Turbulent flow in channels and fractures: conservation laws and Lie group analysis

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university.

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Abstract

The Fanno model that describes turbulent compressible fluid flow in a long channel and a model for turbulent and laminar fluid-driven fracturing in rock in which the fluid is incompressible are considered. Lie point symmetries are derived and used to reduce the partial differential equations to ordinary differential equations. Analytical solutions are derived for both problems.The Lie point symmetry associated with the elementary conserved vector is used to derive the invariant solution of the nonlinear diffusion equation for the mean velocity of the fluid in the channel. Numerical results are obtained for the hydraulic fracture by modifying the shooting method. The ordering of graphs of the half-width and of the length of the fracture under different working conditions at the fracture entry did not change when the fluid flow changed from laminar to turbulent. Conservation laws are derived using the direct method, the characteristic method and the partial Lagrangian method. A review and comparison of the three methods is made. It was found that the partial Lagrangian method was straightforward and less computationally laborious. Unlike the other two methods it did not assume a functional form for the conserved vector but did for the gauge terms. It was also found that when the fluid flow in a fracture changed from laminar to turbulent the number of conservation laws is reduced from two to one.

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Chapter 1

Introduction

1.1 Introduction

Turbulent flow presents itself in almost every instance of practical fluid flow [1] as it often characterises fluid flow around bodies such as airplane wings and flat plates, or flow through conduits such as pipes and fractures. As a result, there has been much research done on turbulent flow and its influence on heat transfer, species transport, drag, vorticity distribution, separation and swirl flow [2]. The two problems considered in this dissertation describe turbulent compressible flow in a long channel and turbulent incompressible flow in a thin fracture in a rock. Both problems have practical applications in the mining industry.

1.2 Turbulent compressible flow in long channels.

The first problem of turbulent compressible flow in a long channel originates from a problem presented to a Mathematics of Industry Study Group (MISG) in South Africa by members of the mining industry concerning air blasts in underground chambers and tunnels in mines. Air blasts are caused by collapsing rock that displaces a large amount of air, forcing it through a void resulting in high pressure air flow that propagates along the tunnels of a mine. They are a source of concern as they can cause death or injury to mine workers and serious damage to mining infrastructure [3]. Studying this phenomenon may provide insight into how to mitigate the effects of an air blast. It was suggested that the Fanno model for turbulent compressible flow be used to model the flow of an air blast in a mining tunnel [4].

The essential assumption in the Fanno flow model is that the main effect of turbulence is to exert a wall drag by way of a boundary layer at the wall. The wall drag dampens flow over large times and distances but it has a small effect locally [5]. The Fanno flow model will be applicable if the flow is turbulent and the tunnel is long enough for wall drag to be important. In the mining situation the diameter of a tunnel may be 4 m and an aspect ratio of 10^{-3} would correspond to a tunnel network of about 4 km in length [3]. In many mining situations there are usually interconnections of an underground excavation by tunnels and shafts to other underground excavations. The Fanno flow model may therefore be applicable.

Ockendon et al [6] considered the problem of turbulent gas flow in a tube when the pressure at the end $x = 0$ is suddenly changed by an amount small compared with the background pressure. The MISG recommended that this problem could model the sudden increase in pressure at the tunnel entrance due to a collapsing rock during mining excavation and therefore may be relevant to the air blast problem. The derivation of the three partial differential equations as presented by Ockendon et al [6] is outlined in Chapter 3. These partial differential equations describe the evolution of the mean velocity and pressure profile of the turbulent compressible flow of gas in the long mining tunnel. In Chapter 3 we will investigate group invariant solutions of these partial differential equations.

Furthermore, air blasts have been observed to travel large distances in tunnel networks and this suggests that conservation laws and conserved quantities may be important in understanding the dynamics of the turbulent air flow in the tunnel [7]. The conserved quantities may also aid in obtaining analytical solutions to partial differential equations that have been reduced to ordinary differential equations using Lie group methods[8]. We will therefore investigate conservation laws as part of our analysis in Chapter 4 and Chapter 5.

An understanding of the mean velocity profile and pressure profile may aid in improving the design of mining tunnels and ventilation systems. Mining engineers may be able to determine which areas are hazardous for workers during an air blast and implement safety measures and procedures to ensure the protection of miners and other personnel during an air blast. It is also important for the mining industry to adopt safer methods of rock extraction in order to reduce the occurrence of air blasts during mining operations. Hydraulic fracturing is recommended as method of rock excavation that can decrease the incidence of air blasts, however hydraulic fracturing does have hazards of its own and therefore the timing of such a mining operation is important [7]. This is the basis of our second problem.

1.3 Turbulent incompressible flow in thin fractures

Hydraulic fracturing involves the propagation of a fracture in brittle material, such as rock, due to incompressible fluid being pumped into the fracture. The crack propagates in order to accommodate the additional fluid [9]. In mining operations, the generation of free surface parallel hydraulic fractures are used as an alternative to explosive blasting [10]. Hydraulic fracturing is considered less likely to cause air blasts as smaller pieces of rock are broken at a time and in most cases are not enough to displace a large amount of air thereby reducing the risk of an air blast occurring [7].

In modelling turbulent incompressible flow in a thin fracture we need to consider the geometry of the fracture. The idealised geometry of the fracture is best described as a finger-like projection. The fracture is one-sided and the fluid enters through one end. The rock is assumed to be impermeable and therefore there is no leak-off of the fracturing fluid. Spence and Turcotte obtained similarity solutions for the propagation of fluid filled fractures. They initially assumed the flow in the fracture was laminar but later extended their results to turbulent flow [9] [11].

In this dissertation we will use their model for laminar and turbulent fluid driven fracture in which they derive a partial differential equation with two variables, the pressure of the fluid and the half width of the fracture. To close the model they assumed that the relationship between the pressure of the fluid and the half width is governed by the static pressure within the crack given by the Cauchy principal value integral [11]. However in this dissertation, we will use the PKN model to close the model instead of the static pressure within the fracture. The PKN model implies that the fluid pressure is proportional to the half-width of the fracture [12] [13]. The problem is formulated in more detail in Chapter 6.

The partial differential equation derived in Chapter 5 describes both turbulent and laminar flow. We will investigate analytical group invariant solutions in Chapter 6 and in Chapter 7 numerical group invariant solutions. We will also study the effect of laminar flow, smooth wall turbulent flow and rough wall turbulent flow on the nature of the solution and its effect on the propagation of the crack. We will complete the analysis by investigating conserved quantities in Chapter 8.

1.4 Aims and outline of the dissertation

The aim of this dissertation is to contribute to the fundamental understanding of turbulent compressible flow in long channels and turbulent incompressible flow in a thin fracture. We will do this by using group analysis methods for reducing partial differential equations to ordinary differential equations and by investigating conserved quantities. Conservation laws used to find conserved quantities will be derived using three different methods. The study may have practical applications in the mining industry in the management and prevention of air blasts an understanding of hydraulic fracturing by laminar and turbulent fluids.

A outline of the dissertation is as follows

- In Chapter 2, the basic operators, definitions and theory used in this dissertation will be briefly explained.
- In Chapter 3 group invariant solutions of equations resulting from the Fanno model for compressible turbulent flow in long channels will be investigated.
- In Chapter 4 we will use a conserved quantity to complete an analytical solution to a partial differential equation that describes the mean velocity of the fluid in a channel.

This chapter will motivate our study of the various approaches to deriving conserved vectors in Chapter 5.

- In Chapter 5 three different methods will be used to derive conservation laws . These methods will be reviewed and compared.
- In Chapter 6 we will investigate group invariant solutions to the partial differential equation that describes laminar and turbulent fluid-driven fractures. Analytical solutions will be determined for two physical conditions that govern the propagation of the fracture.
- In Chapter 7 a numerical solution to the partial differential equation derived in Chapter 6 will be presented. The numerical solution involves the conversion of a boundary value into two initial value problems and the implementation of the shooting method.
- In Chapter 8 we will investigate conserved quantities for laminar and turbulent fluid-driven fracture.
- In Chapter 9 we will summarise our conclusions.

Chapter 2

Formulae and theory

2.1 Introduction

In this chapter we will present results from Lie group analysis that will be used to solve the nonlinear partial differential equations that are described in Chapter 3 and Chapter 6. A concise description of the three methods that will be used in the derivation of conservation laws in Chapter 4 is given. Lastly we will briefly explain how to convert a boundary value problem into two initial value problems because it will be useful in obtaining a numerical solution in Chapter 7.

2.2 Lie point symmetries

We consider the nonlinear second order partial differential equation of the form

$$F(t, x, h, h_t, h_x, h_{tx}, h_{tt}, h_{xx}) = 0, \tag{2.1}$$

where x and t are two independent variables and h the dependent variable. The subscripts denote partial differentiation. The Lie point symmetry generators

$$X = \xi^1(t, x, h)\frac{\partial}{\partial t} + \xi^2(t, x, h)\frac{\partial}{\partial x} + \eta(t, x, h)\frac{\partial}{\partial h} \tag{2.2}$$

of equation (2.1) are derived by solving the determining equation

$$X^{[2]}F(t,x,h,h_t,h_x,h_{tx},h_{tt},h_{xx})|_{F=0}=0, \tag{2.3}$$

for $\xi^1(t,x,h)$, $\xi^2(t,x,h)$ and $\eta(t,x,h)$, where $X^{[2]}$, called the second prolongation of X, is given by

$$X^{[2]}=X+\zeta_1\frac{\partial}{\partial h_t}+\zeta_2\frac{\partial}{\partial h_x}+\zeta_{11}\frac{\partial}{\partial h_{tt}}+\zeta_{12}\frac{\partial}{\partial h_{tx}}+\zeta_{22}\frac{\partial}{\partial h_{xx}}, \tag{2.4}$$

where

$$\zeta_i=D_i(\eta)-h_kD_i(\xi^k), \qquad i=1,2, \tag{2.5}$$

$$\zeta_{ij}=D_j(\zeta_i)-h_{ik}D_j(\xi^k), \qquad i,j=1,2 \tag{2.6}$$

with summation over the repeated index k from 1 to 2. The total derivatives with respect to the independent variables t and x in (2.5) and (2.6) are

$$D_1=D_t=\frac{\partial}{\partial t}+h_t\frac{\partial}{\partial h}+h_{tt}\frac{\partial}{\partial h_t}+h_{xt}\frac{\partial}{\partial h_x}+..., \tag{2.7}$$

$$D_2=D_x=\frac{\partial}{\partial x}+h_x\frac{\partial}{\partial h}+h_{tx}\frac{\partial}{\partial h_t}+h_{xx}\frac{\partial}{\partial h_x}+.... \tag{2.8}$$

The partial differential equations that we need to solve are all second order and therefore the second prolongation of X is needed. The unknown functions $\xi^1(t,x,h)$, $\xi^2(t,x,h)$ and $\eta(t,x,h)$ in the Lie point symmetry do not depend on the derivatives of h. Therefore the determining equation (2.3) can be separated according to derivatives of h and the coefficient of each derivative set to zero which results in expressions for $\xi^1(t,x,h)$, $\xi^2(t,x,h)$ and $\eta(t,x,h)$. By setting all the constants in the aforementioned expression to zero except one in turn, we can obtain the Lie point symmetry generators X_i, $i=1,2,....n$.

2.3 Group invariant solutions

The symmetry generators obtained are of the form

$$X_i = \xi_i^1(t,x,h)\frac{\partial}{\partial t} + \xi_i^2(t,x,h)\frac{\partial}{\partial x} + \eta_i\frac{\partial}{\partial h} \tag{2.9}$$

for $i = 1, 2, ...n$, where n is the number of admitted Lie point symmetries. Any linear combination of the Lie point symmetries is also a Lie point symmetry of the partial differential equation. Denote this linear combination by X_c, where help

$$X_c = c_1X_1 + c_2X_2 + ... + c_nX_n \tag{2.10}$$

and c_i, $i = 1, 2, ...n$.

The group invariant solution, $h = \Phi(t,x)$ of the nonlinear partial differential equation is obtained by solving the equation

$$X_c(h - \Phi(t,x))|_{h=\Phi(t,x)} = 0. \tag{2.11}$$

By substituting the group invariant solution into the nonlinear partial differential equation one can reduce the partial differential equation to an ordinary differential equation in a new variable. This new variable is called the similarity variable.

2.4 Conservation laws

The concept of a conservation law for a partial differential equation is motivated by the conservation of quantities such as mass, angular momentum and energy in fluid mechanics. These quantities are conserved in the sense that they are constant on each trajectory. Consider a kth order differential equation:

$$F(x,u,u_{(1)},u_{(2)},....,u_{(k)}) = 0, \tag{2.12}$$

where x denotes the n independent variables and $u_{(i)}$ denotes all the partial derivatives of order i. For an arbitrary differential equation we write,

$$D_i(T^i) = 0, \tag{2.13}$$

where T^i are differential functions of finite order. We define (2.13) as a conservation law for the differential equation (2.12) if it satisfies the following equation

$$D_i\left[T^i\left(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}\right)\right] = 0. \tag{2.14}$$

This can also be written as

$$D_i T^i|_{F=0} = 0. \tag{2.15}$$

The vector $T = (T^1, ...T^n)$ is called a conserved vector.

A Lie point symmetry generator

$$X = \xi^i(x, u)\frac{\partial}{x^i} + \eta(x, u)\frac{\partial}{\partial u} \tag{2.16}$$

is said to be associated with the conserved vector $T^i = (T^1, ..., T^2)$ for the differential equation (2.1) if [14]

$$X(T^i) + T^i D_k(T^k) - T^k D_k(\xi^i) = 0, \quad i = 1, 2, ..., n \tag{2.17}$$

In (2.17), X is prolongated to as many derivatives as required. Kara and Mahomed [15] pointed out that (2.17) may serve as a determining equation of knowing a conserved vector and finding the Lie point symmetries associated with it. This idea is of practical importance and it will be applied in Chapter 4.

Immediately we ask ourselves, how does one find a conservation law for a partial differential equation? In this paper, we employ three different methods in order to find the conservation laws for the partial differential equations presented in Chapter 3, namely the Direct method, Partial Lagrangian method, and Multiplier method. We will now give a brief overview of each of the methods by presenting the required definitions and a concise description of the method of derivation.

2.4.1 Direct method

The direct method was first used by Laplace [16] and gives all local conservation laws. Equation (2.13) is a local conservation law. This method uses (2.15) as determining equations for the conserved vector. The components $T^1, T^2, ..., T^n$ are obtained separating (2.15) according to powers and products of the derivatives of u.

2.4.2 Partial Lagrangian method

In this method, we use the partial Lagrangian of the differential equation and derive the conservation laws by the partial Noether approach [17]. We first present useful notation, definitions and theorems. Let x_i, $i = 1, 2, ..., n$, be n independent variables and u the dependent variable. The derivative of u with respect to x^i is $u_i = D_i(u)$, where

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + ..., i = 1, 2, ..., n \tag{2.18}$$

is the total derivative operator with respect to x^i. The Euler operator is defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s \geq 1} (-1)^s D_{i_1} ....D_{i_s} \frac{\partial}{\partial u_{i_1 ... i_s}} \tag{2.19}$$

and the Lie-Backlund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \sum_{s \geq 1} \zeta_{i_1 ....i_s} \frac{\partial}{\partial u_{i_1 ... i_s}}, \tag{2.20}$$

where $\zeta_{i_1,...i_s}$ are defined as

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j) \tag{2.21}$$

and

$$\zeta_{i_1...i_s} = D_{i_s}(\zeta_{i_1...i_{s-1}}) - u_{ji_1...i_{s-1}} D_{i_s}(\xi^j), \quad s > 1. \tag{2.22}$$

The Lie-Backlund operator (2.20) in characteristic form is

$$X = \xi^i \frac{\partial}{\partial x^i} + W \frac{\partial}{\partial u} + D_i(W) \frac{\partial}{\partial u_i} + D_i D_j(W) \frac{\partial}{\partial u_{ij}} + ..., \tag{2.23}$$

where

$$W = \eta - \xi^j u_j \tag{2.24}$$

is the Lie characteristic function. The Noether operators associated with a Lie-Backlund operator X are

$$N^i = \xi^i + W \frac{\delta}{\delta u_i} + \sum_{s \geq 1} D_{i_1}...D_{i_s}(W) \frac{\delta}{\delta u_{ij_1...j_s}}, \quad i = 1, ...n, \tag{2.25}$$

where the Euler-Lagrange operator $\frac{\delta}{\delta u_i}$ is

$$\frac{\delta}{\delta u_i} = \frac{\partial}{\partial u_i} + \sum_{s \geq 1} (-1)^s D_{i_1}...D_{i_s} \frac{\partial}{\partial u_{ij_1...j_s}}, \quad i = 1, 2, ...n. \tag{2.26}$$

Now suppose that a $k-th$ order differential can be written as

$$F = F^0 + F^1. \tag{2.27}$$

The function $L = L(x, u, u_{(1)}, u_{(2)}, ...u_{(l)})$, $l \leq k$ is called a partial Lagrangian (2.12) if (2.12) can be expressed as

$$\frac{\delta L}{\delta u} = f F^1 \tag{2.28}$$

provided $F^1 \neq 0$ for some function f. The Lie-Backlund operator X defined in (2.20) is a partial Noether symmetry generator associated with the partial Lagrangian L if there exists a vector $B = (B^1, B^2, ..., B^n)$ such that

$$X(L) + LD_i(\xi^i) = D_i(B^i) + (\eta - \xi^i u_i)\frac{\delta L}{\delta u}. \quad i = 1, 2, ...n, \tag{2.29}$$

The conserved vector, $T = (T^1, T^2, .., T^n)$, of the differential equation (2.12) associated with the a partial Noether operator X corresponding to the partial Lagrangian L is determined from

$$T^i = B^i - \xi^i L - W\frac{\delta L}{\delta u_i} - \sum_{s\geq 1} D_{i_1 \dots i_s}(W)\frac{\delta L}{\delta u_{i_1 \dots i_s}}, \tag{2.30}$$

where W is the characteristic of the conservation law and B^i are the gauge terms. The components of B are called gauge functions.

2.4.3 Characteristic method

This approach involves the variational derivative

$$D_i T^i = \Lambda F. \tag{2.31}$$

where Λ^α are the characteristics. the characteristics are the multipliers which make the equation exact. We sometimes refer to this as the Multiplier method in this dissertation.

2.5 Conversion of boundary value problems to initial value problems.

An interesting and important class of transformations in the solution of boundary value problems is the conversion of a boundary value problem into an initial value problem. This technique when applied simplifies the process of obtaining numerical solutions. We will now present the necessary formulae and methods needed to transform a second order boundary value problems into two initial value problems [18] [19].

Consider the general second-order differential equation

$$\sum_{m,n,r,s} A_{m,n,r,s} y^{''m} y^{'n} y^r x^s = 0, \tag{2.32}$$

subject to the boundary conditions

$$y^{'}(0) = ay(0) + b, \quad y^{(e)}(\infty) = k. \tag{2.33}$$

Here m, n, r, s are arbitrary indices, $A_{m,n,r,s}$ are arbitrary constants and (e) is an arbitrary integer. We can now assume that y can be expressed in the form

$$y = \lambda F(\mu x) \tag{2.34}$$

where $F(x)$ also satisfies (2.32) for arbitrary constants λ, μ, but subject to the initial conditions

$$F(0) = F^{'}(0) = 1. \tag{2.35}$$

In order that both y and $F(x)$ satisfy (2.32) the differential equation must be invariant under the two parameter group of homogenous linear transformations

$$x_1 = \mu x, \quad y_1 = \frac{y}{\lambda} \tag{2.36}$$

The condition that this imposes on the indices m,n,r,s is obtained by substituting (2.34) into (2.32) i.e, that is

$$\sum_{mnrs} A_{mnrs} F^{'}(\mu x)^n F(\mu x)^r (\mu x)^s \lambda^c \mu^d = 0, \tag{2.37}$$

where

$$c = m + n + r, \quad d = 2m + n - s. \tag{2.38}$$

Consequently, c and d must be constant for all sets of indices m, n, r, s and then (2.32) reduces to

$$\sum_{m,n} A_{mn} y^{''m} y^{'n} y^r x^s = 0, \tag{2.39}$$

where r and s are given by (2.38). It follows that

$$y(0) = \lambda, \quad y^{'}(0) = a\lambda + b = \lambda\mu \tag{2.40}$$

and

$$k = \lambda\mu^e F^{(e)}(\infty). \tag{2.41}$$

We now solve the initial value problem for F and determine $F^{(e)}(\infty)$. One can determine λ and μ from (2.41) and (2.40) and as a result obtain $y(0)$ and $y^{'}(0)$. We can determine y from $y = \lambda F(\mu x)$. By assuming the uniqueness of $F(x)$ in $(0, \infty)$ it follows that the existence and uniqueness of y depends on the $existence$ and uniqueness of μ and λ. Eliminating λ in (2.40) and (2.41) gives

$$\mu^e = k^{'}(\mu - a). \tag{2.42}$$

Depending on the relative values of e, $k^{'}$ and a, there can be zero, one, two or three solutions for μ and the same corresponding for y.

We will now look at cases of variations on the boundary conditions that may impact on the nature of the solution.

Case1: Second boundary condition was given by $y^e(L) = k$.
We would then have

$$y^{'}(0) = a\lambda + b = \lambda\mu, \quad k = \lambda\mu^e F^e(\mu L). \tag{2.43}$$

Case2:
The boundary conditions are $y(0) = a$,$y^{(e)}(\infty) = k$. We assume that y can be expressed in the form $y = F(\mu x)$ where $F(x)$ also satisfies (2.32) for arbitrary μ. Therefore in (2.32)

$$2m + n - s = \text{const.} \tag{2.44}$$

Since $y(0) = a$ it follows that $F(0) = a$. By letting $F^{'}(0) = 1$, then $y^{'}(0) = \mu$ which is determined from

$$k = -\mu^e F^{(e)}(\infty) \tag{2.45}$$

provided that $e \neq 0$.

Case3 : For a finite interval with boundary conditions $y^{(e)}(L) = k$ instead of $y^{(e)}(\infty) = k$ we proceed as before.

We will apply the ideas if this section to solve numerically the boundary value problem of a fluid-driven fracture in Chapter 7.

2.6 Concluding remarks

We have now completed the outline of the theory which will be applied in the subsequent chapters. The concepts will become clearer as they are applied to solve problems.

Chapter 3

Fanno model for turbulent compressible flow.

3.1 Introduction

Asymptotic reductions of the Fanno model for one-dimensional turbulent flow of a gas in a long tunnel are derived and investigated. As discussed previously, the Fanno model may be relevant to air blasts in a long tunnel and we will therefore consider turbulent flow in a tube when the pressure at the mouth of the tube is suddenly changed by an amount small compared with the background pressure. We will briefly outline the derivation of the three physical problems that model this phenomenon as presented by Ockendon et al [6] and for each of them derive Lie point symmetries of the governing partial differential equation using the standard procedure described in Chapter 2. Using a linear combination of the Lie point symmetries we aim to investigate the nature of the group invariant solutions that describe the pressure and velocity profiles of turbulent fluid flow in a tunnel. A comparison will be made of our results with those obtained by Ockendon et al [6].

3.2 Formulation of the problem

The mathematical formulation has been summarised by Anthonyrajah et al [8]. The gas is initially at rest with pressure p_0 and density ρ_0 in a semi-infinite tunnel $x > 0$. The pressure at the entrance is changed from p_0 by an amount that is small relative to p_0. Thus the new pressure is $p_0(1 + \epsilon)$ where $\epsilon > 0$. Consider the time (τ) , pressure $(\tilde{p})$ and density variables $(\tilde{\rho})$ as introduced by Ockendon et al [6] that are defined by

$$t = \epsilon\tau \qquad p = \frac{1}{\epsilon}\,\tilde{p} \qquad \rho = 1 + \epsilon\tilde{\rho}. \tag{3.1}$$

The shock moves into the undisturbed region. The equation of the shock in the (x, τ) plane is given by

$$x = \left(1 + \frac{1}{4}\,(1 + \gamma)\right)\tau. \tag{3.2}$$

Ockendon et al [6] considered a sequence of time scales and found that the two most important regimes are $\tau = \mathrm{O}(\epsilon^{-1})$ and $\tau = \mathrm{O}(\epsilon^{-2})$. We will consider the time scale $\tau = \mathrm{O}(\epsilon^{-2})$ as solutions are easier to analyse asymptotically for this time scale.

Consider first the flow near the shock. Introduce the time and length scales, τ_2 and ϵ_2, defined by

$$\tau_2 = \epsilon^2\tau, \qquad x_2 = \epsilon^2 x, \tag{3.3}$$

which implies that $x = \mathrm{O}(\epsilon^{-2})$, and the velocity u_2, pressure p_2 and density ρ_2 defined by

$$u = \epsilon u_2 \qquad \tilde{p} = \epsilon p_2 \qquad \tilde{\rho} = \epsilon\rho_2. \tag{3.4}$$

Then the second order conservation of mass equation and momentum and energy balance equations in x_2 and τ_2 are

$$\frac{\partial \rho_2}{\partial \tau_2} + \frac{\partial u_2}{\partial x_2} = 0, \tag{3.5}$$

$$\frac{\partial u_2}{\partial \tau_2} + \frac{\partial p_2}{\partial x_2} = -u_2^2, \tag{3.6}$$

$$\frac{\partial p_2}{\partial \tau_2} + \frac{\partial u_2}{\partial x_2} = 0. \tag{3.7}$$

The pressure can be eliminated from (3.6) and (3.7) to give a nonlinear wave equation for u_2:

$$\frac{\partial^2 u_2}{\partial \tau_2^2} - \frac{\partial^2 u_2}{\partial x_2^2} = -2u_2 \frac{\partial u_2}{\partial \tau_2}. \tag{3.8}$$

The boundary conditions on u_2 are

$$x_2 = \tau_2 : \qquad u_2 = \frac{2}{\tau_2}, \tag{3.9}$$

$$x_2 \to 0 : \qquad u_2 \sim \frac{3\tau_2}{x_2^2}. \tag{3.10}$$

Also at second order, the pressure $p_2 \to \infty$ as $x_2 \to 0$. By considering how p_2 grows as $x_2 \to 0$, Ockendon et al [6] introduced the variables $\bar{x}_2$ and $\bar{u}_2$ defined by

$$x_2 = \epsilon^{\frac{1}{3}} \bar{x}_2, \qquad u_2 = \epsilon^{-\frac{2}{3}} \bar{u}_2, \tag{3.11}$$

which implies that $x = \mathrm{O}\left(\epsilon^{-5/3}\right)$ and $u = \mathrm{O}\left(\epsilon^{1/3}\right)$, and they transformed from p_2 and ρ_2 back to $\tilde{p}$ and $\tilde{\rho}$ defined by (3.4). Equations (3.5) to (3.7) transform to

$$\frac{\partial \tilde{\rho}}{\partial \tau_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = 0, \tag{3.12}$$

$$\frac{\partial \bar{p}}{\partial \bar{x}_2} = -\bar{u}_2^2, \tag{3.13}$$

$$\frac{\partial \bar{p}}{\partial \tau_2} + \frac{\partial \bar{u}_2}{\partial x_2} = 0. \tag{3.14}$$

The velocity $\bar{u}_2$ can now be eliminated from (3.13) and (3.14) to give a nonlinear diffusion equation for $\tilde{p}$:

$$\frac{\partial \tilde{p}}{\partial \tau_2} = \frac{1}{2(-\frac{\partial \tilde{p}}{\partial \bar{x}})^{\frac{1}{2}}} \frac{\partial^2 \tilde{p}}{\partial \bar{x}^2}, \tag{3.15}$$

subject to the following boundary conditions

$$\bar{x}_2 = 0 : \qquad \bar{p} = 1, \tag{3.16}$$

$$\bar{x}_2 \to \infty : \qquad \tilde{p} \sim \frac{3\tau_2^2}{\bar{x}_2^3}. \tag{3.17}$$

Alternatively. the pressure $\tilde{p}$ can be eliminated from (3.13) and (3.14) to give a nonlinear diffusion equation for $\bar{u}_2$:

$$\frac{\partial^2 \bar{u}_2}{\partial \bar{x}_2^2} = 2\bar{u}_2 \frac{\partial \bar{u}_2}{\partial \tau_2}, \tag{3.18}$$

The velocity $\bar{u}_2$ satisfies the boundary conditions

$$\bar{x}_2 = 0 : \qquad \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = 0, \tag{3.19}$$

$$\bar{x}_2 \to \infty : \qquad \bar{u}_2 \sim \frac{3\tau_2}{\bar{x}_2^2}. \tag{3.20}$$

The structure of the solution of the partial differential equations is as follows. In the region close to the shock, the nonlinear wave equation for velocity (3.8) and boundary conditions (3.9) and (3.10) apply. In the region further from the shock and nearer the entrance the nonlinear diffusion equations for pressure (3.15) and velocity (3.18) and their corresponding boundary conditions apply. We will now investigate Lie point symmetries and group invariant solutions for all three partial differential equations and boundary conditions. The derivation of the Lie point symmetries of the nonlinear diffusion equation for pressure (3.15) is presented in detail in Appendix A. We will state results for the Lie point symmetries of the remaining partial differential equations for velocity.

3.3 Small amplitude waves: nonlinear wave equation

To simplify the notation we will write (3.8) to (3.10) as:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -2u\frac{\partial u}{\partial t}, \tag{3.21}$$

subject to the boundary conditions:

$$\frac{\partial u}{\partial x} = 0 \quad on \quad x = 0, \quad u = \frac{2}{t} \quad on \quad x = t. \tag{3.22}$$

The Lie point symmetries of equation (3.21) are given by

$$X_1 = \frac{\partial}{\partial t}, \tag{3.23}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{3.24}$$

$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{3.25}$$

In order to derive a group invariant solution we consider a linear combination of the Lie point symmetries:

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3, \tag{3.26}$$

where c_1,c_2,c_3,c_4 and c_5 are constants. Now, $u = \Phi(t,x)$is a group invariant solution of equation (3.21) provided

$$X(u - \Phi(t,x)) \mid_{u=\Phi(t,x)} = 0. \tag{3.27}$$

Using the Lie point symmetries given by (3.23) to (3.27) and by using (3.26), equation (3.27) becomes

$$\left[(c_1 + c_3 t)\frac{\partial}{\partial t} + (c_2 + c_3 x)\frac{\partial}{\partial x} - c_3 u\frac{\partial}{\partial u}\right](u - \Phi(t,x)) \mid_{u=\Phi(t,x)} = 0, \tag{3.28}$$

and therefore $\Phi(t,x)$ must satisfy the first order linear partial differential equation

$$(c_1 + c_3 t)\frac{\partial \Phi}{\partial t} + (c_2 + c_3 x)\frac{\partial \Phi}{\partial x} = -c_3 \Phi. \tag{3.29}$$

The differential equations of the characteristic curves are

$$\frac{dt}{(c_1 + c_3 t)} = \frac{dx}{(c_2 + c_3 x)} = -\frac{d\Phi}{c_3 \Phi}. \tag{3.30}$$

Two independent solutions of (3.30) are

$$a_1 = \frac{c_2 + c_3 x}{c_1 + c_3 t} \tag{3.31}$$

and

$$a_2 = (c_1 + c_3 t)\Phi, \tag{3.32}$$

where a_1 and a_2 are constants. The general solution of equation (3.28) is

$$F(a_1) = a_2, \tag{3.33}$$

where F is an arbitrary function. Thus, since $u = \Phi(t, x)$, a group invariant solution of (3.21) is

$$u = \frac{1}{c_1 + c_3 t} F(\xi), \tag{3.34}$$

where

$$\xi = \frac{c_2 + c_3 x}{c_1 + c_3 t}. \tag{3.35}$$

Substitute (3.34) and (3.35) into partial differential equation (3.21) to obtain

$$c_3(\xi^2 - 1)\frac{d^2 F(\xi)}{d\xi^2} + 2\xi(2c_3 - F(\xi))\frac{dF(\xi)}{d\xi} + 2F(\xi)(c_3 - F(\xi)) = 0. \tag{3.36}$$

Since c_2 does not appear in equation (3.36) we can choose $c_2 = 0$ and by considering the boundary condition $u = \frac{2}{t}$ when $x = t$, we choose $c_1 = 0$. In order to remove c_3 from (3.36) we make the transformation

$$F(\xi) = c_3 H(\xi). \tag{3.37}$$

Equation (3.36) becomes

$$(\xi^2 - 1)\frac{d^2 H}{d\xi^2} + 2\xi(2 - H)\frac{dH}{d\xi} + 2H(1 - H) = 0 \tag{3.38}$$

subject to the boundary conditions

$$H(1) = 2 \quad \text{and} \quad \frac{dH}{dx} = 0 \quad \text{when} \quad \xi = 0. \tag{3.39}$$

The velocity $u(t, x)$ is given by

$$u(t, x) = \frac{H(\xi)}{t} \quad \text{where} \quad \xi = \frac{x}{t}. \tag{3.40}$$

We are not able to solve analytically the differential equation (3.38) subject to the boundary conditions (3.39). We recommend using numerical methods to investigate the solution of the problem further.

