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The rainbow connected number of several infinite graph families

Liam B. Baker[1] · **Jonathan Kariv**[2] · **Ronald J. Maartens**[3]

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Abstract
Let G be a nontrivial connected graph with an edge coloring, and let $u, v \in V(G)$. A $u - v$ path in G is said to be a *rainbow path* if no color is repeated on the edges of the path. Similarly, we define a *rainbow geodesic*. A *rainbow connected graph* G is a graph with an edge coloring such that every two vertices in G are connected by a rainbow path. Further, a *strong rainbow connected graph* G is a graph with an edge coloring such that every two vertices in G is connected by a rainbow geodesic. The minimum number of colors needed to make a graph rainbow connected is called the *rainbow connection number*, denoted $\mathrm{rc}(G)$, and the minimum number of colors needed to make a graph strong rainbow connected is called the *strong rainbow connection number*, denoted $\mathrm{src}(G)$. In this paper we determine $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is a *n*-dimensional rectangular grid graph, triangular grid graph, hexagonal grid graph, and a (weak) Bruhat graph, respectively. We show for all these families that $\mathrm{src}(G) = \mathrm{diam}(G)$.

Keywords Rainbow coloring · Rainbow connected · Grid graphs · Bruhat graphs

Mathematics Subject Classification 05C15 · 05C38 · 05C40

1 Introduction

All graphs considered in this paper are simple, implying that the graphs contain no loops, direction, multiple edges or weighted edges. Further, all graphs in this paper are drawn in color. For any undefined terms and concepts, we refer the reader to [3].

✉ Ronald J. Maartens
ronniemaartens@yahoo.co.za

[1] Department of Mathematics, University of Stellenbosch, Stellenbosch, South Africa

[2] School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa

[3] School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa

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Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $u, v \in V(G)$, then the edge from u to v is denoted by uv. A $u-v$ *path* in G is a sequence of non-repeating vertices, except possibly for u and v, starting at u and ending at v, such that any two consecutive vertices have an edge between them in G. The *length* of a path is the number of edges on the path, and the *path graph* with length n is denoted by P_n. A shortest path between two vertices is called a *geodesic*. The *diameter* of a graph G, denoted $\operatorname{diam}(G)$, is the length of the longest geodesic in G. A *cycle graph*, denoted C_n, is a path of length $n \geq 3$ that starts and ends at the same vertex.

An *edge coloring* of a graph is an assignment of a color from a set of possible colors to an edge in the graph, for brevity we will sometimes simply call an edge coloring a *coloring*. A $u-v$ path in G is said to be a *rainbow path* if no color is repeated on the edges of the path. Similarly, a *rainbow geodesic* is defined as a shortest rainbow path between two vertices. That is, for $u, v \in V(G)$, a $u-v$ rainbow geodesic is a rainbow path of length $d(u, v)$, the distance between u and v.

A graph G is said to be *connected* if there is a path between every two vertices in G. A *rainbow connected graph* G is a graph with an edge coloring such that every two vertices in G are connected by a rainbow path. Further, a *strong rainbow connected graph* G is a graph with an edge coloring such that every two vertices in G are connected by a rainbow geodesic. The minimum number of colors needed to make a graph rainbow connected is called the *rainbow connection number*, denoted $\operatorname{rc}(G)$, and the minimum number of colors needed to make a graph strong rainbow connected is called the *strong rainbow connection number*, denoted $\operatorname{src}(G)$.

The rainbow connection number and the strong rainbow connection number of a graph was initiated by Chartrand, Johns, McKeon and Zhang in 2008, see [5]. Since then, the rainbow connection number and strong rainbow connection number of several graph classes have been found. For a dynamic survey of some of these results, see [8] by Li and Sun.

The following results are used later on.

Theorem 1.1 *[5] For any nontrivial connected graph G,*

$$\operatorname{diam}(G) \leq \operatorname{rc}(G) \leq \operatorname{src}(G) \leq |E(G)|.$$

Theorem 1.2 *[5] For the path graph P_n,*

$$\operatorname{rc}(P_n) = \operatorname{src}(P_n) = n = \operatorname{diam}(P_n),$$

and for the cycle graph C_n,

$$\operatorname{rc}(C_n) = \operatorname{src}(C_n) = \lceil n/2 \rceil = \operatorname{diam}(C_n).$$

Notably, in 2011, Chakraborty, Fischer, Matsliah and Yuster [2], confirmed a conjecture of Caro, Lev, Roditty, Tuza and Yuster [1] that the problem to compute $\operatorname{rc}(G)$ in general is NP-hard. Nonetheless, there are specific cases that are amenable to exact computation.

