Markov Processes and Martingale Generalisations on Riesz Spaces

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ABSTRACT

In a series of papers by Wen-Chi Kuo, Coenraad Labuschagne and Bruce Watson results of martingale theory were generalised to the abstract setting of Riesz spaces. This thesis presents a survey of those results proved and aims to expand upon the work of these authors. In particular, independence results will be considered and these will be used to generalise well known results in the theory of Markov processes to Riesz spaces.

Mixingales and quasi-martingales will be translated to the Riesz space setting.

PREFACE

Sections of this PhD. Thesis have been submitted for publication. They are as follows:

Chapter 3, Section 3.2 and Chapter 4, Section 4.3.

Chapter 5, Section 5.2.

Chapter 6, Section 6.2.

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DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. This thesis was converted from a dissertation submitted towards the Degree of Master of Science, under rule 12.2(b). As such, a portion of this work (Chapters 2 - 4) has been examined prior.



Jessica Joy Vardy

Signed on this the 24^{th} day of April 2013, at Johannesburg, South Africa.

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Chapter 1

Introduction

Studying stochastic processes in Riesz spaces gives us insight into the underlying structure of the processes. In this thesis, we focus on Markov processes and martingale generalisations. We continue the work of Kuo, Labuschagne and Watson on the generalisation of stochastic process to the Riesz space setting, see [37, 38]. Other generalisations and studies of martingales and stochastic processes in the setting of Riesz spaces have been given by Boulabiar, Buskes and Triki [13], Dodds, Grobler, Huijsmans and de Pagter [19, 27, 28], Luxemburg and de Pagter [40], Stoica [57, 58], Troitsky [60].

Each chapter begins with an introduction and description of what is to follow, so here we will briefly outline of the structure of the thesis.

In Chapter 2 we present a literature review of results and definitions pertinent to the work that follows. We first define stochastic processes in the classical setting of probability spaces and present a motivation for the definition of stochastic processes

in Riesz spaces. We then give a survey of the properties of Riesz spaces that are needed to define stochastic processes in Riesz spaces. Linear operators in Riesz spaces are then discussed and used to define conditional expectations in Riesz spaces. We conclude Chaper 2 with a review of known results concerning martingales in Riesz spaces.

In [63], Watson proved an Andô-Douglas-Nikodým-Radon type result for conditional expectation operators in Riesz spaces. In Chapter 3 we build on this result and define the notion of T-independent conditional expectation operators. The notion of T-conditional independence is required in order to translate Markov process results to Riesz spaces. The results of Chapter 3 were published by Vardy and Watson in [61].

Markov processes are considered in Chapter 4. Results relevant to the thesis from classical Markov process theory are given. Following this, the theory is generalised to the Riesz space setting. Independent sums and their relationship to Markov processes are also considered. Again, this work has been published by Vardy and Watson in [61].

Generalisations of martingales are considered in Chapters 5 and 6. Mixingales (a combination of the concepts of martingales and mixing processes) are considered in Chapter 5. We define mixingales in a Riesz space and prove a weak law of large numbers for mixingales in this setting. The content of this chapter has been submitted for publication.

In Chapter 6 we generalise quasi-martingales to the Riesz space setting. We show that quasi-martingales in a Riesz space can be decomposed into the sum of a martingale and a quasi-potential (a Riesz decomposition). If, in addition, the quasi-martingale

is right continuous then the martingale and quasi-potential of this decomposition are also right continuous. Further to this, we show that each right continuous quasi-potential can be decomposed into the difference of two positive potentials. Again, the material of this chapter has been submitted for publication.

Finally, we conclude the thesis with a discussion of further work in Chapter 7.

Chapter 2

Preliminaries

2.1 Classical Stochastic Processes

This thesis is concerned with translating results from the classical setting of L^1 probability spaces to the more abstract setting of Riesz spaces and, as such, it is necessary that we first give a brief introduction to probability theory.

Recall that in a measure space, say $(\Omega, \mathcal{F}, \mu)$, a *random variable* is a measurable, real valued map with domain Ω. That is, X is a random variable if

$$X : \Omega \to \mathbb{R}, \quad X^{-1}(B) \in \mathcal{F}$$

for all Borel sets $B \subset \mathbb{R}$.

In this setting, a *filtration* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a collection of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$. The random variables $(X_n)_{n\in\mathbb{N}}$ are said to be *adapted* to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if X_n is $\mathcal{F}_n$ measurable.

2.1.1 Conditioning

Let $(\Omega, \mathcal{F}, P)$ be a probability space (that is, $(\Omega, \mathcal{F}, P)$ is a measure space with $P(\Omega) = 1$). We start by considering *independence*.

(i) We say that measurable sets A and B (i.e $A, B \in \mathcal{F}$) are independent if

$$P(A \cap B) = P(A)P(B).$$

(ii) If $\mathcal{F}_1, \mathcal{F}_2, \ldots,$ are sub-σ-algebras of $\mathcal{F}$ then $\mathcal{F}_1, \mathcal{F}_2, \ldots,$ are independent if for all $i_1 < i_2 < \cdots < i_n,\ n \in \mathbb{N}$, and $A_{i_j} \in \mathcal{F}_j,\ j = 1, 2, \ldots, n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_n}) = \prod_{j=1}^{n} P(A_{i_j}).$$

(iii) The random variables $X_{t_1}, X_{t_2}, \ldots$ are said to be independent if the σ-algebras generated by them are independent. That is, $\sigma(X_{t_1}), \sigma(X_{t_2}), \ldots$ are independent.

Next we consider the concept of *expectation*. Let $f \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$. We define the expectation of f to be

$$\mathbb{E}[f] = \int_\Omega f \, dP.$$

In order to define conditional expectations, we first need to consider conditional probabilities. If $A, B \in \mathcal{F}$ then the probability that B occurs given that event A has occurred is given by

$$P(B \,|\, A) = \frac{P(A \cap B)}{P(A)}.$$

We note, in this case, $P(\cdot\,|A)$ is a measure defined on $\mathcal{F}$ and the *conditional expectation* can be easily built: Let f be a random variable and consider an event $A \in \mathcal{F}$. The

conditional expectation of f given A is

$$\mathbb{E}[f \mid A] = \frac{\int_A f \, dP}{P(A)} = \int_\Omega f \, dP(\cdot \mid A).$$

In particular, $\mathbb{E}[f|A]$ is the average value of f over A relative to the measure P. If $f = \chi_B$ for some $B \in \mathcal{F}$ we see that

$$\mathbb{E}[\chi_B \mid A] = P(B \mid A).$$

We now consider conditioning on a random variable. If we are given a random variable, $\xi : \Omega \to \mathbb{R}$, which takes on countably many distinct values $(a_j)_{j\in\mathbb{N}}$, then setting $A_j = \xi^{-1}(\{a_j\})$, we have that $\Sigma = \{A_j\}_{j\in\mathbb{N}}$ is a partition of Ω and

$$P(B \mid \xi = a_j) = P(B \mid A_j).$$

The above statement can be more effectively expressed by defining the conditional probability of B relative to the random variable ξ as a new random variable which takes the value $P(B|A_j)$ on the set A_j, i.e.

$$P(B \mid \xi)[x] = P(B \mid \xi^{-1}(\{\xi(x)\})).$$

Here it should be observed that the random variable $P(B|\xi)$ is measurable with respect to the σ-algebra generated by Σ, as it is constant on each A_j. Using this interpretation of the conditional probability we obtain that $\mathbb{E}[f|\xi]$ is a random variable that on each A_j takes on the average of f on A_j.

In particular, suppose X and Z are random variables taking the distinct, discrete values $x_1, x_2, \ldots$ and $z_1, z_2, \ldots$ respectively. As seen above, the probability that $X = x_i$ given $Z = z_j$ is

$$P(X = x_i \mid Z = z_j) = \frac{P(\{\omega \mid X(\omega) = x_i\} \cap \{\omega \mid Z(\omega) = z_j\})}{P(Z = z_j)},$$

and the expected value of X given that $Z = z_j$ is

$$\mathbb{E}[X \mid Z = z_j] = \sum_i x_i P(X = x_i \mid Z = z_j).$$

Thus, $\mathbb{E}[X \mid Z]$ defines the random variable

$$Y(\omega) = \mathbb{E}[X \mid Z = Z(\omega)]$$

that is constant on each of the sets $Z^{-1}(z_j)$ and is thus measurable with respect to the minimal σ-algebra under which Z is measurable, $\sigma(Z)$. We also note that

$$\begin{aligned} \int_{\{\omega | Z(\omega)=z_j\}} Y \, dP &= \sum_i x_i P(\{\omega \mid X(\omega) = x_i\} \cap \{\omega \mid Z(\omega) = z_j\}) \\ &= \int_{\{\omega \mid Z(\omega)=z_j\}} X \, dP. \end{aligned}$$

So, for each $A \in \sigma(Z)$,

$$\int_A Y \, dP = \int_A X \, dP$$

and Y is $\sigma(Z)$ measurable. In particular, the only relevance of the random variable Z in the above construction is that it generates $\sigma(Z)$. That is, if Z' is a random variable which also generates the σ-algebra $\sigma(Z)$, then $\mathbb{E}[X \mid Z] = \mathbb{E}[X|Z']$. Hence, it makes sense to denote $Y = \mathbb{E}[X|Z]$ by

$$Y = \mathbb{E}[X \mid \sigma(Z)].$$

This leads us to the final definition of conditional expectation over a sub-σ-algebra of $\mathcal{F}$.

Definition 2.1.1. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and X be a random variable with respect to $\mathcal{F}$. We define the* conditional expectation of X with respect to $\mathcal{G}$ *as the $\mathcal{G}$-measurable function Y with*

$$\int_A Y \, dP = \int_A X \, dP, \quad \textit{for all} \quad A \in \mathcal{G}$$

and denote $Y = \mathbb{E}[X \mid \mathcal{G}]$.

We note that Y has the same expectation (average) over sets in $\mathcal{G}$ as X but is $\mathcal{G}$-measurable where, in general, X is not. In general, the existence of the conditional expectation relies on the Radon-Nikodým Theorem.

2.1.2 Some Measure Theory

In order to make the above discussion rigorous, we need a few results from Measure Theory.

Let $\mathcal{F}$ be a σ-algebra on Ω. By a signed measure λ on $\mathcal{F}$ we mean a real valued countably additive set function on $\mathcal{F}$. A set $A \in \mathcal{F}$ is said to be totally positive (resp. totally negative) with respect to λ if $\lambda(B) \geq$ (resp. $\leq$) 0 for all $B \subset A$ with $B \in \mathcal{F}$. We say that two measures, μ and ν, are mutually singular if there exist disjoint sets, A and B, such that $A \cup B = \Omega$ and μ is zero on all measurable subsets of A and ν is zero on all measurable subsets of B. In this case, we say μ is concentrated on B and ν is concentrated on A.

Theorem 2.1.2. (Hahn Decomposition Theorem)
If λ is a signed measure on $\mathcal{F}$ then there exist sets $A, B \in \mathcal{F}$ that are respectively totally positive and totally negative with respect to λ such that $A \cap B = \emptyset$ and $A \cup B = \Omega$. The pair (A, B) is called the Hahn decomposition of Ω with respect to λ.

Theorem 2.1.3. (Jordan decomposition theorem)
If λ is a signed measure on $\mathcal{F}$ and (A, B) is the Hahn decomposition of Ω with respect to λ let

$$\lambda^+(C) = \lambda(A \cap C) \quad \textit{and} \quad \lambda^-(C) = -\lambda(B \cap C)$$

for all $C \in \mathcal{F}$. Then $\lambda = \lambda^+ - \lambda^-$ and $\lambda^\pm$ are mutually singular measures with λ^+ concentrated on A and λ^- concentrated on B.

Remark: The measures $\lambda^{\pm}$ can be characterised as the minimal measures with $\lambda = \lambda^{+} - \lambda^{-}$.

A signed measure λ is said to be absolutely continuous with respect to a measure μ, denoted $\lambda << \mu$, if

$$\mu(A) = 0 \Rightarrow \lambda(A) = 0 \quad \text{for all } A \in \mathcal{F}.$$

Theorem 2.1.4. (Radon-Nikodým Theorem)

If μ is a σ-finite measure and λ is a signed measure with $\lambda << \mu$ then there exists $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ such that

$$\lambda(A) = \int_A f \, d\mu \quad \textit{for all } A \in \mathcal{F}.$$

The function f is μ almost everywhere unique and is called the Radon-Nikodým derivative of λ with respect to μ and is denoted by

$$f = \frac{d\lambda}{d\mu}.$$

The Radon-Nikodým theorem is needed to prove the existence of the conditional expectation of an $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ random variable, in general. Again a generalisation of the Radon-Nikodým Theorem is needed when considering stochastic processes on Riesz spaces, as will be demonstrated later. A Riesz space analogue of this result can be found in Chapter 3.

2.1.3 Properties of Conditional Expectations

In this section we will give some important and useful results pertaining to conditional expectations on probability spaces.

Theorem 2.1.5. *If $(\Omega, \mathcal{F}, P)$ is a probability space and X is a P-integrable random variable, then $\mathbb{E}[X \,|\, \Sigma]$ exists and is P-almost everywhere unique.*

Let Σ be a sub-σ-algebra of $\mathcal{F}$. We may define the conditional probability of A given Σ in terms of conditional expectations as follows

$$P(A \,|\, \Sigma) = \mathbb{E}[\chi_A \,|\, \Sigma] \quad \text{for all } A \in \mathcal{F}.$$

The following are well known properties of conditional expectations (see, [51, 53, 64] to name a few) on probability spaces and are useful guides in the generalisation of the theory of stochastic processes to a Riesz space setting:

1. If $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, $\alpha, \beta \in \mathbb{R}$ and Σ is a sub-σ-algebra of $\mathcal{F}$, then

   $$\mathbb{E}[\alpha X + \beta Y \,|\, \Sigma] = \alpha \mathbb{E}[X \,|\, \Sigma] + \beta \mathbb{E}[Y \,|\, \Sigma].$$

   That is, $\mathbb{E}[\cdot \,|\, \Sigma]$ is linear.

2. If $X \in \mathcal{L}^1(\Omega, \Sigma, P) \subset \mathcal{L}^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[X \,|\, \Sigma] = X$. i.e. For any $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$

   $$\mathbb{E}\left[\mathbb{E}[Y \,|\, \Sigma] \mid \Sigma\right] = \mathbb{E}[Y \,|\, \Sigma]$$

   making $\mathbb{E}[\cdot \,|\, \Sigma]$ is idempotent.

3. From 1 and 2 above, we have that $\mathbb{E}[\cdot \,|\Sigma]$ is a linear projection.

4. If $f \geq 0$ then $\mathbb{E}[f \,|\, \Sigma] \geq 0$.

5. $\mathbb{E}[\mathbf{1} \,|\, \Sigma] = \mathbf{1}$, where $\mathbf{1}$ is the function with value 1 almost everywhere.

6. If $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $Y \in \mathcal{L}^1(\Omega, \Sigma, P)$ with $XY \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ then

   $$\mathbb{E}[XY \,|\, \Sigma] = Y \mathbb{E}[X \,|\, \Sigma],$$

   i.e. the conditional expectation operator is an averaging operator.

7. If $\Sigma_1 \subset \Sigma_2 \subset \mathcal{F}$ are σ-algebras and $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ then

$$\mathbb{E}\left[\mathbb{E}[X \,|\, \Sigma_1] \mid \Sigma_2\right] = \mathbb{E}[X \,|\, \Sigma_1] = \mathbb{E}\left[\mathbb{E}[X \,|\, \Sigma_2] \mid \Sigma_1\right].$$

8. If $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $\sigma(X)$ and Σ are independent, then

$$\mathbb{E}[X \mid \Sigma] = \mathbb{E}[X].$$

2.2 Riesz Spaces

Riesz spaces were first defined by Frigyes Riesz in 1928, [54]. However, Riesz is not solely responsible for the development of Riesz Space Theory. Independent work done in the mid 1930's by F. Riesz, H. Freudenthal and L. V. Kantorovitch, each with their own methods, founded the theory of Riesz spaces. It is interesting to note that even in the work done in Riesz spaces today, some 70 years later, the different approaches are evident [41]. Riesz's work dealt primarily with the order dual of a given vector space, Freudenthal proved a spectral theorem for Riesz spaces from which the Radon-Nikodým theorem (mentioned in Section 2.1.2 of this thesis) follows, whilst Kantorovich sought insight into the algebraic and convergence properties of Riesz spaces. The research of Freudenthal and Kantorovich has many applications in Operator Theory.

Several other mathematicians, notably A. G. Pinsker, A. I. Judin and B. Z. Vulikh, joined Kantorovich in his research of Riesz spaces. Further contributions to the theory of Riesz spaces were made by H. Nakano, T. Ogasawara, K. Yoshida, S. Kakutani and H. F. Bohnenblust from 1940 to 1944. However, much of the work done by the afore mentioned authors was done independently of one another. It is however possible to

identify three main centres of research: Japan, Russia and the United States. Each centre had its own terminology and it took sometime for the results and terminology to grow together. A good illustration of this is the paper written by I. Amemiya in 1953 [3]. It is not immediately clear that this work is in fact an extension of earlier work by H. Nakano. The reason for this 'obscurity' is the different terminologies used by the different authors. Fortunately there has been an attempt to unify the different results and terminologies in the literature. In particular, the book by Luxemburg and Zaanen, [41] has achieved much in this regard.

A Riesz space is also referred to as a vector lattice or lattice-ordered vector space. As the last name suggests, a Riesz space is an ordered vector space where the order structure is a lattice. Here we shall present a survey of definitions and theorems on lattices. These results form the foundations of Riesz Space Theory.

Many of the proofs in this chapter are of an elementary nature. As a result, proofs are omitted. Details of the proofs can be found in [66, 41].

2.2.1 Partial Orderings

If X is a non-empty set, we shall denote by $x, y, \ldots$ the elements of X (also known as the points of X). $X \times X$ is known as the *Cartesian product* of X and is the set consisting of all ordered pairs (x, y) of points of X. A *relation*, R, in X is any non-empty subset of $X \times X$. We write xRy whenever $(x, y) \in R$. A well known example of a relation and one that is of importance to us is a partial ordering. A relation R is said to be a partial ordering in X if, for all $x,\ y \in X$,

(i) xRx (the relation is reflexive);

(ii) if xRy and yRx then $x = y$ (the relation is anti-symmetric);

(iii) if xRy and yRz then xRz (the relation is transitive).

Note that the relation 'less than or equal to' obeys all the above properties. It is for this reason that the partial ordering R in X is often denoted by $\leq$. Elements x, y of X for which either $x \leq y$ or $y \leq x$ are called comparable. It is important to note that not every element in a partially ordered set X need be comparable to another element. That is, if $x, y \in X$ it is not true, in general, that $x \leq y$ or $y \leq x$.

If X is partially ordered and Y is a non-empty subset of X then Y inherits the partial ordering of X. If $x \in X$ is such that $x \geq y$ for all $y \in Y$, we say that x is an upper bound of Y. If in addition x is such that $x \leq x'$ for any other upper bound x' of Y then x is called the (unique) least upper bound or supremum of Y. We denote the supremum, x, of Y by $x = \sup Y$.

In a similar manner, we define the notions of lower bound and infimum. A lower bound of a non-empty subset Y of a partially ordered set X is an element of X, say x_0, such that $x_0 \leq y$ for all $y \in Y$. The infimum of Y, denoted $\inf Y$, is the greatest lower bound of Y. That is, a lower bound x_0 of Y is the infimum of Y, i.e. $x_0 = \inf Y$, if $x_0 \geq x_0'$ for any lower bound x_0' of Y.

A maximal element, say x_0, of a partially ordered set X is an element that is not smaller than any other element in X. That is, for any $x \in X$, if $x_0 \leq x$ then $x_0 = x$. Since not every pair of elements of a partially ordered set X need be comparable, x_0 being maximal does not imply that $x_0 \geq x$ for all $x \in X$. We define the minimal elements of X in a similar manner. Note that there may be many minimal and maximal elements of X.

If x_0 is a maximal element of partially ordered set X with the property that $x_0 \geq x$ for all $x \in X$ then x_0 is called the largest element of X. In this case, x_0 is the only maximal element. However, even if x_0 is the only maximal element of X it is not necessarily true that x_0 is the largest element of X. The definition of the smallest element of X is similar.

Definition 2.2.1. *Let X be a partially ordered set.*

(i) The set X is said to be Dedekind complete if every non-empty subset of X which is bounded from above has a supremum and every non-empty subset which is bounded from below has an infimum.

(ii) The set X is called Dedekind σ-complete if every non-empty countable subset which is bounded from above has a supremum and every non-empty countable subset which is bounded from below has an infimum.

(iii) The set X is called a lattice if every subset consisting of only two elements has an infimum and a supremum.

Remark: It is clear that every Dedekind complete space is σ-Dedekind complete. However, the reverse implication is not true. We have the following one-sided characterization of Dedekind completeness.

Theorem 2.2.2. *The partially ordered set X is Dedekind complete if and only if every non-empty subset which is bounded from above has a supremum.*

2.2.2 Lattices

A partially ordered set X is a lattice if for each pair of elements, $a, b \in X$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in X. Suppose X is a lattice. We denote the supremum of the set consisting of the elements x, y by $x \vee y$ and the infimum as $x \wedge y$. It is easily seen by induction that in a lattice every finite subset has a supremum and an infimum. If the elements in the finite subset are $x_1, x_2, \ldots, x_n$, then the supremum and infimum of the set are denoted by $\bigvee_{i=1}^{n} x_i$ and $\bigwedge_{i=1}^{n} x_i$.

Definition 2.2.3. *The lattice, X, is called distributive if for all x_1, x_2, y in X,*

$$y \wedge (x_1 \vee x_2) = (y \wedge x_1) \vee (y \wedge x_2).$$

A distributive lattice has the additional property that the operation of the infimum is distributive over the operation of the supremum. The following theorem illustrates this.

Theorem 2.2.4. *The lattice, X, is distributive if and only if for all x_1, x_2, y in X*

$$y \vee (x_1 \wedge x_2) = (y \vee x_1) \wedge (y \vee x_2).$$

The smallest element of lattice X (if it exists) is called the null of X. We denote the null of X by 0. If the lattice X has a largest element, we call this element the unit of X and denote it by $\mathbf{1}$. If X is a distributive lattice with both null and unit and the elements $x, y \in X$ are such that $x \wedge y = 0$ and $x \vee y = \mathbf{1}$ then x, y are called *complements* of one another.

2.2.3 Elementary properties of Riesz Spaces

We now give an outline of Riesz Space Theory. Much of the material appearing in this section was introduced in the period 1935-1942 and is mainly due to Riesz, Kantorovich, Freudenthal, Birkhoff, Yosida, Nakano and Ogasawara.

Definition 2.2.5. *The real, linear space, L, is an ordered vector space if L is partially ordered and for each $f, g, h \in L$,*

(i) if $f \leq g$ then $f + h \leq g + h$;

(ii) if $f \geq 0$ then $af \geq 0$, for each $a \in \mathbb{R}, a \geq 0$.

In other words, an ordered vector space, L, is a real, linear space with a partial ordering compatible with the algebraic structure of L.

The ordered vector space, L, is a Riesz space *if for every pair $f, g \in L$ the supremum, $f \vee g$, with respect to the partial ordering is defined and exists in L.*

Remark: The terminology in the above definition is taken from Bourbaki who used the term *espace de Riesz*. A Riesz space is also referred to as a vector lattice. To illustrate the point made earlier about different centres having different terminology, Nakano and his school called a Riesz space a semi-ordered vector space and in Russian literature a Riesz space is a K-lineal.

We now discuss $\mathcal{L}^p(\Omega, \mathcal{F}, P),\ 1 \leq p \leq \infty$, as a Riesz space. Clearly, $\mathcal{L}^p(\Omega, \mathcal{F}, P)$ is a vector space. We define the partial order on $\mathcal{L}^p(\Omega, \mathcal{F}, P)$ almost everywhere pointwise as follows. For $f,\ g \in \mathcal{L}^p(\Omega, \mathcal{F}, P)$, then $f \leq g$ if and only if $f(x) \leq g(x)$ for almost all $x \in \Omega$. For $f,\ g,\ h \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $\alpha \in \mathbb{R}^+$,

(i) if $f \leq g$ then $f(x) \leq g(x)$ a.e in Ω. Thus, $f(x) + h(x) \leq g(x) + h(x)$ a.e in Ω. Hence, $f \leq g$ implies $f + h \leq g + h$.

(ii) If $f \geq 0$ then it is evident that $\alpha f \geq 0$ for all $\alpha \geq 0$.

