

Gauge Gravity Dualities at Finite N

Wandile Mabanga¹

*School of Physics
University of the Witwatersrand, Johannesburg,
South Africa*

ABSTRACT

In this dissertation we compute the anomalous dimensions for a class of operators, belonging to the $SU(3)$ sector of the theory, that have a bare dimension of order N . For these operators the large N limit and the planar limit are distinct and summing only the planar diagrams will not capture the large N dynamics. Although the spectrum of anomalous dimensions has been computed for this class of operators, previous studies have neglected certain terms which were argued to be small. After dropping these terms diagonalizing the dilatation operator reduces to diagonalizing a set of decoupled oscillators. In this dissertation we explicitly compute the terms which were neglected previously and show that diagonalizing the dilatation operator still reduces to diagonalizing a set of decoupled oscillators.

¹wandile.everwell.mabanga@gmail.com

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1 Introduction

Our current understanding of physics is a patchwork of different theories, each with its own domain of applicability. Sometimes these descriptions overlap. In such a case the overlapping theories have to be unified. Unification often teaches us something non-trivial. Some good examples which illustrate our point include

- When electricity and magnetism were unified into electromagnetism, Maxwell needed to add the correction term $\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ to Ampère's law. No experiment motivated this correction. This term taught us that a changing magnetic field induces an electric field and vice versa. Consequently, it was understood that electromagnetic waves travel through empty space at the speed of light. The fact that light is electromagnetic radiation was a new idea.
- When Einstein unified Special Relativity (SR) with Newtonian gravity into General Relativity (GR), we learnt that spacetime is dynamical and that light bends around heavy objects. We can summarize the basic ideas of GR by saying that matter curves spacetime and spacetime tells matter how to move.¹
- When SR and Quantum Mechanics (QM) were unified into Quantum Field Theory (QFT), we learnt (among other things) that there exists anti-matter and that the vacuum is not empty as a result of pair production.

In summary we learn that in fundamental physics unification is powerful.

In this dissertation I will study the *AdS/CFT* correspondence which is a very dramatic example of unification. The correspondence claims that the $\mathcal{N} = 4$ super Yang-Mills theory, a conformal field theory, is completely equivalent to quantum gravity (realized by type IIB string theory) on the $AdS_5 \times S^5$ background [1]. The correspondence has the potential to provide answers to a number of currently open problems. These problems are

- We do not have a theory of quantum gravity. For example, to understand certain aspects of black hole (BH) physics, one needs to unify GR with QM since both theories make an appearance here.
- Type IIB string theory on $AdS_5 \times S^5$ contains gravitons, strings, membranes and new spacetime geometries. Gravitons and strings are well understood. Membranes and new spacetime geometries are not well understood.
- Quantum Chromodynamics (QCD) is not well understood. The theory at low energies is strongly coupled and although numerical results coming from lattice QCD show good

¹Ford, K., Wheeler, J. A. (2010). *Geons, black holes, and quantum foam: A life in physics*. WW Norton Company.

agreement with experiment, there are no simple analytic approaches to understanding the dynamics.

Non-Abelian Yang-Mills theories are matrix models. A matrix model is a QFT that has matrix valued fields. Denote the dimension of these matrices by N . A very promising approach to the low energy dynamics of a Yang-Mills theory is in terms of an expansion in $\frac{1}{N}$ instead of an expansion in the coupling constant [2]. There are well understood simplifications in the $N \rightarrow \infty$ limit of matrix models. In particular for operators built using $O(1)$ fields, distinct multitrace structures do not mix. For this class of operators the large N answer is given by summing all planar ribbon graphs. This limit is consequently often called the planar limit. To demonstrate that different multitrace structures do not mix, consider the operators²

$$O_J \equiv \frac{\text{Tr} Z^J}{\sqrt{JN^J}} \quad (1.1)$$

which are normalized to have a unit two point function

$$\langle O_J O_J^\dagger \rangle = 1 + O\left(\frac{1}{N^2}\right) \quad (1.2)$$

in the large N limit. To obtain the leading order term in (1.2), we have summed only the planar diagrams. This is perfectly accurate at large N as long as $J^2 \ll N$. Now consider the two point function between a double trace structure given by $O_{J_1} O_{J_2}$ and the single trace $O_{J_1+J_2}$

$$\langle O_{J_1} O_{J_2} O_{J_1+J_2}^\dagger \rangle = \frac{\sqrt{J_1 J_2 (J_1 + J_2)}}{N}. \quad (1.3)$$

If we take $N \rightarrow \infty$ whilst holding J_1 and J_2 fixed, it is clear that the above two point function vanishes. There is no conservation law forcing this correlator to vanish - it is a nontrivial statement about the theory dynamics. The two point function in the planar limit, between two operators that have different multitrace structures, vanishes. This is a general property of matrix models. Thus, if we want to compute anomalous dimensions in the planar limit of the theory, we can focus on single trace operators since these will not mix with operators that have a different trace structure. This property of the planar limit is a crucial ingredient in the arguments for the integrability of the planar limit. Indeed, integrability follows because the planar dilatation operator can be identified with the Hamiltonian of an integrable spin chain. A single trace operator containing K fields can be identified with a spin chain state, where the spin chain lives on a lattice that has K sites. The fields in the single trace operator determine the states of the spins in the lattice. In this way, there is a bijection between single trace operators and the states of a spin chain. If we scale J_1, J_2

²Here J is the number of complex fields. In the dual theory, this is the angular momentum of the giant graviton.

as $N^{\frac{2}{3}}$, the right hand side of (1.3) scales as N^0 at large N and different trace structures start to mix. This mixing sets in even sooner: if we had computed the left hand side of (1.3) exactly, we would find that mixing between different trace structures is no longer suppressed if $J_1, J_2 \gtrsim \sqrt{N}$ [3]. For the case of interest to us $J_i \sim N$ and there is uncontrolled mixing. Consequently, the bijection between single trace operators and the states of a spin chain is not useful at all and the link to the dynamics of a spin chain is lost.

In this dissertation we will study operators built using $O(N)$ fields. There are numerical and analytic results suggesting that this limit is also integrable [4-11]. Further, there is by now convincing evidence that these operators are AdS/CFT dual to giant gravitons [12-20, 4]. Our goal is to compute the spectrum of anomalous dimensions of these operators. We focus on operators constructed from the six scalar Higgs fields ϕ_l , $l = 1, 2, \dots, 6$ that transform in the adjoint of the $U(N)$ gauge group. We can form 3 complex combinations

$$X = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4, \quad Z = \phi_5 + i\phi_6. \quad (1.4)$$

Our operators will be constructed using X , Y and Z . This is not a consistent truncation of the theory, but it is consistent at one loop (see for example [21]). The non-zero free field two point functions are

$$\langle Z_j^i (Z^\dagger)_l^k \rangle = \delta_l^i \delta_j^k = \langle Y_j^i (Y^\dagger)_l^k \rangle = \langle X_j^i (X^\dagger)_l^k \rangle \quad (1.5)$$

where the indices run over the values $1, 2, \dots, N$.³ In writing these two point functions we are not keeping track of spacetime dependence. Conformal invariance determines the coordinate dependence of the correlator and so we focus on the the combinatorial problem of summing all possible contractions of the free field propagators. The operators we consider are constructed using n Z fields, m Y fields and p X fields. We are primarily interested in the case that n, m, p all scale as N in the large N limit, but $n \gg m + p$ and $\frac{m}{p} \sim 1$. Thus, the operators that we consider are a small perturbation of a $\frac{1}{2}$ -BPS operator. The operators that we study are the restricted Schur polynomials. They are defined by

$$\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \frac{1}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma)) X_{i_{\sigma(1)}}^{i_1} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} \\ \times Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}}. \quad (1.6)$$

We call $\text{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\Gamma^R(\sigma))$ the restricted trace of $\Gamma^R(\sigma)$ [14]. When computing this trace, we trace over a subspace of the carrier space of R . R is an irreducible representation of S_{n+m+p} , that is, it is a Young diagram with $m + n + p$ boxes. We write $R \vdash m + n + p$. This subspace we trace over is a carrier space of the subgroup $S_n \times S_m \times S_p$. It is labeled by three Young diagrams $t \vdash p$, $s \vdash m$ and $r \vdash n$. $\vec{\mu}$ and $\vec{\nu}$ are degeneracy labels; they are each two dimensional vectors. Their two components resolve different copies of the two

³The matrices X, Y, Z are $N \times N$ matrices.

representations $s \vdash m$ and $t \vdash p$. To properly understand the role of the degeneracy labels and what they label, we note that the restricted trace can be written as

$$\mathrm{Tr}_{(t,s,r)\vec{\mu}\vec{\nu}}(\cdots) = \mathrm{Tr}_R(P_{(t,s,r)\vec{\mu}\vec{\nu}} \cdots) \quad (1.7)$$

where $P_{(t,s,r)\vec{\mu}\vec{\nu}}$ is an intertwining map. The degeneracy labels $\vec{\mu}$ and $\vec{\nu}$ play an important role in constructing this intertwining map as we now explain. The first step in constructing $P_{(t,s,r)\vec{\mu}\vec{\nu}}$ entails constructing a basis for the (t, s, r) irreducible representation of $S_n \times S_m \times S_p$. To do this start from the Young diagram for irreducible representation R . Remove p boxes in any order such that everytime a box is removed what remains is a valid Young diagram and we remove p_i boxes from row i . Assemble the p_i into a vector \vec{p} ; this vector will play an important role in what follows. Now remove m boxes in any order such that everytime a box is removed what remains is a valid Young diagram and we remove m_i boxes from row i . Assemble the m_i into a vector \vec{m} ; again, this vector will play an important role in what follows. The boxes are labeled according to the order in which they are removed so that the first box removed is box 1, the second box removed is box 2, and so on.⁴ In this way we land up with a partly labeled Young diagram R . The unlabeled boxes have the shape r and each partly labeled Young diagram is a distinct subspace of R that carries the irreducible representation r under the S_n subgroup. Now assemble the vectors with first p boxes labeled into an irrep t of S_p , resolving multiplicities that arise with ν_1 . In this process, the labels of the next m boxes are simply ignored. For each state in a given S_p irreducible representation specified by both t and ν_1 , one has all possible labelings of the next m boxes. Assemble these into vectors in an irreducible representation s of S_m , resolving multiplicities with ν_2 . The two multiplicity labels are assembled to produce the vector $\vec{\nu} = (\nu_1, \nu_2)$. The result of this exercise is a set of vectors labeled with two irreducible representations $t \vdash p$ and $s \vdash m$ each with a multiplicity label ν_1 and ν_2 , and two state labels, a, b , one for each state $|t, \nu_1, a; s, \nu_2, b\rangle$. The boxes that are not labeled stand for vectors that belong to a unique irreducible representation r of S_n . Use c to label states in r . We can make this explicit and write our state as $|t, \nu_1, a; s, \nu_2, b; r, c\rangle$. This gives a basis for the (t, s, r) irreducible representation of $S_n \times S_m \times S_p$. Now, the intertwining map is a matrix so that it has both a row label and a column label. We can use different copies of the (t, s, r) irreducible representation for the rows and columns of the intertwining map. Consequently

$$P_{(t,s,r)\vec{\mu}\vec{\nu}} = \sum_{a,b,c} |t, \mu_1, a; s, \mu_2, b; r, c\rangle \langle t, \nu_1, a; s, \nu_2, b; r, c| \quad (1.8)$$

Since the S_m and S_p actions commute it is clear that

$$|t, \mu_1, a; s, \mu_2, b; r, c\rangle = |t, \mu_1, a\rangle \otimes |s, \mu_2, b\rangle \otimes |r, c\rangle \quad (1.9)$$

where \otimes is the usual tensor product on a vector space. It then also follows that the intertwining maps can be written as a tensor product

$$P_{(t,s,r)\vec{\mu}\vec{\nu}} = \sum_a |t, \mu_1, a\rangle \langle t, \nu_1, a| \otimes \sum_b |s, \mu_2, b\rangle \langle s, \nu_2, b| \otimes \sum_c |r, c\rangle \langle r, c|$$

⁴Please see a discussion of an example of this in Section 2.4.1.

$$\equiv p_{t\mu_1\nu_1} \otimes p_{s\mu_2\nu_2} \otimes \mathbf{1}_r \quad (1.10)$$

The last factor in this product is always a genuine projector.

The restricted Schur polynomials share many of the nice properties that make the Schur polynomials so useful. In particular, the restricted Schur polynomials respect the trace relations and the two point function of the restricted Schur polynomials [17]

$$\langle \chi_{R,(t,s,r)\vec{\mu}\vec{\nu}} \chi_{T,(y,x,w)\vec{\beta}\vec{\alpha}}^\dagger \rangle = \frac{f_R \text{hooks}_R}{\text{hooks}_r \text{hooks}_s \text{hooks}_t} \delta_{RT} \delta_{rw} \delta_{sx} \delta_{ty} \delta_{\vec{\mu}\vec{\beta}} \delta_{\vec{\nu}\vec{\alpha}} \quad (1.11)$$

again diagonalize the free field two point function. The number hooks_R is a product of the hook lengths in Young diagram R . We will often find it convenient to work with operators $\hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}}$ normalized to have a unit two point function. These operators are related to the restricted Schur polynomials $\chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}$ as

$$\hat{O}_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sqrt{\frac{\text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R}} \chi_{R,(t,s,r)\vec{\mu}\vec{\nu}}. \quad (1.12)$$

The key difficulty with working with the restricted Schur polynomials, is in constructing and working with the intertwining maps $P_{(t,s,r)\vec{\mu}\vec{\nu}}$. Convenient methods to accomplish this have been developed for two rows in [6] and in general in [20]. Using these methods, the one loop dilatation operator has been diagonalized in the $su(2)$ sector (obtained by setting $p = 0$) [6, 11, 12, 20]. In this sector, the one loop dilatation operator reduces to a set of decoupled oscillators, which is an integrable system. These results provided perfect confirmation of earlier numerical studies [5, 10]. At two loops the system remains integrable in the $su(2)$ sector [7]. The one loop results were generalized to $p \neq 0$ in [8], but the interactions between the X and Y fields were argued to be subleading and were neglected. The subleading terms are of order $\frac{m}{n}$ relative to the leading terms [8]. It is precisely these terms that we will evaluate in this dissertation.

When interactions between the X and Y fields are neglected, the vectors \vec{p} and \vec{m} defined above are conserved [20]. The dilatation operator only mixes operators that have the same \vec{p} and \vec{m} values. This is not at all surprising - integrable systems are always accompanied with higher conserved quantities. What makes the interaction between the X and Y fields so interesting is that they spoil the conservation of \vec{p} and \vec{m} . This can mean one of two things: either, integrability does not persist beyond the $su(2)$ sector and this large N but non-planar limit is not integrable, or the dynamics remains integrable but the conservation of \vec{p} and \vec{m} is not one of the conservation laws of this (extended) integrable system. Our results are unambiguous - the second case is realized and the one-loop dilatation operator continues to be integrable when extended to act on operators built using all three complex scalars. Indeed, we are able to identify the new terms we have evaluated with elements of the

Lie algebra of a unitary group. Diagonalizing the complete dilatation operator then reduces to a solved problem in representation theory.

This dissertation is organized as follows: In section 2 we introduce the background and technology of group representation theory. We then, in section 3, evaluate the action of the one loop dilatation operator on the fields we study in this dissertation. The description of the operators (the Gauss operators) that diagonalize the terms in the dilatation operator that mix X and Z fields or Y and Z fields is reviewed in section 4. In section 5 we compute the action of the terms in the dilatation operator that mix X and Y , on the Gauss operators. Section 6 is used to argue that the dilatation operator can be written as an element of the Lie algebra of a unitary group. Our conclusions are given in section 7. Finally, we review giant gravitons in appendix A.

The results presented in this dissertation have appeared on the arXiv [21], and have been accepted for publication in the Journal of High Energy Physics (JHEP).

2 Technology and background

We now introduce and discuss the tools we use. We first discuss the dynamics of a free single hermitian matrix model. Our primary goal is to explain simplifications that arise in the $N \rightarrow \infty$ limit. Then, we include interactions and check if our results are modified by these interactions. With these results in hand we then study multimatrix models using group theoretic techniques. In this dissertation we study operators constructed using $O(N)$ fields. For these operators we need to sum a lot more than just the planar diagrams. To accomplish this, new methods have to be developed. These new methods make extensive use of group representation theory. For this reason we devote the remainder of this chapter to reviewing those aspects of group representation theory that are relevant to this project.

2.1 Matrix models

We begin with the simplest possible case of a free single, hermitian matrix. We then generalize our discussion to include interactions. Finally, we consider multimatrix models.

2.1.1 Gaussian Matrix Model

Define M to be an $N \times N$, hermitian matrix (i.e. $M = M^\dagger$). We require our action to be invariant under the ‘‘gauge transformation’’

$$M \rightarrow M' = U^\dagger M U,$$

where U is some $N \times N$, unitary matrix (i.e. $UU^\dagger = 1_N$). The unique free, gauge invariant action (quadratic in M) is

$$S = \frac{\omega}{2} \text{Tr}(M^2) .$$

The dynamics of the theory are captured in correlation functions. We will use the *path integral* to compute the correlation functions. The path integral is given by

$$Z = \int [dM] e^{-S}$$

where $[dM]$ is our measure. The correlation function of a product of observables, denoted $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_k \rangle$ is given by

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_k \rangle = \int [dM] \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_k e^{-S} . \tag{2.1}$$

We must integrate over the complete space of hermitian matrices so that

$$[dM] = \prod_{i=1}^N dM_{ii} \prod_{i < j} dM_{ij} .$$

Since we wish to compute correlation functions, a natural step is to couple a source to the free action, i.e. we define

$$Z[J] = \int [dM] e^{-S + \text{Tr}(JM)} . \quad (2.2)$$

By differentiating with respect to J we will be able to construct any correlation function of interest. We require that $J = J^\dagger$. This condition is necessary to ensure that the action is hermitian. Also, note that

$$\frac{d}{dJ_{ij}} \text{Tr}(JM) = M_{ji} .$$

It is useful to re-write the path integral in the form

$$Z[J] = \int [dM] e^{-\omega \text{Tr} \left(\left(M - \frac{J}{2\omega} \right)^2 \right)} e^{\frac{1}{4\omega} \text{Tr}(J^2)} . \quad (2.3)$$

To evaluate this integral, we recall that

$$\int dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} .$$

Hence, if we define ⁵

$$M_{ij} = \frac{1}{\sqrt{2}}(x_{ij} + iy_{ij}) , \quad i \neq j$$

we have

⁵Note that since

$$M_{ij}^* = M_{ji} ,$$

we have,

$$x_{ij} = x_{ji} \quad \text{and} \quad y_{ij} = -y_{ji} .$$

$$\begin{aligned}
Z[J]\Big|_{J=0} &= \int \prod_{i=1}^N dM_{ii} \prod_{i<j}^N dx_{ij} dy_{ij} \exp\left(-\omega \sum_i M_{ii}^2 - \omega \sum_{i>j} (x_{ij}^2 + y_{ij}^2)\right) \\
&= \left(\sqrt{\frac{\pi}{\omega}}\right)^{N^2}
\end{aligned} \tag{2.4}$$

where M_{ii} are the diagonal elements of M . We require that

$$Z[J]\Big|_{J=0} = 1$$

which can be achieved by rescaling the measure to

$$[dM] = \left(\sqrt{\frac{\omega}{\pi}}\right)^{N^2} \prod_{i=1}^N dM_{ii} \prod_{i<j} dx_{ij} dy_{ij}.$$

Hence

$$Z[J] = e^{\frac{1}{4\omega} \text{Tr}(J^2)}. \tag{2.5}$$

This form is particularly useful for computing correlation functions. We do not usually think about doing perturbation theory in a free theory. Perturbation theory uses a small parameter to develop an approximation scheme. This small parameter is usually taken to be the interaction strength. If a theory is free, there is no interaction strength. In a matrix model it is useful to think of $1/N$ as a small number and to develop perturbation theory in $1/N$. In this way we obtain a sensible and nontrivial perturbation theory for the free matrix model.

We can compute a large class of correlators for this theory, in the large N limit. Indeed, one can show that for large N

$$\langle \text{Tr}(M^{2n+1}) \rangle = 0, \tag{2.6}$$

$$\langle \text{Tr}(M^{2n}) \rangle = \omega \left(\frac{2N}{\omega}\right)^{n+1} \frac{(2n-1)!!}{(2n+2)!!} + O(N^{n-1}). \tag{2.7}$$

Here are some examples that illustrate how correlators are computed using equation (2.5):

$$\begin{aligned}
\langle M_{ij}M_{kl} \rangle &= \left. \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} Z[J] \right|_{J=0} \\
&= \frac{1}{4\omega} \frac{d}{dJ_{ji}} \left(J_{ji} \delta_{ik} \delta_{jl} + J_{ij} \delta_{jk} \delta_{il} \right) e^{\frac{1}{4\omega} J_{ij} J_{ji}} \Big|_{J=0} \\
&= \frac{\delta_{jk} \delta_{il}}{2\omega} ,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\langle M_{ij}M_{kl}M_{pq}M_{mn} \rangle &= \left. \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \frac{d}{dJ_{qp}} \frac{d}{dJ_{nm}} Z[J] \right|_{J=0} \\
&= \frac{1}{(2\omega)^2} (\delta_{jk} \delta_{il} \delta_{qm} \delta_{pn} + \delta_{jp} \delta_{iq} \delta_{lm} \delta_{kn} + \delta_{jm} \delta_{in} \delta_{lp} \delta_{kq}) .
\end{aligned} \tag{2.9}$$

We are only interested in correlation functions of traces of M . Traces of M are “gauge invariant” in the sense that they are invariant under the “gauge transformation” $M \rightarrow U^\dagger M U$. Using the results above it is straight forward to obtain

$$\langle \text{Tr}(M) \text{Tr}(M) \rangle = \langle M_{ii} M_{jj} \rangle = \frac{\delta_{ij} \delta_{ij}}{2\omega} = \frac{\delta_{ii}}{2\omega} = \frac{N}{2\omega} , \tag{2.10}$$

$$\langle \text{Tr}(M^2) \rangle = \langle M_{ij} M_{ji} \rangle = \frac{\delta_{jj} \delta_{ii}}{2\omega} = \frac{N^2}{2\omega} , \tag{2.11}$$

$$\langle \text{Tr}(M^2) \text{Tr}(M^2) \rangle = \langle M_{ij} M_{ji} M_{kl} M_{lk} \rangle = \frac{1}{(2\omega)^2} [N^4 + 2N^2]. \tag{2.12}$$

These are all given by a power of ω multiplied by a polynomial in N . It is now useful to define operators that have a finite expectation value in the large N limit. The operators are

$$\mathcal{O}_{2n} = \frac{\text{Tr}(M^{2n})}{N^{n+1}}$$

so that, for example, we have

$$\langle \mathcal{O}_2 \rangle = \frac{1}{2\omega} , \quad \langle \mathcal{O}_4 \rangle = \frac{2}{(2\omega)^2} \left[1 + O\left(1/N^2\right) \right] \tag{2.13}$$

Also, in the large N limit

$$\begin{aligned}\langle \mathcal{O}_2 \mathcal{O}_2 \rangle &= \frac{1}{(2\omega)^2} \left[1 + \frac{2}{N^2} \right], \\ \langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle &= \frac{1}{(2\omega)^3} \left[1 + O\left(\frac{1}{N^2}\right) \right], \\ \langle \mathcal{O}_2 \mathcal{O}_4 \rangle &= 2 \frac{1}{(2\omega)^3} \left[1 + O\left(\frac{1}{N^2}\right) \right],\end{aligned}$$

and so on.

