}) + \log p}{n(1 - 1/p - g(n_0)))} \\ &\leq \frac{(1 - g(n_0)) \log 2 + g(n_0) \log 2 + \log p}{n(1 - \frac{1}{p} - g(n_0))} = \frac{\log(2p)}{n(1 - 1/p - g(n_0))}. \end{aligned}$$

This together with $n \geq n_0$ and $\alpha \geq \dfrac{1 + \sqrt{5}}{2}$ give (6.24). As in the case of U_n in Section 6.4.3, we a get a contradiction except for $n_0 = p \in \{5, 7\}$. Further, for $n_0 = p \in \{5, 7\}$, putting $n = p\ell$, we obtain similarly

$$\log \left(\frac{1 + \sqrt{5}}{2} \right) \leq \log \alpha \leq \begin{cases} \frac{15.62}{5\ell}, & \text{if } p = 5; \\ \frac{8.11}{7\ell}, & \text{if } p = 7, \end{cases} \quad \text{and} \quad P_{2p\ell}) \leq \begin{cases} 29, & \text{if } p = 5; \\ 19, & \text{if } p = 7. \end{cases}$$

For the pairs (r, s) given by Lemma 6.3 with the above conditions, we check that Equation (6.2) has no solution at $n = p\ell$. Therefore, Equation (6.2) has no solution for α, β real and $n \geq 5,\ n \neq 6$.

Writing $n = p\ell$, we have from $P_{2n} \leq P(B_{m_j})$ for some $j = 1, \ldots, k$, that

$$\log \left(\frac{1 + \sqrt{5}}{2} \right) \leq \log \alpha \leq \begin{cases} \frac{15.62}{5\ell}, & \text{if } p = 5; \\ \frac{8.11}{7\ell}, & \text{if } p = 7, \end{cases} \quad \text{and} \quad P_{2p\ell} \leq \begin{cases} 29, & \text{if } p = 5; \\ 19, & \text{if } p = 7. \end{cases}$$

For the pairs (r, s) given by Lemma 6.3 with the conditions above, we check that Equation (6.1) has no solutions.

□

Chapter 7

Products of Catalan numbers and middle binomial coeffients : the X-coordinates of the Pell equation

We present a corollary regarding X-coordinates of Pell equations which are in $\{B_m, C_m\}$. For a positive integer d which is square-free, let (X_n, Y_n) be the n-th solution of the Pell equation $X^2 - dY^2 = \pm 1$ in positive integers (X, Y)(solution of either $X^2 - dY^2 = 1$ or $X^2 - dY^2 = -1$, not separately). We supplement this with the following result on values of X_n which are in $\{B_m, C_m\}$.

7.1 Main Result

Theorem 7.1 *Let (X_n, Y_n) be the nth solution in positive integers of the equation $X^2 - dY^2 = \pm 1$ for some squarefree integer d. Then $X_n \in \{B_m, C_m\}$ implies $n = 1$. Similarly, let (W_n, Z_n) be the nth solution in positive integers of the equation $W^2 - dZ^2 = \pm 4$ for some squarefree integer d. Then $W_n \in \{B_m, C_m\}$ implies*

$n \in \{1, 3\}$ *or* $n = 2$ *with*

$$d = 2, W_2 = B_2 = 6: \quad 6^2 - 2 \cdot 4^2 = 4, \text{ where } (W_1, Z_1) = (2, 2);$$
$$d = 2, W_2 = C_4 = 14: \quad 14^2 - 2 \cdot 10^2 = -4, \text{ where } (W_1, Z_1) = (2, 2);$$
$$d = 3, W_2 = C_4 = 14: \quad 14^2 - 3 \cdot 8^2 = 4, \text{ where } (W_1, Z_1) = (4, 2).$$

7.2 Preliminaries

First we prove the following result for $s = \pm 1$.

