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Two-step inertial projection and contraction method for variational inequality with quasi-monotonicity

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Abstract

In this article, we propose a new modified projection and contraction algorithm for approximating solutions of a variational inequality problem involving a quasi-monotone and Lipschitz continuous mapping in real Hilbert spaces. We incorporate the technique of two-step inertia into a single projection and contraction method and prove a *weak* convergence theorem for the proposed algorithm. Our weak convergence theorem requires neither the prior knowledge of the Lipschitz constant nor the weak sequential continuity of the associated mapping. Under additional strong pseudomonotonicity, the R-linear convergence rate of the two-step inertial algorithm is derived. Finally, some numerical examples are given to illustrate the effectiveness and competitiveness of the proposed algorithm in comparison with some existing algorithms in the literature.

Keywords Two-step inertial · Double step · Projection and contraction · Variational inequality · Weak and linear convergence

Mathematics Subject Classification 47H09 · 47J20 · 47J25 · 65K15

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Suppose C is a nonempty, closed and convex subset of H. Let $\mathcal{F} : C \to H$ be a continuous operator. A variational inequality problem (for short, $VI(\mathcal{F}, C)$) is to find a point $x \in C$ such that

$$\langle \mathcal{F}x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We shall denote by $\mathcal{S}$, the solution set of $VI(\mathcal{F}, C)$ (1.1). Numerous problems arising from economics, engineering, mechanics, mathematical programming and transportation can be reformulated as $VI(\mathcal{F}, C)$(1.1) (see, e.g., [7, 8, 33]). Let $\mathcal{S}_D$ denote the solution set of the dual variational inequality problem, that is, $\mathcal{S}_D := \{x^* \in C \mid \langle \mathcal{F}y, y - x^* \rangle \geq 0, \ \forall \ y \in C\}$. Then, $\mathcal{S}_D$ is a closed and convex subset of C, and since C is convex and $\mathcal{F}$ is continuous, we have that $\mathcal{S}_D \subset \mathcal{S}$.

The monotonicity of operators has played a vital role in the convergence analysis of algorithms for solving variational inequality problems. Recently, some scholars have solved the variational inequality problem by weakening the conditions of the cost operator in the constructed algorithms using pseudomonotonicity or quasimonotonicity see [15, 16, 19, 32, 45, 64, 66, 67]. In the literature, the extradgradient method is a prominent method for approximating solutions of the $VI(\mathcal{F}, C)$ (1.1). This method was first introduced by Korpelevich [41] for solving saddle point problems and has been well applied extensively (see, e.g. [12, 20–22, 27, 31, 56, 57, 69]). The extragradient method involves computing double projections onto the feasible set C twice per each iteration. However, if the set C is not simple, the computation of the projection onto C can be very complicated and thus impedes the usage of the extragradient method. Considerable efforts have been made by many researchers to modify and improve this method; see, e.g. [2, 13, 14, 18, 23, 24, 39, 55, 61]. One of the prominent modifications of the extragradient method is the subgradient extragradient method introduced by Censor et al. (see [21, 22]). In this method, the authors replaced the second projection onto C with a projection onto a half space which can be calculated explicitly. More so, the subgradient extragradient method requires evaluating the value of the cost operator at two points per iteration. This is also not efficient when the cost operator has a complex structure.

Another modification of extradgradient method is the one proposed in Tseng [41] [63]. Tseng's method is of the form:

$$\begin{cases} y_n = P_C(x_n - \lambda \mathcal{F}x_n), \\ x_{n+1} = y_n - \lambda(\mathcal{F}y_n - \lambda \mathcal{F}x_n), \quad \forall \ n \neq 0, \end{cases} \tag{1.2}$$

where $\lambda \in (0, 1/L)$. Tseng's extragradient method for solving $VI(\mathcal{F}, C)$ (1.1) has received great attention by many authors (see, for example, [3, 9, 36, 65] and the references therein).

From a different point of view, another method for solving the $VI(\mathcal{F}, C)$ (1.1) is the projection and contraction method studied by some authors [35, 60]. The projection and contraction method is of the form:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda \mathcal{F}x_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda(\mathcal{F}x_n - \mathcal{F}y_n), \\ x_{n+1} = x_n - \gamma \beta_n d(x_n, y_n), \quad \forall \ n \geq 0, \end{cases} \tag{1.3}$$

where $\gamma \in (0, 2)$, $\lambda \in (0, 1/L)$, and

$$\begin{cases} \beta_n := \frac{\phi(w_n, y_n)}{\|d(w_n, y_n)\|^2}, \\ \phi(w_n, y_n) := \langle w_n - y_n, d(w_n, y_n) \rangle, \quad \forall n \geq 0. \end{cases}$$

In recent years, the projection and contraction method has received great attention by many authors, who improved it in various ways (see, for example, [11, 29, 30] and the references therein).

Remark 1.1 We remark here that the aforementioned methods require the computation of only one projection onto C per iteration. This may reduce the computational cost of the algorithms compared to the extradgradient and subgradient extradgradient algorithms and their modifications.

Now, let us mention an inertial-type algorithm which is based upon the discrete version of a second order dissipative dynamical system [5, 6] and it can be regarded as a procedure of speeding up the convergence properties (see [1, 4, 44, 46, 50, 68]). In 2001, Alvarez and Attouch [4] applied the inertial technique to obtain an inertial proximal method for solving the problem of finding zero of a maximal monotone operator, which is as follows: for any $x_{n-1}, x_n \in H$ and two parameters $\theta_n \in [0, 1)$, $\lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \big(\lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\big), \quad \forall n \geq 0,$$

which can be written equivalently as follows:

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})), \quad \forall\, n \geq 0,$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$.

In order to improve the convergence speed of the projection and contraction method (1.3), Dong et al. in [28] studied an inertial version of the projection and contraction method for $VI(\mathcal{F}, C)$ (1.1). They introduced the following algorithm:

Algorithm 1.2

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda \mathcal{F} w_n), \\ d_n = w_n - y_n - \lambda(\mathcal{F} w_n - \mathcal{F} y_n), \\ x_{n+1} = w_n - \gamma \eta_n d_n, \end{cases} \tag{1.4}$$

where

$$\eta_n = \begin{cases} \frac{\langle w_n - y_n, d_n}{\|d_n\|^2}, & d_n \neq 0, \\ \theta, & d_n = 0, \end{cases}$$

$0 \leq \theta_n \leq \theta_{n+1} \leq \theta < 1$ *and* $\sigma, \delta > 0$ *such that*

(1) $\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$*;*

(2) $0 < \gamma \leq \frac{2[\delta - \theta[\theta(1+\theta)+\theta\delta+\sigma]]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$.

Then, they proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution of $VI(\mathcal{F}, C)$ (1.1) under conditions (1) and (2).

Furthermore, when the inertial factor $\{\theta_n\}$ is chosen such that $0 \leq \theta_n \leq \overline{\theta_n} < 1$, where

$$\overline{\theta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ \theta, & \text{otherwise} \end{cases} \tag{1.5}$$

with $\theta \in [0, 1)$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, converges results of (1.4) to a solution $VI(\mathcal{F}, C)$ (1.1) have been obtained in [25, 28]. The results in [25, 28, 62] all extend the result of He [35] when $\theta_n = 0$ and they have shown numerically to improve the speed of convergence of the projection and contraction method studied in [11, 35].

Remark 1.3 We remark here that the implementation of the results of [25, 28, 62] involves having the value or an estimate of the Lipschitz constant of the cost operator $\mathcal{F}$. Computing the Lipchitz constant is very challenging in practice.

To address the concern raised in the foregoing remark, Shehu et al. [58], proposed an inertial projection and contraction method and inertial forward-backward-forward method to solve $VI(\mathcal{F}, C)$ (1.1), where $\mathcal{F}$ is Lipschitz continuous and monotone mapping in Hilbert spaces. The inertial factors in the methods of Shehu et al. [58] are relaxed and chosen in such a way that the inertial factor taken as 1 is possible. They obtained weak convergence results of the following algorithm under some mild conditions:

Algorithm 1.4

(S0) *Choose* $\theta \in [0, 1)$, $\mu \in (0, 1)$, $\lambda_1 > 0$, $\gamma \in (0, 2)$ *and* $\alpha \in \left(0, \frac{2(1-\theta)^2}{\gamma\theta(1+\theta)+\gamma(1-\theta)^2}\right)$. *Let* $x_0, x_1 \in H$ *be a given starting point. Set* $n := 1$.

(S1) *compute*

$$\begin{cases} w_n = x_n + \theta(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda_n \mathcal{F} w_n). \end{cases} \tag{1.6}$$

If $w_n = y_n$, *STOP. Otherwise*

(S2) *Compute*

$$d_n = (w_n - y_n) - \lambda_n(\mathcal{F} w_n - \mathcal{F} y_n), \quad \forall n \geq 1.$$

(S3) *Compute*

$$x_{n+1} = (1 - \alpha) w_n + \alpha(w_n - \gamma \rho_n d_n), \ \ n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|\mathcal{F} w_n - \mathcal{F} y_n\|}, \lambda_n\right\}, & \mathcal{F} w_n \neq \mathcal{F} y_n \\ \lambda_n, & \textit{otherwise}. \end{cases} \tag{1.7}$$

(S4) *Set* $n \longleftarrow n + 1$, *and go to* $(S1)$.

Observe that all the variants of the projection and contraction method mentioned above only guarantee weak convergence of the sequences to a solution of $VI(\mathcal{F}, C)$ (1.1). To obtain a strong convergence theorem, Thong et al [62] used the viscosity approximation technique in the projection and contraction method to introduce a new inertial viscosity projection and contraction method for solving $VI(\mathcal{F}, C)$ (1.1). Their algorithm is the following:

Algorithm 1.5

Initialization: Let $\lambda \in (0, 1/L)$ *and* $\alpha > 0$. *Choose two positive sequences* $\{\epsilon_n\} \subset [0, \infty)$, $\{\beta_n\} \subset (0, 1)$ *satisfying*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty,$$

$$\sum_{n=1}^{\infty} \epsilon_n < \infty, \quad \epsilon_n = o(\beta_n).$$

Let $x_0, x_1 \in H$ *be arbitrary.*

Step 1. Given the iterates x_{n-1} *and* x_n $(n \geq 1)$, *choose* α_n *such that*

$$\alpha_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } \ x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{1.8}$$

Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ *and compute*

$$y_n = P_C(w_n - \lambda \mathcal{F} w_n).$$

If $y_n = w_n$ *then stop and* y_n *is a solution of the problem* $VI(\mathcal{F}, C)$ *(1.1). Otherwise, go to* **Step 2.**

Step 2. Compute

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)(w_n - \theta_n d_n),$$

where

$$d_n := w_n - y_n - \lambda(\mathcal{F} w_n - \mathcal{F} y_n)$$

and

$$\theta_n := (1 - \lambda L)\frac{\|w_n - y_n\|^2}{\|d_n\|^2}.$$

Set $n := n + 1$ *and go to* **Step 1.**

They proved that the sequence generated by their algorithm converges weakly to a solution of $VI(\mathcal{F}, C)$ (1.1).