Given the correlators we have computed, we can now argue that this theory simplifies considerably in the large N limit. Indeed, using the methods we have developed above we can verify that in this limit we have

$$\begin{aligned}\langle \mathcal{O}_2 \mathcal{O}_2 \rangle &= \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_2 \rangle + O\left(\frac{1}{N^2}\right) \\ \langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle &= \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_2 \rangle + O\left(\frac{1}{N^2}\right) \\ \langle \mathcal{O}_2 \mathcal{O}_4 \rangle &= \langle \mathcal{O}_2 \rangle \langle \mathcal{O}_4 \rangle + O\left(\frac{1}{N^2}\right).\end{aligned}\tag{2.14}$$

This illustrates what is called factorization. Factorization states that the expectation value of a product is the product of the expectation values, i.e.

$$\left\langle \prod_i \mathcal{O}_i \right\rangle = \prod_i \langle \mathcal{O}_i \rangle.\tag{2.15}$$

This result holds for large N matrix models in any number of spacetime dimensions and for any theory of matrices. What is the interpretation of factorization? Consider a physical system which can be in any of a set of states $\{i\}$. Populate state i with probability μ_i . As usual

$$\sum_i \mu_i = 1.$$

Let \mathcal{O}_n be an observable. The expectation value of \mathcal{O}_n is

$$\langle \mathcal{O}_n \rangle = \sum_i \mu_i \mathcal{O}_n(i)\tag{2.16}$$

where $\mathcal{O}_n(i)$ is the value of \mathcal{O}_n in state i . According to factorization

$$\begin{aligned}
\left\langle \prod_p \mathcal{O}_p \right\rangle &= \sum_i \mu_i \mathcal{O}_1(i) \mathcal{O}_2(i) \cdots \mathcal{O}_p(i) \\
&= \prod_p \langle \mathcal{O}_p \rangle \\
&= \sum_{i_1} \mu_{i_1} \mathcal{O}_1(i_1) \sum_{i_2} \mu_{i_2} \mathcal{O}_2(i_2) \cdots \sum_{i_p} \mu_{i_p} \mathcal{O}_p(i_p)
\end{aligned} \tag{2.17}$$

Since this holds for any product of observables $\prod_p \mathcal{O}_p$, equation (2.17) implies that

$$\mu_{i_j} = 1 \quad , \quad \mu_i = 0 \quad \text{for all } i \neq i_j.$$

This is a classical limit of the theory in the sense that only a single configuration is contributing. In general, it is a highly non-trivial task to construct the classical theory whose dynamics will correctly reproduce the correlation functions of the matrix model in the large N limit. With this insight we could argue that Maldacena [1] conjectured that the classical theory one would use to study the large N limit of $\mathcal{N} = 4$ super Yang-Mills theory is type IIB string theory on the $\text{AdS}_5 \times S^5$ background.

A very efficient way to compute correlators, in many different situations, is through the use of Schwinger-Dyson equations. We will review this method and apply it to the large N limit. First, we note that an exponential vanishes faster than any power so that (don't sum i and j)

$$\begin{aligned}
& \int [dM_{ij}] \frac{\partial}{\partial M_{ij}} \left((M^{n-1})_{ij} e^{-\omega \text{Tr}(M^2)} \right) , \quad [dM_{ij}] = dx_{ij} dy_{ij} \\
&= \int dx_{ij} \int dy_{ij} \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_{ij}} \left((M^{n-1})_{ij} e^{-\omega \text{Tr}(M^2)} \right) \\
&\quad - \int dx_{ij} \int dy_{ij} \frac{i}{\sqrt{2}} \frac{\partial}{\partial y_{ij}} \left((M^{n-1})_{ij} e^{-\omega \text{Tr}(M^2)} \right) \\
&= \int dy_{ij} \frac{1}{\sqrt{2}} \left((M^{n-1})_{ij} e^{-\omega(x_{ij}^2 + y_{ij}^2 + \dots)} \right) \Big|_{x_{ij}=-\infty}^{x_{ij}=\infty} \\
&\quad - \int dx_{ij} \frac{1}{\sqrt{2}} \left((M^{n-1})_{ij} e^{-\omega(x_{ij}^2 + y_{ij}^2 + \dots)} \right) \Big|_{y_{ij}=-\infty}^{y_{ij}=\infty} \\
&= 0 .
\end{aligned}$$

Thus, (i, j) are summed as usual

$$0 = \int [dM] \frac{d}{dM_{ij}} \left((M^{n-1})_{ij} e^{-\omega \text{Tr}(M^2)} \right)$$

and since

$$\begin{aligned}
\frac{d}{dM_{ij}} (M^{n-1})_{ij} &= \frac{d}{dM_{ij}} (M_{ia} M_{ab} \dots M_{lk} M_{kj}) \\
&= (M^0)_{ii} (M^{n-2})_{jj} + (M)_{ii} (M^{n-3})_{jj} + \dots + (M^{n-2})_{ii} (M^0)_{jj} \\
&= \sum_{r=0}^{n-2} \text{Tr}(M^r) \text{Tr}(M^{n-2-r})
\end{aligned}$$

we find

$$0 = \int [dM] e^{-\omega \text{Tr}(M^2)} \left(\sum_{r=0}^{n-2} \text{Tr}(M^r) \text{Tr}(M^{n-2-r}) - 2\omega (M^{n-1})_{ij} M_{ji} \right)$$

which implies that

$$\begin{aligned}
\langle \text{Tr}(M^n) \rangle &= \frac{1}{2\omega} \left\langle \sum_{r=0}^{n-2} \text{Tr}(M^r) \text{Tr}(M^{n-2-r}) \right\rangle \\
&= \frac{1}{2\omega} \sum_{r=0}^{n-2} \langle \text{Tr}(M^r) \rangle \langle \text{Tr}(M^{n-2-r}) \rangle
\end{aligned} \tag{2.18}$$

where we have used factorization in the second step. This equation lets us express the correlator of the trace of n powers of our matrix M in terms of products of correlators of traces of fewer than n powers of M . Since we know that $\langle \text{Tr}(M^n) \rangle = N$ for $n = 0$ we can now recursively construct any correlator we want.

Having illustrated the simplicity of the large N limit in this very simple setting, it is natural to ask if this simplicity is lost when interactions are turned on. We turn to this question next.

2.1.2 Interacting Matrix Model

In the free theory, we were able to compute any correlator exactly at large N by simply evaluating Gaussian moments. In a theory with interactions, we often lose this analytic structure. The integrals are difficult to evaluate in general. Hence, we use perturbative methods. In particular, we evaluate the integrals using Feynman diagrams organized by the interaction parameter(s) of the relevant theory. Using Feynman diagrams, we will argue that factorization continues to hold in the large N limit of the interacting theory. In fact, thanks to factorization, we would again be able to compute any correlator exactly at large N .

We consider an interacting matrix model described by the action

$$S = \omega \text{Tr}(M^2) + g \text{Tr}(M^4). \quad (2.19)$$

Using the methods of the last section, we find

$$\begin{aligned} \langle \text{Tr}(M^2) \rangle &= \int [dM] e^{-\omega \text{Tr}(M^2)} \text{Tr}(M^2) (1 - g \text{Tr}(M^4) + \dots) \\ &= \frac{N^2}{2\omega} - \frac{gN^5}{(2\omega)^3} \left(2 + \frac{9}{N^2} + \frac{4}{N^4} \right) + \dots \end{aligned}$$

Equivalently, we can use Feynman diagrams (also known as *ribbon graphs* for matrix models). The Feynman rules (for this model) are:

- draw a pair of dots for each matrix appearing in the correlator.
- join the dots from different matrices in all possible ways, connecting pairs of dots to pairs of dots using a double line or a ribbon.
- no twisting of double lines is allowed.
- multiply by the factor $(1/2\omega)^n$ for n propagators.

- multiply by the factor N^m for m closed loops/lines.
- multiply by $-g$ for each quartic vertex.

If we require the normalization

$$\langle 1 \rangle = \int [dM] e^{-\omega \text{Tr}(M^2) - g \text{Tr}(M^4)} = 1, \quad (2.20)$$

a standard argument from quantum field theory implies that we only need to sum the Feynman diagrams that don't include vacuum diagrams. For the two point function we study next, this is equivalent to taking the connected piece. The connected piece, to first order in g , of $\langle \text{Tr}(M^2) \rangle$ is given by

$$\langle \text{Tr}(M^2) \rangle_c = \frac{N^2}{2\omega} - \frac{8gN^3}{(2\omega)^3} - \frac{4gN}{(2\omega)^3}.$$

Notice that the first order corrections are not small in the $N \rightarrow \infty$ limit. The problem is even worse for higher order corrections since they appear multiplied by even higher powers of N . To resolve this problem, one needs to study the *double scaling limit* in which $N \rightarrow \infty$, $g \rightarrow 0$ with $\lambda = gN$ fixed. To validate the use of perturbation theory, λ has to be small. The powers in N correspond to the topology of the surface that the ribbon (or Feynman) graph triangulates.⁶ Hence, in this theory, there exists two expansion parameters: λ and $\frac{1}{N^2}$. λ is called the *'t Hooft coupling*. One can also show that in the large N limit factorization emerges even in a theory with interactions. Let us demonstrate this fact with a simple example.

Let

$$\mathcal{O}_{(1,1)} = \frac{1}{N^{5/2}} \text{Tr}(M) \text{Tr}(M)$$

and

$$\mathcal{O}_{(1,1)}^2 = \frac{1}{N^5} \text{Tr}(M) \text{Tr}(M) \text{Tr}(M) \text{Tr}(M).$$

We want to show that in the large N limit, we have

$$\langle \mathcal{O}_{(1,1)} \rangle \langle \mathcal{O}_{(1,1)} \rangle = \langle \mathcal{O}_{(1,1)}^2 \rangle.$$

⁶The topology associated with $O(N^2)$ diagrams is a sphere, $O(1)$ is associated to a torus, $O\left(\frac{1}{N^2}\right)$ a pretzel, and so on. A topology of a genus g is weighted by N^{2-2g} .

There are 25 Feynman diagrams at $O(g)$ for the correlator $\langle \mathcal{O}_{(1,1)}^2 \rangle$. Of these 25 diagrams, 9 are disconnected and 16 are connected. Of the disconnected diagrams, 4 are planar and 5 are non-planar. Of the connected diagrams, 4 are planar and 12 are non-planar. The Feynman diagrams are

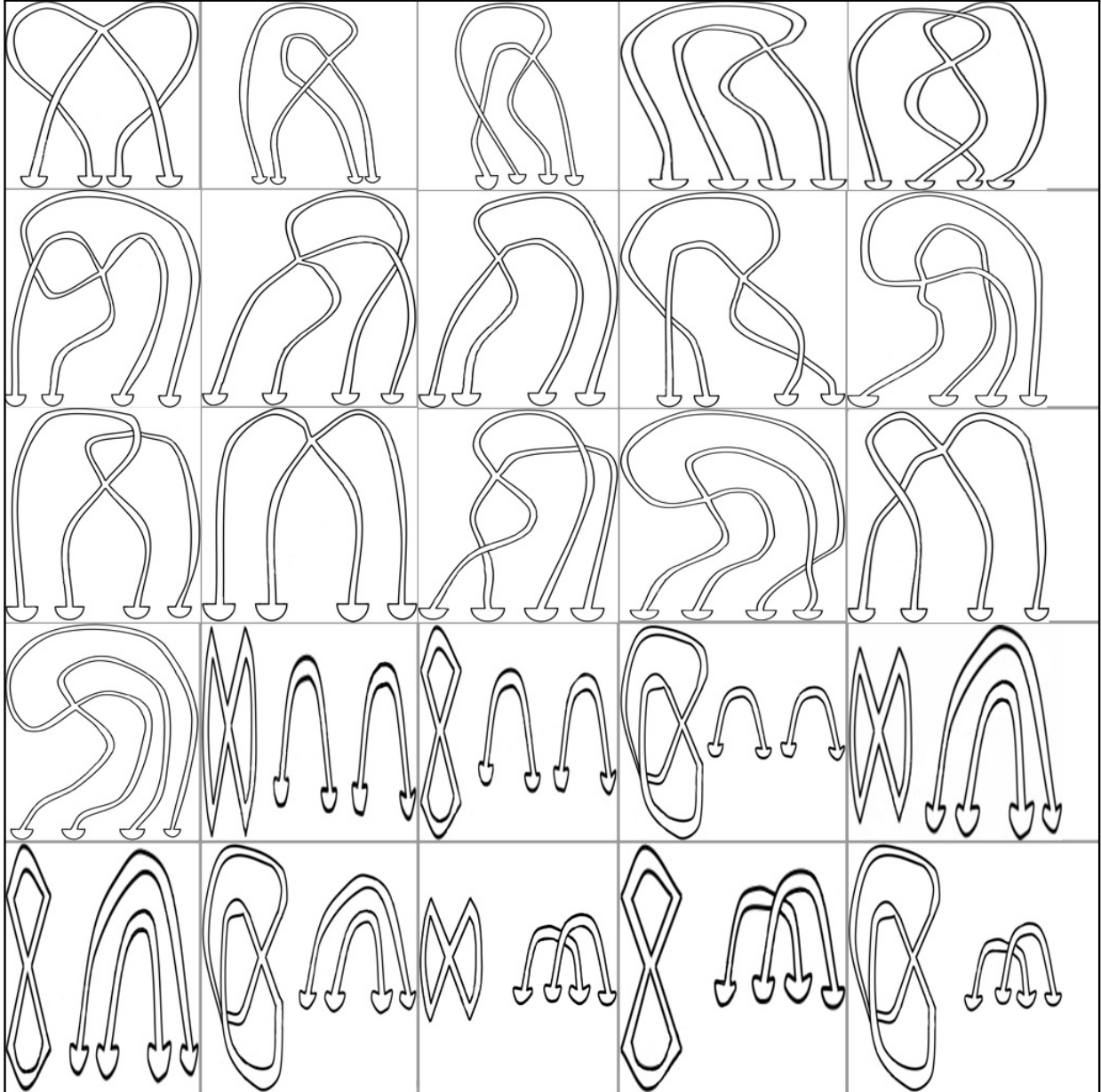


Figure 1: The Feynman diagrams of $\langle \mathcal{O}_{(1,1)}^2 \rangle$ at $O(g)$.

and there are 30 Feynman diagrams at $O(g)$ for the correlator $\langle \mathcal{O}_{(1,1)} \rangle \langle \mathcal{O}_{(1,1)} \rangle$. Of these 30 diagrams, 6 are disconnected and 24 are “connected”. Of the disconnected diagrams, 4 are planar and 2 are non-planar. Of the connected diagrams, 16 are planar and 8 are non-planar.

The Feynman diagrams are (the other half are the same as these but the order in each graph is reversed)

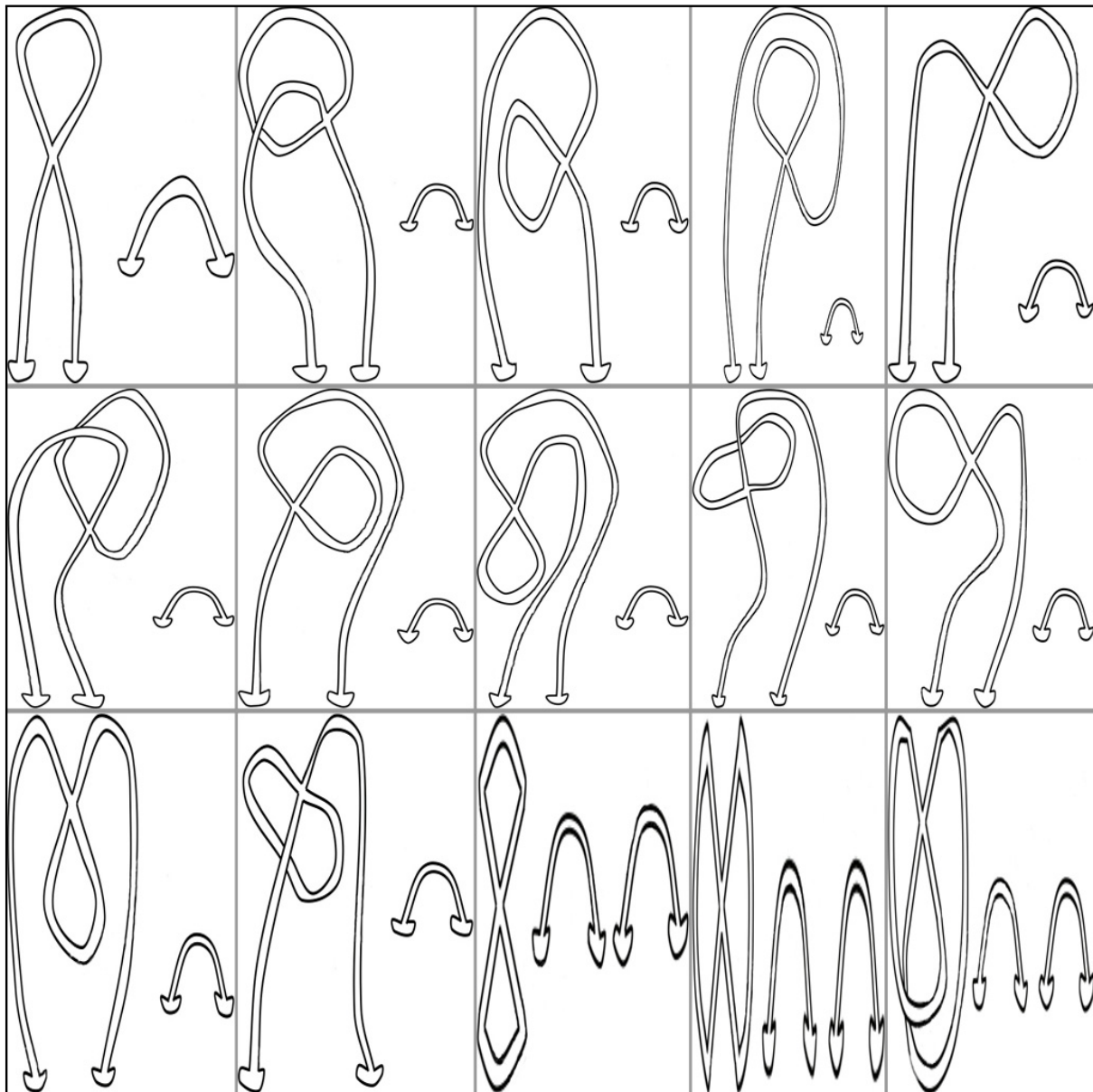


Figure 2: Half of the Feynman diagrams of $\langle \mathcal{O}_{(1,1)} \rangle \langle \mathcal{O}_{(1,1)} \rangle$ at $O(g)$. The other half is similar to these.

Hence, in the large N limit, we have

$$\langle \mathcal{O}_{(1,1)} \rangle \langle \mathcal{O}_{(1,1)} \rangle = -\frac{4g}{(2\omega)^4} = \langle \mathcal{O}_{(1,1)}^2 \rangle \quad (2.21)$$

i.e. factorization continues to hold. One can verify factorization is obeyed for other correlators as well. Let us now summarize the main ideas/lessons:

- to obtain a well defined perturbative expansion one needs to consider the double scaling limit that takes $N \rightarrow \infty$, $g \rightarrow 0$ while holding

$$\lambda = gN \ll 1$$

fixed. Correlation functions are then expanded in terms of two small parameters, λ and $1/N$.

- the expectation value of a product of gauge invariant observables is equal to the product of the expectation value of each observable in the product, in the large N limit. This so-called large N factorization is a signal that the dynamics is classical.
- Maldacena's conjecture implies that the classical theory that governs correlators of the large N limit of $\mathcal{N} = 4$ SYM theory is type IIB string theory on $\text{AdS}_5 \times S^5$ [1].

There is an implicit assumption in the above analysis. Namely, as $N \rightarrow \infty$, the number of fields appearing in the operator we study is held fixed to be $O(1)$. In this case the large N limit and the planar limit coincide. In other sectors (e.g. if we study operators composed of $O(N)$ fields), the large N limit and the planar limit are different. This is because if we allow the number of fields in our operator to scale with N , the number of diagrams of a given topology will also depend on N . This extra N dependence can overpower the suppression of non-planar terms implying that we must sum all Feynman diagrams, both planar and non-planar.

The next natural generalization is to consider interacting multi-matrix models.

2.1.3 Multi-matrix models

The field content of QCD includes four real, hermitean matrices A_μ , $\mu = 0, 1, 2, 3$. In general these matrices do not commute. This has dramatic consequences: to fully understand the dynamics of this theory, we need to study all kinds of local operators, including

$$\text{Tr} \left(D_\alpha^{n_1} D_\beta^{n_2} D_\gamma^{n_3} F_{\mu\nu} \right),$$

where D_α is the covariant derivative. The space of all such operators is often called loop space. It is because our matrices do not commute that we obtain such a rich space of observables. Even just parametrizing loop space is a remarkably non-trivial problem. Instead of proceeding further with QCD we will return to study our simpler but instructive toy model problem.

