# Solitary wave solutions for the magma equation: symmetry methods and conservation laws.

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*Glory be to God.*

# Declaration

I declare that this dissertation is my own, unaided work. The content of the dissertation is original except where due references have been made. It is being submitted as fulfillment of the requirements for the Degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university.

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September 19, 2014

# Abstract

The magma equation which models the migration of melt upwards through the Earth's mantle is considered. The magma equation depends on the permeability and viscosity of the solid mantle which are assumed to be a function of the voidage . It is shown using Lie group analysis that the magma equation admits Lie point symmetries provided the permeability and viscosity satisfy either a power law, or an exponential law for the voidage or are constant. The conservation laws for the magma equation for both power law and exponential law permeability and viscosity are derived using the multiplier method. The conserved vectors are then associated with Lie point symmetries of the magma equation. A rarefactive solitary wave solution for the magma equation is derived in the form of a quadrature for exponential law permeability and viscosity. Finally small amplitude and large amplitude approximate solutions are derived for the magma equation when the permeability and viscosity satisfy exponential laws.

# Acknowledgments

I would like to first thank God for allowing me the opportunity to continue with my studies and giving me strength to complete my dissertation.

I would like to thank my family for their unwavering support. You are the reason I continue to excel in all my endeavours and I thank you for the understanding and patience you had with me while I was pursuing my academic career. To my mom and dad, Nomvula and Themba, your advice, support and investment in my education has really been invaluable and I am so blessed to have such wonderful supportive parents. My siblings, Thandeka and Hlengiwe; and my niece Nokukhanya, my most favourite women in the world, you were my pillars of strength that kept me sane through the rough hustle of my academic career.

To my supervisor, Professor David Mason, thank you for your guidance and support throughout my research. You are certainly the biggest influence in me taking my academic career this far. Your kindness and gentle nature is something I will never forget. I certainly learned a lot from you and your attention to detail is second to none. I feel honoured that I was able to work with you on my dissertation.

A very special thanks to all my colleagues in the School of Computational and Applied Mathematics. Your input and help during my research has been invaluable.

To my friends, thank you for keeping me grounded and bringing laughter and joy to my life.

Lastly I would like to acknowledge the University of the Witwatersrand, National Research Foundation and JCSE Coachlab for their financial support.

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# Chapter 1

# Introduction

## 1.1 Introduction

The development and the study of a certain class of partial differential equations known as evolution equations has taken place for over the last 50 years. This is because of the special type of solutions that these equations posses known as solitary waves or solitons. A solitary wave or "wave of translation" is a wave in nonlinear systems that travels through media without change in shape and loss in energy. This wave occurs as a result of a balance between nonlinear and dispersive effects.

Solitary waves were first observed by John Scott Russell while conducting experiments to determine the most efficient design for canal boats at the Edinburgh to Glasgow canal in 1834. He called it the "great wave of translation". He described the wave in the following words [1]:

I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped- not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

To study this phenomenon further he experimentally generated solitary waves by dropping a weight along the channel of water. From this experiment he was able to deduce empirically that

$$c^2 = g(h + a), \tag{1.1}$$

where $c$ is the speed of the wave, $h$ the undisturbed depth of the water, $a$ the amplitude of the wave and $g$ the acceleration due to gravity. What can be deduced from (1.1) is the special property of solitary waves that larger amplitude waves travel faster.

Boussinesq and Lord Rayleigh put Russell's formula (1.1) into something more concrete by assuming that a solitary wave has a length scale much greater than the depth of water [1]. From this assumption they were able to deduce the following wave profile,

$$\zeta(x, t) = a \ \mathrm{sech}^2(\beta(x - ct)). \tag{1.2}$$

However, the authors did not write down the simple equation for which $\zeta(x, t)$ is a solution. This was completed by Korteweg and de Vries [1].

Solitons, a generalisation of solitary waves, were introduced by Zabusky and Kruskal [1]. A soliton is a solitary wave that behaves like a "particle" and satisfies the following conditions.
1. It maintains its shape when traveling at a constant speed.
2. After interacting with another soliton it emerges unchanged except possibly for a phase change.

In this dissertation we will investigate the magma equation, which describes the phenomenon of solitary waves.

The magma equation,

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[K(\phi)\left(1 - \frac{\partial}{\partial z}\left(G(\phi)\frac{\partial \phi}{\partial t}\right)\right)\right] = 0, \tag{1.3}$$

was proposed by Scott and Stevenson [3] to describe the migration of melt upwards through the mantle of the Earth. In the equation, $\phi(t, z)$ is the voidage or volume fraction of melt, $t$ is time, $z$ is the vertical spatial coordinate, $K$ is the permeability of the medium and $G$ is the viscosity of the matrix phase. The variables $\phi$, $t$, $z$ and the physical quantities $K(\phi)$ and $G(\phi)$ in (1.3) are dimensionless. The voidage $\phi(t, z)$ is scaled by the background voidage $\phi_0$. The background state is therefore defined by $\phi = 1$. The characteristic length in the $z$-direction, which is vertically upwards, is the compaction length $\delta_c$ defined by

$$\delta_c = \left[\frac{K(\phi_0)G(\phi_0)}{\mu}\right]^{\frac{1}{2}}, \tag{1.4}$$

where $\mu$ is the coefficient of shear viscosity of the melt. The characteristic time is $t_0$ defined by

$$t_0 = \frac{\phi_0}{g \triangle \rho}\left[\frac{\mu G(\phi_0)}{K(\phi_0)}\right]^{\frac{1}{2}}, \tag{1.5}$$

where $g$ is the acceleration due to gravity and $\triangle\rho$ is the difference between the density of the solid matrix and the density of the melt. The permeability is scaled by $K(\phi_0)$ and therefore

$$K(1) = 1. \tag{1.6}$$

When the voidage is zero the permeability must also be zero and therefore

$$K(0) = 0. \tag{1.7}$$

The viscosity $G(\phi)$ is scaled by $G(\phi_0)$ so that

$$G(1) = 1 \tag{1.8}$$

and $G(0)$ will be infinite because the viscosity of the solid matrix is infinite when the voidage vanishes. In the derivation of (1.3) it is assumed that the background voidage satisfies $\phi_0 \ll 1$. An outline of the derivation of equation(1.3) is given in Chapter 3.

The partially molten medium consists of a solid matrix and a fluid melt which are modelled as two immiscible fully connected fluids of constant but different densities. The density of the melt is less than the density of the solid matrix and the melt migrates through the compacting medium by the buoyancy force due to the difference in density between the melt and the solid matrix. Changes of phase are not included in the model. It is assumed that the melting has occured and only migration of the melt under gravity is described by (1.3) [2]. In the model proposed by Scott and Stevenson [3]

$$K(\phi) = \phi^n, \quad G(\phi) = \phi^{-m}, \tag{1.9}$$

where $n \geq 0$ and $m \geq 0$. Solitary wave solutions have been obtained for equation (1.3) when the permeability and matrix viscosity satisfy the power laws (1.9), by looking for travelling wave solutions of the form

$$\phi = \psi(z - ct) \tag{1.10}$$

where the constant $c$ is the wave speed $[2 - 10]$.

Lie point symmetries for equation (1.3) when the permeability and matrix viscosity satisfy power laws (1.9) have been investigated by Maluleke and Mason [10]. In Chapter 4 the functional forms of $K(\phi)$ and $G(\phi)$ are investigated using Lie group analysis.

Conservation laws for equation (1.3) when the permeability and matrix viscosity satisfy the power laws (1.9) have been obtained using the direct method by Barcilon and Richter [4] and Harris [11], and using Lie point symmetry generators by Maluleke and Mason [10, 12]. In Chapter 5 the conservation laws for equation (1.3) are derived using the multiplier method.

## 1.2 Aims and outline of the dissertation

The aim of this dissertation is to investigate the different permeability and viscosity models for the magma equation using Lie group analysis. Subsequently this will lead to group invariant solutions which will help us understand better how magma is transported through the Earth's mantle.

A new approach to deriving the conservation laws for the magma equation, namely, the multiplier method is also considered. This is a more systematic approach to deriving conservation laws than the direct method. An outline of the dissertation is as follows. In Chapter 2, the basic operators, definitions and theory used in this dissertation will be briefly explained. In Chapter 3, the mathematical model is derived. The different permeability and viscosity models are investigated using Lie group analysis in Chapter 4. Further in Chapter 5, the conservation laws for equation (1.3) are derived using the multiplier method. Then in Chapter 6, the group invariant solution of equation (1.3) is investigated. Finally the conclusions are summarised in Chapter 7.

# Chapter 2

# Formulae and Theory

## 2.1 Introduction

In this chapter the formulae and theory that we will be used in the forthcoming chapters is presented. Results from Lie group analysis will be used to derive the permeability and viscosity models and solve the partial differential equation (1.3). A description of conservation laws and the method that will be used for the derivation of conservation laws in Chapter 5 is also given.

## 2.2 Lie point symmetries

Consider a k-th order partial differential equation in one dependent variable and $p$ independent variables $x = (x^1, x^2, .., x^p)$

$$F(x, \phi, \phi_{(1)}, ..., \phi_{(k)}) = 0, \tag{2.1}$$

where $\phi_{(k)}$ denotes the collection of $k^{th}$- order partial derivatives of $\phi$. In our case $x^1 = t, x^2 = z$. The Lie point symmetry generators

$$X = \xi^i(x, \phi)\frac{\partial}{\partial x^i} + \eta(x, \phi)\frac{\partial}{\partial \phi}, \tag{2.2}$$

where $i$ runs from 1 to $p$, are derived by solving the invariance criterion

$$X^{[k]}F(x, \phi, \phi_{(1)}, ..., \phi_{(k)})|_{F=0} = 0, \tag{2.3}$$

for $\xi^1(x, \phi), \xi^2(x, \phi)$,..., $\xi^p(x, \phi)$ and $\eta(x, \phi)$ where $X^{[k]}$, called the $k^{th}$ prolongation of $X$, is given by

$$X^{[k]} = X + \zeta_i\frac{\partial}{\partial \phi_i} + \zeta_{ij}\frac{\partial}{\partial \phi_{ij}} + ... + \zeta_{i_1 i_2 ... i_k}\frac{\partial}{\partial \phi_{i_1 i_2 ... i_k}} \tag{2.4}$$

where

$$\begin{aligned}
\zeta_i &= D_i(\eta) - \phi_s D_i(\xi^s), && (2.5)\\
\zeta_{ij} &= D_j(\zeta_i) - \phi_{is} D_j(\xi^s), && (2.6)\\
\zeta_{i_1\ldots i_k} &= D_{i_k}(\zeta_{i_1\ldots i_{k-1}}) - \phi_{i_1\ldots i_{k-1}s} D_{i_k}(\xi^s), && (2.7)
\end{aligned}$$

with summation over the repeated index s, from 1 to $p$ and $i \leq j$, $i_1 \leq i_2 \leq \ldots \leq i_k$ . The total derivatives are defined by

$$D_i = \frac{\partial}{\partial x^i} + \phi_i \frac{\partial}{\partial \phi} + \phi_{il} \frac{\partial}{\partial \phi_l} + \ldots \quad . \tag{2.8}$$

The symmetry generators obtained are of the form

$$X_r = \xi_r^i(x,\phi)\frac{\partial}{\partial x^i} + \eta_r(x,\phi)\frac{\partial}{\partial \phi}, \tag{2.9}$$

for $r = 1, 2, \ldots q$ where $q$ is the number of admitted Lie point symmetries. Any linear combination of Lie point symmetries is also a Lie point symmetry of the partial differential equation.

Denote this linear combination by $X_c$ where

$$X_c = c_1 X_1 + c_2 X_2 + \ldots + c_r X_q. \tag{2.10}$$

The solution $\phi = \Phi(x^1, x^2, \ldots, x^p)$ is an invariant solution of (2.1) provided

$$X(\phi - \Phi(x^1, x^2, \ldots, x^p))|_{\phi=\Phi} = 0. \tag{2.11}$$

By substituting the group invariant solution into the nonlinear partial differential equation in two independent variables one can reduce the partial differential equation to an ordinary differential equation in a new variable. This new variable is called the similarity variable.

## 2.3 Conservation laws

Consider again the s-th order partial differential equation (2.1). The equation

$$D_i T^i = 0, \tag{2.12}$$

evaluated on the surface given by (2.1), where $i$ runs from 1 to $p$ and $D_i$ denotes the total derivative given by (2.8), is called a conservation law for the differential equation (2.1). The vector $T = (T^1, \ldots, T^p)$ is a conserved vector for the partial differential equation and $T^1, \ldots, T^p$ are its components. Thus a conserved vector gives rise to a conservation law.

A Lie point symmetry generator $X$, where $X$ is given by (2.2), and $i$ runs from 1 to $p$, is said to be associated with the conserved vector $T = (T^1, \ldots, T^p)$

for the partial differential equation (2.1) if [13, 14]

$$X(T^i) + T^i D_s(\xi^s) - T^s D_s(\xi^i) = 0, \qquad i = 1, 2, ..., p. \tag{2.13}$$

The association of a Lie point symmetry with a conserved vector can be used to integrate the partial differential equation by the double reduction theorem of Sjöberg [15].

New conserved vectors for a partial differential equation can be generated from known conserved vectors and Lie point symmetries of the partial differential equation. For

$$T^i_* = X(T^i) + T^i D_s(\xi^s) - T^s D_k(\xi^i), \qquad i = 1, 2, ..., p, \tag{2.14}$$

where s runs from 1 to $p$ is a conserved vector for the partial differential equation although it may be a linear combination of known conserved vectors [13, 14].

We now present the multiplier method for the derivation of conservation laws for partial differential equations. We outline its application to the partial differential equation (1.3) in two independent variables.

### 2.3.1 Multiplier method

1. Multiply the partial differential equation (1.3) by the multiplier, $\Lambda$, to obtain the conservation law,

$$\Lambda F = D_1 T^1 + D_2 T^2, \tag{2.15}$$

where $F = 0$ is the partial differential equation (1.3) and

$$D_1 = D_t = \frac{\partial}{\partial t} + \phi_t \frac{\partial}{\partial \phi} + \phi_{tt} \frac{\partial}{\partial \phi_t} + \phi_{zt} \frac{\partial}{\partial \phi_z} + ..., \tag{2.16}$$

$$D_2 = D_z = \frac{\partial}{\partial z} + \phi_z \frac{\partial}{\partial \phi} + \phi_{tz} \frac{\partial}{\partial \phi_t} + \phi_{zz} \frac{\partial}{\partial \phi_z} + ... \,. \tag{2.17}$$

The multiplier depends on $t$, $z$, $\phi$ and the partial derivatives of $\phi$. The more derivatives included in the multiplier the wider the range of conserved vectors that can be derived.

2. The determining equation for the multiplier is obtained by operating on (2.15) by the Euler operator $E_\phi$ defined by [17],

$$E_\phi = \frac{\partial}{\partial \phi} - D_1 \frac{\partial}{\partial \phi_t} - D_2 \frac{\partial}{\partial \phi_z} + D_1^2 \frac{\partial}{\partial \phi_{tt}} + D_1 D_2 \frac{\partial}{\partial \phi_{tz}} + D_2^2 \frac{\partial}{\partial \phi_{zz}} - .... \tag{2.18}$$

Since the Euler operator annihilates divergence expressions, operating on (2.15) with $E_\phi$ gives [17]

$$E_\phi[\Lambda F] = 0, \tag{2.19}$$

3. The determining equation (2.19) is separated by equating the coefficients of like powers and products of the partial derivatives of $\phi$. This partial derivatives of $\phi$ are independent quantities because the condition $F = 0$ has not yet been used.

4. When $\phi$ is a solution of the partial differential equation, $F = 0$, (2.15) becomes a conservation law. The condition $F = 0$ is imposed on (2.15). The product of the multiplier and the partial differential equation is then written in conserved form by elementary manipulations. This yields the conserved vectors by setting all the constants equal to zero except one in turn.

# Chapter 3

# Mathematical Model

## 3.1 Introduction

A derivation of the magma equation is now given. This has been treated by several authors [2,3,4,16,19]. We follow closely the review of Nakayama and Mason [2] but we will keep the permeability, K, and the viscosity, G of the matrix, general.

## 3.2 Derivation of the magma equation

In the derivation of the magma equation there are two important quantities that is, microscopic and macroscopic fields. A microscopic field varies on the length scales of solid grains while a macroscopic field varies on a length scale much greater than the length scale of solid grains [2]. In our formulation we will denote microscopic fields of the melt and solid matrix by $m$ and $s$ respectively and macroscopic fields by (1) and (2) respectively.

By considering the momentum of the melt and the momentum of the solid matrix per unit bulk volume it can be shown that the microscopic velocity, $\mathbf{v}$ and the macroscopic velocity, $\mathbf{V}$, satisfy the following relations [4]

$$\mathbf{v}^{(1)} = \mathbf{v}_m = \mathbf{v}, \tag{3.1}$$

$$\mathbf{V}^{(2)} = \mathbf{V}_s = \mathbf{V}. \tag{3.2}$$

Now consider the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{3.3}$$

The macroscopic continuity equation for the melt is

$$\frac{\partial \rho^{(1)}}{\partial t} + \operatorname{div}(\rho^{(1)} \mathbf{v}) = 0, \tag{3.4}$$

where $\rho$ denotes density. Since $\rho^{(1)} = \phi\rho_m$, where $\rho_m$ is constant, (3.4) takes the form

$$\frac{\partial\phi}{\partial t} + \operatorname{div}(\phi\mathbf{v}) = 0 \tag{3.5}$$

Similarly, since $\rho^{(2)} = (1-\phi)\rho_s$, where $\rho_s$ is constant, the macroscopic conservation of mass equation for the solid matrix can be written as

$$\frac{\partial\phi}{\partial t} - \operatorname{div}((1-\phi)\mathbf{V}) = 0. \tag{3.6}$$

We now consider the momentum balance equations for the melt and the solid matrix. Since the Reynolds number R for the melt is small, ( $R \ll 1$) and for the solid matrix even smaller we can neglect the inertia term in the momentum balance equations for the melt and solid matrix. Now the macroscopic momentum balance equations are

$$\sigma^{(r)}_{ki,k} + \rho^{(r)}b^{(r)}_i + I^{(r)}_i = 0, \quad r = 1, 2, \tag{3.7}$$

where

$$\sigma^{(1)}_{ik} = \phi\sigma_{mik}, \tag{3.8}$$

$$\sigma^{(2)}_{ik} = (1-\phi)\sigma_{mik}, \tag{3.9}$$

$$\sigma_{mik} = -p_m\delta_{ik}, \tag{3.10}$$

$$\sigma_{sik} = [-p_m + (\xi^* - \frac{2}{3}\eta^*)\ \operatorname{div}\mathbf{V}]\delta_{ik} + 2\eta^* D_{ik}(V), \tag{3.11}$$

$$D_{ik}(V) = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i}\right), \tag{3.12}$$

$$\rho^{(1)}b^{(1)}_i = -\phi\rho_s g\delta_{i3}, \tag{3.13}$$

$$\rho^{(2)}b^{(2)}_i = -(1-\phi)\rho_s g\delta_{i3}, \tag{3.14}$$

$$I^{(1)}_i = -I^{(2)}_i = -p_m\frac{\partial}{\partial x_i}(1-\phi) - C(v_i - V_i). \tag{3.15}$$

When $r = 1$, we obtain the momentum balance equation for the melt and when $r = 2$ we get the momentum balance equation for the solid matrix. The Cauchy stress tensor is denote $\sigma_{ik}$. The constitutive equations for the melt and solid matrix are given by (3.10) and (3.11) respectively. In equation (3.10) the viscosity is neglected, although, in other equations such as (3.15) we will see that the shear viscosity of the melt is no neglected. In the constitutive equation for the solid matrix (3.11), the bulk and shear viscosities are denoted by $\xi^*$ and $\eta^*$ respectively and it is assumed that the pressure in the solid matrix equals the pressure in the melt, $p_m$ [18]. The solid matrix is modelled as a non-Stokesian fluid. The body force per unit mass acting on the $r-th$ phase, $b_i^{(r)}$, given by equations (3.13) and (3.14), is due to gravity. The z-axis coincides with the 3-axis which is taken to be positive upwards. The interphase volume force, $I_r^{(1)}$, describes the interaction between the melt and the solid matrix; $I_i^{(1)}$ is the force per unit bulk volume on the melt produced by the solid matrix and $I_i^{(2)}$ is the force per unit volume on the solid matrix produced by the melt. The quantity $C$ in (3.15) is a constant.