(iii) It is easy to check that $\sup\{f, g\}(x) = \max\{f(x), g(x)\}$ is always defined and $\sup\{f, g\} \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$.

Thus, $\mathcal{L}^p(\Omega, \mathcal{F}, P)$ is a Riesz space.

2.2.4 Ordered Vector Space Properties

We define an important subset of an ordered vector space.

Definition 2.2.6. *Let L be an ordered vector space. The positive cone of L, L^+, is the subset of L consisting of all positive elements of L. That is,*

$$L^+ = \{f | f \in L, f \geq 0\}.$$

The following theorem gives some important properties of the positive cone.

Theorem 2.2.7. *Let L be an ordered vector space and L^+ its positive cone.*

(i) If $f, g \in L^+$ then $f + g \in L^+$.

(ii) If $a \geq 0, a \in \mathbb{R}$ and $f \in L^+$ then $af \in L^+$.

(iii) If $f, -f \in L^+$ then $f = 0$.

Conversely, if L^+ is a subset of the real, linear space L such that (i), (ii) and (iii) above are satisfied, it is possible to make L into an ordered vector space through

defining a partial order on L by

$$f \leq g \text{ if and only if } g - f \in L^+.$$

Then L^+ is the positive cone of L with respect to this partial ordering.

Theorem 2.2.8. *Let L be an ordered vector space with positive cone L^+ and let $f, g \in L$. Then,*

(i) $f \geq g$ *if and only if* $f - g \in L^+$.

(ii) $f \geq g$ *if and only if* $f = f \vee g$ *(or* $g = f \wedge g$*).*

(iii) $f \geq g$ *if and only if* $af \geq ag$ *for* $a > 0$ *(or* $af \leq ag$ *if* $a < 0$*),* $a \in \mathbb{R}$.

(iv) if $f \vee g$ *exists then* $(-f) \wedge (-g)$ *exists and*

$$(-f) \wedge (-g) = -(f \vee g).$$

(v) $f \vee g$ *exists in L if and only if* $f \wedge g$ *exists in L. We then have, for any* $h \in L$,

$$\begin{aligned} f + g - (f \vee g) &= f \wedge g, \\ (f + h) \vee (g + h) &= (f \vee g) + h, \\ (f + h) \wedge (g + h) &= (f \wedge g) + h. \end{aligned}$$

In particular, if L is a Riesz space then both $f \vee g$ and $f \wedge g$ exist in L. Furthermore, if L is an ordered vector space such that $f \vee 0$ exists for all $f \in L$ then L is a Riesz space.

(vi) If $f \vee g$ exists then, for all $a > 0, a \in \mathbb{R}$,

$$\begin{aligned} af \vee ag &= a(f \vee g) \\ af \wedge ag &= a(f \wedge g). \end{aligned}$$

(vii) If L is a Riesz space then any finite set of elements in L has a supremum and an infimum in L.

Note 2.2.9. *Part (v) of the theorem above proves that a Riesz space is indeed a lattice.*

Definition 2.2.10. *Let L be an ordered vector space and $f \in L$ be such that $f \vee 0$ exists in L. We define*

$$f^+ = f \vee 0, \quad f^- = (-f) \vee 0, \quad |f| = f \vee (-f).$$

Remark: By the previous theorem, if $f \vee 0$ exists, then $f \wedge 0$ exists, so $(-f) \vee 0$ exists. Furthermore, the element $(2f) \vee 0$ exists. Adding $-f$ to $2f \vee 0$ we have that $f \vee (-f)$ exists.

Theorem 2.2.11. *Let L be an ordered vector space and suppose $f \in L$ is such that $f \wedge 0$ exists. Then,*

(i) $f^+, f^- \in L^+$;

(ii) $f = f^+ - f^-$ and $|f| = f^+ + f^-$;

(iii) for all $a \in \mathbb{R}$ with $a > 0$ we have $(af)^+ = af^+$, $(af)^- = af^-$ and $|af| = a|f|$; and for all $a \in R$ with $a < 0$ we have $(af)^+ = -af^-$, $(af)^- = -af^+$ and $|af| = -a|f|$;

(iv) if $f, g \in L$ and f^+, g^+ exist in L, then $f \leq g$ if and only if $f^+ \leq g^+$ and $g^- \leq f^-$.

Suppose L^+ is the positive cone of an ordered space L. We say L^+ is generating if every element of L can be written as a difference of elements of L^+. That is, L^+ is

generating if for all $f \in L$ we can write $f = u - v$ where $u, v \in L^+$. The positive cone of any Riesz space is always generating. Indeed, if f is an element of a Riesz space then $f = f^+ - f^-$. Furthermore, if the ordered vector space L has positive cone L^+ such that L^+ is generating and $f \vee g$ exists for all $f, g \in L^+$, then we have that L is a Riesz space. To see this, consider arbitrary elements f and g of L. By the hypothesis, we can write $f = u_1 - v_1, g = u_2 - v_2$ where $u_i,\ v_i \in L^+,\ i = 1,\ 2$. Now, let $f_1 = f + (v_1 + v_2)$ and $g_1 = g + (v_1 + v_2)$. Then $f_1, g_1 \in L^+$ and $f_1 \vee g_1$ exists. Let $h_1 = f_1 \vee g_1$. Then,

$$f \vee g = h_1 - (v_1 + v_2),$$

so $f \vee g$ exists and L is a Riesz space.

The following theorem proves that the decomposition $f = f^+ - f^-$ is the decomposition of f as a difference, $u - v$, of elements $u, v \in L^+$ for which u and v are minimal.

Theorem 2.2.12. *Let L be an ordered vector space. If $f = u - v$ where $u, v \in L^+$ then $f^+ \leq u$ and $f^- \leq v$.*

2.2.5 Riesz space inequalities and distributive laws

The following results concern inequalities and distributive laws of Riesz spaces. Let E be a Riesz space.

Theorem 2.2.13. *For all $f, g \in E$ we have*

$$\begin{aligned} (f+g)^+ &\leq f^+ + g^+, \\ (f+g)^- &\leq f^- + g^-, \end{aligned}$$

and

$$||f| - |g|| \leq |f + g| \leq |f| + |g|.$$

Theorem 2.2.14. *If D is a subset of E such that $f_0 = \bigvee_{f \in D} f$ exists in L, then, for all $g \in L$,*

$$f_0 \wedge g = \bigvee_{f \in D} (f \wedge g).$$

The result also remains true when $\vee$ and $\wedge$ are interchanged.

The subsequent result shows that a Riesz space is in fact a distributive lattice with respect to its partial ordering.

Corollary 2.2.15. *For any $f, g, h \in E$,*

$$(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h) \quad \textit{and} \quad (f \wedge g) \vee h = (f \vee h) \wedge (g \vee h).$$

In the previous section we gave a decomposition theorem for elements in an ordered vector space with a generating positive cone. In a Riesz space we have another decomposition property, known as the *Riesz Decomposition Property*.

Theorem 2.2.16. (Riesz Decomposition Property)

Let E be a Riesz space with positive cone E^+. Suppose $u, z_1, z_2 \in E^+$ are such that $u \leq z_1 + z_2$. Then there exist elements $u_1, u_2 \in E^+$ such that $u_i \leq z_i$ for $i = 1, 2$, and

$$u = u_1 + u_2.$$

It will be shown that some stochastic processes admit a Riesz decomposition.

2.2.6 Ideals, Bands and Disjointness

Ideals and Bands

Let E be a Riesz space. There are some particularly important linear subspaces of E which will be the focus of this section. Before we define these spaces, recall that all subsets of E inherit the ordering of E.

Definition 2.2.17. *Let E be a Riesz space.*

(i) The linear subspace V of E is called a Riesz subspace if for $f, g \in V$ we have that $f \vee g$ and $f \wedge g$, in E, belong to V.

(ii) A solid subset S of E is a subset of E such that if $f \in S$ and $g \in E$ with $|g| \leq |f|$ then $g \in S$. That is, if $f \in S$ it follows that set $\{g : -|f| \leq g \leq |f|\}$ in E, is a subset of S.

(iii) The subset A of E is an ideal if A is a solid linear subspace of E.

(iv) An ideal B in E is called a band if for each $D \subset B$ with $\sup D$ existing in E we have that $\sup D \in B$.

We now present a theorem which shows consistency between the ordering in subspaces and that of the space from which the ordering was inherited.

Theorem 2.2.18. *Let E be a Riesz space.*

(i) Every band in E is an ideal and every ideal in E is a Riesz subspace. The trivial spaces - $\{0\}$ consisting only of the null element, and the space E itself - are bands.

(ii) Every Riesz subspace (resp. ideal, band) of a Riesz subspace (resp. ideal, band) of E is itself a Riesz subspace (resp. ideal, band) of E.

For any subset A of E, put $A^+ = E^+ \cap A$ where E^+ denotes the positive cone of E. If D is a subset of E such that for any finite number of elements $f_1, f_2, \ldots, f_n$ of D the supremum, $\bigvee_{i=1}^{n} f_i$ exists in D, we shall say D contains finite suprema (equivalently, D is closed under finite suprema). Similarly for finite infima. Note that if D is bounded above and D_1 is the set of all finite suprema of D then D_1 contains D and both have the same upper bounds.

Theorem 2.2.19. *Let E be a Riesz space.*

(i) Any intersection of Riesz subspaces of E (resp. ideals, bands) is again a Riesz subspace (resp. ideals, bands).

(ii) Suppose that B is an ideal in E. If for each $J \subset B$ with J containing finite suprema we have that $\sup J \in B$ if $\sup J$ exists, then B is a band in E.

We now introduce a few conventions. For any non-empty subset D of E we define the Riesz space generated by D as the intersection of all Riesz subspaces of E containing D. In a similar manner, the ideal and band generated by D can be defined. For ideals, we denote the ideal generated by D as A_D. In the particular case that D consists of only one element, say f, we have $A_D = A_f$ and we call A_f a principal ideal. A principal band is a band generated by a single element.

Definition 2.2.20. *Let E be a Riesz space and let $f \in E$, $f > 0$. We say that f is an order unit if $A_f = E$. We call f a weak order unit if the principal band generated by f is E.*

In a Dedekind complete Riesz space with weak order unit, every band is a principal band, see [41]. Also, if α is a real number then, since A_f is an ideal, every element $g \in E$ such that

$$|g| \leq |\alpha f| \tag{2.2.1}$$

is an element of A_f. Conversely, the set of all g satisfying (2.2.1) for any $\alpha \in \mathbb{R}$ is an ideal in E that contains A_f. Hence, we can write A_f explicitly as follows.

$$A_f = \{g \in E \,|\, |g| \leq |\alpha f|, \alpha \in \mathbb{R}\}.$$

If D is a finite subset of E with elements $f_1, f_2, \ldots, f_n$, we can generalize the explicit formula above and have that

$$A_D = \left\{ g \in E \;\middle|\; |g| \leq \sum_{i=1}^{n} |\alpha_i f_i|, \alpha_i \in \mathbb{R} \right\}.$$

Before we can give a decomposition theorem with respect to ideals, we need to introduce some simple vector space theory. If V_1 and V_2 are subsets of a vector space V, the algebraic sum, $V_1 + V_2$ is given by

$$V_1 + V_2 = \{f_1 + f_2 \,|\, f_1 \in V_1, f_2 \in V_2\}.$$

If V_1 and V_2 are linear subspaces of V then their algebraic sum is also a linear subspace of E. If, in addition to being linear subspaces, we have that $V_1 \cap V_2 = \{0\}$, then $V_1 + V_2$ is the direct sum of V_1 and V_2 and denote this by $V_1 \oplus V_2$. Furthermore, any $f \in V_1 \oplus V_2$ can be written as a unique sum of elements from V_1 and V_2. That is,

$$f = f_1 + f_2 \quad \text{where} \quad f_1 \in V_1, f_2 \in V_2.$$

We now present a decomposition theorem which relies on the Riesz space structure of the space.

Theorem 2.2.21. *If A_1 and A_2 are ideals in E, then $A_1 + A_2$ is an ideal in E. Further, $(A_1 + A_2)^+ = A_1^+ + A_2^+$. Thus, for each $f \in (A_1 + A_2)^+$, we have that $f = f_1 + f_2$, for some $f_1 \in A_1^+, f_2 \in A_2^+$.*

Note that in general f_1 and f_2 in the above theorem are not unique, but when $A_1 + A_2 = A_1 \oplus A_2$, this decomposition is obviously unique. In this case, we also have the following theorem.

Theorem 2.2.22. *If $f, g \in A_1 \oplus A_2$ where A_1, A_2 are ideals of E and f, g have decompositions $f = f_1 + f_2$ and $g = g_1 + g_2$, where $f_i, g_i \in A_i, i = 1, 2$, then $f \leq g$ implies $f_1 \leq g_1$ and $f_2 \leq g_2$.*

Remark: A similar decomposition theorem with respect to bands does not exist since the algebraic sum of bands is not always a band. To see this, consider the example where $E = C([-1, 1])$ the space of all real, continuous functions on the interval $[-1, 1]$. Define the bands B_1 and B_2 by

$$B_1 = \{f \in E \,|\, f = 0 \text{ on } [0, 1]\} \quad \text{and} \quad B_2 = \{f \in E \,|\, f = 0 \text{ on } [-1, 0]\}.$$

Then $B_1 + B_2 = B_1 \oplus B_2 = \{f \in E \,|\, f(0) = 0\}$ is an ideal E in but not a band. The band generated by $B_1 \oplus B_2$ is the entire space E.

Disjointness

Here we give the foundational aspects of disjointness. There are many more results on this topic, but these are not required for the purposes of this thesis. The interested reader can find more details in either [41] or [66].

Elements f, g of the Riesz space E are said to be disjoint if $|f| \wedge |g| = 0$. We write $f \perp g$ if f and g are disjoint. For any non-empty subset D of E we denote the disjoint complement of D by D^d and define

$$D^d = \{f \in E \,|\, f \perp g \text{ for all } g \in D\}.$$

The second disjoint complement of D is the disjoint complement of D^d and is denoted $D^{dd} = (D^d)^d$. If D_1 and D_2 are non-empty sets in E such that for every $d_1 \in D_1$ and $d_2 \in D_2$, then $d_1 \perp d_2$ and we say that D_1 and D_2 are disjoint and write $D_1 \perp D_2$.

Theorem 2.2.23. *Let E be a Riesz space with non-empty subsets D_1 and D_2. If $D_1 \perp D_2$ then $D_1 \cap D_2 = \emptyset$ or $D_1 \cap D_2 = \{0\}$.*

Theorem 2.2.24. *Let D be a non-empty subset of E. We have the following results concerning D:*

(i) D^d is a band in E;

(ii) $D \subset D^{dd}$ and $D^d = D^{ddd}$;

(iii) $D^d \cap D^{dd} = \{0\}$ so $D^d + D^{dd} = D^d \oplus D^{dd}$. In general, this direct sum is an ideal, but not a band.

2.2.7 Order Convergence and Uniform Convergence

We now prove some basic results relating to the convergence of nets of elements of a Riesz space. We start with sequences and then generalize the results to directed sets.

We say a sequence $(f_n)_{n\in\mathbb{N}}$ is increasing if $f_1 \leq f_2 \leq \ldots$ and, for convenience, we denote an increasing sequence by $f_n \uparrow$. Similarly, $(f_n)_{n\in\mathbb{N}}$ is decreasing if $f_1 \geq$

$f_2 \geq \ldots$, denoted $f_n \downarrow$. If f_n is increasing and $f = \sup f_n$ exists, then we write $f_n \uparrow f$. If f_n is decreasing and $f = \inf f_n$ exists, then we write $f_n \downarrow f$. We call this type of convergence *monotone convergence*. The following properties of monotone convergence are well known and easily verified.

(i) If $f_n \uparrow f$ then $f_n + g \uparrow f + g$. Similarly, if $f_n \downarrow f$ then $f_n + g \downarrow f + g$.

(ii) $f_n \uparrow f$ if and only if $(-f_n) \downarrow (-f)$. $f_n \downarrow f$ if and only if $(-f_n) \uparrow (-f)$.

(iii) Finally, $f_n \uparrow f$ if and only if $(f - f_n) \downarrow 0$. $f_n \downarrow f$ if and only if $(f_n - f) \downarrow 0$.

Lemma 2.2.25. *Consider the sequence $(f_n)_{n\in\mathbb{N}}$ and suppose $(f_{n_k})_{k\in\mathbb{N}}$, $n_1 < n_2 < \ldots$, is a subsequence of (f_n).*

(i) If $f_n \uparrow f$ then $f_{n_k} \uparrow f$.

(ii) If $f_n \uparrow f$ then $af_n \uparrow af$ for all $a \geq 0 \in \mathbb{R}$.

(iii) If $p_n \downarrow 0$ and $r_n \downarrow 0$ then $p_n + r_n \downarrow 0$.

(iv) If $0 \leq q_n \leq p_n \downarrow 0$ then $\inf q_n = 0$. Thus, if q_n is decreasing, $q_n \downarrow 0$. On the other hand, if q_n is increasing, then $q_n = 0$ for all n.

Similar results hold for decreasing sequences.

We now define order convergence, a type of convergence that is more general than monotone convergence.

Definition 2.2.26. *A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be order convergent to f if there exists a sequence $p_n \downarrow 0$ such that for all n, $|f - f_n| \leq p_n$. We use $f_n \to f$ to denote order convergence.*

The following properties are easily verified.

(i) If f_n increases or decreases to f (that is, $f_n \uparrow f$ or $f_n \downarrow f$) then $f_n \to f$. That is, monotone convergence implies order convergence.

(ii) If $f_n \to f$, $g_n \to g$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f_n + \beta g_n \to \alpha f + \beta g$. Also, $f_n \vee g_n \to f \vee g$ and $f_n \wedge g_n \to f \wedge g$.

(iii) If $f_n \to f$ and $f_n \geq g$ for all n then $f \geq g$.

(iv) If (f_n) is a monotone sequence and $f_n \to f$, then f_n converges monotonically to f (that is, $f_n \uparrow f$ or $f_n \downarrow f$).

(v) If $f_n \to f$ then $f_{n_k} \to f$ where (f_{n_k}) is a subsequence of (f_n).

We now generalize from sequences to nets. In the following definition, the set Λ is, in general, an infinite set. In the case where Λ is $\mathbb{N}$, we are in the special case of sequences. We now define the notion of a directed set.

Definition 2.2.27. *Let Λ be a non-empty set. We denote the elements of Λ by α. Let E be a Riesz space. Assume that for each element $\alpha \in \Lambda$ there is a mapping $\alpha \mapsto f_\alpha$ which maps from Λ to E. We say that Λ is the index set for the family $(f_\alpha)_{\alpha\in\Lambda}$. The family $(f_\alpha)_{\alpha\in\Lambda}$ is said to be an upwards directed set (denoted $f_\alpha \uparrow_{\alpha\in\Lambda}$) if for any two elements $\alpha_1, \alpha_2 \in \Lambda$ there exists $\alpha_3 \in \Lambda$ such that $f_{\alpha_3} \geq f_{\alpha_1} \vee f_{\alpha_2}$. Finally, if $f_\alpha \uparrow_{\alpha\in\Lambda}$ and $f = \sup_{\alpha\in\Lambda}(f_\alpha)$ we write $f_\alpha \uparrow_{\alpha\in\Lambda} f$. In this case, we say that $(f_\alpha)_{\alpha\in\Lambda}$ is upwards directed with supremum f. Downward directedness (denoted $f_\alpha \downarrow_{\alpha\in\Lambda}$) can be similarly defined but now $f_{\alpha_3} \leq f_{\alpha_1} \wedge f_{\alpha_2}$. If $f_\alpha \downarrow_{\alpha\in\Lambda}$ and $f = \inf_{\alpha\in\Lambda}(f_\alpha)$ we write $f_\alpha \downarrow_{\alpha\in\Lambda} f$. In this case we say that $(f_\alpha)_{\alpha\in\Lambda}$ is downwards directed with infimum f.*

The results for directed sets are analogous to those for sequences, as is shown in the lemma below.

Lemma 2.2.28. *Let Λ be a non-empty set.*

(i) If $f_\alpha \uparrow_{\alpha\in\Lambda} f$ then $(f_\alpha \,|\, f_\alpha \geq f_{\alpha_0}) \uparrow_{\alpha\in\Lambda} f$ for any α_0.

(ii) If $f_\alpha \uparrow_{\alpha\in\Lambda} f$ and $0 \leq a \in \mathbb{R}$ then $af_\alpha \uparrow_{\alpha\in\Lambda} af$.

(iii) If $f_\alpha \uparrow_{\alpha\in\Lambda} f$ then $f_\alpha^+ \uparrow_{\alpha\in\Lambda} f^+$ and $f_\alpha^- \downarrow_{\alpha\in\Lambda} f^-$.

We can also define upwards directedness and downward directedness for subsets D of Riesz space E that are not indexed.

Definition 2.2.29. *Let D be a non-empty subset of Riesz space E. We say that D is upwards directed (denoted $D\uparrow$) if for any two elements $p, q \in D$ there exists $r \in D$ such that $r \geq p \vee q$. Downward directedness (denoted $D\downarrow$) is defined in an analogous manner.*

Finally, we are able to define order convergence for nets. Let (f_α) be an order bounded net in E, then $u_\alpha := \sup\{f_\beta : \alpha \leq \beta\}$ and $\ell_\alpha := \inf\{f_\beta : \alpha \leq \beta\}$ exist in E, for α in the index set of the net. We denote $\limsup f_\alpha := \inf_\alpha u_\alpha$ and $\liminf f_\alpha := \sup_\alpha \ell_\alpha$. Now, (f_α) is order convergent if and only if $\limsup f_\alpha$ and $\liminf f_\alpha$ both exist and are equal. In this case the common value is denoted $\lim f_\alpha$.

2.2.8 Projections and Dedekind Completeness

We have already seen that the algebraic sum of two ideals in a Riesz space is again an ideal, but the algebraic sum (even the direct sum) of two bands is not necessarily

a band. However, if the direct sum of two ideals, say A and B, in Riesz space E is E, the following theorem applies.

Theorem 2.2.30. *Let A and B be disjoint ideals in E. If $A \oplus B = E$ then $B = A^d$ and $A = B^d$. Thus $A = A^{dd}$, $B = B^{dd}$ and A and B are bands.*

Before we state the next important result, we recall some well-known definitions from linear algebra. A mapping between vector spaces, $T : V \to W$, is called linear if for all scalars α, β and $f, g \in V$, we have that

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g).$$

Such a mapping is often known as an operator (or linear operator). For any two operators, T_1, T_2 we define $T_1 T_2 f = T_1(T_2 f)$. An operator $T : V \to V$ is said to be idempotent if $T^2 = TT = T$.

We now come to a theorem which plays an important role in the defining of stochastic processes in Riesz spaces. We first define what is meant by a *projection band*.

Definition 2.2.31. *Let E be a Riesz space. The band $B \in E$ is a projection band if for any $u \in E^+$ we have*

$$u_1 = \bigvee_{v \in B;\ 0 \le v \le u} v$$

exists. We call u_1 the component of u in B.

If

$$u_2 = \bigvee_{w \in B^d;\ 0 \le w \le u} w$$

exists then u_2 is the component of u in B^d and we have that $u = u_1 + u_2$.

Theorem 2.2.32. *Let E be a Riesz space.*

(i) For any $f \in E^+$ we denote the component of f in the projection band B by $P_B f$. P_B is a mapping from E into itself and has the following properties.

(a) P_B is linear and idempotent. (i.e. P_B is a projection);

(b) for all $u \in E^+, 0 \leq P_B u \leq u$.

We call P_B the band projection onto the projection band B.

(ii) We can extend P_B to E by setting $P_B u = P_B u^+ - P_B u^-$ for all $u \in E$.

(iii) If P is a projection from E to E such that for all $u \in E^+$, $0 \leq Pu \leq u$ then there exists a band B such that P is the band projection on B.

(iv) For all $u, v \in E$, if P is a band projection, then $P(u \wedge v) = Pu \wedge Pv$ and $P(u \vee v) = Pu \vee Pv$.

Remark: By the component of f in the projection band B we mean the following. If $f = f_1 + f_2$ where $f_1 \in B$ and $f_2 \in B^d$ then $f_1 = P_B f$. Band projections can be thought of as the 'characteristic functions' of Riesz spaces. In the particular case where E is the Riesz space $\mathcal{L}^1(X, \Omega, P)$ and $A \in \Omega$, the map $Pf = \chi_A \cdot f$ is a band projection in E onto the band $\{f \in E \,|\, f|_{X \setminus A} = 0\}$.