One of the simplest multi-matrix models we could study includes two complex matrix fields, say X and Y . This case has been studied in detail in [4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 18, 19]. The further extension to the case of three scalar, complex matrices X, Y and Z has not been considered in as much detail. A central goal of this study is to include interactions that have been neglected up to now. The theory we study is a sector of $\mathcal{N} = 4$ SYM theory. The piece of the action relevant for us is given by

$$S = \frac{N}{4\pi\lambda} \int_R dt \int_{S_3} \frac{d\Omega_3}{2\pi^2} \text{Tr} \left(\frac{1}{2} (D\phi^a)(D\phi^a) + \frac{1}{4} \left([\phi^a, \phi^b] \right)^2 - \frac{1}{2} \phi^a \phi^a \right) + \dots$$

where $a, b = 1, 2, \dots, 6$. This action has 6 hermitian scalar fields that transform in the adjoint of the gauge group. To obtain the model we study, we combine the 6 hermitian scalars into 3 complex fields

$$Z = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4, \quad X = \phi_5 + i\phi_6.$$

The operators of interest to us are built using X, Y and Z . One of our main goals is to compute the two point functions of these operators, to one loop, at large N . This will allow us to extract the one loop anomalous dimensions of these operators. For the computations that follow, we need the free field propagators which are given by

$$\langle (Z_j^i)^\dagger(t) Z_l^k(t) \rangle = \delta_l^i \delta_j^k = \langle (Y_j^i)^\dagger(t) Y_l^k(t) \rangle = \langle (X_j^i)^\dagger(t) X_l^k(t) \rangle$$

with all of the other possible combinations, for example $\langle (Y_j^i)^\dagger (X_l^k)^\dagger \rangle$, equal to zero.

2.2 Organizing Loop Space

There are a huge set of possible observables in the theory. To provide some structure to this set we will use the notion of \mathcal{R} -charge and study a smaller special set of operators known as Bogomol'nyi-Prasad-Sommerfield (BPS) operators. BPS operators are annihilated by some supersymmetry generators. These are good states to consider because they have features that are protected by supersymmetry. $\frac{1}{2}$ -BPS states preserve half of the supersymmetry generators. Similarly, one can define $\frac{1}{4}$ -BPS and $\frac{1}{8}$ -BPS states. The dynamics of a large class of half BPS states is captured by single, complex matrix models. To go beyond the half-BPS sector, we have to study multi-matrix models. Schur polynomials and restricted Schur polynomials provide a good basis for the local operators of the theory. An importance of this basis is that it diagonalizes the two point function in the free theory to all orders in the $1/N$ expansion. Given the fact that $\mathcal{N} = 4$ SYM theory is dual to type IIB string theory, a natural question to ask is

What are the states in quantum gravity dual to these field theory operators?

The BPS sector of type IIB string theory on $AdS_5 \times S^5$ contains gravitons, strings, membranes and new spacetime geometries. For the $\frac{1}{2}$ -BPS sector it is possible to give a dictionary that relates $\mathcal{N} = 4$ SYM with type IIB string theory on $AdS_5 \times S^5$. We organize our dictionary using \mathcal{R} -charge. The dictionary is given by

Table 1: Dictionary

CFT	String Theory
use $O(1)$ fields	gravitons
$O(J)$ fields, $\frac{J^2}{N} \ll 1$ fixed	strings
$O(N)$ fields	membranes
$O(N^2)$	new spacetime geometries

In the half BPS sector, the number of fields in the operator is equal to the \mathcal{R} -charge of the operator, which we denote J . According to the above dictionary

- for $J \sim O(1)$ we study operators that are dual to Kaluza-Klein gravitons [22]. Gravitons are point-like in string units.
- for $J \sim O(\sqrt{N})$ we study operators dual to fundamental strings [23]. Fundamental strings have a fixed size in string units.
- for $J \sim O(N)$ the field theory operators we study are dual to giant gravitons [20]. Giant gravitons have a size comparable to the radius of the space they occupy.
- and for $J \sim O(N^2)$ we have operators that are dual to objects whose size diverges when measured in units of the radius of curvature of the spacetime. The interpretation of this is that these operators are dual to new spacetime backgrounds. These are the so-called Lin-Lunin-Maldacena (LLM) geometries [24].

2.3 Group representation theory

We begin with some background group theory, afterwhich we introduce group representation theory [25]. Then, we briefly discuss the analysis of representations and discuss the main ideas when working with a group algebra.

2.3.1 Groups

Definition 1:

Let S be a set with n elements, say $1, 2, \dots, n$. One can form a group of permutations, from S , denoted S_n and called the symmetric group. S_n is a finite group of order $n!$. In general, S_n is nonabelian.

□

Permutations are relevant for this dissertation because Wick contractions can be put into correspondence with permutations. Summing the complete set of ribbon graphs is traded for a sum over some group of permutations. A generic permutation σ of the set $S = \{1, 2, \dots, n\}$ is a bijection with $\{\sigma(1), \sigma(2), \dots, \sigma(n)\}$ being the same set S , with the elements reshuffled. After repeated application of σ on some integer $1 \leq i \leq n$, one eventually obtains the initial integer i . This fact motivates the next definition.

Definition 2:

Let $\sigma \in S_n$, $1 \leq i \leq n$, and k the smallest positive integer satisfying the condition

$$\sigma^k(i) = i.$$

We call the set

$$\{\sigma^r(i)\}_{r=0}^{k-1}$$

a cycle of σ . This cycle is of length k or equivalently a k -cycle generated by i .

□

To construct this cycle start with some positive integer i ($1 \leq i \leq n$) and apply σ k -times (to obtain i again). Now, pick another integer j that is not contained in the first cycle and repeat the process. Applying this process repeatedly yields a set of disjoint cycles whose union contains all elements of S , with each appearing once. Thus any permutation can be written as a product of disjoint cycles.

Definition 3:

$\sigma \in S_n$ is called a cyclic permutation of length k if it has exactly one cycle of length k and $n - k$ cycles of length 1.

□

Definition 4:

A cycle of length 2 is called a transposition.

□

- The transposition (ij) switches i and j .

We can decompose a k -cycle (i_1, i_2, \dots, i_k) into a product of transpositions.

Proposition 1:

$$(i_1, i_2, \dots, i_k) = \prod_{r=1}^{k-1} (i_1 i_{k-r+1}).$$

This decomposition is not unique. ■

Definition 5:

If a permutation can be decomposed into an even (odd) number of transpositions, then it is said to have an even (odd) parity. σ is said to be even (odd).

□

We define the parity homomorphism, denoted $sgn(\sigma)$, by

$$sgn(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is even,} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

We will see that permutations can also be used to construct observables. Unlike the connection between Wick contractions and permutations, the connection permutations and observables is many to one. For example, in the 1/2 BPS sector, many permutations correspond to the same observable. We will now introduce the notion of conjugate permutations. It is conjugate permutations that give the same observables.

Theorem 1:

Let σ, τ be two permutations. σ is conjugate to τ if and only if τ and σ have the same cycle structure. For S_n , the distinct conjugacy classes correspond to partitions of n .

Define

$$\lambda_j = \sum_{k=j}^n v_k, \tag{2.22}$$

where v_k is the number of k -cycles in σ . It is not difficult to see that

$$\sum_{i=1}^n \lambda_i = n, \tag{2.23}$$

with $0 \leq \lambda_{i+1} \leq \lambda_i$, $0 \leq i \leq n - 1$. Note that the nonnegative integers

$$(\lambda_1, \lambda_2, \dots, \lambda_n)$$

form a partition of n .

Our discussion has focused mainly on the symmetric group. We also need to know something about the unitary group, a subgroup of the general linear group.

2.3.2 General linear group and unitary groups

Definition 6:

Let \mathcal{V} be a finite vector space. The general linear group of a vector space \mathcal{V} is defined as the set of all homomorphisms from \mathcal{V} back to itself, denoted by $GL(\mathcal{V})$. In simpler language, we consider the set of all invertible matrices acting on \mathcal{V} . If $\mathcal{V} = \mathbb{C}^n$, we use the notation $GL(n, \mathbb{C})$, and similarly, $GL(n, \mathbb{R})$ if $\mathcal{V} = \mathbb{R}^n$.

□

In particular, we are interested in a subgroup of $GL(n, \mathbb{C})$, the set of unitary transformations, denoted $U(n)$. We call $U(n)$ the unitary group. The representation theory of $GL(n, \mathbb{C})$ and $U(n)$ are identical. For certain questions it is easier to study the $GL(n, \mathbb{C})$ problem.

2.3.3 Group representation theory

Definition 7:

Let \mathcal{G} be a group and \mathcal{H} a Hilbert space. A representation of \mathcal{G} on \mathcal{H} is a homomorphism $T : \mathcal{G} \rightarrow GL(\mathcal{H})$. \mathcal{H} is called the carrier space of T . The dimension of \mathcal{H} is called the dimension of the representation T .

□

Since T is a representation, we have

$$T_{g_1 g_2} = T_{g_1} T_{g_2}.$$

We will use a matrix representation of the T_g 's.

Every group has an infinite number of representations. However, there are some very basic representations, called the inequivalent irreducible representations, out of which all other representations are built. Equivalent representations are related by a change of basis for \mathcal{H} . The concept of reducibility is more involved.

Definition 8:

A representation $T : \mathcal{G} \rightarrow GL(\mathcal{H})$ is reducible if it can be expressed as a direct sum of its subspaces, i.e.

$\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$, where \mathcal{A} , \mathcal{B} are invariant subspaces of \mathcal{H} .

Otherwise, \mathcal{H} is said to be irreducible. If \mathcal{B} is the set of all vectors in \mathcal{H} which are orthogonal to all vectors in \mathcal{A} , denoted \mathcal{A}^\perp , then \mathcal{B} is called the orthogonal complement of \mathcal{A} .

□

Our goal is to construct irreducible representations (irreps) of the symmetric group. A well known result from finite group theory proves that every representation of a finite group is equivalent to a unitary representation. Moreover, if \mathcal{A} is an invariant subspace of \mathcal{H} , then so is \mathcal{A}^\perp . i.e.

$$\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^\perp.$$

In general,

$$\mathcal{H} = \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \dots \oplus \mathcal{A}^{(l)} \oplus \dots .$$

For a finite dimensional vector space, \mathcal{V} , we have

$$\mathcal{V} = \bigoplus_{l=1}^k \mathcal{A}^{(l)}.$$

Hence, if we think of $\mathcal{A}^{(l)}$ as the carrier space of some irrep, then

$$T_g = \bigoplus_{\alpha=1}^{\rho} m_\alpha T_g^{(\alpha)}. \quad (2.24)$$

In the above equation, equivalent irreps are grouped together, m_α counts the number of times $T_g^{(\alpha)}$ appears and ρ counts the number of inequivalent irreps. As a matrix representation T_g is block diagonal for all group elements,

$$T_g = \text{diag}(T_g^{(1)}, T_g^{(2)}, \dots, T_g^{(r)}), \forall g \in \mathcal{G} \quad (2.25)$$

where some $T_g^{(l)}$ are equivalent.

Schur's Lemma's are powerful results that can be used to derive the fundamental orthogonality relation. The fundamental orthogonality relation plays a key role in understanding how to decompose a reducible representation into its irreducible components, in defining

projection operators which play an important role in representation theory and finally in proving that the number of inequivalent irreducible representations of a finite group is finite. Given their importance, we will now recall Schur's Lemma's and then use them to derive the fundamental orthogonality relation. Let $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ denote the space of all linear mappings from the vector space \mathcal{V} to the vector space \mathcal{V}' .

Schur's lemmas:

- If T, T' are some irreps with

$$AT_g = T'_g A, \quad \forall g \in \mathcal{G},$$

where $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$, then either $T \equiv T'$ or $A = 0$.

- In particular, if $T = T'$, then

$$A = \lambda \mathbb{I}.$$

■

Let us now use these results to derive the fundamental orthogonality relation. Let $T^{(\alpha)}, T^{(\beta)}$ be irreps (with carrier spaces $\mathcal{A}^{(\alpha)}, \mathcal{A}^{(\beta)}$, respectively) of some group \mathcal{G} , and define

$$A \equiv \sum_{x \in \mathcal{G}} T_x^{(\alpha)} M T_{x^{-1}}^{(\beta)} \tag{2.26}$$

$$= \sum_{x \in \mathcal{G}} T^{(\alpha)}(x) M T^{(\beta)}(x^{-1}) \tag{2.27}$$

where $M \in \mathcal{L}(\mathcal{A}^{(\alpha)}, \mathcal{A}^{(\beta)})$ is some operator. Then,

$$\begin{aligned} T_g^{(\alpha)} A &= \sum_{x \in \mathcal{G}} T^{(\alpha)}(g) T^{(\alpha)}(x) M T^{(\beta)}(x^{-1}) T^{(\beta)}(g^{-1}) T^{(\beta)}(g) \\ &= \sum_{x \in \mathcal{G}} T^{(\alpha)}(gx) M T^{(\beta)}((gx)^{-1}) T^{(\beta)}(g) \\ &= AT_g^{(\beta)}. \end{aligned}$$

In particular, if $T_g^{(\alpha)} = T_g^{(\beta)}$, then $A = \lambda \mathbb{I}$, otherwise $A = 0$. Hence,

$$\sum_{g \in \mathcal{G}} T^{(\alpha)}(g) M T^{(\beta)}(g^{-1}) = \lambda_M \delta_{\alpha\beta} \mathbb{I}. \tag{2.28}$$

If we now choose M to be the matrix whose only nonzero element is a 1, in the l th row, and m th column, then we have

$$\sum_{g \in \mathcal{G}} T_{il}^{(\alpha)}(g) T_{mj}^{(\beta)}(g^{-1}) = \lambda_{lm} \delta_{ij} \delta_{\alpha\beta}.$$

Setting $i = j$, $\alpha = \beta$, and summing over i , we find

$$\lambda_{lm} = \frac{|\mathcal{G}|}{n_\alpha} \delta_{lm},$$

where $|\mathcal{G}|$ is the order of the group \mathcal{G} , and n_α is the dimension of the carrier space $T^{(\alpha)}$.

$$\therefore \sum_{g \in \mathcal{G}} T_{il}^{(\alpha)}(g) T_{mj}^{(\beta)*}(g) = \frac{|\mathcal{G}|}{n_\alpha} \delta_{lm} \delta_{ij} \delta_{\alpha\beta}, \quad (2.29)$$

where we have used the fact that $T_{mj}^{(\beta)}(g^{-1}) = T_{mj}^{(\beta)*}(g)$ which follows because every representation of a finite group is equivalent to a unitary representation. This identity is known as the fundamental orthogonality relation. It would be nice to have a simple algorithm that determines whether two representations are equivalent or not. Given the fact that equivalent representations are related by a change of basis, it is clear that we are looking for a basis independent quantity. A set of quantities that provide a characterization of a representation that is basis independent are the group characters that we study next.

Definition 9:

The character of a representation $T : \mathcal{G} \rightarrow GL(\mathcal{V})$ is a map

$$\chi : \mathcal{G} \rightarrow \mathbb{C} \text{ defined by } \chi(g) = \text{Tr}(T_g) = \sum_i T_{ii}(g).$$

□

Recall that two elements $x, y \in \mathcal{G}$ belong to the same conjugacy class if $\exists g \in \mathcal{G} \mid x = gyg^{-1}$. This same relation holds for the operators representing the group elements: $T_x = T_g T_y T_g^{-1}$. Tracing both sides, and noting that $T_{g^{-1}} = T_g^{-1}$, we see that, all elements of a group belonging to the same conjugacy class have the same character.

Setting $i = l$ and $j = m$ in equation (2.29) and summing over i, j , we obtain

$$\sum_{g \in \mathcal{G}} \chi^{(\alpha)}(g) \chi^{(\beta)}(g^{-1}) = \frac{|\mathcal{G}|}{n_\alpha} \delta_{\alpha\beta} \delta_{ji} \delta_{ij} = \frac{|\mathcal{G}|}{n_\alpha} \delta_{\alpha\beta} \underbrace{\sum_i \delta_{ii}}_{n_\alpha} = |\mathcal{G}| \delta_{\alpha\beta}$$

or

$$\sum_{g \in \mathcal{G}} \chi^{(\alpha)}(g) \chi^{(\beta)*}(g) = |\mathcal{G}| \delta_{\alpha\beta} \quad (2.30)$$

since the representation is unitary. From the above equation, we learn that one can interpret characters as vectors, i.e. $|\chi^{(\alpha)}\rangle \in \mathbb{C}^r$, where in the unitary representation

$$\sum_{i=1}^r c_i \chi_i^{(\alpha)} \chi_i^{(\beta)*} = |\mathcal{G}| \delta_{\alpha\beta} \Rightarrow \langle \chi^{(\beta)} | \chi^{(\alpha)} \rangle = |\mathcal{G}| \delta_{\alpha\beta}, \quad (2.31)$$

with i a label for different conjugacy classes, c_i counts the number of elements in class i , and r is the number of equivalent classes in \mathcal{G} . Noting that the vectors are orthogonal, we have

$$\rho \leq r . \quad (2.32)$$

We now briefly review the representation theory we will need.

2.3.4 Representation Theory

The definition of the trace together with equation (2.24) implies that

$$\chi(g) = \sum_{\alpha=1}^{\rho} m_{\alpha} \chi^{(\alpha)}(g) .$$

Using this together with equation (2.30) we easily find

$$m_{\alpha} = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi(g) \chi^{(\alpha)*}(g) \quad (2.33)$$

where m_{α} is the multiplicity of the representation $T^{(\alpha)}$ in T . In particular, since

$$\sum_{g \in \mathcal{G}} |\chi(g)|^2 = |\mathcal{G}| \sum_{\alpha} m_{\alpha}^2$$

if T is an irrep,

$$\sum_{g \in \mathcal{G}} |\chi(g)|^2 = |\mathcal{G}| . \quad (2.34)$$

This is particularly powerful for low cardinality groups. Let

$$\mathcal{G} = \{g_1, g_2, \dots, g_m\}$$

be a group. Then, recalling that multiplying the elements of \mathcal{G} by some fixed element permutes the elements of \mathcal{G} , we define the regular representation $R : \mathcal{G} \rightarrow GL(m, \mathbb{R})$ by

$$R_{g_i}(x_1, x_2, \dots, x_m) = (x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(m)}) \quad (2.35)$$

where σ_i is a permutation.

The multiplicity of an irrep in a regular representation equals the dimension of the same irrep, i.e.

$$m_\alpha = n_\alpha .$$

For groups of low order, the Cayley table is a useful way to describe a group. A Cayley table describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table reminiscent of an addition or multiplication table. In a Cayley table one can easily “read off” the group properties (whether a group is abelian, the inverses of elements, and so on). A Cayley table for higher order groups has a lot of elements and takes time to write down whereas one can certainly compute the group algebra generators (there is a finite number of these) much more quickly. Thus, for groups of high order, it is better to study the group algebra.

2.3.5 Group algebra

Let $\mathcal{G} = \{g_1, g_2, \dots, g_m\}$ be a group. An element of the group algebra is defined by

$$\mathbf{a} = \sum_{i=1}^m a_i g_i . \quad (2.36)$$

Vector addition is as usual. The product of two vectors is defined by using the group composition law for vector multiplication. As an example

$$\mathbf{a}\mathbf{b} = \sum_{i=1}^m a_i g_i \sum_{j=1}^m b_j g_j = \sum_{i,j=1}^m a_i b_j \underbrace{g_i g_j}_{g_k} = \sum_{k=1}^m c_k g_k = \mathbf{c} . \quad (2.37)$$

We have shown that $\rho \leq r$. With more work one can show

Theorem 1:

The number of inequivalent irreps of a finite group equals the number of its conjugacy classes. ■

With this background in place, we can now study the representations of the symmetric group.

2.4 Representations of the symmetric group

Representations of the symmetric group are labeled by Young diagrams. Apart from this, Young diagrams provide a remarkable description of the states of an irrep and are used in combinatorial descriptions of characters and dimensions of irreps. In this section we will review those aspects of Young diagrams that are relevant to our study.

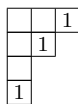
2.4.1 Young-Yamonouchi basis

We will describe a way of labeling Young diagrams. This labeling plays a crucial role in the representation theory we will use. Completely labeled Young diagrams are known as Young tableau / Young-Yamonouchi symbols. Use the integers $i \in \{1, 2, \dots, n\}$ to label the boxes in the Young diagram. Label each box with a unique integer i , $1 \leq i \leq n$. The labeling must respect a rule which is most easily stated as follows: Imagine dropping the box labeled 1, then the box labeled 2, and so on to box n . The rule states that each time a box is dropped, what remains is a valid Young diagram. The Young tableau form a basis for an S_n irrep. We will illustrate this using a specific example.

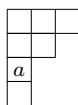
Consider the S_7 irrep



There are three possible boxes we can remove first, labeled one below



As an example of an illegal step, we can't remove say the box labeled a below first



Without loss of generality, we pick the box in the first column. The second set of boxes we can remove are labeled 2 in the figure below

		2
	2	
2		
1		

Again for the sake of this demonstration, we pick a legal box and label it. The third box removed can be any of the boxes labeled 3 in the figure below

		2
	3	
3		
1		

Let us pick the box in the third row to be our box labeled 3. Then, there is no choice for our fourth box. Thus, we now have

		2
	4	
3		
1		

We can now label any of the two boxes with 5 filled in

	5	2
5	4	
3		
1		

We don't have a choice for our sixth label

	5	2
6	4	
3		
1		

Notice that we are only left with one box. Thus our Young tableau is given by

7	5	2
6	4	
3		
1		

This is not the only Young-Yamouchi symbol for this irrep. There are 34 others for this irrep. The corresponding standard basis states are thus 35 dimensional. This example was sufficiently complicated that we were able to describe how Young-Yamouchi labeling works. We will now consider a simpler example for which we can record the complete list of Young-Yamouchi states. Towards this end, consider the irrep

The corresponding Young tableau are

$$\begin{array}{|c|c|} \hline 4 & 1 \\ \hline 3 & \\ \hline 2 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & \\ \hline 1 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 4 & 2 \\ \hline 3 & \\ \hline 1 & \\ \hline \end{array}$$

In the Young tableau basis

$$\left\langle \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \middle| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \right\rangle = \delta_{ij}$$

where i, j label the different Young tableau. One may now ask how the group elements act on the Young-Yamououchi basis, which we denote by \widehat{e}_R for a Young diagram R . This question is answered, in general, in the next section.

