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TOPOLOGIES AND SMOOTH STRUCTURES ON INITIAL AND FINAL OBJECTS IN THE CATEGORY OF FRÖLICHER SPACES

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfillment of the requirements for the degree of Master of Science.

Johannesburg, 2019

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

_____

(Signature of candidate)

_____day of _____20 ___at _____

Abstract

The initial objects (in the category of Frölicher spaces) being studied are Frölicher subspace, product and equalizer's domain; and the final objects are Frölicher quotient, coproduct and coequalizer's codomain. For each object a canonical topology (from the category of topologies) is induced on the underlying set of the object, and Frölicher topologies are induced from the Frölicher structure. There are two Frölicher topologies for each object: a Frölicher topology induced from structure curves and a Frölicher topology induced from structure functions - it's shown that the former Frölicher topology is finer than the latter Frölicher topology for any Frölicher space. It's shown that for each initial object the canonical topology is coarser than the Frölicher topology induced from structure functions, and for each final object the canonical topology is finer than the Frölicher topology induced from structure curves. Furthermore we establish that the building structure for each object is constant and algorithmic.

I dedicate this work to my mother Ntombi Jester MHLANGA, and to the memory of my father Sebushi Johanness MAHUDU.

Acknowledgments

Foremost I acknowledge the supervisor of this dissertation: Dr Mukinayi Herménégilde TSHILOMBO. This dissertation was realized through his meticulous supervision, guidance, mentorship and support. The meetings and long discussions we had on the subject matter of this dissertation, discussions that even extended to politics, socio-economic and cultural issues, and current affairs - these were thought-provoking and made journeying in this dissertation interesting and enthusiastic. I could not have imagined a better supervisor.
We also acknowledge the funds that made this dissertation possible. This dissertation spanned for two years. In its first year it was funded by the NRF-Grant Bursary, and the DAAD-NRF Scholarship in its second year.
My family was my pillar of strength and encouragement in this endevour. Their overwhelming support, prayers, understanding and sacrifices are beyond measure and are unimaginable.
With all and sundry, above all, I give credit and praises to the almighty God for strength and wisdom. Sione weee!

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Chapter 1

Introduction.

Alfred Frölicher, a Swiss mathematician, is the father of Frölicher spaces, that is he was the first to work with and the one who discovered these spaces - Frölicher spaces. But at the time Alfred Frölicher called these spaces smooth spaces (see [12]). Frölicher spaces are sets endowed with a smooth structure - the Frölicher structure. They [Frölicher spaces] were called Frölicher spaces for the first time by P. Chereneck [10] in his paper (submitted in 1996) titled: *Frölicher versus differential spaces: a prelude to cosmology.*

Since Frölicher spaces are endowed with a Frölicher structure, which is a smooth structure, then the morphisms between Frölicher spaces are smooth maps. A collection of Frölicher spaces and smooth maps between the Frölicher spaces build the category of Frölicher spaces. In this dissertation we are "working" in the category of Frölicher spaces. A category can posses initial and final objects [1],[2],[4],[14] and thus from literature there are initial and final objects in the category of Frölicher spaces (see [5],[6],[7] and [10]). Since initial and final objects in the category of Frölicher spaces are themselves Frölicher spaces, then topologies [3],[15] can be induced on the underlying set and from the Frölicher structure of these objects. This dissertation aims to study and investigate to determine the topological relation of topologies in, and the building structure of initial and final objects in the category of Frölicher spaces.

The first chapter is the introduction as usual. The second chapter is preliminaries. In the second chapter we introduce concepts from category theory and general topology which will be used and are needed for the advancement and understanding of this dissertation; and then we build on Frölicher theory - that is we build on the theory of Frölicher spaces and Frölicher topologies, and build the category of Frölicher spaces.

The third chapter is on initial objects, with sections as the initial objects under our study, that is Frölicher subspace, Frölicher product and the domain of Frölicher equalizer. Similarly the fourth chapter is on final objects, with sections

as Frölicher quotient, Frölicher coproduct and the codomain of Frölicher coequalizer - the final objects under our study. For each of the sections in the third and fourth chapter, we have the structure, where we build the object, then the inducing of Frölicher topologies from the Frölicher structure of the object, then the inducing of a canonical topology on the underlying set of the object, and then we have topological relation where we determine the relationship of the canonical topology and the Frölicher topologies of that object.

The main result of this dissertation is the building structure and the topological relations for initial and final objects to be attained from the third and fourth chapters. For whatever topological relation and building structure is attained from these chapters, we then test these results on Frölicher tangent bundles, that is in the fifth chapter.

Chapter 2

Preliminaries.

2.1 Basic concepts from category theory.

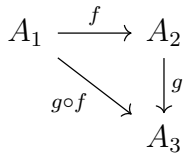
Category theory is a broad theory. Thus we are hereby not discussing the whole of category theory but only limited to concepts of category theory that are relevant and useful to this dissertation. These concepts of category theory to be discussed are important towards the formation of, and the understanding of the category of Frölicher spaces, of which is the category we are working on and working with.

2.1.1 Functions on a set.

We thereby assume that the concept of sets is understood from set theory. We hereby consider functions of sets.

Let A_1 and A_2 be non-empty sets. The arrow $f : A_1 \to A_2$ is a function if all elements of A_1 are assigned to an element of A_2, thus for every $x \in A_1$ there exists $f(x) \in A_2$. The set A_1 is called the domain of f and the set A_2 the codomain of f, denoted $dom(f)$ and $cod(f)$ respectively. The image of $f : A_1 \to A_2$ is given by the set $f(A_1) = \{f(x) \in A_2 \mid x \in A_1\}$, that is $f(X) \subseteq A_2$, while the pre-image is given by the set $f^{-1}(A_2) = \{x \in A_1 \mid f(x) \in A_2\}$, that is $f^{-1}(A_2) \subseteq A_1$.

Suppose now that A_3 is also non-empty and that $g : A_2 \to A_3$ is a function. Note that $cod(f) = Y = dom(g)$. Then there exists a composition function $g \circ f : A_1 \to A_3$ given by $(g \circ f)(x) = g(f(x)) \ \forall \ x \in A_1$. That is the triangle



commutes. That is for every pair of functions f and g such that

$cod(f) = dom(g)$ there exists a composition function $g \circ f : dom(f) \to cod(g)$.

Let A_4 be another non-empty set and $h : A_3 \to A_4$ a function. That is we have functions $f : A_1 \to A_2$, $g : A_2 \to A_3$ and $h : A_3 \to A_4$. Note that $cod(f) = Y = dom(g)$ and $cod(g) = Z = dom(h)$. Then $g \circ f : A_1 \to A_3$ and $h \circ g : A_2 \to A_4$ are defined by $(g \circ f)(x) = g(f(x)) \ \forall \ x \in A_1$ and $(h \circ g)(y) = h(g(y)) \ \forall \ y \in A_2$.
Then

$$\begin{aligned} [h \circ (g \circ f)](x) &= h \circ (g \circ f)(x) \\ &= h \circ (g(f(x))) \\ &= h(g(f(x))) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} [(h \circ g) \circ f](x) &= (h \circ g) \circ f(x) \\ &= h(g(f(x))) \end{aligned} \tag{2.2}$$

That is $[h \circ (g \circ f)](x) = [(h \circ g) \circ f](x), \forall \ x \in A_1$. Therefore $h \circ (g \circ f) = (h \circ g) \circ f$. Thus the operation $\circ$ of composition of functions is associative. That is the diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{f} & A_2 & & \\ & \searrow^{g \circ f} & \downarrow g & \searrow^{h \circ g} & \\ & & A_3 & \xrightarrow[h]{} & A_4 \end{array}$$

commutes.

For every non-empty set A_1 there exists an identity function $I_{A_1} : A_1 \to A_1$ such that $I_{A_1}(x) = x$ for every $x \in A_1$. Thus for every $f : A_1 \to A_2$ we have that $f \circ I_{A_1} = f$ and $I_{A_2} \circ f = f$ where $I_{A_1} : A_1 \to A_1$ and $I_{A_2} : A_2 \to A_2$ are identity functions on A_1 and A_2 respectively. That is, in the sense of abstract algebra, identity functions act as units for the operation $\circ$ of composition of functions. The diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{I_{A_1}} & A_1 & & \\ & \searrow^{f \circ I_{A_1}} & \downarrow f & \searrow^{I_{A_2} \circ f} & \\ & & A_2 & \xrightarrow[I_{A_2}]{} & A_2 \end{array}$$

illustrate this phenomenon, and reckon that the [above] diagram is commutative.

2.1.2 Category

Definition 2.1.1 ***Morphisms.***
A morphism is an arrow between two objects. Objects in this case refers to mathematical objects, for example sets, groups, topologies, algebras, spaces etc. That

is, for every morphism (or arrow) there exist a domain and a codomain, that is the object where the arrow starts and the object where the arrow ends respectively.

For example let g be an arrow from object A to object B, that is g is a morphism from object A to B. Then the morphism g can be represented as $g : A \to B$ or $A \xrightarrow{g} B$. In this case object A is the domain and object B is the codomain. A function or a mapping is an example of a morphism.

Definition 2.1.2 ***Category***.
A category is a collection of mathematical objects and associated morphisms between these mathematical objects. That is a category consists of mathematical objects $X_1, X_2, X_3, ...$ and morphisms $f_1, f_2, f_3, ...$ between these mathematical objects such that the following are satisfied:

1. *Identity: For each mathematical object X_i there exists a morphism $I_{X_i} : X_i \to X_i$ called the identity morphism.*
2. *Composition: For each pair of morphisms f_i and f_j $(i \neq j)$ with $cod(f_i) = dom(f_j)$, there exists a composition $f_j \circ f_i : dom(f_i) \to cod(f_j)$.*

The above conditions are subject to the following conditions:

1. *Associativity: For every morphisms f_i, f_j and f_k $(i \neq j \neq k \neq i)$ such that $f_i : X_1 \to X_2$, $f_j : X_2 \to X_3$ and $f_k : X_3 \to X_4$ where X_1, X_2, X_3 and X_4 are mathematical objects, we have that $f_k \circ (f_j \circ f_i) = (f_k \circ f_j) \circ f_i$.*
2. *Unity: For any morphism $f_n : X_k \to X_j$ we have that $f_n \circ I_{X_k} = f_n$ and $I_{X_j} \circ f_n = f_n$ where $I_{X_k} : X_k \to X_k$ and $I_{X_j} : X_j \to X_j$ are identity morphisms on X_k and X_j respectively.*

From now onwards objects will refer to mathematical objects and morphisms will refer to morphisms between the aforementioned mathematical objects.

The following are examples of categories:

Example 2.1.1 ***The category of sets.***
Let A_1, A_2 and A_3 be sets such that $f : A_1 \to A_2$ and $g : A_2 \to A_3$ are functions. We have seen from Subsection 2.1.1 that there exists a composition function $g \circ f : A_1 \to A_3$, and that for any set A_1 there exists an identity function $I_{A_1} : A_1 \to A_1$ defined as $I_{A_1}(x) = x$, $\forall x \in A_1$. Therefore the collection consisting of sets and functions on sets form a category - the category of sets.

Example 2.1.2 ***The category of posets.***
Let X be a non-empty set with the relation $\alpha \leq_X \beta$ for all $\alpha, \beta \in X$ such that:

1. $\alpha \leq_X \alpha$ *for every* $\alpha \in X$ *(reflexivity).*
2. $\alpha \leq_X \beta$ *and* $\beta \leq_X \gamma \Longrightarrow \alpha \leq_X \gamma \ \forall \alpha, \beta, \gamma \in X$ *(transitivity).*
3. $\alpha \leq_X \beta$ *and* $\beta \leq_X \alpha \Longrightarrow \alpha = \beta \ \forall \alpha, \beta \in X$ *(antisymmetry).*

Then X *is called a poset - a partially-ordered set.*

Let X *and* Y *be posets, then the morphism* $p : X \to Y$ *is a monotone function, that is* $x \leq_X y$ *implies that* $p(x) \leq_X p(y)$ *for every* $x, y \in X$*. Let* X *be a poset and* $I_X : X \to X$ *be an identity function defined by* $I_X(x) = x$ *for every* $x \in X$*. Note that* $x \leq_X x$ *implies that* $x \leq_X x$ *for every* $x \in X$*. That is* $x \leq_X x$ *implies that* $I_X(x) \leq_X I_X(x)$ *for every* $x \in X$*. Thus* I_X *is the identity monotone function. Let* Z *also be a poset and* $f : X \to Y$ *and* $g : Y \to Z$ *be monotone functions. That is* $\alpha \leq_X \beta$ *implies that* $f(\alpha) \leq_Y f(\beta)$ *for every* $\alpha, \beta \in X$ *and* $a \leq_Y b$ *implies that* $g(a) \leq_Z g(b)$ *for every* $a, b \in Y$*. Now consider the function* $g \circ f : X \to Z$*. Let* $\alpha, \beta \in X$ *such that* $\alpha \leq_X \beta$*. Since* $f(\alpha), f(\beta) \in Y$ *then*

$$\begin{aligned} (g \circ f)(\alpha) &= g(f(\alpha)) \\ &\leq_Z g(f(\beta)) \\ &= (g \circ f)(\beta) \end{aligned} \tag{2.3}$$

That is $g \circ f : X \to Z$ *is a monotone function. Thus the category of posets consists of posets as objects and monotone functions as morphisms.*

Example 2.1.3 ***Categories of structured sets.***
A poset is a set endowed with a structure. That is a poset is a structured set where monotone functions preserve the structure. Thus the category of posets is a category of a structured set. Structured sets are sets of which a structure is defined on with functions that preserve that structure (see [4]). Examples of categories of structured sets include but are not limited to:

1. *The category of differential manifolds: Objects are differential manifolds and morphisms are smooth functions.*
2. *The category of groups: Objects are groups and morphisms are group homomorphisms.*
3. *The category of graphs: Objects are graphs and morphisms are graph homomorphisms.*
4. *The category of topological spaces: Objects are topological spaces and morphisms are continuous functions.*
5. *The category of vector spaces: Objects are vector spaces and morphisms are linear mappings.*

2.1.3 Initial and final objects.

Definition 2.1.3 ***Initial object.***
An object I in the category $\mathcal{C}$ is an initial object if the morphism $I \to X$ is unique in $\mathcal{C}$ for every object X in $\mathcal{C}$.

The algebraic dual of initial objects are final objects.

Definition 2.1.4 ***Final object.***
An object F in the category $\mathcal{C}$ is a final object if the morphism $Y \to F$ is unique in $\mathcal{C}$ for every object Y in $\mathcal{C}$.

In other texts final objects are being referred to as terminal objects (see [1], [2] and [4]). Following are examples of initial and final objects in the category of sets and the category of posets:

1. In the category of sets the initial object is the empty set and any singleton is a final object. That is the category of sets has only one initial object and multiple final objects.

2. In the category of posets an object is an initial object if and only if it is the least element, that is the infimum, and an object is a final object if and only if it is the greatest element, that is the supremum. That is the category of posets only has one initial object and one final object.

From these two examples we observe that a category can have at least one initial object and at least one final object. Also a category can have neither an initial and a final object (see [1] and [4]). That is not all categories have initial and final objects. There are categories that have neither initial nor final objects. For example the poset $(\mathbb{Z}, \leq)$ has no initial object and no final object. Also note that there are certain objects which are both initial and final objects. Such objects are called zero objects (see [1]).

Remark 2.1.1 *In this dissertation the initial and final objects are objects with an initial and final structure respectively, not as per definition, this will be illustrated later.*

2.1.4 Functors.

Definition 2.1.5 ***Functor.***
A functor is a morphism between categories. Let $\mathcal{X}$ and $\mathcal{Y}$ be categories and $F : \mathcal{X} \to \mathcal{Y}$. Then $F : \mathcal{X} \to \mathcal{Y}$ is a functor if F preserves composition and identity morphisms. That is $F(I_{\mathcal{X}}) = I_{F(\mathcal{X})}$ where $I_{\mathcal{X}}$ and $I_{F(\mathcal{X})}$ are the identity morphism on $\mathcal{X}$ and $F(\mathcal{X})$ respectively, and $F(f \circ g) = F(f) \circ F(g)$ for any morphims f and g of $\mathcal{X}$.

The functor $F : \mathcal{X} \to \mathcal{Y}$ consists of the object function and the morphism function. The object function assigns every object in $\mathcal{X}$ to an object in $\mathcal{Y}$, and the morphism function assigns every morphism in $\mathcal{X}$ to a morphism in $\mathcal{Y}$. Thus every functor consists of an object and morphism function.

Following are examples of functors:

1. **The identity functor**: For any category $\mathcal{C}$, there is an identity functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ where the object function is given by $id_{\mathcal{C}}(c) = c$ for every object c of $\mathcal{C}$, and the morphism function is given by $id_{\mathcal{C}}(f) = f$ for every morphism f of $\mathcal{C}$.

2. **The forgetful or faithful functor**: For any construct $\mathcal{C}$ (that is a category of a structured set and structure preserving functions, see [1], [2], [4] and [14]), there is a forgetful functor (also called the faithful functor or underlying functor) $U : \mathcal{C} \to \mathbf{Set}$ where **Set** denote the category of sets, where in each case $U(c)$ is the object of **Set** and is the underlying set of $\mathcal{C}$, and $U(f)$ is a morphism in **Set**, and is the underlying function of the morphism f.

2.1.5 Sources and sinks.

Definition 2.1.6 ***Source.***
Let $\mathcal{C}$ be a category and let X and $\{Y_i\}_{i\in I}$ be an object and a family of objects of the category $\mathcal{C}$ respectively. The pair $(X, \{f_i\}_{i\in I})$ where $f_i : X \to Y_i$ is a family of morphisms in $\mathcal{C}$ is a source in $\mathcal{C}$. The family $\{Y_i\}_{i\in I}$ is called the codomain of the source.

The algebraic dual of a source is a sink.

Definition 2.1.7 ***Sink.***
Let $\tilde{\mathcal{C}}$ be a category and B and $\{A_j\}_{j\in J}$ be an object and a family of objects in the category $\tilde{\mathcal{C}}$ respectively. The pair $(\{g_j\}_{j\in J}, B)$ where $g_j : A_j \to B$ is a family of morphisms in $\tilde{\mathcal{C}}$ is a sink in $\tilde{\mathcal{C}}$. The family $\{g_j\}_{j\in J}$ is the domain of the sink.

2.1.6 Limits and colimits.

Definition 2.1.8 ***Diagram.***
A diagram in a category $\mathcal{C}$ is a functor $D : \mathcal{A} \to \mathcal{C}$ where $\mathcal{C}$ is the codomain of D and $\mathcal{A}$ is the domain of D. The category $\mathcal{A}$ is called the scheme of the diagram D.

There is absolutely no technical difference between a diagram and a functor, in actual fact a diagram is a functor but a functor is not necessarily a diagram, and

there is also no difference between a scheme and a category, that is a scheme is a special type of a category, and a category is not necessarily a scheme of a diagram. We however opt to use the aforementioned terminology when treating limits and colimits for reasons of their [limits and colimits] historical development (see [1]). We denote the image of morphism f under diagram D by D_f.

Definition 2.1.9 ***Limit.***
Let $D : \mathcal{A} \to \mathcal{C}$ be a diagram.

1. *A $\mathcal{C}$-source (that is a source in $\mathcal{C}$) $f_i : \mathcal{C} \to D_i$, $i \in Ob(\mathcal{A})$, that is i is an object of category $\mathcal{A}$, is said to be natural for D provided that for each $\mathcal{A}$-morphism (that is a morphism in the category $\mathcal{A}$) $d : i \to j$ we have that $f_j = D_d \circ f_i$ $(i \neq j)$ where $f_j : C \to D_j$, $j \in Ob(\mathcal{A})$. That is the triangle*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f_i} & D_i \\ & \searrow^{f_j} & \downarrow{\scriptstyle D_d} \\ & & D_j \end{array}$$

commutes.

2. *A limit of diagram D is a natural source $l_i : L \to D_i$, $i \in Ob(\mathcal{A})$, for D with the property that each natural source $f_i : \mathcal{C} \to D_i$, $i \in Ob(\mathcal{A})$, for D uniquely factors into it, that is for every such source there exists a unique morphism $f : \mathcal{C} \to L$ such that $f_i = l_i \circ f$. That is the triangle*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & L \\ & \searrow^{l_i \circ f} & \downarrow{\scriptstyle l_i} \\ & & D_i \end{array}$$

commutes.

The algebraic dual of limits is colimits.

Definition 2.1.10 ***Colimits.***
Let $D : A \to \mathcal{C}$ be a diagram.

1. *A $\mathcal{C}$-sink (that is a sink in the category $\mathcal{C}$) $f_i : D_i \to \mathcal{C}$ where i is an object of the category A, is said to be natural for D provided that for each A-morphism (that is a morphism in the category A) $d : i \to j$, we have that $f_i = f_j \circ D_d$, that is the triangle*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & L \\ & \searrow^{l_i \circ f} & \downarrow{\scriptstyle l_i} \\ & & D_i \end{array}$$

commutes.

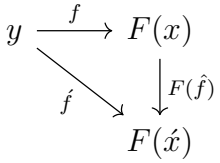
2. *A colimit of D is a natural sink $c_i : D_i \to K$ where i is an object of the category A, for D with the property that each natural sink for D uniquely factors through it.*

2.1.7 Cartesian-closed categories and exponential law.

Definition 2.1.11 *Let $\mathcal{X}$ and $\mathcal{Y}$ be categories, $F : \mathcal{X} \to \mathcal{Y}$ be a functor, and x and y be objects in the category $\mathcal{X}$ and $\mathcal{Y}$ respectively.*

1. *The morphism $f : y \to F(x)$ is called an F-structured morphism with domain y.*

2. *An F-structured morphism $f : y \to F(x)$ with domain y is called an F-universal morphism for y given that for each F-structured morphism $\acute{f} : y \to F(\acute{x})$ with domain y there exists a unique morphism $\hat{f} : x \to \acute{x}$ in the category $\mathcal{X}$ such that $\acute{f} = F(\hat{f}) \circ f$. That is the triangle*



commutes.

Definition 2.1.12 ***Adjoint and co-adjoint functors.***
A functor $F : \mathcal{X} \to \mathcal{Y}$ is an adjoint functor provided that for every object y in the category $\mathcal{Y}$ there exists an F-universal morphism with domain y. Dually, a functor $F : \mathcal{X} \to \mathcal{Y}$ is a co-adjoint functor provided that for every object y in the category $\mathcal{Y}$ the exists an F-universal morphism with codomain y.

In other texts adjoint and co-adjoint functors are refered to as left-adjoint and right-adjoint functors respectively.

Definition 2.1.13 ***Cartesian-closed categories.***
A category $\mathcal{C}$ is said to be cartesian-closed provided that it has final objects and finite products and for each object c in $\mathcal{C}$ the functor $(c \times -) : \mathcal{C} \to \mathcal{C}$ is a co-adjoint functor.

The following are special morphisms and functor isomorphisms in category theory:

Definition 2.1.14 ***Isomorphisms.***
Let $\mathcal{C}$ be a category, the morphism $u : A \to B$ in $\mathcal{C}$ is said to be an isomorphism in $\mathcal{C}$ if there exists a morphism $v : B \to A$ in $\mathcal{C}$ such that $v \circ u = I_A$ and $u \circ v = I_B$, where I_A is the identity morphism on A and I_B is the identity morphism on B in the category $\mathcal{C}$.

Definition 2.1.15 ***Isomorphic objects.***
The objects A and B in the category $\mathcal{C}$ are said to be isomorphic if there exists an isomorphism $u : A \to B$ in $\mathcal{C}$. We write $A \cong B$ to say that A and B are isomorphic.

Definition 2.1.16 ***Sections.***
A morphism $u : A \to B$ in a category $\mathcal{C}$ is called a section if there exists a morphism $v : B \to A$ in $\mathcal{C}$ such that $v \circ u = I_A$.

Definition 2.1.17 ***Retractions.***
A morphism $u : A \to B$ in a category $\mathcal{C}$ is called a retraction if there exists a morphism $v : B \to A$ in $\mathcal{C}$ such that $u \circ v = I_B$.

Following from the definitions of isomorphism, section and retraction we attain the following proposition:

Proposition 2.1.1 *If $u : A \to B$ is an isomorphism in a category $\mathcal{C}$ then u is both a section and a retraction.*

Definition 2.1.18 ***Monomorphism.***
A morphism $u : A \to B$ in a category $\mathcal{C}$ is said to be a monomorphism or an injection if for all pairs $h, k : C \to A$ of morphisms in $\mathcal{C}$ we have that $u \circ h = u \circ k$ implies that $h = k$.

Proposition 2.1.2 *Every section is a monomorphism.*

Proof: Let $u : A \to B$ be a morphism in the category $\mathcal{C}$ such that u is a section. That is there exists a morphism $v : B \to A$ such that $v \circ u = I_A$. Let $h, k : C \to A$ such that $u \circ h = u \circ k$. Then

$$\begin{aligned} h &= I_A \circ h \\ &= (v \circ u) \circ h \\ &= v \circ (u \circ h) \\ &= v \circ (u \circ k) \\ &= (v \circ u) \circ k \\ &= I_A \circ k \\ &= k \end{aligned} \tag{2.4}$$

Hence $u : A \to B$ is a monomorphism. □

Definition 2.1.19 ***Epimorphism.***
A morphism $u : A \to B$ in a category $\mathcal{C}$ is an epimorphism or a surjection given that for all pairs $m, n : B \to C$ of morphisms in $\mathcal{C}$ $m \circ u = n \circ u$ implies that $m = n$.

