

**Enhancing mathematics teachers' mediation
of a selected object of learning through
participation in a learning study:
the case of functions in Grade 10**

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Declaration

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Vasen Pillay

12th day of November in the year 2013

Abstract

This thesis explores the potential of learning study as a teacher professional development model. The learning study is underpinned by variation theory and the judicious use of examples, with the goal of enhancing teachers' mediation of a selected object of learning. The topic of functions at grade 10 provides the context of the content in which the learning study was implemented. More specifically, the object of learning identified by the teachers was to improve the learners' ability to identify and name the class of function given its algebraic representation. In the teaching of mathematics, examples form a key resource for teachers to introduce concepts. Within the context of this study the judicious use of examples means that examples are selected and sequenced in a particular fashion so as to create an example space which keeps the critical feature of the object of learning in focus and thereby provides learners with opportunities to discern the object of learning. In selecting and sequencing examples, I draw on principles inherent in variation theory, particularly the idea of varying one aspect whilst keeping other aspects invariant.

The learning study cycle reported on in this thesis comprised four lessons. The critical feature for the selected object of learning emerged after the second lesson in the cycle and was focused on only in the last lesson of the cycle. Focusing on the highest power of the independent variable emerged as the feature that enabled learners to discern the object of learning.

Describing the enacted object of learning forms the substance of the data analysed in the thesis, with analysis following the production of data from the lesson transcripts. Data production was accomplished by drawing on Bernstein's theory of the pedagogic device, and specifically the evaluative rule since it concerns itself with the transmission of criteria as to what counts as valid knowledge. To construct a more general account of the operation of evaluative judgement I follow the work of Davis who recruits Hegel's theory of judgement. Hegelian judgement is used to elaborate Bernstein's evaluative rule and in doing so, a theory of learning is backgrounded. In this study, I bring a theory of learning into focus by drawing on the principles of variation theory as it provides tools that enable a reading of the opportunities the pedagogy creates that enable learners to see 'something' in a certain way.

This study demonstrates and so confirms that irrespective of the context in which you work: i) when a critical feature is identified, ii) examples are carefully selected with the critical feature in mind, and iii) if the critical feature is in focus for the teacher, increasingly more learners will begin to discern the intended object of learning.

Keywords:

Critical feature

Examples

Functions

Learning study

Object of learning

Teacher Professional Development

Variation Theory

To my wife, Saras, and my daughter, Kimesha

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Abbreviations

CAPS	Curriculum and Assessment Policy Statement
CoP	Community of Practice
CTPD	Continuing Professional Development
DBE	Department of Basic Education
DIPIP	Data Informed Practice Improvement Project
DMJ	Developing Mathematical Judgement
DoV	Dimension of Variation
FET	Further Education and Training
HET	Higher Education and Training
MKT	Mathematical Knowledge for Teaching
MQI	Mathematical Quality of Instruction
PLC	Professional Learning Communities
Pm ⁺	Process with mathematics in focus
Pm ⁻	Process with mathematics obscured
TPD	Teacher Professional Development
WMC-S	Wits Maths Connect: Secondary School Project

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Chapter 1

Introduction to the study and research question

1.1 Introduction

The musicians of the orchestra have taken their places on stage and are tuning their instruments. The sound that emanates from the stage because of what the musicians are doing is now unpleasant to the ear. After a few minutes, the musicians gradually stop tuning, the stage becomes quiet and the conductor of the orchestra steps on to the stage and opens his music score to the first music piece of the performance. To start the performance the conductor taps his baton on his music stand. The musicians then begin to play their instruments in accordance to the conductor's movement of the baton. The sound that now emanates from the stage is melodious to the ear. What lies at the centre of this transition? All the musicians in the orchestra are driven by a goal, which is to play their instruments with excellence. However, there is an even bigger goal. The conductor of the orchestra is the person to control the achievement of this bigger goal, which is to play the whole piece of music as well as possible (the object of performing). By moving the baton, the conductor will encourage musicians to play louder or softer according to how he interprets the composer's musical score. If the conductor of the orchestra does not keep this object in focus, the performance could be a disaster.

One of the essential requirements for the conductor is that he has to be intimately familiar with all the music for the performance. This means that for each piece, the conductor needs to know how each of the instruments relates to the musical score, as well as the individual pieces of music of each section of the orchestra (i.e. strings, woodwind, brass and percussion instruments), and all of this in relation to the full musical score. The successful performance of the music is only possible if the conductor is an expert.

The role of a mathematics teacher is similar to that of the conductor of an orchestra. As the conductor is the person in control of the object of performance, so too is the mathematics teacher the person responsible for the object of teaching mathematics. The object of teaching mathematics is to teach for what Kilpatrick, Swafford and Findell (2001) refer to as mathematical proficiency. In the bid to accomplish this goal the mathematics teacher needs to be intimately familiar with the content of the topics being taught, as well as how his/her students are likely to interact with that content. It also means that the teacher needs to be knowledgeable about the trajectory of the topics being taught, thus bringing to the fore what Shulman (1986) referred to as teacher's subject matter

knowledge, curriculum knowledge and pedagogical content knowledge. In addition, the mathematics teacher needs to know that in his class of diverse learners they are not all going to learn in the same way. To engage all learners and to help them to learn, the use of examples in the mathematics class is an essential ingredient.

The conductor metaphor represents teachers' engagement with the whole class, leading the group while still paying attention to the individual performances. The virtuoso image also illustrates that as the conductor is managing a number of musicians with their different instruments during a performance, the mathematics teacher is managing a diverse group of students. Of course, these are not the same and therefore there are limitations to the metaphor.

The musicians playing in an orchestra are professionals. They are able to play their musical instruments with precision and excellence or else they would not have been selected to play in the orchestra. In addition, all these musicians are able to read music well and are able to engage with the relevant piece of music without being distracted by what the other musicians are doing. Furthermore, while reading their piece of music and playing their instrument in relation to the entire musical piece they are also paying attention to the conductor's movement of the baton. They are able to interpret the movement of the baton and this enables them to maintain the conductor's tempo and play in tune. A school mathematics teacher also has to orchestrate the mutual participation of a whole class of learners. The role of a mathematics teacher thus shares some similarities with that of the conductor of an orchestra, however, there are additional demands placed on the teacher. For instance, unlike the musicians, the learners in the mathematics classroom are not 'professionals' (mathematicians). The conductor of the orchestra is not responsible for teaching the musicians how to read music or play specific instruments, whereas teaching mathematics is a fundamental role of a mathematics teacher.

The metaphor of the conductor of an orchestra is useful in so far as thinking about the orchestrating role of the conductor and relating that to the role of the mathematics teacher. In the mathematics classroom, the teacher will aim to mediate the learners' learning of a specific topic or concept by engaging learners in various activities. For example, the kinds of examples the teacher chooses, and how these are sequenced are some of the many activities a teacher could engage in. In a similar fashion to the conductor, it is the role of the mathematics teacher to keep the focus of the lesson in view. It is also the teacher's role to constantly engage with learners and provide them with opportunities to think about, discuss and complete activities that will enable them to encounter what it is they are to be learning. Following the work of Marton, Runesson and Tsui, I refer to that which

the learners are to be learning as the object of learning which “can be defined by its *critical features*” (Marton, Runesson, & Tsui, 2004, p. 24). The concept, object of learning and critical features as well as the theory of variation more broadly will be elaborated on in Chapter 3. As the conductor is ultimately concerned with the *overall music played together*, the teacher is concerned with the learning opportunities he or she is providing for learners *to discern the object of learning*.

Therefore, what is expected of a teacher to ensure that the activities (including the selection and sequencing of examples) that he/she decides to use in the class are activities that will keep the object of learning in focus? What does the teacher need to know and know how to do with respect to the particular object of learning i.e. the particular content, and the capabilities he or she expects of learners with respect to that content?

Let us move into the classroom of a mathematics teacher, Nash¹, who generously enabled me to study his practice for my master’s research. Nash is a competent teacher and is recognised as such because of his learners’ good results in tests and examinations. In the extracts that follow, I illuminate aspects of Nash’s practice that suggest his orchestration of mathematics has limitations, limitations that in the literature are often referred to as being procedural in orientation. Coupled to his procedural orchestration of mathematics is that some key features of the object of learning are not always in focus for Nash.

The procedural orientation to some of Nash’s orchestration of mathematics has limitations, as will be shown. Consider the following explanation provided by Nash as he goes about demonstrating how to draw the graph of the function $3x - 2y = 6$ by making use of the dual intercept method. The extract which follows illuminates the procedural orientation of his orchestration of mathematics as being an instruction made up of a set of steps that one needs to follow in order to accomplish the task at hand.

Nash: First make your x equal to zero, that gives me my y-intercept. Then the y equal to zero gives me my x-intercept. Put down the two points. We only need two points to draw the graph.

Lr 1: You don’t need all the other parts?

¹ Pseudonym used for the teacher who participated in the study conducted by Pillay (2006). Nash was described as a teacher who was well respected by his colleagues in the mathematics department and other teachers in the school. Nash’s practice was dominated by teacher-driven explanations and the school in which Nash taught was an ‘ordinary’ secondary school. The school’s performance in the grade 12 national mathematics examinations was considered to be successful.

Nash: What's important features of this graph? We can work out from here (points to the graph drawn) we can see what the gradient is. Is this graph a positive or a negative?

Lrs: (chorus) positive.

Nash: It's a positive gradient. We can see there's our y- intercept, there's our x-intercept (points to the points (0;-3) and (2;0) respectively)

(After a brief discussion on the labelling of points on a graph, learners 2 and 3 ask Nash)

Lr 2: Sir, is this the simplest method sir?

Lr 3: How do you identify which side must it go, whether it's the right hand side

(Nash interrupts)

Nash: (response to Lr 2) You just join the two dots.

Lr 2: That's it?

Nash: Yeah the dots will automatically, if it was a positive gradient it will automatically. If this was (refers to the line just drawn) negative, that means this dot (points to the x-intercept) will be on that side (points to the negative x axis) because if the gradient was negative, how could it cut on that side? (points to the positive x axis).

Lr 2: Is this the simplest method sir?

Nash: The simplest method and the most accurate.

Lr 4: Compared to which one?

Nash: Compared to that one (points to the calculation of the previous question where the gradient and y-intercept method was used) because here if you make an error trying to write it in y form, that means it now affects your graph. Whereas here (points to the calculations he has just done on the dual intercept method) you can go and check again. You can substitute. If I substitute for 2 in there (points to the x in $3x - 2y = 6$) I should end up with 0 .

(Pillay, 2006, lesson 3, time interval 33:29)

In his orchestration of the mathematics Nash generalises that the dual intercept method is the simplest and most accurate method for drawing any straight line graph. Absent here, and in the continuation of this lesson, is any engagement with horizontal and vertical straight lines, whose equations stand as examples that do not lend themselves in any straightforward way to this procedure. The general class of lines $y = n$ or $x = n$ (which many learners do not consider to be straight lines as 'they have no slope') are excluded. In his attempt to provide an error-proof method for drawing a line given its equation, and so assist his learners and their performance, he excludes the case of vertical and horizontal lines – the conductor has not fully understood how one section of the orchestra fits in with the full musical score.

In the next extract Nash provides his learners with an explanation as to why a straight line is a function:

Nash: Now why function? All the time we've been saying there's some kind of relationship, now we saying there's simply a function – from a relation it means they were husband and wife – now they having a function – are they getting married now or are they getting married before – a function just basically means that for every x value (points to the table on the board) I've got a unique y value ... for every x value there's a unique y value – one x value doesn't have two different y values [...] So every husband (points to the x value on the table) has got one unique wife (points to the y value on the table) – so that's why we say linear and it is a function.

(Pillay, 2006, lesson 1 time interval 6:17)

Here we see Nash engaging in the use of a metaphor to explain the concept of a function. The use of metaphors is common in teaching and important to the teaching of mathematics (Nolder, 1991) but it is not just any metaphor that can be used, the metaphor actually has to have mathematical integrity i.e. the metaphor for the concept must not only make everyday sense, it must also make mathematical sense. The above extract illustrates that Nash makes up metaphors by drawing on his own understanding, experiences and assumptions. In Nash's attempt to reflect on the concept of function with learners, he shifts out of the mathematical domain and into the non-mathematical domain of marriage, and so provides a context through which learners could make some sense of the mathematics. In the South African context where curriculum change has encouraged such 'relevance' (Nyabanyaba, 1998; Sethole, 2004) it is not surprising that Nash attempts to use a metaphor from a more familiar domain. The problem in this instance is that the metaphor that Nash uses is not mathematically robust since for the metaphor to hold one is forced to consider only a monogamous relationship between a husband and his wife. This precludes same sex marriages and polygamous relationships as allowed in some African cultures (our president Jacob Zuma being a good example). Nash's orchestration of mathematics in this instance illustrates that key features of the concept of function is not in focus for him and so the metaphor he uses does not illuminate the concept of functions for the learners. Indeed it could be argued that such examples increase the demand placed on the learners since they have to grapple with the concept of function and at the same time recognise that the shift into everyday life made by the teacher adds no mathematical value to the concept of function.

The conductor has to know ‘*what*’ the pieces are that make up the musical score and ‘*how*’ these pieces come together to ensure an excellent performance. Similarly, the mathematics teacher needs to know ‘*what*’ the object of learning is, ‘*what*’ its critical features are and ‘*how*’ the object of learning has to be operated on so as to ensure that there is integrity in the mathematics being done. The first example illustrates a problem in relation to the ‘*what*’ for Nash because he gives learners access to a more limited ‘*what*’ and this is in relation to ‘*how*’ he presents it. The second example also illustrates a problem around the ‘*what*’ for Nash in relation to the metaphorical explanation he provides. I argue that both in instances his orchestration of mathematics stands as a challenge to him achieving the bigger goal (improved learner understanding and performance) – ‘the conductor is not fully in tune with the intentions of the composer’, nor with how to connect with his players. In the context of teaching, the overall goal is not only a co-ordinated or orchestrated lesson, but more mathematically proficient learners (Kilpatrick *et al.*, 2001). Despite this, the performance of Nash’s students in the tests and examinations are considered to be successful by both the school and the education authorities.

So, what opportunities could Nash be provided with so as to participate in a practice where he plays more in harmony with what is mathematically accepted within the mathematics community and for his learners? What would it mean to provide Nash with such opportunities? In the section that follows, I provide a possible answer to this question which creates the backdrop for me to introduce the problem statement that will underpin this study.

1.2 The problem statement and research question

Nash is similar to many other mathematics teachers, both nationally and internationally, in that he enables what might be called a limited view or experience of mathematical knowledge and mathematical practice. This is further illuminated by Artigue (2012) who argues that “when the teachers attempt to adapt their practices to the predominant socio-constructivist discourse, for example by setting more open problems for the pupils, ostensibly to engage them in investigative practices, the results are not necessarily satisfactory” (p. 22). This is demonstrated in Nash’s practice when he tries to bring in the everyday context through a metaphor that does not hold the mathematical integrity of the concept in focus. In fact Davis (2010) and Venkat and Adler (2012) show that the manner in which other teachers orchestrate the mathematics are more problematic in comparison to Nash. There are more substantive problems and in some instances the object of learning is not even in focus. In terms of Nash’s practice, one can suggest that he could extend his practice through deeper reflection on metaphors – metaphors which do and do not support the mathematical integrity of a concept. He could also extend the range and sequence of examples he

uses and so illuminate a more elaborate notion and keep it in focus. But Nash does not see that the kinds of explanations he provides, as a result of the metaphors and examples he uses, limits learners' access to the concept in focus. He does not see this because it is possible for his learners to perform successfully if they can display some procedural fluency.

Margolinas, Coulange and Bessot (2005) argue that a teacher's practice is only disturbed when there is an external intervention. Artzt and Armour-Thomas as cited in Margolinas *et al.* (ibid.) argue that an important feature of an external intervention is the type of intervention that will enable "teachers to reflect on their practice from a cognitive perspective" (p. 229). So in Nash's case, if he was provided with opportunities to step back from his practice and then be assisted with looking back at his practice as was done in the previous section, then the process of disturbing his practice could begin. Margolinas *et al.* (ibid.) make an even bolder claim by saying that the conditions needed to stimulate this kind of reflection will not necessarily be found in the practice of teaching alone. In other words, from their perspective, in attempting to gain some principled understanding and a deep grasp of some mathematics education research, a teacher will need to distance himself/herself from his/her practice. To do this, the teacher will require something to assist with creating the distance from practice and then to aid with reflecting back on the practice. They suggest that "significant change may be brought about by external influences when teachers interact in groups with the potential for strong internal dynamics" (Ponte *et al.*, 1994 as cited in Margolinas *et al.*, 2005, p. 229).

There are thus two issues to be confronted in any professional development activity that Nash might wish to embark on. Firstly, how might Nash keep the object of learning in focus, so that when he orchestrates the mathematics for his learners, the lesson is more harmonious? Harmony in this sense refers to both the integrity of the mathematics and the integrity of learners² being taught – the conductor of the orchestra needs to respect the music of the composer as well as the musicians in the orchestra to ensure a successful rendition of the musical piece. This exemplifies to some degree the idea of a critical feature, which is found in the mathematics but in relation to the learners being taught. As indicated earlier the concept of object of learning and critical feature will be elaborated on in Chapter 3. For Nash the intervention needs to focus on the object of learning and features of the object of learning that are critical for the discernment of a more complete notion. Secondly, how might one begin to disturb Nash's practice, to create some kind of self-reflective conflict? In this

² Teaching is often spoken about as being Janus-faced, and in conversation during the AFRICME 4 conference (2013), John Mason described that being a good teacher means respecting both the mathematics and the learners.

study I explore a professional development model that both disturbs practice and brings the object of learning into focus. To this end, my intended study can be encapsulated as follows:

To what extent and how does a learning study professional development model, underpinned by variation theory and the judicious use of examples, enhance mathematics teachers' mediation of a selected object of learning – the case of functions in Grade 10?

In the next section, I locate the gap in which my study is located, that is why this study and the general framing question are worthwhile. I elaborate on and clarify the critical questions that will focus my study after the literature review section.

1.3 The 'gap' in which this study is located

During the mid-eighties Lee Shulman suggested that teacher education research of that era was overlooking the central role of content and subject matter, a phenomenon he called the “missing paradigm”:

The missing paradigm refers to a blind spot with respect to content that now characterizes most research on teaching and, as a consequence, most of our state-level programs of teacher evaluation and teacher certification....What we miss are questions about the *content* of the lessons taught, the questions asked, and the explanations offered. From the perspective of teacher development and teacher education, a host of questions arise. Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?

(Shulman, 1986, pp. 7-8, italics in original)

In coining this phrase, Shulman did not merely call for the inclusion of “both knowledge of general pedagogy and knowledge of subject matter as equally important yet separately engaged components of a teacher’s knowledge base. Rather, he advocated the need to explore the inherent relationship between the two through what he termed ‘pedagogic content knowledge’” (Segall, 2004, p. 489). Shulman (1986) further distinguished between two other important categories of teachers’ knowledge base that includes subject matter, content knowledge and curricular knowledge.

More recently an entire volume in the *International Handbook of Mathematics Teacher Education*, edited by Peter Sullivan and Terry Wood (2008), was devoted to examining the role of teacher knowledge and beliefs, which is an important aspect of mathematics teacher education. In 2009 a

special issue of the journal *For the Learning of Mathematics* which came out as volume 29, number 3 and was edited by Jill Adler and Deborah Ball, is dedicated to teachers' professional knowledge and its use in practice. A book edited by Tim Rowland and Kenneth Ruthven (2011) titled *Mathematical Knowledge in Teaching* is also dedicated to examining the issue of mathematical subject knowledge in teaching.

These three collections that have come out recently show that teachers' knowledge and its use in practice is still the focus of extensive research and policy debate globally. I am only going to refer to two major contributions in this field to demonstrate that while they both engage the field what remains absent is how teachers acquire professional knowledge for use in practice – hence my study.

Building on Shulman's work is the work of Deborah Ball and her colleagues³, which can be seen as two interrelated strands. Firstly, by examining the practice of teaching Ball and her colleagues (Ball, Thames and Phelps, 2008) found concrete ways to elaborate, describe and measure teachers' subject matter knowledge of mathematics. They refined Shulman's categories of content knowledge and this refinement categorises i. common content knowledge, ii. horizon knowledge and iii. specialised content knowledge as *Subject Matter Knowledge* on the one hand and knowledge of i. content and teaching, ii. content and students and iii. content and curriculum as domains of *Pedagogical Content Knowledge* on the other hand. They argue that it is productive to think about the kind of mathematical work that teachers do as a special kind of mathematical problem solving enacted in the practice of teaching (Ball, Bass and Hill, 2004). In Nash's case there is something about what he needs to know about functions i.e. the specialised knowledge he needs as a teacher – suitable examples to keep the object of learning in focus and metaphors that are mathematically robust.

Secondly, Ball and her colleagues (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008), have also tracked the relationship between 'teachers' mathematical knowledge for teaching' (MKT) and the 'mathematical quality of instruction' (MQI). In order for them to get a handle on this relationship they recruited ten teachers to participate in the study based on their (the ten teachers) commitment to attend professional development workshops, a form of an intervention strategy. Ball and her colleagues were not studying the intervention strategy as such, but they were able to use the intervention strategy to look at the relationship between MKT and MQI. They concluded that there

³ For the work of Ball and her colleagues refer to Ball, Thames and Phelps (2008); Hill, Ball and Schilling (2008); Hill, Blunk, Charalambous, Lewis, Phelps, Sleep and Ball (2008) and Hill, Rowan and Ball (2005).

is a powerful relationship between what a teacher knows, how the teacher knows it and what the teacher can do in the context of instruction. In other words, they found a strong positive correlation between the MKT and MQI. They also discuss i. the use of curriculum materials, ii. teachers' beliefs about mathematics, and iii. the effects of teacher professional development as some of the factors that could either support or hinder teachers' use of knowledge in practice.

The resources that Ball and her colleagues provide us with are measures of the mathematical knowledge that teachers hold and use as they go about their work of teaching. In addition, they have used these measures to relate teachers' knowledge to the quality of teaching and learner performance. However, they do not make explicit the theory of pedagogy that underpins the framework that they use for describing MQI, and so their lesson analysis. Furthermore, the methodology that informs their reading of the mathematical work that teachers do during the temporal unfolding of a lesson is also implicit.

Building on the work of Ball and her colleagues, Jill Adler and her colleagues⁴, through the work of QUANTUM⁵, provide us with an explicit methodology for describing the temporal unfolding of mathematical work of teaching. This methodology is rooted in Hegel's theory of judgement as interpreted by Davis (2001; 2005) – an abstract theory of how concepts come to be acquired and Bernstein's theory of pedagogy. Particularly, Bernstein's elaboration of the 'pedagogic device' and the significance of evaluation, in the general sense, in pedagogic practice Bernstein (1990; 1996). Another important aspect of the methodology is the idea of an evaluative event (Davis, 2005); (Parker & Adler, 2012 Online First)⁶.

Adler and her colleagues have engaged with the study of mathematics in teacher education but engaged with how it is produced. While this indicates opportunities to learn specialised knowledge, it does not engage with what is learned, or the lived object of learning (Marton *et al.*, 2004). The work of Ball and her colleagues provide indications that attention to teachers learning mathematics for teaching is under way (Ball and Forzani, 2009). They argue that teaching practice should be at the core of teacher education curriculum. The reference to this kind of work being under way in 2009 is evidenced by a range of studies that take this idea forward, for example the work of

⁴ For the work of Adler and her colleagues see Adler and Davis (2006a); Adler and Davis (2006b); Adler, Davis, Kazima, Parker and Webb (2005); Adler and Pillay (2007); Kazima and Adler (2006) and Kazima, Pillay and Adler (2008).

⁵ QUANTUM is the name given to an R&D project on quality mathematical education for teachers in South Africa. The development arm of QUANTUM focused on **q**ualifications for **t**eachers **u**nderqualified in **m**athematics (hence the name) and completed its tasks in 2003.

⁶ For a detailed discussion of this framework see Davis (2005) and Parker and Adler (2012 Online First) for its elaboration in teacher education.

Prediger (2010) which contributes to the on-going discussion of mathematics for teaching by investigating the case of understanding learners' perspectives on equations and equalities and on meanings of the equal sign; in 2011 an entire issue (issue 3) of the *Journal of Mathematics Teacher Education*, which was made up of four articles, was devoted to illustrating the extensiveness of the knowledge that teachers need and the nature of experiences that might assist teachers in learning that knowledge. In 2012, issue 3 of the same journal was dedicated to papers which reported on studies that demonstrate that effective teaching demands sound teacher knowledge. Most of the studies reported on in these publications were studies which focused on pre-service teachers and the courses they take in their undergraduate studies. My study focuses on work with in-service teachers and so work in professional development which focuses on teachers learning mathematics for teaching is part of the gap in which my study is located.

The work of these two established researchers (Ball and Adler) complement each other and is located in an area of research which is beginning to focus on teachers' learning of mathematics for teaching. My study is located within this emerging area of research, since my concern is related to teachers' professional knowledge and its use in practice – and then the opportunities to learn this specialised knowledge. Since I embarked on this study there has been increasing attention to this area of research, as evidenced by the dedicated issues of journal publications related to this area of study. In the following section, I provide further reasons for undertaking this study.

1.4 Further rationale for undertaking this study

Improvement in learner performance in mathematics is central to ensuring that the economy of the country improves. For example, South Africa is in need of suitably qualified teachers, doctors, scientists and many other scientifically oriented professionals. With the generally poor status of mathematics results it is possible that such a system will not be able to produce enough learners who qualify to enrol at universities to pursue further studies so that they become appropriately qualified to address the professional demands of the country. Furthermore, to provide employment for all, either through job creation or employment in the labour market, a level of scientific and technological advancement that will enable growth and expansion of the economy is needed: South Africa is far from this ideal situation. Therefore, improvement in learner performance is extremely important because the lack of expertise influences the general economic outlook of the country.

The National Policy Framework for teacher education and development in South Africa was approved by the Minister of Education and gazetted in April 2007 (DoE, 2007). One of the purposes of this policy is to provide a strategy for the successful professional development of teachers in the

country. A principle upon which the strategy for continuing professional development (CPTD) is conceived is as follows:

CPTD succeeds best when teachers themselves are integrally involved, reflecting on their own practice; when there is a strong school-based component; when activities are well co-ordinated; and when employers provide sustained leadership and support.

(DoE, 2007, p. 11)

The policy does not elaborate any further on the descriptors of what it envisions for a successful CPTD nor does it make any suggestions of the nature of suitable models of teacher professional development (TPD) that embraces these descriptors. In 2011 the Departments of Basic Education and Higher Education and Training developed the Integrated Strategic Planning Framework for Teacher Education and Development on South Africa (DBE & HET, 2011). This framework provides some practical basis for developing implementation plans for a teacher education development system as a whole. One of the outputs and activities which it envisions to be led by the provincial education departments is to establish Professional Learning Communities (PLCs) to strengthen teacher professionalism. The plan articulates that amongst other activities, PLCs will allow groups of teachers to work “together to learn from video records of practice and other learning materials” (DBE & HET, 2011, p. 14). Absent from the discussion is the expertise that is required for the establishment of PLCs and the knowledge base that is needed to engage with records of practice.

A challenge and possible contribution of this study is that many of the theoretical bases that inform debate in mathematics education are theories of learning, whereas the focus of this study lies in the practices of teaching, which is an emerging field of study. Therefore, another reason for undertaking this study is to gather empirical evidence that speaks back to policy. In speaking back to policy I present a model of TPD that fits with the policy’s vision and in so doing elaborate on what the descriptors of a successful CPTD mean in relation to the model.

In summary, any attempt to support teachers in their bid to strengthen and enhance their practice will need to include disturbance of their practice. Disturbance of practice includes a focus on the object of learning and its related critical features, and how this can be kept in focus and with mathematical integrity in pedagogic activity. This is the link to my intended study, i.e. *to explore whether and how a learning study professional development model, which is underpinned by variation theory and the judicious use of examples, works to enhance mathematics teachers’*

mediation of a selected object of learning. In addition, I have situated this study within the context of relevant and influential research in the field viz. the work of Ball and her colleagues that builds on the work of Lee Shulman, and the work of Adler and her colleagues whose work complements that of Ball and her colleagues. I have also introduced the research question that directs this study and my rationale for wanting to engage with this study.

1.5 Structure of the thesis

Since this study is related to teachers' professional knowledge and its use in practice, I commenced this chapter by highlighting the complex nature of teaching by making use of the metaphor of a conductor of an orchestra. Chapter 1 also illuminates the gap in which this study is located and provides a rationale for undertaking this study. In this chapter the reader is also introduced to the general framing question that drives this study.

This study is about exploring whether a learning study professional development model which is underpinned by variation theory and the judicious use of examples could work to improve teachers' mediation of a selected object of learning. The section of functions at the grade 10 level provides the context of content within which this study is done. In Chapter 2, I discuss the importance of the use of examples in mathematics teaching and what I mean by the judicious use of examples. I also provide reasons for choosing the concept of function to provide a context for this study. The question that drives this study rests on four pillars viz. learning study as professional development model, variation theory, the judicious use of examples, and the teaching of functions. To this end, I start Chapter 2 by reviewing literature relevant to three of the four pillars (*learning study* – a form of a PLC that has the potential to address policy demands in terms of continuous teacher professional development so as to strengthen teacher professionalism; *functions* – its importance and some pedagogical perspective, and *examples* – its significance to the teaching of mathematics). The fourth pillar, viz. variation theory, will be discussed in Chapter 3 as it complements the theoretical framework that underpins this study. I conclude Chapter 2 by introducing the focus question that underpins the general framing question of the study.

Bernstein's theory of the pedagogic device, in particular the evaluative rule, forms the overarching theory that frames this study, which is discussed in Chapter 3. However, Davis (2005) reports that in Bernstein's theory of the pedagogic device there is no detailed account of the workings of the evaluative rule. Following the work of Davis (ibid.) I recruit Hegel's theory of judgement, as interpreted by Davis for use in education, to operationalise the evaluate rule for use in this study.

Absent from the theory thus far is a theory of learning, and it is here that I zoom in on the fourth pillar that supports this study viz. variation theory.

In Chapter 4, I make explicit the methodological approach taken for this study and the reason for its choice. I then move on to setting the scene for this study by describing who the teachers are and the schools in which they teach. I conclude the chapter by providing an overview of how the learning study was implemented and what data collection strategies were employed.

In Chapter 5, I develop the data production tools and provide the external language of description with which to read the enacted object of learning. I also engage with issues of rigour in research and ethical considerations.

Chapter 6 engages with the identification of the object of learning and concludes with the planning of the first lesson of learning study cycle – *the intended object of learning*.

In Chapter 7, I provide a description of each of the lessons. Chapter 8 follows with a description of what comes to be constituted as the enacted object of learning in each of the lessons – *the enacted object of learning*.

Chapter 9 sees a discussion of learning gains made by both the learners and the teachers as a result of their participation in this study – *the lived object of learning*.

Finally, Chapter 10 concludes the study. In this chapter I discuss the value of doing the study and so my contributions to the field. I also provide some recommendations and then conclude the chapter by highlighting the limitation of the study and put forth ideas for further research.

Chapter 2

Review of literature relevant to the study

2.1 Introduction

There are three key components to this study and these are:

- i. Using learning study with its underpinnings in variation theory as a teacher professional development model
- ii. A focus on functions at the grade 10 level
- iii. Examples and their judicious use in practice.

The review of literature that follows illuminates each of these components. I begin with a discussion on TPD with a focus on the South African context and my reason for this focus is explained in the section which follows. The discussion focuses on two TPD projects and illuminates that whilst the three components of any TPD model (the community, artefacts of practice and content) are always present, not all components are in focus simultaneously. I contrast these two projects to a learning study to illuminate which of the three components are foregrounded in a learning study. The content focus for the learning study under examination in this thesis is functions at the grade 10 level and I thus move on to review literature relevant to the function concept. In the teaching of mathematics, and in this case functions, the use of examples plays an important role. In view of this, I also review literature on examples and exemplification in the teaching and learning of mathematics, thus unpacking what I mean by judicious use of examples as discussed in Chapter 1. I conclude this chapter by revisiting the research questions that underpin this study.

2.2 Teacher Professional Development

Improving something as complex and culturally embedded as teaching requires the efforts of all the players, including students, parents, and politicians. **But teachers must be the primary driving force behind change.** They are best positioned to understand the problems that students face and to generate possible solutions.

(Stigler & Hiebert, 1999, p. 135, bold - own emphasis)

It is a well-known fact that a wide range of professionals and business people participate in professional development in order to improve one's skills and knowledge with respect to one's profession, thereby enriching one's practice. This is a phenomenon which occurs worldwide. In

education, there is universal agreement that TPD is an important factor in improving the quality of instruction that in turn impacts learner performance positively. There is, however, less agreement about the nature of professional development which is most effective. Drawing on the work of Little (1993), Matos, Powell and Sztajn (2009) argue that many of the changes and current approaches to TPD are not only a function of the changes that characterise the educational landscape (curriculum reform, new assessments, equity and diversity, approaches to the social organisation of schooling and the professionalisation of teaching) but also a function of the significant changes in understanding of what constitutes learning. Using Sfard's (1998) two metaphors for learning Matos *et al.* (ibid.) describe the shifts in TPD over the years as a move from the '*acquisition metaphor for learning*' to the '*participation metaphor for learning*' and so from a *training model* of professional development to a *practice-based model* of professional development. As a result of this shift, current initiatives in the professional development of mathematics teachers are built around the theory of a *community of practice* (CoP) (Wenger, 1998) or ideas inherent in the concept of a *professional learning community* (PLC) (Stoll & Louis, 2007).

2.2.1 Mathematics teachers' learning as a form of CoP or PLC

There are numerous studies that focus attention on TPD through participation in a CoP across different fields of study, for example mathematics education (Graven, 2002a; 2002b; 2003; 2005); science education (Palincsar, Magnusson, Marano, Ford, & Brown, 1998) and online technology (Schlager & Fusco, 2006). There are also numerous studies across various fields that also focused on TPD through participation in PLC, for example science education (King & Newmann, 2000); mathematics education (Brodie, 2011, 2013; Brodie & Shalem, 2011) and physical education (Armour & Yelling, 2007). Since my study focuses on mathematics education specifically and is conducted in a South African context, for the purposes of the ensuing discussion I focus on the earlier work of Graven which focuses on TPD through a CoP and Brodie's recent work which also focuses on TPD through establishing PLCs, both of which are informed by the wider field. The work of both Graven and Brodie are based in the South African context and rooted in mathematics education and so have direct bearing on my work.

Graven's (2002b) work focused on understanding the nature of teacher learning within an in-service TPD program during the period of major curriculum change which was a result of South Africa moving into a post-apartheid period. She explains that the primary assumptions which informed both the in-service program that she ran as well as her research was that: "(a) teacher learning would be enhanced by stimulating participation within a community of practice where members of the community of practice would provide support for teacher learning; (b) implementation of the

new curriculum would involve changes in teacher roles and teachers' 'ways of being' (identities)" (Graven, 2004, p. 181). Having these as the set of assumptions upon which her work was based, Graven explains that she turned to the work of Lave and Wenger and was most drawn to the central concepts of identity and community of practice (Graven, *ibid.*). The relevance of the concept of identity to teacher learning as argued by Lave and Wenger is that "learning and a sense of identity are aspects of the same phenomenon" (Lave and Wenger, 1991 as cited in Graven, 2002a, p. 23). Graven indicates that although the participants in her study were teaching mathematics, they did not necessarily study or intend to be mathematics teachers. As such, these particular teachers did not identify themselves as mathematics teachers. In view of this, one of the areas of focus in Graven's work was thorough teachers' participation in the CoP to help them "to 'become' *mathematics teachers* in terms of mathematical competence and confident identification with mathematics teaching as their profession" (Graven, 2004, p. 189). To address the challenge of building teachers' identity so that they identify themselves as mathematics teachers, Graven focused the in-service workshops to develop the teachers' mathematics content knowledge for teaching (Graven, *ibid.*).

In summary, teachers' identity, more specifically teachers identifying themselves as mathematics teachers by profession, is in the foreground of Graven's work. In other words, Graven is concerned about teachers as professionals, about teachers working in a CoP to support each other as they travel on a journey aimed at re-aligning their professional identity to teachers of mathematics. This is not to say that in this TPD model the artefacts of teaching⁷ and the specificity of the mathematics content were absent – they are always present but backgrounded.

Figure 2.1 provides a schematic representation of three basic elements that are always present in varying degrees in any form of TPD. The chains between each box illustrates the idea that each of the elements do not exist in isolation during TPD. There are links between these elements and the issue at hand is what is foregrounded and what is shifted into the background in the design and implementation of the TPD model. Now, as already signalled in Graven's work, the professional community was in the foreground whilst the artefacts of practice and the specificity of the content, although present, were backgrounded.

⁷ I use the term 'artefacts' of practice to refer all types of records pertinent to teaching. Some examples of such records include but are not limited to lesson plans; curriculum documents; tests, assessments and tasks; learners' work; classroom teaching etc.

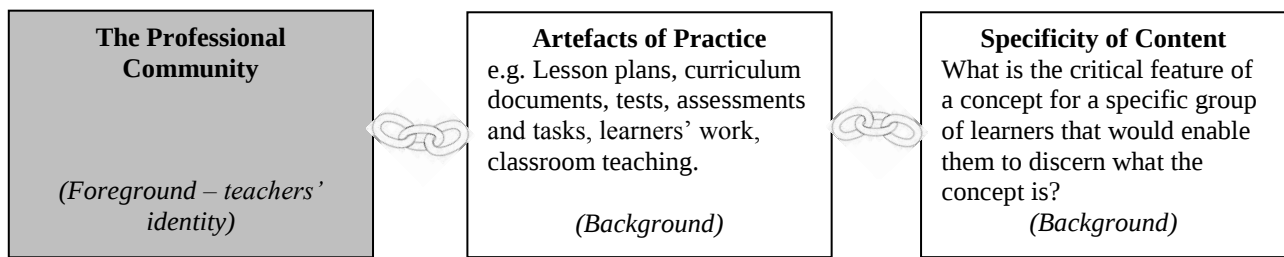


Figure 2.1: A schematic representation of the basic elements of teacher professional development foregrounded in Graven's work within a CoP

In a similar fashion as to how I engaged with the earlier work of Graven to illustrate what was privileged during CoP that she studied, I now turn my attention to the current work of Brodie and her team (Brodie, 2011, 2013; Brodie & Shalem, 2011; Chauraya, 2013) to establish what they are privileging as they go about working with teachers in the PLCs that they set up.

DuFour (2004) argues that the idea of improving schools by developing a PLC is currently in vogue and that the term is used to describe any combination of people with an interest in education. DuFour (ibid.) also argues that as a result of the idea of a PLC being used so ubiquitously, it is in danger of losing all meaning. In describing how PLCs are currently defined, Stoll and Louis indicate that “there is no universal definition of a professional learning community, but there is consensus that you will know that one exists when you can see a group of teachers sharing and critically interrogating their practice in an ongoing, reflective, collaborative, inclusive, learning-orientated, growth-promoting way” (Stoll & Louis, 2007, p. 2). In reviewing a range of literature Stoll and Louis conclude that the term PLC suggests that “focus is not just on individual teachers’ learning but on (1) professional learning; (2) within the context of a cohesive group; (3) that focuses on collective knowledge, and (4) occurs within an ethic of interpersonal caring that permeates the life of teachers, students and school leaders” (Stoll & Louis, 2007, p. 3). It is in line with these suggestions that Brodie and her team set out to establish PLCs.

Brodie (2013) explains that it is the learners’ learning needs that form the object of engagement and so the focus of the work within the PLCs that they set up. Her argument is that a “clear focus on learners’ learning needs informs teachers’ learning needs” (Brodie, 2013, p. 6). To determine the learners’ learning needs, Brodie and her team turn to data, where the data could be obtained from a wide range of sources viz. test results (national, international, class tests), interviews with learners, learners’ work, classroom observations and so on (Brodie, ibid.). In the case of the PLCs that Brodie and her team set up, the kinds of errors that learners make is the data that informs the work within a PLC. Nesher defines learner error as the “systematic, persistent and pervasive mistakes

performed by learners across a range of contexts” (Nesher, 1987 as cited in Brodie, 2013, p. 8). Hence the name of their project – Data Informed Practice Improvement Project (DIPIP). A theoretical assumption that underpins the work of DIPIP, as discussed by Brodie, is that “if teachers search for ways to understand why learners may have made errors, they may come to value learners’ thinking and find ways to engage their current knowledge in order to create new knowledge” (Brodie, 2013, p. 9).

In relating the work of DIPIP back to the three basic elements which were identified earlier as always being present in any form of TPD, the following schematic representation illustrates what is foregrounded and what sits in the background through the work of DIPIP.

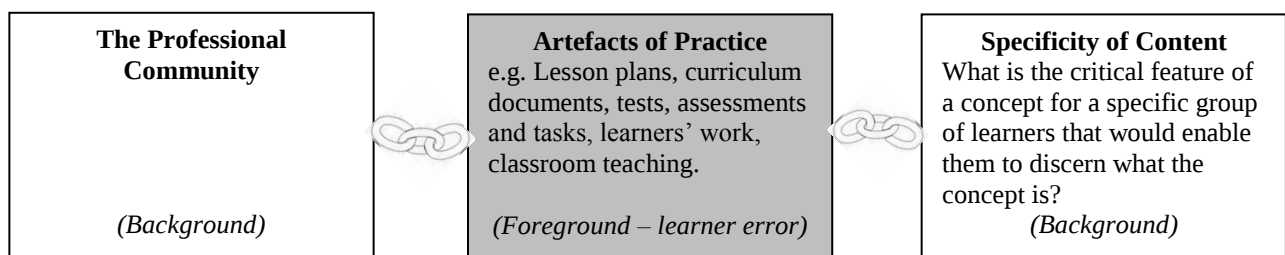


Figure 2.2: A schematic representation of the basic elements of teacher professional development foregrounded in Brodie’s work within PLC’s

In summary, it is important to note that focusing on establishing and sustaining PLCs and focusing on the specificity of content is present and being worked on as members within each of the PLCs within the DIPIP project engage with each other. These elements, although present in the workings of each of the PLCs, are in the background, and learner error (an artefact of practice) is foregrounded.

2.2.2 Lesson study as a form of a PLC

My study focuses on exploring whether a *learning study* TPD model can work to improve teachers’ mediation of a selected object of learning, and this is done within a South African context. Learning study has its roots in the Japanese model of lesson study (Lo, Marton, Pang, & Pong, 2004). There are numerous studies that explore the implementation of versions of lesson study across many countries: United States of America (Perry & Lewis, 2009); China, Singapore, Sweden, Taiwan (Juang, Liu, & Chan, 2008); China (Lee, 2008); England (Dudley, 2012); Indonesia (Suratno, 2012) and South Africa (Coe, Carl, & Frick, 2010; Posthuma, 2012); but there are not as many studies which record the implementation of learning study. In the South African context there is no record of any version of a learning study that has been implemented to date. I start by providing a

review of literature related to the implementation of lesson study so as to form the background against which I present literature relevant to learning study.

In writing about the Japanese approach to improving mathematics teaching and learning Fernandez and Yoshida (2004) provide a brief history of lesson study and highlight that the origins of lesson study can be traced back to the early 1900s. They also indicate that the practice of combining *konaikenshu*⁸ with lesson study is a newer practice which was already well-established by the mid-1960s. Thus lesson study has deep roots in Japanese education as a form of TPD and could be considered as part of the culture of a Japanese teacher. Drawing on the work of other researchers Fernandez and Yoshida signal that Japanese teachers highly value lesson study as a form of TPD that “many of them cannot imagine doing without it” and that “researchers have identified lesson study as being critical to supporting educational change and innovation in Japan” (Fernandez & Yoshida, 2004, pp. 2-3).

To demonstrate how lesson study fits in with the idea of a PLC, I present a brief overview of the lesson study process and in this overview I insert in bold print how each of the four aspects that constitutes a PLC as suggested by Stoll and Louis (2007) has the potential to come into existence during the enactment of the learning study at particular steps in the cycle. The lesson study process commences with the identification of the problem or lesson study goal. This goal is derived from the mission statement of the school and what it suggests about the qualities that they aim to foster in learners. They go on to explain that these goals need not target the development of specific academic skills in learners but should be aimed “at developing in children broader dispositions toward learning, school, peers, and themselves” (Fernandez & Yoshida, 2004, p. 10). Stigler and Hiebert (1999, p. 112) refer to these goals as problems and explain that these problems can start out as general (e.g. “to awaken students’ interest in mathematics”) or it could be more specific (e.g. “to improve students’ understanding of how to add fractions with unlike denominators”). Table 2.1 provides an overview of the lesson study process.

⁸ The term *konaikenshu* is made up of two Japanese words – *konai* which means ‘in school’ and *kenshu* meaning ‘training’. So *konaikenshu* essentially refers to a form of school-based in-service training (Fernandez and Yoshida, 2004, p. 9).

STEPS	OVERVIEW
Step 1: <i>Defining the problem</i>	Teachers need to define the problem that will underpin the work of the lesson study group. The problem could be derived from problems that teachers have experienced from their own practice, something that has posed particular challenges to their learners or even something that arises from problems that have been identified as a national priority. The problem can start out as general or it could be more specific. The problem becomes the object of learning for the lesson study. (The object of learning emerges from some artefact of teaching)
Step 2: <i>Collaboratively planning the study lesson</i>	A study lesson begins by teachers coming together to plan the lesson study. Drawing from their own experiences and observations (past and present) as well as teacher guides and textbooks and other relevant resources, teachers share their ideas. A lesson plan that describes in detail the design the group has decided on marks the end of this step. (Working within the context of a cohesive group focusing on collective knowledge)
Step 3: <i>Seeing the lesson study in action</i>	One of the teachers from the group will teach the lesson to his/her learners. The remaining teachers from the group will observe this lesson. The lesson plan designed in step 1 will be used as a tool to guide what they look for in the lesson.
Step 4: <i>Discussing the study lesson</i>	Now that the group of teachers have seen the lesson unfold in the real world of teaching practice, they begin to reflect on the lesson. They share what they have observed as they watched the lesson and provide their reactions and suggestions. Detailed minutes of these discussion are recorded which will be used for future reference.
Step 5: (optional) <i>Revising the lesson</i>	The group could either stop their work on a study lesson after they have discussed their observations of it; or they could revise and re-teach the lesson so that they could continue to learn from it. This process will lead to an updated version of the lesson plan that will reflect all the adjustments that the group of teachers have decided to make.
Step 6: (optional) <i>Teaching the new version of the lesson</i>	Similar to step 2, the revised lesson is taught. Another member from the group is identified to teach the revised lesson to his/her own learners. The idea of varying the teacher and the learners provides the group with a broader base of experiences to learn from. It is rare for a group to choose to revise and re-teach the lesson for a third time, since there is only so much that could be learned from examining a particular lesson. It would be more productive for the group to move on to an entirely new lesson.
Step 7: <i>Sharing reflections about the new version of the lesson</i>	As in step 3, this step is characterised by teachers from the group meeting to discuss their observations, comments, suggestions and reactions to what they have observed in the teaching of the revised lesson. Once again, detailed minutes of these discussions are recorded for future reference. These minutes are important especially when the group writes a report on the lesson study conducted. The report is shared in various ways: published in a book for the school's teacher resource room, read by the principal, and forwarded to the education authorities. (Not about individual teachers' learning but on professional learning which occurs within an ethic of interpersonal caring that permeates the life of teachers, students and school leaders)

Table 2.1: Overview of the lesson study process

Looking more closely at the steps involved in the lesson study cycle, one is able to see that the core activity in lesson study is for teachers to collaboratively work on what Fernandez and Yoshida (2004) refer to as ‘study lessons’. These lessons are called ‘study lessons’ because they are used to examine the teachers’ practice. The lesson plan forms the backbone of a lesson study since it serves as: i) a *teaching tool* because it provides the script for the activities of the lesson; ii) a *communication tool* since it conveys to others the thinking of the teachers who planned the lesson; and iii) an *observation tool* because it provides the guidelines for what to look for in the enacted lesson (Fernandez & Yoshida, *ibid.*).

Relating the lesson study process back to the three basic elements which was identified earlier as always being present in any form of TPD, Figure 2.3 illustrates what is foregrounded and what sits in the background during the lesson study cycle.

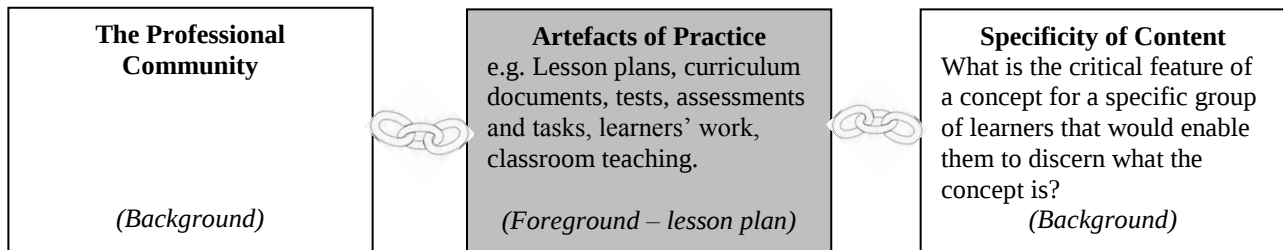


Figure 2.3: A schematic representation of the basic elements of teacher professional development foregrounded in a lesson study

2.2.3 From lesson study to learning study

Lo *et al.* (2004) explain that it was in the spirit of the principles inherent in design experiment⁹ and their need to learn about the potential value of variation theory as a learning tool that they reformulated the Japanese lesson study. The basic principle of variation theory as employed in this study will be illuminated later in this section and elaborated on in Chapter 3. Ference Marton defined learning study in the following way:

A learning study is a systematic attempt to achieve an educational objective and learn from that attempt. It is a design experiment that may or may not be a lesson study. Such a study is a learning study in two senses. First, it aims at bringing learning about, or more correctly, at making learning possible. The students will thus learn, hopefully. Second, those teachers involved try to learn from the literature, from each other, from the students, and not least, from the study itself.

(Marton, 2001 as cited in Lo *et al.*, 2004, p. 192)

⁹ The reference to design experiment is in relation to the work of Brown (1992) and Collins (1992) as cited in Lo *et al.* (2004).

Lo *et al.* (ibid.) further explain that they worked with lesson study since it was an appropriate model that enabled both researchers and teachers to see how the object of learning was being dealt with in the classroom. The fundamental difference between lesson study and learning study is that learning study is underpinned by a theoretical framework of learning viz. phenomenography and variation theory (Lo *et al.*, 2004, p. 193). Building on Marton's (ibid.) initial definition of a learning study, Lo *et al.* (ibid.) argue that learning study is a learning study in three senses: i) it aims at providing learners with opportunities to learn; ii) teachers participating in a learning study have the opportunity to learn from literature, from each other, from the learners and from the study itself; and iii) it provides researchers with opportunities to learn from it as well.

Marton and Pang (2006) characterise a learning study as a group of between two and six teachers working together to find a way of making it possible for learners to “appropriate a specific object of learning”. They identify the object of learning¹⁰ as “a specific insight, skill, or capability that the students are expected to develop during a lesson or during a limited sequence of lessons” (Marton & Pang, 2006, p. 194). In other words, it can be those ‘hard to teach’ or ‘hot spot’ topics and concepts. Within the context of this study, the focus will be on how teachers mediate the object of learning through the judicious use of examples. Once the object of learning has been identified, the implementation of learning study follows the same cyclical process associated with lesson study. Marton and Pang (2007) differentiate between what the learners should learn (the *intended object of learning*) and what they actually learn (the *lived object of learning*). Marton, Runesson and Tsui (2004) further elaborate that the intended object of learning is seen from the teachers’ perspective and “then it is somehow realized in the classroom in the form of a particular space of learning” (p. 22). This is the enacted object of learning as seen from the researcher’s perspective. Marton *et al.* (ibid.) also explain that the “way students see, understand, and make sense of the object of learning when the lesson ends and beyond, is the *lived object of learning*” (Marton *et al.*, 2004, p. 22, italics in original).

Specifically within the context of this study, the enacted object of learning will be influenced by what is varied and what remains invariant. While Marton *et al.* elaborate a framing of this conceptually, a clear methodology for grasping the enacted object of learning needs to be developed. In Chapter 3 I elaborate further. The teachers work together with a researcher or on their own and, in the case of the learning study model that will be explored in this study I, the researcher, will be working in collaboration with the teachers. The teachers choose a specific object of learning

¹⁰ Central to any object of learning is the concept of a critical feature. The concept of a critical feature will be discussed in Chapter 3.

that is central to the curriculum and that is known to cause difficulty for learners. Once the object of learning is identified the group commence with the planning of the lesson(s) with “a special focus on making it possible for the students to appropriate the object of learning” (Marton & Pang, 2006, p. 195). In their planning of the lesson the teachers in the learning study group have as resources their own experiences and previous research. Marton and Pang (ibid.) elaborate this as “the particular framework that they use to identify the necessary conditions for their specific learning target, and – above all – the exploration of the prior understanding of their students or the exploration of the extent to which the object of learning and/or its prerequisites have been appropriated by the students before teaching starts” (Marton & Pang, 2006, p. 195). During the planning of the lesson the focus of the teachers in the group is on the way in which the object of learning will be handled in terms of aspects of the object that will vary and those that will remain invariant during the sequence of the lesson. In the case of this study, what varies and what is invariant is derived from the principles inherent in variation theory, which will be discussed under the theoretical framework.

The next phase or step in the learning study process is the teaching of the lesson. One teacher is selected to teach the lesson while the other members from the group observe. With learning study, the lessons are usually video recorded and what the learners have learnt is probed by written questions and interviews. Marton and Pang (ibid.) explain that the lesson(s) is/are analysed in terms of whether it was possible to “appropriate the object of learning through the pattern of variation and invariance” that was established jointly by the teacher and the learners. Once the lesson is taught, the teachers may come up with suggestions about what can be improved in the lesson and another teacher from the learning study group is selected to teach the updated version of the lesson. According to Marton and Pang (ibid.) this cycle may be repeated up to four times and the study is always concluded by documentation.

Marton and Pang (2006) also describe an alternative way of establishing how different ways of handling the object of learning can affect the learning. Members from the learning study group teach using the same lesson plan, in other words “they deploy the same intended object of learning in parallel” (p. 196). To increase the variation in the ‘enacted object’ of learning, other teachers who are teaching the same object of learning but using a different lesson plan are invited to form a comparison group.

To summarise, in learning study the focus is on providing learners during a lesson with opportunities to discern a selected object of learning. The opportunities that are made available to

the learners are dependent on the dimension of variation that is opened up during the lesson. In a mathematics lesson dimensions of variation are opened up through the careful selection and sequencing of examples. In Chapter 3, I engage with the concept of dimension of variation in greater detail. As is the case with lesson study, in learning study the planning of a lesson is also in focus but in addition there is also a focus on learners' discernment of an object of learning. Now, relating learning study back to the three basic elements which were identified earlier as always being present in any form of TPD, the schematic representation in Figure 2.4 illustrates what is foregrounded and what is in the background during the learning study cycle.

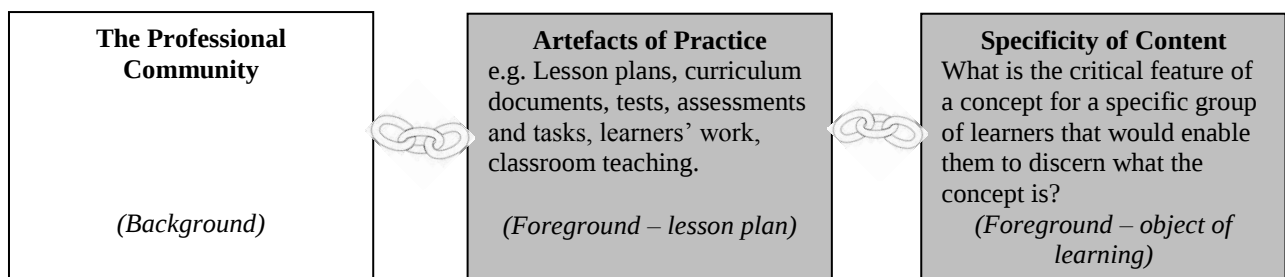


Figure 2.4: A schematic representation of the basic elements of teacher professional development foregrounded in a Learning Study

In this section, I have provided an overview of the earlier work of Graven's CoP and the recent work of Brodie's PLCs to demonstrate what is foregrounded and what is left in the background in each of these TPD models conducted in a South African context. I then illustrated how learning study brings the specificity of the content into focus, which is the hallmark of the Wits Maths CONNECT: Secondary School Project¹¹ (WMC-S project). The learning study model foregrounds the specificity of the content since the aim of learning study is to find ways of making it possible for learners to discern a specific object of learning. The model also foregrounds the lesson plan, since the lesson plan is the tool that provides the script for the activities of the lesson and so is an essential component of the model. The mechanics of the model allows for the intervention to be school-based and it allows for the zooming in on a particular object of learning, and therefore the intervention need not necessarily be a "one size fits all" type of strategy across grades or even schools. The mechanics of the model allow the intervention to be tailor-made for the 'individual' needs. The learning study seems to be an example of the kind of external intervention that Margolinas, Coulange and Bessot (2005) refer to, and one of the distinctive features of the learning

¹¹ This study forms part of a larger study known as the WMC-S project under the directorship of Professor Jill Adler. The WMC-S project is built on three pillars and this study is located within the pillar, developing mathematical judgement (DMJ). DMJ is school-based TPD which focuses on enhancing mathematics teachers' professional knowledge. Using the work of Shulman (1986), teachers' professional knowledge includes their subject matter knowledge, curriculum knowledge and pedagogic content knowledge. The teachers' capacity for informed mathematical judgement that underlies skilful teaching depends on this knowledge base. One of the aims of the DMJ arm of the WMC-S project is to develop teachers' mathematical judgement and to research the process in the FET context.

study model resonates with the idea of a PLC as described by Stoll and Louis (2007). The idea of a PLC is inherent in the learning study model but will not be the focus of this study. The learning study model has the potential to provide the platform for the development of a PLC and the nurturing thereof. This is important in terms of creating a self-sustaining model of teacher professional development within schools even long after this study is completed. This is one of the key aspects of the model since establishing PLCs seems to be the key thing about models today.

As already discussed, a learning study foregrounds the artefacts of teaching together with the specificity of the content. Within the context of this study the specificity of the content will be within the context of functions. It is in view of this that I review literature relevant to the function concept in the section which follows.

2.3 Functions

2.3.1 Functions – their importance

The concept of functions is particularly important since they are ‘all around us’, although we do not always realise this. We exist in a world that is made up of objects that change, and when objects change it is always in relation to a change in another object, and the relationship that connects these two changing objects is a process (Sierpinska, 1992). For example, the change in seasons that we experience is a function of earth’s orbit around the sun, a phone bill is a function of the fixed costs associated with the telephone plus the amount of time spent on making outgoing calls. Similarly, a functional relationship is at play when we are paying for petrol by the litre or food or other items by the gram or kilogram. Algebraic tools allow us to express these functional relationships very efficiently. For example, find the value of one thing (such as the petrol price) when we know the value of the other amount (number of litres), and display the relationship visually in a way that allows us to quickly grasp the direction, magnitude, and rate of change in one variable over the range of values of the other. For ‘simple’ problems such as determining the petrol price, learners’ existing knowledge of multiplication will usually allow them to calculate the cost for a specific amount of petrol once they know the price per litre (say R13 per litre). Learners will know that 2 litres will cost R26 and 10 litres will cost R130 and so on. While we can continue listing each set of values in this fashion, it will be efficient to say that for all values in litres (which we call x , by convention), the total cost (which we call y , by convention), is equal to $13x$. Writing $y = 13x$ is a simple way of saying a great deal. The concept of function allows one the opportunity to represent the same ‘thing’ in different forms. Working with functions also provides opportunities for one to work relationally – you need to understand x in relation to y . It also provides a transition point in

the sense that one moves from working in a discrete way (mathematically) to engaging with the concept of continuity which then leads into the study of calculus.

Within mathematics education the function concept has come to have a broader interpretation that refers not only to the formal definition, but also to the multiple ways in which functions can be written and described (Goldenberg, 1995; Leinhardt, Zaslavsky, & Stein, 1990; Romberg, Fennema, & Carpenter, 1993). Common ways of describing functions include tables, graphs, algebraic symbols, words and problem situations. Each of these representations describes how the value of one variable is determined by the value of another. Teaching for what Kilpatrick *et al.* (2001) refer to as mathematical proficiency implies that learners need to understand that there are different ways of describing the same relationship. This does not only mean developing learners' capacity to perform various procedures such as finding the value of y given the x value or creating a graph given an equation, but should also include assisting learners in developing a conceptual understanding of the function concept. "A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes" (Kilpatrick *et al.*, 2001, p. 119), in other words, the ability to represent a function in a variety of ways and fluency in moving among the multiple representations for specific purposes.

Within the South African context, prior to learners entering grade 10 they would be exposed to working with and solving equations in one unknown. However, during the grade 10 year they are exposed to working with equations containing two unknowns and are then introduced to the process of solving the equations simultaneously. It is during this process that learners come to experience expressing one variable (unknown) in terms of another (e.g. $y = 2x + 1$). This is a transition point for the learners since their conceptions about working with a variable is now being expanded to working with the relationship between two variables where one variable varies in relation to the change in value of the other variable. Metaphorically speaking this transition point could be seen as the 'stepping stone' to the world of functions and relations and is of particular importance to my study since my study will be located within the realm of functions as dealt with at the grade 10 level. Hence the value, in the South African context, of a study focused on functions at this level.

2.3.2 Functions – in the curriculum

At the time of implementing the learning study being discussed in this thesis (in 2011), three of the teachers who participated¹² in this study had in excess of twenty years teaching experience each whilst one teacher had seventeen years of teaching experience. This means that the participants in this study received their school education and commenced their teaching career during the apartheid era in South Africa. It also means that during their teaching career they experienced reform in the education system in 1997¹³. The reform was marked by, amongst other things, changes in the mathematics curriculum. Of particular interest for this study are the changes to the approach to the teaching of the function concept, which then had implications for how textbook writers organised and sequenced the chapter(s) dealing with the function concept. The textbook is a central resource for the participants as they go about their work of teaching (i.e. for planning a lesson and for selecting examples for use during the lesson and homework given to the learners). This very brief background is intended to provide the context and hence relevance for the discussion that follows in this section and the next. Furthermore, the context illustrates that the participants in this study were initially introduced to the function concept in a more ‘traditional’ sense and are now attempting to teach it in a ‘reform’ orientated fashion. I use the terms ‘traditional’ and ‘reform’ in this instance to refer to the approach to teaching functions as modelled in the curriculum, pre- and post-education reform of 1997.

My intention is not to do a content analysis of the curriculum that informed and currently informs the teaching of functions, but to provide a rough sketch of what was and currently is given prominence in the topic and how the topic is sequenced. To accomplish this task I look at how authors of one of the more popular mathematics textbooks¹⁴ that was used across many schools during the period pre-1997 fashion the topic of functions. For purposes of consistency I then compare it to how the authors of the same textbook present the topic of functions to deal with the changes associated with the post-1997 period. My focus on textbooks as curriculum material to inform this discussion is based on the argument that although teachers interpret, understand and use the ideas contained in curriculum materials, their decisions about pacing, sequencing and the tasks that they choose to present to their learners relies heavily on the mathematics textbook that they use (Grouws & Smith, 2000; Tarr, Chávez, Reys, & Reys, 2006).

¹² A detailed description of the sample will be provided in Chapter 4.

¹³ Reform in education in post-apartheid South Africa gave rise to the implementation of an outcomes-based education in 1997. The challenges related to the implementation of this curriculum prompted a review in 2000 and this led to the first curriculum revision in post-apartheid South Africa. Continued implementation challenges resulted in another review in 2009 which led to the implementation of the curriculum and assessment policy statement (CAPS).

¹⁴ The textbook referred to is from the Classroom Mathematics series of textbooks. The intention is not to critique the textbook but to use it as a guide to inform this writing.

Towards the latter part of the pre-1997 era the different classes of functions appeared as separate topics. In grade 10 (then standard 8) the topic of function was introduced by focusing on how input and output values could be represented (function machine, sets, Venn diagrams, sets of ordered pairs). This then provided the platform with which to introduce the concept of domain and range. Since the input and corresponding output values need not necessarily represent a function, the authors introduced the definition of a function. Thereafter, the linear function was introduced and the starting point was to draw the linear graphs from a table of values. The concept of gradient and intercepts were introduced, which set the stage to work with both the dual intercept method and the gradient and y-intercept methods for sketching a linear function. Zooming in on the concept of gradient the authors introduce parallel and perpendicular lines. The section on the linear function is concluded by focusing on questions that require one to determine the equations for lines. The quadratic function of the form $y = ax^2 + q$ is introduced as the next function, and in order to show learners the shape of the quadratic function they are required to complete a table of values and plot the ordered pairs on a Cartesian plane. Features of the quadratic function such as domain and range, axis of symmetry and turning points are introduced next. The hyperbola and the circle follow next in a new chapter.

This very brief description is sufficient to highlight two key issues with regard to how the topic on functions was treated in the school curriculum during the pre-1997 era. I note that the textbook is the authors' interpretation of the intended curriculum as handed down by the education authority of that era but, as indicated earlier, the textbook forms a central resource for the teachers as they go about their work of teaching. The first issue highlighted is that the different classes of functions were treated as separate topics, and the second issue is that there was a strong focus on dealing with functions in a pointwise manner. A pointwise approach to functions will be discussed in the next section.

I now provide an overview of how the authors of the same textbook series develop the function concept to embrace the curriculum demands as a result of the reform in education that characterises the political landscape of South Africa in the post-1997 period. The chapter on functions commences with an activity where learners are to complete a table of values in which they record the number of parts that would be outlined when an A4 piece of paper is folded into 2 parts, 3 parts ... 5 parts and then to plot these points. The set of points obtained are discrete points and so the resulting graph would be a discontinuous function. The next activity is a similar activity but the context in which it is set (investigating how the area of the square changes as the length of the side

changes) results in a continuous function. The authors then focus in on the concept of input and output values and use this to get learners to fill in missing values in a given table of values for some practical context (e.g. a ball is rolled down an inclined plane and the distance it covers is recorded at different times) and to generalise the rule which relates the input value to the output value and to express the rule in algebraic form. Once the table of values is completed the learners are required to draw the graph, and learners are then required to discuss the features of the curves. The contexts provided cut across the different classes of functions and so the different classes of functions do not appear as different sub-topics within the chapter on functions.

As indicated earlier the intention is not to do a content analysis of the curriculum as interpreted in the two textbooks and in so doing analyse the textbooks, but to highlight that there was a shift in thinking about how to present the concept of function to learners. In both the approaches to introducing the concept of function at the grade 10 level it is dealt with in a pointwise manner. However, in the current curriculum there is an attempt to deal with functions in a global fashion, as is noted by the tasks that require learners to explore the features of the graphs representing the different classes of functions. The idea of pointwise and global approach to functions will be discussed in the next section.