3.4 Nonlinear diffusion equation for pressure.

To simplify notation we write (3.15) to (3.17) as

$$\frac{\partial p}{\partial t} = \frac{1}{2(-\frac{\partial p}{\partial x})^{\frac{1}{2}}} \frac{\partial^2 p}{\partial x^2}, \tag{3.41}$$

subject to the boundary conditions

$$p = 1 \quad on \quad x = 0, \quad p \sim \frac{3t^2}{x^3} \quad as \quad x \to \infty. \tag{3.42}$$

The Lie point symmetries which are derived in Appendix A are:

$$X_1 = \frac{\partial}{\partial t}, \tag{3.43}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{3.44}$$

$$X_3 = \frac{\partial}{\partial p}, \tag{3.45}$$

$$X_4 = 3t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}, \tag{3.46}$$

$$X_5 = t\frac{\partial}{\partial t} + 2p\frac{\partial}{\partial p}. \tag{3.47}$$

In order to derive a group invariant solution of (3.41) we consider a linear combination of Lie point symmetries. There are two cases, $c_5 \neq 0$ and $c_5 = 0$. Consider first the case $c_5 \neq 0$. Let

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_1, \tag{3.48}$$

where c_1,c_2,c_3,c_4 and c_5 are constants. Now, $p = \Phi(t, x)$is a group invariant solution of equation (3.41) provided

$$X(p - \Phi(t, x)) \mid_{p=\Phi(t,x)}= 0. \tag{3.49}$$

Using the Lie point symmetries given by (3.43) to (3.47) and by using equation (3.48), equation (3.49) becomes

$$\left[(c_1 + 3c_4t + c_5t)\frac{\partial}{\partial t} + (c_2 + 2c_4x)\frac{\partial}{\partial x} + (c_3 + 2c_5p)\frac{\partial}{\partial p}\right](p - \Phi(t, x)) \mid_{p=\Phi(t,x)}= 0. \tag{3.50}$$

Thus $\Phi(t, x)$ must satisfy the first order linear partial differential equation

$$(c_1 + 3c_4t + c_5t)\frac{\partial \Phi}{\partial t} + (c_2 + 2c_4x)\frac{\partial \Phi}{\partial x} = c_3 + 2c_5\Phi. \tag{3.51}$$

The differential equations of the characteristic curves of (3.51) are

$$\frac{dt}{(c_1 + 3c_4t + c_5t)} = \frac{dx}{(c_2 + 2c_4x)} = \frac{d\Phi}{c_3 + 2c_5\Phi}. \tag{3.52}$$

Two independent solutions of (3.52) are

$$\frac{c_2 + 2c_4x}{(c_1 + (3c_4 + c_5)t)^{\frac{2c_4}{3c_4+c_5}}} = a_1 \tag{3.53}$$

and

$$\frac{c_3 + 2c_5\Phi}{(c_1 + (3c_4 + c_5)t)^{\frac{2c_5}{3c_4+c_5}}} = a_2, \tag{3.54}$$

where a_1 and a_2 are constants. The general solution of (3.51) is

$$a_2 = f(a_1), \tag{3.55}$$

where F is an arbitrary function. Since $p = \Phi(t,x)$, a group invariant solution of equation (3.41) is

$$p(t,x) = \frac{1}{2c_5}\,(c_1 + (3c_4 + c_5)t)^{\frac{2c_5}{3c_4+c_5}}\,F(\xi) - \frac{c_3}{2c_5}, \tag{3.56}$$

where

$$\xi = \frac{c_2 + 2c_4 x}{(c_1 + (3c_4 + c_5)t)^{\frac{2c_4}{3c_4+c_5}}}. \tag{3.57}$$

We can now reduce the partial differential equation (3.41) to an ordinary differential equation. By substituting equations (3.56) and (3.57) into equation (3.41) we find that

$$\frac{d^2F(\xi)}{d\xi^2} + (c_4c_5)^{-\frac{1}{2}}\,\xi\left(-\frac{dF(\xi)}{d\xi}\right)^{\frac{1}{2}}\frac{dF(\xi)}{d\xi} - (\frac{c_5}{c_4^3})^{\frac{1}{2}}F(\xi) = 0. \tag{3.58}$$

We now consider the boundary conditions given by (3.42). The boundary condition $p(t,0) = 1$ becomes

$$\frac{1}{2c_5}\,(c_1 + (3c_4 + c_5)t)^{\frac{2c_5}{3c_4+c_5}}\,F\left(\frac{c_2}{(c_1 + (3c_4 + c_5)t)^{\frac{2c_4}{3c_4+c_5}}}\right) - \frac{c_3}{2c_5} = 1. \tag{3.59}$$

We can set $c_2 = 0$ to give $F(0)$ which is time independent, but the boundary condition (3.42) cannot be satisfied for $c_5 \neq 0$. The case $c_5 \neq 0$ therefore does not give a solution to the problem.

Consider next the case $c_5 = 0$. Then

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4. \tag{3.60}$$

Using equations (3.60) and (3.43) to (3.47), equation (3.49) becomes

$$((c_1 + 3c_4t)\frac{\partial}{\partial t} + (c_2 + 2c_4x)\frac{\partial}{\partial x} + c_3\frac{\partial}{\partial p})(p - \Phi(t,x))\mid_{p=\Phi(t,x)} = 0, \tag{3.61}$$

Thus Φ satisfies the first order linear partial differential equation

$$(c_1 + 3c_4 t)\frac{\partial \Phi}{\partial t} + (c_2 + 2c_4 x)\frac{\partial \Phi}{\partial x} = c_3. \tag{3.62}$$

The differential equations of the characteristic curves are

$$\frac{dt}{(c_1 + 3c_4 t)} = \frac{dx}{(c_2 + 2c_4 x)} = \frac{d\Phi}{c_3}. \tag{3.63}$$

Two independent solutions of equation (3.63) are

$$a_1 = \frac{c_2 + 2c_4 x}{(c_1 + 3c_4 t)^{\frac{2}{3}}} \tag{3.64}$$

and

$$a_2 = \Phi - \frac{c_3}{3c_4}\ln(c_1 + 3c_4 t). \tag{3.65}$$

Thus the general solution of equation (3.63) is

$$a_2 = F(a_1) \tag{3.66}$$

where F is an arbitrary function. Since $p = \Phi(t, x)$, a group invariant solution is given by

$$p = F(\xi) + \frac{c_3}{3c_4}\ln(c_1 + 3c_4 t), \tag{3.67}$$

where

$$\xi = \frac{c_2 + 2c_4 x}{(c_1 + 3c_4 t)^{\frac{2}{3}}}. \tag{3.68}$$

It can be verified that

$$\frac{\partial p}{\partial t} = -\frac{2c_4}{c_1 + 3c_4 t}\xi\frac{dF}{d\xi} + \frac{c_3}{c_1 + 3c_4 t}, \tag{3.69}$$

$$\frac{\partial p}{\partial x} = \frac{2c_4}{(c_1 + 3c_4 t)^{\frac{2}{3}}}\frac{dF}{d\xi}, \tag{3.70}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{4c_4^2}{(c_1 + 3c_4 t)^{\frac{4}{3}}}\frac{d^2 F}{d\xi^2}. \tag{3.71}$$

We substitute equations (3.69) to (3.71) into equation (3.41) to obtain

$$\frac{d^2F}{d\xi^2} + \frac{\sqrt{2}}{\sqrt{c_4}}\xi\frac{dF}{d\xi}\left(-\frac{dF}{d\xi}\right)^{\frac{1}{2}} - \frac{c_3}{\sqrt{2}c_4^{\frac{3}{2}}}\left(-\frac{dF}{d\xi}\right)^{\frac{1}{2}} = 0. \tag{3.72}$$

The boundary condition $p = 1$ when $x = 0$ becomes

$$1 = F\left(\frac{c_2}{(c_1+3c_4t)^{\frac{2}{3}}}\right) + \frac{c_3}{3c_4}\ln(c_1+3c_4t). \tag{3.73}$$

In order to satisfy the boundary condition we choose $c_2 = 0$ and $c_3 = 0$. The second boundary condition in (3.42) becomes

$$F\left(\frac{2c_2x}{(c_1+3c_4t)^{\frac{2}{3}}}\right) \sim \frac{3t^2}{x^3} \quad \text{as} \quad x \to \infty. \tag{3.74}$$

In order to satisfy this boundary condition we choose $c_1 = 0$ and $c_4 = \frac{9}{8}$. Thus (3.67) and (3.68) become

$$p(t,x) = F(\xi), \qquad \xi = \frac{x}{t^{\frac{2}{3}}} \tag{3.75}$$

and equation (3.72) becomes

$$\frac{d^2F(\xi)}{d\xi^2} + \frac{4}{3}\xi\frac{dF(\xi)}{d\xi}\left(-\frac{dF(\xi)}{d\xi}\right)^{\frac{1}{2}} = 0, \tag{3.76}$$

subject the boundary conditions

$$F(0) = 1 \quad \text{and} \quad F(\xi) \sim \frac{3}{\xi^3} \quad as \quad \xi \to \infty. \tag{3.77}$$

Equation (3.76) can be solved analytically by letting

$$G = -\frac{dF}{d\xi} \tag{3.78}$$

and substituting G in (3.76). The resulting equation is a separable first order differential equation

$$\frac{dG}{d\xi} = -\frac{4}{3}\xi G^{\frac{3}{2}} \tag{3.79}$$

which can be solved to give

$$\frac{dF}{d\xi} = -\frac{9}{(\xi^2 + a^2)^2}, \tag{3.80}$$

where a is a constant. By letting $\xi = a\ tan\theta$ in (3.80) and imposing the boundary condition $F(0) = 1$, we obtain

$$F(\xi) = 1 - \frac{9}{2a^3}\left(\frac{a\xi}{\xi^2 + a^2} + tan^{-1}\left(\frac{\xi}{a}\right)\right), \tag{3.81}$$

Now

$$tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + O(\frac{1}{x^5}) + ... \qquad \text{as} \quad x \to \infty \tag{3.82}$$

and therefore

$$F(\xi) = 1 - \frac{9\pi}{4a^2} + \frac{3}{\xi^3} + O(\frac{1}{\xi^5}) \qquad \text{as} \quad x \to \infty. \tag{3.83}$$

From the boundary condition (3.77),

$$a = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} \tag{3.84}$$

and hence from (3.75)

$$p(x,t) = 1 - \frac{3}{\pi}\left(\frac{\zeta}{\zeta^2 + 1} + tan^{-1}\zeta\right), \tag{3.85}$$

where

$$\zeta = \left(\frac{4}{9\pi}\right)^{\frac{1}{3}}\xi = \left(\frac{4}{9\pi}\right)^{\frac{1}{3}}\frac{x}{t^{\frac{2}{3}}}. \tag{3.86}$$

The solution for $p(t,x)$ agrees with the solutions derived by Ockendon et al [6]. Since $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and $c_5 = 0$, it is generated by X_4 given by (3.46). It cannot generated by a linear combination of Lie point symmetries containing X_5.

3.5 Nonlinear diffusion equation for velocity

The solution (3.85) may be used to calculate the fluid velocity u by using

$$\frac{\partial p}{\partial x} = -u^2. \tag{3.87}$$

Alternatively u may be derived without prior knowledge of p by solving the nonlinear diffusion equation for the mean velocity (3.18) subject to boundary conditions given by (3.19) and (3.20). To simplify notation we will write the nonlinear diffusion equations for the mean velocity as

$$\frac{\partial^2 u}{\partial x^2} = 2u\frac{\partial u}{\partial t}, \tag{3.88}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad x = 0, \quad \text{and} \quad u \sim \frac{3t}{x^2} \quad \text{as} \quad x \to \infty. \tag{3.89}$$

The Lie point symmetries of equation (3.88) are

$$X_1 = \frac{\partial}{\partial t}, \tag{3.90}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{3.91}$$

$$X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{3.92}$$

$$X_4 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{3.93}$$

We derive a group invariant solution by considering the linear combination of the Lie point symmetries

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4, \tag{3.94}$$

where c_1,c_2,c_3 and c_4 are constants. Now, $u = \Phi(t, x)$is a group invariant solution of equation (3.88) provided

$$X(u - \Phi(t,x) \mid_{u=\Phi(t,x)}= 0. \tag{3.95}$$

By using the Lie point symmetries given by equations (3.90) to (3.93) and equation (3.94), equation (3.95) becomes

$$\left[(c_1 + 2c_3 t)\frac{\partial}{\partial t} + (c_2 + (c_3 + c_4)x)\frac{\partial}{\partial x} - 2c_4 u\frac{\partial}{\partial u}\right](u - \Phi(t,x)) \mid_{u=\Phi(t,x)}= 0 \tag{3.96}$$

and therefore Φ must satisfy the first order quasi-linear partial differential equation

$$(c_1 + 2c_3 t)\frac{\partial \Phi}{\partial t} + (c_2 + (c_3 + c_4)x)\frac{\partial \Phi}{\partial x} = -2c_4\Phi. \tag{3.97}$$

Two independent solutions of (3.97) are

$$a_1 = \frac{c_2 + (c_3 + c_4)x}{(c_1 + 2c_3 t)^{\frac{1}{2}(1+\frac{c_4}{c_3})}}, \tag{3.98}$$

$$a_2 = (c_1 + 2c_3 t)^{\frac{c_4}{c_3}}\Phi \tag{3.99}$$

where a_1 and a_2 are constants. The general solution of (3.97) is

$$a_2 = F(a_1), \tag{3.100}$$

where F is an arbitrary function. Thus, since $u(t,x) = \Phi(t,x)$, a group invariant solution of equation (3.88) is

$$u(t,x) = (c_1 + 2c_3 t)^{-\frac{c_4}{c_3}} F(\xi), \tag{3.101}$$

where

$$\xi = \frac{c_2 + (c_3 + c_4)\,x}{(c_1 + 2c_3 t)^{\frac{1}{2}(1+\frac{c_4}{c_3})}}. \tag{3.102}$$

Substituting (3.101) and (3.102) into (3.88) we obtain an ordinary differential equation for $F(\xi)$:

$$(c_3 + c_4)^2\frac{d^2F}{d\xi^2} + 2(c_3 + c_4)\xi F\frac{dF}{d\xi} + 4c_4 F^2 = 0. \tag{3.103}$$

We can choose c_2=0 so that $\xi = 0$ when $x = 0$ and $c_1 = 0$ to satisfy the second boundary condition in (3.89). The two boundary conditions in (3.89) become

$$(c_3 + c_4)(2c_3 t)^{-\frac{1}{2}-\frac{3c_4}{2c_3}} \frac{dF(0)}{d\xi} = 0, \tag{3.104}$$

$$F(\xi) \sim \frac{3}{2}\frac{(c_3 + c_4)^2}{c_3}\frac{1}{\xi^2} \qquad \text{as} \qquad \xi \to \infty, \tag{3.105}$$

where

$$\xi = \frac{(c_3 + c_4)x}{(2c_3 t)^{\frac{1}{2}(1+\frac{c_4}{c_3})}} \tag{3.106}$$

The ratio $\frac{c_4}{c_3}$ cannot be obtained from (3.104) because the right hand side of (3.104) vanishes. It also cannot be obtained from (3.105). The constants in equation (3.105) cannot be determined from the boundary conditions.

In deriving similarity solutions using scaling transformations, all parameters in the solution cannot be determined from boundary conditions when the boundary conditions are homogeneous and a further condition is required [20]. The boundary conditions (3.89) may be expressed as

$$\frac{\partial u}{\partial x}(t, 0) = 0 \quad \text{and} \quad u(t, \infty) = 0 \tag{3.107}$$

and are therefore homogeneous. A conserved quantity derived from a conservation law can provide the extra condition required to complete the solution. This will be investigated in the next chapter.

3.6 Conclusions

Lie point symmetries were derived and used to reduce partial differential equations that describe turbulent compressible fluid flow in a long tunnel to ordinary differential equations. We were then able to give an alternative derivation of the analytical solution derived by Ockendon et al

[6] of the nonlinear diffusion equation for pressure using Lie group methods. The nonlinear wave equation was reduced to a second order ordinary differential equation. We were not able to proceed further analytically with this differential equation. It may be best treated numerically. Finally, the need for a conserved quantity was highlighted when an extra condition was required to determine the scaling constants when reducing the nonlinear diffusion equation for the mean fluid velocity to an ordinary differential equation. We will return to this problem in the next chapter. It also motivates the study in Chapter 5 of conservation laws for turbulent compressible flow in a long tunnel.

Chapter 4

Application of conservation laws

4.1 Introduction

We will derive a conserved quantity using the elementary conserved vector of the nonlinear diffusion equation for the mean velocity. This conserved quantity will then be used to complete the analytical solution of the differential equation (3.88). In addition, we will illustrate how one can derive the solution for $u(x,t)$ using a linear combination of the Lie point symmetries associated with the elementary conserved vector. This new method of deriving invariant solutions of problems with conserved quantities was proposed by Kara and Mahomed [15] and is more direct than the standard procedure followed in Chapter 3 of using a linear combination of all the Lie point symmetries of the partial differential equation.

4.2 Conserved quantity for mean velocity

We consider again the problem investigated in Section 3.5 which was to solve the nonlinear diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = 2u\frac{\partial u}{\partial t}, \tag{4.1}$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad x = 0, \quad \text{and} \quad u(t,x) \sim \frac{3t}{x^2} \quad \text{as} \quad x \to \infty. \tag{4.2}$$

The components, T^1, and T^2, of a conserved vector for the partial differential equation (4.1) satisfy the conservation law

$$D_t T^1 + D_x T^2 = 0. \tag{4.3}$$

If T^1 and T^2 are regarded as functions of the independent variables t and x then (4.3) can be written equivalently

$$\frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0. \tag{4.4}$$

Now the nonlinear diffusion equation for the mean velocity (4.1) can be written as

$$\frac{\partial}{\partial t}(u^2) + \frac{\partial}{\partial x}(-u_x) = 0. \tag{4.5}$$

Thus

$$T^1 = u^2, \qquad T^2 = -u_x. \tag{4.6}$$

is a conserved vector for the partial differential equation (4.1). It is referred to as the elementary conserved vector.

We now derive a conserved quantity from the conservation law (4.5). By integrating (4.5) with respect to x along the tunnel from $x = 0$ to $x = \infty$ we obtain

$$\int_0^\infty \frac{\partial}{\partial t}(u^2(t,x))dx = \int_0^\infty \frac{\partial}{\partial x}(u_x(t,x))dx. \tag{4.7}$$

But since the limits of integration are fixed [21]

$$\int_0^\infty \frac{\partial}{\partial t}(u^2(t,x))dx = \frac{d}{dt}\int_0^\infty u^2(t,x)dx. \tag{4.8}$$

Thus from (4.7) and (4.8) we have

$$\frac{d}{dt}\int_0^\infty u(t,x)^2 dx = u_x(t,\infty) - u_x(t,0). \tag{4.9}$$

From the boundary conditions (4.2) we have $u_x(t,0) = 0$ and $u_x(t,\infty) = 0$ and therefore

$$\frac{d}{dt}\int_0^\infty u^2(t,x)dx = 0. \tag{4.10}$$

Hence

$$\int_0^\infty u^2(t,x)dx = k \tag{4.11}$$

where k is a constant independent of t. The left hand side of (4.11) is a conserved quantity for the problem described by the partial differential equation (4.1) and boundary conditions (4.2). To obtain k we note that from (3.13).

$$u^2 = -\frac{\partial p}{\partial x} \tag{4.12}$$

and from (3.16) and (3.17) that $p(t,x)$ satisfies the boundary conditions

$$p(t,0) = 1, \qquad p(t,\infty) = 0. \tag{4.13}$$

Substituting (4.12) into (4.11) and integrating gives $k = 1$. Hence

$$\int_0^\infty u^2(t,x)dx = 1. \tag{4.14}$$

4.3 Group invariant solution for mean velocity

We showed in Section 3.5 that the group invariant solution of (4.1) is of the form

$$u(t,x) = (2c_3 t)^{-\frac{c_4}{c_3}} F(\xi), \tag{4.15}$$

where

$$\xi = \frac{(c_3 + c_4)x}{(2c_3 t)^{\frac{1}{2}(1+\frac{c_4}{c_3})}} \tag{4.16}$$

and that $F(\xi)$ satisfies the ordinary differential equation

$$(c_3 + c_4)^2 \frac{d^2 F}{d\xi^2} + 2(c_3 + c_4)\xi F \frac{dF}{d\xi} + 4c_4 F^2 = 0, \tag{4.17}$$

subject to the boundary conditions

$$\frac{dF(0)}{d\xi} = 0, \quad F(\xi) \sim \frac{3}{2}\frac{(c_3 + c_4)}{c_3}\frac{1}{\xi^2} \quad \text{as} \quad \xi \to \infty. \tag{4.18}$$

We now complete the solution of the problem using conserved quantity (4.14). We substitute for u in the conserved quantity (4.14) and use (4.16) to make the change of variable from x to ξ at fixed time t. Equation (4.14) becomes

$$(c_3 + c_4)^{-1}(c_3 t)^{\frac{1}{2} - \frac{3c_4}{2c_3}} \int_0^\infty F^2(\xi) d\xi = 1. \tag{4.19}$$

For the right hand side of (4.19) to be constant, the exponent of t must vanish. Hence

$$\frac{c_4}{c_3} = \frac{1}{3}. \tag{4.20}$$

The group invariant solution becomes

$$u(t, x) = 2(c_3 t)^{-\frac{1}{3}} F(\xi), \tag{4.21}$$

where

$$\xi = \frac{4}{3}\left(\frac{c_3}{4}\right)^{\frac{1}{3}} \frac{x}{t^{\frac{2}{3}}} \tag{4.22}$$

and $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2 F(\xi)}{d\xi^2} + \frac{3}{4c_3} F(\xi) \frac{dF(\xi)}{d\xi} = 0, \tag{4.23}$$

subject to the boundary conditions

$$\frac{dF(0)}{d\xi} = 0, \qquad F(\xi) \sim \frac{8}{3}\frac{c_3}{\xi^2} \quad \text{as} \quad \xi \to \infty \tag{4.24}$$

and the conserved quantity

$$\int_0^\infty F^2(\xi)d\xi = \frac{4c_3}{3}. \tag{4.25}$$

The problem contains one remaining parameter c_3. Since we have taken $c_1 = 0$ and $c_2 = 0$, the Lie point symmetry which generates the group invariant solution is from (3.92) and (3.93)

$$\frac{1}{c_3}X = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{c_4}{c_3}\left(x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}\right). \tag{4.26}$$

Only the ratio $\frac{c_4}{c_3}$ is significant because a constant multiple of a Lie point symmetry is also a Lie point symmetry. Hence c_3 can be chosen conveniently. It can be verified by direct calculations, keeping c_3 unspecified, that the solution for $u(t,x)$ does not depend on c_3. In order to simplify ξ we choose

$$c_3 = \frac{27}{16}. \tag{4.27}$$

The group invariant solution (4.21) to (4.26) becomes:

$$u(t,x) = \frac{2}{3}\frac{F(\xi)}{t^{\frac{1}{3}}}, \qquad \xi = \frac{x}{t^{\frac{2}{3}}}, \tag{4.28}$$

where $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + \frac{4}{9}\frac{d}{d\xi}(\xi F^2) = 0, \tag{4.29}$$

subject to the boundary conditions

$$\frac{dF(0)}{d\xi} = 0, \qquad F(\xi) \sim \frac{9}{2\xi^2} \qquad \text{as} \quad \xi \to \infty \tag{4.30}$$

and to the conserved quantity

$$\int_0^\infty F^2(\xi)d\xi = \frac{9}{4}. \tag{4.31}$$

By integrating (4.29) with respect to ξ and by imposing the first boundary condition in (4.30) we obtain

$$\frac{dF}{d\xi} = -\frac{4}{9}\xi F^2. \tag{4.32}$$

By using the method of separation of variables we find that integrating (4.32)

$$F(\xi) = \frac{9}{2(\xi^2 + a^2)}, \tag{4.33}$$

where a is a constant. The value of a is determined from the conserved quantity (4.31). This is done by substituting (4.33) into (4.31) and using

$$\int_0^\infty \frac{dw}{(1+w^2)^2} = \frac{\pi}{4}, \tag{4.34}$$

which gives

$$a = (\frac{9}{4\pi})^{\frac{1}{3}}. \tag{4.35}$$

Finally, using (4.28) we have

$$u(t,x) = \frac{3t}{x^2 + (\frac{9\pi}{4})^{\frac{2}{3}} t^{\frac{4}{3}}}. \tag{4.36}$$

It follows from (4.26) with $\frac{c_4}{c_3} = \frac{1}{3}$ that the solution is generated by the Lie point symmetry

$$X = 3t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{4.37}$$

We have shown how the conserved quantity is used to determine the ratio of scaling constants in the reduction of a partial differential equation to an ordinary differential equation. We will now use an alternative method of deriving an invariant solution to the nonlinear diffusion equation for the mean velocity by using a linear combination of Lie point symmetries associated with the conserved vector.

4.4 Conserved vector and associated symmetries

From Section (2.4) the determining equation for the Lie point symmetries X associated with the conserved vector $T = (T^1, T^2)$,

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0. \tag{4.38}$$

Equation (4.38) consists of two components

$$X(T^1) + T^1 D_2 \xi^2 - T^2 D_2 \xi^1 = 0, \tag{4.39}$$

$$X(T^2) + T^2 D_1 \xi^1 - T^1 D_1 \xi^2 = 0. \tag{4.40}$$

Substituting the elementary conserved vector (4.6) into (4.39) and (4.40) gives

$$2u\eta + u^2 \frac{\partial \xi^2}{\partial x} + u^2 u_x \frac{\partial \xi^2}{\partial u} + u_x \frac{\partial \xi^1}{\partial x} + u_x^2 \frac{\partial \xi^1}{\partial u} = 0 \tag{4.41}$$

and

$$-\frac{\partial \eta}{\partial x} - u_x \frac{\partial \eta}{\partial u} + u_x \frac{\partial \xi^2}{\partial x} + u_t \frac{\partial \xi^1}{\partial x} + u_x^2 \frac{\partial \xi^2}{\partial u} - u_x \frac{\partial \xi^1}{\partial t} - u^2 \frac{\partial \xi^2}{\partial t} - u^2 u_t \frac{\partial \xi^2}{\partial u} = 0. \tag{4.42}$$

Separating (4.41) by derivatives of u we find

$$u_x^2: \qquad \frac{\partial \xi^1}{\partial u} = 0, \tag{4.43}$$

$$u_x: \qquad \frac{\partial \xi^1}{\partial x} + u^2 \frac{\partial \xi^2}{\partial u} = 0, \tag{4.44}$$

$$\text{remainder}: \qquad 2u\eta + u^2 \frac{\partial \xi^2}{\partial x} = 0. \tag{4.45}$$

Separating (4.42) by derivatives of u we find

$$u_x^2: \qquad \frac{\partial \xi^2}{\partial u} = 0, \tag{4.46}$$

$$u_x: \qquad -\frac{\partial \eta}{\partial u} + \frac{\partial \xi^2}{\partial x} - \frac{\partial \xi^1}{\partial t} = 0, \tag{4.47}$$

$$u_t: \qquad \frac{\partial \xi^1}{\partial x} = 0, \tag{4.48}$$

$$\text{remainder}: \qquad \frac{\partial \eta}{\partial x} + u^2 \frac{\partial \xi^2}{\partial t} = 0. \tag{4.49}$$

From (4.43) and (4.48)

$$\xi^1 = \xi^1(t) \tag{4.50}$$

and from (4.46)

$$\xi^2 = \xi^2(t, x). \tag{4.51}$$

Differentiating (4.45) with respect to x and substituting for $\frac{\partial \eta}{\partial x}$ in (4.49) yields an equation that is separable by powers of u:

$$u^2 : \qquad \frac{\partial \xi^2}{\partial t} = 0 \tag{4.52}$$

$$u : \qquad \frac{d^2 \xi^2}{d^2 x^2} = 0. \tag{4.53}$$

From (4.52) and (4.53) we find that

$$\xi^2(x) = a_1 x + a_2, \tag{4.54}$$

where a_1 and a_2 are arbitrary constants. Substituting (4.54) for ξ^2 into (4.45) we have

$$\eta(t, u) = -a_1 \frac{u}{2}. \tag{4.55}$$

Substituting (4.54) and (4.55) for ξ^2 and η into (4.47) yields

$$\xi^1(t) = \frac{3}{2} a_1 t + a_3 \tag{4.56}$$

where a_3 is an arbitrary constant. The Lie point symmetries are given by

$$X_1 = \frac{\partial}{\partial t}, \tag{4.57}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{4.58}$$

$$X_3 = \frac{3}{2} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{u}{2} \frac{\partial}{\partial u}. \tag{4.59}$$

We let the generator X be a linear combination of the three Lie point symmetries (4.57) to (4.59). Thus

$$X = (c_1 + 3c_3 t)\frac{\partial}{\partial t} + (c_2 + 2c_3 x)\frac{\partial}{\partial x} - c_3 u\frac{\partial}{\partial u}, \tag{4.60}$$

where c_1, c_2 and c_3 are constants. We say that $u = \Phi(t, x)$ is an invariant solution generated by the symmetries associated with conserved vector (4.47) provided

$$X(u - \Phi(t, x))|_{u=\Phi} = 0 \tag{4.61}$$

that is, provided

$$(c_1 + 3c_3 t)\frac{\partial \Phi}{\partial t} + (c_2 + 2c_3 x)\frac{\partial \Phi}{\partial x} = -c_3 \Phi. \tag{4.62}$$

The two independent solutions of (4.62) are

$$a_1 = \frac{c_2 + 2c_3 x}{(c_1 + 3c_3 t)^{\frac{2}{3}}} \tag{4.63}$$

and

$$a_2 = (c_1 + 3c_3 t)^{\frac{1}{3}}\Phi. \tag{4.64}$$

The general solution of (4.62) is therefore

$$a_2 = F(a_1), \tag{4.65}$$

where F is an arbitrary function. Since $u = \Phi(t, x)$ the invariant solution is

$$u = (c_1 + 3c_3 t)^{-\frac{1}{3}} F(\xi), \tag{4.66}$$

where

$$\xi = \frac{c_2 + 2c_3 x}{(c_1 + 3c_3 t)^{\frac{2}{3}}}. \tag{4.67}$$

Using (4.66) and (4.67) we find that

$$\frac{\partial^2 u}{\partial x^2} = 4c_3^2(c_1 + 3c_3 t)^{-\frac{5}{3}} \frac{d^2 F}{d^2 \xi} \tag{4.68}$$

and

$$\frac{\partial u}{\partial t} = -\frac{c_3}{(c_1 + 3c_3 t)^{\frac{4}{3}}} \left[F(\xi) + 2\xi \frac{dF}{d\xi} \right]. \tag{4.69}$$

Using (4.68) and (4.69) we can reduce the partial differential equation

$$u_{xx} = 2uu_t \tag{4.70}$$

to the ordinary differential equation

$$\frac{d^2 F}{d\xi^2} + \frac{1}{2c_3} \frac{d}{d\xi} \left(\xi F^2 \right) = 0. \tag{4.71}$$

The boundary conditions are given by (4.2) and the conserved quantity by (4.14). We choose $c_2 = 0$ so that $\xi = 0$ when $x = 0$ and we choose $c_1 = 0$ to satisfy the second boundary condition in (4.2). The invariant solution derived so far is :

$$u(t, x) = (3c_3 t)^{-\frac{1}{3}} F(\xi), \tag{4.72}$$

where

$$\xi = 2 \left(\frac{c_3}{9} \right)^{\frac{1}{3}} \frac{x}{t^{\frac{2}{3}}}, \tag{4.73}$$

and $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2 F}{d\xi^2} + \frac{1}{2c_3} \frac{d}{d\xi} (\xi F^2) = 0, \tag{4.74}$$

subject to the boundary conditions

$$\frac{dF(0)}{d\xi} = 0, \qquad F(\xi) \sim \frac{4c_3}{\xi^2} \quad \text{as} \quad \xi \to \infty \tag{4.75}$$

and to the conserved quantity

$$\int_0^\infty F^2(\xi)d\xi = 2c_3.fubble \tag{4.76}$$

When (4.72) was substituted into the conserved quantity (4.14) there was no dependence on t. In comparison, when (4.15) was substituted into (4.14) there was dependence on t and the ratio $\frac{c_4}{c_3}$ was chosen to remove this dependence. Since $c_1 = 0 = c_2$, the invariant solution is generated by $X = c_3 X_3$. Since a constant multiple of a Lie point symmetry is Lie point symmetry the solution will not depend on c_3 which can therefore be chosen conveniently. We again choose c_3 to make the constant factor in ξ equal to unity:

$$c_3 = \frac{9}{8}. \tag{4.77}$$

The invariant solution becomes

$$u(t,x) = \frac{2}{3}\frac{F(\xi)}{t^{\frac{1}{3}}}, \quad \xi = \frac{x}{t^{\frac{2}{3}}}, \tag{4.78}$$

$$\frac{d^2F}{d\xi^2} + \frac{4}{9}\frac{d}{d\xi}(\xi F^2) = 0, \tag{4.79}$$

$$\frac{dF(0)}{d\xi} = 0, \quad F(\xi) \sim \frac{9}{2\xi^2} \quad \text{as} \quad \xi \to \infty, \tag{4.80}$$

$$\int_0^\infty F^2(\xi)d\xi = \frac{9}{4}, \tag{4.81}$$

which is exactly the same as the group invariant solution given by (4.28) to (4.31). The solution is therefore again (4.36):

$$u(t,x) = \frac{3t}{x^2 + (\frac{9\pi}{4})^{\frac{2}{3}} t^{\frac{4}{3}}}. \tag{4.82}$$

The solution is generated by the Lie point symmetry X_3 given by (4.59). It is the same as the Lie point symmetry (4.37) which generated the group invariant solution i the standard procedure.

4.5 Conclusions

We have seen that in problems with a conserved quantity the derivation of the group invariant solution using the Lie point symmetries associated with the conserved vector is more direct than the standard method of using a linear combination of all the Lie point symmetries of the partial differential equation as done in Chapter 3. In addition, in the standard approach, the conserved quantity depends on time unless a ratio of the expansion constants has a certain value, which determines that ratio. Using only the symmetries associated with the conserved vector, the conserved quantity is independent of time as expected.

This chapter has highlighted the importance of conserved quantities in determining invariant solutions and will motivate the search for further conservation laws and the study of various methods for deriving conservation laws in the next chapter.

Chapter 5

Derivation of conservation laws for the Fanno model

5.1 Introduction

In this chapter we will derive conservation laws for the three partial differential equations that describe turbulent fluid flow in a long tunnels as derived in Chapter 3. The equations to be considered are:
nonlinear diffusion equation for pressure

$$\frac{\partial p}{\partial t} = \frac{1}{2(-\frac{\partial p}{\partial x})^{\frac{1}{2}}} \frac{\partial^2 p}{\partial x^2}, \tag{5.1}$$

nonlinear diffusion equation for velocity

$$\frac{\partial^2 u}{\partial x^2} = 2u \frac{\partial u}{\partial t} \tag{5.2}$$

and nonlinear wave equation for velocity

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -2u \frac{\partial u}{\partial t}. \tag{5.3}$$

Conservation laws for (5.1) to (5.3) are derived using the direct, partial Lagrangian and characteristic methods. The most elementary method is the direct method introduced by Laplace [16]. The second method of using a partial Lagrangian to construct conservation laws by means of Noether's theorem is straight forward to apply as the conserved vectors are determined by using a formula [22]. The third method is a systematic way of deriving conservation laws with the help of characteristics [17]. A review and comparison of the three methods is presented in Section 5.5.