Proposition 2.1.3 *Every retraction is an epimorphism.*

Proof: Let $v : A \to B$ be a retraction in a category $\mathcal{C}$ such that $m \circ v = n \circ v$ where $m, n : B \to C$ is a pair of morphisms in $\mathcal{C}$. Since $v : A \to B$ is a retraction then there exists a morphism $z : B \to A$ in $\mathcal{C}$ such that $v \circ z = I_B$. Then

$$\begin{aligned} m &= m \circ I_B \\ &= m \circ (v \circ z) \\ &= (m \circ v) \circ z \\ &= (n \circ v) \circ z \\ &= n \circ (v \circ z) \\ &= n \circ I_B \\ &= n \end{aligned} \tag{2.5}$$

Hence $v : A \to B$ is an epimorphism. □

Then following from propositions 2.1.1, 2.1.2 and 2.1.3 we attain the following proposition:

Proposition 2.1.4 *Any isomorphism in a category is simultaneously an epimorphism and a monomorphism.*

That is by Proposition 2.1.4, any isomorphism is a bijection since any isomorphism is simultaneously an injection and a surjection.

Definition 2.1.20 ***Functor isomorphism.***
Let $\mathcal{X}$ and $\mathcal{Y}$ be categories, the functor $F : \mathcal{X} \to \mathcal{Y}$ is said to be an isomorphism provided that there exists a functor $G : \mathcal{Y} \to \mathcal{X}$ such that $G \circ F = id_{\mathcal{X}}$ and $F \circ G = id_{\mathcal{Y}}$, where $id_{\mathcal{X}}$ and $id_{\mathcal{Y}}$ are identity functors on $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Definition 2.1.21 ***Isomorphic categories.***
The categories $\mathcal{X}$ and $\mathcal{Y}$ are said to be isomorphic if there exists a functor $F : \mathcal{X} \to \mathcal{Y}$ such that F is an isomorphism. If $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic we write $\mathcal{X} \cong \mathcal{Y}$.

The property of cartesian-closure is linked with the exponential law in category theory. We thereby start with the concept of exponential law.

Definition 2.1.22 *Let $\mathcal{C}$ be a category, for every pair (A, B) of objects in $\mathcal{C}$, B^A is the set of all morphisms from A to B. That is*
$B^A = \{u : A \to B \mid u \text{ is a morphism in } \mathcal{C}\}$.

Definition 2.1.23 ***Exponential objects.***
Let $\mathcal{C}$ be a category with products and let A and B be objects in $\mathcal{C}$. We say that the exponential of A and B exists, that is A and B are exponential objects if there is a canonical isomorphism $B^{A\times C} \cong (B^A)^C$.

The canonical isomorphism $B^{A\times C} \cong (B^A)^C$ implies that there is a bijective correspondence between the morphisms $f : A \times C \to B$ and $g : C \to B^A$ and is called the exponential law. That is if the pair (A, B) of objects in a category $\mathcal{C}$ satisfies the exponential law then the objects A and B are called exponential objects or are said to be exponentiable. Furthermore a category is cartesian-closed if all objects in that category are exponential objects. Thus a category with finite products and final objects is cartesian-closed if and only if for each pair (A, B) of objects there exists an object B^A and a morphism $e : A \times B^A \to B$ with the following universal property:
For each morphism $f : A \times C \to B$ there exists a unique morphism $g : C \to B^A$ such that $f = e \circ (I_A \circ g)$, that is the triangle

$$\begin{array}{ccc} A \times C & \xrightarrow{I_A \circ g} & A \times B^A \\ & \searrow^{e} & \downarrow f \\ & & B \end{array}$$

commutes.

We call objects B^A power objects, the morphism $e : A \times B^A \to B$ the evaluation morphism and the morphism $g : C \to B^A$ associated with $f : A \times C \to B$ the exponential morphism for f.

Following are examples of cartesian-closed categories:

1. The category of sets: The category of sets is cartesian-closed with B^A being the set of all functions from A to B where (A, B) is any pair of objects in the category of sets; $e : A \times B^A \to B$ is the usual evaluation map; and $(g(c))(a) = f(a, c)$ where $g : C \to B^A$ is the exponential morphism associated with $f : A \times C \to B$ with A, B and C being objects in the category of sets and $a \in A$ and $c \in C$.

2. The category of posets: The category of posets is cartesian-closed with $(B, \leq_B)^{(A,\leq_A)}$ being the set of all order-preserving functions from A to B with the order as follows: $x \leq_A y$ if and only if $h(x) \leq_B h(y)\ \forall\ x, y \in A$ and $h \in (B, \leq_B)^{(A,\leq_A)}$ where $((B, \leq_B), (A, \leq_A))$ is any pair of objects in the category of posets; $e : (A, \leq_A) \times (B, \leq_B)^{(A,\leq_A)} \to (B, \leq_B)$ is the usual evaluation map; and $(g(c))(a) = f(a, c)$ where
$g : (C, \leq_C) \to (B, \leq_B)^{(A,\leq_A)}$ is the exponential morphism associated with $f : (A, \leq_A) \times (C, \leq_C) \to (B, \leq_B)$ with $(C, \leq_C)$ an object in the category of posets.

2.2 Basic concepts from general topology.

Definition 2.2.1 ***Topology.***
Let X be a non-empty set and τ be a collection of subsets of X, that is $\tau \subseteq P(X)$. The collection τ is a topology on X if:

1. $X, \phi \in \tau$.
2. $U_1, U_2, ..., U_n \in \tau \Longrightarrow \bigcap_{i=1}^{n} U_i \in \tau$.
3. $U_\alpha \in \tau,\ \alpha \in I \Longrightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau$.

If τ is a topology on X then the pair (X, τ) is called a topological space. For simplicity and that is if there is no confusion X oftentimes is used to represent the topological space (X, τ).

Following are examples of topologies:

Example 2.2.1 ***Trivial topology.***
Let X be a non-empty set and consider the collection $\{X, \phi\}$ where ϕ is the empty set. Then:

1. $X, \phi \in \{X, \phi\}$.
2. $X \bigcap \phi = \phi \in \{X, \phi\}$.
3. $X \bigcup \phi = X \in \{X, \phi\}$.

Therefore $\{X, \phi\}$ is a topology. It is called the trivial topology.

Example 2.2.2 ***Discrete topology.***
Let X be a non-empty set. Consider $P(X)$, the power set of X, that is all subsets of X. Thus $P(X) = \{U \mid U \subseteq X\}$. Then:

1. $\phi, X \in P(X)$ *since* $\phi, X \subseteq X$.
2. *Let $U_1, U_2, ..., U_n \in P(X)$, then $U_1, U_2, ..., U_n \subseteq X$, that is $U_i \subseteq X$ for all $i = 1, 2, ..., n$. Therefore $\bigcap_{i=1}^{n} U_i \subseteq X$ which implies that $\bigcap_{i=1}^{n} U_i \in P(X)$.*
3. *Let $\bigcup_{\alpha \in I} U_\alpha \in P(X)$ for $\alpha \in I$, then $U_\alpha \subseteq X\ \forall\ \alpha \in I$. Therefore $\bigcup_{\alpha \in I} U_\alpha \subseteq X$, which implies that $\bigcup_{\alpha \in I} U_\alpha \in P(X)$.*

Therefore $P(X)$ is a topology on X and is called the discrete topology.

Example 2.2.3 ***Finite complement topology.***
Let X be a non-empty set and let $\tau_f := \{U \subseteq X \mid X - U$ is finite or $X - U = X\}$.

1. *Since $\phi, X \in P(X)$ then $\phi, X \subseteq X$ and we have that $X - \phi = X$ and $X - X = \phi$. Therefore $\phi, X \in \tau_f$.*

2. *Let $W_1, W_2, ..., W_m \in \tau_f$, therefore $W_j \subseteq X$ and $X - W_j$ is finite or $X - W_j = X$ for all $j = 1, 2, ..., m$. Since $W_j \subseteq X$ for all $j = 1, 2, ..., m$ then $\bigcap_{j=1}^{m} W_j \subseteq X$. Also $X - \bigcap_{j=1}^{m} W_j = \bigcap_{j=1}^{m}(X - W_j)$. Therefore $X - \bigcap_{j=1}^{m} W_j$ is finite or all of X.*

3. *Let $W_\kappa \in \tau_f$, $\kappa \in K$, then $W_\kappa \subseteq X$ such that $X - W_\kappa$ is finite or all of X for all $\kappa \in K$. Therefore $\bigcup_{\kappa \in K} W_\kappa \subseteq X$ and $X - \bigcup_{\kappa \in K} W_\kappa = \bigcup_{\kappa \in K}(X - W_\kappa)$. Thus $X - \bigcup_{\kappa \in K} W_\kappa$ is finite or all of X.*

Hence τ_f is a topology on X and is called the finite complement topology.

Definition 2.2.2 ***Continuous morphism between topological spaces.*** *Let (X, τ_X) and (Y, τ_Y) be topological spaces, then the morphism $f : X \to Y$ is continuous if $f^{-1}(U) = \{x \in X \mid f(x) \in U\} \subseteq \tau_X$ for every $U \in \tau_Y$.*

Definition 2.2.3 ***Basis of a topology.***
Let X be a non-empty set. A collection β of subsets of X, that is $\beta \subseteq P(X)$, is a basis for a topology on X if:

1. *For every $x \in X$ there exists $B \in \beta$ such that $x \in B$.*

2. *If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subset B_1 \cap B_2$.*

The topology τ_β generated by the basis β of a topology on X is defined as $\tau_\beta = \{U \subseteq X \mid \forall x \in U \exists B \in \beta$ such that $x \in B \subset U\}$

A basis of a topology can be viewed as the building block or the DNA of that topology since a basis generates a topology. Thus every topology has a basis.

Following are examples of bases:

Example 2.2.4 *Consider the open interval $(a, b) = \{x \mid a < x < b\}$ in the real line $\mathbb{R}$. Let $\beta = \{(a, b) \mid a, b \in \mathbb{R}\}$, then the topology generated by β is called the standard topology on the real line, denoted $\tau_{\mathbb{R}}$. Thus β is the basis of the standard topology $\tau_{\mathbb{R}}$.*

Example 2.2.5 *Consider the half-open interval $[a, b) = \{x \mid a \leq x < b\}$ in the real line $\mathbb{R}$. Let $\tilde{\beta} = \{[a, b) \mid a, b \in \mathbb{R}\}$, then the topology generated by $\tilde{\beta}$ is called the lower limit topology. When $\mathbb{R}$ is endowed with the lower limit topology is denoted $\mathbb{R}_\ell$. Thus we denote the lower limit topology by $\tau_{\mathbb{R}_\ell}$. Thus $\tilde{\beta}$ is the basis of the lower limit topology $\tau_{\mathbb{R}_\ell}$. The topological space $(\mathbb{R}_\ell, \tau_{\mathbb{R}_\ell})$ is called the Sorgenfrey line.*

Example 2.2.6 *Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$, and let $a, b \in \mathbb{R}$ and $\hat{\beta}$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$. The topology generated by K is called the K-topology on $\mathbb{R}$. When $\mathbb{R}$ is endowed with the K-topology is denoted $\mathbb{R}_K$. Thus we denote the K-topology on $\mathbb{R}$ by $\tau_{\mathbb{R}_K}$. That is $\hat{\beta}$ is the basis of $\tau_{\mathbb{R}_K}$ and K is the basis of the K-topology on $\mathbb{R}$.*

2.3 Frölicher structures and spaces.

Let X be a non-empty set and let $\mathbb{R}^X$ be the set of all real-valued functions on X and $X^{\mathbb{R}}$ be the set of all curves into X. That is
$\mathbb{R}^X = \{f : X \to \mathbb{R} \mid \ f \text{ is a function}\}$ and $X^{\mathbb{R}} = \{c : \mathbb{R} \to X \mid \ c \text{ is a curve}\}$.
Now consider the power sets $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$, that is all the subsets of $\mathbb{R}^X$ and $X^{\mathbb{R}}$ respectively. We consider the inclusion maps in $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$.

Let $\{A, B, C\} \in P(\mathbb{R}^X) \cup P(X^{\mathbb{R}})$, that is $\{A, B, C\} \subseteq \mathbb{R}^X \cup X^{\mathbb{R}}$. Thus $\{A, B, C\}$ is a collection of sets of real-valued functions on X or sets of curves into X. Let $F : A \hookrightarrow B$ and $G : B \hookrightarrow C$ be inclusion maps. This implies that $A \subseteq B$ and $B \subseteq C$. Note that $cod(F) = B = dom(G)$. Since A, B and C are sets such that $A \subseteq B$ and $B \subseteq C$, then by transitivity we have that $A \subseteq C$. That is we have the composition function $G \circ F : A \hookrightarrow C$. Now consider any $D \in P(\mathbb{R}^X) \cup P(X^{\mathbb{R}})$ and let $I_D : D \to D$ be defined by $I_D(d) = d$ for all $d \in D$. That is d is some function $d : X \to \mathbb{R}$ or some curve $d : \mathbb{R} \to X$. Then $I_D : D \to D$ is an inclusion map since $D \subseteq D$. That is I_D is the identity map in $P(\mathbb{R}^X) \cup P(X^{\mathbb{R}})$. That is the power sets $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$ ordered by inclusion are categories with subsets of $\mathbb{R}^X$ and $X^{\mathbb{R}}$ as objects respectively, and inclusion and identity maps as morphisms.

Let $\Gamma : P(\mathbb{R}^X) \to P(X^{\mathbb{R}})$ and $\Phi : P(X^{\mathbb{R}}) \to P(\mathbb{R}^X)$ be defined by

$$\Gamma F := \{c : \mathbb{R} \to X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F\}$$

and

$$\Phi C := \{f : X \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C\}$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ is the set of all differentiable maps from $\mathbb{R}$ to $\mathbb{R}$. That is $C^\infty(\mathbb{R}, \mathbb{R})$ is a set of all smooth maps from $\mathbb{R}$ to $\mathbb{R}$. Note that F is a set of real-valued functions on X and C is a set of curves into X since $F \in P(\mathbb{R}^X)$ implies that $F \subseteq \mathbb{R}^X$ and $C \in P(X^{\mathbb{R}})$ implies that $C \subseteq X^{\mathbb{R}}$, respectively. That is the rule Γ determines a set of curves ΓF into X for any set of real-valued functions F on X, and similarly the rule Φ determines a set of real-valued functions ΦC on X for any set of curves C into X. That is $\Gamma(Ob(P(\mathbb{R}^X))) = Ob(P(X^{\mathbb{R}}))$ and $\Phi(Ob(P(X^{\mathbb{R}}))) = Ob(P(\mathbb{R}^X))$ where $Ob(P(\mathbb{R}^X))$ and $Ob(P(X^{\mathbb{R}}))$ denotes objects of categories $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$ respectively.

Definition 2.3.1 ***Frölicher structure.***
Let X be a non-empty set. A Frölicher structure on X is the pair (C_X, F_X) where

$C_X \in P(X^{\mathbb{R}})$ and $F_X \in P(\mathbb{R}^X)$ with the duality or compatibility condition that

$$\Gamma F_X = C_X$$

and

$$\Phi C_X = F_X$$

That is

$$C_X = \Gamma F_X = \{c : \mathbb{R} \to X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F_X\} \tag{2.6}$$

and

$$F_X = \Phi C_X = \{f : X \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C_X\} \tag{2.7}$$

where $\Gamma : P(\mathbb{R}^X) \to P(X^{\mathbb{R}})$ and $\Phi : P(X^{\mathbb{R}}) \to P(\mathbb{R}^X)$. Frölicher spaces were previously called smooth spaces (see [12])

Definition 2.3.2 ***Frölicher space.***
Let X be a non-empty set. A Frölicher space is the triple (X, C_X, F_X) where (C_X, F_X) is a Frölicher structure on X. That is a Frölicher space is a non-empty set endowed with a Frölicher structure.

That is if (X, C_X, F_X) is a Frölicher space, then the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{c} & X \\ & \searrow^{f\circ c} & \downarrow f \\ & & \mathbb{R} \end{array}$$

commutes, for every $f \in F_X = \Phi C_X$ and $c \in C_X = \Gamma F_X$.

Remark 2.3.1 *For simplicity and that is if there is no confusion, we write X for the Frölicher space (X, C_X, F_X).*

Lemma 2.3.1 *Let (X, C_X, F_X) be a Frölicher space. Then $c \in C_X \iff F_X \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})$ and $f \in F_X \iff f \circ C_X \subset C^\infty(\mathbb{R}, \mathbb{R})$. And $F_X \circ C_X \subset C^\infty(\mathbb{R}, \mathbb{R})$.*

The proof of Lemma 2.3.1 follows from the definition of Frölicher structure and Frölicher space.

Lemma 2.3.2 *Let (X, C_X, F_X) be a Frölicher space and $\mathbb{R}^X$ and $X^{\mathbb{R}}$ be the set of all real-valued functions on X and the set of all curves into X respectively. Given $F_0, F_1 \in P(\mathbb{R}^X)$ and $C_0, C_1 \in P(X^{\mathbb{R}})$.*

1. *If $F_0 \subseteq F_1$ then $\Gamma F_1 \subseteq \Gamma F_0$, $F_0 \subseteq \Phi\Gamma F_0$ and $\Gamma F_0 = \Gamma\Phi\Gamma F_0$.*
2. *If $C_0 \subseteq C_1$ then $\Phi C_1 \subseteq \Phi C_0$, $C_0 \subseteq \Gamma\Phi C_0$ and $\Phi C_0 = \Phi\Gamma\Phi C_0$.*

Proof: See proof in [12]. □

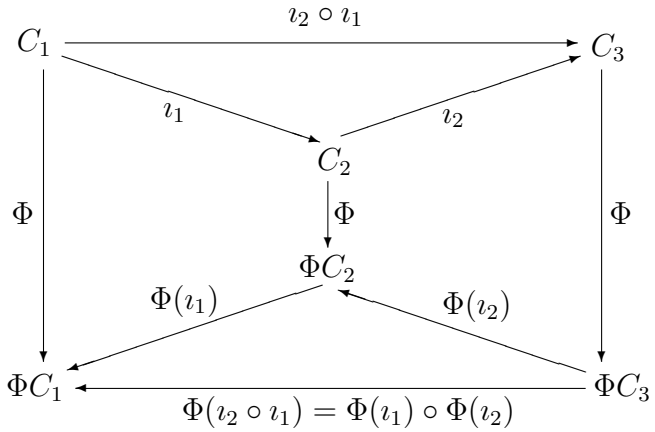
Let $\Gamma : P(\mathbb{R}^X) \to P(X^{\mathbb{R}})$ and $\Phi : P(X^{\mathbb{R}}) \to P(\mathbb{R}^X)$ be morphisms between the cartegories $P(\mathbb{R}^X)$ and $P(X^{\mathbb{R}})$ as defined in Section 2.3, that is

$$\Gamma F := \{c : \mathbb{R} \to X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F\}$$

and

$$\Phi C := \{f : X \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C\}$$

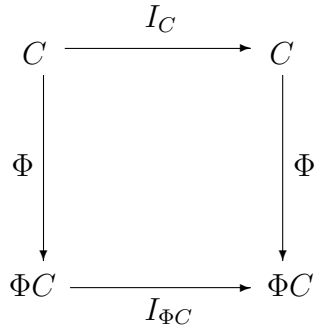
Let $C_1, C_2, C_3 \in P(X^{\mathbb{R}})$ such that $C_1 \subseteq C_2 \subseteq C_3$. That is C_1, C_2 and C_3 are sets of curves into X. Then we have that $\Phi : C_i \mapsto \Phi C_i$ where $\Phi C_i \in P(\mathbb{R}^X)$ for $i = 1, 2, 3$, that is ΦC_1, ΦC_2 and ΦC_3 are sets of real-valued functions on X. Now since $C_1 \subseteq C_2 \subseteq C_3$ in $P(X^{\mathbb{R}})$, then we have the inclusion maps $\imath_1 : C_1 \hookrightarrow C_2$ and $\imath_2 : C_2 \hookrightarrow C_3$ in $P(X^{\mathbb{R}})$. Observe that $\imath_1$ and $\imath_2$ are morphisms in $P(X^{\mathbb{R}})$ since $P(X^{\mathbb{R}})$ is a category. But since $C_1 \subseteq C_2 \subseteq C_3$ then by Lemma 2.3.2 $\Phi C_3 \subseteq \Phi C_2 \subseteq \Phi C_1$, that is we have the inclusion maps $\Phi(\imath_1) : \Phi C_2 \hookrightarrow \Phi C_1$ and $\Phi(\imath_2) : \Phi C_3 \hookrightarrow \Phi C_2$ as morphisms in $P(X^{\mathbb{R}})$. Seeing that $cod(\imath_1) = dom(\imath_2)$, that is the codomain of $\imath_1$ is equivalent to the domain of $\imath_2$, then we have the inclusion $\imath_2 \circ \imath_1 : C_1 \hookrightarrow C_3$. Then since $C_1 \subseteq C_3$ then we have that $\Phi C_3 \subseteq \Phi C_1$ by Lemma 2.3.2, that is the inclusion $\Phi(\imath_2 \circ \imath_1) : \Phi C_3 \hookrightarrow \Phi C_1$ is the corresponding morphism in $P(\mathbb{R}^X)$. But also $cod(\Phi(\imath_2)) = dom(\Phi(\imath_1))$, then we have the inclusion $\Phi(\imath_1) \circ \Phi(\imath_2) : \Phi C_3 \hookrightarrow \Phi C_1$, which is a morphism in $P(\mathbb{R}^X)$. Then it follows that $\Phi(\imath_2 \circ \imath_1) = \Phi(\imath_1) \circ \Phi(\imath_2)$. The whole phenomenon is illustrated in the diagram below:



with the arrows $\imath_1$, $\imath_2$, $\imath_2 \circ \imath_1$, $\Phi(\imath_1)$, $\Phi(\imath_2)$ and $\Phi(\imath_2 \circ \imath_1)$ as inclusion maps.

In the same vein, let C be any element in $P(X^{\mathbb{R}})$, consider the identity map $I_C : C \to C$. Note that $I_C : C \to C$ is a morphism in $P(X^{\mathbb{R}})$ since $C \subseteq C$. Also $\Phi : C \mapsto \Phi C$ and since $C \subseteq C$ then by Lemma 2.3.2 $\Phi C \subseteq \Phi C$ and thus we have the identity map $I_{\Phi C} : \Phi C \to \Phi C$ in $P(\mathbb{R}^X)$. Therefore it follows that

$\Phi(I_C) = I_{\Phi C}$. The diagram



summarizes the aforementioned phenomenon.

The same process can be applied for $\Gamma : P(\mathbb{R}^X) \to P(X^{\mathbb{R}})$, hence Φ and Γ are functors.

Theorem 2.3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then f is smooth on $\mathbb{R}^n$ if and only if $f \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for each curve $c : \mathbb{R} \to \mathbb{R}^n$. That is $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each curve $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$.*

Theorem 2.3.1 is the result of the work by Boman in [8], and thus for the proof of this theorem refer to [8].