2.3.3 Functions – a pedagogical perspective

There is a considerable swathe of research on the teaching and learning of functions in school. The discussion here focuses on research related to the approach to functions as this has direct bearing on the teaching of functions. The approach to functions is what was highlighted as one of the important shifts in the mathematics curriculum reform as discussed above. As already indicated, this study is contextualised at the grade 10 level and so for purposes of this study I restrict my discussion to a pointwise approach and a global approach to functions. A local perspective to functions is a third approach which seems to be relevant only at university level (Vandebrouck, 2011) and so will not be considered here.

Essentially, in a *pointwise approach*, functions are considered as correspondences between two sets of numbers, an element of the first set being associated with a unique element of the second set. The pointwise approach is in accordance with the definition of a function as was emphasised in the curriculum during the pre-1997 period. Even (1998) further explains that to deal with functions pointwise means “to plot, read or deal with discrete points of a functions either because we are interested in some specific points only, or because the function is defined on a discrete set” (p. 109).

The similitude of a *pointwise approach* to functions is what Thompson (1994) describes an *action conception* of a function. This is when learners come to think of an expression as producing a result of calculating. They see the function as a recipe to apply to numbers and this recipe remains the same across numbers, but they must actually apply it to some number before the recipe produces anything. Consider the equation $y = x^2 + 1$: learners with an action view of functions will interpret the equation as a formula for determining an answer for a specific value of x by squaring the number and then adding 1. Oehrtman, Carlson and Thompson (2008) explain that learners whose understanding of functions is limited to an action view experience several difficulties ranging from (but not limited to) an “inability to interpret functions more broadly than by the computations involved in a specific formula” (p. 157), to input and output pairs being considered one at a time. Oehrtman *et al.* (ibid.) further explain that students who perceive input and output pairs one at a time “cannot think of a function as a process that may be reversed (to obtain the inverse of a function) but are limited to understanding only the related procedural tasks such as switching x and y and solving for y ” (p. 157, italics in original). They also indicate that learners with an action view of functions will conceive a function as being static and thus perceive a function’s graph as a geometric figure. On the other hand, Thompson (1994) describes the *process conception* of a function to be when learners build an image of ‘self-evaluating’ expressions. They do not feel compelled to imagine actually evaluating an expression in order to think of the result of its evaluation. Oehrtman *et al.* (2008) explain that students having a process view of a function can imagine a set of input values that are mapped to a set of output values for the expression that defines the function. Thus, such a learner is able to “conceive of the entire process as happening to all values at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values” (Oehrtman *et al.*, 2008, p. 158). The process is independent of the formula and so learners with a process view of functions will conceive a function as being dynamic and will perceive a function’s graph as defined by a mapping of a set of input values to a set of output values. Thompson (1994) further indicates that when learners perceive a function as a correspondence between two sets – a set of possible inputs to the process and a set of possible outputs from the process – then they (the learners) can reason about functions as if they were *objects*. In working with a function as an object one would be able to manipulate the graph of a function¹⁵ without dealing with the graph point by point or shifting to its algebraic representation.

¹⁵ For example, being able to sketch the graph of $y = h'(x)$ when the graph of $h(x)$ is given.

The process conception of function is akin to what Even (1998) refers to as a global approach to functions. She explains that a *global approach* to functions requires one to focus on the behaviour of the function. An example of this is for learners at grade 10 to be able to sketch the graph of a function given in its algebraic form, without having to set up a table of values. As indicated in the previous section, this is the direction towards which the revised curriculum in terms of its approach to the teaching of functions has tried to move. Vandebrouck (2011) argues that for some learners, interpreting an algebraic equation as a function from a global perspective seems relatively natural only for classical functions of the form $y = mx + c$ or $y = x^2$ or $xy = k$ whose global properties are well known. He further argues that for more complex algebraic representation of functions, the most natural perspective is the pointwise approach. This brings into focus the level of complexity with which the learners in this study perceive what Vandebrouck (ibid.) lists as classical functions, bearing in mind that at grade 10 the learners are being introduced to the quadratic, exponential and hyperbolic functions.

There are studies which have argued that many learners deal with functions in a pointwise fashion, but cannot think of functions as they behave over intervals or in a global way (Bell & Janvier, 1981; Lovell, 1971). These studies suggest that a global approach to functions is more powerful than a pointwise approach. On the other hand Even (1998) presents results which demonstrate that for some learners the global approach proved to be more powerful than the pointwise approach, while for other learners the reverse was true. Even (ibid.) therefore argues that in solving problems, a combination of the two approaches is most powerful. Vandebrouck explains that the use of graphical representations should allow for easier connections between the two perspectives however, “a large algebraisation of tasks at the end of the secondary school tends to limit which perspective can be adopted on functions” (Vandebrouck, 2011, p. 2095).

The teachers participating in this study will each be inclined to the use of a particular approach to the teaching of function. The two approaches to functions, as discussed, will thus have a direct bearing on this study as it will inform the planning of the lessons. Secondly, as can be observed in the current approach to presenting functions in the textbook, there is an implicit focus on the multiple representations of functions as evidenced in the kinds of tasks set: i) determining a rule which relates the input values to the output values and expressing the rule *algebraically*; ii) using the rule to complete a *table of values* to represent the input and output values; and iii) producing a *graphical representation*.

During the late eighties and early nineties studies have shown that multiple representations of concepts yielded deeper and more flexible understandings (Harel & Dubinsky, 1992; Skemp, 1987). These studies argue that the selection of the initial representation can be viewed as a function of several variables, one of which is a cognitive preference. In addition to the cognitive preference, Keller and Hirsch (1998) found other factors that influenced the preference for a particular representation, including: i) the nature of the learners' experience with each representation; ii) learners' perceptions of the acceptability of using a representation; and iii) the level of the task. Although these studies had the learners in focus, the relevance to my study is that teachers, too, will have a preference for a particular representation and this could be influenced by their knowledge of the function concept as well as their interpretation of the curriculum as handed down by the education authority and their re-interpretation of the curriculum as interpreted by the textbook authors. The approach to functions and the idea of multiple representations both have a direct bearing on this study as the teachers in this learning study embark on preparing the first lesson, which is discussed in Chapter 6.

Central to the planning of any mathematics lesson is the use of examples, and as Bills, Dreyfus, Mason, Tsamir, Watson and Zaslavsky (2006) explain, the "use of examples is an integral part of the discipline of mathematics and just an aid for teaching and learning" (p. 126).

2.4 Exemplification and its significance

It is common knowledge that we learn through exemplification (Bills *et al.*, 2006). Mathematics texts across all levels are littered with examples because mathematics is an abstract science and a key means for it to be illuminated is through exemplification. In recent research there has been a much more deliberate focus on examples in mathematics, to this end a special issue of the journal *Educational Studies in Mathematics* published in 2008 (volume 69) was dedicated to this topic. So the idea of the significance of the use of examples in mathematics has recently gained currency. Following the work of Watson and Mason (2002), in this study, I take *examples* to include "anything used as raw material for intuiting relationships and inductive reasoning; illustrations of concepts and principles; contexts which illustrate or motivate a particular topic in mathematics; and particular solutions where several are possible" (p. 4). Furthermore, using their work *exemplification* is used to describe "any situation in which something specific is being offered to represent a general class with which the student is expected to become familiar (Watson & Mason, 2002, p. 4).

So what is the role of examples in the teaching and learning process, specifically for mathematics? To reflect on this question, consider explaining the concept of a straight line without having examples of a straight line from one or a combination of representations in which a straight line can exist. It is impossible to consider the teaching and learning of mathematics without considering the use of specific examples. Watson and Mason (2002) indicate that teachers frequently use examples in order to demonstrate and communicate the essence of mathematical concepts and techniques. This is amplified by Zazkis and Chernoff's (2008) discussion that examples are an important component of expert knowledge and that examples are used to verify statements, to illustrate algorithms and procedures, and to provide specific cases that fit the requirements of the definition under discussion. I draw on the data and examples used by Zazkis and Chernoff (ibid.) to illustrate the important role examples play, particularly in dealing with learner error or confusion in the teaching of mathematics. Selina is a prospective elementary school teacher and during an interview, she was presented with the task of simplifying $\frac{13 \times 17}{19 \times 23}$. Selina started by finding the product of 13 and 17 and then went on to find the product of 19 and 23 to obtain $\frac{221}{437}$. She moves on to determine if 221 and 437 are prime numbers by testing for divisibility by 2, 3 and 5 and explains that 2, 3 and 5 are for her the building blocks in determining if other numbers are prime numbers or not. She goes on to explain that once she has tested for divisibility by 2, 3 and 5 and has eliminated these numbers she will know whether she is dealing with a prime number or not. The interviewer pushes her with the question, is 437 a prime number? Selina comes up with a conjecture that 437 is a prime number and draws a conclusion that "two prime numbers multiplied by each other are prime". To invoke a cognitive conflict the interviewer asks whether 15 is a prime number. Selina's response is no to which the interviewer replies "but it's two prime numbers multiplied by each other, 3 and 5". Zazkis and Chernoff (ibid.) explain that the interviewer's choice of 15 is a pivotal example for Selina since it introduced a cognitive conflict and challenged her initial ideas. To resolve the conflict the interviewer asks Selina if 77 is a prime number to which she immediately responds no since it is divisible by 11. Further along in the interview Selina indicates that she was totally wrong by saying that when two prime numbers are multiplied by each other the product will be a prime number. This demonstrates that 77 served as a bridging example in the resolution of Selina's conflict. Zazkis and Chernoff (ibid.) explain that the significance of 77 is that it is small enough, like 15 and unlike 437, because its factors are easier to recognise. On the other hand 77 is not composed of 2, 3 or 5 which were the initial building blocks for Selina.

What is expected from a teacher like Nash to be able to select and sequence appropriate examples when planning a lesson so as to communicate the essence of a concept or illustrate a technique, or

as in the case of Selina during the flow of a lesson to engage with a learner's thinking? What does this imply about the kind of knowledge that a teacher ought to have to be able to engage with learners' ideas by making use of relevant examples with the aim of assisting learners to have a more complete or fuller understanding of the concept in question? Where and how do teachers learn about using examples deliberately, bearing in mind that the use of examples as illustrated above cannot be planned universally and a priori? Indeed, as Watson and Mason (2005) argue, it is about how examples are used in the moment and in practice rather than the examples standing out there as always being a good start-up or model example: for instance, an example might be a good start-up example in one instance but not in another.

Watson and Mason (2005) introduce the idea of example spaces, which are collections of examples that fulfil a specific function. Watson and Mason (ibid.) use the metaphor of a pantry to describe the idea of an example space:

Clustered at the front are frequently used and familiar items. There is a sense that further back there are other items, but reaching them usually means pushing other things aside. When some things run out, they are put on the shopping list, and sometimes while shopping something else catches the eye and is purchased for some imagined future use. This corresponds to the way in which many examples are first encountered from some other source, but then they are appropriated and modified for your own purpose later.

(Watson & Mason, 2005, pp. 61-62)

The contents within any of the example spaces should not be seen as fixed or unchanging but as being fluid, as is suggestive of the pantry metaphor used to describe an example space. The range of examples that a teacher makes use of is drawn from his/her 'pantry' (the example space) and he/she may not be using all the examples within that example space. What prompts the selection of the particular range of examples in use is what Marton, Runesson and Tsui (2004) refer to as the "dimension of variation"¹⁶ that is actually opened up" (p. 24), thus creating the *space of learning*. In addition, Watson and Mason (2005, p. 64) indicate that "to understand mathematics means, among other things, to be familiar with conventional example spaces." The implication for my study then is that the school-based intervention strategy that is being explored should enhance the dimensions of variation that a teacher is able to open up during the flow of a lesson and by so doing, strengthen the example space from which they recruit examples for use.

¹⁶ The idea of variation and dimension of variation will be discussed in section 3.4.

The main theoretical construct that can be seen as a contribution from the work of Watson and Mason (2005) is their idea of an ‘example space’. In their work, they also propose a teaching strategy that hinges on the idea of learner-generated examples, which will be discussed later in this section. The theoretical foundation of the teaching strategy that they propose lies in variation theory and the following is one example to illustrate this:

Consider these sequences:

2,4,6,8, ... 2,5,8,11, ... 2,7,12,17, ...

What is the same and what is different about them? Can you predict later terms? Make up more sequences like these.

Now think about these:

2,4,6,8, ... 7,9,11,13, ... 10,12,14,16, ...

What is the same and what is different about them? Can you predict later terms? Make up more sequences like these.

(Watson & Mason, 2005, p. 109)

Watson and Mason’s (ibid.) valuable contribution is grounded in their own work, including their teaching of mathematics. Watson and Mason have both engaged in studies of advanced mathematics and therefore would be considered as ‘experts’ in this field, and so the question is raised: what would it mean for a teacher like Nash to engage in this kind of practice? With respect to the example above, Watson and Mason (ibid.) explain that the example demonstrates that by varying one parameter at a time we could get learners to focus on the effects of the change without them being distracted by more than one aspect being changed at the same time. It would also assist in distracting learners from making incorrect assumptions associated with the sequence (2,4,6,8, ...) where the first term and the common difference is the same value. In addition to the question about what it takes to recognise these features of variation, there is the additional question of whether and how greater attunement to exemplification in this way will actually enhance the quality of teaching. Furthermore, what is the quality of the opportunity that this example provides for learners to reflect on the concept of sequences so as to transform the concept to something more substantial? These questions become pivotal for my study since it provides the platform for my contribution to the knowledge base. My study will be extending the work of Watson and Mason by subjecting their ideas to an empirical study with teachers in practice in a particular context.

The question that now emerges is how can teachers use examples in a more deliberate and purposeful fashion to provide learners with sufficient opportunities to reflect on a concept (i.e. to

broaden their learners' example spaces and its use in practice)? In an attempt to provide one possible answer to this question I turn to Watson and Mason's (2005) idea of learner-generated examples, which is essentially a process where the teacher gets learners to generate examples according to specified features e.g. 'Write down as many numbers as you can that will leave a remainder of 1 upon dividing by 7. Now write a general form of a number leaving a remainder of 1 upon dividing by 7'. Watson and Mason (2002, p. 2) explain that "students' attempts to create examples according to specified features can be used as starting examples for work on new concepts, and for transforming and developing knowledge of aspects of mathematics already familiar in different, limited or simpler forms." Using learner-generated examples is not a typical practice in South Africa, but it would be valuable for a teacher like Nash to begin to think about examples that are actually good for generating learner-generated examples as this will assist in keeping the critical feature of the object of learning in focus. Watson and Shipman (2008) identify two obstacles to the claim that learners can gain some understanding of concepts that are *new* to them through the idea of learner-generated examples. Firstly, Watson and Shipman (ibid.) cite Fodor's (1980) 'learning paradox' which suggests that learners are unable to construct a conceptually richer system than those they already know. Therefore, Watson and Shipman (ibid.) indicate that some authors argue that it is hard to see how a learner can construct objects without already having an idea of what to construct. Secondly, Watson and Shipman explain that

It is often assumed that learners cannot achieve higher-level understandings empirically, because the actions of mathematical thought required to generate data, perform procedures, and observe similar examples, are not sufficient for conceptualisation. In order to conceptualise there has to be some shift to 'higher mental functioning' which is somehow structured by expert others.

(Watson & Shipman, 2008, pp. 98-99)

In other words, what happens behind the scenes in teachers' use of examples is the teacher's recognition of appropriate examples, and the ordering and organising of the examples for presentation in the class.

Consider the following 'task' developed by the WMC-S Project which is related to the collection of like terms: generate as many examples as you can that would make the following statements true – i) $\square + \square = 5a$; ii) $\square + \square + \square = 5a$ and iii) $\square + \square + \square = 5a - 4$. Watson and Mason (2005) and Watson and Shipman (2008) argue that when learners are presented with representations of relationships and have to discern structure for themselves, as is the case with the example related to the collection of like terms, then in the Vygotskian tradition "this could be described as providing scaffolding, through carefully designed examples, for learners to operate at a higher mental level than they

would otherwise, bringing their spontaneous conceptualisations into contact with the formal culture of mathematics” (Watson & Shipman, 2008, p. 99). They further argue that learners who consistently employ example generation as an integral part of their learning strategy undergo more shifts of concept image, give better explanations, develop broader example spaces and have a more complete understanding of the taught concept. The data from Watson and Shipman's (ibid.) study provides evidence that learners’ “experience can be organised in such a way that shifts of understanding take place as a result of learners’ own actions, including mental acts of organisational reflection on self-generated examples and example spaces” (p. 108). The focus here is on the learner and the learner generating his/her own examples, what is obscured in this discussion thus far is the role of the teacher¹⁷.

Going back to the example based on the collection of like terms, as cited above, was ‘5a’ arbitrarily chosen? Or for that matter, what was the purpose of structuring the sequence to start with the sum of two terms and then moving to the sum of three terms? Why was the constant ‘-4’ introduced in the third part? All of these components were purposefully selected – the coefficient of 5 was to limit the combinations of the coefficients being natural numbers, the three terms in part two was to shift learners to think about adding negative coefficients if they have not already introduced it in the first part. The introduction of the constant was threefold: i) to force learners to add a negative; ii) to engage with the misconception that when adding terms one can merely ignore the unknowns and perform the calculations (e.g. $3a + 4a$, I can ignore the unknown ‘a’ and merely calculate $3 + 4 = 7$, now join the unknown ‘a’ resulting in $7a$); and iii) to engage with the possible learner error of conjoining terms. What this illustrates is that the idea of learner-generated examples starts with the teacher. The teacher needs to design examples that will provoke learners to generate their own examples with the aim of providing learners with opportunities to engage with a concept so as to improve their understanding. Watson and Mason (2005) do not bring into focus the demands and expertise it takes for teachers, like Nash, to generate examples that will be productive in terms of getting learners to generate their own examples and then organising them for presentation in the class.

So what do I mean, within the context of this study, by the judicious use of examples? To illuminate the idea of a judicious use of examples within the context of this study, I commence with a

¹⁷ What happens when the learner-generated examples create an example space that does not take learners forward as planned? The teacher will then need to play a more ‘inputting’ role. In Sfard’s later work, she talks about pushing the communication on in the sense that the teacher will need to create a communication conflict that will enable the learners to change the discourse (workshop conducted by Sfard at the 2011 SAARMSTE conference). I will deal with this issue as I engage with the data.

well-known difficulty that surfaces in senior phase mathematics and which has been documented in other studies (Basbozkurt, 2010; Davis, 2010; Jaffer, 2009) – integer arithmetic. Consider the problem of calculating $-9 + 7$. A common procedure to solving this problem, which is easily recognisable in South African schooling contexts, is: since the signs of -9 and $+7$ are different, ignore the signs in front of each of the numbers (i.e. treat both numbers as though they were positive numbers) then subtract the smaller number from the larger number ($9 - 7 = 2$), now insert the sign of the larger number in front of the answer obtained (-2). Although this answer is correct the procedure followed to arrive at the solution as explained by Jaffer involves using *pseudo-operations* that required the splitting apart (*sundering*) of the signs from -9 and $+7$, subtracting 7 from 9, and then combining (*concatenating*) the minus sign and 2 to arrive at the answer of -2 (Jaffer, 2009).

By doing this, the teacher reduces the problem to whole numbers instead of working with integers, what Davis (2010) refers to a domain shift, and thus misses the object of learning completely. The point that Davis (ibid.) is making is that the intended object of learning is bypassed and the domain shift is a strategy for learners to cope rather than a strategy for them to deal with the complexity of working within the domain of integers. Indeed, Davis (ibid.) argues that the learners now have to remember a more complicated set of steps and that by bypassing the intended object of learning, learners never get to deal with it. The judicious use of examples within the context of this study will mean that the use of examples will zoom in on the critical feature of the object of learning. A judicious use of examples will constrain the object of learning from being acted upon in ways that will reduce or shift the domain; it will keep the domain in focus. In other words, the judicious use of examples will create an example space in which the critical feature of the object of learning is in focus and thereby providing learners with an opportunity to discern the object of learning – learners are given opportunities to learn and use integers.

The teaching of operations with integers poses a challenge, as amplified by Peter’s response during a conversation about the modelling of integer operations. Peter is a mathematician with an interest in mathematics education and he is in conversation with Ann who is a mathematics educator interested in learning from elementary school teachers. This conversation was reported in the journal *For the learning of mathematics* and the following extract is an excerpt from Peter’s response:

I certainly believe that teachers ought to be able to develop the capacity to search for models or heuristics for such procedural examples as $(-3) \times 2$. [...] Notice that I said “the capacity to search.”

It is possible that a teacher will fail to find a good model. That does not make that teacher a failure as a teacher.

(Kajander, Mason, Taylor, Doolittle, Boland, Jarvis, & Maciejewski, 2010, p. 54)

In the absence of a suitable model to deal with $-9 + 7$, what might be a judicious use of examples? To demonstrate a possible application of the judicious use of examples for this particular case, working with the decomposition of -9 and the additive inverse and also employing the idea of variation, consider the following sequence of examples.

- | | |
|--------------------------------------|--------------------------------------|
| a) $-9 + 1 = (-8) + (-1) + (1) = -8$ | b) $-9 + 2 = \square + (-2) + (2) =$ |
| c) $-9 + 3 = \square + (-3) + (3) =$ | d) $-9 + 4 = \square + (-4) + (4) =$ |
| e) $-9 + 5 = \square + (-5) + (5) =$ | f) $-9 + 6 = \square + (-6) + (6) =$ |
| g) $-9 + 7 = \square + (-7) + (7) =$ | h) $-8 + 1 =$ |

This example focuses attention on the use of the additive inverse and decomposing -9, it holds the integrity when working with integer addition. It is merely the starting point because the teacher can put up this example sequence on the board and can ask learners to complete it, but this is not sufficient since it depends on what the teacher does with the learner responses and is also dependent on the nature of the teacher's explanation that follows. If, for instance, the teacher goes through this process and at the end engages in explanations that make use of some pseudo-operations or deals with operations in an extra-mathematical fashion (e.g. using the idea of terms moving across the equal sign 'transposing' without consolidating with the concept of using additive inverses or terms cancel) then the example used is not a judicious use of examples, since the operations that are applied to the domain do not ensure that there is integrity within the domain. Therefore, in the context of this study there are two conditions that need to be satisfied before the use of examples could be classified as 'judicious use'. Firstly, the examples themselves will constrain the manner in which the object of learning will be acted upon in the sense that they will not reduce or shift the domain. It will keep the domain in focus or in other words, these examples will try to ensure that when the critical feature of the object of learning is in focus, and operations are in play the operations act on the object of learning in such a way that it maintains the integrity of the domain. Secondly, it is dependent on what the teacher does with the example and the explanations he/she provides in relation to the example, these need to cohere with the domain.

Since examples are central to the teaching of mathematics, and in this case the teaching of functions, one focus of attention in the learning study under discussion in this thesis is the selection of examples, and their judicious use more generally.

2.5 Problem statement and focus questions revisited

The preceding discussions provide the context in which to restate my problem statement and its areas of focus. To ensure that the areas of focus for this study make sense and are therefore meaningful it was necessary for me to engage with relevant literature to set up a backdrop against which I could pose them. Thus, my intended study is summarised as follows:

Can a learning study professional development model, underpinned by variation theory and the judicious use of examples, work to improve mathematics teachers' mediation of a selected object of learning – the case of function in grade 10?

To explore this question I look at:

1. What was constituted as the intended object of learning for the learning study focused on the function concept in grade 10 and how was it identified?
2. What came to be constituted as the enacted object of learning in each of the lessons in the learning study cycle and what emerged as the critical feature?
3. How does participation in the learning study focused on the function concept in grade 10 impact on participants' learning?

Within the context of this study, mediation means the way in which the object of learning is made available to the learners and the opportunities that the teacher provides to learners so as to transform the object of learning from a level of immediacy to something more substantial (a more elaborated notion). Implicit in these questions are the ideas of variation and the judicious use of examples. I elaborate further on these focus questions after the theory and methodology chapters (chapters 3 and 4).

2.6 Summary

In this chapter I have reviewed literature that highlights the important role examples play in the teaching of mathematics. I have illustrated that it is not a simple task or a 'taken for granted'

capability that teachers will be able, during the flow of a lesson, to spontaneously think of examples that would, for example, create a cognitive conflict and then to also think about pivotal examples that would aid in resolving the conflict. This is dependent on the quality of examples that reside in a teacher's example space as well as the depth and breadth of this example space together with the way ideas are linked within a particular example space and across example spaces. The use of examples and the idea of an example space are important for a knowledge domain like mathematics since, as alluded to earlier, mathematics is abstract and one possible way to get access to the ideas inherent in this knowledge domain is through the process of exemplification. Therefore, examples and example spaces are germane to mathematics and are crucial entities when considering the teaching and learning of mathematics. In spite of the importance of the use of examples and teachers having well-developed example spaces, what empirical evidence is there which demonstrates a correlation between the use of examples and the quality of teaching? In other words, is there empirical evidence that shows that a better use of examples means that there is better quality of teaching and better learning?

I have also highlighted that the work of Watson and Mason (2005) is empirically grounded in their own work. Watson and Mason are 'experts' in the field and so what would it mean for a typical South African teacher, like Nash, to engage with the ideas provided by them? This study will contribute to the extension of the knowledge base by subjecting Watson and Mason's ideas to a different empirical field where the teachers are not as mathematically skilled as Watson and Mason.

Reform in education in post-apartheid South Africa saw a shift in the thinking of how to approach the teaching of functions. These shifts were illuminated by comparing how a group of authors presented the topic of functions in their textbooks that attempted to address the curriculum demands before and after South Africa obtained its democracy. It is in light of this that I focused the review of literature on functions to the approach to teaching of functions and engaged in discussion on a pointwise and global approach to functions and illustrated the similarities to an action and process conception of function. In addition to focusing on the approach to teaching functions, the use of multiple representations is also considered since research has shown that multiple representations of concepts yield deeper and more flexible understanding.

The approach to teaching functions, the use of examples and multiple representations has direct bearing on this study as these aspects will be crucial in thinking about and planning the lessons for the learning study.

The purpose of my study is to explore whether and how a learning study professional development model which is underpinned by variation theory and the judicious use of examples could enhance and strengthen mathematics teachers' mediation of a selected object of learning. In this chapter, I provided an overview of the learning study model as a professional development strategy that I make use of in this study. This model aligns with research done in Sweden (Marton & Pang, 2006) where they have developed the efficacy of the model. Furthermore, in terms of TPD I located my study in relation to key work done and currently being carried out in the field of mathematics TPD in South Africa, as they have direct bearing on this study – the earlier work of Graven and the current work of Brodie.

The review of literature relevant to this study provided the backdrop against which I could pose the questions that will focus the study. As indicated, a further elaboration of these questions is necessary and will therefore be revisited at the end of Chapter 5.

Chapter 3

Theoretical underpinning of the study

3.1 Introduction

Like a lens through which one views the facts; it influences what one sees and what one does not see. ‘Facts’ can only be interpreted in terms of some theory. Without an appropriate theory one cannot even state what the ‘facts’ are.

(Olivier, 1989, p. 10)

This study focuses on exploring the potential of learning study as a school-based teacher professional development model. Many of the theoretical bases that inform debate in mathematics education are theories of learning, whereas the focus of this study lies in the practice of teaching, a less developed field of study, particularly as this relates to what is constituted as knowledge in pedagogic practice. The bulk of the data that will be analysed in this thesis is at the level of the enacted object of learning which is seen from the researcher’s perspective, and for this I draw on a theory of pedagogy which provides me with a language that will assist me in a detailed and thorough analysis of the lessons. Variation theory per se, which will be discussed later in this chapter, is a theory of learning and provides tools that enable a teacher to create opportunities for learners to see ‘something’ in a specific way. Variation theory provides the necessary tools with which to think about and describe the intended and the lived objects of learning, as well as a set of concepts that have been used to study enacted objects. However, as I set out on this study, it was not clear to me how these concepts combined into a theory of pedagogy and how they could translate into a systematic study of the enacted object of learning. In what follows I describe the wider theoretical framing of my study of the enacted object, into which I insert concepts from variation theory.

The theoretical field that informs this study, particularly the enacted object of learning, draws on Bernstein’s structure of the pedagogic device since this structure describes how knowledge is converted into pedagogic communication. Of particular relevance to this study is the evaluative rule, since all pedagogy condenses in evaluation. It is in evaluation that the teacher transmits criteria to learners of what counts as valid knowledge. This is crucial to the study since it is in the transmission of criteria that one is able to observe what comes to be constituted as the enacted object of learning during a lesson. Starting with the premise that teaching always involves teaching something to someone, the question becomes: How, through one’s teaching, does one focus

learners' attention on the object of learning as well as the criteria by which to recognise what counts as valid knowledge?

3.2 Pedagogy condenses in evaluation

Basil Bernstein developed a complex theoretical framework that deals with the conversion of knowledge into pedagogic communication which he refers to as the pedagogic device (Bernstein, 1990, 1996). He described the pedagogic device as a “symbolic ruler of consciousness” (Bernstein, 1990, p. 10). In other words, the pedagogic device as suggested by Bernstein “is the ensemble of rules or procedures via which knowledge (intellectual, practical, expressive, official, or local knowledge) is converted into pedagogic communication (classroom talk)” (Singh, 2002, p. 573). The translation or the conversion of knowledge into pedagogic communication, according to Bernstein, takes place through three interrelated rules: the distributive rule, the recontextualising rule and the evaluative rule. Very briefly, the function of the distributive rule is to distribute different forms of knowledge to different social groups thereby regulating the power relations between social groups – *who gets to learn what*. The recontextualising rules regulate the formation of specific pedagogic discourse. These are rules for “delocating a discourse, for relocating it, for refocusing it” (Bernstein, 1996, p. 47). Through the process of delocating, relocating and refocusing a discourse, a discourse is moved from its original site of production to another site, where it is altered as it is related to other discourses. The recontextualised discourse no longer resembles the original because it has been pedagogised or converted into pedagogic discourse (Singh, 2002). Bernstein was concerned with the distribution of knowledge (who gets to learn what). This study is not about the distribution of knowledge but focuses on the transmission of criteria by which to recognise what counts as valid knowledge in a mathematics lesson. It is in light of this that I focus on the third rule of Bernstein's pedagogic device viz. the evaluative rule.

Singh (2002) indicates that in broad terms the evaluative rules “are concerned with recognising what counts as valid acquisition of instructional (curricular content) and regulative (social conduct, character and manner) texts” (p. 573). Evaluation attempts to control the transmission or acquisition of the available potential meaning. In other words, evaluative rules construct the pedagogic practice by providing the criteria to be transmitted and acquired. This means that the possibilities for meaning are condensed in and through moments of evaluation, and in Bernstein's terms, it is these that will function to specialise consciousness (in this case, knowledge of mathematics and in particular functions at grade 10). In Bernstein's (1990, 1996) terms any pedagogy transmits evaluation rules, this is to say that in any pedagogic practice, teachers transmit criteria to learners of what it is they are to come to know. In other words, at various points in time the teacher needs to

legitimate aspects of the pedagogic discourse (in relation to what it is he wants learners to know), and in order for the teacher to do this he will have to exercise some form of judgement. For Bernstein, the transmission of criteria to learners function at two levels – or what he calls two rules of acquisition: recognition and realisation. In the first instance, the learner must recognise what it is he/she is to attend to as well as the specialised language entailed – *the what*. But recognition (knowing what it is you are meant to know or do) is not sufficient. In Bernstein's terms, recognition needs to translate into realisation, where the learner is able to produce the legitimate text i.e. the kind of response required by the teacher – *the how*. For my purposes and for studying the enacted object, I focus on the criteria a teacher transmits through evaluation, and specifically the actions the teacher engages in or gets the learners to engage in.

In the process of the teacher recontextualising discourse into 'worthwhile' school knowledge and social practice for the learners, the teacher will continuously transmit, through evaluation, the criteria of what would constitute this legitimate school knowledge. So, in his discussion of the pedagogic device, Bernstein concludes that evaluation is key in pedagogic practice (Bernstein, 1996, p. 50). Davis remarks that Bernstein uses the term "*evaluation* rather than *assessment* and is referring to teacher-student interactions as well as questions, problems, tests, projects, examinations and so forth" (Davis, 2005, p. 81). Following Davis's (ibid.) work, it is in this broader and more fundamental sense that the idea of evaluation is used in this study.

Now, although the evaluative rule has a significant role to play in the pedagogic device, Bernstein has not elaborated on the evaluative rule. This is further emphasised by Davis in his search for an answer to the question 'how does evaluative judgement operate in pedagogy?' – a question that is also important to this study so as to get some insights into how concepts come to be acquired. In Davis's search for an answer to this question within Bernstein's theory of pedagogy he fails to find any detailed account of the workings of evaluative judgement. Davis therefore concludes that the "state of the theory on the workings of the evaluative rule is unsatisfactory" (Davis, 2005, p. 82). To construct a more general account of the operation of evaluative judgement Davis (ibid.) recruits Hegel's theory of judgement. What this suggests methodologically is that it is in evaluative moments in pedagogic practice, what Davis (2001, 2005) refers to as evaluative events, that criteria for what is to be acquired become visible. It is through the visibility of the criteria that teachers transmit, that we will be able to observe what comes to be constituted as the enacted object of learning.

3.3 Hegelian judgement

Davis (2001, 2005) turns to Hegel's theory¹⁸ of judgement – an abstract theory of how concepts come to be acquired. My concern here is not just a matter of how concepts come to be acquired, but how they come to be acquired in pedagogic practice. For Hegel, acquiring a concept is bound up with experiences of judgement. In the context of school, whether explicitly or implicitly, teachers continually exercise judgement in their attempt to provide the learners with opportunities to acquire the concept. In other words, as teachers exercise judgement, so learners are afforded opportunities to clarify what it is they are learning i.e. they are offered criteria both implicitly and explicitly.

The four moments of judgement in Hegel's theory are the judgements of: i) existence, ii) reflection, iii) necessity, and iv) notion.

3.3.1 The Judgement of Existence

The essential feature in this judgement is that the judgement of existence has the form of *immediacy*, this is to say that an initial encounter with a concept is one of *immediacy*; it is simply a 'that', an empty signifier: a verbal or written mark, or gesture (Davis, 2005). In other words, there is an absence of adequate predication. Davis indicates that to demonstrate that we understand a concept we need to show a series of predicates which is different from the signifier for the concept itself. For example, to the question 'what is a linear function?' the response 'a linear function is a linear function' is an unacceptable response. To provide a legitimate response one needs to identify the concept of a linear function with some signifier other than itself, for example a linear function is a straight line on a coordinate grid. This response illustrates an attempt to describe the concept of a linear function with the predicate (straight line on a coordinate grid) which is different from the signifier. The predicate 'straight line' is still insufficient in describing a linear function since it includes a vertical line. The concept linear function is still in its *immediacy*, simply a 'that': a sound or a word or a diagram and hence the relationship between subject and predicate is impossible because of the absence of adequate predication.

3.3.2 The Judgement of Reflection

Davis (ibid.) explains that since there is an absence of adequate predication to describe the subject so as to transform it from a mere 'that' into something more discursively substantial, the concept at

¹⁸ In his theory of judgement, Hegel does not specifically refer to schooling as such; he makes reference to the 'everyday'. While I am drawing on Hegel's theory it is important to note that I am drawing on its interpretation by Davis (2001, 2005) and how it is being re-interpreted in education and for education. It is for this reason that I do not refer to Hegel directly, but to Davis's re-interpretation of Hegel for use in education as this is the empirical field in which I am using the constructs inherent in Hegelian Judgement.

the level of immediacy generates the moment of reflection. During reflection there is an attempt to provide adequate predication so that the concept becomes increasingly comprehensible. To accomplish this, it would mean that there needs to be reflection on the notion, in this case a linear function, as it is being represented. It is through several instances and the implied relationship between these instances of reflection that opportunity presents itself for learners to attach different predicates to the given signifier. Extending on the example of a linear function, getting learners to compare the equation of linear function to equations of non-linear functions; or to complete a table of values for linear functions and determine the rate of change and compare it to the rate of change of non-linear functions; or to plot points and sketch the graphs; or to determine the nature of the mapping between the input and output values, are activities that have the potential of filling out of the concept of a linear function for the learners. In other words these activities have the potential to further predicate the concept of a linear function: a linear function has a constant rate of change; a linear function has a one-to-one mapping; a linear function is a non-vertical straight line, and so on. The activities that the teacher would engage the learners in so as to establish these additional predicates is what I refer to as the observable actions in the context of this study and will elaborate further under the analytical framework. Davis explains that the attempt at predication “opens up a space of *possibility* in which an increasingly comprehensible correspondence between subject and predicate(s) is generated” (Davis, 2005, p. 83).

3.3.3 The Judgement of Necessity

Further in his interpretation of Hegel, Davis (2001, 2005) indicates that at some point or another reflection and so the attempt at predication needs to be stopped and when this happens there is a shift in judgement from reflection to necessity. Continuing with the example of the linear function, at some point during the lesson the teacher will need to stop establishing predicates by which to describe a linear function. The teacher will need to draw the learners’ attention to the various predicates already established, since it is through the description provided by these different predicates that the features of a linear function emerge. It is through the combination of these features that we are enabled to distinguish a linear function from other classes of functions. Davis indicates that when this happens “a necessary relation between subject and predicate(s) is established, and the notion is no longer a mere ‘that’” (Davis, 2005, p. 84). Within the context of this study the judgement of necessity will be observed through the teacher’s appeal to some domain of authority (mathematics, the teacher or the everyday) so as to establish the necessary relationship between the subject and predicate(s) for the learners.

3.3.4 The Judgement of the Notion

Through the process of establishing the necessary relationship between the subject and predicate(s) for the learners a concept comes into being. Continuing with the example of the linear function, within the context of schooling the term linear function refers to a first-degree polynomial function of one variable. In more advanced mathematics (post-school), the term linear function refers to a different but related concept viz. a map between two vector spaces that preserves vector addition and scalar multiplication. Under the judgement of the notion Davis (2001, 2005) explains that the concern is not with the filling out of the concept but with the adequacy of the concept itself. In other words, the concern is whether the predicate used to describe the subject is ‘good’ or ‘bad’, ‘elegant’ or ‘clumsy’ and so forth. In Bernstein’s terms the concept that comes into being in a mathematics classroom is a recontextualised concept. As illustrated with the example of a linear function, there exists a gap between the recontextualised concept that comes into being in the mathematics classroom and what mathematicians might regard as viable. These are not necessarily the same thing, but are a function of learners having to learn. Thus, judgement of the notion is related to contingency since “the actuality of the notion depends on the occurrence of an event that is itself irreducibly contingent” (Davis, 2001, p. 10). In a pedagogic practice, the contingency is schooling.

Davis further elaborates that essentially the concept itself is a ‘universal’ and the judgment of the notion is about universality. In reality the only things that we ever experience are about particulars. We never experience the universal in some pure ideal form, for every particular ‘thing’ that we claim is an instance of the universal. It is the element of the contingent that disturbs the realisation of a particular universal that may be desired or claimed and it is the contingent that then is the placeholder for the empirical world. For example, everyone says that they are teaching functions, but we know that everyone is producing something somewhat different. We all pretend that it is the same universal that we are dealing with and we have to do that, this, however, is not a criticism (Davis, personal communication).

Using Hegel’s moments of judgement, as interpreted by Davis, to elaborate Bernstein’s theory of the pedagogic device, specifically the evaluative rule, provides me with the resources for chunking a lesson into moments of pedagogic judgements. These moments of pedagogic judgement provide a picture of the temporal unfolding of a lesson. This temporal unfolding of a lesson is what, following the work of Davis, I refer to as events or episodes which become the unit of analysis in this study, and together the events make up a lesson. The idea of analysing the lesson in chronologically progressive segments has significance for this study, since one will not be able to observe the different attempts made by the teacher to provide his/her learners with adequate predication so as to

transform the concept into something more comprehensible (*judgement of reflection*) in an instant of a lesson. It will also not be possible in an instant of a lesson to observe the knowledge domains a teacher appeals to in his/her attempt to establish the necessary relationship between the subject and predicate(s) for the learners (*judgement of necessity*). It is only over time (*the temporal unfolding of a lesson*) that one would be able to observe what accumulates to form the criteria of what counts as the legitimate text.

Out of focus in the discussion thus far is a theory of learning and the tools that one derives from the theory to assist in understanding pedagogic practice. One could argue that when studying pedagogic practice a theory of learning is always present, even if implicit. In this study, I deliberately and quite explicitly turn to variation theory and argue that variation theory provides tools that have the potential with which to fill out a concept and so shape the possibility for reflection. In other words, variation theory provides the tools that a teacher can use in his/her attempts to provide adequate predication for the learners so as to transform the concept from a mere ‘that’ into something more discursively substantial. It also provides the researcher with tools for interpreting reflection.

3.4 Learning as experiencing variation

Ference Marton and his colleagues¹⁹ have developed a theory for learning²⁰ which they have called ‘variation theory’. The central tenet to this theory lies precisely in the meaning of the word variation i.e. we learn through the experience of difference, rather than the recognition of similarity. Writing about the concept of awareness Marton and Booth explain that “it is impossible to experience anything in total isolation” (Marton & Booth, 1997, p. 96). The experience of anything is always embedded in a context. For example, if you look at a person (say Kim) it is not possible to pay attention to every detail of Kim (ranging from hair style to facial features to the clothing items that Kim is wearing to the various accessories that she is using and the footwear being worn etc.) all at the same time, although it is possible to see these aspects. When one aspect of the Kim’s attire changes whilst all other aspects remain unchanged, some people will begin to notice this change and the more this entity is changed more people will become aware. This is the nature of human awareness (Gurwitch, 1964 as cited in Marton & Booth, 1997). Marton and Booth (ibid.) explain that the fundamental structure of awareness is that we are able to discern entities and aspects and we can be aware of a few entities or aspects at the same time. “Learning to experience the various

¹⁹ See Marton and Booth (1997); Runesson (1999) and Marton and Pang (2006).

²⁰ Although Ference Marton and his colleagues refer to variation theory as a theory of learning, within the context of this study I use variation theory as a pedagogical tool for designing examples for use in practice as well as for purposes of describing and so contributing to the analysis of both the intended and lived objects of learning (see Chapters 6 and 9). I also incorporate some of the principles inherent in variation theory into the analysis of the enacted object of learning (see Chapter 8).

phenomena [...] means becoming capable of discerning certain entities or aspects and having the capability to be simultaneously and focally aware of these certain entities or aspects” (Marton & Booth, 1997, p. 123). Marton and Pang further elaborate that in order to “learn something, the learner must discern what is to be learned (the object of learning)” (Marton & Pang, 2006, p. 193). A fuller description of the idea of ‘object of learning’ will be provided shortly. In Runesson’s words “to be able to see what is the case, I must see what is *not* the case” (Runesson, 1999, p. 3). One can therefore infer that the experience of *variation* is a necessary condition for *discernment*. Thus, the kind of variation embedded in the set of examples a teacher makes use of during his/her teaching is crucial in the teacher’s attempt to keep the object of learning in focus during the course of a lesson.

Bowden and Marton (1998) as cited in Runesson (1999, p. 3) argue, that “when some aspect of a phenomenon or an event vary while another aspect or other aspects remain invariant, the varying aspect will be discerned.” In view of this, the variation that learners are exposed to must be important for their learning. Mason and Watson (2005) argue that an inescapable theme in mathematics is *invariance* in the midst of change. Thus, another way in which the ability to discern can be influenced, is to experience invariance against the backdrop of change. They further argue that if learners are to experience variation and to learn from it, then there must be sufficient variation and in sufficiently quick succession for it to come to their attention. Mason and Watson (ibid.) also indicate that to experience variation means more than simply being exposed to it. It needs to be a repeated experience with “emotional as well as cognitive and affective engagement” (Mason & Watson, 2005, p. 5) . It needs to be repeated until some degree of familiarity develops. The capacity of a teacher to engage his/her learners with tasks, exercises or examples that will allow them to experience variation is dependent on the teacher’s knowledge and understanding of the concept in question. As Mason and Watson put it “it is of major advantage to learners if their teachers are themselves aware of the various dimensions of possible variation” (Mason & Watson, 2005, p. 5). Watson and Mason (2004) refer to the features of an object that can be changed as the *dimension of variation*: for example in the teaching of functions one of the dimensions of variation is the different representations (algebraic, table of values, graphical) in which a function can be presented. The values within a variation are referred to as the *range of change*.

Marton and Pang assert that “one of the main functions of schooling today is to enable students to handle novel situations in powerful ways” (Marton & Pang, 2006, p. 198). They explain that different people may perceive the same situation in different ways and that the same situation will have different meanings for each individual. In addition, they argue that people do not act in relation to a situation but in relation to how they perceive the situation. What does this mean for

teaching and what demands does it place on the teacher? Marton and Pang raise an important question related to this – “What, then, is ‘a way of seeing something,’ and how can we characterize the differences between different ways of seeing the same thing?” (Marton & Pang, 2006, p. 198).

Marton and Pang (ibid.) explain that to make it possible for the learners to develop a certain way of seeing something, the pattern of variation and invariance the learners must experience must be constituted. To this end, Marton and Pang (ibid.) identify four patterns of variation that can facilitate students’ discernment of critical features or aspects of the object. It is important to note that a critical feature is not the same as the difficulty learners are experiencing in relation to the object of learning. Instead, it is the particular feature of the object of learning that the learners must be able to discern in order to experience the object of learning in a certain way. The four patterns of variation as identified by Marton and Pang are ‘contrast’, ‘separation’, ‘generalisation’ and ‘fusion’. Marton and Pang use the example of sight to illuminate these concepts and I use the same examples here to illustrate these constructs. A child who is short-sighted cannot separate his/her short-sightedness from the world that he/she sees. The two are inseparable until the child wears his/her first pair of spectacles. Now the child is able to separate sight and the world because one of the two aspects remained invariant (the world), while the child’s sight varies – sight without spectacles and sight with spectacles. The child is now in a position to discern sight, “the child can potentially discern sight as a dimension of variation within which two different values have been discerned” (Marton & Pang, 2006, p. 199). In the context of this example Marton and Pang unpack the concepts ‘contrast’, ‘separation’, ‘generalisation’ and ‘fusion’.

- Contrast – In order to experience something, one must experience something else to compare it with.

(When this condition is met, the child can discern both sight without spectacles and sight with spectacles.)

- Separation – In order to experience a certain aspect of something (say ‘x’), and in order to separate this aspect from other aspects (separating aspect ‘x’ from other aspects), that specific aspect (aspect ‘x’) must vary whilst other aspects remain invariant.

(Sight for the child changes, but the world remains the same.)

- Generalisation – To fully understand something, we need to experience varying appearances of that something. This is important for one to generalise the meaning of that something.

(The child’s sight with spectacles is invariantly better when the child looks at different, or varying, things.)

- Fusion – If there are several aspects that the learner needs to take into account at the same time, the teacher needs to vary the variations of these different aspects simultaneously (i.e. fusion) so that these aspects will all be experienced simultaneously.

(As the taking off and putting on of the spectacles varies simultaneously with sight, so the simultaneity of the two states can be discerned.)

Marton and Pang (ibid.) explain that learning to perceive ‘situations’ in a certain way is a function of experienced variation and invariance and the four concepts discussed above are necessary conditions for perceptual learning. These are “required for the learner to discern dimensions of variation, the values in those dimensions, the generality of those values, and the simultaneity of the dimensions of variation for the first time” (Marton & Pang, 2006, p. 200).

Marton *et al.* (2004) describe the *object of learning* as a capability that teachers want learners to develop e.g. the capability of being able to identify the class of function that an algebraic equation represents. One of the possible uses of variation could be as a pedagogical tool to help teachers design their lessons and to support learning through the judicious use of examples that keep the object of learning in focus and so attempt to fill out the notion by providing sufficient and appropriate predication. As already indicated in the previous footnote, within the context of this study the constructs inherent in variation theory are also used for purposes of describing and so contributing to the analysis of both the intended and lived objects of learning and incorporated into the analysis of the enacted object of learning.

They argue that when the object of learning goes out of focus, it needs to be brought back into focus but it is not done in and of itself, it is done in relation to the learner. Thus, Marton *et al.* (2004, p. 32) argue that “the kinds of examples and analogies the teacher uses, the stories that the teacher tells, the contexts that the teacher brings in and so on” are important for the constitution of the space of learning.

3.5 Summary

Describing the enacted object of learning constitutes bulk of the data that will be analysed, but before this data can be analysed it needs to be produced from the lesson transcripts. To accomplish this task I recruit Bernstein’s theory of the pedagogic device, more specifically the evaluative rule. Simplistically speaking, the evaluative rule concerns itself with the transmission of criteria of what counts as valid in the process of converting knowledge into pedagogic communication. The transmission of criteria as to what counts as valid knowledge sits at the core of a teacher’s ability to

mediate a particular object of learning. However, in Bernstein's articulation of his theory of the pedagogic device he did not elaborate on the evaluative rule.

To construct a more general account of the operation of evaluative judgement I follow the work of Davis who recruits Hegel's theory of judgement. Methodologically this suggests that it is in evaluative moments in pedagogic practice that criteria for what is to be acquired become visible. It is through the visibility of the criteria that teachers transmit that we will be able to observe what comes to be constituted as the enacted object of learning both at the level of content and at the level of authorisation, and so is described from a researcher's perspective.

Elaborating on Bernstein's evaluative rule by drawing on Hegelian judgement as interpreted by Davis for use in education leaves a theory of learning out of the picture. To this end, I drew on the principles of variation theory as it provides tools that enable a teacher to create opportunities for learners to see 'something' in a certain way. In addition it also provides the necessary tools with which to think about and describe the intended and the lived objects of learning.

Having setup this theory to provide the backdrop for the study, what does it now mean for the theory to interface with the empirical? In the next chapter I describe the empirical space and how the theory will interact with it.

Chapter 4

Methodology and Design

4.1 Introduction

In the preceding chapters I put forth that the aim of this study is to explore whether and how a leaning study professional development model, underpinned by variation theory and the judicious use of examples, could enhance teachers' mediation of a selected object of learning. I also elaborated on the theories that form the backdrop against which this study is conducted. In this chapter, I discuss the design of the study and present the data collection techniques that were employed. I also discuss the sampling strategies applied and provide a description of the sample so as to provide a description of the empirical space in which this study was undertaken.

4.2 The methodological approach

As already indicated, this study is about exploring whether a leaning study professional development model, underpinned by variation theory and the judicious use of examples, could work to improve teachers' mediation of a selected object of learning. Thus, a qualitative approach to research was seen to be most appropriate since this form of research is seen to be:

an effort to understand situations in their uniqueness as part of a particular context and the interactions there. This understanding is an end in itself, so that it is not attempting to predict what may happen in the future necessarily, but to understand the nature of that setting – what it means for teachers to be in that setting, what their lives are like, what's going on for them, what their meanings are, what the world looks like in that particular setting – and in the analysis to be able to communicate that faithfully to others who are interested in that setting [...] The analysis strives for depth of understanding.

(Patton, 1995 as cited in Merriam, 1998, p. 6)

Merriam also explains that qualitative research is an overarching concept that covers several forms of inquiry “that help us to understand and explain the meaning of social phenomena with as little disruption of the natural setting as possible” (Merriam, 1998, p. 5). A case study approach to research is a form of inquiry that can be classified as a qualitative approach to research. Theoretically speaking, the case study approach to inquiry seems to be an appropriate fit under which my study can be categorised. Opie explains that a case study “can be viewed as an in-depth study of interactions of a single instance in an enclosed system [...] the focus of a case study is on a real situation, with real people in an environment often familiar to the researcher” (Opie, 2004, p.

74). The idea of a case study providing for an intensive analysis and description of a single instance in an enclosed system is a characterising feature that differentiates it from other types of qualitative research methods (Merriam, 1998).

In writing about a case study approach to doing research Dowling and Brown use the words of Stake to present a widely held image of a case study approach to inquiry:

Case studies are special because they have a different focus. The case study focus is on a single actor, a single institution, a single enterprise, maybe a classroom under natural conditions so as to understand it – that bounded system – in its natural habitat.

(Stake, 1988 as cited in Dowling & Brown, 2010, p. 171)

The description of a case study as selected by Dowling and Brown (ibid.) emphasises the same characteristics present in the description of a case study as presented by other authors already cited in this section (Merriam, 1998; Opie, 2004): a case study is a study of a single instance within an enclosed system. To this, Opie (ibid.) also adds that the issue of numbers in a case study is meaningless and so the single instance could involve a single person, a group of people in a particular setting, an entire class, a department within a school, a school and so on. Dowling and Brown (ibid.) argue that this kind of description is a “mythologizing of research and a romanticising of the world in general” (p. 171). They make this claim in light of their argument that the description of a case study approach to research presents the ‘natural’ world “as thinkable in terms of a collection of mutually independent (bounded) systems that are nevertheless transparently knowable to us” (Dowling & Brown, 2010, p. 171). They indicate that this in turn places the demand on the observer to “dispense with their preconceptions and motivations” (Dowling and Brown, ibid.). The second aspect which contributes to Dowling and Brown’s argument of the description of a case study mythologising research, is in relation to what Opie refers to as ‘a single instance’ or what Stake (as cited in Dowling and Brown) refers to as ‘a single actor’. In both cases the context is not known, in other words the context in which the interactions of a single instance takes place or the context in which a single actor is involved is not known. Dowling and Brown pose a few questions to illuminate what they mean by the idea of context and at the same time to demonstrate that the single instance or single actor represents or participates in more than one research site at any given time. The questions they pose are:

Will educational research consider the behaviour of the subject in their domestic and leisure activities as well as in the classroom? Will it address the entire life cycle and, indeed, family history

of the subject? Will it be concerned with the physical aspects such as the subject's cardiovascular system?

(Dowling & Brown, 2010, p. 171)

They argue that each of the contexts signalled by the questions posed above are potential research sites and to think of these different research sites as independent of each other “is to constitute a radically schizoid subject” (Dowling & Brown, *ibid.*). They claim a case study approach to inquiry is a good label to use to differentiate it from other approaches that fall under the umbrella of qualitative research (naturalistic inquiry, interpretive research, field study, participant observation, inductive research and ethnography). This differentiation, according to Dowling and Brown (*ibid.*), is useful when considering the curriculum of a research methods course, i.e. in theory. In practice, they argue that there is no such thing as “the case study approach’ [...] Within the context of a specific research study, the use of the term ‘case’ is probably best interpreted as simply a way of describing one’s sampling procedures, which is to say, ‘this is a case of an object that is defined in the following terms...’” (Dowling & Brown, 2010, pp. 171-172). The terms here relate to the selection of teachers in the study.

4.3 Teachers in the study – setting the scene

The quality of a piece of research not only stands or falls by the appropriateness of methodology and instrumentation but also by the suitability of the sampling strategy that has been adopted.

(Cohen, Manion, & Morrison, 2002, p. 92)

The focus of this study is to explore the potential of a learning study intervention as a form of teacher professional development. In view of this, the formation of a learning study group lies at the core of the study and so too do teachers. To identify the teachers for this study I employed a purposive sampling approach (Cohen *et al.*, 2002; Merriam, 1998). The *first criterion* for the selection of teachers was to identify teachers who were teaching grade 10 mathematics as this is the grade level at which this study is located.

In commencing with this learning study, the feature(s) of the object of learning that the learning study group felt was critical for the discernment of the object of learning by the learners were conjectural in nature. These feature(s) needed to be tested and refined so that the feature that enables the discernment of the object of learning (*critical feature*) emerges in relation to the learners participating in the study. The refinement and so the emergence of a critical feature is a result of having to teach a lesson, refine the lesson and re-teach it to a different group of learners.

This cyclical nature which is characteristic of a learning study has implications for the number of teachers to include in my study. Marton and Pang (2006) provide a rough guideline when they describe a learning study as a group of between two and six teachers working together. In the case of my study, I decided to work with a group of four teachers, which meant that the learning study cycle would entail four iterations of the lesson. I felt that working with three teachers and so three iterations of the lesson would not be sufficient to allow for the emergence of the critical feature and then teaching lessons where the critical feature is in focus. The second reason why I opted to work with four teachers will be illuminated a little later in this section.

Having decided that I wanted to establish a learning study group which comprised four teachers teaching mathematics at the grade 10 level, the next step was to identify a school where this criterion would be satisfied. Since my study is located within the larger WMC-S project, as already alluded to in the rationale for this study, I was restricted to identifying schools from the ten schools that were participating in the WMC-S Project. This posed a challenge because none of these schools had assigned four teachers to teach mathematics at grade 10, due to issues around staffing and servicing the curriculum needs of the school. This warranted the need to work across schools and so set up a learning study group that was made up of teachers from different schools. With this task at hand there were two aspects that required careful consideration. Firstly, participating in this study meant that teachers would need to teach aspects of functions at grade 10. The section on functions was already taught in all of the schools and so re-teaching an aspect on functions during school time would be at the expense of teaching other topics as per the work programme. To ensure that my study did not cause any disruptions at any of the schools identified, I decided that the learning study lessons had to be taught after school hours. Secondly, I wanted to minimise the cost involved in transporting teachers from one school to another and so I needed to identify schools that were located in close proximity. Since I was to work across schools, I wanted to minimise the number of schools involved and so opted to work across two schools. This leads to the *second criterion* for this purposive sampling which was based on issues around cost, logistics and convenience. This also illuminates the second reason as to why I opted to work with not more than four teachers in this learning study.

The *third criterion* for selecting the sample was based on the learners' profile in terms of their performance in the mid-year examination, which was a common assessment (refer to section 6.4, identifying the object of learning). Working with the assumption that learners who obtained a similar low score on a particular topic as examined in a common assessment, implied that they experienced similar difficulties and misunderstandings with respect to the concepts within that topic

that were being tested. This was an important consideration in relation to identifying the object of learning, since identifying a suitable object of learning poses a more complex challenge if we are working with classes of learners whose performance with respect to a specific topic in a common assessment was on extreme ends of the performance range.

With this in mind I opted to work with two schools (School M and School R) in a township located within the Johannesburg city municipality. The township lies to the north of the Johannesburg city centre. The township is regarded as one of the poorest urban areas in South Africa and is characterised by a few reasonably well-built houses and a large number of informal dwellings. It is densely populated and has a high unemployment rate. Both School M and School R are ‘no fee’ schools, meaning that learners do not pay any school fees. These schools are dependent on the funding they receive from the Department of Education and business sponsorship. Although the Department of Education has supplied all the relevant textbooks to both schools, there are textbook shortages at these schools as a result of learners not being willing to bring their textbooks to school for fear of them being stolen. Some learners have sold their textbooks to learners who attend schools where textbooks are not supplied and whose parents can afford to purchase the books. The schools do not expect to be able to recover the cost of lost textbooks as the parents of the learners attending School M and School R cannot afford the cost.

I provide this very brief description of the township and the schools in which this study was conducted to provide the reader with some context of the schools and the socioeconomic background of the learners attending the schools. Tables 4.1 and 4.2 display School M’s and School R’s performance in the grade 12 national school leaving examinations.

	2009		2010		2011		2012	
	No. Wrote	Pass %	No. Wrote	Pass %	No. Wrote	Pass %	No. Wrote	Pass %
School M	145	65.5	153	77.9	133	75.2	200	69.0
School R	81	64.2	181	34.3	133	65.4	133	73.7

Table 4.1: Overall grade 12 pass rate for the four year period 2009 to 2012

	2009			2010			2011			2012		
	No. Wrote	Tot Pass	Pass %	No. Wrote	Tot Pass	Pass %	No. Wrote	Tot Pass	Pass %	No. Wrote	Tot Pass	Pass %
School M	84	27	32.1	81	27	33.3	68	14	20.6	100	32	32
School R	53	17	32.1	152	9	5.9	47	18	38.3	24	18	75

Table 4.2: Grade 12 mathematics pass rate for the four year period 2009 to 2012

The information supplied in Tables 4.1 and 4.2 not only adds to the description of the contextual background of Schools M and R but also demonstrates that there is room for enormous improvement in both their mathematics and overall pass rates. It is evident that the overall pass rate from 2009 to 2012 at both schools was above 60%, with the exception of School R which obtained a pass rate of 34.3% in 2010. The reasons for this drop in pass rate goes beyond the scope of this thesis, and therefore will not be engaged with. Having an overall pass rate above 60% does not tell us much about learner performance, so to get a better picture of the learners' performance at both schools let us consider their performance in mathematics. At the outset it is important to note that the pass mark for any subject is a mere 30%. The learners' performance in mathematics for the period 2009 to 2012 does not exceed the 40% mark, again with the exception of School R which obtained a 75% pass rate in mathematics in 2012. Although they obtained a relatively good pass rate here, it was only 18 out of 24 learners and so there was a drastic decline in the number of learners writing mathematics at the end of the grade 12 year. The second important thing to note is that looking at the symbol distribution of the learners who passed mathematics across both schools, the distribution has a mode on the mark range 30% to 39%. This represents a similar picture with respect to learner performance in mathematics in the country as a whole.

Two teachers from School M (Teacher L and Teacher J) and two teachers from School R (Teacher T and Teacher S) volunteered to participate in the study. Across the two schools a total of 109 learners agreed to participate in the study – 60 learners from School M and 49 learners from School R. The number of learners decreased to 96 (School M = 47 learners and School R = 49 learners) when we taught the learning study lessons. The number further decreased when we conducted the delayed post-test (86 learners wrote the delayed post-test). In planning for the implementation of the learning study the four teachers decided to teach across schools. In other words, the two teachers from School M wanted to teach the learners from School R and vice versa.

Table 4.3 displays the teachers' biographical information, and the order in which the teachers' names appear in the columns of the table represents the order in which they taught the lessons in the learning study cycle, starting with the first lesson.

Attribute	Teacher J	Teacher S	Teacher L	Teacher T
Rank at school	Educator	Educator	Head of Department	Educator
Total number of years teaching as at May 2013	26	28	25	17
Nature of appointment	Permanent	Permanent	Permanent	Temporary
Highest Qualification	Post-Graduate Certificate in Education and an Advanced Certificate in Education – both with specialisation in mathematics	BSc honours degree with specialisation in mathematics & mathematics education Currently enrolled for Master's degree with specialisation in mathematics & mathematics education	4-year professional teaching degree with specialisation in mathematics	4-year Higher Diploma in Education with specialisation in economics
Subject and grades taught in 2011 (when the learning study was implemented)	Mathematics grade 11 Mathematical Literacy grade 11	Mathematics grade 10	Mathematics grade 10	Mathematical Literacy grades 11 and 12
Subject and grades taught in 2013	Mathematics grade 12 Mathematical Literacy grade 10	Mathematics grades 10 & 11 Mathematical Literacy grade 11	Mathematics grade 11	Mathematics grades 10 & 11 Mathematical Literacy grade 11

Table 4.3: Teachers' biographical information

Of the four teachers who volunteered to participate in the study, Teacher T is the only teacher whose teacher qualification does not have a specialisation in mathematics. This does not mean that Teacher T is not competent to teach mathematics nor does it mean that the other teachers participating in this study are better at mathematics than Teacher T. Both Teacher T and Teacher J were not allocated to teach any grade 10 mathematics classes at their respective schools in 2011 when the learning study was conducted. As indicated previously, teaching functions at grade 10 formed part of the context in which this study was conducted. As a result of this, necessary arrangements were made for both Teacher J and Teacher T to each have access to one class of grade 10 learners registered for mathematics at the schools in which they were employed to teach. As indicated earlier, this arrangement did not cause any disruptions at the schools because the learning study lessons were planned to be taught at the end of the school day.

The four teachers who agreed to participate in this study had essentially agreed to participate in a learning study. None of these teachers had an idea of what a learning study entailed and so they had to be initiated into the learning study process and its demands. It was through the teachers' participation in the work of the WMC-S project in 2010 and 2011 that they were given the exposure and experience of thinking about and working with i) learner-generated examples (Watson & Mason, 2005) and ii) using the ideas of variation and invariance (Runesson, 2006). Although they were provided with this exposure and experience, it was only at the level of practice rather than any explicit attention to the theoretical underpinnings.

The four teachers together with the researcher set off to embark on a journey where the teachers were 'newcomers' to the entire concept of learning study and variation theory and the researcher possessed only some theoretical knowledge about learning study and variation theory and so was a novice in this regard.

4.4 The process – implementing the learning study

In this section I provide an overview of how I proceeded with the implementation of the learning study. The idea of learning study was a new concept to the teachers and through their participation in the WMC-S project they were exposed to the concept of learner-generated examples and the idea of variation and invariance in relation to its application to examples for use in practice. Their exposure to these concepts was at the level of application and not in relation to any theoretical underpinnings. In implementing this learning study I tried to provide the teachers with more insights into variation theory and its application to examples. To do this I prepared a document in which I provided a brief overview of the central idea inherent in variation theory viz. that we learn through the experience of difference, rather than the recognition of similarity (Marton & Booth, 1997). Using this idea, I then tried to illustrate with the aid of examples how variation could be applied to the examples that we could use with learners so as to bring a particular feature of a concept into focus for them, thus the deliberate use of examples (Pillay, 2011). Drawing on the work of Michener (1978) and Zodik and Zaslavsky (2008) I included in the document some discussion about the different purposes for which examples could be used, thus highlighting the significance of examples to the teaching and learning process. In the document I also provided an overview of the learning study process and at the same time highlighted the idea of an object of learning and the concept of a critical feature. In addition to this document I also selected papers and prepared a reading pack for each teacher. The papers in this reading pack supplemented the ideas

that I put together in the document that I had prepared. The following list is the list of papers that constituted the reading pack that was given to the teachers in this study:

- Sensitivity to Student Learning: A Possible Way to Enhance Teachers' and Students' Learning? (Runesson, Kullberg, & Maunula, 2011)
- Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics (Runesson, 2005)
- Variation and Mathematical Structure (Watson & Mason, 2006)
- Understanding Understanding Mathematics (Michener, 1978)
- Characteristics of teachers' choice of examples in and for the mathematics classroom (Zodik & Zaslavsky, 2008)
- Student-Generated Examples in the Learning of Mathematics (Watson & Mason, 2002)
- Functions, Graphs, and Graphing: Tasks, Learning, and Teaching (Leinhardt *et al.*, 1990)

I worked with the assumption that not all the teachers may read all of these papers, since reading papers of this nature was not necessarily part of their everyday practice as teachers. In view of this, I prepared myself to be in a position to present the ideas that were contained in these papers in a fashion that made them key concepts to the learning study that we were going to implement and in a manner in which I thought it might fit and become applicable. On the other hand, I was also prepared to make other readings related to variation theory, learning study, examples and functions available to the teachers should they request additional reading.

Table 4.4 provides an overview of how the learning study was implemented. The table shows the sequencing of events building up to the implementation of the learning study cycle (preparation phase) and the sequencing of events that constitute the learning study cycle itself (implementation phase). The table also shows the date when each event took place as well as the duration of each session as this provides a more detailed picture of the timeframe involved.