Our contribution in this paper is the computation of the rainbow connection number and the strong rainbow connection number of certain infinite families of graphs. That is, we determine $\operatorname{rc}(G)$ and $\operatorname{src}(G)$ when G is a n-dimensional rectangular grid graph, triangular grid graph, hexagonal grid graph, and a (weak) Bruhat graph, respectively.

We proceed as follows. In Sect. 2, we determine an upper bound for $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is the Cartesian product or box product of graphs. In Sect. 3, we determine $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is a n-dimensional rectangular grid graph, while G being a triangular grid graph is considered in Sect. 4. In Sect. 5, we determine $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is a hexagonal grid graph, while G being a Bruhat graph is considered in Sect. 6. Finally, in Sect. 7, we conclude our work by discussing some related open problems.

2 Preliminary results

The *Cartesian product* or *box product* of two connected graphs G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $\{(u_1, u_2) : u_i \in V(G_i), i = 1, 2\}$, and an edge from vertex (v_1, v_2) to vertex (w_1, w_2) if and only if either $v_1 = w_1$ and $v_2w_2 \in E(G_2)$, or $v_2 = w_2$ and $v_1w_1 \in E(G_1)$.

In general, for $n \geq 2$ and connected graphs G_i with $i = 1, ..., n$, we define

$$G_1 \square ... \square G_n = \overset{n}{\underset{i=1}{\square}} G_i$$

as the graph with vertex set $\{(u_1, ..., u_n) : u_i \in V(G_i)\}$, and an edge from vertex $(v_1, ..., v_n)$ to vertex $(w_1, ..., w_n)$ if and only if $v_jw_j \in E(G_j)$ for exactly one value of $j \in \{1, ..., n\}$ and $v_x = w_x$ for all $x \in \{1, ..., n\} - \{j\}$.

Lui and Hu, the authors of [7], reference the following result by Li and Sun which appears in the preprint [9]. At the time of writing this paper, the status of [9] has not yet changed. Consequently, we provide our own proof here for the following result with the extension to the strong rainbow connection number.

Proposition 2.1 *[9] For $n \geq 2$ and $i = 1, ..., n$, let G_i be connected graphs. Then,*

$$\mathrm{rc}\left(\overset{n}{\underset{i=1}{\square}} G_i\right) \leq \sum_{i=1}^{n} \mathrm{rc}(G_i).$$

Moreover, if $\mathrm{rc}(G_i) = \mathrm{diam}(G_i)$ for each G_i, then the equality holds.

Proposition 2.2 *Let G_1 and G_2 be connected graphs. Then,*

$$\mathrm{rc}(G_1 \square G_2) \leq \mathrm{rc}(G_1) + \mathrm{rc}(G_2) \quad \text{and} \quad \mathrm{src}(G_1 \square G_2) \leq \mathrm{src}(G_1) + \mathrm{src}(G_2).$$

Proof Let G_1 and G_2 be connected graphs together with associated rainbow colorings of $\mathrm{rc}(G_1)$ and $\mathrm{rc}(G_2)$ colors, respectively, such that there is no repeat of colors between these two graphs. We define a coloring on $G_1 \square G_2$ as follows.

1. For an edge from (u, v_2) to (u, w_2), we assign the same color as of the edge v_2w_2 in G_2.
2. For an edge from (v_1, u) to (w_1, u), we assign the same color as of the edge v_1w_1 in G_1.

The graph $G_1 \,\square\, G_2$ now has a rainbow coloring of $\mathrm{rc}\,(G_1) + \mathrm{rc}\,(G_2)$ colors. To see this, consider any two vertices, say (v_1, v_2) and (w_1, w_2), in $G_1 \,\square\, G_2$. Then there are rainbow paths $v_1 = a_0,\ a_1,\ \ldots,\ a_m = w_1$ of *m* colors in G_1, and $v_2 = b_0,\ b_1,\ \ldots,\ b_n = w_2$ of *n* colors in G_2. Using these two paths, we construct the following rainbow path in $G_1 \,\square\, G_2$ with $m + n$ colors:

$$(v_1, v_2) = (a_0, b_0),\ (a_1, b_0),\ \ldots,\ (a_m, b_0),\ (a_m, b_1),\ \ldots,\ (a_m, b_n) = (w_1, w_2).$$

The upper bound on $\mathrm{rc}\,(G_1 \,\square\, G_2)$ is obtained.