Following are examples of Frölicher spaces:

Example 2.3.1 ***The canonical Frölicher space.***
Let $(\mathbb{R})^{\mathbb{R}^n}$ with $n \in \mathbb{N}$ be the set of all functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $(\mathbb{R}^n)^{\mathbb{R}}$ be the set of all curves into $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^n$ and $c : \mathbb{R} \to \mathbb{R}^n$ be a curve into $\mathbb{R}^n$, and let $\Gamma_{\mathbb{R}^n} : P((\mathbb{R})^{\mathbb{R}^n}) \to P((\mathbb{R}^n)^{\mathbb{R}})$ and $\Phi_{\mathbb{R}^n} : P((\mathbb{R}^n)^{\mathbb{R}}) \to P((\mathbb{R})^{\mathbb{R}^n})$ be functors, that is $P((\mathbb{R})^{\mathbb{R}^n})$ and $P((\mathbb{R}^n)^{\mathbb{R}})$ are categories (refer to Section 2.3) where $\Gamma_{\mathbb{R}^n}$ and $\Phi_{\mathbb{R}^n}$ are defined by

$$\Gamma_{\mathbb{R}^n} F = \{c : \mathbb{R} \to \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F\}$$

and

$$\Phi_{\mathbb{R}^n} C = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C\}$$

Let $F = C^\infty(\mathbb{R}, \mathbb{R})$. Then since $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each $f \in F = C^\infty(\mathbb{R}, \mathbb{R})$, then $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ by Theorem 2.3.1. Similarly let $C = C^\infty(\mathbb{R}, \mathbb{R}^n)$, then since $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each $c \in C^\infty(\mathbb{R}, \mathbb{R}^n)$, then $f \in C^\infty(\mathbb{R}^n, \mathbb{R}) = F$. That is

$$\begin{aligned} \Gamma_{\mathbb{R}^n} C^\infty(\mathbb{R}^n, \mathbb{R}) &= \{c : \mathbb{R} \to \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in C^\infty(\mathbb{R}^n, \mathbb{R})\} \\ &= C^\infty(\mathbb{R}, \mathbb{R}^n) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \Phi_{\mathbb{R}^n} C^\infty(\mathbb{R}, \mathbb{R}^n) &= \{f : \mathbb{R}^n \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C^\infty(\mathbb{R}, \mathbb{R}^n)\} \\ &= C^\infty(\mathbb{R}^n, \mathbb{R}) \end{aligned} \tag{2.9}$$

Therefore $(C^\infty(\mathbb{R},\mathbb{R}^n), C^\infty(\mathbb{R}^n,\mathbb{R}))$ *is a Frölicher structure. Hence the triple* $(\mathbb{R}^n, C^\infty(\mathbb{R},\mathbb{R}^n), C^\infty(\mathbb{R}^n,\mathbb{R}))$ *is a Frölicher space. That is the diagram*

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{c} & \mathbb{R}^n \\ & \searrow^{f\circ c} & \downarrow f \\ & & \mathbb{R} \end{array}$$

commutes, for every $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$ *and* $c \in C^\infty(\mathbb{R},\mathbb{R}^n)$*. The Frölicher space* $(\mathbb{R}^n, C^\infty(\mathbb{R},\mathbb{R}^n), C^\infty(\mathbb{R}^n,\mathbb{R}))$ *is called the canonical Frölicher space.*

Example 2.3.2 ***Manifolds.***
Let $\mathbb{R}^M$ *and* $M^{\mathbb{R}}$ *be the set of all real-valued functions on* M *and a set of all curves into* M *respectively, where* M *is a manifold. Let* $f : M \to \mathbb{R}$ *and* $c : \mathbb{R} \to M$ *be a smooth real-valued function on* M *and a smooth curve into* M *respectively. Then* $f \circ c : \mathbb{R} \to \mathbb{R}$ *is a smooth map since* M *is (that is manifolds are) smooth. Let* $\Gamma_M : P(\mathbb{R}^M) \to P(M^{\mathbb{R}})$ *and* $\Phi_M : P(M^{\mathbb{R}}) \to P(\mathbb{R}^M)$ *be functors defined by*

$$\Gamma_M F = \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}), \forall f \in F\}$$

and

$$\Phi_M C = \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}), \forall c \in C\}$$

Then since $c \in C^\infty(\mathbb{R},M)$*,* $f \in C^\infty(M,\mathbb{R})$ *and* $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$*, then we have that*

$$\begin{aligned} \Gamma_M C^\infty(M,\mathbb{R}) &= \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}), \forall f \in C^\infty(M,\mathbb{R})\} \\ &= C^\infty(\mathbb{R},M) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \Phi_M C^\infty(\mathbb{R},M) &= \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R}), \forall c \in C^\infty(\mathbb{R},M)\} \\ &= C^\infty(M,\mathbb{R}) \end{aligned} \tag{2.11}$$

Therefore the pair $(C^\infty(\mathbb{R},M), C^\infty(M,\mathbb{R}))$ *is a Frölicher structure, and thus the triple* $(M, C^\infty(\mathbb{R},M), C^\infty(M,\mathbb{R}))$ *is a Frölicher space. That is the diagram*

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{c} & M \\ & \searrow^{f\circ c} & \downarrow f \\ & & \mathbb{R} \end{array}$$

commutes, for every $f \in C^\infty(M,\mathbb{R})$ *and* $c \in C^\infty(\mathbb{R},M)$*.*

2.4 The category of Frölicher spaces

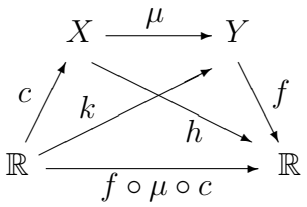
Definition 2.4.1 ***Smooth maps.***
Let $\mu : X \to Y$ *be a mapping where* (X, C_X, F_X) *and* (Y, C_Y, F_Y) *are Frölicher spaces on* X *and* Y *respectively. The mapping* $\mu : X \to Y$ *is a smooth map if*

$$\{\mu \circ c \mid c \in C_X\} \subset C_Y$$

and

$$\{f \circ \mu \mid f \in F_Y\} \subset F_X.$$

That is $\mu : X \to Y$ is a smooth map if $f \circ \mu : X \to \mathbb{R}$, $\mu \circ c : \mathbb{R} \to Y$ and $f \circ \mu \circ c : \mathbb{R} \to \mathbb{R}$ are smooth, that is $f \circ \mu \in C^\infty(X, \mathbb{R})$, $\mu \circ c \in C^\infty(\mathbb{R}, Y)$ and $f \circ \mu \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ where $f \in F_Y \in P(\mathbb{R}^Y)$ and $c \in C_X \in P(X^\mathbb{R})$ where $\mathbb{R}^Y$ is the set of all real-valued functions on Y and $X^\mathbb{R}$ is the set of all curves into X. Then we have the diagram



with $h := f \circ \mu$ and $k := \mu \circ c$.

Let X, Y and Z be Frölicher spaces, that is $X = (X, C_X, F_X)$, $Y = (Y, C_Y, F_Y)$ and $Z = (Z, C_Z, F_Z)$. Let $\mu : X \to Y$ and $\rho : Y \to Z$ be smooth maps, that is $\mu_1 := \{\mu \circ c \mid c \in C_X\} \subset C_Y$, $\mu_2 := \{f \circ \mu \mid f \in F_Y\} \subset F_X$,
$\rho_1 := \{\tilde{f} \circ \rho \mid \tilde{f} \in F_Z\} \subset F_Y$ and $\rho_2 := \{\rho \circ \tilde{c} \mid \tilde{c} \in C_Y\} \subset C_Z$. Note that $cod(\mu) = dom(\rho)$, therefore $\rho \circ \mu : X \to Z$. We have to show that $\rho \circ \mu : X \to Z$ is a smooth map. Let $\acute{c} : \mathbb{R} \to X$ and $\acute{f} : Z \to \mathbb{R}$ be such that $\acute{c} \in C_X$ and $\acute{f} \in F_Z$, and let $f = \acute{f} \circ \rho$ and $\tilde{c} = \mu \circ \acute{c}$ where $f \in F_Y$ and $\tilde{c} \in C_Y$. Then

$$\begin{aligned} \acute{f} \circ (\rho \circ \mu) &= (\acute{f} \circ \rho) \circ \mu \\ &= f \circ \mu \in \mu_2 \subset F_X \end{aligned} \tag{2.12}$$

Therefore $\acute{f} \circ (\rho \circ \mu) \in F_X$. And

$$\begin{aligned} (\rho \circ \mu) \circ \acute{c} &= \rho \circ (\mu \circ \acute{c}) \\ &= \rho \circ \tilde{c} \in \rho_2 \subset C_Z \end{aligned} \tag{2.13}$$

Therefore $(\rho \circ \mu) \circ \acute{c} \in C_Z$. That is $\{\acute{f} \circ (\rho \circ \mu) \mid \acute{f} \in F_Z\} \subset F_X$ and $\{(\rho \circ \mu) \circ \acute{c} \mid \acute{c} \in C_X\} \subset C_Z$. Hence $\rho \circ \mu : X \to Z$ is a smooth map.

Let W be a Frólicher space, that is $W = (W, C_W, F_W)$, such that
$I_W : W \to W$ is defined by $I_W(w) = w$, for all $w \in W$. Let $f \in F_W$ and $c \in C_W$, then $f \circ I_W = f$ and $I_W \circ c = c$. That is $\{f \circ I_W \mid f \in F_W\} \subset F_W$ and $\{I_W \circ c \mid c \in C_W\} \subset C_W$. Thus $I_W : W \to W$ is an identity smooth map.

Therefore Frölicher spaces and smooth maps form a category. We have hereby defined all the tools necessary for the category of Frölicher spaces. The category of Frölicher spaces consists of Frölicher spaces as objects and smooth maps as

morphisms (see [6], [7], [17], [18] and [19]).

The category of Frölicher spaces possesses some interesting properties (see [12]). The main properties are:

1. **The category of Frölicher spaces is complete and cocomplete.** That is the category of Frölicher spaces has limits and colimits (see definitions 2.1.9 and 2.1.10).

2. **The category of Frölicher spaces is Cartesian closed** (see [6] and [13]). Consider the set $C^\infty(Y,Z)$ of all smooth maps from the Frölicher space Y to the Frölicher space Z. For any Frölicher spaces X, Y and Z, the set $C^\infty(Y,Z)$ satisfies the isomorphism

$$C^\infty(X\times Y,Z)\cong C^\infty(X,C^\infty(Y,Z)).$$

If $X=\mathbb{R}$, that is $C^\infty(\mathbb{R}\times Y,Z)\cong C^\infty(\mathbb{R},C^\infty(Y,Z))$, then we construct the set $C_{Y,Z}$ of curves $c:\mathbb{R}\to C^\infty(Y,Z)$ by requiring that the map $\hat{c}:\mathbb{R}\times Y\to Z$ defined by $\hat{c}(t,y):=c(t)(y)$ is smooth.
Let $F_{Y,Z}$ be a set of real-valued functions $\bar{f}:C^\infty(Y,Z)\to\mathbb{R}$ such that $\bar{f}\circ c\in C^\infty(\mathbb{R},\mathbb{R})$ for all $c\in C_{Y,Z}$. Let $\mathbb{R}^{C^\infty(Y,Z)}$ and $(C^\infty(Y,Z))^{\mathbb{R}}$ be the set of all real-valued functions on $C^\infty(Y,Z)$ and the set of all curves into $C^\infty(Y,Z)$ respectively, and let $\Gamma_\infty:P(\mathbb{R}^{C^\infty(Y,Z)})\to P((C^\infty(Y,Z))^{\mathbb{R}})$ and $\Phi_\infty:P((C^\infty(Y,Z))^{\mathbb{R}})\to P(\mathbb{R}^{C^\infty(Y,Z)})$ be functors defined by

$$\Gamma_\infty F:=\{c:\mathbb{R}\to C^\infty(Y,Z)\mid \bar{f}\circ c\in C^\infty(\mathbb{R},\mathbb{R}),\forall\bar{f}\in F\}$$

and

$$\Phi_\infty C:=\{\bar{f}:C^\infty(Y,Z)\to\mathbb{R}\mid \bar{f}\circ c\in C^\infty(\mathbb{R},\mathbb{R}),\forall c\in C\}$$

Now let $F=F_{Y,Z}$ and $C=C_{Y,Z}$, then since $\bar{f}\circ c\in C^\infty(\mathbb{R},\mathbb{R})$ for all $c\in C_{Y,Z}=C$ with $\bar{f}\in F_{Y,Z}$, therefore $\Phi_\infty(C_{Y,Z})=F_{Y,Z}$. Then it follows that

$$\begin{aligned}\Gamma_\infty F_{Y,Z}&=\{c:\mathbb{R}\to C^\infty(Y,Z)\mid \bar{f}\circ c\in C^\infty(\mathbb{R},\mathbb{R}),\forall\bar{f}\in F_{Y,Z}\}\\&=C_{Y,Z}\end{aligned}\tag{2.14}$$

That is the duality or compatibility condition holds. Therefore $(C_{Y,Z},F_{Y,Z})$ is a Frölicher structure on $C^\infty(Y,Z)$. Thus the triple $(C^\infty(Y,Z),C_{Y,Z},F_{Y,Z})$ is a Frölicher space. Then the canonocal isomorphism

$$C^\infty(X\times Y,Z)\cong C^\infty(X,C^\infty(Y,Z))$$

is an exponential law for the category of Frölicher spaces and it holds for every Frölicher space X, Y and Z. Therefore the category of Frölicher spaces is cartesian-closed.

3. **The category of Frölicher spaces is topological over the category of sets.** That is the faithful (forgetful or underlying) functor from the category of Frölicher spaces to the category of sets is faithful (see Subsection 2.1.4) and the category of Frölicher spaces is contained in the category of topological spaces since Frölicher spaces are topological spaces.

4. **Initial and final objects exist in the category of Frölicher spaces** (see [5], [6], [7], [9], [10], [12], [17], [18] and [19]).

2.5 Frölicher topologies.

Let (X, C_X, F_X) be a Frölicher space and consider the collection

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V), f \in F_X, V \in \tau_{\mathbb{R}}\}$$

Since $\phi, X \in P(X)$ then $\phi, X \subseteq X$. We have that $f \in F_X$, therefore $f(x) \in \mathbb{R}$ for every $x \in X$. Then there exists an open interval $\bar{V}$ on $\mathbb{R}$ such that $\bar{V} \in \beta = \{(x, y) \mid x, y \in \mathbb{R}\}$ where β is a basis for a topology on $\mathbb{R}$ such that $f(x) \in \bar{V}$ for every $x \in X$. Therefore $f^{-1}(\bar{V}) = \{x \in X \mid f(x) \in \bar{V}\} = X$. Recall that $\tau_{\mathbb{R}}$ is a topology, that is a standard topology on $\mathbb{R}$, therefore $\phi \in \tau_{\mathbb{R}}$ and $f^{-1}(\phi) = \phi$. That is $\phi, X \in \tau_{F_X}$.

Let $U_1, U_2, ..., U_n \in \tau_{F_X}$, therefore $U_i \subseteq X$ such that $U_i = f^{-1}(V_i)$ where $V_i \in \tau_{\mathbb{R}}$ for all $i = 1, 2, ..., n$ and $f \in F_X$. Then $\bigcap_{i=1}^n U_i \subseteq X$ since $U_i \subseteq X$ for all $i = 1, 2, ..., n$ and $\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n f^{-1}(V_i) = f^{-1}(\bigcap_{i=1}^n V_i)$. But $\bigcap_{i=1}^n V_i \in \tau_{\mathbb{R}}$ since $V_i \in \tau_{\mathbb{R}}$ for all $i = 1, 2, ..., n$, therefore $\bigcap_{i=1} U_i \in \tau_{F_X}$.

Now let $A_\eta \in \tau_{F_X}$, $\eta \in N$. Therefore $A_\eta \subseteq X$ such that $A_\eta = f^{-1}(V_\eta)$ where $V_\eta \in \tau_{\mathbb{R}}$, $\eta \in N$ and $f \in F_X$. Then $\bigcup_{\eta \in N} A_\eta \subseteq X$ since $A_\eta \subseteq X$, $\eta \in N$. Also $\bigcup_{\eta \in N} A_\eta = \bigcup_{\eta \in N} f^{-1}(V_\eta) = f^{-1}(\bigcup_{\eta \in N} V_\eta)$ and $\bigcup_{\eta \in N} V_\eta \in \tau_{\mathbb{R}}$ since $V_\eta \in \tau_{\mathbb{R}}$, $\eta \in N$. Therefore $\bigcup_{\eta \in N} A_\eta \in \tau_{F_X}$.

Hence the collection

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V), f \in F_X, V \in \tau_{\mathbb{R}}\}$$

is a topology from the Frölicher space X.

We now consider the collection

$$\tau_{C_X} = \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}}, c \in C_X\}$$

We have that $\phi, X \subseteq X$ since $\phi, X \in P(X)$. Also $c^{-1}(X) = \{x \in \mathbb{R} \mid c(x) \in X\} \subset \mathbb{R}$. Then there exists an open interval $\hat{V}$ on $\mathbb{R}$ such that $c^{-1}(X) \subset \hat{V}$. But $\hat{V} \in \beta := \{(x, y) \mid x, y \in \mathbb{R}\}$ where β is the basis of a

topology on $\mathbb{R}$. That is $\hat{V} \in \tau_{\mathbb{R}}$. Therefore $c^{-1}(X) \in \tau_{\mathbb{R}}$. Recall that $\phi \in \tau_{\mathbb{R}}$ since $\tau_{\mathbb{R}}$ is a topology, that is the standard topology on $\mathbb{R}$, and $c^{-1}(\phi) = \phi$. Therefore $\phi, X \in \tau_{C_X}$.

Let $R_1, R_2, ..., R_k \in \tau_{C_X}$, then $R_j \subseteq X$ such that $c^{-1}(R_j) \in \tau_{\mathbb{R}}$ for all $j = 1, 2, ..., k$ and $c \in C_X$. Also $\bigcap_{j=1}^{k} R_j \subseteq X$ since $R_j \subseteq X$ for all $j = 1, 2, ..., k$. Also $c^{-1}(\bigcap_{j=1}^{k} R_j) = \bigcap_{j=1}^{k} c^{-1}(R_j) \in \tau_{\mathbb{R}}$ since $c^{-1}(R_j) \in \tau_{\mathbb{R}}$ for all $j = 1, 2, ..., k$. Therefore $\bigcap_{j=1}^{k} R_j \in \tau_{C_X}$.

Let $Q_q \in \tau_{C_X}$, $q \in I$, then $Q_q \subseteq X$ such that $c^{-1}(Q_q) \in \tau_{\mathbb{R}}$ and $c \in C_X$. Also $c^{-1}(\bigcup_{q\in I} Q_q) = \bigcup_{q\in I} c^{-1}(Q_q) \in \tau_{\mathbb{R}}$ since $c^{-1}(Q_q) \in \tau_{\mathbb{R}}$, $q \in I$. Therefore $\bigcup_{q\in I} Q_q \in \tau_{C_X}$ since $\bigcup_{q\in I} Q_q \subseteq X$.

Thus

$$\tau_{C_X} = \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}}, c \in C_X\}$$

is a topology from the Frölicher space X.

That is τ_{F_X} and τ_{C_X} are topologies from the Frölicher space X. That is (X, τ_{F_X}) and (X, τ_{C_X}) are topological spaces. We call τ_{F_X} and τ_{C_X} Frölicher topologies. That is a Frölicher topology is a topology induced from the Frölicher space.

Definition 2.5.1 ***Frölicher topologies.***
Let (X, C_X, F_X) be a Frölicher space. The topology induced by all structure functions on X is the collection

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V), f \in F_X, V \in \tau_{\mathbb{R}}\}$$

and the topology induced by all structure curves into X is the collection

$$\tau_{C_X} = \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}}, c \in C_X\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. The topologies τ_{F_X} and τ_{C_X} are called Frölicher topologies.

Remark 2.5.1 *All structure functions and structure curves, of which are smooth, are continuous in τ_{F_X} and τ_{C_X} respectively.*

Remark 2.5.2 *The standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$ is generated by the basis $\{(x, y) \mid x, y \in \mathbb{R}, x < y\}$ on $\mathbb{R}$.*

Lemma 2.5.1 *For any Frölicher space (X, C_X, F_X), $\tau_{F_X} \subset \tau_{C_X}$.*

Proof: Recall that

$$\tau_{F_X} = \{U \subseteq X \mid U = f^{-1}(V), f \in F_X, V \in \tau_{\mathbb{R}}\}$$

and

$$\tau_{C_X} = \{U \subseteq X \mid c^{-1}(U) \in \tau_{\mathbb{R}}, c \in C_X\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We have to show that $A \in \tau_{F_X}$ implies that $A \in \tau_{C_X}$.
Let $A \in \tau_{F_X}$, then $A \subseteq X$ such that $A = f^{-1}(V)$, $f \in F_X$ and $V \in \tau_{\mathbb{R}}$. Let $c \in C_X$ be arbitrary, then

$$\begin{aligned} c^{-1}(A) &= c^{-1}(f^{-1}(V)) \\ &= (f \circ c)^{-1}(V) \\ &= \{x \in \mathbb{R} \mid (f \circ c)(x) \in V \in \tau_{\mathbb{R}}\} \subset \mathbb{R} \end{aligned} \tag{2.15}$$

Since $f \in F_X$ and $c \in C_X$ then $f \circ c : \mathbb{R} \to \mathbb{R}$ and $f \circ c$ is smooth, that is $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Then there exists an open interval W on $\mathbb{R}$ such that

$$W \in \{(x, y) \mid x, y \in \mathbb{R}, x < y\}$$

with

$$\{x \in \mathbb{R} \mid (f \circ c)(x) \in V \in \tau_{\mathbb{R}}\} \subset W.$$

That is $c^{-1}(A) \subset W$. But $W \in \tau_{\mathbb{R}}$ since $\{(x, y) \mid x, y \in \mathbb{R}, x < y\}$ is a basis for the standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$. Therefore $c^{-1}(A) \in \tau_{\mathbb{R}}$. Thus $A \in \tau_{C_X}$. Hence $\tau_{F_X} \subset \tau_{C_X}$. □

Chapter 3

Intial objects.

3.1 Frölicher subspace.

3.1.1 Structure.

Let X be a Frölicher space, that is $X = (X, C_X, F_X)$ where the pair (C_X, F_X) is Frölicher structure (see Definition 2.3.2). Let S be such that $\phi \neq S \subseteq X$. We hereby induce a Frölicher structure on S generated by the Frölicher structure (C_X, F_X) on X. Now consider the inclusion map $\imath_S : S \hookrightarrow X$ and
$F^* := \{f \circ \imath_S \mid f \in F_X\}$ where $f \circ \imath_S$ is the restriction of the function f on the set S denoted by $f\,|_S$. Since $f \circ c : S \to \mathbb{R}$ then $F^* \subseteq \mathbb{R}^S \Longrightarrow F^* \in P(\mathbb{R}^S)$ where $\mathbb{R}^S$ is the set of all real-valued functions on S and $P(\mathbb{R}^S)$ is the power set of $\mathbb{R}^S$. Let $S^{\mathbb{R}}$ be the set of all curves into S and $P(S^{\mathbb{R}})$ the power set of $S^{\mathbb{R}}$. Then $P(\mathbb{R}^S)$ and $P(S^{\mathbb{R}})$ are categories with subsets of $\mathbb{R}^S$ and $S^{\mathbb{R}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

We now start with the process of building a Frölicher structure on the subset S of X. Consider the inclusion $\imath_S : S \hookrightarrow X$ and let $F^* := \{f \circ \imath_S \mid f \in F_X\}$. Let $\tilde{f} \in F^*$, that is $\tilde{f} = f \circ \imath_S$ with $f \in F_X$, and $\Gamma_S : P(\mathbb{R}^S) \to P(S^{\mathbb{R}})$ be a functor defined by

$$\Gamma_S F^* = \{\tilde{c} : \mathbb{R} \to S \mid \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \tilde{f} \in F^*\} := C_S \tag{3.1}$$

With $c \in C_X$, that is $c : \mathbb{R} \to X$, then we have that $c = \imath_S \circ \tilde{c}$. Thus

$$\begin{aligned} f \circ c &= f \circ (\imath_S \circ \tilde{c}) \\ &= (f \circ \imath_S) \circ \tilde{c} \\ &= \tilde{f} \circ \tilde{c} \end{aligned} \tag{3.2}$$

And $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ since X is a Frölicher space (see equation (2.6) and (2.7)), therefore $\tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$. That is $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ if and only if

$\tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$.

Let $\Phi_S : P(S^{\mathbb{R}}) \to P(\mathbb{R}^S)$ be a functor. Then it follows that

$$\begin{aligned} \Phi_S C_S &= \Phi_S \Gamma_S F^* \\ &= \{\tilde{f} : S \to \mathbb{R} \mid \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \tilde{c} \in \Gamma_S F^* = C_S\} \\ &:= F_S \end{aligned} \tag{3.3}$$

Since $f \circ \imath_S : S \to \mathbb{R}$ and $(f \circ \imath_S) \circ \tilde{c} = \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ (by (3.2)) for every $\tilde{c} \in \Gamma_S F^* = C_S$, therefore

$$F^* \subset \Phi_S C_S = F_S \tag{3.4}$$

Therefore by Lemma 2.2

$$\Gamma_S F^* = \Gamma_S \Phi_S \Gamma_S F^* \tag{3.5}$$

Then by using (3.1), (3.3) and (3.5) we have that

$$\begin{aligned} \Gamma_S F_S &= \Gamma_S \Phi_S C_S \\ &= \Gamma_S \Phi_S \Gamma_S F^* \\ &= \Gamma_S F^* \\ &= C_S \end{aligned} \tag{3.6}$$

Note that $\tilde{f} \in F_S$ since $\tilde{f} \in F^*$ and $F^* \subset F_S$. Then with (3.3) and (3.6) we have that

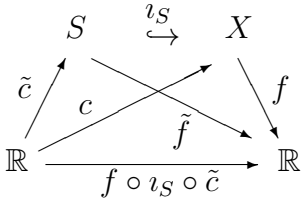
$$\Gamma_S F_S = \{\tilde{c} : \mathbb{R} \to S \mid \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \tilde{f} \in F_S\} = C_S$$

and

$$\Phi_S C_S = \{\tilde{f} : S \to \mathbb{R} \mid \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \tilde{c} \in C_S\} = F_S$$

That is the compatibility or duality condition hold and thus the pair (C_S, F_S) is a Frölicher structure on S. That is $F^* = \{f \circ \imath_S \mid f \in F_X\}$ generates a Frölicher structure on S. Hence the triple $S = (S, C_S, F_S)$ is a Frölicher space.

It is established that $S \subseteq X$ is a Frölicher space with X also being a Frölicher space. That is S and X are objects in the category of Frölicher spaces since they are both Frölicher spaces, then the inclusion $\imath_S : S \hookrightarrow X$ is a morphism between Frölicher spaces. We have seen by (3.4) that $F^* \subset F_S$. Now let
$C^* := \{\imath_S \circ \tilde{c} \mid \tilde{c} \in C_S\}$. Note that $C^* \in P(X^{\mathbb{R}})$. Let $f \in F_X$ then
$f \circ (\imath_S \circ \tilde{c}) = \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ (by (3.2)), therefore $\imath_S \circ \tilde{c} \in C_X$. Hence $C^* \subset C_X$. That is we have that $F^* \subset F_S$ and $C^* \subset C_X$, therefore the morphism $\imath_S : S \hookrightarrow X$ is a smooth map (see Definition 2.4.1). That is we have the diagram



with $\tilde{f} = f \circ \imath_S$ and $c = \imath_S \circ \tilde{c}$.

