The Scattering Problem On Two Half-Lines With Transfer-Matrix Condition At The Origin [1]

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Abstract

The mathematical analysis of scattering theory has been a major area of interest in mathematics and physics research since the latter half of the twentieth century. The aim of this work is to examine, in a functional analytic setting, properties of the differential operator L and solutions involved for scattering on the line $-\infty < x < \infty$. The characterisation of the spectrum of L will provide insight into the physical interpretation of the problem. The study of scattering theory will proceed with the major results in the field being presented with particular focus on reflectionless scattering. Attention is then directed to the inverse reflectionless case. We look at scattering on the line with a matrix transfer condition at the origin in addition an overview of the inverse case is presented.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Richard John Emmett

This the 17^{th} day of October 2007, at Johannesburg, South Africa.

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Contents

Chapter 1

Introduction

1.1 Basic Overview

In this work we investigate the differential equation

$$-\frac{d^2y}{dx^2} + q(x)y = \zeta^2 y, \tag{1.1}$$

and associated differential operator

$$L = -\frac{d^2}{dx^2} + q(x), \tag{1.2}$$

with domain $D(L)$ which will be defined in Chapter 3. Here $q(x)$ is assumed to be a real valued Lebesgue measurable function which obeys the growth condition

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty. \tag{1.3}$$

Two solutions of (1.1), which are of particular interest as they assist with the study of the eigenvalue problem for L, are the, so called, *Jost solutions*. These solutions and various of their properties will be examined.

Once the Jost solutions have been defined, functional analytic aspects of the operator L will be considered, such as location of eigenvalues and continuous spectrum, multiplicity of eigenvalues, scattering coefficients and the reflection coefficient.

The inverse relectionless scattering problem will then be considered. Given two finite sequences $(\eta_i)_{i=1}^N$ and $(c_i)_{i=1}^N$ with $0 < \eta_1 < \ldots < \eta_N$ and $0 < c_i$, $i = 1, \ldots, N$, a reflectionless potential q will be constructed having as scattering data, i.e. eigenvalues and norming constants, $-\eta_1^2, \ldots, -\eta_N^2$ and $c_1^2, \ldots, c_N^2$ respectively.

Following this we consider the scattering problem on the line with a transfer condition at $x = 0$. Here the solutions and their derivatives of (1.1) on $(-\infty, 0)$ and $(0, \infty)$ are related to each other at $x = 0$ by a matrix. The solutions in this case will be related back to the Jost solutions of (1.1) without the transfer condition at $x = 0$.

The notation used is based on that of Weidmann [34], in whose. notation we have $n = 2$, $m = 1$, $p = n \times m = 2$.

1.2 Chapter Structure

In Chapter 2 an overview of the history of scattering theory is presented. The most important and well known contributions to the subject will be discussed.

In Chapter 3 equation (1.1) is put in a functional analytic setting. In particular an appropriate domain for the differential operator L is given and L is shown to be a symmetric operator. Deficiency indices for linear operators are defined and the conditions imposed on the domain of L to ensure that a self-adjoint restriction of L exists, are examined.

The solutions to the scattering problem, known as the Jost solutions, are defined in Chapter 4. Various asymptotic properties of these solutions are presented thereafter and the conjugate solutions for $\zeta = \xi \in \mathbb{R}$ are studied.

In Chapter 5 it is shown that the eigenvalues of L are real and negative. A series of results is then given which show that L has continuous spectrum $[0, \infty)$.

The classical forward scattering problem is studied in Chapter 6. The functions

$a(\zeta)$ and $b(\xi)$, which together give the reflection coefficient, are defined and their properties investigated. Using $a(\zeta)$ and $b(\xi)$ it is shown that L has only a finite number of eigenvalues and that the Jost solutions are linearly dependent at ζ^2 an eigenvalue. The scattering data comprised of the *reflection coefficient*, $r(\xi) = \frac{b(\xi)}{a(\xi)}$, $i\eta_j, j = 1, \ldots, N$, where $-\eta_j^2$ are eigenvalues, and the associated norming constants $c_j, j = 1, \ldots, N$, is studied. Finally we look at the reflectionless, $r(\xi) = 0$, scattering problem.

The focus of Chapter 7 is the inverse scattering problem for reflectionless potentials. This problem is reformulated as a matrix equation and is shown to be solvable. A potential and two solutions functions are defined in terms of the solution vector $\underline{g}(x)$ to the matrix equation. This potential is shown to be reflectioness and the solution function are shown to be Jost solutions associated with is potential. Finally the reconstructed potential and Jost solutions are shown to have the required scattering data.

In Chapter 8 the scattering problem with a transfer matrix condition at the origin is studied. A differential operator L is defined which takes the transfer condition into account. The class of transfer matices which yield a self-adjoint operator L is characterised. Jost solutions for this case are defined and it is shown that the scattering problem with transfer condition at zero, has only a finite number of eigenvalues, as in the classical case.

Chapter 9 gives a brief summary of this work along with an overview of the reflectionless inverse scattering problem with transfer condition. Directions for further study are discussed.

Chapter 2

History

Scattering theory started with the α-particle scattering experiments of Ernest Rutherford in 1911 which brought about the planetary model of the atom. The mathematics of scattering theory is directly related to the spectral theory of second order differential operators.

The first major development in the area of the inverse spectral and scattering problems was the result of Ambartsumyan, see [10], in 1929, which says that, if the eigenvalues of $-y'' + qy = \lambda y$i, on $[0, \pi]$, with boundary conditions $y(0) = y(\pi) = 0$ and q a real continuous function coincide with the set $\{n^2 : n \in \mathbb{N}\}$, then the potential function q is identically zero.

For a general Sturm-Liouville boundary value problem, the spectrum alone does not determine q. This was shown by Borg in 1946, but if q is symmetrical in the sense that $q(x) = q(\pi - x)$, then q is uniquely determined by the spectrum of the boundary value problem. If q is not symmetrical, then it has been shown that the two spectra of the Sturm-Liouville equation with two distinct sets of boundary conditions uniquely determines q. In particular, if $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ denote the eigenvalues of (1.1) with the boundary conditions

$$y'(0) - hy(0) = 0 \quad y'(\pi) + H_1 y(\pi) = 0 \ ,$$

and $\mu_0 < \mu_1 < \mu_2 < \ldots$ denote the eigenvalues of (1.1) with boundary conditions

$$y'(0) - hy(0) = 0 \quad y'(\pi) + H_2 y(\pi) = 0 \ ,$$

where $h, H_1, H_2 \in \mathbb{R}$ and $H_1 \neq H_2$, then the two sets of eigenvalues $\{\lambda_m\}_0^\infty$ and $\{\mu_n\}_0^\infty$ determine $q(x), h, H_1$ and H_2 uniquely, see [10, 23]. Levinson in 1949 showed that one spectrum and the norming constants are sufficient to recover $q(x)$ uniquely on $[0, \pi]$, see [23].

A uniqueness theorem by Tikhonov was published in 1949 [23]. Tikhonov considered the equation

$$u'' + \lambda \rho^2(x) u = 0$$

where $\rho(x)$ is a piecewise analytic function and $\rho(x) \geq \rho_0 > 0$. Tikhonov showed that the Weyl-Titchmarsh m-function

$$m(\lambda) = \frac{u'(0, \lambda)}{q(0, \lambda)}, \quad \lambda < 0$$

uniquely determines $\rho(x)$. Here $m(\lambda)$ is a meromorphic function with simple poles at each eigenvalue.

In 1952, Marčenko, [25], showed that knowing $m(\lambda)$ is equivalent to knowing the spectral measure which is also equivalent to knowing the eigenvalues and norming constants.

Similar results from the same period can be found by Krein, see [18], [19] and [20], by Levitan, see [21], [22], and by Chudov, [6], [7].

In the well known 1951 paper of Gelfand and Levitan, [13], the equation

$$y'' + (\lambda - q(x))y = 0,$$

on $(0, \infty)$, was considered with boundary conditions

$$y(0) = 1 \quad y'(0) = h,$$

for some constant h. Given a function $\rho(\lambda)$ they determined whether there exist an equation of the above form having ρ as its spectral funtion. Eigenfunctions $\varphi(x, \lambda)$, for the problem can be represented by means of a transformation operator

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t\,dt,$$

where $K(x,t)$ is an integral kernel. Let

$$f(x,y) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x\cos\sqrt{\lambda}y\,d\sigma(\lambda),$$

where $\sigma(\lambda)$ is directly related to $\rho(\lambda)$ and can thus be found from the eigenvalues and and norming constants of the Sturm-Liouville problem. The kernels f and K are related by the equation

$$f(x,y) - \int_0^{\infty} f(y,t)K(x,t)\,dt + K(x,y) = 0,$$

now known as the *Gelfand Levitan equation.* This allows one to reconstruct kernel function $K(x,y)$, which, once found, yields the potential by the formula,

$$q(x) = 2\frac{dK(x,x)}{dx}.$$

In the scattering context, the Gelfand-Levitan equation may be used as follows, see [23]. On the line $(-\infty,\infty)$, given two equations

$$\begin{aligned} -y'' + q_1(x)y &= \zeta^2 y, \\ -y'' + q_2(x)y &= \zeta^2 y, \end{aligned}$$

and two sets of scattering data

$$\begin{aligned} &\left\{\frac{b_1(\zeta)}{a_1(\zeta)}, i\eta_1^{(1)}, \ldots, i\eta_{N_1}^{(1)}, c_1^{(1)}, \ldots, c_{N_1}^{(1)}\right\}, \\ &\left\{\frac{b_2(\zeta)}{a_2(\zeta)}, i\eta_1^{(2)}, \ldots, i\eta_{N_2}^{(2)}, c_1^{(2)}, \ldots, c_{N_2}^{(2)}\right\}, \end{aligned}$$

a function $F(x,y)$ can be constructed as follows

$$\begin{aligned} F(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(x,\zeta)g_1(y,\zeta)\left[\frac{b_2(\zeta)}{a_2(\zeta)} - \frac{b_1(\zeta)}{a_1(\zeta)}\right]d\zeta + \\ &\sum_{j=1}^{N_2}\frac{1}{c_j^{(2)}}g_1(x,i\eta_j^{(2)})g_1(y,i\eta_j^{(2)}) - \sum_{j=1}^{N_1}\frac{1}{c_j^{(1)}}g_1(x,i\eta_j^{(1)})g_1(y,i\eta_j^{(1)}), \end{aligned}$$

where $g_1(x,\zeta)$ and $g_2(x,\zeta)$ are Jost solutions. Here, g_2 is related to g_1 by the transformation operator

$$g_2(x,\zeta) = g_1(x,\zeta) + \int_{-\infty}^{x} B(x,y)g_1(y,\zeta)dy.$$

The Gelfand-Levitan equation is set up to solve for $B(x,y)$ from

$$B(x,y) + F(x,y) + \int_{-\infty}^{x} B(x,t)F(t,y)dt = 0, \quad (y < x).$$

Once B is found, the potential functions are found using

$$B(x,x) = \frac{1}{2}\int_{-\infty}^{x} [q_2(t) - q_1(t)]dt.$$

Gasymov in 1963, [11], tackled the inverse problem for the equation

$$-y'' + q(x)y = sy,$$

on the interval $[0,\infty)$ with initial condition

$$y'(0) - hy(0) = 0, \quad h \in \mathbb{R}, \tag{2.1}$$

and $y \in L^2[0,\infty)$. The potential, $q(x)$, considered is such that the self-adjoint extension in $L^2[0,\infty)$, L, of the operator $ly = -y'' + qy$ for $y, y' \in L^2[0,\infty)$ and $ly \in L^2[0,\infty)$ with y obeying (2.1), has only discrete spectrum. The eigenvalues of this problem depend on h and therefore, for $h_1 \neq h_2$, we have two distinct boundary problems with two spectra $\{\lambda_n(h_1)\}$, $\{\lambda_n(h_2)\}$. Gasymov showed that, q, in a suitable class of potentials, is uniquely determined by $\{\lambda_n(h_1)\}$, $\{\lambda_n(h_2)\}$.

An inverse spectral problem for Dirac systems was studied by Gasymov and Levitan in 1966, see [12]. The Dirac system is as follows,

$$Jy' + Q(x)y = \lambda y,$$

on $[0,\infty)$. Here $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}$, p, q, r are real functions and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. It is required here that $y, Jy' + Qy \in L^2[0,\infty)$. Also, boundary conditions of the form $[a, b]y(0) = 0$ are imposed at $x = 0$, where $|a| + |b| \neq 0$. In this setting, an integral equation analogous to the Gelfand-Levitan equation can be formulated allowing the solution of the inverse spectral problem for Dirac operators.

Chapter 3

Self-Adjointness

3.1 Definition of the Differential Operator

Let L be the operator defined by

$$L := -\frac{d^2}{dx^2} + q(x), \tag{3.1}$$

with domain

$$D(L) = \{y \in L^2(-\infty, \infty) \mid y, y', Ly \in L^2(-\infty, \infty), y, y' \in AC(-\infty, \infty)\}. \tag{3.2}$$

Equation (1.1) in $L^2(-\infty, \infty)$ can now be written as the operator equation,

$$Ly = \zeta^2 y. \tag{3.3}$$

The potential $q(x)$ is a real-valued function assumed to satisfy the integrability condition

$$\int_{-\infty}^{\infty} (1 + |x|)q(x)dx < \infty. \tag{3.4}$$

We now show that L is a symmetric operator, i.e.

$$\langle Lf, g\rangle = \langle f, Lg\rangle \quad \text{for all } f, g \in D(L),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product,

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx.$$

We first define a *symmetric operator* which is, in some sense, almost a self-adjoint operator. Indeed, were the operator bounded and symmetric then it would be self-adjoint. Following this, it is shown that L defined above is symmetric.

Definition 3.1 *Let H be a Hilbert space. A linear operator $L : H \to H$ is symmetric if and only if its domain is dense in H and the following holds for all f,$g \in D(L)$*

$$\langle Lf, g\rangle = \langle f, Lg\rangle.$$

For a symmetric operator to be self-adjoint, the domains of the operator and of its adjoint must coincide.

Theorem 3.2 *The operator L is a symmetric operator.*

Proof: Let $f, g \in D(L)$. Then,

$$\begin{aligned}
\langle f, Lg\rangle - \langle Lf, g\rangle &= \int_{-\infty}^{\infty} f\overline{(-g'' + q(x)g)}dx - \int_{-\infty}^{\infty}(-f'' + q(x)f)\overline{g}dx \\
&= \int_{-\infty}^{\infty}(-f\overline{g}'' + f\overline{q(x)g} + f''\overline{g} - q(x)f\overline{g})dx \\
&= \int_{-\infty}^{\infty}(-f\overline{g}'' + f''\overline{g})dx \quad \text{since } q(x) \in \mathbb{R},
\end{aligned}$$

thus, on integrating by part, we have

$$\begin{aligned}
\langle f, Lg\rangle - \langle Lf, g\rangle &= \left[-f\overline{g}' + \int f'\overline{g}'dx + f'\overline{g} - \int \overline{g}'f'dx\right]_{-\infty}^{\infty} \\
&= [f'\overline{g} - f\overline{g}']_{-\infty}^{\infty}
\end{aligned}$$

Since $g \in AC$, there is a function $h(x) \in L^1(-\infty,\infty)$ and a constant c such that

$$g(x) = \int_0^x h(t)\,dt + c.$$

Here $\lim_{x\to\infty} g(x)$ exists and is finite since $h(x) \in L^1(-\infty,\infty)$. Let $\lim_{x\to\infty} g(x) = l^+$. We want to show that $l^+ = 0$. Suppose $l^+ \neq 0$, then there is a point $x_+ > 0$ such that $|g(x)| \geq \frac{1}{2}|l^+|$ for all $x \geq x_+$. Then for $x > x_+$,

$$\begin{aligned} \int_{x_+}^{x} |g(t)|^2\,dt &\geq \frac{1}{4}|l^+|^2|x - x_+| \\ &\to \infty \quad \text{for } x \to \infty. \end{aligned}$$

This contradicts $g \in D(L) \subset L^2(-\infty,\infty)$. Hence $l_+ = 0$. In a similar manner it can be shown g, g', f, f', all tend to zero at $\pm\infty$, since they all lie in $D(L)$. Thus

$$\lim_{x\to\pm\infty} f'\overline{g} = 0 \qquad \lim_{x\to\pm\infty} f\overline{g}' = 0\ ,$$

and

$$\langle f, Lg\rangle - \langle Lf, g\rangle = 0$$

Hence $\langle Lf, g\rangle = \langle f, Lg\rangle$ for all $f, g \in D(L)$ thus L is symmetric. ■

3.2 Deficiency Indices of Symmetric Operators

In this section we consider the theory of deficiency indices in order to construct self-adjoint extensions of symmetric operators and self-adjoint restrictions of operators whose adjoints are symmetric. This is necessary as unbounded operators may be non-self adjoint even though they are symmetric or have a symmetric adjoint. The theory presented here follows the apporoaches of Hutson and Pym [17, Chapter 10], and Weidmann [33, Chapter 4].

For an unbounded operator L in $L^2(-\infty,\infty)$ to have a well defined adjoint, L^*, it is necessary for $D(L)$ be dense in $L^2(-\infty,\infty)$. Even if the unbounded operator L is densely defined, the domain, $D(L)$, of L is not necessarily equal to the domain, $D(L^*)$, of its adjoint L^*. The manner in which a symmetric operator may fail to be necessarily self-adjoint is precisely in the difference between the domains of $D(L)$ and $D(L^*)$.

Prior to considering deficiency spaces we need to consider the concept of a closed operator.

Definition 3.3 *Let $L : H \to H$ be a linear operator on the Hilbert space H. Let (f_n) be a sequence in H. Consider the following three conditions:*

(i) $f_n \in D(L)$ for all n;

(ii) (f_n) is a convergent sequence with limit f;

(iii) (Lf_n) is convergent.

The operator L is **closed** *if and only if for every sequence satisfying (i) – (iii), we have that $f \in D(L)$ and $Lf = \lim Lf_n$.*

Lemma 3.4 *Let $L : H \to H$ be an injective linear operator. Then L^{-1} is a closed operator if and only if L is a closed operator.*

Proof: Let $G(L) = \{[f, Lf] \mid f \in D(L)\} \subset H \times H$ denote the graph of L. Denote the inverse graph by $G'(L) = \{[Lf, f] \mid f \in D(L)\} \subset H \times H$. Now, since L is injective $R(L) = D(L^{-1})$ and $L^{-1}L = I_{|D(L)}$. Hence

$$
\begin{aligned}
G'(L^{-1}) &= \{[L^{-1}g, g] \mid g \in D(L)\} \\
&= \{[L^{-1}g, g] \mid g = Lf, f \in D(L)\} \\
&= \{L^{-1}Lf, Lf] \mid f \in D(L)\} \\
&= G(L).
\end{aligned}
$$

But $G'(L^{-1})$ is closed if and only if $G(L^{-1})$ is closed. Hence L closed if and only if L^{-1} closed. ■

The next result characterises the domains of continuous closed operators, [33, p97].

Lemma 3.5 *Let $L : H \to H$ be a continuous linear operator. Then L is closed if and only if $D(L)$ is closed.*

Lemma 3.6 *Let $L : H \to H$ be a linear operator in a Hilbert space H. If there exists $m > 0$ such that $\|Lf\| \geq m\|f\|$ for all $f \in D(L)$, then L is closed if and only if $R(L)$ is closed.*

Proof: By the condition $\|Lf\| \geq m\|f\|$ and since L is linear it follows that $\|Lf\| = 0$ if and only if $\|f\| = 0$ making L injective. Hence, L^{-1} exists with domain $D(L^{-1}) = R(L)$ and $D(L) = R(L^{-1})$. Let $g \in D(L^{-1})$, then we have,

$$m\|L^{-1}g\| \leq \|LL^{-1}g\| = \|g\|$$

and hence,

$$\|L^{-1}g\| \leq m^{-1}\|g\|.$$

Thus, L^{-1} is bounded, and therefore continuous on $D(L^{-1})$. By Lemma 3.4 L is closed if and only if L^{-1} is closed. Now by Lemma 3.5, L^{-1} is closed if and only if $D(L^{-1})$ is closed and we have seen that $D(L^{-1}) = R(L)$. Therefore, L closed if and only if $R(L)$ is closed. ■

The next result uses the projection theorem for Hilbert spaces[17, p.30], which decomposes a Hilbert space into direct sums of closed orthogonal subspaces.

Theorem 3.7 (Projection Theorem). *Let H be a Hilbert space and M a closed subspace of H. Then $M^{\perp}$ is a closed subspace of H and H can be decomposed as*

$$H = M \oplus M^{\perp}.$$

Lemma 3.8 *Let $L : H \to H$ be a densely defined closed symmetric operator in H, with $\lambda \in \mathbb{C}$, $\Im\lambda \neq 0$. Then*

$$H = R(L - \lambda I) \oplus N(L^* - \overline{\lambda} I) = R(L - \overline{\lambda} I) \oplus N(L^* - \lambda I)$$

and here $R(L - \lambda I)$ and $N(L^* - \overline{\lambda} I)$ are closed orthogonal subspaces of H. Also, $(L - \lambda I)^{-1}$ exists as a bounded operator on $R(L - \lambda I)$.

Proof: Denote $\lambda = \mu + i\nu$, where $\mu, \nu \in \mathbb{R}$ and $\nu \neq 0$. Let $f \in D(L)$. Then

$$\begin{aligned}\|(L-\lambda I)f\|^2 &= \langle((L-\mu I)-i\nu I)f, ((L-\mu I)-i\nu I)f\rangle \\ &= \|(L-\mu I)f\|^2 + \nu^2\|f\|^2 + i\nu\langle(L-\mu I)f, f\rangle - i\nu\langle f, (L-\mu I)f\rangle.\end{aligned}$$

Now, since L is symmetric, so is $(L-\mu I)$ and consequently

$$\langle(L-\mu I)f, f\rangle = \langle f, (L-\mu I)f\rangle$$

and thus

$$\|(L-\lambda I)f\|^2 = \|(L-\mu I)f\|^2 + \nu^2\|f\|^2.$$

As $\|(L-\mu I)f\|^2 \geq 0$, we have

$$\|(L-\lambda I)f\|^2 \geq \nu^2\|f\|^2.$$

Hence we see that $N(L-\lambda I) = \{0\}$ since $\nu \neq 0$. Now, as $N(L-\lambda I) = \{0\}$, $(L-\lambda I)^{-1}$ exists on $R(L-\lambda I)$ and is bounded there. By Lemma 3.6, since L is closed, $R(L-\lambda I)$ is a closed subspace of H. Since L is densely defined, L^* exists and using the result $\overline{R(L)} = N(L^*)^\perp$ [p172, [17]],

$$\begin{aligned}\overline{R(L-\lambda I)} &= R(L-\lambda I) \\ &= N(L^* - \bar{\lambda} I)^\perp.\end{aligned}$$

Hence $N(L^* - \bar{\lambda} I)^\perp$ is closed and using the projection theorem $N(L^* - \bar{\lambda} I)$ is closed, since $L^* - \bar{\lambda} I$ is a closed operator, and is orthogonal to $R(L-\lambda I)$. ■

In the case of $L : H \to H$ densely defined and self-adjoint Lemma 3.8 gives that for $\Im(\lambda) \neq 0$, $R(L-\lambda I) = H$ and $(L-\lambda I)^{-1}$ is thus a bounded operator on H making λ an element of the resolvent set of L. To summarise, we have the following theorem.

Theorem 3.9 *The spectrum of a densely defined self-adjoint operator in a Hilbert space is a subset of* $\mathbb{R}$.

We are now in a position to consider the deficiency spaces. Much of the following is based on section 10.2 from Hutson and Pym [17, pg 253] and is largely presented without proof as we shall mainly be using the results which are well known.