Now by taking $r = 1$ and $r = 2$ respectively the momentum balance equations for the melt and solid matrix may be written as:
$r$=1:

$$v_i - V_i = -\frac{\phi}{C}\frac{\partial P}{\partial x_i}, \tag{3.16}$$

$r$=2:

$$\frac{\partial}{\partial x_i}\left[\left(\xi - \frac{2}{3}\eta\right) div\mathbf{V}\right] + \frac{\partial}{\partial x_k}\left(\eta\frac{\partial V_i}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\eta\frac{\partial V_k}{\partial x_i}\right)$$
$$-(1-\phi)\frac{\partial P}{\partial x_i} - (1-\phi)g \triangle \rho\delta_{i3} + C(v_i - V_i) = 0, \tag{3.17}$$

where

$$C = \frac{\mu\phi^2}{K}, \tag{3.18}$$

$$P = p_m + \rho_m g z, \tag{3.19}$$

$$\xi = (1-\phi)\xi^*, \quad \eta = (1-\phi)\eta^*, \quad \triangle\rho = \rho_s - \rho_m > 0. \tag{3.20}$$

The quantity$\mu$ is the shear viscosity of the melt, $p_m$ is the pressure in the melt, $p_s$ is the pressure in the solid matrix and $K$ is the permeability of the medium. The constant $C$ given by (3.18) is derived by comparison with Darcy's empirical law for flow through a porous medium.
Consider now one-dimensional flow in the z-direction

$$\frac{\partial}{\partial x_i} = \delta_{i3}\frac{\partial}{\partial z}, \quad \mathbf{v} = (0,0,w), \quad \mathbf{V} = (0,0,W). \tag{3.21}$$

Equation (3.5), (3.6), (3.16) and (3.17) then reduce to

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}(\phi w) = 0, \tag{3.22}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}((1-\phi)W) = 0, \tag{3.23}$$

$$w - W = -\frac{K}{\mu\phi}\frac{\partial P}{\partial z}, \tag{3.24}$$

$$\frac{\partial}{\partial z}\left[\left(\xi+\frac{4}{3}\eta\right)\frac{\partial W}{\partial z}\right]-(1-\phi)\frac{\partial P}{\partial z}-(1-\phi)g\triangle\rho+\frac{\mu\phi^2}{K}(w-W)=0. \tag{3.25}$$

By eliminating $P$ from (3.25) using (3.24) we obtain

$$\frac{\partial}{\partial z}\left[\left(\xi+\frac{4}{3}\eta\right)\frac{\partial W}{\partial z}\right]-(1-\phi)g\triangle\rho+\frac{\mu\phi}{K}(w-W)=0. \tag{3.26}$$

In order to eliminate $w$ from (3.26) we first obtain a relation between $w$ and $W$, If (3.23) is subtracted from (3.22) then we find that

$$\frac{\partial w}{\partial z}[\phi w+(1-\phi)W]=0. \tag{3.27}$$

Integrating (3.27) we obtain

$$\phi w+(1-\phi)W=g(t). \tag{3.28}$$

For a compacting medium bounded below by an impermeable boundary we have $w=0=W$ at the boundary [19]. Thus it follows that $g(t)=0$. Scott and Stevenson [3] chose the barycentric frame in which the volume fluxes of the melt and solid matrix through any surface element are equal and opposite. If this local frame is used then,

$$\phi\mathbf{v}+(1-\phi)\mathbf{V}=0. \tag{3.29}$$

Taking $g(t)=0$ and using (3.28) to eliminate $w$ from (3.26) we obtain

$$\frac{\partial}{\partial z}\left[\left(\xi+\frac{4}{3}\eta\right)\frac{\partial W}{\partial z}\right]-\frac{\mu}{K}W-(1-\phi)g\triangle\rho=0. \tag{3.30}$$

Let

$$G(\phi)=\xi(\phi)+\frac{4}{3}\eta(\phi), \tag{3.31}$$

define the dimensionless variables

$$\bar{\phi}=\frac{\phi}{\phi_0},\quad \bar{z}=\frac{z}{\delta_c},\quad \bar{W}=\frac{W}{W_0},\quad \bar{t}=\frac{t}{t_0},\quad \bar{K}(\bar{\phi})=\frac{K(\psi)}{K(\psi_0)},\quad \bar{G}(\bar{\psi})=\frac{G(\psi)}{G(\psi_0)}. \tag{3.32}$$

where $\phi_0$ is the background voidage,at the present non-dimensionalisation the background state is defined by $\phi=1$. The quantity $\delta_c$ is the compaction length given by

$$\delta_c=\left[\frac{G(\phi_0)K(\phi_0)}{\mu}\right]^{\frac{1}{2}}, \tag{3.33}$$

$W_0$ is the characteristic velocity in the z-direction,

$$W_0=\frac{K(\phi_0)(1-\phi_0)g\triangle\rho}{\mu} \tag{3.34}$$

and $t_0$ is the characteristic time,

$$t_0 = \frac{\phi_0}{g \triangle \rho} \left[ \frac{\mu G(\phi_0)}{K(\phi_0)} \right]^{\frac{1}{2}}. \tag{3.35}$$

In order to keep the notation simple the overhead bars in the dimensionless variables will be suppressed. Equations (3.23) and (3.30) become

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial z}[(1 - \phi_0 \phi) W] = 0, \tag{3.36}$$

$$\frac{\partial}{\partial z}\left( G(\phi) \frac{\partial W}{\partial z} \right) - \frac{W}{K(\phi)} - \frac{(1 - \phi_0 \phi)}{(1 - \phi_0)} = 0. \tag{3.37}$$

We now make the approximation that the background voidage $\phi_0 \ll 1$. Equations (3.36 and (3.37) simplify to

$$\frac{\partial \phi}{\partial t} - \frac{\partial W}{\partial z} = 0, \tag{3.38}$$

$$\frac{\partial}{\partial z}\left( G(\phi) \frac{\partial W}{\partial z} \right) - \frac{W}{K(\phi)} - 1 = 0. \tag{3.39}$$

By eliminating W from (3.38) and (3.39) we obtain

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[ K(\phi) \left( 1 - \frac{\partial}{\partial z}\left( G(\phi) \frac{\partial \phi}{\partial t} \right) \right) \right] = 0. \tag{3.40}$$

Equation (3.40) is the partial differential equation for the voidage $\phi$. It generalises the partial differential equation derived by Scott and Stevenson [3] when $K(\phi)$ and $G(\phi)$ are given by power laws.

# Chapter 4

# Lie Point Symmetries

## 4.1 Introduction

In this section we will use Lie point symmetries [21], to investigate the different permeability and viscosity models for equation (1.3). The approach taken is to consider the case in which the magma equation depends only on the permeability (viscosity), that is taking viscosity (permeability) to be constant and finding the functional forms of $K(\phi)$ $(G(\phi))$ for equation (1.3) to admit Lie point symmetries. For arbitrary forms of $K(\phi)$ and $G(\phi)$, (1.3) admits the Lie point symmetries

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial z}. \tag{4.1}$$

We will determine the forms of $K(\phi)$ and $G(\phi)$ for (1.3) to admit other Lie point symmetries besides the Lie point symmetries (4.1).

## 4.2 Permeability models

When $G(\phi) = 1$ the magma equation (1.3) becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[K(\phi)\left(1 - \frac{\partial^2 \phi}{\partial t \partial z}\right)\right] = 0. \tag{4.2}$$

In this section $K(\phi)$ will not be specified initially. We will investigate the Lie point symmetries of (4.2) and the forms of $K(\phi)$ for these to exist.

Equation (4.2) is

$$F = \phi_t + \frac{dK}{d\phi}\phi_z - \frac{dK}{d\phi}\phi_z\phi_{tz} - K\phi_{tzz} = 0 \tag{4.3}$$

where a subscript denotes partial differentiation. We look for Lie point symmetries of the form

$$X = \xi^1(t,z,\phi)\frac{\partial}{\partial t} + \xi^2(t,z,\phi)\frac{\partial}{\partial z} + \eta(t,z,\phi)\frac{\partial}{\partial \phi}. \tag{4.4}$$

The invariance criterion is

$$X^{[3]}\left(\phi_t+\frac{dK}{d\phi}\phi_z-\frac{dK}{d\phi}\phi_z\phi_{tz}-K\phi_{tzz}\right)\Big|_{F=0}=0, \tag{4.5}$$

where the third prolongation of X is of the form

$$X^{[3]}=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial z}+\eta\frac{\partial}{\partial\phi}+\zeta_1\frac{\partial}{\partial\phi_t}+\zeta_2\frac{\partial}{\partial\phi_z}+\zeta_{12}\frac{\partial}{\partial\phi_{tz}}+\zeta_{122}\frac{\partial}{\partial\phi_{tzz}}+.... \tag{4.6}$$

The remaining terms in $X^{[3]}$ are not required in (4.6).

We replace $\phi_{tzz}$ in the invariance criterion (4.5) using the partial differential equation (4.3) and then separate the determining equation according to the following partial derivatives of $\phi$

$$\phi_{ttz}: \qquad \frac{\partial\xi^1}{\partial z}=0, \tag{4.7}$$

$$\phi_z\phi_{ttz}: \qquad \frac{\partial\xi^1}{\partial\phi}=0, \tag{4.8}$$

$$\phi_{zzz}: \qquad \frac{\partial\xi^2}{\partial t}=0, \tag{4.9}$$

$$\phi_t\phi_{zzz}: \qquad \frac{\partial\xi^2}{\partial\phi}=0, \tag{4.10}$$

$$\phi_{zz}: \qquad \frac{\partial^2\eta}{\partial t\partial\phi}=0, \tag{4.11}$$

$$\phi_t\phi_{zz}: \qquad \frac{\partial^2\eta}{\partial\phi^2}=0, \tag{4.12}$$

It follows directly from (4.7) to (4.12) that

$$\xi^1=f(t), \quad \xi^2=g(z), \quad \eta=\phi A(z)+B(t,z), \tag{4.13}$$

where $f(t)$, $g(z)$, $A(z)$ and $B(t,z)$ still have to be determined. Using (4.13) the invariance criterion separates further into the following system of equations:

$$\phi_t\phi_z \quad : \quad \frac{dK}{d\phi}\frac{dA}{dz} = 0, \tag{4.14}$$

$$\phi_{tz} \quad : \quad \left[\phi\frac{dK}{d\phi} + 2K(\phi)\right]\frac{dA}{dz} + \frac{dK}{d\phi}\frac{\partial B}{\partial z} - K(\phi)\frac{d^2 g}{dz^2} = 0, \tag{4.15}$$

$$\phi_t \quad : \quad K(\phi)\frac{d^2 A}{dz^2} + \frac{\phi}{K}\frac{dK}{d\phi}A(z) + \frac{1}{K}\frac{dK}{d\phi}B(t,z) - 2\frac{dg}{dz} = 0, \tag{4.16}$$

$$\begin{aligned}\phi_z\phi_{tz} \quad : \quad & \left[\phi\frac{d^2K}{d\phi^2} - \frac{\phi}{K}\left(\frac{dK}{d\phi}\right)^2 + \frac{dK}{d\phi}\right]A(z) \\ & + \left[\frac{d^2K}{d\phi^2} - \frac{1}{K}\left(\frac{dK}{d\phi}\right)^2\right]B(t,z) = 0,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\phi_z \quad : \quad & \left[\phi\frac{d^2K}{d\phi^2} - \frac{\phi}{K}\left(\frac{dK}{d\phi}\right)^2\right]A(z) + \left[\frac{d^2K}{d\phi^2} - \frac{1}{K}\left(\frac{dK}{d\phi}\right)^2\right]B(t,z) \\ & - \frac{dK}{d\phi}\frac{\partial^2 B}{\partial t\partial z} + \frac{dK}{d\phi}\left(\frac{df}{dt} + \frac{dg}{dz}\right) = 0,\end{aligned} \tag{4.18}$$

$$\phi^0 : \phi\frac{dK}{d\phi}\frac{dA}{dz} - K(\phi)\frac{\partial^3 B}{\partial t\partial z^2} + \frac{dK}{d\phi}\frac{\partial B}{\partial z} + \frac{\partial B}{\partial t} = 0. \tag{4.19}$$

From (4.14) we see that there are two general cases depending on whether the permeability, $K$, depends on $\phi$ or is constant.

### 4.2.1 Permeability depends on voidage

Consider first the case in which $K(\phi)$ depends on the voidage, $\phi$, that is, the permeability is not constant, so that

$$\frac{dK}{d\phi} \neq 0. \tag{4.20}$$

Then from (4.14), $A(z) = A_0$, where $A_0$ is a constant. By differentiating (4.16) with respect to $t$ we find that $B = B(z)$ and from (4.19) it follows that $B(z) = B_0$, where $B_0$ is a constant. Hence, from (4.13),

$$\eta(\phi) = A_0\phi + B_0. \tag{4.21}$$

From (4.15),

$$\xi^2 = g(z) = c_1 z + c_2, \tag{4.22}$$

where $c_1$ and $c_2$ are constants. Also from (4.17) and (4.18),

$$\frac{df}{dt} = A_0 - c_1 \tag{4.23}$$

and therefore

$$\xi^1 = f(t) = (A_0 - c_1)t + c_3, \tag{4.24}$$

where $c_3$ is a constant. Equations (4.14) to (4.19) reduce to

$$(A_0\phi + B_0)\left(\frac{d^2K}{d\phi^2} - \frac{1}{K}\left(\frac{dK}{d\phi}\right)^2\right) + A_0\frac{dK}{d\phi} = 0, \tag{4.25}$$

$$(A_0\phi + B_0)\frac{1}{K}\frac{dK}{d\phi} - 2c_1 = 0. \tag{4.26}$$

It is readily verified that if $K(\phi)$ satisfies (4.26) then $K(\phi)$ satisfies (4.25) identically. This can be done by differentiating (4.26) with respect to $\phi$, which gives (4.25). Thus equation (4.25) therefore does not need to be considered further.

When (4.20) is satisfied Lie point symmetries of (4.2) exist provided $K(\phi)$ satisfies (4.26) and are given by (4.21), (4.22) and (4.24). This case separates into two subcases depending on whether $A_0 \neq 0$ or $A_0 = 0$.

### The Case $A_0 \neq 0$

Consider first $A_0 \neq 0$. The general solution of the ordinary differential equation (4.26) for $K(\phi)$ is

$$K(\phi) = K_0(A_0\phi + B_0)^{\frac{2c_1}{A_0}}, \tag{4.27}$$

where $K_0$ is a constant. But using (1.7) it follows that $B_0 = 0$ and since $K(1) = 1$, by (1.6), we obtain

$$K(\phi) = \phi^n, \quad n = \frac{2c_1}{A_0}. \tag{4.28}$$

Since K is not a constant , $n \neq 0$. Equations (4.24), (4.22) and (4.21) become

$$\xi^1(t) = \frac{c_1}{n}(2-n)t + c_3, \tag{4.29}$$

$$\xi^2(z) = c_1 z + c_2, \tag{4.30}$$

$$\eta(\phi) = \frac{2c_1}{n}\phi. \tag{4.31}$$

The three Lie point symmetries of (4.2) with the power law (4.28) for $K(\phi)$ are presented in Table 4.1.

**The Case $A_0 = 0$, $B_0 \neq 0$**

When $A_0 = 0$ but $B_0 \neq 0$, the general solution of (4.26) is

$$K(\phi) = K_0 \exp(n\phi), \quad n = \frac{2c_1}{B_0}, \tag{4.32}$$

where $K_0$ is a constant. Since $K(\phi)$ is not constant, $n \neq 0$ and because the permeability increases as the voidage increases, $n > 0$. Also, $K(1) = 1$ and therefore

$$K(\phi) = \exp[n(\phi - 1)], \quad n > 0. \tag{4.33}$$

When $\phi = 0$,

$$K(\phi) = \exp(-n) \neq 0. \tag{4.34}$$

If $n$ is large, $K(0)$ is small. However the exponential law (4.33) for $K(\phi)$ does not satisfy the condition $K(0) = 0$. It is not a suitable model for physical phenomena with small values of the voidage. For instance it would not be suitable to describe compressive solitary waves which contain $\phi = 0$ $[6, 7-9]$. It is suitable for describing rarefactive solitary waves which satisfy $\phi \geq 1$ and this will be considered in Section 6. Equations (4.21),(4.22),(4.24) become

$$\xi^1(t) = -c_1 t + c_3, \tag{4.35}$$

$$\xi^2(z) = c_1 z + c_2, \tag{4.36}$$

$$\eta(\phi) = \frac{2c_1}{n}. \tag{4.37}$$

The three Lie point symmetries of (4.2) with the exponential law (4.33) for $K(\phi)$ are presented in Table 4.1.

### 4.2.2 Constant permeability

Finally, consider constant permeability, $K(\phi) = K_0$. Since $K(1) = 1$ it follows that $K_0 = 1$. The model does not satisfy $K(0) = 0$ and it cannot be used to describe physical phenomena in which the permeability depends on voidage.

When $K(\phi) = 1$, equations (4.14) to (4.19) reduce to

$$2\frac{dA}{dz} - \frac{d^2 g}{dz^2} = 0, \tag{4.38}$$

$$\frac{d^2 A}{dz^2} - 2\frac{dg}{dz} = 0, \tag{4.39}$$

$$\frac{\partial B}{\partial t} - \frac{\partial^3 B}{\partial t \partial z^2} = 0. \tag{4.40}$$

Integrating (4.38) and (4.39) once with respect to z gives

$$2A(z) - \frac{dg}{dz} = c_1, \tag{4.41}$$

$$\frac{dA}{dz} - 2g(z) = c_2, \tag{4.42}$$

where $c_1$ and $c_2$ are constants. By eliminating $A(z)$ we obtain

$$\frac{d^2g}{dz^2} - 4g = 2c_2 \tag{4.43}$$

| Permeability | Lie point symmetries |
|---|---|
| $K(\phi) = \phi^n, n > 0$ | $X_1 = (2-n)t\frac{\partial}{\partial t} + nz\frac{\partial}{\partial z} + 2\phi\frac{\partial}{\partial \phi}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \frac{\partial}{\partial t}$ |
| $K(\phi) = \exp[n(\phi - 1)]$<br>$n > 0$ | $X_1 = -nt\frac{\partial}{\partial t} + nz\frac{\partial}{\partial z} + 2\frac{\partial}{\partial \phi}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \frac{\partial}{\partial t}$ |
| $K(\phi) = 1$ | $X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \sinh(2z)\frac{\partial}{\partial z} + \phi\cosh(2z)\frac{\partial}{\partial \phi}$<br>$X_4 = \cosh(2z)\frac{\partial}{\partial z} + \phi\sinh(2z)\frac{\partial}{\partial \phi}$<br>$X_5 = \phi\frac{\partial}{\partial \phi}$<br>$X_A = A(t,z)\frac{\partial}{\partial \phi}$<br>$f(t)$ arbitrary<br>$A(t,z)$ satisfies (1.3) with $n = m = 0$ |

Table 4.1: The permeability $K(\phi)$ and the corresponding Lie point symmetries of (4.2).

and therefore

$$g(z) = c_3\cosh(2z) + c_4\sinh(2z) - \frac{c_2}{2}, \tag{4.44}$$

where $c_3$ and $c_4$ are constants. From (4.42),

$$A(z) = c_3\sinh(2z) + c_4\cosh(2z) + \frac{c_1}{2}. \tag{4.45}$$

Thus, from (4.13)

$$\xi^1(t) = f(t), \tag{4.46}$$

$$\xi^2(z) = c_3\cosh(2z) + c_4\sinh(2z) - \frac{c_2}{2}, \tag{4.47}$$

$$\eta(t,z,\phi) = \phi\left(c_3\sinh(2z) + c_4\cosh(2z) + \frac{c_1}{2}\right) + B(t,z), \tag{4.48}$$

where $f(t)$ is arbitrary and $B(t,z)$ satisfies (4.40) which is the partial differential equation (4.2) with $K(\phi) = 1$. The Lie point symmetries are listed in Table 4.1.

There are therefore three forms of $K(\phi)$ for equation (4.2) to admit Lie point symmetries in addition to (4.1), namely the power law (4.28), the exponential law (4.33), and constant permeability. The Lie point symmetries and the three forms of $K(\phi)$ are presented in Table 4.1. We now consider viscosity models.

## 4.3 Viscosity models

When $K(\phi) = 1$ the magma equation becomes

$$F = \frac{\partial\phi}{\partial t} + \frac{\partial}{\partial z}\left[1 - \frac{\partial}{\partial z}\left(G(\phi)\frac{\partial\phi}{\partial t}\right)\right] = 0. \tag{4.49}$$

We will investigate the Lie point symmetries of (4.49) and the form of $G(\phi)$ for these Lie point symmetries to exist.