We denote the set of all projection bands in E by $\mathcal{B}(E)$. Since $\mathcal{B}(E) \subset 2^E$ we have that $\mathcal{B}(E)$ has set ordering. Consider the mapping from the set of all projection bands to the set of band projections given by $A \to P_A$. This mapping is one-to-one. We define the partial ordering on the set of band projections by $P_A \leq P_B$ if and only if $A \subset B$. Now the mapping $A \to P_A$ is bijective and the map and its inverse are order preserving. Thus, $\mathcal{B}(E)$ and the set of band projections in E are order isomorphic. We now give some properties of band projections.

Before we give further properties of band projections, we require the notion of an *Archimedean* space. These spaces play an important role in the study of Riesz spaces. The interested reader can find more detailed results in [41, 66].

We say that a Riesz space is *Archimedean* if for all elements u in the positive cone of E,

$$\bigwedge_{n\in\mathbb{N}} n^{-1}u = 0.$$

The Archimedean property of Riesz spaces is important as it is this property that gives the uniqueness of order limits in Riesz spaces. In an Archimedean Riesz space, the intersection of projection bands is again a projection band, as stated below.

Theorem 2.2.33. *If B_1, B_2 are projection bands in the Archimedean Riesz space E then $B_3 = B_1 \cap B_2$ is a projection band and the corresponding band projections satisfy*

$$P_1P_2 = P_3 = P_2P_1.$$

As a direct consequence of the previous theorem we have the following.

Theorem 2.2.34. *If E is an Archimedean Riesz space with projection bands B_1, B_2, and corresponding band projections P_1, P_2, then the following are equivalent:*

(i) $B_1 \subset B_2$,

(ii) $P_1P_2 = P_1 = P_2P_1$,

(iii) $P_1 \leq P_2$.

2.2.9 Dedekind Completeness

The notion of a Dedekind complete Riesz space is needed in the definition of stochastic processes in Riesz spaces. This allows us to give a theory which is rich enough to be interesting and agrees with the classical theory when the underlying Riesz space is L^1 over a probability measure. Recall that a Dedekind complete space, as defined in Definition 2.2.1, is a partially ordered set for which every non-empty subset that is bounded above has a supremum.

We now show that it is sufficient to consider only upwards directed sets of positive elements in the definition of Dedekind completeness.

Theorem 2.2.35. *Let E be a Riesz space.*

(i) The space E is Dedekind complete if and only if every non-empty subset of E^+ that is upwards directed and bounded above has a supremum.

(ii) The space E is σ-Dedekind complete if and only if every increasing sequence in E^+ that is bounded above has a supremum.

Theorem 2.2.36. *Let E be a Dedekind complete Riesz space.*

(i) If B_1 and B_2 are disjoint bands in E then $B_1 \oplus B_2$ is a band in E.

(ii) Every band in E is a projection band (i.e. each band B in E has $B \oplus B^d = E$).

We now show that the Riesz space $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ is Dedekind complete. To avoid clumsy notation we will denote the positive cone of $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ by $\mathcal{L}^1_+(\Omega, \mathcal{F}, P)$. Let D be a non-empty subset of $\mathcal{L}^1_+(\Omega, \mathcal{F}, P)$ that is bounded above by $v \in \mathcal{L}^1_+(\Omega, \mathcal{F}, P)$.

Without loss of generality we may assume that D contains finite suprema. Let

$$S = \left\{ \int_\Omega u \; dP \;\middle|\; u \in D \right\}.$$

It is obvious that S is bounded above by $\int_\Omega v \; dP$. Thus, $\beta = \sup S$ exists and there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1_+(\Omega, \mathcal{F}, P)$ such that $\lim\limits_n \int_\Omega v_n \; dP = \beta$. Since finite suprema exist in D we can define $u_n = \sup\limits_{i=1,\ldots,n} v_i$. Then $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence in D with $0 \leq v_n \leq u_n \leq v$ for all $n \in \mathbb{N}$. Let $\lim\limits_{n\to\infty} u_n = u_0$. Then, $u_0 \leq v$ and u_0 is an element of D, by the choice of D. That is, $\lim\limits_{n\to\infty} u_n$ exists in D. Also, $\int_\Omega v_n \, dP \leq \int_\Omega u_n \, dP \leq \beta$ giving $\beta = \lim\limits_{n\to\infty} \int_\Omega v_n \, dP \leq \lim\limits_{n\to\infty} \int_\Omega u_n \, dP \leq \beta$. By Lebesgue's Dominated Convergence Theorem we have

$$\beta = \lim_{n\to\infty} \int_\Omega u_n \, dP = \int_\Omega \lim_{n\to\infty} u_n \, dP = \int_\Omega u_0 \, dP.$$

We show $u_0 = \sup D$. Consider $u^* \in D$. Since D contains finite suprema, we have that $u^* \vee u_n \in D$ for all $n \in \mathbb{N}$ and $u^* \vee u_n \uparrow_n u^* \vee u_0$. Thus,

$$\begin{aligned} \int_\Omega u^* \vee u_0 \; dP &= \lim_n \int_\Omega u^* \vee u_n \; dP \\ &\geq \lim_n \int_\Omega u_n \; dP \\ &= \beta. \end{aligned}$$

But $\int_\Omega u^* \vee u_0 \; dP \in S$ and $(u^* \vee u_0 - u_0) \geq 0$. Thus, $\int_\Omega u^* \vee u_0 \; dP \leq \beta$ and hence, $\int_\Omega (u^* \vee u_0 - u_0) \; dP = 0$ giving $(u^* \vee u_0 - u_0) = 0$ almost everywhere. Thus, $u^* \leq u_0$, in the Riesz space sense, and we have that $u_0 = \sup D$ and $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ is Dedekind complete.

Note 2.2.37. *Every Dedekind complete space is Archimedean.*

2.3 Linear Operators in Riesz Spaces

In this chapter we will consider various properties of linear operators on Riesz spaces and also some spaces of linear operators. Linear operators, in particular positive order continuous linear operators, play a vital role in the generalization of stochastic processes to the Riesz space setting. It is important that we understand these operators.

Let V and W be ordered vector spaces, not necessarily Riesz spaces. We denote by T the mapping $T : V \to W$ and recall that T is a linear operator (operator for brevity) if for all scalars $\alpha,\ \beta$ and for $f,\ g \in V$,

$$T(\alpha f + \beta g) = \alpha T f + \beta T g.$$

Assume that T is an operator. We use $\mathbb{L}(V, W)$ to denote the space of all operators from V into W. $\mathbb{L}(V, W)$ is a vector space. We now define some important classes of operators.

Definition 2.3.1. *Let $V,\ W$ be ordered vector spaces and let $T \in \mathbb{L}(V, W)$.*

(i) T is a positive operator if T maps the positive cone of V into the positive cone of W. We denote this $T \geq 0$.

(ii) T is regular if $T = T_1 - T_2$ for some positive operators $T_1,\ T_2$. We denote the set of all regular operators between V and W by $\mathbb{L}_r(V, W)$.

(iii) The order interval $[g, f]$ is a subset of V of the form $\{h \in V \mid g \leq h \leq f\}$. We say that T is order bounded if T maps order intervals of V into order intervals of W. We denote the set of all order bounded operators from V into W by $\mathbb{L}_b(V, W)$.

Remark: From part (i) of the above definition we have that $\mathbb{L}(V,W)$ becomes an ordered vector space by defining $T_1 \leq T_2$ in $\mathbb{L}(V,W)$ whenever $T_2 - T_1$ is positive.

It is easy to see that T is order bounded if and only if T maps the interval $[0,f]$ into an order bounded subset of W. Thus, if T is a positive operator then T is order bounded since T maps $[0,f]$ into $[0,Tf]$. Furthermore, by (ii) of Definition 2.3.1 it is evident that T is regular if and only if there exists a positive operator T_1 such that $T \leq T_1$. We now have the following theorem.

Theorem 2.3.2. *For ordered vector spaces V,W and operator $T \in \mathbb{L}(V,W)$, if T is regular then T is order bounded.*

We now consider operators between two Riesz spaces which do more than preserve the ordering, they preserve finite suprema and finite infima, i.e. Riesz homomorphisms.

Definition 2.3.3. *The operator T between the Riesz spaces E and F, is said to be a Riesz (or lattice) homomorphism if for all $f,\ g \in E$*

$$T(f \vee g) = Tf \vee Tg.$$

From the above definition it is immediately clear that every Riesz homomorphism is a positive operator.

Consider the band projection P onto projection band B in Riesz space E. Since $0 \leq Pf \leq f$ for every $f \geq 0$ in E, we have that $f \wedge g = 0$ implies that $Pf \wedge Pg = 0$. It is easily verified that this is a characterization of a Riesz homomorphisms, and thus that every band projection is a Riesz homomorphism.

Another important class of operators acting between Riesz spaces is that of *order continuous operators*. Note that by $|Tf|$ we mean the absolute value of Tf. That is to say, $|Tf| = (Tf)^+ + (Tf)^-$ where $(Tf)^\pm = \sup(\pm Tf \vee 0)$.

Definition 2.3.4. *Let E and F be Riesz spaces and T be an operator between E and F. We say that T is an order continuous operator if for any directed set $D \subset E$ with $D \downarrow 0$ we have that $\bigwedge_{f \in D} |Tf| = 0$.*

For the most part, we are interested in order continuous positive operators. However, we note that if F is a Dedekind complete space and $T : E \to F$ is regular then $|T|$ is a well defined positive operator. Order continuous operators have the following properties.

Theorem 2.3.5. *Let E and F be Riesz spaces and consider operators T and S mapping from E into F.*

(i) If T is order continuous then for all scalars, α, αT is order continuous.

(ii) If $T \geq 0$ then T is order continuous if and only if $D \downarrow 0$ in E implies $Tf \downarrow_{f \in D} 0$ in F.

(iii) If $T \geq 0$ is order continuous and $0 \leq S \leq T$, then S is order continuous.

Note that there exist Riesz spaces for which every operator mapping between the spaces is order continuous, see [66].

2.4 Conditional Expectations in Riesz Spaces

Sufficient background has now been presented for us to be able to define the concept of a conditional expectation in a Riesz space setting. In the classical setting, it is through probability measure that stochastic processes are defined. This can be shown to be equivalent to working via conditional expectations, see [52], and this will be our approach in the Riesz space setting. The majority of the results and definitions in this section can be found in [37].

We recall from Section 2.1, some properties of the conditional expectation on $\mathcal{L}^1(\Omega, \mathcal{F}, P)$, conditioned by the sub-σ-algebra Σ of $\mathcal{F}$:

(i) $f \mapsto \mathbb{E}[f \,|\, \Sigma]$ is linear;

(ii) if $f \geq 0$ then $\mathbb{E}[f \,|\, \Sigma] \geq 0$;

(iii) if $\mathbf{1}$ is the function that takes the value 1 almost everywhere, then $\mathbb{E}[\mathbf{1} \,|\, \Sigma] = \mathbf{1}$;

(iv) $\mathbb{E}\left[\mathbb{E}[f \,|\, \Sigma] \;|\; \Sigma\right] = \mathbb{E}[f \,|\, \Sigma]$;

(v) if $f_n \uparrow_n f$ in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[f_n \,|\, \Sigma] \uparrow_n E[f \,|\, \Sigma]$ in $\mathcal{L}^1(\Omega, \Sigma, P)$.

Properties (i), (ii), (iv) and (v) give that $\mathbb{E}[\cdot \,|\, \Sigma]$ is a positive order continuous linear projection.

To make use of (iii), the weak order units in the Dedekind complete Riesz space $\mathcal{L}^1(\Omega, \Sigma, P)$ need to be considered. We provide a sketch of this result, further details can be found in [37]. We show that an element f in $\mathcal{L}^1_+(\Omega, \Sigma, P)$ is a weak order unit if $f > 0$ almost everywhere. Fix $f \in \mathcal{L}^1_+(\Omega, \Sigma, P)$ with $f > 0$ almost everywhere and

let

$$H = \{g \in \mathcal{L}^1_+(\Omega, \Sigma, P) \mid g \leq \alpha f, \alpha \in \mathbb{R}^+\}.$$

To show that f is a weak order unit it is sufficient to prove that the order closure, $\overline{H}$, of H contains $\mathcal{L}^1_+(\Omega, \Sigma, P)$. Let $h \in \mathcal{L}^1_+(\Omega, \Sigma, P)$ and define $h_n = h \wedge nf, n \in \mathbb{N}$. Then $h_n \in H$ and (h_n) is an increasing sequence with $h_n(x) \uparrow_n h(x)$ almost everywhere in Ω. As $h \in \mathcal{L}^1(\Omega, \Sigma, P)$, this gives that h is the order limit of (h_n), thus giving $h \in \overline{H}$.

Note that the function with constant value of 1, denoted $\mathbf{1}$, is a weak order unit of $\mathcal{L}^1(\Omega, \Sigma, P)$ and in addition is invariant under each conditional expectation operator on $\mathcal{L}^1(\Omega, \Sigma, P)$. We can show, moreover, that if f is a weak order unit of $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[f \,|\, \Sigma]$ is a weak order unit by making use of the order continuity of $\mathbb{E}[\cdot \,|\, \Sigma]$. In summary, we take a conditional expectation operator in a Dedekind complete Riesz space with weak order unit to be a positive order continuous projection that maps weak order units to weak order units.

Before we are able to give a formal definition of Riesz space conditional expectation operators, we need two further results.

The first of these is a result by Rao, relating contractive projections to conditional expectations.

Proposition 2.4.1. ([52]) *Let (Ω, Σ, μ) be a finite measure space and $1 \leq p < \infty$. If $T : L^p(\Omega, \Sigma, \mu) \to L^p(\Omega, \Sigma, \mu)$ is a positive contractive projection with $T\mathbf{1} = \mathbf{1}$, then $T = \mathbb{E}[\cdot \,|\, \mathcal{F}]$, for some (unique) σ-algebra $\mathcal{F} \subset \Sigma$.*

In the Riesz space $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ we have that $\mathbb{E}[\cdot \,|\, \Sigma]$ maps weak order units to weak order units and that the $\mathbf{1}$ function remains invariant under $\mathbb{E}[\cdot \,|\, \Sigma]$. The following theorem shows that if either of these conditions is satisfied then the other is too.

Theorem 2.4.2. *[38] Let E be a Riesz space with weak order unit and T be a positive order continuous projection on E. There is a weak order unit e of E with $Te = e$ if and only if Tw is a weak order unit of E for each weak order unit w in E.*

We are now able to define conditional expectations on Riesz spaces. The above two theorems and the properties of conditional expectations motivate this definition.

Definition 2.4.3. *[38] Let E be a Riesz space with weak order unit. A positive order continuous projection T on E with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation if Te is a weak order unit of E for each weak order unit e in E.*

We require that the range of T, $\mathcal{R}(T)$, is Dedekind complete in order to most closely resemble the classical setting. The motivation for this is the subject of [37] and the interested reader can find more details here.

Remark: If T is a conditional expectation operator on E, then, since $\mathcal{R}(T)$ is a Dedekind complete Riesz subspace of E and as T is order continuous, we have immediately that $\mathcal{R}(T)$ is order closed in E.

In order to consider convergence properties of stochastic processes in Riesz spaces, we will need the notion of a $T-$*universally complete Riesz space*. A Riesz space E is said to be *universally complete* if E is Dedekind complete and every subset of E which consists of mutually disjoint elements has a supremum in E. The *universal completion* of Riesz space E, denoted E^u, is a Riesz space that is universally complete and contains E as an order dense Riesz subspace.

Definition 2.4.4. *Let E be a Dedekind complete Riesz space and T be a strictly positive conditional expectation on E. The space E is universally complete with respect*

to T, i.e. T-universally complete, if for each increasing net (f_α) in E^+ with (Tf_α) order bounded, we have that (f_α) is order convergent.

We give a brief outline of the construction here. For more details, the reader is referred to [37]. If E is a Dedekind complete Riesz space and T is a strictly positive conditional expectation operator on E, then E has a T-universal completion, see [37], which is the natural domain of T, denoted $\mathrm{dom}(T)$ in the universal completion, E^u, of E, also see [19, 28, 48, 65]. Here $\mathrm{dom}(T) = D - D$ and $Tx := Tx^+ - Tx^-$ for $x \in \mathrm{dom}(T)$ where

$$D = \{x \in E^u_+ | \exists (x_\alpha) \subset E_+, x_\alpha \uparrow x, (Tx_\alpha) \text{ order bounded in } E^u\},$$

and $Tx := \sup_\alpha Tx_\alpha$, for $x \in D$ with $x_\alpha \uparrow x$, $(x_\alpha) \subset E_+$, (Tx_α) order bounded in E^u.

It is useful to have available the following Riesz space analogues of the L^p spaces as introduced in [39], $\mathcal{L}^1(T) = dom(T)$ and $\mathcal{L}^2(T) = \{x \in \mathcal{L}^1(T) | x^2 \in \mathcal{L}^1(T)\}$. Here we note that for each $x \in E$, x^2 exists and is defined in E^u.

2.4.1 Riesz space conditional expectation operators

In the previous section we considered various properties of classical $\mathcal{L}^1$ conditional expectation operators and then used these properties as the defining properties of conditional expectation operators in Riesz spaces. The Riesz space definition of a condition expectation operator and the classical definition of a conditional expectation coincide when the Riesz space in question is $\mathcal{L}^1$. We now consider whether there are additional properties obeyed by classical conditional expectation operators that are inherited by their Riesz space analogues.

Unless noted otherwise, T will denote a Riesz space conditional expectation operator.

Commutation Properties

In $\mathcal{L}^1(\Omega, \Sigma, P)$ we have that $\mathbb{E}[\cdot \,|\, \mathcal{F}]$ is an averaging operator, i.e. $\mathbb{E}[gf \,|\, \mathcal{F}] = g\mathbb{E}[f \,|\, \mathcal{F}]$ for all $g \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $f \in \mathcal{L}^1(\Omega, \Sigma, P)$ with $gf \in \mathcal{L}^1(\Omega, \Sigma, P)$. Taking $g = \chi_A$ where $A \in \mathcal{F}$, the above averaging property gives that the band projection, $Pf = gf$, has $P\mathbf{1} \in \mathcal{F}$ and commutes with the conditional expectation $\mathbb{E}[\cdot \,|\, \mathcal{F}]$. This can be generalised to the Riesz space setting, as shown in the following lemma.

Lemma 2.4.5. *[37] Let E be a Riesz space with a weak order unit and T be a conditional expectation on E. Let B be the band in E generated by $0 \leq g \in \mathcal{R}(T)$ and P be the band projection onto B. Then:*

(i) *$Tf \in B$ for each $f \in B$;*

(ii) *$Pf, (I - P)f \in \mathcal{R}(T)$ for each $f \in \mathcal{R}(T)$, where I denotes the identity map;*

(iii) *$Tf \in B^d$ for each $f \in B^d$.*

We now have that the conditional expectation, T, and band projections generated by an element in the range of T commute. The averaging properties of conditional expectations follow from this important result.

Theorem 2.4.6. *[37] Let E be a Dedekind complete Riesz space with weak order unit, T a conditional expectation on E and B be the band in E generated by $0 \leq g \in R(T)$, with associated band projection P. Then $TP = PT$.*

If no other structure on our Riesz space is assumed, Theorem 2.4.6 is our Riesz space analogue of the averaging properties of conditional expectations. However, if E is not

just a Riesz space but is an f-algebra, i.e. has a multiplicative structure compatible with the order and additive structures of the space, then the Riesz space averaging property can be strengthened so as to more closely resemble the classic version of the averaging property.

A Riesz space, E, is a Riesz algebra if E has a multiplicative structure that is both associative and distributive over addition. If the Riesz algebra E has the further property that for $u, v \in E$ $u \wedge v = 0$ implies $uw \wedge v = wu \wedge v = 0$, for $w \in E$ then E is called an f-algebra.

On the Dedekind complete Riesz space E^e, where e is a weak order unit in E, there is a natural f-algebra structure generated by setting $(Pe)\cdot(Qe) = PQe = (Qe)\cdot(Pe)$ for all band projections P and Q. In E^e, e is also the multiplicative unity.

We are now able to state a version of Freudenthal's theorem in Riesz spaces which highlights its compatibility with conditional expectation operators.

Theorem 2.4.7. *[33]*(**Freudenthal**)
Let E be a Dedekind complete Riesz space with a weak order unit e and T be a conditional expectation on E with $Te = e$. For each $g \in \mathcal{R}(T)_+$, there exists a sequence (s_n) such that $s_n \uparrow g$ in order. Here, each s_n is of the form $\sum_{i=1}^{k} a_i P_i e$, where $a_i \in \mathbb{R}_+$ and P_i is a band projection which commutes with T.

Using Freudenthal's theorem, the multiplication described above can be extended to the whole E^e (that is, not just elements of E^e that are band projections) and in fact to the universal completion E^u. The interested reader will find more detail on this in [37].

For a general f-algebra, Freudenthal's theorem gives the following averaging property

for conditional expectation operators.

Corollary 2.4.8. *Let E be a Dedekind complete Riesz space with a weak order unit e and T be a conditional expectation on E with $Te = e$. If E is an f-algebra and e is the multiplicative unit of the f-algebra, then $T(gf) = gTf$ for each $g \in R(T)$ and $f \in E$.*

2.5 Martingales in Riesz Spaces

The contents of this chapter are taken from [33, 32] and are included here for completeness. Further, these results, in particular those concerning martingale convergence, will be used in subsequent chapters.

Recall that martingales are traditionally defined in terms of a parametrized family of random variables and a filtration. In order to define a martingale in the Riesz space setting, we first consider the Riesz space analogue of a filtration. A filtration $(\mathcal{F}_i)_{i\in\mathbb{N}}$ in a probability space $(\Omega, \mathcal{F}, P)$ is an increasing family of sub-σ-algebras of $\mathcal{F}$. In terms of conditional expectations, we can write

$$\mathbb{E}\left[\mathbb{E}[f \,|\, \mathcal{F}_i] \,|\, \mathcal{F}_j\right] = \mathbb{E}[f \,|\, \mathcal{F}_i] = \mathbb{E}\left[\mathbb{E}[f \,|\, \mathcal{F}_j] \,|\, \mathcal{F}_i\right],$$

for all $f \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and $i \leq j$. Since to each sub-σ-algebra of $\mathcal{F}$ there corresponds precisely one conditional expectation operator and vice versa, a filtration can also be characterised as a sequence of conditional expectations with increasing range spaces. In light of this, we define a filtration in a Riesz space as follows.

Definition 2.5.1. *Let E be a Riesz space with weak order unit. A filtration on E is a family of conditional expectations, $(T_i)_{i\in\mathbb{N}}$, on E with*

$$T_iT_j = T_i = T_jT_i, \quad \text{for all} \quad i \leq j.$$

The Riesz space analogue of a martingale now follows immediately.

Definition 2.5.2. *Let E be a Riesz space with weak order unit. The pair $(f_i, T_i)_{i\in\mathbb{N}}$ is said to be a martingale on E if $(T_i)_{i\in\mathbb{N}}$ is a filtration, $f_i \in \mathcal{R}(T_i), i \in \mathbb{N}$, and*

$$T_i f_j = f_i, \quad \text{for all} \quad i \leq j.$$

If $T_i f_j \leq (\geq) f_i$ for all $i \leq j$ then $(f_i, T_i)_{i\in\mathbb{N}}$ is a sub(super)-martingale.

We say that a sequence $(f_i)_{i\in\mathbb{N}}$ is predictable (or previsible) with respect to the filtration $(T_i)_{i\in\mathbb{N}}$ if $f_{i+1} \in \mathcal{R}(T_i), i \in \mathbb{N}$.

We will now state the Doob-Meyer Decomposition theorem in Riesz spaces.

Theorem 2.5.3. *[38, Thm 3.3]* **(Doob-Meyer Decomposition)**
Let $(f_i, T_i)_{i\in\mathbb{N}}$ be a submartingale on Riesz space E with weak order unit. Define, for each $j \in \mathbb{N}$,

$$A_j = \sum_{i=1}^{j-1} T_i(f_{i+1} - f_i) \quad \text{and} \quad M_j = f_j - A_j.$$

Then $(M_j, T_j)_{j\in\mathbb{N}}$ is a martingale and $(A_j)_{j\in\mathbb{N}}$ is an increasing, predictable sequence with $A_1 = 0$. The decomposition

$$f_i = M_i + A_i$$

is the unique decomposition of $(f_i, T_i)_{i\in\mathbb{N}}$ into the sum of a martingale and a predictable sequence with starting value zero.

2.5.1 Martingale Convergence

For later reference (cf. Theorem 4.3.8) we now give the main results concerning martingale convergence in Riesz spaces, as proved in [32]. According to Meyer [46], 'It is

natural that martingales should be applied to Markov processes.' The implementation of Meyer's claim is not always obvious, as will be seen later.