DAY	DATE	ACTIVITY & VENUE	Data Collected	START & END TIME	TIME
Wednesday	20/07	All teachers complete a questionnaire (School R)	Completed teacher questionnaire	15:00 – 16:30	1.5Hrs
Monday	25/07	Distributing reading packs to teachers and providing a brief overview of what each paper is about (School M)		15:00 – 17:00	2Hrs
Tuesday	02/08	Discussing ideas related to variation theory, learning study and use of examples (School M)		15:00 – 17:00	2Hrs
Friday	12/08	Discussion of ideas continued (School M)		15:00 – 17:00	2Hrs

Monday	15/08	Brainstorming ideas for lesson 1, analysing learners' results from the mid-year exam, identifying the object of learning (School R)	Video recorded Learners' test scripts	15:00 – 17:00	2Hrs
Friday	19/08	Planning lesson 1 (School R)	Video recorded	15:00 – 17:30	2.5Hrs
Monday	22/08	Finalising lesson 1 (School M)	Video recorded	15:00 – 17:00	2Hrs
Thursday	29/08	Teaching lesson 1; post-test and post-lesson reflection (School R)	Video recorded Post-test answer scripts	15:00 – 17:30	2.5Hrs
Friday	02/09	Reflection on lesson 1 continues Discussing ideas for lesson 2 commences (School R)	Video recorded	15:00 – 17:00	2Hrs
Monday	05/09	Finalising lesson 2 (School M)	Video recorded	15:00 - 16:00	1Hr
Wednesday	07/09	Teaching lesson 2; post-test and post-lesson reflection (School M)	Video recorded Post-test answer scripts	08:00 – 09:30	1.5Hrs
Friday	09/09	Reflection on lesson 2 continues. A critical feature emerges Planning and finalising lesson 3 (School M)	Video recorded	15:00 – 18:00	3Hrs
Monday	12/09	Teaching lesson 3; post-test and post-lesson reflection (School R)	Video recorded Post-test answer scripts	15:00 – 17:30	2.5Hrs
Friday	16/09	Reflection on lesson 3 continues Planning and finalising lesson 4 (School M)	Video recorded	15:00 – 17:30	2.5Hrs
Monday	19/09	Teaching lesson 4; post-test and post-lesson reflection (School M)	Video recorded Post-test answer scripts	15:00 – 17:30	2Hrs
Monday	26/09	Reflection on lesson 4 continues Teachers complete a questionnaire (School M)	Video recorded Teacher Completed teacher questionnaire	15:00 – 17:00	2Hrs
Monday	17/10	Conducting delayed post-test	Learners' solutions	14:00 – 15:00	1Hr

Table 4.4: Overview of how the Learning Study was implemented

In the section that follows, I discuss the data collection strategies employed.

4.5 Data collection strategies employed

In this section I provide an overview of the data collection techniques employed to gather the evidence that was used in this study. The collection of data in this study had three areas of focus. Firstly, learning study forms the nucleus for this study therefore the collection of data was primarily focused on classroom practice, specifically the collaborative planning of lessons (the intended object of learning) and the teaching of the collaboratively planned lessons (the enacted object of learning). The collection of lesson plans provided a record of the intended object of learning. Using some of the principles inherent in variation theory, I was able to analyse and describe these lessons by focusing on the dimensions of variation that were considered during the planning of the lesson.

This provided a way of being able to see what features of the function concept were being emphasised in the lesson.

Secondly, since learners' learning lies at the heart of learning study, I also collected data in the form of learners' performance in a pre-test, post-test and delayed post-test which were based on the function concept (the lived object of learning). It was important to collect this information as it provided me with some means of tracking possible learning gains made by the learners. The idea of a post-test conducted at the end of each learning study lesson and a delayed post-test after the completion of the learning study cycle is in line with the learning study methodology. In this chapter I do not engage in discussion related to the pre-, post- and delayed post-test as data collection strategies employed in this study, since it will only become meaningful when discussed in the context in which it is used as opposed to discussing it in isolation in this chapter. The pre-test is central to the development of the intended object of learning and will be discussed in Chapter 6. The post-test and the delayed post-test are central to describing the lived object of learning and will be elaborated on in Chapter 9.

Thirdly, learning study is also a form of TPD and so I was also interested in seeing what teachers have learnt as a result of their participation in this study. To engage teachers in such reflection, they completed a questionnaire before the implementation of the learning study as well as at the end.

4.5.1 Video recording of lessons

This formed the main source of data collection as indicated in the preceding section. All the lessons taught in this learning study cycle were video recorded and transcribed. In total, there were four lessons. In each case, the video recording of the lessons was done by making use of a digital video recorder that was mounted on a tripod. I made use of only one video recorder, which I operated. The video recorder was positioned at the back of the classroom so as not to obscure any learners' vision of the chalkboard or the teacher, as the teacher moved around the classroom during the lesson. The video recorder was focused on the teacher because the pedagogic practice in relation to the fundamental aspects that underpins this learning study is the object of study. During the time when the teacher got the learners to work on their own or in a group, the camera was focused on the teacher as he worked with the individual learner or group. A limitation of using only one video recorder lies in my inability to have captured the interaction between teacher and learner(s) when the group was not very close to the video recorder. This demonstrates that by using a single video recorder I was unable to capture in detail everything that was enacted during the lesson. Although

this was the case, it was possible to analyse all the lessons since the teaching was enacted in whole class discussions.

I chose to video record the lessons because it provided me with a more permanent record of the lessons as I could replay it as many times as was necessary and this allowed me to focus on events more closely. The video recording of the lesson also provided me with the ability to capture the verbal and non-verbal responses, thus capturing a more complete record of the lesson than would have been possible if I relied solely on using a lesson observation schedule or a mere audio recording of the lesson.

Each of the videoed lessons was transcribed. Cohen *et al.* (2002) argue that transcriptions “inevitably lose data from the original encounter [...] for a transcription represents the translation from one set of rule systems (oral and interpersonal) to another very remote rule system (written language)” (p. 281). Since I was transcribing a video recording of the lessons it was possible to insert in the transcript comments on all the non-verbal communications that took place e.g. what the teacher pointed to on the chalkboard or what the teacher wrote down on the board and so on. Cohen *et al.* (ibid.) argue that as soon as other data are noted, it becomes a matter of interpretation and therefore it is “unrealistic to pretend that the data on transcripts are anything but *already interpreted* data” (p. 281). Bearing this in mind, I took care to ensure that the verbal utterances as captured in the video recordings were transcribed as accurately as possible, and in the process of describing the non-verbal communication I exercised diligence to ensure that the description reflected what transpired.

To analyse the video recordings of the lessons I chunked the transcripts into evaluative events, and these evaluative events became the unit of analysis. This analysis was done only at the completion of the learning study cycle. It was not possible to do a detailed analysis during the unfolding of the learning study cycle. A detailed discussion of how the lessons were segmented into evaluative events and how they were analysed follows in Chapter 5 which deals with the analytic framework used to produce the data and the external language of description for the study.

4.5.2 Teacher questionnaires

While learning study is concerned about the lived object of learning at the level of the learner, I was also concerned about how the experience of participation in the learning study is lived at the level

of the teachers. In this regard, I was particularly interested in determining if there were any shifts²¹ in the teachers' thinking about their teaching practice in relation to the concept of an object of learning, and the deliberate use of examples²² and functions since these aspects were the fundamental ideas that informed the design of the learning study that I wanted to implement. To obtain a sense of the possible shifts in the teachers' thinking about their teaching practice in relation to these aspects, I wanted to know before the study commenced what resources they drew on when planning a lesson, what role they thought examples played and how they chose the examples for use in a lesson. I got the teachers to reflect on these aspects by giving them a questionnaire to complete. The questionnaire comprised three prompts in the form of two open-ended statements and one open-ended question to which they were required to provide a written response. The teachers were given thirty minutes (more time was allowed if required) to write their responses to the following:

- The resources that I use to plan a lesson.
- The role of examples in my practice (planning lessons, during the lesson, homework/assignments given to learners).
- What informs my choice of examples that I use in my practice (in planning lessons, during the lesson, homework to learners)?

The completed questionnaires were collected and filed.

The implementation of the learning study brought more aspects in relation to the planning of a lesson into focus. The aspects referred to are amongst others:

- Identifying the object of learning (which has direct bearing on the resources that teachers could use in planning a lesson);
- Identifying the critical feature in relation to the object of learning; and
- Using the basic principles inherent in variation theory to inform and so structure the selection and sequencing of planned examples. As a result, the example space created ought to provide learners with opportunities to discern the critical feature, thus the deliberateness in the use of examples.

²¹ I use the term shifts to represent any change in teachers' thinking about planning a lesson in relation to an object of learning and the deliberate use of examples as well as their learning in relation to functions and learning in general. Shifts would also include no shift in their considerations regarding these aspects.

²² The deliberate use of examples includes the selection of examples and the sequencing of examples to create the example space during the lesson. The sequencing and selection of examples is done in relation to the object of learning.

I assumed that these aspects were not necessarily in focus for all teachers at the beginning of the learning study cycle. To get a sense of the possible shifts in the teachers thinking about their practice in relation to the key ideas that informed the learning study, I administered another questionnaire at the end of the learning study cycle. From this questionnaire, I wanted to know what the teachers learnt by participating in the study and so I posed questions in relation to the key principles upon which the learning study was based. I asked this quite generally and then followed up with questions related to choosing and using examples. The questions asked:

- What does it mean for me to have participated in a learning study on functions?
- What have I learnt about my own practice/my own teaching through participating in the learning study on functions?
- One of the aspects that we focused on in the work we did was on examples, how we choose and use them for teaching. What do you think you have learnt about choosing and using examples through the work we did together?
- Could you write some comments on the role of examples in your practice (planning lessons, during the lesson, homework/assignments given to learners). What informs your choice of examples that you use (in planning lessons, during the lesson, homework to learners)?
- How would you go about teaching the topic of functions next year, or for that matter any other topic henceforth?

The purpose of collecting data in this fashion provided me with an economical way, in terms of time, to gain some insights into the teachers' thinking about the key aspects that underpin this learning study and how these may have shifted because of their participation in the study. The open-endedness of the questions provided teachers with an opportunity to provide what Opie (2004) refers to as free responses and no preconceived replies. Such replies reflect the spontaneity of the respondents' thinking in relation to the ideas prompted by the questions posed. The questions in both questionnaires acted as prompts that allowed me to do a content analysis of the teachers' responses in a thematic fashion. This made it possible to see shifts in each teacher's thinking about their practice after participating in this learning study.

The disadvantage and so a limitation of this method of data collection lies in the inability of the researcher to pose other questions to better understand what the respondent meant by some of the comments they had written and also the inability to pursue issues that may have not been included

in the set of questions that were given. This is in comparison to a semi-structured interview which “permits flexibility rather than fixity of sequence of discussions” (Cohen *et al.*, 2002, p. 147).

The questions posed in each of the questionnaires provide categories that relate to the fundamental ideas that drive the learning study i.e. (i) planning a lesson with an object of learning and a critical feature in mind, and (ii) the deliberate use of examples. Tables 4.5 and 4.6 provide the analytic tool with which to read the teachers’ responses to the questions in each of the questionnaires.

Questions/prompts <i>before</i> implementation of learning study	Concept in focus	Indicator
The resources that I use to plan a lesson	Object of learning and a critical feature	Is there any reference to learner error or learner difficulty?
The role of examples in my practice (planning lessons, during the lesson, homework/assignments given to learners) <i>Note: the role of examples includes selection and sequencing</i>	Deliberate use of examples	Is there any reference related to the selection and sequencing of examples?
What informs my choice of examples that I use in my practice (in planning lessons, during the lesson, homework to learners)?	Deliberate use of examples	Is there any consideration given to the selection of examples?

Table 4.5: Tools for reading teachers’ responses to questions in the pre-learning study questionnaire

Questions <i>after</i> implementation of learning study	Concept in focus	Indicator
What does it mean for me to have participated in a learning study on functions?	Learning in general	<i>Experience of learning:</i> Is there any reference to working in a group (learning from each other by planning a lesson together, learning by observing a lesson being taught)? So experiences related to collegiality. <i>Content of learning:</i> Is there any reference to object of learning, critical feature, selection and sequencing of examples, functions?
What have I learnt about my own practice/my own teaching through participating in the learning study on functions?	Deliberate use of examples and object of learning	Does the teacher highlight any shifts in his/her thinking about teaching functions and mathematics in general? Particularly, do they highlight shifts in how they talk about the object of learning and examples?
What do you think you have learnt about choosing and using examples through the work we did together?	Deliberate use of examples	Related to the selection and sequencing of examples

What informs your choice of examples that you use (in planning lessons, during the lesson, homework to learners)?	Deliberate use of examples	Related to the selection of examples
How would you go about teaching the topic of functions next year or for that matter any other topic henceforth?	Deliberate use of examples and object of learning	Are there any shifts in teachers' thinking about the teaching of functions and mathematics more generally?

Table 4.6: Tools for reading teachers' responses to questions in the post-learning study questionnaire

It is through asking these questions that I get to see how the teachers' ideas about the key concepts may have shifted. In my discussion about the teachers' learning as a result of participating in this study (see Chapter 9) I select extracts from their responses to questions in both the pre- and post-questionnaires that will help me to see how they reflect on the concept of an object of learning, example use, functions and their general learning from the study.

4.6 My role as researcher

In the literature review section on learning study I alluded to the idea that in the design of a learning study the teachers could either work on their own or with a researcher to find ways to make it possible for the learners to appropriate a particular object of learning. The perspective that I adopted for the implementation of this learning study was for the researcher to work with the teachers. I decided on this perspective because my study is located within the wider WMC-S project, and as a result my data collection sites were restricted to the schools that are participating in the WMC-S project and amongst the range of teachers represented by the domain of schools participating in the WMC-S project, none of the teachers were knowledgeable about learning study nor were they familiar with the principles inherent in variation theory. This does not necessarily mean that beyond the domain of schools participating in the WMC-S project, I would have found teachers who were knowledgeable about the key concepts that drive this study and were willing to participate in my study. Taking these constraints into account therefore meant that in order for me to implement a learning study I had to facilitate the process and so be a 'complete participant' in the research. Opie describes the concept of a complete participant as one were the "researcher is fully immersed in the participant role but uses their position to conduct research" (Opie, 2004, p. 129).

In this study I took on three roles. The first role was that of a researcher with an interest in exploring if a learning study underpinned by variation theory and the deliberate use of examples could assist teachers' mediation of a specific object of learning. It was with this drive in mind that I set out to establish a learning study group and on this journey I took on two additional roles, the second being that of a knowledgeable other within the learning study group with the function of

providing the relevant support in terms of the mathematics and the application of the theoretical constructs from variation theory that had a bearing on this study. The responsibility characterising my third role was that of initiator. Since the teachers in this study were new to learning study, they had to be initiated into learning study. Although I took on these roles it is important to note that I was a relative newcomer to the key constructs that underpin this study viz. learning study, variation theory and the deliberate use of examples through the application of the principles inherent in variation theory. Here too, it was through my participation in the planning and implementation of the professional development work of the WMC-S project in 2010 and 2011 that I was exposed to and gained experience with some of the constructs associated with variation theory and its application to examples for use in practice. In addition, I am also not implying that I have a profound knowledge base in relation to the function concept or that I possess expert facilitation skills. The additional experience that I brought to the learning study was that I have been a secondary school teacher for 10 years: for two of these years I served as the head of the mathematics department and for three years I served as the deputy principal providing support to the mathematics department. For the past 12 years I served several schools in my capacity as an official representing the local education authority. This experience includes mathematics teaching experience as well as knowledge of schools (curriculum, governance and administrative knowledge). Although I commenced this study with this background and took on the role of knowledgeable other, I was seen by the teachers participating in the study as the expert with all the relevant skills and knowledge. Having taken on this role, I never doubted the expertise of the teachers to teach their own learners and I made it clear to them that I was participating in the study to learn from them and from the process as we went about implementing the learning study. The attitude I adopted contributed to the long-term cooperation between me and the teachers and to developing mutual understanding and trust. This kind of relationship was essential because it allowed us to be critical of each other and so inputs made were never seen to be judgmental in nature but rather developmental.

As already discussed in my role of researcher I became a complete participant and so the intention with taking on this role was to have an influence on the teachers participating in the study. The influence that I refer to is not deleterious in nature but to steer the entire learning study process as a knowledgeable other.

4.7 Summary

In considering a methodological approach for this study, a case study approach to research seemed a good fit. However, Dowling and Brown (2010) argue that the description of a case study being a

study of a single instance within an ‘enclosed system’ is in a sense a ‘mythologizing of research’ since it presents the natural world as a collection of mutually independent systems that are visibly known to us, and this is not the case.

This highlights that the individual or group of individuals being studied are participating in a range of systems and in conducting research we are acting selectively with regard to the system to focus on and disregarding the others. According to Dowling and Brown (ibid.) we are sampling and thus the use of the term ‘case’ is best interpreted as simply a way of describing one’s sampling procedures.

In line of this argument, I presented a description of the sample and thereafter I described the data collection techniques employed in this study and I concluded this chapter by describing my role as researcher.

In the next chapter, I engage with the procedures for the production of data and provide the external language of description. I also engage with issues related to rigour in research and ethics.

Chapter 5

Data Production, Rigour and Ethics

5.1 Introduction

Qualitative approaches do not impose probability theory or even, necessarily, the natural number system on their settings. They thereby lose both the power and the rigidity of quantitative methods. The researcher employing qualitative techniques is relieved of the requirement to specify their coding principles sufficiently uniformly to enable their data to be counted. But if it cannot be counted, then it must be represented in some other way.

(Dowling & Brown, 2010, p. 89)

Since this study is located within a qualitative paradigm²³, according to Dowling and Brown (ibid.) the data collected cannot “be counted, it must be represented in some other way” (p. 89). Secondly, they argue that in research related to the field of education “it is no more necessary to resolve your epistemology or ontology [...] than it is to incorporate a declaration of your religious affiliation, though some would consider one or more of these essential [...] the tendency to make a pass at metatheoretical discussion is commonly presented in lieu of adequate theoretical development” (Dowling & Brown, 2010, p. 143). They take a particular position which describes “educational research activity as the production of a coherent set of statements” (Dowling & Brown, ibid.). They further indicate that this coherent set of statements is established and located within explicitly stated theoretical and empirical contexts. So for Dowling and Brown coherence is the “fundamental criteria by which educational research is to be judged [...] this is not to claim that perfect coherence is attainable” (p. 143).

In this study I adopt Dowling and Brown’s perspective with respect to research in education and the purpose of this chapter is to provide the reader with the alternative way in which the data is represented. In other words, the central object of this chapter is to provide a ‘lens’ with which to read the data so as to constitute a coherent set of statements. I commence by providing the descriptors that constitute the external language of description for the study. Thereafter I engage with issues related to my considerations concerning rigour in this research and finally conclude the chapter by elaborating on the issues of ethics relevant to the study.

²³ The idea of this study being located within a qualitative paradigm is elaborated on in Chapter 4, which deals with methodological issues relevant to this study.

5.2 Moving from the theoretical to the empirical

Consider the term ‘Man’. Now pause for a short while and think about the image that immediately comes to your mind? Is it the human species (man versus animal)? Or males of the human species (man versus women)? Or adult males of the human species (man versus boy)? The image that immediately popped up in your mind was dependent on your frame of reference.

Dowling and Brown argue that “the text very definitely does not tell its own story. Rather, its description must be *biased* according to an explicit and coherent organisational language” (Dowling & Brown, 2010, p. 94). In the case of research, the organisational language refers to the theoretical framework that underpins the study. If the theoretical framework is not made explicit and the language for reading the data is not structured coherently, then the reliability of the findings may be compromised.

Dowling and Brown (2010) describe the *theoretical field* as a “nebula of debates, theories and empirical findings” (p. 145) that are used to develop the theoretical underpinning of the study. This study is located within Bernstein’s theory of the pedagogic device, in particular his structuring of the evaluative rule. Since Bernstein has not elaborated on the evaluative rule, to operationalise the workings of the evaluative rule for use in this study I turned to Hegel’s theory of judgment as interpreted by Davis (2001; 2005) for use in education. To assist in understanding pedagogic practice, specifically in relation to providing learners with opportunities to discern an object of learning, I turned to variation theory (refer to Chapter 3, Section 3.4). For Dowling and Brown the *empirical field* “may be glossed as the broad range of practices and experiences to which the research relates [...] the empirical field also constitutes a community or communities” (Dowling & Brown, 2010, p. 147). The empirical field for this study constitutes four teachers from two schools who collaborate with the researcher in implementing a learning study. Included in this are learners from four grade 10 classes who volunteered to participate in the study. The texts that were produced in this study included pre-, post- and delayed post-test of learners; lesson plans generated through the learning study cycle; video records and transcripts of all the lessons taught; and video recording and transcripts of all meetings with the teachers through the learning study cycle.

Dowling and Brown argue that the “theoretical field objectifies the empirical field and not the other way around” (Dowling & Brown, 2010, p. 147). The process through which the theoretical field is related to the empirical field is paramount to ensuring the reliability of the research findings. Bernstein (1996) describes this relationship in terms of a language of description, and he defines a language of description as:

a translation device whereby one language is transformed into another. We can distinguish between internal and external languages of description. [...] A language of description constructs what is to count as an empirical referent, how such referents relate to each other to produce a specific text, and translate these referential relations into theoretical objects or potential theoretical objects. In other words the external language of description (L₂) is the means by which the internal language (L₁) is activated as a reading device or vice versa. A language of description from this point of view, consists of rules for the unambiguous recognition of what is to count as a relevant empirical relation, and rules (realisation rules) for reading the manifest contingent enactments of those empirical relations.

(Bernstein, 1996, pp. 135-137)

In other words, for Bernstein a language of description must always be Janus-faced, with the internal language of description pointing to the theoretical field and the external language of description pointing to the empirical field. As already indicated, in this study the internal language of description is derived from Bernstein (1990; 1996); Hegel as reinterpreted by Davis (2001; 2005) and Marton and Booth (1997) (refer to Chapter 3). Davis explains that the “selection of theoretical referents that inform the production of the internal language of description is done in ‘dialogue’ with the empirical specificity of the object(s) of research” (Davis, 2005, p. 106). Once the internal language of description has been developed, the researcher needs to develop the external language of description which “consists of rules for the unambiguous recognition of what is to count as a relevant empirical relation, and rules (realisation rules) for reading the manifest contingent enactments of those empirical relations” (Bernstein, 1996, p. 137). In other words, the external language of description must construct the rules for what is to count as an empirical referent, how these referents relate to each other and then to translate these referential relations back to the internal language of description i.e. the theoretical field. It is important to note that the external language of description cannot exhaust the empirical field, it cannot capture the specificity of the empirical; the empirical field can only be grasped and interpreted through a theoretical gaze (Davis, 2005).

5.3 Developing the external language of description

My task in this study is to unambiguously describe the enacted and lived objects of learning. In this section I deal with the enacted object of learning – the lessons. Following my theorising of pedagogic discourse, and the significance of evaluation, what I need to describe is how a concept comes into existence in a lesson, how it is reflected on and how the teacher legitimises what counts

as valid knowledge. Implicit at the level of existence and reflection is the idea of variation and invariance and its application to examples for use in practice.

In my attempt to make sense of the lessons and at the same time to provide a description of what comes to be constituted as the enacted object of learning, I chunked the lesson transcripts into units for analysis. The units need to capture the temporal unfolding of the lesson, as well as how evaluative judgment proceeds. Following the work of Davis (2001, 2005) the unit is what is thus referred to as an *evaluative event*, indicating both evaluation and temporality.

5.3.1 The evaluative event

An *evaluative event* is characterised by the teacher introducing a concept (in a mathematics classroom this usually takes place by means of an example). The example could be introduced either through spoken words, what I shall refer to as *verbally*, or through some *written* form (*words*, *symbolically* or *graphically*) or in some combination. These ideas will be elaborated on in the next section.

My intention here is to demonstrate how evaluative events are identified within a lesson. To demonstrate what marks the start and end of an evaluative event consider the following extract:

{EVENT 12 – 50:34}

- Teacher J ... Ok. Now we need to move on. Here on our worksheets, I want you to look at our worksheets.
- Learner Which page? Which one?
- Teacher J Now this is the part where we have got the verbal part, it says here, input value multiplied it by two, gives the output value. That whole thing, that's where we are.
- Learners (*inaudible*) five.
- Teacher J You see that?
- Learners Yes.
- Teacher J Right. Now I want us to try and...you can use your pencil, you can use whatever it is, the input value is multiplied by two to give the output. What will be the equation?
- Learner The what?
- Teacher J The input value...(writes '*Input multiplied by 2 gives the output*')
- Learner Is multiplied by two.
- Teacher J Multiplied by two. It gives the output. Can you give me the algebraic equation there. Algebra. This is verbal. This is in words, can you see that? (writes verbal above the statement '*Input multiplied by two gives the output*').
- Learner Yes. F to x over two x (*talking quietly at the back*).
- Teacher J What about algebraic equations (writes '*Algebraic equation*' alongside '*verbal*'). What would be the algebraic equation there? Let's just do that. We need to do an example I

want you to fill in use your pencil right. What is it going to be? Ok, let me give you a clue. y it's equal to what? (*writes $y =$*)

Learner $f x$.

Teacher J Sorry?

Learner $f x$.

Learner two x .

Teacher J two x (*writes $2x$*). Your input value is x (*points to x*), isn't it?

Learner Yes.

Teacher J And you multiplied it by two, it means it's two x . Do you agree with that?

Learners Yes.

Teacher J Very good. Let's go to the next one. The next one, who can read the next one?

(Lesson 1, event 12)

Since event 12 was one of the shorter events in lesson 1, I selected it for use here as it contributes to the economic use of space for this thesis. The beginning of event 12 is marked by Teacher J referring his learners to another example on the worksheet (input value multiplied by two, gives the output value). In this event the example is introduced in a *written* form (on the worksheet). In order for Teacher J to direct his learners' attention to the correct example, he verbalises it thus bringing the example into existence *verbally* as well. The end of event 12 is marked by Teacher J confirming the learner's response that the algebraic representation is $y = 2x$. Teacher J's utterance 'Let's go on to the next one' also marks the end of event 12.

In some instances when a concept is introduced, the teacher will provide learners with opportunities to *reflect* on the concept so as to fill it out and transform it into something more substantial. In such instances, the teacher could use the same example but shift the focus in terms of what he/she wants the learners to do with the example initially introduced. In this case, each of the aspects in focus would constitute a *sub-event*.

For example, a teacher could introduce the function defined by $y = x$ and in the first sub-event the teacher could get learners to complete a table of values. Using the same example, the teacher could then shift focus and get learners to determine the gradient between different sets of points as captured in the table of values, thus marking another sub-event. Through the process of reflection with the aim of filling out the concept, an evaluative event could comprise multiple examples of the concept across its different representations and therefore multiple evaluative sub-events, or one example with different aspects in focus where each aspect would mark a different evaluative sub-event.

Consider the following extract that constitutes a complete sub-event (event 2.2). For purposes of illustrating the conclusion of the sub-event, I also include the first turn of talk that marks the introduction of the next sub-event (event 2.3). The event number and the time at which the sub-event starts is also provided, thus by calculating the difference in time one is able to determine the duration of an event or sub-event. To put the sub-event into context it is worth noting that in the previous sub-event Teacher J set up a table with two columns. He labelled the first column ‘equations’ and entered $y = 2x$ as the first entry in this column. He then labelled the second column ‘substitution’ and in the next row beneath this heading he shows the working details for the calculation of the y-values after substitution using $x = -4$ and $x = 3$. Sub-event 2.2 starts with Teacher J asking learners to identify the type of function being represented by $y = 2x$.

{EVENT 2.2 – 05:28}

- Teacher J ... What type of a function is this one (*points to the equations in Equation row*)? If you look at this graph. You know it, (*inaudible*) on that graph. What type of a graph is that, if you were to draw a graph?
- Learner (*inaudible*).
- Teacher J Yes!
- Learner (*inaudible*)
- Teacher J Sorry?
- Learner y intercept
- Teacher J y intercept. Um...let me say here, we are looking at the type of the function (*writes ‘Type of function’ at top of third column*). Right, ok. Now let me give you...to try and shorten our time...this type of a function we call it linear, isn’t it? (*few learners mumble yes*) It’s a straight line graph, isn’t it (*few learners mumble yeah*). Ok. So we know this is actually linear (*writes ‘linear’ in third column*)...

{EVENT 2.3 – 06:17}

- Teacher J ...Right do you think you can give me more examples of the type of a graph that is like this one here (*points to equation $y = 2x$ under the equation column*)? y is equal to – it must be a line graph, a straight line graph. Can you give me an example?

(Lesson 1, event 2.2 and start of event 2.3)

All the sub-events within event 2 are aimed at filling out what $y = 2x$ means. In the extracts selected we see Teacher J initially introduced the function $y = 2x$ and got learners to substitute specific input values and calculate its respective output value. Teacher J does not change the example, but instead shifts focus and asks learners to identify the type of function. In this instance the beginning of event 2.2 is marked by the shift in focus in terms of what learners were expected to do with the equation $y = 2x$. The shift in focus is identified by the instruction given by Teacher J

“What type of function is this one?” The end of this sub-event is marked by the teacher’s assertion of the type of function being represented and the assertion in this case is telling. The various forms of assertion and its description will be elaborated on later in this chapter.

The end of an evaluative event is usually marked by i) the introduction of a different example, which marks the beginning of a new event (as in the case of event 12), or ii) a different focus on the same example which indicates a sub-event (as in the case of event 2.2). A sub-event occurs within an evaluative event and it elaborates or exemplifies the concept that was introduced at the start of the specific evaluative event.

5.3.2 Existence

In the previous section I demonstrated that the beginning of an evaluative event is usually marked by the introduction of a concept. The introduction of a concept in a mathematics lesson usually takes place by means of an example. As already indicated in the previous section, an example could be introduced verbally or in some written form.

In any pedagogic setting much occurs verbally and so in most instances when an example is introduced verbally it is also accompanied by some written form. In this study I make a distinction as to what would be considered a verbal introduction of an example. Consider the following extract where Teacher T asks learners to give an example of a linear function:

Teacher T I want you to give me more statements showing linear function, other than the three up here. You have given me one, two and three (*points to $f(x) = x + 1$, $y = 2x$ and $y = \frac{1}{2}x$ respectively*). Can you give me more functions that are linear? More statements that give us linear functions? Another one?

Learner y is equal to negative x plus two.

Teacher T y is equal to negative...x plus two (*write $y = -x + 2$*)...

(Lesson 4, event 2.3)

The extract above illustrates that the example of the linear function $y = -x + 2$ is produced by a learner’s *oral response* to the teacher’s request for learners to generate other examples of a linear function. The example is brought into existence by the learner’s utterance of the words ‘y equal to negative x plus two’. As the learner utters the words, the teacher re-represents the verbal utterances by writing its associated symbols ‘ $y = -x + 2$ ’ on the chalkboard. In this particular extract the example is brought into existence verbally by a learner. As already indicated, in most instances

when examples are introduced verbally, they are also accompanied by some written form, for example, the words representing a function are written on a worksheet and when the teacher refers to the example the words are read out aloud (*verbal*), thus bringing the example into existence both in a written (on the worksheet) and in a verbal form (reading). Consider the following extract:

- Teacher J Now we need to move on. Here on our worksheets, I want you to look at our worksheets.
Learner Which page? Which one?
Teacher J Now this is the part where we have got the verbal part, it says here, input value multiplied it by two, gives the output value. That whole thing, that's where we are.
Learners (*inaudible*) five.
Teacher J You see that?
Learners Yes.
Teacher J Right. Now I want us to try and...you can use your pencil, you can use whatever it is, the input value is multiplied by two to give the output. What will be the equation?

(Lesson 1, event 12)

When examples are provided in a word format (e.g. double the input value gives the output value), as opposed to graphical or symbolic form, there will be a need at some stage to convert the words into algebraic syntax ($y = 2x$) and in this process of talking about and referring to the example one would be required to verbalise the example. As illustrated in the extract above, in order for Teacher J to direct his learners' attention to the relevant example on the worksheet he is forced to read the words and in so doing the example comes into existence *verbally* as well as in a *written* form (words). Using both the verbal utterance and the words representing the example Teacher J now engages his learners in reflection – in filling out the notion – possibly here with the task of re-representing the example algebraically. It is specifically such instances that I refer to as an example coming into existence in a *written* form as well as *verbally*.

The next extract demonstrates an instance when I would categorise an example being introduced both *verbally* as well as *symbolically*. In this extract we find that as Teacher J introduces the example verbally 'f of x is equal to one over two x' he is also expressing the function symbolically by writing its algebraic representation, $f(x) = \frac{1}{2}x$, on the board.

- Teacher J Let's try to look at another one. Suppose you are given f of x is equal to one over a two x (*writes $f(x) = \frac{1}{2}x$*)...ok? (*draws another row on the table*) I want us to take this value that you have taken (*points to $x = -4$*) and this other value (*points to $x = 3$*) that we have

taken, and see what we are going to have. Now what is going to be our f of x if our x is negative four?

(Lesson 1, event 3.2)

To further illustrate when I consider an example coming into existence verbally, I juxtapose the above event with the following event:

Teacher T I'm going to put down here (*writes (1) $y = 2x$ and (2) $y = \frac{1}{2}x$*). Right. Can you all see those two statements, number one and two, that I have written on the chalkboard?

Learners Yes.

Teacher T That's good. I want someone to read statement number one. Who can read statement number one? Yes, girl?

Learner y is equal to two x .

Teacher T y is equal to two x . Right

[...]

Learner y is equal to half x .

Teacher T y is equal to half x . That has been read correctly as well, isn't it?

Learner Yes.

Teacher T y is equal to half x . But when we look at the way we have read these two statements, y equal to two x , and y equal to half x , do they sound the same?

(Lesson 4, event 2.1)

In this event Teacher T introduces an example of two different linear functions by writing their *symbolic (algebraic)* representation on the board. Teacher T then engages his learners with these examples by asking them to read it, thus getting them to re-represent the examples verbally. This extract illustrates an instance when I would regard an example to be introduced *symbolically*. Although the learners were required to verbalise the symbolic representation, the purpose for verbalising is not to introduce the example as was the case in the previous extract but for the learners to compare the two equations.

With regard to bringing examples into existence during the flow of a lesson, the last category that I identified was when a teacher introduces an example through its *graphical* representation. In such instances the teacher would place a pre-drawn graph of the function in question on the board, for example:

Teacher S Right, let's look at the graph that I have on the board. (*On a laminated grid the teacher had drawn two Cartesian planes. The Cartesian plane on the left contained the graph of*

the hyperbola $y = \frac{2}{x}$ labelled A and an exponential graph $y = 2^x$ labelled B). Let's look at this side (refers to the Cartesian plane on the left). A is the pink or whatever colour that is. Who can identify that graph? What do we call that graph, equal to this one (points to the exponential graph)? I've labelled it graph B. What do you call that graph? Anyone?

(Lesson 2, event 1)

The extracts selected thus far and the discussion that followed to provide the indicators of how I categorised the manner in which *an example is introduced* gives one the impression that the teacher is the only person in the pedagogic setting that introduces examples. There are instances when the learners also introduce examples. Across the four lessons that constituted this learning study, whenever learners introduced an example the example was introduced verbally by the learner and then as a way of confirming the learner's response and making the example more accessible to other learners, the teacher re-represents the example in symbolic form by writing its algebraic representation on the board, for example:

- Teacher J Right do you think you can give me more examples of the type of a graph that is like this one here (points to equation $y = 2x$ under the equation column)? y is equal to – it must be a line graph, a straight line graph. Can you give me an example?
- Learner Y is equal to (inaudible)
- Teacher J Y is equal to...? I can't hear.
- Learner Equal to x.
- [...]
- Teacher J Right, just try, just give me any...any, any equation you think is an example of a straight line graph. You want to try? Y is equal to...? (writes $y =$)
- Learner Equals to x. y equals to x.
- Teacher J Y is equal to...?
- Learner x.
- Teacher J Y is equal to x (writes x). y is equal to x. Hmm, that's correct. Another one? Another one? Something that looks like this one here. Yes?
- Learner Y equal to three x plus one.
- Teacher J Right. Let's listen to that one. Let me just move this line here (rubs out horizontal line in third row). y is equal to three x plus one (writes $y = 3x + 1$).

(Lesson 1, event 2.3)

5.3.3 From existence to reflection

When the teacher introduces an example, the teacher is bringing a concept into existence and this concept is housed within the example. This does not mean that the example carries only one concept, there could be various other concepts embedded in the example. The task of the teacher is to now provide learners with opportunities to reflect on the notion that would contribute to them discerning the intended object of learning. During this process of reflection, the teacher would inevitably transmit criteria by which learners are to recognise what the concept is. This does not necessarily mean, however, that teachers accomplish this task or that learners develop an understanding of the concept introduced. By transmitting the criteria the teacher legitimates some form of meaning for the learners. For the purposes of this study the process of reflection is what I refer to as the *observable actions* that the teacher engages his/her learners in with respect to the example introduced. In order to transmit criteria to legitimate meaning, i.e. to show learners where authority lies and what counts as true in the mathematics classroom, the teacher will appeal to some or other domain to authorise meaning. *The transmission of criteria is observed through the teacher talk together with the observable action.* Refer to event 12 under section 5.3.1 in this chapter: the event demonstrates that the observable action engaged with is to re-represent the linear function from words to algebraic form. The criteria transmitted by Teacher J in this instance is a confirmation of the correctness of the learner's response captured by the teacher's utterance (*teacher's talk*) "very good. Let's go to the next one". A description of the indicators for reflection and necessity will be provided in the next two sections.

All sub-categories listed under existence, reflection (*observable actions*) and necessity (*authority*) were empirically established from the data. At the beginning of this section I provided the indicators for how I categorised the introduction of examples as either verbal, symbolic, words or graphical. I now provide indicators for each category under the observable actions and authority.

5.3.4 Indicators of reflection

Once a concept is brought into existence, the teacher needs to provide learners with opportunities to *reflect* on the concept with the purpose of transforming the concept into something more discursively substantial so that the concept becomes increasingly comprehensible for the learners. As discussed previously, *reflection occurs through the observable actions* the teacher engages the learners in during the unfolding of the lesson. The following is a description of the indicators for each of the categories under reflection (observable actions).

- i. *Revising mathematical rules and conventions* – for the purposes of this study a mathematical rule relates to the ‘laws’ that govern the use of mathematics, for example laws of exponents, the syntax for expressing a negative number squared is $(-2)^2$ and not -2^2 ; ‘BODMAS’ rule for the order in which operations are to be evaluated in a given expression or division by zero is undefined. Conventions on the other hand relate to aspects of mathematics that are generally accepted as customary within the community of school mathematics. For example the variable ‘x’ is used to represent the input value and the variable ‘y’ the output value; on a Cartesian plane the horizontal axis is reserved for the independent variable and the vertical axis for the dependent variable; the expression ‘3p’ implies an ‘invisible’ operation and is the convention for representing the operation of multiplication, and $\frac{-2}{3}$; $\frac{2}{-3}$ & $-\frac{2}{3}$ represent equivalent fractions. Conventions, in the context of this study, relate mostly to the manner in which mathematics is notated in the context of school mathematics. The following are extracts that demonstrate instances when I categorised the observable action as revising mathematical rules and conventions.

Teacher J: Do you recall that if you are given f of x (*writes $f(x)$*), this one here (*points to $f(x)$*) is the same as y (*writes $=y$*).

Learner: yes.

Teacher J So you can call it...what is your name?

Learner Lucy.

Teacher J Lucy? I can call it L of x, (*writes $L(x)$*) for Lucy. (*laughter*) And then the other one is Mpho. M of x (*writes $M(x)=y$*), again all this (*points to the functional notation written on the board*) is equal to y, isn’t it?

Learners Yes.

(Lesson 1, event 3.1)

OR

Teacher S: ... which is the input value and which is the output value? All the y’s? All the x’s? ... Anyone? Is y the output or the input value, or x the output? Which is which? This girl here (*points to a learner who raised her hand*)?

Learner: x is the input value.

Teacher S: x is the input value (*writes $x=\text{input value}$*) and y will therefore be...?

Learners: The output value. (Lesson 2, event 3)

OR

Teacher J It's quite interesting, in mathematics what is the denominator of this negative four?
 Learners One.
 Teacher J One. It's not a zero?
 Learners No!
 Teacher J No, it's not allowed in mathematics. It's a mathematical crime. You'll be sued to court
(laughter).

(Lesson 1, event 3.2)

OR

Teacher J Negative four squared (*writes -4^2*). You know what that means?
 Learners Yes.
 Teacher J It means negative four...(*writes -4×-4*)
 Learners Times negative four.
 Teacher J You agree?
 Learners Yes.
 Teacher J Now, if I'm writing this one here...
 Learner It's sixteen.
 Teacher J I will put a bracket there (*writes $(-4)^2$*). Because I am squaring a negative four.

(Lesson 1, event 10.1)

The first two extracts demonstrate that the observable action relates to the teacher revising mathematical conventions with the learners, whilst the last two extracts demonstrate the teacher revising mathematical rules.

- ii. *Changing representation* – the observable action would result in the learners or teacher changing the representation of the example that was introduced. For example:

Teacher L Let's say we have this (*writes $y = 2x$*), can we get a volunteer who does read what I've just written on the board? Just one learner who'll just read what we've just written on the board.

(Lesson 3, event 1.1)

In this example the learners are required to read the algebraic representation of the linear function that was written on the board. In the process of reading, the learners change the representation from algebraic to verbal. This is merely one example where the observable

action results in the changing of a representation. Other examples include changing from the algebraic representation to a table of values and then from a table of value to a mapping between sets or from a table of values to a graphical representation.

- iii. *Substituting and calculating* – observable actions classified under this category entail learners engaged in the process of substituting given x-values into the algebraic representation of a function, performing the related algebraic calculations and finding the corresponding y-values. For example:

Teacher J ... Right, we have got an equation there, right? And now if you would like to complete a table of values (*writes 'Table of values'*), we must use what we call substitution, isn't it?

Learners Yes.

Teacher J Substitution there (*writes 'Substitution' in top middle row next to 'Equation'*). Now, suppose I was given that my x is equal to let me say negative four (*writes $x = -4$ in row under 'Substitution'*), what will be my value of y here (*points to y*)? My x is equal to negative four, what will be the value of my y?

Learners Negative eight.

Teacher J How do you get that?

Learner You substitute.

Teacher J y is equal to two...(*writes $y = 2$*)

Learner Two bracket times...

Teacher J Open bracket...

Learner Negative four.

Teacher J Negative four (*writes -4 in a bracket*). And then if you multiply that one there you get now...(*writes =*)?

Learners Negative eight.

Teacher J Negative eight (*writes -8*)

(Lesson 1, event 2.1)

- iv. *Identifying and naming functions given verbal, algebraic or graphical representation* – learners are provided with either the verbal, algebraic or graphical representation of a particular function or a combination of these representations and then they are required to determine the type of function being represented. In the first extract below Teacher J engages learners in identifying and naming the function defined by $m(x) = 2^x$ – identifying and naming a function given its algebraic representation. The second extract below demonstrates learners being required to identify and name the function given its graphical representation.

Teacher J ... Right. What function is that?

Learner Undefined.

Teacher J (*points to the $m(x) = 2^x$*) Real life situation this

Teacher J Yes

Learner Quadratic

Teacher J Quadratic

Learners Hyperbola.

Learner Parabola

Teacher J Now this is the one that you say is hyperbola (*points to $h(x) = \frac{2}{x}$*). Do you see how it is?

And now do you see this one here (*points to $m(x) = 2^x$*)? Look at these two (*points to $h(x)$ and $m(x)$*).

Learner It's a linear.

Learner It's a quadratic.

Teacher J Ok, right (*writes Exponential in table*). Exponential graph. It's an exponential graph (*learners say oh!*) Now you see how important it is to revise.

(Lesson 1, event 11.2)

OR

Teacher S Right, let's look at the graph that I have on the board. (*On a laminated grid the teacher had drawn two Cartesian planes. The Cartesian plane on the left contained the graph of the hyperbola $y = \frac{2}{x}$ labelled A and an exponential graph $y = 2^x$ labelled B*). Let's look at this side (*refers to the Cartesian plane on the left*). A is the pink or whatever colour that is. Who can identify that graph? What do we call that graph, equal to this one (*points to the exponential graph*)? I've labelled it graph B. What do you call that graph? Anyone?

Learner Parabola.

Teacher S Parabola. Somebody's got parabola here, anyone else? No one, ok. Who can identify graph A? (*beneath the laminated grid the teacher writes A= and B=*) (*using her finger the teacher traces out graph A*)

Learner Hyperbola?

Teacher S It's a hyperbola (*writes hyperbola after A=*). So graph A is a hyperbola.

(Lesson 2, event 1)

- v. *Identifying sameness between equations* – the algebraic representation of two (or more) functions is provided and learners are required to identify aspects that are the same in each of

the equations being compared. For example in the extract that follows Teacher T engages his learners in identifying aspects that are the same in the equations $y = 2x$ and $y = \frac{1}{2}x$:

Teacher T ... Is there anything that is the same with these two statements?

Learner Yes.

Teacher T What do you find to be the same?

Learners The y's...

Teacher T Ah, you raise up your hands, I pick on you, then we get moving. Yes?

Learner y.

Teacher T What you find that is common there, the term y is common for that, that y, that y. Fine. What else? Yes?

Learner x.

Teacher T x again is common. You'll find you've got an x here, and also you've got an x there. What else?

Learner Equals to.

Teacher T The equals to. Right. That's also correct. We've also got an equal sign, they're given to say they are all equations. Alright. Now looking at those two equations, we say this equation is explaining y. y being equal to...

Teacher T ...two times...?

Learner x.

Teacher T x. Right. That's what it means. When you write two x, it means two times x. Two x, if I say this means two times x (*on other side of the board, writes $2x = 2 \times x$*). Then how about half x (*writes $\frac{1}{2}x =$*), what does it mean? Yes?

Learner Half times x.

Teacher T Wonderful. That also means half times x (*writes $\frac{1}{2} \times x$*). So you'll find, as you have already indicated, that's something in common (*circles the unknown x in each equation*). You've got two being multiplied, and again here you've got half being multiplied by...?

Learner x.

Teacher T By x. If I can ask again, what is the exponent of x in both equations? What is the power of x in both equations? Yes, girl?

Learner One.

(Lesson 4, events 1.2, 1.3 and 1.4)

- vi. *Learners generating examples* – dependent on the class of function in focus during an evaluative event, the teacher would ask the learners to provide other examples of that particular class of function, for example consider the following extract:

Teacher T ... y equal to half x is a...linear function. I want you to give me more statements showing linear function, other than the three up here. You have given me one, two and three (*points to $f(x) = x + 1$, $y = 2x$ and $y = \frac{1}{2}x$ respectively*). Can you give me more functions that are linear. More statements that give us linear functions. Another one?

Learner y is equal to negative x plus two.

Teacher T y is equal to negative...x plus two (*writes $y = -x + 2$*). Quite correct. Another one? Let's try to be fast. Y equal to negative x plus two. You see the exponent of x here is...?

Learner One.

Teacher T It's one. That's why we call it a linear function. Let's have another one. Yes, boy?

Learner y is equal to two x plus two.

Teacher T Y is equal to two x plus two (*writes $y = 2x + 2$*). Quite correct.

(Lesson 4, event 2.3)

- vii. *Plotting points* – The teacher demonstrates how to extract the coordinates from the table of values and plot these on the Cartesian plane.

Teacher J Negative two? Ok, fine. Let me do that, x there, y there (*labels y and x on the Cartesian plane*) axis. Ok, fine. Let's go to the next one. This one here (*points to $x = -4$*). I don't have it can you see that, I don't have negative four. That's fine, it doesn't matter. When I come to this one here (*points to the y-value $-\frac{3}{2}$*), it's giving me a fraction, and I don't want to approximate it, it may be here (*refers to approximating the point on the y-axis*) right. Let's come to this one here. Do you see that one? Negative two is going to give us (*points to -1*)?

Learners Negative one.

Teacher J Negative two and negative one (*plots the point*). I'll meet you at the river (*sings and draws meeting lines*) that's where they meet. So this one (*refers to the y-value*) is moving, right, horizontal, this (*refers to the x-value*) vertical that's where they meet, somewhere there (*refers to the plotted point $(-2;-1)$*). Let's take the next one. Let's take the next one. Minus two there (*points to table*), and negative one. Let's take this one here. It's a half, let's leave it. What about zero and zero can you see that?

(Lesson 1, event 6.2)

- viii. *Drawing graphs* – as is the case with the plotting of points, here too this observable action is characterised by the teacher demonstrating the drawing of a graph.

- ix. *Comparing y-values for given x-values* – Learners empirically establish if two functions, presented in their algebraic form, are equivalent. They do this by comparing the respective y-values for the given x-values of the two functions.

Background to the first extract below: the learners were required to find y-values for given x-values for the function $f(x) = \frac{1}{2}x$. Teacher J recorded these values in a table, and then plotted these points on a Cartesian plane and drew the graph so that learners could identify the type of function represented by $f(x)$. Teacher J then moved on to the example $g(x) = \frac{x}{2}$ and asked learners to find y-values for the same x-values that were used when working with $f(x)$. The extract below shows the learners finding the y-values and Teacher J recording these values in a table. Teacher J asks learners what they notice about b and c which refers to the functions $f(x)$ and $g(x)$. The learners' immediate response is that they are the same and so in this instance I inferred that the learners compared the y-values for the two functions as this is what Teacher J brought into focus for the learners through event 7.

Teacher J Let's quickly do the next one ($g(x) = \frac{x}{2}$).

Learner Negative two. Negative three.

Teacher J Negative two (*writes -2 in first square in 3rd row*).

Learners Negative three over two.

Teacher J Negative three over two (*writes $-\frac{3}{2}$ in 2nd square*).

Learners Negative one, negative half, zero; half; one; three over two; two (*as learners are giving the y-values the Teacher J fills in the table of values*).

Teacher J What is it?

Learners It's linear.

Teacher J (*writes linear in the type of graph column*) Very interesting right. How many of you...now what did you notice between b and c?

Learners They are the same.

Teacher J They are the same.

(Lesson 1, events 7.1 & 8)

OR

The next extract illustrates Teacher L deliberately engaging her learners in the action of comparing the y-values for specific x-values to determine if $y = \frac{1}{2} \times x$ and $y = x \div 2$ are equivalent:

Teacher L I want us to look at this two equations (*draws a block around the two equations: $y = \frac{1}{2} \times x$ and $y = x \div 2$*). Using the two examples that we have used, are they different or are they the same? Y is equal to x divided by two and y is equal to half times x.

Learner They are not the same.

Teacher L They are not the same. Can you compare the answers: when x is six, if we get half of six, we get...?

Learners Three.

Teacher L Three. When x is six, if we divide six by two, we still get...?

Learners Three.

Teacher L Three. Are the answers still the same?

Learners They are the same.

(Lesson 3, event 2.3)

- x. *Revisiting effects of parameters* – given a function in its algebraic form, the teacher revises the effect a particular parameter would have on the graphical representation of the function. For example consider the following extract:

Teacher S ...C and G are both parabolae (sic), so why are they not the same? Why don't they look the same? Who can tell me?

Learner (*learner that was just speaking*) Because

Teacher S Anyone else? It's only person here who's talking, what about the others? Are we still sleeping?

Learner (*Learner that was talking*) That one it's because the one that is facing there it's the sad smile. It's because the...

Teacher S Can a smile be sad? A smile?

Learner That one's a happy face.

Teacher S Can I smile when I'm sad?

Learner Ok, the sad face, sad, it's because the exponent was negative.

Teacher S Exponent?

Learner Yes, the exponent was negative. Where that one the exponent is positive. (*comments from class*) No, no, no, the input value was negative. No, it's going to be positive.

- Teacher S Ok what he wants to say. Who can you help him? I know what he wants to say...who can help us here?
- Learner I think he wants to say the coordinates are negative, that's why it's the sad face and the other one it's positive...the happy face is because it's ...the co-ordinates are positive.
- Teacher S The coordinates? Ok let me help you. The value of a (*points to the equation $y=ax^2+q$*) so concentrate on the value of a. Now you probably have done it. So for a sad face what happens there? Is it positive or negative?
- Learner Negative.
- Teacher S Positive.
- Learner Negative.
- Teacher S Sad face, negative? A happy face? (*writes on the board $a > 0$*) When a is positive, what happens? The parabola becomes (*draws a parabola pointing upwards*). Happy face. (*writes $a < 0$*) When a is negative, the parabola (*draws a parabola pointing down*)
- Learner Yes.
- Teacher S Sad.

(Lesson 2, event 7.3)

5.3.5 From reflection to grounding the notion

Having provided learners with opportunities to reflect on a notion, which occurs through the observable actions, the teacher will eventually be required to legitimate some form of meaning for his/her learners. In the process of grounding the notion, reflection needs to be halted and the teacher will provide criteria by which the learners are to recognise what the concept is. As indicated previously, this does not necessarily mean that learners' will understand what the concept is. In legitimating meaning, the teacher appeals to some form of authority. In the context of this study authority lies in mathematics, in the teacher or in everyday experience. The sub-categories under mathematics include the following:

- i. *Definitions* – the utterance by the teacher has the basic entailments of the formal textbook definition of the concept in focus, for example:

Teacher L Each x-value is mapped onto only one y-value [...] so we call it a function

(Lesson 3, event 3.12)

- ii. *Rules and Conventions* – the teacher tells learners that something is the way it is without offering any reasons for example

Teacher J: $f(x)$, this one here is the same as y .

(Lesson 1, event 3.1)

When the teacher does make an attempt to offer a reason, the reason is restricted to what is and what is not allowed by the greater mathematics community, for example:

Teacher J: It's quite interesting, in mathematics what is the denominator of this negative 4?

Learners: One.

Teacher J: One. It's not a zero?

Learners: No!

Teacher J: No, it's not allowed in mathematics. It's a mathematical crime. You'll be sued to court (*laughter*).

(Lesson 1, event 3.2)

iii. *Empirical / Technology* – Empirical means that an outcome or result is based on that which is observed. In a lesson, empirically establishing a result or outcome could be based on a single case as opposed to a general case. Furthermore, the result could be established by making use of a calculator. Empirical and technology is grouped together under one category since in both cases the appeal is to an instance (substituting specific x -values and comparing the related y -values) or an observable result (performing a calculation on a calculator). This is exemplified by the following examples:

a) teacher empirically establishes the validity of the concept introduced e.g. to establish if $y = \frac{1}{2}x$ and $y = \frac{x}{2}$ represents the same function, the teacher gets learners to substitute specific x -values into each of the equations and then compare the resulting output values – refer to the second extract used in number ix above to illustrate the observable action of comparing y -values for given x -values.

b) The extract which follows illustrates instances when I categorised a teachers' appeal to technology as the source of authority:

Teacher J I will put a bracket there (*writes $(-4)^2$*). Because I am squaring a negative four. Are we together?

Learners Yes.

Teacher J Now if you can use your little computer it will tell you

Learner Sixteen.

(Lesson 1, event 10.1)

- iv. *Process* – In categorising a teacher's attempt to legitimate some form of meaning for his/her learners as a process I distinguished between two types of process. In both cases the teacher provides the learners with a set of steps to follow. In the first case the algorithm provided by the teacher has entailments of mathematics in focus and made explicit for the learners. This is a process with mathematics in focus and is referred to as Pm^+ . Consider the following extract to illustrate what I mean:

Teacher J So we want to try and look at now, functions, we want to start with...let me...(draws two long horizontal lines and writes 'Equation' between them. Writes $y = 2x$ underneath. Draws two vertical lines and a horizontal line under $y = 2x$). Right, we have got an equation there, right? And now if you would like to complete a table of values (writes 'Table of values'), we must use what we call substitution, isn't it?

Learners Yes.

Teacher J Substitution there (writes 'Substitution' in top middle row next to 'Equation'). Now, suppose I was given that my x is equal to let me say negative four (writes $x = -4$ in row under 'Substitution'), what will be my value of y here (points to y)? My x is equal to negative four, what will be the value of my y?

Learners Negative eight.

Teacher J How do you get that?

Learner You substitute.

Teacher J y is equal to two...(writes $y = 2$)

Learner Two bracket times...

Teacher J Open bracket...

Learner Negative four.

Teacher J Negative four (writes -4 in a bracket). And then if you multiply that one there you get now...(writes =)?

Learners Negative eight.

Teacher J Negative eight (writes -8).

(Lesson 1, event 2.1)

In the second case, the algorithm provided by the teacher is informed by mathematics but the mathematics is not made explicit for the learners. For example:

Teacher J Right, just try, just give me any...any, any equation you think is an example of a straight line graph. You want to try? Y is equal to...? (*writes $y=$*)

Learner Equals to x. y equals to x.

Teacher J Y is equal to...?

Learner x.

Teacher J Y is equal to x (*writes x*). y is equal to x. Hmm, that's correct. Another one? Another one? Something that looks like this one here. Yes?

Learner Y equal to three x plus one.

Teacher J Y is equal to...?

Learner Three x plus one.

Teacher J Right. Let's listen to that one. Let me just move this line here (*rubs out horizontal line in third row*). y is equal to three x plus one (*writes $y = 3x + 1$*). You agree, that is a straight line? Ok. Anything extra, anything more? Can you give me one extra example? I have got y is equal to x. I understand. (*learner arrives late in class*) y is equal to three x plus one. Again it's a straight line graph. What else? Yes? Ok. (*inaudible*). And this one here (*points to $y = 3x + 1$*) went a bit further. Ok. Do you agree that I can just change this figure here (*points to the 2 in $y = 2x$*), and put more of that, isn't it?

Learners Yes.

Teacher J You can put a three, put a four, put a five.

Learner Half.

Teacher J A half, I can put anything, just there (*points to 2s again in $y = 2x$ sums*), isn't it?

Learners Yes.

(Lesson 1, event 2.3)

Looking at the underlined statement in this extract we see that Teacher J provides learners with an algorithm to generate different equations of a linear function – all that the learners have to do is change the coefficient of x. By changing the coefficient of x the gradient of the line is being altered and this would result in a different linear function. Teacher J is aware of the mathematics underlying the algorithm he makes available for his learners but the mathematics is not brought into focus for the learners. In this case we have a process with the mathematics hidden and this is referred to as Pm̄.

When authority lies with the teacher, the teacher either asserts what the case is, for example:

Teacher J What type of a function is this one (*points to the equations in Equation row*)? If you look at this graph. You know it, (*inaudible*) on that graph. What type of a graph is that, if you were to draw a graph?

Learner (*inaudible*)

Teacher J Yes!

Learner (*inaudible*)

Teacher J Sorry?

Learner y intercept.

Teacher J y intercept. Um...let me say here, we are looking at the type of the function (*writes 'Type of function' at top of third column*). Right, ok. Now let me give you...to try and shorten our time...this type of a function we call it linear, isn't it

(Lesson 1, event 2.2)

Or the teacher merely confirms learners' responses. The confirmation of learner responses could take on different forms:

- i. *Restates and Writes (RW)* – the teacher merely restates what the learner has said and writes it down on the board. In restating the teacher could either repeat the exact same response given by the learner (see first extract) or the teacher could repeat what the learner has said but using the teacher's own words (see second extract):

Teacher J Negative four (*writes -4 in a bracket*). And then if you multiply that one there you get now...(writes =)?

Learners Negative eight.

Teacher J Negative eight (*writes -8*).

(Lesson 1, event 2.1)

OR

Teacher J They said it's equal to, one...

Learner Point five.

Teacher J ...comma five (*writes =1,5*).

(Lesson 1, event 3.3)

- ii. *Restates (R)* – In such instances the teacher merely restates what the learner has said. In restating the teacher endorses the correctness of the learner's response:

Teacher S Now I want you to write it in mapping form. What values are you going to put in the figure on the left (*refers to the input oval on the worksheet*)? (*draws two ovals and writes input above the oval on the left*)

Learner The x values.

Teacher S The x values.

(Lesson 2, event 5.3)

- iii. *Writing (W)* – Instances of teacher confirmation that is categorised as writing is identified when the teacher merely writes down the learner's response, for example:

Teacher J ... What is it?

Learners It's linear.

Teacher J (*writes linear in the type of graph column*)

(Lesson 1, event 7.2)

- iv. *Acknowledging correctness of learner's response (A)* – consider the following extracts to illustrate other instances that constituted a categorisation of teacher confirmation:

Teacher J Ok, class, there are two answers, the one who says it's non-linear, the other one says, it's still a linear. Which one are we going to take?

Learners Linear.

Teacher J Linear? Wow! Let's give ourselves a clap (*claps*) (*writes 'Linear' in 3rd row*)

(Lesson 1, event 3.4)

OR

Teacher J You agree?

Learners Yes.

Teacher J Good – right fine – ok so now can we go to the next one very quickly

(Lesson 1, event 8)

OR

Learner Y equals two x.

Teacher L y...?

Learner Equals two x.

Teacher L Equals two x. Is there any other learner who has another way of reading this out? Any other learner who has another different ways of what he has said? Thank you, it's correct.

(Lesson 3, event 1.1)

OR

Teacher S ... You are so quick. And then you can fill it up. If x is minus three, what will y be equal to?

Learners Negative six.

Teacher S And when y equals minus one?

Learners Negative two.

Teacher S And then when x is zero?

Learners Zero.

Teacher S And for one?

Learners Two.

Teacher S And for two?

Learners Four.

Teacher S Seven?

Learners Fourteen.

Teacher S Good, that's very good.

(Lesson 2, event 5.2)

The last category identified as to where authority lies is in 'everyday' experience. To this end consider the following extracts to illustrate:

Teacher J You see this ruler here (*holds up big ruler*). If you say one over sixteen, in other words, we are going to cut this ruler into sixteen equal parts. Then I'll take only the small part. You understand? We are saying, this thing here, suppose it's a chocolate, and there are sixteen people. You must cut this one so that everyone is going to have an equal piece. Right? So you are going to have the small part that you are going to get is one over sixteen. But otherwise all of that, it's sixteen over sixteen, which is one whole. Isn't it?

(Lesson 1, event 11.2)

OR

Teacher L A function is a relationship in which each x value or each element in the first set is related to only one element of the second set. So for an example, in normal life...right...have you ever heard of a situation where a woman is married to two husbands at the same time²⁴?

(Lesson 3, event 3.12)

To summarise, in moving from the theoretical to the empirical I needed to provide the indicators of the external language of description. The indicators as described in this chapter are for use in analysing the *enacted object of learning* because in order to gain some understanding of what is enacted, data had to be produced from the lessons. To produce the data in a principled fashion I drew on theoretical resources provided by Hegel's theory of judgement as interpreted by Davis (2005) for use in education, as this allows me to see what happens in a lesson over time. To fill out reflection with the aim of elaborating the description of what comes to be constituted as the enacted object of learning I also recruit some of the concepts inherent in variation theory as discussed in Chapter 3.

As already indicated, in order to describe the enacted object of learning it was necessary to produce the data from the lessons first, since using the lesson itself as the unit of analysis obscures what happens over time during the lesson. However, in order to get a description of what comes to be constituted as the *intended and lived objects of learning* there was no need to produce the data as such. To gain some understanding in terms of what comes to be constituted as the *intended object of learning* I was able to use the basic idea in variation theory (varying one aspect whilst keeping other aspects invariant) to determine what dimensions of variation are opened up in the planned example space. It is through these dimensions of variation that one is able to see the features of the concept (in this case functions) that are emphasised in a lesson. To analyse the *lived object of learning* I used the principle of what was varied and what remained invariant across the learning study lessons and linked the differences to the learners' performance in the post-test and delayed post-test. In the sections which follow, I deal with issues of rigour in research and the ethical considerations factored into this study.

5.4 Issues of reliability and validity in this study

Various authors argue that the issue of reliability seems to be a misfit when applied to qualitative studies and so it would be more productive to think about the *dependability* or *consistency* of the

²⁴ It is interesting to note that Nash also made use of the husband and wife metaphor to describe the concept of a mathematical function (refer to Chapter 1).

results obtained from the data (Cohen *et al.*, 2002; Dowling & Brown, 2010; Guba & Lincoln, 1983; Merriam, 1998; Opie, 2004). For Dowling and Brown (*ibid.*) the measure of consistency is in relation to the coding process when the data is coded on different occasions by the researcher or by different researchers. This is only possible if the researcher provides a clear set of instructions for coding the data, in other words if clear descriptors of each category is made available.

In moving from the theoretical to the empirical I provided the descriptors by which to recognise the various categories identified in the data. In this study reliability is related to the external language of description that was made explicit. The process of refining the external language of description to provide a clear set of descriptors was a back and forth movement between the empirical and theoretical fields. To ensure that these descriptors resulted in the constancy of coding by different researchers, my supervisor coded some data and these codes were matched with the codes that I got. I went a step further and got other doctoral fellows within the WMC-S project to also code some of the data and again their codes were compared to my coding. Once I found that there was consistency in the way the various researchers coded the same sets of data I then commenced with coding all the data, thus ensuring a level of reliability.

In discussing the idea of validity in relation to qualitative studies Merriam (1998) distinguishes between internal and external validity. She describes internal validity as having to deal with the question of how research findings match reality. In other words: “How congruent are the findings with reality? Do the findings capture what is really there? Are investigators observing or measuring what they think they are measuring?” (Merriam, 1998, p. 201). Reality is “a multiple set of mental constructions ... made by humans; their constructions are on their minds, and they are, in the main, accessible to the humans who make them” (Lincoln and Guba as cited in Merriam, 1998, p. 203). Merriam (*ibid.*) argues that since the researcher collects and analyses the data, the interpretation of reality is accessed directly through the researchers’ observations. As was explained earlier, the explicit description of the external language of description used to produce data for analysis and interpretation provides the structure to ensure internal validity. This follows in line with Dowling and Brown’s description of validity which they argue “concerns the relationship between the theoretical concept variable (or concepts) and the empirical *indicator variables* (or *indicators*)” (Dowling & Brown, 2010, p. 24, italics in original).

Merriam (1998) describes external validity as “the extent to which the findings of one study can be applied to other situations” (p. 207). This brings into focus the generalisability of the research findings. It is known that within qualitative studies the issue of generalisability poses a challenge

since it is difficult to think about the generalisability in the same way as studies which use an experimental or correlational design. Merriam (ibid.) explains that in experimental or correlation designs the “ability to generalize to other settings or people is ensured through a priori conditions such as assumptions of equivalency between the sample and population from which it was drawn, control of sample size, random sampling, and so on [...] even in these circumstances, generalizations are made within specified levels of confidence” (p. 207). Therefore, a more appropriate approach within a qualitative study is to focus on the transferability of the findings instead of generalisability. For example, in this study, what emerges as the critical feature could be transferrable to other learning situations in which the classification of functions, as dealt with at school level, is in focus. In using this critical feature in another setting it is important to understand that this critical feature might not be critical for all learners in a new setting as the learners in the new setting may have already discerned this critical feature. The critical feature that emerged might not be a new insight of learner difficulty in the field, but its specific location, and its general applicability provides an important example of what learning studies offer. It is also a reminder that something which could be regarded as being so trivial is in fact central to learners being able to identify and name a function given its algebraic representation, again a specific instance of a more general feature of teaching mathematics – that which appears trivial to the teacher could be critical for groups of learners.

5.5 Ethical considerations

At the most elementary level ethics in research has to do with not harming or doing wrong to others but being respectful of others and being fair to others. Ethics has been defined as:

a matter of principled sensitivity to the rights of others. Being ethical limits the choices we can make in the pursuit of truth. Ethics say that while truth is good, respect for human dignity is better, even if, in the extreme case, the respect of human nature leaves one ignorant of human nature.

(Caven, 1977 as cited in Cohen *et al.*, 2002, p. 56)

Merriam (1998) argues that ethical dilemmas in qualitative studies are likely to emerge with regard to the collection of data and in the reporting of the findings. It is in relation to these two areas of the research process that ethical issues were considered in setting up and conducting this study.

In the section which described who the teachers in this study were (refer to Section 4.3) I indicated that there were four teachers from two schools (two per school) who participated in this study. The aim of a learning study is to find ways of getting learners to discern a particular object of learning.