Recent research had shown that the one step inertial method has some shortcomings which affects its efficiency to provide acceleration. It was shown in [52, Sect. 4] by example that the one-step inertial extrapolation

$$w_n = x_n + \theta(x_n - x_{n-1}), \ \theta \in [0, 1)$$

may fail to provide acceleration. It was remarked in [42, Chapter 4] that the use of inertial of more than two points x_n, x_{n-1} could provide acceleration. For example, the following two-step inertial extrapolation

$$y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) \tag{1.9}$$

with $\theta > 0$ and $\delta < 0$ can provide acceleration. The failure of one-step inertial acceleration of ADMM was also discussed in [53, Sect. 3] and adaptive acceleration for ADMM was proposed instead. Polyak [51] also discussed that the multi-step inertial methods can boost the speed of optimization methods even though neither the convergence nor the rate result of such multi-step inertial methods was established in [51]. Some results on multi-step inertial methods and double inertia have recently be studied by many authors see, e.g., [17, 26, 47, 49, 54, 59].

Our aim in this paper is to further improve the inertial projection and contraction method to solve $VI(\mathcal{F}, C)$ (1.1). Specifically, in the present paper we study the inertial projection and contraction method and improve on the results in [28, 58, 62] in the following ways:

- We propose a two-step inertial for projection and contraction method to solve $VI(\mathcal{F}, C)$ (1.1) and we give weak convergence results when $\mathcal{F}$ is quasi-monotone and Lipschitz continuous in real Hilbert spaces. Under additional strong pseudomonotonicity and Lipschitz continuity assumptions, the R-linear convergence rate of the two-step inertial algorithm is presented.
- Our proposed method is applicable in situations where $\mathcal{F}$ is pseudomonotone and Lipchitz continuous. Therefore, our method extends the class of operators considered in [28, 30, 32, 58, 62] and the one-step inertial extragradient method studied in [28, 30, 32, 58, 62].
- We give numerical computations of our proposed method and compare it with the methods in [28, 58, 62]. Our preliminary computational results show that our proposed method is efficient and converges faster (in terms of CPU time and number of iterations) than the one step inertial projection and contraction studied in [28, 58, 62].

2 Preliminaries

Definition 2.1 Let H be a real Hilbert space and $\mathcal{F} : H \to H$ be a mapping. An operator $\mathcal{F}$ is said to be:

(i) *Lipschitz-continuous* on H, if there exists a constant $L > 0$ such that

$$\|\mathcal{F}x - \mathcal{F}y\| \leq L\|x - y\|, \ \forall x, y \in H;$$

If $L \in [0, 1)$, then $\mathcal{F}$ is said to be a contraction mapping.

(ii) *sequentially weakly-strongly continuous*, if for each sequence $\{x_n\}$, we have that $x_n \rightharpoonup x \in H$ implies $\mathcal{F}x_n \to \mathcal{F}x \in H$;

(iii) *monotone*, if

$$\langle \mathcal{F}x - \mathcal{F}y, \ x - y\rangle \geq 0, \ \forall x, y \in H;$$

(iv) β-*strongly pseudomonotone* if there exists a constant $\beta > 0$ such that

$$\langle \mathcal{F}y, \ x - y\rangle \geq 0 \implies \langle \mathcal{F}x, \ x - y\rangle \geq \beta\|x - y\|^2, \ \forall x, y \in H;$$

(v) *pseudomonotone*, if

$$\langle \mathcal{F}y, \ x - y\rangle \geq 0 \implies \langle \mathcal{F}x, \ x - y\rangle \geq 0, \ \forall x, y \in H;$$

(vi) *quasimonotone*, if

$$\langle \mathcal{F}y,\ x-y\rangle > 0 \implies \langle \mathcal{F}x,\ x-y\rangle \geq 0,\ \forall x, y \in H.$$

From the definition above, the following implications hold: $(iv) \implies (v) \implies (vi)$ but the converse is not true in general.

The following circumstances occur when S_D is nonempty.

Lemma 2.2 *[70] Suppose either*

(a) $\mathcal{F}$ is pseudomonotone on C and $\mathcal{S} \neq \emptyset$;
(b) $\mathcal{F}$ is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$ and attains its global minimum on C;
(c) $\mathcal{F}$ is quasimonotone on C, $\mathcal{F} \neq 0$ on C and C is bounded;
(d) $\mathcal{F}$ is quasimonotone on C, $\mathcal{F} \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle \mathcal{F}x,\ y-x\rangle \leq 0$;
(e) $\mathcal{F}$ is quasimonotone on C, int $C \neq \emptyset$ and there exists $x^* \in \mathcal{S}$ such that $\mathcal{F}x^* \neq 0$,

then $\mathcal{S}_D$ is nonempty.

For a nonempty, closed and convex subset C of H, the metric projection $P_C : H \to C$ is defined, for every $v \in H$ as the unique element $P_C v \in C$ such that

$$\|v - P_C v\| = \inf\{\|v-y\| : y \in C\}.$$

It is well known that P_C is characterized by the following inequality:

$$\langle v - P_C v,\ y - P_C v\rangle \leq 0,\ \forall y \in C. \tag{2.1}$$

$$\|P_C v - y\|^2 + \|v - P_C v\|^2 \leq \|v-y\|^2, \quad v \in H, y \in C.$$

More information on the metric projection can be found, for example, in [40].

Lemma 2.3 *[34] Let C be a nonempty closed convex subset of a real Hilbert space H. For any $v \in H$ and $z \in C$, we have*

$$z = P_C v \iff \langle v-z,\ z-y\rangle \geq 0,\ \forall y \in C.$$

Lemma 2.4 *[48] The following identities hold for all $u, v \in H$:*

$$2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u-v\|^2 = \|u+v\|^2 - \|u\|^2 - \|v\|^2.$$

3 Main results

We introduce our proposed projection and contraction method with two-step inertial extrapolation in this section. First, we state the following assumptions necessary for our convergence analysis.

Assumption 3.1 (1) (a) $S_D \neq \emptyset$,

(b) $\mathcal{F}$ is Lipschitz continuous on C with constant $L > 0$,

(c) $\mathcal{F}$ satisfies the following condition: whenever $\{x_n\} \subset C$ and $x_n \rightharpoonup v^*$, one has $\|\mathcal{F}v^*\| \leq \liminf_{n\to\infty} \|\mathcal{F}x_n\|$,

(d) $\mathcal{F}$ is quasi-monotone on H.

(2) (i) $\delta \leq 0 < 1 - \theta - \frac{2\theta}{\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}} \iff \delta \leq 0 < 1 - \theta, \;\; 0 < \rho < 1/\beta.$

(ii)

$$\frac{\theta(1+\theta)}{(1-\theta)^2} < \frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho} \iff 0 < \rho < \frac{2(1-\theta)^2}{\beta\theta(1+\theta) + \beta(1-\theta)^2}$$

(iii)

$$2 + \theta + (1-\theta)\left(\frac{1-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right) > \left(\frac{1-\beta}{\beta\rho} + \frac{1-\rho}{\rho} - 1\right)\delta$$

$$\iff (2+\theta)\beta\rho + (1-\theta)(2-\beta\rho) > (2-2\beta\rho)\delta.$$

Note that if

$$0 < \rho < \frac{2(1-\theta)^2}{\beta\theta(1+\theta) + \beta(1-\theta)^2}$$

then $0 < \rho < \beta$ since

$$\frac{2(1-\theta)^2}{\beta\theta(1+\theta) + \beta(1-\theta)^2} < \frac{1}{\beta}.$$

Prototype. First choose θ and β independently and then evaluate $\frac{2(1-\theta)^2}{\beta\theta(1+\theta)+\beta(1-\theta)^2}$. Then choose ρ and δ. For example, take $\theta = \frac{1}{2}$ and $\beta = \frac{1}{5}$ then $\rho \in (0, 2.5)$. Then choose any delta of your choice negative and then all the conditions given in Assumption 3.1 (2) will be satisfied.