Moreover, if we let G_1 and G_2 have associated strong rainbow colorings of $\mathrm{src}\,(G_1)$ and $\mathrm{src}\,(G_2)$ colors, respectively, such that there is no repeat of colors between these two graphs, then a similar coloring as in the case of $\mathrm{rc}\,(G_1 \,\square\, G_2)$ produces an upper bound for $\mathrm{src}\,(G_1 \,\square\, G_2)$. To motivate this, suppose that the above two rainbow paths from v_1 to w_1, and from v_2 to w_2 are rainbow geodesics within G_1 and G_2, respectively. All that remains to show is that the path between (v_1, v_2) and (w_1, w_2) constructed above is a geodesic in $G_1 \,\square\, G_2$.

In this regard, suppose, to the contrary, that there is another path from (v_1, v_2) to (w_1, w_2) in $G_1 \,\square\, G_2$ of length less than $m + n$. Since each edge on this path involves either a change in the first coordinate of the vertex labels, or in the second coordinate of the vertex labels, and *not* both, we either have a path in G_1 from v_1 to w_1 of length less than *m*, or a path in G_2 from v_2 to w_2 of length less than *n*. Both of these conclusions are a contradiction to the fact that the two rainbow paths $v_1 = a_0,\ a_1,\ \ldots,\ a_m = w_1$ and $v_2 = b_0,\ b_1,\ \ldots,\ b_n = w_2$ are geodesics. The result then follows. □

Using mathematical induction with Proposition 2.2, the following result is obtained.

Corollary 2.3 *For $n \geq 2$ and $i = 1, ..., n$, let G_i be connected graphs. Then*

$$\mathrm{rc}\left(\bigsquare_{i=1}^{n} G_i\right) \leq \sum_{i=1}^{n} \mathrm{rc}\,(G_i) \qquad \textit{and} \qquad \mathrm{src}\left(\bigsquare_{i=1}^{n} G_i\right) \leq \sum_{i=1}^{n} \mathrm{src}\,(G_i).$$

3 Grid graphs

In this section, we determine $\mathrm{rc}\,(G)$ and $\mathrm{src}\,(G)$ when *G* is a rectangular grid graph in $\mathbb{R}^n$.

A *rectangular grid graph*, or in short a *grid graph* (also known as a *lattice graph* or *mesh graph*), is a graph *G* whose drawing forms a regular tiling when embedded in some Euclidean space $\mathbb{R}^n$. That is, $G = \bigsquare_{i=1}^{n} P_{N_i}$ for positive integers N_i.

Let *G* be a grid graph. For $i = 1, \ldots, n$ and positive integers N_i, we label each of the vertices in *G* with an *n*-tuple $(x_1, \ldots, x_n)$ where $0 \leq x_i \leq N_i$. Further, two vertices in *G* are connected by an edge if and only if the two vertex labels differ by one for exactly one value x_j of $j \in \{1, ..., n\}$. To determine $\mathrm{diam}\,(G)$, note that the longest geodesic in *G* is from the vertex with label (0, ..., 0) to $(x_{N_1}, ..., x_{N_n})$, giving

$$\operatorname{diam}\left(\square_{i=1}^{n} P_{N_i}\right) = \sum_{i=1}^{n} N_i.$$

Theorem 3.1 *Let* $G = \square_{i=1}^{n} P_{N_i}$ *for positive integers* N_i. *Then,*

$$\operatorname{rc}\left(\square_{i=1}^{n} P_{N_i}\right) = \operatorname{src}\left(\square_{i=1}^{n} P_{N_i}\right) = \sum_{i=1}^{n} N_i.$$

Proof Let $G = \square_{i=1}^{n} P_{N_i}$. From Theorem 1.1, $\operatorname{src}(G) \geq \operatorname{rc}(G) \geq \operatorname{diam}(G) = \sum_{i=1}^{n} N_i$. For the upper bound, note that $\operatorname{src}(P_{N_i}) = N_i$ from Theorem 1.2. Hence, $\operatorname{src}(G) \leq \sum_{i=1}^{n} \operatorname{src}(P_{N_i}) = \sum_{i=1}^{n} N_i$ from Corollary 2.3. The result then follows. □

4 Triangular grid graphs

In this section, we determine $\operatorname{rc}(G)$ and $\operatorname{src}(G)$ when G is a triangular grid graph [11], also known as a *tetrahedral grid graph* or one of the higher dimensional analogs of these. These higher dimensional analogs are equivalent to those defined in [10], however, we define our own notation here for our proofs to flow more naturally.