We look for symmetries of the form (4.4). The invariance criterion is

$$X^{[3]}\left(\phi_t - \phi_t\phi_z^2\frac{d^2G}{d\phi^2} - \phi_t\phi_{zz}\frac{dG}{d\phi} - 2\phi_z\phi_{tz}\frac{dG}{d\phi} - G(\phi)\phi_{tzz}\right)\bigg|_{F=0} = 0 \tag{4.50}$$

where the terms in the third prolongation which are required in (4.50) are

$$X^{[3]} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial\phi} + \zeta_1\frac{\partial}{\partial\phi_t} + \zeta_2\frac{\partial}{\partial\phi_z} + \zeta_{12}\frac{\partial}{\partial\phi_{tz}} + \zeta_{22}\frac{\partial}{\partial\phi_{zz}} + \zeta_{122}\frac{\partial}{\partial\phi_{tzz}}. \tag{4.51}$$

The remaining terms in $X^{[3]}$ are not needed in (4.51) and $\zeta_i$, $\zeta_{ij}$, $\zeta_{ijk}$, $D_1$, $D_2$ are defined as before. Replace $\phi_{tzz}$ in the invariance criterion (4.50) using

(4.49) and then separate according to the following partial derivatives of $\phi$:

$$\phi_{ttz}: \qquad \frac{\partial \xi^1}{\partial z} = 0, \tag{4.52}$$

$$\phi_z\phi_{ttz}: \qquad \frac{\partial \xi^1}{\partial \phi} = 0, \tag{4.53}$$

$$\phi_{zzz}: \qquad \frac{\partial \xi^2}{\partial t} = 0, \tag{4.54}$$

$$\phi_t\phi_{zzz}: \qquad \frac{\partial \xi^2}{\partial \phi} = 0, \tag{4.55}$$

It follows directly from (4.52) to (4.55) that

$$\xi^1 = f(t), \quad \xi^2 = g(z). \tag{4.56}$$

Using (4.56), the invariance criterion (4.50) separates further into the following system of equations:

$$\phi_t\phi_{zz} \quad : \quad G(\phi)\frac{\partial^2 \eta}{\partial \phi^2} + \frac{dG}{d\phi}\frac{\partial \eta}{\partial \phi} - \left(\frac{1}{G}\left(\frac{dG}{d\phi}\right)^2 - \frac{d^2G}{d\phi^2}\right)\eta = 0, \tag{4.57}$$

$$\phi_{tz} \quad : \quad \frac{d^2 g}{dz^2} - 2G\frac{\partial^2 \eta}{\partial z \partial \phi} - 2\frac{dG}{d\phi}\frac{\partial \eta}{\partial z} = 0, \tag{4.58}$$

$$\phi_z\phi_t \quad : \quad 4\frac{dG}{d\phi}\frac{\partial^2 \eta}{\partial z \partial \phi} + 2\frac{d^2G}{d\phi^2}\frac{\partial \eta}{\partial z} + 2G\frac{\partial^3 \eta}{\partial z \partial \phi^2} - \frac{dG}{d\phi}\frac{d^2 g}{dz^2} = 0, \tag{4.59}$$

$$\begin{aligned}\phi_t\phi_z^2 \quad : \quad & G\frac{\partial^3 \eta}{\partial \phi^3} + 2\frac{d^2G}{d\phi^2}\frac{\partial \eta}{\partial \phi} + 3\frac{dG}{d\phi}\frac{\partial^2 \eta}{\partial \phi^2} \\ & - \left(\frac{1}{G}\frac{dG}{d\phi}\frac{d^2G}{d\phi^2} - \frac{d^3G}{d\phi^3}\right)\eta = 0,\end{aligned} \tag{4.60}$$

$$\phi_{zz} \quad : \quad G\frac{\partial^2 \eta}{\partial t \partial \phi} + \frac{dG}{d\phi}\frac{\partial \eta}{\partial t} = 0, \tag{4.61}$$

$$\phi_z^2 \quad : \quad \frac{d^2G}{d\phi^2}\frac{\partial \eta}{\partial t} + 2\frac{dG}{d\phi}\frac{\partial^2 \eta}{\partial t \partial \phi} + G\frac{\partial^3 \eta}{\partial t \partial \phi^2} = 0, \tag{4.62}$$

$$\phi_z \quad : \quad G\frac{\partial^3 \eta}{\partial t \partial z \partial \phi} + \frac{dG}{d\phi}\frac{\partial^2 \eta}{\partial t \partial z} = 0, \tag{4.63}$$

$$\phi_t \quad : \quad G\frac{\partial^3 \eta}{\partial \phi \partial z^2} + \frac{dG}{d\phi}\frac{\partial^2 \eta}{\partial z^2} + \frac{1}{G}\frac{dG}{d\phi}\eta - 2\frac{dg}{dz} = 0, \tag{4.64}$$

$$\phi^0 \quad : \quad \frac{\partial \eta}{\partial t} - G\frac{\partial^3 \eta}{\partial t \partial z^2} = 0. \tag{4.65}$$

From (4.61)

$$\frac{\partial}{\partial \phi}\left(G\frac{\partial \eta}{\partial t}\right) = 0, \tag{4.66}$$

thus,

$$\frac{\partial \eta}{\partial t} = \frac{1}{G}R(t, z). \tag{4.67}$$

Now substitute (4.67) into (4.65)

$$\frac{1}{G}R(t, z) - \frac{\partial^3 R}{\partial t \partial z^2} = 0 \tag{4.68}$$

and differentiate (4.68) with respect to $\phi$

$$\frac{dG}{d\phi}R(t, z) = 0. \tag{4.69}$$

Thus, from (4.69) we see that there are two general cases depending on whether the viscosity depends on the voidage or is a constant.

### 4.3.1 Viscosity depends on the voidage

If the viscosity depends on the voidage then

$$\frac{dG}{d\phi} \neq 0. \tag{4.70}$$

Then from (4.69) we have $R(t, z) = 0$. Thus, from (4.67)

$$\eta = \eta(z, \phi). \tag{4.71}$$

Hence,

$$\xi^1 = f(t), \quad \xi^2 = g(z), \quad \eta = \eta(z, \phi). \tag{4.72}$$

The remaining equations are (4.57) to (4.64).

It can be shown that any solution of (4.57) is also a solution of (4.59), thus, (4.59) does not need to be considered further. The general solution of (4.57) is

$$\eta(z, \phi) = \frac{H(\phi)}{G(\phi)}d\phi A(z) + \frac{B(z)}{G(\phi)} \tag{4.73}$$

where

$$H(\phi) = \int^{\phi} G(\phi), \quad \frac{dH}{d\phi} = G(\phi). \tag{4.74}$$

Now substitute (4.73) into (4.57). This yields

$$\frac{d^2 g}{dz^2} = 2\frac{dA}{dz}. \tag{4.75}$$

Substituting (4.73) into (4.58) again yields (4.75). Finally, substitute (4.73) into (4.60) which gives

$$G\frac{d^2A}{dz^2} + \frac{H}{G^2}\frac{dG}{d\phi}A(z) + \frac{1}{G^2}\frac{dG}{d\phi}B(z) - 2\frac{dg}{dz} = 0. \tag{4.76}$$

Equation (4.76) is the differential equation for $G(\phi)$. It can be expressed as a second order differential equation for $H(\phi)$. Either $G(\phi)$ can be specified and $A(z)$, $B(z)$, $g(z)$ be calculated to derive the Lie point symmetries for equation (4.49) or some of the functions $A(z)$, $B(z)$, $g(z)$ can be specified and $G(\phi)$ can be calculated. We will consider the approach in which $G(\phi)$ is specified.

### Power law viscosity

Consider the case in which the viscosity is related to the voidage through a power law, that is

$$G(\phi) = \phi^{-m}. \tag{4.77}$$

This model was derived by Scott and Stevenson [3]. Substitute (4.77) into (4.73) and (4.76). We consider the general case in which $m \neq 1$.

$$\eta(z,\phi) = \frac{\phi}{(1-m)}A(z) + B(z)\phi^m, \quad m \neq 1 \tag{4.78}$$

and

$$\phi^{-m}\frac{d^2A}{dz^2} - \frac{m}{1-m}A(z) - m\phi^{m-1}B(z) - 2\frac{dg}{dz} = 0, \quad m \neq 1. \tag{4.79}$$

We can separate (4.79) in powers of $\phi$ if all powers are distinct, that is provided $m \neq 0$, $m \neq 1$, $m \neq \frac{1}{2}$.
We therefore assume that $m \neq 0$, $m \neq 1$, $m \neq \frac{1}{2}$. Splitting (4.79) according to powers of $\phi$ we obtain

$$\phi^{-m}: \quad \frac{d^2A}{dz^2} = 0, \tag{4.80}$$

$$\phi^{m-1}: \quad mB(z) = 0, \tag{4.81}$$

$$\phi^0: \quad \frac{m}{1-m}A(z) + 2\frac{dg}{dz} = 0. \tag{4.82}$$

From (4.80) we have

$$A(z) = c_1 z + c_2 \tag{4.83}$$

where $c_1$ and $c_2$ are constants. From (4.81)

$$B(z) = 0 \tag{4.84}$$

and from (4.82)

$$\frac{dg}{dz} = -\frac{m}{2(1-m)}(c_1 z + c_2). \tag{4.85}$$

Differentiate (4.85) with respect to z

$$\frac{d^2 g}{dz^2} = -\frac{m}{2(1-m)}c_1, \tag{4.86}$$

thus, from (4.75) we have

$$\frac{d^2 g}{dz^2} = 2\frac{dA}{dz} = 2c_1. \tag{4.87}$$

Now equations (4.86) and (4.87) are consistent provided

$$2c_1 = -\frac{m}{2(1-m)}c_1, \tag{4.88}$$

that is, provided

$$c_1[4-3m] = 0. \tag{4.89}$$

Thus, from (4.89) we see that either $c_1 = 0$ or $m = \frac{4}{3}$.

Case (a) $c_1 = 0$

From (4.85)

$$\frac{dg}{dz} = -\frac{mc_2}{2(1-m)}, \tag{4.90}$$

and therefore

$$g(z) = -\frac{mc_2 z}{2(1-m)} + c_3, \tag{4.91}$$

where $c_3$ is a constant. From (4.78), (4.83) and (4.84)

$$\eta(z,\phi) = \frac{c_2\phi}{1-m}. \tag{4.92}$$

Thus, the infinitesimals are given by

$$\xi^1(t) = f(t), \tag{4.93}$$

$$\xi^2(z) = -mc_2^* z + c_3, \tag{4.94}$$

$$\eta(t,z,\phi) = 2c_2^*\phi, \tag{4.95}$$

where

$$c_2^* = \frac{c_2}{2(1-m)}. \tag{4.96}$$

The Lie point symmetries generated by (4.93) to (4.95) are presented in Table 4.2 under Case 1 in the power law.

| Viscosity | Lie point symmetries |
|---|---|
| $G(\phi) = \phi^{-m}$ | Case 1: $m \neq \frac{4}{3}$<br>$X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = -mz\frac{\partial}{\partial z} + 2\phi\frac{\partial}{\partial \phi}$<br>$f(t)$ is arbitrary |
| | Case 2: $m = \frac{4}{3}$<br>$X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = -\frac{2}{3}z\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial \phi}$<br>$X_4 = -z^2\frac{\partial}{\partial z} + 3z\phi\frac{\partial}{\partial \phi}$<br>$f(t)$ is arbitrary |
| $G(\phi) = \exp[-m(\phi - 1)]$ | $X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = z\frac{\partial}{\partial z} + \frac{2}{m}\frac{\partial}{\partial \phi}$<br>$X_3 = \frac{\partial}{\partial z}$ |
| $G(\phi) = 1$ | $X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \sinh(2z)\frac{\partial}{\partial z} + \phi\cosh(2z)\frac{\partial}{\partial \phi}$<br>$X_4 = \cosh(2z)\frac{\partial}{\partial z} + \phi\sinh(2z)\frac{\partial}{\partial \phi}$<br>$X_5 = \phi\frac{\partial}{\partial \phi}$<br>$X_A = A(t,z)\frac{\partial}{\partial \phi}$<br>$f(t)$ arbitrary<br>$A(t,z)$ satisfies (1.3) with $n = m = 0$ |

Table 4.2: The viscosity $G(\phi)$ and the corresponding Lie point symmetries of (4.49).

Case (b) $m = \frac{4}{3}$
Then $c_1 \neq 0$. From (4.83) and (4.85)

$$A(z) = c_1 z + c_2 \tag{4.97}$$

and

$$g(z) = c_1 z^2 + 2c_2 z + c_3, \tag{4.98}$$

where $c_3$ is a constant and from (4.78)

$$\eta(z,\phi) = -3(c_1 z + c_2)\phi \tag{4.99}$$

Thus, from (4.72)

$$\xi^1(t) = f(t), \tag{4.100}$$

$$\xi^2(z) = -c_1^* z^2 - \frac{2}{3} c_2^* z + c_3^*, \tag{4.101}$$

$$\eta(t, z, \phi) = (3c_1^* z + c_2^*)\phi, \tag{4.102}$$

where

$$c_1^* = -c_1, \quad c_2^* = -3c_2, \quad c_3^* = c_3. \tag{4.103}$$

The Lie point symmetries are listed in Table 4.2 under Case 2 in the power law.
The case $m = 0$ corresponds to constant viscosity which we will consider later. It can be verified that the Lie point symmetries for the cases when $m = 1$ and $m = \frac{1}{2}$ can be obtained by substituting $m = 1$ and $m = \frac{1}{2}$ into (4.94).

**Exponential law viscosity**

Consider now the case in which the matrix viscosity is related to the voidage through the exponential law

$$G(\phi) = \exp[-m(\phi - 1)], \quad m > 0 \tag{4.104}$$

Equation (4.104) for $G(\phi)$ has the same form as (4.34) for $K(\phi)$ and satisfies the condition $G(1) = 1$. Also $G(\phi)$ decreases as the voidage increases which is a requirement for matrix viscosity. Equations (4.78) and (4.79) become

$$\eta(z, \phi) = -\frac{1}{m} A(z) + B(z) \exp[m(\phi - 1)] \tag{4.105}$$

and

$$\exp[-m(\phi - 1)] \frac{d^2 A}{dz^2} + A(z) - m \exp[m(\phi - 1)] B(z) - 2\frac{dg}{dz} = 0. \tag{4.106}$$

Differentiate (4.106) with respect to $\phi$ to obtain

$$m \frac{d^2 A}{dz^2} + m^2 \exp[2m(\phi - 1)] B(z) = 0. \tag{4.107}$$

Differentiate once more with respect to $\phi$ which gives

$$2m^3 \exp[2m(\phi - 1)] B(z) = 0. \tag{4.108}$$

We are considering $m > 0$. Hence,

$$B(z) = 0. \tag{4.109}$$

Thus, from (4.106)

$$\frac{d^2A}{dz^2} = 0, \tag{4.110}$$

and therefore

$$A(z) = c_1 z + c_2. \tag{4.111}$$

Equation (4.106) reduces to

$$2\frac{dg}{dz} - A(z) = 0. \tag{4.112}$$

and therefore

$$\frac{dg}{dz} = \frac{1}{2}(c_1 z + c_2). \tag{4.113}$$

There are two equations for $g(z)$, (4.113) and (4.75) using (4.111) in (4.75)

$$\frac{d^2g}{dz^2} = 2c_1. \tag{4.114}$$

But from (4.113),

$$\frac{d^2g}{dz^2} = \frac{1}{2}c_1 \tag{4.115}$$

and (4.114) and (4.115) are compatible provided

$$c_1 = 0. \tag{4.116}$$

Thus, from (4.113)

$$g(z) = \frac{1}{2}c_2 z + c_3 \tag{4.117}$$

and from (4.111)

$$A(z) = c_2. \tag{4.118}$$

Finally, from (4.105)

$$\eta(z,\phi) = -\frac{c_2}{m} \tag{4.119}$$

Hence,

$$\xi^1(t) = f(t), \tag{4.120}$$

$$\xi^2(z) = c_2^* z + c_3, \tag{4.121}$$

$$\eta(t,z,\phi) = \frac{2}{m}c_2^*, \tag{4.122}$$

where

$$c_2^* = \frac{1}{2}c_2. \tag{4.123}$$

The Lie point symmetries with exponential law viscosity are listed in Table 4.2. The case when $m = 0$ corresponds to constant viscosity.

### 4.3.2 Constant viscosity

Finally, consider constant viscosity $G(\phi) = G_0$. Since $G(\phi) = 1$, it follows that $G_0 = 1$. The model does not satisfy the condition that the viscosity of the solid matrix is infinite when the voidage vanishes and cannot be used to describe physical phenomena in which the viscosity depends on voidage.

When $G(\phi) = 1$, this reduces to the case of constant viscosity and constant permeability. The Lie point symmetries of (4.49) with constant viscosity are listed in tables Table 4.1 and Table 4.2.

| Permeability | Viscosity | Lie point symmetries |
|---|---|---|
| $K(\phi) = \phi^n$ | $G(\phi) = \phi^{-m}$ | Case 1: $n > 0, m \geq 0$<br>$X_1 = (2-n-m)t\frac{\partial}{\partial z} + (n-m)z\frac{\partial}{\partial t}$<br>$+ 2\phi\frac{\partial}{\partial \phi}$<br>$X_2 = \frac{\partial}{\partial t}$<br>$X_3 = \frac{\partial}{\partial z}$ |
| | | Case 2: $n = 0, m \neq \frac{4}{3}, m \neq 0$<br>$X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = -mz\frac{\partial}{\partial z} + 2\phi\frac{\partial}{\partial \phi}$<br>$f(t)$ is arbitrary |
| | | Case 3: $n = 0, m = \frac{4}{3}$<br>$X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = z\frac{\partial}{\partial z} - \frac{3}{2}\phi\frac{\partial}{\partial \phi}$<br>$X_4 = -\frac{1}{3}z^2\frac{\partial}{\partial z} + z\phi\frac{\partial}{\partial \phi}$ |

| | | |
|---|---|---|
| | | Case 4: $n = 0, m = 0$<br>$X_1 = \xi(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \sinh(2z)\frac{\partial}{\partial z} + \phi\cosh(2z)\frac{\partial}{\partial \phi}$ |
| | | $X_4 = \cosh(2z)\frac{\partial}{\partial z} + \phi\sinh(2z)\frac{\partial}{\partial \phi}$<br>$X_5 = \phi\frac{\partial}{\partial \phi}$<br>$X_A = A(t,z)\frac{\partial}{\partial \phi}$<br>$A(t,z)$ satisfies (1.3) with $n = m = 0$ |
| $K(\phi) = \exp(n(\phi - 1))$ | $G(\phi) = \exp(-m(\phi - 1))$ | Case 1: $m > 0$, $m \geq 0$<br>$X_1 = (m+n)t\frac{\partial}{\partial t} + (m-n)z\frac{\partial}{\partial z}$<br>$-2\frac{\partial}{\partial \phi}$<br>$X_2 = \frac{\partial}{\partial z}$<br>$X_3 = \frac{\partial}{\partial t}$ |
| | | Case 2: $n = 0, m \neq 0$<br>$X_1 = f(t)\frac{\partial}{\partial t}$<br>$X_2 = \frac{\partial}{\partial t}$<br>$X_3 = z\frac{\partial}{\partial z} - \frac{2}{m}\frac{\partial}{\partial \phi}$ |

Table 4.3: The permeability, $K(\phi)$, viscosity, $G(\phi)$, and the corresponding Lie point symmetries of (1.3)

## 4.4 Combined permeability and viscosity models

We now summarise in Table 4.3 the Lie point symmetries when both permeability and viscosity satisfy power laws and exponential laws in the voidage.

In Table 4.3 Case 1 of the Lie point symmetries when $K(\phi)$ and $G(\phi)$ satisfy power laws were obtained using the SYM package in Mathematica [22]. The results for power law permeability and viscosity agree with Maluleke and Mason [10, 12]. Case 1 when both $K(\phi)$ and $G(\phi)$ satisfy exponential laws was obtained using the SYM package in Mathematica [22].

## 4.5 Concluding remarks

There are therefore three forms of $K(\phi)$ and $G(\phi)$ for which (1.3) has Lie point symmetries, namely, the power laws, (4.27) and (4.77), the exponential laws, (4.33)and (4.104) and constant permeability and viscosity. For the viscosity $G(\phi)$ it remains to be determined if there are other physically realistic forms of $G(\phi)$ which will give significant Lie point symmetries for the partial differential equation (1.3). In Chapter 6 we will investigate the rarefactive solitary wave solution when $K(\phi)$ and $G(\phi)$ are given by exponential laws.

# Chapter 5

# Derivation of conservation laws for the magma equation-multiplier method

## 5.1 Introduction

In this chapter we will derive conservation laws for the magma equation (1.3) using the multiplier method. We will consider the case when we have power law permeability and viscosity and exponential law permeability and viscosity. The conserved vectors are then associated with the Lie point symmetries of the magma equation (1.3).

## 5.2 Conservation laws for the magma equation with power law permeability and viscosity by the multiplier method

When the permeability and viscosity are related to the voidage by the power laws the magma equation becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\phi^n \left(1 - \frac{\partial}{\partial z}\left(\phi^{-m}\frac{\partial \phi}{\partial t}\right)\right)\right] = 0. \tag{5.1}$$

In order to derive conservation laws for (5.1) consider first a multiplier of the form

$$\Lambda = \Lambda(\phi). \tag{5.2}$$

### 5.2.1 Lower order conservation laws

A multiplier for the partial differential equation has the property that

$$\Lambda(\phi)F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) = D_1T^1 + D_2T^2 \tag{5.3}$$

where

$$\begin{aligned} F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) = {} & \phi_t + n\phi^{n-1}\phi_z + m(n-m-1)\phi^{n-m-2}\phi_z^2\phi_t \\ & + m\phi^{n-m-1}\phi_{zz}\phi_t + (2m-n)\phi^{n-m-1}\phi_z\phi_{tz} \\ & - \phi^{n-m}\phi_{tzz}. \end{aligned} \tag{5.4}$$

The determining equation for the multiplier is:

$$E_\phi[\Lambda(\phi)F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})] = 0 \tag{5.5}$$

where the Euler operator $E_\phi$ is defined by (2.19). Separating (5.5) with respect to powers and products of the derivatives of $\phi$ we have the following system of equations

$$\phi_z\phi_{tz} : \phi\frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{5.6}$$

$$\phi_t\phi_{zz} : \phi\frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{5.7}$$

$$\phi_t\phi_z^2 : \phi^2\frac{d^3\Lambda}{d\phi^3} + 2n\phi\frac{d^2\Lambda}{d\phi^2} - (m+n)(m-n+1)\frac{d\Lambda}{d\phi} = 0. \tag{5.8}$$

It is readily verified that every solution of (5.6) is a solution of (5.8). We therefore need to consider only (5.6). The general solution of (5.6) is

$$\Lambda(\phi) = c_2\phi^{1-m-n} + c_1, \quad \text{if} \quad n+m-1 \neq 0. \tag{5.9}$$

and

$$\Lambda(\phi) = c_2\ln\phi + c_1, \quad \text{if} \quad n+m-1 = 0 \tag{5.10}$$

There are several cases to consider depending on the values of $m$ and $n$. The special cases are illustrated as lines and points in the $(m, n)$ plane in Figure 5.1 which is presented later in the chapter.

Case(i) $n+m-1 \neq 0$, $n+m-2 \neq 0$, $m \neq 1$.

From (5.3) and (5.9)

$$\begin{aligned}&\left(c_1 + c_2\phi^{1-m-n}\right)\Big(\phi_t + n\phi^{n-1}\phi_z + m(n-m-1)\phi^{n-m-2}\phi_z^2\phi_t \\ &\qquad + m\phi^{n-m-1}\phi_{zz}\phi_t + (2m-n)\phi^{n-m-1}\phi_z\phi_{tz} - \phi^{n-m}\phi_{tzz}\Big) = \\ &D_1\left[c_1\phi + c_2\left(\frac{1}{2-m-n}(\phi^{2-m-n}-1) + \frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2\right)\right] + \qquad (5.11)\\ &D_2\Big[c_1\Big(\phi^n(1+m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz})\Big) + c_2\Big(\frac{n}{m-1}\phi^{1-m} - \phi^{1-2m}\phi_{tz} \\ &\qquad + m\phi^{-2m}\phi_t\phi_z\Big)\Big].\end{aligned}$$

Equation (5.11) is satisfied for arbitrary functions $\phi(t,z)$. When $\phi(t,z)$ is a solution of the partial differential equation (5.1) then

$$\begin{aligned}&D_1\left[c_1\phi + c_2\left(\frac{1}{2-m-n}(\phi^{2-m-n}-1) + \frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2\right)\right] + \qquad (5.12)\\ &D_2\Big[c_1(\phi^n(1+m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz})) + c_2\Big(\frac{n}{m-1}\phi^{1-m} - \phi^{1-2m}\phi_{tz} \\ &\qquad + m\phi^{-2m}\phi_t\phi_z\Big)\Big] = 0.\end{aligned}$$

Hence any conserved vector of the partial differential equation (5.1) with m and n satisfying the conditions of this case and with multiplier of the form $\Lambda = \Lambda(\phi)$ is a linear combination of the two conserved vectors

$$T^1 = \phi, \qquad (5.13)$$
$$T^2 = \phi^n(1+m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz}), \qquad (5.14)$$

and

$$T^1 = \frac{1}{2-m-n}(\phi^{2-m-n}-1) + \frac{(1-m-n)}{2}\phi^{-2m}\phi_z^2, \qquad (5.15)$$
$$T^2 = \frac{n}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z. \qquad (5.16)$$

The conserved vector with components (5.13) and (5.14) is the elementary conserved vector.