Definition 3.10 *Let $L : H \to H$ be a densely defined operator in the Hilbert space H. The deficiency spaces of L are the following null spaces,*

$$N_+ := N(L^* - iI), \quad N_- := N(L^* + iI).$$

We define the deficiency indices of L by

$$n_+ := Dim(N_+), \quad n_- := Dim(N_-).$$

For the densely defined symmetric operator L, the deficiency indices (n_+, n_-) will be used to measure the "non-self-adjointness" of L. For the Sturm-Liouville operator (3.1), the deficiency spaces consist of the $L^2(-\infty, \infty)$ solutions of the differential equation

$$-y'' + q(x)y = \pm iy.$$

The following result is from Dunford and Schwartz [9, p1232].

Theorem 3.11 *Let L be a symmetric operator and $\lambda \in \mathbb{C}$. Define*

$$M_\lambda = \{x \mid L^*x = \lambda x\}.$$

If $\Im(\lambda) > 0$ then $dim(M_\lambda) = n_+$, the positive deficiency index of L.

The above result gives that the dimension of the deficiency space $N(L^* - \lambda I)$ is constant for λ in the upper half plane. Hence for convenience we choose $\lambda = i$.

For the reader's convenience we quote the following well known result, see [17, p172].

Theorem 3.12 *For a densely defined linear operator $L : H \to H$ in the Hilbert space H, L^* has a dense domain if and only if L is a closable operator and in this case $L^{**} = \overline{L}$, where $\overline{L}$ denotes the closure of L.*

A symmetric operator need not be closed, but the following result guarantees the existence of the closure of a densely defined symmetric operator.

Lemma 3.13 *Let $L : H \to H$ be a densely defined symmetric linear operator in H, then L is closable.*

Proof: As L is symmetric, we have $L \subset L^*$. Since $D(L)$ is dense and $D(L) \subset D(L^*)$ we have that $D(L^*)$ is dense too. Hence by Theorem 3.12, L is closable. As the adjoint operator, L^*, is closed [17, pg 172], we have $\overline{L} \subset L^*$. ■

For a densely defined symmetric linear operator L in H, any self-adjoint extension, A, of L will be a closed operator and a restriction of L^*. Hence it follows that

$$\overline{L} \subset A \subset L^*.$$

Lemma 3.14 *[17, p.254] For a densely defined symmetric linear operator L in H,*

$$D(L^*) = D(\overline{L}) \oplus N_+ \oplus N_-.$$

Therefore the domain of any operator between $\overline{L}$ and L^* will be some subset of $D(\overline{L}) \oplus N_+ \oplus N_-$ and a superset of $D(\overline{L})$. The following corollary states this result more precisely.

Corollary 3.15 *[17, p.255] Let $L : H \to H$ be a densely defined symmetric linear operator in H and let $A : H \to H$ be a linear operator in H such that*

$$\overline{L} \subset A \subset L^*,$$

then, $D(A) = D(\overline{L}) \oplus N$ where $N \subset N_+ \oplus N_-$.

Next is the main result of the section which gives the precise condition on the operator L to ensure that L does indeed have a self-adjoint extension(s).

Theorem 3.16 *[17, p.258] A densely defined symmetric linear operator L with finite deficiency indices has a self-adjoint extension if and only if*

$$n_+ = n_-.$$

Thus we see that the deficiency indices must be equal in order for a symmetric operator to have a self-adjoint extension. This result is very useful especially for ordinary differential operators considered on the domain (3.2) since the deficiency indices are the number of solutions to $Lf = \pm if$ that are in $L^2(-\infty, \infty)$. Note that the deficiency indices will always be finite in our case.

The next definition and theorem give the form of the domain of all extensions of a densely defined symmetric linear operator in H with finite deficiency indices.

Definition 3.17 *For L a densely defined symmetric linear operator in H with finite deficiency indices, we say that the set*

$$\{h_i + \phi_i \mid h_i \in D(L), \phi_i \in N_+ \oplus N_-, i = 1, .., n\}$$

is linearly independent relative to $D(L)$ if the set $\{\phi_1, \dots, \phi_n\}$ is linearly independent.

Theorem 3.18 *[17, p.260] Consider a densely defined symmetric operator L in H with finite deficiency indices*

$$n := n_+ = n_-.$$

Then we have the following cases:

n=0: *$\overline{L}$ is the unique self-adjoint extension of L.*

n $\neq$ 0: *Choose n functions $f_1, .., f_n$ in $D(L^*)$ which are linearly independent relative to $D(L)$ with*

$$\langle L^* f_i, f_j\rangle = \langle f_i, L^* f_j\rangle, \quad \textit{for} \quad i, j = 1, \dots, n,$$

and define $A := L^*|_{D(A)}$ where

$$D(A) := \{f \in D(L^*) \mid \langle L^* f, f_i\rangle = \langle f, L^* f_i\rangle, i = 1, \dots, n\} = D(\overline{L}) \oplus \langle f_1, \dots, f_n\rangle.$$

Then the operator A is a self-adjoint extension of L.

The n conditions we impose on the domain of the extension are what we call the "boundary conditions". A self-adjoint restriction of the adjoint of a symmetric differential operator with deficiency indices (n_+, n_-) where $n_+ = n_-$ will require n_+ boundary conditions on its domain.

Chapter 4

Jost Solutions and Asymptotic Results

4.1 Definition

In this chapter, asymptotic solutions $y(x,\zeta)$, $(x,\zeta) \in \mathbb{R} \times \overline{\mathbb{C}}^{+}$, will be developed for equation (3.3) for large values of $|x| + |\zeta|$, where $\mathbb{C}^{+} = \{x + iy \mid x, y \in \mathbb{R}, y \geq 0\}$. The focus of this chapter will be on the so called Jost solutions of (3.3) and their derivatives. The Jost solutions f_+ and f_- of (3.3) are the solutions of (3.3) defined by their asymptotic behaviour at $\pm\infty$.

Definition 4.1 *[5, p.297] The Jost solutions $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are the solutions of (3.3) with*

$$\begin{aligned} \lim_{x\to\infty} e^{-i\zeta x} f_+(x,\zeta) &= 1, \\ \lim_{x\to-\infty} e^{i\zeta x} f_-(x,\zeta) &= 1. \end{aligned}$$

The existence of the Jost solutions will be proved and it will be shown that they obey the asymptotics

$$f_+(x,\zeta) = e^{i\zeta x} + O\left(C(x)\rho(x)\frac{e^{-\eta x}}{1+|\zeta|}\right), \tag{4.1}$$

$$\frac{df_+}{dx}(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)e^{i\zeta\tau}d\tau + O\left(C(x)\rho(x)\sigma(x)\frac{e^{-\eta x}}{1+|\zeta|}\right), \quad (4.2)$$

for f_+, and

$$f_-(x,\zeta) = e^{-i\zeta x} + O\left(C(-x)\tilde{\rho}(x)\frac{e^{\eta x}}{1+|\zeta|}\right), \quad (4.3)$$

$$\frac{df_-}{dx}(x,\zeta) = -i\zeta e^{-i\zeta x} + \int_{-\infty}^x \cos(\zeta(x-\tau))q(\tau)e^{-i\zeta\tau}d\tau + O\left(\frac{C(-x)\tilde{\rho}(x)\tilde{\sigma}(x)e^{\eta x}}{1+|\zeta|}\right), \quad (4.4)$$

as $|x|+|\zeta| \to \infty$, where $C(x)$ is a non-negative non-increasing function of x and $\rho(x)$, and $\tilde{\rho}(x)$ are as defined in (4.8). Here it is assumed that $q \in L^1$ is real valued and

$$\int_{-\infty}^{\infty}(1+|x|)|q(x)|dx < \infty. \quad (4.5)$$

As q is real it follows that $\overline{f_\pm(x,\overline{\zeta})}$ is again a solution of (3.3). The independence of $f_+(x,\zeta)$ and $\overline{f}_+(x,-\zeta)$ will be shown for $\zeta \neq 0$. The development in this chapter is based on [16].

4.2 Construction of the Jost Solutions

Here we construct by successive approximation the Jost solutions f_+ and f_- of (3.3) as defined in Definition 4.1. The solution of the integral equation

$$y(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)y(\tau,\zeta)d\tau, \quad (4.6)$$

will be constructed and shown to be the Jost solution $f_+(x,\zeta)$. Let the sequence $y_m(x,\zeta),\ x \in \mathbb{R},\ \zeta \in \mathbb{C}$, be defined by:

$$y_0(x,\zeta) = e^{i\zeta x}$$

$$y_{m+1}(x,\zeta) = -\int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)y_m(\tau,\zeta)d\tau, \quad m \geq 0 \quad (4.7)$$

where for $\zeta = 0$, $\frac{\sin(\zeta(x-\tau))}{\zeta} = x-\tau$. We show that the summation

$$y(x,\zeta) = \sum_{m=0}^\infty y_m(x,\zeta),$$

converges everywhere on $\mathbb{R}\times\mathbb{C}$ to a solution of (3.3) and is in fact $f_+(x,\zeta)$.

Lemma 4.2 *The sequence $y_m(x,\zeta)$ obeys the following bound,*

$$|y_m(x,\zeta)| \leq \frac{e^{-\eta x}}{m!}(C(x)\rho(x))^m, \quad x \in \mathbb{R}, \zeta \in \mathbb{C}^+,\ m = 0,1,2,\ldots$$

where $\eta = \Im(\zeta)$, $C(x)$ is a non-negative, non-increasing function and

$$\rho(x) = \int_x^\infty (1+|\tau|)|q(\tau)|d\tau, \quad \tilde{\rho}(x) = \int_{-\infty}^x (1+|\tau|)|q(\tau)|d\tau. \tag{4.8}$$

Proof: It is sufficient to prove the result for

$$C(x) = \begin{cases} 2|x|+2, & \text{for } x \leq 0, \\ 2, & \text{for } x \geq 0. \end{cases}$$

By induction, we prove that

$$|y_m(x,\zeta)| \leq \frac{1}{m!}(C(x)\rho(x))^m e^{-\eta x}, \quad (x,\zeta) \in \mathbb{R}\times\overline{\mathbb{C}}^+.$$

This inequality is obviously true for $m=0$. Assume it holds true for some $m=k$. Then,

$$|y_{k+1}(x,\zeta)| \ \leq \ \frac{1}{k!}\int_x^\infty \left|\frac{\sin(\zeta(x-\tau))}{\zeta}\right| |q(\tau)|C^k(\tau)\rho^k(\tau)e^{-\eta\tau}d\tau.$$

Note that for all $\tau \geq x$ and $\zeta \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned} \frac{\sin(\zeta(x-\tau))}{\zeta} &= \frac{e^{-i\zeta(\tau-x)} - e^{i\zeta(\tau-x)}}{2i\zeta} \\ &= \frac{e^{-i\zeta(\tau-x)}}{2i\zeta}(1-e^{2i\zeta(\tau-x)}) \\ &= -e^{-i\zeta(\tau-x)}\int_0^{\tau-x} e^{2i\zeta z}dz, \end{aligned}$$

and since $\frac{\sin(\zeta(x-\tau))}{\zeta} := x-\tau$ for $\zeta = 0$, the result remains valid for $\zeta = 0$. Hence,

$$\left|\frac{\sin(\zeta(x-\tau))}{\zeta}\right| \leq e^{\eta(\tau-x)}\int_0^{\tau-x} e^{-2\eta z}dz \leq e^{\eta(\tau-x)}(\tau-x), \quad \text{as } \eta \geq 0 \text{ for } \zeta \in \overline{\mathbb{C}}^+.$$

So we get,

$$\begin{aligned} |y_{k+1}(x,\zeta)| &\leq \frac{1}{k!}\int_x^\infty e^{\eta(\tau-x)}(\tau-x)|q(\tau)|C^k(\tau)\rho^k(\tau)e^{-\eta\tau}d\tau \\ &\leq \frac{C^k(x)e^{-\eta x}}{k!}\int_x^\infty (\tau-x)|q(\tau)|\rho^k(\tau)d\tau, \end{aligned}$$

since C is non-increasing.

Case I: $x \geq 0$. For $x \geq 0$ since $\tau \geq x$, we get $\tau - x \leq \tau \leq 1 + \tau \leq C(x)(1+|\tau|)$ and thus

$$\begin{aligned} |y_{k+1}(x,\zeta)| &\leq \frac{C^{k+1}(x)e^{-\eta x}}{k!}\int_x^\infty (1+|\tau|)|q(\tau)|\rho^k(\tau)\, d\tau \\ &= -\frac{C^{k+1}(x)e^{-\eta x}}{k!}\int_x^\infty \rho'(\tau)\rho^k(\tau)d\tau \\ &= -\frac{C^{k+1}(x)e^{-\eta x}}{k!}\left[\frac{\rho^{k+1}(\tau)}{k+1}\right]_x^\infty \\ &= \frac{C^{k+1}(x)e^{-\eta x}\rho^{k+1}(x)}{(k+1)!} \end{aligned}$$

since $\rho'(\tau) = -(1+|\tau|)|q(\tau)|$.

Case II: $x \leq 0$ For $x \leq 0$, we split the integral over $[x, \infty)$ up into the sum of the integrals over the intervals $[x, -x]$ and $[-x, \infty)$, For $\tau \in [x, -x]$ we have $\max(\tau - x) \leq -x - x = 2|x|$, and for $\tau \in [-x, \infty)$ we have $\tau \geq -x$ which gives $2\tau \geq \tau - x$. Hence,

$$\begin{aligned} \int_x^\infty (\tau - x)|q(\tau)|\rho^k(\tau)d\tau &= \int_x^{-x}(\tau - x)|q(\tau)|\rho^k(\tau)d\tau + \int_{-x}^\infty (\tau - x)|q(\tau)|\rho^k(\tau)d\tau \\ &\leq 2|x|\int_x^{-x}|q(\tau)|\rho^k(\tau)d\tau + 2\int_{-x}^\infty \tau|q(\tau)|\rho^k(\tau)d\tau. \end{aligned}$$

Now, for $\tau \in [x, -x]$, since $C(x) \geq 2$,

$$2|x| \leq C(x)(1+|\tau|),$$

and for $\tau \geq -x$,

$$\begin{aligned} 2\tau &\leq 2(1+|\tau|) \\ &\leq C(x)(1+|\tau|). \end{aligned}$$

Combining these results gives

$$\int_x^\infty (\tau - x)|q(\tau)|\rho^k(\tau)d\tau \leq C(x)\int_x^\infty (1+|\tau|)|q(\tau)|\rho^k(\tau)d\tau.$$

So, as in the case for $x \geq 0$, we get,

$$|y_{k+1}(x,\zeta)| \leq \frac{e^{-\eta x}}{(k+1)!}(C(x)\rho(x))^{k+1}$$

and the induction is complete. ■

We now derive another bound for the sequence $y_m(x,\zeta)$, which is useful for $|\zeta|$ large.

Lemma 4.3 *For $\zeta \in \mathbb{C} \setminus \{0\}$, with $\eta = \Im(\zeta) \geq 0$, and $x \in \mathbb{R}$, $m \in \mathbb{N}$, $y_m(x,\zeta)$ obeys the bound*

$$|y_m(x,\zeta)| \leq \frac{1}{m!}\left(\frac{\sigma(x)}{|\zeta|}\right)^m e^{-\eta x}, \tag{4.9}$$

where

$$\sigma(x) = \int_x^\infty |q(\tau)|d\tau. \tag{4.10}$$

Proof: The inequality is true for $m = 0$ since $y_0(x,\zeta) = e^{i\zeta x}$. Assume that (4.9) holds for some $m = k$. Then, by (4.7),

$$\begin{aligned}
|y_{k+1}(x,\zeta)| &\leq \int_x^\infty \left|\frac{\sin(\zeta(x-\tau))}{\zeta}\right| |q(\tau)||y_k(\tau)|d\tau \\
&\leq \frac{1}{k!|\zeta|^{k+1}} \int_x^\infty |\sin(\zeta(x-\tau))||q(\tau)|\sigma^k(\tau)e^{-\eta\tau}d\tau.
\end{aligned}$$

Now for $\tau \geq x$,

$$\begin{aligned}
|\sin(\zeta(x-\tau))e^{-\eta\tau}| &= \left|e^{-\eta\tau}\frac{e^{i\zeta(x-\tau)}}{2i}(1-e^{-2i\zeta(x-\tau)})\right| \\
&= \frac{e^{-\eta x}}{2}|1-e^{-2i\zeta(x-\tau)}| \\
&\leq \frac{e^{-\eta x}}{2}(1+|e^{-2i\zeta(x-\tau)}|) \\
&= \frac{e^{-\eta x}}{2}(1+e^{2\eta(x-\tau)}) \\
&\leq \frac{e^{-\eta x}}{2}(1+1) \quad \text{since } 2\eta(x-\tau) \leq 0 \\
&= e^{-\eta x}.
\end{aligned}$$

But

$$\int_x^\infty |q(\tau)|\sigma^k(\tau)d\tau = -\int_x^\infty \sigma'(\tau)\sigma^k(\tau)d\tau = -\left.\frac{\sigma^{(k+1)}(\tau)}{k+1}\right|_x^\infty = \frac{\sigma^{(k+1)}(x)}{k+1}.$$

and thus

$$|y_{k+1}(x,\zeta)| \leq \frac{1}{k!}\frac{e^{-\eta x}}{|\zeta|^{k+1}}\int_x^\infty |q(\tau)|\sigma^k(\tau)d\tau = \frac{1}{(k+1)!}\left(\frac{\sigma(x)}{|\zeta|}\right)^{k+1} e^{-\eta x},$$

thereby completing the proof. ■

Lemma 4.4 *Let $y_m(x,\zeta)$ be as defined above, then for $\zeta \in \mathbb{C}$ with $\Im(\zeta) \geq 0$, the series*

$$y(x,\zeta) := \sum_{m=0}^{\infty} y_m(x,\zeta)$$

is convergent to a solution of the integral equation

$$y(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)y(\tau,\zeta)d\tau, \tag{4.11}$$

and $f_+(x,\zeta) = y(x,\zeta)$. In particular

$$|f_+(x,\zeta) - e^{i\zeta x}| \leq \frac{e^{-\eta x}e^{C(x)\rho(x)}}{|\zeta|}\int_x^\infty |q(\tau)|d\tau.$$

Proof: By Lemma 4.2, for all $x \in \mathbb{R}$, $\zeta \in \mathbb{C}$ and $m = 0, 1, \ldots$

$$|y_m(x,\zeta)| \leq \frac{1}{m!}(C(x)\rho(x))^m e^{-\eta x},$$

where $C(x)$ is a non-negative non-increasing function of x, $\rho(x) = \int_x^\infty (1+|\tau|)|q(\tau)|d\tau$ and $\eta = \Im(\zeta)$. Then

$$\begin{aligned} \lim_{n\to\infty}\sum_{j=0}^{n} |y_j(x,\zeta)| &\leq \lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}(C(x)\rho(x))^j e^{-\eta x} \\ &= e^{-\eta x}e^{C(x)\rho(x)}. \end{aligned}$$

Thus $y(\tau,\zeta)$ exists and since $C(\tau)\rho(\tau)$ is bounded by $C(x)\rho(x)$ for $\tau \in (x,\infty)$,

$$\begin{aligned} \sum_{j=0}^{n}\left|\frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)y_j(\tau,\zeta)\right| &\leq \sum_{j=0}^{n}\frac{|q(\tau)|}{j!|\zeta|}C^j(\tau)\rho^j(\tau)e^{-\eta x} \\ &\leq \frac{|q(\tau)|}{|\zeta|}e^{-\eta x}e^{C(\tau)\rho(\tau)} \\ &\leq \frac{|q(\tau)|}{|\zeta|}e^{-\eta x}e^{C(x)\rho(x)} \qquad (4.12) \\ &\in L^1(x,\infty) \text{ with respect to } \tau. \end{aligned}$$

Hence, by Lebesgue's Dominated convegence theorem,

$$\begin{aligned} y(x,\zeta) &= e^{i\zeta x} + \sum_{m=0}^{\infty} y_{m+1}(x,\zeta) \\ &= e^{i\zeta x} - \sum_{m=0}^{\infty} \int_x^{\infty} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau) y_m(\tau,\zeta) d\tau \\ &= e^{i\zeta x} - \int_x^{\infty} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau) \sum_{m=0}^{\infty} y_m(\tau,\zeta) d\tau \\ &= e^{i\zeta x} - \int_x^{\infty} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau) y(\tau,\zeta) d\tau, \end{aligned} \tag{4.13}$$

showing that $y(x,\zeta)$ obeys the integral equation (4.11). From (4.12), and Lebesgue's Dominated convergence theorem,

$$\left| \sum_{m=0}^{\infty} \int_x^{\infty} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau) y_m(\tau,\zeta) d\tau \right| \le \frac{e^{-\eta x}}{|\zeta|} e^{C(x)\rho(x)} \int_x^{\infty} |q(\tau)| d\tau$$

and thus

$$|y(x,\zeta) - e^{i\zeta x}| \le \frac{e^{-\eta x} e^{C(x)\rho(x)}}{|\zeta|} \int_x^{\infty} |q(\tau)| d\tau$$

proving that $\lim_{x\to\infty} e^{-i\zeta x} y(x,\zeta) = 1$. Hence $f_+(x,\zeta) = y(x,\zeta)$. ■

Combining the previous two lemmas we obtain the following improved bound on $f_+(x,\zeta) - e^{i\zeta x}$.

Theorem 4.5 *For $x \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ with $\Im(\zeta) \ge 0$,*

$$\left| f_+(x,\zeta) - e^{i\zeta x} \right| \le C(x)\rho(x) \frac{e^{-\eta x}}{1+|\zeta|},$$

where $C(x)$ and $\rho(x)$ are as given earlier.

Proof: From the definition of $\sigma(x)$ given in Lemma 4.3 and the condition

$$\sigma(x) \le \int_{-\infty}^{\infty} (1+|\tau|)|q(\tau)| d\tau := K < \infty,$$

we have that $\sigma(x) \le K$ for some $K \in \mathbb{R}^+$. We now consider the cases $|\zeta| > K+1$ and $|\zeta| \le K+1$.

Case 1: $|\zeta| > K+1$.

By Lemma 4.3 we have,

$$\begin{aligned}
|f_+(x,\zeta) - e^{i\zeta x}| &\leq \sum_{m=1}^{\infty} |y_m(x,\zeta)| \\
&\leq \frac{\sigma(x)}{|\zeta|}\left[\sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{\sigma(x)}{|\zeta|}\right)^{m-1}\right] e^{-\eta x} \\
&\leq \frac{\sigma(x)}{|\zeta|}\left[\sum_{m=1}^{\infty} \frac{1}{(m-1)!}\left(\frac{\sigma(x)}{|\zeta|}\right)^{m-1}\right] e^{-\eta x} \\
&\leq \frac{\sigma(x)}{|\zeta|} e^{\left(\frac{\sigma(x)}{|\zeta|}\right)} e^{-\eta x}.
\end{aligned}$$

Let $C(x)$ be defined as in Lemma 4.2, then $C(x) \geq 2$ and $\sigma(x) \leq \rho(x)$, giving

$$|f_+(x,\zeta) - e^{i\zeta x}| \leq \frac{C(x)\rho(x)}{1+|\zeta|}\left(\frac{1+|\zeta|}{|\zeta|}\right) e^{\left(\frac{\sigma(x)}{|\zeta|}\right)} e^{-\eta x}.$$

Since $e^{\frac{\sigma(x)}{|\zeta|}} < e$ and $\left(\frac{1+|\zeta|}{|\zeta|}\right) < 2$, as $|\zeta| > 1$,

$$|f_+(x,\zeta) - e^{i\zeta x}| \leq \frac{2eC(x)\rho(x)}{1+|\zeta|} e^{-\eta x}.$$

Case 2: $|\zeta| \leq K+1$.