We define a *triangular grid graph* $G = T_n^k$ by labelling the vertices of G with the k-tuples $(x_1, \ldots, x_k)$ where all x_i are non-negative integers such that $\sum_{i=1}^{k} x_i \leq n$. Furthermore, two vertices in G are connected by an edge if and only if either (i) the two vertex labels differ by one for exactly one value x_i with $i \in \{1, ..., k\}$, or (ii) for $i \neq j$ and $i, j \in \{1, ..., k\}$, the two vertex labels differ by one for exactly two values x_i and x_j in opposite directions, that is, one of x_i and x_j increases by one and the other decreases by one.

To determine $\operatorname{diam}(T_n^k)$, note that a longest geodesic in T_k^n is from the vertex with label (0, ..., 0) to the vertex with label $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{k} x_i = n$. Hence, $\operatorname{diam}(T_n^k) = n$.

Theorem 4.1 *For positive integers n and k,*

$$\operatorname{rc}\left(T_n^k\right) = \operatorname{src}\left(T_n^k\right) = n.$$

Proof Let $G = T_n^k$. The lower bound follows from Theorem 1.1, $\operatorname{src}(G) \geq \operatorname{rc}(G) \geq \operatorname{diam}(G) = n$.

To show that $\operatorname{src}(G) \leq n$, define the following coloring on G. Consider any edge in G, say edge st where $s = (s_1, \ldots, s_k)$ and $t = (t_1, \ldots, t_k)$. For $1 \leq j \leq n$, let j be the largest value such that $s_j \neq t_j$. Then, color edge st with the color $\min(s_j, t_j) + \sum_{i=j+1}^{k} s_i$. This color is equal to $\min(s_j, t_j) + \sum_{i=j+1}^{k} t_i$ since $s_i = t_i$ when $i > j$.

This coloring is illustrated in Fig. 1 with the graph T_5^2 using actual colors to represent the colors of the edges on the graph. Note that the coloring used in Fig. 1 is a strong rainbow coloring with $\operatorname{diam}(G) = 5$ colors.

To show that this coloring is in fact a strong rainbow coloring, consider any two vertices in G, say $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Without loss of generality, assume that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$.

As x and y are arbitrary vertices, we need only find a rainbow geodesic from x to y holding the above coloring as fixed. In this regard, classify each of the k coordinates that are used on the $x - y$ path into three types:

- *flat* when $x_i = y_i$,
- *ascents* when $x_i < y_i$, and
- *descents* when $x_i > y_i$

for $i = 1, ..., k$. Notably, these types depend on x and y. The below conditions are sufficient for a path from x to y to be a geodesic:

1. No flat coordinate is ever changed.
2. No ascent is ever decreased.
3. No descent is ever increased.
4. Whenever there are both unresolved ascents and unresolved descents, a move both increases an ascending coordinate and decreases a descending coordinate.

We shall now construct a rainbow path as follows.

First, ignore all flats. Second, bundle the remaining coordinates into blocks of consecutive ascents and descents. For specificity, these blocks are chosen after ignoring the flat coordinates. We shall resolve each block sequentially from the right. That is we shall resolve the block with the highest index coordinates first, and progress to blocks of lower index.

In order to ensure that our constructed path is indeed rainbow, we shall require the two conditions listed below, these are trivially sufficient to ensure that the path is rainbow.

i. The same color is never used while resolving any particular block.

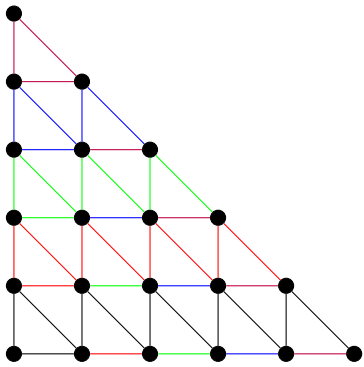

Fig. 1 The triangular grid graph T_5^2

ii. For blocks A and B, where B is to the left of A, that is, B is resolved later in the process than A, every color used to resolve B has higher index than every color used to resolved A.

To resolve an ascending block where no unresolved descending blocks remain, begin from the rightmost unresolved coordinate within the ascending block and increase the coordinate until that coordinate is resolved. To resolve an ascending block where unresolved descending blocks still occur, begin from the rightmost unresolved coordinate within the ascending block and increase the coordinate until that coordinate is resolved. Simultaneously, decrement the leftmost (lowest indexed) descending coordinate.

Move then to the next rightmost unresolved coordinate and repeat the previous step until the block is resolved.