Lemma 7.2 *Let* $s \in \{-1, 1\}$ *and* $r \geq 1$. *Then* $V_n \in \{B_m, C_m, 2B_m, 2C_m\}$ *with* $n > 1$ *and* $m > 1$ *implies* $n = 3$ *or*

$$\begin{aligned} &n = 2 : (r, s; V_2) = (2, 1; B_2); \\ &n = 2 : (r, s; V_2) = (4, -1; C_4); \\ &n = 3 : (r, s; V_3) = (1, 1; 2C_2), (2, 1; C_4), (5, 1; 2B_4). \end{aligned} \tag{7.1}$$

Proof. Let $m \geq 2$ and

$$\mathcal{D}_m := \{B_m, C_m, 2B_m, 2C_m\}.$$

By Theorem 6.1 and $C_2 = 2$, we have $V_n \in \mathcal{D}_m$ implies $n \in \{1, 2, 3, 6\}$. Let $n \in \{2, 6\}$ and $V_n \in \mathcal{D}_m$. Now $V_2 = r^2 + 2s$ and $V_6 = (r^2 + 2s)((r^2 + 2s)^2 - 3)$. Let $p \equiv 5 \pmod{12}$ be a prime such that $p \mid V_2 V_6$. Then we either have $r^2 \equiv -2s \pmod{p}$ or $(r^2 + 2s)^2 \equiv 3 \pmod{p}$. This is not possible since both $\left(\frac{\pm 2}{p}\right) = \left(\frac{3}{p}\right) = -1$, where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Thus, $p \nmid V_2 V_6$ for any prime $p \equiv 5 \pmod{12}$. By Lemma 2.24, we get $m \leq 8$. Further from $s = \pm 1$, we have $\nu_2(r^2 + 2s) \leq 1$ giving $\nu_2(V_2) = \nu_2(V_6) \leq 1$. Using both $5 \nmid V_2 V_6$ and

$\nu_2(V_2) = \nu_2(V_6) \leq 1$, we find that if $V_2, V_6 \in \mathcal{D}_m$ with $2 \leq m \leq 14$, then $V_2, V_6 \in \{C_m\}$ implies $m \in \{2, 4, 5, 7\}$; $V_2, V_6 \in \{2C_m\}$ implies $m = 7$; $V_2, V_6 \in \{B_m\}$ implies $m = 2$ and $V_2, V_6 \notin \{2B_m\}$. Now $V_2 = r^2 + 2s = E_m \in \mathcal{D}_m$ gives $r^2 = E_m - 2s$. We check that for the values $m \in \{2, 4, 5, 7\}$, $C_m - 2s$ is a square only when $r = 2$, $s = -1, C_2 = 2$ and $r = 4, s = -1, C_2 = 2$; $B_2 - 2s = 6 - 2s$ is a square only for $r = 2, s = 1$; and $2C_7 - 2s$ is not a square. These solutions are listed in (7.1) except that we omit $(r, s) = (2, -1)$ since it gives a degenerate characteristic equation. Let $V_6 = (r^2 + 2s)((r^2 + 2s)^2 - 3) = E_m \in \mathcal{D}_m$. We check that for $r \leq 3$, $V_6 = E_m \in \mathcal{D}_m$ only for $r = 2, s = -1, V_6 = C_2 = 2$ and we omit $(r, s) = (2, -1)$. For $r + 2s \geq 4$, we have $r_1 = r^2 + 2s \geq 14$ and hence $(r_1 - 2)^3 < r_1(r_1^2 - 3) = V_6 \leq C_7 = 429$. This gives $r_1 = r^2 + 2s \leq 9$, which is not possible.

Let $n = 3$ and $V_3 = r(r^2 + 3s) = E_m \in \mathcal{D}_m$. For $r \leq 3$, we check that indeed $V_3 = E_m$ only for the pair $(r, s) = (2, 1)$. We now take $r \geq 4$. Using the inequality $r^3 < r(r^2 + 3) = E_m < (r + 1)^3$ when $s = 1$ and $(r - 2)^3 < r(r^2 - 3) = E_m < (r - 1)^3$ when $s = -1$, we get $r = \lfloor E_m^{1/3} \rfloor$ when $s = 1$ and $r = \lfloor E_m^{1/3} + 2 \rfloor$ when $s = -1$. For $m \leq 15$, we find that putting $r = \lfloor E_m^{1/3} \rfloor$ gives $r(r^2 + 3) = E_m$ only when $r = 2$, $E_m = C_4 = 7$ and $r = 5$, $E_m = 2B_4 = 140$ which are already listed in (7.1) (except that we again omit $(r, s) = (2, -1)$) and putting $r = \lfloor E_m^{1/3} + 2 \rfloor$ gives $r(r^2 - 3) \neq E_m$. Thus, we assume that $m \geq 16$.