Algorithm 3.2 *Two-step Inertial for Projection and Contraction method*

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1. *Choose the iterative parameters* $\beta \in (0, 2)$, $\mu \in (0, 1)$ *and* $\lambda_1 > 0$. *Choose* $x_{-1}, x_0, x_1 \in H$ *be given starting points. Set* $n := 1$.
2. *Compute*

$$\begin{cases} w_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}), \\ y_n = P_C(w_n - \lambda_n \mathcal{F}w_n). \end{cases} \tag{3.1}$$

If $w_n = y_n$, *STOP. Otherwise*

3. *Compute*

$$d_n = (w_n - y_n) - \lambda_n(\mathcal{F}w_n - \mathcal{F}y_n), \quad \forall n \geq 1.$$

4. *Compute*

$$x_{n+1} = (1-\rho)w_n + \rho(w_n - \beta\rho_n d_n), \quad n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|\mathcal{F}w_n - \mathcal{F}y_n\|}, \ \lambda_n + a_n\right\}, & \mathcal{F}w_n \neq \mathcal{F}y_n \\ \lambda_n + a_n, & \textit{otherwise}. \end{cases} \tag{3.2}$$

5. *Set* $n \longleftarrow n+1$, *and go to* (2)

Lemma 3.3 *[66] Assume that $\mathcal{F}$ is L-Lipschiptz continuous on H. Let $\{\lambda_n\}$ be the sequence generated by (3.2). Then*

$$\lim_{n\to\infty} \lambda_n = \lambda \ \ \textit{with} \ \ \lambda \in \left[\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + \alpha\right],$$

where $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Moreover

$$\|\mathcal{F}w_n - \mathcal{F}y_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|. \tag{3.3}$$

Lemma 3.4 *Suppose Assumption 3.1 holds and let $u \in \mathcal{S}_D$. Then $\{x_n\}$ generated by Algorithm 3.2 is bounded.*

Proof Let $u_n := w_n - \beta\rho_n d_n, \ \forall n \geq 1$. By Lemma 2.4, we obtain

$$\begin{aligned} \|u_n - u\|^2 &= \|(w_n - u) - \beta\rho_n d_n\|^2 \\ &= \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - u, d_n\rangle + \beta^2\rho_n^2\|d_n\|^2. \end{aligned} \tag{3.4}$$

Observe that

$$\langle w_n - u, d_n\rangle = \langle w_n - y_n, d_n\rangle + \langle y_n - u, d_n\rangle. \tag{3.5}$$

According to $y_n = P_C(w_n - \lambda_n\mathcal{F}w_n)$ and $u \in \mathcal{S}_D \subseteq \mathcal{S} \subseteq \mathcal{C}$, then by Lemma 2.3, we have

$$\langle y_n - u, w_n - y_n - \lambda_n\mathcal{F}w_n\rangle \geq 0. \tag{3.6}$$

Observed that since $y_n \in C$ and $u \in \mathcal{S}_D$, then we have $\langle \mathcal{F}y_n, y_n - u\rangle \geq 0$, for all $n \geq 0$. Hence

$$\langle \lambda_n\mathcal{F}y_n, y_n - u\rangle \geq 0. \tag{3.7}$$

Summing (3.6) and (3.7), we get

$$\langle y_n - u, w_n - y_n - \lambda_n\mathcal{F}w_n + \lambda_n\mathcal{F}y_n\rangle \geq 0.$$

Thus,

$$\langle y_n - u, d_n\rangle \geq 0. \tag{3.8}$$

Combining (3.5) and (3.8), we get

$$\langle w_n - u, d_n\rangle \geq \langle w_n - y_n, d_n\rangle. \tag{3.9}$$

Putting (3.9) into (3.4), we get $\left(\text{with } \rho_n := \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}\right)$

$$\begin{aligned}\|u_n - u\|^2 &\leq \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - y_n, d_n\rangle + \beta^2\rho_n^2\|d_n\|^2 \\ &= \|w_n - u\|^2 - 2\beta\rho_n\langle w_n - y_n, d_n\rangle + \beta^2\rho_n\langle w_n - y_n, d_n\rangle \\ &= \|w_n - u\|^2 - \beta(2-\beta)\rho_n\langle w_n - y_n, d_n\rangle. \end{aligned} \tag{3.10}$$

From the definition of u_n, we get

$$\begin{aligned}\rho_n\langle w_n - y_n, d_n\rangle &= \|\rho_n d_n\|^2 \\ &= \frac{1}{\beta^2}\|u_n - w_n\|^2. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \frac{2-\beta}{\beta}\|u_n - w_n\|^2. \tag{3.12}$$

Now, using Algorithm 3.2, we get

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|(1-\rho)(w_n - u) + \rho(u_n - u)\|^2 \\ &= (1-\rho)\|w_n - u\|^2 + \rho\|u_n - u\|^2 - \rho(1-\rho)\|w_n - u_n\|^2, \end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}\|x_{n+1} - u\|^2 \leq{}& (1-\rho)\|w_n - u\|^2 + \rho\|w_n - u\|^2 \\ & -\rho\frac{(2-\beta)}{\beta}\|u_n - w_n\|^2 - \rho(1-\rho)\|w_n - u_n\|^2. \end{aligned} \tag{3.14}$$

Note that

$$x_{n+1} = (1-\rho)w_n + \rho u_n$$

and this implies

$$u_n - w_n = \frac{1}{\rho}(x_{n+1} - w_n), \quad \forall n. \tag{3.15}$$

Now using (3.15) in (3.14), we get

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq (1-\rho)\|w_n - u\|^2 + \rho\|w_n - u\|^2 \\ &\quad -\frac{(2-\beta)}{\rho\beta}\|x_{n+1} - w_n\|^2 - \frac{(1-\rho)}{\rho}\|x_{n+1} - w_n\|^2 \\ &= \|w_n - u\|^2 - \left[\frac{(2-\beta)}{\rho\beta} + \frac{(1-\rho)}{\rho}\right]\|x_{n+1} - w_n\|^2. \end{aligned} \tag{3.16}$$

Observe that

$$\begin{aligned} w_n - u &= x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) - u \\ &= (1+\theta)(x_n - u) - (\theta - \delta)(x_{n-1} - u) - \delta(x_{n-2} - u). \end{aligned}$$

Observe that for any $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}\|(1+\alpha)x - (\alpha-\beta)y - \beta z\|^2 ={}& (1+\alpha)\|x\|^2 - (\alpha-\beta)\|y\|^2 - \beta\|z\|^2 + (1+\alpha)(\alpha-\beta)\|x-y\|^2 \\ & +\beta(1+\alpha)\|x-z\|^2 - \beta(\alpha-\beta)\|y-z\|^2. \end{aligned}$$

Consequently, we have from Algorithm 3.2 that

$$\begin{aligned}\|w_n-u\|^2 &= \|(1+\theta)(x_n-u)-(\theta-\delta)(x_{n-1}-u)-\delta(x_{n-2}-u)\|^2\\ &= (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n+1}-u\|^2-\delta\|x_{n-2}-u\|^2\\ &\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2\\ &\quad -\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2 \end{aligned}\tag{3.17}$$

Furthermore,

$$\begin{aligned}\|x_{n+1}-w_n\|^2 &= \|x_{n+1}-(x_n+\theta(x_n-x_{n-1})+\delta(x_{n-1}-x_{n-2}))\|^2\\ &= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|^2\\ &= \|x_{n+1}-x_n\|^2-2\theta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle\\ &\quad +2\delta\langle x_n-x_{n+1}, x_{n-1}-x_{n-2}\rangle+\theta^2\|x_n-x_{n-1}\|^2\\ &\quad +2\delta\theta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle+\delta^2\|x_{n-1}-x_{n-2}\|^2\\ &\geq \|x_{n+1}-x_n\|^2-2\theta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|-2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|\\ &\quad +\theta^2\|x_n-x_{n-1}\|^2-2\theta|\delta|\|x_{n-1}-x_n\|\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2\\ &\geq \|x_{n+1}-x_n\|^2-\theta\|x_{n+1}-x_n\|^2-\theta\|x_n-x_{n-1}\|^2-|\delta|\|x_{n+1}-x_n\|^2-|\delta|\|x_{n-1}-x_{n-1}\|^2\\ &\quad +\theta^2\|x_n-x_{n-1}\|^2-|\delta|\theta\|x_{n-1}-x_n\|^2-|\delta|\theta\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2\\ &= (1-|\delta|-\theta)\|x_{n+1}-x_n\|^2+(\theta^2-|\theta|-|\delta|\theta)\|x_n-x_{n-1}\|^2\\ &\quad +(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2. \end{aligned}\tag{3.18}$$

Using (3.17) and (3.18) in (3.16), we get

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\ &\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2\\ &\quad -\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2-\left(\frac{(2-\beta)}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\ &\quad -\left(\frac{(2-\beta)}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2\\ &\quad -\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2\\ &= (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\ &\quad +\left[(1+\theta)(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\right]\|x_n-x_{n-1}\|^2\\ &\quad +\left[\delta(1+\theta)\|x_n-x_{n-2}\|^2-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\right]\|x_{n+1}-x_n\|^2\\ &\quad -\left[\delta(\theta-\delta)+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2\\ &\leq (1+\theta)\|x_n-u\|^2-(\theta-\delta)\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\ &\quad +\left[(1+\theta)(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\theta^2-\theta-|\delta|\theta)\right]\|x_n-x_{n-1}\|^2\\ &\quad +2\delta(1+\theta)\|x_n-x_{n-1}\|^2-\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\ &\quad -\left[\delta(\theta-\delta)+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2\\ &\quad +2\delta(1+\theta)\|x_{n-1}-x_{n-2}\|^2. \end{aligned}$$

Therefore

$$\begin{aligned}
&\|x_{n+1}-u\|^2-\theta\|x_n-u\|^2-\delta\|x_{n-1}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2\\
&\le \|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&+\left((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right)\|x_n-x_{n-1}\|^2\\
&+\left[2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2
\end{aligned}\tag{3.19}$$

For each, $n\ge 0$, define

$$\begin{aligned}
\Gamma_n := &\|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2
\end{aligned}$$

We first show that $\Gamma_n\ge 0\ \forall n\ge 0$. Note that

$$\|x_{n-1}-u\|^2\le 2\|x_n-x_{n-1}\|^2+2\|x_n-u\|^2$$

Hence,

$$\begin{aligned}
\Gamma_n &= \|x_n-u\|^2-\theta\|x_{n-1}-u\|^2-\delta\|x_{n-2}-u\|^2\\
&\quad+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&\ge \|x_n-u\|^2-2\theta\|x_n-x_{n-1}\|^2-2\theta\|x_n-u\|^2\\
&\quad-\delta\|x_{n-2}-u\|^2+\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)\|x_n-x_{n-1}\|^2\\
&=(1-2\theta)\|x_n-u\|^2+\left[\left(\frac{2-\beta}{\beta\rho}+\frac{(1-\rho)}{\rho}\right)(1-|\delta|-\theta)-2\theta\right]\|x_n-x_{n-1}\|^2\\
&\quad-\delta\|x_{n-2}-u\|^2.
\end{aligned}\tag{3.20}$$

Note that from Assumption 3.1(2) (i), we get

$$|\delta|\le 1-\theta-\frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)}\iff\frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)}+\theta-1\le\delta\le 1-\theta-\frac{2\theta}{\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)}.\tag{3.21}$$

We then obtain from (3.20) and (3.21) that $\Gamma_n\ge 0\ \ \forall n\ge 0$. From (3.19), we obtain

$$\begin{aligned}
\Gamma_{n+1}-\Gamma_n \le &\left((\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)\right)\|x_n-x_{n-1}\|^2\\
&+\left[2\delta(1+\theta)-\delta(\theta-\delta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\delta^2-|\delta|-|\delta|\theta)\right]\|x_{n-1}-x_{n-2}\|^2
\end{aligned}$$