The idea of local convergence is core to martingale convergence theorems. By a strictly positive operator on a Riesz space E, we mean an operator that maps to $E^+\backslash\{0\}$, where E^+ denotes the positive cone of E.

Lemma 2.5.4. *[36, Lemma 3.2] Let E be a Dedekind complete Riesz space with weak order unit e. Consider the sub (super) martingale $(f_i, T_i)_{i\in\mathbb{N}}$ where the operators $(T_i)_{i\in\mathbb{N}}$ are strictly positive. If there exists $g \in E^+$ such that $T_1|f_i| \leq g$, for all $i \in \mathbb{N}$, then, for each $n \in \mathbb{N}$, $(ne \wedge f_i \vee (-ne))_{i\in\mathbb{N}}$ is order convergent and the order limit, $F_n \in E$, is given by*

$$\limsup_i(ne \wedge f_i \vee (-ne)) = F_n = \liminf_i(ne \wedge f_i \vee (-ne)).$$

From the above lemma a martingale convergence result for order bounded martingales can be deduced.

Theorem 2.5.5. *[36, Thm 3.3] Let E be a Dedekind complete Riesz space with weak order unit. If there exists $g \in E^+$ such that $|f_i| \leq g$, for all $i \in \mathbb{N}$, then (f_i) is order convergent and the order limit, $f_\infty \in E$, is given by*

$$\limsup_i f_i = f_\infty = \liminf_i f_i.$$

It has been noted in [32] that this result falls short of being a Riesz space analogue of Doob's classical martingale convergence result. However, if we make the additional assumption that E is T-universally complete, we are able to generalize Doob's theorem to Riesz spaces.

Theorem 2.5.6. *[36, Thm 3.5] Let E be a T-universally Riesz space in which T is a strictly positive operator and with filtration $(T_i)_{i\in\mathbb{N}}$ such that $T_iT = T = TT_i$. If*

(f_i, T_i) is a sub (super) martingale on E and there exists $g \in E^+$ with $T|f_i| \leq g$ for all $i \in \mathbb{N}$ then (f_i) is order convergent and the order limit, $f_\infty \in E$, is given by

$$\limsup_i f_i = f_\infty = \liminf_i f_i.$$

Chapter 3

Independence in Riesz Spaces

In probability theory the concept of independence relies on both the presence of a probability measure and the multiplicative properties of $\mathbb{R}^+$. In the Riesz space setting, the role of the probability measure has to be taken on by a conditional expectation operator while the role of multiplication is mirrored at operator level by composition. Recall that an f-algebra is a Riesz space that has a multiplicative structure compatible with the order and additive structures of the space (cf. p.42).

For the remainder of this thesis, we will use wicket brackets, $\langle\cdot\rangle$, to denote the closed Riesz space generated by the elements within the wicket brackets. For example, $\langle f, g\rangle$ denotes the closed Riesz space generated by f and g.

Before we are able to state independence results in Riesz spaces, we first need an Andô-Douglas-Nikodým-Radon Theorem for Riesz spaces.

3.1 Andô-Douglas-Nikodým-Radon Theorem

The Radon-Nikodým theorem is widely known and used in Measure Theory. It is from this theorem that we are able to deduce many results regarding independence and Markov processes. It is important that an analogue of this result can be given in Riesz spaces. In [63] B. A. Watson proves a Radon-Nikodým type theorem in Riesz spaces. This result is fundamental to the work that follows in this thesis. It should be noted that J. J. Grobler has presented a alternative variant of the Radon-Nikodým theorem in Riesz spaces, [26].

Throughout this section T is a strictly positive conditional expectation operator acting on the Dedekind complete Riesz space, E, with weak order unit $e = Te$.

By a Dedekind complete Riesz subspace, F, of E we mean that F is a Riesz subspace of E and that F is Dedekind complete in its own right. Further, we require that if (f_α) is upwards directed and bounded above in F with supremum $f \in F$, then f is also the supremum of F in E.

Let $F \subset E$. We will denote by $\mathcal{B}(F)$ the class of all band projections on E with $Pe \in F$. Before we are able to give the Radon-Nikodým theorem, we require a Hahn-Jordan decomposition theorem in Riesz spaces. This result was also given by Watson in [63]. The result gives a decomposition of the map $\mathcal{B}(F) \to E$ with $P \mapsto TPf$.

Consider $f \in E$. We say that $J \in \mathcal{B}(F)$ is *positive* (resp. negative) with respect to (T, f) if $TPf \geq (resp.\ \leq)\ 0$ for all $P \in \mathcal{B}(F)$ such that $P \leq J$. A band projection, J, is *strictly positive* (resp. strictly negative) if J is positive (resp. negative) with respect to (T, f) and $TJf \neq 0$.

Theorem 3.1.1. *[63, Thm 3.5]* **(Hahn-Jordan Decomposition)**

Let E be a Dedekind complete Riesz space with strictly positive conditional expectation operator, T, and weak order unit, $e = Te$. Let F be a Dedekind complete Riesz subspace of E with $\mathcal{R}(T) \subset F$ and let $f \in E$. There exist band projections $Q_f^+, Q_f^- \in \mathcal{B}(F)$ with $I = Q_f^+ + Q_f^-$ having Q_f^+ positive with respect to (T, f) and $Q_f^- = I - Q_f^+$ negative with respect to (T, f).

Note: The band projections Q_f^+ and Q_f^- of the above theorem can be chosen so that Q_f^+ and Q_f^- are respectively monotonically increasing and decreasing with respect to f. For details of the construction see [63, p.561]

Theorem 3.1.2. *[63, Thm 4.1]* **(Radon-Nikodým)**

Let E be a T-universally complete Riesz space with weak order unit, $e = Te$, where T is a strictly positive conditional expectation operator on E. Let F be a closed Riesz subspace of E with $\mathcal{R}(T) \subset F$. For each $f \in E^+$ there exists a unique $g \in F^+$ such that

$$TPf = TPg, \quad \textit{for all} \quad P \in \mathcal{B}(F).$$

If E is a T-universally complete Riesz space with weak order unit $e = Te$ where T is a strictly positive conditional expectation operator on E, F is a closed Riesz subspace of E with $\mathcal{R}(T) \subset F$ and f and g are as in Theorem 3.1.2, then we denote

$$T_F f = g. \tag{3.1.1}$$

Lemma 3.1.3. *[63, Lemma 5.6] Let T be a strictly positive conditional expectation operator on the T-universally complete Riesz space, E, with weak order unit $e = Te$. Let F be a closed Riesz subspace of E with $\mathcal{R}(T) \subset F$. The map T_F is additive on*

E^+ and has

$$TPT_F(f) = TPf, \quad \text{for all} \quad P \in \mathcal{B}(F), f \in E^+.$$

Remark: The results can be extended to E by noting that each $f \in E$ can be decomposed as $f = f^+ - f^-$ where $f^+, f^- \in E^+$.

Corollary 3.1.4. *[63, Cor. 5.9]* **(Douglas-Andô)**
Let T be a strictly positive conditional expectation operator on the T-universally complete Riesz space, E, with weak order unit, $e = Te$. The subset F of E is a closed Riesz subspace of E with $\mathcal{R}(T) \subset F$ if and only if there is a conditional expectation T_F on E with $\mathcal{R}(T_F) = F$ and $TT_F = T = T_F T$. In this case $T_F f$ for $f \in E^+$ is uniquely determined by the property that

$$TPf = TPT_F f$$

for all band projections on E with $Pe \in F$.

Lemma 3.1.5. *Let E be a Dedekind complete Riesz space with conditional expectation operator, T, and weak order unit, $e = Te$, and E_1 and E_2 be two closed Riesz subspaces of E both containing e. Let $\tilde{E}$ be the closed Riesz subspace of E generated by E_1 and E_2. For band projections P_1, P_2 with $P_1 e \in E_1$ and $P_2 e \in E_2$, we have*

$$P_1 P_2 e \in \tilde{E}.$$

Proof. Consider $P_1 e \in E_1$ and $P_2 e \in E_2$. Then

$$P_1 e \wedge P_2 e = P_1 P_2 e$$

Certainly, $P_1 e \wedge P_2 e \in \tilde{E}$, giving that $P_1 P_2 e \in \tilde{E}$. □

3.2 T-Conditional Independence

Definition 3.2.1. *Let E be a Dedekind complete Riesz space with conditional expectation, T, and weak order unit, $e = Te$. Let P and Q be band projections on E. We say that P and Q are T-conditionally independent (that is, conditionally independent with respect to T) if*

$$TPTQe = TPQe = TQTPe. \tag{3.2.2}$$

We say that two Riesz subspaces E_1 and E_2 of E, where $\mathcal{R}(T) \subset E_i, i = 1, 2$, are T-conditionally independent if all band projections, $P_i, i = 1, 2$, in E with $P_i e \in E_i, i = 1, 2$, are T-conditionally independent.

In the case of $E = L^1(\Omega, \mathcal{A}, \mu)$ where μ is a probability measure, $e = \mathbf{1}$ and T is the expectation operator

$$Tf = \int_\Omega f \, d\mu = \mathbb{E}[f]\mathbf{1},$$

we have that the band projections on E are maps of the form $Pf = f\chi_A$ and $Qf = f\chi_B$ where $A, B \in \mathcal{A}$. Here

$$TPTQe = \mathbb{E}[\chi_A \mathbb{E}[\chi_B]] = \mathbb{E}[\chi_A \mu(B)] = \mu(B)\mathbb{E}[\chi_A] = \mu(B)\mu(A)$$

and similarly

$$TQTPe = \mu(A)\mu(B).$$

Also

$$TPQe = \mathbb{E}[\chi_A \chi_B] = \mathbb{E}[\chi_{A\cap B}] = \mu(A \cap B).$$

Thus, in this case, the Riesz space independence of P and Q corresponds to the classical independence of A and B.

This definition is independent of the choice of the weak order unit e with $e = Te$, as is shown in the following lemma.

It should be noted that the remaining results in this chapter have been published by Vardy and Watson in [61].

Theorem 3.2.2. *Let E be a Dedekind complete Riesz space with conditional expectation, T, and let e be a weak order unit which is invariant under T. The band projections P and Q in E are T-conditionally independent if and only if*

$$TPTQw = TPQw = TQTPw \quad \textit{for all} \quad w \in \mathcal{R}(T). \tag{3.2.3}$$

Proof. That (3.2.3) implies (3.2.2) is obvious. We now show that (3.2.2) implies (3.2.3). From linearity it is sufficient to show that (3.2.3) holds for all $0 \leq w \in \mathcal{R}(T)$. Consider $0 \leq w \in \mathcal{R}(T)$. By Freudenthal's theorem, there exist $a_j^n \in \mathbb{R}^+$ and band projections Q_j^n with $Q_j^n e \in \mathcal{R}(T), j = 0, \ldots, n2^n$, such that

$$s_n = \sum_{j=0}^{n2^n} a_j^n Q_j^n e$$

has

$$w = \lim_{n\to\infty} s_n.$$

Here we can take

$$\begin{aligned} s_n &= \sum_{j=0}^{n2^n} a_j^n Q_j^n e, \\ a_j^n &= \frac{j}{2^n}, \quad j = 0, \ldots, n2^n, \\ Q_{n2^n}^n &= P_{(w-ne)^+}, \\ Q_{j-1}^n &= (I - Q_j^n) P_{(w - a_{j-1}^n e)^+}, \quad j = 1, \ldots, n2^n. \end{aligned}$$

As $e, w \in \mathcal{R}(T)$, $Q_j^n T = TQ_j^n$. Thus

$$TPTQQ_j^n e = Q_j^n TPQe \tag{3.2.4}$$

since Q_j^n commutes with all the factors in the product and therefore with the product itself. Again using the commutation of band projections and the fact that $Q_j^n T = TQ_j^n$ we obtain

$$TPQQ_j^n e = Q_j^n TPQe. \tag{3.2.5}$$

Combining (3.2.4), (3.2.5) and using the linearity of T, P and Q gives

$$TPTQ\sum_{j=0}^{n2^n} a_j^n Q_j^n e \;=\; TPQ\sum_{j=0}^{n2^n} a_j^n Q_j^n e. \tag{3.2.6}$$

Since T, P, Q are order continuous, taking the limit as $n \to \infty$ of (3.2.6) we obtain

$$TPTQw \;=\; TPQw.$$

Interchanging the roles of P and Q gives

$$TQTPw \;=\; TQPw.$$

As band projections commute, we have thus shown that (3.2.3) holds. □

The following corollary to the above theorem shows that T-conditional independence of the band projections P and Q is equivalent to T-conditional independence of the closed Riesz subspaces $\langle Pe, \mathcal{R}(T)\rangle$ and $\langle Qe, \mathcal{R}(T)\rangle$ generated by Pe and $\mathcal{R}(T)$ and by Qe and $\mathcal{R}(T)$ respectively.

Corollary 3.2.3. *Let E be a Dedekind complete Riesz space with conditional expectation, T, and let e be a weak order unit which is invariant under T. Let $P_i, i = 1, 2$, be band projections on E. The band projections, $P_i, i = 1, 2$, are T-conditionally independent if and only if the closed Riesz subspaces $E_i = \langle P_i e, \mathcal{R}(T)\rangle, i = 1, 2$, are T-conditionally independent.*

Proof. The reverse implication is obvious. Assume $P_i, i = 1, 2$, are T-conditionally independent. We show that the closed Riesz subspaces $E_i, i = 1, 2$, are T-conditionally independent. As each element of $\mathcal{R}(T)$ is the limit of a sequence of linear combinations of band projections whose action on e is in $\mathcal{R}(T)$ (see Theorem 2.4.7) it follows from Lemma 3.1.5 that E_i is the closure of the linear span of

$$\{P_iRe, (I - P_i)Re \,|\, R \text{ band projection in } E \text{ with } Re \in \mathcal{R}(T)\}.$$

Consider band projections $R_i, i = 1, 2$, in E with $R_ie \in \mathcal{R}(T), i = 1, 2$. From the linearity and continuity of band projections and conditional expectations, it suffices to prove that each of the band projections P_1R_1 and $(I - P_1)R_1$ are T-conditionally independent of both P_2R_2 and $(I - P_2)R_2$. We will only prove that P_1R_1 is T-conditionally independent of P_2R_2 as the other three cases follow by similar reasoning. From Theorem 3.2.2,

$$TP_1TP_2R_1R_2e = TP_1P_2R_1R_2e = TP_2TP_1R_1R_2e.$$

As band projections commute and since $R_iT = TR_i, i = 1, 2$, we obtain

$$TP_1R_1TP_2R_2e = TP_1R_1P_2R_2e = TP_2R_2TP_1R_1e$$

giving the T-conditional independence of $P_iR_i, i = 1, 2$. □

In the light of the above corollary, when discussing T-conditional independence of Riesz subspaces of E with respect to T, we will assume that they are closed Riesz subspaces containing $\mathcal{R}(T)$.

Theorem 3.2.4. *Let E_1 and E_2 be two closed Riesz subspaces of the T-universally complete Riesz space E with strictly positive conditional expectation operator T and weak order unit $e = Te$. Let S be a conditional expectation on E with $ST = T$. If $\mathcal{R}(T) \subset E_1 \cap E_2$ and $T_{\langle \mathcal{R}(S), E_i \rangle}$ is the conditional expectation having as its range the*

closed Riesz subspace of E generated by $\mathcal{R}(S)$ and E_i, then the spaces E_1 and E_2 are S-conditionally independent, if and only if

$$T_i T_{\langle \mathcal{R}(S), E_{3-i} \rangle} = T_i S T_{\langle \mathcal{R}(S), E_{3-i} \rangle} \quad i = 1, 2,$$

where T_i is the conditional expectation commuting with T and having range E_i.

Proof. Assume first that E_1 and E_2 be S-conditionally independent, i.e. for all band projections P_i with $P_i e \in E_i$ for $i = 1, 2$, we have

$$SP_1SP_2e = SP_1P_2e = SP_2SP_1e.$$

Consider the equation

$$SP_1SP_2e = SP_1P_2e. \tag{3.2.7}$$

Applying T to both sides of the equation gives

$$TP_1P_2e = TP_1SP_2e.$$

Thus, by the Riesz space Radon-Nikodým-Douglas-Andô theorem, Theorem 3.1.2,

$$T_1P_2e = T_1SP_2e.$$

Now, let P_S be a band projection with $P_S e \in \mathcal{R}(S)$. Applying P_S and then T to (3.2.7) gives

$$TP_SSP_1P_2e = TP_SSP_1SP_2e.$$

As $P_S e \in \mathcal{R}(S)$, we have that $SP_S = P_S S$ which, together with the commutation of band projections, yields

$$TP_1P_SP_2e = TP_1SP_SP_2e.$$

Applying the Riesz space Radon-Nikodým-Douglas-Andô theorem now gives

$$T_1P_SP_2e = T_1SP_SP_2e.$$

Each element of $\langle \mathcal{R}(S), E_2 \rangle = \mathcal{R}(T_{\langle \mathcal{R}(S),E_2 \rangle})$ can be expressed as a limit of a net of linear combinations of elements of the form $P_S P_2 e$ where P_S and P_2 are band projections with $P_S e \in \mathcal{R}(S)$ and $P_2 e \in E_2$ respectively. From the continuity of T_1

$$T_1 T_{\langle \mathcal{R}(S),E_2 \rangle} = T_1 S T_{\langle \mathcal{R}(S),E_2 \rangle}.$$

Similarly, if we consider the equation $SP_2P_1e = SP_2SP_1e$ we have

$$T_2 T_{\langle \mathcal{R}(S),E_1 \rangle} = T_2 S T_{\langle \mathcal{R}(S),E_1 \rangle}.$$

Conversely, suppose $T_i T_{\langle \mathcal{R}(S),E_{3-i} \rangle} = T_i S T_{\langle \mathcal{R}(S),E_{3-i} \rangle}$ for all $i = 1, 2$. Again we consider only $T_1 T_{\langle \mathcal{R}(S),E_2 \rangle} = T_1 S T_{\langle \mathcal{R}(S),E_2 \rangle}$. Then, for all $P_2 e \in \mathcal{R}(T_2), P_S e \in \mathcal{R}(S)$,

$$T_1 P_S P_2 e = T_1 S P_S P_2 e.$$

Since $P_S e \in \mathcal{R}(S)$ we have

$$T_1 P_S P_2 e = T_1 P_S S P_2 e.$$

If we apply P_1, where $P_1 e \in \mathcal{R}(T_1)$, and then T to both sides of the above equality we obtain

$$T P_1 T_1 P_S P_2 e = T P_1 T_1 P_S S P_2 e.$$

Commutation of band projections, $T_1 P_1 = P_1 T_1$ and $T = T T_1$, applied to the above equation gives

$$T P_S P_1 P_2 e = T P_S P_1 S P_2 e.$$

Now from the Radon-Nikodým-Douglas-Andô theorem in Riesz spaces we have

$$S P_1 P_2 e = S P_1 S P_2 e.$$

By a similar argument using $T_2 T_{\langle \mathcal{R}(S),E_1 \rangle} = T_2 S T_{\langle \mathcal{R}(S),E_1 \rangle}$, we have

$$S P_2 P_1 e = S P_2 S P_2 e.$$

Since band projections commute we get

$$SP_1SP_2e = SP_1P_2e = SP_2SP_1e$$

which concludes the proof. □

Taking $S = T$ in the above theorem, we obtain the following corollary.

Corollary 3.2.5. *Let E_1 and E_2 be two closed Riesz subspaces of the T-universally complete Riesz space E with strictly positive conditional expectation operator T and weak order unit $e = Te$. If $\mathcal{R}(T) \subset E_1 \cap E_2$, then the spaces E_1 and E_2 are T-conditionally independent, if and only if*

$$T_1T_2 = T = T_2T_1,$$

where T_i is the conditional expectation commuting with T and having range E_i.

The following theorem is useful in the characterization of independent subspaces through conditional expectations.

Corollary 3.2.6. *Under the same conditions as in Corollary 3.2.5, E_1 and E_2 are T-conditionally independent if and only if*

$$T_if = Tf, \quad \text{for all} \quad f \in E_{3-i}, \quad i = 1, 2, \tag{3.2.8}$$

where T_i is the conditional expectation commuting with T and having range E_i.

Proof. Observe that (3.2.8) is equivalent to

$$T_iT_{3-i} = TT_{3-i} = T, \quad i = 1, 2.$$

The corollary now follows directly from Corollary 3.2.5. □

Theorem 3.2.4 can be applied to self-independence to give that the only self-independent band projections with respect to T are those onto bands generated by elements of the range of T.

Corollary 3.2.7. *Let E be a T-universally complete Riesz space E with strictly positive conditional expectation operator T and weak order unit $e = Te$. Let P be a band projection on E which is self-independent with respect to T, then $TP = PT$ and $TPe = Pe$.*

Proof. Let $T_1 = T_{\langle \mathcal{R}(T), Pe \rangle}$, then, by Theorem 3.2.4, $T_1 = T_1^2 = T$ and we obtain $TPe = T_1 Pe$. But $Pe \in \mathcal{R}(T_1)$ so $TPe = Pe$, thus $Pe \in \mathcal{R}(T)$ from which it follows that $TP = PT$. □

In measure theoretic probability, we can define independence of a family of σ-subalgebras. In a similar manner, in the Riesz space setting, we can define the independence with respect to T of a family of closed Dedekind complete Riesz subspaces of E.

For ease of notation, if $(E_\lambda)_{\lambda \in \Lambda}$ is a family of Riesz subspaces of E we put $E_\Lambda = \langle \bigcup_{\lambda \in \Lambda} E_\lambda \rangle$, the Riesz space generated by all $E_\lambda, \lambda \in \Lambda$.

Definition 3.2.8. *Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. Let $E_\lambda, \lambda \in \Lambda$, be a family of closed Dedekind complete Riesz subspaces of E having $\mathcal{R}(T) \subset E_\lambda$ for all $\lambda \in \Lambda$. We say that the family is T-conditionally independent if, for each pair of disjoint sets $\Lambda_1, \Lambda_2 \subset \Lambda$, we have that E_{Λ_1} and E_{Λ_2} are T-conditionally independent.*

Definition 3.2.8 leads naturally to the definition of T-conditional independence for sequences in E, given below.

Definition 3.2.9. *Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$. We say that the sequence (f_n) in E is T-conditionally independent if the family $\langle \{f_n\} \cup \mathcal{R}(T) \rangle, n \in \mathbb{N}$ of Dedekind complete Riesz spaces is T-conditionally independent.*

Chapter 4

Markov Processes

4.1 Introduction

Markov chains were first studied in detail by A. A. Markov in the early part of the 20^{th} century. The notion of Markov chains was born from Markov's desire to show that independence was not a necessary condition for the law of large numbers to hold. That independence was required for the law of large numbers to hold was proposed by P. A. Neskrasov, [8]. In correspondence between Markov and Chuprov [8], a colleague of Markov's, Markov writes: 'The unique service of P. A. Nekrasov is namely this. He brings out sharply his delusion, shared, I believe, by many, that independence is a necessary condition for the law of large numbers. This circumstance has led me to explain, in a series of articles, that the law of large number and Laplace's formula can apply also to dependent variables. In this way a construction of a highly general character was actually arrived at, which P. A. Nekrasov cannot even dream about.' It is interesting to note that Nekrasov was a theologist by training and later took

up mathematics. As a member of the Russian Orthodox Church, Nekrasov was a strong proponent of the religious doctrine of free will. In fact, Nekrasov's belief that independence was a necessary condition for the law of large numbers was used to provide mathematical 'proof' of this belief. Markov, an atheist and excommunicate of the Church, was outspoken in his refutation of this, [8].

Markov's first paper containing the concept of chains was published in 1907 [43]. In this paper he defined the *simple chain* as ' an infinite sequence $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots$ of variables connected in such a way that x_{k+1} for any k is independent of $x_1, x_2, \ldots, x_{k-1}$ in case x_k is known'. However, it was not until 20 years later that the term 'Markov chain' was coined by Bernstein, [10].

Markov, however, was not the first to study such chains. Some of the urn problems studied by Laplace, Bernoulli and Ehrenfests are special cases of Markov chains.

An example of such a process is the following. Consider a deck of cards being shuffled and the order of the cards after each shuffling. To predict the order of the cards after shuffling, all useful information is included in complete knowledge of the current state of the deck. Knowledge of prior states does not make the prediction any more accurate.

Markov processes can be applied to the long term behaviour of systems. For example, the evolution of animal (including human) populations can be described using Markovian models. These models predict only three types of limit behaviour: extinction, equilibrium or explosion.

This chapter is set out in the following manner: we first define Markov processes in the classical setting and state the relevant classical results. The remaining subsections

are dedicated to defining Markov processes in Riesz spaces and providing analogues of classical results in the Riesz space setting.