We observe that $\nu_2(r^2 + 3) \leq 2, \nu_2(r^2 - 3) \leq 1$ and $\nu_3(r^2 \pm 3) \leq 1$. Further we observe that primes $p \mid r^2 + 3s$ with $p > 3$ satisfy $\left(\dfrac{-3s}{p}\right) = 1$. We get $p \equiv 1, 7 \pmod{12}$ if $p > 3$ when $s = 1$ and $p \equiv \pm 1 \pmod{12}$ if $p > 3$ when $s = -1$. We take $n_0 = 12$ and $\ell_0 = 7, -1$, according to whether $s = 1, -1$, respectively. From the equation

$$V_3 = r(r^2 + 3s) = E_m, \quad E_m \in \mathcal{D}_m,$$

we obtain

$$\log(r^2+3s) = 2\log r + \log\left(1+\frac{3s}{r^2}\right) \le \xi(m) + \log 3 + \begin{cases} \log 4, & \text{if} \quad s=1; \\ \log 2, & \text{if} \quad s=-1, \end{cases}$$

where

$$\xi(m) = \sum_{p\equiv 1,\ell_0 \pmod{12}} \nu_p(2B_m)\log p = \sum_{p\equiv 1,\ell_0 \pmod{12}} \nu_p(B_m)\log p.$$

From

$$\log E_m = \log r(r^2+3s) = \frac{3}{2}\left(2\log r + \log\left(1+\frac{3s}{r^2}\right)\right) - \frac{\log(1+3s/r^2)}{2},$$

and

$$\frac{3}{2}\log 4 - \log(1+3s/r^2) < \begin{cases} \frac{3}{2}\log 4, & \text{if} \quad s=1; \\ \frac{3}{2}\log 2 - \log(1-3/16) < \frac{3}{2}\log 4, & \text{if} \quad s=-1, \end{cases}$$

since $r \ge 4$, together with $E_m \ge C_m$, we get

$$\log C_m \le \log E_m < \frac{3}{2}\left(\xi(m)+\log 12\right) \quad \text{implying} \quad \frac{2}{3} < \frac{\xi(m)}{\log C_m} + \frac{\log 12}{\log C_m}.$$

Hence,

$$\log C_m < \frac{\log 12}{\frac{2}{3} - \frac{\xi(m)}{\log C_m}}. \tag{7.2}$$

For $16 \le m \le 35$, we find that $\frac{\xi(m)}{\log C_m} < 0.52$ and therefore

$$\log C_m < \frac{\log 12}{\frac{2}{3}-0.52} < \log C_{16},$$

which is a contradiction. Thus, we have $m \ge 36$. For $36 \le m < 1500$, we check that $\frac{\xi(m)}{\log C_m} < 0.59$ and hence $\log C_m < \frac{\log 12}{\frac{2}{3}-0.59} < \log C_{36}$, which is a contradiction again.

Thus, $m \geq 1500$. As in the proof of Lemma 6.5, we get

$$\begin{aligned}\xi(m) &\leq \sum_{\substack{(2m)^{1/2}<p\leq 2m \\ p\equiv 1,\ell_0 \pmod{12}}} \left(\left\lfloor \frac{2m}{p} \right\rfloor - 2\left\lfloor \frac{m}{p} \right\rfloor\right) \log p \\ &+ \sum_{\substack{p\leq (2m)^{1/2} \\ p\equiv 1,\ell_0 \pmod{12}}} \left\lfloor \frac{\log(2m)}{\log p} \right\rfloor \log p \\ &\leq \sum_{\ell\in\{1,\ell_0\}} \left\{ \psi(2m;12,\ell) + \sum_{t=2}^{3} \theta(2m/t;12,\ell) - \sum_{t=1}^{2} \theta(m/t;12,\ell) \right\}.\end{aligned}$$