Now, observe that

$$(\theta+\delta)(1+\theta)-\left(\frac{2-\beta}{\beta\rho}+\frac{1-\rho}{\rho}\right)(\theta^2-2\theta-|\delta|\theta-|\delta|+1)<0$$

Since $|\delta| = -\delta$ and $\delta \leq 0$, we obtain

$$\iff \theta(1+\theta) + \delta(1+\theta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta^2 - 2\theta + 1) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)\delta(1+\theta) < 0$$

$$\iff \delta < \frac{\left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta-1)^2 - \theta(1+\theta)}{(1+\theta)\left(1 - \frac{2-\beta}{\beta\rho} - \frac{1-\rho}{\rho}\right)} \tag{3.22}$$

(3.22) is obiviously true since $\delta \leq 0$ and

$$\frac{\theta(1+\theta)}{(\theta-1)^2} < \frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}, \quad \text{by (iii)}.$$

Futhermore, from (3.19) again, we get

$$2\delta(1+\theta) - \delta(\theta-\delta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\delta^2 - |\delta| - |\delta|\theta) \leq 0$$

$$\Leftrightarrow \delta\left[2(1+\theta) - (\theta-\delta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(|\delta| - 1 - \theta)\right] \leq 0. \tag{3.23}$$

By Assumption 3.1(2) (iii), we obtain that (3.23) holds. Hence, from (3.19), we obtain that $\{\Gamma_n\}$ is non-increasing and therefore convergent. That is, $\lim_{n\to\infty}\Gamma_n$ exists. Furthermore, (3.19) also implies

$$\lim_{n\to\infty}\left(-(\theta+\delta)(1+\theta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta^2 - 2\theta - |\delta|\theta - |\delta| + 1)\right)\|x_n - x_{n-1}\|^2 - \left(2\delta(1+\theta) - \delta(\theta-\delta) - \left(\frac{2-\beta}{\beta\rho} - \frac{1-\rho}{\rho}\right)(\delta^2 - |\delta| - |\delta|\theta)\right)\|x_{n-1} - x_{n-2}\|^2 = 0$$

Consequently

$$\begin{aligned} 0 &\leq -\left((\theta+\delta)(1+\theta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta^2 - 2\theta - |\delta|\theta - |\delta| + 1)\right)\|x_n - x_{n-1}\|^2 \\ &\leq -\left((\theta+\delta)(1+\theta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta^2 - 2\theta - |\delta|\theta - |\delta| + 1)\right)\|x_n - x_{n-1}\|^2 \\ &\quad - \left(2\delta(1+\theta) - \delta(\theta-\delta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\delta^2 - |\delta| - |\delta|\theta)\right)\|x_{n-1} - x_{n-2}\|^2 \end{aligned} \tag{3.24}$$

Therefore, from (3.24), we get

$$\lim_{n\to\infty} -\left[(\theta+\delta)(1+\theta) - \left(\frac{2-\beta}{\beta\rho} + \frac{1-\rho}{\rho}\right)(\theta^2 - 2\theta - |\delta|\theta - |\delta| + 1)\right]\|x_n - x_{n-2}\|^2 = 0.$$

Thus,

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0.$$

Also,

$$\begin{aligned}\|x_{n+1}-w_n\| &= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|\\ &\le \|x_{n+1}-x_n\|+\theta\|x_n-x_{n-1}\|+|\delta|\|x_{n-1}-x_{n-2}\| \to 0,\ n\to\infty.\end{aligned}$$

Also,

$$\|u_n-w_n\| = \frac{1}{\rho}\|x_{n+1}-w_n\| \to 0,\ n\to\infty, \tag{3.25}$$

and

$$\|u_n-x_{n+1}\| \le \|u_n-w_n\|+\|w_n-x_{n+1}\| \to 0,\ n\to\infty, \tag{3.26}$$

and

$$\|u_n-x_n\| \le \|u_n-x_{n+1}\|+\|x_{n+1}-x_n\| \to 0,\ n\to\infty. \tag{3.27}$$

Since $\lim_{n\to\infty}\Gamma_n$ exists and $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0$, we have from (3.20) that $\{x_n\}$ is bounded. □

Remark 3.5 It is clear to see that the boundedness of $\{x_n\}$ implies that $\{w_n\}$, $\{y_n\}$, $\{\mathcal{F}y_n\}$, and $\{d_n\}$ are all bounded.

Lemma 3.6 *Suppose Assumption 3.1(1) is satisfied and $\{x_n\}$ is generated by Algorithm 3.2. Then, $\lim\limits_{n\to\infty}\|w_n-y_n\|=0$.*

Proof By Algorithm (3.2), we get

$$\begin{aligned}\|d_n\| &= \|w_n-y_n-\lambda_n(\mathcal{F}w_n-\mathcal{F}y_n)\|\\ &\le \|w_n-y_n\|+\lambda_n\|\mathcal{F}w_n-\mathcal{F}y_n\|\\ &\le \left(1+\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n-y_n\|.\end{aligned}$$

So,

$$\frac{1}{\|d_n\|} \ge \frac{1}{\left(1+\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n-y_n\|}. \tag{3.28}$$

Then,

$$\begin{aligned}\langle w_n-y_n,d_n\rangle &= \langle w_n-y_n, w_n-y_n-\lambda_n(\mathcal{F}-\mathcal{F}y_n)\rangle\\ &= \|w_n-y_n\|^2-\langle w_n-y_n,\lambda_n(\mathcal{F}w_n-\mathcal{F}y_n)\rangle\\ &\ge \|w_n-y_n\|^2-\lambda_n\|\mathcal{F}w_n-\mathcal{F}y_n\|\|w_n-y_n\|\\ &\ge \|w_n-y_n\|^2-\frac{\lambda_n\mu}{\lambda_{n+1}}\|w_n-y_n\|^2\\ &= \left(1-\frac{\lambda_n\mu}{\lambda_{n+1}}\right)\|w_n-y_n\|^2.\end{aligned} \tag{3.29}$$

By (3.28) and (3.29), we get

$$\begin{aligned}\|u_n - w_n\| &= \beta\rho_n\|d_n\| \\ &= \beta\frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|} \\ &\geq \beta\left[\frac{\left(1-\frac{\lambda_n\mu}{\lambda_{n+1}}\right)}{\left(1+\frac{\lambda_n\mu}{\lambda_{n+1}}\right)}\right]\|w_n - y_n\|.\end{aligned} \tag{3.30}$$

By (3.25), we get from (3.30) (noting that $\lim_{n\to\infty}\lambda_n = \lambda$.)

$$\lim_{n\to\infty}\|w_n - y_n\| = 0.$$

□

Lemma 3.7 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 such that Assumption 3.1(1) is satisfied. Suppose $\lim\limits_{n\to\infty}\|w_n - y_n\| = 0$. If v^* is one of the weak cluster points of $\{x_n\}$, then we have at least one of the following: $v^* \in \mathcal{S}_D$ or $\mathcal{F}v^* = 0$.*

Proof By Lemma 3.4, $\{x_n\}$ is bounded. Hence, let v^* be be a weak cluster points of $\{x_n\}$. Then, we can choose a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup v^* \in C$.

We consider the following two possible cases.

Case I: Suppose that $\limsup\limits_{n\to\infty}\|\mathcal{F}x_{n_k}\| = 0$. Then, $\lim_{k\to\infty}\|\mathcal{F}x_{n_k}\| = \liminf_{k\to\infty}\|\mathcal{F}x_{n_k}\| = 0$. Thus, we obtain from Assumption that

$$0 < \|\mathcal{F}v^*\| \leq \liminf_{k\to\infty}\|\mathcal{F}x_{n_k}\| = 0. \tag{3.31}$$

This means that $\mathcal{F}v^* = 0$.

Case II: Suppose that $\limsup\limits_{n\to\infty}\|\mathcal{F}x_{n_k}\| > 0$. Then without loss of generality, we can choose a subsequence of $\{\mathcal{F}x_{n_k}\}$ still denoted by $\{\mathcal{F}x_{n_k}\}$ such that $\lim_{k\to\infty}\|\mathcal{F}x_{n_k}\| = M_1 > 0$. Now, using (2.1), we obtain for all $y \in C$ that

$$\langle w_{n_k} - \lambda_{n_k}\mathcal{F}w_{n_k} - y_{n_k}, y - y_{n_k}\rangle \leq 0. \tag{3.32}$$

Adding $\lambda_{n_k}\langle\mathcal{F}y_{n_k}, y - y_{n_k}\rangle$ to both sides of (3.32), we get

$$\lambda n_k\langle\mathcal{F}y_{n_k}, y - y_{n_k}\rangle \geq \langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle\mathcal{F}y_{n_k} - \mathcal{F}w_{n_k}, y - y_{n_k}\rangle. \tag{3.33}$$

Since $\lambda_{n_k} > 0$, we obtain from (3.33) that

$$\langle\mathcal{F}y_{n_k}, y - y_{n_k}\rangle \geq \frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle\mathcal{F}y_{n_k} - \mathcal{F}w_{n_k}, y - y_{n_k}\rangle. \tag{3.34}$$

Since $\lim_{n\to\infty}\|y_n - w_n\| = 0$ and $\mathcal{F}$ is Lipschitz continuous on H, we obtain that $\lim_{n\to\infty}\|\mathcal{F}y_n - \mathcal{F}w_n\| = 0$. Furthermore, we obtain from (3.34) that

$$0 \leq \liminf_{k\to\infty}\langle\mathcal{F}x_{n_k}, y - x_{n_k}\rangle \leq \limsup_{k\to\infty}\langle\mathcal{F}x_{n_k}, y - x_{n_k}\rangle < \infty, \quad \forall\, y \in C. \tag{3.35}$$

Based on (3.35), we consider the following two cases under *Case II:*

Case A: Suppose that $\limsup_{k\to\infty}\langle\mathcal{F}x_{n_k}, y - x_{n_k}\rangle > 0, \quad \forall y \in C$. Then we can choose a subsequence of $\{x_{n_k}\}$ denoted by $\{x_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle\mathcal{F}x_{n_{k_j}}, y - x_{n_{k_j}}\rangle > 0$. Thus, there exists $j_0 \geq 1$ such that $\langle\mathcal{F}x_{n_{k_j}}, y - x_{n_{k_j}}\rangle > 0, \ \forall j \geq j_0$, which by quasimonotonicity of $\mathcal{F}$ on C, implies that $\langle \mathcal{F}(y), y - x_{n_{k_j}} \rangle \geq 0,\ \forall y \in C,\ j \geq j_0$. Hence, letting $j \to \infty$, we get $\langle \mathcal{F}y, y - v^* \rangle \geq 0,\ \forall y \in C$. Therefore, $v^* \in \mathcal{S}_D$.