We will show that a Markov process is a process for which, given the present, the past and future are independent and give a Chapman-Kolmogorov type equation. We show that sums of independent variables form a Markov process and that, under certain conditions, these sums are martingales too.

It should be noted that the material in section 4.3 has been published by Vardy and Watson in [61, 62].

4.2 Classical Markov Processes

Here we present the classic definitions of Markov processes and some well-known results. This section forms the base from which we will develop the theory of Markov processes in Riesz spaces. The results given here will be generalized to the Riesz space setting later in this chapter. We give here only the definitions and results pertinent to the thesis. The interested reader can find further details on these results and many more results in [51].

4.2.1 Classical foundations of Markov processes

We say the collection $\{X_t\}_{t\in\Lambda}$, where $\Lambda \subset \mathbb{R}$, is a *stochastic process* if, for each $t \in \Lambda$, X_t is such that

$$X_t : \Omega \to \mathbb{R}, \quad X_t \in \mathcal{L}^1(\Omega, \mathcal{F}, P).$$

That is, for all $t \in \Lambda$ we have that X_t is a random variable.

Definition 4.2.1. *[51]*

A stochastic process $\{X_t\}_{t\in\Lambda}$ *in* $\mathcal{L}^1(\Omega,\mathcal{F},P)$ *is called a Markov process if for any set of points* $t_1 < t_2 < \cdots < t_{n+1},\ t_i \in \Lambda$, *and* $x \in \mathbb{R}$, *one has*

$$P(X_{t_{n+1}} < x \,|\, X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = P(X_{t_{n+1}} < x \,|\, X_{t_n}) \text{ a.e.}. \tag{4.2.1}$$

It is important to note that the above definition defines the Markov Property in terms of only one state in the future. However, if a stochastic process X is Markov, this property holds for all states in the future. We are thus able to extend the Markov property to all events in entire future, as is given in the following lemma.

Lemma 4.2.2. *[51]*

If a stochastic process $\{X_t\}_{t\in\Lambda}$ *is a Markov process then*

$$P(A \,|\, X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = P(A \,|\, X_{t_n})$$

for any $A \in \sigma(X_s; s \geq t_n), s \in \Lambda$.

The Chapman-Kolmogorov equation is at the base of many aspects of the theory of Markov processes.

Theorem 4.2.3. *[51]* **(Chapman-Kolmogorov Equation)**

Let $\{X_t\}_{t\in\mathbb{N}}$ *be a Markov process in* $\mathcal{L}^1(\Omega,\mathcal{F},P)$ *and* $u < t < v$ *be points from* $\mathbb{N}$. *Then for each* $x \in \mathbb{R}$ *we have*

$$P(X_v < x \,|\, X_u) = E[P(X_v < x \,|\, X_t) \,|\, X_u] \text{ a.e.}.$$

The definition of a Markov process is not uniform throughout the literature. However, the more common of these definitions are equivalent, as shown in the result below.

Theorem 4.2.4. *[51]*

For a process $\{X_t\}_{t\in J\subset\Lambda}$ *in* $\mathcal{L}^1(\Omega,\mathcal{F},P)$ *the following are equivalent.*

(i) The process is a Markov process, as defined in Definition 4.2.1.

(ii) For each $u < v$ in J and $x \in \mathbb{R}$, one has

$$P(X_v < x \,|\, X_s, s \leq u) = P(X_v < x \,|\, X_u) \text{ a.e..}$$

(iii) For $s_1 < s_2 < \cdots < s_m < t < t_1 < t_2 < \cdots < t_n$ from J, and $x_i, y_j \in \mathbb{R}$, almost everywhere one has

$$\begin{aligned} &P(X_{s_i} < x_i, X_{t_j} < y_j; 1 \leq i \leq m, 1 \leq j \leq n \,|\, X_t) \\ &= \quad P(X_{s_i} < x_i; 1 \leq i \leq m \,|\, X_t) P(X_{t_j} < y_j; 1 \leq j \leq n \,|\, X_t). \end{aligned}$$

We note that all the above definitions and theorems concerning Markov processes are stated largely in terms of conditional probabilities. In order to define Markov processes in Riesz spaces we need to translate all these definitions to definitions in terms of conditional expectation operators (as such operators have already been defined in Riesz spaces, cf. [37]). Besides the usual translation ($P(A|B) = \mathbb{E}[\chi_A|B]$) we have that if X is a Markov process, then for all bounded, Borel measurable functions g and for $t > t_n > \cdots > t_1$, we have

$$\mathbb{E}[g(X_t) \,|\, X_{t_1}, X_{t_2}, \ldots, X_{t_n}] = \mathbb{E}[g(X_t) \,|\, X_{t_n}]. \tag{4.2.2}$$

To see that (4.2.1) is equivalent to (4.2.2) translate the probability definition to one involving conditional expectations in the usual manner (using indicator functions) and generalise from there. To see (4.2.2) implies (4.2.1) simply let $g = \chi_B$. This is the case if g is continuous. For a general Borel measurable function, we make use of Lebesgue's Monotone convergence theorem

If the conditional measures are regular, part (iii) of Theorem 4.2.4 may be interpreted as follows. A stochastic process is a Markov process if and only if, given the present

state, the past and future states are independent. However, the condition of regularity of the conditional measures is not a trivial one and requires restrictions on the underlying measure space.

The interpretation of the Markov definition in terms of it's past and present has an interesting, and often useful, consequence. Since the 'past' and the 'present' depend only on the ordering of Λ, we have that if $\{X_t\}_{t\in J}$ is a Markov process then $\{X_t\}_{t\in\tilde{J}}$ is also a Markov process, where $J, \tilde{J}$ have opposite orderings but are the same set. In particular, if $\{X_t\}_{-a\leq t\leq a}$ is a Markov process, then so is $\{X_{-t}\}_{-a\leq t\leq a}$.

Finally, we can relate the sums of independent random variables and Markov processes.

Theorem 4.2.5. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$, then the sequence of partial sums*

$$\left\{ S_n = \sum_{i=1}^{n} X_i \right\}_{n\in\mathbb{N}}$$

forms a Markov process.

Note 4.2.6. *Let $X_1, X_2, \ldots$ be as above. If in addition, $\mathbb{E}(X_i)$ exists and $\mathbb{E}(X_i) = 0$ for all $i \in \mathbb{N}$, then $(S_n, \mathcal{B}_n)_{n\in\mathbb{N}}$ is a martingale, where $\mathcal{B}_n = \sigma(S_1, S_2, \ldots, S_n)$.*

4.3 Markov Processes in Riesz Spaces

4.3.1 Preliminaries

Based on the definition of a Markov process in $\mathcal{L}^1$ by M. M. Rao [51] we define a Markov process in a Riesz space as follows.

Definition 4.3.1. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let Λ be a totally ordered index set. A net $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process in E if for any set of points $t_1 < \cdots < t_n < t,\ t_i,\ t \in \Lambda$, we have*

$$T_{(t_1,\ldots,t_n)}Pe = T_{t_n}Pe \quad \textit{for all} \quad Pe \in \langle \mathcal{R}(T), X_t\rangle\,, \tag{4.3.3}$$

with P a band projection. Here $T_{(t_1,t_2,\ldots,t_n)}$ is the conditional expectation with range $\langle \mathcal{R}(T), X_{t_1}, X_{t_2}, \ldots, X_{t_n}\rangle$.

Note 4.3.2. An application of Lemma 3.1.5 to (4.3.3) yields that (4.3.3) is equivalent to

$$T_{(t_1,\ldots,t_n)}f = T_{t_n}f, \quad \text{for all} \quad f \in \mathcal{R}(T_t),$$

which in turn is equivalent to

$$T_{(t_1,\ldots,t_n)}T_t = T_{t_n}T_t$$

where T_t is the conditional expectation with range $\langle \mathcal{R}(T), X_t\rangle$.

We can extend the Markov property to include the entire future, as is shown below.

Lemma 4.3.3. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let Λ be a totally ordered index set. Suppose $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process in E. If $s_m > \cdots > s_1 > t > t_n > \cdots > t_1$, $t_j, s_j, t \in \Lambda$ and for each $i = 1,\ldots,m$, Q_i is a band projection with $Q_ie \in \langle \mathcal{R}(T), X_{s_i}\rangle$, then*

$$T_{(t_1,\ldots,t_n,t)}Q_1Q_2\ldots Q_me = T_tQ_1Q_2\ldots Q_me. \tag{4.3.4}$$

Proof. Under the assumptions of the lemma, if we denote $s_0 = t$, from Note 4.3.2

$$T_{s_j}Q_{j+1}T_{s_{j+1}} = T_{s_j}T_{s_{j+1}}Q_{j+1} = T_{(t_1,\ldots,t_n,s_0,\ldots,s_j)}T_{s_{j+1}}Q_{j+1} = T_{(t_1,\ldots,t_n,s_0,\ldots,s_j)}Q_{j+1}T_{s_{j+1}},$$

which, if we denote $S_{s_j} = T_{(t_1,\ldots,t_n,s_0,\ldots,s_j)}$, gives

$$T_{s_j} Q_{j+1} T_{s_{j+1}} = S_{s_j} Q_{j+1} T_{s_{j+1}}. \tag{4.3.5}$$

Similarly, if we denote $U_{s_j} = T_{(s_0,\ldots,s_j)}$, then

$$T_{s_j} Q_{j+1} T_{s_{j+1}} = U_{s_j} Q_{j+1} T_{s_{j+1}}. \tag{4.3.6}$$

Applying (4.3.5) recursively we obtain

$$\begin{aligned} T_{s_0} Q_1 T_{s_1} Q_2 T_{s_2} \ldots T_{s_{m-1}} Q_m e &= S_{s_0} Q_1 T_{s_1} Q_2 T_{s_2} \ldots T_{s_{m-1}} Q_m e \\ &= S_{s_0} Q_1 S_{s_1} Q_2 T_{s_2} \ldots T_{s_{m-1}} Q_m e \\ &= \ldots \\ &= S_{s_0} Q_1 S_{s_1} Q_2 S_{s_2} \ldots S_{s_{m-1}} Q_m e. \end{aligned}$$

Here we have also used that $e = T_{s_m} e$. But $Q_i S_{s_j} = S_{s_j} Q_i$ and $S_{s_i} S_{s_j} = S_{s_i}$ for all $i \leq j$ giving

$$S_{s_0} Q_1 S_{s_1} Q_2 S_{s_2} \ldots S_{s_{m-1}} Q_m e = S_{s_0} S_{s_1} \ldots S_{s_{m-1}} Q_1 \ldots Q_m e = S_{s_0} Q_1 \ldots Q_m e.$$

Combining the above two displayed equations gives

$$T_{s_0} Q_1 T_{s_1} Q_2 T_{s_2} \ldots T_{s_{m-1}} Q_m e = S_{s_0} Q_1 \ldots Q_m e.$$

Similarly

$$T_{s_0} Q_1 T_{s_1} Q_2 T_{s_2} \ldots T_{s_{m-1}} Q_m e = U_{s_0} Q_1 \ldots Q_m e.$$

Thus $S_{s_0} Q_1 \ldots Q_m e = U_{s_0} Q_1 \ldots Q_m e$ which proves the lemma. □

Note 4.3.4. From Lemma 3.1.5, we have that the linear span of

$$\{Q_1 \ldots Q_m e \mid Q_i e \in \langle \mathcal{R}(T), X_{s_i} \rangle, Q_i \text{ band projections}, i = 1, \ldots, m\}$$

is dense in $\langle \mathcal{R}(T), X_{s_1}, \ldots, X_{s_m} \rangle$, giving

$$T_{(t_1,\ldots,t_n)} f = T_{t_n} f \quad \text{for all} \quad f \in \langle \mathcal{R}(T), X_{s_1}, \ldots, X_{s_m} \rangle \,, \tag{4.3.7}$$

where $s_1 > s_2 > \cdots > s_m > t > t_n > \cdots > t_1$.

Theorem 4.3.5. (Chapman-Kolmogorov)

Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let Λ be a totally ordered index set. If $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process and $u < t < n$, then

$$T_u X = T_u T_t X, \quad \textit{for all} \quad X \in \mathcal{R}(T_n),$$

where $\mathcal{R}(T_u) = \langle \mathcal{R}(T), X_u \rangle$.

Proof. We recall that $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process if for any set of points $t_1 < \cdots < t_n < t, t, t_i \in \Lambda$ one has

$$T_{(t_1,\ldots,t_n)} X = T_{t_n} X$$

where $X \in \langle \mathcal{R}(T), X_t \rangle$. Thus,

$$T_{(u,t)} X = T_t f, \quad \text{for } X \in \mathcal{R}(T_n).$$

Applying T_u to the above equation gives

$$T_u T_{(u,t)} X = T_u T_t X,$$

and, thus

$$T_u X = T_u T_t X$$

since $\mathcal{R}(T_u) \subset \mathcal{R}(T_{(u,t)})$. □

Under the hypotheses of Theorem 4.3.5, it follows directly from the Chapman-Kolmogov Theorem and Freudenthal's Theorem, as in the proof of Theorem 3.2.2, that if $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process and $u < t < n$, then

$$T_u T_n = T_u T_t T_n.$$

It is often stated that a stochastic process is Markov if and only if the past and future are independent given the present, see [51, p 351]. It is clear that such independence implies, even in the Riesz space setting, that the process is a Markov process. However, the non-commutation of conditional expectations onto non-comparable closed Riesz subspaces (or in the classical setting, the non-commutation of conditional expectations with respect to non-comparable σ-algebras), makes the converse of the above claim more interesting. The proof of this equivalence (part (iii) of the following theorem) relies on the fact that conditional expectation operators are averaging operators and, in the Riesz space setting, that E^e is an f-algebra, and is as such a commutative algebra. Classical versions of the following theorem can be found in [6, 12, 51].

Theorem 4.3.6. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let Λ be a totally ordered index set. For $(X_t)_{t\in\Lambda} \subset E$ the following are equivalent:*

(i) The process, $(X_t)_{t\in\Lambda}$ is a Markov process.

(ii) For conditional expectations $\mathbb{T}_u$ and T_v with $\mathcal{R}(\mathbb{T}_u) = \langle \mathcal{R}(T), X_n; n \leq u\rangle$ and $\mathcal{R}(T_v) = \langle \mathcal{R}(T), X_v\rangle$, $u < v$ in Λ, we have

$$\mathbb{T}_u T_v = T_u T_v.$$

(iii) For any $s_m > \cdots > s_1 > t > t_n > \cdots > t_1$ *from* Λ*, and* P, Q *band projections with* $Qe \in \langle \mathcal{R}(T), X_{s_1}, \ldots, X_{s_m} \rangle$ *and* $Pe \in \langle \mathcal{R}(T), X_{t_1}, \ldots, X_{t_n} \rangle$ *we have*

$$T_t Q T_t P e = T_t Q P e = T_t P Q e = T_t P T_t Q e.$$

Proof. **(i) $\Rightarrow$ (ii)** Let $u < v, u, v \in \Lambda$, and P be a band projection with $Pe \in \langle \mathcal{R}(T), X_v \rangle$. Let P_i be a band projection with $P_i e \in \mathcal{R}(T_{t_i}), t_1 < t_2 < \cdots < t_n = u, n \in \mathbb{N}$. From the definition of a Markov process, for all $t_1 < t_2 < \ldots t_n = u < t = v$ we have $T_{(t_1,\ldots,t_n)} Pe = T_{t_n} Pe$ and $P_i T_{(t_1,\ldots,t_n)} = T_{(t_1,\ldots,t_n)} P_i$ thus

$$T_{(t_1,\ldots,t_n)} P_1 P_2 \ldots P_n Pe = P_1 P_2 \ldots P_n T_{t_n} Pe.$$

Applying T to this equation gives

$$T P_1 P_2 \ldots P_n Pe = T P_1 P_2 \ldots P_n T_{t_n} Pe. \tag{4.3.8}$$

Note, by Lemma 3.1.5, the set of (finite) linear combinations of elements of

$$D = \{P_1 P_2 \ldots P_n e \mid P_i \text{ a band projection}, P_i e \in \mathcal{R}(T_{t_i}), t_1 < t_2 < \cdots < t_n = u, n \in \mathbb{N}\}$$

is dense in $\mathcal{R}(\mathbb{T}_u)$. This, together with (4.3.8), gives

$$TQPe = TQT_{t_n} Pe \tag{4.3.9}$$

for band projections Q with $Qe \in \mathcal{R}(\mathbb{T}_u)$. Applying the Riesz space Radon-Nikodým-Douglas-Andô theorem to (4.3.9) gives

$$\mathbb{T}_u Pe = \mathbb{T}_u T_{t_n} Pe = T_u Pe. \tag{4.3.10}$$

Now Freudenthal's theorem, as in the proof of Theorem 3.2.2, gives

$$\mathbb{T}_u f = T_u f$$

for $f \in \mathcal{R}(T_v)$, or equivalently

$$\mathbb{T}_u T_v = T_u T_v.$$

(ii) $\Rightarrow$ (i) Assume that for $u < v$ we have

$$\mathbb{T}_u T_v = T_u T_v. \tag{4.3.11}$$

Let $t_1 < \cdots < t_n < t$. Taking $v = t$ and $u = t_n$, we have $T_{(t_1,\ldots,t_n)}\mathbb{T}_u = T_{(t_1,\ldots,t_n)}$ and $T_{(t_1,\ldots,t_n)}T_u = T_u = T_{t_n}$. Thus applying $T_{(t_1,\ldots,t_n)}$ to (4.3.11) gives

$$T_{(t_1,\ldots,t_n)}T_t = T_{(t_1,\ldots,t_n)}\mathbb{T}_u T_v = T_{(t_1,\ldots,t_n)}T_u T_v = T_{t_n}T_t.$$

Applying this operator equation to Pe where P is a band projection with $Pe \in \mathcal{R}(T_t)$ gives that $(X_\lambda)_{\lambda\in\Lambda}$ is a Markov process.

(i) $\Rightarrow$ (iii) Let Q be a band projection with $Qe \in \langle \mathcal{R}(T), X_{s_1}, \ldots, X_{s_m}\rangle$ then, from Lemma 4.3.3,

$$T_{(t_1,\ldots,t_n,t)}Qe = T_t Qe.$$

Applying a band projection P with $Pe \in \langle \mathcal{R}(T), X_{t_1}, \ldots, X_{t_n}\rangle$ followed by T_t to this equation gives

$$T_t PQe = T_t T_{(t_1,\ldots,t_n,t)}PQe = T_t P T_{(t_1,\ldots,t_n,t)}Qe = T_t P T_t Qe.$$

To prove $T_t Q T_t Pe = T_t QPe$, we prove $T_t Q T_t Pe = T_t P T_t Qe$ and use the result above. Recall that in the f-algebra we have $Qf = Qe\cdot f$. Using the commutativity of multiplication in the f-algebra E_e and the fact that T_t is an averaging operator in E_e, we

have

$$\begin{aligned} T_tQT_tPe &= T_t((Qe)\cdot(T_tPe)) \\ &= (T_tPe)\cdot(T_tQe) \\ &= (T_tQe)\cdot(T_tPe) \\ &= T_t((Pe\cdot(T_tQe)) \\ &= T_tPT_tQe. \end{aligned}$$

Finally, by the commutation of band projections, $T_tPQe = T_tQPe$.

(iii) ⇒ (i) Suppose $T_tPQe = T_tPT_tQe$ for all band projections P and Q with $Qe \in \langle \mathcal{R}(T), X_{s_1}, \ldots, X_{s_m}\rangle$ and $Pe \in \langle \mathcal{R}(T), X_{t_i}, \ldots, X_{t_n}\rangle$. Let R be a band projection with $Re \in \langle \mathcal{R}(T), X_t\rangle$, then

$$TRPT_{(t_1,\ldots t_n,t)}Qe = TRT_{(t_1,\ldots,t_n,t)}PQe = TT_{(t_1,\ldots,t_n,t)}RPQe$$

as $PT_{(t_1,\ldots t_n,t)} = T_{(t_1,\ldots,t_n,t)}P$ and $RT_{(t_1,\ldots,t_n,t)} = T_{(t_1,\ldots,t_n,t)}R$. But $TT_{(t_1,\ldots,t_n,t)} = T = TT_t$, so

$$TRPT_{(t_1,\ldots t_n,t)}Qe = TRPQe = TT_tRPQe.$$

Since $T_tR = RT_t$ we have $TT_tRPQe = TRT_tPQe$ and the hypothesis gives that $T_tPQe = T_tPT_tQe$ which combine to yield $TT_tRPQe = TRT_tPT_tQe$. Again appealing to the commutation of R and T_t and that $TT_t = T$ we have

$$TRT_tPT_tQe = TT_tRPT_tQe = TRPT_tQe,$$

giving

$$TRPT_{(t_1,\ldots t_n,t)}Qe = TRPT_tQe$$

for all such R and P. As the linear combinations of elements of the form RPe are dense in $\langle \mathcal{R}(T), X_{t_1}, \ldots, X_{t_n}, X_t\rangle$, we have, for all $Se \in \langle \mathcal{R}(T), X_{t_1}, \ldots, X_{t_n}, X_t\rangle$, that

$$TST_{(t_1,\ldots,t_n,t)}Qe = TST_tQe.$$

By (3.1.2) and the unique determination of conditional expectation operators by their range spaces, we have that $T_{(t_1,\ldots,t_n,t)}Qe = T_tQe$, proving the result. □

Note 4.3.7. Proceeding in a similar manner to the proof of (i) ⇒ (ii) in the above proof it follows that (iii) in the above theorem is equivalent to

$$\mathbb{T}_t\mathbb{S}_t = T_t = \mathbb{T}_t\mathbb{S}_t$$

where $\mathbb{S}_t$ is the conditional expectation with range space $\mathcal{R}(\mathbb{S}_u) = \langle \mathcal{R}(T), X_n; n \geq u\rangle$. This shows that a process is a Markov process in a Riesz space if and only if the past and future are conditionally independent on the present.

4.3.2 Independent Sums

There is a natural connection between sums of independent random variables and Markov processes. In the Riesz space case, this is illustrated by the following theorem.

Theorem 4.3.8. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let (f_n) be a sequence in E which is T-conditionally independent then*

$$\left(\sum_{k=1}^{n} f_k\right)$$

is a Markov process.

Proof. Let $S_n = \sum_{k=1}^{n} f_k$. We note that $\langle \mathcal{R}(T), S_1, \ldots, S_n\rangle = \langle \mathcal{R}(T), f_1, \ldots, f_n\rangle$. Let $m > n$ and P and Q be band projections with $Pe \in \langle \mathcal{R}(T), S_n\rangle$ and $Qe \in \langle \mathcal{R}(T), f_{n+1}, \ldots, f_m\rangle$. Since $\langle \mathcal{R}(T), S_n\rangle \subset \langle \mathcal{R}(T), f_1, \ldots, f_n\rangle$ and (f_n) is T-conditionally independent we have that $\langle \mathcal{R}(T), S_n\rangle$ and $\langle \mathcal{R}(T), f_{n+1}, \ldots, f_m\rangle$ are T-conditionally independent. Thus P and Q are T-conditionally independent.