Remark 3.1.1 *The inclusion $\imath_S : S \hookrightarrow X$ is only unique for every Frölicher space X such that $S \subseteq X$. Therefore by definition (see Definition 2.1.3) S is not an initial object in the category of Frölicher spaces. The inclusion $\imath_S : S \hookrightarrow X$ is used to generate the Frölicher structure (C_S, F_S) on S, and since S is the domain of $\imath_S$ then we call the Frölicher structure (C_S, F_S) on S an initial structure. We therefore refer to S as an initial object on the virtue that it is endowed with an initial (Frölicher) structure (see Remark 2.1.1), only for the purpose and work of this dissertation.*

Definition 3.1.1 ***Frölicher subspace***
Let $X = (X, C_X, F_X)$ be a Frölicher space and (C_S, F_S) a Frölicher structure on $S \subseteq X$, $S \neq \phi$. The triple $S = (S, C_S, F_S)$ is a Frölicher subspace of X.

Simply, by Frölicher subspace we are refering to the subspace of a Frölicher space.

3.1.2 Frölicher subspace topologies.

Let $S = (S, C_S, F_S)$ be a Frölicher subspace of the Frölicher space $X = (X, C_X, F_X)$. Since S is a Frölicher space then we have Frölicher topologies on S, that is topologies induced from the Frölicher structure (C_S, F_S) of S (see Section 2.5 in particular Definition 2.5.1). That is the Frölicher topologies induced from the Frölicher structure (C_S, F_S) are denoted τ_{C_S} and τ_{F_S} and are induced respectively by structure curves and structure functions on S, that is being induced from C_S and F_S respectively. Since S is a Frölicher subspace, that is S is also a Frölicher space, then by Definition 2.5.1 the topology induced by all structure functions of S is the collection

$$\tau_{F_S} = \{U \subseteq S \mid U = \tilde{f}^{-1}(V), \tilde{f} \in F_S, V \in \tau_{\mathbb{R}}\}$$

and the topology induced by all structure curves of S is the collection

$$\tau_{C_S} = \{U \subseteq S \mid \tilde{c}^{-1}(U) \in \tau_{\mathbb{R}}, \tilde{c} \in C_S\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. The Frölicher topologies τ_{F_S} and τ_{C_S} are being referred to as Frölicher subspace topologies, and by Lemma 2.5.1

$$\tau_{F_S} \subset \tau_{C_S}.$$

Definition 3.1.2 *Let $X = (X, C_X, F_X)$ be a Frölicher space and $(S, C_S, F_S) = S \subseteq X = (X, C_X, F_X)$. The collection $\{g^{-1}(0,1) \mid g \in F_S\}$ and $\{g^{-1}(0,\infty) \mid g \in F_S\}$ is the basis and subbasis of τ_{F_S} respectively.*

3.1.3 Subspace topology.

The Frölicher subspace topologies τ_{F_X} and τ_{C_X} make the Frölicher space X a topological space, and since S is a Frölicher subspace of X, that is $S \subseteq X$, then we have the subspace topologies

$$\tau_{F_X}(S) := \{S \cap U \mid U \in \tau_{F_X}\}$$

and

$$\tau_{C_X}(S) := \{S \cap U \mid U \in \tau_{C_X}\}$$

on S (see [3], [7] and [15]) . The subspace topologies are canonical topologies on S since they're induced canonically on S and they make the inclusion $\imath_S : S \hookrightarrow X$ continuous.

3.1.4 Topological relations on Frölicher subspace.

Let $(S, C_S, F_S) = S \subseteq X = (X, C_X, F_X)$ where $X = (X, C_X, F_X)$ is a Frölicher space, that is S is a Frölicher subspace of X. We have defined the topologies induced from the Frölicher subspace structure (C_S, F_S), that is τ_{C_S} and τ_{F_S} - these topologies are called Frölicher subspace topologies. We also reckon that $\tau_{F_S} \subset \tau_{C_S}$ by Lemma 2.5.1, that is the topological relation between τ_{F_S} and τ_{C_S}. We also defined the canonical topologies induced on S, and that is the subspace topologies (also called trace or relative topologies, see [3], [7] and [15]) $\tau_{F_X}(S)$ and $\tau_{C_X}(S)$.

Lemma 3.1.1 *Let S be a Frölicher subspace of the Frölicher space X, then $\tau_{F_X}(S) \subset \tau_{C_X}(S)$.*

Proof: Let $S \bigcap U \in \tau_{F_X}(S)$, then $U \in \tau_{F_X}$. But by Lemma 2.5.1 $\tau_{F_X} \subset \tau_{C_X}$, therefore $U \in \tau_{C_X}$, and hence $S \bigcap U \in \tau_{C_X}(S)$. Thus $\tau_{F_X}(S) \subset \tau_{C_X}(S)$. □

Lemma 3.1.1 gives the topological relation between the subspace topologies $\tau_{C_X}(S)$ and $\tau_{F_X}(S)$ on S. We now focus on the relationship between the subspace topologies and Frölicher subspace topologies.

Theorem 3.1.1 *Let S be the Frölicher subspace of the Frölicher space X, then $\tau_{F_X}(S) \subset \tau_{F_S}$.*

Proof: We have to show that if $U \in \tau_{F_X}(S)$ then $U \in \tau_{F_S}$. Recall that $\tau_{F_X}(S) = \{S \bigcap V \mid V \in \tau_{F_X}\}$ and the inclusion map $\imath_S : S \hookrightarrow X$ is a morphism

in the category of Frölicher spaces, that is, it is a smooth map. Let $U \in \tau_{F_X}(S)$, then $U = S \bigcap V$ where $V \in \tau_{F_X}$. Let $V = \bigcup_{\alpha \in I} f_\alpha^{-1}(0,\infty)$, $f_\alpha \in F_X$. Note that $f_\alpha^{-1}(0,\infty)$, $\alpha \in I$, is in the basis of τ_{F_X}, therefore $V = \bigcup_{\alpha \in I} f_\alpha^{-1}(0,\infty) \in \tau_{F_X}$. Then

$$\begin{aligned} U &= S \bigcap V \\ &= S \bigcap (\bigcup_{\alpha \in I} f_\alpha^{-1}(0,\infty)) \\ &= \bigcup_{\alpha \in I} (S \bigcap f_\alpha^{-1}(0,\infty)) \\ &= \bigcup_{\alpha \in I} (\imath_S^{-1}(f_\alpha^{-1}(0,\infty))) \\ &= \bigcup_{\alpha \in I} ((f_\alpha \circ \imath_S)^{-1}(0,\infty)) \end{aligned} \tag{3.7}$$

But since $f_\alpha \in F_X$ then $f_\alpha : X \to \mathbb{R}$ and $f_\alpha \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for every $c \in C_X$. Therefore $f_\alpha \circ \imath_S : S \to \mathbb{R}$. Let $\tilde{c} \in C_S$ be arbitrary, then $(f_\alpha \circ \imath_S) \circ \tilde{c} \in C^\infty(\mathbb{R},\mathbb{R})$ (by (3.2)). Therefore $f_\alpha \circ \imath_S \in F_S$, and thus $(f_\alpha \circ \imath_S)^{-1}(0,\infty)$, $\alpha \in I$, is an element of the basis of τ_{F_S}. Hence $U = S \bigcap V = \bigcup_{\alpha \in I}((f_\alpha \circ \imath_S)^{-1}(0,\infty)) \in \tau_{F_S}$, thus $\tau_{F_X}(S) \subset \tau_{F_S}$. □

Proposition 3.1.1 *Let S be a Frölicher subspace of the Frölicher space X, then $\tau_{F_X}(S) \subset \tau_{F_S} \subset \tau_{C_S}$.*

Proof: Theorem 3.1.1 gives that $\tau_{F_X}(S) \subset \tau_{F_S}$ and by Lemma 2.5.1 $\tau_{F_S} \subset \tau_{C_S}$. Hence $\tau_{F_X}(S) \subset \tau_{F_S} \subset \tau_{C_S}$. □

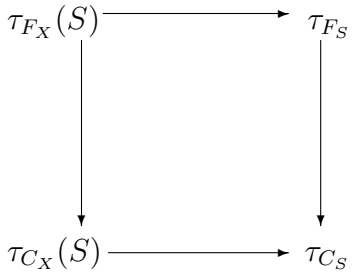
Theorem 3.1.2 *Let S be the Frölicher subspace of the Frölicher space X, then $\tau_{C_X}(S) \subset \tau_{C_S}$.*

Proof: We have to show that $V \in \tau_{C_X}(S)$ implies that $V \in \tau_{C_S}$. Let $V \in \tau_{C_X}(S)$ then $V = S \bigcap U$ where $U \in \tau_{C_X}$. Since $U \in \tau_{C_X}$ then $U \subseteq X$ and $c^{-1}(U) \in \tau_{\mathbb{R}}$ where $c \in C_X$ and $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Observe that
$V = S \bigcap U \subseteq S$, and let $\tilde{c} \in C_S$, that is $\tilde{c} : \mathbb{R} \to S$ such that $\tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R},\mathbb{R})$ for every $\tilde{f} \in F_S$. Then $\tilde{c}^{-1}(V) = \{r \in \mathbb{R} \mid \tilde{c}(r) \in V = S \bigcap U \subseteq S \subseteq X\}$, then there exists $T \in \{(y,z) \mid y,z \in \mathbb{R}, y < z\}$ such that $\tilde{c}^{-1}(V) \in T$ where $(y,z) \in \mathbb{R} \times \mathbb{R}$. But $\{(y,z) \mid y,z \in \mathbb{R}, y < z\}$ is the basis of the standard topology $\tau_{\mathbb{R}}$ on $\mathbb{R}$. Therefore $\tilde{c}^{-1}(V) \in \tau_{\mathbb{R}}$ and hence $\tau_{C_X}(S) \subset \tau_{C_S}$. □

Proposition 3.1.2 *Let S be a Frölicher subspace of the Frölicher space X, then $\tau_{F_X}(S) \subset \tau_{C_X}(S) \subset \tau_{C_S}$.*

Proof: By Lemma 3.1.1 we have that $\tau_{F_X}(S) \subset \tau_{C_X}(S)$ and by Theorem 3.1.2 $\tau_{C_X}(S) \subset \tau_{C_S}$, therefore $\tau_{F_X}(S) \subset \tau_{C_X}(S) \subset \tau_{C_S}$. □

The topological relation between the Frölicher subspace topologies τ_{F_S} and τ_{C_S} and the subspace topologies $\tau_{F_X}(S)$ and $\tau_{C_X}(S)$ is summarized by the diagram



with the arrows representing inclusion maps.

Proposition 3.1.3 *Let X be a Frölicher space and S be a Frölicher subspace X. Then the following hold:*

1. *If $S \in \tau_{F_X}$, then $\tau_{F_S} = \tau_{F_X}(S)$.*
2. *If $S \in \tau_{C_X}$, then $\tau_{C_S} = \tau_{C_X}(S)$.*

Proof:

1. Let $S \in \tau_{F_X}$, then $S \subseteq X$. By Theorem 3.1.1 $\tau_{F_X}(S) \subset \tau_{F_S}$. Then it is sufficient to prove that $\tau_{F_S} \subset \tau_{F_X}(S)$. Let $U \in \tau_{F_S}$ such that $U = \bigcup_{\alpha\in I}((f_\alpha \circ \imath_S)^{-1}(0,\infty))$ where $f_\alpha \in F_X$, $\alpha \in I$. Then it follows that

$$\begin{aligned} U &= \imath_S^{-1}\imath_S(U) \\ &= \imath_S^{-1}\imath_S(\bigcup_{\alpha\in I}((f_\alpha \circ \imath_S)^{-1}(0,\infty))) \\ &= \imath_S^{-1}\bigcup_{\alpha\in I}(\imath_S(f_\alpha \circ \imath_S)^{-1}(0,\infty)) \\ &= \imath_S^{-1}\bigcup_{\alpha\in I}(\imath_S(\imath_S^{-1}(f_\alpha^{-1}(0,\infty)))) \\ &= \imath_S^{-1}\bigcup_{\alpha\in I} f_\alpha^{-1}(0,\infty) \\ &= \bigcup_{\alpha\in I}\imath_S^{-1}(f_\alpha^{-1}(0,\infty)) \\ &= \bigcup_{\alpha\in I}(S\bigcap f_\alpha^{-1}(0,\infty)) \end{aligned} \tag{3.8}$$

   Since $f_\alpha^{-1}(0,\infty)$ is an element of the basis of τ_{F_X} then $U = \bigcup_{\alpha\in I}(S\bigcap f_\alpha^{-1}(0,\infty)) \in \tau_{F_X}(S)$. That is $U \in \tau_{F_S}$ implies that $U \in \tau_{F_X}(S)$, therefore $\tau_{F_S} \subset \tau_{F_X}(S)$, and since $\tau_{F_X}(S) \subset \tau_{F_S}$ hence $\tau_{F_S} = \tau_{F_X}(S)$.

2. Since by Theorem 3.1.2 $\tau_{C_X}(S) \subset \tau_{C_S}$, then it is sufficient to show that $\tau_{C_S} \subset \tau_{C_X}(S)$. That is we have to show that $U \in \tau_{C_S}$ implies that $U \in \tau_{C_X}(S)$. Let $S \in \tau_{C_X}$ and let $U \in \tau_{C_S}$. That is $S \subseteq X$, $c^{-1}(S) \in \tau_{\mathbb{R}}$ where $c \in C_X$ and $U \subseteq S$, $\tilde{c}^{-1}(U) \in \tau_{\mathbb{R}}$ where $\tilde{c} \in C_S$, and $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Then it follows that

$$\begin{aligned} \tilde{c}^{-1}(U) &= \tilde{c}^{-1}(\imath_S^{-1}(\imath_S(U))) \\ &= (\imath_S \circ \tilde{c})^{-1}(\imath_S(U)) \end{aligned} \tag{3.9}$$

Note that $c = \imath_S \circ \tilde{c}$, and since $\imath_S$ is an inclusion map then $\imath_S(U) = U$. Therefore $\tilde{c}^{-1}(U) = c^{-1}(U)$. Then since $\tilde{c}^{-1}(U) \in \tau_{\mathbb{R}}$, therefore $c^{-1}(U) \in \tau_{\mathbb{R}}$. Since $U \subseteq S$ and $S \subseteq X$ then $U \subseteq X$ and we have that $c^{-1}(U) \in \tau_{\mathbb{R}}$ where $c \in C_X$, therefore $U \in \tau_{C_X}$. We also have that

$$\begin{aligned} U &= \imath_S^{-1}(\imath_S(U)) \\ &= S \bigcap \imath_S(U) \\ &= S \bigcap U \end{aligned} \tag{3.10}$$

Then $U = S \bigcap U \in \tau_{C_X}(S)$ since $U \in \tau_{C_X}$. Therefore $\tau_{C_S} \subset \tau_{C_X}(S)$, and since we also have that $\tau_{C_X}(S) \subset \tau_{C_S}$, hence $\tau_{C_S} = \tau_{C_X}(S)$.

□

3.2 Frölicher product

3.2.1 Structure.

Let X_i be non-empty sets for $i \in \mathbb{N}$, $1 \leq i \leq n$, be Frölicher spaces, that is $X_i = (X_i, C_{X_i}, F_{X_i})$, and let $\hat{X}$ be the product of the Frölicher spaces X_i in the cartegory of sets. That is $\hat{X}$ is the cartesian product of X_i, thus $\hat{X} = \prod_{i=1}^{n} X_i$.

From now onwards every subscript i implies that it is for every $i \in \mathbb{N}$ such that $1 \leq i \leq n$, only for this section.

Let $f_i : X_i \to \mathbb{R}$ be a real-valued function on X_i and $c_i : \mathbb{R} \to X_i$ be a curve into X_i. Since $X_i = (X_i, C_{X_i}, F_{X_i})$ is a Frölicher space we have that

$$C_{X_i} = \Gamma_i(F_{X_i}) := \{c_i : \mathbb{R} \to X_i \mid f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f_i \in F_{X_i}\}$$

and

$$F_{X_i} = \Phi_i(C_{X_i}) := \{f_i : X_i \to \mathbb{R} \mid f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c_i \in C_{X_i}\}$$

where $\Gamma_i : P(\mathbb{R}^{X_i}) \to P({X_i}^{\mathbb{R}})$ and $\Phi_i : P({X_i}^{\mathbb{R}}) \to P(\mathbb{R}^{X_i})$ are functors with $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ the set of all real-valued functions on X_i and the set of all curves into

X_i respectively, where $P(\mathbb{R}^{X_i})$ and $P({X_i}^{\mathbb{R}})$ are the power set of $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ respectively, and they [the power sets] are categories with subsets of $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

Let $\hat{f} : \prod_{i=1}^{n} X_i = \hat{X} \to \mathbb{R}$ be a real-valued function on $\hat{X} = \prod_{i=1}^{n} X_i$,
$\hat{c} : \mathbb{R} \to \hat{X} = \prod_{i=1}^{n} X_i$ be a curve into $\hat{X}$ and $p_i : \prod_{i=1}^{n} X_i = \hat{X} \to X_i$ be a projection in the category of sets. Thus is $p_i : \hat{X} = \prod_{i=1}^{n} X_i \ni (x_1, x_2, ..., x_n) \mapsto x_i \in X_i$, and since $f_i : X_i \to \mathbb{R}$ is a function, then every $x_i \in X_i$ is assigned to an element in $\mathbb{R}$. Therefore by using pullback every $(x_1, x_2, ..., x_n) \in \hat{X}$ is assigned to an element in $\mathbb{R}$, thereby making $f_i \circ p_i : \hat{X} \to \mathbb{R}$ a function.

We now start with the process of building a Frölicher structure on $\hat{X} = \prod_{i=1}^{n} X_i$. Let $F_\pi := \{f_i \circ p_i \mid f_i \in F_{X_i}\}$ and $\mathbb{R}^{\hat{X}}$ be the set of all real-valued functions on $\hat{X}$. Then $F_\pi \subset \mathbb{R}^{\hat{X}}$, implying that $F_\pi \in P(\mathbb{R}^{\hat{X}})$ where $P(\mathbb{R}^{\hat{X}})$ is the power set of $\mathbb{R}^{\hat{X}}$. Subsequently let $\hat{X}^{\mathbb{R}}$ be the set of all curves into $\hat{X}$ and $P(\hat{X}^{\mathbb{R}})$ the power set of $\hat{X}^{\mathbb{R}}$. Then $P(\mathbb{R}^{\hat{X}})$ and $P(\hat{X}^{\mathbb{R}})$ are categories with subsets of $\mathbb{R}^{\hat{X}}$ and $\hat{X}^{\mathbb{R}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

Now let $\hat{f} \in F_\pi$, that is $\hat{f} = f_i \circ p_i$ with $f_i \in F_{X_i}$ and let $\hat{\Gamma} : P(\mathbb{R}^{\hat{X}}) \to P(\hat{X}^{\mathbb{R}})$ be a functor defined by

$$\hat{\Gamma} F_\pi := \{\hat{c} : \mathbb{R} \to \hat{X} \mid \hat{f} \circ \hat{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \hat{f} \in F_\pi\} := C_{\hat{X}} \tag{3.11}$$

With $c_i \in C_{X_i}$ we have that $c_i = p_i \circ \hat{c}$, thus

$$\begin{aligned} f_i \circ c_i &= f_i \circ (p_i \circ \hat{c}) \\ &= (f_i \circ p_i) \circ \hat{c} \\ &= \hat{f} \circ \hat{c} \end{aligned} \tag{3.12}$$

Since X_i is a Frölicher space then $f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R})$ as per the definition of C_{X_i} and F_{X_i} above, then by (3.12) we have that $\hat{f} \circ \hat{c} \in C^\infty(\mathbb{R}, \mathbb{R})$. That is $f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R})$ if and only if $\hat{f} \circ \hat{c} \in C^\infty(\mathbb{R}, \mathbb{R})$.

Let $\hat{\Phi} : P(\hat{X}^{\mathbb{R}}) \to P(\mathbb{R}^{\hat{X}})$ be a functor. Then it follows that

$$\begin{aligned} \hat{\Phi} C_{\hat{X}} &= \hat{\Phi} \hat{\Gamma} F_\pi \\ &= \{\hat{f} : \hat{X} \to \mathbb{R} \mid \hat{f} \circ \hat{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \hat{c} \in \hat{\Gamma} F_\pi = C_{\hat{X}}\} := F_{\hat{X}} \end{aligned} \tag{3.13}$$

Since $\hat{f} = f_i \circ p_i : \hat{X} \to \mathbb{R}$ and $(f_i \circ p_i) \circ \hat{c} = \hat{f} \circ \hat{c} \in C^\infty(\mathbb{R}, \mathbb{R})$, then it follows that

$$F_\pi \subset \hat{\Phi} C_{\hat{X}} = F_{\hat{X}} \tag{3.14}$$

Therefore by Lemma 2.3.2

$$\hat{\Gamma} F_\pi = \hat{\Gamma} \hat{\Phi} \hat{\Gamma} F_\pi \tag{3.15}$$

Then by using (3.11), (3.13) and (3.15) we have that

$$\begin{aligned}\hat{\Gamma}F_{\hat{X}} &= \hat{\Gamma}\hat{\Phi}C_{\hat{X}} \\ &= \hat{\Gamma}\hat{\Phi}\hat{\Gamma}F_{\pi} \\ &= \hat{\Gamma}F_{\pi} \\ &= C_{\hat{X}}\end{aligned} \tag{3.16}$$

Note that $\hat{f} \in F_{\hat{X}}$ since $\hat{f} \in F_{\pi}$ and $F_{\pi} \subset F_{\hat{X}}$ (by (3.14)). Then by (3.13) and (3.16) we have that

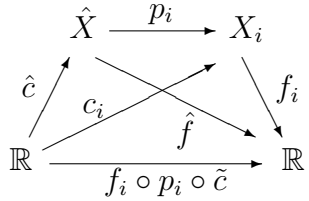
$$C_{\hat{X}} = \hat{\Gamma}F_{\hat{X}} = \{\hat{c} : \mathbb{R} \to \hat{X} \mid \hat{f} \circ \hat{c} \in C^{\infty}(\mathbb{R}, \mathbb{R}), \forall \hat{f} \in F_{\hat{X}}\}$$

and

$$F_{\hat{X}} = \hat{\Phi}C_{\hat{X}} = \{\hat{f} : \hat{X} \to \mathbb{R} \mid \hat{f} \circ \hat{c} \in C^{\infty}(\mathbb{R}, \mathbb{R}), \forall \hat{c} \in C_{\hat{X}}\}$$

That is the compatibility or duality condition holds and thus the pair $(C_{\hat{X}}, F_{\hat{X}})$ is a Frölicher structure on $\hat{X} = \prod_{i=1}^{n} X_i$. That is $F_{\pi} = \{f_i \circ p_i \mid f_i \in F_{X_i}\}$ generates a Frölicher structure on $\hat{X}$. Hence the triple $\hat{X} = (\hat{X}, C_{\hat{X}}, F_{\hat{X}})$ is a Frölicher space.

We have seen that $F_{\pi} \subset F_{\hat{X}}$ (by (3.14)). Now let $C_{\pi} := \{p_i \circ \hat{c} \mid \hat{c} \in C_{\hat{X}}\}$. Note that $p_i \circ \hat{c} : \mathbb{R} \to X_i$. Let $f_i \in F_{X_i}$, that is $f_i : X_i \to \mathbb{R}$ a function such that $f_i \circ c_i \in C^{\infty}(\mathbb{R}, \mathbb{R})$ for every $c_i \in C_{X_i}$. Then by (3.12)
$f_i \circ (p_i \circ \hat{c}) \in C^{\infty}(\mathbb{R}, \mathbb{R})$. Therefore $p_i \circ \hat{c} \in C_{X_i}$ and thus $C_{\pi} \subset C_{X_i}$. That is we have that $F_{\pi} = \{f_i \circ p_i \mid f_i \in F_{X_i}\} \subset F_{\hat{X}}$ and $C_{\pi} = \{p_i \circ \hat{c} \mid \hat{c} \in C_{\hat{X}}\} \subset C_{X_i}$. Therefore the projection $p_i : \hat{X} \to X_i$ is a smooth map (see Definition 2.4.1). That is we have the diagram



with $\tilde{f} = f_i \circ p_i$ and $c_i = p_i \circ \tilde{c}$.

Remark 3.2.1 *The canonical projection $p_i : \hat{X} \to X_i$ is unique for every Frölicher space X_i such that $\hat{X} = \prod_{i=1}^{n} X_i$. Therefore by definition (see Definition 2.1.3) $\hat{X}$ is not an initial object in the category of Frölicher spaces. The projection $p_i : \hat{X} \to X_i$ is used in generation of the Frölicher structure $(C_{\hat{X}}, F_{\hat{X}})$ on $\hat{X}$, and since $\hat{X}$ is the domain of p_i then we call the Frölicher structure $(C_{\hat{X}}, F_{\hat{X}})$ on $\hat{X}$ an initial structure. We therefore refer to $\hat{X}$ as an initial object on the virtue that it is endowed with an initial (Frölicher) structure (see Remark 2.1.1 and 3.1.1). This only for the purpose and work of this dissertation.*

Definition 3.2.1 ***Frölicher product.***
Let $X_i = (X_i, C_{X_i}, F_{X_i})$, with $X_i \neq \phi$, be a Frölicher space and $\hat{X} = \prod_{i=1}^{n} X_i$ in the category of sets with the Frölicher structure $(C_{\hat{X}}, F_{\hat{X}})$ on $\hat{X}$, then the triple $\hat{X} = (\hat{X}, C_{\hat{X}}, F_{\hat{X}})$ is called the Frölicher product of X_i, that is to say the product of Frölicher spaces X_i.