The above inequality with Lemma 2.24, yield

$$\xi(m) \leq \frac{\delta_1 m}{4} = \frac{\delta_1 m}{4\log C_m} \log C_m \quad \text{for} \quad m \geq 1500,$$

where

$$\delta_1 = \frac{47}{15} + \frac{2\sqrt{2}\times 0.863}{\sqrt{1500}}\left(1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}\right) + \frac{2\times 1.5}{\sqrt{1500}}\left(1+\frac{1}{\sqrt{2}}\right).$$

Hence,

$$\frac{\xi(m)}{\log C_m} \leq \frac{\delta_1}{4}\frac{m}{\log C_m} \leq \frac{\delta_1}{4}\frac{1500}{\log C_{1500}} < 0.62,$$

by Lemma 6.4. Inserting this last estimate into (7.2), we get

$$\log C_m < \frac{\log 12}{\frac{2}{3} - 0.62} < 54 < \log C_{50}.$$

This is a contradiction and the proof of Lemma 7.2 is complete.

7.3 Proof of Theorem 7.1

Proof. Let d be a squarefree positive integer and assume $\varepsilon \in \{1, -1\}$. Let (X_n, Y_n) be the nth solution of the equation $X^2 - dY^2 = \varepsilon$. Then $X_n = (\alpha^n + \beta^n)/2$ where

(α, β) are the two roots of the quadratic $x^2 - (2X_1)x + \varepsilon = 0$. Observe that $C_2 = 2$. Thus, $X_n \in \{B_m, C_m\}$ gives $V_n \in \{2C_m, 2B_m\}$ where $V_n = \alpha^n + \beta^n$ and $s = -\alpha\beta = \pm 1$. Then for $n > 1$, V_n is given by Lemma 7.2, namely $n = 3$, $V_n \in \{2C_2, 2B_4\}$. Then $X_n = V_n/2 \in \{C_2 = 2, B_4 = 70\}$. The solutions given by

$$2^2 - 3 \cdot 1^2 = 1, \qquad 2^2 - 5 \cdot 1^2 = -1$$

$$\text{and} \qquad 70^2 - 3 \cdot 23 \cdot 71 \cdot 1^2 = 1, \qquad 70^2 - 29 \cdot 13^2 = -1,$$

are exactly (X_1, Y_1) of the corresponding Pell equations and the assertion of Theorem 7.1 for X_n follows.

We now consider solutions (W_n, Z_n) of $W^2 - dZ^2 = 4\varepsilon$ with $\varepsilon \in \{1, -1\}$. Assume that $W_n \in \{B_m, C_m\}$. Note that by putting

$$\alpha := (W_1 + \sqrt{d}Z_1)/2 \quad \text{and} \quad \beta := (W_1 - \sqrt{d}Z_1)/2,$$

we have that $W_n = \alpha^n + \beta^n = V_n$ and $s = -\alpha\beta = \pm 1$. By Lemma 7.2, we get that either $n = 1$ or $n = 2$ with $V_2 \in \{C_2 = 2, B_2 = 6, C_4 = 14\}$ or $n = 3$ with $V_3 \in \{C_2 = 2, C_4 = 14\}$ or $n = 6$ with $V_6 = C_2 = 2$. For $n \neq 1$, we have solutions

$$d = 2, (W_2, Z_2) = (B_2, 4) \text{ with } 6^2 - 2 \cdot 4^2 = 4 \text{ (and } (W_1, Z_1) = (2, 2));$$

$$d = 2, (W_2, Z_2) = (C_4, 10) \text{ with } 14^2 - 2 \cdot 10^2 = -4 \text{ (and } (W_1, Z_1) = (2, 2));$$

$$d = 3, (W_2, Z_2) = (C_4, 8) \text{ with } 14^2 - 3 \cdot 8^2 = 4 \text{ (and } (W_1, Z_1) = (4, 2)).$$

The solution given by $B_2^2 - 10 \cdot 2^2 = 6^2 - 40 = -4$ is exactly $(W_1, Z_1) = (6, 2)$ for $d = 10$.