5.2 Nonlinear diffusion equation for pressure

5.2.1 Direct method

A conservation law for (5.2) satisfies

$$D_1T^1 + D_2T^2|_{(5.1)} = 0. \tag{5.4}$$

From (2.7) and (2.8)

$$D_t = \frac{\partial}{\partial t} + p_t\frac{\partial}{\partial p} + p_{tt}\frac{\partial}{\partial p_t} + p_{xt}\frac{\partial}{\partial p_x} + ..., \tag{5.5}$$

$$D_x = \frac{\partial}{\partial x} + p_x\frac{\partial}{\partial p} + p_{tx}\frac{\partial}{\partial p_t} + p_{xx}\frac{\partial}{\partial p_x} + ... \tag{5.6}$$

We look for conserved vectors of the form

$$T^1(t, x, p, p_t), \quad T^2(t, x, p, p_x). \tag{5.7}$$

Substitute (5.1) into (5.4) and replace p_{xx} using

$$p_{xx} = 2(-p_x)^{\frac{1}{2}}p_t. \tag{5.8}$$

This gives

$$\frac{\partial T^1}{\partial t} + p_t\frac{\partial T^1}{\partial p} + p_{tt}\frac{\partial T^1}{\partial p_t} + \frac{\partial T^2}{\partial x} + p_x\frac{\partial T^2}{\partial p} + 2(-p_x)^{\frac{1}{2}}p_t\frac{\partial T^2}{\partial p_x} = 0. \tag{5.9}$$

We can separate (5.9) by powers of p_{tt} :

$$p_{tt}: \qquad \frac{\partial T^1}{\partial p_t} = 0, \tag{5.10}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + p_t \frac{\partial T^1}{\partial p} + \frac{\partial T^2}{\partial x} + p_x \frac{\partial T^2}{\partial p} + 2(-p_x)^{\frac{1}{2}} p_t \frac{\partial T^2}{\partial p_x} = 0. \tag{5.11}$$

From (5.10), $T^1 = T^1(t, x, p)$ and we can therefore separate (5.11) according to powers of p_t to obtain:

$$p_t: \qquad \frac{\partial T^1}{\partial p} + 2(-p_x)^{\frac{1}{2}} \frac{\partial T^2}{\partial p_x} = 0, \tag{5.12}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + p_x \frac{\partial T^2}{\partial p} = 0. \tag{5.13}$$

From (5.12),

$$T^2(t, x, p, p_x) = (-p_x)^{\frac{1}{2}} \frac{\partial T^1}{\partial p}(t, x, p) + A(t, x, p). \tag{5.14}$$

Substitute (5.14) into (5.13) and separate according to powers of p_x:

$$(-p_x)^{\frac{3}{2}}: \qquad \frac{\partial^2 T^1}{\partial p^2} = 0, \tag{5.15}$$

$$p_x: \qquad \frac{\partial A}{\partial p} = 0, \tag{5.16}$$

$$(-p_x)^{\frac{1}{2}}: \qquad \frac{\partial^2 T^1}{\partial x \partial p} = 0, \tag{5.17}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + \frac{\partial A}{\partial x} = 0. \tag{5.18}$$

From (5.16), $A = A(t, x)$ and from (5.15) and (5.17) we find that

$$T^1(t, x, p) = pB(t) + C(t, x). \tag{5.19}$$

We then substitute (5.19) into (5.18) which gives

$$p\frac{dB(t)}{dt} + \frac{\partial C(t,x)}{\partial t} + \frac{\partial A(t,x)}{\partial x} = 0. \tag{5.20}$$

Separating (5.20) by powers of p gives $B(t) = B_0$ and

$$\frac{\partial C(t,x)}{\partial t} + \frac{\partial A(t,x)}{\partial x} = 0. \tag{5.21}$$

Hence from (5.19) and (5.12),

$$T^1(t,x,p) = B_0 p + C(t,x), \tag{5.22}$$

$$T^2(t,x,p_x) = B_0(-p_x)^{\frac{1}{2}} + A(t,x), \tag{5.23}$$

where $C(t,x)$ and $A(t,x)$ satisfy (5.21). The conserved vector

$$T^1 = C(t,x), \quad T^2 = A(t,x), \tag{5.24}$$

where $C(t,x)$ and $A(t,x)$ satisfy (5.21) is a trivial conserved vector because

$$D_1T^1 + D_2T^2 = \frac{\partial C(t,x)}{\partial t} + \frac{\partial A(t,x)}{\partial x} \equiv 0 \tag{5.25}$$

without imposing the condition that (5.1) is satisfied. Therefore the only conserved vector of the form (5.7) is the elementary conserved vector that can be found by writing (5.1) in the form

$$D_1(p) + D_2((-p_x)^{\frac{1}{2}})|_{(5.1)} = 0. \tag{5.26}$$

The elementary conserved vector is therefore

$$T = (p, (-p_x)^{\frac{1}{2}}). \tag{5.27}$$

5.2.2 Partial Lagrangian method.

A partial Lagrangian for equation (5.1) is

$$L = -\frac{2}{3}(-p_x)^{\frac{3}{2}}. \tag{5.28}$$

Using equation(5.1) it can be shown that

$$\frac{\delta L}{\delta p} = p_t \tag{5.29}$$

We will now show how the Noether point symmetries for the partial Lagrangian (5.28) can be constructed. As explained in Chapter 2, for first order derivatives the first order prolongation of X is given by

$$\begin{aligned} X =& \xi^1(t,x,p)\frac{\partial}{\partial t} + \xi^2(t,x,p)\frac{\partial}{\partial x} + \eta(t,x,p)\frac{\partial}{\partial p} + \zeta_t(t,x,p,p_t,p_x)\frac{\partial}{\partial p_t} \\ &+ \zeta_x(t,x,p,p_t,p_x)\frac{\partial}{\partial p_x}, \end{aligned} \tag{5.30}$$

where

$$\zeta_t = D_t\eta - p_tD_t\xi^1 - p_xD_t\xi^2, \tag{5.31}$$

$$\zeta_x = D_x\eta - p_tD_x\xi^1 - p_xD_x\xi^2. \tag{5.32}$$

The partial Noether symmetry determining equation is , by (2.29)

$$X^{[1]}L + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2 + (\eta - \xi^1p_t - \xi^2p_x)\frac{\delta L}{\delta p} \tag{5.33}$$

where $B^1 = B^1(t,x,p)$ and $B^2 = B^2(t,x,p)$ are the guage terms. Equation (5.33) for $L = -\frac{2}{3}(-p_x)^{\frac{3}{2}}$ gives

$$\begin{aligned} &(-p_x)^{\frac{1}{2}}\frac{\partial\eta}{\partial x} - (-p_x)^{\frac{3}{2}}\left(\frac{\partial\eta}{\partial p} - \frac{\partial\xi^2}{\partial x}\right) - (-p_x)^{\frac{1}{2}}p_t\frac{\partial\xi^1}{\partial x} + (-p_x)^{\frac{3}{2}}p_t\frac{\partial\xi^1}{\partial p} \\ &- (-p_x)^{\frac{5}{2}}\frac{\partial\xi^2}{\partial p} - \frac{2}{3}(-p_x)^{\frac{3}{2}}\frac{\partial\xi^1}{\partial t} - \frac{2}{3}(-p_x)^{\frac{3}{2}}p_t\frac{\partial\xi^1}{\partial p} - \frac{2}{3}(-p_x)^{\frac{3}{2}}\frac{\partial\xi^2}{\partial x} + \frac{2}{3}(-p_x)^{\frac{5}{2}}\frac{\partial\xi^2}{\partial p} \\ &- p_t\eta + p_t^2\xi^1 - p_t(-p_x)\xi^2 - \frac{\partial B^1}{\partial t} - p_t\frac{\partial B^1}{\partial p} - \frac{\partial B^2}{\partial x} - p_x\frac{\partial B^2}{\partial p} = 0. \end{aligned} \tag{5.34}$$

We split equation (5.34) with respect to the derivatives of p and after simplification we obtain

$$
\begin{aligned}
(-p_x)^{\frac{1}{2}}: &\quad \frac{\partial \eta}{\partial x}=0, && (5.35)\\
(-p_x)^{\frac{3}{2}}: &\quad -\frac{\partial \eta}{\partial p}+\frac{1}{3}\frac{\partial \xi^2}{\partial x}-\frac{2}{3}\frac{\partial \xi^1}{\partial t}=0, && (5.36)\\
(-p_x)^{\frac{1}{2}}p_t: &\quad \frac{\partial \xi^1}{\partial x}=0, && (5.37)\\
(-p_x)^{\frac{3}{2}}p_t: &\quad \frac{\partial \xi^1}{\partial p}=0, && (5.38)\\
(-p_x)^{\frac{5}{2}}: &\quad \frac{\partial \xi^2}{\partial p}=0, && (5.39)\\
p_t^2: &\quad \xi^1=0, && (5.40)\\
p_tp_x: &\quad \xi^2=0, && (5.41)\\
p_t: &\quad \eta+\frac{\partial B^1}{\partial p}=0, && (5.42)\\
p_x: &\quad \frac{\partial B^2}{\partial p}=0, && (5.43)\\
\text{remainder}: &\quad \frac{\partial B^1}{\partial t}+\frac{\partial B^2}{\partial x}=0. && (5.44)
\end{aligned}
$$

The solution of equations (5.35) to (5.41) yields

$$\eta=\eta(t), \qquad \xi^1=0, \qquad \xi^2=0. \tag{5.45}$$

Equations (5.42) and (5.43) give the following expressions for B^1 and B^2 respectively,

$$B^1=-\eta(t)p+A_1(t,x), \quad B^2=A_2(t,x). \tag{5.46}$$

By substituting (5.46) into (5.44) we obtain

$$\frac{d\eta}{dt}p-\frac{\partial A_1}{\partial t}-\frac{\partial A_2}{\partial x}=0. \tag{5.47}$$

Differentiating (5.47) with respect to p we see that

$$\frac{d\eta}{dt}=0, \tag{5.48}$$

which implies

$$\eta(t) = k_1, \tag{5.49}$$

where k_1 is a constant.

A summary of the results is follows:

$$\begin{aligned}
&\xi^1 = 0,\\
&\xi^2 = 0,\\
&\eta = k_1,\\
&B^1 = -k_1 p + A_1(t,x),\\
&B^2 = A_2(t,x),\\
&\frac{\partial A_1}{\partial t} + \frac{\partial A_1}{\partial x} = 0.
\end{aligned} \tag{5.50}$$

The first order Noether conserved vector is $T = (T^1, T^2)$, where, by (2.30).

$$T^1 = B^1 - \xi^1 L - (\eta - \xi^1 p_t - \xi^2 p_x)\frac{\partial L}{\partial p_t}, \tag{5.51}$$

$$T^2 = B^2 - \xi^2 L - (\eta - \xi^1 p_t - \xi^2 p_x)\frac{\partial L}{\partial p_x}, \tag{5.52}$$

which, by using results (5.50), give

$$T^1 = -k_1 p + A_1(t,x), \tag{5.53}$$

$$T^2 = A_2(t,x) - k_1(-p_x)^{\frac{1}{2}}, \tag{5.54}$$

subject to the following condition

$$\frac{\partial A_1}{\partial t} + \frac{\partial A_2}{\partial x} = 0. \tag{5.55}$$

We can choose $A_1(t,x) = A_2(t,x) = 0$ since they satisfy the conservation law identically. The elementary conserved vector (5.27) is again obtained.

5.2.3 Characteristic method

We assume that $\Lambda = \Lambda(t,x,p)$ and $T^i = T^i(t,x,p,p_x,p_t)$. This is more general than the conserved vector (5.7) considered in the direct method. By expanding

$$D_t T^1 + D_x T^2. = \Lambda F \tag{5.56}$$

where

$$F = p_t - \frac{1}{2}(-p_x)^{-\frac{1}{2}} p_{xx} \tag{5.57}$$

we obtain

$$\begin{aligned}\frac{\partial T^1}{\partial t} + \frac{\partial T^1}{\partial p} + p_{tt}\frac{\partial T^1}{\partial p_t} + p_{tx}\frac{\partial T^1}{\partial p_x} + \frac{\partial T^2}{\partial x} + p_x\frac{\partial T^2}{\partial p}\\ + p_{xx}\frac{\partial T^2}{\partial p_x} + p_{xt}\frac{\partial T^2}{\partial p_t} - \Lambda p_t + \frac{1}{2}p_{xx}(-p_x)^{-\frac{1}{2}}\Lambda =&0\ .\end{aligned} \tag{5.58}$$

By separating (5.58) by powers of the second derivatives of p we have

$$p_{xx}: \quad \frac{\partial T^2}{\partial p_x} + \frac{1}{2}(-p_x)^{-\frac{1}{2}}\Lambda = 0, \tag{5.59}$$

$$p_{tt}: \quad \frac{\partial T^1}{\partial p_t} = 0, \tag{5.60}$$

$$p_{tx}: \quad \frac{\partial T^1}{\partial p_t} + \frac{\partial T^1}{\partial p_x} = 0, \tag{5.61}$$

$$\text{remainder}: \quad \frac{\partial T^1}{\partial t} + p_t\frac{\partial T^1}{\partial p} + \frac{\partial T^2}{\partial x} + p_x\frac{\partial T^1}{\partial p} - \Lambda p_t = 0. \tag{5.62}$$

From (5.59) and (5.60) we find that

$$T^1 = T^1(t,x,p,p_x) \tag{5.63}$$

and

$$T^2 = \Lambda(t,x,p)(-p_x)^{\frac{1}{2}} + A(t,x,p,p_t), \tag{5.64}$$

where $A(t, x, p, p_t)$ is an arbitrary function. By substituting (5.64) into (5.61) and differentiating with respect to p_t we obtain

$$\frac{\partial^2 A}{\partial p_t^2} = 0 \tag{5.65}$$

which can be solved to give

$$A = p_t B(t, x, p) + C(t, x, p), \tag{5.66}$$

where functions $B(t, x, p)$ and $C(t, x, p)$ are arbitrary.
By re-substituting (5.66) into (5.61) we obtain

$$T^2 = \Lambda(t, x, p)(-p_x)^{\frac{1}{2}} + p_t B(t, x, p) + C(t, x, p), \tag{5.67}$$

and

$$T^1 = -B(t, x, p)p_x + D(t, x, p), \tag{5.68}$$

It now remains to substitute the results (5.67) and (5.68) into (5.62) and separate by powers of derivatives which yields

$$(-p_x)^{\frac{3}{2}} : \qquad \frac{\partial \Lambda}{\partial p} = 0, \tag{5.69}$$

$$(-p_x)^{\frac{1}{2}} : \qquad \frac{\partial \Lambda}{\partial x} = 0, \tag{5.70}$$

$$p_x : \qquad -\frac{\partial B}{\partial t} + \frac{\partial C}{\partial p} = 0, \tag{5.71}$$

$$p_t : \qquad \frac{\partial D}{\partial p} + \frac{\partial B}{\partial x} - \Lambda = 0 , \tag{5.72}$$

$$\text{remainder} : \qquad \frac{\partial D}{\partial t} + \frac{\partial C}{\partial x} = 0. \tag{5.73}$$

From (5.69) and (5.70) we find that Λ is only a function of t, that is

$$\Lambda = \Lambda(t). \tag{5.74}$$

Differentiating (5.71) with respect to x, (5.72) with respect to t and finally (5.73) with respect to p yields

$$-\frac{\partial^2 B}{\partial t \partial x} + \frac{\partial^2 C}{\partial p \partial x} = 0 \tag{5.75}$$

$$\frac{\partial^2 D}{\partial t \partial p} + \frac{\partial^2 B}{\partial x \partial t} - \frac{d\Lambda}{dt} = 0 \tag{5.76}$$

$$\frac{\partial^2 D}{\partial t \partial p} + \frac{\partial^2 C}{\partial x \partial p} = 0 \tag{5.77}$$

By adding equations (5.75) and (5.76) we obtain

$$\frac{\partial^2 C}{\partial p \partial x} + \frac{\partial^2 D}{\partial t \partial p} - \frac{d\Lambda}{dt} = 0 \tag{5.78}$$

which when using (5.77) reduces to

$$\frac{d\Lambda(t)}{dt} = 0, \tag{5.79}$$

and therefore

$$\Lambda = c_1, \tag{5.80}$$

where c_1 is an arbitrary constant.

We can find an expression for $C(t, x, p)$ by integrating (5.71) with respect to p to find

$$C(t, x, p) = \frac{\partial Q(t, x, p)}{\partial t} + F(t, x), \tag{5.81}$$

where

$$Q(t, x, p) = \int^p B(t, x, p) dp \tag{5.82}$$

Similarly we solve for $D(t, x, p)$ by integrating (5.72) with respect to p to obtain

$$D(t, x, p) = c_1 p - \frac{\partial Q(t, x, p)}{\partial x} + G(t, x), . \tag{5.83}$$

Substituting (5.82) and (5.83) into (5.73) gives

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = 0. \tag{5.84}$$

The components (5.67) and (5.68) can now be written as

$$T^1 = c_1 p + T_*^2 \tag{5.85}$$

$$T^2 = c_1(-p_x)^{\frac{1}{2}} + T_*^2 \tag{5.86}$$

where

$$T_*^1 = -B(t,x,p)p_x - \frac{\partial Q}{\partial x}(t,x,p) + F(t,x) \tag{5.87}$$

$$T_*^2 = B(t,x,p)p_t + \frac{\partial Q}{\partial t}(t,x,p) + G(t,x) \tag{5.88}$$

It can be verified using definition (5.82) for Q(t,x,p) and (5.84) that

$$D_t T_*^1 + D_x T_*^2 \equiv 0 \tag{5.89}$$

and therefore T_*^1 and T_*^2 are the components of a trivial conserved vector. Thus (5.85) and (5.86) reduce to the elementary conserved vector

$$T^1 = p, T^2 = (-p_x)^{\frac{1}{2}} \tag{5.90}$$

We have shown that the only conserved vector with components of the form $T^i = Ti(t,x,p,p_t,p_x)$ and with multipliers of the form $\Lambda = \Lambda(t,x,p)$ is the elementary conserved vector. This result is a little stronger than that derived by the direct method with components T^1 and T^2 which depend on all four variables t, x, p, p_t and p_x were considered.

5.3 Nonlinear diffusion equation for velocity

5.3.1 Direct method

A conservation law for (5.2) satisfies

$$D_1T^1 + D_2T^2|_{(5.2)} = 0. \tag{5.91}$$

We can write equation (5.2) in the form

$$D_1(u^2) + D_2(-u_x)|_{(5.2)} \tag{5.92}$$

The conserved vector

$$T = (u^2, -u_x) \tag{5.93}$$

is the elementary conserved vector for (5.2).

Consider conserved vectors for (5.2) of the form

$$T^1 = T^1(t, x, u, u_t), \quad T^2(t, x, u, u_x). \tag{5.94}$$

Substitute (5.94) into (5.91) and replace u_{xx} with

$$u_{xx} = 2uu_t. \tag{5.95}$$

We obtain

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tt}\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + 2uu_t\frac{\partial T^2}{\partial u_x} = 0. \tag{5.96}$$

Since T^1 and T^2 are independent of u_{tt} we can separate by u_{tt}:

$$u_{tt}: \qquad \frac{\partial T^1}{\partial u_t} = 0, \tag{5.97}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + 2uu_t\frac{\partial T^2}{\partial u_x} = 0. \tag{5.98}$$

From equation (5.97), $T^1 = T^1(t, x, u)$ and since T^1 and T^2 do not depend on u_t, (5.98) can be separated by u_t to give

$$u_t: \qquad \frac{\partial T^1}{\partial u} + 2u\frac{\partial T^2}{\partial u_x} = 0, \tag{5.99}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} = 0. \tag{5.100}$$

By differentiating (5.99) with respect to u_x we obtain

$$\frac{\partial^2 T^2}{\partial u_x^2} = 0, \tag{5.101}$$

which implies

$$T^2(t, x, u, u_x) = u_x A(t, x, u) + B(t, x, u), \tag{5.102}$$

where $A(t, x, u)$ and $B(t, x, u)$ are arbitrary functions. Substituting (5.101) into (5.99) we obtain

$$\frac{\partial T^1}{\partial u} + 2uA(t, x, u) = 0. \tag{5.103}$$

It remains to find $A(t, x, u)$ and $B(t, x, u)$. Substitute (5.102) into (5.100) and separate the resulting equation by powers of u_x to obtain

$$u_x^2: \qquad \frac{\partial A}{\partial u} = 0, \tag{5.104}$$

$$u_x: \qquad \frac{\partial A}{\partial x} + \frac{\partial B}{\partial u} = 0, \tag{5.105}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + \frac{\partial B}{\partial x} = 0. \tag{5.106}$$

From (5.104), $A = A(t, x)$ and therefore from (5.105)

$$B(t, x, u) = -u\frac{\partial A(t, x)}{\partial x} + C(t, x) \tag{5.107}$$

and from (5.103)

$$T^1(t,x,u) = -u^2 A(t,x) + D(t,x), \tag{5.108}$$

where $C(t,x)$ and $D(t,x)$ are arbitrary functions. By substituting (5.107) and (5.108) into (5.106) and separating by powers of u we obtain

$$
\begin{aligned}
u^2: &\qquad \frac{\partial A(t,x)}{\partial t} = 0, &(5.109)\\
u: &\qquad \frac{\partial^2 A(t,x)}{\partial x^2} = 0, &(5.110)\\
\text{remainder}: &\qquad \frac{\partial D(t,x)}{\partial t} + \frac{\partial C(t,x)}{\partial x} = 0. &(5.111)
\end{aligned}
$$

It it follows from(5.109) and (5.110)

$$A(x) = A_1 x + A_2, \tag{5.112}$$

where A_1 and A_2 are constants and hence from (5.107)

$$B(t,x,u) = -A_1 u + C(t,x). \tag{5.113}$$

Let $A_1 = -c_2$ and $A_2 = -c_1$. Using (5.112) and (5.113), equations (5.102) and (5.108) become

$$T^1(t,x,u) = c_1 u^2 + c_2 x u^2 + D(t,x), \tag{5.114}$$

$$T^2 = c_1(-u_x) + c_2(u - x u_x) + C(t,x). \tag{5.115}$$

where $D(t,x)$ and $C(t,x)$ satisfy (5.111).

We can choose $C(t,x) = D(t,x) = 0$ since they satisfy the conservation equation identically. There are therefore two conserved vectors of the form (5.94) for the partial differential equation (5.2):

$$T^1 = u^2, \qquad T^2 = -u_x, \tag{5.116}$$

$$T^1 = xu^2, \qquad T^2 = u = xu_x. \tag{5.117}$$

The conserved vector (5.116) is the elementary conserved vector. This compares with the differential equation for pressure (5.1) for which the only conserved vector of the form (5.7).

5.3.2 Partial Lagrangian

The nonlinear diffusion equation for velocity (5.2) admits the partial Lagrangian

$$L = \frac{1}{2}u_x^2 \tag{5.118}$$

and therefore we have

$$\frac{\delta L}{\delta u} = -2uu_t. \tag{5.119}$$

The partial Noether symmetry determining equation is by (2.29)

$$X^{[1]}L + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2 + (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\delta L}{\delta u} \tag{5.120}$$

where the gauge terms are of the form $B^1 = B^1(t, x, u)$ and $B^2 = B^2(t, x, u)$. The expansion of (5.120) is

$$\begin{aligned}
&\frac{\partial \eta}{\partial x}u_x + u_x^2\frac{\partial \eta}{\partial u} - u_x^2\frac{\partial \xi^2}{\partial x} - u_x^3\frac{\partial \xi^2}{\partial u} - u_x u_t\frac{\partial \xi^1}{\partial x} - u_t u_x^2\frac{\partial \xi^1}{\partial u} + \frac{1}{2}u_x^2\frac{\partial \xi^1}{\partial t} \\
&+ \frac{1}{2}u_x^2 u_t\frac{\partial \xi^1}{\partial u} + \frac{1}{2}u_x^2\frac{\partial \xi^2}{\partial x} + \frac{1}{2}u_x^3\frac{\partial \xi^2}{\partial u} + 2uu_t\eta - 2uu_t^2\xi^1 - 2uu_t u_x\xi^2 \\
&- \frac{\partial B^1}{\partial t} - u_t\frac{\partial B^1}{\partial u} - \frac{\partial B^2}{\partial x} - u_x\frac{\partial B^2}{\partial u} = 0.
\end{aligned} \tag{5.121}$$

By separating (5.121) by powers of derivatives of u we obtain

$$u_x^3: \qquad \frac{\partial \xi^2}{\partial u} = 0, \tag{5.122}$$

$$u_x^2: \qquad \frac{\partial \eta}{\partial u} - \frac{1}{2}\frac{\partial \xi^2}{\partial x} + \frac{1}{2}\frac{\partial \xi^1}{\partial t} = 0, \tag{5.123}$$

$$u_x^2 u_t: \qquad \frac{\partial \xi^1}{\partial u} = 0, \tag{5.124}$$

$$u_x u_t: \qquad 2u\xi^2 + \frac{\partial \xi^1}{\partial x} = 0, \tag{5.125}$$

$$u_t^2: \qquad 2u\xi^1 = 0, \tag{5.126}$$

$$u_t: \qquad -2u\eta + \frac{\partial B^1}{\partial u} = 0, \tag{5.127}$$

$$u_x: \qquad \frac{\partial B^2}{\partial u} - \frac{\partial \eta}{\partial x} = 0, \tag{5.128}$$

$$\text{remainder}: \qquad \frac{\partial B^1}{\partial t} + \frac{\partial B^2}{\partial x} = 0. \tag{5.129}$$

Equations (5.125) and (5.126) imply that

$$\xi^2 = 0, \tag{5.130}$$

and (5.126) that,

$$\xi^1 = 0. \tag{5.131}$$

From (5.123) we see that $\eta = \eta(t, x)$ and by substituting this result into (5.127) and integrating with respect to u we find that

$$B^1 = u^2\eta(t, x) + A(t, x) \tag{5.132}$$

where is $A(t, x)$ is an arbitrary function. Similarly, integrating (5.128) with respect to u gives

$$B^2 = u\frac{\partial \eta}{\partial x} + C(t, x). \tag{5.133}$$

where $C(t, x)$ is an arbitrary function.

It now remains to substitute (5.132) and (5.133) into (5.129) and separate by powers of u. This gives

$$u^2: \quad \frac{\partial \eta}{\partial t} = 0, \tag{5.134}$$

$$u: \quad \frac{\partial^2 \eta}{\partial x^2} = 0, \tag{5.135}$$

$$.\text{remainder}: \quad \frac{\partial A(t,x)}{\partial t} + \frac{\partial C(t,x)}{\partial x} = 0. \tag{5.136}$$

By solving equations (5.134) and (5.135) we find that

$$\eta = c_1 x + c_2 \tag{5.137}$$

and therefore the following results for B^1 and B^2 are obtained:

$$B^1 = u^2(c_1 x + c_2) + A(t,x), \tag{5.138}$$

$$B^2 = c_1 u + C(t,x). \tag{5.139}$$

Note that $A(t,x)$ and $C(t,x)$ are subject to condition (5.136). As a result we can choose $A(t,x) = C(t,x) = 0$ since the conservation law is identically satisfied by $T^1 = A(t,x)$ and $T^2 = C(t,x)$. Therefore from (2.30) the partial Noether conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = B^1 - \xi^1 L - (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\partial L}{\partial u_t}, \tag{5.140}$$

$$T^2 = B^2 - \xi^2 L - (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\partial L}{\partial u_x}. \tag{5.141}$$

Thus

$$T^1 = u^2(c_1 x + c_2), \tag{5.142}$$

and

$$T^2 = c_1 u - (xc_1 + c_2)u_x. \tag{5.143}$$

which agrees with (5.116) (5.117) obtained using the direct method.

5.3.3 Characteristic approach

The conservation law is written in characteristic form as

$$D_t T^1 + D_x T^2. = \Lambda F \tag{5.144}$$

where

$$F(u, u_t, u_{xx}) = u_{xx} - 2uu_t. \tag{5.145}$$

We consider with $\Lambda = \Lambda(t, x, u)$ and T^i having at most first order derivatives:

$$T^1 = T^1(t, x, u, u_t, u_x), \quad T^2 = T^2(t, x, u, u_t, u_x). \tag{5.146}$$

This is more general than the form (5.94) considered in the direct method.

Expanding (5.144) we obtain

$$\begin{aligned}
&- u_{xx}\Lambda + 2\, uu_t\Lambda + \frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tt}\frac{\partial T^1}{\partial u_t} + u_{tx}\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} \\
&+ u_{xx}\frac{\partial T^2}{\partial u_x} + u_{tx}\frac{\partial T^2}{\partial u_t} = 0.
\end{aligned} \tag{5.147}$$

Separation of (5.147) by the second derivatives of u yields

$$\begin{aligned}
u_{xx}: \quad & \frac{\partial T^2}{\partial u_x} - \Lambda = 0\ , && (5.148)\\
u_{tt}: \quad & \frac{\partial T^1}{\partial u_t} = 0, && (5.149)\\
u_{tx}: \quad & \frac{\partial T^2}{\partial u_t} + \frac{\partial T^1}{\partial u_x} = 0, && (5.150)\\
\text{remainder}: \quad & \frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + 2uu_t\Lambda = 0\ . && (5.151)
\end{aligned}$$

Equations (5.149) and (5.148) lead to the following results for T^1, T^2 respectively;

$$T^1 = T^1(t, x, u, u_x) \tag{5.152}$$

and

$$T^2 = u_x\Lambda(t, x, u) + A(t, x, u, u_t), \tag{5.153}$$

where $A(t, x, u, u_t)$ is an arbitrary function. By substituting (5.152) and (5.153) into (5.150) and differentiating with respect to u_t we obtain an expression for A:

$$A = u_tB(t, x, u) + C(t, x, u) \tag{5.154}$$

which when substituted back into (5.150) gives

$$T^1 = -u_xB(t, x, u) + D(t, x, u) \tag{5.155}$$

Where $B(t, x, u)$, $C(t, x, u)$ and $D(t, x, u)$ are arbitrary. Also from (5.153) and (5.154)

$$T^2 = u_x\Lambda(t, x, u) + u_tB(t, x, u) + C(t, x, u). \tag{5.156}$$

It now remains to substitute (5.155) and (5.156) to (5.151) and separate by powers of the first derivatives of u:

$$u_x^2 : \qquad \frac{\partial \Lambda}{\partial u} = 0, \tag{5.157}$$

$$u_x : \qquad -\frac{\partial B}{\partial t} + \frac{\partial \Lambda}{\partial x} + \frac{\partial C}{\partial u} = 0, \tag{5.158}$$

$$u_t : \qquad \frac{\partial D}{\partial u} + \frac{\partial B}{\partial x} + 2u\Lambda = 0\ , \tag{5.159}$$

$$\text{remainder} : \qquad \frac{\partial D}{\partial t} + \frac{\partial C}{\partial x} = 0. \tag{5.160}$$

From (5.157)

$$\Lambda = \Lambda(t, x). \tag{5.161}$$

Differentiating (5.158) with respect to x, (5.159) with respect to t and finally (5.160) with respect to u yields

$$-\frac{\partial^2 B}{\partial t \partial x} + \frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial^2 C}{\partial u \partial x} = 0, \tag{5.162}$$

$$\frac{\partial^2 D}{\partial t \partial u} + \frac{\partial^2 B}{\partial x \partial t} + 2u\frac{\Lambda(t, x)}{\partial t} = 0, \tag{5.163}$$

$$\frac{\partial^2 D}{\partial t \partial u} + \frac{\partial^2 C}{\partial x \partial u} = 0. \tag{5.164}$$

By adding equations (5.162) and (5.163) we obtain

$$\frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial^2 C}{\partial u \partial x} + \frac{\partial^2 D}{\partial t \partial u} + 2u\frac{\Lambda(t, x)}{\partial t} = 0 \tag{5.165}$$

which when using (5.164) reduces to

$$\frac{\partial^2 \Lambda}{\partial x^2} + 2u\frac{\partial \Lambda(t, x)}{\partial t} = 0. \tag{5.166}$$

Separating (5.166) by u gives $\Lambda = \Lambda(x)$ and

$$\frac{d^2 \Lambda}{dx^2} = 0, \tag{5.167}$$

which when integrated gives

$$\Lambda(x) = c_1 x + c_2, \tag{5.168}$$

where c_1, c_2 are arbitrary constants.

We solve for $C(t, x, u)$ by integrating (5.158) with respect to u to find

$$C(t, x, u) = -c_1 u + \frac{\partial Q(t, x, u)}{\partial t} + F(t, x), \tag{5.169}$$

where

$$Q(t, x, u) = \int^u B(t, x, u) du \tag{5.170}$$

and $F(t, x)$ is an arbitrary function.

Similarly we solve for $D(t, x, u)$ by integrating (5.159)

$$D(t, x, u) = -u^2(c_1 x + c_2) - \frac{\partial Q(t, x, u)}{\partial x} + G(t, x), \tag{5.171}$$

where $G(t, x)$ is an arbitrary function. Finally by substituting (5.169) and(5.171) into (5.160) we find that $F(t, x)$ and $G(t, x)$ must satisfy the following condition

$$\frac{\partial F(t, x)}{\partial t} + \frac{\partial G(t, x)}{\partial x} = 0. \tag{5.172}$$

Thus (5.155) and (5.156) become

$$T^1 = -(c_1 x + c_2)u^2 + T^1_* \tag{5.173}$$

$$T^2 = -c_1 u + (c_1 x + c_2)u_x + T^2_* \tag{5.174}$$

where

$$T^1_* = -B(t, x, u)u_x - \frac{\partial Q}{\partial x}(t, x, u) + F(t, x) \tag{5.175}$$

$$T_*^2 = B(t,x,u)u_t + \frac{\partial Q}{\partial t}(t,x,u) + G(t,x) \tag{5.176}$$

The components (5.175) and (5.176) have the same form as (5.87) and (5.88) and therefore satisfy (5.89) identically. Thus T_*^1 and T_*^2 are the components of a trivial conserved vector and (5.173) and (5.174) again give two the conserved vectors (5.116) and (5.117).

We have shown that there are only two conserved vectors for the nonlinear diffusion equations (5.2) given by (5.116) and (5.117) with components of the form $T^i = T^i(t,x,u,u_t,u_x)$ and with multiplier $\Lambda(t,x,u)$. This result is more general than the result derived by the direct method where the conserved vectors of the form (5.94) were considered.

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5.4 Nonlinear wave equation

5.4.1 Direct Method

We know that conserved vectors for (5.3) satisfy

$$D_1 T^1 + D^2 T^2|_{(5.3)} = 0. \tag{5.177}$$

We look for conserved vectors of the form

$$T^1 = T^1(t,x,u,u_t), \quad T^2 = T^2(t,x,u,u_x). \tag{5.178}$$

By substituting (5.178) into (5.177) and replacing u_{xx} by

$$u_{xx} = u_{tt} + 2uu_t, \tag{5.179}$$

we find

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tt}\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + u_{tt}\frac{\partial T^2}{\partial u_x} + 2uu_t\frac{\partial T^2}{\partial u_x} = 0. \tag{5.180}$$

We can separate (5.180) by the derivative u_{tt} because T^1 and T^2 do not depend on u_{tt} and thus obtain

$$u_{tt}: \quad \frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0 \tag{5.181}$$

$$\text{remainder}: \quad \frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + 2uu_t \frac{\partial T^2}{\partial u_x} = 0. \tag{5.182}$$

Differentiating (5.181) with respect to u_t and by integrating the resulting equation twice with respect to u_t gives

$$T^1(t, x, u, u_x) = u_t A(t, x, u) + B(t, x, u); \tag{5.183}$$

where $A(t, x, u)$ and $B(t, x, u)$ are arbitrary functions. We find an expression for T^2 by substituting (5.183) into (5.181),

$$T^2(t, x, u, u_x) = -u_x A(t, x, u) + C(t, x, u) \tag{5.184}$$

where $C(t, x, u)$ is an arbitrary function.

It remains to obtain $A(t, x, u)$, $B(t, x, u)$, and $C(t, x, u)$ by substituting (5.183) and (5.184) into (5.182) and separating by powers of u_t and u_x:

$$u_t^2: \quad \frac{\partial A}{\partial u} = 0, \tag{5.185}$$

$$u_x^2: \quad \frac{\partial A}{\partial u} = 0, \tag{5.186}$$

$$u_t: \quad \frac{\partial A}{\partial t} + \frac{\partial B}{\partial u} - 2uA = 0, \tag{5.187}$$

$$u_x: \quad \frac{\partial A}{\partial x} - \frac{\partial C}{\partial u} = 0, \tag{5.188}$$

$$\text{remainder}: \quad \frac{\partial B}{\partial t} + \frac{\partial C}{\partial x} = 0. \tag{5.189}$$

From (5.185) and (5.186), $A = A(t, x)$ and solving (5.187) for $B(t, x, u)$ we obtain

$$B(t, x, u) = u^2 A(t, x) - u \frac{\partial A(t, x)}{\partial x} + D(t, x), \tag{5.190}$$

where $D(t,x,u)$ is an arbitrary function.