Case (ii) $n+m=1, m \neq 1$.

Proceeding as before we obtain

$$T^1 = \phi, \tag{5.17}$$
$$T^2 = \phi^{1-m}(1 + m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz}), \tag{5.18}$$

and

$$T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi, \tag{5.19}$$
$$T_2 = \phi^{1-2m}\ln\phi - \frac{1}{1-m}\phi^{1-m} - (\phi^{1-2m}\ln\phi)\ \phi_{tz} + (m\phi^{-2m}\ln\phi)\ \phi_t\phi_z. \tag{5.20}$$

The conserved vector with components (5.17) and (5.18) is the elementary conserved vector with $n = 1-m$. The multiplier for the conserved vector with components (5.19) and (5.20) is, from (5.10),

$$\Lambda(\phi) = \ln\phi. \tag{5.21}$$

Case (iii) $n + m = 2$, $m \neq 1$.

We find that

$$T^1 = \phi, \tag{5.22}$$
$$T^2 = \phi^{2-m}(1 + m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz}), \tag{5.23}$$

and

$$T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi, \tag{5.24}$$
$$T^2 = \frac{2-m}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z. \tag{5.25}$$

The conserved vector with components (5.22) and(5.23) is the elementary conserved vector with $n = 2 - m$. The multiplier for the conserved vector with components (5.24) and (5.25) is, from (5.9)

$$\Lambda(\phi) = \frac{1}{\phi}. \tag{5.26}$$

Case (iv) $m = 1, n = 0$.

We obtain

$$T^1 = \phi, \tag{5.27}$$
$$T^2 = 1 + \phi^{-2}\phi_t\phi_z - \phi^{-1}\phi_{tz}, \tag{5.28}$$

and

$$T^1 = -\frac{1}{2}\phi^{-2}\phi_z^2 + \phi\ln\phi - \phi, \tag{5.29}$$
$$T^2 = -(\phi^{-1}\ln\phi)\ \phi_{tz} + (\phi^{-2}\ln\phi)\ \phi_t\phi_z. \tag{5.30}$$

The conserved vector with components (5.27) and(5.28) is the elementary conserved vector with $m = 1, n = 0$. The multiplier for the conserved vector with components (5.29) and (5.30) is given by (5.10) with $c_1 = 0$.

Case (v) $m = n = 1$

We obtain

$$T^1 = \phi, \tag{5.31}$$
$$T^2 = \phi(1 + \phi^{-2}\phi_t\phi_z - \phi^{-1}\phi_{tz}), \tag{5.32}$$

and

$$T^1 = -\frac{1}{2}\phi^{-2}\phi_z^2 + \ln\phi, \tag{5.33}$$
$$T^2 = \ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z. \tag{5.34}$$

The conserved vector with components (5.31) and(5.32) is the elementary conserved vector with $m = n = 1$. The multiplier for the conserved vector with components (5.33) and (5.34) is (5.21).

Case (vi) $m = 1, n \neq 0$, $n \neq 1$.

We obtain

$$T^1 = \phi, \tag{5.35}$$
$$T^2 = \phi^n(1 + \phi^{-2}\phi_t\phi_z - \phi^{-1}\phi_{tz}), \tag{5.36}$$

and

$$T^1 = \frac{1}{1-n}\phi^{1-n} - \frac{n}{2}\phi^{-2}\phi_z^2, \tag{5.37}$$
$$T^2 = \ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z. \tag{5.38}$$

The conserved vector with components (5.35) and (5.36) is the elementary conserved vector with $m = 1$. The multiplier for the conserved vector with components (5.37) and (5.38) is

$$\Lambda(\phi) = \phi^{-n}. \tag{5.39}$$

### 5.2.2 The search for higher order conservation laws

We now consider a multiplier of the form

$$\Lambda = \Lambda(\phi, \phi_z). \tag{5.40}$$

As before the determining equation for the multiplier is

$$E_\phi[\Lambda(\phi, \phi_z)F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})] = 0, \tag{5.41}$$

where $F$ is given by (5.4). By equating the coefficients of the highest order derivative term, $\phi_{tzzz}$, to zero in (5.41) we have

$$\frac{\partial \Lambda}{\partial \phi_z} = 0 \tag{5.42}$$

and therefore

$$\Lambda(\phi, \phi_z) = \Lambda(\phi). \tag{5.43}$$

Hence (5.40) does not give a new multiplier or a new conserved vector.

Consider next the multiplier

$$\Lambda = \Lambda(\phi, \phi_z, \phi_{zz}). \tag{5.44}$$

The determining equation for the multiplier is

$$E_\phi[\Lambda(\phi, \phi_z, \phi_{zz})F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})] = 0, \tag{5.45}$$

where $F$ is given by (5.4). By Equating the coefficients of $\phi_{tz}\phi_{zzzz}$, $\phi_t\phi_{zzzz}$ and $\phi_{zzzz}^2$ to zero in (5.45), we obtain the following system of equations

$$\phi_{tz}\phi_{zzzz}: \quad (2m-n)\phi_z \frac{\partial^2 \Lambda}{\partial \phi_{zz}^2} + \phi \frac{\partial^2 \Lambda}{\partial \phi_z \phi_{zz}} = 0, \tag{5.46}$$

$$\begin{aligned} \phi_t\phi_{zzzz}: \quad & (\phi^{m+2} + m(n-m-1)\phi^n + m\phi^{n+1}) \frac{\partial^2 \Lambda}{\partial \phi_{zz}^2} \\ & + (m+n)\phi^{n+1} \frac{\partial \Lambda}{\partial \phi_{zz}} + \phi^{n+2} \frac{\partial^2 \Lambda}{\partial \phi \partial \phi_{zz}} = 0, \end{aligned} \tag{5.47}$$

$$\phi_{zzzz}^2: \quad \frac{\partial^2 \Lambda}{\partial \phi_{zz}^2} = 0. \tag{5.48}$$

From (5.48) it follows that

$$\Lambda(\phi, \phi_z, \phi_{zz}) = A(\phi, \phi_z)\phi_{zz} + B(\phi, \phi_z). \tag{5.49}$$

Substituting (5.49) into (5.46) we find that

$$\frac{\partial A}{\partial \phi_z} = 0, \tag{5.50}$$

and therefore

$$\Lambda(\phi, \phi_z) = A(\phi). \tag{5.51}$$

Thus (5.49) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = A(\phi)\phi_{zz} + B(\phi, \phi_z). \tag{5.52}$$

Now substitute (5.52) into (5.47) which gives

$$\phi \frac{dA}{d\phi} + (m+n)A = 0. \tag{5.53}$$

The solution of (5.53) is

$$A = c_1 \phi^{-(m+n)} \tag{5.54}$$

where $c_1$ is a constant. Equation(5.52) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1 \phi^{-(m+n)} \phi_{zz} + B(\phi, \phi_z). \tag{5.55}$$

Now substitute (5.55) into (5.45) and then equate the coefficient of $\phi_{tzzz}$ in (5.45) to zero. This gives

$$\frac{\partial B}{\partial \phi_z} = c_1(n-2m)\phi^{-(m+n+1)}\phi_z \tag{5.56}$$

and integrating (5.56) we have

$$B(\phi, \phi_z) = \frac{1}{2} c_1(n-2m)\phi^{-(m+n+1)}\phi_z^2 + P(\phi). \tag{5.57}$$

Thus (5.55) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1 \phi^{-(m+n)} \phi_{zz} + \frac{1}{2} c_1(n-2m)\phi^{-(m+n+1)}\phi_z^2 + P(\phi). \tag{5.58}$$

Lastly, substitute (5.58) into (5.45) and equate the coefficient of $\phi_t \phi_z^2 \phi_{zz}$ to zero, which gives

$$(m-n-1)c_1 = 0. \tag{5.59}$$

It follows from (5.59) that there are two cases.
Case 1: $n \neq m-1$

If $m-n-1 \neq 0$ , then from (5.59) we have $c_1 = 0$. Therefore,

$$\Lambda(\phi, \phi_z, \phi_{zz}) = P(\phi). \tag{5.60}$$

Thus $\Lambda(\phi, \phi_z, \phi_{zz})$ does not give a new multiplier and therefore new conservation laws will not be derived.

Case 2: $n = m-1$

Then (5.59) is satisfied with $c_1 \neq 0$. If $c_1 \neq 0$ then (5.58) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1 \phi^{1-2m} \phi_{zz} - \frac{1}{2}(m+1)c_1 \phi^{-2m}\phi_z^2 + P(\phi). \tag{5.61}$$

Now substitute (5.61) into (5.45) and equate the coefficient of $\phi_z \phi_{tz}$ in (5.45) to zero, which gives

$$\frac{d^2P}{d\phi^2} + \frac{(2m-1)}{\phi}\frac{dP}{d\phi} = (m-2)c_1\phi^{1-2m}. \tag{5.62}$$

Solving (5.62) we have

$$P(\phi) = \frac{m-2}{3-2m}\phi^{3-2m}c_1 + \frac{\phi^{2(1-m)}}{2(1-m)}c_2 + c_3 \tag{5.63}$$

provided $m \neq \frac{3}{2}$ and $m \neq 1$. Since $n = m-1$, these two special cases correspond to the points $\left(\frac{3}{2}, \frac{1}{2}\right)$ and $(1,0)$ on the $(m,n)$ plane in Figure 5.1.

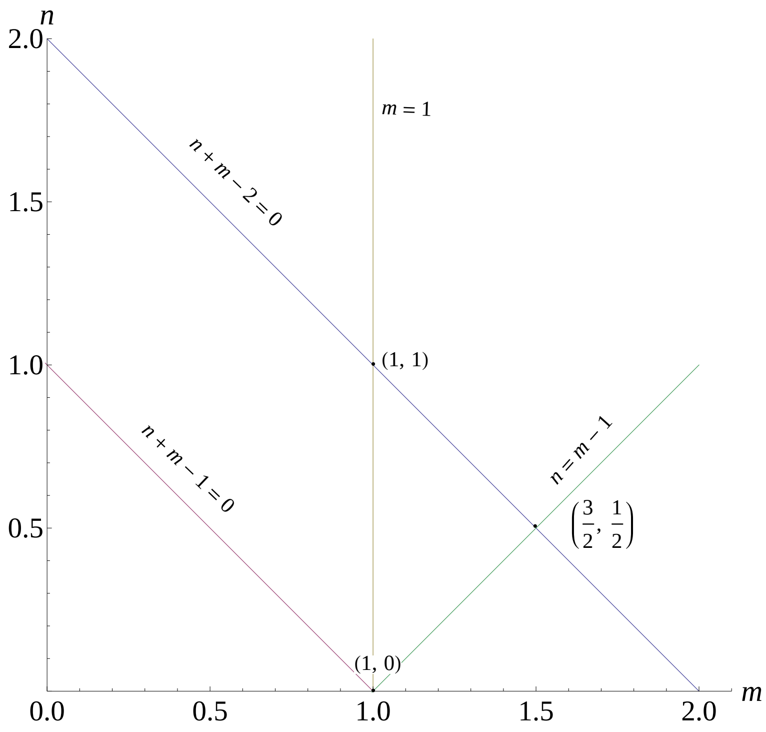


Figure 5.1: The $(m,n)$-plane. The conservation laws are constrained to the region $m \geq 0$, $n \geq 0$. The special cases are the straight lines $n+m-1=0$ , $n+m-2=0$ , $m=1$ and $n=m-1$.

Consider first the general case $n = m - 1$ excluding the points $(\frac{3}{2}, \frac{1}{2})$ and $(1, 0)$. Equation (5.61) becomes

$$\begin{aligned} \Lambda(\phi, \phi_z, \phi_{zz}) = c_1\phi^{1-2m}\phi_{zz} - \frac{1}{2}(m+1)c_1\phi^{-2m}\phi_z^2 + \frac{m-2}{3-2m}\phi^{3-2m}c_1 \\ + \frac{\phi^{2(1-m)}}{2(1-m)}c_2 + c_3. \end{aligned} \tag{5.64}$$

Substituting (5.64) into (5.45) we find that $c_2 = 0$. Hence

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1\phi^{1-2m}\phi_{zz} - \frac{1}{2}(m+1)c_1\phi^{-2m}\phi_z^2 + \frac{m-2}{3-2m}\phi^{3-2m}c_1 + c_3. \tag{5.65}$$

Since the multiplier (5.65) contains two constants, $c_1$ and $c_3$, it leads to two conserved vectors. The conserved vector corresponding to $c_3 = 1, c_1 = 0$ is the elementary conserved vector with components (5.13) and (5.14). The constants $c_1 = 1, c_3 = 0$ lead to the conserved vector

$$\begin{aligned} T_1 = \frac{1}{2}\phi^{-2m}\phi_{zz}^2 - \frac{1}{6}m(m+2)\phi^{-2m-2}\phi_z^4 + \frac{1}{2}(3-m)\phi^{1-2m}\phi_z^2 \\ + \frac{1}{2(3-2m)}\phi^{4-2m}, \end{aligned} \tag{5.66}$$

$$\begin{aligned} T_2 = \left[\frac{1}{6}m(m+5)\phi^{-2m-2}\phi_z^3 - \frac{(m-3)(m-1)}{(3-2m)}\phi^{1-2m}\phi_z\right]\phi_t. \\ - \left[\frac{1}{2}(m+1)\phi^{-2m-1}\phi_z^2 + \frac{(2-m)}{(3-2m)}\phi^{2-2m}\right]\phi_{tz} - \frac{1}{2}(m-1)\phi^{-m-1}\phi_z^2 \\ + \frac{(m-1)}{(3-2m)}\phi^{2-m}. \end{aligned} \tag{5.67}$$

The case $n = m - 1$ with $m = 1$ and $n = 0$ has already been considered. The multiplier is given by (5.10) and the two conserved vectors by (5.27) to (5.28).

Consider $n = m - 1$ with $m = \frac{3}{2}$ and $n = \frac{1}{2}$. The differential equation (5.62) becomes

$$\frac{d^2P}{d\phi^2} + \frac{2}{\phi}\frac{dP}{d\phi} = -\frac{1}{2}\frac{c_1}{\phi^2}. \tag{5.68}$$

The general solution of (5.68) is

$$P(\phi) = -\frac{1}{2}c_1 \ln\phi + \frac{c_3}{\phi} + c_4. \tag{5.69}$$

When $m = \frac{3}{2}$ the multiplier (5.61) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1\phi^{-2}\phi_{zz} - \frac{5}{4}c_1\phi^{-3}\phi_z^2 - \frac{1}{2}c_1\ln\phi + \phi^{-1}c_3 + c_4. \tag{5.70}$$

On substituting (5.70) into the determining equation (5.45) we find that $c_3 = 0$ and the multiplier reduces to

$$\Lambda = c_1\phi^{-2}\phi_{zz}^2 - \frac{5}{4}c_1\phi^{-3}\phi_z^2 - \frac{1}{2}c_1 \ln\phi + c_4. \tag{5.71}$$

The multiplier (5.71) again contains two arbitrary constants, $c_1$ and $c_4$. Setting the constants $c_4 = 1, c_1 = 0$ gives the elementary conserved vector with components (5.13) and (5.14). Setting $c_1 = 1, c_4 = 0$ leads to the conserved vector

$$T^1 = \frac{1}{2}\phi^{-3}\phi_{zz}^2 - \frac{7}{8}\phi^{-5}\phi_z^4 + \frac{3}{4}\phi^{-2}\phi_z^2 + \frac{1}{2}\phi\ln\phi - \frac{1}{2}\phi, \tag{5.72}$$

$$\begin{aligned} T^2 = &\left[\frac{13}{8}\phi^{-5}\phi_z^3 - \phi^{-2}\phi_z + \frac{3}{4}(\phi^{-2}\ln\phi)\ \phi_z\right]\phi_t \\ &- \left[\frac{5}{4}\phi^{-4}\phi_z^2 + \frac{1}{2}\phi^{-1}\ln\phi\right]\phi_{tz} - \frac{1}{4}\phi^{-\frac{5}{2}}\phi_z^2 + \frac{1}{2}\phi^{\frac{1}{2}}\ln\phi. \end{aligned} \tag{5.73}$$

Consider next multipliers of the form

$$\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz}). \tag{5.74}$$

The determining equation for the multiplier is

$$E_\phi(\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz})F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz})] = 0, \tag{5.75}$$

where $F$ is given by (5.4). Equating the coefficient of $\phi_{tzzzzz}$ in (5.75) to zero, we find that

$$\frac{\partial\Lambda}{\partial\phi_{zzz}} = 0 \tag{5.76}$$

and therefore

$$\Lambda = \Lambda(\phi, \phi_z, \phi_{zz}), \tag{5.77}$$

which has already been considered

Harris [6] proved that except possibly for the two special cases, $n = m - 1$ with $m \neq 1$ and $m = 1$ with $n \neq 0$, there are no more independent conserved vectors. She proved this result using the direct method for conservation laws.

We now list in Table 5.1 the multipliers and the corresponding conserved vectors for the partial differential equation (5.1). This table was presented by Maluleke and Mason [12] without the multipliers. These conserved vectors agree with the results obtained by Barcilon and Richter [4] and Harris [11].

| | |
|---|---|
| Case A: $n \geq 0, m \geq 0$<br>Multiplier:$\Lambda = 1$<br>$T^1 = \phi$<br>$T^2 = \phi^n(1 + m\phi^{-m-1}\phi_t\phi_z - \phi^{-m}\phi_{tz})$<br><br>Case B.1: $m \neq 1, m+n \neq 1, m+n \neq 2$<br>Multiplier:$\Lambda = \phi^{1-m-n}$<br>$T^1 = \frac{1}{2-m-n}(\phi^{2-m-n} - 1)$<br>$+\frac{1}{2}(1-m-n)\phi^{-2m}\phi_z^2$<br>$T^2 = \frac{n}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z$<br>Case B.2: $m = 1, n \neq 0,\ n \neq 1$<br>Multiplier: $\Lambda = \phi^{-n}$<br>$T^1 = \frac{1}{1-n} - \frac{n}{2}\phi^{-2}\phi_z^2$<br>$T^2 = n\ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z$<br>Case B.3: $m = n = 1$<br>Multiplier: $\Lambda = \frac{1}{\phi}$<br>$T^1 = -\frac{1}{2}\phi^{-2}\phi_z^2 + \ln\phi$<br>$T^2 = \ln\phi - \phi^{-1}\phi_{tz} + \phi^{-2}\phi_t\phi_z$ | Case B.5: $n+m = 1, m \neq 1$<br>Multiplier: $\Lambda = \phi^{-2(n-2)}$<br>$T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi$<br>$T^2 = \phi^{1-m}\ln\phi - \frac{1}{1-m}\phi^{1-m}$<br>$-(\phi^{1-2m}\ln\phi)\ \phi_{tz} + (m\phi^{-2m}\phi_t\phi_z\ln\phi)$<br><br>Case B.6: $n+m = 2,\ m \neq 1$<br>Multiplier: $\Lambda = \frac{1}{\phi}$<br>$T^1 = -\frac{1}{2}\phi^{-2m}\phi_z^2 + \ln\phi$<br>$T^2 = \frac{2-m}{1-m}\phi^{1-m} - \phi^{1-2m}\phi_{tz} + m\phi^{-2m}\phi_t\phi_z$<br>Case C.1: $n = m-1, m \neq \frac{3}{2}, m \neq 1$<br>Multiplier: $\Lambda = \phi^{1-2m}\phi_{zz}$<br>$-\frac{1}{2}(m+1)\phi^{-2m}\phi_z^2 + \frac{m-2}{3-2m}\phi^{3-2m}$<br>$T^1 = \frac{1}{1-n}\phi^{1-n} - \frac{n}{2}\phi^{-2}\phi_z^2$<br>$+\frac{1}{2}(3-m)\phi^{1-2m}\phi_z^2 + \frac{1}{2(3-2m)}\phi^{4-2m}$<br>$T^2 = \frac{1}{6}m(m+5)\phi^{-2m-2}\phi_z^3\phi_t$<br>$-\frac{(m-3)(m-1)}{(3-2m)}\phi^{1-2m}\phi_z\phi_t$<br>$+\frac{(2-m)}{(3-2m)}\phi^{2-2m}\phi_{tz} - \frac{1}{2}(m+1)\phi^{-2m-1}\phi_z^2\phi_{tz}$<br>$-\frac{1}{2}(m-1)\phi^{-m-1}\phi_z^2 + \frac{(m-1)}{(3-2m)}\phi^{2-m}$ |
| Case B.4: $m = 1, n = 0$<br>Multiplier: $\Lambda = \phi^2$<br><br>$T^1 = \frac{1}{2}\phi^{-2}\phi_z^2 + \phi\ln\phi - \phi$<br>$T^2 = -(\phi^{-1}\ln\phi)\ \ \phi_{tz} + (\phi^{-2}\ln\phi)\ \phi_t\phi_z$ | Case C.2: $n = m-1, m = \frac{3}{2}, n = \frac{1}{2}$<br>Multiplier: $\Lambda = \phi^{-2}\phi_{zz}^2 - \frac{5}{4}\phi^{-3}\phi_z^2$<br>$-\frac{1}{2}\ln\phi$<br>$T^1 = \frac{1}{2}\phi^{-3}\phi_{zz}^2 - \frac{7}{8}\phi^{-5}\phi_z^4 + \frac{3}{4}\phi^{-2}\phi_z^2$<br>$+\frac{1}{2}\phi\ln\phi - \frac{1}{2}\phi$<br>$T^2 = \left[\frac{13}{8}\phi^{-5}\phi_z^3 - \phi^{-2}\phi_z\right]\phi_t$<br>$+\frac{3}{4}(\phi^{-2}\phi_z\phi_t\ln\phi) - \left[\frac{5}{4}\phi^{-4}\phi_z^2\right]\phi_{tz}$<br>$+\frac{1}{2}\phi^{-1}\ln\phi\phi_{tz} - \frac{1}{4}\phi^{-\frac{5}{2}}\phi_z^2 + \frac{1}{2}\phi^{\frac{1}{2}}\ln\phi$ |

Table 5.1: Multipliers and conserved vectors for the partial differential equation (5.1).