Denote by $\mathbb{T}_n, T_n$ and $\mathbb{S}$ the conditional expectations with ranges $\langle \mathcal{R}(T), f_1, \ldots, f_n \rangle$, $\langle \mathcal{R}(T), S_n \rangle$ and $\langle \mathcal{R}(T), f_{n+1}, \ldots, f_m \rangle$ respectively. Now from the independence of (f_n) with respect to T we have, by Corollary 3.2.5,

$$\mathbb{T}_n \mathbb{S} = T = \mathbb{S}\mathbb{T}_n. \tag{4.3.12}$$

As $Pe \in \langle \mathcal{R}(T), S_n \rangle \subset \langle \mathcal{R}(T), S_1, \ldots, S_n \rangle$ and $\mathbb{S}Qe = Qe$ it follows that

$$\mathbb{T}_n PQe = P\mathbb{T}_n Qe = P\mathbb{T}_n \mathbb{S}Qe. \tag{4.3.13}$$

From (4.3.12)

$$P\mathbb{T}_n \mathbb{S}Qe = PTQe. \tag{4.3.14}$$

As $\mathcal{R}(T_n) \subset \mathcal{R}(\mathbb{T}_n)$, which is T-conditionally independent of $\mathbb{S}$,

$$T_n \mathbb{S} = T = \mathbb{S}T_n. \tag{4.3.15}$$

Combining (4.3.14) and (4.3.15) yields

$$PTQe = PT_n \mathbb{S}Qe. \tag{4.3.16}$$

As noted $\mathbb{S}Qe = Qe$, also $T_n P = PT_n$, so

$$PT_n \mathbb{S}Qe = T_n PQe. \tag{4.3.17}$$

Combining (4.3.13), (4.3.14), (4.3.16) and (4.3.17) gives

$$\mathbb{T}_n PQe = P\mathbb{T}_n Qe = P\mathbb{T}_n \mathbb{S}Qe = PTQe = PT_n \mathbb{S}Qe = T_n PQe. \tag{4.3.18}$$

Again, using Lemma 3.1.5 the closure of the linear span of

$$\{PQe \,|\, Pe \in \langle \mathcal{R}(T), S_n \rangle, Qe \in \langle \mathcal{R}(T), f_{n+1}, \ldots, f_m \rangle, P, Q \text{ band projections}\}$$

contains $\mathcal{R}(T_m)$. Thus by the order continuity of T_n and $\mathbb{T}_n$ in (4.3.18),

$$\mathbb{T}_n h = T_n h$$

for all $h \in \langle \mathcal{R}(T), S_m \rangle$, proving that (S_n) is a Markov process. □

Corollary 4.3.9. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let (f_n) be a sequence in E which is T-conditionally independent. If $Tf_i = 0$ for all $i \in \mathbb{N}$, then the sequence of partial sums (S_n), where $S_n = \sum_{k=1}^{n} f_k$, is a martingale with respect the filtration $(\mathbb{T}_n)$ where $\mathbb{T}_n$ is the conditional expectation with range $\langle f_1, \ldots, f_n, \mathcal{R}(T)\rangle$.*

Proof. Since $\mathcal{R}(\mathbb{T}_i) \subset \mathcal{R}(\mathbb{T}_j)$ for all $i \leq j$ we have that

$$\mathbb{T}_i\mathbb{T}_j = \mathbb{T}_i = \mathbb{T}_j\mathbb{T}_i$$

and $(\mathbb{T}_n)$ is a filtration. Further, $f_1, \ldots, f_i \in \mathcal{R}(\mathbb{T}_i)$ for all i by construction of $\mathbb{T}_i$ giving $\mathbb{T}_i S_i = S_i$.

If $i < j$, then from the independence of (f_n) with respect to T we have $\mathbb{T}_i T_j = T = T_j\mathbb{T}_i$ which applied to f_j gives

$$\mathbb{T}_i f_j = \mathbb{T}_i T_j f_j = T f_j = 0, \tag{4.3.19}$$

Thus

$$\mathbb{T}_i S_j = \mathbb{T}_i S_i + \sum_{k=i+1}^{j} \mathbb{T}_i f_k = \mathbb{T}_i S_i = S_i,$$

proving $(f_i, \mathbb{T}_i)$ a martingale. □

From Corollary 4.3.9 and [36, Thm 3.5] we obtain the following result regarding the convergence of sums of independent summands.

Theorem 4.3.10. *Let T be a strictly positive conditional expectation on the T-universally complete Riesz space E with weak order unit $e = Te$. Let (f_n) be a sequence in E which is T-conditionally independent. If $Tf_i = 0$ for all $i \in \mathbb{N}$, and there exists $g \in E$ such that $T\left|\sum_{i=1}^{n} f_i\right| \leq g$ for all $n \in \mathbb{N}$ then the sum $\sum_{k=1}^{\infty} f_k$ is order convergent in the sense that its sequence of partial sums is order convergent.*

It should be noted that the third last line of the proof of the upcrossing theorem ([36, Thm 3.1]) by Kuo, Labuschagne and Watson should be replaced by

'Now, as $S \geq Q$, we have $S_N^n \leq Q_N^n$ and so $Q_N^n - S_N^n \geq 0$, thus ...'.

Chapter 5

Mixingales

Mixingales are a generalisation of martingales and mixing sequences and were first introduced by D.L. McLeish in [45]. McLeish defines mixingales using the L^2-norm and proves invariance principles under strong mixing conditions, [45]. In [44] a strong law for large numbers is given using mixingales with restrictions on the mixingale numbers.

In 1988, Donald W. K. Andrews used an analogue of McLeish's mixingale condition to define L^1-mixingales,[5]. The L^1-mixingale condition is weaker than McLeish's mixingale condition and makes no restriction on the decay rate of the mixingale numbers, as was assumed by McLeish. Andrews used L^1-mixingales to present L^1 and weak laws of large numbers, [5]. The proofs presented in Andrews are remarkably simple and self-contained. Mixingales have also been considered in general L^p spaces ($1 \leq p < \infty$) by, amongst others, de Jong, in [16, 17] and more recently by Hu, see [30].

Examples of L^1-mixingales include martingale difference sequences, integrable M-

dependent sequences and certain stationary Gaussian processes, [5].

In this chapter we define mixingales in a Riesz space and prove a weak law of large numbers for mixingales in this setting. This generalises the results in the L^p setting to a measure free setting. In our approach the proofs rely on the order structure of the Riesz spaces.

We first present the results (in the classical setting) that we will generalise to Riesz spaces.

5.1 Classical Mixingales

For the most part, the contents of this section are due to Andrews, [5]. We follow the usual conventions and let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the random variables $(X_i)_{i\geq 1}$. We deviate from convention by considering the family of non-decreasing sub-σ-algebras of $\mathcal{F}$ indexed by $\{\cdots -2, -1, 0, 1, 2, \dots\}$. We will call this family an *integer indexed filtration*. It is common to let $\mathcal{F}_i = \sigma(X_1, X_2, \dots, X_i)$ for all $i > 0$ and $\mathcal{F}_i = \{\Omega, \phi\}$ for all $i \leq 0$. We denote by $\|\cdot\|_p$ the L^p-norm.

Definition 5.1.1. *Consider the sequence of random variables $(X_i)_{i\geq 1}$ adapted to the integer filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. Let $(c_i)_{i\geq 1}$ and $(\phi_m)_{m\geq 0}$ be sequences of non-negative constants such that $\phi_m \to 0$ as $m \to \infty$. We say that $(X_i, \mathcal{F}_i)_{i\geq 1}$ is a mixingale if for all $i \geq 1, m \geq 0$,*

(i) $\mathbb{E}\,|\mathbb{E}[X_i \,|\, \mathcal{F}_{i-m}]| \leq c_i\phi_m$,

(ii) $\mathbb{E}\,|X_i - \mathbb{E}[X_i \,|\, \mathcal{F}_{i+m}]| \leq c_i\phi_{m+1}$.

We refer to the numbers ϕ_m as the L^1-*mixingale numbers*. These numbers index the temporal dependence of the mixingale. We choose the constants c_i to index the 'magnitude' of the X_i's. In applications of the theory it is common to set $c_i = \mathbb{E}|X_i|$. Further, if the (X_i) are independent and $\sigma(X_1, X_2, \ldots, X_i) \subset \mathcal{F}_i$ we can set $\phi_m = 0$ for all $m \geq 0$.

Recall that a sequence random variables $X = (X_i)_{i\in\mathbb{N}}$ is uniformly integrable if

$$\lim_{C\to\infty} \left\{ \sup_{i\in\mathbb{N}} \mathbb{E}[|X_i| \mid |X_i| \geq C] \right\} = 0.$$

We will now state a Law of Large numbers holds for mixingales.

Theorem 5.1.2. *[5, Theorem 1] Consider the mixingale $(X_i, \mathcal{F}_i)_{i\geq 1}$ and let $(X_i, \mathcal{F}_i)$ be uniformly integrable.*

(a) If $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} c_i < \infty$ *then,*

$$\lim_{n\to\infty} \mathbb{E} \left| \sum_{i=1}^{n} X_i \right| = 0.$$

(b) If the mixingale has constants $c_i = \mathbb{E}|X_i|$ for all i then

$$\lim_{n\to\infty} \mathbb{E} \left| \sum_{i=1}^{n} X_i \right| = 0.$$

We note that (b) is a particular case of (a).

In order to prove Theorem 5.1.2 we make use of an L^p Law of Large numbers for martingale difference sequences. Andrews credits this result to Chow, [15].

Lemma 5.1.3. *Let $(X_i)_{i\geq 1}$ be adapted to the integer filtration $(F_i)_{i\in\mathbb{Z}}$. Define the martingale difference sequence*

$$Y_i = X_i - \mathbb{E}[X_i \,|\, \mathcal{F}_{i-1}], \quad i \geq 1.$$

If $(|Y_i|^p)_{i\geq 1}$ *is a uniformly integrable family for* $1 \leq p \leq 2$*, then*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} Y_i \right\|_p \to 0 \quad \text{as } t \to \infty.$$

5.2 Mixingales in Riesz Spaces

We now formulate a measure free abstract definition of a mixingale in the setting of Riesz spaces with conditional expectation operator. This generalises the above classical definitions.

Definition 5.2.1. *Let* E *be a Dedekind complete Riesz space with conditional expectation operator,* T*, and weak order unit* $e = Te$*. Let* $(T_i)_{i\in\mathbb{Z}}$ *be a filtration on* E *compatible with* T *in that* $T_iT = T = TT_i$ *for all* $i \in \mathbb{Z}$*. Let* $(f_i)_{i\in\mathbb{N}}$ *be a sequence in* E*. We say that* (f_i, T_i) *is a mixingale in* E *compatible with* T *if there exist* $(c_i)_{i\in\mathbb{N}} \subset E^+$ *and* $(\Phi_m)_{m\in\mathbb{N}} \subset \mathbb{R}^+$ *such that* $\Phi_m \to 0$ *as* $m \to \infty$ *and for all* $i, m \in \mathbb{N}$ *we have*

(i) $T|T_{i-m}f_i| \leq \Phi_m c_i$,

(ii) $T|f_i - T_{i+m}f_i| \leq \Phi_{m+1}c_i$.

As in the classical setting, the numbers $\Phi_m, m \in \mathbb{N}$, are referred to as the mixingale numbers. These numbers give a measure of the temporal dependence of the sequence (f_i). The constants (c_i) are chosen to index the 'magnitude' of the the random variables (f_i).

In many applications the sequence (f_i) is adapted to the filtration (T_i). The following theorem sheds more light on the structure of mixingales for this special case.

We recall that if T is a conditional expectation operator on a Riesz space E then $T|g| \geq |Tg|$.

Lemma 5.2.2. *Let E be a Dedekind complete Riesz space with conditional expectation operator, T, and weak order unit $e = Te$. Let (f_i, T_i) be a mixingale in E compatible with T.*

(i) The sequence (f_i) has T-mean zero, i.e. $Tf_i = 0$ for all $i \in \mathbb{N}$.

(ii) If in addition $(f_i)_{i\in\mathbb{N}}$ is T-conditionally independent and $\mathcal{R}(T_i) = \langle f_1, \ldots, f_{i-1}, \mathcal{R}(T)\rangle$ then the mixingale numbers may be taken as zero, where $\langle f_1, \ldots, f_{i-1}, \mathcal{R}(T)\rangle$ is the order closed Riesz subspace of E generated by $f_1, \ldots, f_{i-1}$ and $\mathcal{R}(T)$.

Proof. **(i)** Here we observe that the index set for the filtration (T_i) is $\mathbb{Z}$, thus

$$\begin{aligned} |Tf_i| &= |TT_{i-m}f_i| \\ &\leq T|T_{i-m}f_i| \\ &\leq c_i\Phi_m \\ &\to 0 \quad \text{as } m \to \infty \end{aligned}$$

giving $Tf_i = 0$ for all $i \geq 0$.

(ii) As (f_i) is adapted to the filtration (T_i), $f_i \in \mathcal{R}(T_i)$ for all $i \in \mathbb{N}$ from which it follows that

$$f_i - T_{i+m}f_i = 0, \quad \text{for all} \quad i, m \in \mathbb{N}.$$

As (f_i) is T-conditionally independent, from Corollary 3.2.6 and as (f_i) has T-mean zero (from (i)), we have that

$$T_{i-m}f_i = Tf_i = 0,$$

for $i, m \in \mathbb{N}$. Thus we can choose $\Phi_m = 0$ for all $m \in \mathbb{N}$. □

5.2.1 The Weak Law of Large Numbers

We now show that the above generalisation of mixingales to the measure free Riesz space setting admits a weak law of large numbers.

Lemma 5.2.3. *Let E be a Dedekind complete Riesz space with conditional expectation operator T, weak order unit $e = Te$ and filtration $(T_i)_{i\in\mathbb{N}}$ compatible with T. Let (f_i) be an e-uniformly bounded sequence adapted to the filtration (T_i), and $g_i := f_i - T_{i-1}f_i$, then (g_i, T_i) is a martingale difference sequence with*

$$T|\overline{g}_n| \to 0 \quad \text{as } n \to \infty,$$

where

$$\overline{g}_n := \frac{1}{n}\sum_{i=1}^{n} g_i.$$

Proof. Clearly

$$T_i g_{i+1} = T_i f_{i+1} - T_i^2 f_{i+1} = 0$$

and (g_i) is adapted to (T_i) so indeed (g_i, T_i) is a martingale difference sequence.

As (f_i) is e-uniformly bounded, there exists $B > 0$ be such that $|f_i| \leq Be$, for all $i \in \mathbb{N}$. Let $g_i := f_i - T_{i-1}f_i$. For $j > i$, as $T_jT_i = T_i$ and $T_if_i = f_i$ it follows that $T_ig_i = g_i$ and

$$T_i g_j = T_i f_j - T_i T_{j-1} f_j = T_i f_j - T_i f_j = 0.$$

In addition,

$$|g_i| \leq |f_i| + |T_{i-1}f_i| \leq |f_i| + T_{i-1}|f_i| \leq 2Be, \quad \text{for all} \quad i \in \mathbb{N}.$$

Set

$$s_n = \sum_{i=1}^{n} g_i.$$

As $|g_i| \leq 2Be$ we have that g_i is in the f-algebra E^e. Hence the product $g_i g_j$ is defined in E^e. Now as T_j is an averaging operator, see Corollary 2.4.8, and $g_j \in \mathcal{R}(T_j)$ we have

$$T_i(g_i g_j) = g_i T_i g_j = 0, \quad \text{for} \quad j > i.$$

Combining these results gives

$$\begin{aligned} T(s_n^2) &= \sum_{i,j=1}^{n} T(g_i g_j) \\ &= \sum_{i=1}^{n} T(g_i^2) + 2\sum_{i<j} T(g_i g_j) \\ &= \sum_{i=1}^{n} T(g_i^2) + 2\sum_{i<j} TT_i(g_i g_j) \\ &= \sum_{i=1}^{n} T(g_i^2) + 2\sum_{i<j} T(g_i T_i g_j) \\ &= \sum_{i=1}^{n} T(g_i^2) + 0. \end{aligned}$$

Thus

$$T(s_n^2) = \sum_{i=1}^{n} T(g_i^2).$$

But

$$g_i^2 = |g_i|^2 \leq 4B^2 e$$

as e is the algebraic identity of the f-algebra E^e and $|g_i| \leq 2Be$. Thus

$$T(s_n^2) \leq 4nB^2 e. \tag{5.2.1}$$

Now let

$$J_n = P_{s_n^+} - (I - P_{s_n^+})$$

where $P_{s_n^+}$ is the band projection on the band in E generated by s_n^+. From the definition of the f-algebra structure on E^e, if P and Q are band projections then

$(Pe)(Qe) = PQe$ which together with Freudenthal's Theorem (Theorem 2.4.7) enables us to conclude

$$|s_n| = J_n s_n = (J_n e)s_n$$

and $(J_n e)^2 = e$, as $J_n^2 = I$. But

$$\begin{aligned} 0 &\leq \left(\frac{J_n e}{n^{1/4}} - \frac{s_n}{n^{3/4}}\right)^2 \\ &= \left(\frac{J_n e}{n^{1/4}}\right)^2 + \left(\frac{s_n}{n^{3/4}}\right)^2 - 2\frac{J_n e}{n^{1/4}}\frac{s_n}{n^{3/4}} \\ &= \frac{e}{n^{1/2}} + \frac{s_n^2}{n^{3/2}} - 2\frac{|s_n|}{n} \end{aligned}$$

giving

$$\frac{e}{n^{1/2}} + \frac{s_n^2}{n^{3/2}} \geq 2\frac{|s_n|}{n}.$$

Applying T to this inequality gives

$$\frac{e}{n^{1/2}} + \frac{T(s_n^2)}{n^{3/2}} \geq 2\frac{T|s_n|}{n}$$

Combining the above inequality with (5.2.1) gives

$$\begin{aligned} 2\frac{T|s_n|}{n} &\leq \frac{e}{n^{1/2}} + \frac{T(s_n^2)}{n^{3/2}} \\ &\leq \frac{e}{n^{1/2}} + \frac{4nB^2 e}{n^{3/2}} \\ &= \frac{1+4B^2}{n^{1/2}}e, \end{aligned}$$

and thus

$$T|\overline{g}_n| \leq \frac{1+4B^2}{2n^{1/2}}e. \tag{5.2.2}$$

Since E is an Archimedean Riesz space letting $n \to \infty$ in (5.2.2) gives $T|\overline{g}_n| \to 0$ as $n \to \infty$. □

In order to prove an analogue to the weak law of large numbers for mixingales in Riesz spaces, we first need to give a Riesz space equivalent of a 'uniformly integrable collection of random variables'.

Definition 5.2.4. *Let E be a Dedekind complete Riesz space with conditional expectation operator T and weak order unit $e = Te$. Let $f_\alpha, \alpha \in \Lambda$, be a family in E, where Λ is some index set. We say that $f_\alpha, \alpha \in \Lambda$, is T-uniform if*

$$\sup\left\{TP_{(|f_\alpha|-ce)^+}|f_\alpha| \,:\, \alpha \in \Lambda\right\} \to 0 \quad as \quad c \to \infty, \tag{5.2.3}$$

in E.

Lemma 5.2.5. *Let E be a Dedekind complete Riesz space with conditional expectation T and let e be a weak order unit which is invariant under T. If $f_\alpha \in E, \alpha \in \Lambda$, is a T-uniform family, then the set $\{T|f_\alpha| \,:\, \alpha \in \Lambda\}$ is bounded in E.*

Proof. As the family $f_\alpha, \alpha \in \Lambda$, is T-uniform

$$J_c := \sup\left\{TP_{(|f_\alpha|-ce)^+}|f_i| \,:\, \alpha \in \Lambda\right\} \to 0 \quad \text{as} \quad c \to \infty.$$

In particular, this implies that J_c exists in E for $c > 0$ large and that, for sufficiently large $K > 0$, the set $\{J_c \,:\, c \geq K\}$ is bounded in E. Hence there is $g \in E_+$ so that

$$TP_{(|f_\alpha|-ce)^+}|f_\alpha| \leq g, \quad \text{for all} \quad \alpha \in \Lambda, c \geq K,$$

By the definition of $P_{(|f_\alpha|-ce)^+}$,

$$(I - P_{(|f_\alpha|-ce)^+})|f_\alpha| \leq ce, \quad \text{for} \quad \alpha \in \Lambda, c > 0.$$

Combining the above for $c = K$ gives

$$T|f_\alpha| = TP_{(|f_\alpha|-Ke)^+}|f_\alpha| + T(I - P_{(|f_\alpha|-Ke)^+})|f_\alpha| \leq g + Ke,$$

for all $\alpha \in \Lambda$. □

We are now able to give a Weak Law of Large Numbers for Riesz spaces.

Theorem 5.2.6. (Weak Law of Large Numbers) *Let E be a Dedekind complete Riesz space with conditional expectation operator T, weak order unit $e = Te$ and filtration $(T_i)_{i\in\mathbb{Z}}$. Let $(f_i, T_i)_{i\geq 1}$ be a T-uniform mixingale with c_i and Φ_i as defined in Definition 5.2.1.*

(i) If $\left(\frac{1}{n}\sum_{i=1}^{n} c_i\right)_{n\in\mathbb{N}}$ *is bounded in E then*

$$T|\overline{f}_n| = T\left|\frac{1}{n}\sum_{i=1}^{n} f_i\right| \to 0 \quad \text{as } n \to \infty.$$

(ii) If $c_i = T|f_i|$ for each $i \geq 1$ then

$$T|\overline{f}_n| = T\left|\frac{1}{n}\sum_{i=1}^{n} f_i\right| \to 0 \quad \text{as } n \to \infty.$$

Proof. **(i)** Let

$$y_{m,i} = T_{i+m}f_i - T_{i+m-1}f_i, \quad \text{for} \quad i \geq 1, m \in \mathbb{Z}.$$

Let $h_i = (I - P_{(|f_i|-Be)^+})f_i$ and $d_i = P_{(|f_i|-Be)^+}f_i, i \in \mathbb{N}$, then $f_i = h_i + d_i$. Now $(T_{i+m}h_i)_{i\in\mathbb{N}}$ is e-bounded and adapted to $(T_{i+m})_{i\in\mathbb{N}}$, so from Lemma 5.2.3 $(T_{i+m}h_i - T_{i+m-1}h_i, T_{i+m})_{i\in\mathbb{N}}$ is a martingale difference sequence with

$$T\left|\frac{1}{n}\sum_{i=1}^{n}(T_{i+m}h_i - T_{i+m-1}h_i)\right| \to 0$$

as $n \to \infty$. Thus,

$$\begin{aligned} T\left|\frac{1}{n}\sum_{i=1}^{n}(T_{i+m}d_i - T_{i+m-1}d_i)\right| &\leq T\frac{1}{n}\sum_{i=1}^{n}|T_{i+m}d_i - T_{i+m-1}d_i| \\ &\leq \frac{1}{n}\sum_{i=1}^{n}(TT_{i+m}|d_i| + TT_{i+m-1}|d_i|) \\ &= \frac{2}{n}\sum_{i=1}^{n}T|d_i| \\ &\leq 2\sup\left\{T|d_i| \,:\, i = 1,\ldots,n\right\}. \end{aligned}$$

Using the T-uniformity of (f_i) we can write

$$T\left|\frac{1}{n}\sum_{i=1}^{n}(T_{i+m}d_i - T_{i+m-1}d_i)\right| \leq 2\sup\left\{T|P_{(|f_i|-Be)^+}f_i| \,:\, i \in \mathbb{N}\right\}.$$

Combining the above results gives

$$\begin{aligned} \limsup_{n\to\infty} T\left|\frac{1}{n}\sum_{i=1}^{n}(T_{i+m}f_i - T_{i+m-1}f_i)\right| &\leq 2\sup\left\{TP_{(|f_i|-Be)^+}|f_i| \,:\, i \in \mathbb{N}\right\} \\ &\to 0 \end{aligned}$$

as $B \to \infty$ by the T-uniformity of (f_i). Thus $T|\overline{y}_{m,n}| \to 0$ as $n \to \infty$.

We now make use of a telescoping series to expand $\overline{f}_n$,

$$\begin{aligned} \overline{f}_n &= \frac{1}{n}\sum_{i=1}^{n}f_i \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(f_i - T_{i+M}f_i + \sum_{m=-M+1}^{M}(T_{i+m}f_i - T_{i+m-1}f_i) + T_{i-M}f_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}(f_i - T_{i+M}f_i) + \sum_{m=-M+1}^{M}\frac{1}{n}\sum_{i=1}^{n}(T_{i+m}f_i - T_{i+m-1}f_i) + \frac{1}{n}\sum_{i=1}^{n}T_{i-M}f_i \\ &= \frac{1}{n}\sum_{i=1}^{n}(f_i - T_{i+M}f_i) + \sum_{m=-M+1}^{M}\overline{y}_{m,n} + \frac{1}{n}\sum_{i=1}^{n}T_{i-M}f_i \end{aligned}$$

Applying T to the above expression we can bound $T|\overline{f}_n|$ by means of the defining properties of a mixingales as follows

$$\begin{aligned} T|\overline{f}_n| &\leq \frac{1}{n}\sum_{i=1}^{n} T|f_i - T_{i+M}f_i| + \sum_{m=-M+1}^{M} T|\overline{y}_{m,n}| + \frac{1}{n}\sum_{i=1}^{n} T|T_{i-M}f_i| \\ &\leq \frac{1}{n}\sum_{i=1}^{n} c_i\Phi_{M+1} + \sum_{m=-M+1}^{M} T|\overline{y}_{m,n}| + \frac{1}{n}\sum_{i=1}^{n} c_i\Phi_M. \end{aligned}$$

Since $\left(\frac{1}{n}\sum_{i=1}^{n} c_i\right)_{n\in\mathbb{N}}$ bounded in E there is $q \in E_+$ so that $\frac{1}{n}\sum_{i=1}^{n} c_i \leq q$, for all $n \in \mathbb{N}$, which when combined with the above display yields

$$T|\overline{f}_n| \leq (\Phi_{M+1} + \Phi_M)q + \sum_{m=-M+1}^{M} T|\overline{y}_{m,n}|.$$

Letting $n \to \infty$ gives

$$\limsup_{n\to\infty} T|\overline{f}_n| \leq (\Phi_{M+1} + \Phi_M)q.$$

Now taking $M \to \infty$ gives

$$\limsup_{n\to\infty} T|\overline{f}_n| = 0,$$

completing the proof of (i).