2.4.2 Action of group elements on Young tableau

Our main goal in this subsection is to describe how the group elements act on the Young-Yamououchi basis. We use this to construct the matrix irreps of S_n . We will make use of the notion of the factor of a box.

Definition 10:

A factor [20] of a box in row i , column j is a number $N - i + j$. The positive integer N is arbitrary. When the formula for the factor is used for the representation theory of the symmetric group, the final results are independent of N . Thus, as far as the symmetric group goes, N is an arbitrary parameter. The factor also plays a role in the representation theory of $GL(n, \mathbb{C})$. In this case, N must be set equal to n . The resulting representation theory depends sensitively on n .

□

Let $\sigma \in S_n$. Any element of the group can be written as a product of adjacent permutations $(i \ i + 1)$. For example,

$$(13) = (12)(23)(12).$$

Thus, we only need to understand the action of adjacent permutations on the Young tableau. Young's orthogonal representation formula gives a formula for the action of adjacent permutations on the Young-Yamououchi state. If Γ is some matrix representation of S_n , R is an irrep, c_i a factor of a box labeled i , \widehat{e}_R a Young tableau corresponding to the irrep R , and $\widehat{e}_{R_{i,i+1}}$ the same Young tableau as \widehat{e}_R but with boxes $i, i + 1$ swapped, then

$$\Gamma_R(i \ i + 1)|\widehat{e}_R\rangle = \frac{1}{c_i - c_{i+1}}|\widehat{e}_R\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}|\widehat{e}_{R_{i,i+1}}\rangle.$$

We will use this formula extensively in this dissertation.

2.5 Example of how the symmetric group is used

Given this background, it is useful to return to the model we introduced in section 2.1.3. and to explain how we use the symmetric group. We build our operators using p X s, m Y s, and n Z s. Obviously, X, Y, Z which are matrices, carry a pair of indices. One could use upper indices only, lower indices only, or a mixture

$$X^{ij} \quad , \quad X_{ij} \quad , \quad \text{or} \quad X_j^i$$

where i is a row index and j is a column index. Under the action of the unitary group we know that

$$X \rightarrow U^\dagger X U.$$

Thus, the row and column indices transform differently. We will reflect the difference in how the row and column indices transform by using upper indices for rows and lower indices for columns. This is precisely what we do in special relativity. Just like for single matrix models, we must have gauge invariance. We achieve this by studying trace operators. Examples of trace operators one could study are

$$\mathcal{O}_1 = \text{Tr}(X^p)\text{Tr}(Y^m)\text{Tr}(Z^n)$$

$$\mathcal{O}_2 = \text{Tr}(X^2)\text{Tr}(X^{p-2})\text{Tr}(Y)\text{Tr}(YZ^2Y^3Z)\text{Tr}(Y^{m-5}Z^{n-3})$$

$$\mathcal{O}_k = \text{Tr}(X^{p-3})[\text{Tr}(XYZ)]^2 \underbrace{\text{Tr}(Y) \cdots \text{Tr}(Y)}_{(m-3) \text{ times}} \text{Tr}(YXZ) \underbrace{\text{Tr}(Z) \cdots \text{Tr}(Z)}_{(n-3) \text{ times}}$$

where, writing the indices out, we have

$$(XYZ)_j^i = X_k^i Y_l^k Z_j^l .$$

The complex matrices X_k^i , Y_l^k , Z_j^l are operators acting on some N dimensional vector space V

$$X : V \rightarrow V \quad , \quad Y : V \rightarrow V \quad , \quad Z : V \rightarrow V$$

By tensoring p copies of X with m copies of Y and n copies Z , we obtain an operator

$$X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n} \equiv \underbrace{X \otimes \cdots \otimes X}_{p \text{ times}} \otimes \underbrace{Y \otimes \cdots \otimes Y}_{m \text{ times}} \otimes \underbrace{Z \otimes \cdots \otimes Z}_{n \text{ times}}$$

which acts on the vector space

$$V^{\otimes p+m+n} \equiv \underbrace{V \otimes V \otimes \cdots \otimes V}_{p+m+n \text{ copies}}.$$

The vector space V^{p+m+n} admits a natural action of the symmetric group S_{p+m+n} obtained by allowing $\sigma \in S_{p+m+n}$ to permute the factors of V in $V^{\otimes p+m+n}$. Using the notation

$$(X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n})_J^I = X_{j_1}^{i_1} \cdots X_{j_p}^{i_p} Y_{j_{p+1}}^{i_{p+1}} \cdots Y_{j_{p+m}}^{i_{p+m}} Z_{j_{p+m+1}}^{i_{p+m+1}} \cdots Z_{j_{p+m+n}}^{i_{p+m+n}}$$

we have, for $\sigma \in S_{p+m+n}$, the action

$$(\sigma)_J^I = \delta_{j_{\sigma(1)}}^{i_1} \cdots \delta_{j_{\sigma(p)}}^{i_p} \delta_{j_{\sigma(p+1)}}^{i_{p+1}} \cdots \delta_{j_{\sigma(p+m)}}^{i_{p+m}} \delta_{j_{\sigma(p+m+1)}}^{i_{p+m+1}} \cdots \delta_{j_{\sigma(p+m+n)}}^{i_{p+m+n}}. \quad (2.38)$$

The enlarged vector space V^{p+m+n} provides a unified way of thinking about arbitrary multitrace structures. In particular, a general operator takes the form

$$\sigma_J^I (X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n})_I^J = X_{i_{\sigma(1)}}^{j_1} \cdots X_{i_{\sigma(p)}}^{j_p} Y_{i_{\sigma(p+1)}}^{j_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{j_{p+m}} Z_{i_{\sigma(p+m+1)}}^{j_{p+m+1}} \cdots Z_{i_{\sigma(p+m+n)}}^{j_{p+m+n}}$$

which can also be written as

$$\text{Tr}(\sigma \cdot X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n})$$

with $\sigma \in S_{p+m+n}$. A particular choice of σ would correspond to a specific trace operator.

3 Dilatation Operator

The one loop dilatation operator in the sector we consider, is given by [26]

$$D = -g_{YM}^2 \text{Tr} \left([Y, Z] \left[\frac{d}{dY}, \frac{d}{dZ} \right] + [X, Z] \left[\frac{d}{dX}, \frac{d}{dZ} \right] + [Y, X] \left[\frac{d}{dY}, \frac{d}{dX} \right] \right). \quad (3.1)$$

To be completely explicit, the index structure is

$$\text{Tr} \left([Y, X] \left[\frac{d}{dY}, \frac{d}{dX} \right] \right) = (Y_l^i X_j^l - X_l^i Y_j^l) \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right). \quad (3.2)$$

Our first task is to consider the action of D on restricted Schur polynomials. In what follows we will often need the identity [17]

$$\text{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}) = \sum_{T, t, u, \vec{\nu}} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T, (t, u) \vec{\nu}^*}(\sigma^{-1}) \chi_{T, (t, u) \vec{\nu}}(Z, Y) \quad (3.3)$$

where if $\vec{\nu} = (\nu_1, \nu_2)$ then $\vec{\nu}^* = (\nu_2, \nu_1)$. With a suitable choice of σ , the right hand side above gives any desired multitrace operator. Thus, the above equation expresses an arbitrary multitrace operator as a linear combination of restricted Schur polynomials. The sum above runs over all Young diagrams $T \vdash m+n$, $t \vdash n$ and $u \vdash m$ as well as over the multiplicity labels $\vec{\nu}$. d_T denotes the dimension of the irreducible representation T of S_{n+m} . Similarly, d_t denotes the dimension of irreducible representation t of S_n and d_u the dimension of irreducible representation u of S_m . Finally, $\chi_{T, (t, u) \vec{\nu}^*}(\sigma^{-1})$ is the restricted character obtained by tracing $\Gamma_R(\sigma^{-1})$ over the (t, u) subspace, i.e. $\chi_{T, (t, u) \vec{\nu}^*}(\sigma^{-1}) = \text{Tr}_{(t, u) \vec{\nu}^*}(\Gamma_T(\sigma^{-1}))$. The multiplicity index $\vec{\nu}^* = (\nu_2, \nu_1)$ tells us to trace the row index over the ν_2 copy of (r, s) and the column index over the ν_1 copy. We will consider in detail the subleading term which mixes Y and X . The remaining terms can be evaluated in an identical way. A straight forward computation gives

$$\begin{aligned} & [Y, X]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right) \chi_{R, (t, s, r) \vec{\mu} \vec{\nu}} \\ = & [Y, X]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right) \frac{1}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t, s, r) \vec{\mu} \vec{\nu}}(\Gamma^R(\sigma)) X_{i_{\sigma(1)}}^{i_1} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} \\ & \times Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\ = & \frac{mp}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t, s, r) \vec{\mu} \vec{\nu}}(\Gamma^R(\sigma)) (\delta_{i_{\sigma(1)}}^{i_{p+1}} [Y, X]_{i_{\sigma(p+1)}}^{i_1} - \delta_{i_{\sigma(p+1)}}^{i_1} [Y, X]_{i_{\sigma(1)}}^{i_{p+1}}) \\ & \times X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+2)}}^{i_{p+2}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\ = & \frac{mp}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t, s, r) \vec{\mu} \vec{\nu}}(\Gamma^R([(1, p+1), \sigma])) \delta_{i_{\sigma(1)}}^{i_1} X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p)}}^{i_p} \\ & \times [Y, X]_{i_{\sigma(p+1)}}^{i_{p+1}} Y_{i_{\sigma(p+2)}}^{i_{p+2}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \end{aligned}$$

The delta function in the summand will restrict the sum over S_{n+m+p} to a sum over the $S_{n+m+p-1}$ subgroup. The $S_{n+m+p-1}$ subgroup is obtained by retaining those elements that

hold i_1 inert, i.e. $\sigma(1) = 1$. To see how this happens, introduce the notation $\rho_i = \sigma(i, 1)$ and rewrite the above sum as a sum over $S_{n+m+p-1}$ and its cosets. The result is

$$\begin{aligned}
& [Y, X]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right) \chi_{R, (t, s, r) \bar{\mu} \bar{\nu}} \\
&= \frac{mp}{n! m! p!} \sum_{\sigma \in S_{n+m+p-1}} \sum_{i=1}^{n+m+p} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}(\Gamma^R([(1, p+1), \rho_i])) \delta_{i_{\rho_i(1)}}^{i_1} X_{i_{\rho_i(2)}}^{i_2} \cdots X_{i_{\rho_i(p)}}^{i_p} \\
&\quad \times [Y, X]_{i_{\rho_i(p+1)}}^{i_{p+1}} Y_{i_{\rho_i(p+2)}}^{i_{p+2}} \cdots Y_{i_{\rho_i(p+m)}}^{i_{p+m}} Z_{i_{\rho_i(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\rho_i(m+p+n)}}^{i_{m+p+n}} \\
&= \frac{mp}{n! m! p!} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}(\Gamma^R([(1, p+1), \{N + \sum_{i=2}^{n+m+p} (i, 1)\}])) \\
&\quad \times \text{Tr}_{V^{\otimes n+m+p-1}}(\sigma \cdot X^{\otimes p-1} \otimes [Y, X] \otimes Y^{\otimes m-1} \otimes Z^{\otimes n}) \\
&= \frac{mp}{n! m! p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}([\Gamma^R((1, p+1)), \Gamma^{R'}(\sigma)]) \\
&\quad \times \text{Tr}_{V^{\otimes n+m+p-1}}(\sigma \cdot X^{\otimes p-1} \otimes [Y, X] \otimes Y^{\otimes m-1} \otimes Z^{\otimes n})
\end{aligned}$$

The sum over R' runs over all irreducible representations R' of the $S_{n+m+p-1}$ subgroup that can be subduced from the irreducible representation R of the S_{n+m+p} subgroup. As a Young diagram R' is obtained from R by dropping a single box. A prime on a letter denoting a Young diagram will always indicate that we drop a box. To obtain the last line above, use the fact that $N + \sum_{i=2}^{n+m+p} (i, 1)$ when acting on any state within the subspace R' subduced by R gives $c_{RR'}$. This is proved by noting that $\sum_{i=2}^{n+m+p} (i, 1)$ is a Jucys-Murphy element; see [14] for the details.

$$\begin{aligned}
& [Y, X]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right) \chi_{R, (t, s, r) \bar{\mu} \bar{\nu}} \\
&= \frac{mp}{n! m! p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}([\Gamma^R((1, p+1)), \Gamma^{R'}(\sigma)]) \\
&\quad \times \text{Tr}_{V^{\otimes n+m+p-1}}([(1, p+1), \sigma] \cdot X^{\otimes p} \otimes Y^{\otimes m} \otimes Z^{\otimes n}) \\
&= \frac{mp}{n! m! p!} \sum_{T, (y, x, w) \bar{\alpha} \bar{\beta}} \frac{d_T n! m! p!}{d_w d_x d_y (n+m+p)!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}([\Gamma^R((1, p+1)), \Gamma^{R'}(\sigma)]) \\
&\quad \text{Tr}_{(y, x, w) \bar{\alpha} \bar{\beta}}(\Gamma_T([(1, p+1), \sigma])) \chi_{T, (y, x, w) \bar{\beta} \bar{\alpha}}(X, Y, Z) \\
&= \sum_{R'} c_{RR'} \sum_{T, (y, x, w) \bar{\alpha} \bar{\beta}} \frac{d_T mp}{d_w d_x d_y (n+m+p) d_R} \\
&\quad \text{Tr}_{R \oplus T}([P_{R, (t, s, r) \bar{\mu} \bar{\nu}}, \Gamma^R(1, p+1)] I_{R' T'} [P_{T, (y, x, w) \bar{\alpha} \bar{\beta}}, \Gamma^T(1, p+1)] I_{T' R'}) \chi_{T, (y, x, w) \bar{\beta} \bar{\alpha}}(X, Y, Z).
\end{aligned}$$

To get to the last line sum over $S_{n+m+p-1}$ using the fundamental orthogonality relation. Now consider the second term, which is treated in exactly the same way

$$\begin{aligned}
& [Y, Z]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dZ_k^i} - \frac{d}{dZ_j^k} \frac{d}{dY_k^i} \right) \frac{1}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}(\Gamma^R(\sigma)) X_{i_{\sigma(1)}}^{i_1} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} \\
&\quad \times Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
&= \frac{mn}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t, s, r) \bar{\mu} \bar{\nu}}(\Gamma^R(\sigma)) (\delta_{i_{\sigma(p+1)}}^{i_{p+1}} [Y, Z]_{i_{\sigma(p+1)}}^{i_{p+1}} - \delta_{i_{\sigma(p+1)}}^{i_{p+1}} [Y, Z]_{i_{\sigma(p+1)}}^{i_{p+1}})
\end{aligned}$$

$$\begin{aligned}
& \times X_{i_{\sigma(1)}}^{i_1} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+2)}}^{i_{p+2}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+2)}}^{i_{m+p+2}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{mn}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R((1,p+1)\sigma(1,p+1)))(\delta_{i_{\sigma(p+m+1)}}^{i_1} [Y, Z]_{i_{\sigma(1)}}^{i_{p+m+1}} - \delta_{i_{\sigma(1)}}^{i_{p+m+1}} [Y, Z]_{i_{\sigma(p+m+1)}}^{i_1}) \\
& \times X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p+1)}}^{i_{p+1}} Y_{i_{\sigma(p+2)}}^{i_{p+2}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+2)}}^{i_{m+p+2}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{mn}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R((1,p+1)[\sigma, (1,p+m+1)](1,p+1))\delta_{i_{\sigma(1)}}^{i_1} [Y, Z]_{i_{\sigma(p+m+1)}}^{i_{p+m+1}} \\
& \times X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p+1)}}^{i_{p+1}} Y_{i_{\sigma(p+2)}}^{i_{p+2}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+2)}}^{i_{m+p+2}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{mn}{n! m! p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R(1,p+1)[\Gamma^{R'}(\sigma), \Gamma^R(1,p+m+1)]\Gamma^R(1,p+1)) \\
& \times \text{Tr}_{V^{\otimes n+m+p-1}}([\sigma, (p+1, p+m+1)] \cdot X^{\otimes p} \otimes Y^{\otimes m-1} \otimes [Y, Z] \otimes Z^{\otimes n-1}) \\
= & \frac{mn}{n! m! p!} \sum_{T, (y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T n! m! p!}{d_w d_x d_y (n+m+p)!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \\
& \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R(1,p+1)[\Gamma^{R'}(\sigma), \Gamma^R(1,p+m+1)]\Gamma^R(1,p+1)) \\
& \times \text{Tr}_{(y,x,w)\bar{\alpha}\bar{\beta}}(\Gamma_T((1,p+1)[\sigma, (1,p+m+1)](1,p+1)))\chi_{T, (y,x,w)\bar{\beta}\bar{\alpha}}(X, Y, Z) \\
= & \sum_{R'} c_{RR'} \sum_{T, (y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T mn}{d_w d_x d_y (n+m+p) d_{R'}} \\
& \text{Tr}_{R \oplus T}([\Gamma^R(1,p+1)P_{R, (t,s,r)\bar{\mu}\bar{\nu}}\Gamma^R(1,p+1), \Gamma^R(1,p+m+1)]I_{R'T'}) \\
& \times [\Gamma^T(1,p+1)P_{T, (y,x,w)\bar{\alpha}\bar{\beta}}\Gamma^T(1,p+1), \Gamma^T(1,p+m+1)]I_{T'R'}\chi_{T, (y,x,w)\bar{\beta}\bar{\alpha}}(X, Y, Z)
\end{aligned}$$

Finally, consider the third term

$$\begin{aligned}
& [X, Z]_j^i \left(\frac{d}{dX_j^k} \frac{d}{dZ_k^i} - \frac{d}{dZ_j^k} \frac{d}{dX_k^i} \right) \frac{1}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R(\sigma)) X_{i_{\sigma(1)}}^{i_1} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} \\
& \times Z_{i_{\sigma(m+p+1)}}^{i_{m+p+1}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{pn}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R(\sigma)) (\delta_{i_{\sigma(p+m+1)}}^{i_1} [X, Z]_{i_{\sigma(1)}}^{i_{p+m+1}} - \delta_{i_{\sigma(1)}}^{i_{p+m+1}} [X, Z]_{i_{\sigma(p+m+1)}}^{i_1}) \\
& \times X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+2)}}^{i_{m+p+2}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{pn}{n! m! p!} \sum_{\sigma \in S_{n+m+p}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}(\Gamma^R([\sigma, (1,p+m+1)])) \delta_{i_{\sigma(1)}}^{i_1} [X, Z]_{i_{\sigma(p+m+1)}}^{i_{p+m+1}} \\
& \times X_{i_{\sigma(2)}}^{i_2} \cdots X_{i_{\sigma(p)}}^{i_p} Y_{i_{\sigma(p+1)}}^{i_{p+1}} \cdots Y_{i_{\sigma(p+m)}}^{i_{p+m}} Z_{i_{\sigma(m+p+2)}}^{i_{m+p+2}} \cdots Z_{i_{\sigma(m+p+n)}}^{i_{m+p+n}} \\
= & \frac{pn}{n! m! p!} \sum_{R'} c_{RR'} \sum_{\sigma \in S_{n+m+p-1}} \text{Tr}_{(t,s,r)\bar{\mu}\bar{\nu}}([\Gamma^{R'}(\sigma), \Gamma^R(1,p+m+1)]) \\
& \times \text{Tr}_{V^{\otimes m+n+p-1}}(\sigma \cdot X^{\otimes p-1} \otimes Y^{\otimes m} \otimes [X, Z] \otimes Z^{\otimes n-1}) \\
= & \frac{pn}{n! m! p!} \sum_{R'} c_{RR'} \sum_{T, (y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T n! m! p!}{d_w d_x d_y (n+m+p)!} \\
& \text{Tr}_{R \oplus T}([P_{R, (t,s,r)\bar{\mu}\bar{\nu}}, \Gamma^R(1,p+m+1)]I_{R'T'} [P_{T, (y,x,w)\bar{\alpha}\bar{\beta}}, \Gamma^T(1,p+m+1)]I_{T'R'}) \chi_{T, (y,x,w)\bar{\beta}\bar{\alpha}}(X, Y, Z) \\
= & \sum_{R'} c_{RR'} \sum_{T, (y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T np}{d_w d_x d_y (n+m+p) d_{R'}}
\end{aligned}$$

$$\text{Tr}_{R \oplus T}([P_{R,(t,s,r)\bar{\mu}\bar{\nu}}, \Gamma^R(1, p + m + 1)]I_{R'T'}[P_{T,(y,x,w)\bar{\alpha}\bar{\beta}}, \Gamma^T(1, p + m + 1)]I_{T'R'})\chi_{T,(y,x,w)\bar{\beta}\bar{\alpha}}(X, Y, Z)$$

$$\begin{aligned} & [Y, X]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dX_k^i} - \frac{d}{dX_j^k} \frac{d}{dY_k^i} \right) \hat{O}_{R,(t,s,r)\bar{\mu}\bar{\nu}} \\ &= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T m p}{d_w d_x d_y (n + m + p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_w \text{hooks}_x \text{hooks}_y}} \\ & \text{Tr}_{R \oplus T}([P_{R,(t,s,r)\bar{\mu}\bar{\nu}}, \Gamma^R(1, p + 1)]I_{R'T'}[P_{T,(y,x,w)\bar{\alpha}\bar{\beta}}, \Gamma^T(1, p + 1)]I_{T'R'}) \hat{O}_{T,(y,x,w)\bar{\beta}\bar{\alpha}}. \end{aligned} \quad (3.4)$$