Case B: Suppose that $\limsup_{k\to\infty} \langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle = 0,\ \forall y \in C$. Then, by (3.35), we get

$$\lim_{k\to\infty} \langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle = 0,\quad \forall y \in C, \tag{3.36}$$

from which we get that

$$\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle + |\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1} > 0,\quad \forall y \in C. \tag{3.37}$$

Also, since $\lim_{k\to\infty} \|\mathcal{F}x_{n_k}\| = M_1 > 0$, we can find $k_0 \geq 1$ such that $\|\mathcal{F}x_{n_k}\| > \frac{M_1}{2},\ \forall k \geq k_0$. Hence, we can set $b_{n_k} = \frac{\mathcal{F}x_{n_k}}{\|\mathcal{F}x_{n_k}\|^2},\ \forall k \geq k_0$. Thus, $\langle \mathcal{F}x_{n_k}, b_{n_k} \rangle = 1,\ \forall k \geq k_0$. Therefore, by (3.37), we get

$$\left\langle \mathcal{F}x_{n_k}, y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\rangle > 0,$$

and using the quasimonotonicity of $\mathcal{F}$ on H, we obtain

$$\begin{aligned}&\left\langle \mathcal{F}\left(y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| \frac{1}{k+1}\right]\right),\right.\\ &\quad \left. y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\rangle \geq 0.\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\langle \mathcal{F}y, y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\rangle\\
&\geq \left\langle \mathcal{F}y - \mathcal{F}\left(y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right]\right),\right.\\
&\quad \left. y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\rangle\\
&\geq -\left\| \mathcal{F}y - \mathcal{F}\left(y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right]\right)\right\| \cdot\\
&\quad \left\| y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\|\\
&\geq -L\left\| b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right]\right\|\\
&\quad \left\| y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\|\\
&= \frac{-L}{\|\mathcal{F}x_{n_k}\|}\left(|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right)\\
&\quad \left\| y + b_{n_k}\left[|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right] - x_{n_k} \right\|\\
&\geq \frac{-2L}{M_1}\left(|\langle \mathcal{F}x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1}\right) M_2,
\end{aligned} \tag{3.38}$$

for some $M_2 > 0$, where the existence of M_2 is from the boundedness of

$$\left\{ y + b_{n_k} \left[|\langle \mathcal{F} x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1} \right] - x_{n_k} \right\}.$$

Now, observe that (3.36) implies that

$$\lim_{k\to\infty} \left(|\langle \mathcal{F} x_{n_k}, y - x_{n_k} \rangle| + \frac{1}{k+1} \right) = 0.$$

Thus, as $k \to \infty$ in (3.38), we get that $\langle \mathcal{A} y, y - v^* \rangle \geq 0, \quad \forall y \in C$. Therefore, $v^* \in \mathcal{S}_D$. □

Theorem 3.8 *Suppose $\{x_n\}$ is a sequence generated by Algorithm 3.2. Then under Assumptions 3.1(1) and $\mathcal{F}x \neq 0, \forall x \in C$. Then $\{x_n\}$ converges weakly to an element of $S_D \subset S$.*

Proof Suppose $w_\omega(x_n)$ is the set of weak cluster points of $\{x_n\}$. We show that

$$w_\omega(x_n) \subset S_D.$$

Take $v^* \in w_\omega(x_n)$. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup v^*, \ k \to \infty$. Since C is weakly closed, we have that $v^* \in C$. Since $\mathcal{F}x \neq 0, \ \forall x \in C$, we have $\mathcal{F}v^* \neq 0$. By Lemma, we get $v^* \in S_D$. Therefore, $w_\omega(x_n) \subset S_D$. Since $\lim\limits_{n\to\infty} \Gamma_n$ exists and $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have that

$$\lim_{n\to\infty} [\|x_n - u\|^2 - \theta \|x_{n-1} - u\|^2 - \delta \|x_{n-2} - u\|^2]$$

exists for all $u \in S_D$.
Suppose now that there exists $\{x_{n_j}\} \subset \{x_n\}$ and $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup v^*, \ j \to \infty$ and $x_{n_m} \rightharpoonup x^*, \ m \to \infty$. we show that $v^* = x^*$ Observe that

$$2\langle x_n, x^* - v^* \rangle = \|x_n - v^*\|^2 - \|x_n - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2 \tag{3.39}$$

$$2\langle x_{n-1}, x^* - v^* \rangle = \|x_{n-1} - v^*\|^2 - \|x_{n-1} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

and

$$2\langle x_{n-2}, x^* - v^* \rangle = \|x_{n-2} - v^*\|^2 - \|x_{n-2} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

Therefore

$$2\langle -\theta x_{n-1}, x^* - v^* \rangle = -\theta \|x_{n-1} - v^*\|^2 + \theta \|x_{n-1} - x^*\|^2 + \theta \|v^*\|^2 - \theta \|x^*\|^2. \tag{3.40}$$

and

$$\begin{aligned} 2\langle -\delta x_{n-2}, x^* - v^* \rangle = & -\delta \|x_{n-2} - v^*\|^2 + \delta \|x_{n-2} - x^*\|^2 \\ & +\delta \|v^*\|^2 - \delta \|x^*\|^2. \end{aligned} \tag{3.41}$$

Addition of (3.39), (3.40) and (3.41) gives

$$\begin{aligned} 2\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^* \rangle = & \left(\|x_n - v^*\|^2 - \theta \|x_{n-1} - v^*\|^2 - \delta \|x_{n-2} - v^*\|^2 \right) \\ & -(\|x_n - x^*\|^2 - \theta \|x_{n-1} - x^*\|^2 \\ & -\delta \|x_{n-2} - x^*\|^2) \\ & +(1 - \theta - \delta)(\|x^*\| - \|v^*\|^2). \end{aligned} \tag{3.42}$$

Since $\lim_{n\to\infty}\Gamma_n$ exists, we get

$$\lim_{n\to\infty}[\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2] \tag{3.43}$$

exists and

$$\lim_{n\to\infty}[\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2] \tag{3.44}$$

exists. This implies from (3.42) that $\lim_{n\to\infty}\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle$ exists. Consequently,

$$\begin{aligned}\langle v^* - \theta v^* - \delta v^*, x^* - v^*\rangle &= \lim_{j\to\infty}\langle x_{n_j} - \theta x_{n_{j-1}} - \delta x_{n_{j-2}}, x^* - v^*\rangle\\ &= \lim_{n\to\infty}\langle x_{n-1} - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle\\ &= \lim_{m\to\infty}\langle x_{n_m} - \theta x_{n_{m-1}} - \delta x_{n_{m-2}}, x^* - v^*\rangle\\ &= \langle x^* - \theta x^* - \delta x^*, x^* - v^*\rangle.\end{aligned}$$

Hence

$$(1-\theta-\delta)\|x^* - v^*\|^2 = 0.$$

Since $\delta \le 0 < 1-\theta$, we obtain that $x^* = v^*$. Hence, $\{x_n\}$ converges weakly to a point in $\mathcal{S}_D$. This completes the proof. □

4 Linear convergence

Here, we present the algorithm for our linear convergence under the following assumptions:

Assumption 4.1

(a) $\mathcal{F}$ is ϱ-strongly pseudo-monotone and L-Lipschitz continuous on H.
(b) Let $\mu \in (0,1)$, $\lambda_1 > \frac{\mu}{L}$ and θ, δ, ρ be three real number satisfying

$$0 \le \rho \le \frac{1}{2},\quad 0 \le \theta \le \delta \le \min\left\{\frac{1-2\rho}{1-\rho}, \frac{\rho}{1-\rho}, \frac{\rho\tau}{1-\rho\tau}\right\},$$

where $\tau := \min\left\{\frac{(1-\mu)^2}{2(1+\mu)^2}, \varrho\lambda\frac{(1-\mu)}{(1+\mu)^2}\right\}$.

Algorithm 4.2 *Linear double Step Inertial Projection and Contraction Method*

1. *Given starting points $x_0, x_1 \in H$. Set $n := 1$.*
2. *Compute*

$$\begin{cases} z_n = x_n + \delta(x_n - x_{n-1}),\\ w_n = x_n + \theta(x_n - x_{n-1}),\\ y_n = P_C(w_n - \lambda_n\mathcal{F}w_n),\end{cases}$$

If $w_n = y_n = x_n$, STOP. Otherwise

3. *Compute*

$$d_n = (w_n - y_n) - \lambda_n(\mathcal{F}w_n - \mathcal{F}y_n),\quad \forall n \ge 1.$$

4. *Compute*

$$x_{n+1} = (1-\rho)z_n + \rho(w_n - \beta\rho_n d_n), \quad n \geq 1,$$

where

$$\rho_n = \begin{cases} \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2}, & d_n \neq 0 \\ 0, & d_n = 0 \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\|w_n - y_n\|}{\|\mathcal{F}w_n - \mathcal{F}y_n\|}, \ \lambda_n + a_n \right\}, & \mathcal{F}w_n \neq \mathcal{F}y_n \\ \lambda_n + a_n, & otherwise. \end{cases}$$

5. *Set* $n \longleftarrow n+1$, *and go to* (2).