### 5.2.3 Association of Lie point symmetries with conserved vectors

The Lie point symmetries for the partial differential equation (5.1) are listed in Table 4.3. Using (2.13) we will investigate which of the Lie point symmetries are associated with the conserved vectors for the magma equation (5.1).
Case (i) $0 < n < \infty, 0 \leq m < \infty$.

Consider first the Lie symmetry generator

$$X = [c_1 + (2 - m - n)c_3 t]\frac{\partial}{\partial t} + [c_2 + (n - m)c_3 z]\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi} \tag{5.78}$$

and the elementary conserved vector with components (5.13) and(5.14). Applying (2.13) we find that (5.78) is associated with the elementary conserved vector provided $c_3 = 0$, that is, provided

$$X = c_1\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z}. \tag{5.79}$$

Case (ii) $0 < n < \infty$, $n + m \neq 2, m \neq 1, n + m \neq 1$.

Consider next the Lie point symmetry generator (5.78), with the conserved vector with components (5.15) and (5.16). Applying (2.13) we find that (5.78) is associated with this conserved vector provided $c_3 = 0$, that is, provided $X$ is given by (5.79)

Case (iii) $m = 1, n \neq 0, n \neq 1$.

Now consider (5.78) with $m = 1$,

$$X = (c_1 - (n - 1)c_3\ t)\frac{\partial}{\partial t} + (c_2 + (n - 1)c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi} \tag{5.80}$$

and the conserved vector with components (5.37) and (5.38). Applying (2.13) we find that (5.80) is associated with this conserved vector provided that $c_3 = 0$, that is, provided $X$ is given by (5.79).

Case (iv) $m = n = 1$.

Consider next (5.78) with $m = n = 1$,

$$X = c_1\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi} \tag{5.81}$$

and the conserved vector with components (5.33) and (5.34). Applying (2.13) we find that (5.81) is associated with this conserved vector provided $c_3 = 0$, that is, provided $X$ is (5.79).

Case (v) $n = 0$, $m = 1$.

Consider the Lie point symmetry given by Case 2 with $m = 1$ of Table 4.3,

$$X = f(t)\frac{\partial}{\partial t} + (c_2 - c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi}, \tag{5.82}$$

and the conserved vector with components (5.29) and (5.30). Applying (2.13) we find that (5.82) is associated with this conserved vector provided $c_3 = 0$, that is, provided

$$X = f(t)\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z}. \tag{5.83}$$

Case (vi) $n \neq 0$, $n + m = 1, m \neq 1$.

Consider next (5.78) with $n = m - 1$,

$$X = (c_1 + c_3 t)\frac{\partial}{\partial t} + (c_2 + (1 - 2m)c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi}, \tag{5.84}$$

and the conserved vector with components (5.19) and (5.20). Applying (2.13) we find that (5.84) is associated with this conserved vector provided that $c_3 = 0$, that is, provided $X$ is given by (5.79)

Case (vii) $n \neq 0$ $n + m = 2, m \neq 1$.

Consider (5.78) with $n = 2 - m$,

$$X = c_1\frac{\partial}{\partial t} + (c_2 + 2(1 - m)c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi} \tag{5.85}$$

and the conserved vector with components (5.24) and(5.25). Applying (2.13) we find that (5.85) is associated with this conserved vector provided that $c_3 = 0$,that is, provided $X$ is given by (5.79).

Case (viii) $n = m - 1, m \neq \frac{3}{2}, m \neq 1$.

Consider next (5.78) with $n = m - 1$

$$X = (c_1 + (3 - 2m)c_3)\frac{\partial}{\partial t} + (c_2 - c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi} \tag{5.86}$$

and the conserved vector given by the components (5.66) and (5.67). Applying (2.13) we find that (5.86) is associated with this conserved vector provided that $c_3 = 0$,that is, provided $X$ is given by (5.79).

Case (ix) $m = \frac{3}{2}, n = \frac{1}{2}$.

Finally consider (5.78) with $n = \frac{1}{2}$ and $m = \frac{3}{2}$,

$$X = c_1 \frac{\partial}{\partial t} + (c_2 + c_3 z)\frac{\partial}{\partial z} + 2c_3\phi\frac{\partial}{\partial \phi}, \tag{5.87}$$

and the conserved vector with components (5.72) and (5.73). Applying (2.13) we find that (5.87) is associated with this conserved vector provided that $c_3 = 0$, that is, provided $X$ is given by (5.79).

Except for the conserved vector with components (5.29) and (5.30) ($n = 0, m = 1$) the conserved vectors are associated with the Lie point symmetry (5.79) which generates travelling wave solutions. The Lie point symmetry with which the conserved vector (5.29) and (5.30) is associated contains (5.79) as a special case. In all cases new conserved vectors are not generated by (2.14).

Next we derive the conservation laws for the magma equation with an exponential law for the permeability and matrix viscosity using the multiplier method.

## 5.3 Conservation laws for the magma equation with exponential law permeability and viscosity by the multiplier method

When the permeability and viscosity are related to the voidage by exponential laws the magma equation becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\exp[n(\phi-1)]\left(1 - \frac{\partial}{\partial z}\left(\exp[-m(\phi-1)]\frac{\partial \phi}{\partial t}\right)\right)\right] = 0. \tag{5.88}$$

### 5.3.1 Lower order conservation laws

In order to derive conservation laws for (5.88) consider a multiplier of the form (5.2). A multiplier for the partial differential equation has the property (5.3), where now

$$\begin{aligned} F(\phi, \phi_t, \phi_z, \phi_{tz}, \phi_{zz}, \phi_{tzz}) &= \phi_t + n\phi_z \exp[n(\phi-1)] + mn\phi_t\phi_z^2 \exp((n-m)(\phi-1)) \\ &- n\phi_z\phi_{tz}\exp[(n-m)(\phi-1)] - m^2\phi_t\phi_z^2\exp[(n-m)(\phi-1)] \\ &+ m\phi_t\phi_{zz}\exp[(n-m)(\phi-1)] + 2m\phi_z\phi_{tz}\exp[(n-m)(\phi-1)] \\ &- \exp[(n-m)(\phi-1)]\phi_{tzz}. \end{aligned} \tag{5.89}$$

The determining equation for the multiplier is given by (5.5),where $E_\phi$ is given by (2.18). Separating (5.5) with respect to products and powers of the partial derivatives of $\phi$ we obtain the following system of equations.

$$\phi_z\phi_{tz}: \qquad \frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{5.90}$$

$$\phi_t\phi_{zz}: \qquad \frac{d^2\Lambda}{d\phi^2} + (m+n)\frac{d\Lambda}{d\phi} = 0, \tag{5.91}$$

$$\phi_t\phi_z^2: \qquad \frac{d^3\Lambda}{d\phi^3} + 2n\frac{d^2\Lambda}{d\phi^2} + (n^2-m^2)\frac{d\Lambda}{d\phi} = 0. \tag{5.92}$$

Equation (5.90) is the same as (5.89). It is readily verified that every solution of (5.89) is a solution of (5.91). We therefore need to consider only (5.89). The general solution of (5.89) is

$$\Lambda(\phi) = c_2 \exp[-(m+n)\phi] + c_1, \quad \text{if} \quad n+m \neq 0, \tag{5.93}$$

and

$$\Lambda(\phi) = c_2\phi + c_1, \quad \text{if} \quad n+m = 0. \tag{5.94}$$

We are considering $n \geq 0$ and $m \geq 0$ and therefore $n+m=0$ only if $n=0$ and $m=0$. Proceeding as before we have for various combinations of $m$ and $n$ different conserved vectors.

Case (i) $n > 0, m > 0$.

This gives the conserved vectors

$$T^1 = \phi, \tag{5.95}$$

$$T^2 = \exp[n(\phi-1)] + m\exp[(n-m)(\phi-1)]\phi_t\phi_z - \exp[(n-m)(\phi-1)]\phi_{tz}, \tag{5.96}$$

and

$$T^1 = -\frac{1}{m+n}\exp[-(m+n)\phi] - \frac{(n+m)}{2}\phi_z^2\exp[-2m\phi+m-n], \tag{5.97}$$

$$T^2 = -\frac{n}{m}\exp[-(m\phi+n)] + exp[-2m\phi+m-n](m\phi_t\phi_z - \phi_{tz}). \tag{5.98}$$

The conserved vector with components (5.95) and (5.96) is the elementary conserved vector with the multiplier $c_1$. The multiplier for the conserved vector with components (5.97) and(5.98) is, from (5.93),

$$\Lambda(\phi) = \exp[-(m+n)\phi]. \tag{5.99}$$

Case (ii) $n = m = 0$.

We obtain two conserved vectors,

$$T^1 = \phi, \tag{5.100}$$
$$T^2 = 1 - \phi_{tz}, \tag{5.101}$$

and

$$T^1 = \frac{\phi^2}{2}, \tag{5.102}$$
$$T^2 = \phi_t - \phi_{tz}. \tag{5.103}$$

The conserved vector with components (5.100) and (5.101) is the elementary conserved vector with multiplier $c_1$ and $m = n = 0$. The multiplier for the conserved vector with components (5.102) and (5.103) is, from (5.94),

$$\Lambda(\phi) = \phi. \tag{5.104}$$

Case (iii) $m = 0$ and $n > 0$.

We again obtain two conserved vectors

$$T^1 = \phi, \tag{5.105}$$
$$T^2 = \exp[n(\phi - 1)](1 - \phi_{tz}), \tag{5.106}$$

and

$$T^1 = -\frac{1}{n}\exp[-n\phi] - \frac{1}{2}n\exp[-n\phi]\phi_z^2, \tag{5.107}$$
$$T^2 = n\exp[-n\phi] - \phi_{tz}. \tag{5.108}$$

The conserved vector with components (5.105) and (5.106) is the elementary conserved vector with multiplier $c_1$ and $m = 0$. The multiplier of the conserved vector with components (5.107) and (5.108) is, by (5.93),

$$\Lambda(\phi) = \exp[-n\phi]. \tag{5.109}$$

Case (iv) $n = 0, m > 0$.

Finally we obtain the conserved vectors

$$T^1 = \phi, \tag{5.110}$$
$$T^2 = 1 + m\exp[-m(\phi - 1)]\phi_t\phi_z - \exp[-m(\phi - 1)]\phi_{tz}, \tag{5.111}$$

and

$$T_1 = -\frac{1}{m}\exp(-m\phi) - \frac{1}{2}m\exp[m(1 - 2\phi)]\phi_z^2, \tag{5.112}$$
$$T_2 = \exp[m(1 - 2\phi)](m\phi_t\phi_z - \phi_{tz}). \tag{5.113}$$

The conserved vector with components (5.110) and (5.111) is the elementary conserved vector with multiplier $c_1$ and $n = 0$, while the multiplier for the conserved vector with components (5.112) and (5.113) is by (5.93),

$$\Lambda = \exp[-m\phi]. \tag{5.114}$$

### 5.3.2 The search for higher order conservation laws

We now consider a multiplier of the form

$$\Lambda = \Lambda(\phi, \phi_z). \tag{5.115}$$

The determining equation for the multiplier is equation(5.41) where $F$ is given by (5.88). By equating the coefficient of the highest order derivative term, $\phi_{tzzz}$, to zero in (5.41) we obtain again (5.42). so that $\Lambda(\phi, \phi_z) = \Lambda(\phi)$. The multiplier therefore reduces to that of the previous case and new conserved vectors are not derived.

Consider next the multiplier

$$\Lambda = \Lambda(\phi, \phi_z, \phi_{zz}). \tag{5.116}$$

As before, the determining equation for the multiplier is (5.41),where $F$ is given by (5.88). By equating the coefficients of $\phi_{tz}\phi_{zzzz}$, $\phi_t\phi_{zzzz}$, $\phi^2_{zzzz}$ to zero in (5.45), the following system of equations is obtained:

$$\phi_{tz}\phi_{zzzz}: \quad (2m-n)\phi_z\frac{\partial^2\Lambda}{\partial\phi^2_{zz}} + \frac{\partial^2\Lambda}{\partial\phi_z\phi_{zz}} = 0, \tag{5.117}$$

$$\phi_t\phi_{zzzz}: \quad m((n-m)\phi^2_z + \phi_{zz})\frac{\partial^2\Lambda}{\partial\phi^2_{zz}} + (m+n)\frac{\partial\Lambda}{\partial\phi_{zz}} + \frac{\partial^2\Lambda}{\partial\phi\partial\phi_{zz}} = 0, \tag{5.118}$$

$$\phi^2_{zzzz}: \quad \frac{\partial^2\Lambda}{\partial\phi^2_{zz}} = 0. \tag{5.119}$$

By using (5.117) and (5.119) it is readily shown that (5.50) again holds. Substituting (5.52) into (5.118) gives

$$\frac{dA}{d\phi} + (m+n)A = 0 \tag{5.120}$$

and therefore

$$A(\phi) = c_1\exp[-(m+n)\phi]. \tag{5.121}$$

Equation (5.52) now becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1\exp[-(m+n)\phi]\phi_{zz} + B(\phi, \phi_z). \tag{5.122}$$

Substituting (5.121) into the determining equation(5.45) and then equating the coefficients of $\phi_{tzzz}$ in (5.45) to zero gives

$$\frac{\partial B}{\partial \phi_z} = \frac{1}{2}(n-2m)c_1 \exp[-(m+n)\phi]\phi_z \tag{5.123}$$

and hence

$$B(\phi, \phi_z) = \frac{1}{4}(n-2m)c_1 \ \exp[-(m+n)\phi]\phi_z^2 + P(\phi). \tag{5.124}$$

The multiplier becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1 \exp[-(m+n\phi]\phi_{zz} + \frac{1}{4}(n-2m)c_1 \exp[-(m+n)\phi]\phi_z^2 + P(\phi). \tag{5.125}$$

Finally we substitute (5.125) back into (5.45) and equate the coefficient of $\phi_t\phi_z^2\phi_{zz}$ to zero. This yields

$$m(m-n)c_1 = 0. \tag{5.126}$$

There are three cases to consider, $m = 0$, $m = n$ and $c_1 = 0$. The conserved vector for $m = 0, n > 0$ is given by (5.107) and (5.108). We now consider the two remaining cases.

Case 1: $m \neq n, m > 0$

Then $c_1 = 0$ and (5.125) reduces to

$$\Lambda(\phi, \phi_z, \phi_{zz}) = P(\phi). \tag{5.127}$$

The multiplier is therefore a function of $\phi$ only which does not yield new conserved vectors.

Case 2: $c_1 \neq 0$, $m > 0$

Then $m = n$ and the multiplier (5.125) becomes

$$\Lambda(\phi, \phi_z, \phi_{zz}) = c_1 \exp[-2n\phi] \ \phi_{zz} - \frac{1}{4}nc_1 \exp[-2n\phi] \ \phi_z^2 + P(\phi). \tag{5.128}$$

Equation (5.128) is substituted back into the determining equation (5.45) and by equating the coefficient of $\phi_z\phi_{tz}$ to zero we obtain

$$\frac{d^2P}{d\phi^2} + 2n\frac{dP}{d\phi} = n \exp[-n(2\phi+1)+1)]c_1. \tag{5.129}$$

The general solution to (5.129) is

$$P(\phi) = -\frac{c_1}{4n}(1+2n\phi)\exp[1-n-2n\phi] + c_2\exp[-2n\phi] + c_3, \qquad (5.130)$$

where $n \neq 0$ since $n = m$ and $m \neq 0$. Thus (5.128) becomes

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = {} & c_1\exp[-2n\phi]\phi_{zz} - \frac{1}{4}nc_1\exp[-2n\phi]\phi_z^2 \\ & - \frac{c_1}{4n}(1+2n\phi)\exp[1-n-2n\phi] + c_2\exp[-2n\phi] + c_3.\end{aligned} \qquad (5.131)$$

Finally substituting (5.131) into the determining equation (5.45) gives $c_2 = 0$ and therefore

$$\begin{aligned}\Lambda(\phi,\phi_z,\phi_{zz}) = {} & c_1\exp[-2n\phi]\ \phi_{zz} - \frac{1}{4}nc_1\exp[-2n\phi]\ \phi_z^2 \\ & - \frac{c_1}{4n}(1+2n\phi)\exp[1-n-2n\phi] + c_3.\end{aligned} \qquad (5.132)$$

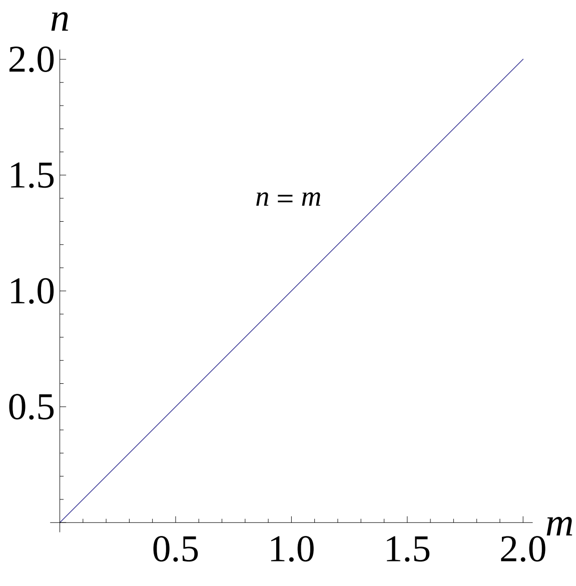


Figure 5.2: The $(m,n)$-plane.The special cases lie on the line $n = m$.

Two conserved vectors are obtained since the multiplier (5.109) contains two arbitrary constants. The constant $c_3$ gives the elementary conserved vector with components (5.94) and (5.95) with $m = n$ while the constant $c_1$ gives the new conserved vector

$$T^1 = \frac{1}{2}\exp[-2n\phi]\ \phi_{zz}^2 - \frac{1}{2}n^2\exp[-2n\phi]\ \phi_z^4 + \frac{1}{4}n(1+2n\phi)\phi_z^2 \qquad (5.133)$$
$$+ \frac{1}{8n^2}(1+2n\phi)\exp[1-n-2n\phi],$$
$$T^2 = \left[n^2\exp[-2n\phi]\ \phi_z^3 - \frac{n}{4}(1+2n\phi)\exp[1-n-2n\phi]\phi_z\right]\phi_t$$
$$- n\exp[-n(\phi+1)]\ \phi_{tz} - \frac{1}{4}(1+2n\phi)\exp[1-n-2n\phi], \qquad (5.134)$$

which exists if $m = n$ and $m > 0$, $n > 0$.

Consider next multipliers of the form (5.74). The determining equation for the multiplier is (5.75),where $F$ is given by (5.88). By equating the coefficient of $\phi_{tzzzzz}$ in (5.75) to zero we again derive (5.76) and the multiplier therefore reduces to the form (5.77) which has already been considered.

We now list the multipliers and the corresponding conserved vectors for the partial differential equation (5.88) in Table 5.2. The $(m, n)$ plane is illustrated in Figure 5.2.

Case A: $n \geq 0, m \geq 0$
Multiplier: $\Lambda(\phi) = 1$
$T^1 = \phi$
$T^2 = \exp[n(\phi-1)] + m\exp[(n-m)(\phi-1)]\ \phi_t\phi_z - \exp[(n-m)(\phi-1)]\ \phi_{tz}$

Case B.1: $m+n \neq 0, m>0, n>0$
Multiplier: $\Lambda(\phi) = \exp[-(m+n)\phi]$
$T^1 = -\frac{1}{m+n}\exp[-(m+n)\phi] - \frac{1}{2}(n+m)\exp[m-n-2m\phi]\phi_z^2$
$T^2 = -\frac{n}{m}\exp[-(m\phi+n)] + \exp[m-n-2m\phi](m\phi_t\phi_z - \phi_{tz})$

Case B.2: $m=0,\ n=0$
Multiplier: $\Lambda(\phi) = \phi$
$T^1 = \frac{1}{2}\phi^2$
$T^2 = \phi_t - \phi_{tz}$

Case B.3: $m=0,\ n>0$
Multiplier: $\Lambda(\phi) = \exp(-n\phi)$
$T^1 = -\frac{1}{n}\exp(-n\phi) - \frac{1}{2}n\exp[-n\phi]\phi_z^2$
$T^2 = n\exp[-n\phi] - \phi_{tz}$

Case B.4: $n=0,\ m>0$
Multiplier: $\Lambda(\phi) = \exp[-m\phi]$
$T^1 = -\frac{1}{m}\exp[-m\phi] - \frac{1}{2}m\exp[m(1-2\phi)]\phi_z^2$
$T^2 = \exp[m(1-2\phi)](m\phi_t\phi_z - \phi_{tz})$

Case C: $m=n,\ n\neq 0$
Multiplier:$\Lambda(\phi,\phi_z,\phi_{zz}) = \exp[-2n\phi]\phi_{zz} - \frac{1}{2}n\exp[-2n\phi]\phi_z^2$
$-\frac{1}{4n}(2n\phi+1)\exp[1-n-2n\phi]$

$$\begin{aligned} T^1 = {} & \tfrac{1}{2}\exp[-2n\phi]\phi_{zz}^2 - \tfrac{1}{2}n^2\exp[-2n\phi]\phi_z^4 + \tfrac{1}{4}n(2n\phi+1)\phi_z^2 \\ & + \tfrac{1}{8n^2}(2n\phi+1)\exp[1-n-2n\phi] \end{aligned}$$

$$\begin{aligned} T^2 = {} & \left[n^2\exp(-2n\phi)\phi_z^3 - \tfrac{1}{4}n(2n\phi+1)\exp[1-n-2n\phi)]\phi_z\right]\phi_t - n\exp(-n(\phi+1))\phi_{tz} \\ & -\tfrac{1}{4}(2n\phi+1)\exp[1-n-2n\phi] \end{aligned}$$

Table 5.2: Multipliers and conserved vectors for the partial differential equation (5.88)

### 5.3.3 Association of Lie point symmetries with conserved vectors

The Lie point symmetries of the partial differential equation (5.87) are given in Table 4.3. We use (2.13) to investigate which Lie point symmetries of (5.88) are associated with the conserved vectors for equation (5.87).

