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Orlicz Spaces and Their Hyperbolic Composition Operators

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Abstract: In this paper, by extending some L^p-norm inequalities to similar inequalities for Orlicz space (L^Φ-norm), we provide equivalent conditions for composition operators to have the shadowing property on the Orlicz space $L^\Phi(\mu)$. Additionally, we show that for composition operators on Orlicz spaces, the concepts of generalized hyperbolicity and the shadowing property are equivalent. These results extend similar findings on L^p-spaces to Orlicz spaces.

Keywords: shadowing property; hyperbolicity; dissipative systems; Orlicz spaces; composition operators

MSC: 37D05; 47A16

1. Introduction and Preliminaries

Mathematicians have achieved many intriguing results in linear dynamics related to dynamical properties, such as transitivity, mixing, Li–Yorke, and many others, in the last two decades. One important question in linear dynamics is finding relations between expansivity, hyperbolicity, the shadowing property, and structural stability. Some classical results on these relations can be found in [1–6].

It is rather well known that hyperbolic operators have the shadowing property. In the setting of finite dimensional Banach spaces, having the shadowing property is equivalent to being hyperbolic. In [7], Bernardes et al. constructed an operator with the shadowing property that is not hyperbolic, settling an open question. In the process, they introduced a class of operators that has come to be known as generalized hyperbolic operators. This class of operators seems to be an important bridge between hyperbolicity and the shadowing property. In [8], the authors show that for a large natural class of operators on L^p-spaces, the notion of generalized hyperbolicity and the shadowing property coincide. They achieved this by giving sufficient and necessary conditions for a certain class of operators to have the shadowing property. They also introduced computational tools that allow the construction of operators with and without the shadowing property. Utilizing these tools, they show how some natural probability distributions, such as the Laplace distribution and the Cauchy distribution, lead to operators with and without the shadowing property on L^p-spaces.

An Orlicz space L^Φ is a natural generalization of L^p spaces and has been considered in various areas, such as probability theory, partial differential equations, and mathematical finance. In this paper, we extend the results provided in [8] on L^p spaces to composition operators on Orlicz spaces. We provide some equivalent conditions for composition operators to have the shadowing property on the Orlicz space $L^\Phi(\mu)$. Additionally, we show that for composition operators on Orlicz spaces, the notions of generalized hyperbolicity and the shadowing property coincide.

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For the convenience of the reader, we recall some essential facts about Orlicz spaces for later use. To obtain more information about Orlicz spaces, consult [9,10].

A function $\Phi : \mathbb{R} \to [0, \infty]$ is called a Young function if Φ is convex, $\Phi(-x) = \Phi(x)$, $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = +\infty$. With each Young function Φ, we can associate another convex function $\Psi : \mathbb{R} \to [0, \infty]$ with similar properties defined by

$$\Psi(y) = \sup\{x \mid y \mid -\Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

The function Ψ is called the complementary function to Φ. It follows from the definition that $\Psi(0) = 0$, $\Psi(-y) = \Psi(y)$, and $\Psi(.)$ is a convex increasing function satisfying $\lim_{y\to\infty} \Psi(y) = +\infty$.

Let Φ be a Young function. Then, we say Φ satisfies the Δ_2-condition or Φ is Δ_2-regular if $\Phi(2x) \leq K\Phi(x)$ for $x \geq x_0$ for some constants $K > 0$ and $x_0 > 0$. Φ satisfies the Δ_2-condition globally if $\Phi(2x) \leq K\Phi(x)$ for $x \geq 0$ for some $K > 0$. Φ is said to satisfy the Δ' condition if there exists $c > 0$ $(b > 0)$ such that

$$\Phi(xy) \leq c\Phi(x)\Phi(y), \quad x, y \geq x_0 \geq 0.$$

If $x_0 = 0$, then it is said to hold globally. If $\Phi \in \Delta'$, then $\Phi \in \Delta_2$.

For a given complete σ-finite measure space $(X, \mathcal{F}, \mu)$, let $L^0(\mathcal{F})$ be the linear space of equivalence classes of $\mathcal{F}$-measurable real-valued functions on X, where functions that are equal μ-almost everywhere on X are identified. The support $S(f)$ of a measurable function f is defined as $S(f) := \{x \in X : f(x) \neq 0\}$.