Theorem 4.3 *Assume that $\mathcal{F}$ is ϱ-strongly pseudo-monotone and L-Lipschitz continuous. Then $\{x_n\}$ be generated by Algorithm (4.2) under the Assumptions (4.1) converges linearly to a unique element of $\mathcal{S}$.*

Proof Let u be the unique element in $\mathcal{S}$. Then, we get $\langle \mathcal{F}u, x-u\rangle \geq 0, \ \forall x \in \mathcal{S}$. In particula, choose $y_n \in \mathcal{S}$ one obtains

$$\lambda_n \langle \mathcal{F}u, y_n - u \rangle \geq 0.$$

Since $\mathcal{F}$ is ϱ-strongly pseudo-monotone, we get

$$\lambda_n \langle \mathcal{F}y_n, y_n - u \rangle \geq \lambda_n \varrho \|y_n - u\|^2. \tag{4.1}$$

Adding of (3.6) and (4.1) gives

$$\langle y_n - u, w_n - y_n - \lambda_n \mathcal{F}w_n + \lambda_n \mathcal{F}y_n \rangle \geq \varrho\lambda_n \|y_n - u\|^2$$

Thus,

$$\langle y_n - u, d_n \rangle \geq \varrho\lambda_n \|y_n - u\|^2, \tag{4.2}$$

we then obtain from (3.5) and (4.2) that

$$\langle w_n - u, d_n \rangle \geq \langle w_n - y_n, d_n \rangle + \varrho\lambda_n \|y_n - u\|^2. \tag{4.3}$$

Also, substituting (4.3) into (3.4) (when $\beta = 1$) gives

$$\begin{aligned} \|u_n - u\|^2 &\leq \|w_n - u\|^2 + \rho_n^2\|d_n\|^2 - 2\rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2 \\ &= \|w_n - u\|^2 + \rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2 \\ &= \|w_n - u\|^2 - \rho_n\langle w_n - y_n, d_n\rangle - 2\rho_n\varrho\lambda_n\|y_n - u\|^2. \end{aligned} \tag{4.4}$$

Adding (3.11) and (4.4), we obtain

$$\|u_n - u\|^2 \leq \|w_n - u\|^2 - \|u_n - w_n\|^2 - 2\rho_n\varrho\lambda_n\|y_n - u\|^2. \tag{4.5}$$

From the definition of ρ_n and (3.29), we get for all $n \geq n_0$

$$\rho_n^2\|d_n\|^2 = \langle w_n - y_n, d_n\rangle \geq \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2$$

and

$$\rho_n = \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2} \geq \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} \tag{4.6}$$

Also, from (3.30), we get

$$-\|u_n - w_n\|^2 \leq -\frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} \|w_n - y_n\|^2 \quad \forall n \geq n_0. \tag{4.7}$$

Substituting (4.7) into (4.5) (noting (4.6)) one has

$$\begin{aligned} \|u_n - u\|^2 &\leq \|w_n - u\|^2 - \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} \|w_n - y_n\|^2 - 2\rho_n \varrho \lambda_n \|y_n - u\|^2 \\ &\leq \|w_n - u\|^2 - \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} \|w_n - y_n\|^2 - 2\varrho \lambda_n \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} \|y_n - u\|^2. \end{aligned} \tag{4.8}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$, we get

$$\lim_{n\to\infty} \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \frac{(1-\mu)^2}{(1+\mu)^2} \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \frac{(1-\mu)}{(1+\mu)^2} \tag{4.9}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|(1-\rho)(z_n - u) + \rho(u_n - u)\|^2 \\ &= (1-\rho)\|z_n - u\|^2 + \rho\|u_n - u\|^2 - \rho(1-\rho)\|u_n - z_n\|^2. \end{aligned} \tag{4.10}$$

Further, $x_{n+1} = (1-\rho)z_n + \rho u_n$, hence $\frac{1}{\rho}(x_{n+1} - z_n) = u_n - z_n$. Substituting this into (4.10), we deduce

$$\|x_{n+1} - u\|^2 \leq (1-\rho)\|z_n - u\|^2 + \rho\|u_n - u\|^2 - \frac{1}{\rho}(1-\rho)\|x_{n+1} - z_n\|^2. \tag{4.11}$$

Substituting (4.8) and (4.9) into (4.11), we obtain

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leq (1-\rho)\|z_n - u\|^2 + \rho\|w_n - u\|^2 - \rho \frac{(1-\mu)^2}{(1+\mu)^2} \|w_n - y_n\|^2 \\ &\quad -2\varrho\lambda_n \frac{(1-\mu)}{(1+\mu)^2} \|y_n - u\|^2 - \frac{1}{\rho}(1-\rho)\|x_{n+1} - z_n\|^2 \end{aligned} \tag{4.12}$$

Recalling that $\tau := \min\left\{\frac{(1-\mu)^2}{2(1+\mu)^2}, \varrho \frac{\mu(1-\mu)}{L(1+\mu)^2}\right\}$, we find

$$\lim_{n\to\infty} \frac{\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \mu \frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \frac{(1-\mu)^2}{(1+\mu)^2} \geq 2\tau.$$

Using Lemma 3.3 and assuming that $\lambda_1 \geq \frac{\mu}{L}$, we have $\lim_{n\to\infty}\lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\} = \frac{\mu}{L}$. Thus

$$\lim_{n\to\infty} \varrho\lambda_n \frac{\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)}{\left(1+\mu\frac{\lambda_n}{\lambda_{n+1}}\right)^2} = \varrho\frac{\mu}{L}\frac{(1-\mu)}{(1+\mu)^2} \geq \tau.$$

Using inequality (4.12), we deduce

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\|z_n-u\|^2 + \rho\|w_n-u\|^2 - 2\rho\tau\|w_n-y_n\|^2 - 2\rho\tau\|y_n-u\|^2 \\ &\quad -\frac{1}{\rho}(1-\rho)\|x_{n+1}-z_n\|^2 \\ &\leq (1-\rho)\|z_n-u\|^2 + \rho(1-\tau)\|w_n-u\|^2 - \frac{1}{\rho}(1-\rho)\|x_{n+1}-z_n\|^2.\end{aligned} \tag{4.13}$$

By the definition of w_n, we have

$$\begin{aligned}\|z_n-u\|^2 &= \|x_n+\delta(x_n-x_{n-1})-u\|^2 \\ &= \|(1+\delta)(x_n-u)-\delta(x_{n-1}-u)\|^2 \\ &= (1+\delta)\|x_n-u\|^2 - \delta\|x_{n-1}-u\|^2 + \delta(1+\delta)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{4.14}$$

By the definition of z_n, we have

$$\begin{aligned}\|w_n-u\|^2 &= \|x_n+\theta(x_n-x_{n-1})-u\|^2 \\ &= \|(1+\theta)(x_n-u)-\theta(x_{n-1}-u)\|^2 \\ &= (1+\theta)\|x_n-u\|^2 - \theta\|x_{n-1}-u\|^2 + \theta(1+\theta)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{4.15}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \|x_{n+1}-x_n-\delta(x_n-x_{n-1})\|^2 \\ &= \|x_{n+1}-x_n\|^2 + \delta^2\|x_n-x_{n-1}\|^2 - 2\delta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle \\ &\geq \|x_{n+1}-x_n\|^2 + \delta^2\|x_n-x_{n-1}\|^2 - 2\delta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\| \\ &\geq (1-\delta)\|x_{n+1}-x_n\|^2 + (\delta^2-\delta)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{4.16}$$

Substituting (4.14), (4.15) and (4.16) into (4.13), we obtain

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\rho)\left[(1+\delta)\|x_n-u\|^2 - \delta\|x_{n-1}-u\|^2 + \delta(1+\delta)\|x_n-x_{n-1}\|^2\right] \\ &\quad +\rho(1-\tau)\left[(1+\theta)\|x_n-u\|^2 - \theta\|x_{n-1}-u\|^2 + \theta(1+\theta)\|x_n-x_{n-1}\|^2\right] \\ &\quad -\frac{1-\rho}{\rho}\left[(1-\delta)\|x_{n+1}-x_n\|^2 + (\delta^2-\delta)\|x_n-x_{n-1}\|^2\right] \\ &= [(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)]\|x_n-u\|^2 - [(1-\rho)\delta+\rho(1-\tau)\theta]\|x_{n-1}-u\|^2 \\ &\quad +\left[(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta) - \frac{1-\rho}{\rho}(\delta^2-\delta)\right]\|x_n-x_{n-1}\|^2 \\ &\quad -\frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2.\end{aligned}$$

This follows that

$$\begin{aligned}&\|x_{n+1}-u\|^2+\frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2\le[(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)]\,\|x_n-u\|^2\\&+\left[(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)\right]\|x_n-x_{n-1}\|^2\\&=\mathcal{M}\left(\|x_n-u\|^2+\frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}\|x_n-x_{n-1}\|^2\right),\end{aligned} \tag{4.17}$$

where $\mathcal{M}:=((1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta))$. From Assumptions 4.1 we have $\delta\le\frac{1-2\rho}{1-\rho}$, and hence

$$\frac{1-\rho}{\rho}(1-\delta)\ge 1.$$

Next, we show that

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)<1.$$

Indeed, from $\theta\le\delta$ we have

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)\le(1-\rho)(1+\delta)+\rho(1-\tau)(1+\delta). \tag{4.18}$$

On the other hand, from $\delta\le\frac{\rho\tau}{1-\rho\tau}$ we find that

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\delta)=1+\delta-\rho\tau(1+\delta)\le 1. \tag{4.19}$$

Combining (4.18) and (4.19) we obtain

$$(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)<1.$$

Next, we prove that

$$\frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}<1. \tag{4.20}$$

The last inequality is equvalent to

$$(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)<(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta). \tag{4.21}$$

That is

$$\frac{1-\rho}{\rho}(\delta-\delta^2)<(1-\delta)(1-\rho)(1+\delta)+\rho(1-\tau)(1-\theta)(1+\theta).$$

This inequality is equivalently to

$$\delta(1-\rho)(1-\delta)<\rho(1-\delta)(1-\rho)(1+\delta)+\rho^2(1-\tau)(1-\theta)(1+\theta). \tag{4.22}$$