3.2.2 Frölicher product topologies

Let $\hat{X} = (\hat{X}, C_{\hat{X}}, F_{\hat{X}})$ be a Frölicher product of X_i. That is $X_i = (X_i, C_{X_i}, F_{X_i})$ is a Frölicher space and $\hat{X} = \prod_{i=1}^{n} X_i$ is the cartesian product of X_i in the category of sets. Since $\hat{X}$ is a Frölicher space then Frölicher topologies can be induced from the Frölicher structure of $\hat{X}$, that is from $(C_{\hat{X}}, F_{\hat{X}})$. We call these topologies [that is Frölicher topologies induced from $(C_{\hat{X}}, F_{\hat{X}})$] Frölicher product topologies. The Frölicher product topologies are denoted $\tau_{C_{\hat{X}}}$ and $\tau_{F_{\hat{X}}}$, that is the topology induced from $C_{\hat{X}}$ and $F_{\hat{X}}$ respectively. By definition (see Definition 2.5.1) we have that

$$\tau_{F_{\hat{X}}} = \{U \subseteq \hat{X} \mid U = \hat{f}^{-1}(V), \hat{f} \in F_{\hat{X}}, V \in \tau_{\mathbb{R}}\}$$

and

$$\tau_{C_{\hat{X}}} = \{U \subseteq \hat{X} \mid \hat{c}^{-1}(U) \in \tau_{\mathbb{R}}, \hat{c} \in C_{\hat{X}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. By Lemma 2.5.1

$$\tau_{F_{\hat{X}}} \subset \tau_{C_{\hat{X}}}$$

since $\hat{X}$ is a Frölicher space.

Definition 3.2.2 *: Let $\hat{X} = (\hat{X}, C_{\hat{X}}, F_{\hat{X}})$ be the Frölicher product of the Frölicher spaces $\{X_i = (X_i, C_{X_i}, F_{X_i})\}$. The families $\{\hat{f}^{-1}(0, \infty) \mid \hat{f} \in F_{\hat{X}}\}$ and $\{\hat{f}^{-1}(0, 1) \mid \hat{f} \in F_{\hat{X}}\}$ are the basis and subbasis for $\tau_{F_{\hat{X}}}$ respectively.*

3.2.3 Product topology.

Since we have the Frölicher topology $\tau_{F_{X_i}}$ induced from F_{X_i}, then we have the topological space $(X_i, \tau_{F_{X_i}})$, that is the Frölicher space $X_i = (X_i, C_{X_i}, F_{X_i})$ is a topological space. And with the Frölicher product $\hat{X} = \prod_{i=1}^{n} X_i$ we have the canonical projection $p_i : \hat{X} \to X_i$, $i \in I$. The product topology on the product $\hat{X} = \prod_{i=1}^{n} X_i$, which we denote by τ_π, is generated by the basis

$$\hat{\beta} = \{\bigcap_{i=1}^{n} {p_i}^{-1}(U_i) \mid U_i \in \tau_{F_{X_i}}, \forall i \in I\}$$

(see Lemma 2.6.1 in [19]). The product topology τ_π is canonical on $\hat{X}$ and makes the projection $p_i : \hat{X} \to X_i$ continuous.

3.2.4 Topological relations on Frölicher product.

Let $X_i = (X_i, C_{X_i}, F_{X_i})$ be a Frölicher space and

$$\hat{X} = \prod_{i=1}^{n} X_i = (\hat{X} = \prod_{i=1}^{n} X_i, C_{\hat{X}}, F_{\hat{X}})$$

be the Frölicher product of X_i. We have defined the Frölicher product topologies on Subsection 3.2.2, that is the topologies $\tau_{C_{\hat{X}}}$ and $\tau_{F_{\hat{X}}}$ - topologies induced from the Frölicher structure $(C_{\hat{X}}, F_{\hat{X}})$ of Frölicher product. Thus $\tau_{C_{\hat{X}}}$ and $\tau_{F_{\hat{X}}}$ are induced from $C_{\hat{X}}$ and $F_{\hat{X}}$ respectively. We have also induced the product topology τ_π on $\hat{X}$ (see Subsection 3.2.3). Now we focus on and determine the relationship between the product topology and Frölicher product topologies.

Theorem 3.2.1 *: Let $\tau_{F_{\hat{X}}}$ be the Frölicher product topology on the cartesian product $\hat{X} = \prod_{i=1}^{n} X_i$ of Frölicher spaces. Let τ_π be the product topology on $\hat{X}$. Then $\tau_\pi = \tau_{F_{\hat{X}}}$.*

Proof: We have to show that $\tau_\pi \subset \tau_{F_{\hat{X}}}$ and that $\tau_{F_{\hat{X}}} \subset \tau_\pi$. We first show that $\tau_\pi \subset \tau_{F_{\hat{X}}}$. Consider the cartesian product $\prod_{i=1}^{n} U_i$ where $U_i \in \tau_{F_{X_i}}$. Then $U_i \subseteq X_i$ such that $U_i = {f_i}^{-1}(V)$ where $f_i \in F_{X_i}$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We have that $\prod_{i=1}^{n} U_i \subseteq \prod_{i=1}^{n} X_i = \hat{X}$ since $U_i \subseteq X_i$. Also $\prod_{i=1}^{n} U_i = \{(u_1, u_2, ..., u_n) \mid u_i \in U_i \subseteq X_i\}$, and consider the function $\hat{f} \in F_{\hat{X}}$, that is $\hat{f} : \prod_{i=1}^{n} X_i = \hat{X} \to \mathbb{R}$. Then there exists $\hat{V} \in \tau_{\mathbb{R}}$ such that $\hat{f}(u_1, u_2, ..., u_n) \in \hat{V}$. Then

$$\begin{aligned}
\hat{f}^{-1}(\hat{V}) &= \{(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} X_i = \hat{X} \mid \hat{f}(u_1, u_2, ..., u_n) \in \hat{V}\} \\
&= \{(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} U_i \mid \hat{f}(u_1, u_2, ..., u_n) \in \hat{V}\} \\
&= \{(u_1, u_2, ..., u_n) \mid u_i \in U_i\} \\
&= \prod_{i=1}^{n} U_i
\end{aligned} \tag{3.17}$$

We have that if $f_i(x_i) \in V$ where $V \in \tau_{\mathbb{R}}$, then by using pull-back $\hat{f}(x_1, x_2, ..., x_n) = (f_i \circ p_i)(x_1, x_2, ..., x_n) \in V$, where $p_i : \prod_{i=1}^{n} X_i = \hat{X} \to X_i$ is a projection onto X_i. Furthermore, since $U_i = {f_i}^{-1}(V)$, $f_i \in F_{X_i}$ for some $V \in \tau_{\mathbb{R}}$

therefore

$$\begin{aligned}\prod_{i=1}^{n} U_i &= \prod_{i=1}^{n} {f_i}^{-1}(V) \\ &= \prod_{i=1}^{n} \{x_i \in X_i \mid f_i(x_i) \in V\} \\ &= \{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X_i = \hat{X} \mid \hat{f}(x_1, x_2, ..., x_n) \in V\} \\ &= \hat{f}^{-1}(V)\end{aligned} \tag{3.18}$$

Thus $\prod_{i=1}^{n} U_i \in \tau_{F_{\hat{X}}}$. But note that $\prod_{i=1}^{n} U_i$ is a basis element of the basis $\hat{\beta} = \{\prod_{i=1}^{n} U_i \mid U_i \in \tau_{F_{X_i}}, \forall i \in \mathbb{N}\}$ of the product topology τ_π. Therefore we have that $\prod_{i=1}^{n} U_i \in \tau_\pi$. That is $\prod_{i=1}^{n} U_i \in \tau_\pi$ implies that $\prod_{i=1}^{n} U_i \in \tau_{F_{\hat{X}}}$, therefore $\tau_\pi \subset \tau_{F_{\hat{X}}}$.

Now we have to show that $\tau_{F_{\hat{X}}} \subset \tau_\pi$. Let $U \in \tau_{F_{\hat{X}}}$, then $U \subseteq \hat{X} = \prod_{i=1}^{n} X_i$, and therefore $U = \prod_{i=1}^{n} Y_i$ where $Y_i \subseteq X_i$. Now consider the projection $p_i : \prod_{i=1}^{n} = \hat{X} \to X_i$ and $f_i \in F_{X_i}$ where X_i is a Frölicher space. Then ${f_i}^{-1}(V) \subseteq X_i$ for every $V \in \tau_{\mathbb{R}}$ and recall that $\hat{f} = f_i \circ p_i$. Since $\prod_{i=1}^{n} Y_i = U \in \tau_{F_{\hat{X}}}$, then we have that $\prod_{i=1}^{n} Y_i = U = \hat{f}^{-1}(W)$ where $W \in \tau_{\mathbb{R}}$ and $\hat{f} \in F_{\hat{X}}$. Therefore

$$\begin{aligned}\prod_{i=1}^{n} Y_i &= U \\ &= \hat{f}^{-1}(W) \\ &= (f_i \circ p_i)^{-1}(W) \\ &= {p_i}^{-1}({f_i}^{-1}(W)) \\ &= \{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X_i = \hat{X} \mid p_i(x_1, x_2, ..., x_n) \in {f_i}^{-1}(W), W \in \tau_{\mathbb{R}}\} \\ &= \{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} X_i = \hat{X} \mid x_i \in {f_i}^{-1}(W), W \in \tau_{\mathbb{R}}\} \\ &= \prod_{i=1}^{n} {f_i}^{-1}(W)\end{aligned} \tag{3.19}$$

Thus $Y_i = {f_i}^{-1}(W)$ where $W \in \tau_{\mathbb{R}}$, therefore $Y_i \in \tau_{F_{X_i}}$. Hence $\prod_{i=1}^{n} Y_i \in \hat{\beta} = \{\prod_{i=1}^{n} U_i \mid U_i \in \tau_{F_{X_i}}\}$ and $\hat{\beta}$ is the basis for the product topology τ_π on $\hat{X}$, therefore $\prod_{i=1}^{n} Y_i \in \tau_\pi$. That is $U = \prod_{i=1}^{n} Y_i \in \tau_{F_{\hat{X}}}$ implies that $U = \prod_{i=1}^{n} Y_i \in \tau_\pi$, therefore $\tau_{F_{\hat{X}}} \subset \tau_\pi$. Since $\tau_\pi \subset \tau_{F_{\hat{X}}}$ and $\tau_{F_{\hat{X}}} \subset \tau_\pi$, hence $\tau_\pi = \tau_{F_{\hat{X}}}$. □

The relationship between the Frölicher product topologies, that is the topologies $\tau_{C_{\hat{X}}}$ and $\tau_{F_{\hat{X}}}$ is given by Lemma 2.5.1. That is since $\hat{X}$ is a Frölicher space, then by Lemma 2.5.1 $\tau_{F_{\hat{X}}} \subset \tau_{C_{\hat{X}}}$. This gives the following proposition:

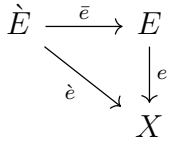
Proposition 3.2.1 *: Let $\tau_{F_{\hat{X}}}$ and $\tau_{C_{\hat{X}}}$ be Frölicher product topologies induced from the Frölicher structure $(C_{\hat{X}}, F_{\hat{X}})$ of Frölicher product, and let τ_π be the product topology on $\hat{X}$. Then $\tau_\pi = \tau_{F_{\hat{X}}} \subset \tau_{C_{\hat{X}}}$.*

3.3 Frölicher equalizer.

3.3.1 Structure.

Definition 3.3.1 ***Equalizer.***
Let (f, g) be a pair of morphisms in the category $\mathcal{C}$ such that $f, g : X \to Y$. The morphism $e : E \to X$ in the category $\mathcal{C}$ is called an equalizer of f and g provided that the following conditions hold:

1. *$f \circ e = g \circ e$.*
2. *For any morphism $\grave{e} : \grave{E} \to X$ in the category $\mathcal{C}$ such that $f \circ \grave{e} = g \circ \grave{e}$ there exists a unique morpism $\bar{e} : \grave{E} \to E$ in the category $\mathcal{C}$ such that $\grave{e} = e \circ \bar{e}$, that is such that the triangle*



commutes.

Proposition 3.3.1 *Let $f, g : X \to Y$ be a pair of morphisms in a category $\mathcal{C}$, then the following hold:*

1. *If $e : E \to X$ and $\grave{e} : \grave{E} \to X$ are both equalizers of f and g in the category $\mathcal{C}$ then there exists an isomorphism $\bar{e} : \grave{E} \to E$ with $\grave{e} = e \circ \bar{e}$.*
2. *If $e : E \to X$ is an equalizer of f and g in the category $\mathcal{C}$ and if $u : \grave{E} \to E$ is an isomorphism the $e \circ u : \grave{E} \to X$ is also an equalizer of f and g in $\mathcal{C}$.*

Proof:

1. Let both $e : E \to X$ and $\grave{e} : \grave{E} \to X$ be equalizers of f and g in the category $\mathcal{C}$, then by definition $f \circ e = g \circ e$ and $f \circ \grave{e} = g \circ \grave{e}$, and there exists morphisms $\bar{e} := \bar{e}_1$ and $\bar{e}_2$ where $\bar{e}_1 : \grave{E} \to E$ and $\bar{e_2} : E \to \grave{E}$ such that

$$\grave{e} = e \circ \bar{e}_1 \tag{3.20}$$

$$e = \grave{e} \circ \bar{e}_2 \tag{3.21}$$

Then by substituting (3.20) into (3.21) we obtain that

$$e = (e \circ \bar{e}_1) \circ \bar{e}_2 = e \circ (\bar{e}_1 \circ \bar{e}_2) \tag{3.22}$$

which implies that

$$\bar{e}_1 \circ \bar{e}_2 = e^{-1} \circ e = I_E \tag{3.23}$$

where $I_E : E \to E$ is the identity morphism on E.
Similarly, by substituting (3.21) into (3.20) we obtain that

$$\grave{e} = (\grave{e} \circ \bar{e}_2) \circ \bar{e}_1 = \grave{e} \circ (\bar{e}_2 \circ \bar{e}_1) \tag{3.24}$$

which implies that

$$\bar{e}_2 \circ \bar{e}_1 = (\grave{e})^{-1} \circ \grave{e} = I_{\grave{E}} \tag{3.25}$$

where $I_{\grave{E}} : \grave{E} \to \grave{E}$ is the identity morphism on $\grave{E}$.
Then from (3.23) and (3.25) it follows that $\bar{e} = \bar{e}_1 : \grave{E} \to E$ is an isomorphism.

2. Let $e : E \to X$ be an equalizer of f and g in the category $\mathcal{C}$ and let $u : \grave{E} \to E$ be an isomorphism in $\mathcal{C}$. Since $e : E \to X$ is an equalizer then there exists a unique morphism $\bar{e} : \grave{E} \to E$ such that $\grave{e} = e \circ e$ where $\grave{e} : \grave{E} \to X$ is a morphism in $\mathcal{C}$ such that $f \circ e = g \circ e$ and $f \circ \grave{e} = g \circ \grave{e}$. Since $\bar{e} : \grave{E} \to E$ is unique the $u^{-1} \circ \bar{e} : \grave{E} \to \grave{E}$ is also unique in $\mathcal{C}$. But

$$\begin{aligned} (e \circ u) \circ (u^{-1} \circ \bar{e}) &= e \circ (u \circ u^{-1}) \circ \bar{e} \\ &= (e \circ I_E) \circ \bar{e} \\ &= e \circ \bar{e} \\ &= \grave{e} \end{aligned} \tag{3.26}$$

Hence $e \circ u : \grave{E} \to X$ is also an equalizer of f and g in the category $\mathcal{C}$.

□

Proposition 3.3.1 gives that equalizers are essentially unique.

Let M and N be sets and $u, v : M \to N$ be a pair of morphisms in the category of sets, and consider the collection

$$T := \{m \in M \mid u(m) = v(m)\}.$$

By definition $T \subseteq M$, thus we have the inclusion $\imath : T \hookrightarrow M$. Also by the definition of the inclusion map for every $m \in T$ we have that

$$(u \circ \imath)(m) = u(\imath(m)) = u(m) = v(m) = v(\imath(m)) = (v \circ \imath)(m),$$

that is $u \circ \imath = v \circ \imath$. Let P be another set such that $q : P \to M$ is a morphism in the category of sets such that $u \circ q = v \circ q$. Let $p \in P$, then $q(p) \in M$, then

$$u(q(p)) = (u \circ q)(p) = (v \circ q)(p) = v(q(p)).$$

Therefore $q(p) \in T \subseteq M$, giving that the morphism $\epsilon : P \to T$ is uniquely defined by $\epsilon : p \mapsto q(p)$. That is the diagram

$$\begin{array}{ccc} P & \xrightarrow{\epsilon} & T \\ & \searrow^{q} & \downarrow{\scriptstyle \imath} \\ & & M \end{array}$$

commutes
Thus the inclusion map $\imath : T \hookrightarrow M$ is an equalizer in the category of sets (see [1]).

We now use the equalizer in the category of sets to determine the equalizer in the category of Frölicher spaces. Let X and Y be Frölicher spaces with $f, g : X \to Y$ where f and g are morphisms in the category of Frölicher spaces, and let

$$E := \{x \in X \mid f(x) = g(x)\}.$$

By definition $E \subseteq X$, so since X is a Frölicher space then E is a Frölicher subspace (see Section 3.1), then the inclusion map $e : E \hookrightarrow X$ is a morphism in the category of Frölicher spaces (see Subsection 3.1.1). Following from the structure of an equalizer in the category of sets, and that the category of Frölicher spaces is topological over the category of sets (see Section 2.4), then the inclusion map $e : E \hookrightarrow X$ is an equalizer in the category of Frölicher spaces.

Definition 3.3.2 ***Frölicher equalizer.***
We call an equalizer in the category of Frölicher spaces a Frölicher equalizer.

By Proposition 3.3.1 Frölicher equalizers are essentially unique, that is Frölicher equalizers are equal up to isomorphism in the category of Frölicher equalizer.

Remark 3.3.1 *Since $E = \{x \in X \mid f(x) = g(x)\}$ is a Frölicher subspace of X then E is not an initial object by definition in the category of Frölicher spaces, however we refer to it as an initial object since it is endowed with an initial (Frölicher) structure (see Remark 3.2.1).*

3.3.2 Topological relations on the domain of Frölicher equalizer.

Having the Frölicher equalizer $e : E \hookrightarrow X$, we have alluded that the domain $E = \{x \in X \mid f(x) = g(x)\}$ is a Frölicher subspace of the Frölicher space X.

That is E is a Frölicher space with the Frölicher structure (C_E, F_E) defined by

$$C_E = \{c_E : \mathbb{R} \to E \mid f_E \circ c_E \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f_E \in F_E\}$$

and

$$F_E = \{f_E : E \to \mathbb{R} \mid f_E \circ c_E \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c_E \in C_E\}$$

(see Definition 2.3.1 in particular equation (2.6) and (2.7), and see Subsection 3.1.1) and (C_E, F_E) is an initial Frölicher structure. Then by Section 2.5 Frölicher topologies, which we denote by τ_{C_E} and τ_{F_E}, are induced from C_E and F_E respectively. By definition

$$\tau_{C_E} = \{U \subseteq E \mid c_E^{-1}(U) \in \tau_{\mathbb{R}}, c_E \in C_E\}$$

and

$$\tau_{F_E} = \{U \subseteq E \mid U = f_E^{-1}(V), f_E \in F_E, V \in \tau_{\mathbb{R}}\}$$

and by Lemma 2.5.1

$$\tau_{F_E} \subset \tau_{C_E}.$$

Since E is a Frölicher subspace of the Frölicher space X then the canonical topology on E is the subspace topology

$$\tau_{F_X}(E) := \{E \cap U \mid U \in \tau_{F_X}\}$$

(see Subsection 3.1.3). By Theorem 3.1.1 $\tau_{F_X}(E) \subset \tau_{F_E}$, then in conjunction with Lemma 2.5.1 we have the topological relation between the canonical topology $\tau_{F_X}(E)$ and the Frölicher topologies τ_{F_E} and τ_{C_E} and that is

$$\tau_{F_X}(E) \subset \tau_{F_E} \subset \tau_{C_E}.$$

Chapter 4

Final objects.

4.1 Frölicher quotient.

4.1.1 Structure.

Definition 4.1.1 ***Equivalence relation.***
Let M be a non-empty set. The relation $\sim$ is an equivalence relation on M if the following conditions hold $\forall\ m, n, p \in M$:

1. *Reflexivity: $m \sim m$.*
2. *Symmetry: $m \sim n$ implies that $n \sim m$.*
3. *Transitivity: $m \sim n$ and $n \sim p$ implies that $m \sim p$.*

Definition 4.1.2 ***Equivalence class.***
Let $\sim$ be an equivalence relation on the non-empty set M. An equivalence class of an element $m \in M$, denoted $[m]$, is a set of all elements in M that are related to or are equivalent to m, that is $[m] = \{n \in M \mid m \sim n\}$.

If M is a non-empty set and $\sim$ is an equivalence relation on M and $m, n \in M$, then we say that m is related to or equivalent to n and vice versa if $m \sim n$. With an equivalence relation $\sim$ on the non-empty set M, the equivalence classes are subsets of M, that is for $m \in M$ we have that $[m] \subseteq M$. Infact the equivalence classes partition M, that is $\bigcup_{m \in M}[m] = M$.

Definition 4.1.3 ***Quotient set.***
If $\sim$ is an equivalence relation on M, the quotient set, denoted $M/_\sim$ is a set of all equivalence classes of order $\sim$, that is $M/_\sim = \{[m] \mid m \in M\}$.

Let $X = (X, C_X, F_X)$ and $Y = (Y, C_Y, F_Y)$ be Frölicher spaces and let $f : X \to Y$ be a morphism in the category of Frölicher spaces, that is $f : X \to Y$ is a smooth

map. Consider the relation $\sim_f$ on X defined by $x \sim_f y \iff f(x) = f(y)$ for every $x, y \in X$. Since $f(x) = f(x)$ is trivial for every $x \in X$, then $x \sim_f x$ for every $x \in X$. Therefore the relation $\sim_f$ is reflexive on X. Also $f(x) = f(y)$ implies that $f(y) = f(x)$ and $y \sim_f x \iff f(y) = f(x)$ then $x \sim_f y$ implies that $y \sim_f x$ since $x \sim_f y \iff f(x) = f(y)$. Thus the relation $\sim_f$ is symmetric on X. Let $x \sim_f y$ and $y \sim_f z$ for every $x, y, z \in X$. That is $f(x) = f(y)$ and $f(y) = f(z)$, therefore $f(x) = f(z)$ and $f(x) = f(z) \iff x \sim_f z$. Giving that the relation $\sim_f$ is transitive on X. Since the relation $\sim_f$ is reflexive, symmetric and transitive on X therefore it is an equivalence relation on X.

Since $\sim_f$ is an equivalence relation on X then we have equivalence classes of elements of X. That is

$$[x] = \{y \in X \mid x \sim_f y\} = \{y \in X \mid f(x) = f(y), \forall f \in C^\infty(X, Y)\}$$

is an equivalence class of $x \in X$. The equivalence classes partition X, that is $X = \bigcup_{x \in X}[x]$. The quotient set of X, denoted $X/_{\sim_f}$ is the set of all equivalence classes on X, that is $X/_{\sim_f} = \{[x] \mid x \in X\}$.