By Lemma 4.2 we have

$$\begin{aligned}
|f_+(x,\zeta) - e^{i\zeta x}| &\leq C(x)\rho(x)\left[\sum_{m=1}^{\infty} \frac{1}{(m-1)!}(C(x)\rho(x))^{m-1}\right] e^{-\eta x} \\
&\leq \frac{C(x)\rho(x)}{1+|\zeta|}(1+|\zeta|)e^{C(x)\rho(x)} e^{-\eta x} \\
&\leq \frac{C(x)\rho(x)}{1+|\zeta|}(K+2)e^{C(x)\rho(x)} e^{-\eta x}.
\end{aligned}$$

Since $0 \leq C(x)$, $\rho(x)$ are non-increasing on $\mathbb{R}$, we have that $C(x)(K+2)e^{C(x)\rho(x)}$ is a non-negative and non-increasing function on $\mathbb{R}$.

We may thus write $\tilde{C}(x) = \max\{C(x)(K+2)e^{C(x)\rho(x)}, 2eC(x)\}$ and the Theorem is proved. ■

The following result shows that the derivative of f_+ exists.

Lemma 4.6 *Let $\zeta \in \mathbb{C}$ with $\eta = \Im(\zeta) \geq 0$. The derivative $f_+'(x,\zeta)$ exists and is given by*

$$\frac{df_+}{dx}(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau.$$

Proof: By Lemma 4.4,

$$f_+(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)f_+(\tau,\zeta)d\tau.$$

Let $x_0 \in \mathbb{R}$ and consider the following limit,

$$\begin{aligned}
\lim_{x\to x_0} \frac{f_+(x,\zeta) - f_+(x_0,\zeta)}{x-x_0} &= \lim_{x\to x_0} \frac{1}{x-x_0}\left[e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)f_+(\tau,\zeta)d\tau \right.\\
&\quad \left. -e^{i\zeta x_0} + \int_{x_0}^\infty \frac{\sin(\zeta(x_0-\tau))}{\zeta} q(\tau)f_+(\tau,\zeta)d\tau\right]\\
&= i\zeta e^{i\zeta x_0} + \lim_{x\to x_0}\left[\int_{x_0}^x \frac{\sin(\zeta(x_0-\tau))}{\zeta(x-x_0)} q(\tau)f_+(\tau,\zeta)d\tau + \right.\\
&\quad \left.\int_x^\infty \frac{\sin(\zeta(x_0-\tau)) - \sin(\zeta(x-\tau))}{\zeta(x-x_0)} q(\tau)f_+(\tau,\zeta)d\tau\right].
\end{aligned}$$

By Theorem 4.5, $|f_+(x,\zeta)| \leq Ke^{-\eta x}$ and $\left|\frac{\sin(x\zeta)}{x\zeta}e^{-\eta x}\right| \leq 1$ for x in a compact set, hence for x in a neighbourhood of x_0,

$$\begin{aligned}
\left|\frac{1}{x-x_0}\int_{x_0}^x \frac{\sin(\zeta(x_0-\tau))}{\zeta} q(\tau)f_+(\tau,\zeta)d\tau\right| &\leq \int_{x_0}^x \left|\frac{\sin(\zeta(x_0-\tau))}{\zeta(x_0-\tau)}\right| |q(\tau)||f_+(\tau,\zeta)|d\tau\\
&\leq K\int_{x_0}^x |q(\tau)|d\tau,
\end{aligned}$$

where $K > 0$ is a constant, but since $q \in L^1$, $\lim_{x\to x_0}\int_{x_0}^x |q(\tau)|d\tau = 0$. Hence

$$\lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^x \frac{\sin(\zeta(x_0-\tau))}{\zeta} q(\tau)f_+(\tau,\zeta)d\tau = 0.$$

Let $\epsilon > 0$, $x < \tau$ and $|x-x_0| < \epsilon$, then using elementary trigonometry,

$$\begin{aligned}
\left|\frac{\sin(\zeta(x_0-\tau)) - \sin(\zeta(x-\tau))}{\zeta(x-x_0)} q(\tau)f_+(x,\zeta)\right| &\leq K\left|q(\tau)\cos\left(\frac{\zeta(x_0+x-2\tau}{2}\right)\right| e^{\eta(-\tau+|x-x_0|)}\\
&\leq K^1|q(\tau)|.
\end{aligned}$$

Making

$$\left|\frac{\sin\zeta(x_0-\tau) - \sin\zeta(x-\tau))}{\zeta(x-x_0)} q(\tau)f_+(\tau,\zeta)\right| \leq K^1|q(\tau)| \in L^1(\mathbb{R}),$$

thus, by Lebesgue's Dominated convergence theorem the limit can be taken through the integral to give,

$$\begin{aligned}
&\lim_{x\to x_0} \frac{f_+(x,\zeta)-f_+(x_0,\zeta)}{x-x_0} \\
&= i\zeta e^{i\zeta x_0} + \int_{x_0}^{\infty} \lim_{x\to x_0}\left[q(\tau)f_+(\tau,\zeta)\cos\left(\frac{\zeta(x_0+x-2\tau)}{2}\right)\sin\left(\frac{\zeta(x_0-x)}{2}\right)\frac{2}{\zeta(x-x_0)}\right]d\tau \\
&= i\zeta e^{i\zeta x_0} + \int_{x_0}^{\infty} q(\tau)f_+(\tau,\zeta)\cos\zeta(x_0-\tau)d\tau.
\end{aligned}$$

Hence

$$\frac{df_+}{dx}(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^{\infty} \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau.$$

The case of $\zeta = 0$ is treated by setting $\frac{\sin\zeta(x-\tau)}{\zeta} = x-\tau$, but is otherwise as above. ■

The next result shows that $\frac{df_+}{dx}(x,\zeta)$ asymptotically can be approximated by $i\zeta e^{i\zeta x}$.

Lemma 4.7 *For $x \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ with $\Im(\zeta) \geq 0$,*

$$\left|\frac{df_+}{dx}(x,\zeta) - i\zeta e^{i\zeta x} + \int_x^{\infty} \cos(\zeta(x-\tau))q(\tau)e^{i\zeta\tau}d\tau\right| \leq C(x)\rho(x)\sigma(x)\frac{e^{-\eta x}}{1+|\zeta|},$$

where $C(x)$ is a non-increasing non-negative function and ρ and σ are as defined in (4.8) and (4.10).

Proof: From Lemma 4.4,

$$f_+(x,\zeta) = e^{i\zeta x} - \int_x^{\infty} \frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)f_+(\tau,\zeta)d\tau, \tag{4.14}$$

while from Lemma 4.6,

$$\frac{df_+}{dx}(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^{\infty} \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau. \tag{4.15}$$

Let

$$I(x,\zeta) = \left|\frac{df_+}{dx}(x,\zeta) - i\zeta e^{i\zeta x} + \int_x^{\infty} \cos(\zeta(x-\tau))q(\tau)e^{i\zeta\tau}d\tau\right|,$$

then substituting (4.14) into (4.15) we obtain,

$$\begin{aligned} I(x,\zeta) &= \left| \int_x^\infty \cos(\zeta(x-\tau))q(\tau)(e^{i\zeta\tau} - f_+(\tau,\zeta))d\tau \right| \\ &\leq \int_x^\infty |\cos(\zeta(x-\tau))||q(\tau)| \left(C(\tau)\rho(\tau)\frac{e^{-\eta\tau}}{1+|\zeta|} \right) d\tau, \end{aligned}$$

by Theorem 4.5. Note that, for $\tau > x$,

$$\begin{aligned} |\cos(\zeta(x-\tau))e^{-\eta\tau}| &= \left| e^{-\eta\tau}\frac{e^{i\zeta(x-\tau)} + e^{-i\zeta(x-\tau)}}{2} \right| \\ &\leq e^{-\eta\tau}e^{\eta(\tau-x)} \\ &= e^{-\eta x}, \end{aligned}$$

giving

$$I(x,\zeta) \leq \frac{e^{-\eta x}}{1+|\zeta|} \int_x^\infty |q(\tau)|C(\tau)\rho(\tau)d\tau.$$

Since $C(x)$ and $\rho(x)$ are non-increasing functions, the previous inequality yields,

$$\begin{aligned} I(x,\zeta) &\leq C(x)\rho(x)\frac{e^{-\eta x}}{1+|\zeta|} \int_x^\infty |q(\tau)|d\tau \\ &= C(x)\rho(x)\sigma(x)\frac{e^{-\eta x}}{1+|\zeta|}. \quad \blacksquare \end{aligned}$$

So far we have established (4.1) and (4.2),

$$\begin{aligned} f_+(x,\zeta) &= e^{i\zeta x} + O\left(C(x)\rho(x)\frac{e^{-\eta x}}{1+|\zeta|} \right), \\ \frac{df_+}{dx}(x,\zeta) &= i\zeta e^{i\zeta x} + \int_x^\infty \cos(\zeta(x-\tau))q(\tau)e^{i\zeta x}d\tau + O\left(\frac{C(x)\rho(x)\sigma(x)}{1+|\zeta|}e^{-\eta x} \right). \end{aligned}$$

The next lemma gives the existence of the second derivative of f_+.

Lemma 4.8 *For $\zeta \in \overline{\mathbb{C}}^+$ and $x \in \mathbb{R}$, the second derivative of $f_+(x,\zeta)$ with respect to x exists and is given by*

$$f_+''(x,\zeta) = -\zeta^2 e^{i\zeta x} + q(x)f_+(x,\zeta) + \int_x^\infty \zeta\sin(\zeta(x-\tau)q(\tau)f_+(\tau,\zeta)d\tau.$$

Proof: We have shown that the derivative of $f'_+(x,\zeta)$ exists for each $x \in \mathbb{R}$. Recall that $f_+(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau$. Consider the limit

$$\begin{aligned}
&\lim_{x\to x_0} \frac{f'_+(x,\zeta) - f'_+(x_0,\zeta)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{1}{x-x_0}\Bigg[i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau - \\
&\quad i\zeta e^{i\zeta x_0} + \int_{x_0}^\infty \cos(\zeta(x_0-\tau))q(\tau)f_+(\tau,\zeta)d\tau\Bigg] \\
&= -\zeta^2 e^{i\zeta x_0} - \lim_{x\to x_0} \frac{1}{x-x_0}\int_x^{x_0} \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau + \\
&\quad \lim_{x\to x_0} \int_{x_0}^\infty \frac{\cos(\zeta(x_0-\tau)) - \cos(\zeta(x-\tau))}{x-x_0} q(\tau)f_+(\tau,\zeta)d\tau.
\end{aligned}$$

The last term above can be shown, in a similar manner to that used in Lemma 4.6, to be equal to

$$\int_{x_0}^\infty \zeta \sin(\zeta(x_0-\tau))q(\tau)f_+(\tau,\zeta)d\tau.$$

Considering the term

$$\lim_{x\to x_0} \frac{1}{x-x_0}\int_x^{x_0} \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau, \tag{4.16}$$

let

$$F(x) := \int_x^{x_0} \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau.$$

Then $F(x)$ is AC on $[a,\infty)$ for all $a \in \mathbb{R}$, since

$$\begin{aligned}
F(x) &= \cos(\zeta x)\int_x^{x_0} \cos(\zeta\tau)q(\tau)f_+(\tau,\zeta)d\tau + \sin(\zeta x)\int_x^{x_0} \sin(\zeta\tau)q(\tau)f_+(\tau,\zeta)d\tau. \\
&= \cos(\zeta x)F_1(x) + \sin(\zeta x)F_2(x).
\end{aligned}$$

where

$$\begin{aligned}
F_1(x) &= \int_x^{x_0} \cos(\zeta\tau)q(\tau)f_+(\tau,\zeta)d\tau, \\
F_2(x) &= \int_x^{x_0} \sin(\zeta\tau)q(\tau)f_+(\tau,\zeta)d\tau,
\end{aligned}$$

are both AC with

$$\begin{aligned}
F'_1(x) &= -\cos(\zeta x)q(x)f_+(x,\zeta) \quad \text{a.e.} \\
F'_2(x) &= -\sin(\zeta x)q(x)f_+(x,\zeta) \quad \text{a.e..}
\end{aligned}$$

Hence, $F(x) = \cos(\zeta x)F_1(x) + \sin(\zeta x)F_2(x)$ is AC on each finite interval, and

$$\begin{aligned} F'(x) &= -\zeta\sin(\zeta x)F_1(x) + \cos(\zeta x)F_1'(x) + \zeta\cos(\zeta x)F_2(x) + \sin(\zeta x)F_2'(x) \\ &= \zeta(\cos(\zeta x)F_2(x) - \sin(\zeta x)F_1(x)) - q(x)f_+(x,\zeta). \end{aligned}$$

Hence, $F'(x_0) = -q(x_0)f_+(x_0,\zeta)$ since $F_1(x_0) = F_2(x_0) = 0$. Combining the above we get that

$$f_+''(x,\zeta) = -\zeta^2 e^{i\zeta x} + q(x)f_+(x,\zeta) + \int_x^\infty \zeta\sin(\zeta(x-\tau)q(\tau)f_+(\tau,\zeta)d\tau. \quad \blacksquare$$

We now show that f_+ is a solution to equation (3.3).

Corollary 4.9 *The function $f_+(x,\zeta)$ is a solution to the equation*

$$-\frac{d^2y}{dx^2} + q(x)y = \zeta^2 y.$$

Proof: From Lemmas 4.4, 4.6 and 4.8, it follows that

$$f_+(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)f_+(\tau,\zeta)d\tau, \tag{4.17}$$

$$f_+'(x,\zeta) = i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau,$$

$$f_+''(x,\zeta) = -\zeta^2 e^{i\zeta x} + q(x)f_+(x,\zeta) + \int_x^\infty \zeta\sin(\zeta(x-\tau))q(\tau)f_+(\tau,\zeta)d\tau.$$

Hence

$$\begin{aligned} f_+''(x,\zeta) &= q(x)f_+(x,\zeta) + \zeta^2\left[\int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta}q(\tau)f_+(\tau,\zeta)d\tau - e^{i\zeta x}\right] \\ &= q(x)f_+(x,\zeta) - \zeta^2 f_+(x,\zeta). \quad \blacksquare \end{aligned}$$

It is now shown that solutions to (4.17) are unique. To show this we first give a version of Grönwall's Lemma on the semi-axis.

Lemma 4.10 *On $[a,\infty)$, $a \in \mathbb{R}$, let $\varphi(t) \geq 0$, $\psi(t) \geq 0$ be measurable functions with with $\int_a^\infty \psi(\tau)d\tau < \infty$ and*

$$\varphi(t) \leq \int_t^\infty \varphi(\tau)\psi(\tau)d\tau < \infty, \text{ for } t \in [a,\infty),$$

then $\varphi(t) = 0$ for all $t \in [a,\infty)$.

Proof: Let $R(t) := \int_t^\infty \varphi(\tau)\psi(\tau)d\tau$.

(i) If $t \in [a,\infty)$ and $R(t) = 0$ then $\varphi\psi = 0$ on $[t,\infty)$.

(ii) If $t \in [a,\infty)$ and $R(t) \neq 0$ then

$$R'(t) = -\varphi(t)\psi(t).$$

By assumption we then have $R'(t) \geq -R(t)\psi(t)$. That is,

$$\frac{R'(t)}{R(t)} \geq -\psi(t).$$

Integrating from t to A where $A \geq t$ and $R(A) \neq 0$ gives

$$\log\left(\frac{R(A)}{R(t)}\right) \geq -\int_t^A \psi(\tau)d\tau.$$

Taking expontentials gives

$$\frac{R(A)}{R(t)} \geq \exp\left(-\int_t^A \psi(\tau)d\tau\right)$$

Then the following inequality holds

$$R(A)\exp\left(\int_t^A \psi(\tau)d\tau\right) \geq R(t) \geq 0 \tag{4.18}$$

If $R(t) \neq 0$ for all $t \in [a,\infty)$ then the expontential term is bounded and $\lim_{A\to\infty} R(A) = 0$. Hence $R(t) = 0$ for all t, while if $R(s) = 0$ for some $s \in (t,\infty)$, let s^* be the least such s. Letting $A \to s^{*-}$ in (4.18) gives $R(t) = 0$ for all $t \in [a, s^*)$ and case (i) applies for $t \geq s^*$ giving $R(t) = 0$ for all $t \geq s^*$. By assumption, $0 \leq \varphi(t) \leq R(t) = 0$, hence $\varphi(t) = 0$ for all $t \in [a,\infty)$. ■

Lemma 4.11 *The solutions to integral equation (4.17) are unique.*

Proof: Let f_+ and $\tilde{f}_+$ be solutions to equation (4.17). Let $g := f_+ - \tilde{f}_+$. Then

$$g(x,\zeta) = -\int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)g(\tau,\zeta)d\tau.$$

So

$$|g(x,\zeta)| \leq \int_x^\infty \frac{e^{\eta(\tau-x)}}{|\zeta|}|q(\tau)||g(\tau,\zeta)|d\tau.$$

i.e.

$$|e^{\eta x}g(x,\zeta)| \leq \int_x^\infty \frac{|q(\tau)|}{|\zeta|}|e^{\eta\tau}g(\tau,\zeta)|d\tau.$$

Hence

$$|e^{\eta x}g(x,\zeta)| \leq \int_x^\infty \frac{|q(\tau)|}{|\zeta|}|e^{\eta\tau}g(\tau,\zeta)|d\tau < \infty, \text{ for all } x \in \mathbb{R}.$$

and Lemma 4.10 may be applied to give $e^{\eta x}g(x,\zeta) = 0$. That is,

$$f_+(x,\zeta) - \tilde{f}_+(x,\zeta) = g(x,\zeta) = 0 \text{ a.e.}$$

and hence uniqueness. ■

Having constructed $f_+(x,\zeta)$, the other Jost solution $f_-(x,\zeta)$ can easily be constructed, as follows. Let $F_+(x,\zeta)$ be the Jost solution as constructed before but for the equation

$$-F''(x) + q(-x)F(x) = \zeta^2 F(x).$$

Let

$$f_-(x,\zeta) := F_+(-x,\zeta),$$

then $f_-(x,\zeta)$ is a solution of equation (3.3) and obeys the asymptotics

$$|f_-(-x,\zeta) - e^{i\zeta x}| \leq C(x)\tilde{\rho}(x)\frac{e^{-\eta x}}{1+|\zeta|},$$

where $\tilde{\rho}(x)$ is as in (4.8). Hence

$$|f_-(x,\zeta) - e^{-i\zeta x}| \leq C(-x)\tilde{\rho}(x)\frac{e^{\eta x}}{1+|\zeta|}$$

for which it follows that f_- is as required.

4.3 The Conjugate Jost Solutions $\overline{f}_+$ and $\overline{f}_-$

Let us now assume that the second argument in $f_+(x,\zeta)$ is the real number $\zeta = \xi$. Then $\overline{f}_+(x,\xi)$, the complex conjugate of f_+ obeys equation (3.3) as $q(x)$ is real.

Taking the conjugate of

$$f_+(x,\xi) = e^{i\xi x} - \int_x^\infty \frac{\sin(\xi(x-\tau))}{\xi} q(\tau) f_+(\tau,\xi)d\tau, \tag{4.19}$$

gives

$$\overline{f}_+(x,\xi) = e^{-i\xi x} - \int_x^\infty \frac{\sin(-\xi(x-\tau))}{(-\xi)} q(\tau) \overline{f}_+(\tau,\xi)d\tau.$$

From the above two expressions and the uniqueness of the solution of (4.19), we may conclude that

$$\overline{f}_+(x,\xi) = f_+(x,-\xi), \quad \xi \in \mathbb{R}. \tag{4.20}$$

The asymptotic results earlier in this chapter obviously hold for the conjugate solution $\overline{f}_+(x,\xi)$ but with simplification as $\eta = 0$ in this case, in particular

$$\overline{f}_+(x,\xi) = e^{-i\xi x} + O\left(\frac{C(x)\rho(x)}{1+|\xi|}\right), \tag{4.21}$$

$$\frac{d\overline{f}_+}{dx}(x,\xi) = -i\xi e^{-i\xi x} + \int_x^\infty \cos(\xi(x-\tau))q(\tau)e^{-i\xi\tau}d\tau + O\left(\frac{C(x)\rho(x)\sigma(x)}{1+|\xi|}\right) \tag{4.22}$$

where $C(x)$ is non-increasing and ρ and σ are as defined earlier.

Theorem 4.12 *Let the Wronskian of f_+ and $\overline{f}_+$ be $W[f_+,\overline{f}_+]$ then for $\xi \in \mathbb{R}$ and $x \in \mathbb{R}$,*

$$W[f_+,\overline{f}_+](x,\xi) = -2i\xi.$$

Proof: Recall that

$$W[f_+,\overline{f}_+] = f_+\overline{f}'_+ - f'_+\overline{f}_+.$$

For $\xi \in \mathbb{R}$ and large $x \in \mathbb{R}$, from Theorem 4.5, Lemma 4.7, (4.21) and (4.22) it follows that,

$$f_+\overline{f}'_+(x,\zeta) = -i\xi e^{-i\xi x}e^{i\xi x} - e^{i\xi x}\int_x^\infty \cos(\xi(x-\tau))q(\tau)e^{i\xi\tau}d\tau + o(1),$$

$$f'_+\overline{f}_+(x,\zeta) = i\xi e^{i\xi x}e^{-i\xi x} - e^{-i\xi x}\int_x^\infty \cos(\xi(x-\tau))q(\tau)e^{i\xi\tau}d\tau + o(1).$$

Since $f_+(x,\zeta)$ and $\overline{f}_+(x,\zeta)$ both solve (3.3) with $\zeta = \xi$, their Wronskian is independent of x, so letting $x \to \infty$ gives,

$$f_+\overline{f}'_+ - f'_+\overline{f}_+ = -2i\xi - \lim_{x\to\infty}\left[(e^{i\xi x} - e^{-i\xi x})\int_x^\infty \cos(\xi(x-\tau))q(\tau)e^{i\xi\tau}d\tau + o(1)\right].$$

Now since $q(x)$ is in L^1,

$$\int_x^\infty \cos(\xi(x-\tau))q(\tau)e^{i\xi\tau}d\tau \to 0 \text{ as } x \to \infty.$$

Thus $W[f_+, \overline{f}_+] = -2i\xi$. ■

By construction, $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are solutions of (1.1). For $\zeta = \xi \in \mathbb{R}$, $\overline{f}_+(x,\zeta)$ is also a solution of (1.1) and from Theorem 4.12, $f_+(x,\xi)$ and $\overline{f}_+(x,\xi)$ are linearly independent for $\xi \neq 0$. Since the solution space of (1.1) is 2 dimensional, there exist coefficients $a(\xi)$ and $b(\xi)$ such that,

$$f_-(x,\xi) = a(\xi)\overline{f}_+(x,\xi) + b(\xi)f_+(x,\xi), \text{ for all } x \in \mathbb{R}. \tag{4.23}$$

Chapter 5

Location of Eigenvalues

Since the differential operator, L, considered is self-adjoint, see Chapter 3, the eigenvalues of L are real. The more precise location of the eigenvalues in special cases will now be investigated, i.e. whether in these cases the eigenvalues are all negative or all positive.

5.1 Properties of Eigenvalues

We now give the well-known proof that since L is self-adjoint, the point spectrum of L is contained in $\mathbb{R}$. We will also show that the continuous spectrum is contained in $\mathbb{R}$ and that the residual spectrum is empty.