From $\delta\le\frac{\rho}{1-\rho}$ we obtain

$$\delta(1-\rho)(1-\delta)<\rho(1-\delta)(1-\rho)(1+\delta). \tag{4.23}$$

Combining (4.21),(4.22) and (4.23) we deduce that inequality (4.20) holds. Now, we prove that the sequence $\{x_n\}$ converges to u with an R-linear rate. Indeed from (4.17) we deduce

$$\begin{aligned}
&\|x_{n+1}-u\|^2+\|x_{n+1}-x_n\|^2 \le \|x_{n+1}-u\|^2+\frac{1-\rho}{\rho}(1-\delta)\|x_{n+1}-x_n\|^2\\
&\le \mathcal{M}\left(\|x_n-u\|^2+\frac{(1-\rho)\delta(1+\delta)+\rho(1-\tau)\theta(1+\theta)+\frac{1-\rho}{\rho}(\delta-\delta^2)}{(1-\rho)(1+\delta)+\rho(1-\tau)(1+\theta)}\|x_n-x_{n-1}\|^2\right)\\
&\le \mathcal{M}\left(\|x_n-u\|^2+\|x_n-x_{n+1}\|^2\right).
\end{aligned} \tag{4.24}$$

Letting $v_n := \|x_n-u\|^2+\|x_n-x_{n-1}\|^2$ and $\omega := (1-\rho(1-(1-\tau)(1+\delta)))$, from we get

$$\|x_{n+1}-u\|^2 \le v_{n+1} \le \omega v_n \le \omega^{n-N+1}v_N = \frac{\omega}{\omega^N}v^N\omega^n.$$

Thus, the sequence $\{x_n\}$ converges R-linearly to u, as desired. □

5 Numerical illustrations

In this section, we compare the performance of different algorithms to tackle some classical benchmark problems that have been considered by some authors (see, e.g., [10, 58, 62]). We specifically evaluate Algorithm 3.1 of Dong et al. [28] (DCZR Alg. 3.1 for short), Algorithm 3 of Izuchukwu and Shehu [37] (IS Alg. 3 for short), Algorithm 1 of Izuchukwu et al. [38] (ISY Alg. 1 for short), Algorithm 1 of Thong et al. [62] (TVC Alg 1 for short), Algorithm 3.2 of Shehu et al. [58] (SLMD Alg. 3.2 for short), and compare their performance with our proposed Algorithm 3.2 (OAAO Alg. 3.2 for short) in terms of the required number of iterations and computational time to satisfy the stopping criteria.

Example 5.1 Let $\mathcal{F}:\mathbb{R}\to\mathbb{R}$ be defined by

$$\mathcal{F}(x)=\begin{cases}2x-1, & x>1;\\ x^2, & x\in[-1,1];\\ -2x-1, & x<1.\end{cases}$$

Let $C=[-1,1]$. It is well known that $\mathcal{F}$ is quasimonotone on C and Lipschitz continuous but not pseudomonotone or monotone (see, e.g., [43]). In DCZR Alg. 3.1, we take $\alpha_n=\frac{1}{n+1}$, $\tau=10^{-7}$, $\gamma=0.0021$. In IS Alg. 3, we take $\delta=0.9$, $\theta=-0.45$, $\lambda_0=0.8$ and $\bar{\lambda}=2$. In ISY Alg. 1, we take $\delta=0.01$, $\mu=\frac{0.99(1-2\delta)}{2}$, $\theta=0.01(\min(\frac{\delta}{2},\frac{0.5-\mu}{2}))$, $a_n=\frac{1}{(n+1)^8}$, $\gamma_0=0.1$ and $\gamma_1=0.5$. In TVC Alg 1, we take $\lambda=10^{-9}$, $\beta_n=\frac{1}{10^4n+1}$, $\epsilon_n=\frac{1}{(n+1)^2}$, $\alpha=0.9$. In SLMD Alg. 3.2, we take $\gamma=0.2$, $\theta=\frac{15}{16}$, $\alpha=\frac{1}{150}$, $\mu=10^{-6}$ and $\lambda_1=0.1$. In our proposed OAAO Alg. 3.2, we take $\mu=10^{-6}$, $\lambda_1=0.001$, $\beta=0.2$, $\theta=\frac{15}{16}$, $\rho=\frac{1}{150}$, $\delta=-0.1$ and $\alpha_n=\frac{1}{(n+1)^4}$. The initial points x_0,x_1,x_2 are generated randomly in $\mathbb{R}$. We consider two cases for the tolerance $tol=10^{-6}$ and 10^{-8}. For each case, the simulation is terminated when the number of iterations (Iter for short) $n=3001$ or error $E_n=\|x_{n+1}-x_n\|>tol$. The result of the numerical simulations are presented in Table 1 and Fig. 1 below.

Discussion of Results. In this example, we consider a special variational inequality in which the cost function is quasi-monotone and not pseudomonotone or monotone. From Fig. 1, we observe that the iterates of ISY Alg. 1 are spinning even though they approach the solution

Table 1 Example 5.1 for Different Cases

Example 5.1 for $tol = 10^{-6}$ and 10^{-8}

Tol	DCZR Alg. 3.1		IS Alg. 3		ISY Alg. 1		TVC Alg 1		SLMD Alg. 3.2		OAAO Alg. 3.2	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$Tol = 10^{-6}$	94	0.0075	119	0.0131	153	0.0157	83	0.0088	170	0.0151	55	0.0019
$Tol = 10^{-8}$	186	0.0226	198	0.0224	237	0.0245	390	0.0316	375	0.0267	80	0.0124

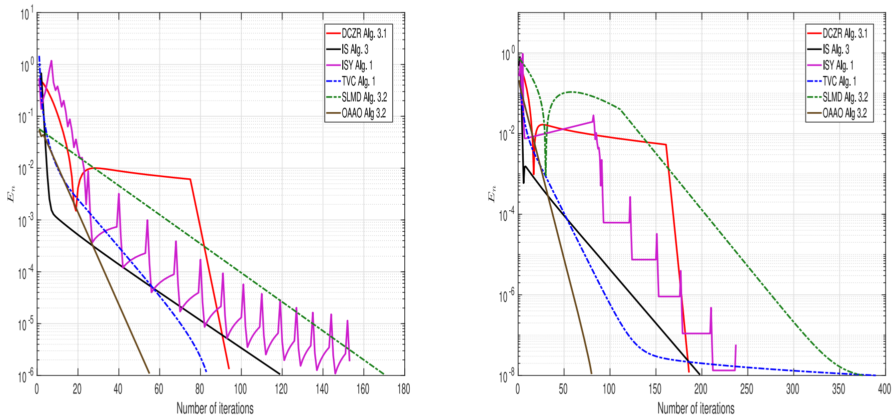

Fig. 1 Graph the iterates for $tol = 10^{-6}$ and $tol = 10^{-8}$

as the iteration process progresses and there is an abrupt drop to the solution as the iteration process progresses in DCZR Alg. 3.1. Even though the iterates generated by IS Alg. 3, TVC Alg. 1 and SLMD Alg. 3.2 progresses smoothly towards the solution, our proposed Algorithm 3.2 (OAAO Alg. 3.2) requires fewer number of iterations and CPU time to satisfy the prescribed stopping criterion in the two cases. Thus, in this example, our proposed OAAO Alg. 3.2 demonstrates its competence and promise over these algorithms in solving variational inequalities involving quasi-monotone operators.

Example 5.2 Define $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\mathcal{F}x = \left(e^{-x^T Qx} + \alpha\right)(Px + r),$$

where Q is a positive definite matrix, P is a positive semi-definite matrix, $r \in \mathbb{R}^N$ and $\alpha > 0$. Observe that $\mathcal{F}$ is Lipschitz continuous and pseudo-monotone but not monotone (see, e.g., [10, Example 2.1]). In MATLAB, we generate P, Q and r randomly and took $\alpha = 5$. In addition, we set $C := \{x \in \mathbb{R}^N : \|x\| \leq 10\}$. In DCZR Alg. 3.1, In IS Alg. 3, In ISY Alg. 1 and SLMD Alg. 3.2 we use the same parameters used in Example 5.1. In TVC Alg 1, we take $\lambda = 10^{-8}$, $\beta_n = \frac{1}{10^5 n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\alpha = 0.9$. In our proposed OAAO Alg. 3.2, we take $\mu = 10^{-6}$, $\lambda_1 = 0.001$, $\beta = \frac{1}{1.1}$, $\theta = \frac{15}{16}$, $\rho = \frac{1}{150}$, $\delta = -\frac{1}{50}$ and $\alpha_n = \frac{1}{(n+1)^4}$. The initial points x_0, x_1, x_2 are generated randomly in $\mathbb{R}^N$. To test the robustness of the algorithms we consider four (4) different dimensions, i.e., $\mathbb{R}^{50}$, $\mathbb{R}^{100}$, $\mathbb{R}^{500}$, and $\mathbb{R}^{1000}$. For each case, the simulation is terminated when the number of iterations (Iter for short) $n = 3001$ or $\|x_{n+1} - x_n\| > 10^{-6}$. The result of the numerical simulations are presented in Table 2 and Figs. 2 and 3 below.

Discussion of Results. In this example, we considered a variational inequality problem involving a pseudomonotone operator. From Table 2 we observe that as we increase the dimension of the space to 1000, both DCZR Alg. 3.1 and TVC Alg. 1 fail to satisfy the tolerance before the prescribed maximum number of iterations was exhausted. This makes them less attractive for solving pseudomonotone monotone variational inequality on higher dimensional spaces. Also, we observe that in this example, the spinning nature of ISY Alg. 1 did not occur as the iterates move smoothly towards the solution. Although IS Alg. 3 and SLMD Alg. 3.2 were very competitive, the choice of $\beta = \frac{1}{1.1}$ in our proposed OAAO Alg.