So in addition to teachers being participants in the study, there were also learners who participated in the study. Access to both the teachers and learners had to be through the schools. Since this study forms part of the larger WMC-S project and both School M and School R are project schools, permission to conduct this study in these two schools was obtained through the ethics protocol received for the WMC-S project. This included access to all the project schools participating in the WMC-S project and the teachers in the respective mathematics departments but excluded access to the learners. To this end, I obtained what Cohen *et al.* refer to as ‘informed consent’ which they explain as “the procedures in which individuals choose whether to participate in an investigation after being informed of facts that would be likely to influence their decisions” (Cohen *et al.*, 2002, p. 51). This informed consent was obtained in writing through letters addressed to the learners as well as the parents. In the letters to the learners and their parents I addressed, amongst other things, the following issues:

- A brief outline of what my study was about and for what purposes I was conducting the study.
- A guarantee of autonomy and an explanation that anonymity would be maintained, especially in the reporting of the findings.
- The fact that participation in the study was purely voluntary and that the learners could withdraw their consent at any time without any penalty or prejudice.
- The need for the video recording of lessons as a data collection strategy and so the request for their permission to use such a strategy since they may appear on the video.
- A request for permission to make copies of their classwork, homework or assessments that they might produce as part of the lessons that constitute the learning study.

As will be shown in the chapter dealing with learner performance, there was a percentage of learners who dropped out of the study²⁵ as the learning study progressed from pre-test to post-test to the delayed post-testing. The decline in learner numbers did not adversely affect the study as the teachers were at the centre of the study. In this regard, I was fortunate that none of the teachers opted out of the study.

Lesson observations formed the core data collection strategy employed in this study. By teachers agreeing to participate in the study, they were essentially agreeing to being observed as they went

²⁵ Since this learning study was conducted after school hours and learners were not compelled to participate, the decline in the number of learners who wrote the delayed post-test was beyond my control. Reasons for learners not writing the delayed post-test are unknown to me and attempting to understand their reasons is beyond the scope of this study.

about teaching a lesson as this is a key feature of learning study. The teachers were made aware of what the study entailed and for what purpose the study was being conducted. This was done in a meeting with the teachers before they decided to participate in the study. As already indicated I adopted a complete participant role as researcher and as I worked with the teachers we established a relationship of collegiality. This contributed to each teacher being comfortable with the remaining members of the learning study group sitting in and observing their lesson. Although teachers were comfortable with this arrangement the concern for me was not knowing what to do in an instance where I witnessed utterly ineffective or perhaps potentially damaging teacher behaviour. Merriam (1998) indicates that knowing when and how to intervene is perhaps the most perplexing dilemma facing qualitative researchers and that failure not to act is in itself an ethical and political choice. Fortunately, I was not placed in any situation where a teacher acted in a manner that would bring harm to the learners in either a physical or a content manner.

In conducting this study I ensured that the implementation of the learning study at both School M and School R caused the least amount of disruption to the normal function of the school. In the design of the study all learning study lessons were scheduled to be taught after school hours.

With regard to the issue of anonymity I assigned letters to refer to the two schools (School M, School R) as well as to the teachers that participated in the study (Teacher J, Teacher S, Teacher L, Teacher T). In this thesis or any other publication that may come out from this study or any presentation made about this study or aspects of it, the schools as well as the teachers will be referred to by these letters. Of course, the tension with respect to anonymity in in-depth studies is well documented (Adler & Lerman, 2003). They explain that the richer the description is, the more recognisable the participants become particularly to themselves and close colleagues.

Finally before commencing with this study I obtained ethics clearance from the University with protocol number 2011ECE005C (see Appendix A).

5.6 Revisiting the research questions

The problem statement for this study is:

To what extent and how does a learning study professional development model, underpinned by variation theory and the judicious use of examples, enhance mathematics teachers' mediation of a selected object of learning – the case of functions in Grade 10?

The questions that focus this study are:

1. What was constituted as the intended object of learning for the learning study focused on the function concept in grade 10 and how was it identified?
2. What came to be constituted as the enacted object of learning in each of the lessons in the learning study cycle and what emerged as the critical feature?
 - 2.1 How were the examples introduced across the lesson?
 - 2.2 What were the observable actions on the examples that were introduced?
 - 2.3 To what domains of authority does the teacher appeal in his/her attempt to establish meaning for the learners?
 - 2.4 What emerged as the critical feature and what was the role of variation in this?
3. How does participation in the learning study focused on the function concept in grade 10 impact on participants learning?
 - 3.1 How do learners perform on pre-, post- and delayed tests focused on aspects of the function concept in grade 10?
 - 3.2 Does teachers' thinking about the teaching of functions in grade 10 shift in any way?

To summarise, I present the following model (Figure 5.1) which maps out how the external language of description interfaces with the internal language of description for use in analysing and describing the *enacted object of learning*.

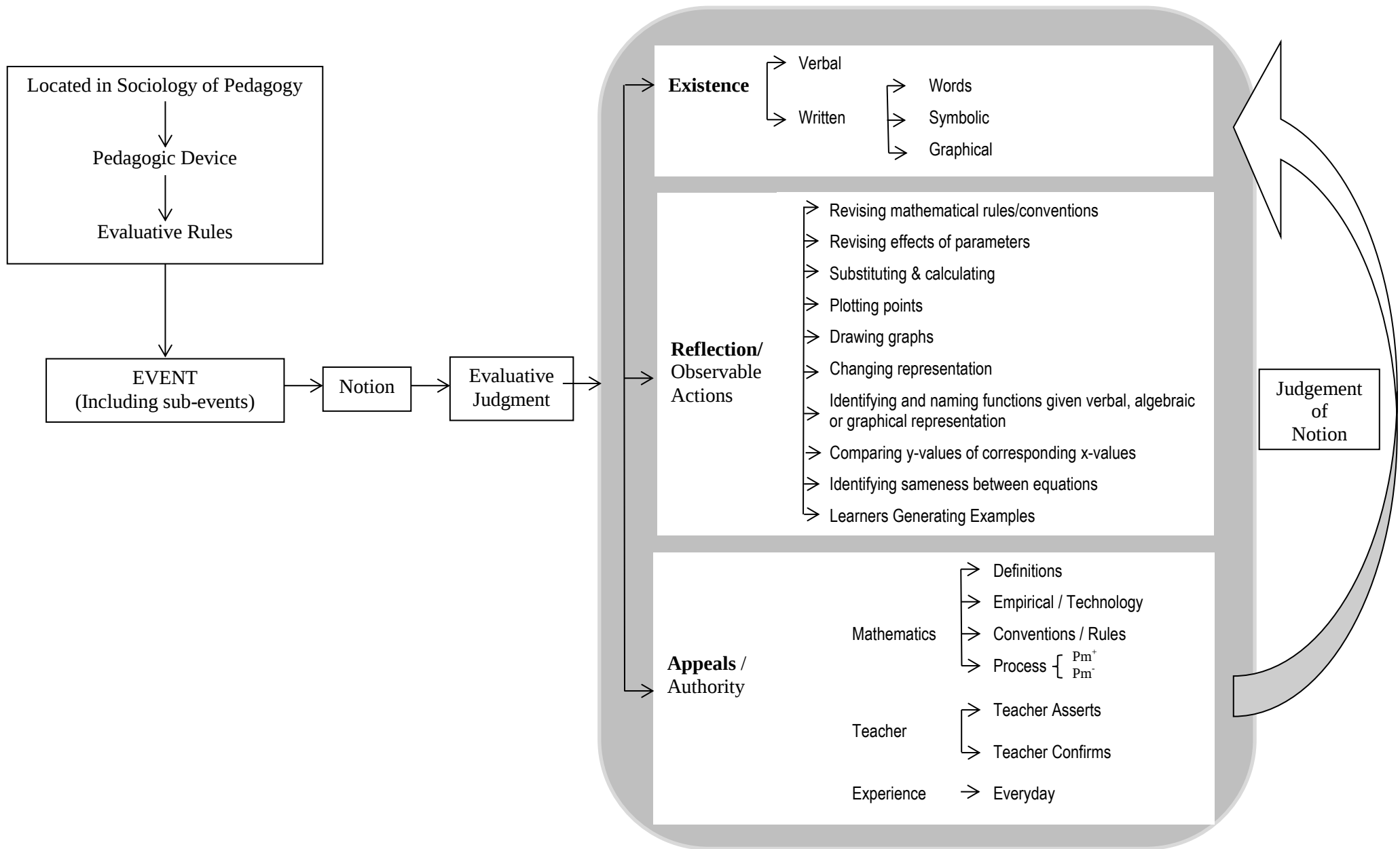


Figure 5.1: A model mapping how the external language of description interfaces with the internal language of description

Chapter 6

The Intended Object of Learning

6.1 Introduction

The aim of this chapter is to provide an answer to the first focus question that underpins this study i.e. what was constituted as the intended object of learning for the learning study focused on the function concept in grade 10 and how was it identified?

To identify the object of learning, the learning study group used the learners' performance in the mid-year examinations. The mid-year examination was set by the Gauteng Department of Education's Johannesburg Cluster. All the learners who participated in this study wrote the same mathematics test paper in June, which made the comparison of learners' performance across the two schools possible. In view of this, the learning study group decided to use the learners' performance in this test as the pre-test for the learning study that we were going to embark on. Before I engage in discussion about learner performance in the pre-test, I present a brief overview of the test paper.

6.2 Overview of the pre-test

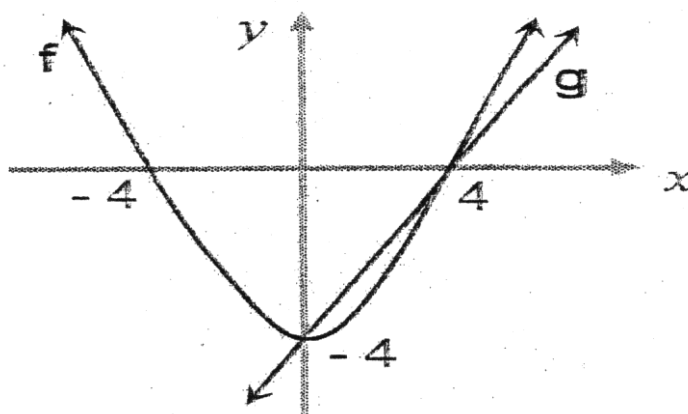
The test paper covered various aspects of mathematics that the teachers were expected to have covered during the first half of the year. This was as per the work programme supplied by the education authorities governing the district. The aspects of mathematics covered in the test paper included algebra, financial mathematics, functions, trigonometry and transformational geometry. The duration of the test was two hours; the paper was made up of nine questions and was worth 100 marks. Since the focus of this study is on functions, I now focus my attention on questions related to functions as tested in the June examination.

There was only one question in the test paper that focused on functions viz. question 7 (see Figure 6.1). This question focused on the linear, quadratic, exponential and hyperbolic functions and comprised two independent sub-questions. In the first part of the question (question 7.1) the graphs of the quadratic and linear functions are drawn on the same set of axes and the graphs intersect at the points $(0; -4)$ and $(4; 0)$. The questions that follow required learners to extract information from the graphical representation and determine its algebraic equivalence. There was one question which was related to finding the range and another related to finding x-values for which the y-values satisfied a given condition in relation to the two functions. In the second sub-question (question 7.2), the learners were required to draw the exponential function ($y = 2^x$) and then find

the value of the abscissa given the ordinate of a point which lies on the graph. Thereafter, the learners were required to find the equation of a hyperbola that passes through this coordinate. Aspects of transformations and asymptotes are also tested. Question 7 carried 22 marks out of the total 100 marks. Figure 6.1 is an extract taken from the test paper.

Question 7

7.1 In the sketch below, you are given graphs of f and g



- 7.1.1 Determine the equation of g . (4)
- 7.1.2 If the equation of f is defined by $f(x) = ax^2 + c$, calculate the values of a and c . (4)
- 7.1.3 Write down the values of x for which $f(x) \leq g(x)$ (2)
- 7.1.4 What is the range of f ? (2)

7.2 In your answer script draw a neat graph of the equation $f(x) = 2^x$
(NB: Show any intercepts) (3)

- 7.2.1 If $(m; 8)$ lies on the graph of f , calculate the value of m . (3)
- 7.2.2 If the hyperbola defined by $g(x) = \frac{k}{x}$ passes through point B, calculate the value of k . (2)
- 7.2.3 Write down the equation of the asymptote of $h(x) = f(x) + 1$ (1)
- 7.2.4 Write down the equation of q the reflection of f along the y -axis. (1)

[22]

Figure 6.1: Question on functions – Grade 10 mid-year examination of 2011

The above description of question 7 is to illustrate the interwoven manner in which the examiners tested various aspects related to functions. This fashion of questioning is not new to the learners, as they have encountered these types of examples during their interaction with their teachers in class.

Furthermore, they were also exposed to these types of questions in the form of homework, assignments and class tests, as these examples are typical textbook type questions that appear at the end of the chapter on functions for consolidation and revision purposes. At the time the learners wrote the examination, their teachers were to have completed teaching the section of functions as per the work schedule set out by the Department of Education. Although this was the case, the learners across both schools performed very poorly in these mid-year examinations. Since question 7 was used as the pre-test for this learning study, the teachers in this learning study group re-marked this question for the learners who agreed to participate in the study. For the purposes of analysing learner performance in the pre-test (question 7), the questions were marked using the following codes:

- Right (R): learner gets the answer correct.
- Partial (P): learner gets at least 50% of the answer correct.
- Wrong (W): learner gets the answer completely incorrect.
- Missing (M): learner did not attempt the question.

6.3 Learners' performance in the pre-test

The bar charts which follow (Figures 6.2 to 6.5) display the learners' performance in the pre-test. The horizontal axis represents the question numbers and the vertical axis represents the learners' mark as a percentage. The number of learners who wrote the test is indicated on the label of the vertical axis. The learner performance in the pre-test is displayed per class and the order in which the bar charts are presented are in the order in which the classes were taught in this learning study.

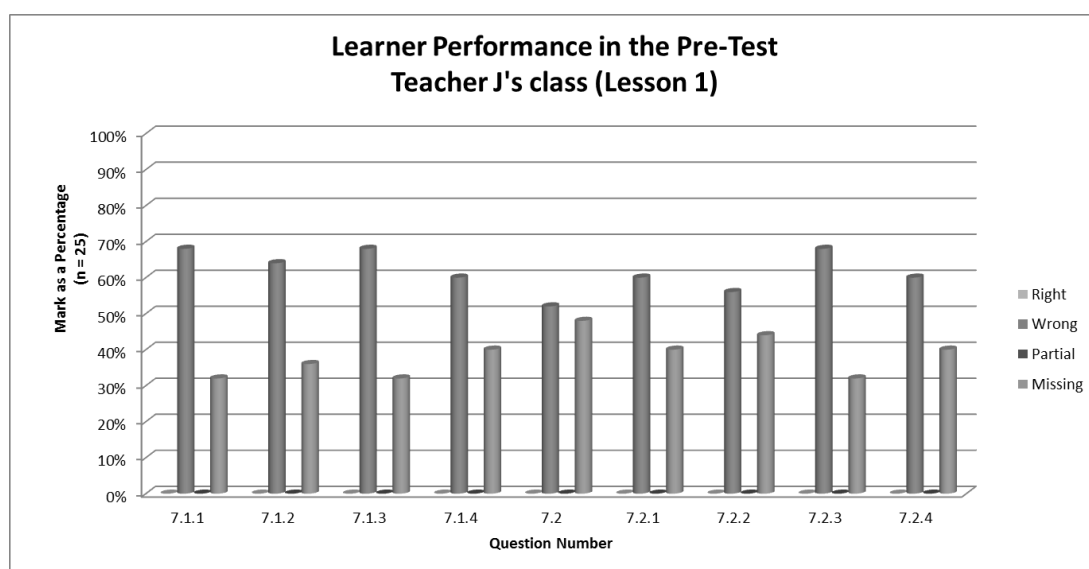


Figure 6.2: Learner performance in pre-test – Teacher J's class

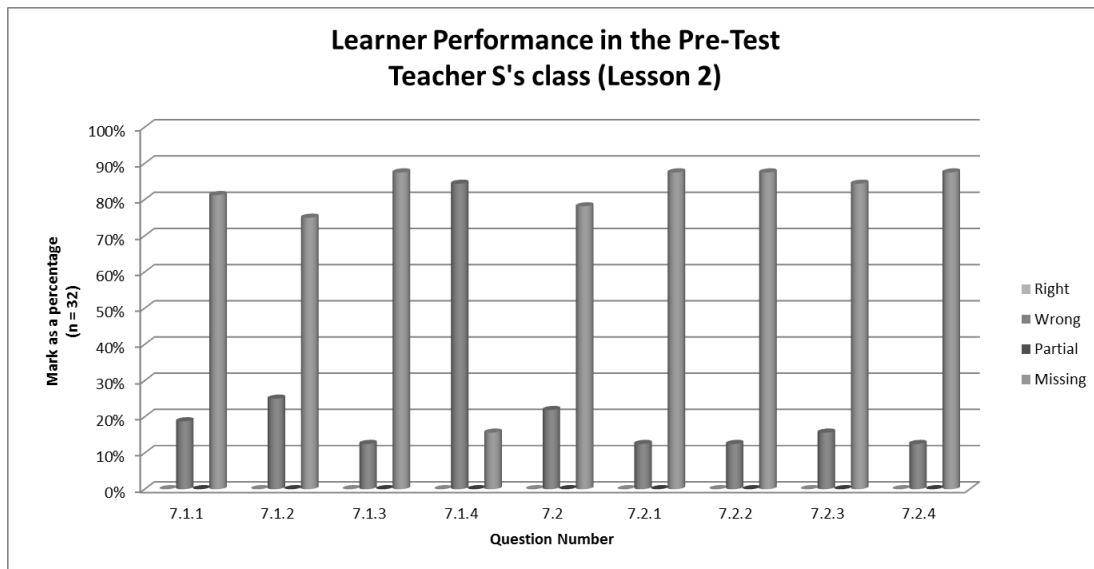


Figure 6.3: Learner performance in pre-test – Teacher S's class

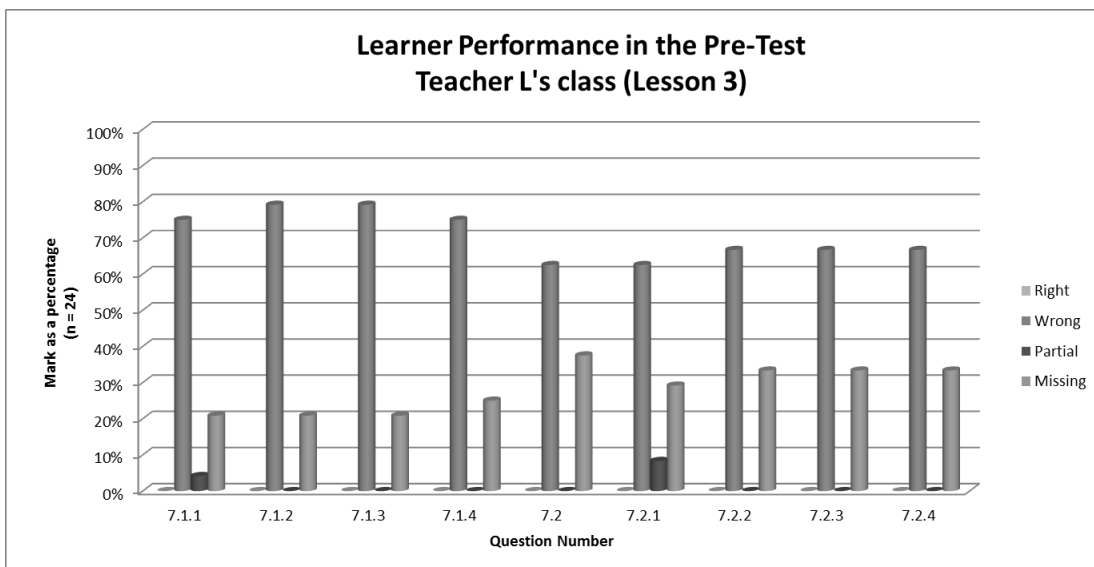


Figure 6.4: Learner performance in pre-test – Teacher L's class

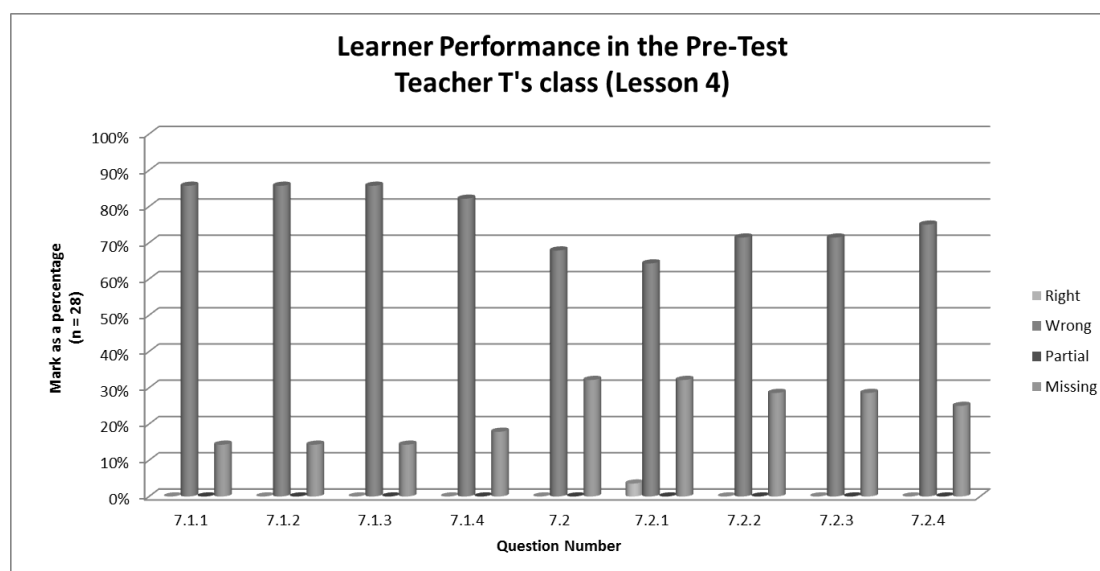


Figure 6.5: Learner performance in pre-test - Teacher T's class

Looking at the bar charts above, the number of incorrect responses and unanswered questions across the four classes is glaring and distressing. These bar charts are also characterised by the absence of bars representing correct and partially correct answers. This simplistic representation of the learners' performance on the pre-test across the four classes begins to illustrate their poor performance in the mid-year examinations for mathematics specifically on questions related to the function concept.

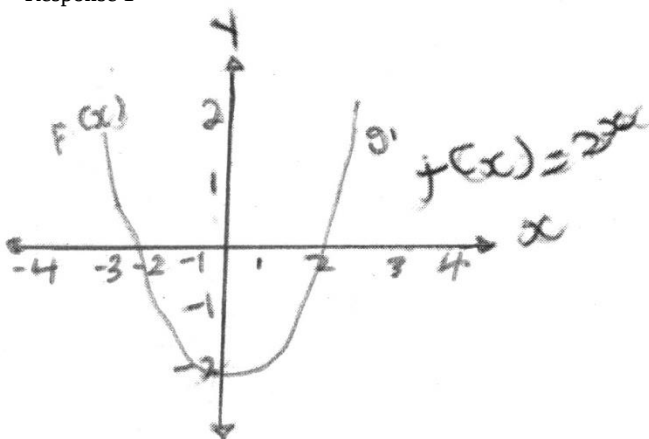
Looking at the learners' performance in relation to the questions that constituted the pre-test, the teachers in the learning study group indicated that there are so many concepts that need attention, some of which are assumed for the study of functions in grade 10 e.g. laws of exponents, substitution, algebraic manipulation of expressions, performing routine arithmetic calculations, understanding of inequality signs and knowing how to use them. Entering the realm of functions the list of concepts that require attention continues to grow e.g. understanding functional notation and what it means, being able to identify a class of functions from an equation, being able to identify a function from the graphical representation, plotting of points on a Cartesian plane and reading coordinates, domain and range. These are just some of the concepts that were identified as needing attention.

6.4 Identifying the object of learning

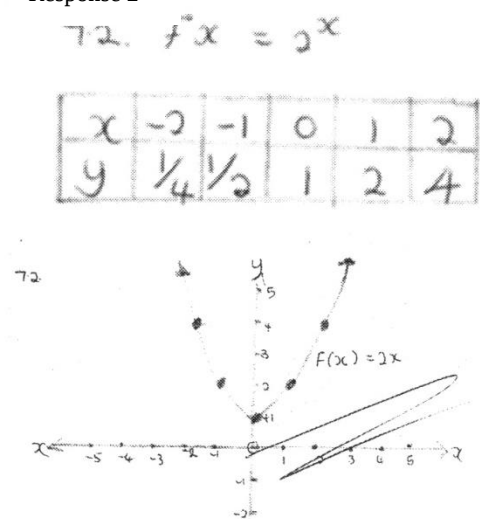
Although the teachers identified a wide range of concepts their learners had difficulty with, what stood out for the teachers was what seemed to be the learners' disconnected view between the different representations of a function. This translates to, amongst other things, learners not being

able to recognise the general shape of a graph given the equation, or to correctly identify the structure of an equation for a given graph. Figure 6.6, which shows learners' solutions to question 7.2 from the pre-test, demonstrates the (mis)recognition of the features of a function through different representations. In this question the learners were required to draw the graph of the equation $f(x) = 2^x$. The teachers felt that this should have been a fairly simple task for the learners because they could take x-values and substitute into the equation $f(x) = 2^x$ to find y-values, and then plot these points. The teachers indicated that this is something that the learners ought to be familiar with since this is the process that they introduced learners to for the drawing of graphs.

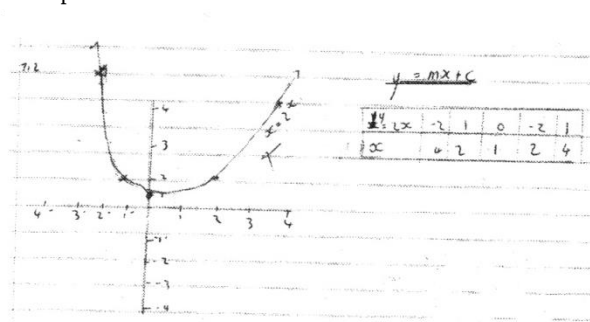
Response 1



Response 2



Response 3



Response 4

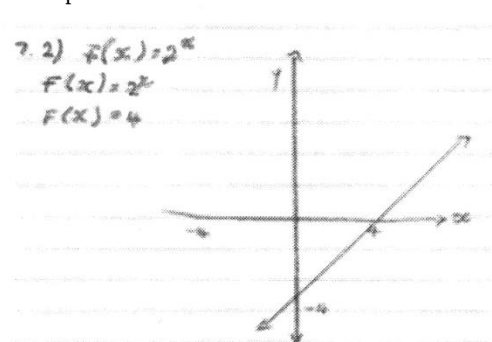


Figure 6.6: (Mis)recognition between multiple representations of a function

In Figure 6.6, responses 1, 2 and 3 suggest that the learners associated the function defined by $f(x) = 2^x$ with the quadratic function. They possibly (mis)recognised the structure and characteristics between 2^x and x^2 . In response 1, the learner did not show the derivation of the coordinates but drew a parabola with x-intercepts $(-2; 0)$ and $(2; 0)$ and y-intercept $(0; -2)$. Interestingly, in response 2 learners were able to perform the correct arithmetic calculations after substituting various values of x into the given equation and represented these values correctly in a table, but plotted some of the points (where x is negative) incorrectly to form the shape of a

parabola. In responses 1 and 2, it seems that the learners saw x^2 and 2^x as representing the same function. In response 4, the learners associated 2^x with a linear function which has a positive gradient and it is unclear as to how they arrived at the x-intercept (4; 0) and y-intercept (0; -4). In response 3, the learners signalled that they are interpreting $f(x) = 2^x$ as being linear by writing the general equation of a linear function $y = mx + c$. Furthermore, in completing the table of values the learners changed 2^x to $2x$, which is then written in the cell to indicate x-values in the table of values. The rule used to relate input values with their respective output values is not clear and does not satisfy the equation defined by $y = 2x$. However, after representing the coordinates as a table of values the learners proceeded to draw a quadratic function. One can again assume that for these learners as well, $f(x) = 2^x$ represents the graph of a parabola.

The range of learner responses above represents the kind of responses teachers found when they re-marked the scripts of the group of learners who agreed to participate in the study. The learners' responses illustrate that there are no consistent or appropriate connections between their conception of different classes of functions (a linear, quadratic and exponential function) in terms of their algebraic and graphical representations.

Indeed there are indications that there are deeper difficulties for learners. Relatively simple algebraic forms appear to have little meaning for these learners. This raises some questions that are perhaps prior to multiple representations of functions: do learners see 2^x ; x^2 ; and $2x$ as one and the same thing? What about $\frac{x}{2}$ or $\frac{2}{x}$? Do they understand the relationship between the variable and the constant in each instance? Do the learners recognise the mathematical operations implied in each of these instances and are they able to evaluate each of the expressions for a particular value of x ?

The learners' responses to the questions related to functions in the mid-year examinations that functioned as the pre-test suggest at this point, a lack of coherence and meaning in terms of their understanding of aspects related to different classes of functions. This includes amongst other things working with functional notation, understanding the difference in meaning between expressions such as 2^x ; x^2 ; $2x$; $\frac{1}{2}x$; $\frac{x}{2}$; $\frac{2}{x}$ the concept of domain and range, asymptotes and intercepts. In some instances starting with the number 2 poses a challenge e.g. $2 + 2 = 2 \times 2 = 2^2$ and so therefore it becomes unproductive to use the number 2. Within the context of functions at grade 10 level, starting with the number 2 is precisely the right place to start. Using a number such as 3 would pose additional challenges because the cubic function is not dealt with at grade 10.

In light of the analysis of the test responses, the learning study group decided that the *intended object of learning* should focus on enhancing learners' ability distinguish between the different classes of functions specifically in terms of the different ways in which the function can be represented (verbally, algebraically and graphically). Related to this is the discernment of algebraic notation and the different meanings that the changing position of the '2' and the 'x' would bring about. This is what the learners would need to do simultaneously if they are to get the relevant tools to deal with the kind of errors and (mis)conceptions they have about the different families of functions as evident in their performance in the pre-test. Therefore, the learning study group articulated the intended object of learning for the learning study cycle as:

- To enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential functions across their different representations viz. verbal, equation (symbolic), table of values or sets, and graphical.
- To relate the varying equation forms to the different relationship between input and output for each function (the simultaneity).
- To read, interpret and use functional notation and evaluate a function at a value in its domain.

In relation to the object of learning another important aspect that needed to be considered is what would it mean for English second language learners when we as teachers talk about 'two to the x' or 'x over two': do they hear what we are intending. Thus, during the lesson the teachers were required to pay deliberate attention to the use of language. Language was to be used in a more precise manner – 'two to the power of x' or 'x divided by two'. In addition, the teachers indicated that in terms of the learners' performance it seems that there is so much that is lacking in terms of learners' knowledge and so what would it mean for them if during the lesson there are other problems that need attention before they can get to the object of learning? All members in the learning study group agreed that these challenges will need to be addressed: they cannot be ignored as they are important to achieving the object of learning. So, it would be interesting to see what the teachers do in terms of the kinds of examples they choose to use as they proceed, since this will be unplanned.

The extent (depth and breadth) of the errors in learners' test performance might provoke the question why deal with functions at all when their responses suggest that they are experiencing more elementary difficulties? One of the principles which informs the work of the WMC-S project, and so also informs this study, is to work with teachers in their current contexts. The context in which the participating teachers in this study find themselves is one where learner errors are so widespread that it makes it difficult to decide on a starting point to begin to address the errors.

Despite the widespread nature of learner error, the focus for the teachers remained that of functions, a key concept their grade 10 learners would again be assessed on at year end.

6.5 Planning the first lesson of the learning study cycle

As already discussed, the learners' responses to question 7.2 in the pre-test (draw a graph of $f(x) = 2^x$) were pivotal in assisting the teachers with the identification of the object of learning. The teachers felt that being able to draw a graph was one of the basic skills that learners ought to have had since the drawing of graphs is an exercise that learners were introduced to whilst they were in grade 9 and when they were introduced to the topic of functions. Furthermore, it is an activity that has been practiced each time the learners were required to draw a graph. Engaging with the learners' responses to question 7.2, the teachers identified two inter-related areas of difficulty displayed by the learners: i) they had difficulty in distinguishing between the different classes of functions, and ii) they have a disconnected view of a function across its different representations. Using this as the point of departure for the planning of the first lesson the learning study group conjectured that if they planned a lesson which focused on the multiple representations of a function they would be providing learners with opportunities to distinguish between the different classes of functions.

Having agreed on a focus for the lesson the next step was to plan the examples to be used during the lesson. The learners' responses to the question, draw the graph of $f(x) = 2^x$, seemed to suggest that these learners were comfortable in treating 2^x ; $2x$ and x^2 as representing the same function. Informed by the central idea inherent in variation theory, viz. to discern a specific feature of an object, we need to vary that particular feature whilst keeping other features of the object invariant, the learning study group planned the examples for use in the lesson. The group identified the following examples for use in the lesson: $y = 2x$; $f(x) = \frac{1}{2}x$; $g(x) = \frac{x}{2}$; $y = \frac{2}{x}$; $p(x) = x^2$; $g(x) = 2^x$.

Using these examples a worksheet was designed where each of the planned examples were given in its verbal form. Provision was also made for learners to express the verbal statement in i) an algebraic form, ii) a table of values with given x-values, iii) a mapping between sets, and iv) graphically. Figure 6.7 is an extract taken from the worksheet.

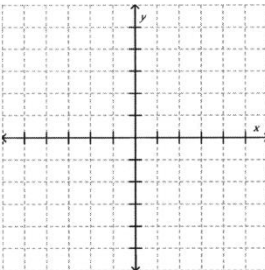
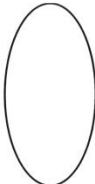
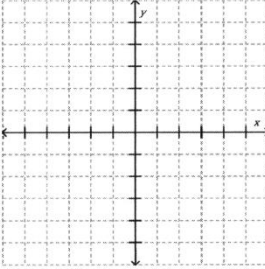
Verbal	Equation	Table of values	Mapping of sets	Graph																
Two divided by the input value gives the output value		<table><tr><td>x</td><td>-4</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>4</td></tr><tr><td>y</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr></table>	x	-4	-2	-1	0	1	2	4	y								Input (x)  Output (y) 	
x	-4	-2	-1	0	1	2	4													
y																				
Raise the input value to the power of two to get the output value OR Square the input value to get the output value		<table><tr><td>x</td><td>p(x)</td></tr><tr><td>-4</td><td></td></tr><tr><td>-3</td><td></td></tr><tr><td>-2</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>3</td><td></td></tr><tr><td>4</td><td></td></tr></table>	x	p(x)	-4		-3		-2		0		1		3		4		Input (x)  Output p(x) 	
x	p(x)																			
-4																				
-3																				
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4																				

Figure 6.7: Extract from worksheet used during the learning study lessons

The worksheet was to be used as a resource so that the teacher could direct the learners' attention to specific aspects of the worksheet at various instances during the lesson. The intention was not for the completion of the worksheet in a sequential manner in which it was constructed. The idea behind the lesson was for the teacher to introduce a function algebraically by writing it on the board. The learners were required to read the equation and to explain the relationship between the variable 'x' and the constant '2'. The intention here was to provide learners with an opportunity to engage with the algebraic notation and so the potential for them to discern how the changing position of the '2' and the 'x' mean different things. Since the group started the planning of the lesson with the difficulty learners had in drawing a graph in the pre-test, the intention of the planned lesson was to then move onto the process of drawing a graph with the next activity being to complete a table of values and thereafter to plot these points. The group also felt that once a table of values is obtained it would not be very time-consuming to re-represent the ordered pairs as a mapping between two sets. In doing so it would provide the teacher with some opportunity to briefly engage with the concept of domain and range before moving on to draw the graph.

In planning the first lesson there were thus two areas of focus:

i. *The deliberate use of examples*

Informed by the basic principles inherent in variation theory viz. what stays the same and what changes (the idea of sameness and difference), examples were planned for use in the lesson. As already indicated, the following examples were planned $y = 2x$; $f(x) = \frac{1}{2}x$;

$g(x) = \frac{x}{2}$; $y = \frac{2}{x}$; $p(x) = x^2$; $g(x) = 2^x$. The dimension of variation (DoV) contained in this example space is the index of the independent variable. This DoV was present in the example space as a result of the different classes of functions that the teachers in the learning study group wanted to engage with. This range of functions that was being considered by the teachers was a result of the different classes of functions the learners were already exposed to already. Although this DoV was present in the example space it was not in focus for the members of the learning study group. In other words, the examples were not selected because they open the index of the independent variable as a DoV, they were selected for reasons already mentioned.

ii. *Multiple representation of functions*

The different ways in which a function can be represented was a planned DoV that was focused on in planning the lesson. It is in view of this DoV that a worksheet was developed for use as a resource during the lesson.

When one stands back and looks at range of aspects to be covered in a single lesson, one might argue that there is just too much to cover and so show an over-ambitious intention of what the teachers in the group consider as doable within a lesson. This issue was raised during the planning of the lesson and the teachers felt that it was possible to achieve working across the different classes of functions and across multiple representations in a lesson, since this is revision for the learners and so they are not teaching anything new. Although the lesson was planned in collaboration with all the members in the learning study group, it was essential for Teacher J, who was to teach the first lesson, to internalise the planned lesson and make sense of it before teaching the lesson. This was not only applicable to Teacher J but to all the teachers in the learning study group when it was their turn to teach the lesson. The full lesson plan and worksheet is given as Appendix B.

6.6 The Post-Test

In the design of this learning study the administration of a post-test was planned for the end of each lesson. The post-test comprised of two questions with sub-questions and learners were given thirty minutes to complete the test. The post-test was set before the first lesson in the learning study cycle was taught since it was to be conducted immediately after the lesson. Secondly, because the intention was to compare learner performance across the four classes in which the learning study lessons were taught, the learning study group decided not to change the questions in the post-test.

In designing the post-test, the teachers tried to include a range of questions that mirrored to some degree the range of questions that appeared in the pre-test (refer to Section 6.2). The inclusion of questions in this test was restricted to aspects covered in the lesson plan for lesson 1. As can be seen from the test paper (see Figure 6.8 on the next page), there was a focus on the following aspects in relation to functions at grade 10:

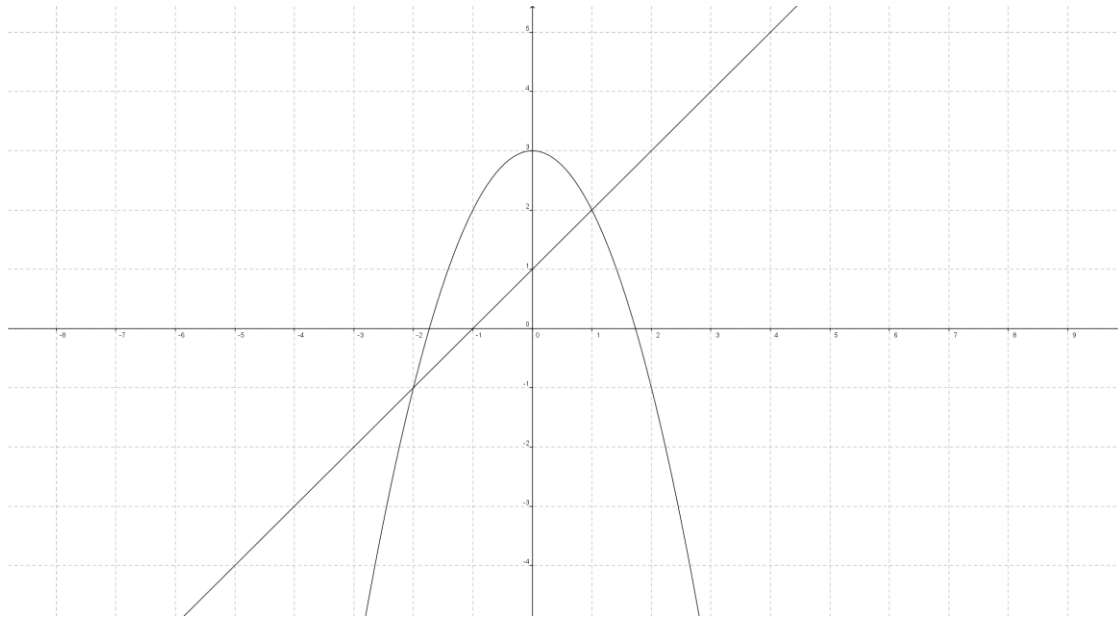
- i. Identifying and naming a function given its graphical and/or algebraic representation – questions 1.1 and 2.1.
- ii. Drawing a graph given its algebraic representation.
- iii. Working with functional notation – reading off co-ordinates from the graph or substituting and calculating – questions 1.2, 1.3 and 2.2.
- iv. Determining domain and range – questions 1.5 and 1.6.
- v. Working with features of graphs viz. points of intersection of two graphs and the concept of an asymptote – questions 1.4 and 2.3.

Post-Test

Name: _____ Grade: 10 _____ Date: _____

School: _____

1.



The equations of the two graphs sketched above are: $f(x) = x + 1$ and $g(x) = -x^2 + 3$

1.1 Write down the equation of the parabola.

1.2 Find $f(-2)$ and $g(-2)$.

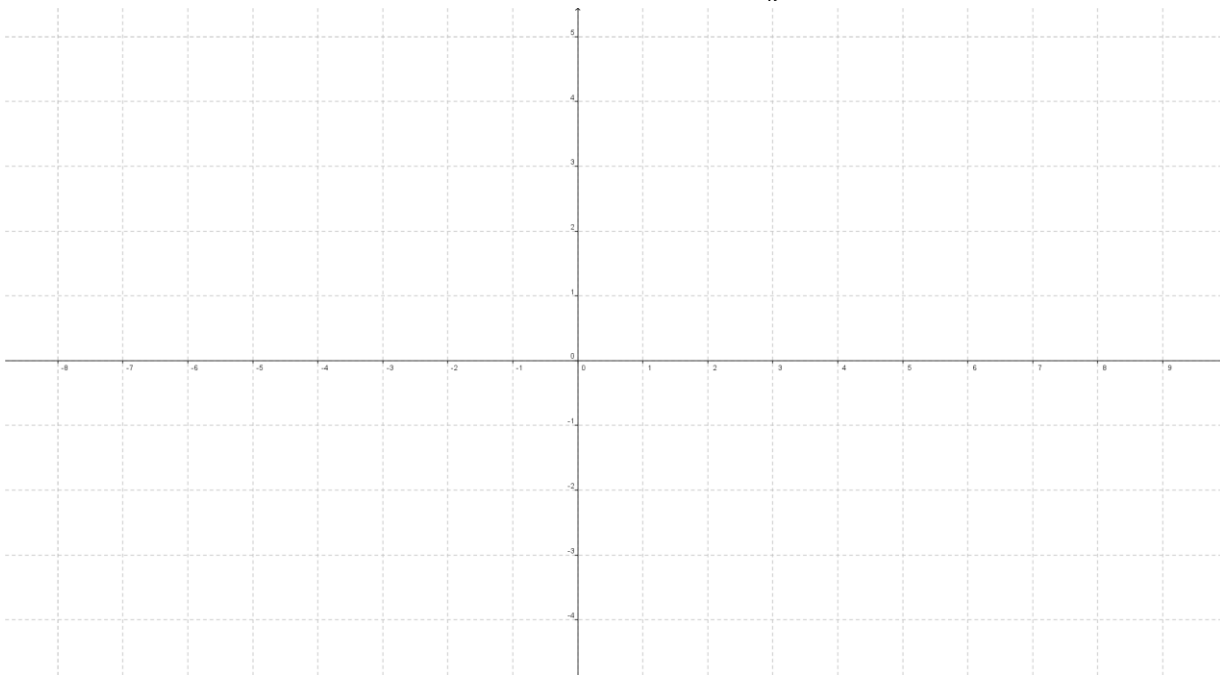
1.3 Find $f(1)$ and $g(1)$.

1.4 From your answers in questions 1.2 and 1.3 what can you conclude about the two graphs $f(x)$ and $g(x)$?

1.5 What is the domain of $f(x)$?

1.6 What is the range of $g(x)$?

2. On the Cartesian plane provided, draw the graph $p(x) = \frac{-4}{x}$



2.1 What is this graph called?

2.2 Find $p(0)$.

2.3 Using your answer to question 2.2, what can you conclude?

Figure 6.8: Post-test items administered after each lesson.

In planning the first lesson the focus was on multiple representations of a function, and two of the representations that were included on the worksheet that was used in the lesson, were to express the input and output values in a table and then to represent these values as a mapping between two sets. These two representations provided opportunities for the teacher to engage with the elementary ideas inherent in the concept of domain and range. It is with this in mind that questions 1.5 and 1.6 were asked.

As already indicated, the post-test was set before the first lesson was taught and the same test was administered at the end of each lesson. As can be seen in the post-lesson discussions in Chapter 8, (Sections 8.3, 8.5, 8.7 and 8.9) the post-test did not function significantly in the development of the next iteration of the lessons and in the identification of the critical feature. In light of this, I provide only an outline of the post-test in this chapter but will discuss the post-test in relation to learner performance in Chapter 9 (Section 9.2) when I look at the lived object of learning, and also discuss the delayed post-test. In the next chapter, I provide a description of each lesson. The description focuses on the observable actions that the teacher and/or learners engaged with across each of the lessons. This is done in relation to the sequencing of events as the lesson was enacted and so illuminates the temporal unfolding of the lesson.

Chapter 7

An overview of the learning study and a description of the lessons

7.1 Introduction

The purpose of this chapter is to provide a description of what transpired in each of the lessons that constituted the learning study. I commence by providing an overview of how the learning study unfolded and then I provide a description of each lesson. To structure the description of how each lesson unfolded I focus mainly on the observable actions that the teacher and/or learners engaged with. It is through the observable actions that one is able to see what happens in a lesson.

Implicit in what happens in a lesson are the examples used. Of particular interest in this study is how they are introduced and also the criteria a teacher transmits of what counts as valid knowledge. In this chapter when I describe the lessons, I do not engage with the authorisation of meaning. A more detailed description of the representations used to introduce examples and the legitimisation of what counts as valid knowledge in the mathematics class is provided in Chapter 8 and will contribute to describing what comes to be constituted as the enacted object of learning in each of the lessons.

7.2 Overview of the learning study

As discussed in Chapter 4, four teachers agreed to participate in this study and this had a direct bearing on the number of iterations this learning study would undergo i.e. in terms of teaching and revising lessons. The teachers decided that they were not going to teach their own learners. This meant that Teacher J and Teacher L taught the learners from School R whilst Teacher S and Teacher T taught the learners from School M. Teacher J volunteered to teach the first lesson, and because they wanted to alternate between the two schools Teacher S decided to teach the second lesson. This followed with Teacher L and Teacher T teaching lessons 3 and 4 respectively. All the learning study lessons were scheduled to be taught after school hours as the intention was to conduct the lessons without causing any disruption to day-to-day functioning of the schools.

In commencing with learning study the first step is for teachers to identify the object of learning. As discussed in Chapter 6, in this learning study the teachers did not set a pre-test as such to identify the object of learning, but used the learners' performance in the mid-year examination, which was a common assessment set by the local education authority. This decision was based on the fact that

the teachers were disappointed with their learners' performance as they were expecting the learners to have performed better. The teachers zoomed in on their learners' performance in questions related to the function concept and were surprised as to how poorly all the learners performed. Of particular concern for the teachers was the learners' inability to sketch the graph of $f(x) = 2^x$. The main concern for the teachers was whether the learners are able to distinguish between the expressions: 2^x ; x^2 and $2x$. What about $\frac{1}{2}x$; $\frac{x}{2}$ and $\frac{2}{x}$? Do they recognise the mathematical operations embedded in each expression? This was a serious concern for the teachers because in terms of syllabus coverage, they had completed teaching the section on functions which included the different classes of functions as represented by each of the algebraic expressions listed and yet the learners performed so poorly.

In preparing for the first lesson the teachers decided to use $y = 2^x$; $y = x^2$; $y = 2x$; $y = \frac{1}{2}x$; $y = \frac{x}{2}$ and $y = \frac{2}{x}$ as the planned examples. Secondly, as already discussed in Chapter 6, they decided to focus on the multiple representations of functions and to this end they developed a worksheet for use during the lesson (an extract and a discussion of the worksheet was provided in Chapter 6). In setting up this learning study provision was made for conducting a post-test at the end of each lesson. In this chapter I do not engage in discussion of this post-test. The significance of the learners' performance in the post-test in informing the discussion for the next iteration of the lesson will be reflected on as I describe the enacted object of learning in Chapter 8 and again in Chapter 9 when I describe the lived object of learning. As indicated in Chapter 6, the volume of aspects in relation to the function concept that has been included in the planning of the first lesson seems an overambitious task to achieve within an hour long lesson. But the teachers argued that the lessons in this learning study cycle would be revision lessons and so the concepts being introduced are not new to the learners.

Having briefly described how this learning study commenced, I now move on to describe what transpired in each of the lessons. To accomplish this task I describe what took place in each of the events by focusing on the observable action that characterises each of the events in relation to the example(s) introduced. In describing the lesson in this fashion I obscure the fact that some of the events are made up of sub-events, together with a backgrounding of how knowledge is legitimated. For a detailed account of the representation used to introduce each example per event as well as its related sub-events, the observable action engaged with in relation to the example, and the appeals to knowledge domains, refer to Appendix C.

7.3 Lesson 1

This lesson was chunked into 15 events and Teacher J starts this lesson by telling learners that the lesson is on functions. Having announced this focus, Teacher J goes onto an unplanned activity (multiplication with 11) which is unrelated to the topic on functions. Consider the following extract:

Teacher J Today we want to look at a topic called functions. We want to look at functions. I know you have done this one before, but what we want to try and find out is what is it that you still remember on these functions. Are we together?

Learners Yes, sir.

Teacher J Ok. It's very, very important that before I – before I proceed – can I have a ruler (*Teacher J asks one of the teachers from school R who is an observer for a ruler – chalkboard ruler*) ... Right, so we want to quickly go over. Now (*inaudible*) to see that there is something that happens in mathematics, and once you know that one there you don't even need a calculator because it's part of you, it's very interesting and you know it. Ok, before we start, what we need to start looking at today, because we are looking at functions... (*writes 'Functions' on chalkboard*) we are looking at functions. Who can tell me what is eleven times eleven? You multiply eleven times eleven, what is the answer?

[A few minutes later]

Teacher J ... So it's very interesting. So if there's (*inaudible*) you are (*inaudible*) very fast, right? So that is very interesting. Now we need you students to try and formulate clubs, whether you are School M you are School R you are what-what form clubs, and then you do mathematics, and you'll see that, ah, there (*Teacher T from school R brings in a chalkboard ruler*) are certain things that we think they are difficult but they are not difficult at all. Are we together?

Learners Yes.

Teacher J So we want to try and look at (*inaudible*) functions, we want to start with...let me... (*draws two long horizontal lines and writes 'Equation' between them. Writes $y=2x$ underneath*).

(Lesson 1, event 1)

This activity consumed the first 2 minutes and 31 seconds of the lesson and there are two possible reasons for this shift from the planned lesson. Firstly, in the extract above we can see that Teacher J asked for a chalkboard ruler and another teacher from the learning study group (Teacher T) left the classroom to get a chalkboard ruler. It is at this moment that Teacher J made a decision to deviate from the planned lesson. A few minutes later Teacher T enters the classroom with the chalkboard ruler in hand and at that moment, Teacher J rounded off the activity of multiplying with 11 and started to refocus learners' attention on the topic of functions. So, a possible reason for the deviation

from the planned lesson could be that Teacher J used this activity to keep learners occupied whilst he waited for the ruler. A second possible reason, which is inferred, is that the multiplication by 11 activity is a starter activity for the lesson with the aim of motivating learners. The motivating features of the activity is visible by the meta-mathematical narrative provided by Teacher J, for example ‘it’s very interesting’; ‘it’s very fast’ (these comments in relation to the multiplication strategy); ‘there are certain things that we think are difficult but they are not’; ‘you need to form maths clubs’. The first reason provided seems to be a more plausible reason for the deviation from the planned lesson. I did not discuss the reason for the deviation with Teacher J since at the time it seemed unimportant, but this is not to say that its level of importance has increased in any way since.

Figure 7.1 is a snapshot of the work done on the chalkboard as the lesson progressed to act as a visual tool to aid in the narration that follows.

Equation	Substitution	Type of function
$y = 2x$	$x = -4$ $y = 2(-4) = -8$	Linear $y = x$
$y = 2x$	$x = 3$ $y = 2(3) = 6$	$y = 5x + 1$
$f(x) = \frac{1}{2}x$	$f(-4) = -2$ $f(3) = \frac{3}{2}$	Linear
$g(x) = \frac{x}{2}$		
$h(x) = \frac{2}{x}$		
$k(x) = x^2$		
$m(x) = 2^x$		

Figure 7.1: Identifying and naming functions given its algebraic representation & through substitution – events 2 and 3

In event 2, which lasted 5:56, Teacher J introduced the liner function $y=2x$ in symbolic form and engaged in substituting x -values ($x = -4$ and $x = 3$) and calculating the corresponding y -values. Once this was done, Teacher J wanted the learners to identify i.e. name the class of function that they were working with and then asked them to generate other examples of a linear function.

In event 3, Teacher J introduced the function $f(x)=\frac{1}{2}x$ also in its symbolic form and once again engaged in the process of substitution (again $x = -4$ and $x = 3$). Here too, as visible in Figure 7.1, the working details for the substitution process are recorded on the chalkboard but no table of values is yet completed. Again, Teacher J then asked learners to identify the type of function; this event consumed 7 minutes and 55 seconds of the lesson.

For the next 8 minutes and 8 seconds (event 4) Teacher J asked the learners to find y-values when $x = -4$ and $x = 3$ for the functions $g(x) = \frac{x}{2}$; $h(x) = \frac{2}{x}$; $k(x) = x^2$ and $m(x) = 2^x$. The learners were required to complete this task on their worksheets and whilst they engaged in this task Teacher J walked around the class and worked with individual learners. Teacher J did not prescribe the order in which the examples were to be completed and so learners did not necessarily start with the same example, nor did they progress to the next example in the same sequence. The end of this event was marked by Teacher J's decision to complete a table of values for each of the functions on the chalkboard as a class collective.

Figure 7.2 is an image of the chalkboard which was taken towards the end of the lesson. Once again I present this figure to act as visual tool to aid with my description of this lesson.

func	x	-4	-3	-2	-1	0	1	2	3	4	Type of graph
a	$y = 2x$	-8	-6	-4	-2	0	2	4	6	8	Linear
b	$f(x) = \frac{1}{2}x$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	Linear
c	$g(x) = \frac{x}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	Linear
d	$h(x) = \frac{2}{x}$	$-\frac{1}{2}$	$-\frac{2}{3}$	-1	-2	ud	2	1	$\frac{2}{3}$	$\frac{1}{2}$	Hyperbola
e	$k(x) = x^2$	16	9	4	1	0	1	4	9	16	quadratic
f	$m(x) = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	Exponential

Figure 7.2: Identifying and naming functions given its algebraic representation & table of values – events 5, 6, 7, 9, 10 & 11

Event 5, which lasted 1 minute and 45 seconds, saw Teacher J returning to the linear function $y = 2x$ and again he got learners to find y-values for given x-values, but now he wanted the learners to represent the ordered pairs in the table of values. Once this was done he again wanted learners to identify the type of function.

Event 6, which lasted 4 minutes and 52 seconds, focused on the function $f(x) = \frac{1}{2}x$ which was introduced symbolically and here too, Teacher J got learners find y-values for a given set of x-values. The ordered pairs were recorded in a table of values and using the table of values Teacher J demonstrated the plotting of the points and the sketching of the graph. With this information at hand the learners were then required to identify the type of function.

The next event (event 7) was based on the same linear function but expressed differently $g(x) = \frac{x}{2}$. In this event, which lasted only 32 seconds, Teacher J wanted learners to find y-values for given x-values and thereafter they were required to identify the type of function.

Event 8 followed with Teacher J wanting learners to compare the two functions $f(x)=\frac{1}{2}x$ and $g(x)=\frac{x}{2}$. The comparison of the two equations was not at the level of what is the same and different with respect to the symbols used to structure the equations. The comparison was only at the level of the ordered pairs as represented in the table of values.

Event 9, which lasted 4 minutes and 8 seconds, saw the introduction of the two functions $h(x) = \frac{2}{x}$ and $g(x) = \frac{x}{2}$. These examples were introduced symbolically and once again the teacher got the learners to complete the table of values for $h(x)$ as the y-values for $g(x)$ were obtained in the previous event. Using the table of values, learners were required to establish if $h(x)$ and $g(x)$ represented the same function. Then based on the table of values the learners were required to identify the class of function being represented by $h(x)$.

In event 10, the quadratic function defined by $k(x)=x^2$ was introduced symbolically and once again the learners were required to complete the table of values. Based on the table of values the learners were then required to identify the class of function being represented by $k(x)$. This event consumed 6 minutes and 10 seconds of the teaching time.

In event 11, the exponential function $m(x) = 2^x$ was introduced in its algebraic form and here too the learners were required to complete the table of values and based on this information they were required to identify the class of function.

The next three events (12, 13 & 14) saw a return to the functions $y = 2x$ which was introduced in event 2; $f(x)=\frac{1}{2}x$ introduced in event 3 and $g(x) = \frac{x}{2}$ introduced in event 7. In events 12, 13 and 14 the functions were introduced verbally and then the learners were required to express them in algebraic form. These events together consumed 10 minutes and 25 seconds of the teaching time.

The last event of this lesson (event 15) consumed 5 minutes and 16 seconds and focused on the function $y = 2x$. In this event Teacher J engaged with representing the function as a mapping between two sets.

Table 7.1 displays the observable actions that were in play during each event (including the sub-events). The table also displays the main example(s) used during each event. I draw your attention to the fact that in some events other examples were introduced, such as in instances where learners

had to generate other examples of the function under discussion. There were also events (e.g. 12, 13 and 14) where the example was introduced using a different representation to the one displayed in the table.

	Events (including sub-events)														
Observable Actions	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Main example(s) used		$y=2x$	$f(x) = \frac{1}{2}x$		$y=2x$	$f(x) = \frac{1}{2}x$	$g(x) = \frac{x}{2}$	$f(x) = \frac{1}{2}x$ & $g(x) = \frac{x}{2}$	$h(x) = \frac{2}{1}x$ & $g(x) = \frac{x}{2}$	$k(x) = x^2$	$m(x) = 2^x$	$y=2x$	$f(x) = \frac{1}{2}x$	$g(x) = \frac{x}{2}$	$y=2x$
Revising mathematical rules/conventions			✓	Learners working on their worksheets					✓	✓	✓				
Revisiting effects of parameters															
Substituting and calculating		✓	✓		✓	✓	✓		✓	✓	✓				✓
Plotting points						✓									
Drawing graphs						✓									
Changing Representation					✓	✓	✓		✓	✓	✓	✓	✓	✓	✓
Identifying and naming functions given verbal, algebraic or graphical representation		✓	✓		✓	✓	✓		✓	✓	✓				
Comparing y-values for given x-values								✓	✓						
Identifying sameness between equations															
Learners generating examples		✓													

Table 7.1: Summary of Observable Actions across Events – Lesson 1

The immediate reaction of tallying the number of ticks per observable action shows that *changing representation* is the most frequently occurring observable activity during this lesson. This however provides a false first impression of what is given prominence in this lesson, to get a true reflection one needs to look at this activity in relation to the observable activity *substituting and calculating*. Included in the observable action *changing representation* is the process of representing x-values and their associated y-values in a table of values. The precondition for completion of this task is to substitute given x-values into an equation, perform the necessary arithmetic manipulations to find

the corresponding y-values and then represent it in a table of values. Now looking across events 5, 6, 7, 9, 10 and 11 Teacher J engaged learners in the process of substituting given x-values into an equation and finding its associated y-values and thereafter representing them in a table of values. In event 15, the learners are also engaged in the process of finding y-values for given x-values but in this case the values are represented as a mapping between two sets. What this means is that in events 2, 3, 5, 6, 7, 9, 10, 11 and 15 the changing of representation is a secondary result of the activity of finding y-values for given x-values and representing the ordered pair in a table of values or as a mapping between two sets. It is only in events 12, 13 and 14 where changing representation is the primary result of the activity that learners are engaged in. In these events the functions were introduced in a verbal form and the first thing that Teacher J gets the learners to do is to change its representation to its algebraic equivalence.

So, in overview, and as captured in table 7.1 while all the planned functions were dealt with in the sequence as recorded in table 7.1, the observable action was largely substitution of values of x to set up the table of values. From this learners were expected to identify and name the different functions. It is only in event 6 that Teacher J used the table of values to demonstrate the process of plotting points and then drawing the graph.

7.4 Lesson 2

Like all the lessons that constituted this learning study, lesson 2 was also planned to last for an hour after which a post-test was to be written by the learners. The necessary arrangements were made and lesson 2 was scheduled to be taught at School M on Monday, 05 September 2011 at 15:00. Only five learners turned up for the lesson on this day and so the lesson had to be rescheduled. The lesson was taught on Wednesday, 07 September 2011 at 08:00 and was chunked into 9 events with some events having sub-events. Teacher S started this lesson by placing pre-drawn graphs of a hyperbola, exponential, linear and quadratic functions on the board (see Figure 7.3). The function examples used in Lesson 1 are used in this lesson as well. The focus here on the graphs of the functions was a result of the teachers' reflection on Lesson 1.

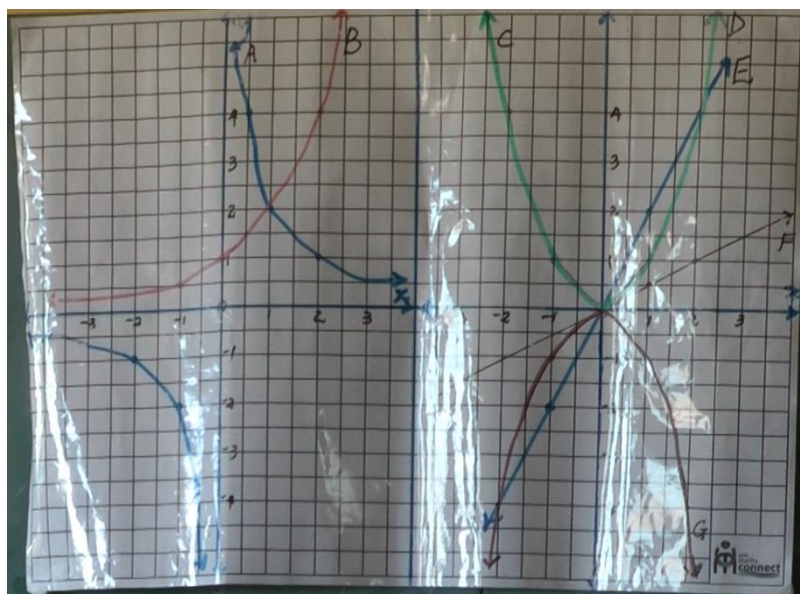


Figure 7.3: Multiple functions used to introduce lesson 2

The equations for these graphs are as follows:

A: $y = \frac{2}{x}$

B: $y = 2^x$

C: $y = x^2$

D: same graph as C

E: $y = 2x$

F: $y = \frac{1}{2}x$

G: $y = -x^2$

In the first event, which lasted 1 minute and 22 seconds, the learners were required to identify and name the function represented by the graph labelled B – refer to the extract below:

Teacher S Right, let's look at the graph that I have on the board. (*On a laminated grid the teacher had drawn two Cartesian planes. The Cartesian plane on the left contained the graph of the hyperbola $y = \frac{2}{x}$ labelled A and an exponential graph $y = 2^x$ labelled B*). Let's look at this side (*refers to the Cartesian plane on the left*). A is the pink or whatever colour that is. Who can identify that graph? What do we call that graph, equal to this one (*points to the exponential graph*)? I've labelled it graph B. What do you call that graph? Anyone?

Learner Parabola.

Teacher S Parabola. Somebody's got parabola here, anyone else? No one, ok. Who can identify graph A? (*beneath the laminated grid the teacher writes A= and B=*) (*using her finger the teacher traces out graph A*)

Learner Hyperbola?

Teacher S It's a hyperbola (*writes hyperbola after A=*). So graph A is a hyperbola. Can anyone remember what graph B is? We heard one say it's a parabola. Who else has a different identification, a different name for that one?

Since learners had difficulty in correctly identifying and naming the exponential function, Teacher S moves on to the graph labelled A.

The difficulty that the learners experienced in their attempt to identify the class of function marked as graph A resulted in Teacher S asking learners to identify and name any of the graphs that appeared on the board. Event 2 consumed 5 minutes and 20 seconds of the lesson and learners experienced difficulty in identifying the class of functions from the graphical representation.

Event 3 lasted for 4 minutes and 13 seconds and started with Teacher S encouraging the learners to provide the equation for any of the graphs that appeared on the board. To complete this task the learners were expected to provide a specific equation which was a more complex task that required some working out as opposed to merely identifying the class of function being represented. Teacher S then shifted to asking learners to provide the general form of the equation associated with any of the graphs drawn e.g. $y = mx + c$ represents a linear function.

Event 4 was a short event which lasted 29 seconds and in this event Teacher S revised the mathematical convention that x-values represent input values and y-values represent output values.

In event 5, which took 18 minutes and 36 seconds of the lesson, Teacher S directed the learners' attention to the worksheet and asked them to re-represent the statement 'input value multiplied by two gives the output value' in its algebraic form. Once the learners had done this, the teacher moved on to show the learners how to represent the ordered pairs associated with the function $y = 2x$ as a mapping between two sets. This then lead to the plotting of points on the Cartesian plane and the sketching of the graph. The event is concluded with learners being asked to identify the class of function being represented.

Event 6 consumed 5 minutes and 10 seconds of the teaching time and unfolded in a similar fashion to event 5. However in this event Teacher S introduced the function $f(x) = \frac{1}{2}x$ and once the table of values was completed, she moved on to sketching the graph.

In the next event, Teacher S introduced the quadratic function $y = x^2$ in its verbal form and commenced the event by asking learners to provide its algebraic form. On the board the graph of $y = -x^2$ was drawn and the teacher used this to engage with the effect of the parameter a on the

graph $y = ax^2$. Event 7 continued to include working with functional notation and in total this event consumed 9 minutes and 17 seconds of the lesson.

Event 8 lasted for 1 minute and 14 seconds and in this event Teacher S introduced the function $g(x) = 2^x$ in its verbal form and then asked learners to change the representation to algebraic.

The last event in this lesson consumed 2 minutes and 37 seconds of the teaching time and in this event the hyperbola $y = \frac{2}{x}$ was introduced in its verbal form and once again the learners were required to provide the algebraic representation. Teacher S then went on to focus on determining the output value of $y = \frac{2}{x}$ when $x = 0$.