Case (i) $n > 0, m > 0$
Consider first the Lie point symmetry generator

$$X = (c_3(m+n)t + c_1)\frac{\partial}{\partial t} + (c_3(m-n)z + c_2)\frac{\partial}{\partial z} - 2c_3\frac{\partial}{\partial \phi}, \tag{5.135}$$

and the elementary conserved vector with components (5.95) and (5.96). We find that (5.135) is associated with the elementary conserved vector provided $c_3 = 0$, that is, provided $X$ is given by (5.79).

Case (ii) $m > 0, n > 0$
Consider next the Lie point symmetry generator (5.135) and the conserved vector with components (5.97) and (5.98). It can be verified that this conserved vector is associated with (5.135) provided $c_3 = 0$, that is, provided $X$ is given by (5.79).

Case (iii) $m = 0, n > 0$
Now consider the Lie point symmetry from Case 1 of Table 4.3

$$X = (c_1 - c_3 nt)\frac{\partial}{\partial t} + (c_3 nz + c_2)\frac{\partial}{\partial z} + 2c_3\frac{\partial}{\partial \phi} \tag{5.136}$$

and the conserved vector with components (5.107) and (5.108). Using again (2.13) we find that (5.135) is associated with this conserved vector provided that $c_3 = 0$, that is, provided $X$ is given by (5.79).

Case (iv) $n = 0, m > 0$
Consider the Lie point symmetry from Case 2 of Table 4.3,

$$X = f(t)\frac{\partial}{\partial t} + (c_2 + c_3 z)\frac{\partial}{\partial z} - \frac{2c_3}{m}\frac{\partial}{\partial \phi} \tag{5.137}$$

and the conserved vector with components (5.112) and (5.113). We find that (5.137) is associated with this conserved vector provided $c_3 = 0$, that is, provided

$$X = f(t)\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial z}. \tag{5.138}$$

We see that, except for (5.138) ($n = 0, m > 0$), the conserved vectors are associated with the Lie point symmetry (5.79) which generates a travelling wave solution. The conserved vector with components (5.112) and (5.113) is associated with (5.138) which includes (5.79) as a special case. In all cases, (2.14) does not yield a new conserved vector.

## 5.4 Concluding remarks

The multiplier method was found to be a systematic method to re-derive the conservation laws for the magma equation for power laws relating the permeability and viscosity to the voidage and new conservation laws for the exponential law relations. We were not able to go beyond the results derived by Harris [11] for the power law relations either by using the multiplier method or by generating new conserved vectors from known conserved vectors and Lie point symmetries of the magma equation using (2.14).

We have also shown that the only Lie point symmetries associated with the conserved vectors for both power law and exponential law permeability and viscosity are the ones which generate travelling wave solutions. By Sj öberg double reduction theorem the associated Lie point symmetry will guarantee a double reduction in the partial differential equation.

# Chapter 6

# Group Invariant Solution for the magma equation

## 6.1 Introduction

The rarefactive solitary wave solution has been derived for $K(\psi)$ given by an exponential law and $G(\psi) = 1$ by Mindu and Mason [20]. In this chapter we investigate the group invariant solution of the magma equation when the permeability and viscosity, $K(\phi)$ and $G(\phi)$ respectively, are given by exponential laws. We also investigate the existence of the rarefactive solitary wave solution and consider large and small amplitude approximations of the solitary wave solution.

## 6.2 Rarefactive solitary wave

When the permeability and viscosity both satisfy an exponential law, the partial differential equation (1.3) becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\exp[n(\phi - 1)]\left(1 - \frac{\partial}{\partial z}\left(\exp[-m(\phi - 1)]\frac{\partial \phi}{\partial t}\right)\right)\right] = 0. \qquad (6.1)$$

We now derive a rarefactive solitary wave solution with $\phi \geq 1$ of the partial differential equation (6.1).

The solution $\phi = \Phi(t, z)$ is an invariant solution of (6.1) provided that

$$X(\phi - \Phi(t, z))|_{\phi=\Phi} = 0, \qquad (6.2)$$

where $X$ is a linear combination of Lie point symmetries of (6.1). The Lie point symmetries of (6.1) are given in Table 4.3. Consider the invariant solution generated by the linear combination

$$X = c_1 \frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial z}. \qquad (6.3)$$

Equation (6.2) becomes

$$c_1\frac{\partial\Phi}{\partial t}+c_2\frac{\partial\Phi}{\partial z}=0, \tag{6.4}$$

where $c_1$ and $c_2$ are constants. The general solution of (6.4) is readily derived and since $\phi=\Phi(t,z)$ we obtain

$$\phi=\psi(\zeta),\quad \zeta=z-ct, \tag{6.5}$$

where $c=c_2/c_1$. The group invariant solution (6.5) is a travelling wave solution and the constant $c$ is the dimensionless speed of the wave.

Substitute (6.5) into (6.1). This gives the third order differential equation

$$c\frac{d\psi}{d\zeta}-\frac{d}{d\zeta}\left[\exp(n(\psi-1))\left(1+c\frac{d}{d\zeta}\left(\exp[-m(\psi-1)]\frac{d\psi}{d\zeta}\right)\right)\right]=0. \tag{6.6}$$

Integrating (6.6) with respect to $\zeta$ we have

$$c\frac{d}{d\zeta}\left(\exp[-m(\psi-1)]\frac{d\psi}{d\zeta}\right)=c\psi\exp[-n(\phi-1)]+A\exp[-n(\psi-1)]n-1, \tag{6.7}$$

where A is a constant now since

$$\frac{d}{d\zeta}\left(\exp([-m(\psi-1)]\frac{d\psi}{d\zeta}\right)=\frac{1}{2}\exp[m(\psi-1)]\frac{d}{d\psi}\left(\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2\right)$$

equation (6.7) becomes

$$\begin{aligned}\frac{c}{2}\frac{d}{d\psi}\left(\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2\right)&=c\psi\exp[-(n+m)(\psi-1)]\\&\quad+A\exp[-(n+m)(\psi-1)]\\&\quad-\exp[-m(\psi-1)].\end{aligned} \tag{6.8}$$

Now integrate (6.8) with respect to $\psi$ from $\psi=1$ to $\psi=\psi$:

$$\begin{aligned}&\frac{c}{2}\left[\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2\right]-\frac{c}{2}\left[\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2\right]\Bigg|_{\psi=1}=\\&c\int_1^{\psi}\psi\exp[-(n+m)(\psi-1)]d\psi+A\int_1^{\psi}\exp[-(n+m)(\psi-1)]d\psi\\&-\int_1^{\psi}\exp[-m(\psi-1)]d\psi.\end{aligned} \tag{6.9}$$

Let

$$\int_1^\psi \psi \exp[-(n+m)(\psi-1)]d\psi = I_1, \tag{6.10}$$

$$\int_1^\psi \exp[-(n+m)(\psi-1)]d\psi = I_2, \tag{6.11}$$

$$\int_1^\psi \exp[-m(\psi-1)] = I_3, \tag{6.12}$$

Then

$$\begin{aligned} I_1 = & \frac{1}{n+m}[1-\psi\exp[-(n+m)(\psi-1))] \\ & + \frac{1}{(n+m)^2}[1-\exp[-(n+m)(\psi-1)]], \end{aligned} \tag{6.13}$$

$$I_2 = \frac{1}{n+m}[1-\exp[-(n+m)(\psi-1)]], \tag{6.14}$$

$$I_3 = \frac{1}{m}[1-\exp[-m(\psi-1)]]. \tag{6.15}$$

Using (6.13) to (6.15), (6.9) becomes

$$\begin{aligned} & \frac{c}{2}\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2 - \frac{c}{2}\exp[-2m(\psi-1)]\left(\frac{d\psi}{d\zeta}\right)^2\Bigg|_{\psi=1} \\ & = \frac{A}{n+m}[1-\exp[-(n+m)(\psi-1)]] + \frac{c}{n+m}[1-\psi\exp[-(n+m)(\psi-1)]] \\ & + \frac{c}{(n+m)^2}[1-\exp[-(n+m)(\psi-1)]] - \frac{1}{m}[1-\exp[-m(\psi-1)]]. \end{aligned} \tag{6.16}$$

where $A$ and $c$ are constants.

In order to obtain the two arbitrary constants, $A$ and $c$, we impose boundary conditions suitable for a rarefactive solitary wave:

$$\psi = 1 \quad : \quad \frac{d\psi}{d\zeta} = 0, \qquad \frac{d^2\psi}{d\zeta^2} = 0, \tag{6.17}$$

$$\psi = \Psi \quad : \quad \frac{d\psi}{d\zeta} = 0, \tag{6.18}$$

where $\Phi$ is the amplitude. since the background state is $\psi = 1$, the amplitude of the solitary wave is $\Psi - 1$. Now by (6.17) and (6.18) using (6.8) and (6.16) we have

$$A + c - 1 = 0 \tag{6.19}$$

$$\begin{aligned} &\frac{A}{n+m}[1 - \exp[-(n+m)(\Psi - 1)]] + \frac{c}{n+m}[1 - \Psi \exp[-(n+m)(\Psi - 1)]] \\ &+ \frac{c}{(n+m)^2}[1 - \exp[-(n+m)(\Psi - 1)]] - \frac{1}{m}[1 - \exp[-m(\Psi - 1)]] = 0. \end{aligned} \tag{6.20}$$

Solving for c in (6.20) we have

$$c = \frac{(n+m)[1 - \frac{(n+m)}{m}\exp[n(\Psi - 1)] + \frac{n}{m}\exp[(n+m)(\Psi - 1)]}{\exp[(n+m)(\Psi - 1)] - (n+m)(\Psi - 1) - 1} \tag{6.21}$$

Now from (6.16) we have

$$\left(\frac{d\psi}{d\zeta}\right)^2 = f(\psi) \tag{6.22}$$

where

$$\begin{aligned} f(\psi) = \frac{2}{c}\Bigg[&\frac{(1-c)}{n+m}\{\exp[2m(\psi - 1)][1 - \exp[-(n+m)(\psi - 1)]]\} \\ &+ \frac{c}{(n+m)^2}\{\exp[2m(\psi - 1)][1 - \exp[-(n+m)(\psi - 1)]]\} \\ &+ \frac{c}{n+m}\{\exp[2m(\psi - 1)][1 - \psi\exp[-(n+m)(\psi - 1)]]\} \\ &- \frac{1}{m}\{\exp(2m[\psi - 1)][1 - \exp[-m(\psi - 1)]]\}\Bigg]. \end{aligned} \tag{6.23}$$

We now investigate the existence of a rarefactive solitary wave solution.

Since the left hand side of (6.22) is nonegative, for a solitary wave to exist it is necessary that $f(\psi) > 0$ for $1 < \psi < \Psi$.

Before proceeding further with the solution, we first investigate the properties of $f(\psi)$. In order to do that we define

$$F(\psi) = \frac{(n+m)[1 - \frac{n+m}{m}\exp[n(\psi - 1)] + \frac{n}{m}\exp[(n+m)(\psi - 1)]}{\exp[(n+m)(\psi - 1)] - (n+m)(\psi - 1) - 1}. \tag{6.24}$$

From (6.21), it follows that

$$F(\Psi) = c. \tag{6.25}$$

equation (6.23) can be written in terms of $F(\psi)$ as follows

$$f(\Psi) = \frac{2}{(n+m)^2}\Bigg[\exp[2m(\psi-1)] \tag{6.26}$$
$$+ (n+m-1-(n+m)\psi)\exp[(m-n)(\psi-1))]\Bigg]\left[\frac{F(\Psi)-F(\Psi)}{F(\Psi)}\right].$$

Now differentiating (6.24) with respect to $\psi$, we have

$$\frac{dF}{d\psi} = \frac{2(n+m)^2\exp[n+\frac{1}{2}m](\psi-1)}{[\exp[(n+m)(\psi-1)]-(n+m)(\psi-1)-1]^2}\Bigg[\cosh((n+\frac{1}{2}m)(\psi-1))$$
$$-\cosh(\frac{1}{2}m(\psi-1)) - \frac{n}{m}(n+m)(\psi-1)\sinh(\frac{1}{2}m(\psi-1))\Bigg]. \tag{6.27}$$

Let

$$G(\psi) = \cosh\left(n+\frac{1}{2}m)(\psi-1)\right) - \cosh\left(\frac{1}{2}m(\psi-1)\right)$$
$$-\frac{n}{m}(n+m)(\psi-1)\sinh\left(\frac{1}{2}m(\psi-1)\right) \tag{6.28}$$

Using the identities

$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B, \tag{6.29}$$
$$\cosh 2A = 2\sinh^2 A + 1, \tag{6.30}$$
$$\cosh 2A = 2\sinh A\cosh B, \tag{6.31}$$

it can be shown that

$$G(\psi) = 2\cosh\left(\frac{m}{2}(\psi-1)\right)\Bigg[\sinh^2\left(\frac{n}{2}(\psi-1)\right)$$
$$-\frac{1}{2}\tanh\left(\frac{m}{2}(\psi-1)\right)[\frac{n}{m}(n+m)(\psi-1)-\sinh(n(\psi-1))]\Bigg] \tag{6.32}$$
$$= 2\cosh\left(\frac{m}{2}(\psi-1)\right)H(\psi)$$

where

$$H(\psi) = \sinh^2\left(\frac{n}{2}(\psi-1)\right) - \frac{1}{2}\tanh\left(\frac{m}{2}(\psi-1)\right)\left(\frac{n}{m}(n+m)(\psi-1)-\sinh(n(\psi-1))\right). \tag{6.33}$$

We observe that $\sinh^2\left[\frac{n}{2}(\psi-1)\right]$ does not depend on $m$. We will show that

$$Q(m) = \frac{1}{2}\tanh\left(\frac{m}{2}(\psi-1)\right)\left(\frac{n}{m}(n+m)(\psi-1) - \sinh(n(\psi-1))\right) \quad (6.34)$$

is a decreasing function of $m$ for $n > 0$ and $m \geq 0$, $\psi \geq 1$ and that

$$H(\psi) = \sinh^2\left(\frac{n}{2}(\psi-1)\right) - Q(m) > \sinh^2\left(\frac{n}{2}(\psi-1)\right) - Q(0) > 0. \quad (6.35)$$

Now

$$\begin{aligned}\frac{dQ}{dm} &= -\frac{1}{4}(\psi-1)^2\operatorname{sech}^2\left(\frac{m}{2}(\psi-1)\right)(\sinh(n(\psi-1)) - n(\psi-1)) \\ &\quad + \frac{n^2}{m^2}\Bigg(\sinh(m(\psi-1)) - m(\psi-1)\Bigg) \\ &< 0\end{aligned} \quad (6.36)$$

since $\sinh x > x$. Therefore $Q(m)$ is a decreasing function of $m$. Now expanding $\tanh\left(\frac{m}{2}(\psi-1)\right)$ for small m and then letting $m \to 0$ we find that

$$Q(0) = \left[\frac{n(\psi-1)}{2}\right]^2. \quad (6.37)$$

Hence,

$$H(\psi) > \sinh^2\left(\frac{n}{2}(\psi-1)\right) - \left[\frac{n(\psi-1)}{2}\right]^2 > 0. \quad (6.38)$$

Since $\sinh^2 x > x$, it then follows from (6.37) and (6.38) that for $n > 0, m > 0$ and $\psi > 1$

$$\frac{dF}{d\psi} > 0 \quad (6.39)$$

Hence, $F(\psi)$ is an increasing function $\psi$. Using l'Hopital's rule it can be shown that

$$F(1) = n. \quad (6.40)$$

Consider first $n > 0$ and $m \geq 0$. Since $F(\psi)$ is an increasing function of $\psi$, it follows that $F(\psi) > n > 0$ and $F(\Psi) - F(\psi) > 0$ for $1 < \psi < \Psi$. Also

$$\begin{aligned}&\exp[2m(\psi-1)] + (n+m-1-(n+m)\psi)\exp[(m-n)(\psi-1)] \\ &\exp[2m(\psi-1)]\left[1 - \frac{(1+(n+m)(\psi-1)}{\exp[(n+m)(\psi-1)]}\right] \\ &> 0\end{aligned} \quad (6.41)$$

for $\psi > 1$. Hence from (6.26), $f(\psi) > 0$ for $1 < \psi < \Psi$ and a rarefactive solitary solitary wave solution exist.

Consider next $n < 0$. Then from (6.24), $F(\psi) \to 0$ as $\psi \to \infty$, and since $F(1) = n < 0$ and $F(\psi)$ is an increasing function of $\psi$, it follows that $F(\Psi) < 0$. We still have that $F(\psi) - F(\Psi) > 0$ by (6.39) and (6.41) is still satisfied for $n < 0$. Hence, from (6.27) $f(\psi) < 0$ for $1 < \psi < \Psi$. Thus, a rarefactive solitary wave solution does not exists.

Thus we have shown that for $n > 0$ and $m \geq 0$ rarefactive solitary wave solutions that satisfy the boundary conditions (6.17) and (6.18) exist.

Graphs of $f(\psi)$ plotted against $\psi$ are presented in Figure 6.1 for $n = 3$ and $m = 0.4, 0.5$ and $0.6$ while in Figure 6.2 the graphs are plotted for $n = 1, 2$ and $3$ with $m = 0.25$

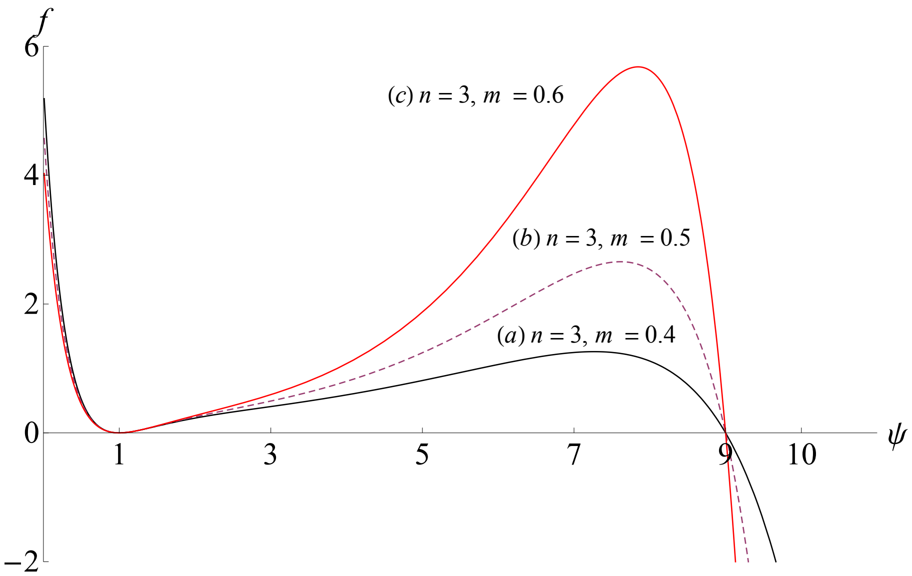


Figure 6.1: The function defined by (6.23) plotted against $\psi$ for $\Psi = 10$ and (a) $n = 3, m = 0.4$ (b) $n = 3, m = 0.5$ and (c) $n = 3, m = 0.6$

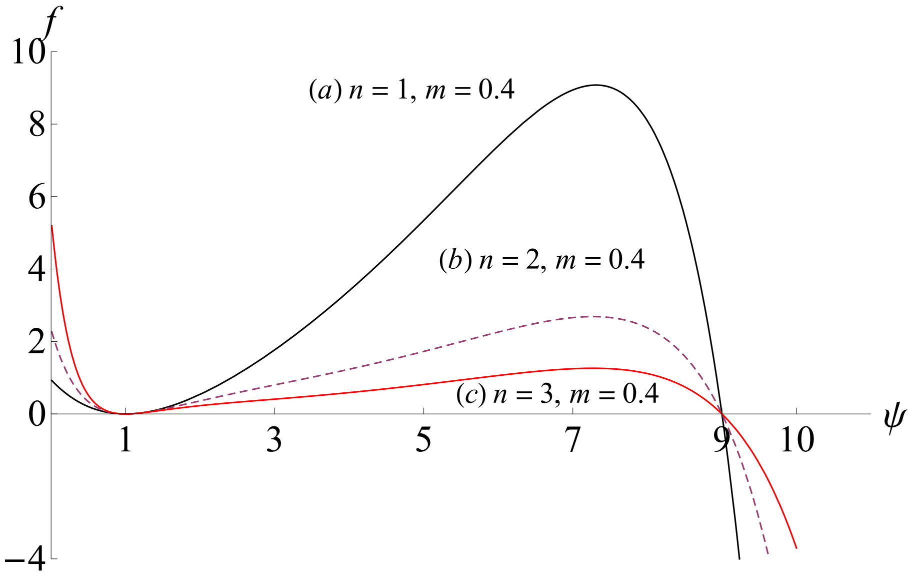


Figure 6.2: The function defined by (6.23) plotted against $\psi$ for $\Psi = 10$ and (a) $n = 1, m = 0.4$ (b) $n = 2, m = 0.4$ and (c) $n = 3, m = 0.4$

Now from (6.22) and (6.23), the rarefactive solitary wave solution is given by

$$\zeta = \pm(n+m)\left(\frac{c}{2}\right)^{\frac{1}{2}} \int_{\psi}^{\Psi} \frac{d\psi}{[P(\psi)]^{\frac{1}{2}}} \tag{6.42}$$

where

$$\begin{aligned} P(\psi) = (n+m+c)\exp(2m[\psi-1)] - \frac{(n+m)^2}{m}\Big(&\exp[2m(\psi-1)] \\ - \exp[m(\psi-1)]\Big) - \Big(&n+m-c(n+m-1) \\ + (n+m)c\psi\Big)&\exp[(m-n)(\psi-1)] \end{aligned} \tag{6.43}$$

and the wave speed $c$ is given by (6.21). In Figure 6.3, the solitary wave solution (6.42) is plotted against $\zeta$ for a range of values of $n$ an $m$. The quadrature that occurs on the right hand side of the analytical solution (6.42) was integrated numerically. In order to compare the solutions for different values of $n$ and $m$, the length $\zeta$ is scaled in all cases by the same choice of characteristic length, $\delta_c$. We see that the width of the solitary wave increases when $n$ increases for fixed $m$. Thus the permeability has the effect of spreading the wave. Also the width of the solitary wave decreases when $m$ increases for fixed $n$. Thus the viscosity has the effect of preventing the wave from spreading.