Remark 4.1.1 *As per definition of the relation $\sim_f$ with relation to the morphism $f : X \to Y$ in the category of Frölicher spaces, that is $x \sim_f y \iff f(x) = f(y)$, then we say the relation $\sim_f$ is consistent with f. Thus the relation $\sim_f$ is called the kernel equivalence of f (see [7]).*

Let $X = (X, C_X, F_X)$ be a Frölicher space and $\tilde{X} = X/_{\sim_f}$ be a quotient set of X. Consider the canonical projection $\pi_\sim : X \to \tilde{X}$ defined by $\pi_\sim(x) = [x]$ for every $x \in X$. Let $\tilde{C} := \{\pi_\sim \circ c \mid c \in C_X\}$ and let $\tilde{X}^{\mathbb{R}}$ and $\mathbb{R}^{\tilde{X}}$ be the set of all curves into $\tilde{X} = X/_{\sim_f}$ and the set of all real-valued functions on $\tilde{X} = X/_{\sim_f}$ respectively. Since $\pi_\sim \circ c : \mathbb{R} \to \tilde{X}$ therefore $\tilde{C} \subseteq \tilde{X}^{\mathbb{R}}$ which implies that $\tilde{C} \in P(\tilde{X}^{\mathbb{R}})$ where $P(\tilde{X}^{\mathbb{R}})$ is the power set of $\tilde{X}^{\mathbb{R}}$. The power set of $\mathbb{R}^{\tilde{X}}$, that is $P(\mathbb{R}^{\tilde{X}})$, and the power set $P(\tilde{X}^{\mathbb{R}})$ are categories with subsets of $\mathbb{R}^{\tilde{X}}$ and $\tilde{X}^{\mathbb{R}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

We now start with the process of building a Frölicher structure on $\tilde{X} = X/_{\sim_f}$. Let $c_\sim = \pi_\sim \circ c$ with $c \in C_X$, and let $\Phi_\sim : P(\tilde{X}^{\mathbb{R}}) \to P(\mathbb{R}^{\tilde{X}})$ be a functor defined by

$$\Phi_\sim \tilde{C} = \{f_\sim : \tilde{X} \to \mathbb{R} \mid f_\sim \circ c_\sim \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c_\sim \in \tilde{C}\} := F_{\tilde{X}} \tag{4.1}$$

With $f \in F_X$ we have that $f = f_\sim \circ \pi_\sim$, then we have that

$$\begin{aligned} f \circ c &= (f_\sim \circ \pi_\sim) \circ c \\ &= f_\sim \circ (\pi_\sim \circ c) \\ &= f_\sim \circ c_\sim \end{aligned} \tag{4.2}$$

Since X is a Frölicher space then $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ (see equation (2.6) and (2.7)), therefore also $f_\sim \circ c_\sim \in C^\infty(\mathbb{R}, \mathbb{R})$. That is

$$f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \iff f_\sim \circ c_\sim \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Let $\Gamma_\sim : P(\mathbb{R}^{\tilde{X}}) \to P(\tilde{X}^{\mathbb{R}})$ also be a functor. Then it follows that

$$\begin{aligned} C_{\tilde{X}} &:= \Gamma_\sim F_{\tilde{X}} \\ &= \Gamma_\sim \Phi_\sim \tilde{C} \\ &= \Gamma_\sim \{f_\sim : \tilde{X} \to \mathbb{R} \mid f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R}), \forall c_\sim \in \tilde{C}\} \\ &= \{c_\sim : \mathbb{R} \to \tilde{X} \mid f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R}), \forall f_\sim \in \Phi_\sim \tilde{C} = F_{\tilde{X}}\} \end{aligned} \tag{4.3}$$

Since $c_\sim = \pi_\sim \circ c : \mathbb{R} \to \tilde{X}$ and $f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R})$ for every $f_\sim \in \Phi_\sim \tilde{C} = F_{\tilde{X}}$, therefore $\tilde{C} \subset \Gamma_\sim F_{\tilde{X}} = C_{\tilde{X}}$. Therefore by Lemma 2.3.2

$$\Phi_\sim \tilde{C} = \Phi_\sim \Gamma_\sim \Phi_\sim \tilde{C} \tag{4.4}$$

Then by using (4.1), (4.3) and (4.4) we have that

$$\begin{aligned} \Phi_\sim C_{\tilde{X}} &= \Phi_\sim \Gamma_\sim F_{\tilde{X}} \\ &= \Phi_\sim \Gamma_\sim \Phi_\sim \tilde{C} \\ &= \Phi_\sim \tilde{C} \\ &= F_{\tilde{X}} \end{aligned} \tag{4.5}$$

Since $\tilde{C} \subset C_{\tilde{X}}$ and $c_\sim = \pi_\sim \circ c \in \tilde{C}$ therefore $c_\sim \in C_{\tilde{X}}$. Then from (4.3) and (4.5) we have that

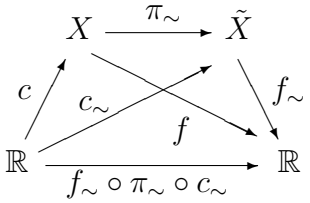
$$\Gamma_\sim F_{\tilde{X}} = C_{\tilde{X}} = \{c_\sim : \mathbb{R} \to \tilde{X} \mid f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R}), \forall f_\sim \in F_{\tilde{X}}\}$$

and

$$\Phi_\sim C_{\tilde{X}} = F_{\tilde{X}} = \{f_\sim : \tilde{X} \to \mathbb{R} \mid f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R}), \forall c_\sim \in C_{\tilde{X}}\}$$

That is the compatibility condition hold (see Definition 2.3.1), of which makes the pair $(C_{\tilde{X}}, F_{\tilde{X}})$ a Frölicher structure on $\tilde{X}$. That is $\tilde{C} = \{\pi_\sim \circ c \mid c \in C_X\}$ generates a Frölicher structure on $\tilde{X}$. Hence the triple $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ with $\tilde{X} = X/_{\sim_f}$ is a Frölicher space (see Definition 2.3.2).

With X and $\tilde{X}$ Frölicher spaces then the canonical projection $\pi_\sim : X \to \tilde{X}$ is a morphism between Frölicher spaces. With $f_\sim \in F_{\tilde{X}}$, that is $f_\sim : \tilde{X} \to \mathbb{R}$ and $f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R})$ for every $c_\sim \in C_{\tilde{X}}$, then $f_\sim \circ \pi_\sim$ is a real-valued function on X, that is $f_\sim \circ \pi_\sim : X \to \mathbb{R}$ is a function. Then for any $c \in C_X$ we have that $(f_\sim \circ c_\sim) \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ by (4.2). Therefore $f_\sim \circ \pi_\sim \in F_X$, implying that $\tilde{F} := \{f_\sim \circ \pi_\sim \mid f_\sim \in F_{\tilde{X}}\} \subset F_X$. We have also shown that $\tilde{C} = \{\pi_\sim \circ c \mid c \in C_X\} \subset C_{\tilde{X}}$. With $\tilde{F} \subset F_X$ and $\tilde{C} \subset C_{\tilde{X}}$, therefore the canonical projection $\pi_\sim : X \to \tilde{X}$ is a smooth map, and thus a morphism in the category of Frölicher spaces. That is we have the diagram



with $c_\sim = \pi_\sim \circ c$ and $f = f_\sim \circ \pi_\sim$.

The morphism $\mathbb{F} \to \tilde{X}$ is not unique for every Frölicher space $\mathbb{F}$. That is $\pi_\sim : X \to \tilde{X}$ is unique only for X and the quotient set $\tilde{X} = X/_{\sim_f}$ of X. For example, consider the Frölicher spaces X and Y such that $Y \subset X$, then the cartesian product $X \times Y$ is also a Frölicher space (see Section 3.2). Then $\pi^* : X \times Y \to \tilde{X}$ is a morphism between Frölicher spaces. Let $f_\sim \in F_{\tilde{X}}$, then we have that $f_\sim \circ \pi^* : X \times Y \to \mathbb{R}$. Since $X \times Y$ is a Frölicher space then $(f_\sim \circ \pi^*) \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ for every $\bar{c} \in C_{X\times Y}$, therefore $f_\sim \circ \pi^* \in F_{X\times Y}$. Similarly, we have that $\pi^* \circ \bar{c} : \mathbb{R} \to \tilde{X}$ for $\bar{c} \in C_{X\times Y}$. Then since $\tilde{X} = X/_{\sim_f}$ is a Frölicher space $f_\sim \circ (\pi^* \circ \bar{c}) \in C^\infty(\mathbb{R}, \mathbb{R})$ for every $f_\sim \in F_{\tilde{X}}$, therefore $\pi^* \circ \bar{c} \in C_{\tilde{X}}$. Hence the morphism $\pi^* : X \times Y \to \tilde{X}$ is a smooth map and thus a morphism in the category of Frölicher spaces. But $\pi^* : X \times Y \to \tilde{X}$ can be defined as $\pi^*(x, y) = [x]$ or $\pi^*(x, y) = [y]$, $[x] \neq [y]$, for $(x, y) \in X \times Y$. Therefore $\pi^* : X \times Y \to \tilde{X}$ is not unique.

Remark 4.1.2 *The morphism $\pi_\sim : X \to \tilde{X}$ is used in generation of the Frölicher structure $(C_{\tilde{X}}, F_{\tilde{X}})$ on $\tilde{X}$, and since $\tilde{X}$ is the codomain of $\pi_\sim$ then we call $(C_{\tilde{X}}, F_{\tilde{X}})$ a final structure. The Frölicher quotient $\tilde{X}$ is not a final object in the category of Frölicher spaces by definition (see Definition 2.1.4) but we refer to it as a final object on the virtue that it is endowed with a final (Frölicher) structure (see Remark 2.1.1), only for the purpose and work of this dissertation.*

Remark 4.1.3 *For any $f \in F_X$ there exists a unique map $g_\sim \in F_{\tilde{X}}$ such that $f = g_\sim \circ \pi_\sim$.*

Definition 4.1.4 ***Frölicher quotient.***
Let $X = (X, C_X, F_X)$ be a Frölicher space, $\sim_f$ a kernel equivalence of $f \in F_X$ and $\tilde{X} = X/_{\sim_f}$ the quotient set on X, with the Frölicher structure $(C_{\tilde{X}}, F_{\tilde{X}})$ on $\tilde{X}$. The triple $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ is called a Frölicher quotient.

4.1.2 Frölicher quotient topologies.

Since the triple $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ with $\tilde{X} = X/_{\sim_f}$, is a Frölicher space, that is a Frölicher quotient, then we have that the pair $(C_{\tilde{X}}, F_{\tilde{X}})$ is a Frölicher structure

as alluded before - the Frölicher quotient structure. Frölicher quotient topologies are topologies induced from the Frölicher quotient structure $(C_{\tilde{X}}, F_{\tilde{X}})$ (see Section 2.5). The Frölicher quotient topologies are denoted $\tau_{C_{\tilde{X}}}$ and $\tau_{F_{\tilde{X}}}$ and are induced respectively by structure curves and structure functions on $\tilde{X}$, that is being induced from $C_{\tilde{X}}$ and $F_{\tilde{X}}$ respectively. With the Frölicher quotient $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$, by Definition 2.5.1 the topology induced by all structure curves of $\tilde{X} = X/_{\sim_f}$ is the collection

$$\tau_{C_{\tilde{X}}} = \{U \subseteq \tilde{X} \mid {c_\sim}^{-1}(U) \in \tau_{\mathbb{R}}, c_\sim \in C_{\tilde{X}}\}$$

and the topology induced by all structure functions of $\tilde{X} = X/_{\sim_f}$ is the collection

$$\tau_{F_{\tilde{X}}} = \{U \subseteq \tilde{X} \mid U = {f_\sim}^{-1}(V), f_\sim \in F_{\tilde{X}}, V \in \tau_{\mathbb{R}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. By Lemma 2.5.1 $\tau_{C_{\tilde{X}}}$ is finer than $\tau_{F_{\tilde{X}}}$, that is

$$\tau_{F_{\tilde{X}}} \subset \tau_{C_{\tilde{X}}}.$$

Definition 4.1.5 *Let $X = (X, C_X, F_X)$ be a Frölicher space and $\sim_f$ a kernel equivalence of $f \in F_X$ on X and the Frölicher quotient $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ with $\tilde{X} = X/_{\sim_f}$ a quotient set on X, then the collection $\{{f_\sim}^{-1}(0,1) \mid f_\sim \in F_{\tilde{X}}\}$ and $\{{f_\sim}^{-1}(0,+\infty) \mid f_\sim \in F_{\tilde{X}}\}$ are the basis and subbasis of $\tau_{F_{\tilde{X}}}$ respectively.*

4.1.3 Quotient topology.

Since (X, τ_{F_X}) is a topological space and we have the relation $\sim_f$ on X, that is the equivalence kernel of $f \in F_X$, the quotient set $\tilde{X} = X/_\sim$ and the canonical projection $\pi_\sim : X \to \tilde{X}$, then the collection

$$\tau_\sim = \{V \subseteq \tilde{X} \mid \pi_\sim^{-1}(V) \in \tau_{F_X}\}$$

is a quotient topology on $\tilde{X}$ (see [3] and [15]). The quotient topology $\tau_\sim$ is a canonical topology on $\tilde{X}$ and makes the canonical projection $\pi_\sim : X \to \tilde{X}$ continuous. That is the quotient topology is the largest, that is the finest topology on $\tilde{X}$ for which $\pi_\sim$ is continuous.

4.1.4 Topological relations on Frölicher quotient.

With $X = (X, C_X, F_X)$ a Frölicher space and $\sim_f$ a kernel equivalence of $f \in F_X$ we have the quotient set $\tilde{X} = X/_{\sim_f}$ and the Frölicher quotient $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$. In Subsection 4.1.2 Frölicher quotient topologies were defined, that is $\tau_{C_{\tilde{X}}}$ and $\tau_{F_{\tilde{X}}}$, of which are topologies induced from the Frölicher quotient structure $(C_{\tilde{X}}, F_{\tilde{X}})$. In Subsection 4.1.3 the quotient topology $\tau_\sim$ was induced on $\tilde{X}$ - this is the standard or canonical topology on $\tilde{X}$. This subsection determines the relationship between Frölicher quotient topologies and the quotient topology. Lemma 2.5.1 gives the relationship between the Frölicher quotient topologies, that is the relationship between $\tau_{C_{\tilde{X}}}$ and $\tau_{F_{\tilde{X}}}$, given as thus $\tau_{F_{\tilde{X}}} \subset \tau_{C_{\tilde{X}}}$.

Lemma 4.1.1 *Let $X = (X, C_X, F_X)$ be a Frölicher space, $\sim_f$ be the kernel equivalence of $f \in F_X$, $\tilde{X} = X/_{\sim_f}$ be the quotient set on X and $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ the Frölicher quotient, then $\tau_{F_{\tilde{X}}} \subset \tau_\sim$.*

Proof: Let $U \in \tau_{F_{\tilde{X}}}$, then $U \subseteq \tilde{X}$ such that $U = {f_\sim}^{-1}(V)$, $f_\sim \in F_{\tilde{X}}$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Recall that $f = f_\sim \circ \pi_\sim$ where $f \in F_X$ and $\pi_\sim : X \to \tilde{X}$ (see Subsection 4.1.1). Therefore

$$\begin{aligned} {\pi_\sim}^{-1}(U) &= {\pi_\sim}^{-1}({f_\sim}^{-1}(V)) \\ &= ({\pi_\sim}^{-1} \circ {f_\sim}^{-1})(V) \\ &= (f_\sim \circ \pi_\sim)^{-1}(V) \\ &= f^{-1}(V) \end{aligned} \tag{4.6}$$

Since ${\pi_\sim}^{-1}(U) \subseteq X$, therefore ${\pi_\sim}^{-1}(U) \in \tau_{F_X}$. Thus $U \in \tau_\sim$, and hence $\tau_{F_{\tilde{X}}} \subset \tau_\sim$. □

Lemma 4.1.2 *Let $X = (X, C_X, F_X)$ be a Frölicher space, $\sim_f$ the kernel equivalence of $f \in F_X$, $\tilde{X} = X/_{\sim_f}$ the quotient set on X and $\tilde{X} = (\tilde{X}, C_{\tilde{X}}, F_{\tilde{X}})$ the Frölicher quotient, then $\tau_{C_{\tilde{X}}} \subset \tau_\sim$.*

Proof: Let $U \in \tau_{C_{\tilde{X}}}$, then $U \subseteq \tilde{X}$ such that ${c_\sim}^{-1}(U) \in \tau_{\mathbb{R}}$ for $c_\sim \in C_{\tilde{X}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Since $f = f_\sim \circ \pi_\sim$ then ${f_\sim}^{-1} = \pi_\sim \circ f^{-1}$. Therefore

$$\begin{aligned} (f \circ {\pi_\sim}^{-1})(U) &= (\pi_\sim \circ f^{-1})^{-1}(U) \\ &= (f^{-1})^{-1}(U) \\ &= f_\sim(U) \\ &= V \end{aligned} \tag{4.7}$$

for some $V \in \tau_{\mathbb{R}}$. Thus ${\pi_\sim}^{-1}(U) = f^{-1}(V)$ with $f \in F_X$. Therefore ${\pi_\sim}^{-1}(U) \in \tau_{F_X}$ which implies that $U \in \tau_\sim$, hence $\tau_{C_{\tilde{X}}} \subset \tau_\sim$. □

Corollary 4.1.1 *Let τ_{F_X} be defined as usual, then the collection $\beta_\sim = \{\pi_\sim(U) \mid U \in \tau_{F_X}\}$ is a basis for the quotient topology $\tau_\sim$.*

Proof: Let $V = \pi_\sim(U)$ where $U \in \tau_{F_X}$, that is $V \in \tilde{X}$ since $\pi_\sim : X \to \tilde{X}$. Then ${\pi_\sim}^{-1}(V) = ({\pi_\sim}^{-1} \circ \pi_\sim)(U) = U \in \tau_{F_X}$. Thus $\beta_\sim = \pi_\sim$ is a trivial basis. □

Lemma 4.1.3 *Let $\tau_{F_{\tilde{X}}}$ and $\tau_{C_{\tilde{X}}}$ be defined in the usual way, then $\tau_{C_{\tilde{X}}} \subset \tau_{F_{\tilde{X}}}$.*

Proof: Let $U \in \tau_{C_{\tilde{X}}}$, the $U \subseteq \tilde{X}$ such that ${c_\sim}^{-1}(U) \in \tau_{\mathbb{R}}$ with $c_\sim \in C_{\tilde{X}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Let $f_\sim \in F_{\tilde{X}}$ and $V \in \tau_{\mathbb{R}}$ then we have that

$$\begin{aligned} {c_\sim}^{-1}({f_\sim}^{-1}(V)) &= ({c_\sim}^{-1} \circ {f_\sim}^{-1})(V) \\ &= (f_\sim \circ c_\sim)^{-1}(V) \end{aligned} \tag{4.8}$$

Since $f_\sim \circ c_\sim \in C^\infty(\mathbb{R},\mathbb{R})$, that is $f_\sim \circ c_\sim : \mathbb{R} \to \mathbb{R}$ is differentiable and thus smooth, then $f_\sim \circ c_\sim : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, therefore

$$\tau_{\mathbb{R}} \ni (f_\sim \circ c_\sim)^{-1}(V) = {c_\sim}^{-1}({f_\sim}^{-1}(V)).$$

From the definition of $\tau_{C_{\tilde{X}}}$ we have that ${c_\sim}^{-1}(U) \in \tau_{\mathbb{R}}$, then there exists $V \in \tau_{\mathbb{R}}$ such that $U = {f_\sim}^{-1}(V)$. Therefore $U \in \tau_{F_{\tilde{X}}}$ of which implies that $\tau_{C_{\tilde{X}}} \subset \tau_{F_{\tilde{X}}}$. □

Corollary 4.1.2 *Let $\tau_{F_{\tilde{X}}}$ and $\tau_{C_{\tilde{X}}}$ be defined as usual, then $\tau_{C_{\tilde{X}}} = \tau_{F_{\tilde{X}}}$.*

Proof: Lemma 2.5.1 gives that $\tau_{F_{\tilde{X}}} \subset \tau_{C_{\tilde{X}}}$, and Lemma 4.1.3 gives that $\tau_{C_{\tilde{X}}} \subset \tau_{F_{\tilde{X}}}$. Hence $\tau_{C_{\tilde{X}}} = \tau_{F_{\tilde{X}}}$. □

Lemma 4.1.4 *Let the quotient topology $\tau_\sim$ and the Frölicher topology $\tau_{F_{\tilde{X}}}$ be defined as usual, then $\tau_\sim \subset \tau_{F_{\tilde{X}}}$.*

Proof: Let $W \in \tau_\sim$, that implies that $W \subseteq \tilde{X}$ such that ${\pi_\sim}^{-1}(W) \in \tau_{F_X}$. That is ${\pi_\sim}^{-1}(W) \subseteq X$ such that ${\pi_\sim}^{-1}(W) = f^{-1}(V)$, $f \in F_X$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Since ${\pi_\sim}^{-1}(W) = f^{-1}(V)$, then we have that $W = \pi_\sim(f^{-1}(V)) = (\pi_\sim \circ f^{-1})(V)$. By Remark 4.1.3 there exists a unique map $g_\sim \in F_{\tilde{X}}$ such that $f = g_\sim \circ \pi_\sim$. Therefore

$$\begin{aligned} W &= (\pi_\sim \circ f^{-1})(V) \\ &= \pi_\sim(f^{-1}(V)) \\ &= \pi_\sim((g_\sim \circ \pi_\sim)^{-1}(V)) \\ &= \pi_\sim(({\pi_\sim}^{-1} \circ {g_\sim}^{-1})(V)) \\ &= \pi_\sim({\pi_\sim}^{-1}({g_\sim}^{-1}(V))) \\ &= (\pi_\sim \circ {\pi_\sim}^{-1}) \circ {g_\sim}^{-1}(V) \\ &= {g_\sim}^{-1}(V) \end{aligned} \tag{4.9}$$

That is $W \subseteq \tilde{X}$ such that $W = {g_\sim}^{-1}(V)$ where $g_\sim \in F_{\tilde{X}}$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Therefore $W \in \tau_{F_{\tilde{X}}}$ and hence $\tau_\sim \subset \tau_{F_{\tilde{X}}}$. □

Corollary 4.1.3 *Let the quotient topology $\tau_\sim$ and the Frölicher topology $\tau_{F_{\tilde{X}}}$ be defined as usual, then $\tau_\sim = \tau_{F_{\tilde{X}}}$.*

Proof: Lemma 4.1.1 gives that $\tau_{F_{\tilde{X}}} \subset \tau_\sim$ and Lemma 4.1.4 gives that $\tau_\sim \subset \tau_{F_{\tilde{X}}}$, hence $\tau_\sim = \tau_{F_{\tilde{X}}}$. □

Lemma 4.1.5 *Let the quotient topology $\tau_\sim$ and the Frölicher topology $\tau_{C_{\tilde{X}}}$ be defined as usual, then $\tau_\sim \subset \tau_{C_{\tilde{X}}}$.*

Proof: Let $W \in \tau_\sim$, that is $W \subseteq \tilde{X}$ such that ${\pi_\sim}^{-1}(W) \in \tau_{F_X}$ and ${\pi_\sim}^{-1}(W) \in \tau_{F_X}$ implies that ${\pi_\sim}^{-1}(W) \subseteq X$ such that ${\pi_\sim}^{-1}(W) = f^{-1}(V)$, $f \in F_X$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$, and ${\pi_\sim}^{-1}(W) = f^{-1}(V)$ implies that $W = \pi_\sim(f^{-1}(V))$. By Remark 4.1.3 there exists a unique map $g_\sim \in F_{\tilde{X}}$ such that $f = g_\sim \circ \pi_\sim$, thus

$$f^{-1}(V) = (g_\sim \circ \pi_\sim)^{-1}(V) = ({\pi_\sim}^{-1} \circ {g_\sim}^{-1})(V).$$

Therefore

$$\begin{aligned} W &= \pi_\sim(f^{-1}(V)) \\ &= \pi_\sim(({\pi_\sim}^{-1} \circ {g_\sim}^{-1})(V)) \\ &= \pi_\sim({\pi_\sim}^{-1}({g_\sim}^{-1}(V))) \\ &= {g_\sim}^{-1}(V) \end{aligned} \tag{4.10}$$

Then for $c_\sim \in C_{\tilde{X}}$ we have that

$$\begin{aligned} {c_\sim}^{-1}(W) &= {c_\sim}^{-1}({g_\sim}^{-1}(V)) \\ &= ({c_\sim}^{-1} \circ {g_\sim}^{-1})(V) \\ &= (g_\sim \circ c_\sim)^{-1}(V) \end{aligned} \tag{4.11}$$

But $g_\sim \circ c_\sim : \mathbb{R} \to \mathbb{R}$ is continuous since $g_\sim \in F_{\tilde{X}}$, $c_\sim \in C_{\tilde{X}}$ and $g_\sim \circ c_\sim \in C^\infty(\mathbb{R}, \mathbb{R})$. Therefore $(g_\sim \circ c_\sim)^{-1}(V) \in \tau_{\mathbb{R}}$ implying that ${c_\sim}^{-1}(W) \in \tau_{\mathbb{R}}$. Therefore $W \in \tau_{C_{\tilde{X}}}$ and hence $\tau_\sim \subset \tau_{C_{\tilde{X}}}$. □

Corollary 4.1.4 *Let the quotient topology $\tau_\sim$ and the Frölicher topology $\tau_{C_{\tilde{X}}}$ be defined as usual, then $\tau_\sim = \tau_{C_{\tilde{X}}}$.*

Proof: Lemma 4.1.2 gives that $\tau_{C_{\tilde{X}}} \subset \tau_\sim$ and Lemma 4.1.5 gives that $\tau_\sim \subset \tau_{C_{\tilde{X}}}$, hence $\tau_\sim = \tau_{C_{\tilde{X}}}$. □

By Corollary 4.1.2, Corollary 4.1.3 and Corollary 4.1.4 we attain the following Proposition:

Proposition 4.1.1 *Let the quotient topology $\tau_\sim$ and the Frölicher topologies $\tau_{F_{\tilde{X}}}$ and $\tau_{C_{\tilde{X}}}$ be defined as usual, then $\tau_{F_{\tilde{X}}} = \tau_{C_{\tilde{X}}} = \tau_\sim$.*

4.2 Frölicher coproduct.

We hereby foremost define the notion of coproducts from set theory, that is coproducts as objects in the category of sets.