(ii) By Lemma 5.2.5, $(T|f_i|)$ is bounded, say by $q \in E_+$, so

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_i = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} T|f_i| \leq q,$$

making (i) applicable. □

Chapter 6

Quasi-martingales

Quasi-martingales were first introduced by H. Rubin in an invited lecture at the Institute of Mathematical Statistics in 1956. In [25], by Fisk, quasi-martingales were formally introduced and defined. Fisk gave the necessary and sufficient conditions under which a quasi-martingale with continuous sample paths could be decomposed as the sum of a martingale and a process having almost every sample path of bounded variation. Orey, in [49] was able to generalise Fisk's results to right continuous processes (or F-processes, in Orey's terminology). Finally, in [50], Rao gave a greatly simplified and elegant proof of Orey's result. Rao was also able to prove that every right-continuous martingale can be written as the sum of two positive supermartingales.

In [23], L. Egghe gives an application of quasi-martingales to a real world model. Egghe constructs a stochastic process that 'describes the evolution of a set of source journals', for example the set of ISI (Institute for Scientific Information) journals. Under certain conditions, the model is a quasi-martingale. Other examples of quasi-

martingales include sub- and super-martingales.

We begin with a review of the classical results for quasi-martingales. Unless otherwise stated, the results in the following section can be found in [50].

6.1 Classical Quasi-martingales

Throughout this section we will assume we are working in the probability space $(\Omega, \mathcal{F}, P)$ with right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. We consider the stochastic process $(X_t)_{t\geq 0} = (X(t))_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$. Unless otherwise stated $\mathbb{E}[X_t] < \infty$.

We aim to provide decompositions for quasi-martingales. One such decomposition is the Riesz decomposition.

Definition 6.1.1. *A process $X = (X_t)$ admits a Riesz decomposition if there exists a martingale $Y = Y_t$ and a process $Z = Z_t$ with $\mathbb{E}[|Z_t|] \to 0$ as $t \to \infty$ (that is, Z_t is a potential) such that for all t,*

$$X_t = Y_t + Z_t.$$

We note that the Riesz decomposition is almost everywhere unique. To see this, suppose

$$X(t) = Y_1(t) + Z_1(t) \quad \text{and} \quad X(t) = Y_2(t) + Z_2(t).$$

Now,

$$Y_1(t) + Z_1(t) = Y_2(t) + Z_2(t) \quad \text{and} \quad Y_1(t) - Y_2(t) = Z_2(t) - Z_1(t).$$

Thus,

$$\lim_{t\to\infty} \mathbb{E}[|Y_1(t) - Y_2(t)|] = \lim_{t\to\infty} \mathbb{E}[|Z_2(t) - Z_1(t)|],$$

giving

$$\lim_{t\to\infty} \mathbb{E}[|Y_1(t) - Y_2(t)|] = 0$$

from the assumption that $Z_1(t)$ and $Z_2(t)$ are potentials. $Y_1(t)$ and $Y_2(t)$ are martingales, so $|Y_1(t) - Y_2(t)|$ is a sub-martingale. Thus, $\mathbb{E}[|Y_1(t) - Y_2(t)|] \geq \mathbb{E}[|Y_1(s) - Y_2(s)|] \geq 0$ for all $t \geq s$. Taking $t \to \infty$ gives $\mathbb{E}[|Y_1(s) - Y_2(s)|] \equiv 0$ and $Y_1 = Y_2$.

Definition 6.1.2. *A process $X = X_t$ is said to be a quasi-martingale if there exists a constant M such that*

$$\sup_{\{t_1<t_2<\cdots<t_n\}\in\mathbb{R}^+} \sum_{i=1}^{n} \mathbb{E}\left[\left|X_{t_i} - \mathbb{E}[X_{t_{i+1}} \,|\, \mathcal{F}_{t_i}]\right|\right] \leq M.$$

A quasi-martingale $X = X_t$ is said to be a quasi-potential if

$$\lim_{t\to\infty} \mathbb{E}[|X_t|] = 0.$$

We shall call the number M that satisfies the defining inequality of quasi-martingales a *quasi-bound*.

Note 6.1.3. *In [25], Fisk defines a quasi-martingale as follows:* 'Let $T \subset \mathbb{R}$. A process $(X_t)_{t\in T}$ will be called a quasi-martingale if there exists a martingale $(X_1(t))_{t\in T}$ and a process $(X_2(t))_{t\in T}$ with an almost everywhere sample function of bounded total variation on T such that

$$P\left([X(t) = X_1(t) + X_2(t)\,;\, t \in T]\right) = 1,$$

where $[\dots]$ denotes the subset of Ω where '$\dots$ ' is true.'

However in [25, Lemma 3.1.2] it is shown that the definitions of Fisk and Rao are the same. Further, Fisk's definition is essentially the same as assuming that every quasi-martingale admits a Riesz decomposition.

In [50] it is shown that every quasi-martingale admits a Riesz decomposition. We now state the relevant theorem.

Theorem 6.1.4. *[50, Theorem 1.1] Every quasi-martingale, $(X_t)_{t\geq 0}$ can be written in the form*

$$X_t = Y_t + Z_t$$

where Y_t is a martingale and Z_t is a quasi-potential. This decomposition is unique up to sets of measure zero.

In [49], Orey makes the statement that any super(sub)-martingale, $(Y(t))_{t\in[0,a]}$, is a quasi-martingale. He continues 'From the analogy of functions with bounded variation, one is prompted to ask whether every F-process (quasi-martingale) is the difference of two super-martingales. We do not know the answer. ' Rao provides an answer to this question.

Theorem 6.1.5. *[50, Theorem 1.2] If $X(t)$ is a quasi-potential such that, for all t,*

$$\lim_{h\downarrow 0} \mathbb{E}[|X(t+h) - X(t)|] = 0$$

then there exist two super-martingales $X_+(t), X_-(t)$ such that $\lim_{t\to\infty} \mathbb{E}[X_\pm(t)] = 0$ and

$$E[|X(t) - (X_+(t) - X_-(t))|] = 0 \quad \textit{for all } t.$$

That is, $X(t) = X_+(t) - X_-(t)$ almost surely.

Rao gives the following construction of super-martingales that satisfy Theorem 6.1.5.

Let $X(t)$ be a quasi-potential. For all $k = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots$ define

$$\begin{aligned}
\Delta(k,n) &= \mathbb{E}[X(k2^{-n}) - X\left((k+1)2^{-n}\right) \mid \mathcal{F}_{k2^{-n}}], \\
X_+^n(t) &= \sum_{k \geq \lfloor 2^n t \rfloor + 1} \mathbb{E}[\Delta^+(k,n) \,|\, \mathcal{F}_t], \\
X_-^n(t) &= \sum_{k \geq \lfloor 2^n t \rfloor + 1} \mathbb{E}[\Delta^-(k,n) \,|\, \mathcal{F}_t],
\end{aligned}$$

where $\lfloor 2^n t \rfloor$ denotes the greatest integer less than or equal to $2^n t$ and $\Delta^+ = \max(\Delta, 0)$, $\Delta^- = \max(-\Delta, 0)$. Here, $X_\pm^n(t)$ are strictly increasing super-martingales with $\mathbb{E}[|X_\pm^n(t)|] \to 0$ as $t \to \infty$. Finally, we define

$$X_+(t) = \sup_n X_+^n(t) \quad \text{and} \quad X_-(t) = \sup_n X_-^n(t).$$

Theorem 6.1.5 suggests a link between quasi-martingales and super(sub)-martingales. Further evidence to support this idea is given in the form of the following super(sub)-martingale inequalities which have been generalised to quasi-martingales.

The first inequality is given by Orey in [49] and is a generalisation of Doob's sub-martingale inequality.

Lemma 6.1.6. *[49, Lemma 2.1] Let $(X_k)_{k=1,\ldots,n}$ be a quasi-martingale with quasi-bound M. Then for $\lambda \geq 0$,*

$$\lambda P(\max_k X_k \geq \lambda) \leq \mathbb{E}|X_n| + M; \quad \lambda P(\min_k X_k \leq -\lambda) \leq \mathbb{E}|X_n| + M.$$

From the above discrete time result, one can deduce the Kolmogorov-Doob inequality, a continuous time result. It is interesting to note that the proof follows from Lemma 6.1.6 as in the super-martingale case, see [49].

Theorem 6.1.7. *[49, Theorem 2.1] Let $0 < a < \infty$ and let $(X_t)_{t\in[0,a]}$ be a quasi-martingale with quasi-bound M. Then, for $\lambda \geq 0$,*

$$\lambda P(\sup_{0\leq s\leq a} X_s \geq \lambda) \leq \mathbb{E}|X_a| + K; \quad \lambda P(\inf_{0\leq s\leq a} X_s \leq -\lambda) \leq \mathbb{E}|X_a| + K.$$

Rao also gives an inequality similar to the super-martingale inequalities. This inequality, under suitable conditions on the random variables, yields the Hajek-Renyi inequality, see [50]. Before we state Rao's result, we note that it is clear from the quasi-martingale definition that every finite collection of random variables with finite expectation is a quasi-martingale.

Lemma 6.1.8. *Consider a set of random variables (X_i) adapted to the σ-algebras $\mathcal{F}_i$, $1 \leq i \leq n$. Assume that the X_i's have finite expectation. We define:*

$$\Delta_i = X_i - \mathbb{E}[X_{i+1} \,|\, \mathcal{F}_i] \quad \text{for } 1 \leq i \leq n-1; \quad \Delta_n = X_n,$$

$$A_n^+ = \sum_{i=1}^{n} \Delta_i^+; \quad A_n^- = \sum_{i=1}^{n} \Delta_i^-,$$

Then, for all $\lambda > 0$,

$$\lambda P(\max X_i \geq \lambda) \leq \sum_{i=1}^{n-1} \mathbb{E}[\Delta_i^+] + \mathbb{E}[X_n \,|\, \max X_i \geq \lambda] \leq \mathbb{E}[A_n^+]$$

6.2 Quasi-martingales in Riesz spaces

In order to translate quasi-martingales to the setting of a Riesz space, we first need to define continuous time martingales in Riesz spaces.

The following definition is from Grobler, [27], where a more in depth discussion of the properties of continuous time martingales (under slightly different assumptions to ours) can be found.

Definition 6.2.1. *Let E be a Dedekind complete Riesz space with weak order unit and conditional expectation T. Let $T_t, t \in [0,\infty)$, be a family of conditional expectations on E with $T_tT = T = TT_t$. Denote by T_{s^+} the conditional expectation with range $\bigcap_{t>s} \mathcal{R}(T_t)$.*

(i) The family $(T_t)_{t\in[0,\infty)}$ of conditional expectations is said to be a filtration if $T_tT_s = T_s = T_sT_t$ for all $s \leq t$.

(ii) We say that the filtration, $(T_t)_{t\in[0,\infty)}$, is right continuous if $T_{s^+} = T_s$ for all $s \in [0,\infty)$.

Note 6.2.2. The existence of T_{s^+} is guaranteed by the Radon-Nikodým Theorem, [63]. Here T_{s^+} commutes with T.

We now give a slightly stronger notion of right continuity than that mentioned above. We call this 'joint weak right continuity'.

Definition 6.2.3. *Let E be a Dedekind complete Riesz space with weak order unit and conditional expectation T. Let $T_t, t \in [0,\infty)$, be a family of conditional expectations on E with $T_tT = T = TT_t$. Let (f_t) be a family in E. We say that the filtration (T_t) is a joint weak right continuous filtration if*

$$\lim_{t\downarrow s} T_t f_t = T_s f_s.$$

It must be noted that joint weak right continuity certainly implies right continuity. Furthermore, we believe that right continuity, as defined above, implies uniform weak right continuity. However, in order to prove this, we need to generalise the convergence of martingales from the discrete setting to the continuous time setting. This work is currently being undertaken by J. J. Grobler.

Definition 6.2.4. *Let E be a Dedekind complete Riesz space with weak order unit. We say that $(f_s, T_s)_{s\in[0,\infty)}$ is a martingale if (T_s) is a filtration, as defined in Definition 6.2.1, and $T_s f_t = f_s$ for all $s \leq t$.*

Definition 6.2.5. *Let E be a Dedekind complete Riesz space with conditional expectation operator T and weak order unit $e = Te$. Let $(T_t)_{t\in[0,\infty)}$ be a filtration on E with $TT_t = T = T_t T$. We say a process $(f_t)_{t\in[0,\infty)}$ is a T-quasi-martingale if (f_t) is adapted to (T_t) and there exists $M \in E^+$ such that*

$$\sup_{(t_1,t_2,\ldots,t_{n+1})\in\Pi} \sum_{i=1}^{n} T|f_{t_i} - T_{t_i} f_{t_{i+1}}| \leq M,$$

where Π is the collection of all finite sequences of real numbers, $(t_1, t_2, \ldots, t_{n+1}), n \in \mathbb{N}$, with $0 \leq t_1 < t_2 < t_3 < \cdots < t_{n+1}$.

If $(f_t)_{t\in[0,\infty)}$ is a T-quasi-martingale, then we say $(f_t)_{t\in[0,\infty)}$ is a T-quasi-potential if

$$\lim_{t\to\infty} T|f_t| = 0.$$

Theorem 6.2.6 (Riesz Decomposition Theorem)**.** *Let E be a T-universally complete Riesz space where T is a conditional expectation operator and e is a weak order unit such that $e = Te$. Every T-quasi-martingale can be written as the sum of a martingale and a T-quasi-potential. If T is strictly positive, then the decomposition is unique.*

If, in addition, the T-quasi-martingale is right continuous and we have joint weak right continuity of the filtration, then the martingale and the T-quasi-potential resulting from the decomposition are both right continuous.

Proof. Let (s_n) be a strictly increasing sequence in $[0, \infty)$ with $\lim_{i\to\infty} s_i = \infty$. Set

$$\Delta_i = f_{s_i} - T_{s_i} f_{s_{i+1}}, \qquad \text{for } i \in \mathbb{N}.$$

Let

$$Y_{t,i} = T_t f_{s_i}, \qquad t \in [0, \infty), \quad i \in \mathbb{N}.$$

For $s_i \geq t$,

$$T_t \Delta_i = Y_{t,i} - Y_{t,i+1}. \tag{6.2.1}$$

Applying T to $|T_t \Delta_i|$ from (6.2.1) we get

$$T|\Delta_i| = TT_t|\Delta_i| \geq T|T_t \Delta_i| = T|Y_{t,i} - Y_{t,i+1}|. \tag{6.2.2}$$

Let $M \in E^+$ be as in Definition 6.2.4, then

$$T \sum_{s_i \geq t}^{i \leq n} |Y_{t,i} - Y_{t,i+1}| \leq \sum_{s_i \geq t}^{i \leq n} T|\Delta_i| \leq M,$$

for all $n \in \mathbb{N}$. Thus, from the T-universal completeness of E, $\sum(Y_{t,i} - Y_{t,i+1})$ is absolutely convergent in E and, as this a telescoping series, $(Y_{t,i})$ in convergent in E as $i \to \infty$. Denote

$$q_t = \lim_{i \to \infty} Y_{t,i}. \tag{6.2.3}$$

We show that q_t is independent of the sequence (s_i) chosen above. Let $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ be two increasing, unbounded sequences and

$$\lim_{i \to \infty} T_t f_{u_i} = \tilde{q}_t \quad \text{and} \quad \lim_{i \to \infty} T_t f_{v_i} = \overline{q}_t.$$

Construct the increasing, unbounded sequence $(s_i)_{i \in \mathbb{N}}$ such that $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ are subsequences. By the above construction, there exists q_t such that

$$\lim_{i \to \infty} T_t f_{s_i} = q_t. \tag{6.2.4}$$

By the uniqueness of limits and the construction of $(s_i)_{i \in \mathbb{N}}$, we have that

$$\lim_{i \to \infty} T_t f_{u_i} = \lim_{i \to \infty} T_t f_{s_i} = \lim_{i \to \infty} T_t f_{v_i}.$$

That is,

$$\tilde{q}_t = q_t = \overline{q}_t.$$

We now show q_t is a martingale. Let $s \leq t$. From the definition of q_t and as T_s is order continuous, it follows that

$$T_s q_t = T_s \lim_{i\to\infty} Y_{t,i} = \lim_{i\to\infty} T_s Y_{t,i}.$$

As $T_s T_t = T_s$ and $Y_{t,i} = T_t f_{s_i}$, from the above we have

$$\begin{aligned} T_s Y_{t,i} &= T_s T_t f_{s_i} \\ &= T_s f_{s_i} \\ &= Y_{s,i} \\ &\to q_s, \qquad \text{as } i \to \infty. \end{aligned}$$

So $T_s q_t = q_s$ for $s \leq t$.

We now show that $(f_{s_i} - q_{s_i})$ is a T-quasi-potential, that is $T|f_{s_i} - q_{s_i}| \to 0$ as $i \to \infty$. Note that $\sum_{i=1}^{n} T|\Delta_i| \leq M$ for all $n \in N$. Thus, $\left(\sum_{i\geq k} T|\Delta_i|\right)_k$ converges in order to 0 as $k \to \infty$. That is $(x_k) \downarrow 0$ where

$$\sum_{i\geq k} T|\Delta_i| = x_k, \quad k \in \mathbb{N}.$$

Consider $s_i \geq t$. From (6.2.1) and (6.2.2) we have $T_t\Delta_i = Y_{t,i} - Y_{t,i+1}$ and $T|\Delta_i| \geq T|Y_{t,i} - Y_{t,i+1}|$. Now

$$\sum_{k\leq i\leq j} (Y_{s_k,i} - Y_{s_k,i+1}) = Y_{s_k,k} - Y_{s_k,j+1} = f_{s_k} - Y_{s_k,j+1},$$

as $f_{s_k} \in \mathcal{R}(T_{s_k})$. So

$$T|f_{s_k} - Y_{s_k,j+1}| \leq \sum_{k\leq i\leq j} T|Y_{s_k,i} - Y_{s_k,i+1}| \leq \sum_{k\leq i\leq j} T|\Delta_i|,$$

and thus,

$$T|f_{s_k} - Y_{s_k,j+1}| \leq \sum_{i\geq k} T|\Delta_i| = x_k, \quad \text{for all } j \geq k.$$

But $Y_{s_k,j+1} \to q_{s_k}$ as $j \to \infty$, thus

$$T|f_{s_k} - q_{s_k}| \leq x_k,$$

and $T|f_{s_k} - q_{s_k}| \to 0$, in order. Hence, $(f_{s_k} - q_{s_k})$ is a T-quasi-potential.

We have proved that for each sequence, $s_1 < s_2 < \cdots < s_n < \ldots$, $s_n \uparrow \infty$, there exists a martingale $(q_t)_{t\in[0,\infty)}$ such that

$$T|f_t - q_t| \to 0, \qquad t \to \infty, \qquad t \in \{s_1, s_2, \ldots\}.$$

We now extend the result to all $t \in [0,\infty)$. Consider the T-quasi-martingale $(f_t)_{t\in[0,\infty)}$ and the martingale q_t constructed as above. We suppose

$$T|f_t - q_t| \nrightarrow 0 \quad \text{as } t \to \infty \text{ in } [0,\infty).$$

As $T|f_s - T_s f_t| \leq M$, we have

$$\limsup_{t\to\infty} T|f_t - q_t| = h,$$

for some $h > 0$. Let Π denote the collection of all finite partitions $(t_1, t_2, \ldots, t_{n+1})$ of $[0,\infty)$ with $t_1 < t_2 \cdots < t_{n+1}$. The definition of a T-quasi-martingale gives

$$\sup_{(t_1,t_2,\ldots,t_{n+1})\in\Pi} \sum_{i=1}^{n} T|f_{t_i} - T_{t_i} f_{t_{i+1}}| \leq M.$$

Let $(t_1, \ldots, t_{2n+1}) \in \Pi$, then $\sum_{i=1}^{2n} T|f_{t_i} - T_{t_i} f_{t_{i+1}}| \leq M$. Thus,

$$M \geq \sum_{i=1}^{n} T|f_{t_{2i}} - T_{t_{2i}} f_{t_{2i+1}}|. \tag{6.2.5}$$

By construction $q_t = \lim_{s_i \to \infty} T_t f_{s_i}$, for all $t \in [0, \infty)$, where $(s_i)_{i \in \mathbb{N}}$, $s_i \to \infty$ as $i \to \infty$. We recall that this construction is independent of the sequence chosen and that the limit is a sequential limit. Consider the net $(t_\alpha) = [0, \infty)$ with $t_\alpha \uparrow \infty$. Then,

$$\begin{aligned} T|f_t - q_t| &= \lim_{i \to \infty} T|T_t f_{s_i} - f_t| \\ &= \limsup_{i \to \infty} T|T_t f_{s_i} - f_t| \\ &\leq \limsup_{t_\alpha \to \infty} T|T_t f_{t_\alpha} - f_t|, \end{aligned} \tag{6.2.6}$$

as $(s_i) \subset (t_\alpha)$.

Taking the limit supremum as t_{2n+1} tends to infinity of (6.2.5) and using (6.2.6) above, gives

$$M \geq \sum_{i=1}^{n-1} T|f_{t_{2i}} - T_{t_{2i}} f_{t_{2i+1}}| + \limsup_{t_{2n+1} \to \infty} T|f_{t_{2n}} - T_{t_{2n}} f_{t_{2n+1}}|.$$

Thus,

$$M \geq \sum_{i=1}^{n-1} T|f_{t_{2i}} - T_{t_{2i}} f_{t_{2i+1}}| + T|f_{t_{2n}} - q_{t_{2n}}|. \tag{6.2.7}$$

Taking the limit supremum as t_{2n} tends to infinity in (6.2.7), gives

$$M \geq \sum_{i=1}^{n-1} T|f_{t_{2i}} - T_{t_{2i}} f_{t_{2i+1}}| + h.$$

Repeating this process inductively, we obtain $M \geq nh$. But the choice of $n \in \mathbb{N}$ was arbitrary and so $M \geq nh$ for all $n \in \mathbb{N}$. As E is an Archimedean Riesz space, it now follows that $h = 0$ and

$$T|f_t - q_t| \to 0, \qquad \text{as } t \to \infty \text{ with } t \in [0, \infty).$$

Setting $Z_t = f_t - q_t$, for all $t \in [0, \infty)$, we have that Z_t is a T-quasi-potential and $f_t = q_t + Z_t$ for $t \in [0, \infty)$.

To see the uniqueness of the Riesz decomposition, consider the quasi-martingale f_t with decompositions

$$q_t + Z_t = f_t = \overline{q}_t + \overline{Z}_t.$$

Rearranging the equation gives

$$q_t - \overline{q}_t = \overline{Z}_t - Z_t,$$

taking absolute values and T on both sides of the equation and making use of the fact that $\overline{Z_t}, Z_t$ are quasi-potentials we get

$$\begin{aligned} T|q_t - \overline{q}_t| &= T|Z_t - \overline{Z}_t| \\ &\leq T|Z_t| + T|\overline{Z}_t| \\ &\to 0, \end{aligned}$$

as $t \to \infty$. But $(|q_t - \overline{q}_t|)$ is a sub-martingale and, therefore, $T|q_t - \overline{q}_t|$ is non-decreasing in t. Thus,

$$T|q_t - \overline{q}_t| = 0.$$

Hence, $q_t = \overline{q}_t$, $Z_t = \overline{Z}_t$ and the Riesz decomposition is unique.

If the T-quasi-martingale $(f_t)_{t\in[0,\infty)}$ is right continuous and the filtration $(T_t)_{t\in[0,\infty)}$ is a joint weak right continuous filtration, then, if $t > \tau > s$,

$$q_\tau = T_\tau q_t \to T_s q_t = q_s,$$

as $\tau \downarrow s$. This gives that the T-quasi-potential, $(f_t - q_t)$ is the difference of two right continuous elements and so is right continuous. □

Recall that a process X_t is a *potential* if X_t is an adapted super-martingale and $T|X_t| \to 0$ as $t \to \infty$.

Theorem 6.2.7. *Let E be a T-universally complete Riesz space where T is a strictly positive conditional expectation operator and e is a weak order unit with $e = Te$. Let $(T_t)_{t\in[0,\infty)}$ be a joint weak right continuous filtration on E, where $T_tT = T = TT_t$, $t \in [0,\infty)$. If $(X_t)_{t\in[0,\infty)} \subset E$ is a T-quasi-potential such that*

$$\lim_{h\downarrow 0} T|X_{t+h} - X_t| = 0, \qquad \text{for all } t \in [0,\infty), \tag{6.2.8}$$

then there exist two potentials X_t^p, X_t^m such that

$$X_t = X_t^p - X_t^m \qquad \text{for all } t \in [0,\infty). \tag{6.2.9}$$

Proof. We first construct X_t^p and X_t^m.