Using identical methods it is straight forward to find

$$\begin{aligned} & [Y, Z]_j^i \left(\frac{d}{dY_j^k} \frac{d}{dZ_k^i} - \frac{d}{dZ_j^k} \frac{d}{dY_k^i} \right) \hat{O}_{R,(t,s,r)\bar{\mu}\bar{\nu}} \\ &= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T m n}{d_w d_x d_y (n + m + p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_w \text{hooks}_x \text{hooks}_y}} \\ & \quad \text{Tr}_{R \oplus T}([\Gamma^R(1, p + 1)P_{R,(t,s,r)\bar{\mu}\bar{\nu}}\Gamma^R(1, p + 1), \Gamma^R(1, p + m + 1)]I_{R'T'} \\ & \quad \times [\Gamma^T(1, p + 1)P_{T,(y,x,w)\bar{\alpha}\bar{\beta}}\Gamma^T(1, p + 1), \Gamma^T(1, p + m + 1)]I_{T'R'}) \hat{O}_{T,(y,x,w)\bar{\beta}\bar{\alpha}}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} & [X, Z]_j^i \left(\frac{d}{dX_j^k} \frac{d}{dZ_k^i} - \frac{d}{dZ_j^k} \frac{d}{dX_k^i} \right) \hat{O}_{R,(t,s,r)\bar{\mu}\bar{\nu}} \\ &= \sum_{R'} c_{RR'} \sum_{T,(y,x,w)\bar{\alpha}\bar{\beta}} \frac{d_T p n}{d_w d_x d_y (n + m + p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_w \text{hooks}_x \text{hooks}_y}} \\ & \text{Tr}_{R \oplus T}([P_{R,(t,s,r)\bar{\mu}\bar{\nu}}, \Gamma^R(1, p + m + 1)]I_{R'T'}[P_{T,(y,x,w)\bar{\alpha}\bar{\beta}}, \Gamma^T(1, p + m + 1)]I_{T'R'}) \hat{O}_{T,(y,x,w)\bar{\beta}\bar{\alpha}}. \end{aligned} \quad (3.6)$$

The next step in the evaluation of the action of the dilatation operator entails computing the traces over $R \oplus T$ that have appeared in our results above. Our results for the action of the one loop dilatation operator given above are exact. From this point on we assume the displaced corners approximation so that our answers for the traces are only valid in the large N limit. The reader wanting to follow all of the details in this section should consult [20] for background. For the term in the one loop dilatation operator that mixes X and Y the trace that needs to be computed is

$$\mathbb{T} = \text{Tr}_{R \oplus T}([P_{R,(t,s,r)\bar{\mu}\bar{\nu}}, \Gamma^R(1, p + 1)]I_{R'T'}[P_{T,(y,x,w)\bar{\alpha}\bar{\beta}}, \Gamma^T(1, p + 1)]I_{T'R'}). \quad (3.7)$$

To ease the notation we will use the following shorthand

$$P_{T,(y,x,w)\bar{\alpha}\bar{\beta}} \equiv p_y \otimes p_x \otimes \mathbf{1}_w. \quad (3.8)$$

Consider the case that R' is obtained from R by dropping a box in row i and that T' is obtained from T by dropping a box from row j . The intertwiner is only non-zero if $T' = R'$. In this case the intertwiners are

$$I_{R'T'} = E_{ij}^{(1)}, \quad I_{T'R'} = E_{ji}^{(1)}. \quad (3.9)$$

Since the trace \mathbb{T} is a product of two commutators, when we expand things out we get a total of four terms. Since both the swaps $\Gamma^R(1, p+1)$ and $\Gamma^T(1, p+1)$ have a trivial action on the Z indices, we know that the result will be proportional to δ_{rw} and that the trace over the Z indices produce a factor d_r . Thus, after tracing over the Z indices we have

$$\begin{aligned} \mathbb{T} = & \left(\text{Tr}(p_t \otimes p_s \Gamma^R(1, p+1) E_{ij}^{(1)} p_y \otimes p_x \Gamma^T(1, p+1) E_{ji}^{(1)}) \right. \\ & - \text{Tr}(p_t \otimes p_s \Gamma^R(1, p+1) E_{ij}^{(1)} \Gamma^T(1, p+1) p_y \otimes p_x E_{ji}^{(1)}) \\ & - \text{Tr}(\Gamma^R(1, p+1) p_t \otimes p_s E_{ij}^{(1)} p_y \otimes p_x \Gamma^T(1, p+1) E_{ji}^{(1)}) \\ & \left. + \text{Tr}(\Gamma^R(1, p+1) p_t \otimes p_s E_{ij}^{(1)} \Gamma^T(1, p+1) p_y \otimes p_x E_{ji}^{(1)}) \right) \delta_{rw} d_r. \end{aligned} \quad (3.10)$$

Allow the swaps to act on the intertwiners

$$(1, p+1) E_{ij}^{(1)} = E_{lj}^{(1)} E_{il}^{(p+1)}, \quad (1, p+1) E_{ji}^{(1)} = E_{li}^{(1)} E_{jl}^{(p+1)} \quad (3.11)$$

to obtain

$$\begin{aligned} \mathbb{T} = & \left(\langle \vec{p}, t, \nu_1; a | E_{lj}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ki}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\ & \times \langle \vec{m}, s, \nu_2; c | E_{il}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{jk}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\ & - \langle \vec{p}, t, \nu_1; a | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\ & \times \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | \vec{m}, s, \mu_2; c \rangle \\ & - \langle \vec{p}, t, \nu_1; a | E_{ij}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | \vec{p}, t, \mu_1; a \rangle \\ & \times \langle \vec{m}, s, \nu_2; c | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{ji}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\ & + \langle \vec{p}, t, \nu_1; a | E_{il}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{jk}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\ & \times \langle \vec{m}, s, \nu_2; c | E_{lj}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{ki}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \left. \right) \delta_{rw} d_r \\ & (3.12) \\ = & \left(\langle \vec{p}, t, \nu_1; a | E_{lj}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ki}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\ & \times \langle \vec{m}, s, \nu_2; c | E_{il}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{jk}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\ & - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\nu_1 \alpha_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\beta_2 \mu_2} \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\ & \times \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \\ & - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\mu_1 \beta_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\alpha_2 \nu_2} \langle \vec{p}, t, \nu_1; a | E_{ij}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \\ & \times \langle \vec{m}', x, \beta_2; d | E_{ji}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\ & + \langle \vec{p}, t, \nu_1; a | E_{il}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{jk}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\ & \times \langle \vec{m}, s, \nu_2; c | E_{lj}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{ki}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \left. \right) \delta_{rw} d_r. \end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
& \text{Tr}_{R \oplus T}([\Gamma^R(1, p+1)P_{R,(t,s,r)}\bar{\mu}\bar{\nu}\Gamma^R(1, p+1), \Gamma^R(1, p+m+1)]I_{R'T'}) \\
& \times [\Gamma^T(1, p+1)P_{T,(y,x,w)}\bar{\alpha}\bar{\beta}\Gamma^T(1, p+1), \Gamma^T(1, p+m+1)]I_{T'R'} \\
& = \delta_{ty}d_t\delta_{\bar{p}\bar{p}'}\delta_{\nu_1\alpha_1}\delta_{\beta_1\mu_1}d_{r'_i}\delta_{r'_i w'_j}\delta_{\vec{m}\vec{m}'}\left[\langle\vec{m}', x, \beta_2; d|E_{ii}^{(p+1)}|\vec{m}, s, \mu_2; c\rangle\langle\vec{m}, s\nu_2; c|E_{jj}^{(p+1)}|\vec{m}', x, \alpha_2; d\rangle\right. \\
& \left. + \langle\vec{m}', x, \beta_2; d|E_{jj}^{(p+1)}|\vec{m}, s, \mu_2; c\rangle\langle\vec{m}, s\nu_2; c|E_{ii}^{(p+1)}|\vec{m}', x, \alpha_2; d\rangle\right] \\
& - \delta_{ij}\delta_{ty}d_t\delta_{\bar{p}\bar{p}'}\delta_{\nu_1\alpha_1}\delta_{\beta_1\mu_1}d_{r'_i}\delta_{r_w}\delta_{sx}\delta_{\vec{m}\vec{m}'}\left[\delta_{\nu_2\alpha_2}\langle\vec{m}, x, \beta_2; c|E_{jj}^{(p+1)}|\vec{m}, s, \mu_2; c\rangle\right. \\
& \left. + \delta_{\beta_2\mu_2}\langle\vec{m}, s\nu_2; c|E_{ii}^{(p+1)}|\vec{m}, x, \alpha_2; c\rangle\right]
\end{aligned} \tag{3.13}$$

relevant for the term in the one loop dilatation operator that mixes Z and Y and

$$\begin{aligned}
& \text{Tr}_{R \oplus T}([P_{R,(t,s,r)}\bar{\mu}\bar{\nu}, \Gamma^R(1, p+m+1)]I_{R'T'}[P_{T,(y,x,w)}\bar{\alpha}\bar{\beta}, \Gamma^T(1, p+m+1)]I_{T'R'}) \\
& = \delta_{sx}d_s\delta_{\bar{p}\bar{p}'}\delta_{\nu_2\alpha_2}\delta_{\beta_2\mu_2}d_{r'_i}\delta_{r'_i w'_j}\delta_{\vec{m}\vec{m}'}\left[\langle\vec{p}', y, \beta_1; d|E_{ii}^{(1)}|\vec{p}, t, \mu_1; c\rangle\langle\vec{p}, t, \nu_1; c|E_{jj}^{(1)}|\vec{p}', y, \alpha_1; d\rangle\right. \\
& \left. + \langle\vec{p}', y, \beta_1; d|E_{jj}^{(1)}|\vec{p}, t, \mu_1; c\rangle\langle\vec{p}, t, \nu_1; c|E_{ii}^{(1)}|\vec{p}', y, \alpha_1; d\rangle\right] \\
& - \delta_{ij}\delta_{ty}d_s\delta_{\bar{p}\bar{p}'}\delta_{\nu_2\alpha_2}\delta_{\beta_2\mu_2}d_{r'_i}\delta_{r_w}\delta_{sx}\delta_{\vec{m}\vec{m}'}\left[\delta_{\nu_1\alpha_1}\langle\vec{p}, y, \beta_1; c|E_{jj}^{(1)}|\vec{p}, t, \mu_1; c\rangle\right. \\
& \left. + \delta_{\beta_1\mu_1}\langle\vec{p}, t, \nu_1; c|E_{ii}^{(1)}|\vec{p}, y, \alpha_1; c\rangle\right]
\end{aligned} \tag{3.14}$$

which is relevant for the term in the one loop dilatation operator that mixes X and Z .

This completes our discussion of the action of the one loop dilatation operator.

4 Gauss Operators

The problem of diagonalizing the terms in the dilatation operator that mix the X and Z fields and the terms that mix the Y and Z fields has been solved [6, 7, 12, 20]. The operators that have a good scaling dimension are the Gauss operators. Our ultimate goal is to write the action of the terms in the dilatation operator that mix X and Y fields, on the Gauss operators, which amounts to a change of basis from restricted Schur polynomials to Gauss operators. Towards this end we describe how to construct Gauss operators for operators built from three complex scalar fields and develop the tools we will need to change basis. The results of this section are a simple generalization of [12].

Natural hints for the construction of the Gauss operators come from the AdS/CFT correspondence. Indeed, the correspondence implies an equivalence between quantum states in the quantum gravity and operators in the $\mathcal{N} = 4$ super-Yang-Mills theory. In particular, the restricted Schur polynomials $\chi_{R,(t,s,r),\vec{\mu}\vec{\nu}}(X, Y, Z)$ are dual to multiple giant graviton systems [27, 28, 29] consisting of large branes in the AdS_5 space when R has order one rows each of length order N , or to systems consisting of large branes in the S^5 space when R has order one columns each of length order N . A giant graviton has a compact world volume so that the Gauss Law forces the total charge on the giant's world volume to vanish. Since the string end points are charged, this gives a constraint on the possible open string configurations that are allowed: the number of strings leaving the giant must equal the number of strings arriving at the giant. The matrices X and Y generate two species of 1-bit strings [30, 31, 32, 33]. Each row of R corresponds to a giant graviton. Each open string configuration corresponds to a pair of graphs - one for each open string species. We will refer to these as the X graph and the Y graph. The vertices of the graph represent the branes and the directed links represent the (oriented) strings. Motivated by [34] a useful combinatoric description of these graphs is to divide each string into two halves and label each half. Using the orientation of the string, label the outgoing ends with numbers $\{1, \dots, p\}$ for the X graph or $\{1, \dots, m\}$ for the Y graph and the ingoing ends with these same numbers. A permutation $\sigma \in S_p \times S_m$ is then determined by how the halves are joined. We will often decompose $\sigma = \sigma^X \circ \sigma^Y$ with $\sigma^X \in S_p$ and $\sigma^Y \in S_m$. Given a permutation, we can reconstruct the graphs. A graph is not associated to a unique permutation because the strings leaving the i 'th vertex are indistinguishable, and the strings arriving at the i 'th vertex are indistinguishable.

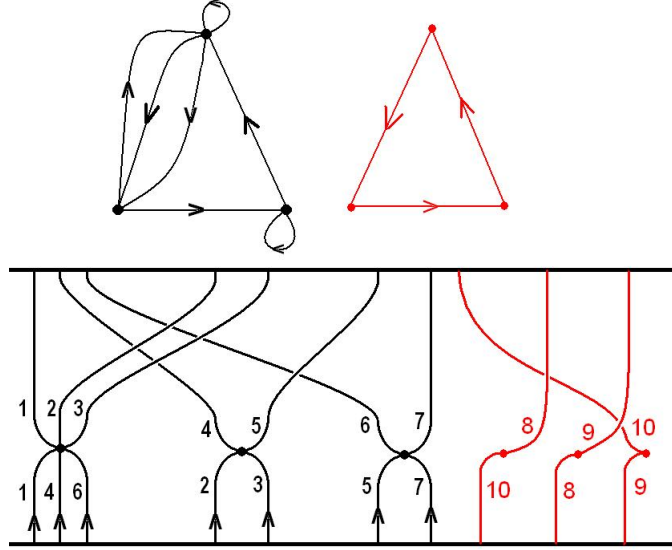


Figure 3: Any open string configuration can be mapped to a pair of labeled graphs. The black graph describes the X matrices and the red graph the Y matrices. The two bold horizontal lines are identified. The graphs determine a permutation, so each open string configuration is mapped to a permutation. For the graph shown the permutation in cycle notation is $\sigma = (2, 4)(5, 3, 6)(8, 10, 9)$. The figure shows a configuration for a three giant system with ten open strings attached. Equivalently, this is an operator whose Young diagram describing the Z fields has 3 long rows/columns and $p = 7$, $m = 3$. The vectors \vec{p} and \vec{m} describe the number of strings leaving each node. Thus, $\vec{p} = (3, 2, 2)$, $\vec{m} = (1, 1, 1)$.

We will make use of two subgroups in what follows

$$H_Y = S_{m_1} \times \cdots \times S_{m_g} \quad H_X = S_{p_1} \times \cdots \times S_{p_g}. \quad (4.1)$$

H_X acts on boxes in the partly labeled Young diagrams that are labeled with an integer $i < p + 1$, i.e. on the boxes associated to X s. H_Y acts on boxes associated to Y s. These two subgroups leave all partly labeled Young diagrams invariant. Consequently, the partly labeled Young diagrams belong to $S_p \times S_m / H_X \times H_Y$. The Gauss graphs themselves are in one-to-one correspondence with elements of the double coset

$$H_X \times H_Y \setminus S_p \times S_m / H_X \times H_Y \quad (4.2)$$

Introduce the states (these states span $V^{\otimes p+m}$)

$$|v, \vec{p}, \vec{m}\rangle \equiv |v_1^{\otimes p_1} \otimes v_2^{\otimes p_2} \otimes \cdots \otimes v_g^{\otimes p_g} \otimes v_{g+1}^{\otimes m_1} \otimes v_{g+2}^{\otimes m_2} \otimes \cdots \otimes v_{2g}^{\otimes m_g}\rangle. \quad (4.3)$$

There is an action of the $S_p \times S_m$ group defined on this space by

$$\sigma |v_{i_1} \otimes \cdots \otimes v_{i_{m+p}}\rangle = |v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(m+p)}}\rangle. \quad (4.4)$$

This can trivially be enlarged to obtain an action of S_{p+m} , but we want to consider only permutations that mix X indices with each other and Y indices with each other, but not X and Y indices. Introduce the notation $|v_\sigma\rangle \equiv \sigma|v, \vec{p}, \vec{m}\rangle$. Invariance under the $H_X \times H_Y$ subgroup can be written as

$$|v_\sigma\rangle = |v_{\sigma\gamma}\rangle \quad \gamma \in H_X \times H_Y \quad (4.5)$$

or even

$$|v_\sigma\rangle = \frac{1}{|H_X \times H_Y|} \sum_{\gamma \in H_X \times H_Y} |v_{\sigma\gamma}\rangle. \quad (4.6)$$

Recall that the operator that projects onto representation r of a group \mathcal{G} is given by [35]

$$P_r = \frac{d_r}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi_r(g) g. \quad (4.7)$$

By the identity representation we mean the representation for which all the elements of $H_X \times H_Y$ are represented by 1. We want to project onto the identity representation of $H_X \times H_Y$ within the carrier space (s, t) organizing the X s and Y s. Recall that $t \vdash p$ and $s \vdash m$. The characters in the identity representation are of course all equal to 1. The identity representation may appear more than once in (s, t) . Resolve these different copies with a multiplicity label $\vec{\mu}$. The multiplicity label has two components, one that refers to s and one that refers to t . Introduce branching coefficients that resolve these projectors into a set of projectors onto each of the one dimensional spaces labeled by $\vec{\mu}$

$$\frac{1}{|H_X \times H_Y|} \sum_{\gamma \in H_X \times H_Y} \Gamma^{(s,t)}(\gamma)_{ik} = \sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\vec{\mu}}^{s \rightarrow 1_{H_X \times H_Y}}. \quad (4.8)$$

Thus, for example, $B_{i\vec{\nu}}^{(s,t) \rightarrow 1_H} B_{k\vec{\nu}}^{(s,t) \rightarrow 1_H}$ projects onto the copy ν_1 of the identity representation of H_X in s and onto the copy ν_2 of the identity representation of H_Y in t . The branching coefficient $B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}}$ can be understood as the one dimensional vector that spans the $\vec{\nu}$ copy of $1_{H_X \times H_Y}$ inside the carrier space of (s, t)

$$|\vec{\nu}\rangle_i = B_{i\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}}. \quad (4.9)$$

Vector orthogonality says

$$\langle \vec{\nu} | \vec{\mu} \rangle = \delta_{\vec{\mu}\vec{\nu}} = \sum_i B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{i\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \quad (4.10)$$

whilst vector completeness says

$$\sum_{\vec{\mu}} |\vec{\mu}\rangle \langle \vec{\mu}| = 1_{H_X \times H_Y} \quad (4.11)$$

or, displaying all indices,

$$\sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} = (1_{H_X \times H_Y})_{ij}. \quad (4.12)$$

Together (4.10) and (4.12) allow us to think of the branching coefficients $B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}}$ as a matrix that implements a change of basis

$$|i\rangle = \sum_{\vec{\mu}} B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} |\vec{\mu}\rangle \quad |\vec{\mu}\rangle = \sum_i B_{i\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} |i\rangle. \quad (4.13)$$

We are now ready to argue that the Gauss operators are simply an alternative basis to the restricted Schur polynomials. First, following [12], we will show that the number of restricted Schur polynomials is equal to the number of Gauss operators. Towards this end consider

$$|v_{(s,t)}, i, j\rangle = \sum_{\sigma \in S_m \times S_p} \Gamma_{ij}^{(s,t)}(\sigma) \sigma |v, \vec{p}, \vec{m}\rangle. \quad (4.14)$$

Above we have projected onto the representation (s, t) of $S_m \times S_p$. You can think of j as a label for different vectors and of i as the components of the vector. According to [20] this space is organized by the Schur-Weyl duality between $S_m \times S_p$ and $U(m) \times U(p)$. Concretely, we can trade the index j for a Gelfand-Tsetlin pattern. Thus, using [20] we know that we can decompose this space as

$$\begin{aligned} V_g^{\otimes p+m} &= \bigoplus_{\substack{s+m \ t+p \\ c_1(s) \leq g \ c_1(t) \leq g}} V_{(s,t)}^{U(m) \times U(p)} \otimes V_{(s,t)}^{S_m \times S_p} \\ &= \bigoplus_{\substack{s+m \ t+p \\ c_1(s) \leq g \ c_1(t) \leq g}} \bigoplus_{\vec{m} \ \vec{p}} V_{(s,t) \rightarrow (\vec{m}, \vec{p})}^{U(g) \times U(g) \rightarrow U(1)^g \times U(1)^g} \otimes V_{(s,t)}^{S_m \times S_p} \end{aligned} \quad (4.15)$$

The first factor in the last line above is the space of Gelfand-Tsetlin patterns. Now, lets consider a different decomposition of this space, as follows [12]

$$\begin{aligned} |v_{(s,t)}, i, j\rangle &= \sum_{\sigma \in S_m \times S_p} \Gamma_{ij}^{(s,t)}(\sigma) \sigma |v, \vec{p}, \vec{m}\rangle \\ &= \sum_{\sigma \in S_m \times S_p} \Gamma_{ij}^{(s,t)}(\sigma) |v_\sigma\rangle \\ &= \frac{1}{|H_X \times H_Y|} \sum_{\sigma \in S_m \times S_p} \sum_{\gamma \in H_X \times H_Y} \Gamma^{(s,t)}(\sigma \gamma)_{ij} |v_\sigma\rangle \\ &= \frac{1}{|H_X \times H_Y|} \sum_{\sigma \in S_m \times S_p} \sum_{\gamma \in H_X \times H_Y} \Gamma^{(s,t)}(\sigma)_{ik} \Gamma^{(s,t)}(\gamma)_{kj} |v_\sigma\rangle \\ &= \sum_{\sigma \in S_m \times S_p} \Gamma^{(s,t)}(\sigma)_{ik} \sum_{\vec{\mu}} B_{k\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} |v_\sigma\rangle. \end{aligned} \quad (4.16)$$

As we have already discussed, the branching coefficients provide a natural change of basis from one space to the other

$$|\vec{m}, \vec{p}, (s, t), \vec{\mu}; i\rangle = \sum_j B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \sum_{\sigma \in S_m \times S_p} \Gamma_{ij}^{(s,t)}(\sigma) \sigma |v_\sigma\rangle. \quad (4.17)$$