Consider now the dependence of the wave velocity, c, on the amplitude of the solitary wave,$\Psi - 1$. Since $F(\Psi) = c$ it follows from (6.39) that for $n > 0$ and $m \geq 0$,

$$\frac{dc}{d\Psi} > 0. \tag{6.44}$$

The velocity increases as the amplitude of the solitary wave increases and therefore larger amplitude waves travel faster. This property also holds for the solitary waves described by the power law for $n > 1$ [5]. Equation (6.44) is a special case of the general result that $F(\psi)$ is an increasing function of $\psi$ for $1 \leq \psi \leq \Psi$ which was central to the proof of the existence of solitary wave solutions for $n > 0$. There is therefore a close connection between

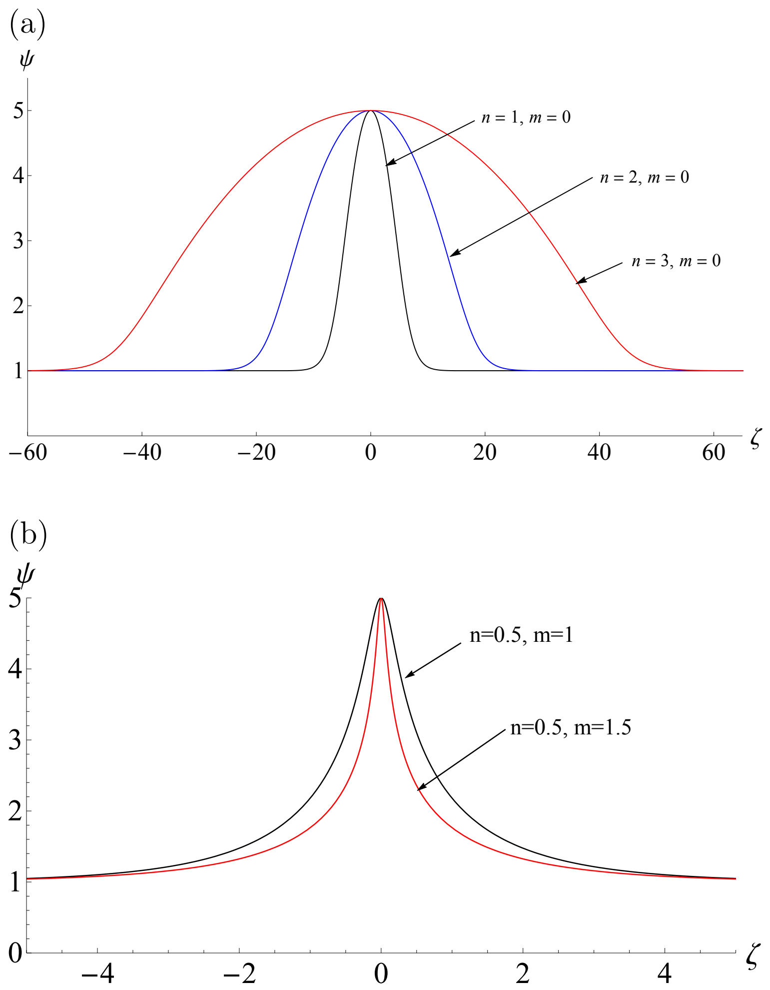


Figure 6.3: Comparison of solitary wave solutions plotted against $\psi$ for different values of $n$ and $m$. $(a)$ $n = 1.8, m = 0.5$ ;$n = 1, m = 0$;$n = 2, m = 0$ and $n = 2.5, m = 0$ . $(b)$ $n = 0.5, m = 1$ and $n = 0.5, m = 1.5$ with $\Psi = 5$. The length is scaled with the characteristic length $\delta_c$ with n=1 and m=0.

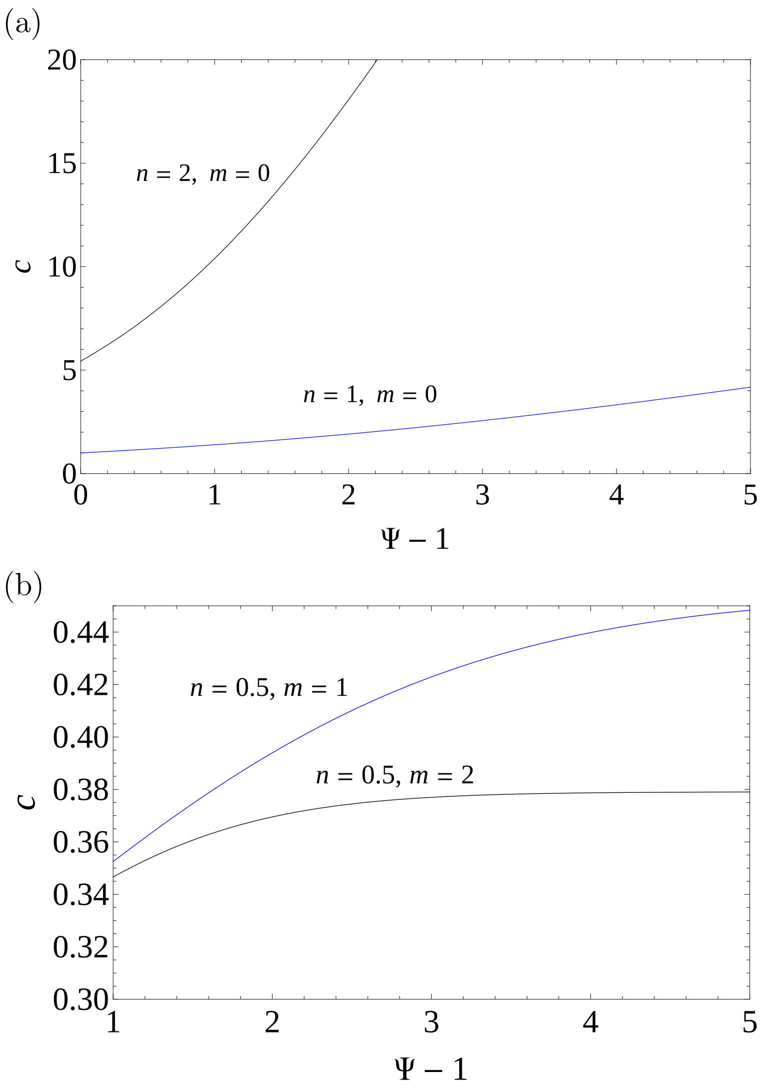


Figure 6.4: Velocity, $c$, of a rarefactive solitary wave for $(a)$ $n = 1$, $m = 0$; $n = 2$, $m = 0$ and $(b)$ $n = 0.5$, $m = 1$; $n = 0.5$, $m = 2$.

between the property that larger amplitude solitary waves travel faster and the existence of solitary wave solutions. We also determine from (6.40) the limiting value

$$\Psi - 1 = 0 : \qquad c = n. \tag{6.45}$$

This limiting value also holds for the dimensionless velocity of rarefactive solitary waves for the power law with $n > 1$.

In Figure 6.4 the dimensionless wave velocity, $c$, of the solitary wave, given by (6.21), is plotted against the amplitude $\Psi - 1$ . In order to compare $c$ for different values of $n$ the wave velocity is scaled for each value of $n$ by the same characteristic velocity $v_0 = \delta_c/t_0$. We see that $c$ increases steadily with $\Psi - 1$ for both cases, in agreement with (6.39). We also see that as $n$ increases the velocity $c$ increases and at an increasing rate. As m increases, $c$ increases. The rate of increase of $c$ also decreases. The increase in the permeability as $n$ increases allows the melt to propagate at a greater speed through the solid matrix and a decrease in viscosity of the solid matrix as $m$ increases, decreases the speed at which the melt propagates through the solid matrix.

### 6.2.1 Small amplitude approximation

We now consider small-amplitude solitary waves and derive the solitary wave solution and velocity given by this small amplitude approximation. Small amplitude wave solutions have been derived by Nakayama and Mason [23] for power law permeability and matrix viscosity. We will follow their calculations quite closely in order to compare the solution at each step and the final result.

Let $\varepsilon$ denote the amplitude of the solitary wave so that

$$\Psi = 1 + \varepsilon, \quad 0 < \varepsilon < 1 \tag{6.46}$$

The perturbation parameter is $\varepsilon$. By substituting (6.46) into (6.21) and expanding in powers of $\varepsilon$ we get the following expression for the velocity:

$$c = n[1 + P\varepsilon + Q\varepsilon^2 + R\varepsilon^3 + O(\varepsilon^4)] \tag{6.47}$$

as $\varepsilon \to 0$ where

$$P = \frac{1}{3}n \tag{6.48}$$

$$Q = \frac{1}{36}n(2n - m) \tag{6.49}$$

$$R = -\frac{1}{540}n(2n^2 - 8nm - m^2) \tag{6.50}$$

In Figure 6.5 we compare the small amplitude approximation of the velocity $c$ to orders, $O(\varepsilon)$, $O(\varepsilon^2)$ and $O(\varepsilon^3)$ with the exact solution for $c$ for $n = 3, m = 0.5$. We see that the approximation $O(\varepsilon)$ underestimates the exact value of c while the approximation $O(\varepsilon^3)$ overestimates the exact solution. The $O(\varepsilon^2)$ and the analytical value are indistinguishable. For the values of $n$ and $m$ considered there is no improved accuracy by going beyond $O(\varepsilon^2)$.

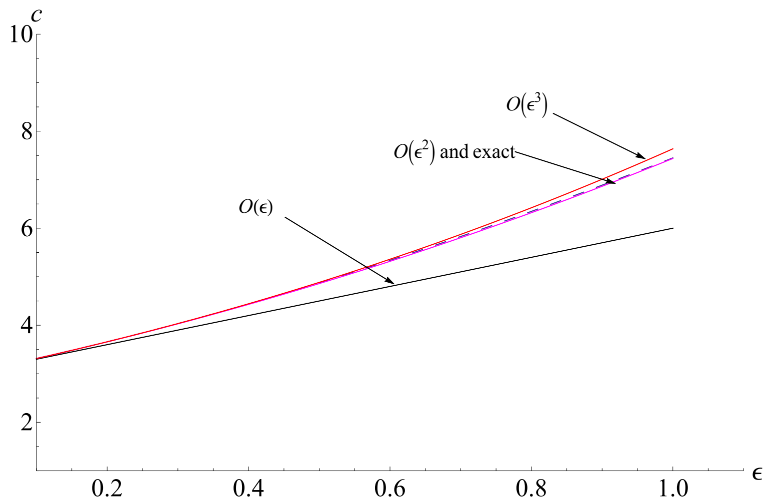


Figure 6.5: Comparison of perturbation velocity (6.47) with the exact velocity (6.21)

Next in Figure 6.6 we compare the accuracy of the approximate solution of $c$ for varying $n$ with $m$ fixed and varying $m$ and fixed $n$. In Figure 6.6(a) for $m = 0$, $0 \leq n \leq 5$ with $\varepsilon = 0.8$, we see that the perturbation solution becomes less accurate as $n$ increases but the perturbation solution to $O(\varepsilon^3)$ is more accurate than that to $O(\varepsilon^2)$. In figure 6.5 (b) for $n = 1$, $0 \leq m \leq 3$ with $\varepsilon = 0.8$, we see that the perturbation solution becomes less accurate as m increases and the perturbation solution to $O(\varepsilon^3)$ is more accurate than the perturbation solution to $O(\varepsilon^2)$. For small of $m$ the perturbation solution to $O(\varepsilon^3)$ slightly overestimates the exact solution while for large values of $m$ it underestimates the analytical solution We can see from Figure 6.6 (b) that $c$ is a decreasing function of $m$.

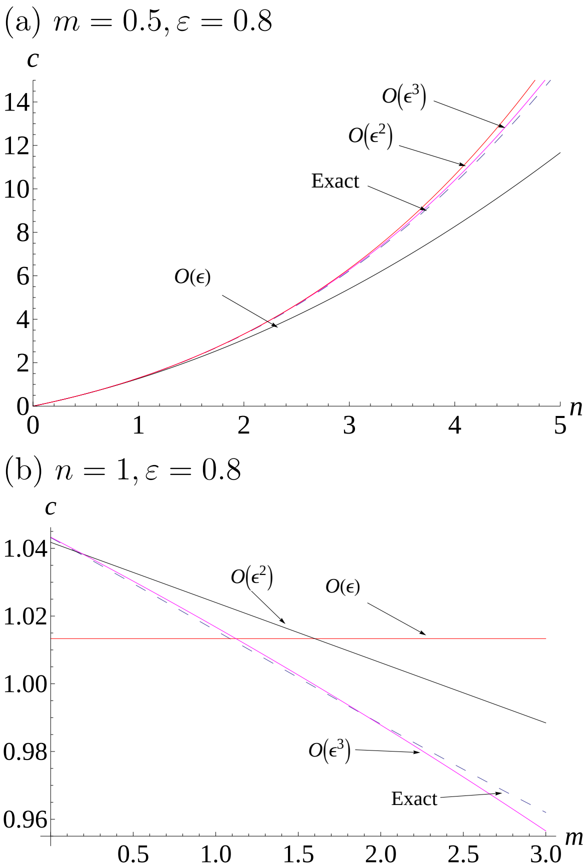


Figure 6.6: Comparison of perturbation solution, (6.47) and exact solution(6.21), for c with $\varepsilon = 0.8$:(a) $m = 0.5, 0 \leq n \leq 5$; (b) $n = 1, 0 \leq m \leq 3$

Next consider the small amplitude approximation of the solitary wave $\psi$. We look for a perturbation solution. Consider a straightforward perturbation expansion of the form

$$\zeta(\psi, \varepsilon) = \frac{1}{\varepsilon^{\alpha}}(\zeta_0(\psi) + \varepsilon\zeta_1(\psi) + \varepsilon^2\xi_2(\psi) + O(\varepsilon^3)) \tag{6.51}$$

as $\varepsilon \to 0$ , where $\psi$ is kept fixed in the limiting process and the exponent $\alpha$ is to be determined during the analysis.The straightforward perturbation expansion breaks down. This also occurred in the perturbation solution when the permeability and viscosity are expressed as power laws. An explanation for the breakdown and how to obtain a uniformly valid perturbation expression has been given by Nakayama and Mason [23] and is now summarised.
The function $f(\psi)$ has a factor $(\Psi - \psi)$, which occurs when the boundary condition $f(\Psi) = 0$ is imposed. Now, $\frac{d\psi}{d\zeta}$ has the factor $(\Psi - \psi)^{\frac{1}{2}}$ which can be written as $(1 - \psi + \varepsilon)^{\frac{1}{2}}$. The lowest order term in the straightforward perturbation of $\frac{d\psi}{d\zeta}$ is found by setting $\varepsilon = 0$. This term has the factor $(1 - \psi)^{\frac{1}{2}}$

which is imaginary as $\psi > 1$. The reason why (6.51) does not work is due to the sharp change in the dependent variable $\zeta$, from either $\zeta = \infty$ or $\zeta = -\infty$ to $\zeta = 0$ in the domain $1 \leq \psi \leq 1 + \varepsilon$ of the variable $\psi$ [23].

This sharp change region is $\psi - 1 = O(\varepsilon)$. To remedy this sharp change consider the transformation

$$u = \frac{\psi - 1}{\varepsilon}. \tag{6.52}$$

The variable $u$ has the domain of $0 \leq u < 1$. Using this transformation, (6.53) becomes

$$\zeta(u, \varepsilon) = \frac{1}{\varepsilon^{\alpha}} \left( \zeta_0(u) + \varepsilon \zeta_1(u) + \varepsilon^2 \zeta_2(u) \right) \quad \text{as} \quad \varepsilon \to 0, \tag{6.53}$$

where $u$ is fixed. The exponent $\alpha$ will be determined by balancing the dominant terms in the analysis.

Now

$$\left(\frac{d\psi}{d\zeta}\right)^2 = \frac{2}{c(n+m)^2} \Bigg[ (n+m+c) \exp[2m(\psi - 1)] \tag{6.54}$$
$$- (n + m - c(n+m-1) + (n+m)c\psi) \exp[(m-n)(\psi - 1)]$$
$$- \frac{(n+m)^2}{m} \left( \exp[2m(\psi - 1)] - \exp[m(\psi - 1)] \right) \Bigg].$$

But

$$\psi = 1 + \varepsilon u, \tag{6.55}$$

Substitute (6.55) into (6.54). This gives

$$\left(\frac{du}{d\zeta}\right)^2 = \frac{\varepsilon n(m+n)(1-u)u^2}{3} \Bigg[ 1 - \frac{\varepsilon}{24} \varepsilon (3(2n - 5m)u + 2n + m)$$
$$+ \frac{\varepsilon^2}{360} (mn(36m(4m - 3n)u^2 - 3m(7m + 11n)u + m(2n - m)) + O(\varepsilon^3) \Bigg] \tag{6.56}$$

as $\varepsilon \to 0$. The factor $n$ occurs on the right hand side of (6.56), thus this illustrates the condition we showed earlier that for solitary wave solutions to exist we need $n > 0$. It follows directly from (6.56) and the perturbation expansion (6.53) for $\zeta(u, \varepsilon)$ that

$$\frac{1}{\varepsilon^{\alpha}} (d\zeta_0 + \varepsilon d\zeta_1 + \varepsilon^2 d\zeta_2 + O(\varepsilon^3))$$
$$= \pm \frac{1}{\varepsilon^{\frac{1}{2}}} \left(\frac{3}{n}\right)^{\frac{1}{2}} \Bigg[ \frac{1}{u(1-u)^{\frac{1}{2}}} + \varepsilon \left( \frac{A(n,m)}{2(1-u)^{\frac{1}{2}}} + \frac{B(n,m)}{u(1-u)^{\frac{1}{2}}} \right) + \tag{6.57}$$
$$\varepsilon^2 \left( \frac{3}{2} \frac{D(n,m)u}{(1-u)^{\frac{1}{2}}} + \frac{E(n,m)}{2(1-u)^{\frac{1}{2}}} + \frac{F(n,m)}{u(1-u)^{\frac{1}{2}}} + O(\varepsilon^3) \right) \Bigg] du$$

where

$$A(n,m) = \frac{(2n-5m)}{4}, \tag{6.58}$$
$$B(n,m) = \frac{2n+m}{24}, \tag{6.59}$$
$$D(n,m) = \frac{12n^2-108nm+119m^2}{960}, \tag{6.60}$$
$$E(n,m) = \frac{28n^2-32nm-19m^2}{480}, \tag{6.61}$$
$$F(n,m) = \frac{28n^2+28nm+31m^2}{5760}. \tag{6.62}$$

Balancing the dominant terms on the right and left-hand sides yields $\alpha = \frac{1}{2}$. Thus equation (6.57) becomes

$$\begin{aligned} &d\zeta_0 + \varepsilon d\zeta_1 + \varepsilon^2 d\zeta_2 + O(\varepsilon^3) \\ &= \pm\left(\frac{3}{n}\right)^{\frac{1}{2}}\left[\frac{1}{u(1-u)^{\frac{1}{2}}} + \varepsilon\left(\frac{A(n,m)}{2(1-u)^{\frac{1}{2}}} + \frac{B(n,m)}{u(1-u)^{\frac{1}{2}}}\right) + \right. \\ &\left.\varepsilon^2\left(\frac{3}{2}\frac{D(n,m)u}{(1-u)^{\frac{1}{2}}} + \frac{E(n,m)}{(1-u)^{\frac{1}{2}}} + \frac{F(n,m)}{u(1-u)^{\frac{1}{2}}} + O(\varepsilon^3)\right)\right]du. \end{aligned} \tag{6.63}$$

Now $\zeta(1) = 0$ which corresponds to $\psi = \Psi$ [23]. Thus

$$\zeta_n(1) = 0, \quad n \geq 0 \tag{6.64}$$

Equation (6.63) is solved subject to the initial condition (6.65) by equating the coefficients of like powers of $\varepsilon$.
Consider first the zero order in $\varepsilon$ terms in (6.63). Now

$$d\zeta_0 = \pm\left(\frac{3}{n}\right)^{\frac{1}{2}}\frac{du}{u(1-u)^{\frac{1}{2}}} \tag{6.65}$$

For the zero-order we obtain the solution in explicit and implicit form. Now let $u = \operatorname{sech}^2(\theta)$. By imposing the initial condition the explicit solution is

$$u = \operatorname{sech}^2\left(\frac{1}{2}\left(\frac{n}{3}\right)^{\frac{1}{2}}\zeta_0\right) \tag{6.66}$$