Table 7.2 provides an overview of the observable actions across the 9 events (including the sub-events) that make up lesson 2. Once again the main example(s) used during each event is also displayed bearing in mind that in some events there were other examples introduced, such as in instances where learners had to generate other examples of the function under discussion. Furthermore, there are instances where the main example was introduced using another representation other than being introduced algebraically as depicted in the table. In addition, under events 2 and 4 the main example used is captured as ‘multiple functions’ since in these events there were 4 to 5 different examples engaged with and space in the table does not allow for the recording of all these functions.

Observable Actions	Events (including sub-events)								
	1	2	3	4	5	6	7	8	9
Main example(s) used	$y = 2x$	Multiple functions	$y = 2^x$	Multiple functions	$y = 2x$	$f(x) = \frac{1}{2}x$	$p(x) = x^2$	$g(x) = 2^x$	$y = \frac{2}{x}$
Revising mathematical rules/conventions			✓	✓		✓	✓		✓
Revisiting effects of parameters							✓		
Substituting and calculating					✓	✓	✓		✓
Plotting points					✓	✓	✓		
Drawing graphs					✓				

Changing representation			✓		✓	✓	✓	✓	✓
Identifying and naming functions given verbal, algebraic or graphical representation	✓	✓			✓		✓	✓	
Comparing y-values for given x-values									
Identifying sameness between equations									
Learners Generating Examples									

Table 7.2: Summary of Observable Actions across Events – Lesson 2

As already described the starting events (1 and 2) illustrate that this lesson starts with the graphical representation of functions in focus. The observable action that characterised these events was to identify and name the functions presented.

Scanning Table 7.2 quickly one is able to see that there is a clustering of ticks around events 5, 6, and 7. Taking a closer look at these events and focusing on the columns for each of these events, one is able to trace the observable actions in play per event which shows that the observable action of *changing representation* is common across all three events. In each of these events the functions $y = 2x$, $f(x) = \frac{1}{2}x$ and $p(x) = x^2$ appeared as words on the worksheet and so were brought into existence in the lesson verbally e.g. input value multiplied by two gives the output value. Since these functions came into existence verbally the first step was for Teacher S to get learners to change the representation and rewrite these functions in symbolic form before she could proceed. Following the structure of the worksheet, the next activity Teacher S engaged her learners with was for them to find y-values for given x-values and to express these values in a table of values – *substituting and calculating*. The next aspect for completion in the worksheet was to represent the ordered pairs as a mapping between two sets – *changing representation*.

In overview, the teacher introduced the lesson by attempting to have learners identify graphs of functions and provide equations or general forms for all those which were pre-drawn. The learners experienced difficulty with this task and in further enactment of the planned lesson the teacher followed the worksheet and so introduced functions in their written form. For learners to engage with these functions they were required to change its representation to algebraic and their engagement with the functions is at the level of substituting values of x to obtain the associated y-values.

7.5 Lesson 3

Teacher L taught this lesson and it was chunked into four main events. Event 1 consumed 6 minutes and 45 seconds of the lesson and saw Teacher L introducing the linear function $y = 2x$ in its symbolic form, she then asked the learners to express it verbally. Once the learners articulated the symbols $y = 2x$ as words, the teacher moved on to focus their attention on the mathematical operation implied in '2x'. The event was then extended to the learners being required to find output values for specific input values.

Event 2 lasted for 12 minutes and 54 seconds and in this event Teacher L focused on the linear function $y = \frac{1}{2}x$. Once again she revised that $\frac{1}{2}x$ meant $\frac{1}{2} \times x$ and so focused the learners' attention on the mathematical operation embedded in the expression $\frac{1}{2}x$. Thereafter, she engaged in the process of finding output values for specific input values and concluded the event by establishing the equivalence between the equations $y = \frac{1}{2} \times x$ and $y = x \div 2$. To do this she got the learners to compare input values and their corresponding output values as obtained by substituting into each of the equations.

Event 3, which took 10 minutes and 54 seconds of the teaching time, saw a return to the function $y = 2x$, but in this event it was introduced verbally and learners were required to provide its algebraic equivalence. In the process of changing the representation to its algebraic form, the equations $y = 2x$ and $y = x + x$ emerged. Teacher L then engaged with the process of substituting specific x-values into each of the equations to demonstrate that both equations would return the same corresponding y-value. The ordered pairs obtained were recorded in a table of values and thereafter Teacher L plotted the points and drew the graph. At this point in time $y = 2x$ has been represented verbally, algebraically, as a table of values and graphically. Having these different representations available, the learners were asked to identify the type of function represented by $y = 2x$. Once the learners identified it as linear, Teacher L asked them to generate other examples of a linear function.

Event 4 consumed 11 minutes and 54 seconds of the lesson and in this event Teacher L introduced the hyperbola $y = \frac{2}{x}$ in its verbal form and once again learners were required to provide its algebraic representation. As in the other events here too Teacher L focused on the mathematical operation between '2' and 'x' and then engaged in the substitution process to complete a table of values. The

points were plotted and the graph was drawn. Thereafter, the learners were asked to identify the class of function represented.

Table 7.3 provides an overview of the observable actions across the 4 events (including the sub-events) that make up this lesson. Here again the main example(s) that were used during each event is displayed. In some events other examples were introduced, for example in instances where learners had to generate other examples of the function under discussion. There are also instances where the main example was introduced using a representation other than algebraic.

Lesson 3	Events (including sub-events)			
Observable Actions	1	2	3	4
Main example(s) used	$y = 2x$	$y = \frac{1}{2}x$ $y = \frac{x}{2}$	$y = 2x$ $y = x+x$	$y = \frac{2}{x}$
Revising mathematical rules/conventions	✓	✓	✓	✓
Revisiting effects of parameters				
Substituting and calculating	✓	✓	✓	✓
Plotting points			✓	✓
Drawing graphs			✓	✓
Changing representation	✓		✓	✓
Identifying and naming functions given verbal, algebraic or graphical representation			✓	✓
Comparing y-values for given x-values		✓		
Identifying sameness between equations				
Learners generating examples			✓	

Table 7.3: Summary of Observable Actions across Events – Lesson 3

Looking horizontally across the rows it becomes immediately noticeable that the observable actions *revising mathematical rules or conventions* and *substituting and calculating* are actions that are engaged with across all events. The conventions that Teacher L revises with her learners centres around aspects such as:

- The mathematical operation between the ‘2’ and ‘x’ in $y = 2x$ is multiplication, so $y = 2x$ is the same as $y = 2 \times x$
- $y = 2x$ is the same as $y = x + x$, the meaning of doubling
- In an equation like $y = 2x$, the variable x represents input values whilst the variable y represents output values
- The notation for representing input and output values is a mapping between two sets.

In summary, the observable action *substituting and calculating* is prevalent across all the events and the amount of time that Teacher L spends on this specific activity is in excess of a third of the lesson. It is through the process of substituting and calculating that Teacher L focuses on the observable action of revising mathematical rules and conventions. These two observable actions were dominant in this lesson.

7.6 Lesson 4

This was the last lesson which was taught in this learning study cycle. The lesson commenced with Teacher T introducing two examples by writing their algebraic representation on the chalkboard viz. $y = 2x$ and $y = \frac{1}{2}x$. He then asked learners to read these equations: the verbalisation of these equations, for obvious reasons, would sound different. Nonetheless, based on the sound of the verbal utterances he asked learners if the verbal utterances of the two statements sounded the same. Teacher T then moved on to asking learners to identify the sameness between the two equations. In this process he focused the learners’ attention on the values of the exponent of x . Event 1, which lasted 5 minutes and 28 seconds, was concluded with the learners merely identifying the sameness between the two equations.

Event 2, which consumed the next 4 minutes and 20 seconds of the lesson, saw Teacher T introducing the linear function defined by $f(x) = x + 1$ in both its algebraic and graphical form. He then revised the concept of functional notation, and the event was concluded with Teacher T asking learners to generate other equations that represent a linear function.

Event 3 was marked by the introduction of two more linear functions: $y = \frac{1}{2}x$ and $f(x) = \frac{x}{2}$. Once again Teacher T got the learners to read the two equations and, based on the sound of the verbal utterances, he wanted learners to determine if the utterances sounded the same. He then proceeded to demonstrate the equivalence between the two equations by getting learners to find the output values for particular input values with respect to both the equations. This event took 5 minutes and

21 seconds of the lesson and was concluded with Teacher T asking the learners to compare the two equations and then to identify the class of function they each represented.

In event 4, Teacher T introduced the functions $h(x) = \frac{2}{x}$ and $f(x) = \frac{x}{2}$ through their algebraic representation and once again requested learners to read the statements and then to compare the sound of the read statements. Once this was done he engaged the learners with the process of identifying the sameness between the two equations and in this process he revised the rules of exponents related to expressing $h(x)$ with a denominator of 1 ($h(x) = 2x^{-1}$). This event, which lasted 6 minutes and 51 seconds, was concluded with the identification of the class of function defined by $h(x)$.

Event 5, which took only 2 minutes and 29 seconds of the lesson, saw the introduction of the function $p(x) = \frac{-2}{x}$ in both its graphical and algebraic form. The event progressed and ended with the rewriting of $\frac{-2}{x}$ as $-2x^{-1}$

$p(x) = x^2$ and $g(x) = 2^x$ are introduced in event 6, which consumed 3 minutes and 15 seconds of the teaching time. The functions were introduced symbolically and the event commenced once again with the learners having to read the equations and then compare the sound of the verbal utterances. The event progressed with Teacher T then getting learners to compare the exponents of x in each equation. This then led to the identification of the type of function being represented by $p(x)$.

In event 7, Teacher T introduced the quadratic function $g(x) = x^2 - 2$ in both its graphical and symbolic forms. In this event, which lasted for 1 minute and 53 seconds, the learners were only required to identify the type of function that $g(x)$ represented.

The focus of event 8 was for learners to generate other examples of equations that represented a quadratic function. This event consumed only 1 minute and 20 seconds of the lesson.

Through the process of learners generating examples of a quadratic function, a function defined by $q(x) = 2^2 + x$ was generated. The focus in event 9 was to identify the class of function being represented by the equation $q(x) = 2^2 + x$. This event was the shortest event in this lesson and only consumed 25 seconds of the lesson.

Once the function $q(x) = 2^2 + x$ was identified as linear, the learner who generated the equation defined by $q(x)$ as an example of a quadratic function wanted confirmation of her new realisation and so asked the question “Must x always be having an exponent of two?” (Lesson 4, event 10.1). To deal with this question, Teacher T introduced the equation $q(r) = r^2 + 2^2$ and asked the learners if the equation represented a quadratic function and so reinforced that if the polynomial is of the second degree then a quadratic function is being signalled. Teacher T then asked the learner to provide another example of an equation that represents a quadratic function. This event (event 10) consumed 1 minute and 59 seconds of the lesson.

Event 11, which lasted 4 minutes and 15 seconds, saw Teacher T introducing the function $g(x) = 2^x$ in its algebraic form and the event commenced by identifying the type of function. Once the function had been identified the teacher introduced the graph of $g(x) = 2^x$ and then he asked learners to generate more examples of an exponential function.

In event 12, Teacher T introduced a range of functions: $y = 2x$; $h(x) = \frac{2}{x}$; $y = \frac{1}{2}x$; $p(x) = x^2$ and $p(x) = 4^x$. In this event he got his learners to identify the type of function being represented in each example. The event consumed 2 minutes and 35 seconds of the teaching time.

Event 13 was marked by the return of the linear function $y = 2x$, but in this instance the function was introduced as words. First the learners had to re-represent the function in its algebraic form and then they were required to complete a table of values. Using the table of values, Teacher T then engaged with the concept of domain and range, representing the x and y values as a mapping between two sets and finally plotting the points and drawing the graph. The duration of this event was 17 minutes and 41 seconds.

Event 14 saw the introduction of the rational function defined by $f(x) = \frac{2}{x}$ in the form of words. The learners were then required to re-represent the function algebraically and thereafter Teacher T engaged with the process of substitution so as to complete a table of values. The event was concluded with a discussion on finding $f(0)$. This event took 8 minutes and 13 seconds of the lesson.

The last event was characterised by the introduction of the function $g(x) = x^2 - 2$ in its algebraic form. Teacher T used this example to engage in a discussion focusing on functional notation. This event consumed the last 1 minute and 23 seconds of the lesson.

Table 7.4 provides an overview of the observable actions across of the 15 events (including the sub-events) that make up this lesson. Here again the main example(s) that were used during each event is displayed. In some events other examples were introduced, for example in instances where learners had to generate other examples of the function under discussion. There were also instances where the main example was introduced using a representation other than algebraic. In event 12 the main example used is captured as ‘multiple functions’ since in this event there was 5 different examples engaged with and space in the table does not allow for the recording of all these functions.

Lesson 4	Events (including sub-events)														
Observable Actions	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Main example(s) used	$y=2x$ $y=\frac{1}{2}x$	$f(x)=x+1$	$y=\frac{1}{2}x$ $f(x)=\frac{x}{2}$	$h(x)=\frac{1}{2}x$ $f(x)=\frac{x}{2}$	$p(x)=\frac{-2}{x}$	$p(x)=x^2$ $g(x)=2^x$	$g(x)=x^2-2$	LGE's	$q(x)=2^{2+x}$	$q(t)=t^2+2^2$	$g(x)=2^x$	Multiple functions	$y=2x$	$f(x)=\frac{2}{x}$	$g(x)=x^2-2$
Revising mathematical rules/conventions	✓	✓		✓	✓	✓			✓	✓			✓	✓	✓
Revisiting effects of parameters															
Substituting and calculating			✓										✓	✓	✓
Plotting points														✓	
Sketching graphs														✓	
Changing representation	✓		✓	✓		✓							✓	✓	
Identifying and naming functions given verbal, algebraic or graphical representation		✓		✓	✓	✓	✓		✓		✓	✓	✓		
Comparing y-values for given x-values			✓												
Identifying sameness between equations	✓		✓	✓		✓									
Learners generating examples		✓				✓		✓		✓	✓				

Table 7.4: Summary of Observable Actions across Events – Lesson 4

In overview the observable actions that dominate this lesson were the revision of mathematical rules and conventions and identifying and naming functions. Teacher T focuses on the mathematical rules

and conventions because he wanted learners to identify the sameness and differences in the algebraic representation of given pairs of functions. Some examples of the mathematical rules and conventions that Teacher T focused on during the lesson were:

- the mathematical operation implied in the expression $2x$ is multiplication;
- revising the laws of exponents that would allow us to express $\frac{-2}{x}$ with a denominator of 1;
- the power of x in $2x$ is 1.

To conclude, the observable actions in each of the lessons give us a sense of kinds of tasks the teacher got the learners to grapple with or with which the teacher him/herself engaged. In this chapter, I merely focused on the observable action that was dominant in each of the lessons. By focusing on the dominant observable action, we see at a glance the crux of each lesson and so get a sense of how the lessons of this learning study cycle unfolded. In summary, the focus of each lesson can be described as follows:

- Lesson 1 – Substituting input values into the equation of a function, calculating the output value, and recording these results in a table of values. Thereafter identifying and naming the function.
- Lesson 2 – Identifying and naming a function given its graphical representation.
- Lesson 3 – Substituting input values into the equation of a function and calculating the output value.
- Lesson 4 – Identifying and naming a function given its algebraic representation.

In the next chapter, I provide a description of what comes to be constituted as the enacted object of learning in each of the lessons by elaborating on the observable actions. In providing this description I also describe what emerges as the critical feature in this learning study.

Chapter 8

The Enacted Object of Learning

8.1 Introduction

I commence this chapter by noting the object of learning upon which the learning study was based viz. to enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential functions across their different representations i.e. verbal, equation (symbolic), table of values or sets, and graphical.

The articulation of this object of learning is what the teachers of this learning study group intended for learners to learn, thus becoming the *intended object of learning*. This intended object of learning was 'realised' in each of the learning study lessons, as described by Marton *et al.* (2004), in the 'form of a particular space of learning'. Within the context of this study I interpret the space of learning as being characterised by the opportunities the teacher makes available for learners to *reflect* on notions and their associated sub-notions so as to transform them from a level of immediacy to something more substantial. In addition, the space of learning is also characterised by the *appeals* the teacher makes in order to legitimate some meaning for his/her learners. This space of learning is referred to as the *enacted object of learning* which is described from the researcher's perspective (Marton *et al.*, 2004). As discussed in the theoretical framework, I will use the principles inherent in variation theory to gain insights into what aspects (features) of the content are brought into focus thus providing learners with an opportunity to discern this aspect (feature). In other words, in describing this space of learning my focus will be on "how the teacher structures the conditions of learning so that it is possible for the object of learning to come to the fore of the learners' awareness" (Marton *et al.*, 2004, p. 4) or, in Kullberg's words, the enacted object of learning "describes what features of the content it is possible to experience during a lesson" (Kullberg, 2010, p. 34).

In providing learners with opportunities to experience the object of learning there are two things that need clarification. Firstly, as Marton *et al.* (2004) indicate, the assumption made is that teachers are trying to work toward an object of learning. They elucidate further that the object of learning that a teacher works towards may "be more or less conscious for the teacher and it may be more or less elaborated" (Marton *et al.*, 2004, p. 4). They further elaborate that this object of learning which is what the teacher is aware of is the *intended object of learning* (seen from the teacher's perspective) and it may change dynamically during the course of a lesson. Secondly, providing learners with opportunities to experience the object of learning does not necessarily mean that they

would discern that which was intended. Marton *et al.* (ibid.) explain that learners could fail to discern that which was intended and so focus on some other aspect of the object of learning. They further explain that the best case scenario is if the *critical feature* of the object of learning is in focus for the teacher and if the space of learning which the teacher opens up brings the critical feature into the focal awareness of the learners, and then the learners will discern that which was intended. Marton *et al.* (2004, p. 24) state that “the critical feature is critical in distinguishing one way of thinking from another, and is relative to the group participating in the study, or to the population represented by the sample.” A critical feature cannot be derived from the mathematics alone; it is derived in relation to the learners’ experiences and the object of learning:

The critical features have, at least in part, to be found empirically – for instance, through interviews with learners and through the analysis of what is happening in the classroom – and they also have to be found for every object of learning specifically, because the critical features are critical features of specific objects of learning.

(Marton *et al.*, 2004, p. 24)

But irrespective of the scenario, that which the learners actually learn is what is referred to as the *lived object of learning*.

As discussed in the theoretical framework (Chapter 3, Section 3.4), one way of bringing an aspect of an object of learning into focus is through the patterns of variation and invariance. So in analysing a lesson the focus of the analysis will be on whether it was possible for learners to experience the object of learning through the pattern of variation and invariance that is constituted by the teacher and learners jointly (Runesson, 1999). Essentially this is the basic idea of the variation framework that underpins a learning study.

When I provided a description of the four lessons in Chapter 7, I did not focus on the appeals because the purpose there was merely to provide a narration of the lessons and illustrate the temporal unfolding of the lesson – what happened in each lesson over time. A description of the appeals becomes central to describing what comes to be constituted as the enacted object of learning. To provide this description, I start by listing the range of examples introduced during the lesson (both planned and unplanned). In addition to the range of examples introduced, I also take into account the accompanying observable action(s) engaged in by the teacher and his/her learners in relation to the examples introduced, as this is central to describing the opportunities for learning that were made available during the lesson. It is through these observable actions that the teacher provides learners with opportunities to transform the notion from a mere ‘that’ into something more

substantial. On one level the observable action tells us what aspects of mathematics are being engaged with during the lesson, but this is insufficient. It does not illuminate the nature of the opportunity that is provided. The nature of the opportunity that is provided becomes visible through the nature of the teacher's appeal i.e. in his/her attempt to legitimate some form of meaning for the learners. In other words, the observable action does not really tell us whether the teacher merely asserts some concept and expects learners to grasp the concept or whether the teacher links the concept introduced to previous mathematics that the learners would have experienced and so build learners' understanding, or by engaging in an aspect of mathematics whether the teacher provides learners with opportunities to empirically establish the validity of the idea that is being engaged with.

The opportunities for learning that are made available during a lesson are seen, in Hegelian terms, through the *judgment of existence* (the range of examples used to introduce a concept), the *judgement of reflection* (the accompanying observable action(s) carried out on the example) and the *judgment of necessity* (the appeals the teacher makes in an attempt to legitimate some kind of meaning). To illustrate further consider the following example: the teacher writes on the chalkboard $y = 2x$ (bringing the *concept* of a linear function into *existence* through its algebraic representation). The teacher then asks learners to find the y-values for given x-values – the *observable action* performed on the example is substituting x-values and calculating to find the corresponding y-values (part of the *reflection* process that the teacher thinks will assist in filling out the concept of a linear function for his learners). Thereafter, the teacher demonstrates the process of substituting an x-value and calculating to find the corresponding y-value (teacher *appeals* to mathematics as some kind of process that one needs to follow, and the process in this case is called 'substitution').

To enhance the discussion of reflection I also look at what the teacher varies and what the teacher keeps invariant in relation to the examples used. As argued previously (Section 3.4), it is through the process of varying a specific aspect of the object of learning whilst keeping other aspects invariant that the specific aspect is brought into focus. Runesson also explains that "studying a learning situation from the point of view of what varies and what is 'invariant' is an efficient way to describe the promoted space of learning" (Runesson, 2008, p. 157). To commence with a description of the enacted object of learning I re-organise the events per lesson according to the examples used. What this means is that by re-organising the events according to the example introduced the temporal unfolding of the lesson is lost. For instance, the teacher could commence the lesson by introducing the example of the linear function $y = 2x$ and thereafter examples of

other functions could be introduced and towards the end of the lesson the teacher could re-introduce the example $y = 2x$. In re-organising the events according to the example introduced, all events that deal with the same example will be recorded as a group, thus distorting the temporal unfolding of the lesson. When the temporal unfolding of the lesson is distorted in this fashion it creates a false impression that there was deliberate planning to introduce the examples in a particular sequence so that learners are to take into account several aspects embedded within the examples at the same time (simultaneity). In re-representing the events pivoted around the example introduced (notion), the table also displays: i) how the example was introduced; ii) the duration of the event; iii) a description of the sub-notion; iv) the observable action; and v) the domain of authority to which the teacher appeals in order to legitimate some form of meaning. To reiterate, as discussed in Chapter 5, (Section 5.3.2), in a mathematics lesson or any other lesson much occurs verbally. In this study the idea of verbal is used specifically to describe instances where examples are provided in a text format (e.g. double the input value gives the output value), as opposed to graphical or symbolic form. In talking about this example with the learners there will be a need at some stage to convert the words into an algebraic syntax ($y = 2x$) and in this process of talking about and referring to the example one is required to verbalise the example. It is specifically in such instances that I refer to the example as coming into existence *verbally*.

In the section that follows, I provide a description of what comes to be constituted as the enacted object of learning in each of the lessons in this learning study. At this juncture it is important to reiterate that the intended object of learning for this learning study was to enhance the learners' ability to differentiate between the different families of functions. To get a description of what comes to be constituted as the enacted object of learning one has to look at the observable actions identified. In doing this, one is reminded that for the purposes of this description only the primary observable action that was identified per event and its related sub-events would be considered. For example, in instances where y-values were found for the given x-values and then represented in a table of values the observable action 'changing representation' was marked. In such instances the observable action of changing representation could be considered as the secondary action whilst the primary observable action is finding y-values for given x-values. It is in relation to these aspects that I engage in discussion about the enacted object of learning across the lessons in this learning study. It is also important to note that in some instances across a lesson the observable action that the teacher and learners are engaged with is in relation to more than one example. For instance a teacher could introduce the algebraic representation of the functions defined by $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$ and then get learners to compare the sameness and differences between the two equations

with the aim of focusing learners' attention to the exponent of x . Another instance where more than one example is in focus is in events where the teacher asks learners to generate other examples of equations representing a particular class of function.

In addition, ranking the observable actions from those that consumed the most amount of time to those observable actions that consumed the least amount of time is not sufficient in describing the enacted object of learning. The nature of the appeals will also be considered when describing the enacted object of learning since it tells us about the kind of opportunity learners were provided with when engaged with the observable action. Consider the following two examples to illustrate the point: i) in identifying and naming a function the teacher merely asserts the class of function, and ii) in identifying and naming a function the teacher goes through the process of sketching the graph, thus empirically establishing the class of function with his learners. In the first scenario the teacher merely asserts by telling the learners what type of function is being represented. In the second scenario, the teacher attempts to establish some form of meaning for the learners by providing them with an opportunity to determine the type of function by sketching the graph and so identifying the function through its graphical representation. To get a handle on the range of appeals I tally the frequency of each appeal in relation to a specific observable action and express it as a percentage of the number of events that deal with that specific observable action. The purpose of expressing the range of appeals per observable action as a percentage is merely to illustrate the prominence of each appeal per observable action. It is important to note that in some events the teacher appeals to more than one domain of authority and so all the appeals will be taken into account when determining the frequency of its occurrence across this lesson.

8.2 The enacted object of learning – lesson 1

Table 8.1 displays the re-ordering of events as explained above.

Example	How was the example introduced?	Duration	Sub-notion	Observable action	Legitimizing meaning (where does authority lie?)
$y=2x$	Symbolically	2:57	Finding y-values for given x-values	Substituting and calculating	1. P_m^+ 2. Restates and writes
	Symbolically	0:49	Type of function	Identifying and naming	Teacher asserts
	Symbolically	1:39	Finding y-values for given x-values & completing a table of values	1. Substituting and calculating 2. Changing representation	Restates and writes
	Symbolically	0:06	Type of function	Identifying and naming	Restates and writes

	1. Words 2. Verbally	2:47	Write in algebraic form	Changing representation	1. Restates and writes 2. Mathematical conventions
	Symbolically	5:16	Expressing x and y values as a mapping between sets	1. Substituting and calculating 2. Changing representation	1. Mathematical conventions 2. Pm ⁺ 3. Restates and writes
$y = x$ $y = 3x + 1$	Verbally	2:10	Examples of equations of linear function	Learners generating examples	1. Pm ⁻ 2. Restates & writes
$f(x) = y$ $M(x) = y$	Symbolically	0:35	Convention for writing functional notation	Revising mathematical conventions	Mathematical convention
$f(x) = \frac{1}{2}x$	Verbally & Symbolically	3:56	Finding y-values for given x-values	1. Revising mathematical conventions 2. Substituting & calculating	1. Mathematical rules 2. Pm ⁺ 3. Restates and writes
	Symbolically	0:31	Improper fraction: $f(3) = \frac{3}{2}$	Revising mathematical rules	1. Mathematical rules 2. Teacher asserts 3. Restates and writes
	Symbolically	2:53	Type of function	Identifying and naming	Acknowledging correctness of learner's response
	Symbolically	1:11	Finding y-values for given x-values & completing a table of values	1. Substituting & calculating 2. Changing representation	Restates and writes
	Symbolically	3:41	Type of function	1. Plotting points 2. Sketching graphs 3. Changing representations 4. Identifying and naming	1. Empirical 2. Mathematical conventions 3. Teacher asserts 4. Restates and writes
	1. Words 2. Verbally	0:36	Write in algebraic form	Changing representation	Restates and writes
$g(x) = \frac{x}{2}$ $h(x) = \frac{x}{2}$ $k(x) = x^2$ $m(x) = 2^x$	Symbolically	8:08	Finding y-values for $x = -4$ & $x = 3$	Learners complete this task on their worksheets	
$g(x) = \frac{x}{2}$	Symbolically	0:22	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Changing representation	Restates and writes
	Symbolically	0:10	Type of function	Identifying and naming	Writes
$f(x) = \frac{1}{2}x$	Symbolically	0:16	Establishing equivalence between	Comparing y-values for given x-values	1. Empirical 2. Acknowledging

$g(x) = \frac{x}{2}$			$f(x)$ and $g(x)$		correctness of learners' response
$h(x) = \frac{2}{x}$ $g(x) = \frac{x}{2}$	Symbolically	1:33	Establishing equivalence between $h(x)$ and $g(x)$	1. Substituting and calculating 2. Comparing y-values for given x-values	1. Empirical 2. Pm ⁺ 3. Restates and writes
$\frac{2}{-3}, \frac{-2}{3}, \frac{2}{3}$	Symbolically	0:54	Notation for writing negative fractions	Revising mathematical rules	1. Mathematical Rules 2. Restates and writes
$h(x) = \frac{2}{x}$	Symbolically	1:34	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Changing representation	Restates and writes
	Symbolically	1:07	Type of function	Identifying and naming	Teacher asserts
	1. Words 2. Verbally	7:02	Write in algebraic form	Changing representation	Acknowledging correctness of learners' response
$k(x) = x^2$ find $k(-4)$	Symbolically	2:58	Squaring a negative number	1. Revising mathematical rules 2. Substituting and Calculating	1. Technology 2. Mathematical rules 3. Pm ⁺ 4. Teacher asserts
	Symbolically	1:22	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Changing representation	Acknowledging correctness of learners' response
	Symbolically	1:50	Type of function	Identifying and naming	Restates and writes
$m(x) = 2^x$	Symbolically	2:37	Finding a y-value for a given x-value	1. Revising mathematical rules 2. Substituting and calculating	1. Mathematical rules 2. Teacher asserts
$m(x) = 2^x$	Symbolically	2:52	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Changing representation	1. Mathematical rules 2. Teacher asserts 3. Restates and writes 4. Everyday
	Symbolically	1:52	Type of function	1. Substituting and calculating 2. Changing representation	Teacher asserts

Table 8.1: Range of examples introduced – Lesson 1

Looking down the first column of table 8.1 it is evident that Teacher J introduced all the planned examples for purposes of progressing with the lesson. In instances where Teacher J wanted to revise a mathematical rule or to remind learners about the acceptable conventions in mathematics he introduced his own examples which were based on the planned example used. For example, in

revising the notation for expressing a negative fraction ($\frac{-2}{3} = \frac{2}{-3} = -\frac{2}{3}$) Teacher J based it on the planned example $g(x) = \frac{x}{2}$ and finding $g(-2)$. In terms of the planned examples ($y = 2x$; $f(x) = \frac{1}{2}x$; $g(x) = \frac{x}{2}$; $h(x) = \frac{2}{x}$; $k(x) = x^2$ and $m(x) = 2^x$), the position of the variable 'x' and the constant '2' varied from one equation to the next. In terms of the planned lesson, the learning study group thought that this level of variation in conjunction with the different representations of a function (table of values, mapping between sets and graphical) would provide some opportunities during the lesson to focus learners' attention on the class of function represented by each equation. However, in terms of the enacted lesson these examples were introduced one at a time and Teacher J did not provide opportunities for learners to make comparisons between the different examples so that they could compare them simultaneously – what Marton and Pang (2006) refer to as the simultaneity of the two states.

To commence with describing what comes to be constituted as the enacted object of learning in this lesson I invite you to examine the column labelled 'observable actions' in the table 8.1. Perusing through this column reveals that the act of substituting given x-values into an equation and finding its associated y-values occurs most frequently. This is followed by the observable actions identifying and naming; revising rules and conventions; changing representation and finally learner generated examples. In table 8.2, I present the amount of time that was spent on each of these observable actions together with the frequency of the range of appeals made by Teacher J in his attempts to legitimate some form of meaning for his learners.

Observable Action	% of time spent	Number of occurrences	Mathematics				Teacher		Experience
			Definitions/ theorems	Empirical/ technology	Conventions/ rules	Process	Teacher asserts	Teacher confirms	Everyday
Substituting and calculating	50.5	11		9%	36%	36%	18%	82%	9%
Identifying and naming functions	18.8	8		13%	13%		50%	63%	
Changing representation	15.7	3			33%			100%	
Revising mathematical rules and conventions	7.5	4		25%	100%	25%	50%	50%	
Learners generating examples	7.5	1				100%		100%	

Table 8.2: Percentage of time spent per observable action and the range of appeals – Lesson 1

As already mentioned, the observable activity of substituting and calculating consumed the most amount of time. In terms of the percentage distribution across the various observable actions, half the lesson was spent engaging in the observable action of substituting and calculating. Looking at this observable action in relation to the nature of the appeals, one can see that Teacher J appealed to various domains of authority in his attempt to ensure that his learners were sufficiently equipped to find output values for given input values. In most instances learners were able to provide the correct output value for the given input value and Teacher J merely confirmed the learners' responses by writing their responses. For instance consider the following example which illustrates Teacher J confirming learners' responses by writing and in so doing completes a table of values:

- Teacher J Who would like to complete this one for me (*refers to finding the y-value ($y = 2x$) when $x = -4$) very fast. What are we going to have there? Here, for this one (*points to first square in table*)?*
- Learners Negative eight.
- Teacher J Negative eight (*writes -8*). And here?
- Learners Negative six.
- Teacher J Negative six (*writes -6 in 2nd square*).
- Learners Negative four.
- Teacher J Negative four (*writes -4 in 3rd square*).
- Learners Negative two.
- Teacher J Negative two (*writes -2 in 4th square*).
- Learners Zero.
- Teacher J Aha (*writes 0 in 5th square*).
- Learners Two (*writes 2 in 6th square*). Four (*writes 4 in 7th square*). Six (*writes 6 in 8th square*). Eight (*writes 8 in 9th square*).

(Lesson 1, event 5.1)

Teacher J also went to the extent of revising mathematical rules for performing arithmetic calculations which one would consider as being elementary considering the learners being taught are at the end of their grade 10 year. For instance, consider the following extract:

- Teacher J Negative four squared (*writes -4 squared*). You know what that means?
- Learners Yes.
- Teacher J It means negative four...(*writes -4×-4*)
- Learners Times negative four.
- Teacher J You agree?
- Learners Yes.

Teacher J Now, if I'm writing this one here...

Learner It's sixteen.

Teacher J I will put a bracket there (*writes $(-4)^2$*). Because I am squaring a negative four. Are we together?

(Lesson 1, event 10.1)

Or for that matter demonstrating the mathematical process of substituting given x-values in an equation such as $y = 2x$ and calculating the corresponding y-value. Once again, engaging in this fashion with learners who are at the end of their grade 10 academic year could be considered as elementary work.

Teacher J Substitution there (*writes 'Substitution' in top middle row next to 'Equation'*). Now, suppose I was given that my x is equal to let me say negative four (*writes $x = -4$ in row under 'Substitution'*), what will be my value of y here (*points to y*)? My x is equal to negative four, what will be the value of my y?

Learners Negative eight.

Teacher J How do you get that?

Learner You substitute.

Teacher J y is equal to two...(*writes $y = 2$*)

Learner Two bracket times...

Teacher J Open bracket...

Learner Negative four.

Teacher J Negative four (*writes -4 in a bracket*). And then if you multiply that one there you get now...(*writes =*)?

Learners Negative eight.

Teacher J Negative eight (*writes -8*).

(Lesson 1, event 2.1)

I have indicated that going through such processes with learners who are at the end of their grade 10 year is an elementary exercise, more so when one considers the simplicity of the examples used. Yet Teacher J felt that this was important and so provided learners with opportunities to revise these mathematical processes and by so doing increasing their capacity to accurately determine output values for given input values.

In the process of engaging with the observable action of finding y-values for specific x-values Teacher J also goes to the extent of providing an explanation of what the fraction $\frac{1}{16}$ means. He does

this by appealing to the everyday knowledge of the concept of a fraction. The extract which follows is in relation to the function $m(x) = 2^x$, find $m(-3)$:

Learner Can you still write it as like sixteen over?

Teacher J Which one? This one (*points to $\frac{1}{16}$ in the table*)?

Learner (*inaudible*).

Teacher J You see this ruler here (*holds up big ruler*). If you say one over sixteen, in other words, we are going to cut this ruler into sixteen equal parts. Then I'll take only the small part. You understand? We are saying, this thing here, suppose it's a chocolate, and there are sixteen people. You must cut this one so that everyone is going to have an equal piece. Right? So you are going to have the small part that you are going to get is one over sixteen. But otherwise all of that, it's sixteen over sixteen, which is one whole. Isn't it?

Learners Yes.

Teacher J One whole ruler. So now if you had this one here (*points $\frac{1}{16}$ in table*)...if you write this one here as sixteen, that's very wrong, isn't it?

Learners Yes.

Teacher J Because you are a having a small piece. Are we together?

Learners Yes.

(Lesson 1, event 11.2)

By inserting these extracts, I have tried to illustrate the extent to which Teacher J went to ensure that his learners were well prepared to engage in the process of substituting x-values into an equation and finding the corresponding y-values.

Identifying and naming functions was the second most frequently occurring observable action. In most instances learners were required to identify and name the function after ordered pairs were found and represented in a table of values. This did not provide learners with the appropriate criteria for identifying and naming a function given its algebraic representation, as the critical feature for establishing the class of function was not in focus. This is evidenced by the teacher's attempts to legitimate meaning for the learners. In each of these instances, Teacher J either acknowledges that the learners have correctly identified the given function or asserts and so tells the learners what function is being represented. In addition there are instances where Teacher J confirms a learner's response by restating and writing. In terms of identifying and naming the given function there was only one instance in this lesson when Teacher J acknowledged the correctness of a learner's response. It is important to note that in this instance the acknowledgement made by the teacher does

not necessarily mean that the learners understood why the equation represented a specific class of function. In fact one would associate this interaction as a guess between two given options (refer to underlined text in the extract below)

- Teacher J Now, from what we have done, can you tell me what type of a function is this one?
- Learner Which one, sir?
- Teacher J This (*points to $f(x) = \frac{1}{2}x$*)...this one we said it's a linear (*points to $y = 2x$*), what about this one (*points to $f(x) = \frac{1}{2}x$*)?
- Learner It's still a linear.
- Learner Non-linear.
- Teacher J Yes?
- Learner It's still a linear.
- Teacher J Yes?
- Learner Non-linear.
- Teacher J It's non-linear.
- Learners It's still a linear, sir.
- Teacher J Yes.
- Learner It's still a linear.
- Teacher J Ok, class, there are two answers, the one who says it's non-linear, the other one says, it's still a linear. Which one are we going to take?
- Learners Linear.
- Teacher J Linear? Wow! Let's give ourselves a clap

(Lesson 1, event 3.4, underline my emphasis)

In most of the remaining instances where learners were required to identify and name the given function which was represented algebraically, Teacher J merely asserted the class of function being represented. Teacher J realised that engaging learners with the process of finding ordered pairs and representing them in a table of values was insufficient in providing the learners with the appropriate criteria for identifying the class of function given its algebraic form and that the learners needed something else. The extract which follows demonstrates Teacher J providing learners with additional criteria to assist them in identifying the class of function defined by $f(x) = \frac{1}{2}x$.

- Teacher J What type of a graph is that? Put up your hands.
- Learners Non-linear.
- Teacher J Non-linear? If it's non-linear, what is it now?
- Learner Linear.

Learner Something...

Teacher J Non-linear or linear...?

Learner Undefined.

Teacher J A...?

Learner It's curved. It's curved like this (*learner gestures with hand*).

Teacher J It's curved. Are you sure?

Learners (*comments from learners*)

Teacher J Alright, ok. Sometimes we need to try and find out if we can give the answer for ourselves. Let me do this one here very fast; allow me to do it fast but although it's still got to be in order (*draws Cartesian plane*) ...

(Lesson 1, event 6.2)

Event 6.2 continues with Teacher J plotting the points on the Cartesian plane and then sketching the graph and finally with learners correctly identifying the function being represented. The portion of the extract taken from event 6.2 is sufficient to highlight that Teacher J realises that thus far the learners were not given sufficient criteria to be able to identify and name a function which appears as an equation. Teacher J tells his learners that they need to try and find out for themselves, and here he refers to the identification and naming of a function. Teacher J demonstrates to his learners that the route to follow in becoming, in a sense, 'self-sufficient' in being able to identify and name a function given its defining equation is to draw its graph. This is a route that could lead to learners being able to identify and name a function, provided that they are able to recognise the class of function being represented graphically. This route is not the most efficient process to follow in order to identify the class of function given its algebraic representation. It illustrates that the critical feature that would enable these learners to identify and name a function given its algebraic representation is not yet in focus. The critical feature is emergent and thus was not in focus for the members in this learning study whilst they planned this lesson.

The next observable action that consumed the third highest amount of time in this lesson was that of changing representation. This excludes the amount of time taken to represent ordered pairs in a table of values, as this aspect was taken into account under the observable action finding ordered pairs and representing them in a table of values. There are three instances where changing representation was the primary observable action identified across an event or sub-event. Two of these instances are when the examples were introduced verbally because they appeared as words e.g. the input value divided by two gives the output value. In the third instance the learners were also required to change the representation from words to algebraic form but in this case the learners worked on their worksheets and Teacher J worked with learners on an individual basis. This

specific instance contributed the largest chunk of time to the amount of time spent on this observable action.

In addition to varying the examples (specifically the mathematical relationship between the constant '2' and the variable 'x'), the second dimension of variation that was built into the lesson plan was the multiple representations of a function (verbal, algebraic, table of values, mapping between sets and the graph). In this lesson Teacher J opened up this dimension of variation but restricted it to verbal, algebraic and a table of values. Having focused only on these three representations resulted in Teacher J not providing his learners with appropriate criteria to be able to discern between the different classes of functions. In relation to the planned lesson the criteria that could assist learners to be able to discern between the different classes of functions was the graphical representation. In this lesson, representing functions graphically was restricted to the linear function defined by $f(x) = \frac{1}{2}x$. This limited exposure to the graphical representation of a function did not provide learners with an opportunity to experience a linear function in relation to what other functions would look like graphically, thus allowing them an opportunity to contrast a linear function with a non-linear linear function.

Teacher confirmation and assertions dominated the kind of appeals made by Teacher J in his attempts to legitimate some form of meaning for his learners during this lesson. The teacher assertions and confirmations were done in the absence of mathematical criteria that could enable learners to distinguish between the different families of functions, thus the assertions and confirmations were not geared towards making the intended object of learning distinct. Teacher J's appeal to mathematical processes was also limited to enhancing learners' ability to substitute x-values into an equation and calculating the resulting output value. Here again the nature of such appeals did not focus on making the intended object of learning discernable for the learners. The appeal in this case contributes to enhancing the learners' skill of substituting values for x and performing the relevant arithmetic calculations. It is in this process that Teacher J also appealed to rules and conventions in mathematics to assist the learner in performing the correct arithmetic calculations. For example Teacher J appealed to rules and conventions in mathematics for expressing an improper fraction as a mixed number fraction. Other instances where Teacher J appealed to rules and conventions are as a result of the example given in the worksheet. Since some examples foregrounded the functional notation for expressing the dependent variable, Teacher J then appealed to the rules and conventions related to functional notation. The appeal to rules and conventions was once again not directed at bringing the intended object of learning into focus but rather to focus learners' attention on the aspects of mathematics that were required to complete the

task at hand as required by the given example. This is not surprising because there was no critical feature to focus on. Although this was the case the nature of appeals could be described as mere legitimization of meaning through enforcement of rules to be followed and the acceptance of some aspects of mathematics to be true because the teacher said so.

To reiterate, the intended object of learning was to enhance the learners' ability to differentiate between the different families of functions. In describing what comes to be constituted as the enacted object of learning in this lesson as seen through *reflection*, some principles of *variation theory (contrast and simultaneity)* to elaborate reflection and the *legitimizing of meaning*, the enacted object of learning could be summarised as:

i. In relation to the observable action (reflection)

- Substituting x-values into an equation and finding its associated y-value(s) and representing these values in a table
- Using the equation and its associated table of values to identify and name the given function
- Changing representation
- Revising mathematical rules and conventions

ii. In relation to variation theory (elaborating reflection)

No opportunities were made available for the learners to compare different classes of functions.

iii. In relation to legitimating meaning (Authority)

- Enforcement of mathematical rules and conventions to be followed
- Teacher assertion and confirmation

In summary, the planned lesson placed emphasis on the mathematical relationship between the variable 'x' and the number '2' across the various planned examples. In Teacher J's attempt to provide his learners with opportunities to discern the relationship between the variable 'x' and the number '2', he focused learners' actions on finding output values for the given input values for each of the planned examples and then they were required to name the class of function being represented. No criteria were made available for discerning the class of function represented and so the legitimization of meaning was based on assertion. In addition, the planned lesson had as the point of departure functions represented algebraically and this had an important role to play in Teacher J's enactment of the lesson. Teacher J focused on the algebraic representation and learners were not provided with the appropriate criteria to determine the class of function being engaged with. The

learners were only able to classify an equation as representing a linear function once they were given access to its graphical representation. Having a planned lesson which starts by focusing on the algebraic representation of a function on the one hand and the absence of an appropriate critical feature on the other hand formed the basis from which this lesson was taught.

8.3 The post-lesson discussion – lesson 1

The learners did not perform well in the post-test and their results were made available to the teachers at the beginning of this discussion. To start the conversation, I asked Teacher J to reflect on the lesson he taught and talk to us about what he would do differently if he was to re-teach the lesson. Teacher J highlighted the pace of the lesson:

Teacher J So the pace again, I think my pace was a bit slow. I think I took too much time on this one.

VP When you say this one, you're referring to the table?

Teacher J Yes. I could have moved a bit faster than that, so that at least we could have ended up at least drawing some of these graphs, so that they see that, ok, these are the drawings.

(Post-lesson 1 discussion, utterances 6 to 8)

Refer to Figure 8.1 for the table that Teacher J refers to.

math	x	-4	-2	-1	0	1	2	3	4	Type of graph
a	$y = 2x$	-8	-4	-2	0	2	4	6	8	Linear
b	$y = \frac{1}{2}x$	-2	-1	-0.5	0	0.5	1	1.5	2	Linear
c	$y(x) = x^2$	16	4	1	0	1	4	9	16	Quadratic
d	$k(x) = 2^x$	1/16	1/4	1/2	1	2	4	8	16	Exponential

Figure 8.1: Table of values – Lesson 1

In setting up these tables of values, Teacher J focused on substitution and calculation, which was not a central aspect of the intended lesson. In Teacher J's comment above, he indicates that if he had increased the pace of this activity there would have been more time to focus on the graphical representation of the functions. Teacher J could have chosen to use the time on other aspects related to the concept of function but signals that he would focus on the graphical representation since he realised that without the 'picture' of the function learners were having difficulty in classifying a function. Teacher T entered the discussion at this point:

Teacher T Yes, maybe myself, I picked up one thing as the lesson was going on. The aspect of asking learners to identify the type of graph. I think it was a bit difficult to give the correct type of graph with only the equation and the table of value completed. There was need to have drawn the graph...or as you later did, you later drew that linear function, they said, oh, yes, it's a linear function. So from there, learners I think they should have been encouraged to say, try to make a sketch. I think identifying what type of graph it is, it's more appealing in picture form than in algebraic all the time.

(Post-lesson 1 discussion, utterance 12)

From Teacher T's input the group agreed that including the graphical representation is crucial for learners to discern classes of functions. This agreement was made devoid of the learners' performance in the post-test²⁶. Instead the teachers were focusing on the difficulty the learners had in identifying the classes of functions as observed when the lesson was enacted. Secondly, the teachers' attention to the visual representation is a result of its general absence in the enacted lesson, although it was factored into the planning of the lesson. What emerges as a critical feature for the teachers is the graphical representation of a function. In planning for the next iteration of the lesson, the group decided to follow Teacher T's suggestion:

Teacher T Yes, I was thinking in terms of the pictures, then we need to pre-draw some of the pictures before the lesson, so that maybe after completion of table of values, maybe we just show how we draw one or two, then from there the rest of it we just paste them up.

(Post-lesson 1 discussion, utterance 56)

The members of the group also felt that since Teacher J did not use the worksheet as support material in the lesson, the use of the worksheet needed to be in focus in the next iteration of the lesson. The group did not see the need to make any amendments to the planned lesson as such, as the graphical representation of the functions was part of the initial planning of the lesson.

8.4 The enacted object of learning – lesson 2

To commence with a description of the enacted object of learning in this lesson, Table 8.3 displays the re-ordering of the events as discussed in Chapter 7, Section 7.4.

²⁶ A discussion on the learners' performance in the post-test is provided in Chapter 9 when the lived object of learning is discussed.

Example	How was the example introduced?	Duration	Sub-notion	Observable action	Legitimizing meaning (where does authority lie?)
$y = \frac{2}{x}$	Graphically	1:22	Type of function	Identifying and naming	Restates and writes
	1. Words 2. Verbally	0:43	Write in algebraic form	1. Revising mathematical conventions 2. Changing representation	1. Mathematical conventions 2. Teacher asserts
	Symbolic	1:54	Finding the y-values when $x=0$	Substitution and calculating	1. Technology 2. Restates
$y = 2^x$ $y = x^2$ $y = -x^2$ $y = 2x$ $y = \frac{1}{2}x$	Graphically	5:20	Type of function	Identifying and naming	1. Teacher asserts 2. Writes 3. Restates and writes
	1. Graphically 2. Verbally	2:53	Providing equations of specific cases	Changing representation	1. Teacher asserts 2. Restates and writes
	1. Symbolically 2. Graphically	1:20	Providing the general equation	Changing representation	3. Mathematical conventions 4. Teacher asserts
	Symbolically	0:29	Input and output values	Revising mathematical conventions	1. Mathematical conventions 2. Restates and writes
$y = 2x$	1. Words 2. Verbally	2:58	Write in algebraic form	Changing representation	Restates and writes
	Symbolically	1:12	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Changing representation	1. Writes 2. Acknowledging correctness of learners' response
	Symbolically	2:06	Expressing x and y values as a mapping between two sets	Changing representation	1. Mathematical Conventions 2. Restates 3. Writes
	Symbolically	3:54	Plotting points and drawing the graph	1. Plotting points 2. Sketching graphs 3. Changing representation	Acknowledging correctness of learners' response
	1. Graphically 2. Symbolically	0:51	Type of function	Identifying and naming	Restates
	1. Symbolically 2. Graphically	7:35	Testing of points lie on the graph	Plotting points	1. Empirical 2. Teacher asserts 3. Restates
$f(x) = \frac{1}{2}x$	1. Words 2. Verbally	1:12	Write in algebraic form	Changing representation	Restates and writes
	Symbolically	2:53	Finding y-values for given x-values and completing a table of values	1. Revising mathematical conventions 2. Substituting & calculating 3. Changing representation	1. Mathematical conventions 2. Pm ⁺ 3. Teacher asserts 4. Restates and writes 5. Acknowledging correctness of

					learners' response
	Symbolically	1:05	Sketch the graph	Learners are told to complete this task at home	
$p(x) = x^2$	1. Words 2. Verbally	0:21	Write in algebraic form	Changing representation	Write
	Symbolically	4:03	Type of function	Identifying and naming	1. Teacher asserts 2. Restates and writes 3. Everyday
	Graphically	2:21	Effects of the coefficient of x^2	Revising effects of parameters	1. Pm 2. Teacher asserts 3. Everyday
	Symbolically	1:01	Meaning of p(1)	1. Revising mathematical conventions 2. Substituting and calculating	1. Mathematical conventions 2. Teacher asserts
	Symbolically	0:54	Plotting the point (1;1)	1. Revising mathematical conventions 2. Substituting and calculating	1. Mathematical conventions 2. Acknowledging correctness of learners' response
	Symbolic	0:37	The meaning of p(x)	Revising mathematical conventions	1. Mathematical conventions 2. Acknowledging correctness of learners' response
$g(x) = 2^x$	1. Words 2. Verbally	0:48	Write in algebraic form	Changing representation	Restates and writes
	Symbolic	0:26	Type of function	Identifying and naming	Teacher asserts

Table 8.3: Range of examples introduced – Lesson 2

Looking at the range of examples that were introduced during this lesson (refer to column titled 'Example' in Table 8.3) one is able to see that Teacher S also followed the planned lesson closely in terms of the examples that she used. As discussed when providing the description of this lesson, Teacher S started this lesson by introducing a range of examples graphically. The graphs introduced were graphs of all the planned examples as well as the graph $y = -x^2$. Having highlighted this it is important to note that the observable actions identified across this lesson focused mainly on the planned examples. To proceed with the description of the enacted object of learning Table 8.4 displays the amount of time that was spent on each of the observable actions and the frequency of the range of appeals made by Teacher S in her attempt to legitimate meaning for her learners.

Observable Action	% of time spent	Number of occurrences	Mathematics				Teacher		Experience
			Definitions/Theorems	Empirical/Technology	Conventions/Rules	Process	Teacher Asserts	Teacher Confirms	Everyday
Plotting points and drawing graphs	26.2	2		50%			50%	100%	
Changing representation	24.0	8			38%		38%	88%	
Identifying and naming functions	23.4	5					60%	100%	20%
Substituting and calculating	13.6	4		25%	50%	25%	50%	100%	
Effects of parameters	4.6	1				100%	100%		100%
Revising mathematical conventions	2.1	8			38%		38%	88%	

Table 8.4: Percentage of time spent per observable action and the range of appeals – Lesson 2

Firstly, when one adds the percentage of time spent on each of the observable actions listed in table 8.4 there is a shortfall of 5.97% with respect to the distribution of time across the different observable actions. This difference accounts for the time spent on setting up and organising the class at the beginning of the period (remember this lesson was conducted during school hours).

From this table it becomes evident that engaging in activities related to the *plotting of points and sketching of graphs* consumed just over a quarter of the lesson (26.2% of the time). Although this category of observable action consumed fractionally the most amount of time it was not the most frequently occurring observable action across the lesson. This stems from the fact that in getting learners to complete the task of plotting points and sketching a graph by themselves takes much longer than activities that are led by the teacher who is able to control the pace. There is only one occurrence in this lesson where Teacher S got learners to go through the process of plotting points and sketching the graph. This activity provided learners with an opportunity to practice the process of plotting points on a Cartesian plane and then drawing the graph. The activity allowed learners to demonstrate their ability to successfully plot points as Teacher S was able to engage with the learners' work as she moved around the class. Included in the process of plotting points and sketching a graph was the activity of determining if a point lies on the graph sketched. In this instance Teacher S provides two coordinates, one which lies on the graph and one which does not, consider the following extract:

- Teacher S Now if you got a graph already, give me, I want you to show me a point, a marking point P, which has these values. (*points to the coordinate $P(1;2)$ which was written on the board*) P has the value one, two. (*Teacher now writes $Q(2;1)$ beneath the coordinate of P*) and another one which has two, one. (*Teacher fills in the remaining x-values in the table on board*) (*walks around class*)
- Learner Straight line.
- Teacher S Straight line.
- Learner Yes (*camera zooms to learner's graph drawn on paper – the learner plotted all the points and traces over the points with his finger and says*) straight line.
- Teacher S Join the points (*teacher walks to the front of the class, then checks on learners work*).
- Teacher S For P and for Q show me on that graph that you got where are these points (*points to the coordinates of P and Q that were written on the board*)? (*teacher walks around the class and checks on learners' work*).
- Teacher S Anyone else who got the point P and Q? On P and on Q. (*teacher continues checking on learners' work and engages with them*)
- Teacher S For point Q people (*points to the coordinate of Q which is written on the board*) do not look for the values on the table, just give me the point two one from the graph as you've drawn it (*teacher looks at learners' work and engages with them*).
- Teacher S And where does that point lie?
- Teacher S (*engaging with learner at the back near the camera*) Where is the point? Q, where is it? (*learner shows the teacher the point $(-2;-1)$ on his graph*) I didn't say Q is minus two, one. It says Q is two, one. So where is the point two, one, from there? (*learner erases the coordinate $(-2;-1)$, teacher moves to the front of the class and checks on other learners' work*)
- Teacher S So what do you notice between point P and point Q? What can you say about point P and point Q? (*inaudible*) (*looks at learners' work*). You have a graph and you found that point one two lies on the graph. What about point Q? It lies on a straight line, what about point Q?
- Learner It doesn't lie on the straight line.
- Teacher S It doesn't lie on the straight line. So what does it tell you? No-one can tell me what it is. Point Q, (*teacher points to the coordinate written on the board*) these values do not fall into the value of tables...table of values, as you (*points to a learner*) have pointed out. So point Q is not in your graph. It doesn't lie on your graph, it's a point outside of the graph. So what does it mean when x is equals to two, from your equations, y equals to two x (*writes $y = 2x$*). What happens when x is equal to two? Two...(Writes $=2(2)$)
- Learner It's four.

Teacher S (Writes =4) So this point (*points to the coordinate Q(2;1)*) doesn't satisfy the given equation (*points to $y = 2x$*), so it means it doesn't lie on the graph.

(Lesson 2, event 5.6)

The extract illustrates the nature of the opportunities that Teacher S has provided to ensure that her learners are capable of plotting points and drawing the respective graph. Teacher S worked with the learners individually as they engaged in this task. This provided Teacher S with some opportunity to engage directly with individual learner's challenges. The nature of the opportunity provided in this instance was a 'hands on' approach for the learners. The activity provided the learners with an empirical means of establishing the validity of what they were required to do and so to establish some form of meaning.

The observable action *changing representations* was the second highest consumer of time in this lesson. A total of 12 minutes and 21 seconds was spent on activities in which changing representation was the primary activity. Approximately 50% of this time (6 minutes 2 seconds) was spent on changing the verbal representation of the function to its algebraic equivalence. For example consider the following extract:

Teacher S Can you give me an equation for the first one...that's written here. Input value multiplied by two is the output value. Or in other words, double the input value to get the output value. Write it now in the table form.

Learner It's going to be, $f(x)$ equals to x . Two times x .

Teacher S Yes. The one you gave me. So write down in your [...] in your handouts, there is a column there that's the verbal (*points the columns on the handout the she is holding*) now you write down the equation. Now you see there is a clue there, they gave you write it in this form – y will be equal to – what? (*camera zooms to learner's worksheet*) There are the x values, you said the x is the input value and the y is the output value. So now write an equation using these variables. y equals to whatever, or x equals to whichever way you look at it? Who has an answer? Where's your handout (*directed to learner in front*)? Write it in there, give me an answer...as soon as you've written down an answer give it to us so we all know. (*teacher walks slowly to the opposite side of the class*). Anyone? (*pauses for a moment*) What's the first one?

Learner The first one.

Teacher S Yes.

Learner It's negative ten.

Teacher S (*inaudible*) (*walks to back of class to learner who answered*)

Learner It's negative ten.

Teacher S Negative ten.

Learner Yes.

Learner The first one.

Teacher S Oh, ok, ok, ok. (*goes back to front of class*) What I want is, I want you to write the equation that represents this verbal form (*points to the verbal column in the worksheet*). An equation representing those letters in words. Now, this what we call the verbal form, now I want you to write it in equation form. How can you represent that in equation form? Ok, I'm coming to you (*talking to the learner who gave the previous answer of -10, teacher walks towards the learner*).

Learner It's y is equals to two times x.

Teacher S y equals to two x (*writes $y = 2x$*). Do you all agree with that?

(Lesson 2, event 5.1)

From the extract above we see that when Teacher S asks learners to re-represent the verbal form of the equation in its algebraic representation the learners are able to complete the task relatively fast, but the conversion at this stage is only verbal. Teacher S further instructs the learners to write down their responses on the worksheet, which she refers to as the handout. In asking learners to write down their responses Teacher S provides all learners with an opportunity to commit their thinking to paper and write down the sense they made of what they may have heard when a learner provided the re-representation in a verbal form at the beginning of this event. Whether the learners were listening or not, the opportunity provided in this instance was valuable for learners to make sense for themselves of what they were being asked to do. At the end of the event we see that Teacher S, after giving learners some time to write down their response, writes down the equation on the board. At this juncture she asks learners if they agree, again providing them with an opportunity to reflect on their answer and thus confirming the correctness of their answers. As indicated earlier there was a quick response by a learner, Teacher S did not merely accept the response but requested learners to write down their responses. It was this action of changing from a verbal representation of a function to its algebraic equivalence that consumed so much time.

Changing representation also included one instance across this lesson where Teacher S engaged learners with representing ordered pairs as a mapping between two sets. In this case the opportunity provided by Teacher S could be regarded as a quick demonstration by the teacher. This claim is amplified by the amount of time taken to engage in this specific activity (2 minutes and 6 seconds) and the nature of the appeals made by Teacher S in her attempt to legitimate some form of meaning for her learners. Teacher S appealed to conventions in mathematics in terms of demonstrating the labelling of the two sets, the placing of input and output values in these sets and the relationship

between these values by means of arrows. Another aspect that was included under the observable action of changing representation as it played out in this lesson was in an instance where learners were required to provide the equations for each of the graphs that were represented. In the second instance, learners were required to provide the general equations of the same graphs. Both instances together lasted 4 minutes and 13 seconds and in her attempt to legitimate some form of meaning for her learners, Teacher S merely asserted the relevant equation in instances where learners had difficulty in identifying the correct equation. I will pick up on this aspect again as I conclude this section. Having inserted the equations through an assertion, specifically in instances where learners had difficulty in identifying the graphs, Teacher S has not provided the appropriate criteria for identifying the general equation for the graphs in question.

Identifying and naming of functions ranks as the third highest consumer of time in this lesson. In most of the events learners were required to identify and name a function from its graphical representation, for example:

- Teacher S Ok, give me any graph that you identify. We've got graph A, graph B...*(teacher now also includes the graphs that are drawn on the Cartesian plane lying on the right. Sketched are graphs C – a parabola $y = x^2$, D – same as graph C, E – straight line $y = 2x$, F – straight line $y = \frac{1}{2}x$ & G – parabola $y = -x^2$ – she points to each graph)*. Here graph C is the green one, graph E is that blue one there, graph G is that brown one, and graph F is the black one. Just give me the letter and the name that you can identify.
- Learner Graph F is a straight line?
- Teacher S Graph F...can you all see graph F?
- Learners Yes.
- Teacher S Is it a straight line? *(teacher continues to enumerate the labels for each graph below each other and writes on the board C=, D=, E =, F=straight line)*. Anyone else?
- Learner Graph G.
- Teacher S Graph G.
- Learner Parabola.
- Teacher S Graph G *(writes G=)*...G is a ... G
- Learner *(inaudible)*
- Teacher S Brown one. G is a parabola *(writes G =parabola)*, is that correct?
- [...]
- Learner Graph E is a straight line graph.
- Teacher S Graph.
- Learner E.

- Teacher S Graph E, straight line graph (*writes straight line after E=*). Otherwise we got B and D (*refers to graphs not yet classified*)...graph B? ... And graph D?
- Learner (*inaudible*)
- Teacher S (*teacher looks at the graphs and realises that graph C and D is the same graph*) C and D are parabolae (sic), fine (*writes parabola after D=*). Now B? The last one.
- Learner Hyperbola.
- Teacher S A is a hyperbola and B is a hyperbola.
- Learner Yes.
- Teacher S A and B are both hyperbolae? Ok, graph B (*writes exponential graph after B=*) is an exponential graph. Now I'd like you to look at your handout. Let's start with...there are columns there, the verbal, the equation, the table of value, the mappings, and graph...
(Lesson 2, event 2)

The extract demonstrates that learners were able to identify the linear and quadratic function from the range of graphs presented. In instances where they had difficulty in identifying the graph, Teacher S merely asserted the type of graph. By asserting what should be and confirming learners' answers, learners are once again not provided with the appropriate criteria to identify and name the function, given its graphical representation in this case.

The observable action of *substituting and calculating* follows next on the list. Teacher S engaged learners with this task immediately after learners changed the representation of the given function from words and so verbal to algebraic. The pattern here follows the sequence for the completion of the worksheet. In the process of learners finding y-values for specific x-values Teacher S provides her learners with an opportunity to develop their ability to substitute and calculate correctly. In this regard she appealed to various domains of authority to build her learners' skill, for example consider the following extract:

- Teacher S What happens then if x equals to negative five, what will f of x be equal to?
- Learner It's going to be two point five.
- Teacher S Two point five?
- Learner Yes ... negative two point five.
- Teacher S Negative two point five. So how did you get that? F of (*writes f(, on board*)... what do I put inside?
- Learner f of x...no, it's f it's negative five...
- Teacher S (*writes -5 in brackets to get f(-5)*) But x now is no longer x it's negative five. It's equal to half times...? (*writes $\frac{1}{2}$ to get $f(-5) = \frac{1}{2}$*)

Learners Negative five.

Teacher S (*writes -5 inside brackets to get $f(-5) = \frac{1}{2}(-5)$*) And what's the answer?

Learner The answer, negative two point five.

Learner On that side (*refers to the right hand side of the equation*) it going to be negative two point five.

Teacher S (*writes = -2,5*) negative two point five. (*what appears on the board thus far:*

$$f(x) = \frac{1}{2}x$$

$$f(-5) = \frac{1}{2}(-5)$$

$$= -2.5$$

(Lesson 2, event 6.2)

This extract illustrates the extent to which Teacher S has gone in order to assist her learners in the process of substituting and calculating. Bear in mind these learners are at the end of their grade 10 academic year. From the extract above we see Teacher S focusing on the mathematical process of substitution and the multiplication of negative five by half. We also see Teacher S focusing on the concept of functional notation so that learners are able to interpret the notation correctly.

As was the case in lesson 1, in this lesson Teacher S also worked with the planned examples. The focus in this lesson was on the graphical representation and in examining each of the events we see that once again the examples were introduced one at a time. Introducing the examples in this fashion resulted in Teacher S focusing on one class of function at a time, thus presenting learners with no opportunity to compare different classes of function given either their algebraic or graphical representations. The lesson starts with Teacher S displaying the graphical representation of the different functions viz. ($y = \frac{2}{x}$; $y = 2^x$; $y = x^2$; $y=2x$; $y = \frac{1}{2}x$ and $y = -x^2$) and asking learners to identify and name the functions. The exercise that Teacher S engaged with lies at heart of the difficulty that these learners are experiencing – hence the intended object of learning which characterises this learning study. If Teacher S had provided the algebraic representations of these functions together with the graphical representation, the space of learning that would have been opened would have provided learners with an opportunity to compare the different classes of functions (algebraically and graphically) and so opportunities to engage with the simultaneity of the two states. This episode, which contributes to the description of the enacted object of learning, illustrates a missed opportunity for opening up a learning space that could have had the potential to focus learners' attention on the intended object of learning.

As already mentioned in the previous section, a dimension of variation that was built into the lesson plan was the multiple representations of a function. In this lesson this dimension of variation was opened but no opportunities were provided in the lesson which allowed learners to compare across the representations of a particular function. Trying to identify what makes a function linear from a set of ordered pairs implies determining if the rate of change is constant. This poses a challenge in determining the class of function if the rate of change is not constant. In comparing the different representations a productive comparison would have been to compare the algebraic with the graphical representation, which would have shown the learners what the function is. However, in order for them discern the particular class of function they would then require opportunities to compare it with another class of function so that they can see what it is not (Runesson, 1999). This dimension of variation with respect to multiple representations was built into the lesson plan and so was opened up intentionally by Teacher S but it was opened up in a fashion that did not enable comparisons to be made, thus an opportunity for learners to discern between the different classes of functions was missed.

In the absence of suitable mathematical criteria by which to identify the different classes of functions, I now focus on how Teacher S attempts to legitimate meaning for her learners. Looking at the range of appeals one is able to see that the domains of authority to which Teacher S appealed in her attempt to legitimate meaning for her learners are clustered around teacher assertions and teacher confirmations. Confirmation of learner responses merely signal to the learner(s) who provided the response that they are on the right track. For the learners who did not provide an acceptable response the teacher's confirmation could be perceived by them as a mere statement of truth and so must be accepted as such since no explanations are provided. This also holds true for instances where the teacher asserted. The nature of appeal does not provide learners with any opportunities to make sense of the aspect of mathematics in focus but rather they accept it as truth because the teacher said so. For example in the extract below Teacher S asks the learners to identify the graphs which appear on the board. One learner identifies the graph defined by the equation $y = \frac{2}{x}$ as a parabola and another identifies it as a hyperbola. Teacher S merely confirms that it is a hyperbola but does not provide reasons for identifying it as such. No appropriate criteria are made available to the learners as to why that specific graph represents a hyperbola.