This decomposition is [12]

$$\begin{aligned}
V_g^{\otimes p+m} &= \bigoplus_{\substack{s \vdash m \quad t \vdash p \\ c_1(s) \leq g \quad c_1(t) \leq g}} V_{(s,t)}^{U(m) \times U(p)} \otimes V_{(s,t)}^{S_m \times S_p} \\
&= \bigoplus_{\substack{s \vdash m \quad t \vdash p \\ c_1(s) \leq g \quad c_1(t) \leq g}} \bigoplus_{\vec{m} \vec{p}} V_{(s,t) \rightarrow 1_{H_X \times H_Y}}^{S_m \times S_p \rightarrow H_X \times H_Y} \otimes V_{(s,t)}^{S_m \times S_p}.
\end{aligned} \tag{4.18}$$

Comparing (4.15) to (4.18) we conclude that

$$|V_{(s,t) \rightarrow (\vec{m}, \vec{p})}^{U(g) \times U(g) \rightarrow U(1)^g \times U(1)^g}| = |V_{(s,t) \rightarrow 1_{H_X \times H_Y}}^{S_m \times S_p \rightarrow H_X \times H_Y}|. \tag{4.19}$$

Using the idea that the branching coefficients provide a transformation between two bases, we easily write the Gauss operators

$$\begin{aligned}
O_{R,r}(\sigma_X, \sigma_Y) &= \frac{|H_X \times H_Y|}{\sqrt{m! p!}} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\vec{\mu}} \sum_{\vec{\nu}} \sqrt{d_s d_t} \Gamma^{(s,t)}(\sigma_X \circ \sigma_Y)_{jk} \\
&\quad \times B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} O_{R,(t,s,r)\vec{\mu}\vec{\nu}}.
\end{aligned} \tag{4.20}$$

Note that the factor $\sqrt{d_s d_t}$ can not be determined by group theory alone. It is chosen so that the group theoretic coefficients

$$C_{\vec{\mu}\vec{\nu}}^{(s,t)}(\sigma_X \circ \sigma_Y) = \frac{|H_X \times H_Y|}{\sqrt{m! p!}} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sqrt{d_s d_t} \Gamma^{(s,t)}(\sigma_X \circ \sigma_Y)_{jk} B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \tag{4.21}$$

provide an orthogonal transformation between the restricted Schur polynomials and the Gauss graph basis. Indeed,

$$\sum_{(s,t)} \sum_{\vec{\mu}} \sum_{\vec{\nu}} C_{\vec{\mu}\vec{\nu}}^{(s,t)}(\sigma_X \circ \sigma_Y) C_{\vec{\mu}\vec{\nu}}^{(s,t)}(\tau_X \circ \tau_Y) = \sum_{\gamma \in H_X \times H_Y} \delta(\gamma_1 \sigma_X \circ \sigma_Y \gamma_2 \tau_X^{-1} \circ \tau_Y^{-1}). \tag{4.22}$$

There is an important point that is worth stressing here: Our Gauss operators are normalized as

$$\langle O_{R,r}(\sigma_X, \sigma_Y) O_{R,r}(\tau_X, \tau_Y)^\dagger \rangle = \sum_{\gamma \in H_X \times H_Y} \delta(\gamma_1 \sigma_X \circ \sigma_Y \gamma_2 \tau_X^{-1} \circ \tau_Y^{-1}). \tag{4.23}$$

These operators certainly do not have unit two point function. For example, if we set both σ_X, σ_Y and τ_X, τ_Y equal to the identity permutation, the right hand side evaluates to $|H_X \times H_Y|$. Our final answer is simplest when expressed in terms of normalized operators

$$\hat{O}_{R,r}(\sigma_X, \sigma_Y) \equiv \frac{1}{N_{\sigma_X, \sigma_Y}} O_{R,r}(\sigma_X, \sigma_Y), \tag{4.24}$$

$$N_{\sigma_X, \sigma_Y}^2 = \langle O_{R,r}(\sigma_X, \sigma_Y) O_{R,r}(\sigma_X, \sigma_Y)^\dagger \rangle. \tag{4.25}$$

We will not obtain or need the explicit form of N_{σ_X, σ_Y} .

5 Dilatation Operator in the Gauss Graph Basis

We will now write the term in the dilatation operator that mixes X and Y in the Gauss graph basis, i.e. we will write this term in the basis provided by (4.20). We already know that the other two terms are diagonal in this basis and we know their detailed form [12, 20].

Towards this end, transform the intertwining operator used to construct the restricted Schur polynomial

$$P_{R,(t,s,r)\vec{\mu}\vec{\nu}} = \sum_i |\vec{m}, \vec{p}, (s, t), \vec{\mu}; i\rangle \langle \vec{m}, \vec{p}, (s, t), \vec{\nu}; i| \otimes \mathbf{1}_r \quad (5.1)$$

to the Gauss graph basis. Of course, it is only

$$p_{(t,s)\vec{\mu}\vec{\nu}}^{\vec{m}\vec{p}} = \sum_i |\vec{m}, \vec{p}, (s, t), \vec{\mu}; i\rangle \langle \vec{m}, \vec{p}, (s, t), \vec{\nu}; i| \quad (5.2)$$

that we need to consider. The transformation is a simple computation

$$\begin{aligned} & \sum_{(s,t)} |\vec{m}, \vec{p}, (s, t), \vec{\mu}; i\rangle \langle \vec{m}, \vec{p}, (s, t), \vec{\nu}; i| B_{l\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{m\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \Gamma_{lm}^{(s,t)}(\sigma_2) \\ &= \frac{1}{|H_X \times H_Y| m! p!} \sum_{(s,t)} \sum_{\sigma, \tau \in S_m \times S_p} d_s d_t B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \Gamma_{bj}^{(s,t)}(\sigma) |v_\sigma\rangle \langle v_\tau | \Gamma_{bk}^{(s,t)}(\tau) B_{k\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \\ & \quad \times B_{l\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{m\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \Gamma_{lm}^{(s,t)}(\sigma_2) \\ &= \frac{1}{|H_X \times H_Y| m! p!} \sum_{(s,t)} \sum_{\sigma, \tau \in S_m \times S_p} d_s d_t |v_\sigma\rangle \langle v_\tau | \Gamma_{jk}^{(s,t)}(\sigma^{-1}\tau) B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \\ & \quad B_{l\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{m\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} \Gamma_{lm}(\sigma_2) \\ &= \frac{1}{|H_X \times H_Y| m! p!} \sum_{(s,t)} \sum_{\sigma, \tau \in S_m \times S_p} \frac{1}{|H_X \times H_Y|^2} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} d_s d_t \Gamma_{jm}^{(s,t)}(\gamma_1) \Gamma_{kl}^{(s,t)}(\gamma_2) \Gamma_{lm}^{(s,t)}(\sigma_2) \\ & \quad \times \Gamma_{jk}^{(s,t)}(\sigma^{-1}\tau) |v_\sigma\rangle \langle v_\tau | \\ &= \frac{1}{m! p! |H_X \times H_Y|} \sum_{(s,t)} \sum_{\sigma, \tau \in S_m \times S_p} \frac{1}{|H_X \times H_Y|^2} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} d_s d_t \chi_u(\gamma_1 \sigma_2^{-1} \gamma_2^{-1} \tau^{-1} \sigma) |v_\sigma\rangle \langle v_\tau | \\ &= \frac{1}{|H_X \times H_Y|^3} \sum_{\sigma, \tau \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} \delta(\gamma_1 \sigma_2^{-1} \gamma_2^{-1} \tau^{-1} \sigma) |v_\sigma\rangle \langle v_\tau |. \end{aligned} \quad (5.3)$$

Notice that up to normalization this is a sum over all $\sigma, \tau \in S_m \times S_p$ of $|v_\sigma\rangle \langle v_\tau|$ with the condition that $\tau^{-1}\sigma$ belongs to the same coset as σ_2 does. With this result in hand, we easily find

$$\begin{aligned} & \langle O_{T,w}^\dagger(\sigma_2) D_{XY} O_{R,r}(\sigma_1) \rangle = \\ &= \frac{|H_X \times H_Y| |H'_X \times H'_Y|}{m! p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\vec{\mu}\vec{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\vec{\alpha}\vec{\beta}} \sqrt{d_s d_t} \sqrt{d_x d_y} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \end{aligned}$$

$$\begin{aligned}
& \times B_{j\bar{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\bar{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\bar{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\bar{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \langle O_{T,(y,x,w)\bar{\alpha}\bar{\beta}}^\dagger D_{XY} O_{R,(t,s,r)\bar{\mu}\bar{\nu}} \rangle \\
&= \frac{|H_X \times H_Y| |H'_X \times H'_Y|}{m! p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\bar{\mu}\bar{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\bar{\alpha}\bar{\beta}} \sqrt{d_s d_t} \sqrt{d_x d_y} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \\
& \quad \times B_{j\bar{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\bar{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\bar{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\bar{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \\
& \quad \times \sum_{R'} c_{RR'} \frac{d_T m p}{d_x d_y (n+m+p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_x \text{hooks}_y}} \delta_{rw} \\
& \quad \times \left(\langle \vec{p}, t, \nu_1; a | E_{lj}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ki}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\
& \quad \times \langle \vec{m}, s, \nu_2; c | E_{il}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{jk}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\
& \quad - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\nu_1 \alpha_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\beta_2 \mu_2} \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \\
& \quad - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\mu_1 \beta_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\alpha_2 \nu_2} \langle \vec{p}, t, \nu_1; a | E_{ij}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{m}', x, \beta_2; d | E_{ji}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\
& \quad \quad \left. + \langle \vec{p}, t, \nu_1; a | E_{il}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{jk}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\
& \quad \left. \times \langle \vec{m}, s, \nu_2; c | E_{lj}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{ki}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \right) \\
&= \frac{|H_X \times H_Y| |H'_X \times H'_Y|}{m! p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\bar{\mu}\bar{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\bar{\alpha}\bar{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \\
& \quad \times B_{j\bar{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\bar{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\bar{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\bar{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \\
& \quad \sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \left(\langle \vec{p}, t, \nu_1; a | E_{lj}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ki}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\
& \quad \times \langle \vec{m}, s, \nu_2; c | E_{il}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{jk}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\
& \quad - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\nu_1 \alpha_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\beta_2 \mu_2} \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \\
& \quad - \delta_{\vec{p}\vec{p}'} \delta_{yt} \delta_{\mu_1 \beta_1} \delta_{\vec{m}\vec{m}'} \delta_{sx} \delta_{\alpha_2 \nu_2} \langle \vec{p}, t, \nu_1; a | E_{ij}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{m}', x, \beta_2; d | E_{ji}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \\
& \quad \quad \left. + \langle \vec{p}, t, \nu_1; a | E_{il}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{jk}^{(1)} | \vec{p}, t, \mu_1; a \rangle \right. \\
& \quad \left. \times \langle \vec{m}, s, \nu_2; c | E_{lj}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{ki}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle \right).
\end{aligned}$$

There are four terms in the above expression. We will deal with each term, one at a time.

5.1 First term

Focus on the first term for now

$$\begin{aligned}
&= \frac{|H_X \times H_Y| |H'_X \times H'_Y|}{m! p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\bar{\mu}\bar{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\bar{\alpha}\bar{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \\
& \quad \times B_{j\bar{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\bar{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\bar{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\bar{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \\
& \quad \sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p) d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \langle \vec{p}, t, \nu_1; a | E_{lj}^{(1)} | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ki}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\
& \quad \times \langle \vec{m}, s, \nu_2; c | E_{il}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | E_{jk}^{(p+1)} | \vec{m}, s, \mu_2; c \rangle
\end{aligned}$$

$$\begin{aligned}
& \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
& \times \langle v, \vec{p}', \vec{m}' | E_{ji}^{\tau^{-1}(p+1)} \tau^{-1}(1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \langle v, \vec{p}, \vec{m} | E_{ij}^{\phi^{-1}(p+1)} \phi^{-1}(1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle
\end{aligned}
= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!}$$

Now, lets study the case that $i = j$. To find a simple condition on \vec{p}', \vec{m}' and \vec{p}, \vec{m} that tells us when this matrix element is non-zero, focus on

$$\langle v_\tau | E_{ji}^{(p+1)}(1, p+1) | v_\psi \rangle \langle v_\phi | (1, p+1) E_{ij}^{(1)} | v_\sigma \rangle. \quad (5.4)$$

If $i = j$, the matrix element $\langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(p+1)}(1, p+1) \psi | v, \vec{p}, \vec{m} \rangle$ forces $\vec{p} + \vec{m} = \vec{p}' + \vec{m}'$. Indeed, $E_{ii}^{(p+1)}$ is one if the vector in the first slot of $\psi | v, \vec{p}, \vec{m} \rangle$ is v_1 and it is zero otherwise, so it clearly does not change the identity of any vectors. The remaining elements between the two states (i.e. τ^{-1} and $(1, p+1) \psi$) can swap vectors around but not change the identity of any vector. Thus, the identity of the collection of vectors used to construct $|v, \vec{p}, \vec{m} \rangle$ must match the identity of the collection of vectors used to construct $|v, \vec{p}', \vec{m}' \rangle$. This then proves that $\vec{p} + \vec{m} = \vec{p}' + \vec{m}'$. We can argue for this conclusion in a second way: recall that we obtain R' from R by dropping box in row i and we obtain T' from T by dropping a box in row j . Thus, if $i = j$, since $R' = T'$ we are saying that $R = T$. We already know that $r = w$. $\vec{p} + \vec{m}$ tells us the collection of boxes that needs to be dropped from R to get r and $\vec{p}' + \vec{m}'$ tells us the collection of boxes that needs to be dropped from T to get w . Since $R = T$ and $r = w$, this then again proves that $\vec{p} + \vec{m} = \vec{p}' + \vec{m}'$. We can say a bit more. Consider

$$\langle v, \vec{p}, \vec{m} | \phi^{-1}(1, p+1) E_{ii}^{(1)} \sigma | v, \vec{p}', \vec{m}' \rangle. \quad (5.5)$$

This tells you that if you take the state $|v, \vec{p}', \vec{m}' \rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (σ does this shuffling) keeping only states with vector v_i in their first slot, and then swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}, \vec{m} \rangle$ by shuffling (according to ϕ^{-1}) what we have. **Thus, to get $|v, \vec{p}, \vec{m} \rangle$ from $|v, \vec{p}', \vec{m}' \rangle$ we removed v_i from an X slot of $|v, \vec{p}', \vec{m}' \rangle$ and inserted it into a Y slot of $|v, \vec{p}, \vec{m} \rangle$.**

Now consider

$$\langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ii}^{(p+1)}(1, p+1) \psi | v, \vec{p}, \vec{m} \rangle. \quad (5.6)$$

This tells you that if you take the state $|v, \vec{p}, \vec{m} \rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (ψ does this shuffling) keeping only states with vector v_i in their first slot, and then, swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}', \vec{m}' \rangle$ by shuffling (according to τ^{-1}) what we have. **Thus, to get $|v, \vec{p}', \vec{m}' \rangle$ from $|v, \vec{p}, \vec{m} \rangle$ we removed v_i from an X slot of $|v, \vec{p}, \vec{m} \rangle$ and inserted it into a Y slot of $|v, \vec{p}', \vec{m}' \rangle$.**

Thus, the two vectors we are swapping have the *same* identity. This implies that we must have $\vec{p} = \vec{p}'$ and $\vec{m} = \vec{m}'$. Since we must have $\vec{p} = \vec{p}'$ and $\vec{m} = \vec{m}'$ we find

$$\begin{aligned}
&= \frac{1}{|H_X \times H_Y|^{4m!} p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2 \in H_X \times H_Y} \\
&\times \langle v, \vec{p}, \vec{m} | E_{ji}^{\tau^{-1}(p+1)} \tau^{-1}(1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \langle v, \vec{p}, \vec{m} | E_{ij}^{\phi^{-1}(p+1)} \phi^{-1}(1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}, \vec{m} \rangle \\
&= \frac{1}{|H_X \times H_Y|^{4m!} p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2, \rho_1, \rho_2 \in H_X \times H_Y} \\
&\times \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \rho_1) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \rho_2) \\
&\times \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\tau^{-1}(p+1), k) \delta(\tau^{-1}(1), l) \delta(\phi^{-1}(p+1), q) \delta(\phi^{-1}(1), r).
\end{aligned}$$

Now, set $\tau = \alpha \tilde{\tau}$ and $\beta = \alpha \tilde{\beta}$ with $\alpha \in Z_m \times Z_p$, with $Z_m \times Z_p$ a product of cyclic groups. The above expression becomes

$$\begin{aligned}
&= \frac{1}{|H_X \times H_Y|^{4m!} p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \\
&\times \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2, \rho_1, \rho_2 \in H_X \times H_Y} \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \rho_1) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \rho_2) \\
&\times \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\tau^{-1}(\alpha(p+1)), k) \delta(\tau^{-1}(\alpha(1)), l) \delta(\phi^{-1}(\alpha(p+1)), q) \delta(\phi^{-1}(\alpha(1)), r) \\
&= \sum_{\alpha \in Z_m \times Z_p} \frac{1}{|H_X \times H_Y|^{4m!} p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \\
&\times \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2, \rho_1, \rho_2 \in H_X \times H_Y} \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \rho_1) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \rho_2) \\
&\times \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\tau^{-1}(\alpha(p+1)), k) \delta(\tau^{-1}(\alpha(1)), l) \delta(\phi^{-1}(\alpha(p+1)), q) \delta(\phi^{-1}(\alpha(1)), r) \\
&= \frac{1}{|H_X \times H_Y|^{4m!} p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \\
&\times \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2, \rho_1, \rho_2 \in H_X \times H_Y} \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \rho_1) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \rho_2) \\
&\times \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\tau^{-1}(\phi(q)), k) \delta(\tau^{-1}(\phi(r)), l) \\
&= \frac{1}{|H_X \times H_Y|^4} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\phi \in S_m \times S_p} \\
&\times \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2, \rho_1, \rho_2 \in H_X \times H_Y} \delta(\phi \beta_2 \sigma_2 \beta_1^{-1} \rho_1) \delta(\phi^{-1} \gamma_2 \sigma_1 \gamma_1^{-1} \rho_2) \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\phi(q), k) \delta(\phi(r), l) \\
&= \frac{1}{|H_X \times H_Y|^2} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2 \in H_X \times H_Y} \\
&\times \delta(\phi \beta_2 \sigma_2 \beta_1^{-1}) \delta(\phi^{-1} \gamma_2 \sigma_1 \gamma_1^{-1}) \sum_{k, q \in S_{i,m}} \sum_{l, r \in S_{i,p}} \delta(\phi(q), k) \delta(\phi(r), l)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|H_X \times H_Y|^2} \sum_{R'} c_{RR'} \text{hooks}_{R'} \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\gamma_1, \gamma_2, \beta_1, \beta_2 \in H_X \times H_Y} \\
&\quad \times \delta(\gamma_2 \sigma_1 \gamma_1^{-1} \beta_2 \sigma_2 \beta_1^{-1}) n_{ii}^X(\sigma_1) n_{ii}^Y(\sigma_1) \\
&= \sum_{R'} c_{RR'} \text{hooks}_{R'} \sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \delta(\sigma_1 \beta_2 \sigma_2 \beta_1^{-1}) n_{ii}^X(\sigma_1) n_{ii}^Y(\sigma_1).
\end{aligned} \tag{5.7}$$

Now, return to the case that $i \neq j$. The matrix element $\langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(p+1)} (1, p+1) \psi | v, \vec{p}, \vec{m} \rangle$ forces $\vec{p} + \vec{m} \neq \vec{p}' + \vec{m}'$. Indeed, $(1, p+1) \psi$ shuffles vectors, $E_{ji}^{(p+1)}$ removes v_i and inserts v_j and τ^{-1} does some more shuffling. Thus, using an obvious notation, we have

$$\vec{p} + \vec{m} - \hat{i} = \vec{p}' + \vec{m}' - \hat{j}. \tag{5.8}$$

We can also see this by noting that since $i \neq j$ we know that $R \neq T$. We still have $r = w$ so that the collection of boxes that needs to be dropped from R to get r (described by $\vec{p} + \vec{m}$) and the collection of boxes that needs to be dropped from T to get w (described by $\vec{p}' + \vec{m}'$) can't possibly be equal.

Again, we can say more. Consider

$$\langle v, \vec{p}, \vec{m} | \phi^{-1} (1, p+1) E_{ij}^{(1)} \sigma | v, \vec{p}', \vec{m}' \rangle. \tag{5.9}$$

This tells you that if you take the state $|v, \vec{p}', \vec{m}'\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (σ does this shuffling) keeping only states with vector v_j in their first slot, replacing this vector v_j with another vector v_i and then swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}, \vec{m}\rangle$ by shuffling (according to ϕ^{-1}) what we have. We can summarize this as

$$\begin{aligned}
\vec{p}' - \hat{j} &= \vec{p} - \hat{a} \\
\vec{m}' - \hat{a} &= \vec{m} - \hat{i}.
\end{aligned} \tag{5.10}$$

Now consider

$$\langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(p+1)} (1, p+1) \psi | v, \vec{p}, \vec{m} \rangle. \tag{5.11}$$

This tells you that if you take the state $|v, \vec{p}, \vec{m}\rangle$ and shuffle some of the X slots amongst each other and some of the Y slots amongst each other (ψ does this shuffling) keeping only states with vector v_i in their first slot, replacing this vector v_i with v_j and then, swapping the vectors in slots 1 and $p+1$, we can get the vector $|v, \vec{p}', \vec{m}'\rangle$ by shuffling (according to τ^{-1}) what we have. We can summarize this as

$$\vec{p}' - \hat{i} = \vec{p} - \hat{b}$$

$$\vec{m} - \hat{b} = \vec{m}' - \hat{j}. \quad (5.12)$$

The equations (5.10) and (5.12) only have two solutions. If we choose $\hat{a} = \hat{i}$, we must have $\hat{b} = \hat{j}$ and then

$$\begin{aligned} \vec{m} &= \vec{m}' \\ \vec{p} - \hat{i} &= \vec{p}' - \hat{j}. \end{aligned} \quad (5.13)$$

If we choose $\hat{a} = \hat{j}$, we must have $\hat{b} = \hat{i}$ and then

$$\begin{aligned} \vec{p} &= \vec{p}' \\ \vec{m} - \hat{i} &= \vec{m}' - \hat{j}. \end{aligned} \quad (5.14)$$

Thus, only \vec{m} or \vec{p} can change - but not both. In fact, only one of the Gauss graphs (there is one graph for the X s and one for the Y s) change - but not both.