Table 2 Example 5.2 for Different Dimensions

Example 5.2 for the four (4) dimensions

$\mathbb{R}^N$	DCZR Alg. 3.1		IS Alg. 3		ISY Alg. 1		TVC Alg. 1		SLMD Alg. 3.2		OAAO Alg. 3.2	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$N = 50$	194	0.0341	595	0.0869	953	0.0944	493	0.0478	325	0.0394	188	0.0317
$N = 100$	308	0.0478	608	0.1423	1045	0.2722	699	0.1569	332	0.1022	211	0.0632
$N = 500$	3000	5.3317	670	1.3810	1169	2.4395	1494	2.9460	350	0.8373	236	0.5588
$N = 1000$	3000	18.8286	1099	7.0129	1117	9.0283	3000	19.0328	358	3.5853	242	2.4767

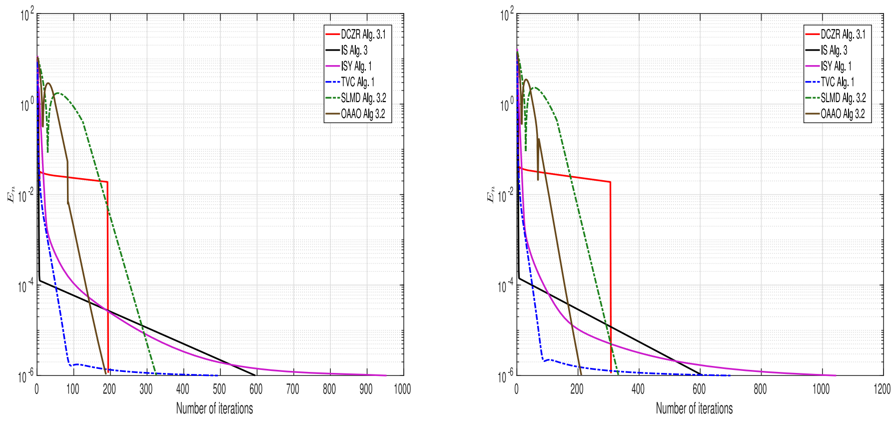

Fig. 2 Graph the Iterates for Dimensions $N = 50$ and 100

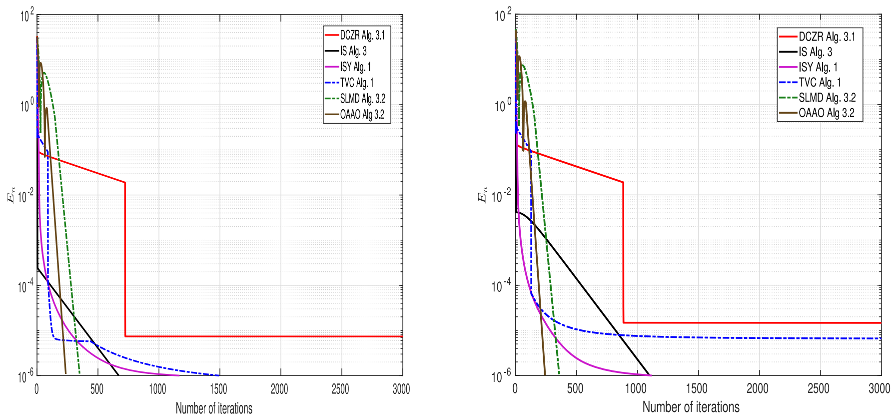

Fig. 3 Graph the Iterates for Dimensions $N = 500$ and 1000

3.2 greatly influence the computational advantage of our algorithm. Again, from the results in Table 2 and, Figs. 2 and 3, our proposed algorithm can be considered as a better alternative for solving pseudomonotone variational inequality problems.

Example 5.3 Consider the function $F : \mathbb{R}^N \to \mathbb{R}^N$ defined by $Fx = Ax + b$, where $b \in \mathbb{R}^N$,

$$A = MM^T + S + D,$$

M is an $N \times N$ matrix, S and D are $N \times N$ skew-symmetric matrix and diagonal matrix, respectively. Then, in A is monotone and Lipschiptz continuous. In MATLAB, we generated M, S and D randomly with entries in $[1, 100]$. We generated b randomly with entries in $[-100, 0]$. Furthermore, we take

$$C = \{x \in \mathbb{R}^N : -1 \leq x_k \leq 1\}, \quad k = 1, 2, \ldots, N.$$

In DCZR, we take $\alpha_n = \frac{1}{n+1}$, $\tau = 10^{-10}$, $\gamma = 10^{-3}$. In IS Alg. 3, we take $\delta = 0.9$, $\theta_0 = 0.1$, $\theta = -0.5$, $\lambda_0 = 10^{-7}$ and $\bar{\lambda} = 2$. In ISY Alg. 1, we take $\delta = 0.01$, $\mu = \frac{0.99(1-2\delta)}{2}$, $\theta = 0.01(\min(\frac{\delta}{2}, \frac{0.5-\mu}{2}))$, $a_n = \frac{1}{(n+1)^8}$, $\gamma_0 = 1.9$ and $\gamma_1 = 1$. In TVC Alg 1, we take $\lambda =$

Table 3 Example 5.3 for Different Dimensions

Example 5.3 for the four (4) dimensions

$\mathbb{R}^N$	DCZR Alg. 3.1		IS Alg. 3		ISY Alg. 1		TVC Alg. 1		SLMD Alg. 3.2		OAAO Alg. 3.2	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$N = 50$	1644	0.1939	240	0.0364	789	0.1045	1677	0.3023	173	0.0242	26	0.0041
$N = 100$	2099	0.5342	339	0.1084	864	0.1436	1687	0.2366	178	0.0785	26	0.0043
$N = 500$	2972	2.3601	702	0.4807	3000	2.8269	3000	2.0425	190	0.2036	28	0.0288
$N = 1000$	3000	11.3548	318	1.1915	3000	13.5323	3000	10.8433	194	1.0727	28	0.1427

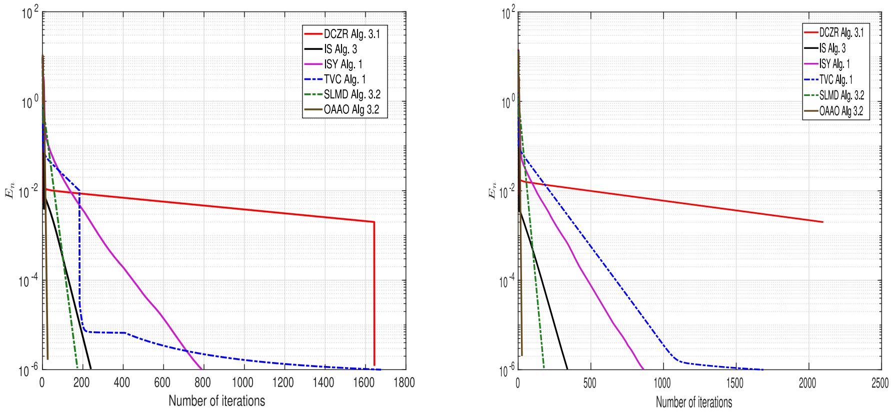

Fig. 4 Graph the Iterates for Dimensions $N = 50$ and 100

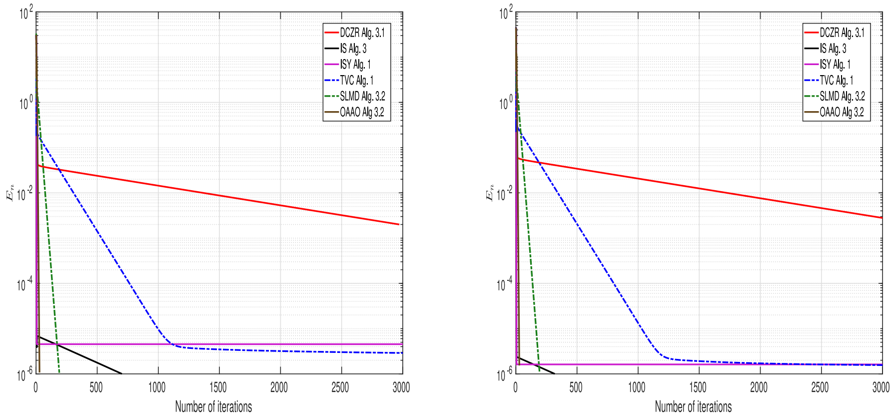

Fig. 5 Graph the Iterates for Dimensions $N = 500$ and 1000

10^{-7}, $\beta_n = \frac{1}{10^4 n+1}$, $\epsilon_n = \frac{1}{(n+1)^2}$, $\alpha = 0.9$. In SLMD Alg. 3.2 and our proposed OAAO Alg. 3.2, we take the same parameters used in Example 5.2. The initial points x_0, x_1, x_2 are generated randomly in $\mathbb{R}^N$. To test the robustness of the algorithms we consider four (4) different dimensions, i.e., $\mathbb{R}^{50}$, $\mathbb{R}^{100}$, $\mathbb{R}^{500}$ and $\mathbb{R}^{1000}$. For each case, the simulation is terminated when the number of iterations $n = 3001$ or $\|x_{n+1} - x_n\| > 10^{-6}$. The result of the numerical simulations are presented in Table 3 and Figs. 4 and 5 below.

Discussion of Results. In this example, we considered a variational inequality problem involving a monotone operator. From Table 3, we observe that as the dimension of the space was increased to 1000, the DCZR Alg. 3.1, ISY Alg. 1 and TVC Alg. 1 fail to satisfy the tolerance before the specified maximum number of iterations was reached. Furthermore, from Figures 4 and 5 we observe that the successive iterates generated by DCZR Alg. 3.1 are far apart resulting in the nature of the graph we saw in these figures. Although IS Alg. 3 and SLMD Alg. 3.2 satisfy the stopping criterion for all the dimensions considered, our proposed OAAO Alg. 3.2 outperformed them as it satisfies the stopping criterion in fewer iterations and computational time. Again, our proposed AOOA Alg. 3.2 has demonstrated its competence and superiority in solving monotone variational inequality problems over these algorithms.

6 Conclusion

This paper provides a projection and contraction method using two-step inertial procedure for solving variational inequality problems involving quasimonotone and Lipchitz continuous mapping. A nice feature of the proposed method is the fact it requires neither the prior knowledge of the Lipschitz constant nor the weak sequential continuity of the associated mapping to obtain convergence. Under additional strong pseudomonotonicity, the R-linear convergence rate of the proposed two-step inertial algorithm is presented. Based on the numerical simulations conducted in Examples 5.1, 5.2 and 5.3, the proposed Algorithm 3.2 has demonstrated its competitiveness and promise in comparison to existing algorithms. Specifically, it has outperformed Algorithm 3.1 of Dong et al. [28], Algorithm 3 of Izuchukwu and Shehu [37], Algorithm 1 of Izuchukwu et al. [38], Algorithm 1 of Thong et al. [62], and Algorithm 3.2 of Shehu et al. [58] in terms of computational time and the number of iterations required to satisfy the specified stopping criterion.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Use of AI The authors declare that they did not use AI to generate any part of the paper.

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