When resolving a descending block, begin from the leftmost unresolved coordinate within the block and decrease the coordinate until that coordinate is resolved. Move then to the next leftmost unresolved coordinate and repeat the previous step until the block is resolved. If an unresolved ascending coordinate still exists, then we increment the rightmost such coordinate.

Our algorithm satisfies the two conditions (i.) and (ii.) above. Trivially, within each block resolution, no color is repeated. To see that the smallest color used in each block exceeds the largest color used in any previous block, observe that all descents have been used to increment the rightmost unresolved ascent. Thus, the sum associated to the moves after resolving this coordinate is still increasing. It is easy to see that ascents increase the value of future colors used. □

To illustrate, consider the path from (2, 3, 4, 5, 6, 7, 8, 9, 10) to (12, 3, 24, 5, 4, 5, 8, 6, 7) in T_n^9 where $n \geq 74$. Here, the second, fourth, and seventh coordinates are flats, that is, $(2, 3, 4, 5, 6, 7, 8, 9, 10)$. We proceed as follows where the changed coordinates are made bold:

$$
\begin{aligned}
& (2,3,4,5,6,7,8,9,10) \\
\longrightarrow\ & (2,3,\mathbf{5},5,5,7,8,9,10) && \text{uses color } 5+7+8+9+10=39 \\
\longrightarrow\ & (2,3,\mathbf{6},5,\mathbf{4},7,8,9,10) && \text{uses color } 4+7+8+9+10=38 \\
\longrightarrow\ & (2,3,\mathbf{7},5,\mathbf{4},\mathbf{6},8,9,10) && \text{uses color } 6+8+9+10=33 \\
\longrightarrow\ & (2,3,\mathbf{8},5,\mathbf{4},\mathbf{5},8,9,10) && \text{uses color } 5+8+9+10=32 \\
\longrightarrow\ & (2,3,\mathbf{9},5,\mathbf{4},\mathbf{5},\mathbf{8},8,10) && \text{uses color } 8+10=18 \\
\longrightarrow\ & (2,3,\mathbf{10},5,\mathbf{4},\mathbf{5},\mathbf{8},\mathbf{7},10) && \text{uses color } 7+10=17 \\
\longrightarrow\ & (2,3,\mathbf{11},5,\mathbf{4},\mathbf{5},\mathbf{8},\mathbf{6},10) && \text{uses color } 6+10=16 \\
\longrightarrow\ & (2,3,\mathbf{12},5,\mathbf{4},\mathbf{5},\mathbf{8},\mathbf{6},\mathbf{9}) && \text{uses color } 6+9=15 \\
\longrightarrow\ & (2,3,\mathbf{13},5,\mathbf{4},\mathbf{5},\mathbf{8},\mathbf{6},\mathbf{8}) && \text{uses color } 6+8=14 \\
\longrightarrow\ & (2,3,\mathbf{14},5,\mathbf{4},\mathbf{5},\mathbf{8},\mathbf{6},\mathbf{7}) && \text{uses color } 6+7=13
\end{aligned}
$$

The next block is the descending block given by the first and third coordinates, currently occupied by 2 and 14. As we are out of active descents, just simply increase these.

5 Hexagonal grid graphs

In this section, we determine $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is a hexagonal grid graph.

Let $G = H_n$ denote the *hexagonal grid graph* which is the hexagonally shaped triangular grid graph with n edges on each of the sides of the hexagon. That is, G is a subgraph of some two dimensional version of the triangular grid graph covered in Sect. 4. The graph H_4 is illustrated in Fig. 2.

Although G is a subgraph of some two dimensional version of the triangular grid graph, note that Theorem 5.1 below does not follow trivially from Theorem 4.1 as the diameter of G is only two thirds of the diameter of the smallest triangular grid it can be embedded in. In particular, H_n is embedded in T^2_{3n} and, using the triangular coloring of T^2_{3n} as in the proof of Theorem 4.1, all $3n$ colors are assigned to H_n while only $2n$ colors are required.

We label the vertices of G with the integer valued 2-tuples (x, y) where $-n \leq x \leq n$ and $-n \leq y \leq n$ such that $-n \leq x + y \leq n$. Furthermore, two vertices in G are connected by an edge if and only if either (i) the two vertex labels differ by one for the x or y coordinate while the other coordinate stays the same, or (ii) both vertex labels differ by one but in opposite directions, that is, one of x and y increases by one and the other decreases by one. Trivially, $\mathrm{diam}(H_n) = 2n$.

Theorem 5.1 *For positive integer n,*

$$\mathrm{rc}(H_n) = \mathrm{src}(H_n) = 2n.$$

Proof Let $G = H_n$. From Theorem 1.1, $\mathrm{src}(G) \geq \mathrm{rc}(G) \geq \mathrm{diam}(G) = 2n$. We proceed to show that $\mathrm{src}(G) \leq 2n$.