To obtain $C(t,x,u)$ we integrate (5.188) with respect to u:

$$C(t,x,u) = u\frac{\partial A(t,x)}{\partial x} + E(t,x), \tag{5.191}$$

where $E(t,x,u)$ is an arbitrary function. We then substitute (5.190) and (5.191) into (5.189) and separate by powers of u to obtain:

$$u^2: \qquad \frac{\partial A(t,x)}{\partial t} = 0, \tag{5.192}$$

$$u: \qquad \frac{\partial^2 A(t,x)}{\partial t^2} - \frac{\partial^2 A(t,x)}{\partial x^2} = 0. \tag{5.193}$$

From (5.192) and (5.193) we can deduce that

$$A(x) = c_1 + c_2 x, \tag{5.194}$$

where c_1 and c_2 are constants and thus from (5.190) and (5.191)

$$B(t,x,u) = (c_1 + c_2 x)u^2 + D(t,x) \tag{5.195}$$

and

$$C(t,x,u) = c_2 u + E(t,x). \tag{5.196}$$

Equations (5.183) and (5.184) become

$$T^1 = c_1(u^2 + u_t) + c_2(xu^2 + xu_t) + D(t,x), \tag{5.197}$$

$$T^2 = c_1(-u_x) + c_2(u - xu_x) + E(t,x), \tag{5.198}$$

where $D(t,x)$ and $E(t,x)$ satisfy (5.189).

$$\frac{\partial D(t,x)}{\partial t} + \frac{\partial E(t,x)}{\partial x} = 0. \tag{5.199}$$

The conserved vector

$$T^1 = D(t,x), \quad T^2 = E(t,x), \tag{5.200}$$

is a trivial conserved vector because the conservation law (5.177) is identically satisfied.The conserved vector is therefore a linear combination of two conserved vectors

$$T^1 = u^2 + u_t, \quad T^2 = -u_x, \tag{5.201}$$

$$T^1 = x(u^2 + u_t), \quad T^2 = u - xu_x. \tag{5.202}$$

The conserved vector (5.201) is the elementary conserved vector.

5.4.2 Partial Lagrangian method

Consider the partial Lagrangian

$$L = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2, \tag{5.203}$$

so that (5.3) becomes

$$\frac{\delta L}{\delta u} = -2uu_t. \tag{5.204}$$

The partial Noether symmetry determining equation is, by (2.29)

$$X^{[1]}L + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2 + (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\delta L}{\delta u}. \tag{5.205}$$

Substituting (5.203) in (5.205) gives

$$
\begin{aligned}
& u_x\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi^2}{\partial x}\right) u_x^2 - u_t u_x \frac{\partial \xi^1}{\partial x} - u_t u_x^2 \frac{\partial \xi^1}{\partial u} - u_x^3 \frac{\partial \xi^2}{\partial u} - u_t \frac{\partial \eta}{\partial t} \\
& - \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi^1}{\partial t}\right) u_t^2 + u_t^3 \frac{\partial \xi^1}{\partial u} + u_x u_t \frac{\partial \xi^2}{\partial t} + u_x u_t^2 \frac{\partial \xi^2}{\partial u} + \frac{1}{2} u_x^2 \frac{\partial \xi^1}{\partial t} \\
& + \frac{1}{2} u_x^2 u_t \frac{\partial \xi^1}{\partial u} + \frac{1}{2} u_x^2 \frac{\partial \xi^2}{\partial x} + \frac{1}{2} u_x^3 \frac{\partial \xi^2}{\partial u} - \frac{1}{2} u_t^2 \frac{\partial \xi^1}{\partial t} - \frac{1}{2} u_t^3 \frac{\partial \xi^1}{\partial u} - \frac{1}{2} u_t^2 \frac{\partial \xi^2}{\partial x} \\
& - \frac{1}{2} u_t^2 u_x \frac{\partial \xi^2}{\partial u} + 2 u u_t \eta - 2 u u_t^2 \xi^1 - 2 u u_t u_x \xi^2 - \frac{\partial B^1}{\partial t} - u_t \frac{\partial B^1}{\partial u} \\
& - \frac{\partial B^2}{\partial x} - u_x \frac{\partial B^2}{\partial u} = 0. \qquad (5.206)
\end{aligned}
$$

where $B^1 = B^1(t, x, u)$ and $B^2 = B^2(t, x, u)$ are the gauge terms. Separating equation (5.206) with respect to deirvatives of u, we obtain

$$u_x^3: \quad \frac{\partial \xi^2}{\partial u} = 0, \qquad (5.207)$$

$$u_t^3: \quad \frac{\partial \xi^1}{\partial u} = 0, \qquad (5.208)$$

$$u_x^2: \quad \frac{\partial \eta}{\partial u} - \frac{1}{2}\left(\frac{\partial \xi^2}{\partial x} - \frac{\partial \xi^1}{\partial t}\right) = 0, \qquad (5.209)$$

$$u_t^2: \quad \frac{1}{2}\frac{\partial \xi^2}{\partial x} - \frac{1}{2}\frac{\partial \xi^1}{\partial t} + \frac{\partial \eta}{\partial u} + 2u\xi^1 = 0, \qquad (5.210)$$

$$u_t u_x: \quad -\frac{\partial \xi^1}{\partial x} + \frac{\partial \xi^2}{\partial t} - 2u\xi^2 = 0, \qquad (5.211)$$

$$u_t u_x^2: \quad \frac{\partial \xi^1}{\partial u} = 0, \qquad (5.212)$$

$$u_t^2 u_x: \quad \frac{\partial \xi^2}{\partial u} = 0, \qquad (5.213)$$

$$u_x: \quad \frac{\partial \eta}{\partial x} - \frac{\partial B^2}{\partial u} = 0, \qquad (5.214)$$

$$u_t: \quad -\frac{\partial \eta}{\partial t} - \frac{\partial B^1}{\partial u} + 2u\eta = 0, \qquad (5.215)$$

$$\text{remainder}: \quad \frac{\partial B^1}{\partial t} + \frac{\partial B^2}{\partial x} = 0. \qquad (5.216)$$

From (5.207) and (5.213)

$$\xi^2 = \xi^2(t, x), \qquad (5.217)$$

and from (5.208) and (5.212) we find that

$$\xi^1 = \xi^1(t,x). \tag{5.218}$$

Consider (5.211) which can be separated by powers of u to yield

$$u: \qquad \xi^2 = 0 \tag{5.219}$$

$$u: \qquad \frac{\partial \xi^1}{\partial x} = 0. \tag{5.220}$$

Therefore $\xi^1 = \xi^1(t)$, using this and $\xi^2 = 0$ and subtracting (5.210) from (5.209) we obtain

$$\frac{d\xi^1}{\partial t} = 2u\xi^1. \tag{5.221}$$

Separating (5.221) by u it follows that $\xi^1 = 0$ and using this result and $\xi^2 = 0$ in (5.209), we find that .

$$\eta = \eta(t,x). \tag{5.222}$$

Now consider (5.214) and result (5.222) which give

$$B^2 = u\frac{\partial \eta}{\partial x} + A(t,x), \tag{5.223}$$

where $A(t,x)$ is an arbitrary function. Equations (5.215) and (5.222) give

$$B^1 = u^2\eta - u\frac{\partial \eta}{\partial t} + C(t,x), \tag{5.224}$$

where $C(t,x)$ is an arbitrary function. Substituting (5.223) and (5.224) into (5.216) and separating by powers of u gives

$$u^2: \qquad \frac{\partial \eta}{\partial t} = 0, \tag{5.225}$$

$$u: \qquad \frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2} = 0, \tag{5.226}$$

$$\text{remainder}: \qquad \frac{\partial C}{\partial t} + \frac{\partial A}{\partial x} = 0. \tag{5.227}$$

Equations (5.225) and (5.226) give

$$\eta = c_1 x + c_2. \tag{5.228}$$

From (5.223), (5.224) and (5.228)

$$B^1(x,u) = (c_1 x + c_2)u^2 + C(t,x), \quad B^2(u) = c_1 u + A(t,x) \tag{5.229}$$

We choose $A(t,x) = C(t,x) = 0$ as they contribute to the trivial part of the conserved vector. The first order Noether conserved vector is $T = (T^1, T^2)$, where, by (2.30),

$$T^1 = B^1 - \xi^1 L - (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\partial L}{\partial u_t}, \tag{5.230}$$

$$T^2 = B^2 - \xi^2 L - (\eta - \xi^1 u_t - \xi^2 u_x)\frac{\partial L}{\partial u_x}, \tag{5.231}$$

which when substituting results for ξ^1, ξ^2, η and L yield

$$T^1 = (c_1 x + c_2)u^2 + (c_1 x + c_2)u_t, \tag{5.232}$$

and

$$T^2 = c_1 u - (c_1 x + c_2)u_x \tag{5.233}$$

which agrees with (5.201) and (5.202) obtained using the direct method.

5.4.3 Characteristic Approach

We look for conserved vectors of the form $T^i = T^i(t, x, u, u_x.u_t)$ and multiplier $\Lambda = \Lambda(t, x, u)$. This is more general than in the direct method where we looked for conserved vectors of the form (5.178).

Expanding

$$(u_{tt} - u_{xx} + 2uu_t)\Lambda(t, x, u) = D_t T^1(t, x, u, u_x, u_t) + D_x T^2(t, x, u, u_x, u_t) \tag{5.234}$$

results in the following determining equation:

$$\begin{aligned} &- u_{tt}\Lambda + u_{xx}\Lambda - 2uu_t\Lambda + \frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tt}\frac{\partial T^1}{\partial u_t} + u_{tx}\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} \\ &+ u_{xx}\frac{\partial T^2}{\partial u_x} + u_{tx}\frac{\partial T^2}{\partial u_t} = 0. \end{aligned} \tag{5.235}$$

We can now separate (5.235) by the second derivatives of u which gives

$$u_{tt}: \qquad \frac{\partial T^1}{\partial u_t} - \Lambda = 0\ , \tag{5.236}$$

$$u_{xx}: \qquad \frac{\partial T^2}{\partial u_x} + \Lambda = 0, \tag{5.237}$$

$$u_{tx}: \qquad \frac{\partial T^2}{\partial u_t} + \frac{\partial T^1}{\partial u_x} = 0, \tag{5.238}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} - 2uu_t\Lambda = 0\ . \tag{5.239}$$

By integrating (5.236) with respect to u_t and (5.237) with respect to u_x

$$T^1(t, x, u, u_x, u_t) = u_t\Lambda(t, x, u) + A(t, x, u, u_x) \tag{5.240}$$

$$T^2(t, x, u, u_x, u_t) = -u_x\Lambda(t, x, u) + B(t, x, u, u_x), \tag{5.241}$$

where A and B are arbitrary functions.

Substitute (5.240) and (5.241) into (5.238) and differentiate both sides with respect to u_x and solve for A:

$$A = u_x C(t, x, u) + D(t, x, u), \tag{5.242}$$

where $C(t, x, u)$ and $D(t, x, u)$ are arbitrary functions.

By substituting (5.242) back into (5.238) we find an expression for B:

$$B = -u_t C(t, x, u) + E(t, x, u), \tag{5.243}$$

where $E(t, x, u)$ is an arbitrary function.

We now substitute results (5.240) to (5.243) into (5.239) and separate by powers of the first derivatives of u:

$$u_t^2 : \qquad \frac{\partial \Lambda}{\partial u} = 0, \tag{5.244}$$

$$u_x^2 : \qquad \frac{\partial \Lambda}{\partial u} = 0, \tag{5.245}$$

$$u_t : \qquad \frac{\partial \Lambda}{\partial t} + \frac{\partial D}{\partial u} - \frac{\partial C}{\partial x} - 2u\Lambda = 0\ , \tag{5.246}$$

$$u_x : \qquad \frac{\partial C}{\partial t} - \frac{\partial \Lambda}{\partial x} + \frac{\partial E}{\partial u} = 0, \tag{5.247}$$

$$\text{remainder} : \qquad \frac{\partial D}{\partial t} + \frac{\partial E}{\partial x} = 0, \tag{5.248}$$

From (5.244) and (5.245) we find that $\Lambda = \Lambda(t, x)$. By differentiating (5.246) with respect to t, (5.247) with respect to x and (5.248) with respect to u we have the following:

$$\frac{\partial^2 \Lambda}{\partial t^2} + \frac{\partial^2 D}{\partial u \partial t} - \frac{\partial^2 C}{\partial x \partial t} - 2u\frac{\partial \Lambda}{\partial t} = 0, \tag{5.249}$$

$$\frac{\partial^2 C}{\partial t \partial x} - \frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial^2 E}{\partial u \partial x} = 0, \tag{5.250}$$

$$\frac{\partial^2 D}{\partial t \partial u} + \frac{\partial E}{\partial x \partial u} = 0. \tag{5.251}$$

By adding (5.249) and (5.250) we find

$$\frac{\partial^2 \Lambda}{\partial t^2} + \frac{\partial^2 D}{\partial u \partial t} - 2u\frac{\partial \Lambda}{\partial t} - \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 E}{\partial u \partial x} = 0, \tag{5.252}$$

which using (5.251) can be further reduced to

$$\frac{\partial^2 \Lambda}{\partial t^2} - 2u\frac{\partial \Lambda}{\partial t} - \frac{\partial^2 \lambda}{\partial x^2} = 0. \tag{5.253}$$

Since Λ is independent of u, separating (5.253) by powers of u leads to

$$\Lambda = \Lambda(x) \tag{5.254}$$

and

$$\frac{d^2 \Lambda}{dx^2} = 0. \tag{5.255}$$

Therefore from (5.255)

$$\Lambda = c_1 x + c_2, \tag{5.256}$$

where c_1 and c_2 are constants. We can now write from (5.240) and (5.241)

$$T^1(t, x, u, u_{x,} u_t) = (c_1 x + c_2)u_t + C(t, x, u)u_x + D(t, x, u) \tag{5.257}$$

and

$$T^2(t, x, u, u_{x,} u_t) = -(c_1 x + c_2)u_x - u_x C(t, x, u) + E(t, x, u) \tag{5.258}$$

We substitute for Λ in (5.246) and (5.247) and solve for D and E respectively. By integrating (5.246) and (5.247) with respect to u we obtain

$$D(t, x, u) = (c_1 x + c_2)u^2 + \frac{\partial Q(t, x, u)}{\partial x} + F(t, x), \tag{5.259}$$

and

$$E(t, x, u) = uc_1 - \frac{\partial Q}{\partial t}(t, x, u) + G(t, x), \tag{5.260}$$

where

$$Q(t,x,u) = \int^u C(t,x,u)du, . \tag{5.261}$$

Lastly we see by substituting (5.259) into (5.260) (5.248) that F and G must satisfy the condition

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = 0. \tag{5.262}$$

The conserved vectors (5.257) and (5.258)

$$T^1 = (c_1x + c_2)u_t + (c_1x + c_2)u^2 + T^1_*, \tag{5.263}$$

$$T^2 = -(c_1x + c_2)u_x + c_1u + T^2_*, \tag{5.264}$$

and

$$T^1_* = C(t,x,p)u_x + \frac{\partial Q}{\partial x}(t,x,u) + F(t,x) \tag{5.265}$$

$$T^2_* = -C(t,x,u)u_t + \frac{\partial Q}{\partial x}(t,x,u) + G(t,x) \tag{5.266}$$

and $Q(t,x,u)$ is defined in terms of $C(t,x,u)$ by (5.261). The components (5.265) and (5.266) have the same form as (5.175) and (5.176) with $B(t,x,u)$ replaced by $-C(t,x,u)$.
The conservation equation

$$D_tT^1_* + D_xT^2_* \equiv 0 \tag{5.267}$$

is identically satisfied and T^1_* and T^2_* are the components of a trivial conserved vector. The conserved vectors (5.173) and (5.174) derived by the direct method are again obtained.
We have shown that there are only two conserved vectors for the nonlinear wave equation (5.2) of the form $T^i = T^i(t,x,u,u_t,u_x)$ with multiplier $\Lambda(t,x,u)$ given by (5.232) and (5.233). The conserved vectors considered in the multiplier approach were a little more general than the conserved vectors (5.94) investigated in the direct method.

5.5 Concluding Remarks

The three methods considered, the direct method, the characteristic method and the partial Lagrangian method, all gave the same results for conservation laws for turbulent compressible flow in a long tunnel. The direct method and the characteristic method were done in an almost algorithmic fashion. Once a partial Lagrangian had been found, the partial Lagrangian method was straightforward and by using differential operators the conserved vector could be calculated in a systematic way.

For the nonlinear diffusion equation and non linear waver equation for the fluid velocity, two conservation laws were derived, one of which was the elementary conservation law. For the nonlinear diffusion equation for the pressure only the elementary conservation law was obtained. It is not clear why there was only one conservation law for the pressure under the same assumptions on the conserved vector, multiplier and gauge terms. It would be of interest to investigate further conservation laws for the nonlinear diffusion equation for the pressure.

In the direct method and the characteristic method conserved vectors which depended on at most first order partial derivatives were considered. The characteristic method was used to investigate conserved vectors which depended on first order partial derivatives with respect to both t and x, but the direct method could also have been used although it may have been longer computationally. In the characteristic method and the partial Lagrangian method the multiplier and the gauge terms did not depend on partial derivatives. Higher order conservation laws could be investigated by considering conserved vectors which depend on higher order spatial derivatives and multipliers and gauge terms which depend on partial derivatives. The analysis will be computationally laborious and may best be done with the aid of computer programs.

Chapter 6

Group invariant solutions for laminar and turbulent fluid fracture

6.1 Introduction

In this chapter we will briefly outline the derivation of the partial differential equation which models turbulent and laminar fluid driven fracturing in a rock in which the flow is incompressible. Lie point symmetries are derived and used to reduce the partial differential equation to an ordinary differential equation for which analytical solutions exist for two physical conditions governing the propagation of the fracture. The first solution results from assuming the total volume of the fluid in the fracture is constant and the second solution results from assuming that the speed at which the fracture propagates is constant. For general physical conditions at the fracture entrance the ordinary differential equation is solved numerically.

6.2 Mathematical formulation

We consider the two-dimensional model of turbulent fluid fracture as presented by Spence and Turcotte [9]. It is assumed that the crack is sufficiently thin so that the lubrication approximation applies,

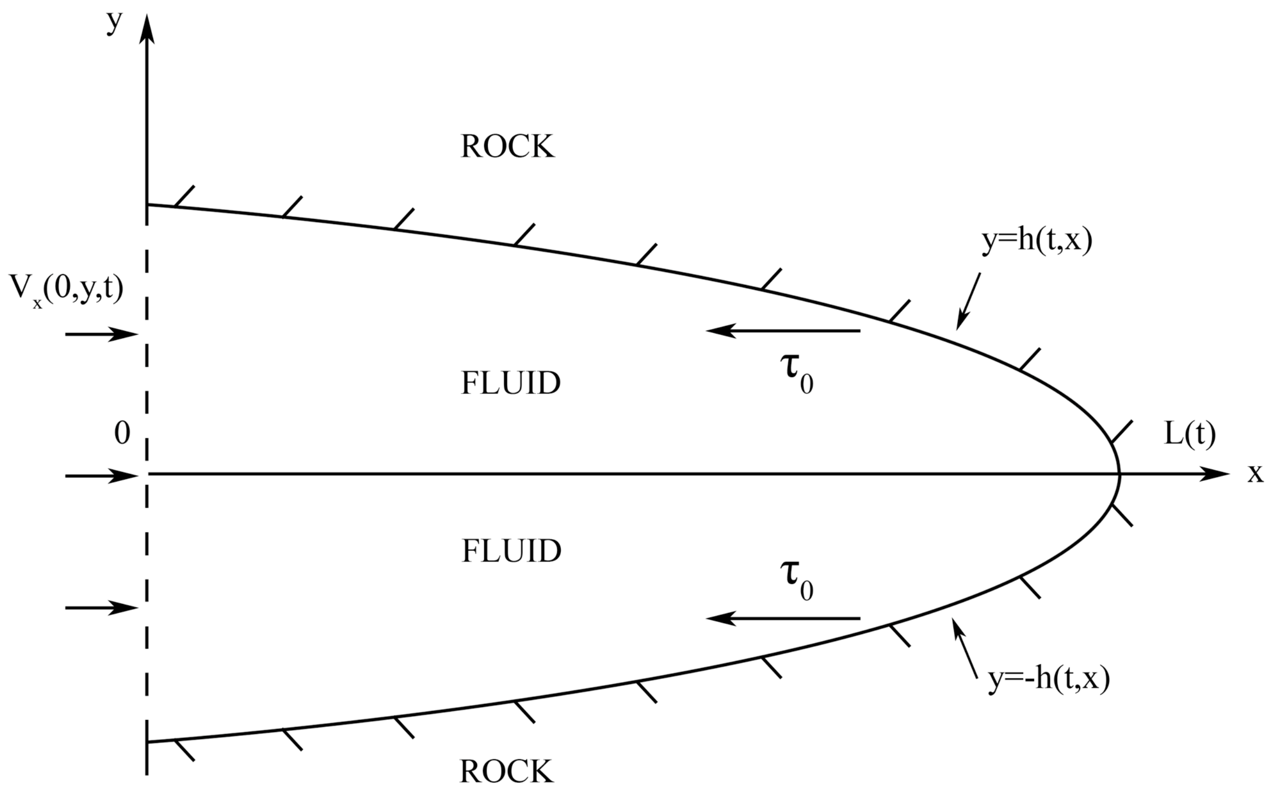


Figure 6.1: Fluid-driven fracture in impermeable rock

$$\frac{H}{L} << 1, \qquad Re\left(\frac{H}{L}\right)^2 << 1, \tag{6.1}$$

where H is the characteristic length in the across the fracture and L is the characteristic length along the fracture. We therefore neglect the inertia terms.

The fracture is illustrated in Figure 6.1. The rock is impermeable and therefore there is no leak-off of fluid into the surrounding rock. The x-axis is along the length of the fracture and the y-axis is in the direction of the width of the fracture. The fracture is symmetrical about the x-axis. The fluid-rock interface is $y = \pm h(t, x)$.

To model laminar and turbulent flows Spence and Turcotte (**??**) utilized the bulk flow model and the result

$$\frac{\tau_0}{\frac{1}{2}\rho\bar{u}^2} = n\left(\frac{2\rho h\bar{u}}{\eta}\right)^m, \tag{6.2}$$

where

- τ_0=wall shear stress
- $\bar{u}(x)$ = mean flow velocity averaged over the width of fracture
- ρ = density of the incompressible fluid
- η = coefficient of dynamic viscosity
- $h(t, x)$= half-width
- n,m = constants.

Empirical evidence indicates that for laminar flow $m = -1$, $n = \frac{1}{2}$.

When $m = -1$, (6.2) becomes

$$\tau_0 = \frac{n}{2}\eta\frac{\bar{u}}{2h} \tag{6.3}$$

The wall shear stress, τ_0, is linearly proportional to the mean velocity gradient $\frac{\bar{u}}{2h}$ as for laminar flow of a Newtonian fluid. The constant of proportionality depends on the velocity η. For smooth wall turbulent flow experimental results indicate that $m = -\frac{1}{4}, n = \frac{1}{15}$. The Reynolds number of the fluid in the fracture is

$$Re = \frac{\rho}{\eta} 2h\bar{u} \tag{6.4}$$

and (6.2) can be written as

$$\frac{\tau_0}{\frac{1}{2}\rho\bar{u}^2} = nRe^m) \tag{6.5}$$

When $m = -\frac{1}{4}$, (6.5) is the Blasius result for the wall shear stress in smooth pipe flow [24]. Emerman, Turcotte and Spence [25] argue that turbulent fow over a rough surface is described by setting $m = 0$ in (6.2). In Chapter 6 and 7 we will derive the results for general values of m. We will use $m = -1$ for laminar flow in the fracture, $m = -\frac{1}{4}$ for turbulent flow in a smooth wall fracture and $m = 0$ for turbulent flow in a rough wall fracture.

For turbulent flows $m = \frac{-1}{4}$, $n = \frac{1}{15}$ but other values can be considered.

We now derive a partial differential equation for the half-width of the fracture $h(t, x)$. We first average the momentum equation over the width of the fracture. The momentum balance equation is given by

$$\rho \frac{Dv_i}{Dt} = \tau_{ki,k}, \tag{6.6}$$

where $\frac{D}{Dt}$ denotes the material time derivative, τ_{ik} is the Cauchy stress tensor and the comma denotes partial differentiation. We assume that the fracture is sufficiently that the lubrication approximation can be applied and as a result we can neglect the inertia term $\rho \frac{Dv_i}{Dt}$. Therefore

$$\tau_{ki,k} = 0. \tag{6.7}$$

Consider the x-component of (6.7)

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0. \tag{6.8}$$

Since the flow is considered to be two-dimensional we have that $\frac{\partial}{\partial z} = 0$, and by operating with $\int_{-h(t,x)}^{h(t,x)} dy$ on equation (6.8) in order to average over the width of the fracture, we obtain

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} \tau_{xx} dy + \int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial y} \tau_{yx} dy = 0. \tag{6.9}$$

We now consider the first term of equation (6.9) and use the theorem for differentiation under the integral sign [21]:

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} \tau_{xx} dy = \frac{\partial}{\partial x} \int_{-h(t,x)}^{h(t,x)} \tau_{xx} dy - \tau(x,h) \frac{\partial h}{\partial x} + \tau(x,-h) \left(-\frac{\partial h}{\partial x} \right) = 0. \tag{6.10}$$

However $\frac{\partial h}{\partial x} = O\left(\frac{H}{L}\right)$ and neglecting terms of order $\frac{H}{L}$ we find that equation (6.10) becomes

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} \tau_{xx} dy = \frac{\partial}{\partial x} \int_{-h(t,x)}^{h(t,x)} \tau_{xx} dy. \tag{6.11}$$

However,

$$\tau_{xx} = -p + O\left(\frac{H}{L}\right)^2 \tag{6.12}$$

and therefore

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} \tau_{xx} dy = -\frac{\partial}{\partial x} \int_{-h(t,x)}^{h(t,x)} p(x,y) dy. \tag{6.13}$$

If we let $\bar{p}(t,x)$ denote the average pressure across the width of the fracture, then

$$\bar{p}(t,x) = \frac{1}{2h} \int_{-h(t,x)}^{h(t,x)} p(x,y,t) dy. \tag{6.14}$$

By substituting equation (6.14) into equation (6.13) we obtain the following result

$$\begin{aligned} \int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} \tau_{xx} dy &= -2h \frac{\partial \bar{p}}{\partial x} - 2\bar{p} \frac{\partial h}{\partial x} \\ &= -2h \frac{\partial \bar{p}}{\partial x} + O\left(\frac{H}{L}\right) \end{aligned} \tag{6.15}$$

Finally, consider now the second term in equation (6.9):

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial y}\tau_{yx} dy = [\tau_{yx}(x,y)]_{-h(t,x)}^{h(t,x)} = \tau_{xy}(x,h) - \tau(x,-h) \\ = -\tau_0 - (\tau_0) = -2\tau_0. \tag{6.16}$$

where from Figure 6.1

$$\tau_0 = -\tau_{yx}(x,h) = \tau_{yx}(x,-h). \tag{6.17}$$

We substitute (6.15) and (6.16) into equation (6.9) and neglect terms of order $\frac{H}{L}$. This gives

$$\tau_0 = -h\frac{\partial \bar{p}}{\partial x}. \tag{6.18}$$

Also assuming an incompressible fluid,

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{6.19}$$

Averaging equation (6.19) over the thickness of the fracture by operating with $\int_{-h(t,x)}^{h(t,x)} dy$ gives

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial v_x}{\partial x} dy + \int_{-h(t,x)}^{h(t,x)} \frac{\partial v_y}{\partial y} dy = 0. \tag{6.20}$$

We consider the first term in equation (6.20) and find that it can be written as

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial v_x}{\partial x} dy = \frac{\partial}{\partial x}\int_{-h(t,x)}^{h(t,x)} v_x(x,y) dy + O\left(\frac{H}{L}\right) \tag{6.21}$$

Denote by $\bar{v}_x(x,y)$ the average of $v_x(x,y)$ across the width of the fracture. Then

$$\bar{u} = \bar{v}_x = \frac{1}{2h}\int_{-h(t,x)}^{h(t,x)} v_x(x,y) dy. \tag{6.22}$$

From equations (6.21) and (6.22) we can see that, neglecting terms of order $\frac{H}{L}$,

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial}{\partial x} v_x(x,y) dy = \frac{\partial}{\partial x}\left(2h\bar{u}\right). \tag{6.23}$$

In addition, the second term of equation (6.20) gives

$$\begin{aligned}\int_{-h(t,x)}^{h(t,x)} \frac{\partial v_y(x,y)}{\partial y} dy &= [v_y]_{-h(t,x)}^{h(t,x)} \\ &= 2\frac{Dh}{Dt} \\ &= 2\left(\frac{\partial h}{\partial t} + v_x(x,h,t)\frac{\partial h}{\partial x}\right)\end{aligned} \tag{6.24}$$

and if we neglect the terms of $O\left(\frac{H}{L}\right)$ equation (6.24) reduces to

$$\int_{-h(t,x)}^{h(t,x)} \frac{\partial v_y(x,y)}{\partial y} dy = 2\frac{\partial h}{\partial t}. \tag{6.25}$$

Hence, substituting results from equations (6.23) and (6.25) into (6.20) we find that

$$\frac{\partial}{\partial x}(\bar{u}h) + \frac{\partial h}{\partial t} = 0. \tag{6.26}$$

To summarise, equations (6.2), (6.18) and (6.26) are three equations for the four quantities τ_0, h, $\bar{u}$ and p. A fourth equation will be added later. In order to obtain the partial differential equation fir $h(t,x)$ we substitute equation (6.2) into equation (6.18) and solve for $h\bar{u}$:

$$h\bar{u} = D\left(-h^3\frac{\partial p}{\partial x}\right)^{\frac{1}{m+2}} \tag{6.27}$$

where

$$D = \left(2^{1-m}\frac{1}{n}\eta^m\rho^{-(1+m)}\right)^{\frac{1}{m+2}} \tag{6.28}$$

We then substitute equation (6.27) into equation (6.26) to obtain

$$\frac{\partial h}{\partial t} = -D\frac{\partial}{\partial x}\left(\left(-h^3\frac{\partial p}{\partial x}\right)^{\frac{1}{m+2}}\right).. \tag{6.29}$$

It remains to relate $p(t,x)$ to $h(t,x)$. We will make the PKN approximation [13] [23] that the fluid pressure is proportional to the half-width $h(t,x)$, therefore,

$$p(t,x) = \Lambda h(t,x), \tag{6.30}$$

where Λ is a constant.

We have already used Λ to denote the multiplier in the characteristic method for the derivation of conservation laws. There should be no confusion in the notation because Λ in (6.30) will later be included in a rescaling of time. The PKN approximation is widely used in the oil and gas industry. Adachi and Peirce [12], have investigated the validity of the PKN approximation. They found that it applies in an outer region away from the fracture tips with a small correction term that depends on the second derivative of the half-width $h(t, x)$. Equation (6.30) is the fourth equation required to complete the system.

Substituting for $p(t, x)$ in equation (6.29) using (6.30)we finally obtain the non-linear diffusion equation for the half-width $h(t, x)$:

$$\frac{\partial h}{\partial t} = -D^* \frac{\partial}{\partial x} \left(\left(-h^3 \frac{\partial h}{\partial x} \right)^{\frac{1}{m+2}} \right), \tag{6.31}$$

where

$$D^* = D\Lambda^{\frac{1}{m+2}}. \tag{6.32}$$

The boundary condition at the fracture tip, $x = L(t)$, is

$$h(L(t), t) = 0. \tag{6.33}$$

The model stipulates that the rock has a pre-existing fracture. A further condition that may be specified is the initial fracture shape:

$$t = 0: \qquad h(x, 0) = h_0(x). \tag{6.34}$$

We will find later that an arbitrary initial fracture shape cannot be prescribed by similarity solution. Consider now the balance law for the total volume of the fluid in the fracture. Let $V(t)$ denote the total volume of the fracture per unitl length in the z-dircetion:

$$V(t) = 2 \int_0^{L(t)} h(t, x) dx. \tag{6.35}$$

Since the fluid is incompressible and the rock is impermeable, the time rate of increase of the total volume of the fracture per unit length in the z-direction is equal to the rate of flow of fluid into the fracture per unit length in the z-direction at $x = 0$:

$$\frac{dV}{dt} = 2h(0,t)\bar{u}(0,t). \tag{6.36}$$

Using equation (6.27) for $h(0,t)\bar{u}(0,t)$ and by substituting $p = \Lambda h(t,x)$ into equation (6.27), equation (6.36) becomes

$$\frac{dV}{dt} = 2D^* \left(-h^3(0,t)\frac{\partial h(0,t)}{\partial x} \right)^{\frac{1}{m+2}}. \tag{6.37}$$

Equation (6.37) gives the time rate of fluid injection into the fracture per unit length in the y-direction. The characteristic length has not yet been specified. We take as the characteristic length the initial of the fracture : Thus

$$t = 0: \qquad L(0) = 1, \tag{6.38}$$

$$t = 0: \qquad V(0) = V_0. \tag{6.39}$$

6.3 Lie point symmetries and group invariant solution

We will use a linear combination of the Lie point symmetries of the nonlinear diffusion equation

$$\frac{\partial h}{\partial t} = -D^* \frac{\partial}{\partial x} \left(\left(-h^3 \frac{\partial h}{\partial x} \right)^{\frac{1}{m+2}} \right), \tag{6.40}$$

to construct a group invariant solution for $h(t,x)$,$L(t)$, $p(t,x)$ and $V(t)$. The general procedure for the derivation of the Lie point symmetries and group invariant solution of a partial differential equation was described in Section 2.2. We will present here a brief summary of the derivation of the Lie point symmetries of (6.40). The outline of the derivation of the Lie point symmetries is presented in Appendix B.

The initial step in deriving the Lie point symmetries is to simplify the form of equation (6.40) by elminating D^* using the transformation, $t^* = D^* t$. The resulting equation is in terms of the independent variables x and t^*, however, for simplicity we drop the $*$ and write the nonlinear diffusion equation as

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(\left(-h^3\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}\right), \tag{6.41}$$

The Lie point symmetries of (6.41) are of the form

$$X = \xi^1(t,x,h)\frac{\partial}{\partial t} + \xi^2(t,x,h)\frac{\partial}{\partial x} + \eta(t,x,h)\frac{\partial}{\partial h}. \tag{6.42}$$

Since (6.41) depends on h_t,h_x and h_{xx} the second prolongation $X^{[2]}$ has to be considered:

$$\begin{aligned} X^{[2]} = {} & \xi^1(t,x,h)\frac{\partial}{\partial t} + \xi^2(t,x,h)\frac{\partial}{\partial x} + \eta(t,x,h)\frac{\partial}{\partial h} + \zeta_t(t,x,h)\frac{\partial}{\partial h_t} + \zeta_x(t,x,h)\frac{\partial}{\partial h_x} \\ & + \zeta_{xx}(t,x,h)\frac{\partial}{\partial h_{xx}}, \end{aligned} \tag{6.43}$$

where ζ_t, ζ_x and ζ_{xx} are defined in Appendix B. The determining equation is

$$X^{[2]}F|_{F=0} = 0 \tag{6.44}$$

where

$$\begin{aligned} F(h,h_t,h_x,h_{xx}) = & \frac{\partial h}{\partial t} + \frac{3k}{(m+2)}h^{\frac{1-m}{m+2}}\left(\frac{\partial h}{\partial x}\right)^{\frac{m+3}{m+2}} \\ + & \frac{k}{(m+2)}h^{\frac{3}{m+2}}\left(\frac{\partial h}{\partial x}\right)^{-\frac{1+m}{m+2}}\frac{\partial^2}{\partial x^2} \end{aligned} \tag{6.45}$$

and $k = (-1)^{\frac{1}{m+2}}$.