Definition 4.2.1 *Let A and B be non-empty sets, the coproduct of A and B denoted $A \oplus B$, or $A + B$ or $A \coprod B$ is the disjoint union of A and B defined as $A \coprod B = \{(a, 1) \mid a \in A\} \bigcup \{(b, 2) \mid b \in B\}$.*

That is in set theory the disjoint union of a family of non-empty sets is a union operation that takes the union of indexed cartesian products which indexes the elements according to which set they are an element of. Thus let $\{A_i \mid i \in I\}$ be a family of non-empty sets indexed by I, then the coproduct of the sets A_i, $i \in I$, that is the disjoint union of the aforementioned family of sets is the set $\coprod_{i\in I} A_i = \bigcup_{i\in I}\{(x,i) \mid x \in A_i\}$. In actual fact $\coprod_{i\in I} A_i = A \times I$.

4.2.1 Structure.

Let $\{X_i \mid i \in I\}$ be a family of Frölicher spaces, that is $X_i = (X_i, C_{X_i}, F_{X_i})$ where (C_{X_i}, F_{X_i}) is a Frölicher structure, $i \in I$, on X_i. Let $\{f_i : X_i \to \mathbb{R} \mid i \in I\}$ and $\{c_i : \mathbb{R} \to X_i \mid i \in I\}$ be a family of real-valued functions on X_i and a family of curves into X_i, $i \in I$, respectively. Then since $X_i = (X_i, C_{X_i}, F_{X_i})$ is a Frölicher space then

$$C_{X_i} = \Gamma_i(F_{X_i}) = \{c_i : \mathbb{R} \to X_i \mid f_i \circ c_i \in C^\infty(\mathbb{R},\mathbb{R}), \forall f_i \in F_{X_i}, i \in I\}$$

and

$$F_{X_i} = \Phi_i(C_{X_i}) = \{f_i : X_i \to \mathbb{R} \mid f_i \circ c_i \in C^\infty(\mathbb{R},\mathbb{R}), \forall c_i \in C_{X_i}, i \in I\}$$

where $\Gamma_i : P(\mathbb{R}^{X_i}) \to P({X_i}^{\mathbb{R}})$ and $\Phi_i : P({X_i}^{\mathbb{R}}) \to P(\mathbb{R}^{X_i})$ are functors with $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ the set of all real-valued functions on X_i and the set of all curves into X_i respectively. The power sets $P(\mathbb{R}^{X_i})$ and $P({X_i}^{\mathbb{R}})$ of $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ respectively are categories with subsets of $\mathbb{R}^{X_i}$ and ${X_i}^{\mathbb{R}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

Consider the coproduct $\bar{X} := \coprod_{i\in I} X_i$ of the family $\{X_i \mid i \in I\}$ of Frölicher spaces, and consider the canonical inclusion map $\varphi_i : X_i \to \bar{X}$, $i \in I$. We now start with the process of building a Frölicher structure on $\bar{X}$. Let $\bar{X}^{\mathbb{R}}$ and $\mathbb{R}^{\bar{X}}$ be the set of all curves into $\bar{X}$ and the set of all real-valued functions on $\bar{X}$ respectively. Then the power sets $P(\bar{X}^{\mathbb{R}})$ and $P(\mathbb{R}^{\bar{X}})$ are categories with subsets of $\bar{X}^{\mathbb{R}}$ and $\mathbb{R}^{\bar{X}}$ as objects respectively, and inclusion maps as morphisms (see Section 2.3).

Let $C_{\amalg} := \{\varphi_i \circ c_i \mid c_i \in C_{X_i}\}$, then since $\varphi_i \circ c_i : \mathbb{R} \to \bar{X}$, $i \in I$, then we have that $C_{\amalg} \subset \bar{X}^{\mathbb{R}}$ which implies that $C_{\amalg} \in P(\bar{X}^{\mathbb{R}})$. Let $\bar{f} = \varphi_i \circ c_i$ such that $c_i \in C_{X_i}$, that is $\bar{f} \in C_{\coprod}$, and let $\bar{\Phi} : P(\bar{X}^{\mathbb{R}}) \to P(\mathbb{R}^{\bar{X}})$ be a functor defined by

$$\bar{\Phi}C_{\amalg} := \{\bar{f} : \bar{X} \to \mathbb{R} \mid \bar{f} \circ \bar{c} \in C^\infty(\mathbb{R},\mathbb{R}), \forall \bar{c} \in C_{\amalg}\} := F_{\bar{X}} \tag{4.12}$$

With $f_i \in F_{X_i}$ we have that $f_i = \bar{f} \circ \varphi_i$, $i \in I$, therefore

$$\begin{aligned} f_i \circ c_i &= (\bar{f} \circ \varphi_i) \circ c_i \\ &= \bar{f} \circ (\varphi_i \circ c_i) \\ &= \bar{f} \circ \bar{c} \end{aligned} \tag{4.13}$$

Since X_i is a Frölicher space then $f_i \circ c_i \in C^\infty(\mathbb{R},\mathbb{R})$ (see the definition of C_{X_i} and F_{X_i}), therefore $\bar{f}\circ\bar{c} \in C^\infty(\mathbb{R},\mathbb{R})$. Hence by (4.13)

$$f_i \circ c_i \in C^\infty(\mathbb{R},\mathbb{R}) \Longleftrightarrow \bar{f}\circ\bar{c} \in C^\infty(\mathbb{R},\mathbb{R}).$$

Let $\bar{\Gamma} : P(\mathbb{R}^{\bar{X}}) \to P(\bar{X}^{\mathbb{R}})$ be a functor, then it follows that

$$\begin{aligned}\bar{\Gamma}F_{\bar{X}} &= \bar{\Gamma}\bar{\Phi}C_{\amalg}\\ &= \{\bar{c}:\mathbb{R}\to\bar{X} \mid \bar{f}\circ\bar{c}\in C^\infty(\mathbb{R},\mathbb{R}), \forall \bar{f}\in \bar{\Phi}C_{\amalg} = F_{\bar{X}}\}\\ &:= C_{\bar{X}}\end{aligned} \tag{4.14}$$

Since $\bar{c} = \varphi_i \circ c_i \in \bar{X}^{\mathbb{R}}$ and $\bar{f}\circ\bar{c} \in C^\infty(\mathbb{R},\mathbb{R})$ then it follows that

$$C_{\amalg} \subset \bar{\Gamma}F_{\bar{X}} = C_{\bar{X}} \tag{4.15}$$

Therefore by Lemma 2.3.2

$$\bar{\Phi}C_{\amalg} = \bar{\Phi}\bar{\Gamma}\bar{\Phi}C_{\amalg} \tag{4.16}$$

Then by using (4.12), (4.14) and (4.16) we have that

$$\begin{aligned}\bar{\Phi}C_{\bar{X}} &= \bar{\Phi}\bar{\Gamma}F_{\bar{X}}\\ &= \bar{\Phi}\bar{\Gamma}\bar{\Phi}C_{\amalg}\\ &= \bar{\Phi}C_{\amalg}\\ &= F_{\bar{X}}\end{aligned} \tag{4.17}$$

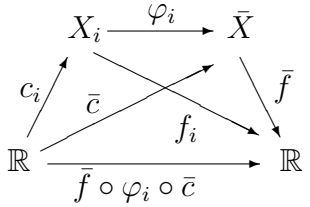
And $\bar{c} \in C_{\bar{X}}$ since $\bar{c} \in C_{\amalg}$ and $C_{\amalg} \subset C_{\bar{X}}$ by (4.15), then by (4.14) and (4.17) we have that

$$C_{\bar{X}} = \bar{\Gamma}F_{\bar{X}} = \{\bar{c}:\mathbb{R}\to\bar{X} \mid \bar{f}\circ\bar{c}\in C^\infty(\mathbb{R},\mathbb{R}), \forall \bar{f}\in F_{\bar{X}}\}$$

and

$$F_{\bar{X}} = \bar{\Phi}C_{\bar{X}} = \{\bar{f}:\bar{X}\to\mathbb{R} \mid \bar{f}\circ\bar{c}\in C^\infty(\mathbb{R},\mathbb{R}), \forall \bar{c}\in C_{\bar{X}}\}$$

That is the compatibility or duality condition holds and thus the pair $(C_{\bar{X}}, F_{\bar{X}})$ is a Frölicher structure on $\bar{X} = \coprod_{i\in I} X_i$ (see Definition 2.3.1). Thus
$C_{\amalg} = \{\varphi_i \circ c_i \mid c_i \in C_{X_i}, i \in I\}$ generates a Frölicher structure on $\bar{X}$. Hence the triple $\bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ is a Frölicher space. Therefore the canonical inclusion $\varphi_i : X_i \to \bar{X}$, $i \in I$, is a morphism between Frölicher spaces. Now let
$F_{\amalg} := \{\bar{f}\circ\varphi_i \mid \bar{f} \in F_{\bar{X}}\}$, then $\bar{f}\circ\varphi_i : X_i \to \mathbb{R}$ is a real-valued function on X_i and by (4.13) we have that $(\bar{f}\circ\varphi_i)\circ c_i \in C^\infty(\mathbb{R},\mathbb{R})$ for every $c_i \in C_{X_i}$, hence $\bar{f}\circ\varphi_i \in F_{X_i}$, therefore $F_{\amalg} \subset F_{X_i}$. Thus since we have that $C_{\amalg} = \{\varphi_i\circ c_i \mid c_i \in C_{X_i}, i \in I\} \subset C_{\bar{X}}$ and $F_{\amalg} \subset F_{X_i}$ then the canonical inclusion map $\varphi_i : X_i \to \bar{X}$, $i \in I$, is a smooth map (see Definition 2.4.1). That is we have the diagram



where $\bar{c} = \varphi_i \circ c_i$ and $f_i = \bar{f} \circ \varphi_i$, $i \in I$. Therefore $\varphi_i : X_i \to \bar{X}$, $i \in I$, is a morphism in the category of Frölicher spaces.

The morphism $\varphi_i : X_i \to \bar{X}$ is unique only for the Frölicher space X_i and the coproduct $\bar{X}$ of X_i, $i \in I$. That is to say the morphism $\mathbb{F} \to \bar{X}$ is not unique for every Frölicher space $\mathbb{F}$. For example let $\{Y_j \mid j \in J\}$ be a family of Frölicher spaces and suppose that Y_j is a group under the binary operation $*$ for every $j \in J$. That is Y_j is closed and associative under the operation $*$, and there's an inverse and an identity in Y_j under the operation $*$. Thus for every $j \in J$

1. $a * b \in Y_j$ for every $a, b \in Y_j$.
2. $a * (b * c) = (a * b) * c$ for every $a, b, c \in Y_j$.
3. For every $a \in Y_j$ there exists an element $e \in Y_j$ such that $a * e = e * a = a$.
4. For every $a \in Y_j$ there exists $a^{-1} \in Y_j$ such that $a^{-1} * a = a * a^{-1}$

Consider the Frölicher product $Y_m \times Y_m$ and the morphism $\psi : Y_m \times Y_m \to \coprod_{j \in J} Y_j$. Let $x, y \in Y_m$ then ψ can be defined as $\psi(x, y) = (x, m)$, $\psi(x, y) = (y, m)$ and $\psi(x, y) = (x * y, m)$. Therefore ψ is not unique. Thus $\coprod_{i \in I} X_i = \bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ is not necessarily a final object in the category of Frölicher spaces.

Remark 4.2.1 *The morphism $\varphi_i : X_i \to \bar{X}$ is used in generation of the Frölicher structure $(C_{\bar{X}}, F_{\bar{X}})$ on $\bar{X}$, and since $\bar{X}$ is the codomain of φ_i then we call the Frölicher structure $(C_{\bar{X}}, F_{\bar{X}})$ on $\bar{X}$ a final structure. The Frölicher space $\bar{X}$ is not a final object in the category of Frölicher spaces by definition (see Definition 2.1.4) however we refer to it as a final object on the vitue that $\bar{X}$ is endowed with a final (Frölicher) structure (see Remark 2.1.1). This is only for the purpose and work of this dissertation.*

Definition 4.2.2 ***Frölicher coproduct.***
Let $\{X_i = (X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces with the coproduct $\bar{X} = \coprod_{i \in I} X_i$, and $(C_{\bar{X}}, F_{\bar{X}})$ be a Frölicher structure on $\bar{X}$. The triple $\bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ is called the Frölicher coproduct.

4.2.2 Frölicher coproduct topologies.

In the previuos subsection we have built a Frölicher structure $(C_{\bar{X}}, F_{\bar{X}})$, that is the Frölicher coproduct structure on the coproduct $\bar{X} = \coprod_{i \in I} X_i$ of the family $\{X_i \mid i \in I\}$ of Frölicher spaces, giving the Frölicher space $\bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ called the Frölicher coproduct. As such we can induce Frölicher topologies from the Frölicher coproduct structure $(C_{\bar{X}}, F_{\bar{X}})$, these are the Frölicher coproduct

topologies denoted $\tau_{C_{\bar{X}}}$ and $\tau_{F_{\bar{X}}}$ induced from $C_{\bar{X}}$ and $F_{\bar{X}}$ respectively and they are defined as

$$\tau_{C_{\bar{X}}} = \{U \subseteq \bar{X} \mid \bar{c}^{-1}(U) \in \tau_{\mathbb{R}}, \bar{c} \in C_{\bar{X}}\}$$

and

$$\tau_{F_{\bar{X}}} = \{U \subseteq \bar{X} \mid U = \bar{f}^{-1}(V), \bar{f} \in F_{\bar{X}}, V \in \tau_{\mathbb{R}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Since $\bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ is a Frölicher space, then Lemma 2.5.1 gives that

$$\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}}$$

that is the topology $\tau_{C_{\bar{X}}}$ is finer than the topology $\tau_{F_{\bar{X}}}$.

Definition 4.2.3 *Let $\{X_i = (X_i, C_{X_i}, F_{X_i}) \mid i \in I\}$ be a family of Frölicher spaces and $\bar{X} = (\bar{X}, C_{\bar{X}}, F_{\bar{X}})$ be the Frölicher coproduct with $\bar{X} = \coprod_{i\in I} X_i$, the families $\{\bar{f}^{-1}(0,\infty) \mid \bar{f} \in F_{\bar{X}}\}$ and $\{\bar{f}^{-1}(0,1) \mid \bar{f} \in F_{\bar{X}}\}$ are the basis and subbasis for the Frölicher topology $\tau_{F_{\bar{X}}}$ respectively.*

4.2.3 Coproduct topology.

Let $\bar{X} = \coprod_{i\in I} X_i$ be the Frölicher coproduct, then with the Frölicher topology $\tau_{F_{\bar{X}}}$ induced from structure functions $F_{\bar{X}}$ we have the topological space $(\bar{X}, \tau_{F_{\bar{X}}})$. The coproduct topology on $\bar{X}$ is the finest topology on $\bar{X}$ to which the canonical morphism $\varphi_i : X_i \to \bar{X}$ is continuous. That is the coproduct topology is given by the collection

$$\tau_{\amalg} = \{U \subseteq \bar{X} \mid {\varphi_i}^{-1}(U) \in \tau_{F_{X_i}}, i \in I\}$$

(see [3] and [15]).

4.2.4 Topological relations on Frölicher coproduct.

Lemma 2.5.1 gives the relationship between the Frölicher coproduct topologies $\tau_{F_{\bar{X}}}$ and $\tau_{C_{\bar{X}}}$, and that is that $\tau_{C_{\bar{X}}}$ is finer than $\tau_{F_{\bar{X}}}$. In this subsection we determine the relationship between the Frölicher coproduct topologies and the coproduct topology.

Lemma 4.2.1 *The coproduct topology $\tau_{\amalg}$ is the finest topology in which all canonical inclusions $\varphi_i : X_i \to \bar{X} = \coprod_{i\in I} X_i$ are continuous and $\tau_{C_{\bar{X}}} \subset \tau_{\amalg}$.*

Proof: Let τ be an arbitrary topology on $\bar{X} = \coprod_{i\in I} X_i$ for which
$\varphi_i : X_i \to \bar{X}$ is continuous for every $i \in I$. That is for every open set $V \in \tau$ we have that ${\varphi_i}^{-1}(V) = U_i \in \tau_{F_{X_i}}$, that is ${\varphi_i}^{-1}(V)$ is open in $\tau_{F_{X_i}}$, $i \in I$. Then by applying φ_i on both sides of $U_i = {\varphi_i}^{-1}(V)$ we have that

$$\varphi_i(U_i) = \varphi_i({\varphi_i}^{-1}(V)) = V \bigcap \varphi_i(U_i).$$

Since $U_i \in \tau_{F_{X_i}}$ then $U_i \subseteq X_i$ implying that $\varphi_i(U_i) \subseteq \bar{X}$ since $\varphi_i : X_i \to \bar{X}$, $i \in I$. But ${\varphi_i}^{-1}(\varphi_i(U_i)) = U_i \in \tau_{F_{X_i}}$, therefore $\varphi_i(U_i) \in \tau_{\amalg}$. Thus
$\tau_{\amalg} \ni \varphi_i(U_i) = V \bigcap \varphi_i(U_i)$. Since $\tau_{\amalg}$ is a topology and thus closed under finite intersection therefore $V \in \tau_{\amalg}$. Hence $\tau \subset \tau_{\amalg}$ which verifies that $\tau_{\amalg}$ is the finest topology in which φ_i is continuous. Since τ was arbitrary then by letting $\tau = \tau_{C_{\bar{X}}}$, then we have that $\tau_{C_{\bar{X}}} \subset \tau_{\amalg}$. □

Lemma 4.2.2 *Let the Frölicher coproduct topologies $\tau_{C_{\bar{X}}}$ and $\tau_{F_{\bar{X}}}$ and the coproduct topology $\tau_{\amalg}$ be defined as usual, then $\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}} \subset \tau_{\amalg}$.*

Proof: By Lemma 2.5.1 we have that $\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}}$ and Lemma 4.2.1 gives that $\tau_{C_{\bar{X}}} \subset \tau_{\amalg}$, therefore $\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}} \subset \tau_{\amalg}$. □

Lemma 4.2.3 *Let the Frölicher coproduct topology $\tau_{F_{\bar{X}}}$ and the coproduct topology $\tau_{\amalg}$ be defined as usual, then $\tau_{F_{\bar{X}}} = \tau_{\amalg}$.*

Proof: From Lemma 4.2.2 we have that $\tau_{F_{\bar{X}}} \subset \tau_{\amalg}$ since $\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}} \subset \tau_{\amalg}$. Recall that

$$\tau_{F_{\bar{X}}} = \{U \subseteq \bar{X} \mid U = \bar{f}^{-1}(V), \bar{f} \in F_{\bar{X}}, V \in \tau_{\mathbb{R}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$, and

$$\tau_{\amalg} = \{U \subseteq \bar{X} \mid {\varphi_i}^{-1}(U) \in \tau_{F_{X_i}}, i \in I\}$$

with the canonical inclusion $\varphi_i : X_i \to \bar{X} = \coprod_{i \in I} X_i$, $i \in I$.
It is sufficient to show that $\tau_{\amalg} \subset \tau_{F_{\bar{X}}}$. Let $U \in \tau_{\amalg}$, that is $U \subseteq \bar{X}$ such that ${\varphi_i}^{-1}(U) \in \tau_{F_{X_i}}$, $i \in I$. Recall also that

$$\tau_{F_{X_i}} = \{W \subseteq X_i \mid W = {f_i}^{-1}(V), f_i \in F_{X_i}, V \in \tau_{\mathbb{R}}\}$$

Then ${\varphi_i}^{-1}(U) \in \tau_{F_{X_i}}$ implies that ${\varphi_i}^{-1}(U) \subseteq X_i$ such that
${\varphi_i}^{-1}(U) = {f_i}^{-1}(V)$, $f_i \in F_{X_i}$, $i \in I$ and $V \in \tau_{\mathbb{R}}$. Applying φ_i on both sides of ${\varphi_i}^{-1}(U) = {f_i}^{-1}(V)$ we have that $U = \varphi_i({f_i}^{-1}(V))$. But since $f_i = \bar{f} \circ \varphi_i$, $\bar{f} \in F_{\bar{X}}$, then we have that

$$\begin{aligned} U &= \varphi_i({f_i}^{-1}(V)) \\ &= \varphi_i((\bar{f} \circ \varphi_i)^{-1}(V)) \\ &= \varphi_i(({\varphi_i}^{-1} \circ \bar{f}^{-1})(V)) \\ &= \varphi_i({\varphi_i}^{-1}(\bar{f}^{-1}(V))) \\ &= \bar{f}^{-1}(V) \end{aligned} \tag{4.18}$$

Since $U \subseteq \bar{X}$ therefore we have that $U \in \tau_{F_{\bar{X}}}$ and hence $\tau_{\amalg} \subset \tau_{F_{\bar{X}}}$. □

Theorem 4.2.1 *Let the Frölicher coproduct topologies $\tau_{C_{\bar{X}}}$ and $\tau_{F_{\bar{X}}}$ and the coproduct topology $\tau_{\amalg}$ be defined as usual, then $\tau_{F_{\bar{X}}} = \tau_{C_{\bar{X}}} = \tau_{\amalg}$.*

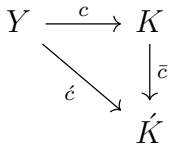
Proof: Since $\tau_{F_{\bar{X}}} \subset \tau_{C_{\bar{X}}} \subset \tau_{\amalg}$ by Lemma 4.2.2, and $\tau_{F_{\bar{X}}} = \tau_{\amalg}$ by Lemma 4.2.3 then it follows that $\tau_{F_{\bar{X}}} = \tau_{C_{\bar{X}}} = \tau_{\amalg}$. □

4.3 Frölicher coequalizer.

4.3.1 Structure.

Definition 4.3.1 ***Coequalizer***
Let $\mathcal{C}$ be a category and (u, v) be a pair of morphisms in the category $\mathcal{C}$ with $u, v : X \to Y$, then the morphism $c : Y \to K$ in the category $\mathcal{C}$ is called a coequalizer of u and v provided that the following conditions hold:

1. $c \circ u = c \circ v$.
2. *For any morphism $\acute{c} : Y \to \acute{K}$ in the category $\mathcal{C}$ with $\acute{c} \circ u = \acute{c} \circ v$, there exists a unique morphism $\bar{c} : K \to \acute{K}$ in the category $\mathcal{C}$ such that $\acute{c} = \bar{c} \circ c$, that is the triangle*



commutes.

Proposition 4.3.1 *Coequalizers are essentially unique. That is given $u, v : X \to Y$ as morphisms in the category $\mathcal{C}$, then the following hold:*

1. *If $c : Y \to K$ and $\acute{c} : Y \to \acute{K}$ are both coequalizers of u and v in the category $\mathcal{C}$ then there is an isomorphism $\bar{c} : K \to \acute{K}$ with $\acute{c} = \bar{c} \circ c$.*
2. *If $c : Y \to K$ is a coequalizer of u and v in the category $\mathcal{C}$ and if $\kappa : K \to \acute{K}$ is an isomorphism then $\kappa \circ c : Y \to \acute{K}$ is also a coequalizer of u and v in the category $\mathcal{C}$.*

Proof:

1. Let $c : Y \to K$ and $\acute{c} : Y \to \acute{K}$ be coequalizers of u and v in the category $\mathcal{C}$, then by definition there exist morphisms $\bar{c} := \bar{c}_1$ and $\bar{c}_2$ where $\bar{c}_1 : K \to \acute{K}$ and $\bar{c}_2 : \acute{K} \to K$ such that
$$\acute{c} = \bar{c}_1 \circ c \tag{4.19}$$
and
$$c = \bar{c}_2 \circ \acute{c} \tag{4.20}$$
Then by substituting (4.19) into (4.20) we obtain that
$$c = \bar{c}_2 \circ (\bar{c}_1 \circ c) = (\bar{c}_2 \circ \bar{c}_1) \circ c$$
Which implies that
$$\bar{c}_1 \circ \bar{c}_2 = c \circ c^{-1} = I_K \tag{4.21}$$

Where I_K is an identity morphism on K.
Similarly by substituting (4.20) into (4.19) we obtain that

$$\acute{c} = \bar{c}_1 \circ (\bar{c}_2 \circ \acute{c}) = (\bar{c}_1 \circ \bar{c}_2) \circ \acute{c}$$

Which implies that

$$\bar{c}_1 \circ \bar{c}_2 = \acute{c} \circ (\acute{c})^{-1} = I_{\acute{K}} \tag{4.22}$$

Where $I_{\acute{K}}$ is an identity morphism on $\acute{K}$.
Then from (4.21) and (4.22) it follows that $\bar{c} := \bar{c}_1$ is an isomorphism in the category $\mathcal{C}$.

2. Let $c : Y \to K$ be a coequalizer of u and v and $\kappa : K \to \acute{K}$ be an isomorphism in the category $\mathcal{C}$. Since $c : Y \to K$ is a coequalizer of u and v then by definition there exists a unique morphism $\bar{c} : K \to \acute{K}$ in the category $\mathcal{C}$ such that $\acute{c} = \bar{c} \circ c$ where $\acute{c} : Y \to \acute{K}$ is a morphism in the category $\mathcal{C}$ such that $\acute{c} \circ u = \acute{c} \circ v$. Since $\bar{c} : K \to \acute{K}$ is unique then it follows that $\bar{c} \circ \kappa^{-1} : \acute{K} \to \acute{K}$ is also unique. But

$$(\bar{c} \circ \kappa^{-1}) \circ (\kappa \circ c) = \bar{c} \circ (\kappa^{-1} \circ \kappa) \circ c = \bar{c} \circ I_K \circ c = \bar{c} \circ c = \acute{c}$$

Hence $\kappa \circ c : Y \to \acute{K}$ is also a coequalizer of u and v in the category $\mathcal{C}$.