For a Young function Φ, let $\rho_\Phi : L^\Phi(\mu) \to \mathbb{R}^+$ be defined as $\rho_\Phi(f) = \int_X \Phi(f)d\mu$ for all $f \in L^\Phi(\mu)$. Then, the space

$$L^\Phi(\mu) = \left\{ f \in L^0(\mathcal{F}) : \exists k > 0, \rho_\Phi(kf) < \infty \right\}$$

is called an Orlicz space. The functional

$$N_\Phi(f) = \inf\{k > 0 : \rho_\Phi(\frac{f}{k}) \leq 1\}$$

is a norm on $L^\Phi(\mu)$ and is called the gauge norm (or Luxemburg norm). Also, $(L^\Phi(\mu), N_\Phi(.))$ is a normed linear space. If functions that are equal almost everywhere are identified, then $(L^\Phi(\mu), N_\Phi(.))$ is a Banach space, and the basic measure space $(X, \mathcal{F}, \mu)$ is unrestricted. Hence, every element of $L^\Phi(\mu)$ is a class of measurable functions that are equal almost everywhere. There is also another norm on $L^\Phi(\mu)$ defined as follows:

$$\|f\|_\Phi = \sup\{\int_X \mid fg \mid d\mu : g \in B_\Psi\} = \sup\{\mid \int_X fgd\mu \mid : g \in B_\Psi\}, \tag{1}$$

where $B_\Psi = \{g \in L^\Psi(\mu) : \int_X \Psi(\mid g \mid)d\mu \leq 1\}$. The norm $\|.\|_\Phi$ is called the Orlicz norm. For any $f \in L^\Phi(\mu)$, where Φ is a Young function, we have

$$N_\Phi(f) \leq \|f\|_\Phi \leq 2N_\Phi(f). \tag{2}$$

For every $F \in \mathcal{F}$ with $0 < \mu(F) < \infty$, we have $N_\Phi(\chi_F) = \frac{1}{\Phi^{-1}(\frac{1}{\mu(F)})}$.

Throughout this paper, $(X, \mathcal{F}, \mu)$ will be a measure space, $\varphi : X \to X$ will be a non-singular measurable transformation, and $\mathcal{F}$ will be a sigma algebra on X. A non-singular measurable transformation $\varphi : X \to X$ is defined as $\varphi^{-1}(F) \in \mathcal{F}$ for every $F \in \mathcal{F}$ and $\mu(\varphi^{-1}(F)) = 0$ if $\mu(F) = 0$. If there exists a positive constant c satisfying

$$\mu(\varphi^{-1}(F)) \leq c\mu(F), \quad \text{for every} F \in \mathcal{F}, \tag{3}$$

then the linear operator

$$C_\varphi : L^\Phi(\mu) \to L^\Phi(\mu), \quad C_\varphi(f) = f \circ \varphi,$$

is well defined and continuous on the Orlicz space $L^\Phi(\mu)$ and is called a composition operator. For more details on the basic properties of composition operators on Orlicz spaces and function spaces, refer to [5,11–16]. The reference [5] is pioneering in composition operators on function spaces. The class of composition operators is wide and includes, for instance, weighted shifts, which are a fundamental tool for the construction of examples in linear dynamics. Additionally, it contains composition operators induced by odometers, as shown in [17]. Recent advances in the dynamic properties of composition operators in various contexts are collected in [18,19].

2. Hyperbolicity and Shadowing Property

In this section, we first rewrite the definitions of the shadowing property, hyperbolicity, wandering set, dissipative system, and dissipative system of bounded distortion in Orlicz spaces, which were defined for L^p spaces in [8].

Definition 1. *Let $T : X \to X$ be an operator on a Banach space X. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in X is called a δ-pseudotrajectory of T, where $\delta > 0$, if*

$$\|Tx_n - x_{n+1}\| \leq \delta, \quad \text{for all} \quad n \in \mathbb{Z}.$$

Now, we can define the shadowing property.