Theorem 5.1 *Let L be the differential operator, $L = -\frac{d^2}{dx^2} + q(x)$, with domain $D(L)$ as defined in Section 3.1, then all eigenvalues of L are real.*

Proof: Let λ be an eigenvalue of L with corresponding normalised eigenfunction φ, then from the self-adjointness of L,

$$\begin{aligned} \lambda &= \lambda\langle\varphi,\varphi\rangle = \langle\lambda\varphi,\varphi\rangle = \langle L\varphi,\varphi\rangle \\ &= \langle\varphi,L\varphi\rangle = \langle\varphi,\lambda\varphi\rangle = \overline{\lambda}\langle\varphi,\varphi\rangle = \overline{\lambda}. \quad \blacksquare \end{aligned}$$

Theorem 5.2 *All eigenvalues of L as defined in (3.1)-(3.3) are negative.*

Proof: From Chapter 4, for $\zeta \neq 0$, f_+ and $\overline{f}_+$ constitute an independent set of solutions to equation (1.1). If L has a positive eigenvalue $\lambda = \zeta^2 > 0$ where $\zeta > 0$, then L has an eigenfunction of the form

$$F(x,\zeta) = c_1 f_+(x,\zeta) + c_2\overline{f}_+(x,\zeta).$$

Theorem 4.5 gives the asymptotic approximation

$$\begin{aligned} f_+(x,\zeta) &= e^{i\zeta x} + O\left(\frac{C(x)\rho(x)e^{-\eta x}}{1+|\zeta|}\right), \\ \overline{f}_+(x,\zeta) &= e^{-i\zeta x} + O\left(\frac{C(x)\rho(x)e^{\eta x}}{1+|\zeta|}\right). \end{aligned}$$

for $|\zeta| + x$ large, $\zeta \in \mathbb{C}$, $x \in \mathbb{R}$, where $\eta = \Im(\zeta)$. Hence

$$F(x,\zeta) \notin L^2(-\infty,\infty),$$

if $|c_1| + |c_2| \neq 0$. Hence L has no positive eigenvalues.

For $\zeta = 0$, consider $f_+(x,\zeta)$ as $\zeta \to 0^+$. From the definition of f_+,

$$f_+(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau) f_+(\tau,\zeta)dx.$$

Using Theorem 4.5 the above expression for f_+ becomes

$$f_+(x,\zeta) = e^{i\zeta x} - \int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)\left(e^{i\zeta\tau} + O\left(\frac{C(\tau)\rho(\tau)e^{-\eta\tau}}{1+|\zeta|}\right)\right)dx,$$

for all $x \in \mathbb{R}$, $\zeta \in \mathbb{C}\setminus\{0\}$. Therefore, in the limit as $\zeta \to 0^+$ we have

$$\lim_{\zeta\to 0^+} f_+(x,\zeta) = 1 - \lim_{\zeta\to 0^+}\int_x^\infty \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)\left(e^{i\zeta\tau} + O\left(\frac{C(\tau)\rho(\tau)e^{-\eta\tau}}{1+|\zeta|}\right)\right)dx.$$

Since $q(x) \in L^1(\mathbb{R})$,

$$\begin{aligned}\lim_{\zeta\to 0^+} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)\left(e^{i\zeta\tau} + O\left(\frac{C(\tau)\rho(\tau)}{1+|\zeta|}\right)\right) &= (x-\tau)q(\tau)(1+O(C(\tau)\rho(\tau)) \\ &\in L^1(x,\infty),\end{aligned}$$

and

$$\left|\frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)\left(e^{i\zeta\tau} + O\left(\frac{C(\tau)\rho(\tau)}{1+|\zeta|}\right)\right)\right| \le K|q(\tau)| \in L^1(x,\infty) \text{ for } 0 < \zeta < 1,$$

where $K > 0$ is a constant. Hence Lebesgue's dominated convergence theorem can be applied, allowing the limit to be taken through the integral, yielding

$$\begin{aligned}\lim_{\zeta\to 0^+} f_+(x,\zeta) &= 1 - \int_x^\infty \lim_{|\zeta|\to 0} \frac{\sin(\zeta(x-\tau))}{\zeta} q(\tau)\left(e^{i\zeta\tau} + O\left(\frac{C(\tau)\rho(\tau)e^{-\eta\tau}}{1+|\zeta|}\right)\right) dx \\ &= 1 - \int_x^\infty (x-\tau)q(\tau)(1+O(C(\tau)\rho(\tau))dx.\end{aligned}$$

Since $\int_{-\infty}^\infty (1+|x|)|q(x)|dx < \infty$, $f_+(x,0) := \lim_{\zeta\to 0^+} f_+(x,\zeta)$ exists and is a Jost solution with $\lim_{x\to\infty}\lim_{\zeta\to 0^+} f_+(x,\zeta) = 1$.

Hence $f_+(x,0) \notin L^2(-\infty,\infty)$. Observe that $g(x) := f_+(x,0)\int_c^x \frac{d\tau}{(f_+(\tau,0))^2}$ is a solution of (1.1) which is asymptotic to x for $x \to \infty$, [16, p.173], and therefore independent of $f_+(x,0)$. But $af_+(x,0) + bg(x)$ is asymptotic to $a + bx$ as $x \to \infty$, and thus not in $L^2(-\infty,\infty)$ unless $|a| + |b| = 0$. Hence $\zeta^2 = 0$ is not an eigenvalue of L. ■

5.2 Continuous Spectrum

This section contains results whcih culminate in a characterisation of the continuous spectrum of L.

Definition 5.3 *[34, p.54] A function $f : (a,b) \to \mathbb{C}$ is said to 'lie left in $L^2(a,b)$' if for every $c \in (a,b)$ we have that $f \in L^2(a,c)$, and is said to 'lie right in $L^2(a,b)$' if for every $c \in (a,b)$, $f \in L^2(c,b)$.*

Definition 5.4 *Let $L : H \to H$ be a densely defined linear operator. Let*

$$\Gamma(L) := \{z \in \mathbb{C} \mid \exists\ k(z) > 0 \text{ such that } \|(zI - L)f\| \geq k(z)\|f\| \quad \text{for all} \;\; f \in D(L)\},$$

then $\Gamma(L)$ is known as the regularity domain of L.

Definition 5.5 *Let $L : H \to H$ be a densely defined linear operator. We call $\mathbb{C}\backslash\Gamma(L)$ the spectral kernel of L and denote it by $S(L)$.*

Lemma 5.6 *Let $L : H \to H$ be a closed densely defined symmetric linear operator, then $S(L) \subset \sigma(L)$. If, in addition, L is self-adjoint, then $S(L) = \sigma(L)$. In either case, $S(L) \subset \mathbb{R}$.*

Proof: Observe that, the open upper and lower half-planes are contained in $\Gamma(L)$. To see this let $z \in \mathbb{C} \setminus \mathbb{R}$, then for each $f \in D(L)$,

$$\begin{aligned} \|(zI - L)f\|^2 &= \|(\Re(z)I - L)f\|^2 + |\Im(z)|^2\|f\|^2 \\ &\geq |\Im(z)|^2\|f\|^2, \end{aligned}$$

showing that $z \in \Gamma(L)$, i.e. $z \in \mathbb{C} \setminus S(L)$. Let $\lambda \in S(L)$ then for each $n \in \mathbb{N}$, there exists $f_n \in D(L)$ with

$$\|(L - \lambda I)f_n\| \leq \frac{1}{n}\|f_n\|.$$

Thus, if $(L - \lambda I)^{-1}$ exists and it is not continuous by the closed graph theorem nor is it everywhere defined on H. Hence $S(L) \subset \sigma(L)$.

If L is self-adjoint, then $\lambda \in \rho(L)$, the resolvent set of L, if and only if there exists a constant $c > 0$ such that $\|(L - \lambda I)f\| \geq c\|f\|$ for all $f \in D(L)$, [17, p.167], making, $\Gamma(L) = \rho(L)$ and

$$\sigma(L) = \mathbb{C} \setminus \rho(L) = \mathbb{C} \setminus \Gamma(L) = S(L). \quad \blacksquare$$

Definition 5.7 *Let M be a subspace of a vector space V. A set of linearly independent vectors $\{x_1, \ldots, x_n\}$ is said to be linearly independent modulo M if for all $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}$ the linear combination $\sum_{i=1}^n c_i x_i \notin M$. In this case, the set of vectors $\{x_1, \ldots, x_n\}$ is said to have dimension n modulo M.*

The next definition concerns the minimal and maximal operators generated by a formal differential operator.

Definition 5.8 *[34, p.41]* **Maximal operator.** *Given a differential expression τ of order p defined on an interval (a,b), $-\infty \leq a < b \leq \infty$, the maximal operator L generated by τ is defined by*

$$\begin{aligned} D(L) &:= \{f \in L^2(a,b) | f^{(0)}, f^{(1)}, \ldots, f^{(p-1)} \in AC(a,b), \tau f \in L^2(a,b)\}, \\ Lf &:= \tau f \quad \text{for } f \in D(L). \end{aligned}$$

The maximal operator L generated by τ is the operator with the largest domain in $L^2(a,b)$ which is mapped into $L^2(a,b)$. If, in addition τ is formally self-adjoint, i.e. $\langle \tau f, g\rangle = \langle f, \tau g\rangle$ for all $f, g \in C_0^\infty(a,b)$ then the minimal operator, L_0, is the closure of $\tau|_{C_0^\infty(a,b)}$, which is also the adjoint of the maximal operator, i.e. $L_0 = L^*$.

The following result, from Weidman [34, p.162], is useful for characterising the continuous spectrum of our operator L.

Theorem 5.9 *Let L and L_0 be the maximal and minimal operators, as defined above, and suppose L_0 to have equal deficiency indices (n,n). Let $c \in (a,b)$, $\lambda \in \mathbb{R}$, and set*

$$n_{a,\lambda} := dimN((L-\lambda I)|_{D(L)\cap L^2(a,c)})$$

and

$$n_{b,\lambda} := dimN((L-\lambda I)|_{D(L)\cap L^2(c,b)}).$$

Here is should be noted that $n_{a,\lambda}$ and $n_{b,\lambda}$ are independent of c. If $\lambda \in \mathbb{R}$ and

$$n_{a,\lambda} + n_{b,\lambda} < pn + defect(L_0),$$

then $\lambda \in \sigma(A)$ for every self-adjoint extension A of L_0.

Proof: Let $\lambda \notin \sigma(A)$ where A is a self-adjoint extension of L_0. Then, by Lemma 5.6, $\lambda \in \Gamma(A)$. But since L_0 is a restriction of A, $\Gamma(A) \subset \Gamma(L_0)$, making $\lambda \in \Gamma(L_0)$. Define the operator $L_{c,0}$ to be the restriction of L_0 with the domain

$$D(L_{c,0}) = \{f \in D(L_0) \mid \quad f^{(j)}(c) = 0,\ j = 0, \ldots, p-1\}.$$

Then, $L_{c,0}$ being a restriction of L_0, we have $\lambda \in \Gamma(L_{c,0})$, giving that $\lambda I - L_{c,0}$ and $\lambda I - L_0$ are one-to-one maps.

Let $\{(\alpha_0^l, \alpha_1^l, \ldots, \alpha_{p-1}^l) \in \mathbb{C}^{n\times p} \mid l = 1, \ldots, pn\}$ be a basis of $\mathbb{C}^{n\times p}$ and consider the interval (α, β) where $a < \alpha < c < \beta < b$. By Naimark, Lemma 2 in Section 17.3 [28], for each $l = 1, \ldots, pn$, functions $u_l \in D(L_0)$ can be constructed such that

$$\begin{aligned} u_l^{(j)}(c) &= \alpha_j^l, \quad j = 0, \ldots, p-1, \\ u_l(x) &= 0, \quad x \in (a, \alpha] \cup [\beta, b). \end{aligned}$$

Hence $D(L_0) = D(L_{c,0}) \oplus span\{u_1, \ldots, u_{pn}\}$. Since the set of functions $\{u_1, \ldots, u_{pn}\}$ are linearly independent modulo $D(L_{c,0})$, we have

$$dim D(L_0)/D(L_{c,0}) = pn.$$

Since $\lambda I - l_0$ is one-to-one

$$R(\lambda I - L_0) = R(\lambda I - L_{c,0}) \oplus (\lambda I - L_0)\mathrm{span}\{u_1, \ldots, u_{pn}\}.$$

Thus $dim\,(R(\lambda I - L_0)/R(\lambda I - L_{c,0})) = pn$ giving

$$\mathrm{defect}(L_{c,0}) = \mathrm{defect}(L_0) + pn.$$

Let $L_{a,c}$ and $L_{c,b}$ be the maximal operators generated by our formal differential operator τ in $L^2(a, c)$ and $L^2(c, b)$ respectively. Then $n_{a,\lambda} = dimN(\lambda I - L_{a,c})$ and $n_{b,\lambda} = dimN(\lambda I - L_{b,c})$. It can also be verified directly that the maximal operator, $L_{c,0}^*$, is the direct sum of operators $L_{a,c}$ and $L_{b,c}$, i.e.

$$L_{c,0}^* = \begin{pmatrix} L_{a,c} & 0 \\ 0 & L_{c,b} \end{pmatrix},$$

where we have identified $L^2(a,b)$ with $L^2(a,c) \oplus L^2(c,b)$. Since $R(\overline{L})^{\perp} = N(L^*)$, [17, p.172], we have

$$\begin{aligned}
\text{defect}(L_{c,0}) &= dimR(\lambda I - L_{c,0})^{\perp} \\
&= dimN(\lambda I - L_{c,0}^{*}) \\
&= dimN(\lambda I - L_{a,c}) + dimN(\lambda I - L_{c,b}) \\
&= n_{a,\lambda} + n_{b,\lambda}.
\end{aligned}$$

Thus $n_{a,\lambda} + n_{b,\lambda} = \text{defect}(L_0) + pn$, contrary to the assumption of the theorem, and hence L is self-adjoint. ■

Thus we have seen that $[0,\infty) \subset \sigma(L)$ and, since for each $\lambda \in [0,\infty)$ and $a, b \in \mathbb{R}$ that $n_{a,\lambda} = 0 = n_{b,\lambda}$, that L is self-adjoint.

Chapter 6

Classical Scattering Problems

Scattering problems arise from various experiments in physics, particularly in quantum particle physics. Often the objects under study are on a microscopic scale and scattering experiments provide the only viable data, see [30]. Waves, or particles, are projected at an object, the reflected and transmitted waves or particles, are observed giving the scattering amplitude and phase shift which we call the scattering data.

Scattering is described by the Schrödinger wave equation, which, in one spatial dimension, after separation of variables, can be studied via the Sturm-Liouville equation, as seen in Chapter 1. Taking into account various boundary conditions, this can be formulated as a Sturm-Liouville boundary value problem. The presentation of the classical scattering problem is based largely on [16].

6.1 Preliminaries

As we saw in Chapter 4, if the potential function satisfies the integration condition (1.3), there exist two solutions, which we denote by f_+ and f_-, of equation (3.3) on the line.

Let the second parameter of the Jost solution be the real number ξ, i.e. the spectral parameter $\lambda = \xi^2 \geq 0, \xi \in \mathbb{R}$. We recall from (4.20) that

$$\overline{f}_+(x,\xi) = f_+(x,-\xi), \tag{6.1}$$

and, similarly,

$$\overline{f}_-(x,\xi) = f_-(x,-\xi), \quad \xi \in \mathbb{R}. \tag{6.2}$$

By Theorem 4.12, for $\xi \neq 0$, $f_+(x,\xi)$ and $\overline{f}_+(x,\xi)$ are linearly independent solutions of equation (3.1)-(3.3) and thus span the two-dimensional solution space of (3.1)-(3.3). Hence, since $f_-(x,\xi)$ is also a solution of (3.1)-(3.3), we have that for each $\xi \neq 0,\ \xi \in \mathbb{R}$, there are constants $a(\xi)$ and $b(\xi)$ for which

$$f_-(x,\xi) = a(\xi)\overline{f}_+(x,\xi) + b(\xi)f_+(x,\xi), \text{ for all } x \in \mathbb{R}. \tag{6.3}$$

Computing the Wronskian of f_+ and f_- gives,

$$\begin{aligned} W[f_+, f_-] &= W[f_+, a(\xi)\overline{f}_+] + W[f_+, b(\xi)f_+] \\ &= a(\xi)W[f_+, \overline{f}_+], \end{aligned}$$

but, from Theorem 4.12,

$$W[f_+(x,\xi), \overline{f}_+(x,\xi)] = W[f_+(x,\xi), f_+(x,-\xi)] = -2i\xi$$

and thus

$$W[f_+(x,\xi), f_-(x,\xi)] = -2i\xi a(\xi). \tag{6.4}$$

Since $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are analytic for $\Im(\zeta) \geq 0$, we can use (6.4) to define $a(\zeta)$ analytically for $\Im(\zeta) \geq 0$. Hence,

$$a(\zeta) = \frac{i}{2\zeta} W[f_+(x,\zeta), f_-(x,\zeta)], \text{ for all } \zeta \text{ with } \Im(\zeta) \geq 0. \tag{6.5}$$

From the definition of $a(\zeta)$ and $b(\zeta)$,

$$f_-(x,\zeta) = a(\zeta)f_+(x,-\zeta) + b(\zeta)f_+(x,\zeta), \text{ for all } \zeta \text{ with } \Im(\zeta) \geq 0.$$

For $\Im(\zeta) \geq 0$ by analytic continuation,

$$W[f_+(x,\zeta), f_+(x,-\zeta)] = -2i\zeta.$$

Hence $a(\zeta)$ can be written as,

$$a(\zeta) = \frac{i}{2\zeta} W[f_+(x,\zeta), f_-(x,\zeta)], \quad \text{for } \Im(\zeta) \geq 0, \zeta \neq 0. \tag{6.6}$$

Similarly, we have for $b(\xi)$,

$$b(\xi) = \frac{-i}{2\xi} W[f_+(x,-\xi), f_-(x,\xi)], \quad \xi \in \mathbb{R}, \tag{6.7}$$

which extends, by analyticity, to $\xi = \zeta \in \mathbb{C}$ with $\Im(\zeta) \geq 0$. We now give the form of $a(\zeta)$ and $b(\xi)$ for large values of ζ and ξ respectively.

Lemma 6.1 *For $\zeta \in \mathbb{C}$, with $\Im(\zeta) \geq 0$,*

$$a(\zeta) = 1 + O\left(\frac{1}{\zeta}\right) \text{ as } |\zeta| \to \infty.$$

Proof: In order to prove this lemma we use the asymptotics for the Jost solutions from Chapter 4. By (6.6) we have

$$a(\zeta) = \frac{i}{2\zeta}(f_+ f_-' - f_+' f_-)(x,\zeta).$$

Recall from Theorem 4.5 and Lemma 4.7, we have

$$\begin{aligned} f_+(x,\zeta) &= e^{i\zeta x} + O\left(\frac{C(x)\rho(x)e^{-\eta x}}{1+|\zeta|}\right), \\ f_+'(x,\zeta) &= i\zeta e^{i\zeta x} - \int_x^\infty \cos(\zeta(x-\tau))q(\tau)e^{i\zeta\tau}d\tau + O\left(\frac{C(x)\rho(x)\sigma(x)e^{-\eta x}}{1+|\zeta|}\right), \end{aligned}$$

and, similarly,

$$\begin{aligned} f_-(x,\zeta) &= e^{-i\zeta x} + O\left(\frac{C(-x)\tilde{\rho}(x)e^{\eta x}}{1+|\zeta|}\right), \\ f_-'(x,\zeta) &= -i\zeta e^{-i\zeta x} - \int_{-\infty}^x \cos(\zeta(x-\tau))q(\tau)e^{-i\zeta\tau}d\tau + O\left(\frac{C(-x)\tilde{\rho}(x)\sigma(x)e^{\eta x}}{1+|\zeta|}\right). \end{aligned}$$

Since the Wronskian of $f_+(x,\zeta)$ and $f_-(x,\zeta)$ is constant, in order to evaluate it, we may take $x = 0$. From the above asymptotics,

$$f_+(0,\zeta) = 1 + O\left(\frac{C(0)\rho(0)}{1+|\zeta|}\right),$$

where $C(0)$ and

$$\rho(0) = \int_0^\infty (1+|\tau|)|q(\tau)|d\tau$$

are finite real numbers independent of ζ. Hence there exists $K_1 \in \mathbb{R}^+$ so that

$$|f_+(0,\zeta) - 1| \leq \frac{K_1}{1+|\zeta|},$$

for $|\zeta|$ large. The derivative of f_+ at $x=0$ takes the form

$$f_+'(0,\zeta) = i\zeta - \int_0^\infty \cos(\zeta\tau)q(\tau)e^{i\zeta\tau}d\tau + O\left(\frac{C(0)\rho(0)\sigma(0)}{1+|\zeta|}\right),$$

for $|\zeta|$ large, where $\sigma(0)$ is a real number independent of ζ. Observe that

$$\begin{aligned} |\cos(\zeta\tau)e^{i\zeta\tau}| &= \left|\frac{e^{i\zeta\tau}+e^{-i\zeta\tau}}{2}e^{i\zeta\tau}\right| \\ &= \left|\frac{e^{2i\zeta\tau}+1}{2}\right| \\ &\leq \frac{1}{2}(e^{-2\eta\tau}+1), \end{aligned}$$

where $\eta = \Im(\zeta)$ and $\eta \geq 0$. Hence

$$\begin{aligned} \left|\int_0^\infty \cos(\zeta\tau)q(\tau)e^{i\zeta\tau}d\tau\right| &\leq \int_0^\infty \frac{1}{2}(e^{-2\eta\tau}+1)|q(\tau)|d\tau \\ &\leq \int_0^\infty |q(\tau)|d\tau, \end{aligned}$$

independent of ζ and we may thus write $f_+'(0,\zeta)$ as

$$f_+'(0,\zeta) = i\zeta + O(1),$$

asymptotically for large $|\zeta|$. Similarly it can be shown that

$$\begin{aligned} f_-(0,\zeta) &= 1 + O\left(\frac{1}{1+|\zeta|}\right), \\ f_-'(0,\zeta) &= -i\zeta + O(1), \end{aligned}$$

for $|\zeta|$ large with $\Im(\zeta) \geq 0$. The Wronskian now becomes

$$f_+f_-' - f_+'f_- = -2i\zeta + O(1),$$

for $|\zeta|$ large with $\Im(\zeta) \geq 0$. Since $a(\zeta) = \frac{-1}{2i\zeta}W[f_+, f_-]$ it follows that

$$a(\zeta) = 1 + O\left(\frac{1}{\zeta}\right). \quad \blacksquare$$

By a similar calculation, for $\xi \in \mathbb{R}$, it can be shown that

$$b(\xi) = O\left(\frac{1}{\xi}\right)$$

for $|\xi|$ large. We now relate the complex conjugates of $a(\xi)$ and $b(\xi)$ back to $a(\xi)$ and $b(\xi)$. Recall from (6.3) that, for $\xi \in \mathbb{R} \setminus \{0\}$,

$$f_-(x,\xi) = a(\xi)\overline{f}_+(x,\xi) + b(\xi)f_+(x,\xi),$$

and thus

$$\overline{f}_-(x,\xi) = \overline{a}(\xi)f_+(x,\xi) + \overline{b}(\xi)\overline{f}_+(x,\xi). \tag{6.8}$$

Hence, from (6.1), (6.2),(6.3) and (6.8) we obtain

$$\begin{aligned} f_-(x,-\xi) &= a(-\xi)\overline{f}_+(x,-\xi) + b(-\xi)f_+(x,-\xi) \\ &= a(-\xi)f_+(x,\xi) + b(-\xi)\overline{f}_+(x,\xi). \end{aligned}$$

Since $f_+(x,\xi)$ and $\overline{f}_+(x,\xi)$ are linearly independent for $\xi \in \mathbb{R} \setminus \{0\}$, comparing the above expression for $f_-(x,\xi)$ and (6.8) gives

$$\overline{a}(\xi) = a(-\xi) \text{ and } \overline{b}(\xi) = b(-\xi). \tag{6.9}$$

We now relate the squares of the magnitudes of the coefficients $a(\xi)$ and $b(\xi)$. This will be seen to be consistent with the asymptotics given in the previous lemma.