The Lie point symmetries are derived by separating the determining equation by powers and products of the derivatives of h and solving the resulting equations for the unknown functions ξ^1, ξ^2, and η. However, in the case of the nonlinear diffusion equation (6.41) it is sightly more complex as for special values of m, the powers of certain derivatives are equal. An example of such a case occurs when comparing the derivative terms $h_x^{\frac{2m+5}{m+2}}$ and h_x. The powers of these two terms are the same for $m = -3$, therefore when separating the determining equation according to powers of derivatives, the coefficients of the two terms are be grouped together for $m = -3$ and treated separately otherwise. In Appendix B we will present the derivation of Lie point symmetries for a general m, where $m \neq -2, 2, -1, -3$. The results for the laminar case ($m = -1$) have been derived by Fit et al [13]. Values for $m = 2, -3$ do not have any physical significance. We will quote the results for Lie point symmetry derivation for these values $m = -2, -3$.

The Lie point symmetries for special cases of m and the general case are presented in Table (6.1).

Table 6.1: Derived Lie point symmetries.

Case	Lie Point Symmetries
General	$X_1 = \frac{\partial}{\partial t}$
$m \neq 2$	$X_2 = t\frac{\partial}{\partial t} + (\frac{m+2}{m-2})h\frac{\partial}{\partial h}$
$m \neq -2$	$X_3 = x\frac{\partial}{\partial x} - (\frac{m+3}{m-2})h\frac{\partial}{\partial h}$
$m \neq -3$	$X_4 = \frac{\partial}{\partial x}$
$m = 2$	$X_1 = \frac{\partial}{\partial t}$
	$X_2 = 5t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial x}$
	$X_3 = h\frac{\partial}{\partial h}$
	$X_4 = \frac{\partial}{\partial x}$
$m = -3$	$X_1 = \frac{\partial}{\partial t}$
	$X_2 = t\frac{\partial}{\partial t} + \frac{1}{5}h\frac{\partial}{\partial h}$
	$X_3 = t^2\frac{\partial}{\partial t} + \frac{1}{25}(h^5 - 5t)x\frac{\partial}{\partial x} + \frac{2}{5}th\frac{\partial}{\partial h}$
	$X_4 = x\frac{\partial}{\partial x}$
	$X_5 = B(h,t)\frac{\partial}{\partial x}$
	where $B(t,h)$ satisfies
	$\frac{\partial B}{\partial t} + \frac{\partial}{\partial h}\left(\frac{1}{h^3}\frac{\partial B}{\partial h}\right) = 0$

We will now derive the group invariant solution for the general case where $m \neq 2, -2, -3$. Now $h(t,x) = \Phi(t,x)$ is a group invariant solution provided

$$X\left(h(t,x) - \Phi(t,x)\right)|_{h=\Phi} = 0, \tag{6.46}$$

where X is defined by

$$X = (c_1 + c_2 t)\frac{\partial}{\partial t} + (c_4 + c_3 x)\frac{\partial}{\partial x} - \left(c_2\frac{(m+2)}{(m-2)} - c_3\frac{(m+3)}{(m-2)}\right)h\frac{\partial}{\partial h}. \tag{6.47}$$

Equation (6.46) becomes a first order linear partial differential equation for $\Phi(t,x)$:

$$(c_1 + c_2)t\frac{\partial \Phi}{\partial t} + (c_3 x + c_4)\frac{\partial \Phi}{\partial x} = \left(c_2\frac{(m+2)}{(m-2)} - c_3\frac{(m+3)}{(m-2)}\right)\Phi. \tag{6.48}$$

The differential equations of the characteristic curves of equation (6.48) are

$$\frac{dt}{c_1 + c_2 t} = \frac{dx}{c_4 + c_3 x} = \frac{d\Phi}{\left(c_2\frac{(m+2)}{(m-2)} - c_3\frac{(m+3)}{(m-2)}\right)\Phi}. \tag{6.49}$$

It is rewritten equivalently as

$$\frac{dt}{c_1 + c_2 t} = \frac{dx}{c_4 + c_3 x}, \qquad \frac{dt}{c_1 + c_2 t} = \frac{d\Phi}{\left(c_2\frac{(m+2)}{(m-2)} - c_3\frac{(m+3)}{(m-2)}\right)\Phi}. \tag{6.50}$$

On integrating each of the differential equations in (6.50), one arrives at the following independent first integrals:

$$I_1 = \frac{c_4 + c_3 x}{(c_1 + c_2 t)^{\frac{c_3}{c_2}}}, \qquad I_2 = \frac{\Phi}{(c_1 + c_2 t)^{\left(\frac{(m+2)}{(m-2)} - \frac{(m+3)}{(m-2)}\frac{c_3}{c_2}\right)}}. \tag{6.51}$$

The constants I_1 and I_2 form a basis of invariants of (6.48) since they are independent. The general form of the solution of (6.48) is

$$I_2 = F(I_1), \tag{6.52}$$

where F is an arbitrary function. Hence

$$\Phi(t,x) = F(\xi)(c_1 + c_2 t)^{\left(\frac{(m+2)}{(m-2)} - \frac{(m+3)}{(m-2)}\frac{c_3}{c_2}\right)}, \tag{6.53}$$

where

$$\xi = \frac{c_4 + c_3 x}{(c_1 + c_2 t)^{\frac{c_3}{c_2}}}. \tag{6.54}$$

Since $\Phi = h(t, x)$, it follows that

$$h(t,x) = (c_1 + c_2 t)^{\frac{(m+2)}{(m-2)} - \frac{c_3}{c_2}\frac{(m+3)}{(m-2)}} F(\xi), \tag{6.55}$$

where $F(\xi)$ is an arbitrary function of ξ. We have obtained the general form of the group invariant solution for the fracture half-width, $h(t, x)$. We therefore substitute (6.55) into equation (6.41) to obtain a second order nonlinear ordinary differential equation for $F(\xi)$:

$$c_3^{\frac{1}{m+2}} \frac{d}{d\xi}\left(\left(-F^3(\xi)\frac{dF(\xi)}{d\xi}\right)^{\frac{1}{m+2}}\right) - \frac{d}{d\xi}\left(\xi F(\xi)\right) + \frac{m+2}{m-2}\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)F(\xi) = 0. \tag{6.56}$$

We observe that equation (6.56) does not depend on c_4. As a result we can choose $c_4 = 0$ in equation (6.54) so that $\xi = 0$ when $x = 0$.

Consider the next boundary condition (6.33),

$$h(L(t), t) = 0. \tag{6.57}$$

From equation (6.55), the boundary condition (6.56) becomes

$$F(A(t)) = 0, \tag{6.58}$$

where

$$A(t) = \frac{c_3 L(t)}{(c_1 + c_2 t)^{\frac{c_3}{c_2}}}. \tag{6.59}$$

In oder to determine $A(t)$, differentiate (6.58) with respect to t. This gives

$$\frac{dF}{d\xi}(A)\frac{dA(t)}{dt} = 0. \tag{6.60}$$

Since in general

$$\frac{dF}{d\xi}(A) \neq 0 \tag{6.61}$$

It follows that

$$\frac{dA(t)}{dt} = 0. \tag{6.62}$$

and therefore

$$A(t) = A_0. \tag{6.63}$$

where A_0 is constant. Thus

$$\frac{c_3 L(t)}{(c_1 + c_2 t)^{\frac{c_3}{c_2}}} = A_0, \tag{6.64}$$

and therefore

$$L(t) = \frac{A_0}{c_3}(c_1 + c_2 t)^{\frac{c_3}{c_2}}. \tag{6.65}$$

Imposing the initial condition, $L(0) = 1$, leads to

$$A = c_3 c_1^{\frac{-c_3}{c_2}}. \tag{6.66}$$

And therefore

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}}. \tag{6.67}$$

The boundary condition (6.58) becomes

$$F(c_3 c_1^{-\frac{c_3}{c_2}}) = 0 \tag{6.68}$$

and the similarity variable (6.53) is

$$\xi = c_3 c_1^{-\frac{c_3}{c_2}} \frac{x}{L(t)}. \tag{6.69}$$

Consider next the balance law for fluid volume, by substituting (6.55) into equation (6.35) and using (6.67) for $L(t)$ we obtain

$$V(t) = \frac{2}{c_3}\left(c_1 + c_2 t\right)^{\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}} \int_0^{c_3 c_1^{\frac{-c_3}{c_2}}} F(\xi)d\xi. \tag{6.70}$$

Differentiating (6.70) with respect to t gives

$$\frac{dV}{dt} = 2\frac{c_2}{c_3}\left(\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}\right)\left(c_1 + c_2 t\right)^{\frac{4}{m+2} - \frac{5}{m-2}\frac{c_3}{c_2}} \int_0^{c_3 c_1^{\frac{-c_3}{c_2}}} F(\xi)d\xi. \tag{6.71}$$

Next we consider equation (6.37), and apply let $t^* = D^* t$:

$$\frac{dV}{dt^*} = 2\left(-h^3(0,t^*)\frac{\partial h(0,t^*)}{\partial x}\right)^{\frac{1}{m+2}}. \tag{6.72}$$

For simplicity we will drop the superscript $*$ in equation (6.72) and substitute (6.55) into (6.72) to find a second expression for $\frac{dV}{dt}$:

$$\frac{dV}{dt} = 2\left(c_1 + c_2 t\right)^{\frac{4}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}} c_3^{\frac{1}{m+2}}\left(-F^3(0)\frac{dF(0)}{d\xi}\right)^{\frac{1}{m+2}}. \tag{6.73}$$

Equating (6.71) and (6.73) yields,

$$\left(-c_3 F^3(0)\frac{dF(0)}{d\xi}\right)^{\frac{1}{m+2}} = \frac{m+2}{m-2}\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)\int_0^{c_3 c_1^{\frac{-c_3}{c_2}}} F(\xi)d\xi. \tag{6.74}$$

Equation (6.74) is the balance equation for fluid volume in the fracture.

Lastly, from equation (6.70), the total volume $V(t)$ of the fracture per unit length in the y-direction can be expressed as

$$V(t) = V_0\left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}}, \tag{6.75}$$

where

$$V_0 = \frac{2}{c_3} c_1^{\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}} \int_0^{c_3 c_1^{\frac{-c_3}{c_2}}} F(\xi)d\xi. \tag{6.76}$$

A summary of the mathematical formulation is as follows: $m \neq -1$, $m \neq -2$, $m \neq 2$:

$$\frac{d}{d\xi}\left(\left(-c_3F^3(\xi)\frac{dF(\xi)}{d\xi}\right)^{\frac{1}{m+2}}\right)-\frac{d}{d\xi}\left(\xi F(\xi)\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)F(\xi)=0, \quad (6.77)$$

$$F\left(c_3c_1^{\frac{-c_3}{c_2}}\right)=0, \quad (6.78)$$

$$\left(-c_3F^3(0)\frac{dF(0)}{d\xi}\right)^{\frac{1}{m+2}}=\left(\frac{m+2}{m-2}\right)\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\int_0^{c_3c_1^{\frac{-c_3}{c_2}}}F(\xi)d\xi, \quad (6.79)$$

$$V_0=\frac{2}{c_3}c_1^{\frac{m+2}{m-2}-\frac{5}{m-2}\frac{c_3}{c_2}}\int_0^{c_3c_1^{\frac{-c_3}{c_2}}}F(\xi)d\xi, \quad (6.80)$$

$$\frac{c_2}{c_1}=\frac{c_2}{c_3}\frac{c_3}{c_1}, \quad (6.81)$$

$$V(t)=V_0\left(1+\frac{c_2}{c_1}t\right)^{\frac{m+2}{m-2}-\frac{5}{m-2}\frac{c_3}{c_2}}, \quad (6.82)$$

$$L(t)=\left(1+\frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}}, \quad (6.83)$$

$$h(t,x)=(c_1+c_2t)^{\frac{(m+2)}{(m-2)}-\frac{c_3}{c_2}\frac{(m+3)}{(m-2)}}F(\xi), \quad (6.84)$$

$$\xi=c_3c_1^{-\frac{c_3}{c_2}}\frac{x}{L(t)}. \quad (6.85)$$

We simplify equations (6.77) through to (6.85) by making a change of variables. We let

$$u=\frac{x}{L(t)}, \quad (6.86)$$

where $0 \leq u \leq 1$. Then

$$\xi = c_3 c_1^{-\frac{c_3}{c_2}} u \tag{6.87}$$

and we introduce $G(u)$ defined by,

$$F(\xi) = c_3^{-(\frac{m+2}{m-2})} c_1^{\frac{c_3}{c_2}(\frac{m+3}{m-2})} G(u). \tag{6.88}$$

Equations (6.77) and (6.85) are transformed from $(\xi, F(\xi))$ to $(u, G(u))$ as follows

$m \neq -3, m \neq -2, m \neq 2$:

$$\frac{d}{du}\left(\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}}\right) - \frac{d}{du}\left(uG(u)\right) + \left(\frac{m+2}{m-2}\right)\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)G(u) = 0, \tag{6.89}$$

$$G(1) = 0, \tag{6.90}$$

$$\left(-G^3(0)\frac{dG(0)}{du}\right)^{\frac{1}{m+2}} = \left(\frac{m+2}{m-2}\right)\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)\int_0^1 G(u)du, \tag{6.91}$$

$$V_0 = 2\left(\frac{c_1}{c_3}\right)^{\frac{m+2}{m-2}}\int_0^1 G(u)du, \tag{6.92}$$

$$\frac{c_2}{c_1} = \frac{c_2}{c_3}\frac{c_3}{c_1}, \tag{6.93}$$

$$V(t) = V_0\left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}}, \tag{6.94}$$

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}}, \tag{6.95}$$

$$h(t,x) = \left(\frac{c_1}{c_3}\right)^{\frac{m+2}{m-2}}\left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{m-2} - \frac{c_3}{c_2}\frac{m+3}{m-2}} G(u), \tag{6.96}$$

$$u = \frac{x}{L(t)}. \tag{6.97}$$

Obtaining the solution to the problem can be found in an algorithmic manner. The first step is to choose values for $\frac{c_2}{c_3}$, V_0 and m. Next, solve (6.89) subject to the boundary conditions

(6.90) and (6.91). Using (6.92) one can calculate $\frac{c_3}{c_1}$ and from (6.93) the value of $\frac{c_2}{c_1}$. Lastly, $V(t)$, $L(t)$, and $h(t, x)$ follow from (6.94), (6.95) and (6.96), respectively.

6.4 Special values for the ratios $\frac{c_3}{c_2}$

We will now investigate the effect of imposing various physical conditions on the fluid-driven fracture, in particular the effect on the ratio $\frac{c_3}{c_2}$. From empirical studies it has been shown that for laminar flow $m = -1$, for smooth wall turbulent flow $m = -\frac{1}{4}$ and for rough wall turbulent flow $m = 0$. For certain physical conditions $\frac{c_3}{c_2}$ will depend on m and as a result will vary according to the different flow conditions.

6.4.1 Length of the fracture

The length of the fracture is given by equation (6.83) and $L(t) \to 1$ as $\frac{c_2}{c_3} \to 0$. The rate at which the length of the fracture grows is given by

$$\frac{dL}{dt} = \frac{c_3}{c_1}\left(1 + \frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}-1}. \tag{6.98}$$

If

$$\frac{c_3}{c_2} = 1. \tag{6.99}$$

then the speed of propagation of the fracture is constant which is independent of m.

6.4.2 Total volume of the fracture

The total volume of the fracture per unit length in the z-direction, $V(t)$, is given by (6.94). The total volume of the fracture remains constant if

$$\frac{c_3}{c_2} = \frac{m+2}{5}. \tag{6.100}$$

We have that

$$\frac{dV}{dt} = V_0 \left(\frac{c_2}{c_1}\frac{m+2}{m-2} - \frac{5}{m-2}\frac{c_3}{c_1} \right) \left(1 + \frac{c_2}{c_1} t \right)^{\frac{4}{m-2} - \frac{5}{m-2}\frac{c_3}{c_2}} \tag{6.101}$$

The rate of change of total volume of the fracture is constant if

$$\frac{c_3}{c_2} = \frac{4}{5}, \tag{6.102}$$

which is independent of m.

6.4.3 Pressure at the fracture entry

From the PKN model

$$p(t,x) = \Lambda h(t,x). \tag{6.103}$$

Thus using (6.84) and (6.103) the pressure at the fracture entry, $x = 0$, is

$$p(0,t) = \Lambda \frac{c_1}{c_3}^{\frac{m+2}{m-2}} (1 + \frac{c_2}{c_1} t)^{\frac{(m+2)}{(m-2)} - \frac{c_3}{c_2}\frac{(m+3)}{(m-2)}} G(0). \tag{6.104}$$

The pressure at the entry to the fracture therefore remains constant if

$$\frac{c_3}{c_2} = \frac{m+2}{m+3}. \tag{6.105}$$

6.4.4 Rate of working of the pressure at the fracture entry

The rate of working of the pressure at the fracture entry per unit length in the y-direction, $W(t)$, is

$$W(t) = p(0,t)\frac{dV}{dt}. \tag{6.106}$$

Thus from (6.101) and (6.104) we obtain

$$W(t) = \Lambda V_0 \left(\frac{c_1}{c_3} \right)^{\frac{m+2}{m-2}} \left[\left(\frac{m+2}{m-2} \right) \frac{c_2}{c_1} - \frac{5}{m-2}\frac{c_3}{c_1} \right] \left(1 + \frac{c_2}{c_1} t \right)^{\frac{m+6}{m+2} - \left(\frac{m+8}{m-2} \right) \frac{c_3}{c_2}}. \tag{6.107}$$

Thus the rate of working of the pressure at the entry per unit length in the z-direction is constant if

$$\frac{c_3}{c_2} = \frac{m+6}{m+8}. \tag{6.108}$$

Table 6.2: Physical significance of values of $\frac{c_3}{c_2}$.

$\frac{c_3}{c_2}$ for physical cases	Laminar $m=-1$	Smooth wall turbulent $m=-\frac{1}{4}$	Rough wall turbulent $m=0$	General
Total volume of fluid in fracture is constant.	0.2	0.35	0.4	$\frac{c_3}{c_2}=\frac{m+3}{5}$
Pressure at fracture entry is constant.	0.5	0.636	0.66	$\frac{c_3}{c_2}=\frac{m+2}{m+3}$
Rate of working of pressure at fracture entry is constant.	0.714	0.742	0.75	$\frac{c_3}{c_2}=\frac{m+6}{m+8}$
Rate of change of total volume of fracture is constant.	0.8	0.8	0.8	$\frac{c_3}{c_2}=\frac{4}{5}$
Speed of propagation of fracture is constant.	1	1	1	$\frac{c_3}{c_2}=1$

The results derived in this section and corresponding values of $\frac{c_3}{c_2}$ for laminar smooth, turbulent smooth wall and rough wall turbulent flow are presented in the Table 6.2.

6.5 Solution to boundary value problem

In this section we will derive the solution to the boundary value problem presented by equations (6.89) through to (6.97) for two special cases of $\frac{c_3}{c_2}$. The first case that will be considered when the total volume of fluid in the fracture is constant. In the second case we will investigate the solution when the speed at which the fracture propagates is constant.

6.5.1 Special solution 1: Constant volume of the fracture

If the total volume of fluid in the fracture is constant,

$$\frac{c_3}{c_2} = \frac{m+2}{5}. \tag{6.109}$$

From (6.109) we see that the differential equation (6.89) reduces to

$$\frac{d}{du}\left(\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}}\right) - \frac{d}{du}\left(uG(u)\right) = 0, \tag{6.110}$$

subject to the following boundary conditions (6.90) and (6.91):

$$G(1) = 0, \tag{6.111}$$

$$\left(-G^3(0)\frac{dG(0)}{du}\right)^{\frac{1}{m+2}} = 0. \tag{6.112}$$

By integrating equation (6.110) with respect to u we obtain

$$\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}} - uG(u) = A, \tag{6.113}$$

where A is an arbitrary constant. Imposing the boundary condition (6.112) at $u = 0$ gives $A = 0$. Equation (6.113) becomes the separable differential equation

$$G^{1-m} dG = -u^{m+2} du. \tag{6.114}$$

We solve the resulting equation

$$\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}} - uG(u) = 0 \tag{6.115}$$

Thus since $m \neq 2$ and $m \neq = 3$

$$G^{2-m} = B - \left(\frac{2-m}{m+3}\right) u^{m+3,} \tag{6.116}$$

where B is an arbitrary constant. Imposing the boundary (6.111), we find that

$$B = \frac{2-m}{m+3.} \tag{6.117}$$

and therefore for $m \neq 2$ and $m \neq 3$,

$$G(u) = \left(\frac{2-m}{m+3}\right)^{\frac{1}{2-m}} \left(1 - u^{m+3}\right)^{\frac{1}{2-m}}. \tag{6.118}$$

In order to investigate if boundary condition (refpup5) is satisfied we calculate using (6.118)

$$\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}} = \left(\frac{2-m}{3+m}\right)^{\frac{1}{2-m}} u \left(1 - u^{m+3}\right)^{\frac{1}{2-m}} \tag{6.119}$$

We see that boundary condition (6.112) is satisfied provided $-3 < m < 2$. The remaining quantities can now be calculated.

Substituting equation (6.118) into (6.92) gives

$$\frac{c_3}{c_1} = \left(\frac{V_0}{2I(m)}\right)^{\frac{2-m}{m+2}} \left(\frac{m+3}{2-m}\right)^{\frac{1}{m+2}}, \tag{6.120}$$

where

$$I(m) = \int_0^1 \left(1 - u^{m+3}\right)^{\frac{1}{2-m}} du \tag{6.121}$$

and therefore from (6.93)

$$\frac{c_2}{c_1} = \frac{5}{m+2} \left(\frac{V_0}{2I(m)}\right)^{\frac{2-m}{m+2}} \left(\frac{m+3}{2-m}\right)^{\frac{1}{m+2}}. \tag{6.122}$$

From equations (6.94) to (6.96)

$$V(t) = V_0, \tag{6.123}$$

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{5}}, \tag{6.124}$$

$$h(t,x) = \frac{V_0}{2I(m)L(t)}\left(1 - u^{m+3}\right)^{\frac{1}{2-m}}, \tag{6.125}$$

The results for this special case are summarized as follows: $-3 < m < 2$

$$G(u) = \left(\frac{2-m}{m+3}\right)^{\frac{1}{2-m}}\left(1 - u^{m+3}\right)^{\frac{1}{2-m}}. \tag{6.126}$$

$$\frac{c_3}{c_2} = \frac{m+2}{5}, \tag{6.127}$$

$$\frac{c_2}{c_1} = \frac{5}{m+2}\left(\frac{3+m}{2-m}\right)^{\frac{1}{2+m}}\left[\frac{V_0}{2I(m)}\right]^{\frac{2-m}{2+m}} \tag{6.128}$$

$$I(m) = \int_0^1 \left[1 - u^{m+3}\right]^{\frac{1}{2-m}} du \tag{6.129}$$

$$V = V_0 \tag{6.130}$$

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{5}} \tag{6.131}$$

$$h(t,x) = \frac{V_0}{2L(t)}\left[1 - u^{m+3}\right]^{\frac{1}{2-m}} \tag{6.132}$$

where

$$u = \frac{x}{L(t)} \qquad 0 \le u \le 1 \tag{6.133}$$

6.5.2 Special solution 2: Constant speed of propagation of fracture

We look for a solution of (6.89) for $G(u)$ of the form

$$G(u) = A(1-u)^n, \tag{6.134}$$

Where A and n are constants to be determinded. It can be verified by substituting (6.134) into (6.89) that

$$\begin{aligned} &\frac{4n-1}{m+2}\left(nA^4\right)^{\frac{1}{m+2}}(1-u)^{\frac{4n-m-3}{m+2}} + nA\,(1-u)^{n-1} \\ &+\frac{m+2}{m-2}\left(\frac{c_2}{c_3} - \frac{5}{m+2} - \left(\frac{m-2}{m+2}\right)(1+n)\right)A(1-u)^n = 0 \end{aligned} \tag{6.135}$$

Equating the powers of $(1-u)$ of the first two terms in (6.135) and solving for n we obtain

$$n = \frac{1}{2-m}. \tag{6.136}$$

The constant A can be found by equating the first two terms of (6.135) and substituting for n, which gives

$$A = (2-m)^{\frac{1}{2-m}}. \tag{6.137}$$

provided $m < 2$. The remaining term we find that

$$\frac{c_3}{c_2} = 1, \tag{6.138}$$

which is independent of m. Thus the solution can be expressed as

$$G(u) = (2-m)^{\frac{1}{2-m}}(1-u)^{\frac{1}{2-m}}. \tag{6.139}$$

The boundary condition (6.91) was not used in the derivation of (6.139) satisfies the boundary condition (6.105). The solution is completed as in Special Case I. The integral in (6.92) can be evaluated for the Special Case II.

The results are summarised as follows

$$m \neq -3, \quad m \neq -2, \quad m \neq 2 \tag{6.140}$$

$$G(u) = (2-m)^{\frac{1}{2-m}} (1-u)^{\frac{1}{2-m}} \tag{6.141}$$

$$\frac{c_3}{c_1} = \frac{c_2}{c_1} = \left(\frac{3-m}{2}V_0\right)^{\frac{2-m}{2+m}} (2-m)^{\frac{m-3}{m+2}}, \tag{6.142}$$

$$\frac{c_3}{c_2} = 1 \tag{6.143}$$

$$V(t) = V_0\left(1 + \frac{c_2}{c_1}t\right)^{\frac{3-m}{2-m}}, \tag{6.144}$$

$$L(t) = 1 + \frac{c_2}{c_1}t, \tag{6.145}$$

$$h(t,x) = \frac{1}{2}\left(\frac{3-m}{2-m}\right) V_0 L(t)^{\frac{1}{2-m}} (1-u)^{\frac{1}{2-m}} \tag{6.146}$$

$$u = \frac{x}{L(t)} \qquad 0 \leq u \leq 1. \tag{6.147}$$

In the next chapter we will investigate the properties of the two special solutions by looking at different values of m and their related physical significance.

6.6 Concluding remarks

We were able to reduce the problem of a fluid-driven fracture to a boundary value problem for a second order ordinary differential equation. Laminar and turbulent fluid flow in the fracture could be treated simultaneously. We were able to find two exact analytical solutions. The general case will have to be treated numerically and this will be considered in the next chapter. The two exact analytical solutions will be a useful check on the numerical methods.

The results of this chapter for laminar in the fracture agree with those derived by Fitt et al [13].

Chapter 7

Numerical solution to the boundary value problem of laminar and turbulent fluid fracture

In this chapter we will present a numerical solution to the boundary value problem presented by equations (6.89) through to (6.97). First we will transform the boundary value problem into two initial value problems. It has been shown that the invariance of a boundary value problem for an ordinary differential equation under a scaling transformation allows the boundary value problem to be transformed to two initial value problems which makes the process of obtaining a numerical solution easier [18] [19]. The boundary value problem will be solved for values of m that describe laminar flow, smooth wall turbulent and rough wall turbulent flow. Under each of these flow conditions we will examine the profiles of the length of the fracture ($L(t)$), and the half-width of the fracture ($h(t, x,)$) for the cases of $\frac{c_3}{c_2}$ that have physical significance. The values of $\frac{c_3}{c_2}$ under consideration are presented in Table (6.2). We will also compare the numerical solution with the analytical solutions obtained for the cases when the total volume of the the fluid in the fracture is constant and when the speed at which the fracture propagates is constant.

7.1 Conversion of boundary value problem into two initial value problems.

The boundary value problem is to solve the ordinary differential equation

$$\frac{d}{du}\left(\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}}\right)-\frac{d}{du}\left(uG(u)\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)G(u)=0, \tag{7.1}$$

subject to the boundary conditions;

$$G(1)=0, \tag{7.2}$$

and

$$\left(-G^3(0)\frac{dG(0)}{du}\right)^{\frac{1}{m+2}}=\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\int_0^1 G(u)du. \tag{7.3}$$

Consider the scaling transformation

$$\bar{u}=\lambda^a u, \qquad \bar{G}=\lambda^b G \tag{7.4}$$

Then equation (7.1) transforms to

$$\lambda^{\frac{a-4b}{m+2}+a+b}\frac{d}{d\bar{u}}\left(\left(-\bar{G}^3\frac{d\bar{G}}{d\bar{u}}\right)^{\frac{1}{m+2}}\right)-\frac{d}{d\bar{u}}\left(\bar{u}\bar{G}\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\bar{G}=0. \tag{7.5}$$

Thus the differential equation (7.5) is from invariant under the transformation provided (7.4)

$$b=-a\left(\frac{m+3}{m-2}\right). \tag{7.6}$$

Therefore by replacing λ by λ^a we have

$$\bar{u}=\lambda u \qquad \bar{G}(\bar{u})=\lambda^{-\frac{m+3}{m-2}}G(u) \tag{7.7}$$

and substituting (7.7) into the ordinary differential equation (7.1) transforms it to

$$\frac{d}{d\bar{u}}\left(\left(-\bar{G}^3\frac{d\bar{G}}{d\bar{u}}\right)^{\frac{1}{m+2}}\right)-\frac{d}{d\bar{u}}\left(\bar{u}\bar{G}\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\bar{G}=0, \tag{7.8}$$

subject to the boundary conditions

$$\bar{G}(\lambda)=0, \tag{7.9}$$

$$\left[-\bar{G}^3(0)\frac{d\bar{G}(0)}{d\bar{u}}\right]^{\frac{1}{m+2}}=\frac{m+2}{m-2}\left[\frac{c_2}{c_3}-\frac{5}{m+2}\right]\int_0^\lambda \bar{G}(\bar{u})d\bar{u}. \tag{7.10}$$

We choose

$$\bar{G}(0)=1. \tag{7.11}$$

Then (7.10) becomes

$$\left[-\frac{d\bar{G}(0)}{d\bar{u}}\right]^{\frac{1}{m+2}}=\frac{m+2}{m-2}\left[\frac{c_2}{c_3}-\frac{5}{m+2}\right]\int_0^\lambda \bar{G}(\bar{u})d\bar{u}. \tag{7.12}$$

Also, since $\bar{G}(0)=1$ it follows from (7.7) that

$$G(0)=\lambda^{\frac{m+3}{-m-2}} \tag{7.13}$$

and (7.3) becomes

$$\left[-\frac{dG(0)}{d\bar{u}}\right]^{\frac{1}{m+2}}=\lambda^{\frac{-3(m+3)}{(m-2)(m+2)}}\left(\frac{m+2}{m-2}\right)\left[\frac{c_2}{c_3}-\frac{5}{m+2}\right]\int_0^1 G(u)du, \tag{7.14}$$

where λ is obtained from (7.9).

We now state the two initial value problems:

Initial Value Problem 1:

$$\frac{d}{d\bar{u}}\left(\left(-\bar{G}^3(\bar{u})\frac{d\bar{G}(\bar{u})}{d\bar{u}}\right)^{\frac{1}{m+2}}\right)-\frac{d}{d\bar{u}}\left(\bar{u}\bar{G}(\bar{u})\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\bar{G}(\bar{u})=0, \tag{7.15}$$

$$\bar{G}(0)=1, \tag{7.16}$$

$$\left[-\frac{d\bar{G}(0)}{d\bar{u}}\right]^{\frac{1}{m+2}} = \frac{m+2}{m-2}\left[\frac{c_2}{c_3} - \frac{5}{m+2}\right]\int_0^{\lambda} \bar{G}(\bar{u})d\bar{u}, \tag{7.17}$$

where λ is obtained from

$$\bar{G}(\lambda) = 0. \tag{7.18}$$

Initial Value Problem 2:

$$\frac{d}{du}\left(\left(-G^3(u)\frac{dG(u)}{du}\right)^{\frac{1}{m+2}}\right)-\frac{d}{du}\left(uG(u)\right)+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)G(u)=0, \quad (7.19)$$

$$G(0)=\lambda^{\frac{m+3}{m-2}}, \quad (7.20)$$

$$\left(-\frac{dG(0)}{du}\right)^{\frac{1}{m+2}}=\lambda^{-3\frac{m+3}{(m-2)(m+2)}}\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)\int_0^1 G(u)du. \quad (7.21)$$

In the next section we will give a brief outline of the derivation of the asymptotic solution to boundary value problem (7.1) to (7.3) as $u \to 1$ because it will be needed to solve the initial value problems numerically.

7.2 Asymptotic solution

We look for an asymptotic solution of the form

$$G(u)=A(1-u)^p, \quad (7.22)$$

as $u \to 1$ where A and p are constants to be determined. The boundary condition (7.2) is satisfied by (7.22) when $u=1$.

Substituting (7.22) into (7.1) we obtain

$$\begin{aligned}&-\frac{(4p-1)}{(m+2)}p^{\frac{1}{m+2}}A^{\frac{4}{m+2}}(1-u)^{\frac{4p-m-3}{m+2}}-(p+1)A(1-u)^p\\&+pA(1-u)^{p-1}+\frac{m+2}{m-2}\left(\frac{c_2}{c_3}-\frac{5}{m+2}\right)A(1-u)^p=0. \quad (7.23)\end{aligned}$$

If we equate the exponents of $(1-u)$ in the first and third terms in (7.23)we find that

$$\frac{4p-m-3}{m+2}=p-1, \quad (7.24)$$

and therefore that

$$p=\frac{1}{2-m}. \quad (7.25)$$

If we let $p = \frac{1}{2-m}$ in each term in (7.23) and ensure that all exponents $(1-u)$ are expressed in terms of p we have

$$-(2-m)^{-\frac{(m+3)}{(m+2)}} A^{\frac{4}{m+2}} (1-u)^{p-1} + \frac{A}{(2-m)}(1-u)^{p-1} + A\frac{(m+2)}{(m-2)}\left(\frac{c_2}{c_3}-1\right)(1-u)^p = 0, \tag{7.26}$$

which after multiplying through by $(1-u)^{1-p}$ can be written as

$$-(2-m)^{-\frac{m+3}{m+2}} A^{\frac{4}{m+2}} + \frac{A}{2-m} + A\frac{(m+2)}{(m-2)}\left(\frac{c_2}{c_3}-1\right)(1-u). \tag{7.27}$$

If we let $u \to 1$, then (7.27) becomes

$$-(2-m)^{(\frac{m+3}{m-2})} A^{\frac{4}{m+2}} + \frac{A}{2-m} = 0, \tag{7.28}$$

which leads to the result

$$A = (2-m)^{\frac{1}{m+2}}. \tag{7.29}$$

Thus

$$G(u) \sim (2-m)^{\frac{1}{2-m}}(1-u)^{\frac{1}{2-m}} \quad \text{as} \quad u \to 1. \tag{7.30}$$

This asymptotic result is valid for all $\frac{c_2}{c_3}$ because as $u \to 1$ the term in (7.27) which depends on $\frac{c_2}{c_3}$ vanishes.