Teacher S Right, let's look at the graph that I have on the board. (*On a laminated grid the teacher had drawn two Cartesian planes. The Cartesian plane on the left contained the graph of the hyperbola $y = \frac{2}{x}$ labelled A and an exponential graph $y = 2^x$ labelled B*). Let's look

at this side (*refers to the Cartesian plane on the left*). A is the pink or whatever colour that is. Who can identify that graph? What do we call that graph, equal to this one (*points to the exponential graph*)? I've labelled it graph B. What do you call that graph? Anyone?

Learner Parabola.

Teacher S Parabola. Somebody's got parabola here, anyone else? No one, ok. Who can identify graph A? (*beneath the laminated grid the teacher writes $A=$ and $B=$*) (*using her finger the teacher traces out graph A*)

Learner Hyperbola?

Teacher S It's a hyperbola (*writes hyperbola after $A=$*). So graph A is a hyperbola.

(Lesson 2, event 1)

Teacher S appealed to other domains of authority as well but having done so she did not provide her learners with appropriate criteria to be able to distinguish between the different classes of functions.

In describing what comes to be constituted as the enacted object of learning in this lesson as seen through *reflection*, some principles of *variation theory* (*contrast and simultaneity*) to elaborate reflection and the *legitimizing of meaning*, the enacted object of learning could be summarised as:

i. In relation to the observable action (reflection)

- Plotting points and drawing graphs;
- Changing representation;
- Identifying and naming functions given its graphical representation and
- Substituting x-values into an equation and finding its associated y-value(s).

ii. In relation to variation theory (elaborating reflection)

No opportunities were made available for the learners to compare different classes of functions.

iii. In relation to legitimating meaning (Authority)

- Teacher assertion and confirmation

To conclude, the post-lesson discussion after lesson 1 put the spotlight on the need for the graphical representation of the functions introduced. Lesson 2 started by focusing on the graphical representation as described in Chapter 7. Although the graphs were drawn and presented simultaneously (on the same Cartesian plane), they were not focused on together. Instead there was a focus on re-representing the function in an algebraic form. We see in the extract below that Teacher S wants learners to provide the general equation of the graphs represented. Instead learners begin to provide specific equations.

Teacher S Anyone who can give me a general formula for any of these graphs that you know of? A general formula of any of the graphs? Anyone? Graph A, there's a general formula that looks like...? Yes?

Learner Yes. For the straight one, for the straight line it's two times x.

Teacher S Two times x.

Learner Yes.

Teacher S Two times x.

Learner Yes. It's f x, it's f of x is equal to two times x. Then you put an equals to...and then you say f of x.

Teacher S (*next to E writes* $f(x) = 2x$) f of x equals two x. Another one?

Learner B. y is equal to x squared.

Teacher S y equals to x squared, for B? Do you all agree? (*no learners respond*) An exponential graph has the formula, y equals to x to the power of 2? Which one has the formula, y equals to x to the power of two? Is it the hyperbola that you say is B, or any other one?

Learner D.

Teacher S (*writes* $y = x^2$ *next to D*) A parabola has this formula, y equals to x to the power of two (*writes* $y = x^2$ *next to C*)

(Lesson 2, event 3.1)

Firstly, there is no evidence of learners performing any calculations such as finding the gradient of the straight line yet they provide $y = 2x$ as the defining equation of the straight line. Secondly, learners seem to relate with the linear and quadratic functions as these are the functions that they engage with as they don't attempt to offer equations for the hyperbolic and exponential functions drawn. How can the learners be sure that $y = 2x$ and $y = x^2$ represent the linear and the one quadratic function drawn? Is providing the graphical representation of the function sufficient criteria for identifying a function? What if learners were given a 'magnified view' or a zoomed in view of the graph $y = 2^x$ as seen in the picture below with its equation – would they identify the graph as an exponential graph or linear graph? Although there was an attempt to focus in on the graphical representation of the function in this lesson, the appropriate criteria for being able to differentiate between the different classes of function is still out of focus.

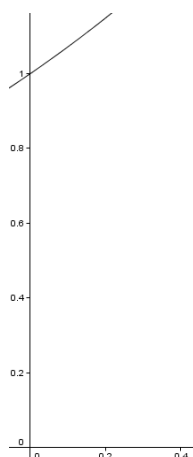


Figure 8.2: The graph of $y=2^x$ for $0 \leq x \leq 0.4$

8.5 Post-lesson 2 discussion and the emergence of a critical feature

I started the discussion by asking Teacher S if there was anything that she thinks she would do differently if she had to re-teach this lesson. To which she replied:

Teacher S Yes, I would have done things a lot differently. If I had foreseen that there would be problems with it. The drawing of the graph and then when they're claiming that this one was too small. The one I gave them.

(Post-lesson 2 discussion, utterance 5)

Here Teacher S refers to the Cartesian plane that was provided on the worksheet and the range of x-values that were provided on the table of values. To sketch the graph on the Cartesian plane provided for the range of x-values given, implied that not all the points on the table could be plotted. This was deliberate when planning the worksheet so as to demonstrate to the learners that the values in the table of values are a subset of all the points that make up the graph and so they did not need to plot all the points to draw the graph. To deal with the challenge during the lesson, Teacher S told learners that they could draw the graph on another sheet of paper.

Teacher S continued to express her concerns:

Teacher S My concern again was with these stupid errors that they made, with writing the equation, y equals to two x and they say, it's a hyperbola. I still feel let down because my lesson didn't produce what I wanted it to. Learners cannot identify the graphs and it is seen in the post-test.

[Later in the discussion Teacher S further explains]

Teacher S The last thing I noticed is that, the learners, most of them, could only describe or give a description of the graph after it has been drawn.

VP Say more about this.

Teacher S Because after writing there from the verbal to the equation, y plus two x , and I asked them what type of a graph it was, someone said, it's a parabola. And I was so (*inaudible*), I couldn't believe it and it was there (*inaudible*).

(Post-lesson 2 discussion, utterances 36 and 142 to 144)

Although Teacher S indicates that the learners were able to identify the type of function given its graphical representation (*response underlined*) this was not the case in the lesson. Consider the following extract taken from the lesson:

Teacher S Right, let's look at the graph that I have on the board. (*On a laminated grid the teacher had drawn two Cartesian planes. The Cartesian plane on the left contained the graph of the hyperbola $y = \frac{2}{x}$ labelled A and an exponential graph $y = 2^x$ labelled B*). Let's look at this side (*refers to the Cartesian plane on the left*). A is the pink or whatever colour that is. Who can identify that graph? What do we call that graph, equal to this one (*points to the exponential graph*)? I've labelled it graph B. What do you call that graph? Anyone?

Learner Parabola.

Teacher S Parabola. Somebody's got parabola here, anyone else? No one, ok. Who can identify graph A? (*beneath the laminated grid the teacher writes A= and B=*) (*using her finger the teacher traces out graph A*)

Learner Hyperbola?

Teacher S It's a hyperbola (*writes hyperbola after A=*). So graph A is a hyperbola. Can anyone remember what graph B is? We heard one say it's a parabola. Who else has a different identification, a different name for that one? ...

Teacher S ... Ok, give me any graph that you identify.

(Lesson 1, event 1 going into event 2)

The extract shows that the learners identify the function $y = 2^x$ (graph B) as a parabola, and it is because of this difficulty that Teacher S (at the end of the extract) asks learners to identify any of the graphs that the learners are able to identify. In responding to the teacher's request learners begin to provide names of function in an ad hoc fashion. This illustrates that even in this lesson learners were not provided with the appropriate criteria to be able to distinguish between the different classes of functions. Focusing on the graphical representation was also not sufficient in providing

appropriate criteria to assist learners in distinguishing between the different classes of functions. This demonstrates that the critical feature for enabling these learners to distinguish between the different classes of functions has still not yet emerged in this learning study.

The learners performed poorly in the post-test and their results were made available to the teachers at the beginning of this post-lesson discussion. In planning for the next lesson in the cycle, the teachers did not focus on the learners' performance in the post-test and so did not consider the challenges and errors the learners made. Due to the insignificant role played by the post-test in planning for the next lesson I will leave the discussion of the learners' performance in the post-test for the next chapter. In talking about how to progress with the next iteration of the lesson the teachers started focusing on aspects of lesson 2 that could be left out so as to have more time available in the lesson. Teacher J suggested taking out the aspect dealing with the mapping between sets since it is not really essential for drawing the graph:

Teacher J Maybe you can leave out the mapping. You see, these kids, they think what we are doing there, we will show them, yes, maybe one or two and say, the rest you can do. Because they will think, if they are to write a test, they must do the graph, they must do the mapping, they must...that's what they are going to think. If I do my table of values, I must do then the mapping, then draw the graph.

(Post-lesson 2 discussion, utterance 269)

Teacher T suggested removing some of the examples:

Teacher T There was even in the worksheet there is a second example, the one on y equal to half x , which is not very different from, $g(x)$ equal to x over two. Maybe to reduce the amount of work, you can leave the first one completely.

(Post-lesson 2 discussion, utterance 282)

The changes being suggested by both Teacher J and Teacher T were aimed at removing aspects from lesson 2 which they felt were not really important and in so doing they would have more time available in the lesson to focus on other aspects. The changes being suggested were not related to the criteria that would facilitate learners being able to discern between the different classes of functions. In response to Teacher T's comment I responded:

VP So there's not much difference between the two. We know it. But the learners they don't see half x and x over two as being the same. You had that come out in the lesson. So let

us just put a pause on the changes to make. I started off the session by looking at the starting of Teacher J's lesson, and the starting of Teacher S's lesson. And the starting is completely different. So Teacher J's lesson, if you remember, he set up the table where he had equation, the substitution part, and he had the type of function. Do you want me to write on the flip chart?

Teachers No.

VP Then he started with equation, y equal to two x . Then the next column he said, x equal to four, he did the substitution, and he worked it out. And then on the type of function he was expecting kids to say it's a linear function, but he had inserted and said this is linear. And then he asked learners then to generate other examples of linear, and he was putting it in that third table. Then Teacher J went on to the half x . And after a while they could figure out that it was a linear function. And I was asking myself the question, so they said half x is a linear function, but what made it linear? In the same way, y equal to two x is a linear function, but can the kids tell us why it's a linear function?

Teacher T (*inaudible*) the variable

[Teacher T then talks about Teacher S wanting to focus on this aspect when she asked learners to provide the general equation to the graphs presented. At this point Teacher S had already left the meeting as she had attend to other important issues]

VP But the question is then, what still makes it quadratic?

Teacher T Exactly. They need to know why, is it still a quadratic?

Teacher L The exponent.

(Post-lesson 2 discussion, utterances 285 to 288 and 307 to 309)

At this juncture the teachers' thinking about how to change the lesson has shifted and they are now focused on an aspect that could be considered as being critical for enabling the learners to differentiate between the different classes of functions and so have a rule by which to recognise the class of function given its algebraic representation. The conversation amongst the teachers continues:

Teacher T Very good. I don't know why I did not mention it, I looked at - on my lesson and I said, oh my goodness, here I mentioned linear, which is two x ... y is equal to two x . What if I were to say to the learners, what about if it's y is equal to two x squared?

Teacher J It changes completely.

Teacher T It changes completely but the learners maybe they will still tell you it's linear. Because they see a two and they see an x there. This exponent, the two there, to them it doesn't matter, it doesn't work much. So I think VP there is saying...actually the two lessons did you notice that we did not actually put clearly to these learners, to say, why do we say this is linear? It was not made clear.

(Post-lesson 2 discussion, utterances 310 to 312)

The conversation as illustrated in the extracts above illustrates the possible emergence of the critical feature that would characterise this learning study. In planning for the third iteration of the lesson focusing on the highest power of the independent variable became a key feature.

Now it is important to note that mathematically speaking $y = 2x$ is not linear because it is a polynomial function of the first degree, it is linear because it has a constant rate of change and $y = x^2$ is quadratic because the rate of change of the rate of change is a constant and not that it is a polynomial function of the second degree. In terms of relating the syntax of the algebraic representation of a function to its graph, focusing on the value of the exponent of the independent variable provides some criteria for learners to be able to identify the class of function being represented by a given equation.

In planning for the third iteration of the lesson I engaged in a quick demonstration with the teachers in the group, I wrote the functions $y = 2x$ and $y = \frac{1}{2}x$ on the board and asked them to identify what is the same and what is different between the equations. This was an attempt to demonstrate that the examples did not have to be introduced one at a time but in pairs with the intention of providing learners with an opportunity to compare the equations and by doing this one could open up a dimension of variation through contrast. In the process of comparing equations, opportunities could present themselves for learners to discern, for example, what an exponential equation looks like and how it differs from a quadratic equation.

8.6 The enacted object of learning – lesson 3

As was done when describing the enacted object of learning for lessons 1 and 2, I also commence this description by providing Table 8.5 which displays the re-ordering of the events as discussed in Chapter 7, Section 7.5.

Example	How was the example introduced?	Duration	Sub-Notion	Observable action	Legitimizing meaning (where does authority lie?)
$y = 2x$	Symbolic	1:08	Express verbally	Changing representation	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	2:06	Relationship between '2' & 'x'	Revising mathematical conventions	1. Mathematical conventions 2. Acknowledging correctness of

					learners' response
	Symbolic	3:31	Finding y-values for given x-values	Substituting and calculating	1. Pm ⁺ 2. Restates and writes
	1. Words 2. Verbal	5:51	Write in algebraic form	1. Changing representation 2. Revising mathematical conventions	1. Mathematical conventions 2. Teacher Asserts 3. Restates and writes
$y = x + x$	Symbolic	1:31	Finding y-values for given x-values	Substituting and calculating	1. Restates and writes 2. Acknowledging correctness of learners' response
$y = 2x$ $y = x + x$	Symbolic	5:15	Finding y-values for given x-values and completing a table of values	1. Substituting and calculating 2. Revising mathematical conventions 3. Changing representation	1. Mathematical conventions 2. Pm ⁺ 3. Restates and writes
$y = 2x$	Symbolic	4:52	Plotting points & sketching the graph	1. Revising mathematical conventions 2. Plotting points 3. Sketching graphs 4. Changing representation	1. Mathematical conventions 2. Teacher asserts
	1. Symbolic 2. Graphical	0:55	Type of function	Identifying and naming	1. Empirical 2. Teacher asserts
	1. Symbolic 2. Graphical	1:49	x and y intercepts	Revising mathematical rules	1. Mathematical definition 2. Teacher asserts 3. Acknowledging correctness of learners' response
	Symbolic	3:54	Plotting points and sketching the graph	Learners complete this task on their worksheets	
	1. Symbolic 2. Graphical	1:30	Writing x-values and their associated y-values as ordered pairs	1. Revising mathematical conventions 2. Changing representation	1. Mathematical conventions 2. Restates and writes
	1. Symbolic 2. Graphical	3:08	Expressing x and y-values as a mapping between two sets	1. Revising mathematical conventions 2. Changing representation	1. Mathematical conventions 2. Teacher asserts 3. Writes
	Symbolic	3:23	Using the mapping between two sets to determine if a function is represented	Revising mathematical rules	1. Mathematical definition 2. Teacher asserts 3. Everyday

$y = 3x$ $y = 4x$ $y = \frac{1}{2}x$	Verbally	0:54	Give other equations of a linear function	Learners generating examples	1. Restates and writes 2. Acknowledging correctness of learners' response
$y = \frac{1}{2}x$	Symbolic	2:44	Relationship between ' $\frac{1}{2}$ ' and ' x '	Revising mathematical conventions	1. Mathematical conventions 2. Restates 3. Acknowledging correctness of learners' response
	Symbolic	1:52	Is it a linear function?	1. Substituting and calculating 2. Plotting points 3. Sketching graphs 4. Changing representation	Empirical
$y = \frac{1}{2}x$ $y = \frac{x}{2}$	Symbolic	5:46	Finding y-values for given x-values	Substituting and calculating	1. Pm ⁺ 2. Teacher asserts 3. Acknowledging correctness of learners' response 4. Writing 5. Everyday
$y = \frac{1}{2}x$ $y = x \div 2$	Symbolic	4:24	Establishing equivalence between equations	1. Substituting and calculating 2. Comparing y-values for given x-values	1. Empirical 2. Restates and writes
$y = \frac{2}{x}$	1. Words 2. Verbal	1:45	Write in algebraic form	Changing representation	1. Teacher asserts 2. Restates and writes
	Symbolic	0:45	Relationship between ' 2 ' & ' x '	Revising mathematical conventions	1. Mathematical conventions 2. Restates
	Symbolic	6:30	Finding y-values for given x-values and completing a table of values	1. Revising mathematical rules 2. Substituting and calculating	1. Mathematical rules 2. Pm ⁺ 3. Teacher asserts 4. Restates and writes 5. Acknowledging correctness of learners' response
	Symbolic	2:11	Plotting points and sketching the graph	1. Substituting and calculating 2. Plotting points 3. Sketching graphs 4. Changing representation	Teacher asserts

	1. Symbolic 2. Graphical	0:43	Type of function	Identifying and naming	Acknowledging correctness of learners' response
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Table 8.5: Range of examples introduced – Lesson 3

Looking down the first column of Table 8.5, the range of examples that Teacher L introduced during this lesson was restricted to linear and hyperbolic functions, more specifically, the functions represented by the equations $y = 2x$, $y = x + x$, $y = \frac{1}{2}x$, $y = \frac{x}{2}$, $y = x \div 2$ and $y = \frac{2}{x}$. Essentially Teacher L only worked with three of the planned examples viz. $y = 2x$, $y = \frac{1}{2}x$ and $y = \frac{2}{x}$ and so provided her learners with opportunities to engage with the linear function and the hyperbolic function. To assist in providing a description of what comes to be constituted as the enacted object of learning in this lesson I provide table 8.6 that displays the amount of time spent on each of the observable actions as well as the range of appeals made in the teacher's attempt to legitimate meaning for her learners.

Observable Action	% of time spent	Number of occurrences	Mathematics				Teacher		Experience
			Definitions/Theorems	Empirical/Technology	Conventions/Rules	Process	Teacher Asserts	Teacher Confirms	Everyday
Substituting and calculating	40.6	6		17%	33%	67%	33%	100%	17%
Changing representation	20.1	5			60%		60%	100%	
Plotting points and sketching graphs	19.3	3		33%	33%		67%		
Revising mathematical rules and conventions	16.2	5	40%		60%		40%	100%	20%
Identifying and naming a function	2.5	2		50%			50%	50%	
Learner generated examples	1.4	1						100%	

Table 8.6: Percentage of time spent per observable action and the range of appeals – Lesson 3

The observable action that dominated this lesson was that of *substituting and calculating*. In the process of substituting and calculating Teacher L demonstrated the process for her learners by showing all the steps that one would follow to accomplish the task. The extract below is in relation

to finding y-values for given x-values for the function $y = 2x$. Before Teacher L engaged in this activity she spent time engaging learners with the meaning of $2x$. Consider the following extract:

- Teacher L And then can we get someone who can try and explain to us, how is this two and the x related? What is the relationship between the two and the x [...] What must happen with the two and the x?
- Learner The relationship is to multiply the x value by two in order to get the y value.
- Teacher L Correct. Well done. I hope you have heard what he has just said. Could you just repeat what you have just said?
- Learner The relationship is to multiply the x value by two to get the y value.
- Teacher L We multiply the x value...?
- Learner By two.
- Teacher L By two. Right

(Lesson 3, event 1.2)

The extract demonstrates that learners understand what $2x$ means, that each x value must be multiplied by two in order to obtain the corresponding y-value. To reiterate, these learners are in grade 10 and multiplication by two should not pose a challenge. Despite this, Teacher L still demonstrates the process of substituting and calculating in the equation $y = 2x$ as illustrated in the extract below:

- Teacher L How did you get negative six?
- Learner Negative six times two...ah, negative three times two.
- Teacher L So y will be equal to...a negative...(y=-)?
- Learners Six.
- Teacher L How can we get negative six? We said two multiplied by...
- Learners Negative three.
- Teacher L Negative three (*writes* $2 \times (-3)$). And that gives us...(writes =)?
- Learner Negative six.

(Lesson 3, event 1.3)

Teacher L goes through this process for various values of x. In doing so, she appeals to a mathematical process known as substitution. A little later in the lesson, Teacher L introduces the function in its verbal form and wants to know if there is another way of writing $y = 2x$:

- Teacher L Is there any other person who has another way of writing this as y is equal to two x (*points to the equation written on the board: $y = 2x$*)?

Learner Y equals to x plus x.

Teacher L Y is equal to x plus x. Y is equal to x plus x (*writes $y = x + x$*). Right, that is also correct. If I double a thing, it means I can add or multiply it by two, I can add x to another x. And we know that when any like terms, x plus x will still give us?

Learners Two x.

Teacher L Two x. Right.

(Lesson 3, event 3.1)

At this level learners understand that $x + x = 2x$ as this is a basic collection of like terms that learners at the end of their grade 8 year ought to understand. Nonetheless, Teacher L decides to extend the idea of substituting and calculating to include $y = x + x$ to provide empirical evidence that the two expressions, $y = 2x$ and $y = x + x$, are indeed the same. Refer to the extract below:

Teacher L Let's just extend it and get what we were busy with, when we were looking at number one. What will happen if we have, y is equal to x, plus x, and then when x...when x is seven (*writes: $y = x + x$ when $x = 7$*)...using this one (*points to $y = x + x$*). y will be equal to what (*writes then $y =$*)? When x is seven. We are using x plus x.

Learner Two x

Learner Fourteen.

Learners Fourteen.

Teacher L It will be fourteen. How did you get fourteen?

Learner Seven times two

Learners Two times seven.

Learner Seven multiplied by two

Learner Seven plus seven.

Teacher L We are using this one now (*points to $y = x + x$*).

(Lesson 3, event 3.2)

Teacher L also uses substitution as a means for learners to empirically establish the equivalence between two equations such as $y = \frac{1}{2} \times x$ and $y = x \div 2$ by getting learners to compare the y-values for specific x-values. The range of extracts provided above with the accompanying elaboration is aimed at providing the reader with some insights into why the observable action of substituting and calculation consumed just over 40% of the lesson.

Having illustrated the various reasons for which Teacher L engaged her learners with the action of substituting and calculating is insufficient to get a handle on the nature of the opportunities that she

provided. To accomplish this task I now focus on the nature of the appeals that Teacher L made in her attempts to legitimate some form of meaning for her learners. To ensure that learners were able to substitute and calculate sufficiently well Teacher L appealed to various authority domains to establish some form of legitimacy and meaning for the learners. The authority domains ranged from mathematical processes, to rules and conventions in mathematics, to teacher confirmation and assertions, to the everyday. For example, to establish some meaning for learners in their understanding of how to substitute and calculate in the two equations $y = \frac{1}{2}x$ and $y = \frac{x}{2}$ consider the following extract where Teacher L goes to the extent of bringing in an ‘everyday’ context to explain division as sharing:

Teacher L If I had three loaves, six loaves...I have six loaves of bread, I have two children, I want to divide the six loaves of bread equally amongst my children.

Learner *(inaudible)*

Teacher L I have two children. I have six loaves of bread. I want to divide the six loaves equally amongst my children. So each one will get?

Learners Three.

Teacher L Three. Three. And then I think that is what he was trying to say, you are going to divide each x value into two equal...

(refers to the calculation on the board:

$$y = \frac{1}{2}x, \text{ when } x = 6 \text{ then}$$

$$y = \frac{6}{2} = 3)$$

Learner Parts.

Teacher L Parts. Divide by two. And then here, if one says, get half of each x value.

(refers to the calculation:

$$y = \frac{1}{2}x, \text{ when } x = 6 \text{ then}$$

$$y = \frac{1}{2} \times \frac{6}{1} = \frac{6}{2} = 3)$$

Teacher L Let's say I say get half of eight. What is half of eight?

Learners Four.

Teacher L Four. And then if here, meaning that I will say half times eight divided by one (*writes*

$$y = \frac{1}{2} \times \frac{8}{1} =), \text{ which will give me eight times one is}$$

Learners Eight

Teacher L Eight, two, divided by (*inserts the division line for the fraction*), two times one (*writes* $\frac{8}{2}$)?

Learners Two.

Teacher L Two. And then this one says, divide eight equally into (*writes* $y=$) two halves, or by two. Divide eight sweets equally amongst two children. There are eight sweets on the table,

there are two children, I want to share them equally amongst the two children. Each child is going to get?

Learners Four.

Teacher L Four. Right? So we have eight divided by...?

Learners Two.

Teacher L Two (*writes $\frac{8}{2}$*). Then this gives us?

Learners Four. (*writes 4 to get $y = \frac{8}{2} = 4$*)

Teacher L Four.

(Lesson 3, event 2.2)

Changing representation ranks as the second highest consumer of time in this lesson. This observable activity is constituted by events in the lesson where Teacher L introduced the function in the form of words and so in order to engage further with the function, the function had to be expressed algebraically, thus changing representation. The amount of time spent on this observable action does not translate into it being a frequently occurring observable action. In the context of this lesson it merely means that Teacher L chose to spend time on this activity. To illustrate this point refer to the extract that follows.

Teacher L Can we look at...unfortunately our worksheet is without the numbers. The one with the verbal representation where it says, input value multiplied by two gives the output value, or double the input value to get the output value. You see that verbal representation.

Learners Yes.

Teacher L Ok. We have input value multiplied by two, gives the output value. Right, or, double the input value to get the output value. How can we write this as an equation? Input value multiplied by output value, gives...or double the input value to get your input (*sic*) value.

Learner That can be written as y equals to x times two.

Teacher L Y is equal (*writing on board while speaking*)

Learner x times 2.

Teacher L To x times two (*writes $y = x \times 2$*). Clear? The first one it's, y is equal to...that is the input value multiplied by the output value. Which one is the input value?

Learner Two.

Learner It's the x value.

Learner X value.

Teacher L The input value, so we have input value...our input value is? (*writes input (x)*)

Learners x.

Teacher L x. And then our output value?

Learners y.

Teacher L y (*writes output (y)*). So it says, input value, that is x , multiplied by two, gives

Learner Y value.

Teacher L Output value. Is there any other person who has another way of writing this as an equation?

(Lesson 3, event 3.1)

We see that as soon as the learners' attention is focused on the relevant example on the worksheet and once the example is introduced a learner immediately responds with the correct algebraic representation. Instead of accepting this response and moving on with the next aspect of the lesson Teacher L probes further for alternate ways of representing the example algebraically to which learners respond: $y = x \times 2$, $x \times 2 = y$ and $y = x + x$. This is merely one example to highlight the unnecessary use of time and the redundancy of the observable action. I refer to the re-representation of $y = 2x$, as $y = x \times 2$, $x \times 2 = y$ and $y = x + x$ as being redundant because it does not move the lesson on in terms of focusing learners' attention on the critical feature for being able to distinguish between different classes of functions.

Changing representation also included representing ordered pairs as a mapping between two sets. In this event Teacher L merely asserts and so demonstrates the mapping between the ordered pairs by writing on the board as is illuminated by this tiny extract taken from the episode that deals with the representation of the ordered pairs as a mapping between sets:

Teacher L Fourteen. Clear? So, you can draw the arrows to say, when x is this, this is the value, your output. When your input is zero, your output is...? Is zero. When your input is one, your output is two. When your input is four, your output is...and when your input is two your output is four. When your input is seven, your output is...? (*draws arrows to show the mapping between the input and output*)

(Lesson 3, event 3.11)

Following from this activity Teacher L goes on to define a function for the learners by making use of the one-to-one correspondence represented on the board:

Teacher L When we talk about a function, from what we have seen here (*points to mappings*), you realise that for each input value, there is one output value. When you substitute x by a negative five, when y is equal to two x , you'll get a negative ten. And then when x is negative three, your y is negative six. So if it's like this, you have one x value, which is related to one y value, we call that a function (*writes function*). So when we talk about the

function, we are referring to a relationship between two sets. The first set must be the set of the output or your x value...the input or your x values, and the other set must be your...output.

Learner Output.

Teacher L So for it to qualify as a function, it means, from the first set, which is your input, each element...remember this is just a subset, it is not a whole set. We have just chosen a few values of x. Right? Because we could go on and on and on for the whole year drawing the graph, without finishing to draw the graph. So we just selected a few values as a subset. So it represents there...if you look at this, each x value is mapped onto only one y value. I hope this...from...this example you have seen (*points to the mapping between the two sets*), there's no x value that has two answers. I hope you realise. There's no x value that has two answers. Right. So we call it a...function.

Learner Function.

Teacher L A function is a relationship in which each x value or each element in the first set is related to only one element of the second set. So for an example, in normal life...right...have you ever heard of a situation where a woman is married to two husbands at the same time?

(Lesson 3, event 3.12)

Providing a definition of a function was not planned for. Although it did not feature in the planned lesson, learners were not provided with sufficient opportunities to make sense of the definition of a function as asserted by Teacher L. Perhaps for learners to make sense of this definition it would have been productive for them to contrast this one-to-one mapping with a many-to-one mapping so that they could see the unique association between input and output values. To increase learners' chances of discerning, Teacher L could then contrast these mappings to a many-to-many or one-to-many mapping to show what a function is not. Instead Teacher L moves into the everyday realm of the relationship between a husband and a wife to illuminate the concept of a function²⁷. The everyday example is not suitable since it begs the question of whose everyday conception of marriage is being considered here. This extract further illustrates the shifting of focus away from the critical feature and thus an unproductive use of time to provide learners with the appropriate criteria to be able to distinguish between the different classes of functions.

Plotting points and sketching the graph also featured as an observable action that consumed a substantial amount of time in this lesson. Of importance here is for me to examine the nature of the opportunities that Teacher L has provided for her learners to discern the critical feature through the

²⁷ It is interesting that Nash used the same metaphor to describe the concept of a function (refer to Chapter 1). This is indicative of the wider context where contextualising mathematics with everyday life is being emphasised.

process of plotting points and sketching graphs. A quick background picture to contextualise the extract below: Teacher L has the table of values for the function $y = 2x$ on the board. A Cartesian plane is set up on the board and Teacher L announces the x-value from the table of values and the learner chorus the corresponding y-values, at this juncture Teacher L plots the point on the Cartesian plane:

Teacher L So if we join these points (*places ruler through points on graph*), what type of a graph are we going to have?

Learners A straight line.

Teacher L A straight line

Learners Graph.

Teacher L Graph. Right. If we join these points with a line (*draws the graph*), we get a straight line graph. This is the straight line defined by the equation, y is equal to two x (*labels the line $y=2x$*). Right. In this straight line, y is equal to two x. I want us to...why do we say it is a straight line? How do we know? Is this really straight?
(*teacher pauses for a very short while*)

Teacher L We call this is a straight line, right?

Learners Yes.

Teacher L Because we can see it. The line is straight.

(Lesson 3, event 3.5, underline my emphasis)

The extract illustrates that the learners were able to identify the graph drawn as that of a linear function. Teacher L asks an important question ‘why is it a straight line?’ and this question had the potential for Teacher L to bring the critical feature into focus. Asking this question may have been perceived as an odd question to ask because in the presence of the graph one is able to see that $y = 2x$ does represent a linear function. This is exactly the response asserted by Teacher L; this further demonstrates that the critical feature was not in focus for Teacher L as she went about teaching this lesson. In this case, learners were provided with an opportunity to empirically establish that $y = 2x$ represents a linear function and the criteria for identifying it as such lies in its visual appearance. I draw your attention back to the discussion of the enacted object of learning for lesson 2 where I asked how learners who are able to discern the class of function from its graphical representation would identify and name the function $y = 2^x$ when the graph is drawn for the interval $0 \leq x \leq 0.4$, probably learners would identify it as being linear since it looks like a straight line. This demonstrates that the graphical representation may not be an appropriate criterion for being able to successfully distinguish between the different classes of functions, and so other criteria become necessary.

The observable action of *revising mathematical rules and conventions* is the last observable action in this lesson that consumed a significant amount of time. One of the intentions of the planned lesson was for the teacher to introduce two examples at a time (e.g. $y = 2x$ and $y = 2^x$ or $y = x^2$ and $y = 2^x$). The purpose behind this was for learners to compare the sameness and differences between the pair of examples and through these comparisons to look at the relationship between the '2' and the 'x' and to focus on the critical feature. In operationalising this thinking in the lesson Teacher L introduced one equation at a time and got learners to focus on the relationship between the '2' and the 'x':

- Teacher L And then can we get someone who can try and explain to us, how is this two and the x related? What is the relationship between the two and the x...that we have? In the equation...remember this is the equation, it consists of the terms...a term, the y term on the left, and the x term on the right, separated by that equal sign. So we call it an equation. It describes a particular relationship. Right, so who can tell us, what is the relationship between the two and the x? The variable x and the constant two. What must happen with the two and the x? According to this equation, how are we going to have a relationship between the...what type of relationship are we going to end up with when we are to work with two and an x?
- Learner The relationship is to multiply the x value by two in order to get the y value.
- Teacher L Correct. Well done. I hope you have heard what he has just said. Could you just repeat what you have just said?
- Learner The relationship is to multiply the x value by two to get the y value.
- Teacher L We multiply the x value...?
- Learner By two.

(Lesson 3, event 1.2)

Since one equation was introduced at a time, Teacher L could not ask learners to compare the equations. In attempting to legitimate meaning for her learners, Teacher L could only appeal to mathematical conventions. This again illustrates that the critical feature was not in focus for Teacher L and so the opportunity that she makes available in such instances is an opportunity to discern something other than the critical feature identified for this lesson.

During the planning of this lesson the importance of introducing two functions at a time so as to allow learners an opportunity to compare the sameness and differences between given pairs of functions was highlighted. The opportunity to compare would in turn facilitate the opening up of a

dimension of variation through contrast. In this case the dimension of variation is the index of the equations representing the different classes of functions which then focus on the critical feature that was identified. As already indicated Teacher L introduced the functions one at a time and so did not open this dimension of variation during the lesson.

As discussed previously another dimension of variation that was built into the lesson was the multiple representations of a function (verbal, algebraic, table of values, mapping between sets and the graph). In this lesson Teacher L focused mostly on changing representation from verbal to algebraic and then on setting up a table of values. Working across these limited representations did not provide opportunities to focus learners' attention on the critical feature, which again illustrates that the critical feature was not in focus for Teacher L as she taught this lesson.

In describing what comes to be constituted as the enacted object of learning in this lesson as seen through *reflection*, some principles of *variation theory (contrast and simultaneity)* to elaborate reflection and the *legitimizing of meaning*, the enacted object of learning could be summarised as:

- i. In relation to the observable action (reflection)
 - Substituting x-values into an equation and finding its associated y-value(s) and representing these values in a table,
 - Changing representation,
 - Plotting points and sketching graphs, and
 - Revising mathematical rules and conventions
- ii. In relation to variation theory (elaborating reflection)

No opportunities were made available for the learners to compare different classes of functions.
- iii. In relation to legitimating meaning (Authority)
 - Teacher assertion and confirmation

8.7 The post-lesson discussion – lesson 3

The discussion started with Teacher L reflecting on the lesson she had delivered. Teacher L starts by talking about the difficulty the learners had with performing basic calculations – here she refers to the second example that was introduced viz. $y = \frac{1}{2}x$:

Teacher L Yes, with the two x's. So that was good. I really appreciated that. Then we moved on to the second example, y is equal to half x. That's when it opened up y is equal two divided

by x . Then the other one came up with...ok, one over two times x . Is there any difference between these two? Looking at if we or substitute x by the value...the different x values. So fortunately they were able to see that, no, this...the answers are going to be the same. Y is equal to half of x . That's when I had to bring in the practical example of six loaves, sharing it equally amongst two children.

VP So you started off by saying, when you came to the second example that's when you realised a lot of things. What were some of the things that you realised whilst you were teaching?

Teacher L Whilst teaching I realised that...no, fractions, here learners have a problem.

VP Fractions

Teacher L Fractions, especially half of...because I had to ask them...they were able to get...(inaudible) to half of negative four correct. Some of them were able to get...they got the answer as negative half, and because those who were using the calculator got the answer as negative zero comma five. I needed to draw their attention to the fact that this negative half and negative zero comma five, is one and the same thing. But I was more interested in them (inaudible) formulas how did they get the negative? That's when I realised that I'll say the bigger number over the smaller number, but it was not that clear. Like the boy against the wall at first when he was saying that, he thought when I realised that, he thought that now the bigger number is a negative four. And the smaller number is a negative two. So I had to purposely ask him, ok, between two and negative four, which one is bigger than the other one? Now he said, no, two is bigger than negative.

(Post-lesson 3 discussion, utterances 10 to 14)

This raises the dilemma that when teaching a topic like functions, for instance, one is not teaching the section in isolation from other aspects of mathematics. In this example we see that working with fractions and negative numbers becomes an integral part of the lesson. The teacher then needs to make a decision as to what to do in terms of steering the lesson and the decision needs to be taken in the moment. In this lesson, Teacher L decided to engage with the challenges the learners experienced and this is why the observable action of substituting and calculating consumed so much of time. This does not mean that Teacher L has successfully assisted these learners to overcome the challenges that she has identified.

After the second lesson of this learning study a critical feature for enabling learners to discern between the different classes of functions given their algebraic representation emerged, viz. to focus on the highest power on the independent variable. This was the first lesson in this learning study cycle which had a critical feature in focus when planning the lesson. During the enactment of this lesson however, the critical feature was out of focus for the teacher as she explains:

- Teacher L Substitution is not the object of learning, but we had to use it as a tool to arrive at...what's that...
- VP Table of values
- Teacher L The table of values
- VP ... substitution is not the object in focus, but it is an integral part of the work that we're doing. But we don't want the substitution to hijack us from what we are doing...If we ask the kids, so what makes it a straight line? Because I think when you put up the graph of the straight line, you were saying, ok, so the points are all on the straight line, or we can see it is straight.
- Teacher L No, I just asked by what type of a graph, after joining them. They said straight line.
- VP They said straight line.
- Teacher L They said it was a straight line. I did not even...I recalled later I should have asked them why? They said that because they saw it over...on the graph. But what if it was not drawn?

(Post-lesson 3 discussion, utterances 44 to 50)

In terms of the enactment of this particular episode Teacher L does in fact ask the learners how they would know that it was a straight line, thus opening up the space to engage with the critical feature, but she does not use the opportunity productively – refer to the extract from the lesson (lesson 3, event 3.5) which appears in the previous section under the discussion of the observable action plotting points and sketching the graph. The extract from the lesson shows that Teacher L is the one who asserts that it is a straight line because one is able to see it from the graph. This demonstrates that in teaching this lesson, the critical feature was not in focus for Teacher L.

In discussing the lesson Teacher J highlighted the use of language – consider the following extracts taken from the lesson and focus on the use of the word 'into':

- Teacher L Two. And then this one says, divide eight equally into (writes $y =$) two halves, or by two. Divide eight sweets equally amongst two children. There are eight sweets on the table, there are two children, I want to share them equally amongst the two children. Each child is going to get?

(Lesson 3, event 2.2)

- Learner it's two into negative five.
- Teacher L Y is equal to...two into...
- Learner Negative five.
- Teacher L Negative five (writes $y = 2(-5) =$). And that gives us?

In the first extract ‘into’ refers to division whilst in the second extract ‘into’ refers to multiplication. So in planning for the last iteration of the lesson Teacher J suggested that when teaching we need to be cautious about the use of words and that we should be using the correct mathematical language.

Once again, the learners’ performance in the post-test was made available to the teachers at the beginning of this post-lesson discussion. However, in planning for the last lesson in the cycle the teachers were concerned with bringing the critical feature into focus during the next iteration of the lesson and did not consider the kinds of errors the learners made in the post-test²⁸. Since the critical feature was not in focus during lesson 3, the group decided not to change the lesson plan. However, emphasis was placed on ensuring that during the next lesson the teacher needs to provide learners with opportunities to experience the critical feature through the examples used and to direct the learners’ attention to it.

8.8 The enacted object of learning – lesson 4

Table 8.7 displays the re-ordering of the events as discussed in Chapter 7, Section 7.6.

Example	How was the example introduced?	Duration	Sub-notion	Observable action	Legitimizing meaning (where does authority lie?)
$y = 2x$ $y = \frac{1}{2}x$	Symbolic	1:33	Reading the given equations	Changing representation	1. Restates 2. Acknowledging correctness of learners’ response
	Symbolic	0:24	Compare the sound of the verbal utterances	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners’ response
	Symbolic	0:54	Comparing variables	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners’ response
	Symbolic	0:59	Comparing the mathematical operation between ‘x’ & its coefficient	1. Identifying sameness between equations 2. Revising mathematical	1. Mathematical conventions 2. Teacher asserts 3. Restates and writes 4. Acknowledging

²⁸ As indicated previously, because the learners’ performance in the post-test did not feature in any significant way in terms of planning for the next lesson, I will pick up on this discussion in Chapter 9.

				conventions	correctness of learners' response
	Symbolic	1:38	Comparing the exponents of 'x'	1. Identifying sameness between equations 2. Revising mathematical rules and conventions	1. Mathematical rules 2. Mathematical Conventions 3. Teacher asserts 4. Restates 5. Acknowledging correctness of learners' response
$f(x) = x + 1$	1. Symbolic 2. Graphical	2:08	Graph of a linear function	Identifying and naming a function	1. Teacher asserts 2. Acknowledging correctness of learners' response
	Symbolic	0:30	Functional Notation	Revising mathematical conventions	1. Mathematical Conventions 2. Acknowledging correctness of learners' response
$y = -x + 2$ $y = 2x + 2$ $f(x) = 3x + 16$	Verbal	1:42	Give other equations of a linear function	Learners generating examples	1. Mathematical rule 2. Teacher asserts 3. Restates and writes 4. Acknowledging correctness of learners' response
$y = \frac{1}{2}x$ $f(x) = \frac{x}{2}$	Symbolic	1:31	Reading the given equations	Changing representation	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	0:26	Compare the sound of the verbal utterances	Identifying sameness between equations	1. Teacher asserts 2. Restates
	Symbolic	2:04	Finding y-values for given x-values to establish equivalence between the two equations	1. Substituting and calculating 2. Comparing y-values for given x-values 3. Identifying sameness between equations	1. Empirical 2. Restates
	Symbolic	0:50	Comparing the given equations	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	0:30	Type of function	Identifying and naming	1. Mathematical rules

					2. Restates 3. Acknowledging correctness of learners' response
$h(x) = \frac{2}{x}$ $f(x) = \frac{x}{2}$	Symbolic	0:44	Reading the given equation	Changing representation	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	0:12	Compare the sound of the verbal utterances	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	0:55	Comparing the given equations	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners' response
$h(x) = \frac{2}{x}$	Symbolic	3:05	Rewriting the given equation	Revising mathematical rules	1. Mathematical rules 2. Teacher asserts 3. Restates and writes 4. Acknowledging correctness of learners' response
	1. Words 2. Verbal	1:40	Write in algebraic form	Changing representation	Teacher asserts
	Symbolic	3:40	Finding y-values for given x-values & completing a table of values	Learners are completing this task on their worksheet	
	Symbolic	2:53	Finding f(0)	1. Substituting and calculating 2. Revising mathematical rules	1. Mathematical rules 2. Teacher asserts
$h(x) = 2x^{-1}$	Symbolic	1:55	Type of function	1. Revising mathematical rules 2. Identifying and naming	1. Mathematical rules 2. Teacher asserts
$p(x) = \frac{-2}{x}$	1. Symbolic 2. Graphical	1:48	Graph of the hyperbola	Identifying and naming	Teacher asserts
	Symbolic	0:41	Rewriting the given equation	Revising mathematical rules	1. Mathematical rules 2. Writes 3. Acknowledging correctness of learners' response
$p(x) = x^2$ $g(x) = 2^x$	Symbolic	0:52	Reading the given equations	Changing representation	1. Restates 2. Acknowledging correctness of

					learners' response
	Symbolic	0:31	Compare the sound of the verbal utterances	Identifying sameness between equations	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	0:40	Identifying and comparing exponents	Identifying sameness between equations	Acknowledging correctness of learners' response
$p(x) = x^2$	Symbolic	1:12	Type of function	1. Revising mathematical rules 2. Identifying and naming	1. Mathematical rules 2. Teacher asserts 3. Restates 4. Acknowledging correctness of learners' response
$g(x)=x^2-2$	1. Symbolic 2. Graphical	1:53	Graph of the parabola	Identifying and naming	Teacher asserts
	Symbolic	1:23	Functional notation	1. Substituting and calculating 2. Revising mathematical rules	1. Mathematical rules 2. Pm ⁺ 3. Teacher asserts 4. Acknowledging correctness of learners' response
$h(x)=2x^2+1$ $y=x^2$ $q(x)=2^2+x$	Verbal	1:20	Give other equations of a quadratic function	Learners generating examples	1. Writes 2. Acknowledging correctness of learners' response
$q(x)=2^2+x$	Symbolic	0:25	Type of function	1. Revising mathematical rules 2. Identifying and naming	1. Mathematical rules 2. Acknowledging correctness of learners' response
$q(r)=r^2+2^2$	Symbolic	1:32	Exponent of the independent variable	Revising mathematical rules	1. Mathematical rules 2. Teacher asserts
$g(s)=s^2+3^2$	Verbal	0:27	Give other equations of a quadratic equation	1. Revising mathematical rules 2. Learners generating examples	1. Writes 2. Acknowledging correctness of learners' response
$g(x) = 2^x$	Symbolic	1:14	Type of function	Identifying and naming	1. Teacher asserts 2. Restates 3. Acknowledging correctness of learners' response
$h(x) = 2^x$	1.Symbolic 2.Graphical	1:46	Graph of the exponential function	Identifying and naming	Teacher asserts

$y=3^x+2$ $g(x)=3^x-1$	Verbal	1:15	Give other equations of an exponential function	Learners generating examples	1. Restates and writes 2. Acknowledging correctness of learners' response
$y = 2x$ $h(x) = \frac{2}{x}$ $y = \frac{1}{2}x$ $p(x) = x^2$ $p(x) = -4^x$	Symbolic	2:35	Type of function	Identifying and naming	1. Writes 2. Acknowledging correctness of learners' response
$y = 2x$	1. Words 2. Verbal	5:01	Write in algebraic form	1. Changing representation 2. Revising mathematical conventions	1. Mathematical conventions 2. Teacher asserts 3. Acknowledging correctness of learners' response
	Symbolic	0:09	Type of function	Identifying and naming	1. Restates 2. Acknowledging correctness of learners' response
	Symbolic	3:50	Finding y-values for given x-values and completing a table of values	1. Changing representation 2. Substituting and calculating 3. Revising mathematical conventions	1. Mathematical conventions 2. Pm ⁺ 3. Acknowledging correctness of learners' response
	Symbolic	0:22	Domain and Range	Revising mathematical conventions	Teacher asserts
	Symbolic	1:54	Expressing x and y values as a mapping between two sets	Changing representation	1. Mathematical conventions 2. Teacher asserts
	Symbolic	6:25	Sketching the graph	1. Revising mathematical conventions 2. Plotting points 3. Sketching graphs 4. Changing representation	1. Mathematical conventions 2. Teacher asserts 3. Restates and writes 4. Acknowledging correctness of learners' response

Table 8.7: Range of examples introduced – Lesson 4

In planning this lesson a strong emphasis was placed on ensuring that the critical feature is brought into focus. To reiterate, focusing on the highest power of the independent variable emerged as the critical feature for learners to be able to discern between different families of functions given their algebraic representation. Looking down the column of examples (refer to the table 8.7) it becomes immediately noticeable that, unlike in the previous lessons, in this lesson Teacher T introduces two examples at a time in most of the events across the lesson. The pairs of examples introduced in these events are: a) $y = 2x$ and $y = \frac{1}{2}x$; b) $y = \frac{1}{2}x$ and $f(x) = \frac{x}{2}$; c) $f(x) = \frac{x}{2}$ and $h(x) = \frac{2}{x}$; d) $p(x) = x^2$ and $g(x) = 2^x$. Introducing two functions at a time provided learners with an opportunity to compare the given equations and in so doing it provided Teacher T with an opportunity to bring the critical feature into focus for his learners. These examples are the planned examples and they were not the only examples that were introduced during this lesson. To provide a description of what comes to be constituted as the enacted object of learning in this lesson I commence by providing Table 8.8 which displays the amount of time that was spent on each of the observable actions as well as the range of appeals made by the teacher in his attempt to legitimate meaning for his learners.

Observable Action	% of time spent	Number of occurrences	Mathematics				Teacher		Experience
			Definitions/ Theorems	Empirical/ Technology	Conventions/ Rules	Process	Teacher Asserts	Teacher Confirms	Everyday
Identifying and naming a function	23.1	11			36%		64%	64%	
Changing representation	19.6	7			43%	14%	43%	86%	
Substituting and calculating	15.1	4		25%	50%	25%	50%	50%	
Identifying sameness between equations	11.1	10			20%		30%	100%	
Plotting points and sketching graphs	9.5	1			100%		100%	100%	
Revising mathematical rules and conventions	9.1	5			100%		60%	60%	
Learners generating examples	7.0	4			25%		25%	100%	

Table 8.8: Percentage of time spent per observable action and the range of appeals – Lesson 4

In each of the four instances, as listed above, where Teacher T introduced a pair of functions simultaneously he got them to compare the two equations. Since the equations in each instance were introduced algebraically, the first observable action that he got his learners to engage with was to *change the representation* from algebraic to verbal. The first comparison, and so an engagement with the observable action *identifying sameness between equations*, that Teacher T got the learners to make was to compare the sound of the verbal utterances. As already discussed under the description of this lesson, getting learners to compare the verbal utterances seduced them into saying that the two equations were not the same. This was not a productive comparison to make since in the case when the two equations $y = \frac{1}{2}x$ and $f(x) = \frac{x}{2}$ were introduced the learners indicated that they were not the same. Although they are different in terms of the symbols used to make up the equations they both represent the same function. The next level of comparison that Teacher T got his learners to make, was for them to compare the elements that constituted the algebraic representation of each of the functions, for example consider the following extracts which illustrate the learners identifying the symbols that make up the equations e.g. variables and equal sign:

- Teacher T Are they the same, these two statements, y equal to two x, and y equal to half x (*Refers to $y = 2x$ and $y = \frac{1}{2}x$*). Are they the same?
- Learners No.
- Teacher T No, they're not the same. Right. Is there anything that is the same with these two statements?
- Learner Yes.
- Teacher T What do you find to be the same?
- Learners The y's...
- Teacher T Ah, you raise up your hands, I pick on you, then we get moving. Yes?
- Learner y.
- Teacher T What you find that is common there, the term y is common for that, that y, that y. Fine. What else? Yes?
- Learner x.
- Teacher T x again is common. You'll find you've got an x here, and also you've got an x there. What else?
- Learner Equals to.
- Teacher T The equals to. Right. That's also correct. We've also got an equal sign, they're given to say they are all equations. Alright.

(Lesson 4, event 1.3)

Teacher T What else is the same with these two? What else is the same? ... Yes?

Learner y and f of x.

Teacher T y and f of x. Ok. Saying this one is just as good as f of x. Correct. What else? What else is the same? Let's try to be quick.

Learner One over two x (*inaudible*).

Teacher T Yes?

Learner One over two times x is the same as x over two.

Teacher T Right, one over two x is the same as x over two. It's only that this has been written differently, isn't it?

Learner Yes.

Teacher T Right, this has been written differently but it is...?

Learners The same.

Teacher T The same.

(Lesson 4, event 3.4)

The observable action of *comparing the sameness between the given pair of equations* starts with the learners being able to compare the symbols between the two equations that are explicit viz. the variables and the equal sign. A comparison of the implicit features embedded within the examples had to be scaffolded by the teacher e.g. the mathematical operation embedded in the given equations and the power of x in $y = 2x$ for example. The two extracts which follow illustrate this:

Teacher T Now looking at those two equations, we say this equation is explaining y. y being equal to...

Teacher T ...two times...?

Learner x.

Teacher T x. Right. That's what it means. When you write two x, it means two times x. Two x, if I say this means two times x (*on other side of the board, writes $2x = 2 \times x$*). Then how about half x (*writes $\frac{1}{2}x =$*), what does it mean? Yes?

Learner Half times x.

Teacher T Wonderful. That also means half times x (*writes $\frac{1}{2} \times x$*). So you'll find, as you have already indicated, that's something in common (*circles the unknown x in each equation*). You've got two being multiplied, and again here you've got half being multiplied by...?

Learner x.

Teacher T By x.

(Lesson4, event 1.4)

Teacher T If I can ask again, what is the exponent of x in both equations? What is the power of x in both equations? Yes, girl?

Learner One.

Teacher T If you can speak up.

Learner One.

Teacher T One. Do we all agree?

Learners Yes.

Teacher T Correct. The exponent of x here is (*points to $y = 2x$*)...?

Learners One.

Teacher T Is one. So which means, we can actually even put a one here (*writes $= 2x^1$*), but we don't usually write it, ok? What is the exponent of x in this other one? (*door bangs*) Sorry for that. What was the exponent of x in the second one? Yes?

Learner One.

(Lesson 4, event 1.5)

The comparisons in each instance were limited to that which was the same between the two equations and not to the differences between the equations. By engaging in the observable action of determining the sameness between the given pairs of equations, Teacher T has opened up the power of the variable x as a dimension of variation and thus provides his learners with some criteria for being able to identify the class of function given its algebraic representation.

After identifying the power of x in the pairs of equations that were introduced, Teacher T had to assert in a sense a rule so that the learners could make some sense in relating the power of ' x ' to the class of function being represented. For instance consider the following extracts to exemplify the point and refer to the underlined text:

Teacher T It's one again (*writes $y = \frac{1}{2}x^1$*). Now, you'd find that this is what is the same with these two equations. That the exponent of x , the exponent of this variable, x is...?

Learners One.

Teacher T Is one. And each time we have an equation where the exponent of the variable is one, we describe that type of equation as a linear equation. In the language of functions, we then call it a linear function. Let me write linear (*writes linear*), which is one type of function that I'm going to talk about. We call it a linear function. (*inaudible*) I've already made some presentation.

(Lesson 4, event 1.5)

- Teacher T Ok, if you divide it, it will still have the exponent of one. That is his argument. How about the others? What do you say? But what is the exponent of x here (*points to $2x^{-1}$*)?
- Learners Negative one
- Teacher T What is the exponent of x ? Yes?
- Learner Negative one.
- Teacher T It's negative one. And now, that brings us to another type of function. Where we are saying each time we raise a variable to a negative exponent, we have what we call an inverse (writes inverse)...we have what we call an inverse function, or we call it a hyper...(writes hyperbola)
- Learner Hyperbola.
- Teacher T A hyperbola. So we can talk of it as an inverse function or a...hyperbola²⁹.
- (Lesson 4, event 4.5)

Throughout this lesson the observable action of *identifying and naming a function* was restricted to identifying and naming a function given its algebraic representation. This is a result of what was emphasised in the lesson i.e. to focus on the highest power of the variable ' x ' and therefore an emphasis on the algebraic representation of functions. Having asserted the different rules as described above provided learners with the appropriate criteria to be able to identify and name a function given its algebraic representation. In one event Teacher T wrote the equation of a function and learners were required to identify and name the function:

- Teacher T But let us go through very quick exercise here (*cleans a portion of the chalkboard*). Where we will be identifying all the given functions. What type of a function they are. Right, I'll give you (*writes $y=2x$*)...what type of a function is this? y equal to two x . yes?
- Learner Linear.
- Teacher T That's linear. Linear, I'll write L, alright? (*writes L*)
- Learners Yes.
- Teacher T Then we have another one, h of x equal to two over x (*writes $h(x) = \frac{2}{x}$*). What type of a function is this? You said this one is linear (*puts brackets around L that was used to categorise $y = 2x$*), we all agree, yes?
- Learner Hyperbola.
- Teacher T Hyperbola. Do you all agree?
- Learners Yes.

²⁹ It is important to note that in Teacher T's explanation he confuses the concept of reciprocal and inverse – $\frac{1}{x}$ can be written as x^{-1} and this is the reciprocal of x because when the two are multiplied the product will be one, whereas the inverse of $y = \frac{2}{x}$ is $x = \frac{2}{y}$. This incorrect use of the term inverse was discussed during the post-lesson discussion (refer to post-lesson 4 discussion, utterances 7 to 18).

- Teacher T That's a hyperbola (*writes H in brackets*). This one is a hyperbola, H. Then we have another one, y equal to half x (*writes $y = \frac{1}{2}x$*). What type of a function is this? Yes, boy?
- Learner Linear.
- Teacher T Correct. That's linear (*writes L in brackets*). That's good. We have p of x equal to x squared (*writes $p(x) = x^2$*). What type of a function? Let's all take part, so that at least we've gone through all this, right? Yes, my girl?
- Learner Quadratic.
- Teacher T Quadratic, correct. Or?
- Learners Parabola.
- Teacher T Parabola. Right. So we write
- Learner P.
- Teacher T Let's write p (*writes P in brackets*). Parabola. And we have another one. Let me write the last one down. P of x equal to negative four the exponent x (*writes $p(x) = -4^x$*). What type of a function is it? Yes.
- Learner Exponential.
- Teacher T That's also exponential (*writes E in brackets*). And that is quite correct. That's wonderful. By the end of this lesson I'm going to give you four functions, and when you go home you can actually go and try out. Will you take one (*teacher hands out worksheets*). Just take one and pass back. Take one and pass back. Right, don't write anything on those worksheets until I explain about how we should work through them, alright.

(Lesson 4, event 12)

One might argue that the examples used in this event are the same examples that Teacher T had introduced previously in the lesson and so the learners were familiar with the examples. This familiarity facilitated in them being able to identify and name the functions correctly. This is a possibility which is not disputed, what is important to note in this extract is that there is no guessing of answers and so the mere shouting out of inappropriate answers is absent from the learners' responses. This could suggest that learners have some criteria that they are using to successfully identify and name the given equations.

Changing representation as an observable action in this lesson comprises learners being asked to read the algebraic equations that the teacher introduces by writing them on the board. In the process of reading the learners then change the representation from written to verbal. This observable action is also related to instances in the lesson where the teacher introduces the function by means of words and in order to engage with the function the first step is for learners to change the representation to algebraic. This occurs in instances where Teacher T decides to follow the

worksheet that was used in the previous lesson as well. In following the sequence of the task as set out in the worksheet, Teacher T also takes learners through the process of finding y-values for given x-values and completing a table of values for the function $y = 2x$. Thereafter, Teacher T demonstrates how to represent these values as a mapping between sets and finally the sketching of the graph. Going through the process of changing representation from algebraic to graphical was not the focus of this lesson. It was a once-off occurrence towards the end of the lesson with the teacher's intention of orientating his learners to what was required of them to complete the worksheet. By orientating the learners in terms of what was expected of them when completing the worksheet, Teacher T also engaged in the observable actions of *plotting points and sketching graphs*. Once again these observable actions were not given prominence in this lesson.

The observable action of *substituting and calculating* was engaged in for two purposes. Firstly, since completing a table of values was one of the aspects required for the completion of the worksheet, a necessary process to engage with was substituting and calculating. As already indicated, this happened only once in the lesson and it was for purposes for orientating learners with respect to the completion of the worksheet. Secondly, this observable action was again called into play when Teacher T provided an empirical justification to show that $y = \frac{1}{2}x$ and $f(x) = \frac{x}{2}$ yield the same y-values for given x-values. The observable action of substituting and calculating was another observable action that was not given prominence in this lesson.

Learners generating examples of equations of a particular function was another observable action that characterised this lesson. The space of learning that was opened when Teacher T engaged with this type of activity and the potential learning opportunities that prevailed was discussed in Chapter 7, Section 7.6 which dealt with the description of the lesson. This observable action only contributed to 7.02% of the teaching time and so was not given much prominence in the lesson.

Having described the range of observable actions that prevailed across this lesson is insufficient to provide a description of what came to be constituted as the enacted object of learning. This description needs to be considered in relation to the nature of appeals in Teacher T's attempt to legitimate some form of meaning for his learners. Referring to Table 8.8, which displays the range of appeals, it is clear that the appeals are clustered around teacher assertions and confirmations and the appeal to rules and conventions in mathematics. The appeal to mathematical rules and conventions was for purposes of reminding learners about previously learnt mathematics as was described above. As described previously, Teacher T introduced two examples simultaneously so as to enable learners to compare the equations. The purpose was to provide learners with an

opportunity to identify sameness and differences in the index of the input variable of the given equations and in so doing to focus on the critical feature. Having identified the index of the input variable meant that the learners identified an element between the two equations that were either the same or different. To legitimate some form of meaning for the learners in terms of their ability to identify and name a function given its algebraic representation, Teacher T asserted rules. For example if the index is one then it is a linear function and if the index is two then the equation represents a quadratic function and so on.

In this lesson Teacher T managed to open the dimension of variation related to the index of the input variable. In so doing, Teacher T provided his learners with suitable criteria for being able to identify and name a function given its algebraic representation. What is important to note at this juncture is that the space of learning that was opened up in this lesson provided learners with opportunities to discern the critical feature. This does not necessarily translate into them actually discerning the critical feature. As discussed previously, another dimension of variation that was built into the lesson was the multiple representations of a function (verbal, algebraic, table of values, mapping between sets, and the graph). In this lesson Teacher T did not focus much on this dimension as it was limited to only one instance towards the latter part of the lesson. Although the learners were provided with criteria to identify and name a function given its algebraic representation, the only link to its graphical representation was through the pre-drawn graphs that the teacher placed on the board. With regard to this there was a mismatch between the functions engaged with algebraically and the function represented graphically. Prior to the extract which follows Teacher T got learners to compare the functions defined by $y = 2x$ and $y = \frac{1}{2}x$, but when he decides to show the learners what a linear function looks like he presents them with the graph of $f(x) = x + 1$.

Teacher T How does the graph of the linear function look like? Right, (inaudible) (*puts prestik on the laminated Cartesian plane on which the graph $f(x) = x + 1$ is pre drawn*) let me put it here (*sticks the Cartesian plane on the chalkboard*). From time to time we will be looking at this. Right, that should deal with that. There was supposed to be an L here (*rewrites the letter L for linear got erased – this is the title of Cartesian plane stuck to chalkboard*), I think basically everything is still there. Right, so we are saying, this is how the graph of a linear function looks like. We have of course, this blue graph there (*points to the line $f(x) = x + 1$*), which is f of x equal to x plus one. Can you see that the exponent of x here is still one. No wonder why we call it a linear

Learner Function.

- Teacher T Function. Alright. Right, and of course what type of line, or what type of graph, if you can further describe, that we are having there?
- Learner A straight line.
- Teacher T That's wonderful. We have got a...straight line graph. Saying it's f of x equal to x plus one (*writes $f(x) = x + 1$*).

(Lesson 4, event 2.1)

In describing what comes to be constituted as the enacted object of learning in this lesson as seen through *reflection*, some principles of *variation theory (contrast and simultaneity)* to elaborate reflection and the *legitimizing of meaning*, the enacted object of learning could be summarised as:

- i. In relation to the observable action (reflection)
 - Identifying sameness between equations
 - Identifying and naming a function
 - Changing representation
 - Substituting x -values into an equation and finding its associated y -value and representing these values in a table
 - Plotting points and drawing graphs
 - Learners generating other examples of equations defining specific classes of functions.
- ii. In relation to variation theory (elaborating reflection)
 - Comparing sameness and differences between the algebraic representations of functions.
- iii. In relation to legitimating meaning (Authority)
 - Enforcement of mathematical rules and conventions to be followed
 - Teacher assertion and confirmation.

8.9 The post-lesson discussion – lesson 4

We started off this session by looking at the learners' performance in the post-test, in Chapter 9 I discuss their performance in this test as I describe the lived object of learning. Once again to commence the discussion I asked Teacher T to reflect on the lesson he taught. In reflecting on his lesson Teacher T indicated that he tried to incorporate the ideas of the planned lesson in terms of focusing on critical feature, thus providing learners with opportunities to discern the object of learning. At the same time, he was also trying to focus on aspects of functions that were covered in the post-test, for example being able to find y -values for specific x -values and being able to sketch a graph:

Teacher T Yes, looking at the lesson that we have just been to, of course I tried my best to prepare, looking at the previous reflections, I was trying to address as much as possible, but all the same I also noted I couldn't go as far as I was expected in terms of reflecting on the post-test that I later gave. You find it's important even at normal day-to-day teaching, that after the teaching process we should reflect on what the learners would need in the examination, so there was need for me to do even more than just that one example that would assist learners in going through the post-test. But I only know I tried my best to look at the object of learning and follow the agreed lesson plan, and of course do some extras.

(Post-lesson 4 discussion, utterance 4)

Teacher T's comment sheds more light in terms of him providing a reason for the once-off occurrence of taking his learners through the process of sketching a graph. His comment also illustrates that there was too much to focus on in the lesson and that the lesson should have not covered such a wide area within the topic of functions. This was a concern that was raised when the first lesson was planned, as well as in the previous post-lesson discussions, but because the previous lesson did not go as planned it was decided that not many changes should be made in terms of the content to be covered. However, in planning the first lesson, and notwithstanding the poor test results, there was a sense that learners were not seeing functions for the first time and thus varying the class and the representations could take place together. Teacher T's comment focuses the spotlight on the learning study process itself and shows that as one gets clearer on what to focus on in the lesson (*critical feature*) it begins to stand in tension with content coverage.

The space of learning that was opened up when getting learners to generate other examples of quadratic equations as described under the description of this lesson stood out for the teachers:

Teacher T Yes, and I found this class to be a very good class. Like the question that girl asked me. I was preparing, I never thought of it, just because of course I was also guided by what we set down in that grade. But that question shows it's a question coming from a bright learner, to say

Teacher L x to the exponent one, is it always linear?

Teacher S x to the?

Teacher L y is equal to x ...if it has to be linear...for an equation to be linear, is the x ...should the x always be to the exponent one?

Teacher T Yes, and this girl, she also asked to say, should you always have x ...is it always the case that all these functions should always say, y equal to x or $f(x)$ equal to x ? Then I realised, no, we don't restrict ourselves only to x and y . You wonder why we use $g(x)$, you can also talk

- of $g(r)$. Then it means this is a function, if r then you should have an r there. Then the other one who gave me, was it $h(s)$. Then s equal to r and I said, oh these numbers are
- Teacher L She is the same like that, because you said, ok, give me another example.
- Teacher S But when she asked that question, I thought maybe she meant, is it always the x that tends to be squared?
- Teacher L No, to the exponent one.
- Teacher T Yes, but is it always the x that has to be squared or it's any other letter.
- Teacher S Yes, because now (they said?), the other learner, I don't know who, y equals to two to the power of two plus x . The two was squared there, is that also a parabola? So this question followed after that one.
- Teacher L Yes.
- Teacher S So I thought maybe this girl wasn't sure what should be squared.
- Teacher T Is the number or the variable?
- Teacher S The number or the variable? So I thought that's what she was asking.
- Teacher T It was an intelligent question.
- VP Remember this learner asked this question because of what the class was asked to do.