Lemma 6.2 *For $\xi \in \mathbb{R} \setminus \{0\}$, $a(\xi)$ and $b(\xi)$ satisfy the following equality*

$$|a(\xi)|^2 - |b(\xi)|^2 = 1.$$

Proof: We begin by obtaining an expression for the solution $f_+(x,\xi)$ in terms of the conjugate solutions $f_-(x,\xi)$ and $\overline{f}_-(x,\xi)$, for $\xi \in \mathbb{R} \setminus \{0\}$. Using the asymptotics for f_- and a computation similar to that given in Theorem 4.12 we obtain

$$\det \begin{bmatrix} \overline{f}_- & f_- \\ \overline{f}'_- & f'_- \end{bmatrix} = W[\overline{f}_-, f_-] = -2i\xi.$$

Thus, $\overline{f}_-(x,\xi)$ and $f_-(x,\xi)$ are linearly independent for $\xi \in \mathbb{R}\setminus\{0\}$ and consequently

$$f_+(x,\xi) = C_1(\xi)\overline{f}_-(x,\xi) + C_2(\xi)f_-(x,\xi), \tag{6.10}$$

where $C_1(\xi), C_2(\xi)$ are independent of x. Equation (6.10) and its x-derivative give the matrix equation

$$\begin{bmatrix} f_+ \\ f_+' \end{bmatrix} = \begin{bmatrix} \overline{f}_- & f_- \\ \overline{f}_-' & f_-' \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

which has solution

$$\begin{aligned} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} \overline{f}_- & f_- \\ \overline{f}_-' & f_-' \end{bmatrix}^{-1} \begin{bmatrix} f_+ \\ f_+' \end{bmatrix} \\ &= \frac{i}{2\xi} \begin{bmatrix} f_-' & -f_- \\ -\overline{f}_-' & \overline{f}_- \end{bmatrix} \begin{bmatrix} f_+ \\ f_+' \end{bmatrix}. \end{aligned}$$

Thus, from (6.6),

$$\begin{aligned} C_1(\xi) &= \frac{i}{2\xi}(f_+f_-' - f_-f_+') \\ &= \frac{i}{2\xi}W[f_+, f_-] \\ &= a(\xi), \end{aligned}$$

and, by (6.7),

$$\begin{aligned} C_2(\xi) &= \frac{i}{2\xi}(\overline{f}_-f_+' - \overline{f}_-'f_+) \\ &= \frac{i}{2\xi}W[\overline{f}_-, f_+] \\ &= -\overline{b}(\xi). \end{aligned}$$

Combining the above expressions for $C_1(\xi)$ and $C_2(\xi)$ with (6.10) gives

$$f_+(x,\xi) = a(\xi)\overline{f}_-(x,\xi) - \overline{b}(\xi)f_-(x,\xi). \tag{6.11}$$

Substituting (6.11) into (4.23) gives

$$\begin{aligned} f_- &= a\overline{f}_+ + bf_+ \\ &= a(\overline{a}f_- - b\overline{f}_-) + b(a\overline{f}_- - \overline{b}f_-) \\ &= (|a|^2 - |b|^2)f_-. \end{aligned}$$

Now, since $f_-(x,\xi) \not\equiv 0$ for $\xi \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}$,

$$|a|^2 - |b|^2 = 1. \quad \blacksquare$$

By Theorem 5.2, all eigenvalues, ζ^2, for the scattering problem (3.1)-(3.3) on the line are negative. So each eigenvalue, ζ is of the form

$$\zeta = i\eta \qquad \text{for some } \eta \in \mathbb{R}.$$

The eigen-condition on solutions of (3.3) is that they be in $L^2(\mathbb{R})$. For $\zeta \in \mathbb{R}\setminus\{0\}$, ζ is not an eigenvalue, but the solution space of $-y'' + qy = \zeta^2 y$ is 2 dimensional but the space of solutions of (3.3) lying left in $L^2(\mathbb{R})$ for $\zeta \in \mathbb{R} \setminus \{0\}$ contains $f_-(x,\zeta)$. Similarly the space of solutions of (3.3) lying right in $L^2(\mathbb{R})$ contains $f_+(x,\zeta)$. Hence each of these spaces is 1 dimensional (as otherwise we would be at an eigenvalue). Thus for ζ^2 an eigenvalue, $f_-(x,\zeta) = kf_+(x,\zeta)$ for all $x \in \mathbb{R}$ and some $k \in \mathbb{C}$. Consequently, from (6.3), ζ^2 is an eigenvalue if and only if $a(\zeta) = 0$, and in this case

$$f_-(x,\zeta) = b(\zeta)f_+(x,\zeta).$$

Hence for each $\zeta = i\eta$, $\eta \in \mathbb{R} \setminus \{0\}$, where ζ^2 is an eigenvalue of (3.1)-(3.3), we have

$$a(i\eta) = 0. \tag{6.12}$$

Conversely, if $a(\zeta) = 0$, then $f_-(x,\zeta)$ and $f_+(x,\zeta)$ are linearly dependent making $f_+ \in L^2(\mathbb{R})$ and ζ^2 is an eigenvalue of (3.1)-(3.3). Thus $\zeta = i\eta$ for some $\eta \in \mathbb{R} \setminus \{0\}$.

Since $a(\zeta)$ is analytic for $\Im(\zeta) \geq 0$ and $a(\zeta) = 1 + O(1/|\zeta|)$, by the identity theorem, $a(\zeta)$ has only a finite number of zeros in the half plane $\mathbb{C}^+ = \{x+iy \mid x, y \in \mathbb{R}, y \geq 0\}$. Hence the scattering problem (3.1)-(3.3) has only a finite number of eigenvalues.

Let the eigenvalues of (3.1)-(3.3) be $\{\zeta_k^2 \mid k = 1, 2, \ldots, N\}$ where $\Im(\zeta_k) > 0$. Let

$$\begin{aligned} \frac{1}{c_k} &= \int_{-\infty}^{\infty} f_+(x, i\eta_k)^2 dx, \quad k = 1, \ldots, N, \\ r(\xi) &= \frac{b(\xi)}{a(\xi)}, \quad \xi \in \mathbb{R}. \end{aligned}$$

The set, $\{r(\xi),\ \xi \in \mathbb{R},\ \eta_k,\ c_k,\ k = 1, \ldots, N\}$ is called the *scattering data* for the potential $q(x)$ [16, p.175]. We assume $0 < \eta_1 < \eta_2 < \eta_3 < \cdots < \eta_N$. The function $r(\xi)$ is called the *reflection coefficient* for the potential $q(x)$ and $\frac{1}{c_k}$ is the norming constant for the eigenfunction $f_+(x, i\eta_k)$. The above definition of the norming constants is equivalent to $\frac{1}{c_k} = \int_{-\infty}^{\infty} |f_+(x, i\eta_k)|^2 dx$ since for ζ^2 an eigenvalue with $(i\eta_k)^2 = \zeta^2$, it is easily seen that $f_+(x, i\eta_k) \in \mathbb{R}$, for all $x \in \mathbb{R}$.

We next show that the eigenvalues of the scattering problem are simple.

Lemma 6.3 *The eigenvalues of the scattering problem, (3.1)-(3.3), are simple. The positive square roots of the eigenvalues, ζ^2, are the zeros of $a(\zeta)$ and the order of the zero of $a(\zeta)$ is the multiplicity of the eigenvalue ζ^2. In addition, if $\zeta^2 = -\eta_k^2$ is an eigenvalue of (3.1)-(3.3), then*

$$\frac{\partial a(\zeta)}{\partial \zeta}\Big|_{\zeta = i\eta_k} = -i\frac{d_k}{c_k}$$

where $f_-(x, i\eta_k) = d_k f_+(x, i\eta_k)$ for all $x \in \mathbb{R}$, and $d_k \neq 0$, $k = 1, \ldots, N$.

Proof: As shown in Chapter 3, (3.1)-(3.3) is a self-adjoint boundary value problem which is equivalent to the self-adjoint operator eigenvalue problem $Ly = \lambda y$, $y \in D(L)$, where L and $D(L)$ are as defined by (3.1)-(3.3). Thus, by Theorem 5.1, the eigenvalues are real and semisimple. The solution space of (3.1)-(3.3) is of dimension at most 2, giving the maximum geometric multiplicity of an eigenvalue as 2. From the asymptotic expansions of $\overline{f}_+(x, \xi) = f_+(x, -\xi)$, $\xi \in \mathbb{R} \setminus \{0\}$, given in Section 4.3, we see that $\overline{f}_+$ is a solution of (3.1)-(3.3) which, when restricted to $(0, \infty)$, is not in $L^2(0, \infty)$. Hence $f_+(x, -\xi)$ is not in $L^2(-\infty, \infty)$ giving the maximum geometric multiplicity of an eigenvalue as 1. If ζ^2 is an eigenvalue then the eigenspace must have dimension at least 1, and so the geometric multiplicity is exactly 1. Since the eigenvalues are semisimple, the geometric multiplicity of an eigenvalue is equal to its algebraic multiplicity. Thus the algebraic multiplicity of each eigenvalue is 1.

Next it is shown that the order of the zeros of $a(\zeta)$ when ζ^2 is an eigenvalue is 1.

From (6.5) we have

$$a(\zeta) = \frac{i}{2\zeta} W[f_+, f_-].$$

By (6.12), given that $-\eta_k^2 \neq 0$ is an eigenvalue of (3.1)-(3.3), we have that $a(i\eta_k) = 0$. It remains to be shown that $\frac{da(\zeta)}{d\zeta}|_{\zeta=i\eta_k} \neq 0$.

$$\frac{da}{d\zeta}(\zeta) = \frac{-i}{2\zeta^2} W[f_+, f_-] + \frac{i}{2\zeta} W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right](x,\zeta) + \frac{i}{2\zeta} W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right](x,\zeta).$$

Then evaluated at $\zeta = i\eta_k$, the above becomes

$$\frac{da}{d\zeta}(\zeta) = \frac{1}{2\eta_k} W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right](x, i\eta_k) + \frac{1}{2\eta_k} W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right](x, i\eta_k)$$

where the above expression is independent of x, but each of its terms may depend on x. As f_+ and f_- are solutions of

$$-f'' + qf = \zeta^2 f,$$

on differentiating the above equation by ζ, we obtain,

$$-\frac{\partial f_+''}{\partial \zeta} + q(x)\frac{\partial f_+}{\partial \zeta} = 2\zeta f_+ + \zeta^2 \frac{\partial f_+}{\partial \zeta},$$

and

$$-\frac{\partial f_-''}{\partial \zeta} + q(x)\frac{\partial f_-}{\partial \zeta} = 2\zeta f_- + \zeta^2 \frac{\partial f_-}{\partial \zeta}.$$

Thus

$$\begin{aligned}
\frac{d}{dx} W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right] &= \frac{d}{dx}\left[\frac{\partial f_+}{\partial \zeta} f_-' - \frac{\partial f_+'}{\partial \zeta} f_-\right] \\
&= \frac{\partial f_+'}{\partial \zeta} f_-' + \frac{\partial f_+}{\partial \zeta} f_-'' - \frac{\partial f_+''}{\partial \zeta} f_- - \frac{\partial f_+'}{\partial \zeta} f_-' \\
&= \frac{\partial f_+}{\partial \zeta}(q(x) f_- - \zeta^2 f_-) + f_-\left(2\zeta f_+ + \zeta^2 \frac{\partial f_+}{\partial \zeta} - q(x)\frac{\partial f_+}{\partial \zeta}\right),
\end{aligned}$$

giving

$$\frac{d}{dx} W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right] = 2\zeta f_+ f_-. \tag{6.13}$$

Similarly

$$\frac{d}{dx} W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right] = -2\zeta f_+ f_-. \tag{6.14}$$

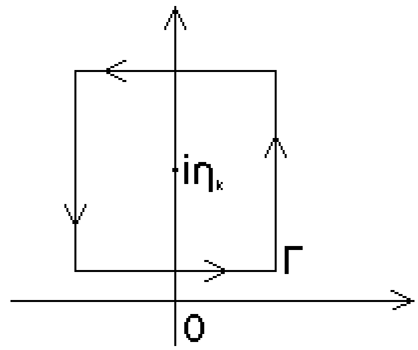

Figure 6.1: The square contour with centre $i\eta_k$.

Integrating (6.13) and (6.14) over the intervals $[x, A]$ and $[-A, x]$ respectively gives

$$W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right](A) - W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right](x) = 2\zeta \int_x^A f_+ f_- dx, \tag{6.15}$$

$$W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right](x) - W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right](-A) = -2\zeta \int_{-A}^x f_+ f_- dx. \tag{6.16}$$

It is next shown that for $\zeta = i\eta_k$, $k = 1, \ldots, N$, $\frac{\partial f_+}{\partial \zeta}(x, i\eta_k)$ and its x-derivative are bounded functions. Let Γ be the positively oriented square with sides of length 2ϵ and centre $i\eta_k$, where $\min\{\eta_N - \eta_{N-1}, \ldots, \eta_2 - \eta_1, \eta_1\} > \epsilon > 0$, as shown in Figure 6.1.

Then by Cauchy's integral representation for analytic functions and their derivatives we have

$$\frac{\partial f_+}{\partial \zeta}(x, i\eta_k) = \frac{1}{2\pi i} \int_\Gamma \frac{f_+(x, z)}{(z - i\eta_k)^2} dz.$$

For $z \in \Gamma$, $|z - i\eta_k| \geq \frac{\epsilon}{2}$ giving $\frac{1}{|z - i\eta_k|^2} \leq \frac{4}{\epsilon^2}$. Thus

$$\begin{aligned}
\left|\frac{\partial f_+}{\partial \zeta}(x, i\eta_k)\right| &\leq \frac{1}{2\pi}\left|\int_\Gamma \frac{f_+(x, z)}{(z - i\eta_k)^2} dz\right| \\
&\leq \frac{1}{2\pi}(4\epsilon)\left(\frac{4}{\epsilon^2}\right) \max_{z \in \Gamma} |f_+(x, z)| \\
&\leq \frac{8}{\pi\epsilon} \max_{z \in \Gamma} |f_+(x, z)|.
\end{aligned}$$

Hence by Theorem 4.5,

$$\left|\frac{\partial f_+}{\partial \zeta}(x, i\eta_k)\right| \leq \frac{8}{\pi\epsilon} \max_{z \in \Gamma}\left(e^{-\Im(z)x} + \left|C(x)\rho(x)\frac{e^{-\Im(z)x}}{1 + |z|}\right|\right)$$

which is bounded on each interval of the form $[A, \infty)$. Similarly, as $f_+(x, \zeta)$ is analytic in the upper half plane and second differentiable with respect to x on $\mathbb{R}$, we have

$$\frac{d}{dx}\frac{\partial f_+}{\partial \zeta}(x, i\eta_k) = \frac{\partial f_+'}{\partial \zeta}(x, i\eta_k).$$

Proceeding as above, we obtain

$$\left|\frac{\partial f_+'}{\partial \zeta}(x, i\eta_k)\right| \leq \frac{8}{\pi\epsilon}\max_{z\in\Gamma}|f_+'(x, z)|,$$

which by Lemma 4.7, is bounded on each interval of the form $[A, \infty)$. In exactly the same way, it can be shown that $\frac{\partial f_-}{\partial \zeta}(x, i\eta_k)$ and $\frac{\partial f_-'}{\partial \zeta}(x, i\eta_k)$ are bounded functions on each interval of the form $(-\infty, -A]$.

Letting $\zeta = i\eta_k$ and $A \to \infty$ in (6.15), (6.16), gives

$$\begin{aligned} W\left[\frac{\partial f_+}{\partial \zeta}, f_-\right](x) &= -2\zeta\int_x^\infty f_+f_-dx, \\ W\left[f_+, \frac{\partial f_-}{\partial \zeta}\right](x) &= -2\zeta\int_{-\infty}^x f_+f_-dx. \end{aligned}$$

Since $\zeta^2 = -\eta_k^2$ is an eigenvalue and the eigenspace is 1 dimensional, the Jost solutions must be dependent, that is, there must exist a constant, $d_k \neq 0$ such that

$$f_-(x, i\eta_k) = d_k f_+(x, i\eta_k), \text{ for all } x \in \mathbb{R}.$$

Hence

$$\begin{aligned} \frac{da}{d\zeta}(i\eta_k) &= \frac{1}{2\eta_k}\left(-2i\eta_k\int_x^\infty f_+f_-|_{\zeta=i\eta_k}dx\right) + \frac{1}{2\eta_k}\left(-2i\eta_k\int_{-\infty}^x f_+f_-|_{\zeta=i\eta_k}dx\right) \\ &= -i\int_{-\infty}^\infty f_+f_-|_{\zeta=i\eta_k}dx \\ &= -id_k\int_{-\infty}^\infty f_+^2(x, i\eta_k)dx \qquad (6.17) \\ &= -i\frac{d_k}{c_k} \\ &\neq 0. \end{aligned}$$

Hence, the zeros of $a(\zeta)$ are simple. ■

So, for the classical scattering problem, we have a finite set of simple eigenvalues and the zeros of $a(\zeta)$, $\zeta \neq 0$ are simple and occur at the eigenvalues.

6.2 Scattering Data

As previously mentioned, the function $r(\xi) = \frac{b(\xi)}{a(\xi)}$, $\xi \in \mathbb{R}$, along with the square roots of the eigenvalues $i\eta_j$ and their norming constants c_j, $j = 1, \ldots, N$, constitute the so called scattering data for (3.1)-(3.3). We now show that the function $a(\zeta)$ can be obtained from the scattering data, and hence the function $b(\xi)$ can be found. The first lemma gives a representation of analytic functions in the upper half-plane in terms of their values on the real axis, see Hsieh [16].

Lemma 6.4 *Let f be a function analytic in the upper half-plane and having*

1. *$f(\zeta)$ continuous for $\zeta \neq 0$ with $\Im(\zeta) \geq 0$;*
2. *$\zeta(f(\zeta) - 1)$ bounded on $\Im(\zeta) \geq 0$;*
3. *$f(\zeta) \neq 0$ for $\zeta \neq 0$ with $\Im(\zeta) \geq 0$;*
4. *$\zeta = 0$ as a first order pole.*

Let $F(\zeta) = \log f(\zeta)$, then

$$F(\zeta) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\Re(F(\xi))}{\xi - \zeta} d\xi,$$

for $\zeta \in \mathbb{C}$ with $\Im(\zeta) > 0$.

Proof: Consider the contour Γ_R consisting of the intervals, $I_{-R,-\epsilon}$,$I_{\epsilon,R}$, on the real axis from $-R$ to $-\epsilon$ and ϵ to R and the semi-circles, S_R and S_ϵ^-, in the upper-plane from R to $-R$ and from $-\epsilon$ to ϵ, as seen in Figure 6.2. Here we assume $0 < \epsilon < |\zeta|/2$. That is $\Gamma_R = I_{-R,-\epsilon} \cup S_\epsilon^- \cup I_{\epsilon,R} \cup S_R$.

By Cauchy's integral formula, for $|\zeta| < R$, we have

$$\begin{aligned} F(\zeta) &= \frac{1}{2\pi i} \int_{\Gamma_R} \frac{F(z)}{z - \zeta} dz \\ &= \frac{1}{2\pi i} \int_{-R}^{-\epsilon} \frac{F(\xi)}{\xi - \zeta} d\xi + \frac{1}{2\pi i} \int_{S_R} \frac{F(z)}{z - \zeta} dz + \frac{1}{2\pi i} \int_{-\epsilon}^{-R} \frac{F(\xi)}{\xi - \zeta} d\xi - \frac{1}{2\pi i} \int_{S_\epsilon} \frac{F(z)}{z - \zeta} dz. \end{aligned}$$

We now show that as $|R| \to \infty$ the integral of $\frac{F(z)}{z-\zeta}$ over S_R tends to zero.

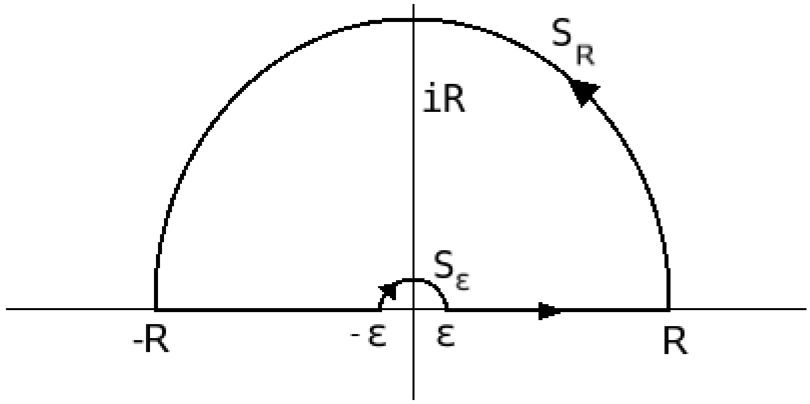

Figure 6.2: The contour

Since, by assumption, $z(f(z)-1)$ is bounded by a constant, say $K_1 \in \mathbb{R}^+$, from the triangle inequality it follows that for $|z| > 2K_1$,

$$1 - \frac{K_1}{|z|} \leq |f(z)| \leq \frac{K_1}{|z|} + 1.$$

Taking the logarithm of this expression gives that for $|z| > 2K_1$,

$$\frac{-2K_1}{|z|} \leq \log\left(1 - \frac{K_1}{|z|}\right) \leq \log|f(z)| \leq \log\left(\frac{K_1}{|z|} + 1\right) \leq \frac{2K_1}{|z|}.$$

Now since $|f(z) - 1| \leq \frac{K_1}{|z|}$, it follows that

$$\begin{aligned} |\arg f(z)| &\leq \arcsin\left(\frac{K_1}{|z|}\right) \\ &\leq \frac{2K_1}{|z|} \quad \text{for } |z| \text{ large.} \end{aligned}$$

Thus

$$|F(z)| \leq |\log|f(z)|| + |\arg f(z)| \leq \frac{4K_1}{|z|},$$

for $|z|$ large.

Hence

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_{S_R} \frac{F(z)}{z-\zeta} dz\right| &\leq \frac{2K_1}{R - |\zeta|} \\ &\rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

We show that $\frac{1}{2\pi i}\int_{S_\epsilon}\frac{F(z)}{z-\zeta}dz \to 0$ as $\epsilon \to 0^+$.

$$\begin{aligned}\left|\int_{S_\epsilon}\frac{F(z)}{z-\zeta}dz\right| &= \left|\int_0^\pi \frac{F(\epsilon e^{i\theta})}{\epsilon e^{i\theta}-\zeta} i\epsilon e^{i\theta}d\theta\right| \\ &\leq \frac{2\epsilon}{|\zeta|}\int_0^\pi |F(\epsilon e^{i\theta})|d\theta.\end{aligned}$$

By standard results for the complex logarithm function we have

$$\begin{aligned}\Re(F(z)) &= \log|f(z)|, \\ \Im(F(z)) &= \arg f(z).\end{aligned}$$

Since $\zeta = 0$ is a first order pole of f we have

$$f(z) = \frac{\alpha}{z} + \beta + O(z)$$

near $z = 0$, where $\alpha \neq 0$. In particular, if $z = \epsilon e^{i\theta}$, then

$$f(\epsilon e^{i\theta}) = \frac{\alpha}{\epsilon}e^{-i\theta} + \beta + O(\epsilon)$$

as $\epsilon \to 0^+$ and $\Im(\log f(\epsilon e^{i\theta})) = \arg(\alpha) - \theta + O(\epsilon)$ as $\epsilon \to 0^+$. Hence

$$\frac{2\epsilon}{|\zeta|}\int_0^\pi |\Im(F(\epsilon e^{i\theta}))|d\theta \to 0 \qquad \text{as } \epsilon \to 0^+.$$

Also $|f(\epsilon e^{i\theta})| \leq \frac{2|\alpha|}{\epsilon}$ for ϵ near 0, $(\epsilon > 0)$. Thus

$$\begin{aligned}0 &\leq \lim_{\epsilon\to 0^+}\frac{2\epsilon}{|\zeta|}\int_0^\pi |\Re(F(\epsilon e^{i\theta}))|d\theta \\ &\leq \lim_{\epsilon\to 0^+}\frac{2\epsilon}{|\zeta|}\int_0^\pi |\log 2|\alpha| - \log\epsilon|d\theta \\ &= \lim_{\epsilon\to 0^+} -\frac{2\epsilon}{|\zeta|}\pi\log\epsilon \\ &= 0 \quad \text{by l'Hôpital's rule.}\end{aligned}$$

Hence we have

$$F(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(\xi)}{\xi-\zeta}d\xi. \tag{6.18}$$

Since $\Im(\overline{\zeta}) \leq 0$, $\overline{\zeta}$ is not enclosed by the contour Γ and so, reasoning as above, we obtain

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(\xi)}{\xi-\overline{\zeta}}d\xi = 0. \tag{6.19}$$

Adding the conjugate of (6.19) to (6.18) gives

$$\begin{aligned} F(\zeta) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(\xi)}{\xi-\zeta}d\xi + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{F(\xi)}}{\xi-\zeta}d\xi \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2\Re(F(\xi))}{\xi-\zeta}d\xi. \quad \blacksquare \end{aligned}$$

Using Lemma 6.4 we can now give an expression for $a(\zeta)$ in terms of the scattering data.