The asymptotic result (7.30) can be used to show that the average velocity of the fluid at the fracture tip, $x = L(t)$, equals the velocity of the fracture tip. From (6.27)

$$h\bar{u} = D\left(-h^3\frac{\partial p}{\partial x}\right)^{\frac{1}{m+2}} \tag{7.31}$$

and using the scaled time $t^* = D^* t$ and suppressing the star it follows that

$$\bar{u}(t,x) = h^{\frac{1-m}{2-m}}\left(-\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}. \tag{7.32}$$

But from (6.96)

$$h(t,x) = \left(\frac{c_1}{c_3}\right)^{\frac{m+2}{m-2}} \left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{m-1} - \frac{(m+3)}{(m-2)}\frac{c_3}{c_2}} G(u) \tag{7.33}$$

and it follows that

$$h(t,x) \sim \left(\frac{c_1}{c_3}\right)^{\frac{m+2}{m-2}} \left(1 + \frac{c_2}{c_1}t\right)^{\frac{m+2}{m-1} - \frac{(m+3)}{(m-2)}\frac{c_3}{c_2}} (2-m)^{\frac{1}{2-m}} (1-u)^{\frac{1}{2-m}}, \tag{7.34}$$

as $u \to 1$. Using (7.32) and (7.34) it can be shown that in the limit as $u \to 1$,

$$\bar{u}(t, L(t)) = \frac{c_3}{c_1}\left(1 + \frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}-1}. \tag{7.35}$$

Since from (6.95),

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{\frac{c_3}{c_2}} \tag{7.36}$$

we see that

$$\bar{u}(t, L(t)) = \frac{dL}{dt}. \tag{7.37}$$

There is therefore no fluid lag in the fracture and the average velocity of the fluid in the fracture at the fracture tip does not exceed the velocity of propagation of the fracture.

7.3 Numerical solution

The algorithm used to obtain the numerical results is as follows. We successively use the backward and forward shooting method to successively solve initial value problem 1 (IVP1) and initial value problem 2 (IVP2).

STEP 1

We solve IVP1 using the ordinary differential equation solver function ODE113 with MATLAB. We use a star to refer to the variables of the backward shooting method. λ^* refers to the approximation of λ at the end of STEP 1. In order to evaluate $\bar{G}$ in STEP 2, we first need to have an approximation of the slope of $\bar{G}$ at $x = 0$: $y_1(0) = \frac{d\bar{G}}{d\bar{u}} = \bar{G}'(0)$. To do this, a recursive algorithm is used to find λ^*, until the condition $|\bar{G}(0) - 1| < \varepsilon_1$ is met. In our case, $\varepsilon_1 = 10^{-5}$, which gave us satisfying results.

Due to the singularity for $\bar{u} = \lambda^*$, the integration of IVP1 is done backwards over the interval $[\lambda^*, 0]$. The initial conditions are obtained from the asymptotic solution of $\bar{G}(\bar{u})$ and $y_1(\bar{u}) = \bar{G}'(\bar{u})$ in a ε-neighborhood of λ. By using the transformation (7.7) and the asymptotic solution (7.30) it can be shown that

$$\bar{G}(\bar{u}) \sim (2-m)^{\frac{1}{2-m}} \lambda^{\frac{2+m}{2-m}} (\lambda - \bar{u})^{\frac{1}{2-m}} = A_\lambda (\lambda - \bar{u})^n \quad \text{as} \quad \bar{u} \to \lambda, \tag{7.38}$$

where

$$A_\lambda = A(\lambda) = (2-m)^{\frac{1}{2-m}} \lambda^{\frac{2+m}{2-m}}, \qquad n = \frac{1}{2-m}. \tag{7.39}$$

We now solve IVP1 backwards on the interval $[\lambda^* - \varepsilon, 0]$ with:

$$\bar{G}(\lambda^* - \varepsilon) = A(\lambda^*)\varepsilon^n, \qquad y_1(\lambda^* - \varepsilon) = A(\lambda^*) n \varepsilon^{n-1}. \tag{7.40}$$

In order to obtain a rapid convergence of $\bar{G}$ a bisection algorithm was used on λ^*. Examination of IVP1 shows that $\bar{G}(0)$ is an increasing function of λ^*. We set a and b as an interval of confidence such that λ^* is in [a,b], for instance [0,2]. We then compute $m = \frac{a+b}{2}$ by solving IVP1 with the initial conditions derived previously over the interval $[m - \varepsilon, 0]$; if $\bar{G}(0) > 1$ we set $b = m$, otherwise we set $a = m$. Using k as the algorithm step, we are confident that $|\bar{G}_{k+1}(0) - 1| < |\bar{G}_k(0) - 1|$. In this method, we know that the interval in which λ^* lies is divided by 2 at each step. From this first step we retain the slope of $\bar{G}$ at $x = 0$: $y_1^*(0) = \bar{G}'(0)$. We now have an approximation of $\bar{G}$ but the condition $|\bar{\beta}_1 - \bar{\beta}_2| < \varepsilon$ is not necessarily met, where

$$\bar{\beta}_1 = (-y_1(0))^{\frac{1}{m+2}}, \tag{7.41}$$

$$\bar{\beta}_2 = \left(\frac{m+2}{m-2}\right)\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)\int_0^{\lambda^*} \bar{G}(\bar{u})\,d\bar{u}. \tag{7.42}$$

STEP 2

In order to find $\bar{G}$ and λ such that $|\bar{\beta}_1 - \bar{\beta}_2| < \varepsilon$, we now use the forward shooting method, that is, starting from $\bar{u} = 0$ with the initial conditions $\bar{G}(0) = 1$ and $y_1(0) = K_1$ we make the parameter K_1 vary to obtain $\bar{G}$.
A brief description of the bisection method is as follows:

- Set K_{1a} and K_{1b} as an interval of confidence such that the solution of IVP1 $\bar{G}$ verifies $K_{1a} < \bar{G}'(0) < K_{1b}$, with $\bar{G}' = y_1$.
- Choose $[y_1^*(0) - \varepsilon_2, y_1^*(0) + \varepsilon_2]$ where $y_1^*(0)$ is obtained in STEP 1 and $\varepsilon_2 = 1$.
- Compute $K_{1m} = \frac{K_{1a}+K_{1b}}{2}$.
- While $|\bar{\beta}_1 - \bar{\beta}_2| > \varepsilon$, we solve IPV1 over the interval $[0, \lambda]$ where $\bar{G}(\lambda) = 0$ with the initial conditions $\bar{G}(0) = 1$ and $y_1(0) = K_{1m}$.
- If $\bar{\beta}_1 - \bar{\beta}_2 > 0$ then set $K_{1b} = K_{1m}$ otherwise we choose $K_{1a} = K_{1m}$.
  Indeed the bisection method also works here since $K_1 \to \beta_1$ is a positive increasing function of K_1 and $K_1 \to \beta_2$ is a decreasing function of K_1. As a result, $K_1 \to \bar{\beta}_1 - \bar{\beta}_2$ is an increasing function of K_1.

With STEP 2 we have obtained $\bar{G}$ and λ, where $\bar{G}(\lambda) = 0$.

STEP 3

We now solve IVP2 in a similar fashion. Using the same asymptotic representation and initial conditions, "shooting backwards", we find λ^* with the bisection method to find G(u) such that $|G(0) - \lambda^{*\frac{m+3}{m-2}}| < \varepsilon_1$. We then retain $y_2^*(0)$ for STEP 4. We expect to find λ^* to be close to 1.

STEP 4

We finish solving IVP2 using the bisection method described in step 2, but now over the interval $[y_2^*(0) - \varepsilon_2, y_2^*(0) + \varepsilon_2]$. The latter gives us G such that $|\beta_1 - \beta_2| < \varepsilon_3$ where:

$$\beta_1 = (-y_2(0))^{\frac{1}{m+2}}\,, \tag{7.43}$$

$$\beta_2 = \lambda^{-\frac{3(m+3)}{(m-2)(m+2)}}\left(\frac{m+2}{m-2}\right)\left(\frac{c_2}{c_3} - \frac{5}{m+2}\right)\int_0^1 G(u)du. \tag{7.44}$$

From this last step, we now have G(u).

RESULTS

We then compute the ratio $\frac{c_1}{c_3} = \left(\frac{V_0}{2\int_0^1 G(u)du}\right)^{\frac{m-2}{m+2}}$ with $V_0 = 1$. Since $\frac{c_3}{c_2}$ is given, the ratio $\frac{c_2}{c_1}$ can be calculated.

The fracture length is $L(t) = (1 + \frac{c_2}{c_1}t)^{\frac{c3}{c2}}$.

The variable $u(t,x) = \frac{x}{L(t)}$, where $0 \le u \le 1$

The fracture half width is $h(t,x) = \left(\frac{c_1}{c_3}\right)^{\frac{m+2}{m-2}}(1 + \frac{c_2}{c_1}t)^{\frac{m+2}{m-2} - \left(\frac{m+3}{m-2}\right)\frac{c_3}{c_2}}G(u)$

7.4 Analysis of numerical results

A comparison of the numerical results and the analytical solutions obtained for the two special cases is discussed. In addition, we present a discussion of graphs obtained from the numerical solutions of the two initial value problems presented in Section 7.1.

7.4.1 Validity of numerical results

We will consider results for the special cases derived in Chapter 6 under the three different flow conditions, rough wall turbulent, smooth wall turbulent and laminar flow.

Case 1: Total volume of fluid in fracture is constant.

The numerical and analytical results where computed for the case in which the total volume of fluid in the fracture remains constant. The analytical solution is given by (6.127) to (6.132). A comparison of the numerical and analytical results is presented in Figure 7.1. The numerical values are consistent with the analytical results for laminar and turbulent flow. However we observe the numerical values deviate slightly near $x = 0$ in the case of rough wall turbulent flow.

Case 2: Speed of the propagation of the fracture is constant.

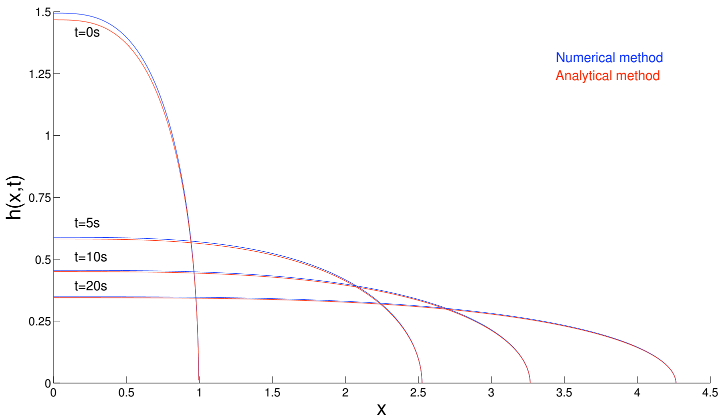
From Figure 7.2 we see that, as with Case 1 the numerical results are consistent with the analytical results. Slight deviations at $x = 0$ occur for when $m = 0$.

7.4.2 Analysis of results

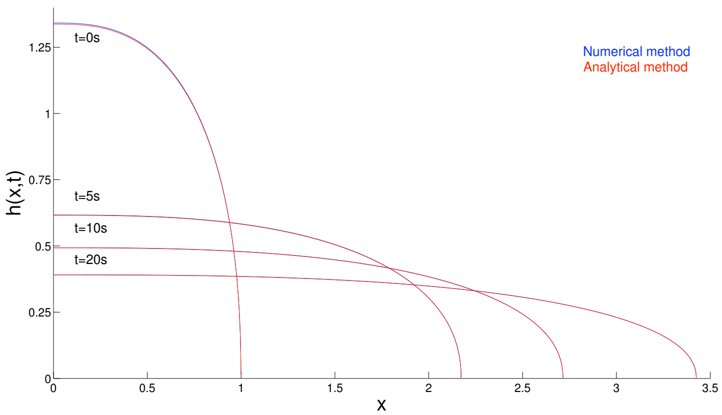
We present in this section a discussion of the graphs obtained from the numerical solutions of the two initial problems presented in Section 7.1. In all figures, curves labeled (a) to (e) are associated with values of $\frac{c_3}{c_2}$ which have physical significance as listed in Table 6.2. To be precise:

- Curve (a) is plotted for $\frac{c_3}{c_2} = \frac{m+2}{5}$ when the total volume of fluid in the fracture is constant.
- Curve (b) is plotted for $\frac{c_3}{c_2} = \frac{m+2}{m+3}$ when the pressure at the entry is kept constant.
- Curve (c) is plotted for $\frac{c_3}{c_2} = \frac{m+6}{m+8}$ when the rate of working of the pressure at the fracture entry is constant.
- Curve (d) is plotted for $\frac{c_3}{c_2} = 0.8$ when the rate of change of total volume of the fracture is constant.
- Curve (e) is plotted for the value of $\frac{c_3}{c_2} = 1$ when the speed of propagation fracture is constant.

In Figure 7.3, the length of the fracture $L(t)$ is plotted against time t for the five values of $\frac{c_3}{c_2}$ stated above. It can be seen from Figure 7.3 that for the five flows considered, the length of the fracture is greatest when the speed of propagation of the fracture is constant. Also, the fracture length is the least when the total volume of fluid in the fracture is constant. When the total volume of the fluid in the fracture is constant no fluid enters the fracture at $x = 0$ and the pumping of fluid into the fracture has ceased. We see that as the fracture relaxes the length of



(1)



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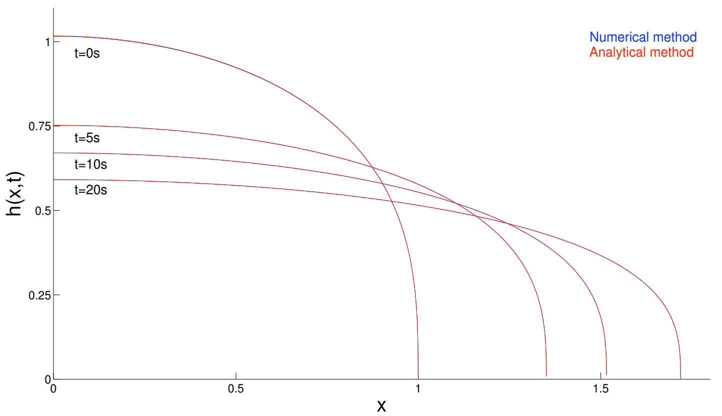
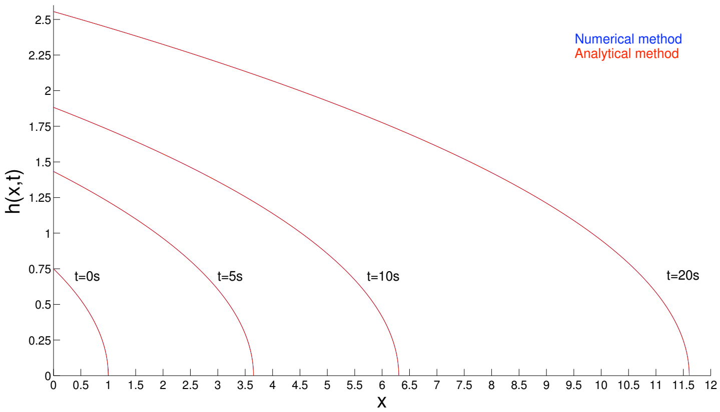
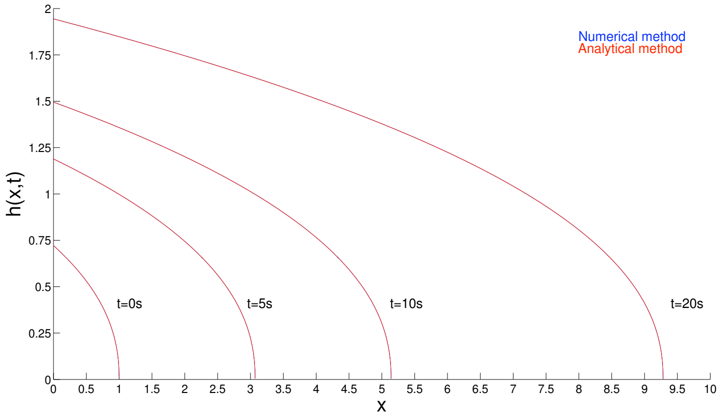


(3)

Figure 7.1: Case 1. Total volume of fluid in the fracture is constant. Analytical results and numerical results for $h(t, x)$ plotted against x : (1) rough wall turbulent flow; (2) smooth wall turbulent flow; (3) laminar flow.



(1)



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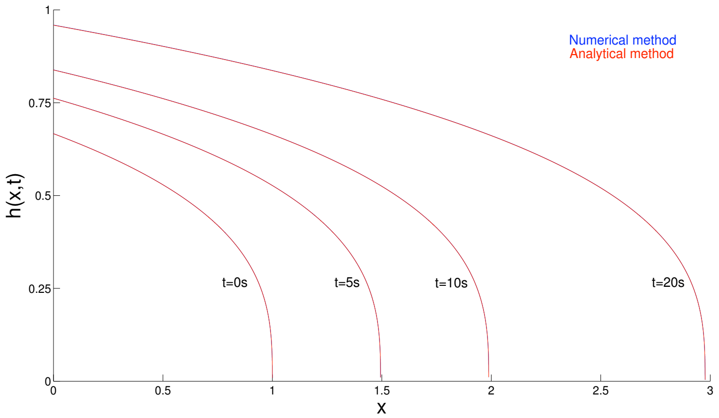
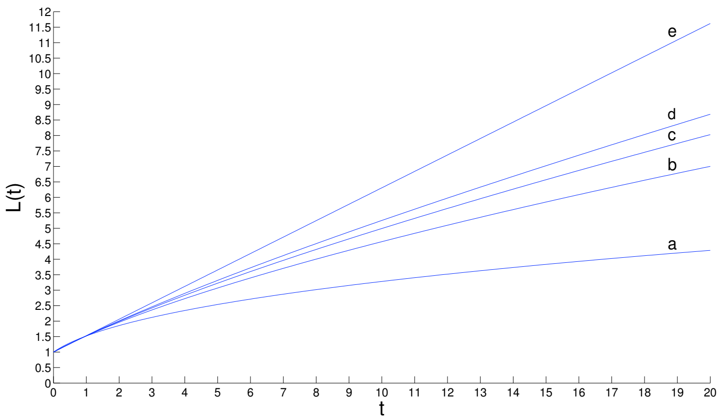
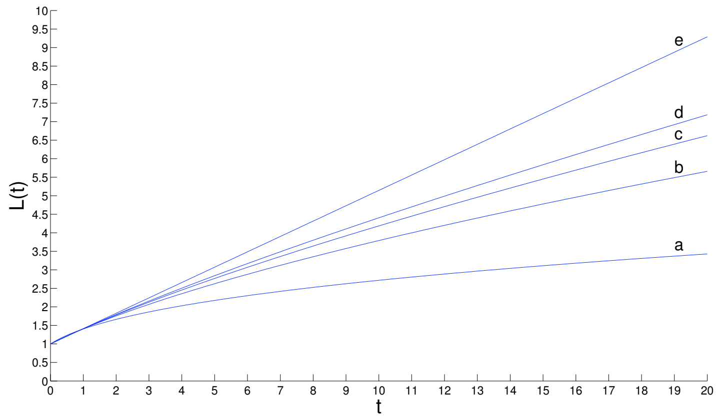


(3)

Figure 7.2: Case 2. Speed of propagation of the fracture is constant. Analytical results and numerical results for $h(t,x)$ plotted against x : (1) rough wall turbulent flow; (2) smooth wall turbulent flow; (3) laminar flow.



(1)



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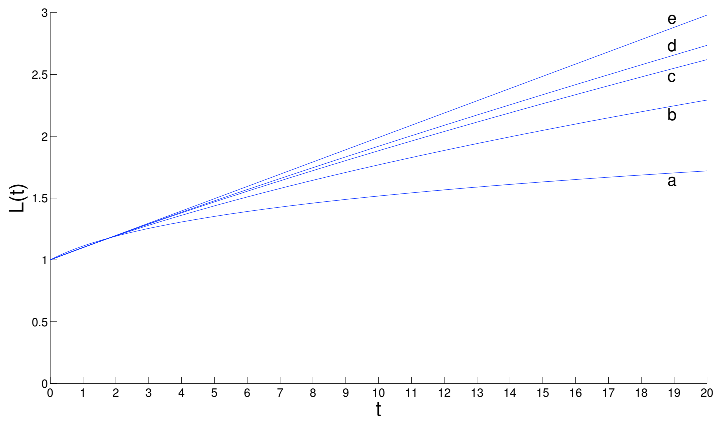
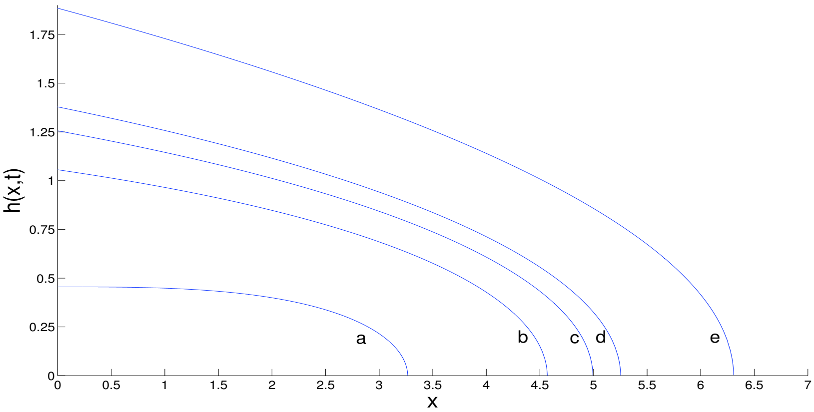


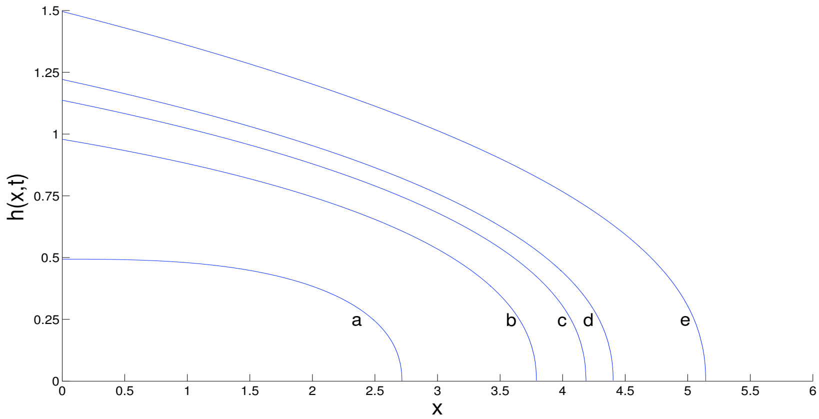
(3)

Figure 7.3: Graphs of fracture length $L(t)$ plotted against time t for a selection of values of $\frac{c_3}{c_2}$: (1) rough wall turbulent flow; (2) smooth wall turbulent flow; (3) laminar flow. The physical significance of curves (a) to (e) is explained in Section 7.4.2.

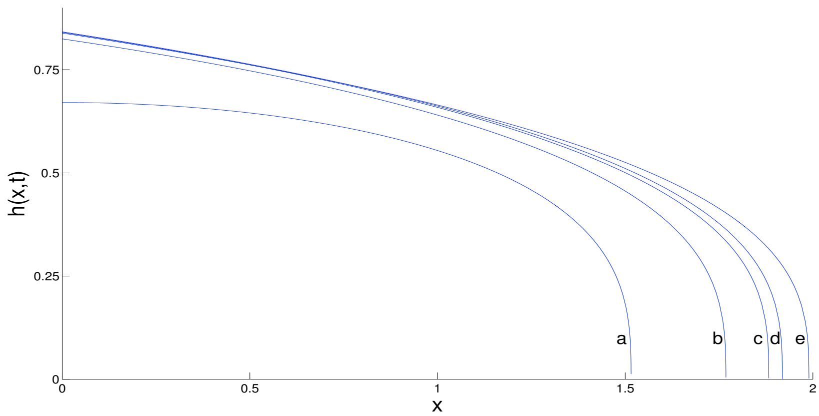
the fracture continues to increase. In Figure 7.4 the half-width of the fracture $h(t, x)$ is plotted against x at time $t = 10$ for the same five values of $\frac{c_2}{c_3}$. We see that the half-width is greatest when the speed of propagation of the fracture is constant and is lowest when the total volume of fluid in the fracture is constant. The curves for a turbulent fluid-driven fracture are more spaced out than for a laminar fluid-driven fracture. This indicates that there is a bigger difference due to different working conditions at the fracture entry in a turbulent fracture than in a laminar fracture.



(1)



(2)



(3)

Figure 7.4: Graphs of half-width $h(t,x)$ plotted against x for $t = 10$ and a selection of values of $\frac{c_3}{c_2}$: (1) rough wall turbulent flow; (2) smooth wall turbulent flow; (3) laminar flow. The physical significance of curves (a) to (e) is explained on page 115.

Since the scaled time t^* defined by

$$t^* = D^* t \tag{7.45}$$

where

$$D^* = D\Lambda^{\frac{1}{m+2}}, \qquad D = \left[2^{1-m}\frac{1}{n}\eta^m\rho^{-(1+m)}\right]^{\frac{1}{m+2}} \tag{7.46}$$

is used, the time scales in Figures 7.3 and 7.4 for rough wall turbulent flow, smooth wall turbulent flow and laminar flow are different. Comparison of the three types of flow cannot therefore be made from Figures 7.3 and 7.4. We do see, however, that the order of the graphs in the figures for the three types of flow is preserved.

7.5 Conclusions

Conversion of the the boundary value problem for the fracture to a set of two initial value problems simplified the process of obtaining numerical solutions. Numerical solutions to the initial value problems were found using the shooting method and asymptotic approximations near the tip of the fracture. They were consistent with analytical results for certain values of $\frac{c_3}{c_2}$ obtained in Chapter 6. We found that the order of the graphs in the figures are preserved as we moved from rough wall turbulent to smooth wall turbulent flow to laminar flow. This means, for instance, that the rate of growth of the length of the fracture is greater when the rate of change of volume of the fracture is constant than when the pressure at the fracture entry is constant for all three flows, rough wall turbulent, smooth wall turbulent and laminar.

It now remains to investigate the existence of conserved quantities for a fluid-driven fracture.

Chapter 8

Conservation laws for laminar and turbulent fluid-driven fractures

In Chapter 5 we saw the important role conservation laws could play in determining similarity solutions of partial differential equations. It was also shown that conserved quantities for flow in a tunnel could be determined by utilizing conservation laws. In this chapter we will investigate if conservation laws, aside from the elementary conservation law, do indeed exist for fluid-driven fractures. We will investigate the existence of conservation laws under rough wall turbulent, smooth wall turbulent and laminar flow conditions.

As in Chapter 4 we will compare the direct, characteristic and partial Lagrangian methods approach for deriving conservation laws.

8.1 Elementary conservation law

As shown in Chapter 6, the nonlinear diffusion equation describing a fluid-driven fracture can be written in the form of a conservation law

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\left(-h^3\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}\right) = 0, \tag{8.1}$$

where

$$T^1 = h \tag{8.2}$$

and

$$T^2 = \left(-h^3 \frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}. \tag{8.3}$$

Equation (8.1) is referred to as the elementary conservation law and (8.2) and (8.3) are the components of the elementary conserved vector.

8.2 Direct method

Consider (8.1) which can be written as

$$F = \frac{\partial h}{\partial t} + \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} \left(\frac{\partial h}{\partial x}\right)^{\frac{m+3}{m+2}} + \frac{k}{m+2} h^{\frac{3}{m+2}} \left(\frac{\partial h}{\partial x}\right)^{\frac{-1-m}{m+2}} \frac{\partial^2 h}{\partial x^2} = 0 \tag{8.4}$$

where $k = (-1)^{\frac{1}{m+2}}$. We will look for conserved vectors of the form $T^i = T^i(t, x, h, h_x)$ which satisfy the condition

$$D_t T^1 + D_x T^2|_{(8.4)} = 0, \tag{8.5}$$

The total derivatives D_t and D_x are:

$$D_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{xt} \frac{\partial}{\partial h_x} + ..., \tag{8.6}$$

$$D_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{tx} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + ... \tag{8.7}$$

Using (8.6) and (8.7) and replacing h_t using the partial differential equation (8.4), (8.5) becomes

$$\begin{aligned} &\frac{\partial T^1}{\partial t} - \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} h_x^{\frac{m+3}{m+2}} \frac{\partial T^1}{\partial h} - \frac{k}{m+2} (h)^{\frac{3}{m+2}} (h_x)^{\frac{-1-m}{m+2}} h_{xx} \frac{\partial T^1}{\partial h} \\ &h_{tx} \frac{\partial T^1}{\partial h_x} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} + h_{xx} \frac{\partial T^2}{\partial h_x} = 0 \end{aligned} \tag{8.8}$$

which can be separated by second order derivatives of h to give

$$h_{xx}: \qquad -\frac{k}{m+2}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}\frac{\partial T^1}{\partial h}+\frac{\partial T^2}{\partial h_x}=0, \tag{8.9}$$

$$h_{tx}: \qquad \frac{\partial T^1}{\partial h_x}=0, \tag{8.10}$$

$$\text{remainder}: \qquad \frac{\partial T^1}{\partial t}-\frac{3k}{(m+2)}h^{\frac{1-m}{m+2}}h_x^{\frac{m+3}{m+2}}\frac{\partial T^1}{\partial h}+\frac{\partial T^2}{\partial x}+h_x\frac{\partial T^2}{\partial h}=0. \tag{8.11}$$

From (8.10) we see that

$$T^1=T^1(t,x,h) \tag{8.12}$$

and therefore we can integrate (8.9) with respect to h_x to obtain

$$T^2=k\,(h)^{\frac{3}{m+2}}\,(h_x)^{\frac{1}{m+2}}\frac{\partial T^1}{\partial h}+A(t,x,h) \tag{8.13}$$

where A is an arbitrary function.

Substituting (8.12) and(8.13) into (8.11) yields

$$\frac{\partial T^1}{\partial t}+k\,(h)^{\frac{3}{m+2}}\,(h_x)^{\frac{1}{m+2}}\frac{\partial^2 T^1}{\partial h\partial x}+\frac{\partial A}{\partial x}+k\,(h)^{\frac{3}{m+2}}\,(h_x)^{\frac{m+3}{m+2}}\frac{\partial^2 T^1}{\partial h^2}+h_x\frac{\partial A(t,x,h)}{\partial h}=0. \tag{8.14}$$

It now remains to separate (8.14) by powers of h_x, however one must first identify values of m for which certain powers of h_x might be equal. Upon examination of $h_x^{\frac{1}{m+2}}$ and h_x it can be determined that the powers of h_x are equal for $m=-1$ and thus their coefficients should be grouped together. We will therefore look for conserved vectors under two different cases, the first of which is when the flow in the fracture is laminar ($m=-1$), the second is for general m, $m\neq -1,-2$. The general case includes smooth wall turbulent flow ($m=-1/4$) and rough wall turbulent flow ($m=0$).

8.2.1 Case $m=-1$

We separate (8.14) as follows:

$$
\begin{aligned}
h_x^2: &\qquad \frac{\partial^2 T^1}{\partial h^2} = 0, && (8.15)\\
h_x: &\qquad \frac{\partial A}{\partial h} - h^3 \frac{\partial^2 T^1}{\partial h \partial x} = 0, && (8.16)\\
\text{remainder}: &\qquad \frac{\partial A}{\partial x} + \frac{\partial T^1}{\partial t} = 0. && (8.17)
\end{aligned}
$$

Integrating (8.15) twice with respect to h gives

$$T^1 = hB(t,x) + C(t,x). \tag{8.18}$$

When (8.18) is substituted into (8.16), (8.16) can also be integrated with respect to h to obtain an expression for A:

$$A(x,t,h) = \frac{h^4}{4}\frac{\partial B}{\partial x} + D(t,x), \tag{8.19}$$

where D is an arbitrary function. It now remains to substitute results (8.18) and (8.19) into (8.17) and separate by powers of h which yields

$$
\begin{aligned}
h^4: &\qquad \frac{\partial^2 B}{\partial x^2} = 0, && (8.20)\\
h: &\qquad \frac{\partial B}{\partial t} = 0, && (8.21)\\
\text{remainder}: &\qquad \frac{\partial D}{\partial x} + \frac{\partial C}{\partial t} = 0. && (8.22)
\end{aligned}
$$

From (8.21) we find that B is a function of x only and by integrating (8.20) we have

$$B(x) = xc_1 + c_2, \tag{8.23}$$

where c_1, c_2 are constants. We can chose $D(t,x) = C(t,x) = 0$ as condition (8.22) implies that they are the components of a trivial conserved vector for which the conservation law is identically satisfied. Thus

$$T^1 = (c_1 x + c_2)h, \tag{8.24}$$

$$T^2 = -h^3 h_x(c_1 x + c_2) + \frac{h^4}{4} c_1. \tag{8.25}$$

The following conserved vectors for laminar flow in a fracture have therefore been derived:

$$T^1 = h, \qquad T^2 = -h^3 h_x, \tag{8.26}$$

$$T^2 = xh \qquad T^2 = -xh^3 h_x + \frac{h^4}{4}. \tag{8.27}$$

The conserved vector (8.26) is the elementary conserved vector.

8.2.2 General case: $m \neq -1, -2$

We can separate equation (8.14) by powers of h_x as follows

$$h_x^{\frac{1}{m+2}} : \qquad \frac{\partial^2 T^1}{\partial h \partial x} = 0, \tag{8.28}$$

$$h_x^{\frac{m+3}{m+2}} : \qquad \frac{\partial^2 T^1}{\partial h^2} = 0, \tag{8.29}$$

$$h_x : \qquad \frac{\partial A}{\partial h} = 0, \tag{8.30}$$

$$\text{remainder} : \qquad \frac{\partial A}{\partial x} + \frac{\partial T^1}{\partial t} = 0. \tag{8.31}$$

Integrating (8.28) with respect to x yields

$$\frac{\partial T^1}{\partial h} = B(t, h), \tag{8.32}$$

which when substituted into (8.29) implies that the arbitrary function B is a function of t only. By integrating (8.32)with respect to h we find that,

$$T^1 = hB(t) + C(t, x), . \tag{8.33}$$

where C is an arbitrary function. From (8.30)

$$A = A(t, x), \tag{8.34}$$

which when substituted with (8.33) in (8.31) yields

$$\frac{\partial A(t,x)}{\partial x} + h\frac{dB}{dt} + \frac{\partial C(t,x)}{\partial t} = 0. \tag{8.35}$$

We can separate (8.35) by powers of h to find

$$B(t) = c_1 \tag{8.36}$$

and

$$\frac{\partial C(t,x)}{\partial x} + \frac{\partial A(t,x)}{\partial x} = 0, \tag{8.37}$$

where c_1 an arbitrary constant.