□

Let A and B be sets and let (m, n) be a pair of morphims in the category of sets such that $m, n : A \to B$, that is $dom(m) = A = dom(n)$ and $cod(m) = B = cod(n)$. Let $\sim$ be the smallest equivalence relation (see Definition 4.1.1) on B such that $m(a) \sim n(a)$ for all $a \in A$. We now consider the quotient of B by the relation $\sim$, that is the quotient set $B/_\sim$ (see Definition 4.1.3), and the morphism $\sigma : B \to B/_\sim$ defined by $\sigma(b) = [b]$ for every $b \in B$. We have $\sigma \circ m, \sigma \circ n : A \to B/_\sim$ and since $m(a), n(a) \in B$ for every $a \in A$, therefore

$$(\sigma \circ m)(a) = \sigma(m(a)) = [m(a)] \tag{4.23}$$

and

$$(\sigma \circ n)(a) = \sigma(n(a)) = [n(a)] \tag{4.24}$$

But since $m(a) \sim n(a)$ for every $a \in A$, then $[m(a)] = [n(a)]$, that is
(4.23) = (4.24), thus we have that $(\sigma \circ m)(a) = (\sigma \circ n)(a)$ for every $a \in A$, therefore $\sigma \circ m = \sigma \circ n$.

Let $\acute{\sigma} : B \to \acute{B}$ be a morphism in the category of sets such that $\acute{\sigma} \circ m = \acute{\sigma} \circ n$ and $\acute{\sigma} = \bar{\sigma} \circ \sigma$. By right-multiplying $\acute{\sigma} = \bar{\sigma} \circ \sigma$ both sides by σ^{-1} we obtain $\bar{\sigma} = \acute{\sigma} \circ \sigma^{-1}$. Since $[m(a)] \in B/_\sim$ then

$$\bar{\sigma}([m(a)]) = (\acute{\sigma} \circ \sigma^{-1})([m(a)]) = \acute{\sigma}(\sigma^{-1}([m(a)])) = \acute{\sigma}(m(a)) \in \acute{B}$$

and furthermore $\bar{\sigma}([m(a)]) = \bar{\sigma}([n(a)])$ since $\acute{\sigma}(m(a)) = \acute{\sigma}(n(a))$. Thus the triangle

$$\begin{array}{ccc} B & \xrightarrow{\sigma} & B/_{\sim} \\ & \searrow^{\acute{\sigma}} & \downarrow{\scriptstyle \bar{\sigma}} \\ & & \acute{B} \end{array}$$

commutes. Therefore the morphism $\acute{\sigma} : B \to \acute{B}$ is unique, which makes the morphism $\sigma : B \to B/_{\sim}$ a coequalizer in the category of sets.

We now translate from the coequalizer in the category of sets to a coequalizer in the category of Frölicher spaces and we use the coequalizer in the category of sets to determine the coequalizer in the category of Frölicher spaces. Let X and Y be Frölicher spaces, (f, g) be a pair of morphisms in the category of Frölicher spaces such that $f, g : X \to Y$, and $\sim$ be the smallest equivalence relation on Y such that $f(x) \sim g(x)$ for every $x \in X$. The quotient set $Y/_{\sim}$ is a Frölicher quotient since Y is a Frölicher space, and the morphism $e : Y \to Y/_{\sim}$ is a morphism in the category of Frölicher spaces (see Subsection 4.1.1). Following from the structure of the coequalizer in the category of sets and since the category of Frölicher spaces is topological over the category of sets (see Section 2.4) then the morphism $e : Y \to Y/_{\sim}$ is a coequalizer in the category of Frölicher spaces.

Definition 4.3.2 ***Frölicher coequalizer.***
We call the coequalizer in the category of Frölicher spaces the Frölicher coequalizer.

By Proposition 4.3.1 the Frölicher coequalizer $e : Y \to Y/_{\sim}$ is unique up to isomorphism in the category of Frölicher spaces.

Remark 4.3.1 *Since $Y/_{\sim}$ is a Frölicher quotient then $Y/_{\sim}$ is not a final object by definition in the category of Frölicher spaces, however we refer to it as a final object since it is endowed with a final (Frölicher) structure (see Remark 4.1.2).*

4.3.2 Topological relations on the codomain of Frölicher coequalizer.

Let $K := Y/_{\sim}$. We have alluded that K is a Frölicher quotient, therefore K is endowed with a final Frölicher structure (C_K, F_K) defined by

$$C_K = \{\dot{c} : \mathbb{R} \to K \mid \dot{f} \circ \dot{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \dot{f} \in F_K\}$$

and

$$F_K = \{\dot{f} : K \to \mathbb{R} \mid \dot{f} \circ \dot{c} \in C^\infty(\mathbb{R}, \mathbb{R}), \forall \dot{c} \in C_K\}$$

(see Definition 2.3.1 in particular equation (2.6) and (2.7), and see Subsection 3.1.1). Frölicher topologies, which we denote by τ_{C_K} and τ_{F_K} are induced from

the Frölicher structure (C_K, F_K), that is induced from C_K and F_K respectively (see Section 2.5). By definition (see Definition 2.5.1)

$$\tau_{C_K} = \{W \subseteq K \mid \dot{c}^{-1}(W) \in \tau_{\mathbb{R}}, \dot{c} \in C_K\}$$

and

$$\tau_{F_K} = \{W \subseteq K \mid W = \dot{f}^{-1}(V), \dot{f} \in F_K, V \in \tau_{\mathbb{R}}\}$$

and by Lemma 2.5.1

$$\tau_{F_K} \subset \tau_{C_K}.$$

With K a Frölicher quotient then the canonical topology on K is the quotient topology

$$\tau_K := \{U \subseteq K \mid e^{-1}(U) \in \tau_{F_X}\}$$

(see Subsection 4.1.3). By Proposition 4.1.1 the topological relation between the quotient topology τ_K and the Frölicher topologies τ_{C_K} and τ_{F_K} is given by $\tau_{F_K} = \tau_{C_K} = \tau_K$.

Chapter 5

Application: Frölicher tangent bundles.

5.1 Structure.

Definition 5.1.1 ***Operational tangent vector.***
Let (X, C_X, F_X) be a Frölicher space and $x \in X$. The operational tangent vector on X at the point $x \in X$ is the mapping $v_x : F_X \to \mathbb{R}$ such that for all $f, g \in F_X$ and $\alpha \in \mathbb{R}$ we have that:

1. $v_x(f + g) = v_x(f) + v_x(g)$
2. $v_x(\alpha f) = \alpha v_x(f)$
3. $v_x(fg) = v_x(f)g(x) + v_x(g)f(x)$

Conditions 1 and 2 indicates that the mapping $v_x : F_X \to \mathbb{R}$ is a linear map or linear operator, and condition 3 is called the Leibniz rule, that is $v_x : F_X \to \mathbb{R}$ satisfies the Leibniz rule. Also recall that if (X, F_X, C_X) is a Frölicher space then F_X and C_X is the set of smooth real-valued functions on X and the set of smooth curves into X defined as

$$F_X = \{f : X \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall c \in C_X\}$$

and

$$C_X = \{c : \mathbb{R} \to X \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \forall f \in F_X\}$$

Definition 5.1.2 ***Operational tangent space.***
Let (X, C_X, F_X) be a Frölicher space and v_x be an operational tangent vector on X at the point $x \in X$. The set of all operational tangent vectors on X at the point $x \in X$, denoted T_xX, is called the operational tangent space and thus is defined as

$$T_xX = \{v_x : F_X \to \mathbb{R} \mid v_x \in (\texttt{LIN}) \cap (\texttt{LBNIZ})\}$$

where `LIN` *is the set of all linear mappings and* `LBNIZ` *is the set of all linear mappings satisfying the Leibniz rule.*

The expression $v_x \in (\texttt{LIN}) \cap (\texttt{LBNIZ})$ means that $v_x \in (\texttt{LIN})$ and $v_x \in (\texttt{LBNIZ})$, that is $v_x : F_X \to \mathbb{R}$ is linear and satisfies the Leibniz rule, thus $v_x : F_X \to \mathbb{R}$ is an operational tangent vector.

Definition 5.1.3 ***Tangent bundle.***
Let (X, C_X, F_X) be a Frölicher space and T_xX be the operational tangent space, then the operational tangent bundle on X denoted TX is defined as

$$\begin{aligned} TX &= \coprod_{x \in X} T_xX \\ &= \bigcup_{x \in X} \{x\} \times T_xX \\ &= \{(x, v_x) \mid x \in X, v_x \in T_xX\} \end{aligned} \tag{5.1}$$

We now build a Frölicher structure on the tangent bundle TX. The morphism $t : TX \to X$ is a canonical projection defined by $t(x, v_x) = x$ for every $x \in X$, then we have that $f \circ t : TX \to \mathbb{R}$ for $f \in F_X$. We also have a smooth derivation $d : F_X \to F_X$ such that $df : TX \to \mathbb{R}$ for $f \in F_X$ (see [19], in particular Remark 4.1.2). Consider the collection

$$G_0 = \{df : TX \to \mathbb{R} \mid f \in F_X\} \cup \{f \circ t : TX \to \mathbb{R} \mid f \in F_X\}.$$

Let $\mathbb{R}^{TX}$ and $TX^{\mathbb{R}}$ be the set of all real-valued functions on TX and the set of curves into TX, respectively. The power sets $P(\mathbb{R}^{TX})$ and $P(TX^{\mathbb{R}})$ are categories with objects as subsets of $\mathbb{R}^{TX}$ and subsets of $TX^{\mathbb{R}}$, respectively, and inclusion maps as morphisms (see Section 2.3).

Let $\Gamma_{TX} : P(\mathbb{R}^{TX}) \to P(TX^{\mathbb{R}})$ be a functor defined by

$$\Gamma_{TX}G_0 = \{s : \mathbb{R} \to TX \mid g \circ s \in C^\infty(\mathbb{R}, \mathbb{R}), \forall g \in G_0\} := C_{TX} \tag{5.2}$$

Let $\Phi_{TX} : P(TX^{\mathbb{R}}) \to P(\mathbb{R}^{TX})$ be a functor. Then it follows that

$$\begin{aligned} \Phi_{TX}C_{TX} &= \Phi_{TX}\Gamma_{TX}G_0 \\ &= \{g : TX \to \mathbb{R} \mid g \circ s \in C^\infty(\mathbb{R}, \mathbb{R}), \forall s \in \Gamma_{TX}G_0 = C_{TX}\} \\ &:= F_{TX} \end{aligned} \tag{5.3}$$

Let $g \in G_0$ and $s \in C_{TX}$ then by definition $g \circ s \in C^\infty(\mathbb{R}, \mathbb{R})$, therefore $g \in F_{TX}$, hence

$$G_0 \subset \Phi_{TX}C_{TX} = F_{TX} \tag{5.4}$$

Following from (5.4) then Lemma 2.3.2 gives that

$$\Gamma_{TX}G_0 = \Gamma_{TX}\Phi_{TX}\Gamma_{TX}G_0 \tag{5.5}$$

Then by using the equations (5.3), (5.2), (5.4) and (5.2) in that order, we obtain

$$\begin{aligned} \Gamma_{TX}F_{TX} &= \Gamma_{TX}\Phi_{TX}C_{TX} \\ &= \Gamma_{TX}\Phi_{TX}\Gamma_{TX}G_0 \\ &= \Gamma_{TX}G_0 \\ &= C_{TX} \end{aligned} \tag{5.6}$$

Equation (5.3) and (5.6) gives the compatibility condition (see Definition 2.3.1). That is $(C_{TX}, F_{TX}) = (\Gamma_{TX}G_0, \Phi_{TX}\Gamma_{TX}G_0)$ is a Frölicher structure on TX, thus G_0 generates a Frölicher structure on TX. Then the triple (TX, C_{TX}, F_{TX}) is a Frölicher space.

Definition 5.1.4 ***Frölicher tangent bundle.***
Let TX be a tangent bundle with the Frölicher structure (C_{TX}, F_{TX}). The Frölicher space (TX, C_{TX}, F_{TX}) is called a Frölicher tangent bundle.

Let $g \in F_{TX}$ such that $h = g|_{T_xX}$, that is h is the restriction of g on T_xX, then the collection

$$F_{T_xX} = \{h = g|_{T_xX} \mid g \in F_{T_xX}\}$$

is a set of structure functions on T_xX, and is a restriction of F_{TX} on T_xX. Therefore $(F_{T_xX}, \Phi_{TX}F_{T_xX})$ is a Frölicher structure on T_xX and thus $(T_xX, F_{T_xX}, \Phi_{TX}F_{T_xX})$ is a Frölicher space.

By (5.4) we have that

$$\{f \circ t : TX \to \mathbb{R} \mid t : TX \to X, f \in F_X\} \subset F_{TX} \tag{5.7}$$

Let $s \in C_{TX}$, and since $t : TX \to X$ then we have $t \circ s : \mathbb{R} \to X$. Let $f \in F_X$ then $f \circ (t \circ s) = (f \circ t) \circ s$, but $f \circ t \in F_{TX}$ by (5.7), and $s \in C_{TX}$, therefore $C^\infty(\mathbb{R}, \mathbb{R}) \ni (f \circ t) \circ s = f \circ (t \circ s)$, hence $t \circ s \in C_X$ which implies that

$$\{t \circ s \mid s \in C_{TX}, t : TX \to X\} \subset C_X \tag{5.8}$$

Then by (5.7) and (5.8) it follows that the canonical projection $t : TX \to X$ is a smooth map, that is $t : TX \to X$ is a morphism in the category of Frölicher spaces.

Remark 5.1.1 *The projection $t : TX \to X$ is unique and is used in generation of the Frölicher structure (C_{TX}, F_{TX}) on TX, then since TX is the domain of t then we call (C_{TX}, F_{TX}) an initial structure. The Frölicher tangent bundle TX is not an initial object by definition (see Definition 2.1.4) but we refer to it as an initial object on the virtue that it is endowed with an initial (Frölicher) structure (see Remark 2.1.1), only for the purpose and work of this dissertation.*

5.2 Frölicher tangent bundle topologies.

Consider the Frölicher tangent bundle (TX, C_{TX}, F_{TX}), then by Section 2.5 Frölicher topologies can be induced from the Frölicher structure (C_{TX}, F_{TX}). We call these topologies Frölicher tangent bundle topologies, denoted $\tau_{C_{TX}}$ and $\tau_{F_{TX}}$, induced from C_{TX} and F_{TX} respectively. By definition (see Definition 2.5.1)

$$\tau_{C_{TX}} = \{U \subseteq TX \mid s^{-1}(U) \in \tau_{\mathbb{R}}, s \in C_{TX}\}$$

and

$$\tau_{F_{TX}} = \{U \subseteq TX \mid U = g^{-1}(V), g \in F_{TX}, V \in \tau_{\mathbb{R}}\}$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$, and by Lemma 2.5.1

$$\tau_{F_{TX}} \subset \tau_{C_{TX}}.$$

5.3 Canonical topology on Frölicher tangent bundle.

Lemma 5.3.1 *Let*

$$\tau_{TX} := \{t^{-1}(U) \mid U \in \tau_{F_X}, t : TX \to X\},$$

then τ_{TX} is a topology on TX.

Proof: Since $\phi, X \in \tau_{F_X}$ [τ_{F_X} is a topology] then $t^{-1}(\phi) = \phi \in \tau_{TX}$ and $t^{-1}(X) = TX \in \tau_{TX}$.

Let $t^{-1}(U_i) \in \tau_{F_{TX}}$ for all $i = 1, 2, ..., n$, thus $U_i \in \tau_{F_X}$ for $i = 1, 2, ..., n$. But $\bigcap_{i=1}^n t^{-1}(U_i) = t^{-1}(\bigcap_{i=1}^n U_i)$, and since τ_{F_X} is closed under finite intersection then $\bigcap_{i=1}^n U_i \in \tau_{F_X} \Longrightarrow t^{-1}(\bigcap_{i=1}^n U_i) \in \tau_{TX} \Longrightarrow \bigcap_{i=1}^n t^{-1}(U_i) \in \tau_{TX}$.

Let $t^{-1}(U_\alpha) \in \tau_{TX}$, $\alpha \in I$, therefore $U_\alpha \in \tau_{F_X}$ for all $\alpha \in I$. Since τ_{F_X} is closed under infinite union then $\bigcup_{\alpha\in I} U_\alpha \in \tau_{F_X} \Longrightarrow \tau_{TX} \ni t^{-1}(\bigcup_{\alpha\in I} U_\alpha) = \bigcup_{\alpha\in I} t^{-1}(U_\alpha)$

Therefore τ_{TX} is a topology on TX. □

5.4 Topological relations on Frölicher tangent bundle.

We now determine the topological relation between the canonical topology τ_{TX} and the Frölicher tangent bundles topologies $\tau_{C_{TX}}$ and $\tau_{F_{TX}}$. Following from the topological relation on initial objects (on Chapter 3) we have the following Lemma:

Lemma 5.4.1 *Let TX be the Frölicher tangent bundle, and τ_{TX}, $\tau_{F_{TX}}$ and $\tau_{C_{TX}}$ be defined in the usual way, then $\tau_{TX} \subset \tau_{F_{TX}} \subset \tau_{C_{TX}}$.*

Proof: It suffices to show that $\tau_{TX} \subset \tau_{F_{TX}}$ since $\tau_{F_{TX}} \subset \tau_{C_{TX}}$ by Lemma 2.5.1. Let $t^{-1}(U) \in \tau_{TX}$, therefore $U \in \tau_{F_X}$. But $U \in \tau_{F_X}$ implies that $U \subseteq X$ such that $U = f^{-1}(V)$, $f \in F_X$ and $V \in \tau_{\mathbb{R}}$ where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. Therefore since $U = f^{-1}(V)$, $f \in F_X$ then

$$\begin{aligned} t^{-1}(U) &= t^{-1}(f^{-1}V) \\ &= (t^{-1} \circ f^{-1})(V) \\ &= (f \circ t)^{-1}(V) \end{aligned} \tag{5.9}$$

By (5.7) we have that $f \circ t \in F_{TX}$, therefore $t^{-1}(U) \in \tau_{F_{TX}}$. Thus $\tau_{TX} \subset \tau_{F_{TX}}$. Hence $\tau_{TX} \subset \tau_{F_{TX}} \subset \tau_{C_{TX}}$. □

Conclusion

Frölicher spaces are structured sets, that is sets endowed with a structure - the Frölicher structure (see Definition 2.3.1), and the Frölicher structure is a smooth structure. Given a Frölicher space (X, C_X, F_X) then the pair (C_X, F_X) is a Frölicher structure where X is a non-empty set, C_X a set of all curves into X and F_X the set all real-valued functions on X. We have built the category of Frölicher spaces, and it consists of Frölicher spaces as objects and smooth maps as the associated morphisms. From literature (see [5],[6],[7],[9],[10],[11],[12]) the category of Frölicher spaces possesses initial and final objects (see Definition 2.1.3 and 2.1.4), and furthermore it [the category of Frölicher spaces] is complete, co-complete, cartesian-closed and topological over sets (see [6],[10]). Intial and final objects in the category of Frölicher spaces are themselves Frölicher spaces, that is they are structured sets, thus topologies were induced on initial and final objects. Our research title aimed at studying and investigating the building structure of, and the topological relations or the relations of topologies on initial and final objects in the category of Frölicher spaces.

The initial objects in the category of Frölicher spaces under our study were Frölicher subspace, Frölicher product and the domain of Frölicher equilizer. The final objects were Frölicher quotient, Frölicher coproduct and the codomain of Frölicher coequalizer. It should be noted that the aforementioned objects are not the only initial and final objects in the category of Frölicher spaces, and by definition (see definitions 2.1.3 and 2.1.4) are not initial and final objects in the category Frölicher spaces, however for the purpose and work of this dissertation we refer to them as initial and final objects in the category of Frölicher spaces on the virtue that they are endowed with an initial and final structure respectively (see Remark 2.1.1).

The Frölicher structure on each object is generated from an existing Frölicher space - we'll call this Frölicher space the generating Frölicher space. For each object there's a generating set and a canonical morphism used in generation of the Frölicher structure on the object. For initial objects, the initial object is the domain and the generating Frölicher space is the codomain of this canonical morphism, and contrary for final objects this canonical morphism has the generating Frölicher space as the domain and the final object as the codomain. The generating set generating a Frölicher structure on the initial objects is built from the

structure functions of the generating Frölicher space, and the generating set for final objects is built from the structure curves of the generating Frölicher spaces.

For each initial and final object, Frölicher topologies were induced from the Frölicher structure, and there are two Frölicher topologies for each object: a Frölicher topology induced from the structure functions of the object and a Frölicher topology induced from the structure curves of the object. Also for each object a canonical topolgy was induced from the underlying set of the object. Thus for each object we have three topologies.

For any Frölicher space the Frölicher topology induced from the structure functions is contained in the Frölicher topology induced from the structure curves (see Lemma 2.5.1). The topological relation between the canonical topology and the Frölicher topologies can be summarised as follows: If (X, C_X, F_X) is any initial object (under our study) in the category of Frölicher spaces, and let τ_X, τ_{C_X} and τ_{F_X} denote the canonical topology on X, the Frölicher topology induced from structure curves into X and the Frölicher topology induced from the structure funtions on X, then $\tau_X \subseteq \tau_{F_X} \subset \tau_{C_X}$; and if (Y, C_Y, F_Y) is any final object (under our study) with τ_Y, τ_{C_Y} and τ_{F_Y} the canonical topology on Y, the Frölicher topology induced from the structure curves into Y and the Frölicher topology induced from the structure functions on Y respectively, then $\tau_{F_Y} = \tau_{C_Y} = \tau_Y$.

The aforementioned topological relation only hold for the initial and final objects under our study, and these objects are or might not be the only initial and final objects in the category of Frölicher spaces, thus a further study can be conducted to determine whether the aforementioned topological relation holds for any initial and final object in the category of Frölicher spaces. For application we used tangent bundles in the category of Frölicher spaces and the aforementioned topological relation does hold, and a further application can be applied on the cotangent bundles in the category of Frölicher space. The initial and final objects under our study are not initial and final objects as per categorical definitions (definitions 2.1.3 and 2.1.4), however we refer to them as initial and final objects since they are endowed with an initial and final structure respectively, as in literature ([6], [7]). Therefore a topological relation can be investigated on real initial and final objects (as per definition) in the category of Frölicher spaces. At this point real initial and final objects in the category of Frölicher spaces are unknown to us.

Bibliography

[1] J. Adámek, H. Herrlich, G. Strecker, *Abstract and concrete categories*, Wiley-Interscience Publication, John Wiley and Sons Inc, 1990.

[2] M. Arbib, E. Manes, *Arrows, structures and functors: The categorical imperative*, Academic Press Inc, 1975.

[3] A. Arkangel'skii, L. Pontryagin, *General topology I*, Springer-Verlag, Berlin, 1990.

[4] S. Awodey, *Category theory ,2nd edition*, Oxford University Press, 2010, Reprinted 2011.

[5] A. Batubenge, P. Ntumba, *On the way to Frölicher Lie groups*, Quaestiones Mathematicae 28 (2005), 73 - 93.

[6] A. Batubenge, H. Tshilombo, *Topologies on product and coproduct Frölicher spaces*, Demonstratio Mathematica 47 no 4 (2014), 1012 - 1024.

[7] A. Batubenge, H. Tshilombo, *Topologies and smooth maps on initial and final objects in the category of Frölicher spaces*, Demonstratio Mathematica 42 no 3 (2009), 641 - 655.

[8] J. Boman, *Differentiability of a function and of its compositions with functions of one variable*, Math. Scand. 20 (1967), 249 - 268.

[9] P. Cherenack, *Applications of Frölicher spaces to cosmology*, Annals Univ. Sc. Budapest 41 (1998), 63 - 91.

[10] P. Cherenack, *Frölicher versus Differential spaces: A prelude to cosmology*, G. Brümmer and C. Gilmour (eds), Papers in Honour of Bernhard Banaschewski, 2000 Kluwer Academic Publishers, Printed in the Netherlands (1998), 63 - 91.

[11] P. Cherenack, P. Ntumba, *Spaces with differential structure and an application to cosmology*, Demonstratio Mathematica 34 no 1 (2001), 161 - 180.

[12] A. Frölicher, A. Kriegl, *Linear spaces with differentiation theory*, Wisley and Sons, 1988.

[13] J. Lawvere, S. Schanuel, W. Zane, *On C^∞-functions*, Preprint, State University of New York, Buffalo, 1981.

[14] S. Maclane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971.

[15] J. Munkres, *Topology*, Upper Prentice Hall, Inc. c2000 2nd edition, 1930.

[16] W. Sasin, *The De Rham Cohomology of Differential Spaces*, Demonstratio Mathematica 22 no 1 (1989), 249 - 270.

[17] W. Toko, *Bundles in the category of Frölicher spaces and symplectic structures*, MSc Dissertation, University of the Witwatersrand, Johannesburg, 2006.

[18] H. Tshilombo, *Cohomologies on symplectic quotients of locally Euclidean Frölicher spaces*, PhD thesis, University of South Africa, Pretoria, 2015.

[19] H. Tshilombo, *Symplectic reduction on pseudomanifolds*, MSc Dissertation, University of the Witwatersrand, Johannesburg, 2009.