(Post-lesson 4 discussion, utterances 29 to 44)

This discussion between the teachers highlights Teacher T's misinterpretation of the learner's question: 'must it always have an x to an exponent two?' – this issue was discussed under the description of the lesson and so I do not engage in further discussion here. Secondly, the discussion between the teachers highlights that it was only possible for the learner to ask this question because of the space of learning that was opened up at that particular point in time. More specifically, it was due to learners providing examples of quadratic equations that this opportunity presented itself. The discussion further illustrates that during a lesson opportunities that are critical for learners' learning may present themselves but the opportunity is dependent on the teacher's ability to realise it. Realising such opportunities and using them for the benefit of the learners implies that the teacher needs to make the appropriate moves instantaneously as the opportunity emerges. In the case of Teacher T we see that such moves by the teacher are not easy especially as this is all happening in real time.

In reflecting about this lesson I could not resist expressing my wish of having to start the learning study cycle with this lesson as the first lesson:

- VP I was thinking, it would have been so nice if this was the first lesson. I really wish that this lesson that Teacher T taught was the very first lesson.
- Teacher J Why?

- Teacher L So that we could see what follows and see how the next lesson develops.
- VP Yes, that precisely. And the second thing is, you see, we didn't know...after Teacher S lesson, we didn't know what we knew after Teacher S's lesson, before Teacher J started teaching. We never knew that. I don't know if I'm making sense.
- Teacher T Ok, yes, to say after Teacher S's lesson, we all went through the process, right? So after Teacher S had taught her lesson, what we knew then, after Teacher S had taught the lesson, what we knew at that point in time, we never knew prior to Teacher J teaching.
- VP Because after Teacher S's lesson, using Teacher S and Teacher J's lesson, we decided to change the lesson and focus because we said...so we're saying this is linear, this is quadratic, but what is making it linear, what is making it quadratic? If we had come to that at the very beginning, if we had that in focus, it would have been nice to have your lesson, what you did today, first, and to see some of the things that we're trying to tease apart, to see how would it then play out in lesson two, three, four, and see what would the end product would be. But nonetheless, I think we would not have been able to come to where we came today if it wasn't through the lesson that Teacher J started teaching, and that Teacher S took forward, and then Teacher L went on to do, and up till today's lesson.

(Post-lesson 4 discussion, utterances 147 to 152)

The critical feature which emerged from this learning study seemed so trivial and was even overlooked when planning the first and second lessons of this learning study cycle. Although the critical feature that emerged may seem trivial, it was central for this group of learners to be able to identify and name a function given its algebraic representation. What this reflection alludes to is that in order for us, the participants of this learning study, to have started the cycle where we wished to have started, we needed to have a fuller understanding of the object of learning. This does not mean that the participants of this learning study have a lack of understanding of the topic of functions as dealt with in the school curriculum. All participants are successfully able to distinguish between the different classes of functions across the various representations, which implies that they are able to focus on various features of a function all at the same time and so some level of fusion exists. To have been able to identify the critical feature at the beginning of the cycle we were required to 'suspend our natural attitude' and 'analyse the object of learning carefully' (Lo, 2012). Having made this statement it is also important to note that the critical feature could not only come from the mathematics alone, it emerges in relation to the learners being taught (Runesson, 2006). Factoring these two comments retrospectively into this learning study could have meant that the critical feature could have emerged much sooner than what it did and so the possibility of commencing this learning study with this critical feature in focus.

8.10 Summary

In this chapter I focused on answering the question: what came to be constituted as the enacted object of learning in each of the lessons in the learning study cycle and what emerged as the critical feature? In describing the enacted object of learning I focused on three aspects: i) the observable actions that the teacher and learners engaged with during the lesson (reflection); ii) to elaborate on the description of reflection I drew on constructs from variation theory (comparing sameness and differences, contrast and simultaneity); and iii) how meaning was legitimated.

This learning study, as with all other learning studies, was implemented without having identified a critical feature for the object of learning to start with. As a result in lesson 1 Teacher J had no critical feature to focus on. Instead he followed some aspects of the collaboratively planned lesson and focused on the algebraic representation of the different classes of function by engaging in the observable action of substitution. This is what dominated in lesson 1. The graphical representation of the functions did feature in the planned lesson, but was not really focused on in lesson 1.

The absence of attention to the graphical representation of the different classes of functions marked the shift for what was to be the focus in lesson 2. Teacher S started the lesson by emphasising the graphical representation of the various classes of functions. At the end of lesson 2 the critical feature emerged and with the emergence of the critical feature there was also a refining of the object of learning. The object of learning shifted from being able to classify different classes of functions across their various representations to being able to classify the different classes of functions given their algebraic representation. The critical feature for this object of learning was to focus learners' attention on the highest power of the independent variable.

In planning for lesson 3 emphasis was placed on the refined object of learning and its critical feature. The object of learning and its critical feature was not in focus for Teacher L. In the enactment of lesson 3, Teacher L focused on the algebraic representation with an emphasis on the observable action of substitution. In the enactment of lesson 4, Teacher T managed to zoom in on the object of learning and its critical features.

Across all four lessons we see that teacher assertions and confirmation form the predominant means for the legitimation of meaning. This illustrates that teacher authority resides in the lessons and although this is the case there are still opportunities for some discernment to take place.

Using the degree of an equation as the criteria by which to identify and name the class of function being represented by the equation emerged as the critical feature in this learning study. One may argue that this critical feature is a visual cue which is insufficient since it is not grounded in mathematics, therefore the critical feature may be inadequate for the learners' understanding of functions and may lead to learners developing partially formed ideas and possibly what literature refers to as 'prototypical' thinking (Tall & Bakar, 1992). Although what emerged as the critical feature in this study may be inadequate for learners developing a deeper understanding, it is nevertheless crucial for them, since it provides them with some resources with which to go forward in terms of their learning of mathematics and if they do not even have this level of knowledge, it is more of a problem. Of course, if all they stay with is this level of thinking it is a problem.

In the next chapter I engage in describing what the participants have gained by participating in this study and so what comes to be the lived object of learning.

Chapter 9

The Lived Object of Learning

9.1 Introduction

In any learning study there are two groups of participants (excluding the researcher) i.e. the teachers and the learners. In addition, the eventual aim of any learning study is for teachers to find ways of providing learners with opportunities to discern an object of learning. Therefore, it is the learners' learning that is in focus during a learning study.

This chapter focuses on answering the question – how does participation in the learning study impact on participants learning? In answering this question I will focus on:

- The *learners' learning*, which consumes bulk of the discussion in this chapter. The learners' learning is measured through: i) their performance in the post-test which was conducted at the end of each lesson, and ii) their performance in the delayed post-test which was conducted about a month after the completion of the learning study cycle.
- The *teachers' learning*, which is measured through their responses to questions that they answered in a pre- and post-learning study questionnaire. The yardstick for measurement in this instance is to look at shifts in their thinking about: i) the use of examples in their practice; ii) the considerations they would make when planning a lesson in future; and iii) their learning about the function concept.

9.2 The post-test

For a description of the post-test refer to Chapter 6: typically when we talk about a post-test we compare it with a pre-test. The issue here is that at the level of learner error and the extent of the error in what was considered to be an appropriate pre-test renders it not that useful. Therefore I look at the post-test in and of itself in terms of what learners were able to do. In Chapter 8, I described what came to be constituted as the enacted object of learning in each lesson. In this description, I indicated that across each of the post-lesson discussions the learners' performance in the post-test did not feature in any significant way, that is, in terms of informing the teachers how the learners experienced what is to be learned and so impacting on the planning for the next lesson. It is through an analysis of the lesson and the learners' performance in the post-test that one is able to explore what is 'critical' for the learners' learning. As indicated in the post-lesson discussions, the changes the teachers made to the lesson for its next iteration was based on the difficulties the learners had in

responding to questions during the lesson as well as on what they felt was lacking in the lesson just taught. In this section I discuss the post-test in relation to learners' performance.

Across all the lessons taught in this learning study cycle there was no emphasis placed on the concept of functional notation. The ideas of functional notation that emerged across the lessons were relegated to merely ' $f(x)$ ' or ' $g(x)$ ' or ' $h(x)$ ' etc. as being the same as ' y '. The concept of an asymptote did not feature in any of the planned lessons nor did it feature in any of the lessons taught. Reflecting on the range of questions posed in the post-test (refer to Figure 6.8), in particular questions 1.2, 1.3, 1.4, 2.2 and 2.3, one is able to see that there is a disconnectedness between what was taught and emphasised in the lessons and what was tested. This is an issue that was also raised by Teacher J during the post-lesson discussion after the last lesson in the cycle:

Teacher J ... there are certain things that they [learners] will see in the exam. And when we teach we try to sneak those things in so that they see them. Ok, this is it. For instance, the test, I think they are asked the domain there, and range. And from what my marking they said, we don't know what is domain. We have never done the range and domain. You see that? Now to them they are saying, but these people [teachers], they teach but surely they did not teach us these things but they are testing us. Then they begin to raise concerns and say, but they always test us on something they did not teach us this. And we need to give them so that we can now test them.

VP So what I'm thinking is, in the delayed post-test...we're going to do a delayed post-test with these learners. [...] include items specifically that focus on the child's ability to identify a class or function.

[Post-lesson 4 discussion, utterances 141 to 142]

Due to the dissonance between what was emphasised during each of the lessons and what was tested, as discussed above, it was on the learning study team's agenda to set a delayed post-test which would focus on the aspects that were emphasised during each of the lessons. But first I present the learner performance on the post-test and display their performance using bar charts and sequence these bar charts in the order of the lessons taught during the learning study cycle. It is also important to note that test was marked by coding the learner's response. The codes that were used:

Right – the response provided was correct

Wrong – the response provided was incorrect

- Partial – the response provided was partly correct, where partly is quantified to be a response that is 50% or more correct but not 100% correct e.g. where learners are to find $f(-2)$ and $g(-2)$ but only correctly find $f(-2)$.
- Missing – The learner made no attempt to answer the question.

In each of the graphs presented, the horizontal axis displays the question number and the vertical axis displays the learners' performance as a percentage. The label of the vertical axis also indicates the number of learners in each of the classes who wrote the test.

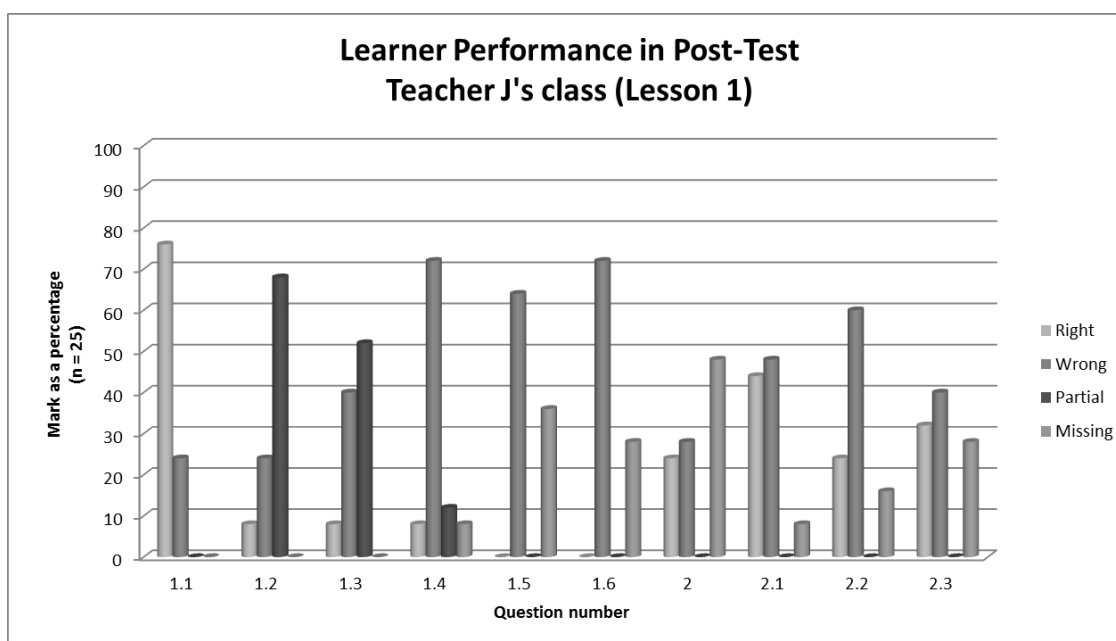


Figure 9.1: Learner performance in post-test – Lesson 1

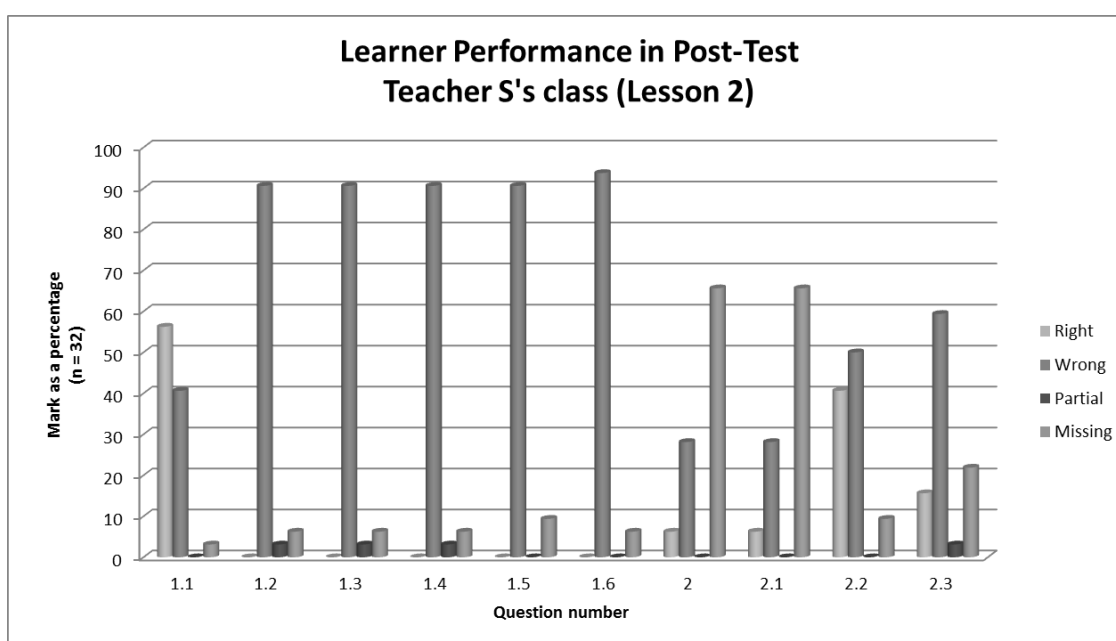


Figure 9.2: Learner performance in post-test – Lesson 2

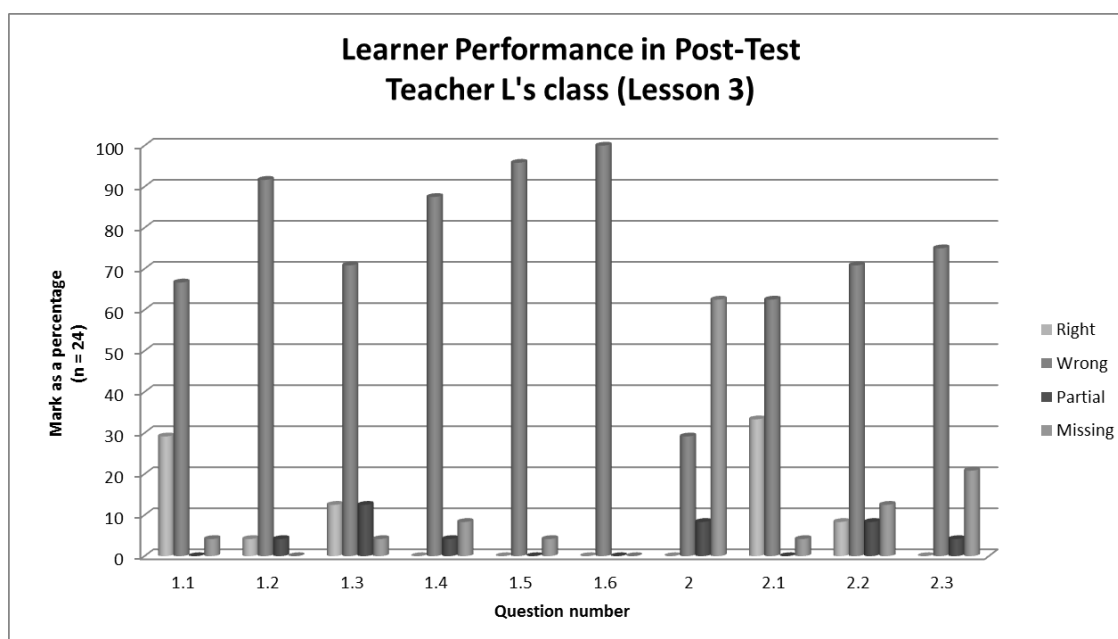


Figure 9.3: Learner performance in post-test – Lesson 3

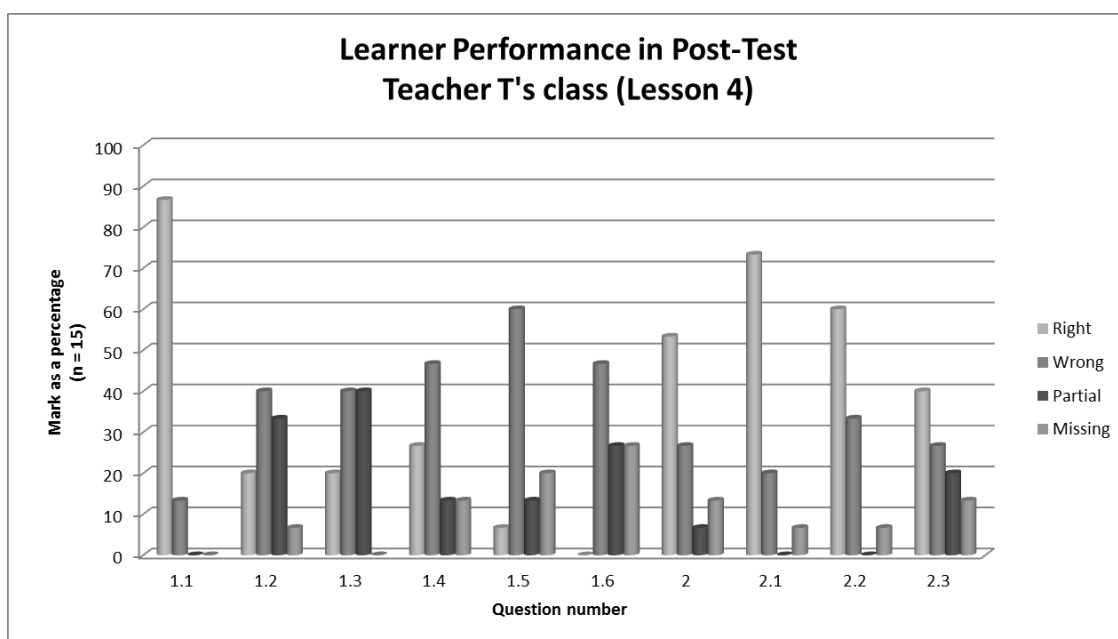


Figure 9.4: Learner performance in post-test – Lesson 4

Before commenting on the learners' performance as represented on the bar charts above, I reiterate the rationale for not engaging in their performance across all the questions in the post-test. In commencing the journey of this learning study, members of the learning study group identified the intended object of learning to be:

- To enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential functions across their different representations viz. verbal, equation (symbolic), table of values or sets, and graphical.

- To relate the varying equation forms to the different relationship between input and output for each function (the simultaneity).
- To read, interpret and use functional notation and evaluate a function at a value in its domain.

Based on this intended object of learning, lesson 1 was planned and taught. Using the same intended object of learning, lesson 2 was planned and taught. The reason for not amending the intended object of learning was discussed in Chapter 8 which deals with the description of the enacted object of learning. After the second lesson, the critical feature emerged and the focus for lessons 3 and 4 was to zoom onto identifying and naming a class of function given its algebraic representation – a refinement of the intended object of learning. The ability for learners to differentiate between the different families of functions is the common capability that characterises the intended object of learning at the beginning of the learning study cycle and its revision after the second lesson. In view of this, in comparing learner performance across all the lessons I start by focusing on their performance in question numbers 1.1 and 2.1 as these questions are directly related to learners being able to identify and name the function that is given. In both these questions the algebraic representation is provided and in question 1 the graphical representation is also provided. Thereafter, I look at learner performance in questions 1.2, 1.3 and 2.2 since these questions have the element of substitution embedded in them although learners were required to answer questions 1.2 and 1.3 by merely reading the values off the graph. The same is true for question 2.2 provided that they had sketched the graph correctly. But irrespective of reading the values off the graph, an alternate way of determining the values is through the process of substitution and calculating. The learners' performance on these particular questions is displayed in Table 9.1 as a percentage:

Lesson	Question 1.1				Question 2.1				Question 1.2				Question 1.3				Question 2.2			
	Right	Partial	Missing	Wrong	Right	Partial	Missing	Wrong	Right	Partial	Missing	Wrong	Right	Partial	Missing	Wrong	Right	Partial	Missing	Wrong
1	76	0	0	24	44	0	8	48	8	68	0	24	8	52	0	40	24	0	16	60
2	56	0	3	41	6	0	66	28	0	3	6	91	0	3	6	91	41	0	9	50
3	29	0	4	67	33	0	4	63	4	4	0	92	13	13	4	70	8	8	13	71
4	87	0	0	13	73	0	7	20	20	33	7	40	20	40	0	40	60	0	7	33

Table 9.1: Learner performance in post-test for selected items

From the table 9.1 and the bar charts (Figures 9.1 to 9.4) it is clear that with regard to questions 1.1 and 2.1, the learners from lesson 4 performed better than the learners from the other lessons. Remember, in these two questions the learners were required to identify the class of function being

represented and in lesson 4 the critical feature for being able to do so when the function was given in its algebraic form was in focus.

The learner performance in questions 1.2, 1.3 and 2.2 came as a surprise. As discussed earlier, substituting and calculating is an approach that could be employed to answer these questions although it is not the most efficient approach given the context of these questions. Across all four lessons in this learning study the learners were provided with opportunities to experience functional notation as being the same as y and so the signal for substitution. Since substitution was emphasised in lessons 1 and 3 one would have expected that the learners from these lessons would perform much better in questions 1.2, 1.3 and 2.2. However, this was not the case as is evidenced by the percentage of learners who got these items correct. Perhaps a reason for this is that the only criteria operating in lessons 1 and 3 in terms of what justifies substitution is at the level of teacher confirmation.

Looking at both the bar charts and at Table 9.1 one is able to conclude that the learners in lesson 4 performed better in the post-test than the learners from the other lessons that constituted this learning study. These results demonstrate that when the critical feature is in focus for a teacher, the teacher is able to provide learners with opportunities to reflect on the critical feature through the examples introduced. Through this process of reflection, aspects of which depend on simultaneity and contrast, the teacher will provide learners with the criteria by which to recognise the critical feature and this in turn will lead to the discernment of the object of learning by more learners. In the next section I present the learners' performance in the delayed post-test.

9.3 The delayed post-test

In the previous section, I engaged in a discussion of learners' performance in the post-test on its own and provided a reason for not comparing it to their performance in what was considered to be the pre-test. In this section I discuss learners' performance in the delayed post-test in terms of what they were able to do, because the end result is about whether they are able to do what we wanted them to do.

The last lesson of this learning study cycle was taught on the 19th of September 2011. At the end of September, the schools closed for the third quarter vacations that lasted for about a week. The delayed post-test was only written towards the end of October into the first week of November. The delayed post-test across the four classes that participated in this study was not written on the same day as teachers had to identify dates and times that were convenient for them so as not to cause any

disruptions to their work programme. I sketch this background to illustrate the time lapse between the post-test, which was conducted at the end of each lesson, and the delayed post-test.

For the purpose of providing a rationale for the structure and nature of the delayed post-test, I now briefly recap what was focused on in each of the lessons. Remember, the intention of this learning study was to improve the learners' ability to distinguish between the different classes of functions. In the first lesson, Teacher J tried to achieve this by getting learners to identify and name a function given its algebraic representation through the process of substitution. In the second lesson, Teacher S tried to achieve the intended object of learning by providing learners with a graphical representation of an example of each class of function. In the third lesson, Teacher L tried to achieve the intended object of learning by focusing on substitution. In the fourth lesson, Teacher T tried to achieve the intended object of learning by focusing on the highest power of the input variable, thus getting learners to identify and name a function from its algebraic representation. Since there was a different focus in each of the lessons, in designing the delayed post-test my focus was on setting questions that specifically required learners to identify and name a function. In relation to this intention, I provided learners with the graphical representation of a range of functions as well as the related equations. Learners were then required to match the algebraic and graphical representations and to name the type of graph that was being represented. So essentially given the algebraic representation: i) match it with an appropriate graph, and ii) name the type of function (refer to Figure 9.5).

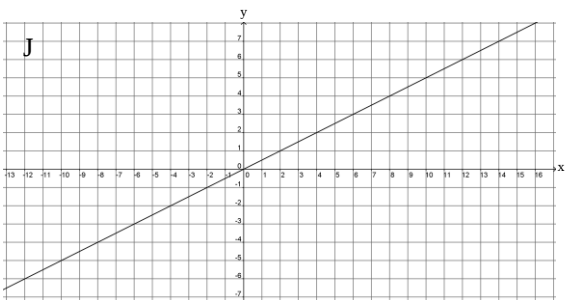
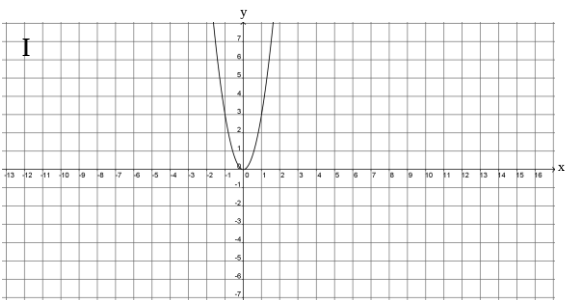
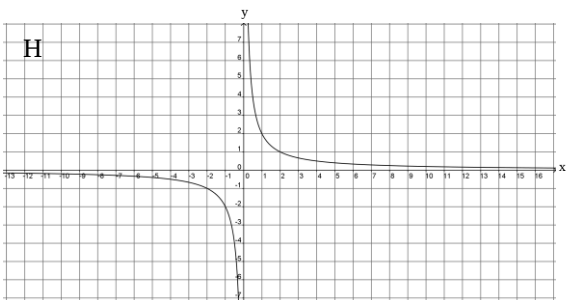
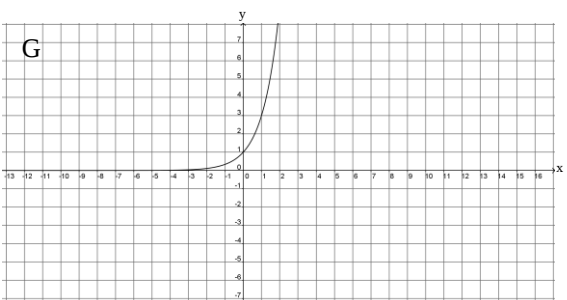
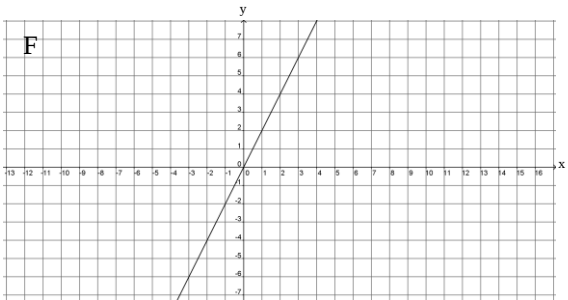
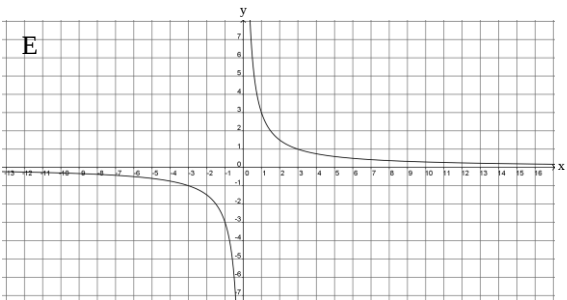
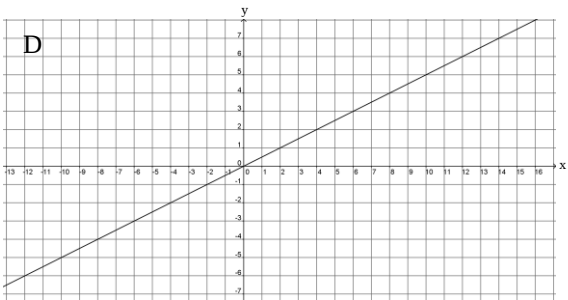
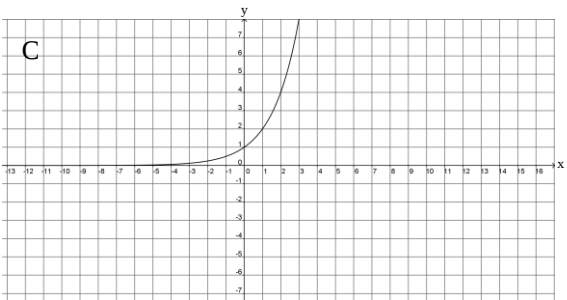
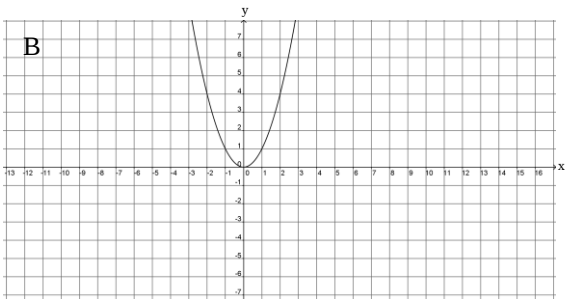
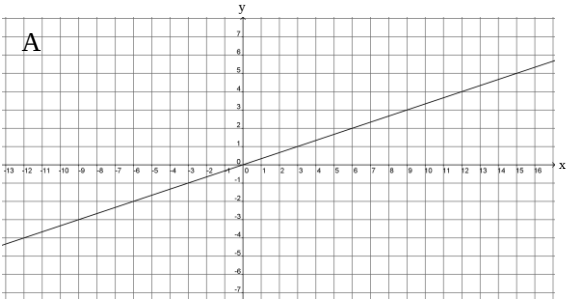
Delayed Post-Test

Name: _____

Grade: 10 _____ Date: _____

School: _____

Duration: 30 minutes



On page 1 you are given 10 graphs, they are labelled A, B, C, D, E, F, G, H, I & J. The equation for each graph is given in the table below. Match each graph to an equation and write down only the letter of your choice in the middle column which is titled 'Graph'. In the last column you are required to write down the type of graph – is it a parabola or straight line or hyperbola or exponential?

The first one has been done as an example.

No.	<u>Equation</u>	<u>Graph</u> - Only write down the letter of your choice	<u>Type</u> of Graph
1	$y = 2x$	F	Straight line
2	$y = 2^x$		
3	$y = x^2$		
4	$y = \frac{2}{x}$		
5	$y = \frac{x}{2}$		
6	$y = \frac{1}{2}x$		
7	$y = \frac{x}{3}$		
8	$y = \frac{3}{x}$		
9	$y = 3x^2$		
10	$y = 3^x$		

Figure 9.5: Delayed Post-Test

This test was marked using the codes right, wrong and missing, and the results are shown as bar charts in Figures 9.6 to 9.13. I first present the learners performance on their ability to match the algebraic and graphical representations and then their performance on their ability to identify and name a function given its algebraic and/or graphical representation.

1. Learner performance – matching equation and graph:

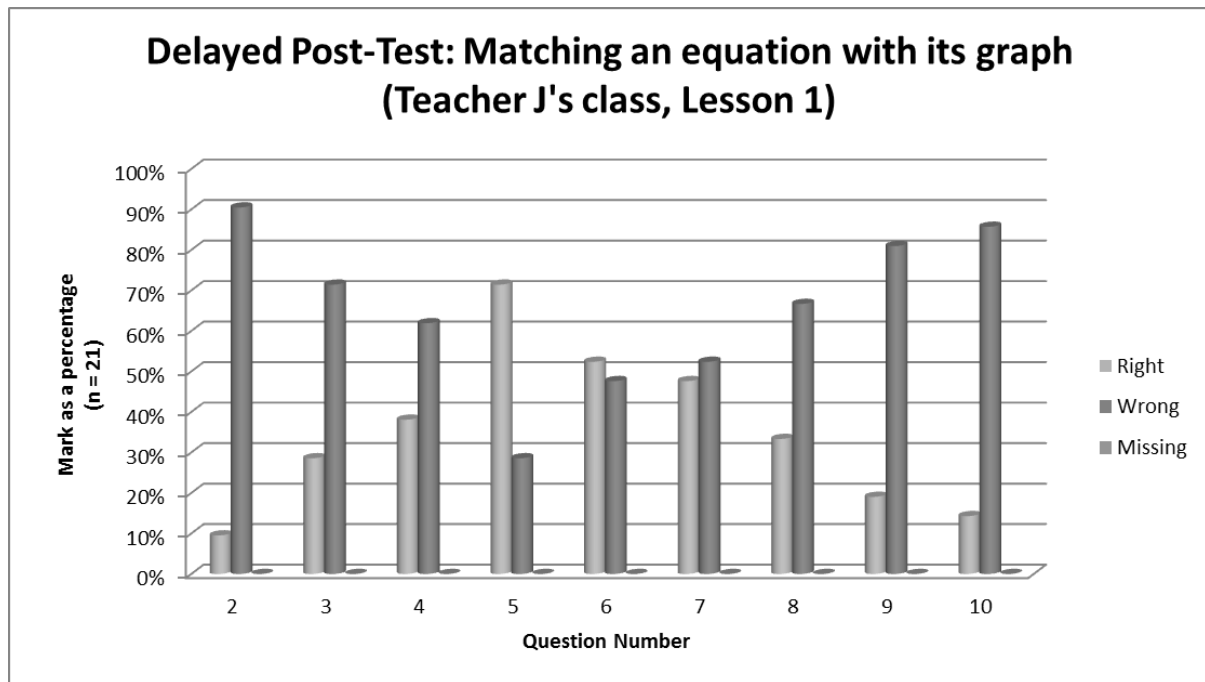


Figure 9.6: Delayed Post-Test: Matching equation and graph – Lesson 1

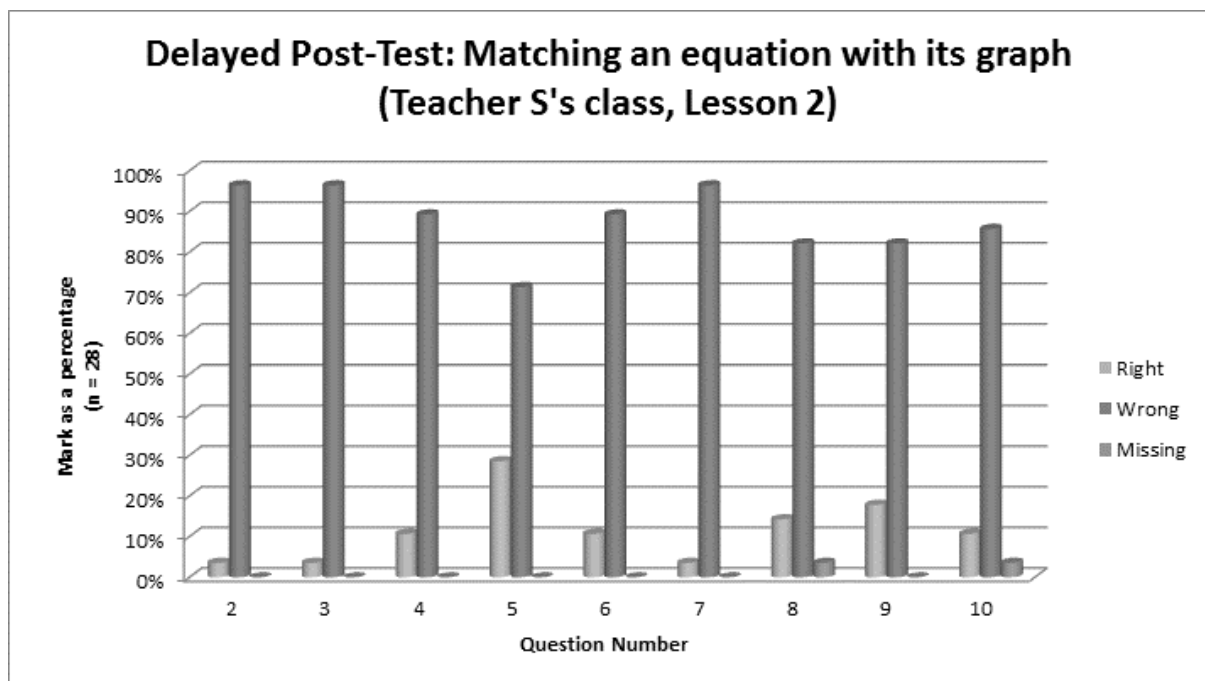


Figure 9.7: Delayed Post-Test: Matching equation and graph – Lesson 2

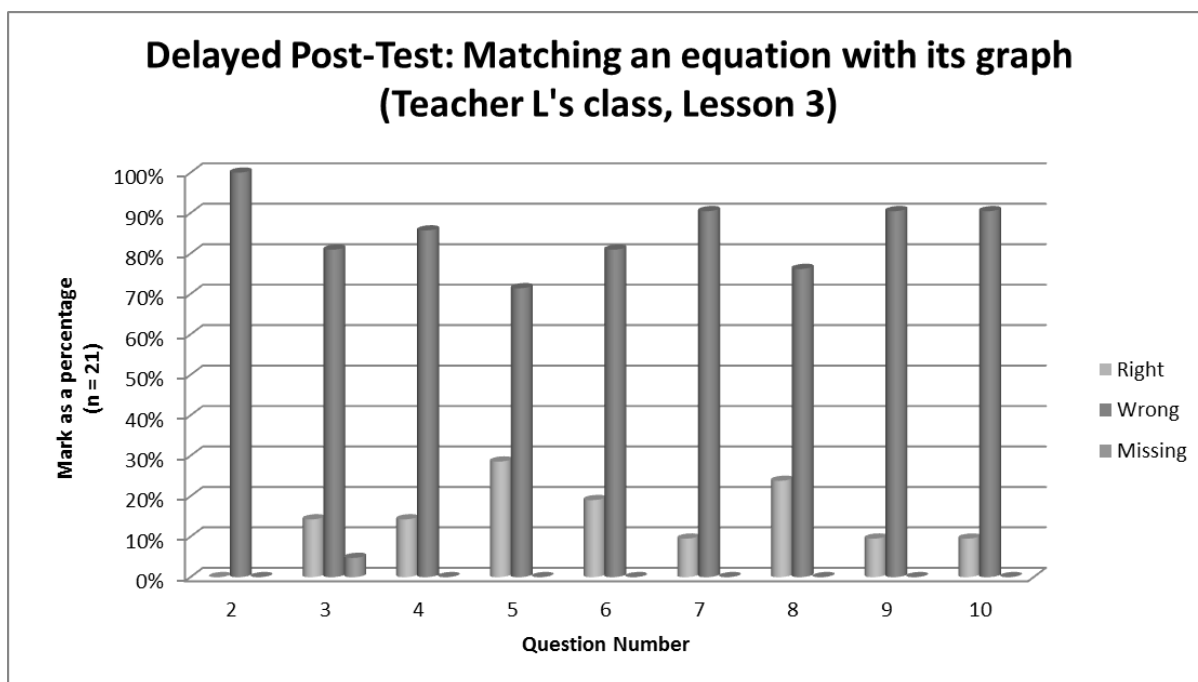


Figure 9.8: Delayed Post-Test: Matching equation and graph – Lesson 3

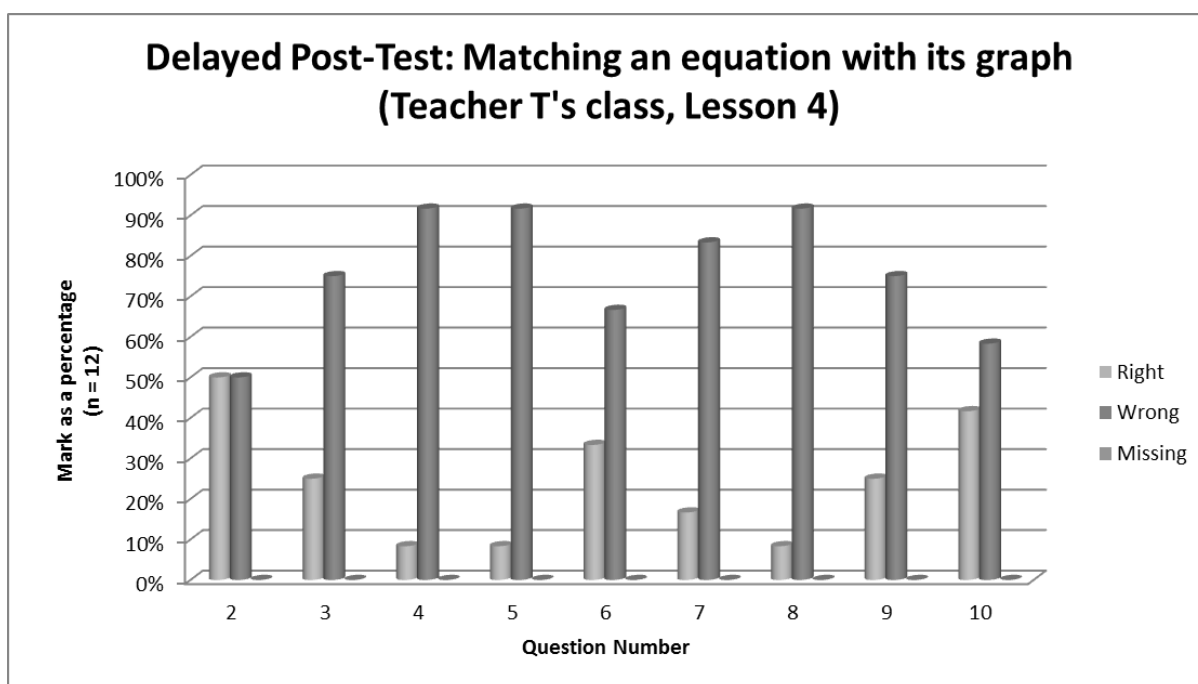


Figure 9.9: Delayed Post-Test: Matching equation and graph – Lesson 4

2. Learner performance – Identifying and naming a function given its algebraic representation:

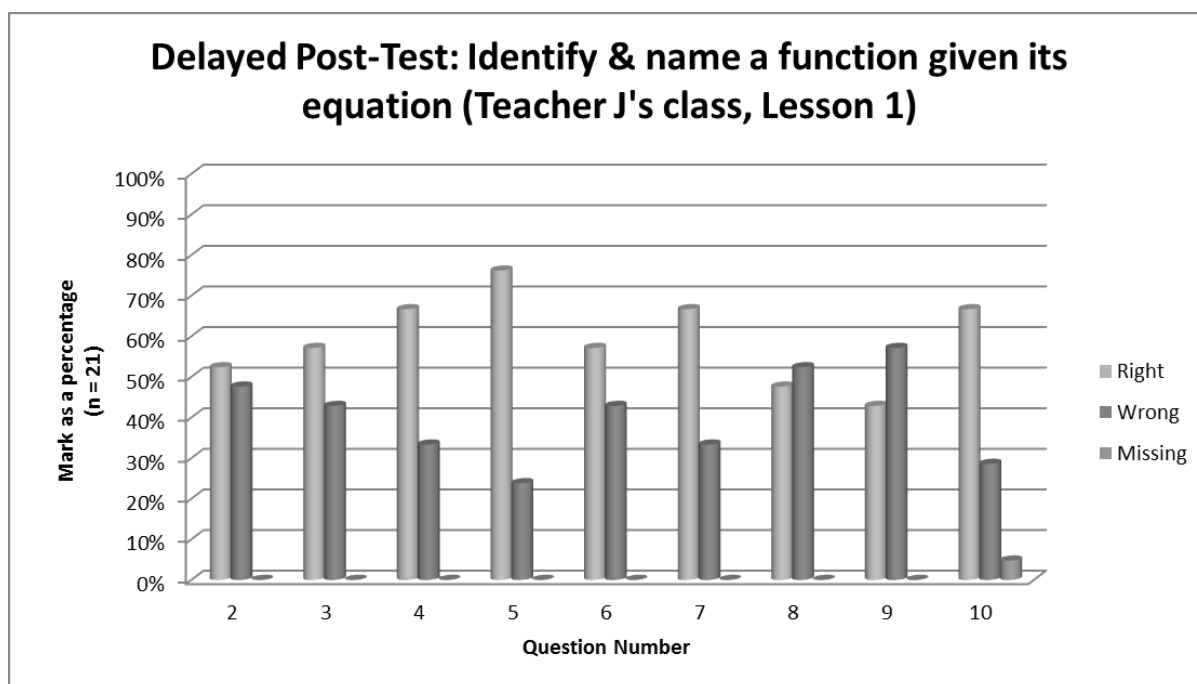


Figure 9.10: Delayed Post-Test: Identifying a function given its equation – Lesson 1

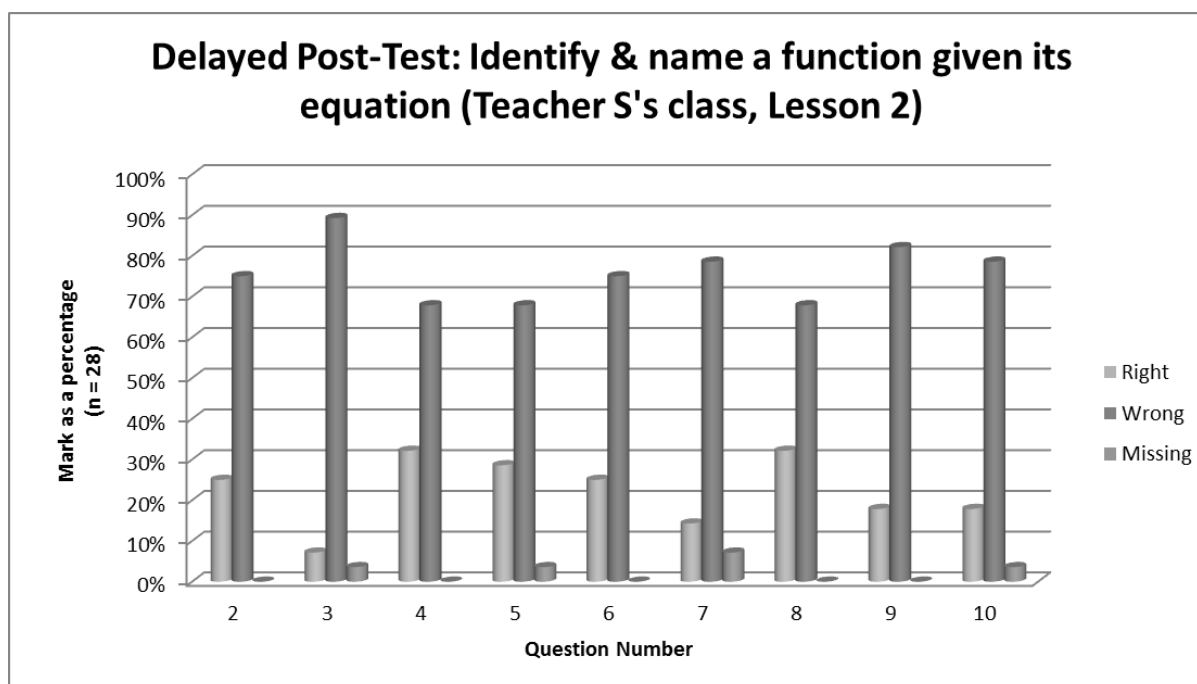


Figure 9.11: Delayed Post-Test: Identifying a function given its equation – Lesson 2

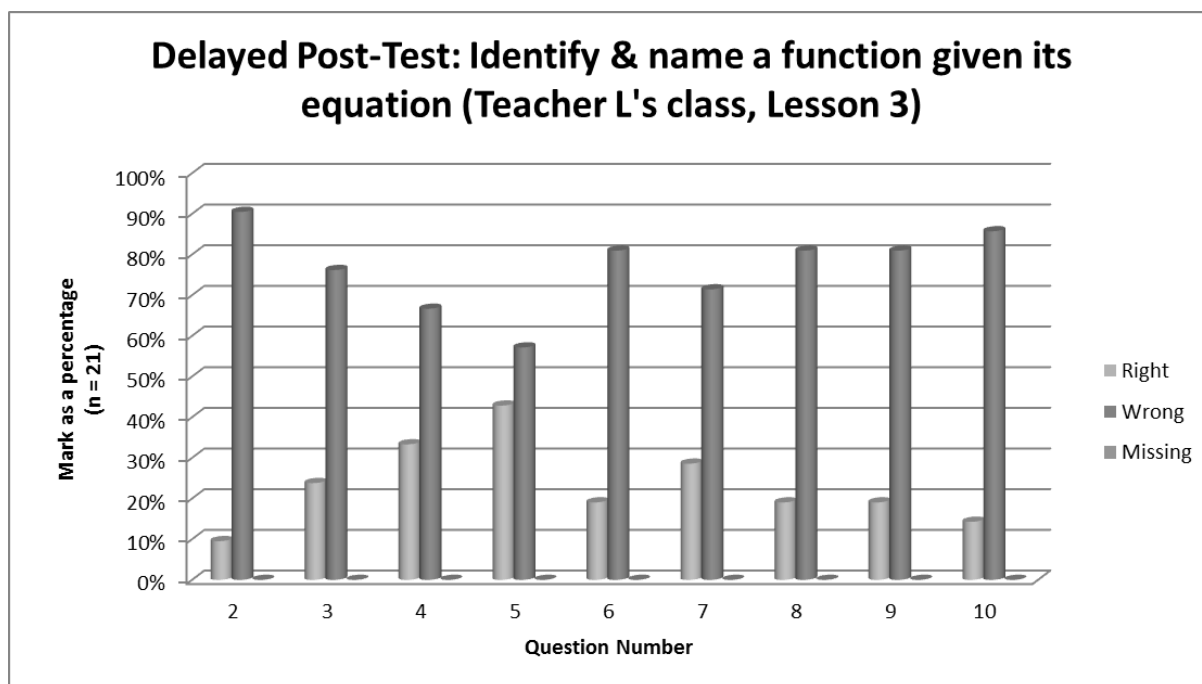


Figure 9.12: Delayed Post-Test: Identifying a function given its equation – Lesson 3

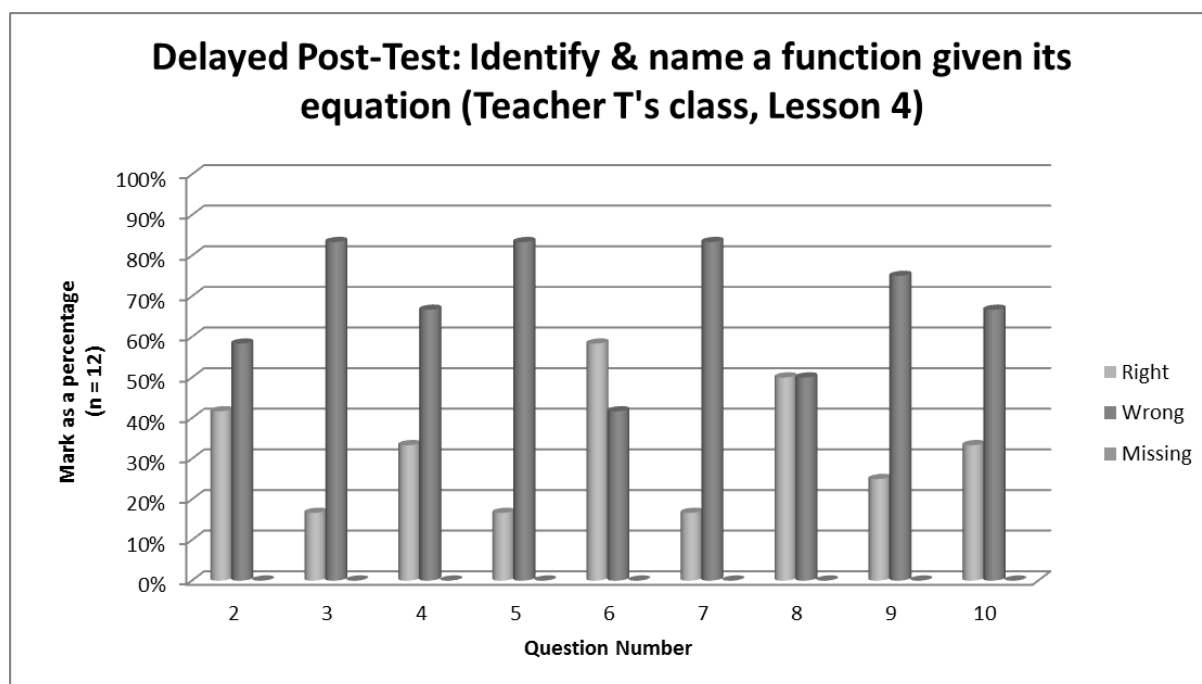


Figure 9.13: Delayed Post-Test: Identifying a function given its equation – Lesson 4

Not all the learners who wrote the post-test wrote the delayed post-test. The difference in learner numbers is not large and so does not play any significant role – the number of learners who wrote each of these tests in each of the classes is displayed on the label for the vertical axis in the bar charts above.

From the bar charts it is evident that learners across all four lessons performed relatively better on naming the type of function being represented than on matching the algebraic and graphical representations. A possible reason for this stems from the fact that learners could not merely guess a match since more than one example of a particular class of function was represented. For example, two equations representing quadratic functions were given $y = x^2$ and $y = 3x^2$. In order to match these equations to the appropriate graphs, merely identifying them as representing a parabola was not sufficient. In this particular case learners needed either to understand the effect of the coefficient of x^2 on the graph or they would have had to take a point through which the graph passed and test to see if it satisfied the equation. In this way the learners could correctly match the equation with the graph.

If a learner correctly matched the equation to the graph, then in their attempt to identify and name a function they would have access to both the algebraic and graphical representations. Otherwise, they were required to identify and name the function from its algebraic form. In this regard my expectation was that the learners who were taught in lesson 4 by Teacher T would have performed much better than the learners who were taught in the first three lessons. This expectation was grounded in the fact that it was only in lesson 4 that the learners were provided with appropriate criteria to identify and name a function from its algebraic representation. Secondly, on the post-test the learners from this class performed better than the other learners. However, when one compares the performance of learners in the delayed post-test across the four lessons, the learners from lesson 1 performed better than all the other learners. This came as a surprise because in lesson 1, Teacher J focused mainly on substitution: although he also engaged his learners with identifying and naming a function, he did not provide them with appropriate criteria to do so.

During a group discussion after the delayed post-test was conducted it came to light that Teacher S revised the work with her learners. To recap, in setting up this learning study two teachers from each of two schools volunteered to participate in the study. The group of teachers decided to teach across schools so during this learning study the teachers were not teaching learners from their own school. When reference is made to Teacher S revising work with her learners, it implies that these are the learners whom she is responsible for teaching in terms of her employment at School R. In relation to this learning study this is the group of learners who were taught by Teacher J during the first lesson of the learning study cycle. Earlier in the discussion I asked the teachers if they had noticed any improvements with the learners' performance in the delayed post-test in comparison to their performance in the post-test. Teachers commented that generally they could not see any improvement but then Teacher S indicated that she could see some improvement:

Teacher S No, I can see some improvement.

VP Tell us more.

Teacher S The improvement is matching the equations and the, and the type of graph

VP Matching the equation with the type of graph.

Teacher S Yes, telling us this is (*inaudible*) a straight line. There's an improvement on that. And the other test to this one.

VP The post-test and the delayed post-test.

Teacher S Yes.

VP So between the post-test and the delayed post-test, in terms of from the equation to the type of graph...so moving from the algebraic representation to label or to classify the type of graph...

Teacher S The classification there.

VP The kids in that lot, that you are busy with, they've improved, they can. So you can see that.

Teacher S Not all but the majority can, yes. More than 50% at least.

VP But still from the equation to match to the picture, it is then a problem.

Teacher S It's still a problem.

(Discussion – session 11, utterances 41 to 53)

It was a pleasant surprise that improvement was noticed, but of immediate concern to me was that the improvement was not noticed with the learners who were taught in the last lesson where a critical feature was in focus. A short while later in this discussion Teacher S illuminated possible reasons for the improvement that she noticed:

VP And Teacher S, your class was the first class that wrote.

Teacher S Yes, the first class.

VP But then you were telling me that...

Teacher S I did go through this with them in class, after the...

VP After the...

Teacher T First test.

VP After the first learning study lesson. You did work with the kids on these things again.

Teacher S Mm and then also after the second lesson because after that lesson you were focusing on what makes...what makes for example y equal to two x linear.

Teacher T So it's a positive development. At least...because if they can reproduce that again, they have learnt.

- VP No, that's good but what I'm saying is we can't attach success there totally to saying it's because of this learning study. Because there is also the teacher engaging with the learners in between.
- Teacher S But it wasn't for long. I just took two lessons to recap. I said, ok, this is what we should have done. That's it. So those who attended understood what they're talking about.
- VP Those that attended the...?
- Teacher S The afternoon.
- VP That first lesson of the learning study.
- Teacher S Yes. They could follow what was happening. I didn't explain like that starting from scratch. I just explained to those who were present.
- VP So when you engaged with those learners, can you tell us more about that lesson that you had. Can you put us into the picture.
- Teacher S I just... I think I just wrote the...what (*inaudible*), the first...
- Teacher T The equations.
- Teacher S Yes, the equations and then I asked them to...
- Teacher T Identify what type of ...
- Teacher S To classify them, what type of graph, and then I showed them the power of x. When it is one we get a linear and two we get a parabola. And then I showed them the graphs, I had it drawn on the grid.
- VP So you focused on the power of the dependent variable.
- Teacher S Yes. And I just showed them where the x and the two and the two...(*inaudible*) I just focused on where the x and the two is. Like for example, two to the power of x and x squared.

(Discussion – session 11, utterances 85 to 107)

The learners from lesson 1 performed better than the other learners on the delayed post-test and this is contrary to what was focused on in each of the learning study lessons and what came to be constituted as the enacted object of learning in each of these lessons. Although this is the case, it is important to note that during the external intervention³⁰ conducted by Teacher S she focused these learners' attention on the critical feature that emerged through this learning study. The details of the revision lessons taught by Teacher S were not recorded and so it is possible that over the two lessons she directed the learners' attention to other features in addition to the critical feature. One can therefore argue that the learners' performance is a result of something else. Since there is no record and analysis of the two revision lessons taught by Teacher S, it is difficult to counter-argue this potential claim. In the group discussion Teacher S explained briefly to the group what she focused on in the two revision lessons. She states that she focused on the critical feature that

³⁰ I refer to the revision done by Teacher S as external intervention since it did not form part of this learning study.

emerged through this learning study which illustrates that the critical feature was in her focal awareness after the second lesson. This stands in contrast to Teacher L and what she focused on in lesson 3. As highlighted in the discussion of what came to be constituted as the enacted object of learning in lesson 3, the critical feature was not in focus for Teacher L and so during the lesson she could not provide her learners with opportunities to engage with the critical feature although it formed the platform for planning the lesson. So, what is important to note is that a teacher is only able to focus his/her learners' attention on a feature of an object if that feature is in the focal awareness of the teacher himself/herself. So if Teacher S, during the revision lessons, focused on some other feature that was significant in terms of enhancing the learners' ability to identify and name a function, this feature would have also been at the centre of her discussion since it would have been at the centre of her awareness.

Having discussed the learners' performance in the post-test as well as the delayed post-test, I now draw your attention back to the learners performance in the pre-test (refer to Figure 6.1). The bar charts representing learner performance in the pre-test across all classes are characterised by the bars representing incorrect and missing responses. It is also characterised by the absence of bars that would represent correct responses and partially correct responses. One of the critical questions in the pre-test that focused our attention in describing the intended object of learning as we began the journey of this learning study was when learners were asked to draw the graph of $y = 2^x$, the range of incorrect responses suggested that this cohort of learners could not distinguish between the different classes of functions. Since the learners performed so poorly in the pre-test one could argue that focusing learners' attention on any aspect related to functions in relation to aspects of the planned lesson would result in learning gains. This claim could have some value because in an instance when one 'has nothing' (pre-test), one's situation can only improve (post- and delayed post-testing) if one was given access to 'something positive' (a lesson on functions in relation to aspects that were identified as being problematic for the learners). So it is precisely due to the learners' poor performance in the pre-test that one can claim that any form of revision could spark learner improvement.

Consider the following sets of bar charts (Figures 9.14 and 9.15) which illustrate the learners' performance in the delayed post-test specifically for questions that required learners to identify and name a function given its equation. These bar charts are different from the ones presented earlier in this chapter (Figures 9.10 to 9.13) since in the bar charts that follow the learners' performance is presented as groupings of classes. The learners who were taught in the first lesson in the learning study cycle are referred to as group 1 learners, the learners who were taught in the second lesson are

referred to as group 2 learners, the learners who were taught in the third lesson of the cycle are referred to as group 3 learners and the learners who were taught in the last lesson are referred to as group 4 learners.

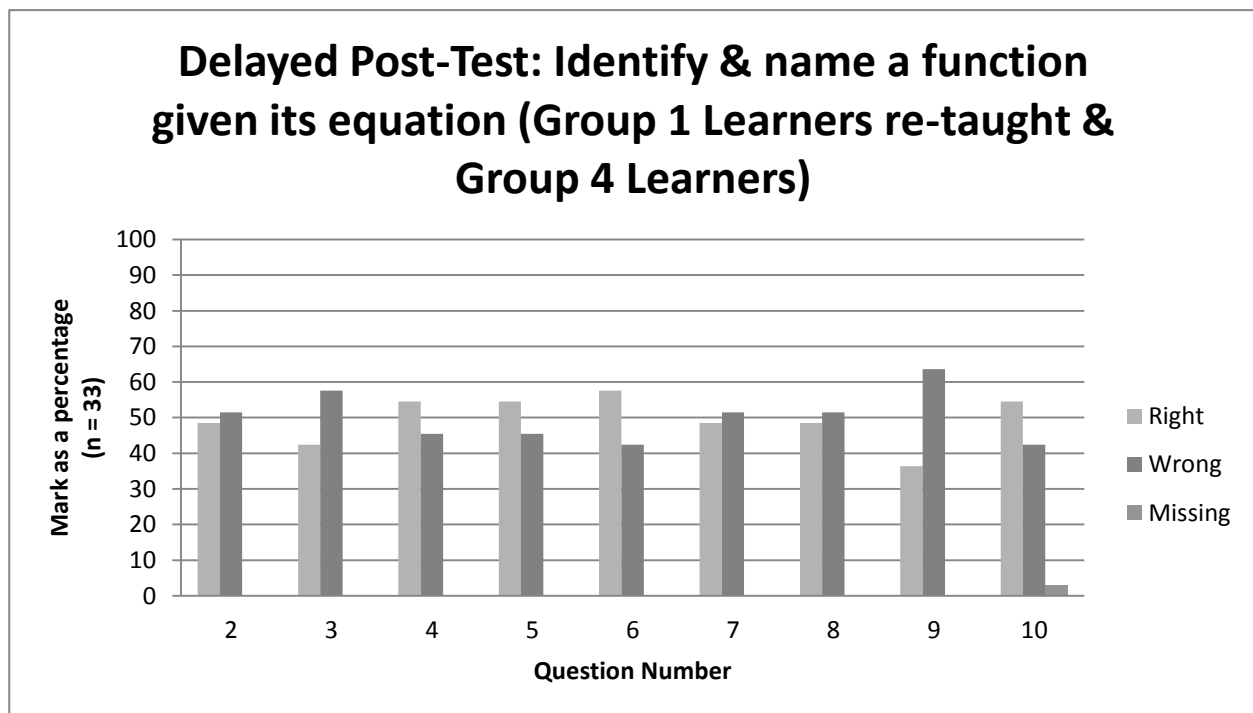


Figure 9.14: Delayed Post-Test: Identifying a function given its equation – Group 1 and Group 4 learners.

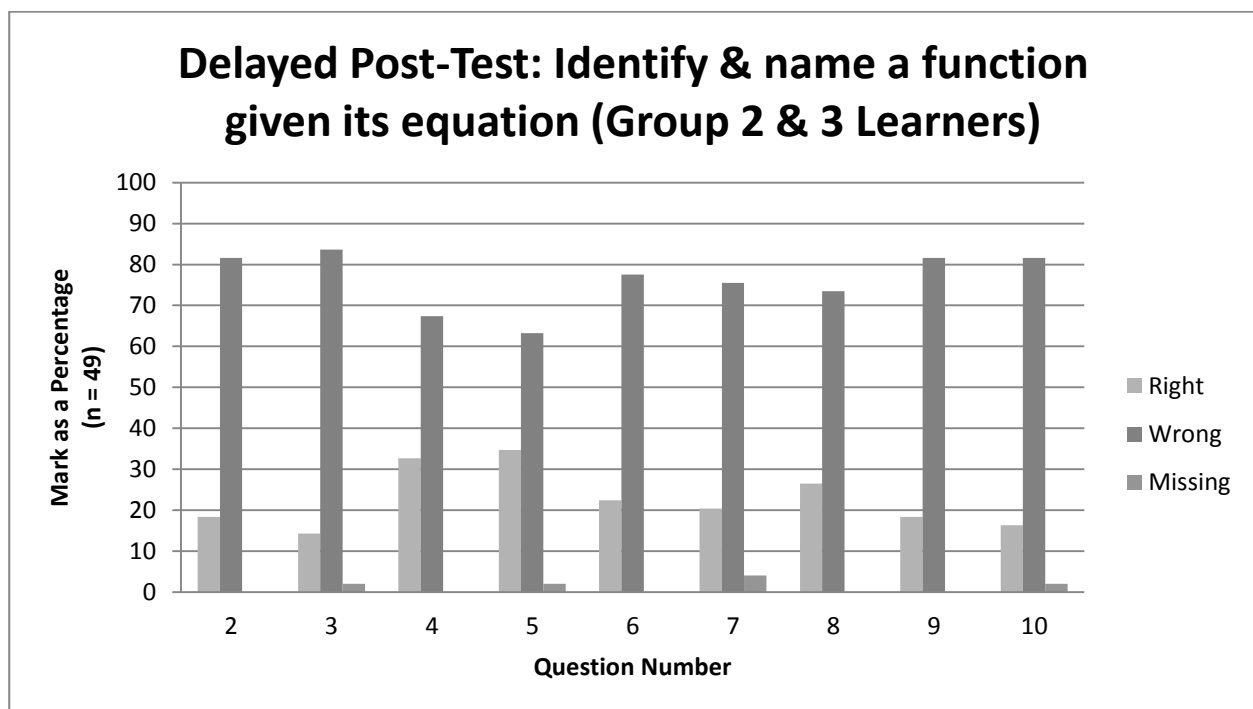


Figure 9.15: Delayed Post-Test: Identifying a function given its equation – Group 2 and Group 3 learners.