It is now rather simple to write the relation between $|v, \vec{p}', \vec{m}'\rangle$ and $|v, \vec{p}, \vec{m}\rangle$. Consider for example, (5.13). Let $S_{j,p}$ denote the collection of slots that

(i) are X slots and (ii) are occupied by v_j . There are similar definitions for $S_{j,m}$, $S'_{j,p}$ and $S'_{j,m}$. To go from \vec{p} to \vec{p}' , we want to remove a v_i and replace it with a v_j and then reorder the slots into the order prescribed by (4.3). We can do this as

$$|v, \vec{p}', \vec{m}'\rangle = \zeta E_{ji}^{(q)} |v, \vec{p}, \vec{m}\rangle \quad q \in S_{i,p} \quad \zeta \in S_m \times S_p. \quad (5.15)$$

Consequently we can again write a definite relation between $|v, \vec{p}', \vec{m}'\rangle$ and $|v, \vec{p}, \vec{m}\rangle$. This allows us to simplify the matrix element expressions to the structure of elements we have already evaluated. Now, consider

$$\begin{aligned} A &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} c_{RR'} \text{hooks}_{R'} m p \\ &\sqrt{\frac{f_T}{f_R \text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\ &\quad \times \langle v, \vec{p}', \vec{m}' | E_{ji}^{\tau^{-1}(p+1)} \tau^{-1} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\ &\quad \times \langle v, \vec{p}, \vec{m} | E_{ij}^{\phi^{-1}(p+1)} \phi^{-1} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\ &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \\ &\times \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\tau, \phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\ &\quad \times \langle v, \vec{p}', \vec{m}' | E_{ji}^{\tau^{-1}(p+1)} \tau^{-1} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\ &\quad \times \langle v, \vec{p}, \vec{m} | E_{ij}^{\phi^{-1}(p+1)} \phi^{-1} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle. \end{aligned} \quad (5.16)$$

To start, study

$$\langle v, \vec{p}', \vec{m}' | E_{ji}^{\tau^{-1}(p+1)} \tau^{-1} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle$$

$$= \langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(p+1)} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \quad (5.17)$$

and consider a matrix element for which $\vec{m} = \vec{m}'$ and $\vec{p} = \vec{p}' - \hat{j} + \hat{i}$. In this case, we know that we can write

$$|v, \vec{p}', \vec{m}'\rangle = \zeta E_{ji}^{(q)} |v, \vec{p}, \vec{m}\rangle \quad \zeta \in S_p \quad q \in S_{i,p} \quad (5.18)$$

We can choose any basis for the vectors $|v, \vec{p}, \vec{m}\rangle$, $|v, \vec{p}', \vec{m}'\rangle$ that we like - the result will be independent of the choice we make. In (4.3) choose the i and j vectors to sit in adjacent slots, and always choose q to lie on the border between the two. In this case we can always choose ζ_q to be the identity. With this choice understood, we have

$$|v, \vec{p}', \vec{m}'\rangle = E_{ji}^{(q)} |v, \vec{p}, \vec{m}\rangle \quad q \in S_{i,p}. \quad (5.19)$$

In a similar way

$$\begin{aligned} \langle v, \vec{p}, \vec{m} | E_{ij}^{\phi^{-1}(p+1)} \phi^{-1} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\ = \langle v, \vec{p}, \vec{m} | \phi^{-1} E_{ij}^{(p+1)} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \end{aligned} \quad (5.20)$$

and, from (5.19) we have

$$|v, \vec{p}, \vec{m}\rangle = E_{ij}^{(q)} |v, \vec{p}', \vec{m}'\rangle. \quad (5.21)$$

Consequently

$$\begin{aligned} A &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \\ &\quad \times \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau, \phi \in S_m \times S_p} \langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(p+1)} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\ &\quad \quad \quad \times \langle v, \vec{p}, \vec{m} | \phi^{-1} E_{ij}^{(p+1)} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\ &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \\ &\quad \times \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau, \phi \in S_m \times S_p} \langle v, \vec{p}, \vec{m} | E_{ij}^{(q)} \tau^{-1} E_{ji}^{(p+1)} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\ &\quad \quad \quad \times \langle v, \vec{p}', \vec{m}' | E_{ji}^{(q)} \phi^{-1} E_{ij}^{(p+1)} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\ &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \\ &\quad \times \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau, \phi \in S_m \times S_p} \langle v, \vec{p}, \vec{m} | \tau^{-1} E_{ij}^{\tau(q)} E_{ji}^{(p+1)} (1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\ &\quad \quad \quad \times \langle v, \vec{p}', \vec{m}' | \phi^{-1} E_{ji}^{\phi(q)} E_{ij}^{(p+1)} (1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle \\ &= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \end{aligned}$$

$$\begin{aligned} & \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau, \phi \in S_m \times S_p} \langle v_\tau | E_{ij}^{\tau(q)} E_{ji}^{(p+1)}(1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1} | v_\tau \rangle \\ & \quad \times \langle v'_\phi | E_{ji}^{\phi(q)} E_{ij}^{(p+1)}(1, p+1) \tau \gamma_2 \sigma_1 \gamma_1^{-1} \phi^{-1} | v'_\phi \rangle \end{aligned} \quad (5.22)$$

We need to understand $\langle v_\tau | E_{ij}^{\tau(q)} E_{ji}^{(p+1)}(1, p+1)$ and $\langle v'_\phi | E_{ji}^{\phi(q)} E_{ij}^{(p+1)}(1, p+1)$ better. Consider $\langle v_\tau | E_{ij}^{\tau(q)} E_{ji}^{(p+1)}(1, p+1)$ first. Turn this into a ket state

$$\begin{aligned} (1, p+1) E_{ij}^{(p+1)} E_{ji}^{\tau(q)} \tau | v \rangle &= (1, p+1) \tau E_{ij}^{\tau^{-1}(p+1)} E_{ji}^{(q)} | v \rangle \\ &= \tau (\tau^{-1}(1), \tau^{-1}(p+1)) E_{ij}^{\tau^{-1}(p+1)} E_{ji}^{(q)} | v \rangle \end{aligned} \quad (5.23)$$

Now, there are a few things we should note. First, recall that $i \neq j$. In the matrix element

$$\langle v_\tau | E_{ij}^{\tau(q)} E_{ji}^{(p+1)}(1, p+1) \phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1} | v_\tau \rangle \quad (5.24)$$

we know that $\phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1}$ is an element in $S_m \times S_p$ and thus it is not able to swap vectors between the Y and X slots. The product $E_{ij}^{\tau(q)} E_{ji}^{(p+1)}$ makes an X slot change as (imagine acting to the right) $j \rightarrow i$ and a Y slot change as $i \rightarrow j$. This amounts to exchanging an i vector from X with a j vector from Y . The only way that the above matrix element can be non-zero, is if $(1, p+1)$ is able to swap these two back again. Thus, we can write

$$\begin{aligned} & \tau (\tau^{-1}(1), \tau^{-1}(p+1)) E_{ij}^{\tau^{-1}(p+1)} E_{ji}^{(q)} | v \rangle \\ &= \tau E_{ij}^{\tau^{-1}(1)} E_{ji}^{\tau^{-1}(p+1)} E_{ij}^{\tau^{-1}(p+1)} E_{ji}^{(q)} | v \rangle \\ &= \sum_{l \in S_{j,m}} \delta(\tau^{-1}(p+1), l) \tau E_{ij}^{\tau^{-1}(1)} E_{ji}^{(q)} | v \rangle \\ &= \sum_{l \in S_{j,m}} \delta(\tau^{-1}(p+1), l) \left(\delta(\tau^{-1}(1), q) + \sum_{r \in S_{j,p}} \delta(\tau^{-1}(1), r) (q, r) \right) \tau | v \rangle \end{aligned} \quad (5.25)$$

Now, each of the terms in round brackets for which index r belongs to a string that loops back to node j above makes the same contribution so that we have

$$(1, p+1) E_{ij}^{(p+1)} E_{ji}^{\tau(q)} | v_\tau \rangle = n_{jj}^X(\sigma_2) \sum_{l \in S_{j,m}} \delta(\tau^{-1}(p+1), l) \delta(\tau^{-1}(1), q) | v_\tau \rangle. \quad (5.26)$$

The above equation is not exactly true (certain terms on the RHS have been dropped) but it gives the correct result when plugged into (5.22). A very similar argument implies that we can replace

$$(1, p+1) E_{ji}^{(p+1)} E_{ij}^{(\phi \zeta_q)(q)} | v'_\phi \rangle = n_{ii}^X(\sigma_1) \sum_{w \in S'_{i,m}} \delta(\phi^{-1}(p+1), w) \delta(\phi^{-1}(1), \zeta_q(q)) | v'_\phi \rangle \quad (5.27)$$

We can now use these results to compute

$$A = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{C_{RR'} C_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw}$$

$$\begin{aligned}
& \times \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\tau^{-1}(p+1), l) \delta(\phi^{-1}(p+1), w) \\
& \quad \delta(\tau^{-1}(1), q) \delta(\phi^{-1}(1), \zeta_q(q)) \langle v_\tau | \phi \beta_2 \sigma_2 \beta_1^{-1} \tau^{-1} | v_\tau \rangle \langle v'_\phi | \tau \gamma_2 \sigma_1 \gamma_1^{-1} \phi^{-1} | v'_\phi \rangle \\
& = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \\
& \times \sum_{\gamma_1, \gamma_2, \gamma \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2, \beta \in H_X \times H_Y} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\tau^{-1}(p+1), l) \delta(\phi^{-1}(p+1), w) \\
& \quad \delta(\tau^{-1}(1), q) \delta(\phi^{-1}(1), q) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \beta) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \gamma) \\
& = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y| |H'_X \times H'_Y| m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} m p \delta_{rw} \\
& \times \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\tau^{-1}(p+1), l) \delta(\phi^{-1}(p+1), w) \\
& \quad \delta(\tau^{-1}(1), q) \delta(\phi^{-1}(1), q) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1}) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& \tag{5.28}
\end{aligned}$$

Now, we do the same trick as before, setting $\phi = \alpha \tilde{\phi}$ and $\tau = \alpha \tilde{\tau}$. After performing manipulations just like we did before we find

$$\begin{aligned}
A & = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y| |H'_X \times H'_Y| m! p!} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} \delta_{rw} \\
& \quad \times \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \\
& \quad \times \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\tau^{-1} \phi(w), l) \delta(\tau^{-1} \phi(q), q) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1}) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y| |H'_X \times H'_Y|} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \sum_{\tau \in S_m \times S_p} \\
& \quad \times \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\tau^{-1}(w), l) \delta(\tau^{-1}(q), q) \delta(\tau^{-1} \beta_2 \sigma_2 \beta_1^{-1}) \delta(\tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& = \frac{n_{ii}^X(\sigma_1) n_{jj}^X(\sigma_2)}{|H_X \times H_Y| |H'_X \times H'_Y|} \sum_{R'} \sqrt{c_{RR'} c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
& \quad \times \sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\gamma_2 \sigma_1 \gamma_1^{-1}(w), l) \delta(\gamma_2 \sigma_1 \gamma_1^{-1}(q), q) \delta(\beta_2 \sigma_2 \beta_1^{-1} \gamma_2 \sigma_1 \gamma_1^{-1}). \\
& \tag{5.29}
\end{aligned}$$

Consider

$$\sum_{l \in S_{j,m}} \sum_{w \in S'_{i,m}} \delta(\gamma_2 \sigma_1 \gamma_1^{-1}(w), l) = n_{ij}^{Y+}(\sigma_1) \tag{5.30}$$

where $n_{ij}^{Y+}(\sigma_2) = n_{ij}^{Y+}(\sigma_1)$ is the number of strings going from i to j in the Gauss graph associated to the Y s. The fact that this $n_{ij}^{Y+}(\sigma_2)$ appears suggests that the strings stretching

between i and j in the Gauss graph associated to the Y s are participating, even though it is the X Gauss graph that undergoes the transition. Thus

$$A = \frac{n_{ii}^X(\sigma_1)n_{jj}^X(\sigma_2)}{|H_X \times H_Y||H'_X \times H'_Y|} \sum_{R'} \sqrt{c_{RR'}c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} n_{ij}^{Y+}(\sigma_2) \delta(\gamma_2 \sigma_1 \gamma_1^{-1}(q), q) \delta(\beta_2 \sigma_2 \beta_1^{-1} \gamma_2 \sigma_1 \gamma_1^{-1}) \quad (5.31)$$

Next, consider the role of $\delta(\gamma_2 \sigma_1 \gamma_1^{-1}(q), q)$. This tells us that a single string, which loops back to the same brane, is plucked from brane i (or j) and reattached to brane j (or i). This follows because the string which is plucked has startpoint q and end point $\gamma_2 \sigma_1 \gamma_1^{-1}(q)$. So the delta function is setting the start point equal to the end point. Another way to say it is that states with different values for the n_{ij}^Y or n_{ij}^X don't mix - and this is why the terms in the dilatation operator that mix X and Y commute with terms that mix X and Z and the terms that mix Y and Z . The role of this delta function is also easy to interpret in terms of the Gauss graph: the two Gauss graphs that mix, σ_1 and σ_2 , are related by peeling a closed loop from node i of σ_1^X (or σ_1^Y) and reattaching it to node j of σ_2^X (or σ_2^Y). This implies that, as permutations, σ_1 and σ_2 are identical (recall that closed loops are 1 cycles).

If we peel a string from node i of σ_1^X and reattach it to node j of σ_2^X , the factor $n_{ii}^X(\sigma_1)n_{jj}^X(\sigma_2) = n_{ii}^X(\sigma_1)(n_{jj}^X(\sigma_1) + 1) = (n_{ii}^X(\sigma_2) + 1)n_{jj}^X(\sigma_2)$ is the number of strings starting and ending at node i before we peel a string off, multiplied by the number of strings starting and ending at node j after we have attached the string.

Notice that the delta function $\delta(\gamma_2 \sigma_1 \gamma_1^{-1}(q), q)$ reduces the full sum over γ_1 and γ_2 to those elements of $H'_X \times H'_Y$ that leave q inert. This is a subgroup of $(H'_X \times H'_Y) \cap (H_X \times H_Y)$ that we will denote $\mathcal{G}_{\sigma_1, q}$. Consequently, the size of this matrix element is

$$n_{ii}^X(\sigma_1)n_{jj}^X(\sigma_2)n_{ij}^{Y+}(\sigma_2)|\mathcal{G}_{\sigma_1, q}| \quad (5.32)$$

Notice that

$$\frac{|\mathcal{G}_{\sigma_1, q}|}{N_{\sigma_1}} = n_{ii}^X(\sigma_1) \quad \frac{|\mathcal{G}_{\sigma_1, q}|}{N_{\sigma_2}} = n_{jj}^X(\sigma_2). \quad (5.33)$$

These two formulas follow because N_σ counts the number of symmetries of the Gauss graph, while $|\mathcal{G}_{\sigma, q}|$ counts the number of symmetries that don't include permutations of the closed loop corresponding to q . Thus, we finally see that the normalized matrix element is nothing but

$$\sqrt{c_{RR'}c_{TR'}} \frac{\text{hooks}_{R'}}{\sqrt{\text{hooks}_R \text{hooks}_T}} \delta_{rw} \sqrt{n_{ii}^X(\sigma_1)n_{jj}^X(\sigma_2)n_{ij}^{Y+}(\sigma_2)}. \quad (5.34)$$

The evaluation of the fourth term is practically identical and will not be discussed.

5.2 Second Term

Now consider the second term

$$\begin{aligned}
B &= \frac{|H_X \times H_Y||H'_X \times H'_Y|}{m!p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\bar{\mu}\bar{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\bar{\alpha}\bar{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \\
&\quad \times B_{j\bar{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\bar{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\bar{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\bar{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \\
\sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p)d_{R'}} &\sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \langle \vec{p}, t, \nu_1; a | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\
&\quad \times \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | \vec{m}, s, \mu_2; c \rangle \\
&= \frac{1}{|H_X \times H_Y|^2 |H'_X \times H'_Y|^2 m! p!} \\
\sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p)d_{R'}} &\sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H'_X \times H'_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
&\quad \times \delta_{\vec{m}\vec{m}'} \delta_{\vec{p}\vec{p}'} \langle v, \vec{p}', \vec{m}' | \tau^{-1} E_{ji}^{(1)} \phi \beta_2 \sigma_2 \beta_1^{-1} | v, \vec{p}, \vec{m} \rangle \\
&\quad \times \langle v, \vec{p}, \vec{m} | \phi^{-1} E_{ij}^{(p+1)} \tau \gamma_2 \sigma_1 \gamma_1^{-1} | v, \vec{p}', \vec{m}' \rangle.
\end{aligned} \tag{5.35}$$

For the above result to be nonzero it is clear that we need

$$\vec{p} = \vec{p}' \quad \vec{m} - \hat{i} + \hat{j} = \vec{m}' \tag{5.36}$$

as well as

$$\vec{p} = \vec{p}' \quad \vec{m} = \vec{m}'. \tag{5.37}$$

Consequently, this term is only non-zero when $i = j$. In this case

$$\begin{aligned}
B &= \frac{1}{|H_X \times H_Y|^4 m! p!} \\
\sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p)d_{R'}} &\sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2, \gamma \in H_X \times H_Y} \sum_{\beta_1, \beta_2, \beta \in H_X \times H_Y} \\
&\quad \times \sum_{k \in S_{j,m}} \delta(\tau^{-1}(1), k) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1} \beta) \\
&\quad \times \sum_{l \in S_{i,m}} \delta(\phi^{-1}(p+1), l) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1} \gamma) \\
&= \frac{1}{|H_X \times H_Y|^2 m! p!} \\
\sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p)d_{R'}} &\sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
&\quad \times \sum_{k \in S_{j,m}} \delta(\tau^{-1}(1), k) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1})
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{l \in S_{i,m}} \delta(\phi^{-1}(p+1), l) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& \qquad \qquad \qquad = \frac{1}{|H_X \times H_Y|^2 m! p!} \\
& \sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\phi \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
& \times \sum_{k \in S_{j,m}} \sum_{r=1}^m \delta(\tau^{-1}(m), k) \sum_{l \in S_{i,m}} \sum_{s=p+1}^{p+m} \delta(\phi^{-1}(s), l) \delta(\tau^{-1} \phi \beta_2 \sigma_2 \beta_1^{-1}) \delta(\phi^{-1} \tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& \qquad \qquad \qquad = \frac{1}{|H_X \times H_Y|^2} \\
& \sum_{R'} c_{RR'} \frac{d_T p}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\tau \in S_m \times S_p} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
& \times n_i^{+,X} n_i^{+,Y} \delta(\tau^{-1} \beta_2 \sigma_2 \beta_1^{-1}) \delta(\tau \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& \qquad \qquad \qquad = \frac{1}{|H_X \times H_Y|^2} \\
& \sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\gamma_1, \gamma_2 \in H_X \times H_Y} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} \\
& \times n_i^{+,X} n_i^{+,Y} \delta(\beta_2 \sigma_2 \beta_1^{-1} \gamma_2 \sigma_1 \gamma_1^{-1}) \\
& = \sum_{R'} c_{RR'} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} n_{ii}^{+,X} n_{ii}^{+,Y} \delta(\beta_2 \sigma_2 \beta_1^{-1} \sigma_1)
\end{aligned} \tag{5.38}$$

In summary, and perhaps writing it a bit more clearly, we have

$$\begin{aligned}
B &= \frac{|H_X \times H_Y| |H'_X \times H'_Y|}{m! p!} \sum_{jk} \sum_{s \vdash m} \sum_{t \vdash p} \sum_{\vec{\mu} \vec{\nu}} \sum_{lm} \sum_{x \vdash m} \sum_{y \vdash p} \sum_{\vec{\alpha} \vec{\beta}} \Gamma^{(s,t)}(\sigma_1)_{jk} \Gamma^{(x,y)}(\sigma_2)_{lm} \\
& \times B_{j\vec{\mu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{k\vec{\nu}}^{(s,t) \rightarrow 1_{H_X \times H_Y}} B_{l\vec{\alpha}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} B_{m\vec{\beta}}^{(x,y) \rightarrow 1_{H'_X \times H'_Y}} \\
& \sum_{R'} c_{RR'} \frac{d_T m p}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \langle \vec{p}, t, \nu_1; a | \vec{p}', y, \alpha_1; b \rangle \langle \vec{p}', y, \beta_1; b | E_{ji}^{(1)} | \vec{p}, t, \mu_1; a \rangle \\
& \times \langle \vec{m}, s, \nu_2; c | E_{ij}^{(p+1)} | \vec{m}', x, \alpha_2; d \rangle \langle \vec{m}', x, \beta_2; d | \vec{m}, s, \mu_2; c \rangle \\
& = \delta_{\vec{p}\vec{p}'} \delta_{\vec{m}\vec{m}'} \sum_{R'_i} c_{RR'_i} \frac{d_T}{(n+m+p)d_{R'}} \sqrt{\frac{f_T \text{hooks}_T}{f_R \text{hooks}_R}} \delta_{rw} \sum_{\beta_1, \beta_2 \in H_X \times H_Y} n_i^{+,X} n_i^{+,Y} \delta(\beta_2 \sigma_2 \beta_1^{-1} \sigma_1)
\end{aligned} \tag{5.39}$$

Notice that this term is already diagonal in the Gauss graph basis. Notation: $n_i^{+,X}$ is the number of strings ending on node i of the X Gauss graph; n_{ii}^X is the number of strings starting on and then looping back to end on node i of the X Gauss graph.

The evaluation of the third term is practically identical and will not be discussed.

5.3 Final Answer

In this section we will summarize the action of the term in the dilatation operator that mixes X s and Y s on the Gauss operators.

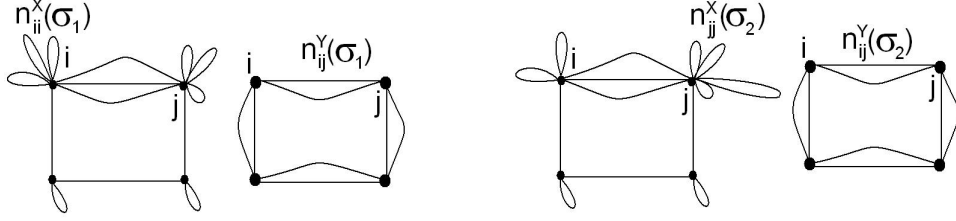


Figure 4: The Gauss graph on the left is described by σ_1 , while the Gauss graph on the right is described by σ_2 . To make a transition between the two pairs of Gauss graphs shown, we pluck a string from node i of the X graph on the left and attach it to node j of the X graph on the right. The numbers which participate are (i) the number of strings n_{ij}^Y stretching between nodes i and j of the Y graph, (ii) the number of strings attached to node i of the X graph before a string is removed $n_{ii}^X(\sigma_1) = n_{ii}^X(\sigma_2) + 1$ and (iii) the number of strings attached to the node j of the X graph after a string is attached $n_{jj}^X(\sigma_1) + 1 = n_{jj}^X(\sigma_2)$.