Since $\psi = 1 + \varepsilon u$ and $\varepsilon$, $\zeta_0 = \varepsilon^{\frac{1}{2}}\zeta$. Equation (6.66) becomes

$$\psi = 1 + (\Psi - 1)\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{n(\Psi-1)}{3}\right)^{\frac{1}{2}}\zeta\right] \tag{6.67}$$

This approximate solution has the same $\text{sech}^2$ form as the single-soliton solution of the KdV equation [23].
Now consider the solution in implicit form. Let $u = 1 - w^2$ Then it can be verified that

$$\int \frac{du}{u(1-u)^{\frac{1}{2}}} = -\ln\left(\frac{1+(1-u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right) + \text{const.} \tag{6.68}$$

Thus using (6.68) and imposing the initial condition (6.64) The solution to (6.65) is

$$\zeta_0(u) = \mp\left(\frac{3}{n(\Psi-1)}\right)^{\frac{1}{2}} \ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right). \tag{6.69}$$

Consider next the first order in $\varepsilon$ terms. Equating coefficients of $\varepsilon$ in (6.57) we have

$$d\zeta_1 = \pm\left(\frac{3}{n}\right)^{\frac{1}{2}} \left[\frac{A(n,m)}{2(1-u)^{\frac{1}{2}}} + \frac{B(n,m)}{u(1-u)^{\frac{1}{2}}}\right] du. \tag{6.70}$$

Now integrating (6.70) with the aid of (6.68) and imposing the initial condition (6.61) we obtain

$$\zeta_1 = \mp\left(\frac{3}{n}\right)^{\frac{1}{2}} \left[A(n,m)(1-u)^{\frac{1}{2}} + B(n,m)\ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right)\right] \tag{6.71}$$

Finally consider the second order in $\varepsilon$ terms. Equating the coefficient of $\varepsilon^2$ in (6.57) we have

$$d\zeta_2 = \pm\left(\frac{3}{n}\right)^{\frac{1}{2}} \left[\frac{3}{2}\frac{D(n,m)u}{(1-u)^{\frac{1}{2}}} + \frac{E(n,m)}{2(1-u)^{\frac{1}{2}}} + \frac{F(n,m)}{u(1-u)^{\frac{1}{2}}}\right] du \tag{6.72}$$

Integrating (6.72) using

$$\int \frac{udu}{(1-u)^{\frac{1}{2}}} = -\frac{2}{3}(2+u)(1-u)^{\frac{1}{2}} + \text{const} \tag{6.73}$$

we obtain

$$\begin{aligned}\zeta_2 = \mp\left(\frac{3}{n}\right)^{\frac{1}{2}} \Bigg[&(1-u)^{\frac{1}{2}}(2+u)D(n,m) + (1-u)^{\frac{1}{2}}E(n,m) \\ &+F(n,m)\ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right)\Bigg]\end{aligned} \tag{6.74}$$

Thus the solution correct to order $\varepsilon^2$ is given by

$$\begin{aligned}\zeta = \mp\left(\frac{3}{n\varepsilon}\right)^{\frac{1}{2}}\Bigg[&\ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right)\\ &+\varepsilon\left(A(n,m)(1-u)^{\frac{1}{2}}+B(n,m)\ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right)\right)\\ &+\varepsilon^2\Bigg((1-u)^{\frac{1}{2}}(2+u)D(n,m)+(1-u)^{\frac{1}{2}}E(n,m)\\ &+F(n,m)\ln\left(\frac{1+(1+u)^{\frac{1}{2}}}{1-(1-u)^{\frac{1}{2}}}\right)\Bigg)+O(\varepsilon^3)\Bigg]\end{aligned} \tag{6.75}$$

as $\varepsilon \to 0$,where

$$\psi = 1+\varepsilon u \tag{6.76}$$

Note that

$$u = \frac{1}{\varepsilon}(\psi-1), \tag{6.77}$$

thus to obtain an implicit solution in the form $\zeta = \zeta(\psi)$ we can substitute $u = \frac{1}{\varepsilon}(\psi-1)$.

In Figure 6.7 the perturbation solutions for different orders $O(\varepsilon), O(\varepsilon^2)$ and $O(\varepsilon^3)$ are compared with the analytical solution when $\varepsilon = 0.8$. We see that for, $n = 3, m = 0$,$n = m = 1$ and $n = 7, m = 2$ the approximate solution to $O(\varepsilon^2)$ overlaps the analytical solution and for $m = n$ it is even more accurate . The perturbation solution to $O(\varepsilon^2)$ is less accurate for $n = 1, m = 2$ that is the solution for $m < n$.

### 6.2.2 Large amplitude approximation

We now consider large amplitude solitary waves and derive the solitary wave solution and velocity given by this large amplitude approximation.
As before we recall from (6.21) that

$$c = \frac{(n+m)[1-\frac{(n+m)}{m}\exp[n(\Psi-1))]+\frac{n}{m}\exp[(n+m)(\Psi-1)]}{\exp((n+m)(\Psi-1))-(n+m)(\Psi-1)-1} \tag{6.78}$$

For $\Psi - 1 \gg 1$,

$$c = (n+m)\frac{n}{m}, \quad n>0 \quad \text{and} m>0. \tag{6.79}$$

To derive the large amplitude solitary wave solution we make use of the transformation

$$w = \frac{\psi-1}{\Psi-1} \tag{6.80}$$

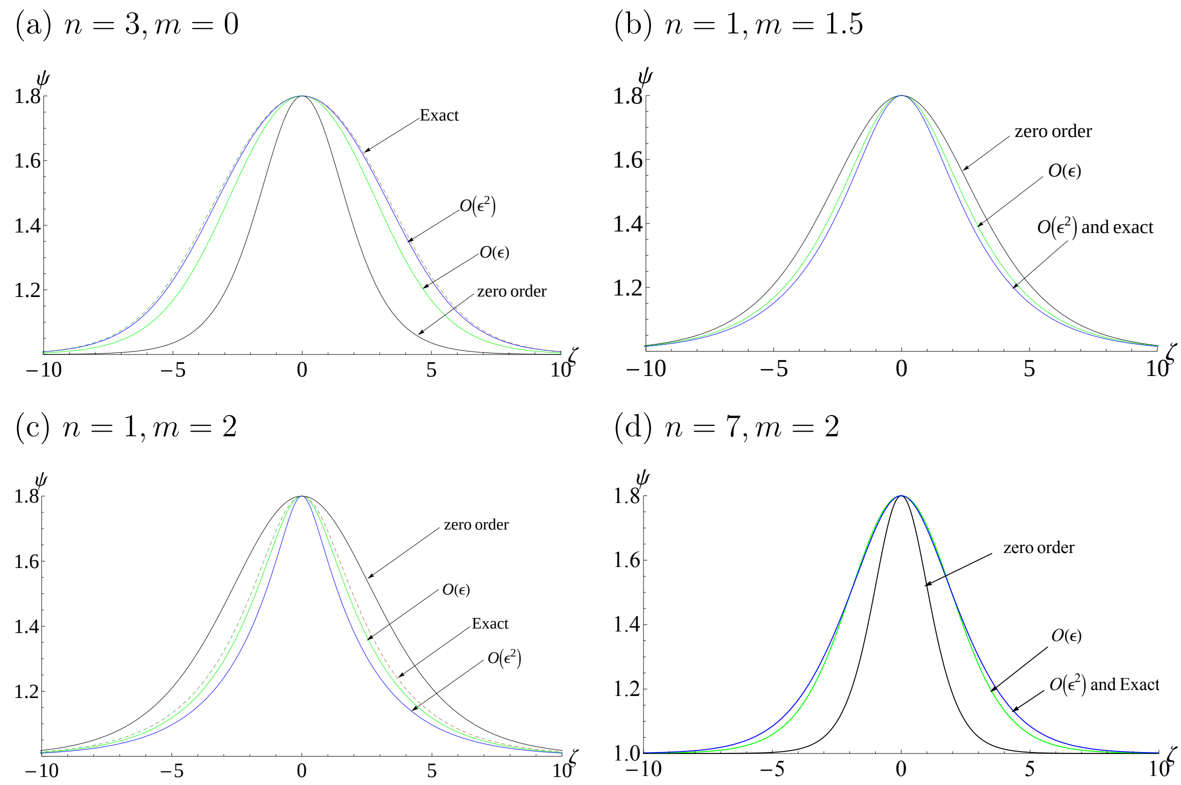


Figure 6.7: Comparison of perturbation solution, (6.69) and (6.70), for $\varepsilon = 0.8$ with the analytical solution $(- - -)$.(a) $n = 3, m = 0$; (b) $n = 1, m = 1.5$; (c) $n = 1, m = 2$; (d) $n = 7, m = 2$.

Now from (6.42) and (6.43) the solitary wave solution is given by

$$\zeta = \pm(n+m)\left(\frac{c}{2}\right)^{\frac{1}{2}} \int_{\psi}^{\Psi} \frac{d\psi}{[P(\psi)]^{\frac{1}{2}}} \tag{6.81}$$

where

$$\begin{aligned} P(\psi) = (n+m+c)\exp(2m(\psi-1)) - \frac{(n+m)^2}{m}\Bigg(&\exp(2m(\psi-1)) \\ - \exp(m(\psi-1))\Bigg) - \Bigg(&n+m-c(n+m-1) \\ + (n+m)c\psi\Bigg)\exp((m-n)(\psi-1))) \end{aligned} \tag{6.82}$$

By using (6.21) for $c$, making the transformation (6.52) and the large amplitude approximation $\Psi - 1 \gg 1$, equation (6.81) reduces for $n > 0$ and $m > 0$ to

$$\zeta = \pm\frac{1}{m}[2n(n+m)]^{\frac{1}{2}}[\exp[\frac{m}{2}(\Psi-1] - \exp[\frac{m}{2}(\psi-1)] \tag{6.83}$$

The large amplitude approximation of the solitary wave solution is presented in Figure 6.8.

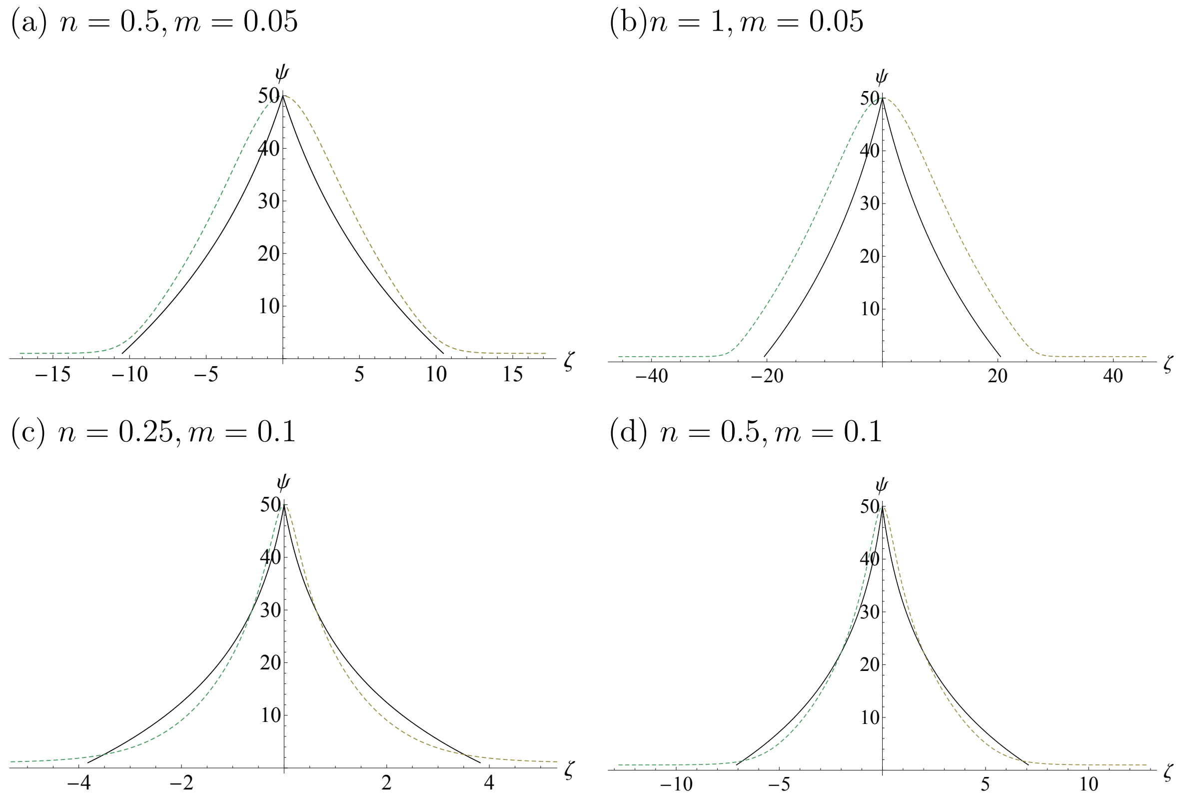


Figure 6.8: Comparison of Large Amplitude Approximation solution (dashed) , (6.83) with the analytical solution: (a) $n = 0.5, m = 0.05$; (b)$n = 1, m = 0.05$;(c)$n = 0.25, m = 0.1$;(d)$n = 0.5, m = 0.1$.

We can see from the comparisons that for small $m$ and $n$ the large amplitude approximation solution is close to the analytical solution.

## 6.3 Concluding remarks

We found that the magma equation with exponential laws relating the permeability and viscosity to the voidage admits solitary wave solutions. A double reduction was performed on the magma equation giving the solution in the form of a quadrature. The existence of solitary wave solutions is closely related to the physical property that larger amplitude solitary waves travel faster.

The small amplitude approximate solution for the solitary wave to $O(\varepsilon^2)$

was accurate for $\varepsilon$ as large as 0.8. The contribution of this approximate solution is that it applies for a large range of values of $n$ and $m$ and not only for a few specific values.

# Chapter 7

# Conclusions

We have shown that as well as a power law, the permeability, $K(\phi)$, and the viscosity, $G(\phi)$, can also be related to the voidage by an exponential law. The magma equation for the voidage still admits a range of Lie point symmetries when permeability and viscosity are related to the voidage by means of an exponential law. The results for the Lie point symmetries were verified using the Mathematica package SYM [22].

A rarefactive solitary wave solution was derived for when both $K$ and $G$ are related to the voidage by exponential laws. This solution exists for $n > 0$ and $m \geq 0$. There is a close connection between the existence rarefactive solitary wave solutions and the result that larger amplitude waves travel faster. The velocity of the solitary wave increases at an increasing rate as $n$ increases and increases at a decreasing rate as $m$ increases. The width of the solitary waves increases as $n$ increases and decreases as $m$ increases. Thus an increase in $n$ has the effect of spreading the solitary wave and an increase in $m$ has the effect of preventing the wave from spreading.

The perturbation solution for the solitary wave applies for all values of $n > 0$ and $m \geq 0$. By comparing the perturbation solution with the analytical solution it was found that the perturbation solution to $O(\varepsilon^2)$ is a good approximation to the small-amplitude solitary wave solution. In all cases, the graphs of the perturbation solution to second-order in $\varepsilon$ and the analytical solution overlapped. The lowest-order perturbation solution has the same $\text{sech}^2$ form and similar properties to the single-soliton solution of the KdV equation [1]. To this approximation the width of the solitary wave is inversely proportional to the square root of the amplitude and the speed of the solitary wave increases linearly with the amplitude. For the large amplitude approximation, we have shown that for $n > 0$ and $m > 0$, $c$ tends to a constant value dependent on $n$ and $m$. The approximate large amplitude solitary wave solution becomes more accurate for smaller values of $n$ and $m$.

The multiplier method was used to derive the conserved vectors for the magma equation when the permeability and matrix viscosity satisfy a power law. The results agree with those of Harris [11], who derived the conserved vectors using the direct method. Unlike the direct method the functional form of the conserved vectors does not need to be assumed with the multiplier method. Instead the variables on which the multiplier depends, have to be chosen but this can be done by starting with a simple form and including higher order partial derivatives later to derive higher order conservation laws. The determining equation for the multiplier is readily obtained with the aid of the Euler operator.

Conserved vectors for the magma equation when the permeability and matrix viscosity are related to the voidage by exponential laws were derived using the multiplier method.

We investigated the association of Lie point symmetries of the magma equation with the conserved vectors. For all conserved vectors considered except two the associated Lie point symmetry was the Lie point symmetry which generates travelling wave solutions. The associated symmetry was used to reduce the partial differential equation twice. This was a consequence of Sj öberg's double reduction theorem.

We were not able to derive new conservation laws for the magma equation with power law relation for permeability and viscosity, or determine if the number of conservation laws for (5.1) are finite or infinite. Harris [11] has proved that except possibly for the two special cases, $n = m - 1$ with $m \neq 1$ and $m = 1$ with $n \neq 0$, there are no more independent conserved vectors. Our results derived using multipliers are consistent with the results of Harris. All known conserved vectors of the magma equation with power law permeability and viscosity and also the new conserved vectors for the magma equation with exponential relations law permeability and viscosity, can be derived from multipliers which depend only on $\phi$ and the partial derivatives of $\phi$ with respect to z.It remains to be investigated if there exists multipliers which depend on $\phi_t, \phi_{tz}, \phi_{tt}, \ldots$ . We found that the multipliers, $\Lambda(\phi, \phi_z)$ and $\Lambda(\phi, \phi_z, \phi_{zz}, \phi_{zzz})$, whose variables ended in odd order partial derivatives of $\phi$ with respect to $z$ did not generate new conserved vectors but instead reduced to the multipliers $\Lambda(\phi)$ and $\Lambda(\phi, \phi_z, \phi_{zz})$ respectively. This also applied to the multipliers for the conserved vectors with exponential law permeability and viscosity.

In some instances such us Lie point Symmetries and conservation laws, SYMS has been used to verify the results derived by hand or vice versa.

# Appendix A

# Expansions in powers of $\varepsilon$

We present the intermediate results in the derivation of equation (6.56). These simplifications will be used in the derivation of the small amplitude wave equation. For more complicated expressions Mathematica was used.

We have from (6.54),

$$\left(\frac{d\psi}{d\zeta}\right)^2 = \frac{2}{(n+m)^2}\Bigg[\left(\frac{n+m}{c}+1\right)\exp(2m(\psi-1))$$
$$-\left(\frac{n+m}{c}+1+(n+m)(\psi-1)\right)\exp((m-n)(\psi-1)) \tag{A.1}$$
$$-\frac{(n+m)^2}{cm}\Big(\exp(2m(\psi-1))-\exp(m(\psi-1))\Big)\Bigg]$$

where from (6.21)

$$c = \frac{(n+m)[1-\frac{(n+m)}{m}\exp(n(\Psi-1))+\frac{n}{m}\exp((n+m)(\Psi-1))]}{\exp((n+m)(\Psi-1))-(n+m)(\Psi-1)-1} \tag{A.2}$$

For small amplitude waves

$$c = n[1+P\varepsilon+Q\varepsilon^2+R\varepsilon^3+O(\varepsilon^4)] \tag{A.3}$$

where

$$P = \frac{1}{3}n \tag{A.4}$$
$$Q = \frac{1}{36}n(2n-m) \tag{A.5}$$
$$R = \frac{1}{540}n(2n^2-8nm-m^2). \tag{A.6}$$

Now when

$$\psi = 1+\varepsilon u, \quad \varepsilon = \Psi - 1, \tag{A.7}$$

equation (A.1) becomes

$$\begin{aligned}\left(\frac{d\psi}{d\zeta}\right)^2 &= \frac{2}{(n+m)^2}\Bigg[\left(\frac{n+m}{c}+1\right)\exp(2m\varepsilon u)\\ &-\left(\frac{n+m}{c}+1+(n+m)\varepsilon u\right)\exp((m-n)\varepsilon u)\\ &-\frac{(n+m)^2}{cm}(\exp(2m\varepsilon u)-\exp(m\varepsilon u))\Bigg].\end{aligned} \tag{A.8}$$

Now

$$\begin{aligned}\frac{1}{c} &= \frac{1}{n}\Bigg[1-\frac{n}{3}\varepsilon+\frac{n}{36}(m+2n)\varepsilon^2\\ &+\frac{n}{540}(m^2-2mn-2n^2)\varepsilon^3+\frac{1}{6480}(3m^2+24mn+4n^2)\\ &+\frac{1}{19440}(2m^3+9m^2n+40mn^2+4n^3)\varepsilon^5+O(\varepsilon^6)\Bigg],\end{aligned} \tag{A.9}$$

and therefore

$$\begin{aligned}\frac{n+m}{c}+1 &= \left(2+\frac{m}{n}\right)-\frac{1}{3}(m+n)\varepsilon+\frac{1}{36}(n+m)(m+2n)\varepsilon^2 \qquad \text{(A.10)}\\ &+\frac{1}{540}(m+n)(m^2-2mn-2n^2)\varepsilon^3-\frac{1}{6480}n(m+n)(3m^2+24mn+4n^2)\varepsilon^4\\ &+\frac{1}{19440}(n(n+m)(2m^3+9m^2n+40mn^2+4n^3))\varepsilon^5+O(\varepsilon^6),\end{aligned}$$

and

$$\begin{aligned}\frac{(n+m)^2}{cm} &= \frac{(n+m)^2}{mn}\Bigg[1-\frac{n}{3}\varepsilon+\frac{n}{36}(m+2n)\varepsilon^2\\ &+\frac{n}{540}(m^2-2mn-2n^2)\varepsilon^3+\frac{1}{6480}(3m^2+24mn+4n^2)\\ &+\frac{1}{19440}(2m^3+9m^2n+40mn^2+4n^3)\varepsilon^5+O(\varepsilon^6)\Bigg].\end{aligned} \tag{A.11}$$

If (A.10) and (A.11) are substituted into (A.8) and the exponential terms are expanded in powers of $\varepsilon$ then equation (6.56) is derived.

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