Definition 2. *Let $T : X \to X$ be an invertible operator on a Banach space X. Then, T is said to have the shadowing property if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudotrajectory $\{x_n\}_{n\in\mathbb{Z}}$ of T is ϵ-shadowed by a real trajectory of T, that is, there exists $x \in X$ such that*

$$\|T^n x - x_n\| < \epsilon, \quad \text{for all} \quad n \in \mathbb{Z}.$$

We can define the notion of positive shadowing for an operator T by replacing the set $\mathbb{Z}$ by $\mathbb{N}$ in the above definition. In such a "positive" case, T does not need to be invertible. A fundamental notion in linear dynamics is that of hyperbolicity.

Definition 3. *An invertible operator T is said to be hyperbolic if $\sigma(T) \cap \mathbb{T} = \emptyset$.*

In the following, we recall the definition of a wandering set.

Definition 4. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $\varphi : X \to X$ be an invertible non-singular transformation. A measurable set $W \subseteq X$ is called a wandering set for φ if the sets $\{\varphi^{-n}(W)\}_{n\in\mathbb{Z}}$ are pairwise disjoint.*

Definition 5. *Let $(X, \mathcal{F}, \mu)$ be a measure space, $\Phi :\to [0, \infty]$ a Young's function, and $\varphi : X \to X$ be an invertible non-singular transformation. The quadruple $(X, \mathcal{F}, \mu, \varphi)$ is called the following:*

- *A dissipative system generated by W, if $X = \dot{\cup}_{k\in\mathbb{Z}}\varphi^k(W)$ for some $W \in \mathcal{F}$ with $0 < \mu(W) < \infty$ (the symbol $\dot{\cup}$ denotes pairwise disjoint union);*
- *A dissipative system of bounded distortion in L^Φ, generated by W, if there exists $K > 0$, such that*

$$\frac{1}{K} N_\Phi(C_\varphi^k(\chi_W)) N_\Phi(\chi_F) \leq N_\Phi(C_\varphi^k(\chi_F)) N_\Phi(\chi_W) \leq K N_\Phi(C_\varphi^k(\chi_W)) N_\Phi(\chi_F), \quad (4)$$

for all $k \in \mathbb{Z}$ and $F \in \mathcal{F}_W = \{F \cap W : F \in \mathcal{F}\}$. If we replace N_Φ by the norm of L^p, then we will have the bounded distortion property in L^p-spaces (Definition 2.6.4, [8]).

Definition 6. *A composition dynamical system $(X, \mathcal{F}, \mu, \varphi, C_\varphi)$ is called the following:*

- *A dissipative composition dynamical system generated by W, if $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system generated by W;*
- *A dissipative composition dynamical system of bounded distortion in L^Φ generated by W, if $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system of bounded distortion in L^Φ generated by W.*

In the following, we have a proposition in Orlicz spaces similar to Proposition 2.6.5 of [1].

Proposition 1. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system of bounded distortion in L^Φ generated by W. Then, the following hold:*

1. *There exists $H > 0$ such that*

$$\frac{1}{H}\frac{N_\Phi(C_\varphi^{t+s}(\chi_W))}{N_\Phi(C_\varphi^s(\chi_W))} \leq \frac{N_\Phi(C_\varphi^{t+s}(\chi_F))}{N_\Phi(C_\varphi^s(\chi_F))} \leq H\frac{N_\Phi(C_\varphi^{t+s}(\chi_W))}{N_\Phi(C_\varphi^s(\chi_W))}, \tag{5}$$

   for all $F \in \mathcal{F}_W$ with $\mu(F) > 0$ and all $s, t \in \mathbb{Z}$;
2. *Let $\Phi \in \Delta_2$. If*

$$\sup\left\{\frac{N_\Phi(C_\varphi^{k-1}(\chi_W))}{N_\Phi(C_\varphi^k(\chi_W))}, \frac{N_\Phi(C_\varphi^{k+1}(\chi_W))}{N_\Phi(C_\varphi^k(\chi_W))} : k \in \mathbb{Z}\right\}$$

   is finite and φ is bijective, then φ and φ^{-1} satisfy the (3) condition.