This finishes the proof of Theorem 7.1. □

Chapter 8

Conclusion

As observed in Capter 3, if $\{U_n\}_{n\geq 0}$ is a Lucas sequence, then the largest n such that $|U_n| = m_1!m_2!\cdots m_k!$ with $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ satisfies $n < 62000$. In Chapter 4 we also see that if the roots of the Lucas sequence are real, we have $n \in \{1, 2, 3, 4, 6, 12\}$. As a consequence, we showed in Chapter 5 that if $\{X_n\}_{n\geq 1}$ is the sequence of X- coordinates of a Pell equation $X^2 - dY^2 = \pm 1$ with a nonsquare integer $d > 1$, then $X_n = m!$ implies $n = 1$. In Chapter 6 we proved that the largest n such that $|U_n| = C_{m_1}C_{m_2}\cdots C_{m_k}$ with $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, where C_m is the mth Catalan number, satisfies $n < 6500$ and in case the roots of the Lucas sequence are real, we have $n \in \{1, 2, 3, 4, 6, 8, 12\}$. Of consequence, in Chapter 7 we showed that if $\{X_n\}_{n\geq 1}$ is the sequence of the X- coordinates of a Pell equation $X^2 - dY^2 = \pm 1$ with a nonsquare integer $d > 1$, then $X_n = C_m$ implies $n = 1$.

Here we provide two conjectures that have arisen from the work in Articles [11] and [12]

Conjecture 1 : There are only finitely many solutions of Equation (3.1) with $n \in \{6, 12\}$ regardless or whether α, β are real or complex conjugates unconditionally. The only solutions (r, s, n) of (3.2) with $n \geq 4$ are given by
$(n; r, s) \in \{(4; \pm 1, -2), (4; \pm 2, -7), (5; \pm 2, -3), (6; \pm 1, 1)\}$.

The reader will recall that the case of α, β real with $n \in \{6, 12\}$, was proven under the *abc* conjecture in Chapter 4.

What lends more credence to Conjecture 1 is the fact that solutions of Equation (3.1) were checked for $|r| \leq 100, |s| \leq 1000$ and $5 \leq n \leq 30$, to find that

$$n = 12, \quad (r, s) = (\pm 1, 1), \quad U_{12} = \pm 144;$$

$$(n; r, s) \in \begin{array}{l} \{(5; \pm 1, -2), (5; 1, -3), (5; \pm 12, -55), \\ (5; 12, -377), (7; \pm 1, -5), (13; \pm 1, -2)\}, \quad \text{for which } U_n = \pm 1; \end{array}$$

$n = 6$ and

$$(r, s) \in \begin{array}{c} \{(\pm 1, 5), (\pm 1, -7), (\pm 1, 53), (\pm 3, -1), (\pm 3, -5), (\pm 3, \pm 7), \\ (\pm 3, -11), (\pm 3, -19), (\pm 3, 61), (\pm 3, -73), (\pm 5, -19), (\pm 5, 29), \\ (\pm 6, -97), (\pm 9, -11), (\pm 9, -17), (\pm 9, -91), (\pm 9, -97)\}. \end{array}$$

If α, β are real, then $r^2 + 4s > 0$, and the only such solution from the above list are for

$$n = 12 : (r, s) = (\pm 1, 1);$$

and

$$n = 6 : (r, s) \in \{(\pm 1, 5), (\pm 1, 53), (\pm 2, -1), (\pm 3, -1)\}.$$

We conjecture that these are the only solutions (r, s, n) of (3.1) with $n \geq 5$.

Further computations served as impetus to the following conjecture.

Conjecture 2 : There are only finitely many solutions of (6.1) such that $n \in \{6, 8, 12\}$ regardless of whether α, β are real or complex conjugates. There are only finitely many solutions of (6.2) with $n \in \{3, 6\}$.

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