In this regard, consider the following coloring.

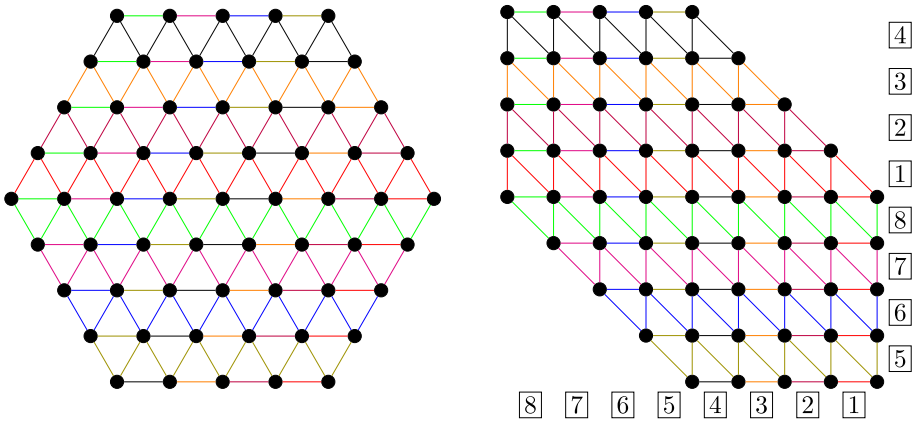

Fig. 2 The Hexagonal Grid Graph H_4, and the same graph displayed in Cartesian coordinates. The colors used for respective changes to the x and y coordinates are displayed numerically on the right and bottom sides

i. For all edges that alter a y coordinate, irrespective of whether or not the x coordinate is also changed, between $y = a$ and $y = a + 1$, apply the color $a + 1$ when $a \geq 0$ and the color $2n + 1 + a$ when $a < 0$.
ii. For all remaining edges, that is, the edges that affect only the x coordinate between $x = b$ and $x = b + 1$, apply the color $n - b$.

Note that:

1. Color $c \in \{1, 2, .., n\}$ is used when the y coordinate changes from $c - 1$ to c.
2. Color $c \in \{n + 1, n + 2, .., 2n\}$ is used when the y coordinate changes from $c - 2n - 1$ to $c - 2n$. This is equivalent to the above case when colors are considered modulo $2n$.
3. Color c is used when the y coordinate is unchanged and the x coordinate is changed between $n - c$ and $n - c + 1$.

All that remains to show is that this coloring is in fact strong rainbow. Therefore, consider the two vertices $u = (x_u, y_u)$ and $v = (x_v, y_v)$. Without loss of generality, assume that $x_u \leq x_v$. Thus, we consider only paths which travels towards the right, paths that travels towards the left are simply the same paths reversed. □

Fact 1. The graph automorphism $(x, y) \to (-x, -y)$ does not preserve the coloring, but the graph automorphism does preserve whether or not two edges have the same color.

We proceed by considering cases on the y coordinate.

Case 1: $y_u \leq y_v$.

For $y_u \leq y_v$, note that all geodesics from $u = (x_u, y_u)$ to $v = (x_v, y_v)$ are moving up and to the right. A geodesic then consist of a path whereon only one of the coordinates are incremented while the other stays the same. Also, every geodesic between u and v use an identical collection of colors as both u and v are incremented through the same ranges.

Note that the colors used to increment the y coordinate from y_u to y_v are given in the set $Y = \{y_u + 1, y_u + 2, \ldots, y_v\} \pmod{2n}$, and the colors used to increment the x coordinate are given in the set $X = \{n - x_u + 1, n - x_u + 2, \ldots, n - x_v\}$. The colors within each of these sets are consecutive and so distinct. All that remains to show is that $X \cap Y = \emptyset$. To see this we proceed with a proof by contradiction.

Assume, to the contrary, that $X \cap Y \neq \emptyset$ and that a repeating color occurs on the edge which starts with the vertex (x^*, y^*) on the $u - v$ geodesic. Then, $x_u \leq x^* < x_v$ and $y_u \leq y^* < y_v$ with $y^* + 1 \equiv n - x^* \pmod{2n}$. Hence, $x^* + y^* = n - 1 \pmod{2n}$, and so, $x + y = n - 1$ from the boundary conditions. The equation, $x + y = n - 1$, describes a valid set of coordinates along a diagonal path which is a single move away from the $x + y = n$ boundary. As both x^* and y^* do not increase at the same time, see above, the vertex (x^*, y^*) cannot be the start of two different edges on the $u - v$ geodesic. The assumption is then false, and the result follows.