Theorem 6.5 *In terms of the scattering data $\{r(\xi), \eta_1, \ldots, \eta_N\}$, the function $a(\zeta)$ is as follows*

$$a(\zeta) = \left[\prod_{j=1}^{N}\frac{\zeta - i\eta_j}{\zeta + i\eta_j}\right]\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1-|r(\xi)|^2)}{\zeta-\xi}d\xi\right].$$

Proof: Let

$$f(\zeta) = a(\zeta)\prod_{j=1}^{N}\frac{\zeta+i\eta_j}{\zeta-i\eta_j},$$

for $\zeta \in \mathbb{C}$, $\Im(\zeta) \geq 0$. By Lemma 6.1,

$$\begin{aligned} f(\zeta) &= a(\zeta)\prod_{j=1}^{N}\frac{\zeta+i\eta_j}{\zeta-i\eta_j} \\ &= a(\zeta)\prod_{j=1}^{N}\left(1+\frac{2i\eta_j}{\zeta-i\eta_j}\right) \\ &= 1 + O\left(\frac{1}{\zeta}\right), \end{aligned}$$

as $|\zeta| \to \infty$ with $\Im(\zeta) \geq 0$. Since the zeros of $a(\zeta)$ are simple and occur at $i\eta_j, j = 1, \ldots, N$, f is continuous and non-zero for $\Im(\zeta) \geq 0$, $\zeta \neq 0$, (see Lemma 6.3). From the definition of $a(\zeta)$ as given in (6.5), we see that $a(\zeta)$ has a first order pole at $\zeta = 0$. Hence we have that $\zeta(f(\zeta) - 1)$ is bounded and is analytic in the upper plane. Thus, $f(\zeta)$ satisfies the requirements of Lemma 6.4. Putting $F(\zeta) = \log f(\zeta)$, where the branch of $\log f(\zeta)$ chosen is that which has $F(\zeta) \to 0$ as $|\zeta| \to \infty$, we get

$$\begin{aligned} F(\zeta) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{2\Re(F(\xi))}{\xi - \zeta} d\xi \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{2\log|f(\xi)|}{\xi - \zeta} d\xi \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log|f(\xi)|^2}{\xi - \zeta} d\xi. \end{aligned}$$

Observe that for $\xi \in \mathbb{R}$, $|f(\xi)| = |a(\xi)|$, now Lemma 6.2 gives $|a(\xi)|^2 - |b(\xi)|^2 = 1$, for $\xi \in \mathbb{R}$, from which it follows that

$$\begin{aligned} \log|f(\xi)|^2 &= \log|a(\xi)|^2 \\ &= -\log\left(\frac{|a(\xi)|^2 - |b(\xi)|^2}{|a(\xi)|^2}\right) \\ &= -\log(1 - |r(\xi)|^2). \end{aligned}$$

Hence

$$\log f(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{-\log(1 - |r(\xi)|)^2}{\xi - \zeta} d\xi$$

and

$$f(\zeta) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 - |r(\xi)|^2}{\zeta - \xi} d\xi\right],$$

which, with the definition of $f(\zeta)$ gives

$$a(\zeta) = \left[\prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}\right] \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 - |r(\xi)|^2)}{\zeta - \xi} d\xi\right]. \quad \blacksquare$$

The expression for $a(\zeta)$, given above, in the physics literature is usually referred to as a *dispersion relation.*

6.3 Reflectionless Potential Case

The potential q in (3.1)-(3.3) is said to be reflectionless if the reflection coefficient $r(\xi)$ is zero for all $\xi \in \mathbb{R}$, resulting in pure transmission. In this case the inverse scattering problem is substantially simpler than in the general case. We will look at this case here. This section stems from Section VI-7 of Hsieh and Sibuya, [16].

Recall, from the definitions of a and b, that, for $\xi \in \mathbb{R} \setminus \{0\}$,

$$f_-(x,\xi) = a(\xi)\overline{f}_+(x,\xi) + b(\xi)f_+(x,\xi).$$

Since we are now considering the reflectionless case, $b(\xi) = 0$ for all $\xi \in \mathbb{R}$, since $r(\xi) = \frac{b(\xi)}{a(\xi)}$, hence the above equation becomes

$$f_-(x,\xi) = a(\xi)\overline{f}_+(x,\xi), \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R}.$$

As $a(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, rearranging the previous formula and using (4.20) we get

$$f_+(x,\xi) = \frac{f_-(x,-\xi)}{a(-\xi)}, \text{ for } \xi, x \in \mathbb{R}.$$

The above relation can be used to extend f_+ into the lower half-plane as follows. Recall that $a(\zeta)$ is an analytic function for $\Im(\zeta) \geq 0$ with a finite number of zeros, $i\eta_j$, $\eta_j \in \mathbb{R}$, $j = 1, \ldots, N$, each of which is simple.

For $\Im(\zeta) \leq 0$ let $f_+(x,\zeta)$ be defined by

$$f_+(x,\zeta) = \frac{f_-(x,-\zeta)}{a(-\zeta)},$$

then $f_+(x,\zeta)$ has simple poles at the points $-i\eta_j$, $j = 1, \ldots, N$, and is analytic in $\mathbb{C}$.

The residue of f_+ at each pole $-i\eta_j$ is given by

$$\begin{aligned}
\operatorname{Res}(f_+, -i\eta_j) &= \lim_{\zeta \to -i\eta_j} \frac{f_-(x,-\zeta)}{a(-\zeta)}(\zeta + i\eta_j) \\
&= \lim_{\zeta \to i\eta_j} f_-(x,\zeta)\frac{-\zeta + i\eta_j}{a(\zeta) - a(i\eta_j)}, \qquad \text{as } a(i\eta_j) = 0 \\
&= -f_-(x,i\eta_j)\frac{1}{a_\zeta(i\eta_j)}
\end{aligned}$$

where $a_\zeta(i\eta_j) = \frac{da(\zeta)}{d\zeta}|_{\zeta=i\eta_j}$. From Lemma 6.3, $a_\zeta(i\eta_j) = -\frac{id_j}{c_j}$ and by definition of d_j $f_-(x, i\eta_j) = d_j f_+(x, i\eta_j)$, giving

$$\begin{aligned} \mathrm{Res}(f_+, -i\eta_j) &= f_-(x, i\eta_j)\frac{c_j}{id_j} && (6.20)\\ &= -ic_j f_+(x, i\eta_j). && (6.21) \end{aligned}$$

From the asymptotics of the Jost solutions, $f_+(x,\zeta) = e^{i\zeta x}(1 + o(1))$, as $|\zeta| \to \infty$. Defining

$$H(x,\zeta) := e^{-i\zeta x} f_+(x,\zeta) - 1 = e^{-i\zeta x}\frac{f_-(x,-\zeta)}{a(-\zeta)} - 1, \quad \text{for } \Im(\zeta) \le 0,$$

we have that $H(x,\zeta) \to 0$, for $|\zeta| \to \infty$ and $H(x,\zeta)$ is an analytic function in the lower half plane with simple poles at the points $-i\eta_j, j = 1,\ldots,N$. From (6.21) it follows that the residue of $H(x,\zeta)$ at $-i\eta_j$ is $-ie^{-\eta_j x}c_j f_+(x, i\eta_j)$.

Subtracting from $H(x,\zeta)$ the principal parts of the Laurent series expansions of $H(x,\zeta)$ about each of the poles, using the residues found above, we obtain that the function $G(x,\zeta)$ defined by

$$G(x,\zeta) := e^{-i\zeta x} f_+(x,\zeta) - 1 + i\sum_{j=1}^{N}\frac{g_j(x)}{\zeta + i\eta_j},$$

where $g_j(x) = e^{-\eta_j x}c_j f_+(x, i\eta_j)$ is entire. Now since $H(x,\zeta) \to 0$ as $|\zeta| \to \infty$ and the definition of $G(x,\zeta)$, it follows that $G(x,\zeta) \to 0$ as $|\zeta| \to \infty$. Hence by Liouville's theorem, $G(x,\zeta) = 0$ and

$$f_+(x,\zeta) = e^{i\zeta x}\left[1 - i\sum_{j=1}^{N}\frac{g_j(x)}{\zeta + i\eta_j}\right]. \tag{6.22}$$

If for each eigenvalue $\zeta^2 = (i\eta_k)^2$ we let $\zeta = i\eta_k$ in equation (6.22), then

$$\frac{e^{2\eta_k x}}{c_k}g_k(x) + \sum_{j=1}^{N}\frac{1}{\eta_k + \eta_j}g_j(x) = 1, \quad k = 1,\ldots,N. \tag{6.23}$$

The above set of N equations will be used in matrix form in the next chapter to solve the inverse scattering problem. The following result will also be needed in the next chapter.

Lemma 6.6 *Let $F(x,\zeta) = e^{-i\zeta x} f_+(x,\zeta)$, then $F(x,\zeta)$ satisfies the equation*

$$-F'' - 2i\zeta F' + q(x)F = 0.$$

Proof: From the definition of $F(x,\zeta)$ we have

$$\begin{aligned} F(x,\zeta) &= e^{-i\zeta x} f_+, \\ F'(x,\zeta) &= -i\zeta e^{-i\zeta x} f_+ + e^{-i\zeta x} f_+', \\ F''(x,\zeta) &= -\zeta^2 e^{-i\zeta x} f_+ - 2i\zeta e^{-i\zeta x} f_+' + e^{-i\zeta x} f_+''. \end{aligned}$$

Combining the above three equations gives

$$\begin{aligned} &-F'' - 2i\zeta F' + q(x)F \\ &= \zeta^2 e^{-i\zeta x} f_+ + 2i\zeta e^{-i\zeta x} f_+' - e^{-i\zeta x} f_+'' - 2i\zeta[-i\zeta e^{-i\zeta} f_+ + e^{-i\zeta x} f_+'] + q(x)e^{-i\zeta x} f_+ \\ &= e^{-i\zeta x}[\zeta^2 f_+ + 2i\zeta f_+' - f_+'' - 2\zeta^2 f_+ - 2i\zeta f_+' + q(x) f_+] \\ &= e^{-i\zeta x}[-\zeta^2 f_+ - f_+'' + q(x) f_+] \\ &= 0. \quad \blacksquare \end{aligned}$$

From the above Lemma, we can now give an expression for the potential function in the reflectionless case.

Lemma 6.7 *The potential function $q(x)$ may be computed as follows.*

$$q(x) = 2\sum_{j=1}^{N} g_j'(x). \tag{6.24}$$

Proof: We have in the reflectionless potential case by (6.22),

$$f_+(x,\zeta) = e^{i\zeta x}\left[1 - i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta + i\eta_j}\right].$$

The function $F(x,\zeta)$, as defined in Lemma 6.6, and its derivatives can now be written as

$$F(x,\zeta) = 1 - i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta + i\eta_j},$$
$$F'(x,\zeta) = -i\sum_{j=1}^{N} \frac{g_j'(x)}{\zeta + i\eta_j},$$
$$F''(x,\zeta) = -i\sum_{j=1}^{N} \frac{g_j''(x)}{\zeta + i\eta_j}.$$

Substituting the above expressions for F and its derivatives into the equation of Lemma 6.6, gives

$$\begin{aligned} 0 &= i\sum_{j=1}^{N} \frac{g_j''(x)}{\zeta + i\eta_j} + (2i\zeta)i\sum_{j=1}^{N} \frac{g_j'(x)}{\zeta + i\eta_j} + q(x)\left[1 - i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta + i\eta_j}\right] \\ &= i\sum_{j=1}^{N} \frac{g_j''(x) - q(x)g_j(x)}{\zeta + i\eta_j} - 2\zeta\sum_{j=1}^{N} \frac{g_j'(x)}{\zeta + i\eta_j} + q(x) \\ &= i\sum_{j=1}^{N} \frac{g_j''(x) - q(x)g_j(x)}{\zeta + i\eta_j} - 2\sum_{j=1}^{N} \frac{g_j'(x)}{1 + \frac{i\eta_j}{\zeta}} + q(x). \end{aligned}$$

Letting $|\zeta| \to \infty$, we get

$$q(x) = 2\sum_{j=1}^{N} g_j'(x). \qquad \blacksquare$$

We now show that the scattering operator (3.1)-(3.3) on $\mathbb{R}$ is closed and self-adjoint.

Let $-\infty \le a < b \le \infty$ and define the set $A_n(a,b)$ to be the set of all complex valued functions on (a,b) with $f, f', \dots, f^{(n-2)}$ continuously differentiable on (a,b) and having $f^{(n-1)} \in AC(a,b)$.

The following theorem from Section 6.3 of Weidmann [33, p.157] is needed.

Theorem 6.8 *Let $\epsilon > 0$ and $n \in \mathbb{N}$. Then for every interval (a,b) there exists a constant $C > 0$ such that*

$$\int_a^b |f^{(j)}(x)|^2 dx \le \epsilon \int_a^b |f^{(n)}(x)|^2 dx + C\int_x^b |f(x)|^2 dx,$$

for all $j \in \{0, 1, \dots, n-1\}$ and all $f \in A_n(a,b)$.

Theorem 6.9 *The operator*

$$L = -\frac{d^2}{dx^2} + q(x),$$

with domain

$$D(L) = \{f \in L^2(-\infty, \infty)|\ f, f' \in AC, Lf \in L^2(-\infty, \infty)\}$$

and a potential function $q \in L^\infty(-\infty, \infty)$ *satisfying the scattering condition*

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty,$$

is closed and self-adjoint considered in $L^2(\mathbb{R})$*.*

Proof: As $C_0^\infty(-\infty, \infty) \subset D(L)$, it follows that $D(L)$ is dense in $L^2(-\infty, \infty)$. Also, by Theorem 3.2, L is symmetric on its domain. We now show that L is a closed operator.

Let (f_n) be a sequence in $D(L)$ which converges in $L^2(-\infty, \infty)$ to f and has Lf_n convergent in $L^2(-\infty, \infty)$ with limit h. To show L is closed, we need to show that $f \in D(L)$ and $Lf = \lim Lf_n$. Since each $f_n \in D(L)$, we have that $h_n := Lf_n \in L^2(-\infty, \infty)$ and $f_n'' = q(x)f_n - h_n \in L^2(-\infty, \infty)$. Then

$$\begin{aligned} \|f_n'' - f_m''\|_{L^2} &= \|q(x)f_n - h_n - q(x)f_m + h_m\|_{L^2} \\ &\leq \|q(x)\|_{L^\infty}\|f_n - f_m\|_{L^2} + \|h_n - h_m\|_{L^2}. \end{aligned}$$

Since $\|f_n - f_m\|_{L^2} \to 0$ and $\|h_n - h_m\|_{L^2} \to 0$ as $n, m \to \infty$, (f_n'') is a Cauchy sequence in $L^2(-\infty, \infty)$. The conditions for Theorem 6.8 above are satisfied by f_n and $f_n - f_m$ giving that for each $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\int_{-\infty}^{\infty} |f_n'(x)|^2 dx \leq \epsilon \int_{-\infty}^{\infty} |f_n''(x)|^2 dx + C \int_{-\infty}^{\infty} |f_n(x)|^2 dx,$$

and

$$\int_{-\infty}^{\infty} |f_n'(x) - f_m'(x)|^2 dx \ \leq\ \epsilon \int_{-\infty}^{\infty} |f_n''(x) - f_m''(x)|^2 dx + C \int_{-\infty}^{\infty} |f_n(x) - f_m(x)|^2 dx$$

for all $m, n \in \mathbb{N}$. Letting $m, n \to \infty$ gives $\|f_n' - f_m'\|_{L^2} \to 0$.

Therefore (f_n') is Cauchy in $L^2(-\infty,\infty)$. Thus (f_n) is a Cauchy sequence in the Sobolev space $H^2 = \{f \in AC_2(\mathbb{R}) \mid \|f\|_{H^2} < \infty\}$ where the H^2 norm is defined by

$$\|f\|_{H^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx + \int_{-\infty}^{\infty} |f'(x)|^2 dx + \int_{-\infty}^{\infty} |f''(x)|^2 dx.$$

Since H^2 is a Hilbert space, and in particular, is complete, (f_n) converges in H^2 to say g. Hence

$$\begin{aligned} f_n &\to g \text{ in } L^2(-\infty,\infty) \text{ giving } g = f, \\ f_n' &\to g' \text{ in } L^2(-\infty,\infty) \text{ giving } g' = f', \\ f_n'' &\to g'' \text{ in } L^2(-\infty,\infty) \text{ giving } g'' = f''. \end{aligned}$$

So $qf_n - h_n \to g''$ and $qf_n - h_n \to qg - h$ giving $h = -g'' + qg$ where $g \in D(L)$ and thus $f \in D(L)$ and $h = \lim Lf_n = Lf = -f'' + q(x)f \in L^2(-\infty,\infty)$. Therefore, L is a closed operator.

As shown in Chapter 4, the Jost solutions of the differential equation $-f'' + q(x)f = \zeta^2 f$, $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are linearly independent for $\zeta^2 \in \mathbb{C} \setminus \mathbb{R}$. Recall from Theorem 3.11 that the deficiency index is constant in the upper half-plane, this enables us to take $\zeta^2 = 2k^2 i$ in the equation where $k > 0$. If y is an $L^2(-\infty,\infty)$ solution, then $\overline{y}$ is an $L^2(-\infty,\infty)$ of the equation $-f'' + qf = -2k^2 if$, hence Dim $N_+ =$ Dim N_-. That is $n_+ = n_- \le 2$. Consider the equation $Ly = 2k^2 iy$. We then have that $\zeta = k(1+i)$ and Theorem 4.5 gives

$$f_+(x, k(1+i)) = e^{-kx}\left(e^{ikx} + O\left(\frac{C(x)\rho(x)}{1+\sqrt{2}k}\right)\right),$$

and similarly,

$$f_-(x, k(1+i)) = e^{kx}\left(e^{-ikx} + O\left(\frac{C(-x)\tilde{\rho}(x)}{1+\sqrt{2}k}\right)\right).$$

The general solution to the equation $Ly = 2k^2 iy$ is then

$$y = af_+(x, k(1+i)) + bf_-(x, k(1+i))$$

where $a, b \in \mathbb{C}$. Now for $k > 0$ large, for $x > 0$,

$$\begin{aligned} |y| &\sim |b||f_-(x, k(1+i))| \\ &= |b|e^{kx}\left(e^{-ikx} + O\left(\frac{C(-x)\tilde{\rho}(x)}{1+\sqrt{2}k}\right)\right) \notin L^2(0, \infty) \end{aligned}$$

making $b = 0$. Hence $n_+ \leq 1$. Again for $k > 0$ large, for $x < 0$,

$$\begin{aligned} |y| &\sim |a||f_+(x, k(1+i))| \\ &= |a|e^{-kx}\left(e^{ikx} + O\left(\frac{C(x)\rho(x)}{1+\sqrt{2}k}\right)\right) \notin L^2(-\infty, 0) \end{aligned}$$

making $a = 0$. Hence $n_+ = 0$ and by the above $n_- = 0$. Hence the deficiency indices are $(n_+, n_-) = (0, 0)$. Now since $L = \overline{L}$, L is self-adjoint and no extra boundary conditions are needed. $\blacksquare$

Chapter 7

Inverse Scattering - Reflectionless Case

The reflectionless inverse scattering problem involves starting with the two sequences $0 < \eta_1 < \eta_2 < \ldots < \eta_N$ and $c_i > 0, i = 1, \ldots, N$ and showing the existence of a unique reflectionless potential q; so that $-y'' + qy = \zeta^2 y,\ y \in L^2(\mathbb{R})$ has eigenvalues $-\eta_1^2 > -\eta_2^2 > \ldots > -\eta_N^2$ with corresponding norming constants $c_1, c_2, \ldots, c_N$. The method used in this chapter is based on that of Hsieh and Sibuya [16].

7.1 Setting up the Basic Inverse Problem Equation

The theory of reflectionless potentials in Section 6.3 and in particular equation (6.23) is used as a guide in setting up the framework for the inverse problem. Given the reflectionless scattering data, $\{\eta_j, c_j\}_{j=1,\ldots,N}$ with the η_j's ordered as above, we use equation (6.23) as the defining equation for functions $\{g_j(x) |\ j = 1, \ldots, N\}$.

Let the matrix $A(x)$ be defined by

$$A_{i,j}(x) = \frac{e^{2\eta_j x}}{c_j}\delta_{ij} + \frac{1}{\eta_i + \eta_j}, \quad i, j = 1, \ldots, N, \tag{7.1}$$

where δ_{ij} denotes Kronecker delta.

Definition 7.1 *An $N \times N$ matrix M is positive definite if for all vectors $\underline{x} \in \mathbb{C}^N \setminus \{0\}$ we have*

$$\underline{x}^T M \overline{\underline{x}} > 0.$$

Equivalently M should be Hermitian and have strictly positive eigenvalues.

Theorem 7.2 *The matrix $A(x)$ defined in (7.1) is positive definite.*

Theorem 7.2 is a direct consequence of the following three lemmas.

From [36], we have the following well known results regarding positive definite matrices.

Lemma 7.3 *If matrices A and B are positive definite, then their sum $A + B$ is positive definite.*

Lemma 7.4 *If the matrix A is positive definite then its inverse, A^{-1}, is also positive definite.*

The matrix $A(x)$, above, can be written as

$$A(x) = E(x) + C,$$

where

$$E(x) = \operatorname{diag}\left(\frac{e^{2\eta_1 x}}{c_1}, \ldots, \frac{e^{2\eta_N x}}{c_N}\right),$$

and

$$C = \begin{pmatrix} \frac{1}{\eta_1+\eta_1} & \frac{1}{\eta_1+\eta_2} & ,\dots, & \frac{1}{\eta_1+\eta_N} \\ \frac{1}{\eta_2+\eta_1} & \frac{1}{\eta_2+\eta_2} & ,\dots, & \frac{1}{\eta_2+\eta_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\eta_N+\eta_1} & \frac{1}{\eta_N+\eta_2} & ,\dots, & \frac{1}{\eta_N+\eta_N} \end{pmatrix}.$$

Here, since $E(x)$ is a diagonal matrix with strictly positive diagonal entries, it is positive definite. Now by Lemma 7.3, in order to show that $A(x)$ is positive definite it is sufficient to show that C is positive definite.