For $k = 0, 1, 2, \dots$ and $n = 0, 1, 2, \dots$ define

$$\Delta(k,n) = T_{k2^{-n}}(X_{k2^{-n}} - X_{(k+1)2^{-n}}) = X_{k2^{-n}} - T_{k2^{-n}}X_{(k+1)2^{-n}}.$$

By the definition of a T-quasi-potential there exists $M \in E^+$ so that for all $n \in \{0, 1, \dots\}$ and $\kappa \in \mathbb{N}$ we have

$$\sum_{k=0}^{\kappa} T|\Delta(k,n)| \le M.$$

Hence, by the T-universal completeness of E, $\sum_{k=i}^{\infty} |\Delta(k,n)|$ converges. Thus, the following sums converge in E

$$\sum_{k=i}^{\infty} \Delta(k,n), \quad \sum_{k=i}^{\infty} \Delta^{\pm}(k,n), \tag{6.2.10}$$

where $\Delta^{+(-)}(k,n) = \sup\{(-)\Delta(k,n), 0\}$. We can make the following definitions, for all $t \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$,

$$\overline{X}_{t,n} := T_t \sum_{k\ge\lfloor 2^n t\rfloor+1} \Delta^+(k,n)$$

$$\underline{X}_{t,n} := T_t \sum_{k\ge\lfloor 2^n t\rfloor+1} \Delta^-(k,n),$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

We will now show that $\overline{X}_{t,n}$ is a potential. The proof for $\underline{X}_{t,n}$ is similar. Firstly, $\overline{X}_{t,n}$ is a super-martingale. To see this, let $s \leq t$ then,

$$\begin{aligned} T_s \overline{X}_{t,n} &= T_s T_t \sum_{k \geq \lfloor 2^n t \rfloor + 1} \Delta^+(k,n) \\ &= T_s \sum_{k \geq \lfloor 2^n t \rfloor + 1} \Delta^+(k,n) \\ &\leq T_s \sum_{k \geq \lfloor 2^n s \rfloor + 1} \Delta^+(k,n) \\ &= \overline{X}_{s,n}. \end{aligned}$$

We now show that $T|\overline{X}_{t,n}| \to 0$ as $t \to \infty$. As $\overline{X}_{t,n} \geq 0$, $T|\overline{X}_{t,n}| = T\overline{X}_{t,n}$ and

$$\begin{aligned} T\overline{X}_{t,n} &= TT_t \sum_{k \geq \lfloor 2^n t \rfloor + 1} \Delta^+(k,n) \\ &= T \sum_{k \geq \lfloor 2^n t \rfloor + 1} \Delta^+(k,n) \\ &\to 0 \qquad \text{as } t \to \infty, \end{aligned}$$

by (6.2.10). Thus, we have shown that $\overline{X}_{t,n}$ is a potential.

We now show $\overline{X}_{t,n}$ is increasing in n. If $i2^{-n} \leq t < (i+1)2^{-n}$ (that is, $i \leq t2^n < (i+1)$), then

$$\overline{X}_{t,n} = T_t \sum_{k \geq i+1} \Delta^+(k,n). \tag{6.2.11}$$

Suppose $2i2^{-(n+1)} \leq t < (2i+1)2^{-(n+1)}$ (that is, $2i \leq t2^{n+1} < 2i+1$). Then,

$$\begin{aligned}\overline{X}_{t,n+1} &= T_t \sum_{k\geq 2i+1} \Delta^+(k, n+1) \\ &= T_t\Delta^+(2i+1, n+1) + T_t \sum_{k\geq 2i+2} \Delta^+(k, n+1) \\ &\geq T_t \sum_{k\geq i+1} \left(\Delta^+(2k, n+1) + \Delta^+(2k+1, n+1)\right).\end{aligned}$$

But, as $\Delta^+(2k, n+1) + \Delta^+(2k+1, n+1) \geq (\Delta(2k, n+1) + \Delta(2k+1, n+1))^+$ and $t < 2k2^{-(n+1)}$, for $k \geq i+1$, so

$$\begin{aligned}\overline{X}_{t,n+1} &\geq \sum_{k\geq i+1} T_t \left(\Delta(2k, n+1) + \Delta(2k+1, n+1)\right)^+ \\ &= \sum_{k\geq i+1} T_t T_{2k2^{-(n+1)}} \left(\Delta(2k, n+1) + \Delta(2k+1, n+1)\right)^+.\end{aligned}$$

From $T_{2k2^{-(n+1)}}(f^+) \geq (T_{2k2^{-(n+1)}} f)^+$ it follows that

$$\begin{aligned}\overline{X}_{t,n+1} &\geq \sum_{k\geq i+1} T_t \left(T_{2k2^{-(n+1)}} \{\Delta(2k, n+1) + \Delta(2k+1, n+1)\}\right)^+ \\ &= \sum_{k\geq i+1} T_t\Delta^+(k, n) \\ &= \overline{X}_{t,n} \qquad \text{from (6.2.11)}.\end{aligned}$$

So, for all $i \leq t2^n < i + \frac{1}{2}$, $\overline{X}_{t,n+1} \geq \overline{X}_{t,n}$. The proof for the case where $i + \frac{1}{2} \leq t2^n < i+1$ is similar with the exception that the term $T_t\Delta^+(2i+1, n+1)$ does not occur.

We now define X^p_t, X^m_t by

$$\begin{aligned}X^p_t &:= \sup_n \overline{X}_{t,n} = \lim_{n\to\infty} \overline{X}_{t,n}, \\ X^m_t &:= \sup_n \underline{X}_{t,n} = \lim_{n\to\infty} \underline{X}_{t,n}.\end{aligned}$$

Here we note that these suprema and limits exist in E since E is T-universally complete, $\overline{X}_{t,n}$ ($\underline{X}_{t,n}$) is increasing and $T\overline{X}_{t,n}(\underline{X}_{t,n}) \leq M$. Also, X^p_t and X^m_t are right

continuous in t, since $\sum_{k\geq\lfloor 2^n t\rfloor+1}\Delta^{\pm}(k,n)$ are right continuous in t and T_t is jointly weakly right continuous (by assumption).

In addition, X_t^p, X_t^m are super-martingales as they are the suprema of super-martingales.

We show that X_t^p, X_t^m obey (6.2.9). Let $r = i2^{-k}$ be a dyadic rational, then, using (6.2.8) and as $T|X_t| \to 0$ as $t \to \infty$ we have

$$\begin{aligned} T|X_r - (X_r^p - X_r^m)| &= \lim_{n\to\infty} T\left|X_r - T_r \sum_{j\geq\lfloor 2^n r\rfloor+1}\{\Delta^+(j,n) - \Delta^-(j,n)\}\right| \\ &= \lim_{n\to\infty} T\left|X_r - T_r \sum_{j\geq 2^n r+1}\Delta(j,n)\right| \end{aligned}$$

Thus,

$$\begin{aligned} T|X_r - (X_r^p - X_r^m)| &= \lim_{n\to\infty} T\left|X_r - T_r \sum_{j\geq 2^n r+1} T_{j2^{-n}}\left(X_{j2^{-n}} - X_{(j+1)2^{-n}}\right)\right| \\ &= \lim_{n\to\infty} T\left|X_r - \lim_{N\to\infty}\sum_{j=2^n r+1}^{N} T_r T_{j2^{-n}}\left(X_{j2^{-n}} - X_{(j+1)2^{-n}}\right)\right| \\ &= \lim_{n\to\infty}\lim_{N\to\infty} T|X_r - T_r\left(X_{r+2^{-n}} - X_{(N+1)2^{-n}}\right)| \\ &\leq \lim_{n\to\infty}\lim_{N\to\infty}\left\{T|X_r - T_r X_{r+2^{-n}}| + T|X_{(N+1)2^{-n}}|\right\} \\ &= \lim_{n\to\infty} T|X_r - T_r X_{r+2^{-n}}| \\ &= \lim_{n\to\infty} T|T_r(X_r - X_{r+2^{-n}})| \\ &\leq \lim_{n\to\infty} T|X_r - X_{r+2^{-n}}| \\ &= 0. \end{aligned}$$

Thus, $X_r = X_r^p - X_r^m$. Again using (6.2.8) and the right continuity of X_t^m and X_t^p we obtain $X_t = X_t^p - X_t^m$ for all $t \in [0,\infty)$.

Finally, we show that X_t^m and X_t^p are potentials. We prove $T|X_t^m| \to 0$ as $t \to \infty$. The proof that X_t^p is a potential is similar.

Recall

$$\underline{X}_{t,n} = \sum_{k \geq \lfloor 2^n t \rfloor + 1} T_t \Delta^-(k,n). \tag{6.2.12}$$

If $k < \lfloor 2^n t \rfloor + 1$ then $k \leq \lfloor 2^n t \rfloor$. In particular, $k \leq 2^n t$ giving that $t \geq 2^{-n}k$, and

$$\begin{aligned} T_t \Delta(k,n) &= T_t T_{k2^{-n}} \left(X_{k2^{-n}} - X_{(k+1)2^{-n}} \right) \\ &= T_{k2^{-n}} \left(X_{k2^{-n}} - X_{(k+1)2^{-n}} \right) \\ &= \Delta(k,n). \end{aligned}$$

Also, $\Delta(k,n) = \Delta^+(k,n) - \Delta^-(k,n)$, so, for $t \geq 2^{-n}k$,

$$T_t \Delta^\pm(k,n) = \Delta^\pm(k,n), \tag{6.2.13}$$

as $\mathcal{R}(T_t)$ is a Riesz subspace of E. By (6.2.10), we have

$$\Delta^-_{\infty,n} := \sum_{k=0}^{\infty} \Delta^-(k,n) \quad \text{exists in } E. \tag{6.2.14}$$

Now, by (6.2.12), (6.2.13) and (6.2.14),

$$\underline{X}_{t,n} + \sum_{k=0}^{\lfloor 2^n t \rfloor} \Delta^-(k,n) = T_t \Delta^-_{\infty,n}.$$

We note that $\underline{X}_{t,n}$ is increasing in n (proved prior), as is $\displaystyle\sum_{k=0}^{\lfloor 2^n t \rfloor} \Delta^-(k,n)$. Further, the limit of $\displaystyle\sum_{k=0}^{\lfloor 2^n t \rfloor} \Delta^-(k,n)$, as $n \to \infty$, exists in E. Now, $\Delta^-_{\infty,n}$ is increasing in n and, as $T\Delta^-_{n,\infty} \leq M$, the T-universal completeness of E gives that the limit as $n \to \infty$ of $\Delta^-_{n,\infty}$ exists in E. Let

$$\lim_{n \to \infty} \Delta^-_{\infty,n} =: \Delta^-_\infty.$$

Hence,

$$X_t^m = T_t\Delta_\infty^- - \lim_{n\to\infty}\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n),$$

so,

$$T|X_t^m| = TX_t^m = T\Delta_\infty^- - T\lim_{n\to\infty}\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n).$$

Also, $\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n)$ is increasing in n and t and its limit as $t\to\infty$ exists in E, so

$$\begin{aligned}\lim_{t\to\infty}\lim_{n\to\infty}\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n) &= \sup_{t,n}\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n)\\ &= \lim_{n\to\infty}\lim_{t\to\infty}\sum_{k=0}^{\lfloor 2^n t\rfloor}\Delta^-(k,n)\\ &= \lim_{n\to\infty}\Delta_{\infty,n}^-\\ &= \Delta_\infty^-\end{aligned}$$

Hence,

$$\lim_{t\to\infty} T|X_t^m| = T\Delta_\infty^- - T\Delta_\infty^- = 0,$$

giving that X_t^m is a potential, as desired. □

From Theorems 6.2.7 and 6.2.6 we have the following corollary.

Corollary 6.2.8. *Let E be a T-universally complete Riesz space where T is a strictly positive conditional expectation operator and e is a weak order unit with $e = Te$. Let $(T_t)_{t\in[0,\infty)}$ be a joint weak right continuous filtration on E, where $T_tT = T = TT_t$, $t\in[0,\infty)$. If $(X_t)_{t\in[0,\infty)}\subset E$ is a T-quasi-martingale such that*

$$\lim_{h\downarrow 0} T|X_{t+h} - X_t| = 0 \qquad \text{for all } t\in[0,\infty),$$

then $(X_t)_{t\in[0,\infty)}$ can be decomposed as the sum of a right continuous martingale and the difference of two right continuous positive potentials.

In the particular case where the Riesz space $E = \mathcal{L}^1(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space, Theorem 6.2.7 gives the following result. We believe this result to be new in the classical setting. This result extends Rao's in that his expectation operator has been replaced by a conditional expectation operator.

Corollary 6.2.9. *Consider the probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a right continuous filtration in $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}_0 \subset \mathcal{F}_t \subset \mathcal{F}$ for all $t \in [0, \infty)$, and $(X_t)_{t\in[0,\infty)}$ an $\mathcal{F}_0$-quasi-martingale in $\mathcal{L}^1(\Omega, \mathcal{F}, P)$. If (X_t) is such that*

$$\lim_{h\downarrow 0} \mathbb{E}[|X_{t+h} - X_t| \mid \mathcal{F}_0] = 0 \quad \text{for all } t \in [0, \infty),$$

then there exist two right continuous positive super-martingales, X_t^p and X_t^m, and a right continuous martingale, (Y_t), such that $X_t = Y_t + X_t^p - X_t^m$ and

$$\lim_{t\to\infty} \mathbb{E}[|(X_t - Y_t) - (X_t^p - X_t^m)| \mid \mathcal{F}_0] = 0.$$

We now give an inequality for quasi-martingales. This equality is similar to that for super-martingales.

Theorem 6.2.10. *Let E be a Dedekind complete Riesz space with conditional expectation operator T. Let E be T-universally complete with filtration (T_i). Consider a sequence (f_i) in E adapted to (T_i). Set*

$$\Delta_i = f_i - T_i f_{i+1}, \quad 1 \le i \le n-1, \qquad \Delta_n = f_n,$$
$$A_n^+ = \sum_{i=1}^n \Delta_i^+, \quad A_n^- = \sum_{i=1}^n \Delta_i^-,$$

and let

$$P = I - P_{\left(\lambda e - \bigvee_{i=1}^n f_i\right)^+} \tag{6.2.15}$$

where $P_{\left(\lambda e - \bigvee_{i=1}^n f_i\right)^+}$ denotes band projection onto the band generated by $\left(\lambda e - \bigvee_{i=1}^n f_i\right)^+$. Then, for each $\lambda > 0$,

$$\lambda TPe \le \sum_{i=1}^{n-1} T\Delta_i^+ + TPf_n \le TA_n^+. \tag{6.2.16}$$

We note that since $(-x)^+ = x^-$ it follows from (6.2.16) (changing f_i to $-f_i$) that

$$\lambda TPe \leq \sum_{i=1}^{n-1} T\Delta_i^- - TPf_n \leq TA_n^-,$$

where $P = I - P_{(\lambda e + \bigwedge_{i=1}^n f_i)^+}$.

Proof. We note that for $i \leq j$,

$$\begin{aligned} T_i(\Delta_j) &= T_i(f_j - T_j f_{j+1}) \\ &= T_i f_j - T_i f_{j+1}. \end{aligned}$$

This gives, for $i \leq j$,

$$\begin{aligned} T_i\left(\sum_{j=i}^{n} \Delta_j\right) &= \sum_{j=i}^{n-1}(T_i f_j - T_i f_{j+1}) + T_i\Delta_n \\ &= T_i f_i \\ &= f_i. \end{aligned} \tag{6.2.17}$$

Now,

$$\sum_{j=1}^{i-1} \Delta_j^+ + \sum_{j=i}^{n} \Delta_j \leq \sum_{j=1}^{n-1} \Delta_j^+ + f_n.$$

So, making use of (6.2.17),

$$\begin{aligned} T_i\left(\sum_{j=1}^{n-1} \Delta_j^+ + f_n\right) &\geq T_i\left(\sum_{j=1}^{i-1} \Delta_j^+ + \sum_{j=i}^{n} \Delta_j\right) \\ &= \sum_{j=1}^{i-1} \Delta_j^+ + f_i \\ &\geq f_i \end{aligned} \tag{6.2.18}$$

Let

$$\begin{aligned}
P_i &= I - P_{(\lambda e - f_i)^+} \quad i = 1, 2, \dots \\
\tilde{P}_1 &= P_1 \\
\tilde{P}_2 &= P_2(I - P_1) \\
\tilde{P}_3 &= P_3(I - P_2)(I - P_1) \\
&\vdots
\end{aligned}$$

Then $\sum_{i=1}^n \tilde{P}_i = P$, where P is as in (6.2.15). From (6.2.18) and the construction of $\tilde{P}_i$ we have

$$\begin{aligned}
T \sum_{i=1}^n \tilde{P}_i T_i \left(\sum_{j=1}^{n-1} \Delta_j^+ + f_n \right) &\geq T \sum_{j=1}^n \tilde{P}_i f_i \\
&\geq T \sum_{i=1}^n \tilde{P}_i \lambda e \\
&= \lambda T P e.
\end{aligned}$$

Now, as $TT_i = T$ and $T_i \tilde{P}_i = \tilde{P}_i T_i$, we have

$$\begin{aligned}
\lambda T P e &\leq T \sum_{i=1}^n \tilde{P}_i T_i \left(\sum_{j=1}^{n-1} \Delta_j^+ + f_n \right) \\
&= TP \sum_{j=1}^{n-1} (\Delta_j^+ + f_n).
\end{aligned}$$

But $P\Delta_j^+ \leq \Delta_j^+$, so

$$\begin{aligned}
\lambda T P e &\leq T \sum_{j=1}^{n-1} \Delta_j^+ + TPf_n \\
&= T \sum_{j=1}^{n} \Delta_j^+ - T\Delta_n^+ + TPf_n \\
&= TA_n^+ - T(Pf_n^- + (I - P)f_n^+) \\
&\leq TA_n^+,
\end{aligned}$$

giving the desired inequality. □

From Theorem 6.2.10 we are able to deduce the Hájek-Rényi inequality, [29]. Indeed, let E be a Riesz space with strictly positive conditional expectation operator T and weak-order unit $e = Te$. Consider the space $\mathcal{L}^2(T) = \{x \in \mathcal{L}^1(T) | x^2 \in \mathcal{L}^1(T)\}$ (see Section 2.4, p40). Let $(\eta_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables in $\mathcal{L}^2(T)$ with T-mean zero (that is, $T\eta_i = 0$ for all $i \in \mathbb{N}$). Let $C_1, C_2, \ldots$ be a decreasing (not necessarily strictly) sequence of positive real numbers. Fix $N \in \mathbb{R}$. Let

$$f_i = C_{N+i}^2 \left(\eta_1 + \eta_2 + \cdots + \eta_{N+i}\right)^2$$

and define the filtration $(T_i)_{i\in\mathbb{N}}$ such that T_i is the conditional expectation with range the closed Riesz subspace of E generated by $\mathcal{R}(T)$ and $f_1, f_2, \ldots, f_i$.

Making use of independence and the T-mean of the random variables, η_i, we have, for all $1 \leq i \leq n-1$,

$$T_i \eta_{N+i+1} \eta_j = 0 \quad \text{for all } 1 \leq j \leq N+i$$

and

$$\begin{aligned}
\Delta_i &= f_i - T_i f_{i+1} \\
&= C_{N+i}^2 \left(\eta_1 + \eta_2 + \cdots + \eta_{N+i}\right)^2 - T_i C_{N+i+1}^2 \left(\eta_1 + \eta_2 + \cdots + \eta_{N+i+1}\right)^2 \\
&= C_{N+i}^2 \left(\eta_1 + \eta_2 + \cdots + \eta_{N+i}\right)^2 - T_i C_{N+i+1}^2 \left(\left(\eta_1 + \eta_2 + \cdots + \eta_{N+i}\right)^2\right) \\
&\quad - T_i C_{N+i+1}^2 \left(2(\eta_1 + \eta_2 + \cdots + \eta_{N+i})\eta_{N+i+1}\right) - T_i C_{N+i+1}^2 \left(\eta_{N+i+1}^2\right) \\
&= (C_{N+i}^2 - C_{N+i+1}^2)(\eta_1 + \eta_2 + \cdots + \eta_{N+i})^2 - T_i \eta_{N+i+1}^2.
\end{aligned}$$

Thus,

$$\Delta_i^+ \leq (C_{N+i}^2 - C_{N+i+1}^2)(\eta_1 + \eta_2 + \cdots + \eta_{N+i})^2. \tag{6.2.19}$$

Let $Q_{\left(\lambda e-\bigvee_{k=N}^{N+n} C_k|\eta_1+\eta_2+\cdots+\eta_k|\right)^+}$ be the band projection generated by $\left(\lambda e-\bigvee_{k=N}^{N+n} C_k|\eta_1+\eta_2+\cdots+\eta_k|\right)^+$ and $P_{\left(\lambda^2 e-\bigvee_{i=1}^{n} f_i\right)^+}$ be the band projection generated by $(\lambda^2 e-\bigvee_{i=1}^{n} f_i)^+$. Consider $Q=I-Q_{\left(\lambda e-\bigvee_{k=N}^{N+n} C_k|\eta_1+\eta_2+\cdots+\eta_k|\right)^+}$ and $P=I-P_{\left(\lambda^2 e-\bigvee_{i=1}^{n} f_i\right)^+}$, then $P=Q$.

By Theorem 6.2.10, (6.2.19), independence and the T-mean property of the random variables, we have

$$\begin{aligned}
\lambda^2 TQe &= \lambda^2 TPe \\
&\leq \sum_{i=0}^{n-1} T\Delta_i^+ + TPf_n \\
&\leq \sum_{i=0}^{n-1} T\left[(C_{N+i}^2 - C_{N+i+1}^2)(\eta_1+\eta_2+\cdots+\eta_{N+i})^2\right] + T\left[C_{N+n}^2(\eta_1+\eta_2+\cdots+\eta_{N+n})^2\right] \\
&= \sum_{i=0}^{n-1} (C_{N+i}^2 - C_{N+i+1}^2) T(\eta_1^2+\eta_2^2+\cdots+\eta_{N+i}^2) + C_{N+n}^2 T(\eta_1^2+\eta_2^2+\cdots+\eta_{N+n}^2) \\
&= C_N^2 T(\eta_1^2+\eta_2^2+\cdots+\eta_N^2) + \sum_{k=N+1}^{N+n} C_k^2 T\eta_k^2,
\end{aligned}$$

which is the Hájek-Rényi inequality.

Chapter 7

Further Work

This thesis, building on the work done by Kuo et. al. in [34, 35, 36, 37, 38], presents the foundations of the theory of Markov processes, quasi-martingales and mixingales in Riesz spaces and some of their fundamental properties. Much more remains to be done.

7.1 Markov Processes

In the case of martingales, many of the classical resuts were shown to hold in Riesz spaces by Kuo et.al., [34, 35, 36, 37, 38], for Markov processes deeper aspects of the theory, such as convergence, generating functions and the uses of stopped Markov processes still need attention. In [33, 32] stopping times in Riesz spaces were defined and used these to analyse the convergence of Riesz space martingales. However, this method of approach presents non-trivial hurdles for Markov processes in Riesz spaces.

Another difficulty that arises is the absence of transition kernels for Markov processes in Riesz spaces. This makes many of the classical approaches to Markov processes unusable in the Riesz space setting.

As was mentioned earlier, it is often said that the convergence of Markov processes can be studied via convergence results of martingales. Beside the statement itself, we have yet to find evidence of the validity of the claim. It is hoped that future work will shed some light on this.

7.2 Quasi-Martingales

We have shown that a quasi-martingale can be decomposed as the sum of a martingale and two positive supermartingales. In the classical setting, Rao uses this result to decompose a quasi-martingale into the sum of a local martingale and a process with finite expected total variation, [50]. In order to translate this result to Riesz spaces, we need the notion of local martingales on Riesz spaces. However, difficulties arise in constructing local martingales in Riesz spaces. One such difficulty is that of continuous stopping times. We have yet to successfully construct continuous stopping times on Riesz spaces.

Egghe has shown in [24] that quasi-martingales are uniform amarts. It was noted by Bellow, [9], that any $\mathcal{L}^1$-uniform amart converges. In [34], Kuo, Labuschagne and Watson construct amarts in Riesz spaces. Thus, the structures of amarts and quasi-martingales exist in Riesz spaces, but the link between the two remains to be studied.

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