From (8.33) and (8.25) the conserved vector is

$$T^1 = hc_1 + C(t,x), \tag{8.38}$$

$$T^2 = \left(-h^3 h_x\right)^{\frac{1}{m+2}} c_1 + A(t,x). \tag{8.39}$$

We can set $C(t,x) = 0 = A(t,x)$ because from (8.37) they are components of a trivial conserved vector. Thus for the general case there is only one conserved vector of the form $T^i = T^i(t, x, h, h_x)$, namely

$$T^1 = h, \qquad T^2 = \left(-h^3 h_x\right)^{\frac{1}{m+2}}, \tag{8.40}$$

which is the elementary conserved vector.

8.3 Characteristic method

We will look for conserved vectors of the form $T^i = T^i(t, x, h, h_x, h_t)$ which is more general than in Section 8.2 because the components can depend on h_t and a multiplier of the from

$\Lambda = \Lambda(t, x, h)$ that satisfy

$$D_t T^1 + D_x T^2 = \Lambda F. \tag{8.41}$$

We expand (8.41) using (8.6) and (8.7) to find

$$\begin{aligned} &\frac{\partial T^1}{\partial t} + h_t \frac{\partial T^1}{\partial h} + h_{tt} \frac{\partial T^1}{\partial h_t} + h_{tx} \frac{\partial T^1}{\partial h_x} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} \\ &+ h_{xx} \frac{\partial T^2}{\partial h_x} + h_{tx} \frac{\partial T^2}{\partial h_t} - \frac{\partial h}{\partial t}\Lambda - \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} (h_x)^{\frac{m+3}{m+2}} \Lambda \\ &- \frac{k}{m+2} h^{\frac{3}{m+2}} (h_x)^{\frac{-1-m}{m+2}} h_{xx} \Lambda = 0 \ . \end{aligned} \tag{8.42}$$

By separating (8.42) by powers of second order derivatives of h we have

$$h_{tx}: \qquad \frac{\partial T^1}{\partial h_x} + \frac{\partial T^2}{\partial h_t} = 0, \tag{8.43}$$

$$h_{xx}: \qquad \frac{\partial T^2}{\partial h_x} - \Lambda \frac{k}{m+2} h^{\frac{3}{m+2}} (h_x)^{\frac{-1-m}{m+2}} = 0, \tag{8.44}$$

$$h_{tt}: \qquad \frac{\partial T^1}{\partial h_t} = 0, \tag{8.45}$$

$$\begin{aligned} \text{remainder}: \qquad &\frac{\partial T^1}{\partial t} + h_t \frac{\partial T^1}{\partial h} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} - \Lambda h_t \\ &- \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} (h_x)^{\frac{m+3}{m+2}} \Lambda = 0 \ . \end{aligned} \tag{8.46}$$

From (8.45)

$$T^1 = T^1(t, x, h, h_x) \tag{8.47}$$

and by integrating (8.44) with respect to h_x we find that

$$T^2 = k\Lambda \,(h)^{\frac{3}{m+2}} (h_x)^{\frac{1}{m+2}} + A(t, x, h, h_t). \tag{8.48}$$

where $A(t, x, h, h_t)$ is an arbitrary function. Substituting (8.47) and (8.48) into (8.43) and differentiating the resulting equation with respect to h_t we have the following

$$\frac{\partial^2 A}{\partial h_t^2} = 0. \tag{8.49}$$

Integrating (8.49) twice with respect to h_t leads to

$$A = h_t B(t, x, h) + C(t, x, h), \tag{8.50}$$

where B and C are arbitrary functions. Equation(8.48) becomes

$$T^2 = k\,(h)^{\frac{3}{m+2}}\,(h_x)^{\frac{1}{m+2}}\,\Lambda + h_t B(t, x, h) + C(t, x, h), \tag{8.51}$$

By substituting (8.51) into (8.43) and integrating with respect to h_x we obtain

$$T^1 = -h_x B(t, x, h) + D(t, x, h) \tag{8.52}$$

where D is an arbitrary function. It now remains to substitute our results for T^1 and T^2 into (8.46):

$$\begin{aligned} &- h_x \frac{\partial B}{\partial t} + \frac{\partial D}{\partial t} + h_t \frac{\partial D}{\partial h} + k(h)^{\frac{3}{m+2}} (h_x)^{\frac{1}{m+2}} \frac{\partial \Lambda}{\partial x} + h_t \frac{\partial B}{\partial x} + \frac{\partial C}{\partial x} \\ &+ k(h)^{\frac{3}{m+2}} (h_x)^{\frac{m+3}{m+2}} \frac{\partial \Lambda}{\partial h} + h_x \frac{\partial C}{\partial h} - \Lambda h_t = 0. \end{aligned} \tag{8.53}$$

As in the case of the direct method, separation by powers of h_x in (8.53) could result in a different set of equations for certain values of m. Once again in the case for laminar flow where $m = -1$, the equations resulting from separating by powers h_x will be different to that of the general case where $m \neq -1, -2$. We will consider each case and find its conserved vectors.

8.3.1 Case m=-1

For $m = -1$ the coefficients of $h_x^{\frac{1}{m+2}}$ and h_x can be grouped together and separated according to the first derivatives of h to yield

$$
\begin{aligned}
h_x^2: &\qquad \frac{\partial \Lambda(t,x,h)}{\partial h} = 0, && (8.54)\\
h_x: &\qquad -\frac{\partial B(t,x,h)}{\partial t} + \frac{\partial C(t,x,h)}{\partial h} - h^3 \frac{\partial \Lambda}{\partial x} = 0, && (8.55)\\
h_t: &\qquad \frac{\partial D(t,x,h)}{\partial h} + \frac{\partial B(t,x,h)}{\partial x} - \Lambda(t,x,h) = 0, && (8.56)\\
\text{remainder}: &\qquad \frac{\partial D(t,x,h)}{\partial t} + \frac{\partial C(t,x,h)}{\partial x} = 0. && (8.57)
\end{aligned}
$$

From (8.54) we have

$$\Lambda = \Lambda(t,x). \tag{8.58}$$

Differentiating (8.55) with respect to x, (8.56) with respect to t and summing the two resulting equations gives

$$\frac{\partial^2 D(t,x,h)}{\partial t \partial h} + \frac{\partial^2 C(t,x,h)}{\partial x \partial h} - h^3 \frac{\partial^2 \Lambda}{\partial x^2} - \frac{\partial \Lambda}{\partial t} = 0. \tag{8.59}$$

Finally we differentiate (8.57) with respect to h and substitute the resulting equation into (8.59) which yields an equation that can be separated by powers of h to give:

$$
\begin{aligned}
h^3: &\qquad \frac{\partial^2 \Lambda(t,x)}{\partial x^2} = 0, && (8.60)\\
\text{remainder}: &\qquad \frac{\partial \Lambda(t,x)}{\partial t} = 0. && (8.61)
\end{aligned}
$$

From (8.60) and (8.61),

$$\Lambda = c_1 x + c_2, \tag{8.62}$$

where c_1 and c_2 are constants.

Re-substituting Λ into (8.55) and solving for $C(t,x,h)$ by integrating with respect to h yields

$$C(t,x,h) = \frac{h^4}{4} c_1 + \frac{\partial Q(t,x,h)}{\partial t} + F(t,x), \tag{8.63}$$

where $F(t,x)$ is an arbitrary function and $Q(t,x,h)$ is defined by

$$Q(t,x,h) = \int^h B(t,x,h)dh. \tag{8.64}$$

We solve for $D(t,x,h)$ in a similar way by integrating (8.56) with respect to h:

$$D(t,x,h) = (c_1 x + c_2)h - \frac{\partial Q(t,x,h)}{\partial t} + G(t,x), \tag{8.65}$$

where $G(t,x)$ is an arbitrary function. Finally, substituting (8.63) and (8.65) into (8.57) gives

$$\frac{\partial G(t,x)}{\partial t} + \frac{\partial F(t,x)}{\partial x} = 0. \tag{8.66}$$

Hence from (8.51) with $m = -1$ and (8.52)

$$T^1 = (c_1 x + c_2)h + T^1_*, \tag{8.67}$$

$$T^2 = -(c_1 x + c_2)h^3 h_x + \frac{c_1}{4}h^4 + T^2_*, \tag{8.68}$$

where

$$T^1_* = -h_x B(t,x,h) - \frac{\partial Q(x,t,h)}{\partial x} + G(t,x), \tag{8.69}$$

$$T^2_* = h_t B(t,x,h) + \frac{Q(x,t,h)}{\partial t} + F(t,x). \tag{8.70}$$

However, by using the definition (8.64) for $Q(t,x,h)$ and (8.66) it can be shown that

$$D_t T^1_* + D_x T^2_* \equiv 0 \tag{8.71}$$

and therefore T^1_* and T^2_* are the components of a trivial conserved vector. Thus (8.67) and (8.68) reduce to (8.26) and (8.27). We have therefore shown that for laminar flow the only conserved vectors with components of the form $T^i = T^i(t,x,h,h_t,h_x)$ and with multipliers of the form $\Lambda = \Lambda(t,x,h)$ are

$$T^1 = h, \qquad T^2 = -h^3 h_x, \tag{8.72}$$

$$T^1 = xh, \qquad T^2 = -xh^3 h_x + \frac{h^4}{4}. \tag{8.73}$$

8.3.2 General case: $m \neq -1, -2$

We separate (8.53) as follows:

$$\begin{aligned}
h_x : &\qquad -\frac{\partial B(t,x,h)}{\partial t} + \frac{\partial C(t,x,h)}{\partial h} = 0, && (8.74)\\
(h_x)^{\frac{1}{m+2}} : &\qquad \frac{\partial \Lambda(t,x,h)}{\partial x} = 0, && (8.75)\\
(h_x)^{\frac{m+3}{m+2}} : &\qquad \frac{\partial \Lambda(t,x,h)}{\partial h} = 0, && (8.76)\\
(h_t) : &\qquad \frac{\partial D(t,x,h)}{\partial h} + \frac{\partial B(t,x,h)}{\partial x} - \Lambda(t,x,h) = 0, && (8.77)\\
\text{remainder} : &\qquad \frac{\partial D(t,x,h)}{\partial t} + \frac{\partial C(t,x,h)}{\partial x} = 0. && (8.78)
\end{aligned}$$

From (8.75) and (8.76)

$$\Lambda = \Lambda(t). \tag{8.79}$$

We proceed to differentiate (8.74) with respect to x, (8.77) with respect to t and finally (8.78) with respect to h to obtain

$$\frac{\partial^2 B}{\partial x \partial t} = \frac{\partial^2 C}{\partial x \partial h}, \tag{8.80}$$

$$\frac{\partial^2 B}{\partial t \partial x} = \frac{d\Lambda}{dt} - \frac{\partial^2 D}{\partial t \partial h} \tag{8.81}$$

and

$$\frac{\partial^2 D}{\partial t \partial h} + \frac{\partial^2 C}{\partial x \partial h} = 0. \tag{8.82}$$

Subtracting (8.80) from (8.81) using (8.82) gives

$$\frac{d\Lambda}{dt} = 0. \tag{8.83}$$

Therefore

$$\Lambda = c_1 \tag{8.84}$$

where c_1 is a constant. In order to find an expression for $C(t,x,h)$ and $D(t,x,h)$ we integrate (8.74) and (8.77) with respect to h and find

$$C = \frac{\partial Q(t,x,h)}{\partial t} + F(t,x) \tag{8.85}$$

and

$$D = c_1 h - \frac{\partial Q(t,x,h)}{\partial x} + G(t,x), \tag{8.86}$$

where $F(t,x)$ and $G(t,x)$ are arbitrary functions and $Q(t,x,h)$ is defined by

$$Q = \int^h B(t,x,h)dh. \tag{8.87}$$

Substituting (8.85) and (8.86) into (8.78) gives again (8.66). From (8.51) and (8.52)

$$T^1 = c_1 h + T^1_*, \tag{8.88}$$

$$T^2 = c_1(-h^3 h_x)^{\frac{1}{m+2}} + T^2_*. \tag{8.89}$$

where T^1_* and T^2_* are given by (8.69) and (8.70). Since T^1_* and T^2_* are components of a trivial conserved vector it follows that the elementary conserved vector (8.40) is again obtained. When $m \neq -1, -2$ only one conserved vector of the form $T^i(t,x,h,h_t,h_x)$ with multiplier Λ is obtained.

8.4 Partial Lagrangian method

Consider the partial Lagrangian

$$L = \left(\frac{m+2}{m+3}\right) h^{\frac{3}{m+2}} (-h_x)^{\frac{m+3}{m+2}} = k \left(\frac{m+2}{m+3}\right) h^{\frac{3}{m+2}} (h_x)^{\frac{m+3}{m+2}} \tag{8.90}$$

where $k = (-1)^{\frac{1}{m+2}}$,

then

$$\frac{\delta L}{\delta h} = -3\frac{k}{m+3} (h)^{\frac{1-m}{m+2}} (h_x)^{\frac{m+3}{m+2}} - h_t. \tag{8.91}$$

The partial Noether symmetry given by (5.205) is

$$X^{[1]}L + L(D_t\xi^1 + D_x\xi^2) = D_tB^1 + D_xB^2 + (\eta - \xi^1 h_t - \xi^2 h_x)\frac{\delta L}{\delta h} \tag{8.92}$$

We consider gauge functions of the form $B^1 = B^1(t,x,h)$ and $B^2 = B^2(t,x,h)$. When expanded for partial Lagrangian (8.90), (8.91) becomes

$$\begin{aligned}
&- k(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}\frac{\partial \eta}{\partial x} - k\frac{\partial \eta}{\partial h}(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}} + k(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}h_t\frac{\partial \xi^1}{\partial x}\\
&+ k(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}h_t\frac{\partial \xi^1}{\partial h} + k(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}\frac{\partial \xi^2}{\partial x} - k\frac{m+2}{m+3}(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}\frac{\partial \xi^1}{\partial t}\\
&- k\frac{m+2}{m+3}(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}h_t\frac{\partial \xi^1}{\partial h} - k\frac{m+2}{m+3}(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}\frac{\partial \xi^2}{\partial x}\\
&- k\frac{m+2}{m+3}(h)^{\frac{3}{m+2}}(h_x)^{\frac{2m+5}{m+2}}\frac{\partial \xi^2}{\partial h} - \left(\frac{3k}{m+3}\right)(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{m+3}{m+2}}h_t\xi^1\\
&- \left(\frac{3k}{m+3}\right)(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{2m+5}{m+2}}\xi^2 + \eta h_t - h_t^2\xi^1 - h_t h_x\xi^2\\
&- \frac{\partial B^1}{\partial t} - h_t\frac{\partial B^1}{\partial h} - \frac{\partial B^2}{\partial x} - h_x\frac{\partial B^2}{\partial h} = 0.
\end{aligned} \tag{8.93}$$

We first separate (8.93) by the derivatives h_t^2. This gives

$$\xi^1(t,x,h) = 0. \tag{8.94}$$

Setting $\xi^1 = 0$ in (8.93) and then separating by $h_x h_t$ gives

$$\xi^2(t,x,h) = 0. \tag{8.95}$$

These results apply for all values of m. We then set $\xi^1 = 0 = \xi^2$ in (8.93) and separate the equation by first derivatives of h. The case $m = -1$ has to be treated separately. We do not consider $m = -2$.

8.4.1 Case m=-1

When $m = -1$, $k = -1$. We can separate (8.93) as follows

$$h_x^2 : \qquad \frac{\partial \eta}{\partial h} = 0, \tag{8.96}$$

$$h_x : \qquad h^3 \frac{\partial \eta}{\partial x} - \frac{\partial B^2}{\partial h} = 0, \tag{8.97}$$

$$h_t : \qquad \eta - \frac{\partial B^1}{\partial h} = 0, \tag{8.98}$$

$$\text{remainder} : \qquad \frac{\partial B^1}{\partial t} + \frac{\partial B^2}{\partial x} = 0. \tag{8.99}$$

From (8.96) $\eta = \eta(t, x)$ and we can integrate (8.97) and (8.98) with respect to h to obtain expressions for B^1 and B^2. This gives

$$B^1 = h\eta + A(t, x), \tag{8.100}$$

and

$$B^2 = \frac{h^4}{4} \frac{\partial \eta}{\partial x} + C(t, x), \tag{8.101}$$

where A and C are arbitrary functions.

Substituting (8.100) and (8.101) into (8.99) yields,

$$h \frac{\partial \eta}{\partial t} + \frac{\partial A}{\partial t} + \frac{h^4}{4} \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial C}{\partial x} = 0. \tag{8.102}$$

It now remains to separate (8.102) by powers of h which results in equations that can be easily solved to give

$$\eta = c_1 x + c_2, \tag{8.103}$$

$$B^1 = (c_1 x + c_2)h + A(t, x), \tag{8.104}$$

$$B^2 = \frac{h^4}{4}c_1 + C(t,x). \tag{8.105}$$

The remainder in (8.102) gives

$$\frac{\partial A(t,x)}{\partial t} + \frac{\partial C(t,x)}{\partial x} = 0. \tag{8.106}$$

Now

$$T^1 = B^1 - \xi^1 L - (\eta - \xi^1 h_t - \xi^2 h_x)\frac{\partial L}{\partial h_t}, \tag{8.107}$$

$$T^2 = B^2 - \xi^2 L - (\eta - \xi^1 h_t - \xi^2 h_x)\frac{\partial L}{\partial h_x} \tag{8.108}$$

therefore

$$T^1 = (c_1 x + c_2)h + A(t,x), \tag{8.109}$$

and

$$T^2 = -(c_1 x + c_2)h^3 h_x + \frac{c_1}{4}h^4 C(t,x). \tag{8.110}$$

We can choose $A(t,x) = C(t,x) = 0$ since by (8.106) they satisfy the conservation law identically. The two conserved vectors for laminar flow, (8.109) and (8.110), derived in this section are equivalent to the conserved vectors (8.72) and (8.73) obtained using the direct and characteristic methods.

8.4.2 Case $m \neq -1, -2$

Equation(8.93) can be separated according to derivatives of h to give

$$h_x^{\frac{1}{m+2}}: \qquad \frac{\partial \eta}{\partial x} = 0, \tag{8.111}$$

$$h_x^{\frac{m+3}{m+2}}: \qquad \frac{\partial \eta}{\partial h} = 0, \tag{8.112}$$

$$h_t: \qquad \eta - \frac{\partial B^1}{\partial h} = 0, \tag{8.113}$$

$$h_x: \qquad \frac{\partial B^2}{\partial h} = 0. \tag{8.114}$$

$$\text{remainder}: \qquad \frac{\partial B^1}{\partial t} + \frac{\partial B^2}{\partial x} = 0. \tag{8.115}$$

From (8.111) and (8.112) we have that $\eta = \eta(t)$ and we can therefore integrate (8.113) with respect to h which gives

$$B^1 = h\eta(t) + D(t,x) \tag{8.116}$$

where D is an arbitrary function. From (8.114)

$$B^2 = E(t,x) \tag{8.117}$$

where E is an arbitrary function. Substituting (8.116) and (8.117) into (8.115) yields an equation that can separated by h:

$$h: \qquad \frac{d\eta}{dt} = 0, \tag{8.118}$$

$$\text{remainder}: \qquad \frac{\partial D(x,t)}{\partial t} + \frac{\partial E(t,x)}{\partial x} = 0. \tag{8.119}$$

From (8.118)

$$\eta = c_1 \tag{8.120}$$

where c_1 is a constant. We can choose $D(t,x) = E(t,x) = 0$ because by (8.119) they satisfy the conservation law identically. Using the formula (8.107) and (8.108) the conserved vector for the general case for m is

$$T^1 = c_1 h, \qquad T^2 = c_1(-h^3 h_x)^{\frac{1}{m+2}}. \tag{8.121}$$

which is the elementary conserved vector. This is in agreement with the results obatined using the direct and characteristic methods. We have seen that for the general case ($m \neq -1, -2$) that describes turbulent flow in a fracture when $m = -1$ and $m = -\frac{1}{4}$ we obtain only the elementary conserved vector unlike in the case for laminar flow when $m = -1$ where the elementary conserved vector and a second conserved are obtained. Tis is a significant difference between laminar and turbulent flow in a fracture. A conservation law is "lost" due to turbulence.

We will now use the conservation laws to investigate conserved quantities and balance laws for laminar and turbulent hydraulic fracturing.

8.5 Conserved quantities for fluid flow in a fracture

The conservation laws are derived from the partial differential equation and they do not depend on boundary conditions. They apply to any problem described by the partial differential equation. Conserved quantities and balance laws are derived from conservation laws and boundary conditions and as shown in Chapter 4 they may be useful in determining invariant solutions. Also in Chapter 4, we saw that the boundary conditions for a problem determine which conservation law applies for that problem. When the components, T^1 and T^2, are regarded as functions of t,x,h,h_t,h_x,... as independent variables, the conservation law is written in terms of the total derivatives D_t and D_x as

$$D_t T^1 + D_x T^2 = 0. \tag{8.122}$$

If we regard T^1 and T^2 as functions of t and x as independent variables then it can be verified that the conservation law can be written equivalently as

$$\frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0. \tag{8.123}$$

We now investigate conserved quantities and balance laws for hydraulic fracturing. We first consider the elementary conservation law which is valid for both laminar and turbulent flow and then the second conservation law which applies only for laminar flow.

8.5.1 Elementary conservation law for laminar and turbulent fluid fracture

The elementary conserved vector is

$$T^1 = h, \qquad T^2 = (-h^3 h_x)^{\frac{1}{m+2}}. \tag{8.124}$$

Therefore we have the conservation law

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[(-h^3 h_x)^{\frac{1}{m+2}}\right] = 0. \tag{8.125}$$

We integrate (8.125) with respect to x from 0 to $L(t)$ keeping t fixed during the integration which yields

$$\int_0^{L(t)} \frac{\partial h(x,t)}{\partial t} dx + \int_0^{L(t)} \frac{\partial}{\partial x}\left[(-h^3 h_x)^{\frac{1}{m+2}}\right] dx = 0. \tag{8.126}$$

Using the theorem for differentiation under the integral sign [21] it follows that

$$\frac{d}{dt}\int_0^{L(t)} h(t,x)dx = \int_0^{L(t)} \frac{\partial h(t,x))}{\partial t} dx + h(t,L(t))\frac{dL(t)}{dt} - 0. \tag{8.127}$$

Since the width of the fracture vanishes at the fracture tip,

$$h(t,L(t)) = 0. \tag{8.128}$$

Hence (8.12) becomes

$$\frac{d}{dt}\int_0^{L(t)} h(t,x)dx + \int_0^{L(t)} \frac{\partial}{\partial x}\left[(-h^3 h_x)^{\frac{1}{m+2}}\right] dx = 0. \tag{8.129}$$

Integrating the second term of (8.129) with respect to x yields

$$\frac{d}{dt}\int_0^{L(t)} h(t,x)dx + (-h^3 h_x)^{\frac{1}{m+2}}|_{x=L(t)} - (-h^3 h_x)^{\frac{1}{m+2}}|_{x=0} = 0. \tag{8.130}$$

But from (6.27) expressed in terms of the scaled time $D^* t$,

$$h(t,x)\bar{u}(t,x) = \left(-h^3 \frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}. \tag{8.131}$$

Therefore

$$\left(-h^3 \frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}} |_{x=L(t)} = h(t, L(t))\bar{u}(t, L(t)) = 0. \tag{8.132}$$

by (8.128) and

$$\left(-h^3 \frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}} |_{x=0} = h(t,0)\bar{u}(t,0). \tag{8.133}$$

Thus (8.130) becomes

$$\frac{d}{dt}\int_0^{L(t)} h(t,x)dx = h(t,0)\bar{u}(t,0), \tag{8.134}$$

but the total volume of the fracture at time t is

$$V(t) = 2\int_0^{L(t)} h(t,x)dx. \tag{8.135}$$

Thus (8.131) becomes

$$\frac{dV}{dt} = 2h(t,0)\bar{u}(t,0), \tag{8.136}$$

which states that the rate of change of the volume of the fracture with respect to time equals the rate of inflow of fluid at the fracture entrance. If there is no inflow of fluid at the fracture entrance then the volume of the fracture is conserved. The volume of the fracture V is the conserved quantity.

The elementary conservation law with boundary condition (8.128) therefore corresponds to the balance law for fluid volume.

8.5.2 Second conservation law for laminar fluid-driven fracture

We have seen that laminar fluid flow in a thin fracture has two conserved vectors. It now remains to find the second conserved quantity or balance law associated with the non-elementary conservation law which is given by

$$T^1 = xh, \qquad T^2 = -xh^3 h_x + \frac{h^4}{4}. \tag{8.137}$$

We have

$$\frac{\partial}{\partial t}(xh) + \frac{\partial}{\partial x}(-xh^3 h_x + \frac{h^4}{4}) = 0 \tag{8.138}$$

which when integrated with respect to x from 0 to $L(t)$ keeping t fixed becomes

$$\int_0^{L(t)} \frac{\partial}{\partial t}(xh(t,x))dx + \int_0^{L(t)} \frac{\partial}{\partial x}\left(-h(t,x)^3 h_x(t,x)x + \frac{h^4}{4}\right) dx = 0. \tag{8.139}$$

Using once again the theorem for differentiation under the integral sign [21] and the boundary condition (8.128) we find

$$\frac{d}{dt}\int_0^{L(t)} xh(x,t)dx = \int_0^{L(t)} \frac{\partial}{\partial t}(xh(t,x))dx. \tag{8.140}$$

Also using (8.131) for $h\bar{u}$ and the boundary condition (8.128) we obtain

$$\int_0^{L(t)} \frac{\partial}{\partial x}\left[-xh^3 h_x + \frac{h^4}{4}\right] dx = \frac{-1}{4}h^4(t,0). \tag{8.141}$$

Substituting (8.140) and (8.141) into (8.139) gives

$$\frac{d}{dt}\int_0^{L(t)} xh(t,x)dx = \frac{1}{4}h^4(t,0). \tag{8.142}$$

Equation (8.142) is the balance law associated with the second conserved vector (8.137). The right hand side depends on $h(t,0)$ which shows the important part played by the working conditions at the fracture entrance on the evolution of the fracture. The physical interpretation of the balance law (8.142) is not clear. It may be useful as a check on the accuracy of numerical solutions.

8.6 Conclusion

The conserved vectors were derived using three different methods which all yielded the same result and, as in Chapter 5, it was found that the partial Lagrangian method was an effective method and computationally less laborious compared to the direct and characteristic methods. We were able to derive another conservation law besides the elementary conservation law for laminar flow in a thin fracture ($m = -1$). We were only able to find the elementary conserved vector for other values of m. It may indicate that as fluid flow changes from laminar to turbulent there is a loss of conserved vectors. This may have been the case in the Fanno model nonlinear diffusion equation for pressure in which the flow was turbulent and the nonlinear diffusion equation for pressure equation yielded only the elementary conservation law. Perhaps, had the flow been laminar, one could have been able to find a non-elementary conserved vector. In the next and final chapter we will present a summary of our findings.

Chapter 9

Conclusions

In this dissertation Lie group analysis proved to be a powerful tool in the study of partial differential equations. We were able to derive group invariant solutions for turbulent compressible flow in a long channel as well as for turbulent incompressible flow in a thin fracture. Lie point symmetries were derived for both problems and used to reduce the partial differential equations to ordinary differential equations.

For turbulent fluid-driven fractures, Lie point symmetries were used to reduce the partial differential equation to a boundary value problem for an ordinary differential equation. Analytical solutions for the boundary value problem were obtained for special cases of physical significance that govern the propagation of fracture. The first special solution is derived for the total volume of fluid in the fracture being constant and the second special solution for the speed of propagation of the fracture being constant. The boundary value problem was solved by converting it into two initial value problems that in turn was solved using a variation on the shooting method. The ordering of graphs of the half-width and of the length of the fracture under different working conditions at the fracture entry did not change when the fluid flow changed from laminar to turbulent

An analytical solution for the mean pressure and mean velocity of the fluid in a long channel was obtained. Ockendon et al [6] first solved for the mean pressure and then obtain the mean velocity using the relation between the two. We were able to give an alternative derivation to the mean velocity that does not require any prior knowledge of the mean pressure.The investigation of conserved vectors allowed for a more flexible approach to the method of solution of certain differential equations. It was shown that an invariant solution to the nonlinear diffusion equation for the mean velocity could be found using the Lie point symmetry associated with the elementary conserved vector. Conserved vectors proved to be useful in the solution of partial differential equation and we undertook to review three methods to derive them.

The direct method, characteristic method and partial Lagrangian method were used to derive conserved vectors. The direct method and the characteristic method were done in an almost algorithmic manner. The partial Lagrangian method was straightforward and by using differential operators, a conserved vector could be calculated in a systematic way. The partial Lagrangian method also did not make any assumptions on the functional form of the conserved vector but did on the gauge terms. For the nonlinear diffusion equation and the nonlinear wave equation for fluid velocity, two conservation laws were were derived, one of which was the elementary conservation law. However, for the nonlinear diffusion equation for the pressure, only the elementary conservation law was obtained. The existence of additional conserved vectors may be investigated by considering conserved vectors that depend on higher order derivatives.

Using the same methods, conserved quantities for fluid-driven fractures were derived. As for the nonlinear diffusion equation for the pressure of the fluid in a long channel, we were only able to find the elementary conserved vector when the fluid flow was turbulent. However, when the flow was laminar, we were able to derive another conserved vector besides the elementary conserved vector. It would seem that as the fluid becomes turbulent, conserved vectors are not preserved.

Further investigation into the reduction in the number of conserved vectors as the fluid changes from laminar to turbulent flow should include the study of conserved vectors with higher order derivatives. It is also of interest to understand why in laminar fluid flow there are more conserved vectors than if when the fluid flow is turbulent.

Appendix A

Lie point symmetries for the nonlinear diffusion equation or pressure.

The derivation of the Lie point symmetries of the nonlinear diffusion equation for pressure,

$$\frac{\partial p}{\partial t} = \frac{1}{2(-\frac{\partial p}{\partial x})^{\frac{1}{2}}}\frac{\partial^2 p}{\partial x^2}, \tag{A.1}$$

will be outlined.

Equation (A.1) can be written as

$$F(p, p_t, p_x, p_{xx}) = 0, \tag{A.2}$$

where

$$F = \frac{\partial p}{\partial t} - \frac{1}{2(-\frac{\partial p}{\partial x})^{\frac{1}{2}}}\frac{\partial^2 p}{\partial x^2}. \tag{A.3}$$

The Lie point symmetry generator is of the form

$$X = \xi^1(t,x,p)\frac{\partial}{\partial t} + \xi^2(t,x,p)\frac{\partial}{\partial x} + \eta(t,x,p)\frac{\partial}{\partial p}. \tag{A.4}$$

The determining equation is

$$X^{[2]}F \mid_{F=0} = 0, \tag{A.5}$$

where $X^{[2]}$ is the second prolongation of X given by

$$X^{[2]} = X + \zeta_t \frac{\partial}{\partial p_t} + \zeta_x \frac{\partial}{\partial p_x} + \zeta_{xx} \frac{\partial}{\partial p_{xx}}, \tag{A.6}$$

and ζ_i and ζ_{ij} are defined by

$$\zeta_i = D_i(\eta) - p_k D_i(\xi^k), \tag{A.7}$$

$$\zeta_{ij} = D_j(\eta_i) - p_{ik} D_j(\xi^k), \tag{A.8}$$

where

$$D_t = \frac{\partial}{\partial t} + p_t \frac{\partial}{\partial p} + p_{tt} \frac{\partial}{\partial p_t} + p_{tx} \frac{\partial}{\partial p_x} \tag{A.9}$$

$$D_x = \frac{\partial}{\partial x} + p_x \frac{\partial}{\partial p} + p_{xx} \frac{\partial}{\partial p_x} + p_{tx} \frac{\partial}{\partial p_t}. \tag{A.10}$$

The determining equation(A.5) becomes

$$\zeta_t + \zeta_x \left(\frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} \right) + \zeta_{xx} \left(-\frac{1}{2(kp_x)^{\frac{1}{2}}} \right) = 0, \tag{A.11}$$

where $k = -1$.