Lemma 7.5 *The matrix C, as defined above, is positive definite.*

Proof: Let $\underline{\gamma} \in \mathbb{C}^N \setminus \{\underline{0}\}$, then since $e^{-\eta_1 x}, \dots, e^{-\eta_N x}$ are linearly independent in $L^2(\mathbb{R})$ it follows that $\gamma_1 e^{-\eta_1 x} + \cdots + \gamma_N e^{-\eta_N x}$ is not the zero function in $L^2(\mathbb{R})$ giving

$$\int_0^\infty \left| \sum_{j=1}^N \gamma_j e^{-\eta_j x} \right|^2 dx > 0.$$

Now

$$\begin{aligned} \int_0^\infty \left| \sum_{j=1}^N \gamma_j e^{-\eta_j x} \right|^2 dx &= \int_0^\infty \sum_{j=1}^N \sum_{k=1}^N \gamma_j \overline{\gamma_k} e^{-(\eta_j+\eta_k)x} dx \\ &= \sum_{j=1}^N \sum_{k=1}^N \int_0^\infty \gamma_j \overline{\gamma_k} e^{-(\eta_j+\eta_k)x} dx \\ &= \sum_{j=1}^N \sum_{k=1}^N \left(\frac{-1}{\eta_j+\eta_k} \right) \gamma_j \overline{\gamma_k} \left[e^{-(\eta_j+\eta_k)x} \right]_0^\infty \\ &= \sum_{j=1}^N \sum_{k=1}^N \frac{\gamma_j \overline{\gamma_k}}{\eta_j+\eta_k} \\ &= \underline{\gamma}^T C \overline{\underline{\gamma}}. \end{aligned}$$

Hence we have that $\underline{\gamma}^T C \underline{\gamma} > 0$ and C is positive definite. ■

By Lemma 7.3, $A(x) = E(x) + C$ is positive definite. Thus Theorem 7.2 is proved. Since $A(x)$ is positive definite, by Lemma 7.4, it is also invertible, thus we may define the functions $\{g_j(x)|\ j = 1, \dots, N\}$ by the equation

$$A(x)\underline{g}(x) = \underline{1}, \tag{7.2}$$

where $\underline{1}$ is the $\mathbb{C}^N$ vector, each component of which is 1. i.e.

$$\underline{g}(x) = A^{-1}(x)\underline{1}.$$

Since the functions $\{g_j(x)|\ j = 1, \dots, N\}$ are well defined by the above, we define the potential $q(x)$ in an analagous way to that in Lemma 6.7 by

$$q(x) = 2\sum_{j=1}^{N} g_j'(x). \tag{7.3}$$

7.2 Uniqueness

The next theorem shows that the potential recovered is unique.

Theorem 7.6 *If $-y'' + qy = \lambda y$, $y \in L^2(\mathbb{R})$ and $-y'' + \tilde{q}y = \lambda y$, $y \in L^2(\mathbb{R})$, where q and $\tilde{q}$ are reflectionless potentials with*

$$\int_{-\infty}^{\infty}(1 + |x|)|q(x)|dx < \infty$$

and

$$\int_{-\infty}^{\infty}(1 + |x|)|\tilde{q}(x)|dx < \infty,$$

where $q, \tilde{q} \in L^\infty(-\infty, \infty)$, are real valued, both giving rise to eigenvalue problems with eigenvalues $-\eta_1^2 > -\eta_2^2 > \dots > -\eta_N^2$, and norming constants $c_1, \dots, c_N$, then $q \equiv \tilde{q}$.

Proof: Since q and $\tilde{q}$ are reflectionless potentials with N eigenvalues, there exist functions $\{g_j(x)|j = 1, \dots, N\}$ and $\{\tilde{g}_j(x)|j = 1, \dots, N\}$ such that

$$q(x) = 2\textstyle\sum_{j=1}^N g_j'(x) \qquad \tilde{q}(x) = 2\textstyle\sum_{j=1}^N \tilde{g}_j'(x)$$

Since, by assumption, $\eta_1 = \tilde{\eta}_1, \ldots, \eta_N = \tilde{\eta}_N$ and also $c_1 = \tilde{c}_1, \ldots, c_N = \tilde{c}_N$, the defining matrix equations for $\underline{g}$ and $\underline{\tilde{g}}$ are

$$A\underline{g} = \underline{1} \qquad A\underline{\tilde{g}} = \underline{1}.$$

That is

$$A\underline{g} = A\underline{\tilde{g}}$$

and since A is invertible, $\underline{g} \equiv \underline{\tilde{g}}$. Hence $q(x) \equiv \tilde{q}(x)$. ∎

Corollary 7.7 *If q is a reflectionless potential and $\tilde{q}(x) = q(-x)$ for a.e. $x \in \mathbb{R}$. Then $\tilde{q}$ is a reflectionless potential with $\eta_1 = \tilde{\eta}_1, \ldots, \eta_N = \tilde{\eta}_N$, here $N = \tilde{N}$. Also*

$$\frac{\tilde{c}_j}{c_j} = \frac{\|f_-(x, i\eta_j)\|^2}{\|f_+(x, i\eta_j)\|^2}, \quad \textit{by } (7.3).$$

7.3 Existence

It is required to show that the potential $q(x)$ is a real valued function, is in $L^1(-\infty, \infty)$ and satisfies the integrability condition

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty.$$

The next lemma gives asymptotic results for $\underline{g}(x)$.

Lemma 7.8 *On $[0, \infty)$, $\underline{g}(x)$ is exponentially decreasing, while on $(-\infty, 0]$, $\underline{g}(x) = O(1)$.*

Proof: Using Cramer's rule on the equation (7.2) for the functions $g_j(x)$, we see that each $g_j(x)$ is given by

$$g_j(x) = \frac{\Delta_j(x)}{\Delta(x)},$$

where $\Delta(x) = \det A(x)$ and $\Delta_j(x)$ is the subdeterminant of the matrix A, the j^{th} column of which has been replaced by the $\underline{1}$ vector. Since the subdeterminant $\Delta_j(x)$

has a column replaced by $\underline{1}$, the column vector each entry of which is 1, the order of the exponential in the denominator of the expression for each $g_j(x)$ must be greater than that of the numerator, and hence for large $x > 0$ we must have that $g_j(x) = O(e^{-\eta_j x})$ and hence for large x, $\underline{g}(x)$ is expontentially bounded.

For $(-\infty, 0]$, consider again equation (7.2) where $A(x) = E(x) + C$ as before. For $x << 0$, the matrix $E(x)$ is expontentially small and can be considered a perturbation of the matrix C. That is, $A(x) \sim C$ for $x \to -\infty$ and then we have $\underline{g}(x) \sim C^{-1}\underline{1}$. Thus $\underline{g}(x) = O(1)$ for large negative values of x. ■

Using the previous result, for $x \to -\infty$, equation (7.2) gives

$$\lim_{x \to -\infty} \sum_{k=1}^{N} \frac{g_k(x)}{\eta_j + \eta_k} = 1, \quad j = 1, \ldots, N. \tag{7.4}$$

Since $q(x)$ is defined in terms of $\underline{g}'(x)$, we need asymptotic results for $\underline{g}'(x)$.

Lemma 7.9 *On* $(-\infty, \infty)$, $\underline{g}'(x) = o(e^{-\eta_1 |x|})$.

Proof: Differentiating equation (7.2) gives

$$A'(x)\underline{g}(x) + A(x)\underline{g}'(x) = \underline{0}. \tag{7.5}$$

First consider the interval $[0, \infty)$. From (7.5) and since $A'(x) = 2[\underline{\eta}]E(x)$, where $[\underline{\eta}]$ is the diagonal matrix i with diagonal entries $\eta_1, \ldots, \eta_N$, we get

$$(E(x) + C)\underline{g}'(x) = -2[\underline{\eta}]E(x)\underline{g}(x).$$

Since $[\underline{\eta}]$ and $E(x)$ are both diagonal matrices, multiplication order may be interchanged to give

$$E^{-1}(x)(E(x) + C)\underline{g}'(x) = -2[\underline{\eta}]\underline{g}(x),$$

and therefore

$$(I + E^{-1}(x)C)\underline{g}'(x) = -2[\underline{\eta}]\underline{g}(x).$$

Since, for $x \to \infty$, $E^{-1}(x)C \to 0$ we can consider $E^{-1}(x)C$ as a perturbation of the identity matrix in the equation above. Therefore, for large values of x the above expression becomes

$$\underline{g}'(x) = -2[\underline{\eta}](I + O(e^{-\eta_1 x}))\underline{g}(x).$$

By Lemma 7.8, $\underline{g}(x) = O(e^{-\eta_1 x})$ on $[0, \infty)$, hence on $[0, \infty)$ we have that

$$\underline{g}'(x) = O(e^{-\eta_1 x}).$$

Next consider $(-\infty, 0]$. We have

$$(E(x) + C)\underline{g}'(x) = -2[\underline{\eta}]E(x)\underline{g}(x).$$

For $x \to -\infty$, $(E(x) + C) \to C$ using the same perturbation argument as before. This gives,

$$\underline{g}'(x) = -2(I + O(e^{\eta_1 x}))C^{-1}[\underline{\eta}]E(x)\underline{g}(x).$$

Now, since C and $[\underline{\eta}]$ are constants, $\underline{g}(x)$ is bounded by Lemma 7.8 and $E(x)$ is exponentially small on $(-\infty, 0]$. Therefore $\underline{g}'(x) = O(e^{\eta_1 x})$ for large negative values of x. Hence the result holds. ■

Therefore

$$q(x) := 2\sum_{j=1}^{N} g_j'(x) \in L^\infty(-\infty, \infty)$$

and

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty.$$

Lemma 7.10 *The function $q(x)$ as defined above is real valued.*

Proof: For any x, the matrix $A(x)$ in equation (7.2) is real valued. Hence $A^{-1}(x)$ is a real valued matrix, and

$$\underline{g}(x) = A^{-1}(x)\underline{1}$$

is real valued. Therefore, by Lemma 7.9, each $g_j(x)$ is real valued and hence each $g_j'(x)$ is real valued. ■

7.4 Construction of the Jost solutions

The first theorem in this section shows that the functions $g_j(x)$, $j = 1, \ldots, N$, satisfy a certain differential equation, which will be needed later.

Theorem 7.11 *Let* $q(x) = 2\sum_{j=1}^{N} g_j'(x)$, *then*

$$g_j''(x) + 2\eta_j g_j'(x) - q(x)g_j(x) = 0, \quad j = 1, \ldots, N.$$

Proof: Note that for the matrix $E(x)$ above, we have that $E' = 2[\underline{\eta}]E$ where $[\underline{\eta}] = \mathrm{diag}(\eta_1, \ldots, \eta_N)$ and so $A' = 2[\underline{\eta}]E$. Let $\mathbb{I} = [1]_{i,j}$ be the $N \times N$ matrix each in which each entry is 1. Then by direct matrix multiplication it is seen that

$$[\underline{\eta}]C = \left[\frac{\eta_i}{\eta_i + \eta_j}\right]_{i,j} = \left[1 - \frac{\eta_j}{\eta_i + \eta_j}\right]_{i,j} = \mathbb{I} - C[\underline{\eta}].$$

Thus,

$$\begin{aligned} 2[\underline{\eta}]C\underline{g}' + 2C[\underline{\eta}]\underline{g}' &= 2[\mathbb{I} - C[\underline{\eta}]]\underline{g}' + 2C[\underline{\eta}]\underline{g}' \\ &= 2\mathbb{I}\underline{g}' \\ &= q\underline{1} \end{aligned} \tag{7.6}$$

since each component of $2\mathbb{I}\underline{g}'$ is equal to $2\sum_{j=1}^{N} g_j'$. Differentiating the defining equation $A\underline{g} = \underline{1}$ gives

$$\underline{0} = A'\underline{g} + A\underline{g}'.$$

Hence

$$\begin{aligned} A\underline{g}' &= -2[\underline{\eta}]E\underline{g} \\ &= -2[\underline{\eta}](A - C)\underline{g} \\ &= -2[\underline{\eta}](A\underline{g} - C\underline{g}) \\ &= -2[\underline{\eta}](\underline{1} - C\underline{g}). \end{aligned}$$

Differentiating the above gives

$$\begin{aligned} 2[\underline{\eta}]E\underline{g}' + A\underline{g}'' &= 2[\underline{\eta}]C\underline{g}' \\ &= q\underline{1} - 2C[\underline{\eta}]\underline{g}' \quad \text{by (7.6)} \end{aligned}$$

which after reorganisation becomes, since $[\underline{\eta}]E = E[\underline{\eta}]$,

$$2A[\underline{\eta}]\underline{g}' + A\underline{g}'' = q\underline{1}.$$

Thus $\underline{g}'' + 2[\underline{\eta}]\underline{g}' = qA^{-1}\underline{1} = q\underline{g}$ since $A\underline{g} = \underline{1}$. Hence for each $j = 1, \ldots, N$,

$$g_j'' + 2\eta_j g_j' - qg_j = 0. \quad \blacksquare$$

Let

$$F_+(x,\zeta) = e^{i\zeta x}\left[1 - i\sum_{j=1}^{N}\frac{g_j(x)}{\zeta + i\eta_j}\right], \quad x \in \mathbb{R}, \zeta \in \mathbb{C}\setminus\{-i\eta_1, \ldots, -i\eta_N\}. \tag{7.7}$$

We now show that $F_+(x,\zeta)$ is in fact the Jost solution of $-y'' + qy = \zeta^2 y$ where q is as defined before. Observe that, since, by Lemma 7.8, $g_j(x) = O(e^{-\eta_1 x})$ as $x \to \infty$, $F_+(x,\zeta)e^{-i\zeta x} \to 1$ as $x \to \infty$. Hence $F_+(x,\zeta) = e^{i\zeta x}(1 + o(1))$ for $\Im(\zeta) > 0$, making $F_+ \in L^2(0,\infty)$.

Lemma 7.12 *If q is as defined in (7.3), then F_+, as defined in (7.7), satisfies*

$$-F_+'' + qF_+ = \zeta^2 F_+.$$

Proof: Consider the function defined by

$$F(x,\zeta) := e^{-i\zeta x}F_+(x,\zeta).$$

By Theorem 7.11,

$$\begin{aligned}
&F''(x,\zeta) + 2i\zeta F'(x,\zeta) - q(x)F(x,\zeta)\\
&= -i\sum_{j=1}^{N}\frac{g_j''(x)}{\zeta + i\eta_j} + 2\zeta\sum_{j=1}^{N}\frac{g_j'(x)}{\zeta + i\eta_j} - q(x) + iq(x)\sum_{j=1}^{N}\frac{g_j(x)}{\zeta + i\eta_j}\\
&= -i\sum_{j=1}^{N}\frac{1}{\zeta + i\eta_j}[g_j''(x) + 2i\zeta g_j'(x) - q(x)g_j(x)] - q(x)\\
&= -i\sum_{j=1}^{N}\frac{1}{\zeta + i\eta_j}[g_j''(x) + 2i\zeta g_j'(x) - q(x)g_j(x) - 2i(\zeta + i\eta_j)g_j'(x)]\\
&= -i\sum_{j=1}^{N}\frac{1}{\zeta + i\eta_j}(g_j''(x) + 2\eta_j g_j'(x) - q(x)g_j(x))\\
&= 0.
\end{aligned}$$

From the expression

$$F''(x,\zeta) + 2i\zeta F'(x,\zeta) - q(x)F(x,\zeta) = 0,$$

it now follows by straight forward computation that

$$-F_+'' + qF_+ = \zeta^2 F_+. \quad \blacksquare$$

Thus we have shown that $F_+(x,\zeta) = f_+(x,\zeta)$, the Jost solution with $q = -2\sum_{j=1}^N g_j'$.

We now define the $f_-(x,\zeta)$ function and show that it is the Jost solution which its symbols suggests. Define

$$f_-(x,\zeta) := e^{-i\zeta x}\left[\prod_{j=1}^N \frac{\zeta - i\eta_j}{\zeta + i\eta_j} + i\sum_{j=1}^N \frac{g_j(x)}{\zeta + i\eta_j}\prod_{\substack{k=1\\k\neq j}}^N \frac{\zeta - i\eta_k}{\zeta + i\eta_k}\right]. \tag{7.8}$$

Let $F(x,\zeta) := e^{i\zeta x} f_-(x,\zeta)$. Following the proof of Lemma 7.12, it is seen that $f_-(x,\zeta)$ as defined above is a satisfies, $-f_-'' + qf_- = \zeta^2 f_-$. From the asymptotic properties of $g_j(x)$, it now follows that $\lim_{x\to-\infty} F(x,\zeta) = 1 + O(1/\zeta)$, thus $f_-(x,\zeta)$ is the Jost solution that its symbolic name claims it to be.

Theorem 7.13 *The function $a(\zeta)$ is given by*

$$a(\zeta) = \prod_{j=1}^N \frac{\zeta - i\eta_j}{\zeta + i\eta_j}.$$

Proof: From (6.6) and the definition of $f_\pm(x,\zeta)$, $a(\zeta)$ will be found. By (7.7) we have,

$$f_+'(x,\zeta) = i\zeta e^{i\zeta x}\left[1 - i\sum_{j=1}^n \frac{g_j(x)}{\zeta + i\eta_j}\right] - ie^{i\zeta x}\sum_{j=1}^N \frac{g_j'(x)}{\zeta + i\eta_j},$$

and by (7.8),

$$\begin{aligned} f_-'(x,\zeta) = -i\zeta e^{-i\zeta x} &\left[\prod_{j=1}^n \frac{\zeta - i\eta_j}{\zeta + i\eta_j} + i\sum_{j=1}^N \frac{g_j(x)}{\zeta + i\eta_j}\prod_{\substack{k=1\\k\neq j}}^N \frac{\zeta - i\eta_k}{\zeta + i\eta_k}\right] \\ &+ e^{-i\zeta x}\left[i\sum_{j=1}^N \frac{g_j'(x)}{\zeta + i\eta_j}\prod_{\substack{k=1\\k\neq j}}^N \frac{\zeta - i\eta_k}{\zeta + i\eta_k}\right]. \end{aligned}$$

Computing the Wronskian, $W[f_+, f_-]$, gives,

$$\begin{aligned} W[f_+, f_-] &= f_+f_-' - f_+'f_- \\ &= \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j} \left\{ -2i\zeta \left[1 - i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta + i\eta_j} \right] \left[1 + i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta - i\eta_j} \right] + \right. \\ & \left. i\sum_{j=1}^{N} g_j'(x) \left[\frac{1}{\zeta - i\eta_j} + \frac{1}{\zeta + i\eta_j} \right] \right\}. \end{aligned}$$

Letting $x \to \infty$ in $W[f_+, f_-]$, as it's independent of x, by the asymptotic properties of $g_j(x)$ and $g_j'(x)$ we get

$$W[f_+, f_-] = -2i\zeta \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}.$$

Comparing this with (6.6), we conclude that

$$a(\zeta) = \prod_{j=1}^{N} \frac{\zeta - i\eta_j}{\zeta + i\eta_j}. \quad \blacksquare$$

Factoring (7.8), we see that we have

$$\begin{aligned} f_-(x, \zeta) &= \left(\prod_{k=1}^{N} \frac{\zeta - i\eta_k}{\zeta + i\eta_k} \right) e^{-i\zeta x} \left[1 + i\sum_{j=1}^{N} \frac{g_j(x)}{\zeta - i\eta_j} \right] & (7.9) \\ &= a(\zeta) f_+(x, -\zeta) & (7.10) \end{aligned}$$

By the previous theorem we see that $a(\zeta)$ has a finite set of simple zeros, $\{i\eta_j \mid j = 1, \ldots, N\}$ and a finite set of poles, $\{-i\eta_j \mid j = 1, \ldots, N\}$. Also, f_+ and f_- are linearly dependent functions if and only if $a(\zeta) = 0$ by (6.6). Therefore we have reconstructed a potential, for which the eigenvalues of (3.1)-(3.3) are $-\eta_1^2, \ldots, -\eta_N^2$, as required. We therefore have the following result.

Theorem 7.14 *In the reflectionless case, the function $f_+(x, \zeta)$ is an eigenfunction of (3.1)-(3.3) if and only if $\zeta = i\eta_j, j = 1, \ldots, N$.*

Proof: $f_+(x, \zeta)$ is an eigenfunction of (3.1)-(3.3) if and only if $f_+(x, \zeta)$ and $f_-(x, \zeta)$ are linearly dependent functions. This is true if and only if $a(\zeta) = 0$ which occurs if and only if $\zeta = i\eta_j, j = 1, \ldots, N$. ■

It remains to be shown that $\{c_1, \ldots, c_N\}$ are the norming constants as required.

Lemma 7.15 *For the reconstructed Jost solution $f_+(x, i\eta_k)$, we have that*

$$\frac{1}{c_k} = \int_{-\infty}^{\infty} f_+^2(x, i\eta_k)dx, \;\; k = 1, \ldots, N.$$

Proof: Let $\zeta = i\eta_k$ such that $\zeta^2 = -\eta_k^2$ is an eigenvalue. Then we have the following representations for f_+ and f_- as defined,

$$\begin{aligned} f_+(x, i\eta_j) &= e^{-\eta_j x}\left[1 - \sum_{k=1}^{N} \frac{g_k(x)}{\eta_k + \eta_j}\right] \\ &= e^{-\eta_j x}\left[\frac{e^{2\eta_j x}}{c_j} g_j(x)\right] \qquad \text{by (6.23)} \\ &= \frac{e^{\eta_j x}}{c_j} g_j(x). \end{aligned}$$

Putting $\zeta = i\eta_j$ in (7.9) gives

$$f_-(x, i\eta_j) = e^{\eta_j x}\left[\frac{g_j(x)}{2\eta_j} \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\eta_j - \eta_k}{\eta_j + \eta_k}\right] \tag{7.11}$$

Since we are at an eigenvalue, we have that there exists a constant d_j such that $f_-(x, i\eta_j) = d_j f_+(x, i\eta_j)$. We can use the above expressions to find d_j. Dividing f_- by f_+ gives

$$d_j = \frac{c_j}{2\eta_j} \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\eta_j - \eta_k}{\eta_j + \eta_k}. \tag{7.12}$$

Since the $a(\zeta)$ function is in fact the correct '$a(\zeta)$' for the problem, see (7.10), the result from (6.17) may be used. Therefore we have

$$\frac{da}{d\zeta}(i\eta_j) = -id_j \int_{-\infty}^{\infty} f_+^2(x, i\eta_j)dx. \tag{7.13}$$

Taking the derivative of $a(\zeta)$ as given in Theorem 7.13 gives

$$\frac{da}{d\zeta}(\zeta) = \sum_{k=1}^{N} \frac{2i\eta_k}{(\zeta + i\eta_k)^2} \prod_{\substack{m=1 \\ m\neq k}}^{N} \frac{\zeta - i\eta_m}{\zeta + i\eta_m},$$

hence for $\zeta = i\eta_j$,

$$\begin{aligned}
\frac{da}{d\zeta}(i\eta_j) &= \frac{2i\eta_j}{(2i\eta_j)^2} \prod_{\substack{m=1 \\ m\neq k}}^{N} \frac{\eta_j - \eta_m}{\eta_j + \eta_m} \\
&= \frac{-i}{2\eta_j} \prod_{\substack{m=1 \\ m\neq k}}^{N} \frac{\eta_j - \eta_m}{\eta_j + \eta_m} \\
&= -i\frac{d_j}{c_j}, \qquad \text{by (7.12).}
\end{aligned}$$

Comparing the above result with (7.13) gives

$$\frac{1}{c_j} = \int_{-\infty}^{\infty} f_+^2(x, i\eta_j)dx. \quad \blacksquare$$

Chapter 8

Scattering with Transfer Condition

8.1 Preliminaries

Here we consider the scattering problem on the line but with a transfer condition at $x = 0$. The condition is stated by use of a transfer matrix M - in this case, a 2×2 invertible matrix - relating a solution, F, on the positive-axis and its derivative $\begin{pmatrix} F \\ F' \end{pmatrix}$ to its continuation by setting

$$\begin{pmatrix} F(0^+, \zeta) \\ F'(0^+, \zeta) \end{pmatrix} = M \begin{pmatrix} F(0^-, \zeta) \\ F'(0^-, \zeta) \end{pmatrix} \tag{8.1}$$

as a solution on the negative-axis.