Here is the final answer for matrix elements of D taken with *normalized operators*. The diagonal terms are

$$\langle O_{R,r}^\dagger(\sigma) D_{XY} O_{R,r}(\sigma) \rangle = 2 \sum_{i=1}^p \frac{c_{RR'}}{l_i} \left(n(\sigma)_i^{+,X} n(\sigma)_i^{+,Y} - n(\sigma)_{ii}^{+X} n(\sigma)_{ii}^{+Y} \right) \quad (5.40)$$

Now, consider an off diagonal term. One possible non-zero matrix element corresponds to the case that the X Gauss graph changes, by detaching a loop from node i of the σ_1^X Gauss graph and reattaching it to node j . The matrix element describing this process is (recall that we only ever get a non-zero matrix element if $n_{ij}^Y(\sigma_1) = n_{ij}^Y(\sigma_2)$ and $n_{ij}^X(\sigma_1) = n_{ij}^X(\sigma_2)$)

$$\langle O_{R,r}^\dagger(\sigma_1) D_{XY} O_{R,r}(\sigma_2) \rangle = -\sqrt{\frac{c_{RR'} c_{TR'}}{l_i l_j}} n_{ij}^Y(\sigma_1) \sqrt{n_{ii}^X(\sigma_1) (n_{jj}^X(\sigma_1) + 1)} \quad (5.41)$$

Another non-zero matrix element is obtained when the Y Gauss graph changes, by detaching a loop from node i of the σ_1^Y Gauss graph and reattaching it to node j . The matrix element describing this process is

$$\langle O_{R,r}^\dagger(\sigma_1) D_{XY} O_{R,r}(\sigma_2) \rangle = -\sqrt{\frac{c_{RR'} c_{TR'}}{l_i l_j}} n_{ij}^X \sqrt{n_{ii}^Y(\sigma_1) (n_{jj}^Y(\sigma_1) + 1)} \quad (5.42)$$

This gives a complete description of the action of the term in the dilatation operator that mixes X s and Y s on the Gauss operators.

6 Diagonalization

To understand the structure of the diagonalization problem, let's start off with a warm up problem. This will also be an example of the use of the formulas (5.40), (5.41) and (5.42), which will allow the reader to test her understanding of our result. Consider the Gauss graphs shown in figure 5.

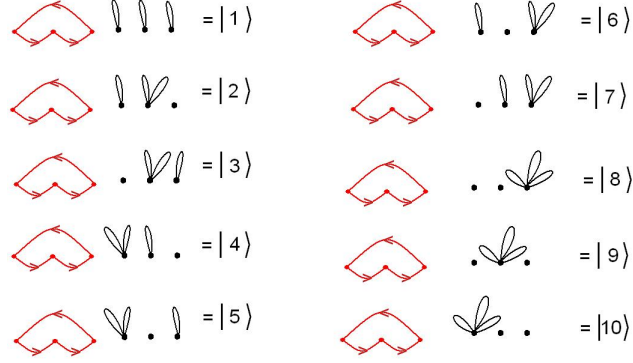


Figure 5: The 10 states that appear in our first example are defined in the figure above.

Using the formulas from the previous section, there is a transition between $|1\rangle$ and $|2\rangle$. To understand how we have labeled the dots, we must detach a loop from black node 3 of $|1\rangle$ and attach it to black node 2 of $|2\rangle$. Denote the Gauss graph corresponding to $|1\rangle$ by σ_1 and the Gauss graph of $|2\rangle$ by σ_2 . We have $n_{23}^X(\sigma_1) = 1$ (read from the red Gauss graph), $n_{22}^Y(\sigma_1) + 1 = 2$ read from $|1\rangle$ and $n_{33}^Y(\sigma_1) = 1$ read from $|1\rangle$. Thus, in total the matrix element is

$$-\sqrt{\frac{(N+l_2)(N+l_3)}{l_2 l_3}} \sqrt{2} \quad (6.1)$$

As a second example, the matrix element for the transition between $|2\rangle$ and $|3\rangle$ is

$$-\sqrt{\frac{(N+l_1)(N+l_3)}{l_1 l_3}} \quad (6.2)$$

For the 10 states shown, we have the off diagonal piece of the dilatation operator given by

$$-\sqrt{\frac{(N+l_1)(N+l_2)}{l_1 l_2}} M_{12} - \sqrt{\frac{(N+l_2)(N+l_3)}{l_2 l_3}} M_{23} - \sqrt{\frac{(N+l_1)(N+l_3)}{l_1 l_3}} M_{13} \quad (6.3)$$

where

$$M_{12} = \begin{bmatrix} 0 & 0 & -\sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.4)$$

$$M_{23} = \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.5)$$

$$M_{13} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.6)$$

and we have the on diagonal piece of the dilatation operator given by

$$\frac{(N + l_1)}{l_1} M_{11} + \frac{(N + l_2)}{l_2} M_{22} + \frac{(N + l_3)}{l_3} M_{13} \quad (6.7)$$

where

$$M_{11} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \quad (6.8)$$

$$M_{22} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.9)$$

$$M_{33} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.10)$$

To get some insight into the structure of these matrices, note that the matrix

$$M = M_{11} + M_{22} + M_{33} + M_{12} + M_{23} + M_{13} \quad (6.11)$$

has eigenvalues $0, 3, 3, 6, 6, 6, 9, 9, 9, 9$. The even spacing and the degeneracy of the eigenvalues matches the weights of the symmetric representation $\square\square\square$ of $SU(3)$. This strongly

suggests that, we can understand the off diagonal pieces of the dilatation operator as raising/lowering operators of some $SU(k)$ representations, with $k \leq g$. Recall that g is the number of rows in our restricted Schur polynomials. This guess turns out to be correct as we now explain.

First, we need to define a bijection between the Gauss graphs that mix and the states of a particular unitary group representation. Lets start by considering a situation for which the Y Gauss graph is fixed and we have transitions between different X Gauss graphs. We can only have transitions of closed loops between nodes i and j in the X Gauss graph if $n_{ij}^Y(\sigma) \neq 0$. Denote the number of connected components of σ^Y by C . Each connected component is a set of directed line segments running between nodes. Let c denote the number of connected components that have more than a single node. Let the number of nodes in each of these connected components be n_i , $i = 1, \dots, c$. The irreducible representation that organizes the σ^X graphs is an irreducible representation of the group

$$SU(n_1) \times SU(n_2) \times \cdots \times SU(n_c) \quad (6.12)$$

Focus on one of the connected components, say the j^{th} connected component. Assume that there are a total of \tilde{n} closed loops attached to nodes of σ^X that belong to this connected component. The irreducible representation of the $SU(n_j)$ factor in the above group that plays a role is labeled by a Young diagram that has a single row containing \tilde{n} boxes. We now want to give the map between different X Gauss graphs and states of this representation. Number the nodes in the j^{th} connected component from 1 up to n_j . Consider an X Gauss graph that has n_{11} strings attached to node 1, n_{22} to node 2, and so on up to $n_{n_j n_j}$ attached to node n_j . The Gelfand-Tsetlin pattern for this state

$$|M\rangle = \begin{bmatrix} m_{1,n_j} & m_{2,n_j} & \cdots & m_{n_j-1,n_j} & m_{n_j n_j} \\ & m_{1,n_j-1} & m_{2,n_j-1} & \cdots & m_{n_j-1,n_j-1} \\ & & \cdots & \cdots & \cdots \\ & & m_{1,2} & m_{2,2} & \\ & & & m_{1,1} & \end{bmatrix}$$

has $m_{p,q} = 0$ for $p > 1$ and

$$m_{1,q} = \sum_{i=1}^q n_{ii} \quad (6.13)$$

This completes our discussion of how the Gauss graphs are organized, for a fixed σ^Y . To complete the discussion note that there is a completely parallel argument with the roles of σ^X and σ^Y switched.

As a concrete example, the Gauss graph in figure 6 corresponds to the Gelfand-Tsetlin pattern

$$|M\rangle = \begin{bmatrix} n_{11} + n_{22} + n_{33} & 0 & 0 \\ & n_{11} + n_{22} & 0 \\ & & n_{11} \end{bmatrix}$$

This map between Gelfand-Tsetlin patterns and operators labeled by Gauss graphs turns out to be useful because we know the matrix elements of the Lie algebra elements in the Gelfand-Tsetlin basis. For example, let us consider the lowering operator $E_{i,i+1}$. This will shift $n_{ii} \rightarrow n_{ii} - 1$ and $n_{i+1,i+1} \rightarrow n_{i+1,i+1} + 1$. The net effect of these shifts in the Gelfand-Tsetlin pattern is to replace $m_{i,k} \rightarrow m_{i,k} - 1$; we will denote this pattern by M_i^- . The Gauss graph corresponding to M_i^- is obtained from the Gauss graph corresponding to M by peeling a closed loop from node i and reattaching it to node $i + 1$. We have already studied the matrix element of the dilatation operator that mixes these two Gauss graphs and have found

$$-\sqrt{\frac{CRR'CTR'}{l_i l_{i+1}}} n_{i,i+1}^Y(\sigma) \sqrt{n_{ii}^X(\sigma)(n_{i+1,i+1}^X(\sigma) + 1)} \quad (6.14)$$

where σ describes the state with Gelfand-Tsetlin pattern M . According to [36] the matrix element for the lowering operator, written in terms of the entries of the Gelfand-Tsetlin pattern, is

$$\langle M_i^- | E_{i,i+1} | M \rangle = \sqrt{-\frac{\prod_{k'=1}^{l+1} (m_{k',l+1} - m_{k,l} + k - k' + 1) \prod_{k'=1}^{l-1} (m_{k',l-1} - m_{k,l} + k - k')}{\prod_{k'=1, k' \neq k}^l (m_{k',l+1} - m_{k,l} + k - k' + 1)(m_{k',l+1} - m_{k,l} + k - k')}}} \quad (6.15)$$

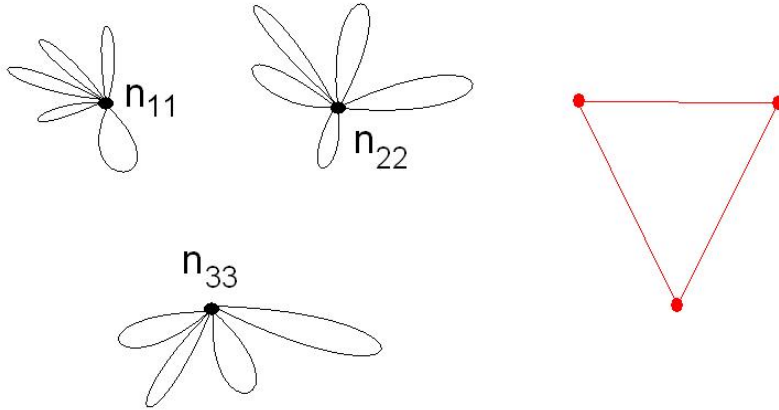


Figure 6: The Gauss graph is shown in black. Closed loops can detach from a node and reattach to another node.

Plugging in the patterns for the two Gauss graphs that mix, it is straight forward to see that

(6.15) evaluates to

$$\sqrt{n_{ii}^X(\sigma)(n_{i+1,i+1}^X(\sigma) + 1)} \quad (6.16)$$

Comparing to (6.14) we see that the off diagonal term of the dilatation operator that we are considering is in fact

$$-\sqrt{\frac{c_{RR'}c_{TR'}}{l_i l_{i+1}}} n_{i,i+1}^Y(\sigma) E_{i,i+1} \quad (6.17)$$

We will state the result for the general case, for which loops move on both the X and Y Gauss graphs, using an example for illustration. The Gauss graph relevant for this example is shown in figure 7.

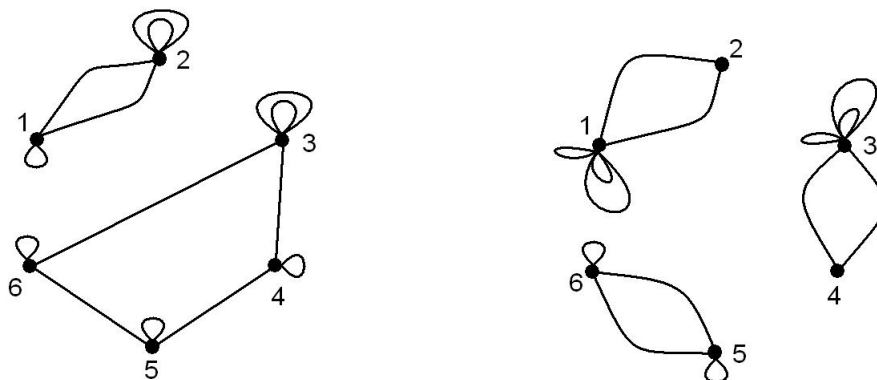


Figure 7: The graph on the left is σ^X . The graph on the right is σ^Y . Each node label in the above diagrams corresponds to a row number of Young diagram R in the restricted Schur polynomial $\chi_{R,(t,s,r)\bar{\mu}\bar{\nu}}$. The Gauss graphs shown correspond to an R with 6 long rows.

Note that σ^X has two connected components, one which has 2 nodes and one which has 4 nodes. Consequently the group relevant for the organization of the Y s is $SU(2) \times SU(4)$. Counting closed loops on the nodes in σ^Y grouped by the connected components of σ^X we find that the representation of $SU(2)$ we need is \square while the representation of $SU(4)$ we need is $\square\square\square\square$. Also, σ^Y has three connected components, each of which has 2 nodes. Consequently the group relevant for the organization of the Y s is $SU(2) \times SU(2) \times SU(2)$. Counting closed loops on the nodes in σ^X grouped by the connected components of σ^Y we find that the three representations for the three different $SU(2)$ groups we have are $\square\square$, $\square\square$ and \square . Denoting the groups that appear with a superscript

$$G^{(1)} \times G^{(2)} \times G^{(3)} \times G^{(4)} \times G^{(5)} = SU(2) \times SU(4) \times SU(2) \times SU(2) \times SU(2) \quad (6.18)$$

we can write the off diagonal terms in the dilatation operator as (the superscript on the Lie algebra element tells you which group it belongs to)

$$\begin{aligned}
D_{\text{off diagonal}} = & - \sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}} (E_{12}^{(2)} + E_{21}^{(2)}) - \sqrt{\frac{(N+l_4)(N+l_5)}{l_4 l_5}} (E_{23}^{(2)} + E_{32}^{(2)}) \\
& - \sqrt{\frac{(N+l_5)(N+l_6)}{l_5 l_6}} (E_{34}^{(2)} + E_{43}^{(2)}) - \sqrt{\frac{(N+l_6)(N+l_3)}{l_6 l_3}} (E_{14}^{(2)} + E_{41}^{(2)}) \\
& - 2\sqrt{\frac{(N+l_1)(N+l_2)}{l_1 l_2}} (E_{12}^{(1)} + E_{21}^{(1)}) - 2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}} (E_{12}^{(3)} + E_{21}^{(3)}) \\
& - 2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}} (E_{12}^{(4)} + E_{21}^{(4)}) - 2\sqrt{\frac{(N+l_3)(N+l_4)}{l_3 l_4}} (E_{12}^{(5)} + E_{21}^{(5)})
\end{aligned} \tag{6.19}$$

The specific representation we should use for each Lie algebra has been spelt out above.

7 Conclusions and Discussion

In this dissertation we have evaluated certain subleading terms in the action of the dilatation operator in the $SU(3)$ sector. The operators we have studied have a classical dimension that scales as N . Consequently, even at large N , non-planar diagrams need to be summed and the limit we study is quite distinct from the planar limit. There is by now growing evidence that the dilatation operator in the large N but non-planar limit can be mapped into the Hamiltonian of a set of decoupled oscillators and hence that this limit of the theory continues to enjoy integrability.

In the $SU(2)$ sector, a new conservation law has been found [20]. The corrections that we have evaluated spoil this new conservation law⁷ and consequently, these terms may be the first indications that the limit we consider is not integrable.

Our results clearly show that although the new terms do spoil the old conservation law, the system that emerges continues to be integrable. Indeed, the terms in the Hamiltonian that mix X and Z or Y and Z commute with the terms that mix X and Y , so that we simply need to change basis inside eigenspaces of fixed anomalous dimension. This change of basis has been reduced to the problem of diagonalizing certain elements in the Lie algebra of a well defined representation of a definite product of special unitary groups (the specific representation and product can be read off of the Gauss graphs as we explained in the last section). This is a solved problem in group theory.

The term in the dilatation operator that mixes X and Y does not act on the Z labels. The eigenproblem in the Z label, after moving to the Gauss operator basis, reduces to an oscillator problem [11, 20]. The eigenvalues of the term in the dilatation operator that mixes X and Y sets the ground state energy of these oscillators. Note however that the BPS operators, which correspond to Gauss graphs with loops that start and end on the same node but no directed segments between nodes, are annihilated by the term in the dilatation operator that mixes X and Y , so that these operators remain BPS even when the corrections we have computed are included.

Finally, it will be interesting to extend our analysis to the $SU(2|3)$ sector (and beyond) and to see if integrability persists. We leave these projects for the future.

⁷See equation (5.8).

A Giant gravitons

In this appendix we will first review the behaviour of a dipole moving in a uniform magnetic field. Then we review giant gravitons moving in the bulk in the presence of a non-zero Ramond-Ramond five form field strength and compare these two pictures. This appendix relies heavily on [27].

A.1 Dipole moving in a uniform magnetic field

Suppose we have a dipole (made of a positive charge and a negative charge) moving through a uniform magnetic field, \vec{B} , with a velocity \vec{v} . Suppose the \vec{B} field is pointing towards the page, and the dipole is moving south in the page. The positive charge experiences a magnetic force $\vec{F} = |q|\vec{v} \times \vec{B}$ and the negative charge experience a force equal to $-\vec{F}$. We are interested in the case where the \vec{B} field is constant and uniform. *In this case the dipole gets bigger the faster it moves.* A dipole moving perpendicular to the magnetic field is stretched to a size [27]

$$R = m_d|v|/B \tag{A.1}$$

where m_d is the mass of the dipole and $|v|$ is its speed. However, if we switch off the magnetic field, the charges attract and snap together. Hence, we can imagine a spring connecting the two charges, with a spring constant k . This is summarized in the figure below

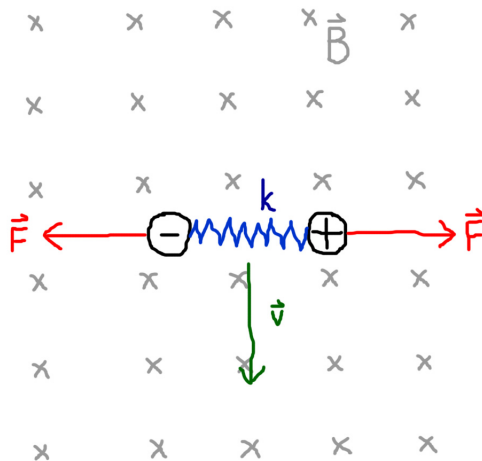


Figure 8: A permanent dipole moving through a constant and uniform magnetic field with a speed v .

Let us now imagine a dipole moving on the surface of a sphere of radius R . Suppose $+|q|$ is located at the coordinates (R, θ, ϕ) and $-|q|$ is located at $(R, \theta, -\phi)$. Here θ is the azimuthal angle and $\phi \in [0, \phi]$. Let us suppose a flux magnetic N leaves the surface of the sphere. The Lagrangian of this system is [27]

$$\mathcal{L} = -N \sin \phi \dot{\theta} - \frac{k}{2} R^2 \sin^2 \phi.$$

Thus, the angular momentum is given by

$$L = -N \sin \phi$$

which is *bounded above* by N .

A.2 Giant gravitons

The bulk (i.e. the $AdS \times S$ background) has a non-zero Ramond-Ramond five-form field strength switched on. This flux couples to D3-branes. Gravitons carry a D3 dipole charge and are hence polarised by the background flux [37].

A graviton moving on the S^5 of $AdS \times S$ expands into a spherical D3-brane [27]. The D3-brane wraps an S^3 which moves in S^5 . The size of the membrane increases with increasing angular momentum until it has a radius equal to the radius of the S^5 at which point it can't grow any further [27]. At this point ($R = R_{S^5}$), the angular momentum is maximal and equal to the number of five-form flux units on the S^5 , N .⁸ These are called *sphere giant gravitons*. Sphere giant gravitons are identified with Schur polynomials [22] with completely antisymmetric irreps, i.e.

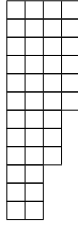


Irreps can have at most N rows, where N is the dimension of the gauge group $SU(N)$. If we have c columns, then we have c sphere giant gravitons. Hence, there is no upper bound on the number of sphere giant gravitons. For example, the Young diagram

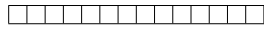
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$$\int_{S^5} F_{\alpha\mu\nu\rho\sigma} dx^\alpha \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = N \tag{A.2}$$

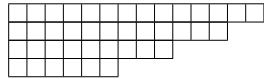
where $F_{\alpha\mu\nu\rho\sigma}$ is the Ramond-Ramond five-form flux and \wedge denotes the wedge product.



can be identified with a system of four sphere giant gravitons, two having the same angular momentum L and the other two with different, smaller angular momentum. One can also have a graviton moving in the AdS_5 or $AdS \times S$. In this case, the gravitons can expand to an arbitrary size. As a result, AdS giant gravitons have no upper bound on their angular momentum. However, one can only have at most N AdS giant gravitons. AdS giant gravitons are identified with Schur polynomials with completely symmetric irreps, i.e.



Completely symmetric irreps can have an arbitrary number of boxes. If we have g rows, then we have g AdS giant gravitons. For example, the Young diagram



can be identified with a system of four AdS giant gravitons, all with different angular momenta.

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