Case 2: $y_u > y_v$.

For $y_u > y_v$, note that in this case all geodesics from $u = (x_u, y_u)$ to $v = (x_v, y_v)$ are moving down and to the right.

We begin by finding a rainbow geodesic in the case where $x_v - x_u < y_u - y_v$.

Here, all $u - v$ geodesics are moving down and to the right with the restriction that the geodesic has more downward moves than moves to the right.

Consequently, all $u - v$ geodesics are a combination of vertical and diagonal moves. These moves all make mutually distinct changes to the y coordinate and are therefore of different colors.

The remainder of the proof deals with the case where $x_v - x_u \geq y_u - y_v$. With $x_v - x_u \geq y_u - y_v$, more movement towards the right is made than down, and the $u - v$ geodesic is a combination of horizontal and diagonal moves.

The colors associated to the horizontal moves are affected by *when* the horizontal moves are made and this affects which geodesics are rainbow under our coloring.

We proceed by investigating sub-cases.

Case 2.1: $y_v \leq y_u \leq 0$.

For $y_v \leq y_u \leq 0$, note that we are below the *x*-axis on the Cartesian Plane. We consider the $u - v$ geodesic begins by moving diagonally down and to the right until the $y = y_v$ line is reached. Thereafter, the geodesic moves horizontally towards *v*.

The diagonal moves on the geodesic from (x_u, y_u) to $(x_u + (y_v - y_u), y_v)$ use colors $\{y_u - 1, y_u - 2, .., y_v\}$, while the horizontal moves on the geodesic from $(x_u + (y_v - y_u), y_v)$ to (x_v, y_v) use the colours $\{y_u - 1, y_u - 2, .., y_v\}$. At each step along the geodesic the value of the color used decreased by 1 except possibly at the change of direction where $x_u + y_u - n - 1$ colors are skipped. This is sufficient to show that this geodesic is rainbow.

Case 2.2: $0 \leq y_v \leq y_u$.

The case for $0 \leq y_v \leq y_u$ is equivalent to Case 2.1 under the involution $(x, y) \to (-x, -y)$ with the geodesic from (x_u, y_u) to (x_v, y_v) replaced by $(-x_v, -y_v)$ to $(-x_u, -y_u)$.

Case 2.3: $y_v \leq 0 \leq y_u$.

For $y_v \leq 0 \leq y_u$, the $u - v$ geodesic is moving down and to the right while moving more right than down and crossing the $y = 0$ line.

Case 2.3.1: $x_v \leq x_u \leq 0$.

For the rainbow $u - v$ geodesic, begin with diagonal steps from (x_u, y_u) to $(x_u - y_u + y_v, y_v)$, followed by horizontal steps to (x_v, y_v). Note that the horizontal steps use colors of index at least $n + 1$ and form part of a path dealt with in Case 2.1 above.

Case 2.3.2: $0 \leq x_v \leq x_u$.

The case for $0 \leq x_v \leq x_u$ is equivalent to Case 2.3.1 under the involution $(x, y) \to (-x, -y)$ with the geodesic from (x_u, y_u) to (x_v, y_v) replaced by $(-x_v, -y_v)$ to $(-x_u, -y_u)$.

Case 2.3.3: $x_v \leq 0 \leq x_u$.

Let $x_v \leq 0 \leq x_u$ and recall that $y_v \leq 0 \leq y_u$. Note that any $u - v$ geodesic in this case consist exclusively of horizontal and diagonal moves.

Consider the $u - v$ geodesic that moves diagonally from (x_u, y_u) to $(x_u - y_v, y_u + y_v)$, then horizontally to $(x_v - y_u, y_u + y_v)$, and then diagonally again to (x_v, y_v). Assign colors to this geodesic whereby diagonal moves above the *x*-axis uses all the colors in the set $A = \{1, 2, ..., y_u\}$, and diagonal moves below the *x*-axis uses all the colors in the set $C = \{2n + y_v + 1, 2n + y_v + 2, ..., 2n\}$. Lastly, use all the colors in the set $B = \{n - (x_u - y_v), n - (x_u - y_v) + 1, ..., n - (x_v - y_u) + 1\}$ for the horizontal moves.