In order for $-y'' + qy = \lambda y$ on $(-\infty, 0)$ and $(0, \infty)$, with $y \in L^2((-\infty, 0) \cup (0, \infty))$ with (8.1) to be self-adjoint, the transfer matrix M will have to be of a certain form, which will be investigated in the next section. The scattering problem can then be treated as two classical half-line problems joined at the origin by the matrix condition (8.1). We begin by giving the definition of a transfer matrix.

Definition 8.1 *A transfer matrix on an interval* $[a, b]$ *is an invertible matrix* M

which relates the values of a function f and its derivative f' at one endpoint of the interval to the function and its derivative evaluated at the other endpoint.

We shall only be dealing with *point transfer matrices* which can be constructed by taking 'shrinking' intervals $[a_n, b_n]$ and considering operators T_n on these intervals which converge to an operator T as done in Gordon and Pearson [15]. Thus, the following definition is all that we will need for our work here.

Definition 8.2 *A Point Transfer Matrix is an invertible matrix M at a point a which relates the values of the left and right limits of a function f and its derivative f' at the point a, i.e.*

$$\lim_{x\to a^-}\begin{pmatrix} f(x) \\ f'(x) \end{pmatrix} = \lim_{x\to a^+} M \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix}.$$

We will only consider point transfer matrices at the origin and henceforth will refer to them as *transfer matrices*. The transfer matrix represents a change of medium which affects the solution and its derivative as represented by components of the matrix. Our transfer matrices will be real constant transfer matrices - all components will be constants. The theory could be extended to eigenparameter dependent transfer matrices, this will be considered in future studies.

8.2 Functional Analytic Setting

Consider the operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

on

$$J = (-\infty, 0) \cup (0, \infty),$$

where $q \in L^\infty(J)$ is real valued, with domain

$$D(L) = \{f \in L^2(J) | f|_{(-\infty,0)}, f'_|(-\infty, 0), f_|(0,\infty), f'|_{(0,\infty)} \in AC, Lf \in L^2(J)\}.$$

The operator eigenvalue problem associated with L can be reformulated as a system eigenvalue problem as follows. Let $\underline{\mathrm{Y}}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and consider the differential operator in $L^2(0,\infty) \oplus L^2(0,\infty)$ given by

$$\underline{L\mathrm{Y}} := -\frac{d^2\underline{\mathrm{Y}}}{dx^2} + Q\underline{\mathrm{Y}} = \zeta^2\underline{\mathrm{Y}}, \tag{8.2}$$

where $Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & q(-t) \end{pmatrix}$. As the norm on $L^2(0,\infty) \oplus L^2(0,\infty)$ we take

$$\|\underline{\mathrm{Y}}\|^2 = \int_0^\infty \underline{\mathrm{Y}}^T \cdot \overline{\underline{Y}} dx.$$

It should be noted here that $Ly = \lambda y \Leftrightarrow \underline{LY} = \lambda \underline{Y}$ where $\underline{\mathrm{Y}}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $t \in (0,\infty)$, and

$$y(t) = \begin{cases} y_1(t), & t > 0 \\ y_2(t), & t < 0 \end{cases}.$$

Let $\underline{\mathrm{Y}}, \underline{\mathrm{Z}} \in \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \middle| f_i, f_i'' \in L^2(0,\infty),\ f_i, f_i' \in AC, i = 1,2 \right\}$, then

$$\begin{aligned} \langle \underline{\mathrm{LY}}, \underline{\mathrm{Z}} \rangle - \langle \underline{\mathrm{Y}}, \underline{\mathrm{LZ}} \rangle &= \int_0^\infty (-y_1''\overline{z_1} - y_2''\overline{z_2})dx - \int_0^\infty (-y_1\overline{z_1''} - y_2\overline{z_2''})dx \\ &= \int_0^\infty (y_1\overline{z_1''} - y_1''\overline{z_1})dx + \int_0^\infty (y_2\overline{z_2''} - y_2''\overline{z_2})dx \\ &= [y_1\overline{z_1'} - y_1'\overline{z_1}]_0^\infty + [y_2\overline{z_2'} - y_2'\overline{z_2}]_0^\infty. \end{aligned}$$

As seen in Chapter 3, $(y_i\overline{z}_i' - y_i'\overline{z}_i)(t) \to 0$ as $t \to \infty$, $i = 1,2$. Thus, in order for L (or equivalently $\underline{\mathrm{L}}$) to be symmetric, a pair of domain conditions at 0 are needed in order to ensure that $(y_1\overline{z}_1' - y_1'\overline{z}_1 + y_2\overline{z}_2' - y_2'\overline{z}_2)(0) = 0$.

Let $\zeta^2 = 2k^2 i, k \in \mathbb{R}^+$, so that the differential equation becomes

$$\underline{LY} = 2k^2 i\underline{Y}.$$

Since such a $\zeta^2 \notin \mathbb{R}$, ζ^2 is not an eigenvalue of the classical problem and hence $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are linearly independent. The equation has the solution $\underline{\mathrm{Y}} =$

$\begin{pmatrix} af_+(x,\zeta)+bf_-(x,\zeta) \\ cf_+(x,\zeta)+df_-(x,\zeta) \end{pmatrix}$ where $\zeta = k(1+i)$ and f_+, f_- are the Jost solutions of the classical scattering problem. By Lemma 3.11, the deficiency indices n_+, n_- are constant in the upper and lower half planes, hence, letting $k \to \infty$ does not alter the deficiency indices. Thus we may use the asymptotic approximation for f_+ and f_-, giving that the solution becomes

$$\underline{\mathrm{Y}} = \begin{pmatrix} ae^{i\zeta x} + be^{-i\zeta x} + o(1) \\ ce^{i\zeta x} + de^{-i\zeta x} + o(1) \end{pmatrix}.$$

Now, $\|\underline{\mathrm{Y}}\| < \infty$ if and only if $b = 0$ and $d = 0$ giving deficiency indices $(2, 2)$. So we need 2 boundary conditions for the above system, or considering the original problem on the interval J, we need to restrict the domain of $\underline{L}$ by two boundary conditions. For the operator to be symmetric with two linear domain conditions, they must ensure that $(y_1\overline{z}_1' - y_1'\overline{z}_1 + y_2\overline{z}_2' - y_2'\overline{z}_2)(0) = 0$, but $y_1(t) = y(t)$, $y_2(t) = y(-t)$, $z_1(t) = z(t)$, $z_2(t) = z(-t)$ for $t > 0$, so this equation becomes, $(y\overline{z}' - y'\overline{z})(0^+) = (y\overline{z}' - y'\overline{z})(0^-)$.

In this chapter, we use a transfer matrix to impose the required domain conditions. In the next section we will reconsider the forward scattering problem with a transfer condition present.

8.3 The Forward Scattering Problem

We now consider the scattering problem on two half-lines connected by a transfer condition at 0. As the form of the differential equation on each half-line has not changed and we have merely replaced continuity conditions at $x = 0$ by a transfer condition, hence we expect that most of the results seen thus far still hold.

Some properties of the transfer matrix which will simplify the theory will now be considered.

As mentioned in the introduction to this chapter, we need to relate the values of the Jost solutions $f_+(x)$ and $f_+'(x)$ as they approach 0 from the right to the values of $f_-(x)$ and $f_-'(x)$ as they approach 0 from the left by means of a transfer matrix condition. In particular, the eigencondition becomes

$$k \begin{pmatrix} f_+(0^+) \\ f_+'(0^+) \end{pmatrix} = M \begin{pmatrix} f_-(0^-) \\ f_-'(0^-) \end{pmatrix},$$

for some $k \in \mathbb{C} \setminus \{0\}$, where M is the transfer matrix, $f_+(x,\zeta)$ and $f_-(x,\zeta)$ are the Jost solutions as before.

Let M be given by

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad p, q, r, s \in \mathbb{R}.$$

We require that M be invertible as otherwise the boundary value problems on $(-\infty, 0)$ and $(0, \infty)$ are separate problems.

As the solution space is at most a 2-dimensional space and $F_+(x,\xi)$ and $\overline{F}_+(x,\xi)$ are linearly independent solutions for $\xi \in \mathbb{R} \setminus \{0\}$,, we must have that,

$$F_-(x,\xi) = a(\xi)\overline{F}_+(x,\xi) + b(\xi)F_+(x,\xi), \quad \xi \in \mathbb{R} \setminus \{0\}$$

We now define a domain for the operator T which incorporates the transfer condition and results in a self-adjoint operator in $L^2(J)$. As before, we require $f, f' \in L^2(J)$, f and f' absolutely continuous on $(0, \infty)$ and $(-\infty, 0)$ and of course the transfer condition

$$\begin{pmatrix} f(0^+) \\ f'(0^+) \end{pmatrix} = M \begin{pmatrix} f(0^-) \\ f'(0^-) \end{pmatrix}.$$

Hence we set

$$D(T) = \left\{ f \in D(L) \;\middle|\; \begin{pmatrix} f(0^+) \\ f'(0^+) \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} f(0^-) \\ f'(0^-) \end{pmatrix} \right\}.$$

The transfer matrix scattering problem can now be posed as

$$Ty = \zeta^2 y, \quad y \in D(T), \tag{8.3}$$

$T := -\frac{d^2}{dx^2} + q(x)$, where $q(x)$ obeys (3.4).

Let $f, g \in D(T)$. Define

$$S(f,g) \ := \ \langle Tf, g\rangle - \langle f, Tg\rangle \quad \text{for all } f, g \in D(T).$$

We would like to find conditions on the transfer matrix such that T is a formally self-adjoint operator, i.e. $\langle Tf, g\rangle - \langle f, Tg\rangle = 0 \quad$ for all $f, g \in D(T)$.

Theorem 8.3 *The operator T is a formally self-adjoint operator if and only if the transfer matrix M is such that* $\det M = 1$.

Proof: Let $f, g \in D(T)$. Then, as in Theorem 3.2,

$$\begin{aligned}
S(f,g) &= \langle Tf, g\rangle - \langle f, Tg\rangle \\
&= \int_{-\infty}^{\infty} (-f'' + qf)\overline{g}dt - \int_{-\infty}^{\infty} f(-\overline{g}'' + q\overline{g})dt \\
&= \int_{-\infty}^{\infty} -f''\overline{g} + f\overline{g}''dt \\
&= \int_{-\infty}^{0^-} [-f'\overline{g} + f\overline{g}']'dt + \int_{0^+}^{\infty} [-f'\overline{g} + f\overline{g}']'dt \\
&= (f\overline{g}' - f'\overline{g})(0^-) - (f\overline{g}' - f'\overline{g})(0^+) \\
&= [f, f']\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{bmatrix} \overline{g} \\ \overline{g}' \end{bmatrix}(0^-) - [f, f']\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{bmatrix} \overline{g} \\ \overline{g}' \end{bmatrix}(0^+).
\end{aligned}$$

Denote $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then we have that $J^T = -J$ and $J^2 = -I$.

Now,

$$\begin{bmatrix} g \\ g' \end{bmatrix}(0^+) = M\begin{bmatrix} g \\ g' \end{bmatrix}(0^-)$$

and

$$\begin{bmatrix} f & f' \end{bmatrix}(0^+) = \begin{bmatrix} f \\ f' \end{bmatrix}^T(0^+) = \begin{bmatrix} f & f' \end{bmatrix}(0^-)\ M^T.$$

So

$$\begin{aligned} S(f,g) &= [\ f \quad f'\ ](0^-)J\begin{bmatrix}\overline{g}\\ \overline{g}'\end{bmatrix}(0^-) - [\ f \quad f'\ ](0^-)M^TJM\begin{bmatrix}\overline{g}\\ \overline{g}'\end{bmatrix}(0^-) \\ &= [\ f \quad f'\ ](0^-)[J - M^TJM]\begin{bmatrix}\overline{g}\\ \overline{g}'\end{bmatrix}(0^-). \end{aligned}$$

Since $f(0^-)$, $f'(0^-)$, $g(0^-)$ and $g'(0^-)$ are arbitrary for $f, g \in D(T)$, we need

$$J = M^TJM. \tag{8.4}$$

The above expression becomes

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} p & r \\ q & s \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &= \begin{pmatrix} 0 & ps - rq \\ rq - ps & 0 \end{pmatrix}. \end{aligned}$$

Hence it is required that

$$\det M = ps - rq = 1. \quad \blacksquare$$

We have established that a transfer matrix will have determinant 1 for T to be a self-adjoint operator. We now describe the various types of transfer matrices within this class,

$$\begin{aligned} ps - rq = 1 \quad &: \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ ps = 1 \quad &: \quad \begin{pmatrix} p & 0 \\ r & s \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} \\ rq = -1 \quad &: \quad \begin{pmatrix} p & q \\ r & 0 \end{pmatrix}, \begin{pmatrix} 0 & q \\ r & s \end{pmatrix}, \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \end{aligned}$$

So there are 7 basic types of constant transfer matrices for the problem.

We now define the Jost solutions to problem (8.3). Since this problem can be considered as two half-line problems, solutions to (8.3) can be defined in terms of the Jost solutions $f_+(x,\zeta)$ and $f_-(x,\zeta)$ of the classical problem (3.1)-(3.3).

Definition 8.4 *The solutions $F_+(x,\zeta)$ and $F_-(x,\zeta)$ to (8.3) are defined by*

$$F_+(x,\zeta) := \begin{cases} f_+(x,\zeta), & x > 0 \\ h_1(x,\zeta), & x < 0 \end{cases} \tag{8.5}$$

$$F_-(x,\zeta) := \begin{cases} f_-(x,\zeta), & x < 0 \\ h_2(x,\zeta), & x > 0 \end{cases} \tag{8.6}$$

where $h_1(x,\zeta)$ is a solution of (1.1) on $(-\infty,0)$ obeying

$$\begin{pmatrix} h_1(0^-,\zeta) \\ h_1'(0^-,\zeta) \end{pmatrix} = M^{-1} \begin{pmatrix} f_+(0^+,\zeta) \\ f_+'(0^+,\zeta) \end{pmatrix}$$

and $h_2(x,\zeta)$ is a solution of (1.1) on $(0,\infty)$ obeying

$$\begin{pmatrix} h_2(0^+,\zeta) \\ h_2'(0^+,\zeta) \end{pmatrix} = M \begin{pmatrix} f_-(0^-,\zeta) \\ f_-'(0^-,\zeta) \end{pmatrix}.$$

8.4 Conjugate Solutions

As in the classical case, for $\zeta = \xi \in \mathbb{R}$, we may define the conjugate Jost solution. In this case we have for $\overline{F}_+(x,\xi)$,

$$\overline{F}_+(x,\xi) = \begin{cases} \overline{f}_+(x,\xi) = f(x,-\xi), & x > 0 \\ \overline{h}_1(x,\xi) = h_1(x,-\xi), & x < 0 \end{cases} \tag{8.7}$$

where the transfer condition holds at $x = 0$.

Since $F_+(x,\xi)$ and $\overline{F}_+(x,\xi)$ are equal to $f_+(x,\xi)$ and $\overline{f}_+(x,\xi)$ on $(0,\infty)$, we see from Theorem 4.12 that F_+ and $\overline{F}_+$ are linearly independent for all $\xi \in \mathbb{R}\setminus\{0\}$. Therefore, as in the classical case, there must exist functions $A(\xi)$ and $B(\xi)$ such that

$$F_-(x,\xi) = A(\xi)\overline{F}_+(x,\xi) + B(\xi)F_+(x,\xi). \tag{8.8}$$

As in (6.6) and (6.7), we have

$$A(\zeta) = \frac{-1}{2i\zeta} W[F_+(x,\zeta), F_-(x,\zeta)], \quad \Im(\zeta) \geq 0, \zeta \neq 0, \tag{8.9}$$

after analytic continuation and

$$B(\xi) = \frac{1}{2i\xi} W[\overline{F}_+(x,\zeta), F_-(x,\zeta)], \quad \xi \in \mathbb{R}. \tag{8.10}$$

The reflection coefficient can be defined for this case as

$$R(\xi) = \frac{B(\xi)}{A(\xi)}, \quad \xi \in \mathbb{R}. \tag{8.11}$$

As in (6.12), the eigencondition for the transfer condition scattering problem is

$$A(\zeta) = 0.$$

We now investigate some of the properties of $A(\zeta)$.

Lemma 8.5 *The function $A(\zeta)$ has a finite set of zeros.*

Proof: As in the classical case, in order for T to be self-adjoint, it can, in a manner similar to the classical case, be shown that ζ must be of the form $i\eta$ with $\eta \in \mathbb{R}^+$.

We now compute the asymptotic form of $A(\zeta)$. For $x > 0$, $x \to 0^+$ and $|\zeta|$ large,

$$\begin{aligned}
A(\zeta) &= \frac{-1}{2i\zeta} \det \begin{bmatrix} F_+(0^+,\zeta) & F_-(0^+,\zeta) \\ F_+'(0^+,\zeta) & F_-'(0^+,\zeta) \end{bmatrix} \\
&= \frac{-1}{2i\zeta} \det \begin{bmatrix} F_+(0^+,\zeta) & M \begin{pmatrix} F_-(0^-,\zeta) \\ F_-'(0^-,\zeta) \end{pmatrix} \\ F_+'(0^+,\zeta) & \end{bmatrix} \\
&= \frac{-1}{2i\zeta} \det \begin{bmatrix} f_+(0^+,\zeta) & pf_-(0^-,\zeta) + qf_-'(0^-,\zeta) \\ f_+'(0^+,\zeta) & rf_-(0^-,\zeta) + sf_-'(0^-,\zeta) \end{bmatrix} \\
&= \frac{-1}{2i\zeta} f_+(0^+,\zeta)[rf_-(0^-,\zeta) + sf_-'(0^-,\zeta)] - f_+'(0^+,\zeta)[pf_-(0^-,\zeta) + qf_-'(0^-,\zeta)].
\end{aligned}$$

Using the asymptotics results for the Jost solutions for large values of $|\zeta|$ we get

$$\begin{aligned}
A(\zeta) &= \frac{-1}{2i\zeta}(-q\zeta^2 - (p+s)i\zeta + r + O(1/\zeta)) \\
&= \frac{1}{2}(-iq\zeta + (p+s)) + O\left(\frac{1}{\zeta^2}\right) \qquad (8.12)
\end{aligned}$$

Now, $A(\zeta)$ is analytic on the upper half plane and thus on the imaginary axis, therefore, by the Identity theorem it can have at most a countable number of zeros. By the properties of the Jost solutions f_+ and f_-, it is seen that $A(\zeta) \not\to 0$ as $\zeta \to 0$ on the imaginary axis. By (8.12) and the fact that p, q, r, s are not all zero, $A(\zeta)$ does not have any zeros for large $|\zeta|$. Hence, $A(\zeta)$ has only a finite number of zeros. ■

Theorem 8.6 *The Scattering Problem with transfer condition has a finite number of eigenvalues.*

Proof: As in the classical case, the zeros of $A(\zeta)$ are precisely the eigenvalues of the problem. The result follows from Lemma 8.5. ■

Chapter 9

Conclusion

In this chapter we give a short summary of what has been achieved in this work and an overview of an approach for solving the inverse scattering problem with transfer condition at zero.

9.1 Summary

In this work, the first development was to define a suitable domain for the differential operator L to put equation (1.1) in the proper setting. We showed that L was symmetric. Theory of deficiency indices tells us that for a linear operator to have a self-adjoint extension it is necessary that the deficiency indices n_+ and n_- be equal and if $n_+ = n_- = 0$, then $\overline{L} = L^*$.

The Jost solutions to (3.3) were defined. These solutions have particularly nice properties being very close to exponential functions and indeed are precisely exponential functions in the degenerate case $q \equiv 0$.

The Jost solutions were used to solve the forward scattering problem. It was shown that L has only a finite number of simple eigenvalues. The functions $a(\zeta)$, $b(\xi)$ and the reflection coefficient $r(\xi)$ were defined. Although no explicit mention was made of the S-matrix found in the scattering theory literature, it can easily be incorporated into our study and can be defined as

$$S = \begin{pmatrix} \frac{1}{a(\xi)} & \frac{b(-\xi)}{a(\xi)} \\ \frac{-b(xi)}{a(\xi)} & \frac{1}{a(\xi)} \end{pmatrix}.$$

Using the S-matrix identical results would be obtained.

The reflectionless inverse case was solved. Since $r(\xi) = 0$, the integral part of the Gelfand-Levitan equation is zero and hence the became the matrix equation (7.2). Jost solutions f_+ and f_- were reconstructed together with the $a(\zeta)$ function.

For the transfer condition scattering problem, Jost solutions which obeyed the transfer condition at the origin were defined. $A(\zeta)$ which is the analog to $a(\zeta)$ was defined and shown to have a finite number of zeros which correspond to the square roots of the eigenvalues of the problem.

9.2 Overview of Reflectionless Inverse Transfer Condition Scattering Problem

For a reflectionless potential, we require $R(\xi) = \frac{B(\xi)}{A(\xi)} = 0$ and hence that $B(\xi) = 0$. We can then write

$$F_-(x, \xi) = A(\xi)\overline{F}_+(x, \xi).$$

As in the classical case, this can be used to define F_+ for $\Im(\zeta) < 0$ since F_- is analytic,

$$F_+(x, \zeta) = \frac{F_-(x, -\zeta)}{A(-\zeta)}.$$

Then $F_+(x, \zeta)$ has a finite set of poles at the zeros of $A(-\zeta)$ of the form $\zeta = -i\eta_j,\ j = 1, \ldots, N$.

A matrix equation can be set up as in (7.2) for this problem. Let us examine the function $B(\xi)$ more closely. $B(\xi)$ was defined in (8.10) as

$$B(\xi) = \frac{1}{2i\xi} W[\overline{F}_+(x,\xi)\ F_-(x,\xi)], \quad \xi \in \mathbb{R}.$$

Then for $x > 0$, $x \to 0^+$,

$$\begin{aligned} B(\xi) &= \frac{-1}{2i\xi} \det \begin{bmatrix} \overline{F}_+(0^+,\xi) & F_-(0^+,\xi) \\ \overline{F}'_+(0^+,\xi) & F'_-(0^+,\xi) \end{bmatrix} \\ &= \frac{-1}{2i\xi} \det \begin{bmatrix} \overline{F}_+(0^+,\xi) & M \begin{pmatrix} F_-(0^-,\xi) \\ F'_-(0^-,\xi) \end{pmatrix} \\ \overline{F}'_+(0^+,\xi) & \end{bmatrix}. \end{aligned}$$

For $|\xi|$ large, we get

$$B(\xi) = \frac{-1}{2i\xi}(q\xi^2 + i\xi(p-s) + r) + O\left(\frac{1}{\xi}\right). \tag{9.1}$$

By the above we see that $B(\xi)$ is of the same order as $A(\xi)$ in (8.12) for the transfer condition case and hence we have that $R(\xi) \not\equiv 0$. So in some sense there is always reflection present in the scattering when a transfer matrix condition exists. Due to the different cases of matrix for the problem, quite a few cases occur for the matrix equation to be solved. This will be considered in further studies.

9.3 Future Directions

As mentioned above, the inverse reflectionless case with transfer condition will be looked at considering all the cases for constant transfer matrix. Following that, eigenparameter dependent transfer matrices can be looked at. Eventually the full non-reflectionless case must be considered which will involve setting up the Gelfand-Levitan equation and to take into account the transfer conditon.

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