Proof. The proof of (1) follows, with slight changes, as in (1) of [5] (Proposition 2.6.5) (2): Let

$$M = \sup\{\frac{N_\Phi(C_\varphi^{k-1}(\chi_W))}{N_\Phi(C_\varphi^k(\chi_W))}, \frac{N_\Phi(C_\varphi^{k+1}(\chi_W))}{N_\Phi(C_\varphi^k(\chi_W))} : k \in \mathbb{Z}\}.$$

For each $B \in \mathcal{F}$, we have $B = \cup_{k=-\infty}^{+\infty}\varphi^k(W) \cap B$. Let $B_k = \varphi^k(W) \cap B$. As B_k's are disjoint and the measure mu is countably additive, we just prove Condition (3) for B_k. If $\mu(B_k) = 0$, then $N_\Phi(\chi_{B_k}) = 0$ and if we set $F = \varphi^{-k}(B_k)$, then $C_\varphi^{-k}(\chi_{\varphi^{-k}(B_k)}) = \chi_{B_k}$ and so by Condition (4) as:

$$\frac{1}{K}N_\Phi(C_\varphi^{-k}(\chi_W))N_\Phi(\chi_F) \leq N_\Phi(C_\varphi^{-k}(\chi_F))N_\Phi(\chi_W) \leq KN_\Phi(C_\varphi^{-k}(\chi_W))N_\Phi(\chi_F),$$

we obtain that $N_\Phi(\chi_{\varphi^{-k}(B_k)}) = 0$ and so $\chi_{\varphi^{-k}(B_k)} = 0$. Thus, $\mu(\varphi^{-k}(B_k)) = 0$ and, therefore, $\mu(\varphi^{-1}(B_k)) = 0$. For the case $\mu(B_k) > 0$, we apply Condition (5) to $F = \varphi B_k$, and we obtain that

$$\frac{N_\Phi(C_\varphi(\chi_{B_k}))}{N_\Phi(\chi_{B_k})} \leq H\frac{N_\Phi(C_\varphi^{-1+k}(\chi_W))}{N_\Phi(C_\varphi^k(\chi_W))} \leq HM.$$

If we let $c = HM$, then

$$N_\Phi(C_\varphi(\chi_{B_k})) \leq cN_\Phi(\chi_{B_k})$$

and so

$$\Phi^{-1}(\frac{1}{\mu(B_k)}) \leq c\Phi^{-1}(\frac{1}{\mu(\varphi^{-1}(B_k))}).$$

Since $\Phi \in \Delta_2$ and Φ is increasing, then there exists $n > 0$ such that

$$\frac{1}{\mu(B_k)} = \Phi\Phi^{-1}(\frac{1}{\mu(B_k)}) \leq \Phi(c\Phi^{-1}(\frac{1}{\mu(\varphi^{-1}(B_k))})) \leq \frac{cK^n}{2^n}\frac{1}{\mu(\varphi^{-1}(B_k))}.$$

Now, by taking $d = \frac{cK^n}{2^n}$, we have $\mu(\varphi^{-1}(B_k)) \leq d\mu(B_k)$. So, Condition (3) holds for φ. Similarly, we obtain the thesis for φ^{-1}. □

Let $h_k = \frac{d(\mu\circ\varphi^{-k})}{d\mu}$ denote the Radon–Nikodym derivative of $\mu \circ \varphi^{-k}$ with respect to μ, where $\mu \circ \varphi^{-k}(F) = \mu(\varphi^{-k}(F))$, for every $F \in \mathcal{F}$. The proof of the following result adapts an argument used in [20] (Proposition 2.6.6).

Proposition 2. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system generated by W and φ is injective. We put $m_k = ess\,inf_{x\in W} h_k(x)$, and $M_k = ess\,sup_{x\in W} h_k(x)$. Without loss of generality, we can assume that $m_k \leq 1$ and $M_k \geq 1$. If the sequence $\{\frac{M_k}{m_k}\}_{k\in\mathbb{Z}}$ is bounded, then φ is of bounded distortion on W.*

Proof. Suppose that $\{\frac{M_k}{m_k}\}_{k\in\mathbb{Z}}$ is bounded, so there exists $K > 0$ such that $\frac{M_k}{m_k} \leq K$. Let $F \in \mathcal{F}$ with $\mu(F) > 0$. Then, for every $k \in \mathbb{Z}$,

$$\begin{aligned}\int_X \Phi(\frac{C_\varphi^k(\chi_F)}{M_k N_\Phi(\chi_F)})d\mu &\leq \int_X \Phi(\frac{\chi_F}{M_k N_\Phi(\chi_F)})h_k d\mu \\ &\leq \int_X \Phi(\frac{\chi_F}{M_k N_\Phi(\chi_F)})M_k d\mu \\ &\int_X \Phi(\frac{\chi_F}{N_\Phi(\chi_F)})d\mu \leq 1.\end{aligned}$$