To recap, the critical feature emerged at the end of lesson 2 and was not in focus for Teacher L as she taught the third lesson in the cycle. When the critical feature emerged, Teacher S decided to re-teach the lesson to her learners (these were the learners who were taught in the first lesson of the learning study cycle). This lesson did not form part of the learning study cycle and was thus not observed. Lesson 4 was the only lesson in the learning study cycle where the critical feature was in focus for the teacher and opportunities made available during the lesson for the learners to discern the critical feature. This meant that two of the four classes that participated in this learning study were provided with opportunities to discern the critical feature. In other words, in lesson 4 and the re-taught lesson by Teacher S the critical feature was in focus for the teachers and so during their teaching they provided learners with opportunities to discern the critical feature. Figure 9.14 illustrates group 1 and group 4 learners' performance on questions that required them to identify and name a function from its algebraic representation. It illustrates that across the questions between 40% and 60% of the learners were able to correctly identify the class of function being represented by the given equation. Figure 9.15 illustrates that across all the questions between 60% and 85% of the learners from groups 2 and 3 were unable to correctly identify and name the function given its algebraic representation. These two bar charts show that when the critical feature of the object of learning is in focus for the teacher (Figure 9.14) they are able to provide opportunities for learners to discern the object of learning and this is evidenced by a significant improvement in learners' performance in relation to questions pertaining to the object of learning.

Comparing learner performance in the pre-test to their performance in both the post-test and the delayed post-test it is evident that there has been a significant improvement in their ability to differentiate between the different classes of functions. What this learning study managed to achieve was to place on the table and so bring into focus a critical feature that would facilitate in improving learners' ability to identify and name different classes of functions, as experienced in grade 10, given their algebraic representation. It also demonstrates a practice in which teachers become deliberate about identifying learner difficulty and deliberately intervene to attend to the learner difficulty.

9.4 Shifts in teachers' thinking

In conducting this study my interest went beyond merely finding out if the implementation of this learning study improved teachers' mediation of a particular object of learning. I was also interested in finding out what they may have learnt by participating in the study, this however constituted a small aspect of the study. As discussed in the methodology chapter (Chapter 4), I got each of the teachers to complete a questionnaire before we implemented the learning study and another one at

the end of the learning study cycle. As already explained, the implementation of this learning study brought specific aspects in relation to the planning a lesson into focus and these included: i) identifying an object of learning; ii) determining which of its features are critical for mediating the object of learning for the specific group of learners; and iii) using the basic principles of variation theory to structure and sequence the examples used. It also brought particular aspects of the content into focus. It is in relation to these aspects that I intended to track if there were possible shifts in the teachers' thinking about such aspects. To demonstrate the shifts in teachers' thinking I provide anecdotal accounts to highlight some of the shifts in each of the teachers' thinking in relation to the key constructs that drive this study and their reflections about their learning in general.

9.4.1 Shifts in Teacher J's thinking

Prior to the learning study, Teacher J indicated that he used the textbook to plan his lessons and the examples that he used during the lesson was chosen by the learners. He explained that he gave the learners the freedom to choose any examples from the textbook which they felt were difficult and these were the examples that he would work out during the lesson. He also indicated that he selected questions from past exam papers and gave these to his learners to work on. The reason for doing this was motivational in nature as he wrote: "This will help learners to see that what they do in class is what comes at the end of the year [...] These also boost the morale of the learners when they know they can answer exam questions".

Teacher J went on to indicate that he used the mistakes that learners made as a resource for selecting examples for use in his teaching. He believed that these examples will highlight what he referred to as 'critical steps' where learners make the mistake. This indicates that Teacher J did not do any error analysis as such to diagnose what 'feature' of the concept was the source of the error. Instead Teacher J's reference to the errors that learners make is in relation to procedural error made when solving a problem which he referred to as the 'critical steps'. In other words, Teacher J's reference to learner error was not in relation to the conceptual challenges that impede learners' understanding of a concept but rather in relation to the procedural error made when writing out the solution to a problem.

At the end of the study, in reflecting about the choice of examples for use in practice, Teacher J indicated that "one must think deeply about variations so that the examples will clear up misunderstanding". I am inferring here that Teacher J's comment is linked to the sequencing of examples and the idea of only changing one aspect in the example whilst keeping other aspects the same when moving from one example to the next. In reflecting about how he would go about

teaching any topic Teacher J started by writing: “My focus will be the object of learning and I will make it a point that the lesson will be achieved if more than 50% understood the purpose of the lesson”.

In his reflection about what it meant for him to have participated in the study, Teacher J indicated: “I have taught functions for many years but never noticed that exponents play an important role on functions. This learning study on functions has become an eye opener for me. This has given me a clue on why learners fail to identify linear, parabola, exponential etc.” As discussed in Chapter 8, the critical feature that emerged through this learning study is crucial for the group of learners who participated in the study because it provides them with a visual cue or a recognition rule by which to identify and name the class of function from its algebraic representation. Without this particular recognition rule in the forefront of Teacher J’s awareness, he was capable of identifying and naming the class of function given its algebraic representation. One can only assume that over the many years of exposure to the various classes of functions Teacher J developed his own recognition rules for classifying functions given their algebraic representation, and implicit to this recognition rule was the degree of the equation in relation to the independent variable. Bringing this feature into the focal awareness of learners as they are being introduced to the different classes of functions will contribute to them engaging with functions in a more meaningful manner.

9.4.2 Shifts in Teacher S’s thinking

At the beginning the textbook formed the main resource for Teacher S in terms of it guiding her in the sequencing of her lessons and acting as a source for examples. She also perceived the role of examples as providing “direction to the learners as to the applicability of the concept(s) that are dealt with” and to “show learners different ways of solving the same problems”. In other words, for Teacher S the role of examples was merely to provide learners with exposure to different conditions under which a concept has to be recruited when solving a problem.

In responding to what informed her choice of examples prior to the study, Teacher S referred to the level of the learners. She was not explicit about what she meant by the level of the learners, and as already discussed in the methodology chapter, the limitation of using this method of data collection rests in not being able to query this with her. In referring to the level of the learners Teacher S did not refer to the errors that learners make and so possibly level refers to how well learners perform in mathematics assessments. For learners who perform well, Teacher S selects more complex examples from the textbook and for weaker learners she selects less complex examples. In the post-

study questionnaire, Teacher S indicated that previously she selected examples that ranged from the simplest to the most difficult and that she chose the examples randomly from the textbook.

After her participation in this study Teacher S indicated that: “this learning study has helped me in discovering that the object of learning should be the driving force for any lesson [...] I have been made aware of the fact that examples do play an important / if not [a] major role in the planning and teaching of the lesson”. In her response to a later question on the questionnaire, Teacher S explained that “the examples used should be such that they tend to reinforce the object of learning and are chosen in such a way that they make the learners discern the object of learning”.

Teacher S seems to see some value in working collaboratively as we did through this learning study and commented that the “community of practice should be able to periodically meet to iron out errors, misconceptions and problems encountered by teachers in the teaching of different mathematical topics”. This comment reinforces that as a result of Teacher S participating in this study, she now attaches value to working with learner error or misconceptions as the point of departure in planning a lesson – we think about and plan a lesson (including the selection of examples) with an object of learning in mind.

9.4.3 Shifts in Teacher L’s thinking

In the pre-learning study questionnaire Teacher L indicated that she used the textbook and past exam papers as a source for examples and that she selected examples for use in the classroom in relation to the level of the learners. Once again what is meant by the level of the learners is not explained, as was the case with Teacher S. Here again the inference made is that the level of the learners is in relation to their performance in the various assessments done in mathematics. The inference made is confirmed by her response to the role of examples in her practice: “I choose examples that will enable me to present content knowledge that will be to the learners’ level of understanding”. So the level of learners in no way refers to the kinds of errors that learners make or the difficulty that learners have with a particular concept.

In the post-learning study questionnaire, Teacher L indicated that when planning for any future lesson she must “ensure that there is a clear, realistic and achievable object of learning”. In addition, she indicated that she has also come to realise that she needs to carefully consider “the relation between what the learner learns and what is critical for learning to take place”. So in addition to merely having an object of learning as the starting point for planning a lesson one needs to know what the critical features are for that particular object of learning.

During the lesson Teacher L uses mathematical language inappropriately, for example she writes down the equation $y = 2x$ on the board and she wants learners to focus in on the operation between '2' and 'x'

Teacher T: What is the relationship between the two and the x...that we have? In the equation...remember this is the equation, it consists of the terms...a term, the y term on the left, and the x term on the right, separated by that equal sign...

(Lesson 3, event 1.2)

In grade 8, learners learn that it is the operations of addition and subtraction that separate to create terms but in the above extract we see Teacher L referring to the variable 'x' and 'y' as being terms and inappropriately refers to them as terms that are being separated by the equal sign. Her inaccurate use and lack of rigour in which she used mathematical language in the lesson was brought to her attention during the post-lesson discussion. Now, in reflecting about what she has learnt about her own practice, Teacher L indicated: I've also learnt to be more rigorous in terms of the mathematical language I use when planning and teaching (i.e. it is imperative that I use correct mathematics language and correct notation at all times".

9.4.4 Shifts in Teacher T's thinking

Prior to his participation in this learning study, Teacher T indicated that he used the textbook to guide him with the content that had to be taught. The textbook also provided him with a source from which to choose examples and he supplemented this source with past exam papers. As was the case with the other teachers, Teacher T also took into account the level of his learners in terms of their performance in the various assessments when choosing examples for use during the lesson and for homework purposes. In reflecting on the role of examples he saw examples as a means through which to explain the content that he had to teach, and after the lesson the examples given to learners as homework were for practice and for purposes of establishing for themselves the degree to which they understood the concepts taught.

In reflecting about the study, he indicated that: "it was an eye opener in as far as the relationship between planning and lesson execution is concerned". He went on to explain this as follows: "I have learnt that in planning one has to be clear of the steps to follow and the sequencing of activities. Examples used should be in some chronology that builds up to the full attainment of the lesson objectives". I can only infer that his reference to the chronology of the examples is the use of

variation to structure and sequence of the examples so that the resulting example space brings the object of learning into focus. Teacher T also indicated that what he had learnt was that the “object of learning should always be the reference point in the whole process of teaching”. In his reflection about examples, Teacher T expressed the need to ensure that when examples are selected they must be done so with the object of learning in mind and that the examples should be carefully sequenced during the delivery of the lesson.

9.5 Summary

It is important to note that these teachers were only exposed to one cycle of a learning study which focused on the deliberate use of examples, where the deliberate use of examples refers to the use of the principles inherent in variation theory to structure and sequence examples so as to bring the object of learning into focus. Through their participation in this study the teachers were provided with a limited exposure to the ideas inherent in variation theory and its application to examples for use in practice. Their experience in relation to variation theory and its application to examples was further restricted by the delayed emergence of the critical feature, which constrained the opportunities to explore the elaboration of the example space and hence the further application of the tools that variation theory provides in terms of example development. I return to this issue in the concluding chapter.

It is through the teachers’ participation in this learning study that they come to realise that focusing on a critical feature of an object of learning and then planning a lesson with this in mind is central to providing learners with opportunities to discern that which is intended. More specifically, the object of learning needs to be in focus when selecting and sequencing the examples for use in practice. These tie in with the fundamental principle that underpins the work within the WMC-S project, which is that professional development work remains focused on objects of learning. It is also through the learners’ performance in the delayed post-test and through teachers’ reflections in the post-learning study questionnaire that we begin to see how significant a small move was in the teaching practice of the teachers in this study – a teaching practice that is dominant across many of our schools.

The learners’ performance in the delayed post-test and the teachers’ reflection in the post-learning study questionnaire demonstrate that, even in a context characterised by serious lack of resources and an extremely low knowledge base of learners, engaging in this learning study had positive outputs.

Chapter 10

Conclusion

10.1 Introduction

I embarked on this study because of my interest in the use of examples in the teaching of mathematics. In the literature review on examples I elaborated on the importance of the use of examples in the teaching of mathematics, irrespective of the level at which it is taught, since it is through examples that the essence of mathematical concepts and techniques can be communicated. In reading about the use of examples in the teaching of mathematics, the work of Watson and Mason (2006) gave me access to the concept of variation theory and its application to the structuring and sequencing of examples to create an example space. The concept of variation theory and its use in structuring examples for use in practice as discussed by Watson and Mason (ibid.) informed the earlier TPD work of the WMC-S project. Through my participation in the TPD work of the WMC-S project my exposure to these key ideas was taken from the level of reading knowledge to first-hand experience in working with the ideas. This formed the gateway for the extension of my reading into the area of variation theory which then afforded me access into learning study. It was with this knowledge base that I set out on a journey to explore whether a learning study that is underpinned by variation theory and the judicious use of examples could work to assist teachers in mediating a particular object of learning. Within the context of this study mediating an object of learning refers to the process in which:

- i. The teacher introduces a concept which is contained within the example selected – *Judgement of existence*
- ii. The observable actions performed on the example (by the teacher and/or learners) with the intention of elaborating the concept so as to move it from a level of immediacy to something more substantial – *Judgement of reflection*
- iii. The teacher's attempt at legitimating or fixing meaning for the learners by appealing to some knowledge domain – *Judgement of necessity*

It is the cumulative outcome of the above process together with the principles of *contrast* and *simultaneity* in relation to the range of examples used in the lesson (*the example space*) that the different concepts as exemplified by each of the examples contribute towards providing opportunities for learners to discern a particular object of learning.

Returning to my journey that is being reported on in this thesis, the simple answer to my quest is yes, it does work and it did work³¹. That is, a learning study which is underpinned by variation theory and the judicious use of examples does work in assisting teachers to mediate a particular object of learning. As illustrated in Chapter 9, in the two classes (groups 1 and 4) where the critical feature of the object of learning was in focus for the teachers, there was a substantial improvement in the learners' performance in the delayed post-test. This stands in contrast to the learners' performance in the delayed post-test where the critical feature of the object of learning was not in focus for the teachers (groups 2 and 3). In terms of reporting that this learning study has worked it is important to understand that it was not simply a matter of more teaching. Through participation in this learning study all four classes received additional teaching, but what made the difference is that in the lessons where the critical feature of the object of learning was in focus for the teacher, the teacher was able to provide his/her learners with better opportunities to discern the object of learning.

Returning to the opening chapter of this thesis, and discussion of teaching in relation to orchestration, this study suggests that a lesson where the object of learning and its critical feature are in focus for the teacher and he/she brings it into focus for the learners is akin to an orchestra playing in harmony. All four teachers in this study used the same example space that was planned and what they were doing during the observed lessons was to provide their learners with opportunities to discern the object of learning. Through the four observed lessons, it is only in the case of Teacher T where we see that the teacher's orchestration of the lesson brings the critical feature of the object of learning into focus for the learners. This is a direct result of the teacher himself having a clear focus and understanding of the object of learning and the critical feature.

What then of the story of Nash in Chapter 1? For Nash to participate in a practice where he plays more in harmony with what is mathematically accepted within the mathematics community and for his learners, he would benefit from a focus on the object of learning and its critical feature. The question that remains is how might Nash come to focus on the object of learning and the issue of a critical feature? A possible answer to this question lies in Nash's participation in a learning study as a form of professional development, as this could act as an entry point to get Nash to think about and so focus on an object of learning and its critical feature in his practice. Of course, within the practice of teaching, more harmonious orchestration does not necessary stretch to learning but only

³¹ The reference to 'work' is also in relation to the context in which this learning study was developed: a context characterised by the lack of resources and the very poor knowledge base of the learners.

to a good joint sound. Thus, like all metaphors, the metaphor of teacher as orchestra conductor also has its limits.

In this study I demonstrated that by participating in the learning study, learners' performance in relation to the object of learning, as measured by the pre-, post- and delayed post-testing, improved. In addition, there are recorded shifts in teachers' thinking about planning a lesson with an object of learning in mind as well as being more deliberate about the selection and sequencing of examples in a lesson in relation to the object of learning. While teachers' learning is not in focus in a learning study, as a form of professional development, I was interested and concerned with teachers' learning. This formed a more limited aspect of the study and requires more thinking and further consideration. Firstly, questionnaires are limited, if you wish to study teachers' learning in a learning study as professional development then more attention would be needed both through detailed analysis of the post-lesson discussions and interviews with the teachers. This is recommended for further study if the focus is on professional development. In the remainder of this chapter I reflect on the contribution that this study makes towards the knowledge base of the field in which it is located. I also engage in discussion related to recommendations that emanate as a result of my undertaking of this journey, the limitations of the study and ideas for further research.

10.2 The value of doing this study

All the research done thus far related to learning studies are studies in which researchers studied learning study cycles that were implemented in schools in developed countries such as Sweden and high income cities such as Hong Kong. The learning study reported on in this thesis was conducted in South Africa, in two schools in a community where unemployment is rife and the learners attending these schools are from a low socio-economic background. As highlighted in the methodology chapter (Chapter 4), both School M and School R do not charge school fees and so are dependent on funding from the education authority. Some of the learners attending these schools are from child-headed households and, like most of the other families residing in the area, are dependent on social grants from the state. This means that the learning study being reported on in this thesis was conducted in schools with very little functional resources and therefore different from the schools in which other learning studies were conducted. This study contributes to the field by telling the field that whilst learning study is labour intensive, and has assumptions about knowledge, skills and resources that are in place, it was relatively successful within this context.

This study demonstrates and so confirms that irrespective of the context in which you work, when a critical feature is identified and examples are carefully selected with the critical feature in mind and

if the critical feature is in focus for the teacher, increasingly more learners will begin to discern the intended object of learning. Of course, there were also significant adaptations to the model when the learning study was implemented. For instance, the teachers did not teach their own learners during the study, they wanted to teach learners from the other school. The lessons were taught after school hours and all teachers in the learning study group were present in all of the lessons. Post-lesson discussions and planning for the next iteration of the lesson was also done after school hours. This stands in contrast to how a learning study cycle is implemented in Sweden. For example, a group of teachers working in the municipality of Fagersta in Sweden explain³² that firstly in implementing learning studies there is funding from each of the municipalities. This allows the principal of the participating schools to reduce the workload of the teachers by up to 10% so that they can engage in learning study activities. Secondly, learning study lessons are taught within the school day and are videoed. Other teachers participating in the learning study cycle need not necessarily be present during the learning study lessons as the group of teachers watch the videoed lessons together during time set aside for learning study. So although the learning study reported on in this thesis did work there were some differences in terms of how it was implemented. However, irrespective of context and the adaptations made to the implementation, what we see is that when the critical feature is in focus for the teacher and the teacher is able to focus his/her learners' attention to the critical feature, learning is enabled.

As discussed in the methodology section, I took on three roles in this study: firstly to initiate the teachers into the learning study process, secondly that of researcher, and thirdly that of knowledgeable other within the learning study group. My role as knowledgeable other was to influence the group positively by providing them with the necessary inputs – mathematically, conceptually and theoretically. When implementing the learning study we (the learning study group) identified the object of learning but we had no idea of which feature of the object of learning would be critical for the learners' learning. After the first lesson we still did not have an idea of what the critical feature could be. Theoretically, I knew that the critical feature for the object of learning had to emerge from the mathematics in relation to the learners being taught. Although I knew this, it was still a disconcerting feeling for me as the researcher since I was concerned about the possibility of going through the entire learning study cycle without a critical feature being identified. Through supervision support, my supervisor provided some comfort, but learning study per se was new to her as well, and hence our joint interest in pursuing a learning study and its

³² This information was obtained by talking to these teachers (Thomas Bergström, Tobias Nordin, Sara Fritz, Lena Lindberg, Linda Eldstahl, Sofia Sander and Emma Rönning) at the World Association of Lesson and Learning Studies which was held in Gothenburg, Sweden during September 2013.

potential. It is obvious that all researchers worry about their research and the direction that it is taking. Perhaps as knowledgeable other in a learning study this could be exacerbated. As indicated previously, I was worried about the possible non-emergence of the critical feature and whether this was my fault. Being the knowledgeable other in this learning study essentially meant that I was researching myself and so following the work of Graven (2002b) all that I could do was to be reflexive in my approach i.e. understanding that as the researcher who is also a participant in the study, I have an impact on those being researched. My concerns and anxieties did not derail or disrupt the study. How they might have influenced it is for a different study.

I had no previous experience in conducting a learning study and so was inducted into learning study through my reading knowledge of other learning studies that had been conducted and reported on. My concern was whether I was implementing the learning study by merely engaging with the form of the model. Although this was a concern, I knew that the essential features of the model such as identifying the object of learning and its critical features were always in focus for me. Such issues were not elaborated on in the reports of other learning studies that were conducted. In other words, the knowledge base of the knowledgeable other is assumed to be in place and therefore not focused on in other studies. This suggests that some reflection on learning to do learning study across contexts would be important for the field. It is important because what learning studies from different contexts do amongst other things is to bring different facets of the study, different challenges of the study and different issues around critical features into focus.

This study underscores the knowledge base of the knowledgeable other and marks it as an essential component to be in place before a learning study is implemented. The knowledge base that I am referring to comprises two components: i) that the knowledgeable other is skilled and equipped with the relevant facilitation skills, and ii) that the facilitator has what Ma (1999) refers to as a *profound understanding of fundamental mathematics*. In attempting to address the demands of policy, this suggests that it would be good practice for teachers to have participated in at least one learning study organised and implemented by an experienced knowledgeable other so as to gain some experience before implementing his/her own learning study. Alternatively, for the newcomer to learning study it would certainly add value if there is another researcher who is willing to collaborate whilst implementing a learning study. This provides a kind of support base for the sharing of ideas and contributes towards building confidence in leading a learning study.

As seen through this study, a learning study approach to TPD is school-based and the object of learning that directs the learning study is derived through a process of reflection on one's practice

and so requires that teachers become active participants in the professional development model. As illustrated in this study, reflection on practice within a learning study is driven by the need to establish the critical feature for a specific object of learning and then finding ways for learners to experience the object of learning in a specific way. In particular, this study illustrates how the judicious use of examples provides a possible mechanism that provides learners with opportunities to experience an object of learning in a particular way. Essentially the design that underpins a learning study embraces the principles as envisioned through the National Policy Framework (DBE & HET, 2011). However, a key feature of learning study, which is silent in the policy, is that the completion of one learning study cycle consumes a considerable amount of time. Unlike the usual workshop type approach which characterises most of the TPD initiatives driven by the Department of Education, TPD models, like learning study, require an investment of extended periods of time. The Department of Education is trying to move away from this through the revised framework (DBE&HET, 2011) however, the time investment needed is underscored in the policy. It is interesting that in a context such as Sweden, the local municipal authorities invest money so that the teachers' workload can be reduced, thus providing them with time to engage in learning study activities. In the Swedish context participation in TPD is not added onto the workload of the teachers, as is currently the case in South Africa.

Learning study is about critical features, their emergence and then using the critical feature to open up a dimension of variation to bring a specific feature of an object of learning into focus. In this study it took a relatively long time for the learning study group to both identify the critical feature and then to pedagogically bring it into focus. However when this did occur the learners' performance improved, i.e. in relation to questions focused on the object of learning. Thus aside from the constraints of learning study as such, this study is one which contributes to the confirmation of the premise upon which the professional development work with teachers in the WMC-S project is developed i.e. focusing on bringing objects of learning into focus is a promising development.

In her study of the significance of critical features, Kullberg (2010) concludes that the critical features identified from two different learning studies were transferrable to other learning study groups. In view of this, the critical feature that emerged through this study, viz. to focus learners' attention on the highest power of the independent variable in order for them to discern the class of function given its algebraic representation, adds to the knowledge base of critical features for a specific object of learning. In other words, this finding contributes to mathematics education research by suggesting how a content specific feature made a difference for both learners' and

teachers' learning in the learning study group being reported on in this thesis. The kind of collaborative planning and teaching required in a learning study attunes teachers to critical features and the kind of work entailed in identifying these in areas where learners are experiencing extensive difficulty.

In my role as a researcher, I could see that using Bernstein's pedagogic device, and particularly the idea of the evaluative event, was critical for enabling me to produce the data (chunking the lesson into evaluative events, classifying it, coding it and then describing it). The production of data from the lessons allowed me to see not only what was enacted at the level of content, but also what was enacted at the level of authorisation, and this is a key contribution of Bernstein's work. By focusing on the operational activity (*reflection*) one gets to see what aspects of the content are engaged with during a lesson. This method of analysis thus works from the empirical and enables a relatively 'clean'³³ reading of what comes to be constituted as mathematics during the lesson. This was very productive in terms of authorisation, but left a weakness in terms of reflection. To get a more elaborate description of reflection, it was necessary to draw on tools from variation theory, since these tools allow one to describe the opportunities that are made available for learners to discern the object of learning. For instance, one is able to see what is varying and what is kept constant, and one is also able to see how the teacher introduces contrast and simultaneity to facilitate in the learners' discernment of the object of learning by zooming in on the critical feature through these patterns of variation. To enhance the description of what comes to be constituted as the enacted object of learning, I also considered how meaning was legitimated (*authorisation*). The content that is engaged with could be substantiated in mathematical ways or they could be substantiated on the basis of the teacher. Interestingly, across the four lessons in this study, the legitimating of meaning remained largely at the level of the teacher. It is these three things together (observable actions, tools from variation theory and legitimating of meaning) that provide a powerful description of both what is afforded and what is constrained during a lesson. Variation theory per se does not provide the tools to deal with authorisation but it could possibly give a richer and a thicker description of reflection. Hence, I would advocate for the continuation of doing this kind of analysis, as it remains an open question whether the tools of variation theory have developed to be able to do the data production as well as the data analysis. This provides scope for further study on the methodological tools for describing the enacted object of learning and what they illuminate and/or obscure.

³³ Attempting to describe what came to be constituted as mathematics in the lesson by drawing on the literature on functions, in particular, a pointwise and global approach to functions meant that I had to impose the inherent constructs onto the empirical. The literature on functions was not used typologically to analyse the data. The data was analysed inductively and from the empirical to produce the description.

When we look at the literature on functions and classes of functions it seems that the significance of the algebraic form is assumed: this study suggests that for many learners, it requires some attention. It may not require attention in all cases, but in this study it certainly required attention. What this suggests is that perhaps if studies are done where certain levels of learning and teaching are assumed and taken for granted then it is possible to slide over basic ideas. In this case I could not. This study highlights that in work with functions and classes of functions the recognition of the algebraic form is critical. It is interesting that while the learners in this study, for example in Teachers J's class, could do the substitution in the various classes of functions they still had a problem in identifying it. It is also interesting that while I set out and in fact the initial planning of the lesson drew on the literature e.g. pointwise and global approach to functions, it disappeared into the background as the critical feature emerged which is not one identified in the wealth of literature on functions. Despite the wealth of literature related to the teaching and learning of functions none of it brings this feature of the algebraic representation of a function into focus. It is for this reason that the literature on functions discussed in chapter 2 recedes into the background in this concluding chapter.

10.3 Recommendations – attempting to address some policy demands

The implications of this study are policy related. As already discussed in Chapter 1, the Integrated Strategic Planning Framework for Teacher Education and Development on South Africa (DBE & HET, 2011) provides some practical basis for developing implementation plans for a teacher education development system as a whole. One of the outputs and activities which it envisions is the establishment of PLCs which will allow groups of teachers to work together. In working together they are expected to learn from video records of practice and other learning materials.

It is important to note that being an experienced teacher (like the teachers in this study) does not necessarily translate into being a good teacher educator. For teachers to work together within a PLC implies that they would be required to engage with the artefacts of their practice with some perspective and knowledge about research and so to some extent stepping into the shoes of a teacher educator. For instance three of the teachers in this study each had in excess of twenty years teaching experience but yet when they identified the difficulty their learners were experiencing (*object of learning*) they did not know how to address the problem (*identification of a critical feature*). In suggesting possible ways to engage with the learner difficulty, they defaulted to strategies and approaches to teaching functions as captured in a textbook. This did not come as a surprise since these approaches made sense to them and so to shift their thinking, the role and support of the knowledgeable other became crucial. The identification of a critical feature is the crux of a learning

study and it is not a simple task, as is demonstrated through this study. The low knowledge base of the learners in this study as seen by the wide and varied errors made in the pre-test resulted in the emergence of a critical feature that could easily be taken for granted and so overlooked. Nonetheless, the critical feature that emerged was crucial for this group of learners to discern the object of learning. In conversation with other people conducting learning studies, also in the context of functions, in Sweden we see similar kinds of critical features emerging, critical features that are not grounded in mathematics per se, but in some visual cue. These are the kinds of prompts which literature argues could lead to prototypical thinking (Tall & Bakar, 1992). As alluded to in Chapter 8 in starting to learn about a concept, prototypical ways of thinking are a good starting point but if left at that level, it is a problem. However, literature related to the teaching of functions emphasises ongoing proficiency of deeper conceptual understanding of the function concept. The literature does not get to grips with what the key things are that act as a catalyst to promote this proficient and deeper conceptual understanding. It takes research, such as the one reported on in this thesis, to allow us to see and so confront the problems that are obscured because they are so taken for granted.

The concern and challenge, as already alluded to, is the teachers' dependence on professional developers and on sustained leadership and support by the education authority and school management teams. Rolling out models of TPD as envisioned by the National Policy Framework requires that the starting point is in ensuring that there is a layer of professional developers/facilitators who are proficiently equipped to lead TPD models such as learning study. This is to ensure that in the implementation of such a model the focus will not be on the structural aspects of the model, because if that is the case then all that will happen is that teachers would merely be mimicking the superficial features of the model without paying attention to the essential features of the model.

Within the context of the preceding discussion this study demonstrates two things:

1. As a newcomer to the idea of learning study and variation theory myself, I was able to lead a group of teachers through a learning study cycle. This was made possible through the support of my supervisor and external collaboration with individuals in the field. At first, it seemed that I implemented the model by engaging with the structural aspects. However, by collaborating with individuals in the field I received confirmation that the essential features of the model, such as identifying the object of learning and its critical features, were in focus for me. Therefore, I would argue that for a newcomer to successfully engage in the

implementation of a learning study there needs to be a level of collaboration with experienced others to ensure that the essential features of the model are focused on as the learning study cycle is implemented.

2. I would also argue that there was value in conducting this learning study because the end product resulted in positive shifts in some of the learners' performance as measured and compared through the pre-, post- and delayed post-tests. This was also coupled with positive shifts in teachers' thinking about their practice. In addition, a critical feature emerged which also contributed to improving the content knowledge of teachers – consider the case of Teacher J who was bold enough to reflect on this in the post learning study questionnaire.

Although there are challenges in implementing a learning study, one of which is with respect to the knowledgeable other in the group which then impacts on the scalability of the model, there is nonetheless value in engaging with this form of TPD. This then raises the question of how the Department of Education could deal with such challenges and so consider learning study as a possible model of TPD that aligns with their vision as expressed in the National Policy Framework for teacher education and development in South Africa.

Drawing on the work of Chokshi and Fernandez (2004) I provide possible ways in which we should begin to look at learning study as a possible model of TPD within the South African context. This is done with the intention of shifting our thinking that learning study would be difficult to implement because of: i) the lack of skilled facilitators to lead learning study groups; ii) insufficient time to conduct learning study; and iii) learning study being a model of TPD imported from a country with a teaching and learning culture different from ours.

1. Since the vision of the National Policy Framework is for teachers to take responsibility for their own professional development, the school's management team will need to make provisions for learning study groups to meet after school hours and to teach learning study lessons after school hours as well – *providing leadership*. In this way, the implementation of a learning study at a school will not affect curriculum delivery as per the demands of the education authority. Learning study is time-consuming but, as this study demonstrates, there are rewards. Thus, finding time to do learning study is not an impossible task once the teachers have made a commitment to it – *establishing a professional culture and not being workshopped on what to do*. As already indicated and to reiterate, in Sweden different municipalities invest money for teachers to participate in learning studies. The money

invested allows the principal to reduce the workload of participating teachers so that they have time during the official school day to engage in learning study. Thus, participating in learning study is not an add-on to the teachers' workload but becomes a part of their job.

2. Although learning study is a model of TPD that was developed elsewhere, cultural differences with respect to teaching and learning should not play a negative role to the implementation of learning study. Learning study places the teacher at the centre of the process and so teachers work collaboratively on identifying an object of learning, its critical features and possible ways of providing their learners with opportunities to discern the object of learning. Within a learning study group teachers work together with a common purpose and can draw from one another's knowledge and experience and so build on their content knowledge.
3. In addressing the demands of policy there are no panaceas, and learning study is not a panacea. However, learning study is a model that aligns with the National Policy Framework and has potential over time to build knowledge and practice.
4. To deal with the issue of the inexperienced 'knowledgeable other' Chokshi and Fernandez (ibid.) are of the opinion that content knowledge (including facilitation skills) should not be the gatekeeper that prevents participation or implementation of learning study. Rather, learning study should be seen as a vehicle through which teachers can deepen their understanding. This does not detract from the fact that a level of teachers/facilitators need to be skilled and equipped with the principles inherent in learning study and variation theory to lead learning study groups. To this end, the Department of Education could forge links with researchers and other individuals skilled enough to implement learning study groups that are comprised of subject advisors and selected mathematics teachers. Once a learning study cycle or two is completed with these individuals they could then take on the role of 'inexperienced' knowledgeable other and implement their own learning study groups. After all, this is part of the road that I travelled in order to implement the learning study reported on in this thesis, the difference being that I had to rely on my reading knowledge of learning study and variation theory and support from my supervisor to implement a learning study. I am of the opinion that if an individual is provided with opportunities to lead more than one learning study cycle, his/her knowledge base and experience will grow, thus gradually moving from 'inexperienced' knowledgeable other to experienced knowledgeable other. Using the idea of a cascading model to provide other teachers with experience of learning

study that is underpinned by variation theory we get to some level of teachers taking responsibility for their own professional development. This can never happen in an express fashion overnight, but requires long-term commitment. Over time, learning study has the potential to decentralise the support base from the researchers, other skilled individuals and subject advisors to learning study groups within the mathematics department of a school and across schools.

10.4 Limitation of this study and ideas for further research

In identifying the object of learning it was evident that the learners in this study experienced difficulty in distinguishing between the different classes of functions across its algebraic and graphical representations. The critical feature that emerged was in relation to identifying the class of function given its algebraic representation. Secondly, the critical feature only emerged after the second lesson and came into focus for the teacher in the last lesson of the cycle. This was a limitation in the sense that there was no opportunity within this cycle to expand on the example space to focus on the graphical representations as well.

In terms of further research, it would be valuable to conduct another learning study cycle where the first lesson is planned using the same example space that was created in this study with a focus on the critical feature that emerged from this study. The aim of the new study would be to elaborate on the critical feature that emerged from the current study so that it provides a way of being able to distinguish between the different classes of functions across the different representations.

A second area for further research emerges from a recommendation made in the previous section. It would be interesting to pilot a learning study in which the participants are a mix of subject advisors and selected teachers from a school. This pilot study would serve to induct the participants into learning study and the principles that underpin it. Thereafter, the readiness of the participants to lead a learning study group could be investigated by getting them to lead their own learning study groups. Through this process, the ability to scale up this model of TPD could be explored and the feasibility do so examined.

Finally, the message from this study is rather a simple one: if it is clear for teachers and learners what it is they are to discern and this is made possible, learning is enabled. As with an orchestra, the coordinated sound belies its complexity – the contribution of learning study is to make the simple explicit. This may seem easy, but finding the key that unlocks the next level for the learners in terms of their understanding a particular concept (critical feature) is by no means an effortless task.

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Appendix A: Ethics Clearance

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STUDENT NUMBER: 8710172X
Protocol number: 2011ECE005C

05 April 2011

Mr. V Pillay
Vassen.pillay@iburst.co.za

Dear Mr. Pillay

Application for Ethics Clearance: Doctor of Philosophy

I have a pleasure in advising you that the Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has agreed to approve your application for ethics clearance submitted for your proposal entitled

Exploring the potential of a learning study intervention to strengthen and enhance teacher's mediation of a selected object of learning – the case of functions in grade 10

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

Yours sincerely


Matsie Mabeta
Wits School of Education

Cc Supervisor: Prof, J Adler (via email)

Appendix B: Lesson Plans and Worksheet

Learning Study Lesson 1 & 2

Object of learning

To enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential functions across their different representations viz. verbal, equation (symbolic), table of values or sets and graphical.

Teacher Activity	Learner Activity	Notes / Comments
<u>Attending to verbal and symbolic representations:</u> Teacher writes down the first equation on the board and then engages with learner activity 1. Ask learners to evaluate the equation for a specific value of x to reinforce the meanings. Thereafter the second equation is written on the board and the above process is repeated. Engage in very brief discussion of functional notation and use functional notation to show substitution. $y = 2x$; $y = \frac{1}{2}x$; $f(x) = \frac{x}{2}$; $h(x) = \frac{2}{x}$; $p(x) = x^2$; $g(x) = 2^x$ Identify for the learners the name of the class of function that each example represents i.e. linear, quadratic etc. Write down a list. Pay careful attention to chalkboard layout Ask learners to write in the class of function in the space under the verbal description. Now do learner	1. Learners are required to read the equation and to explain the relationship between the constant and the variable. 2. Learners to complete the equation column of the worksheet What about $y = 2^{-x}$ and $y = \frac{-x}{2}$? Perhaps a homework task.	This is an attempt to provide learners with an opportunity to discern the algebraic notation and how the changing position of the '2' and the 'x' mean different things. This is what the learners need to do simultaneously if they are to get a firmer idea of the object of learning. Teacher to pay careful attention to how the learners read each of the equations. The meaning of each equation needs to be reinforced with the actual evaluation of the equation for specific value(s) of x. Need to pay careful attention to rigour in which mathematical language is used. Need to use reference examples,

activity 2. Engage in a brief explanation about the concepts 'input' and 'output' and how they relate to the equation.		counter examples or other types of examples to redirect learner's thinking if needed.
<u>The relationship between input and output:</u> Explain the relationship between 'input' and 'output' with respect to the table of values. Complete table of values - refer to worksheet. Teacher to direct the completion of one ordered pair per example in the table as well as a mapping.	Learners to complete table of values.	If the need arises teacher to show all the steps to calculate the output value. With the example $y = \frac{2}{x}$ select x-values such that the y-value is an integer. This provides opportunity to reinforce the idea that for the table of values we could skip some x-values.
<u>Relating input and output to table of values and mapping of sets:</u> Explain how the table of values relates to the mapping between two sets - included in this explanation is the concept of 'input' and 'output'	Learners to complete the mappings between the two sets	Possible opportunity to introduce the concepts "domain" and "range" Domain - all the x-values (input values) for which the function exists and Range is the set of related y-values (output values).
<u>Extraction of coordinates and sketching the graph:</u> Reiterate that the table of values and the mappings between two sets come from the equation. Also highlight how the table and the mappings give us ordered pairs (coordinates) which are to be plotted on the Cartesian plane.	Learners to draw the graphs	
Ask learners to generate other examples for each class of function If learners provide inappropriate responses ask other learners to comment and identify the possible class of function. Pay careful attention to chalkboard layout.	Learners to write down these examples at the back of the worksheet.	Linear: $y=2x$; $f(x)=-x$; $p(x)=\frac{-1}{2}x+5$ Quadratic: $f(x)=-x^2-1$; $p(x)=\frac{1}{4}x^2+\frac{1}{2}$ Exponential: $q(x)=-4^x$; $p(x)=3^x-4$ Hyperbolic: $t(x)=\frac{-3}{x}$; $s(x)=\frac{4}{x}+1$

Learning Study Lesson 3 and 4

Object of learning

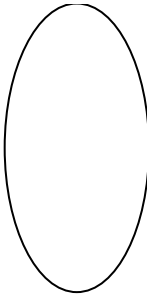
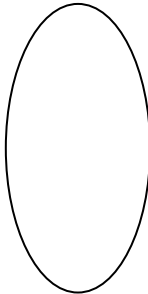
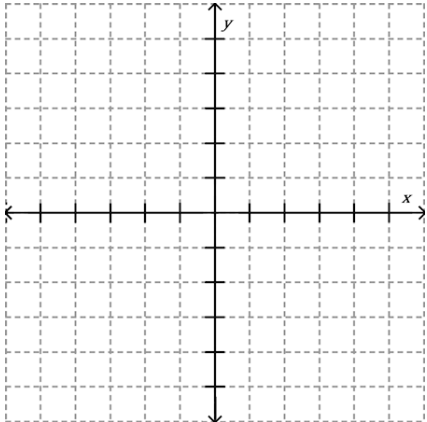
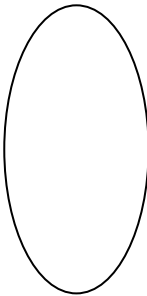
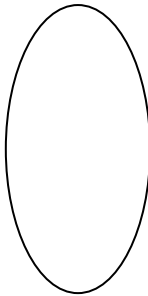
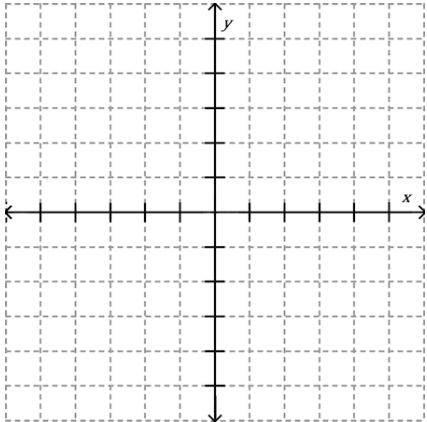
To enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential functions across their different representations viz. verbal, equation (symbolic), table of values or sets and graphical.

Teacher Activity	Learner Activity	Notes / Comments
<u>Attending to verbal and symbolic representations:</u> Teacher writes down (one pair at a time) $y = 2x; y = \frac{1}{2}x$ Engage with activity 1 $y = \frac{1}{2}x; f(x) = \frac{x}{2};$ Engage with activity 1 & 2 $y = 2x; y = \frac{1}{2}x; f(x) = \frac{x}{2}$ Engage with activity 1 (but don't do substitution) $f(x) = \frac{x}{2}; h(x) = \frac{2}{x};$ Engage with activity 1 & 2 (knowledge of exponents important: x^{-1}, may need to insert this) Teacher writes down the next pair $p(x) = x^2; g(x) = 2^x$ Engage with activity 1 & 2 $g(x) = 2^x; y = 2^{-x}$ Engage with activity 1 & 2, if necessary $f(x) = \frac{x}{2}; y = \frac{-x}{2}, \text{ if necessary}$ Task:	The pace of this section is fast - these are not new to the learners but not at the expense of 'loosing' learners 1. Learners are required to read the equation and to explain what is different and what is same between the equations. The teacher could reinforce the difference by doing one substitution and going through the actual operation/process that relates '2' and 'x'. (Use a value or more, if needed, from the table on the worksheet and learners are to fill in these values on the table). Link to verbal part of worksheet & get learners to complete this part only after they have articulated the verbal correctly. 2. Insert the meaning of functional notation - be specific about the substitution - the y value for a given x value (keep in mind that when you get to any graphical representation to reinforce this idea). What about $y = 2^{-x}$ and $y = \frac{-x}{2}$? Perhaps a homework task. Possible chalkboard layout:	This is an attempt to provide learners with an opportunity to discern the algebraic notation and how the changing position of the '2' and the 'x' mean different things. This is what the learners need to do simultaneously if they are to get a firmer idea of the object of learning. Teacher to pay careful attention to how the learners read each of the equations. The meaning of each equation needs to be reinforced with the actual evaluation of the equation for specific value(s) of x. Focus on the power of x as one possible identifier that distinguishes between the different families of functions. Need to pay careful attention to rigour in which mathematical language is used.

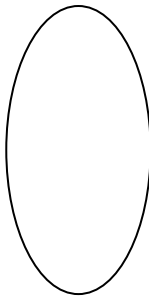
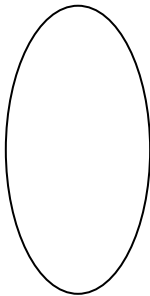
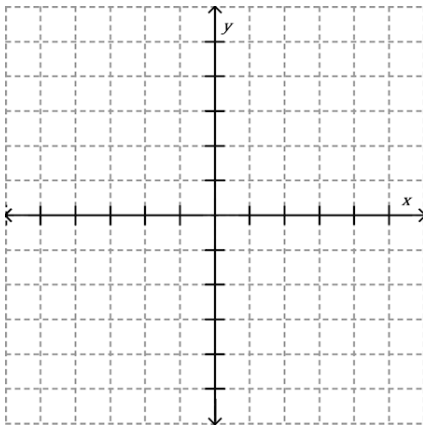
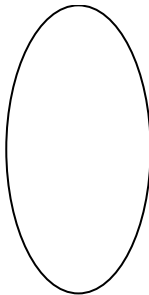
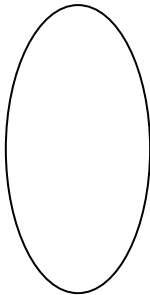
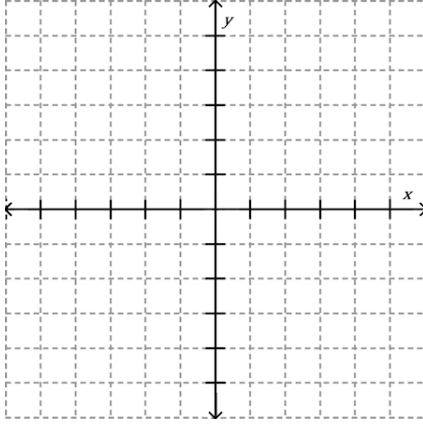
Group the equations that belong together: $y = 2x ; h(x) = \frac{2}{x} ; y = \frac{1}{2}x ; p(x) = x^2 ; f(x) = \frac{x}{2} ;$ $g(x) = 2^x ; y = \frac{-x}{2} ; y = 2^{-x}$	1. $y = 2x ; y = \frac{1}{2}x ; f(x) = \frac{x}{2} ; y = \frac{-x}{2}$ } Straight line/Linear 2. $p(x) = x^2$ } Parabola/Quadratic 3. $g(x) = 2^x ; y = 2^{-x}$ } Exponential 4. $h(x) = \frac{2}{x}$ } Hyperbola	Need to use reference examples, counter examples or other types of examples to redirect learner's thinking if needed.
<u>The relationship between input and output:</u> x-values (input) are randomly selected from a set of numbers, in this lesson we use the set of real numbers. So we just pick a few numbers. But to the value for y (output value) depends on the value you choose for x. Refer to the x and y values that were used in the previous part of the lesson. Complete table of values - refer to worksheet. Teacher to direct the completion of one ordered pair per example in the table.	Explain the relationship between 'input' and 'output' with respect to the table of values: The x-values on the table of values in the worksheet was provided by the teacher. But what informed the teacher when selecting these values? Randomly selected May need to explain the idea of random very quickly Learners to complete table of values.	If the need arises teacher to show all the steps to calculate the output value. With the example $y = \frac{2}{x}$ select x-values such that the y-value is an integer. This provides opportunity to reinforce the idea that for the table of values we could skip some x-values.
<u>Extraction of coordinates and sketching the graph:</u> Reiterate that the table of values come from the equation. Also highlight how the table of values give us ordered pairs (coordinates) which are to be plotted on the Cartesian plane.		Learners to draw the graphs on the worksheet - even if they say that it is small (this was an issue in lesson 2). 1 block represents 1 unit
Ask learners to generate other examples for each class of function If learners provide inappropriate responses ask other learners to comment and identify the possible class of function. Pay careful attention to chalkboard layout.	Learners to write down these examples at the back of the worksheet. Possible chalkboard layout (following from what is already on the board)	Linear: $y=2x$; $f(x)=-x$; $p(x)=\frac{-1}{2}x+5$ Quadratic: $f(x)=-x^2-1$; $p(x)=\frac{1}{4}x^2+\frac{1}{2}$ Exponential: $q(x)=-4^x$; $p(x)=3^x-4$ Hyperbolic: $t(x)=\frac{-3}{x}$; $s(x)=\frac{4}{x}+1$

	1. $y = 2x$; $y = \frac{1}{2}x$; $f(x) = \frac{x}{2}$; $y = \frac{-x}{2}$ }Straight line/Linear LGE's 2. $p(x) = x^2$ } Parabola/Quadratic LGE's 3. $g(x) = 2^x$; $y = 2^{-x}$ } Exponential LGE's 4. $h(x) = \frac{2}{x}$ } Hyperbola LGE's	
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Worksheet

Verbal	Equation	Table of values	Mapping of sets	Graph																
Input value multiplied by two gives the output value OR Double the input value to get the output value.		<table border="1"> <tr> <td>x</td><td>-5</td><td>-3</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>7</td></tr> <tr> <td>y</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>	x	-5	-3	-1	0	1	2	7	y								Input (x)  Output (y) 	
x	-5	-3	-1	0	1	2	7													
y																				
Input value multiplied by half gives the output value		<table border="1"> <tr> <td>x</td><td>-5</td><td>-3</td><td>-1</td><td>0</td><td>1</td><td>4</td><td>5</td></tr> <tr> <td>f(x)</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>	x	-5	-3	-1	0	1	4	5	f(x)								Input (x)  Output f(x) 	
x	-5	-3	-1	0	1	4	5													
f(x)																				

Input value divided by two gives the output value		<table border="1"> <thead> <tr> <th>x</th> <th>g(x)</th> </tr> </thead> <tbody> <tr><td>-5</td><td></td></tr> <tr><td>-3</td><td></td></tr> <tr><td>-1</td><td></td></tr> <tr><td>0</td><td></td></tr> <tr><td>1</td><td></td></tr> <tr><td>4</td><td></td></tr> <tr><td>5</td><td></td></tr> </tbody> </table>	x	g(x)	-5		-3		-1		0		1		4		5		Input (x) Output g(x)	
x	g(x)																			
-5																				
-3																				
-1																				
0																				
1																				
4																				
5																				
Two divided by the input value gives the output value		<table border="1"> <thead> <tr> <th>x</th> <th>-4</th> <th>-2</th> <th>-1</th> <th>0</th> <th>1</th> <th>2</th> <th>4</th> </tr> </thead> <tbody> <tr> <th>y</th> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> </tr> </tbody> </table>	x	-4	-2	-1	0	1	2	4	y								Input (x) Output (y)	
x	-4	-2	-1	0	1	2	4													
y																				

Raise the input value to the power of two to get the output value OR Square the input value to get the output value		<table><tr><td>x</td><td>p(x)</td></tr><tr><td>-4</td><td></td></tr><tr><td>-3</td><td></td></tr><tr><td>-2</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>3</td><td></td></tr><tr><td>4</td><td></td></tr></table>	x	p(x)	-4		-3		-2		0		1		3		4		Input (x) Output p(x)  	
x	p(x)																			
-4																				
-3																				
-2																				
0																				
1																				
3																				
4																				
Two raised to the power of the input value gives the output value.		<table><tr><td>x</td><td>g(x)</td></tr><tr><td>-4</td><td></td></tr><tr><td>-3</td><td></td></tr><tr><td>-1</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr></table>	x	g(x)	-4		-3		-1		0		1		2		3		Input (x) Output g(x)  	
x	g(x)																			
-4																				
-3																				
-1																				
0																				
1																				
2																				
3																				

Appendix C: Lessons chunked into events

Lesson 1						Judgement of Notion																			
Event	Duration					EXISTENCE		REFLECTION Observable Actions										NECESSITY - AUTHORITY Accompanying Teacher Talk							
		Notion	Sub- Notion	Examples Generated by:		Verbal	Written			Revising mathematical rules/Conventions	Revisiting effects of parameters	Substituting and calculating	Plotting points	Drawing graphs	Changing representation	Identifying and naming functions given verbal, algebraic or graphical representation	Comparing y-values for given x-values	Identifying sameness between equations	Learners generating examples	Mathematics				Teacher	
Teacher	Learner			Words	Symbolic		Graphical	Definition / Theorems	Empirical / Technology											Conventions / Rules	Process	Teacher asserts	Teacher confirms	Everyday	
1	2:31	Multiplicatio n with 11		11×11 = 11×12 = 11×14 =		↙		↙																	
2.1	2:57	The Linear Function	Finding y-values for given x-values	y=2x				↙			↙									Pm +		R W			
2.2	0:49	The Linear Function	Type of function	y=2x				↙							↙						↙				

2.3	2:10	The Linear Function	Give other equations of a linear functions		y=x y=3x+1	✓												✓				Pm-		R W				
3.1	0:35	Functional Notation	Convention for writing in functional notation	f(x) = y M(x) = y				✓		✓											C		✓					
3.2	3:56	Functional Notation	Finding y-values for given x-values	f(x) = $\frac{1}{2}$ x Find f(-4) and f(3)		✓		✓		✓		✓									R	Pm ₊		R W				
3.3	0:31	Functional Notation	Improper fraction	f(3) = $\frac{3}{2}$ =1,5				✓		✓											R		✓	R W				
3.4	2:53	The linear function	Type of function	f(x) = $\frac{1}{2}$ x				✓							✓									A				
4	8:08	Linear, Hyperbolic, Quadratic & Exponential Function	Finding y-values for given x-values (x=-4 & x=3)	g(x) = $\frac{x}{2}$ h(x) = $\frac{2}{x}$ k(x) = x ² m(x) = 2 ^x				✓		Learners are completing the work on their worksheets																		

5.1	1:39	The Linear Function	Finding y-values for given x-values & completing a table of values	y=2x				✓			✓			✓										R	W		
5.2	0:06	The Linear Function	Type of function	y=2x				✓							✓									R	W		
6.1	1:11	The Linear Function	Finding y-values for given x-values & completing a table of values	$f(x) = \frac{1}{2}x$				✓			✓			✓										R	W		
6.2	3:41	The Linear Function	Type of function	$f(x) = \frac{1}{2}x$				✓			✓	✓	✓	✓							E	C		✓	R	W	
7.1	0:22	The Linear Function	Finding y-values for given x-values & completing a table of values	$g(x) = \frac{x}{2}$				✓			✓			✓											R	W	
7.2	0:10	The Linear Function	Type of function	$g(x) = \frac{x}{2}$				✓							✓										W		
8	0:16	The Linear Function	Establishing g equivalence between f(x) and g(x)	$f(x) = \frac{1}{2}x$				✓		Observable action is not explicit, it is inferred that the learners compared the y values for the given x values since the table of values is on the chalkboard															A		
				$g(x) = \frac{x}{2}$														✓									

9.1	1:33	The Hyperbolic & Linear Function	Establishin g equivalence between h(x) and g(x)	$h(x) = \frac{2}{x}$ $g(x) = \frac{x}{2}$ $f(x) = \frac{1}{2}x$				✓			✓					✓					E		Pm ₊		R W	
9.2	0:54	The Hyperbolic Function	Notation for writing negative fractions	$\frac{2}{-3}$; $-\frac{2}{3}$; $-\frac{2}{3}$				✓		✓												R			R W	
9.3	1:34	The Hyperbolic Function	Finding y-values for given x-values & completing a table of values	$h(x) = \frac{2}{x}$				✓			✓			✓											R W	
9.4	1:07	The hyperbolic function	Type of function	$h(x) = \frac{2}{x}$				✓						✓										✓		
10.1	2:58	The Quadratic Function	Squaring a negative number	k(x)=x ² Find k(-4)				✓		✓		✓									T	R	Pm ₊	✓		
10.2	1:22	The Quadratic Function	Finding y-values for given x-values & completing a table of values	k(x) = x ²				✓			✓			✓											A	

10.3	1:50	The Quadratic Function	Type of function	$k(x) = x^2$				✓							✓									R W	
11.1	2:37	The Exponential Function	Finding a y value for a given x value	$m(x)=2^x$ Find $m(-4)$				✓		✓		✓									R		✓		
11.2	2:52	The Exponential Function	Finding y-values for given x-values & completing a table of values	$m(x) = 2^x$				✓			✓			✓							R		✓	R W	✓
11.3	1:52	The Exponential Function	Type of function	$m(x) = 2^x$				✓							✓								✓		
12	2:47	The Linear Function	Write in algebraic form	Input value multiplied by two gives the output value		✓	✓							✓							C			R W	
13	0:36	The Linear Function	Write in algebraic form	Input value multiplied by half gives the output value		✓	✓							✓										R W	
14	7:02	The Hyperbolic Function	Write in algebraic form	Input value divided by two gives the output value		✓	✓							✓										A	

Lesson 2

Lesson 2

Event

Duration

1

1:22

Hyperbolic Function

Type of function

$y = \frac{2}{x}$

2

5:20

Exponential, Parabolic & Linear Function

Type of function

$y=2^x$
 $y=x^2$
 $y=-x^2$
 $y=2x$
 $y = \frac{1}{2}x$

3.1

2:53

Exponential, Parabola & Linear Function

Providing equations of specific cases

$y=x^2$

$y=2x$

EXISTENCE

Verbal

Words

Symbolic

Graphical

REFLECTION

Observable Actions

Revising mathematical rules/Conventions

Revisiting effects of parameters

Substituting and calculating

Plotting points

Drawing graphs

Changing representation

Identifying and naming functions given verbal, algebraic or graphical representation

Comparing y-values for given x-values

Identifying sameness between equations

Learners generating examples

NECESSITY - AUTHORITY

Accompanying Teacher Talk

Mathematics

Definition / Theorems

Empirical / Technology

Conventions / Rules

Process

Teacher

Teacher asserts

Teacher confirms

Everyday

Experience

↻

Judgement of Notion

↻

3.2	1:20	Hyperbola, Exponential, Parabola & Linear Function	Providing the general equation	$y=ax^2+q$ $f(x)=mx+c$ $y = \frac{k}{x}$ $f(x)=a^x$				✓	✓	✓					✓							C		✓		
4	0:29	Dependent & independent variables	Input & output values	$y = \frac{k}{x}$ $f(x)=a^x$ $y=ax^2+q$ $y=mx+c$			✓		✓												C				RW	
5.1	2:58	The Linear Function	Write in algebraic form	Input value multiplied by two gives the output value OR Double the input value to get the output value		✓	✓							✓												RW
5.2	1:12	The Linear Function	Finding y-values for given x-values & completing a table of values	$y=2x$				✓			✓			✓												W A
5.3	2:06	The Linear Function	Expressing x and y values as a mapping between two sets	$y=2x$				✓						✓							C					R W

9.2	1:54	The Hyperbolic function	Finding the y-value when x=0	$y = \frac{2}{x}$				✓			✓										T					R	
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Lesson 3

Lesson 3						Judgement of Notion																				
Event	Duration					EXISTENCE		REFLECTION Observable Actions												NECESSITY - AUTHORITY Accompanying Teacher Talk						
		Notion	Sub-Notion	Examples Generated by:		Verbal	Written			Revising mathematical rules/Conventions	Revisiting effects of parameters	Substituting and calculating	Plotting points	Drawing graphs	Changing representation	Identifying and naming functions given verbal, algebraic or graphical representation	Comparing y-values for given x-values	Identifying sameness between equations	Learners generating examples	Mathematics				Teacher		Experience
				Teacher	Learner		Words	Symbolic	Graphical											Definition / Theorems	Empirical / Technology	Conventions / Rules	Process	Teacher asserts	Teacher confirms	Everyday
1.1	1:08	The Linear Function	Express verbally	y=2x			◀							◀									R A			
1.2	2:06	The Linear Function	Relationship between '2' & 'x'	y=2x			◀		◀												C		A			
1.3	3:31	The Linear Function	Finding y-values for given x-values	y=2x			◀			◀											Pm ⁺		RW			
2.1	2:44	The Linear Function	Relationship between '½' and 'x'	$y = \frac{1}{2}x$			◀		◀												C		R A			

2.2	5:46	The Linear Function	Finding y-values for given x-values	$y = \frac{1}{2}x$ $y = \frac{x}{2}$				✓			✓											Pm+	✓	A W	✓
2.3	4:24	The Linear Function	Establishing equivalence between equations	$y = \frac{1}{2} \times x$ $y = x \div 2$				✓			✓					✓				E				RW	
3.1	5:51	The Linear Function	Write in algebraic form	Input value multiplied by two gives the output value OR Double the input value to get the output value		✓	✓			✓					✓							C		✓	RW
3.2	1:31	The Linear Function	Finding y-values for given x-values	$y=x + x$				✓			✓														RW A
3.3	5:15	The Linear Function	Finding y-values for given x-values & completing a table of values	$y=2x$ $y=x + x$				✓		✓	✓				✓							C	Pm+		RW
3.4	4:52	The Linear Function	Plotting points and sketching the graph	$y=2x$				✓		✓		✓	✓	✓								C		✓	

3.5	0:55	The Linear Function	Type of function	$y=2x$				✓	✓							✓					E			✓		
3.6	1:49	The Linear Function	x and y intercepts	$y=2x$				✓	✓	✓											D			✓	A	
3.7	0:54	The Linear Function	Give other equations of a linear function		$y=3x$ $y=4x$ $y = \frac{1}{2}x$	✓													✓						RW A	
3.8	1:52	The Linear Function	Is it a linear function?	$y = \frac{1}{2}x$				✓				✓	✓	✓	✓							E				
3.9	3:54	The Linear Function	Plotting points & sketching the graph	$y=2x$				✓		Learners are completing this task on their worksheets. Note this specific activity was done by the teacher (see event 3.4)														✓		
3.10	1:30	The Linear Function	Writing x-values & its associated y-values as ordered pairs	$y=2x$				✓	✓	✓					✓								C		RW	
3.11	3:08	The Linear Function	Expressing x & y-values as a mapping between two sets	$y=2x$				✓	✓	✓					✓								C	✓	W	

3.12	3:23	The Linear Function	Using the mapping between two sets to determine if a function is represented	$y=2x$				✓		✓										D				✓		✓
4.1	1:45	The Hyperbola	Write in algebraic form	Two divided by the input value gives the output value		✓	✓							✓										✓	RW	
4.2	0:45	The Hyperbola	Relationship between the '2' & 'x'	$y = \frac{2}{x}$				✓		✓												C			R	
4.3	6:30	The Hyperbola	Finding y-values for given x-values & completing a table of values	$y = \frac{2}{x}$				✓		✓		✓										R	Pm ⁺	✓	RW A	
4.4	2:11	The Hyperbola	Plotting points & sketching the graph	$y = \frac{2}{x}$				✓			✓	✓	✓	✓										✓		
4.5	0:43	The Hyperbola	Type of Function	$y = \frac{2}{x}$				✓	✓						✓										A	

Lesson 4

Lesson 4						Judgement of Notion																					
Event	Duration					EXISTENCE		REFLECTION Observable Actions										NECESSITY - AUTHORITY Accompanying Teacher Talk									
		Notion	Sub-Notion	Examples Generated by:		Verbal	Written			Revising Mathematical rules/Conventions	Revisiting effects of parameters	Substituting and calculating	Plotting points	Drawing graphs	Changing representation	Identifying and naming functions given verbal, algebraic or graphical representation	Comparing y-values for given x-values	Identifying sameness between equations	Learners generating examples	Mathematics				Teacher		Experience	
				Teacher	Learner		Words	Symbolic	Graphical											Definition / Theorems	Empirical / Technology	Conventions / Rules	Process	Teacher asserts	Teacher confirms		Everyday
1.1	1:33	The Linear Function	Reading the given equations	$y=2x$ $y = \frac{1}{2}x$				◀						◀												R A	
1.2	0:24	The Linear Function	Compare the sound of the verbal utterances	$y=2x$ $y = \frac{1}{2}x$				◀										◀								R A	
1.3	0:54	The Linear Function	Comparing variables	$y=2x$ $y = \frac{1}{2}x$				◀										◀								R A	

1.4	0:59	The Linear Function	Comparing the mathematical operation between 'x' & its coefficient	$y=2x$ $y = \frac{1}{2}x$				✓		✓							✓				C		✓	RW A	
1.5	1:38	The Linear Function	Comparing the exponents of 'x'	$y=2x$ $y = \frac{1}{2}x$				✓		✓							✓				C R		✓	R A	
2.1	2:08	The Linear Function	Graph of a linear function	$f(x)=x+1$				✓		✓					✓								✓	A	
2.2	0:30	The Linear Function	Functional Notation	$f(x)=x+1$				✓		✓											C			A	
2.3	1:42	The Linear Function	Give other equations of a linear function	$y=-x+2$ $y=2x+2$ $f(x)=3x+1$ 6		✓											✓				R		✓	RW A	
3.1	1:31	The Linear Function	Reading the given equations	$y = \frac{1}{2}x$ $f(x) = \frac{x}{2}$				✓						✓										R A	
3.2	0:26	The Linear Function	Compare the sound of the verbal utterances	$y = \frac{1}{2}x$ $f(x) = \frac{x}{2}$				✓									✓						✓	R	

3.3	2:04	The Linear Function	Finding y-values for given x-values to establish equivalence between the two equations	$y = \frac{1}{2}x$ $f(x) = \frac{x}{2}$				✓				✓					✓	✓			E					R	
3.4	0:50	The Linear Function	Comparing the given equations	$y = \frac{1}{2}x$ $f(x) = \frac{x}{2}$				✓										✓								R	A
3.5	0:30	The Linear Function	Type of function	$y = \frac{1}{2}x$ $y = \frac{x}{2}$				✓		✓					✓							R				R	A
4.1	0:44	Hyperbolic & Linear Functions	Reading the given equations	$h(x) = \frac{2}{x}$ $f(x) = \frac{x}{2}$				✓					✓													R	A
4.2	0:12	Hyperbolic & Linear Functions	Compare the sound of the verbal utterances	$h(x) = \frac{2}{x}$ $f(x) = \frac{x}{2}$				✓										✓								R	A
4.3	0:55	Hyperbolic & Linear Functions	Comparing the given equations	$h(x) = \frac{2}{x}$ $f(x) = \frac{x}{2}$				✓										✓								R	A
4.4	3:05	The Hyperbolic Function	Rewriting the given equation	$h(x) = \frac{2}{x}$				✓		✓												R		✓		RW	A

4.5	1:55	The Hyperbolic Function	Type of function	$h(x) = 2x^{-1}$				✓		✓						✓						R		✓		
5.1	1:48	The Hyperbolic Function	Graph of the hyperbola	$p(x) = \frac{-2}{x}$				✓		✓						✓								✓		
5.2	0:41	The Hyperbolic Function	Rewriting the given equation	$p(x) = \frac{-2}{x}$				✓		✓												R				W A
6.1	0:52	Parabolic & Exponential Function	Reading the given equations	$p(x) = x^2$ $g(x) = 2^x$				✓						✓												R A
6.2	0:31	Parabolic & Exponential Function	Compare the sound of the verbal utterances	$p(x) = x^2$ $g(x) = 2^x$				✓									✓									R A
6.3	0:40	Parabolic & Exponential Function	Identifying and comparing exponents	$p(x) = x^2$ $g(x) = 2^x$				✓										✓								A
6.4	1:12	The Parabolic Function	Type of function	$p(x) = x^2$				✓		✓						✓						R		✓		R A
7	1:53	The Parabolic Function	Graph of the parabola	$g(x) = x^2 - 2$				✓		✓						✓								✓		

13.5	1:54	The Linear Function	Expressing x & y values as a mapping between two sets	$y=2x$				✓						✓							C		✓		
13.6	6:25	The Linear Function	Sketching the graph	$y=2x$				✓				✓	✓	✓							C		✓		RW A
14.1	1:40	The Hyperbola	Write in algebraic form	Two divided by the input value gives the output value		✓	✓							✓									✓		
14.2	3:40	The Hyperbola	Finding y-values for given x-values & completing a table of values	$f(x) = \frac{2}{x}$				✓	Learners are completing this task on their worksheets																
14.3	2:53	The Hyperbola	Finding $f(0)$	$f(x) = \frac{2}{x}$				✓	✓		✓										R		✓		
15	1:23	The Quadratic Function	Functional notation	$g(x) = x^2 - 2$				✓	✓		✓										R	Pm +	✓		A