Expansions for ζ_t, ζ_x and ζ_{xx} are required:

$$\zeta_t = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial p} - \frac{\partial \xi^1}{\partial t} \right) p_t - p_t^2 \frac{\partial \xi^1}{\partial p} - p_x \frac{\partial \xi^2}{\partial t} - p_x p_t \frac{\partial \xi^2}{\partial p}, \tag{A.12}$$

$$\zeta_t = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial p} - \frac{\partial \xi^2}{\partial x} \right) p_x - p_t \frac{\partial \xi^1}{\partial x} - p_x p_t \frac{\partial \xi^1}{\partial p} - p_x^2 \frac{\partial \xi^2}{\partial p}, \tag{A.13}$$

$$\begin{aligned} \zeta_{xx} &= \frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial p \partial x} - \frac{\partial^2 \xi^2}{\partial x^2} \right) p_x + \left(-\frac{\partial^2 \xi^1}{\partial x^2} \right) p_t + \left(\frac{\partial \eta}{\partial p} - 2\frac{\partial \xi^2}{\partial x} \right) p_{xx} + \left(-2\frac{\partial \xi^1}{\partial x} \right) p_{xt} \\ &\quad + \left(\frac{\partial^2 \eta}{\partial p^2} - 2\frac{\partial^2 \xi^2}{\partial p \partial x} \right) p_x^2 + \left(-2\frac{\partial^2 \xi^1}{\partial x \partial p} \right) p_x p_t + \left(-\frac{\partial^2 \xi^2}{\partial p^2} \right) p_x^3 + \left(-\frac{\partial^2 \xi^1}{\partial p^2} \right) p_t p_x^2 \\ &\quad + \left(-3\frac{\partial \xi^2}{\partial p} \right) p_x p_{xx} + \left(-\frac{\partial^2 \xi^1}{\partial p} \right) p_t p_{xx} + \left(-2\frac{\partial \xi^1}{\partial p} \right) p_x p_{xt}. \end{aligned} \tag{A.14}$$

The determining equation (A.14), using expansions (A.12),(A.13) and (A.14) becomes

$$\begin{aligned}
X^{[2]}F &= \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial p} - \frac{\partial \xi^1}{\partial t}\right) p_t - \frac{\partial \xi^1}{\partial p} p_t^2 - \frac{\partial \xi^2}{\partial t} p_x - \frac{\partial \xi^2}{\partial p} p_x p_t \\
&+ \frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} \left(\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial p} - \frac{\partial \xi^2}{\partial x}\right) p_x - p_t \frac{\partial \xi^1}{\partial x} - p_t p_x \frac{\partial \xi^1}{\partial p} - p_x^2 \frac{\partial \xi^2}{\partial p}\right) \\
&- \frac{\partial 1}{\partial 2(kp_x)^{\frac{1}{2}}} \left(\frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial p \partial x} - \frac{\partial^2 \xi^2}{\partial x^2}\right) p_x - \frac{\partial^2 \xi^1}{\partial x^2} p_t + \left(\frac{\partial \eta}{\partial p} - 2\frac{\partial \xi^2}{\partial x}\right) p_{xx}\right) \\
&- 2\frac{\partial \xi^1}{\partial x} p_{xt} + \left(\frac{\partial^2 \eta}{\partial p^2} - 2\frac{\partial^2 \xi^2}{\partial p \partial x}\right) p_x^2 + \left(-2\frac{\partial^2 \xi^1}{\partial x \partial p}\right) p_x p_t - \frac{\partial^2 \xi^2}{\partial p^2} p_x^3 - \frac{\partial^2 \xi^1}{\partial p^2} p_t p_x^2 \\
&- 3\frac{\partial \xi^2}{\partial p} p_x p_{xx} - \frac{\partial \xi^1}{\partial p} p_t p_{xx} - 2\frac{\partial \xi^1}{\partial p} p_x p_{xt} \qquad \text{(A.15)}
\end{aligned}$$

To impose the condition $F = 0$, replace p_t in equation (A.15) using

$$p_t = \frac{1}{2(kp_x)^{\frac{1}{2}}}. \qquad \text{(A.16)}$$

Thus equation (A.15) becomes

$$\begin{aligned}
X^{[2]}F &= \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial p} \frac{p_{xx}}{2(kp_x)^{\frac{1}{2}}} - \frac{\partial \xi^1}{\partial p} \frac{p_{xx}}{2(kp_x)^{\frac{1}{2}}} - \frac{\partial \xi^1}{\partial p} \frac{p_{xx}^2}{4kp_x} - \frac{\partial \xi^2}{\partial t} p_x - \frac{\partial \xi^2}{\partial p} p_x \frac{p_{xx}}{2(kp_x)^{\frac{1}{2}}} \\
&+ \frac{\partial \eta}{\partial x} \frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} + \frac{\partial \eta}{\partial p} + \frac{\partial \eta}{\partial p} p_x \frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} - \frac{\partial \xi^2}{\partial x} p_x \frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} - \frac{\partial \xi^2}{\partial x} \frac{p_{xx}^2}{8(kp_x)^2} \\
&- \frac{\partial \xi^1}{\partial p} p_x \frac{p_{xx}}{8(kp_x)^{\frac{1}{2}}} - \frac{\partial \xi^2}{\partial p} p_x^2 \frac{p_{xx}}{4(kp_x)^{\frac{3}{2}}} - \frac{\partial^2 \eta}{\partial x^2} \frac{1}{2(kp_x)^{\frac{1}{2}}} - \frac{\partial^2 \eta}{\partial p \partial x} \frac{p_x}{(kp_x)^{\frac{1}{2}}} \\
&+ \frac{\partial^2 \xi^2}{\partial x^2} \frac{p_x}{2(kp_x)^{\frac{1}{2}}} + \frac{\partial^2 \xi^1}{\partial x^2} \frac{p_{xx}}{4(kp_x)} - \frac{\partial \eta}{\partial p} \frac{p_{xx}}{2(kp_x)^{\frac{1}{2}}} + \frac{\partial \xi^2}{\partial x} \frac{p_{xx}}{(kp_x)^{\frac{1}{2}}} + \frac{\partial \xi^2}{\partial x} \frac{p_{xt}}{(kp_x)^{\frac{1}{2}}} \\
&- \frac{\partial^2 \eta}{\partial p^2} \frac{p_x^2}{2(kp_x)^{\frac{1}{2}}} + \frac{\partial^2 \xi^2}{\partial p \partial x} \frac{p_x^2}{(kp_x)^{\frac{1}{2}}} + \frac{\partial^2 \xi^1}{\partial x \partial p} \frac{p_{xx}}{2k} + \frac{\partial^2 \xi^2}{\partial p^2} \frac{p_x^3}{2(kp_x)^{\frac{1}{2}}} + \frac{\partial^2 \xi^1}{\partial p^2} \frac{p_{xx} p_x^2}{4(kp_x)} \\
&+ \frac{\partial \xi^2}{\partial p} \frac{3p_x p_{xx}}{2(kp_x)^{\frac{1}{2}}} + \frac{\partial \xi^1}{\partial p} \frac{p_{xx}^2}{4(kp_x)} + \frac{\partial \xi^1}{\partial p} \frac{p_x p_{xt}}{(kp_x)^{\frac{1}{2}}} = 0. \qquad \text{(A.17)}
\end{aligned}$$

Equate the coefficients of separate powers of the derivatives of p to zero.

$$\text{remainder} \quad : \quad \frac{\partial \eta}{\partial t} = 0 \tag{A.18}$$

$$p_{xx}(-p_x)^{\frac{1}{2}} \quad : \quad \frac{1}{4}\frac{\partial \eta}{\partial p} - \frac{1}{2}\frac{\partial \xi^1}{\partial t} + \frac{3}{4}\frac{\partial \xi^2}{\partial x} = 0 \tag{A.19}$$

$$p_x \quad : \quad \frac{\partial \xi^2}{\partial t} = 0 \tag{A.20}$$

$$p_{xx}^2(-p_x) \quad : \quad \frac{\partial \xi^1}{\partial p} = 0, \tag{A.21}$$

$$p_{xx}(-p_x)^{\frac{1}{2}} \quad : \quad \frac{\partial \xi^2}{\partial p} = 0, \tag{A.22}$$

$$p_{xx}(-p_x)^{-\frac{3}{2}} \quad : \quad \frac{\partial \eta}{\partial x} = 0, \tag{A.23}$$

$$p_{xx}^2(-p_x)^{-2} \quad : \quad \frac{\partial \xi^1}{\partial x} = 0, \tag{A.24}$$

$$p_{xx}^2(-p_x)^{-1} \quad : \quad \frac{\partial \xi^1}{\partial p} = 0, \tag{A.25}$$

$$(-p_x)^{-\frac{1}{2}} \quad : \quad \frac{\partial^2 \eta}{\partial x^2} = 0, \tag{A.26}$$

$$(-p_x)^{\frac{1}{2}} \quad : \quad \frac{\partial^2 \eta}{\partial x \partial p} + \frac{1}{2}\frac{\partial^2 \xi^2}{\partial x^2} = 0, \tag{A.27}$$

$$p_{xx}(-p_x)^{-1} \quad : \quad \frac{\partial^2 \xi^1}{\partial x^2} = 0, \tag{A.28}$$

$$p_{xt}(-p_x)^{-\frac{1}{2}} \quad : \quad \frac{\partial \xi^1}{\partial x} = 0, \tag{A.29}$$

$$(-p_x)^{\frac{3}{2}} \quad : \quad -\frac{1}{2}\frac{\partial^2 \eta}{\partial p^2} + \frac{\partial \xi^2}{\partial p \partial x} = 0, \tag{A.30}$$

$$p_t(-p_x)^{-\frac{1}{2}} \quad : \quad \frac{\partial^2 \xi^1}{\partial x \partial p} = 0, \tag{A.31}$$

$$(-p_x)^{\frac{5}{2}} \quad : \quad \frac{\partial^2 \xi^2}{\partial p^2} = 0, \tag{A.32}$$

$$\begin{aligned} p_{xx}(-p_x) \quad &: \quad \frac{\partial^2 \xi^1}{\partial p^2} = 0 \\ p_{xt}(-p_x)^{\frac{1}{2}} \quad &: \quad -\frac{\partial \xi^1}{\partial p} = 0. \end{aligned} \tag{A.33}$$

Consider equations (A.21),(A.24),(A.25),(A.29),(A.30),(A.33) and (A.34), It follows that

$$\frac{\partial \xi^1}{\partial p} = 0 \tag{A.34}$$

and

$$\frac{\partial \xi^1}{\partial x} = 0, \tag{A.35}$$

therefore

$$\xi^1 = \xi^1(t). \tag{A.36}$$

From equations (A.20),(A.22) and (A.33) it follows that

$$\frac{\partial \xi^2}{\partial t} = 0, \tag{A.37}$$

and

$$\frac{\partial \xi^2}{\partial p} = 0 \tag{A.38}$$

and thus

$$\xi^2 = \xi^2(x). \tag{A.39}$$

From equations (A.18),(A.23) and (A.26) we obtain the following result

$$\eta = \eta(p) \tag{A.40}$$

since

$$\frac{\partial \eta}{\partial t} = 0 \tag{A.41}$$

and

$$\frac{\partial \eta}{\partial x} = 0. \tag{A.42}$$

By substituting equation (A.39) into equation (A.28) we see that

$$\frac{d^2 \eta}{dp^2} = 0, \tag{A.43}$$

and therefore

$$\eta(p) = c_1 p + c_2. \tag{A.44}$$

By substituting equation (A.40 into equation (A.31) we see that it simplifies to

$$\frac{d^2\xi^2}{dx^2} = 0, \tag{A.45}$$

and therefore by solving equation (A.45) we find that

$$\xi^2(x) = c_3 x + c_4. \tag{A.46}$$

Substitute (A.44) and (A.46) into equation (A.19). This gives

$$\frac{\partial \xi^1}{\partial t} = \frac{1}{2}c_1 + \frac{3}{2}c_3 \tag{A.47}$$

and hence

$$\xi^1(t) = \frac{1}{2}(c_1 + 3c_3)t + c_5. \tag{A.48}$$

The Lie point symmetry generator is

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p} \tag{A.49}$$

$$= \frac{1}{2}c_1 t \frac{\partial}{\partial t} + c_1 p \frac{\partial}{\partial p} + c_2 \frac{\partial}{\partial p} + \frac{3}{2}c_3 t.\frac{\partial}{\partial t} + c_3 x \frac{\partial}{\partial x}, + c_5 \frac{\partial}{\partial t} + c_4 \frac{\partial}{\partial x}. \tag{A.50}$$

and therefore the resulting Lie point symmetries are

$$X_1 = t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}, \tag{A.51}$$

$$X_2 = 3t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}, \tag{A.52}$$

$$X_3 = \frac{\partial}{\partial p}, \tag{A.53}$$

$$X_4 = \frac{\partial}{\partial x}, \tag{A.54}$$

$$X_5 = \frac{\partial}{\partial t}. \tag{A.55}$$

Appendix B

Lie point symmetries for the nonlinear diffusion equation for turbulent fluid driven fracture in rock

The nonlinear diffusion equation for a fluid driven fracture is given by

$$\frac{\partial h}{\partial t} = -D^{*}\frac{\partial}{\partial x}\left[\left(-h^{3}\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}\right], \tag{B.1}$$

where

$$D = D\Lambda^{\frac{1}{m+2}}. \tag{B.2}$$

Let $t^{*} = Dt^{*}$ and as a result we can reduce equation (B.1) to

$$\frac{\partial h}{\partial t^{*}} = -\frac{\partial}{\partial x}\left[\left(-h^{3}\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}\right]. \tag{B.3}$$

For simplicity we suppress the star, let $t = t^{*}$. The partial differential equation becomes

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left[\left(-h^{3}\frac{\partial h}{\partial x}\right)^{\frac{1}{m+2}}\right]. \tag{B.4}$$

Equation (B.4) can be written as

$$F\left(h, h_t, h_x, h_{xx}\right) = 0, \tag{B.5}$$

where

$$F = h_t + \frac{3k}{(m+2)}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{m+3}{m+2}} + \frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-m-1}{m+2}}h_{xx} \tag{B.6}$$

and $k = (-1)^{\frac{1}{m+2}}$.

The Lie point symmetry generator is of the form

$$X = \xi^1(t,x,h)\frac{\partial}{\partial t} + \xi^2(t,x,h)\frac{\partial}{\partial x} + \eta(t,x,h)\frac{\partial}{\partial h}. \tag{B.7}$$

The determining equation is

$$X^{[2]}F \mid_{F=0} = 0, \tag{B.8}$$

where $X^{[2]}$ is the second prolongation of X given by

$$X^{[2]}|F_{F=0} = \xi^1(t,x,h)\frac{\partial}{\partial t} + \xi^2(t,x,h)\frac{\partial}{\partial x} + \eta(t,x,h)\frac{\partial}{\partial h} + \zeta_t\frac{\partial}{\partial h_t} + \zeta_x\frac{\partial}{\partial h_x} + \zeta_{xx}\frac{\partial}{\partial h_{xx}}, \tag{B.9}$$

and ζ_i and ζ_{ij} are defined by

$$\zeta_i = D_i(\eta) - h_k D_i(\xi^k), \tag{B.10}$$

$$\zeta_{ij} = D_j(\zeta_i) - h_{ik} D_j(\xi^k), \tag{B.11}$$

where

$$D_t = \frac{\partial}{\partial t} + h_t\frac{\partial}{\partial h} + h_{tt}\frac{\partial}{\partial h_t} + h_{tx}\frac{\partial}{\partial h_x} \tag{B.12}$$

$$D_x = \frac{\partial}{\partial x} + h_x\frac{\partial}{\partial h} + h_{xx}\frac{\partial}{\partial h_x} + h_{tx}\frac{\partial}{\partial h_t}. \tag{B.13}$$

The determining equation (B.8) becomes

$$\begin{aligned}
= \quad & \frac{3(1-m)k}{(m+2)^2}(h)^{\frac{-1-2m}{m+2}}(h_x)^{\frac{m+3}{m+2}} + \eta\frac{3k}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{-m-1}{m+2}}h_{xx} + \zeta_t \\
& + \zeta_x\left(\frac{3(m+3)k}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{1}{m+2}} - \frac{(1+m)k}{(m+2)^2}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-3-2m}{m+2}}h_{xx}\right) \\
& + \zeta_{xx}\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-1-m}{m+2}}.
\end{aligned} \tag{B.14}$$

Expansions for ζ_t, ζ_x and ζ_{xx} are required:

$$\zeta_t = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \eta}{\partial h} - \frac{\partial \xi^1}{\partial t}\right) h_t - h_t^2 \frac{\partial \xi^1}{\partial h} - h_x \frac{\partial \xi^2}{\partial t} - h_x h_t \frac{\partial \xi^2}{\partial h} \tag{B.15}$$

$$\zeta_t = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial h} - \frac{\partial \xi^2}{\partial x}\right) h_x - h_t \frac{\partial \xi^1}{\partial x} - h_x h_t \frac{\partial \xi^1}{\partial h} - h_x^2 \frac{\partial \xi^2}{\partial h} \tag{B.16}$$

$$\begin{aligned}
\zeta_{xx} \;=\; & \frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial h \partial x} - \frac{\partial^2 \xi^2}{\partial x^2}\right) h_x + \left(-\frac{\partial^2 \xi^1}{\partial x^2}\right) h_t + \left(\frac{\partial \eta}{\partial h} - 2\frac{\partial \xi^2}{\partial x}\right) h_{xx} + \left(-2\frac{\partial \xi^1}{\partial x}\right) h_{xt} \\
& + \left(\frac{\partial^2 \eta}{\partial h^2} - 2\frac{\partial^2 \xi^2}{\partial h \partial x}\right) h_x^2 + \left(-2\frac{\partial^2 \xi^1}{\partial x \partial h}\right) h_x h_t + \left(-\frac{\partial^2 \xi^2}{\partial h^2}\right) h_x^3 + \left(-\frac{\partial^2 \xi^1}{\partial h^2}\right) h_t h_x^2 \\
& + \left(-3\frac{\partial \xi^2}{\partial h}\right) h_x h_{xx} + \left(-\frac{\partial^2 \xi^1}{\partial h}\right) h_t h_{xx} + \left(-2\frac{\partial \xi^1}{\partial h}\right) h_x h_{xt}
\end{aligned} \tag{B.17}$$

The determining equation (B.14), using expansions (B.15),(B.16) and (B.17) becomes

$$
\begin{aligned}
X^{[2]}|_{F=0} \;=\;& \frac{3(1-m)k}{(m+2)^2}h^{\frac{-1-2m}{m+2}}h_x^{\frac{m+3}{m+2}}[\eta] + \frac{3k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{-1-m}{m+2}}h_{xx}[\eta] \\
&+\frac{\partial\eta}{\partial t} + h_t\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^1}{\partial t}\right] - h_x\left[\frac{\partial\xi^2}{\partial t}\right] - h_t^2\left[\frac{\partial\xi^1}{\partial h}\right] - h_x h_t\left[\frac{\partial\xi^2}{\partial h}\right] \\
&+\frac{3(m+3)k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{1}{m+2}}\left[\frac{\partial\eta}{\partial x}\right] + \frac{3(m+3)k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{m+3}{m+2}}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^2}{\partial x}\right] \\
&-\frac{3(m+3)k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{1}{m+2}}h_t\left[\frac{\partial\xi^1}{\partial x}\right] - \frac{3(m+3)k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{5+2m}{m+2}}\left[\frac{\partial\xi^2}{\partial h}\right] \\
&-\frac{3(m+3)k}{(m+2)^2}h^{\frac{1-m}{m+2}}h_x^{\frac{m+3}{m+2}}h_t\left[\frac{\partial\xi^1}{\partial h}\right] - \frac{(m+1)k}{(m+2)^2}h^{\frac{3}{m+2}}h_x^{\frac{-3-2m}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial x}\right] \\
&-\frac{(m+1)k}{(m+2)^2}h^{\frac{3}{m+2}}h_x^{\frac{-m-1}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^2}{\partial x}\right] + \frac{(m+1)k}{(m+2)^2}h^{\frac{3}{m+2}}h_x^{\frac{-3-2m}{m+2}}h_{xx}h_t\left[\frac{\partial\xi^1}{\partial x}\right] \\
&+\frac{(m+1)k}{(m+2)^2}h^{\frac{3}{m+2}}h_x^{\frac{1}{m+2}}h_{xx}\left[\frac{\partial\xi^2}{\partial h}\right] + \frac{(m+1)k}{(m+2)^2}h^{\frac{3}{m+2}}h_x^{\frac{-m-1}{m+2}}h_t\left[\frac{\partial\xi^1}{\partial h}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}\left[\frac{\partial^2\eta}{\partial x^2}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{1}{m+2}}\left[2\frac{\partial^2\eta}{\partial h\partial x} - \frac{\partial^2\xi^2}{\partial x^2}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}h_t\left[-\frac{\partial^2\xi^2}{\partial x^2}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial h} - 2\frac{\partial\xi^2}{\partial x}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}h_{xt}\left[-2\frac{\partial\xi^1}{\partial x}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{m+3}{m+2}}\left[\frac{\partial^2\eta}{\partial h^2} - 2\frac{\partial^2\xi^2}{\partial h\partial x}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{1}{m+2}}h_t\left[-2\frac{\partial^2\xi^1}{\partial x\partial h}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{2m+5}{m+2}}\left[-\frac{\partial^2\xi^2}{\partial h^2}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{m+3}{m+2}}h_t\left[-\frac{\partial^2\xi^1}{\partial h^2}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{1}{m+2}}h_{xx}\left[-3\frac{\partial\xi^2}{\partial h}\right] \\
&+\frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}h_t h_{xx}\left[-\frac{\partial\xi^1}{\partial h}\right] + \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{1}{m+2}}h_{xt}\left[-2\frac{\partial\xi^1}{\partial h}\right] \quad \text{(B.18)}
\end{aligned}
$$

Since $X^{[2]}$ is evaluated at $F = 0$, we replace h_t with

$$h_t = -\frac{3k}{(m+2)}h^{\frac{1-m}{m+2}}h_x^{\frac{m+3}{m+2}} - \frac{k}{(m+2)}h^{\frac{3}{m+2}}h_x^{\frac{-1-m}{m+2}}h_{xx}. \quad \text{(B.19)}$$

Thus equation(B.18) becomes

$$
\begin{aligned}
X^{[2]} \;=\;& \frac{1-m}{(m+2)^2}k(h)^{\frac{-1-2m}{m+2}}(h_x)^{\frac{m+3}{m+2}}[\eta] + \frac{3k}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{-1-m}{m+2}}h_{xx}[\eta] \\
&+\frac{\partial\eta}{\partial t} - \frac{3k}{(m+2)}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{m+3}{m+2}}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^1}{\partial t}\right] \\
&-\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-1-m}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^1}{\partial t}\right] - h_x\left[\frac{\partial\xi^2}{\partial t}\right] \\
&-\frac{-9k^2}{(m+2)^2}(h)^{\frac{2-2m}{m+2}}(h_x)^{\frac{2m+6}{m+2}}\left[\frac{\partial\xi^1}{\partial h}\right] - \frac{-6k^2}{(m+2)^2}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}h_{xx}\left[\frac{\partial\xi^1}{\partial h}\right] \\
&-\frac{k^2}{(m+2)^2}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-2m-2}{m+2}}(h_{xx})^2\left[\frac{\partial\xi^1}{\partial h}\right] + \frac{3k}{(m+2)}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{2m+5}{m+2}}\left[\frac{\partial\xi^2}{\partial h}\right] \\
&-\frac{(1+m)k}{(m+2)^2}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-3-2m}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial x}\right] - \frac{(1+m)k}{(m+2)^2}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-m-1}{m+2}}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^2}{\partial x}\right] \\
&-\frac{3k^2(1+m)}{(m+2)^3}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{-m}{m+2}}h_{xx}\left[\frac{\partial\xi^1}{\partial h}\right] - \frac{3k^2(1+m)}{(m+2)^3}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-3m-4}{m+2}}(h_{xx})^2\left[\frac{\partial\xi^1}{\partial h}\right] \\
&+\frac{(1+m)k}{(m+2)^2}(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}h_{xx}\left[\frac{\partial\xi^2}{\partial h}\right] - \frac{3k^2(1+m)}{(m+2)^3}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}\left[\frac{\partial\xi^2}{\partial h}\right] \\
&-\frac{3k^2(1+m)}{(m+2)^3}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-2m-2}{m+2}}h_{xx}\left[\frac{\partial\xi^1}{\partial x}\right] + \frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-1-m}{m+2}}\left[\frac{\partial^2\eta}{\partial x^2}\right] \\
&+\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}h_{xx}\left[\frac{\partial\xi^2}{\partial h}\right] + \frac{3k(m+3)}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{1}{m+2}}\left[\frac{\partial\eta}{\partial x}\right] \\
&+\frac{3k(m+3)}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{m+3}{m+2}}\left[\frac{\partial\eta}{\partial h} - \frac{\partial\xi^2}{\partial x}\right] + \frac{9k^2(m+2)}{(m+2)^3}(h)^{\frac{2-2m}{m+2}}(h_x)^{\frac{m+4}{m+2}}\left[\frac{\partial\xi^1}{\partial x}\right] \\
&+\frac{3k^2(m+3)}{(m+2)^3}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{-m}{m+2}}h_{xx}\left[\frac{\partial\xi^1}{\partial x}\right] - \frac{3k(m+3)}{(m+2)^2}(h)^{\frac{1-m}{m+2}}(h_x)^{\frac{5+2m}{m+2}}\left[\frac{\partial\xi^2}{\partial h}\right] \\
&+\frac{9k^2(m+3)}{(m+2)^2}(h)^{\frac{2-2m}{m+2}}(h_x)^{\frac{2m+6}{m+2}}\left[\frac{\partial\xi^1}{\partial h}\right] + \frac{3k^2(m+3)}{(m+2)^2}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}h_{xx}\left[\frac{\partial\xi^1}{\partial h}\right] \\
&+\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}\left[2\frac{\partial^2\eta}{\partial x\partial h} - \frac{\partial^2\xi^2}{\partial x^2}\right] - \frac{3k^2}{(m+2)}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}\left[-\frac{\partial^2\xi^1}{\partial x^2}\right] \\
&-\frac{3k^2}{(m+2)}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-2m-2}{m+2}}h_{xx}\left[-\frac{\partial^2\xi^1}{\partial x^2}\right] + \frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-1-m}{m+2}}h_{xx}\left[\frac{\partial\eta}{\partial h} - \frac{2\partial\xi^2}{\partial x}\right] \\
&+\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{-1-m}{m+2}}h_{xt}\left[-2\frac{\partial\xi^1}{\partial x}\right] + \frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{m+3}{m+2}}\left[\frac{\partial^2\eta}{\partial h^2} - 2\frac{\partial^2\xi^2}{\partial x\partial h}\right] \\
&-\frac{3k^2}{(m+2)^2}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}h_{xx}\left[-\frac{\partial\xi^1}{\partial h}\right] - \frac{k^2}{(m+2)^2}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-2m-2}{m+2}}h_{xx}^2\left[-\frac{\partial\xi^1}{\partial h}\right] \\
&+\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{2m+5}{m+2}}\left[-\frac{\partial^2\xi^2}{\partial h^2}\right] - \frac{3k^2}{(m+2)^2}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2m+6}{m+2}}\left[-2\frac{\partial^2\xi^1}{\partial x\partial h}\right] \\
&-\frac{k^2}{(m+2)^2}(h)^{\frac{6}{m+2}}(h_x)^{\frac{2}{m+2}}h_{xx}\left[-2\frac{\partial^2\xi^1}{\partial x\partial h}\right] + \frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}h_{xx}\left[-3\frac{\partial\xi^2}{\partial h}\right] \\
&-\frac{3k^2}{(m+2)^2}(h)^{\frac{4-m}{m+2}}(h_x)^{\frac{2}{m+2}}h_{xx}\left[-\frac{\partial\xi^1}{\partial h}\right] - \frac{k^2}{(m+2)^2}(h)^{\frac{6}{m+2}}(h_x)^{\frac{-2m-2}{m+2}}h_{xx}^2\left[-\frac{\partial\xi^1}{\partial h}\right] \\
&+\frac{k}{(m+2)}(h)^{\frac{3}{m+2}}(h_x)^{\frac{1}{m+2}}h_{xt}\left[-2\frac{\partial\xi^1}{\partial h}\right] = 0. \qquad \text{(B.20)}
\end{aligned}
$$

Now, equate the coefficients of each power and product of derivatives of h to zero. We start by considering the powers and products of derivatives that contain h_{xt}.

$$h_x^{\frac{1}{m+2}} h_{xt} \quad : \quad \frac{k}{m+2} h^{\frac{3}{m+2}} \left[-2 \frac{\partial \xi^1}{\partial h} \right] = 0, \tag{B.21}$$

$$h_x^{\frac{-1-m}{m+2}} h_{xt} \quad : \quad \frac{k}{m+2} h^{\frac{3}{m+2}} \left[-2 \frac{\partial \xi^1}{\partial x} \right] = 0, \tag{B.22}$$

From equation(B.21)

$$\frac{\partial \xi^1}{\partial h} = 0. \tag{B.23}$$

and from equation(B.22)

$$\frac{\partial \xi^1}{\partial x} = 0. \tag{B.24}$$

By solving equations(B.23) and (B.24) we obtain the result

$$\xi^1 = \xi^1(t). \tag{B.25}$$

Substitute equation(B.25) into equationB.20 and separate by powers and products of deivatives to obtain the following equations:

$$\begin{aligned} h_x^{\frac{m+3}{m+2}} \quad : \quad & \frac{3k(1-m)}{(m+2)^2} h^{\frac{-1-2m}{m+2}} [\eta] - \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} \left[\frac{\partial \eta}{\partial h} - \frac{\partial \xi^1}{\partial t} \right] \\ & + \frac{3(m+3)k}{(m+2)^2} h^{\frac{1-m}{m+2}} \left[\frac{\partial \eta}{\partial h} - \frac{\partial \xi^2}{\partial x} \right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial^2 \eta}{\partial h^2} - 2 \frac{\partial^2 \xi^2}{\partial t \partial x} \right] = 0, \end{aligned} \tag{B.26}$$

$$\begin{aligned} h_x^{\frac{-1-m}{m+2}} h_{xx} \quad : \quad & \frac{3k}{(m+2)^2} h^{\frac{1-m}{m+2}} [\eta] - \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial \eta}{\partial h} - \frac{\partial \xi^1}{\partial t} \right] \\ & - \frac{(m+1)k}{(m+2)^2} h^{\frac{3}{m+2}} \left[\frac{\partial \eta}{\partial h} - \frac{\partial \xi^2}{\partial x} \right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial \eta}{\partial h} - 2 \frac{\partial \xi^2}{\partial x} \right] = 0, \end{aligned} \tag{B.27}$$

$$h_x^{\frac{2m+5}{m+2}} \quad : \quad \frac{3k}{(m+2)} h^{\frac{1-m}{m+2}} \left[\frac{\partial \xi^2}{\partial h}\right] - \frac{3k(m+3)}{(m+2)^2} h^{\frac{1-m}{m+2}} \left[\frac{\partial \xi^2}{\partial h}\right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[-\frac{\partial^2 \xi^2}{\partial h^2}\right] = 0, \tag{B.28}$$

$$h_x^{\frac{-3-2m}{m+2}} h_{xx} \quad : \quad \frac{-(1+m)}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial \eta}{\partial x}\right] = 0, \tag{B.29}$$

$$h_x^{\frac{1}{m+2}} h_{xx} \quad : \quad \frac{k(1+m)}{(m+2)^2} h^{\frac{3}{m+2}} \left[\frac{\partial \xi^2}{\partial h}\right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial \xi^2}{\partial h}\right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[-3\frac{\partial \xi^2}{\partial h}\right] = 0, \tag{B.30}$$

$$h_x^{\frac{-1-m}{m+2}} \quad : \quad \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[\frac{\partial^2 \eta}{\partial x^2}\right] = 0, \tag{B.31}$$

$$h_x^{\frac{1}{m+2}} \quad : \quad \frac{3(m+3)k}{(m+2)^2} h^{\frac{1-m}{m+2}} \left[\frac{\partial \eta}{\partial x}\right] + \frac{k}{(m+2)} h^{\frac{3}{m+2}} \left[2\frac{\partial^2 \eta}{\partial x \partial h} - \frac{\partial^2 \xi^2}{\partial x^2}\right] = 0, \tag{B.32}$$

$$h_x \quad : \quad \frac{\partial \xi^2}{\partial t} = 0. \tag{B.33}$$

The remainder is :

$$\frac{\partial \eta}{\partial t} = 0. \tag{B.34}$$

Special considerations have to be made for different values of m, as they can result in a different set of equations after separating equation(B.18) by powers and products of derivatives. In order to find at which values of m this occurs, we compare the powers of derivatives and try to find values of m for which they are equal.We start our analysis by considering powers and products of derivatives of the form $h_x^* h_{xx}$.

Table B.1: Derivatives of the form $h_x^{*}h_{xx}$

Derivatives	$h_x^{\frac{-1-m}{m+2}}h_{xx}$	$h_x^{\frac{1}{m+2}}h_{xx}$	$h_x^{\frac{-3-2m}{m+2}}h_{xx}$
$h_x^{\frac{-1-m}{m+2}}h_{xx}$	-	$m=-2$	$m=-2$
$h_x^{\frac{1}{m+2}}h_{xx}$	$m=-2$	-	$m=-2$
$h_x^{\frac{-3-2m}{m+2}}h_{xx}$	$m=-2$	$m=-2$	$-$

From Table B.1 the only value for m at which powers of derivatives are equal is $m = -2$. However, from the form of equation(B.4) we see that $m \neq -2$, therefore equations (B.27), (B.29) and (B.30) hold for all possible vaues of m. Consider equation(B.29), which simplifies to

$$\frac{\partial \eta}{\partial x} = 0, \tag{B.35}$$

and leads to the result

$$\eta = \eta(t, h). \tag{B.36}$$

By also considering equation(B.29) simplifies to

$$\frac{\partial \xi^2}{\partial h} = 0, \tag{B.37}$$

and leads to the result

$$\xi^2 = \xi^2(x, t). \tag{B.38}$$

Using equations (B.36) and (B.38) we find that equations (B.28) and (B.31) are identically satisfied.

We now consider derivatives of the form h_{xx}^{*} and find special cases of m for which their powers are equal.

We will now derive the Lie point symmetries for the general case ($m \neq -2, -1, -3$).

Table B.2: Derivatives of the form h_{xx}^{*}

Derivatives	$h_x^{\frac{m+3}{m+2}}$	$h_x^{\frac{1}{m+2}}$	h_x	h_x^0
$h_x^{\frac{m+3}{m+2}}$	-	$m=-2$	-	$m=-3$
$h_x^{\frac{1}{m+2}}$	$m=-2$	-	$m=-1$	-
h_x	-	$m=-1$	-	-
h_x^0	$m=-3$	-	-	-

B.0.1 Lie point symmetries for the general case ($m \neq -2, -1, -3$)

From equation (B.33) we further deduce that

$$\xi^2 = \xi^2(x). \tag{B.39}$$

Equation (B.32) reduces to

$$\frac{\partial^2 \xi^2}{\partial x^2} = 0 \tag{B.40}$$

and thus

$$\xi^2 = c_1 x + c_2. \tag{B.41}$$

By differentiating equation (B.27) with respect to t we obtain

$$\frac{\partial^2 \xi^1}{\partial t^2} = 0, \tag{B.42}$$

which has the solution

$$\xi^1 = c_3 t + c_4. \tag{B.43}$$

We substitute the results

$$\xi^1 = c_3 t + c_4, \tag{B.44}$$

$$\xi^2 = c_1 x + c_2, \tag{B.45}$$

$$\eta = \eta(h). \tag{B.46}$$

into equation (B.27), which simplifies to

$$-\frac{3}{h(1+m)}\eta + \frac{d\eta}{dh} = -\frac{(m+3)}{(m+1)}c_1 + \frac{(m+2)}{(m+1)}c_3. \tag{B.47}$$

Equation (B.47) is a first order ordinary differential equation that can be solved using an integrating factor.

The integrating factor is

$$\begin{aligned} IF &= e^{-\int \frac{3}{h(m+1)}dh} \\ &= h^{-\frac{3}{1+m}}. \end{aligned} \tag{B.48}$$

which can be used as $m \neq 1$. The integrating factor we obtained is only valid for all $m \neq -1$, we would therefore have to consider the case for when $m = -1$ separately.

Case $m \neq 2, -1, -2, -3$**.**

We multiply the integrating factor given by equation(B.48), through equation (B.47) which in turn gives us the expression

$$\frac{d}{dh}\left[h^{-\frac{3}{1+m}}\eta dh\right] = \left[\frac{(m+2)}{(m+1)}c_3 - \frac{(m+3)}{(m+1)}c_1\right] h^{\frac{-3}{1+m}}dh. \tag{B.49}$$

The solution to equation (B.47) is given by

$$\eta(h) = \left[\frac{(m+2)}{(m-2)}c_3 - \frac{(m+3)}{(m-2)}c_1\right] h + c_5 h^{\frac{3}{1+m}}. \tag{B.50}$$

Now substitute the results for ξ^1, ξ^2, and η into the following equation

$$(1-m)\eta + \frac{\partial \eta}{\partial h}h + (m+2)\frac{\partial \xi^1}{\partial t}h - (m+3)\frac{\partial \xi^2}{\partial x}h + \frac{1}{3}(m+2)h^2\frac{\partial^2 \eta}{\partial h^2} \quad = \quad 0. \tag{B.51}$$

Equation (B.51) was obtained by simplifying equation (B.26). We now find that $c_5 = 0$. It follows that,

$$\eta(h) = \left[\frac{(m+2)}{(m-2)}c_3 - \frac{(m+3)}{(m-2)}c_1\right]h. \tag{B.52}$$

The Lie point symmetry is of the form

$$X = c_1X_1 + c_2X_2 + c_2X_3 + c_4X_4, \tag{B.53}$$

where

$$X_1 \quad = \quad t\frac{\partial}{\partial t} + \frac{(m+2)}{(m-2)}h\frac{\partial}{\partial h}, \tag{B.54}$$

$$X_2 \quad = \quad x\frac{\partial}{\partial x} - \frac{(m+3)}{(m-2)}h\frac{\partial}{\partial h}, \tag{B.55}$$

$$X_3 \quad = \quad \frac{\partial}{\partial x} \tag{B.56}$$

$$X_4 \quad = \quad \frac{\partial}{\partial t}. \tag{B.57}$$

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