If sets *A*, *B* and *C* are pairwise disjoint, then the $u - v$ geodesic described above is rainbow. To see this first note that *A* and *C* are trivially disjoint. Further, note that $\max A = y_u < \min B = n - (x_v - y_u) + 1$ and $\max B = n - (x_u - y_v) < \min C = 2n + y_v + 1$. These two inequalities reduce to the

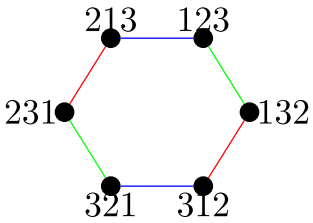

Fig. 3 The Bruhat Graph B_3

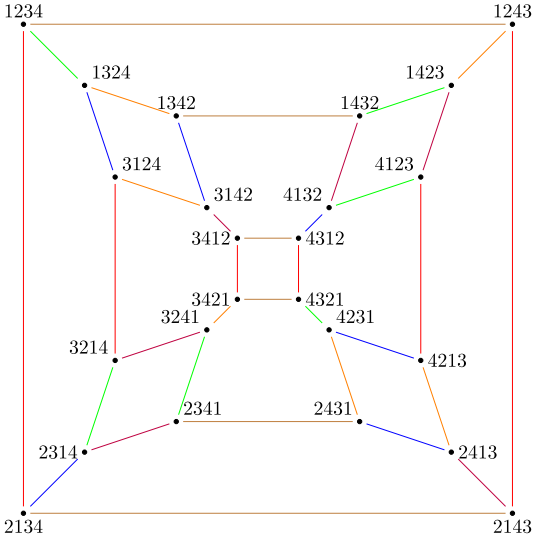

Fig. 4 The Bruhat Graph B_4

trivially true statements $x_u < n+1$ and $x_v < n+1$. Hence, A, B and C are pairwise disjoint, and so, the $u-v$ geodesic described above is rainbow. □

6 Bruhat graphs

In this last section, we determine $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ when G is a (weak) Bruhat graph.

The Bruhat graph B_n is a graph with $n!$ vertices, each uniquely labelled with a permutation of all the numbers in the set $\{1, 2, \ldots, n\}$. An edge occurs between two vertices if their vertex labels differ by a transposition.

Graphs B_3 and B_4 are drawn in Fig. 3 and Fig. 4, respectively, with a strong rainbow coloring of the edges. Note that the number of colors used for the strong rainbow coloring is equal to the diameter of each graph. Also, graph B_3 is isomorphic to graph C_6, and graph B_4 is isomorphic to the Truncated Octahedral Graph. In general, graphs B_n are the Cayley graphs of the symmetric group S_n generated by adjacent transpositions [6].

Trivially, $\mathrm{diam}(B_n) = \binom{n}{2} = \frac{n(n-1)}{2}$.

Theorem 6.1 *For positive integer $n \geq 2$,*

$$\mathrm{rc}\,(B_n) = \mathrm{src}\,(B_n) = \binom{n}{2}.$$

Proof Let $G = B_n$. From Theorem 1.1, $\mathrm{src}\,(G) \geq \mathrm{rc}\,(G) \geq \mathrm{diam}\,(G) = \binom{n}{2}$. We proceed to show that $\mathrm{src}\,(G) \leq \binom{n}{2}$.

As stated above, an edge occurs between two vertices if their vertex labels differ by a transposition. Color an edge in G with the color denoted by the two numbers that transposed between the two vertices incident to that edge. In total, $\binom{n}{2}$ colors are then used to color G.

Consider now a $u - v$ geodesic where vertices u and v have vertex labels σ and ρ, respectively. Let $N_{\sigma,\rho}$ be the number of pairs in different orders within the two permutations σ and ρ. Observe that a transposition decreases $N_{\sigma,\rho}$ by at most 1. In particular, if $\sigma \neq \rho$, then there is always a transposition that decreases $N_{\sigma,\rho}$ by exactly one. Otherwise, $\sigma = \rho$. Finally, observe that on a $u - v$ geodesic, the same two numbers are never transposed twice. □

7 Conclusion and further work

In this paper, we showed that $\mathrm{rc}\,(G) = \mathrm{src}\,(G) = \mathrm{diam}\,(G)$ when G is a n-dimensional rectangular grid graph, triangular grid graph, hexagonal grid graph, and a (weak) Bruhat graph, respectively. Similar families may also be investigated.

A natural generalization to these results is to determine the rainbow k-connection number for each of these graphs G. Let G be a l-connected graph and k an integer such that $1 \leq k \leq l$. The *rainbow k-connection number* of G, denoted $rc_k(G)$, is defined as the smallest integer r for which there exists a r-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. For more on the rainbow k-connection number, see [4] and [8].

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Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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