This implies that

$$N_\Phi(C_\varphi^k(\chi_F)) \leq M_k N_\Phi(\chi_F).$$

Also,

$$\begin{aligned}\int_X \Phi(\frac{m_k\chi_F}{N_\Phi(C_\varphi^k(\chi_F))})d\mu &\leq \int_X \Phi(\frac{m_k\chi_F}{N_\Phi(C_\varphi^k(\chi_F))})\frac{h_k}{m_k}d\mu \\ &\int_X \Phi(\frac{m_k C_\varphi^k(\chi_F)}{m_k N_\Phi(C_\varphi^k(\chi_F))})d\mu \leq 1.\end{aligned}$$

This implies that

$$m_k N_\Phi(\chi_W) = N_\Phi(m_k\chi_W) \leq N_\Phi(C_\varphi^k(\chi_W)).$$

By the above observations, we have

$$\frac{1}{K}\frac{N_\Phi(C_\varphi^k(\chi_W))}{N_\Phi(\chi_W)} \leq \frac{N_\Phi(C_\varphi^k(\chi_F))}{N_\Phi(\chi_F)} \leq K\frac{N_\Phi(C_\varphi^k(\chi_W))}{N_\Phi(\chi_W)}.$$

So, we obtain the result. □

Definition 7. *Let φ, φ^{-1} satisfy Condition (3) and $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system generated by W. We define the following conditions*

$$\overline{\lim}_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}\right)^{\frac{1}{n}} < 1 \tag{6}$$

$$\underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{Z}} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}\right)^{\frac{1}{n}} > 1 \tag{7}$$

$$\overline{\lim}_{n\to\infty} \sup_{k\in-\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k-n}(W))})}\right)^{\frac{1}{n}} < 1 \text{ and } \underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}\right)^{\frac{1}{n}} > 1 \tag{8}$$

From now on, we assume that $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system generated by W such that the associated composition operator C_φ is an invertible operator on Orlicz space $L^\Phi(\mu)$. Now, we rewrite some notations and terminologies for Orlicz spaces similar to those one that were introduced for $L^p(\mu)$-spaces in [8].

Definition 8. *Let $f \in L^\Phi(\mu)$. Then, we define $f = f_+ + f_-$, where*

$$f_+(x) = \begin{cases} 0, & \text{if } x \in \cup_{k=0}^{\infty} \varphi^k(W) \\ f(x), & \text{otherwise,} \end{cases}$$

and

$$f_-(x) = \begin{cases} 0, & \text{if } x \in \cup_{k=1}^{\infty} \varphi^{-k}(W) \\ f(x), & \text{otherwise.} \end{cases}$$

Let $L^\Phi_+ = \{f_+ : f \in L^\Phi(\mu)\}$ and $L^\Phi_- = \{f_- : f \in L^\Phi(\mu)\}$. It is clear that $L^\Phi(\mu) = L^\Phi_+ \oplus L^\Phi_-$ and $C_\varphi(L^\Phi_+) \subseteq L^\Phi_+$ and $C_\varphi^{-1}(L^\Phi_-) \subseteq L^\Phi_-$.

Definition 9. *Let $K, t > 0$ and let $\mathcal{U}^\Phi_C(K,t)$ and $\mathcal{U}^\Phi_D(K,t)$ be the set of all $f \in L^\Phi(\mu)$, which satisfy the following conditions, respectively:*

$$\sup_{k\in\mathbb{Z}} \frac{N_\Phi(C^k_{\varphi^{-1}} f)}{N_\Phi(C^{k+n}_{\varphi^{-1}}(f))} \leq Kt^n, \quad \forall n \in \mathbb{N} \tag{9}$$

and

$$\inf_{k\in\mathbb{Z}} \frac{N_\Phi(C^k_{\varphi^{-1}} f)}{N_\Phi(C^{k+n}_{\varphi^{-1}}(f))} \geq Kt^n, \quad \forall n \in \mathbb{N}. \tag{10}$$

Also, we let $\mathcal{U}^\Phi_{GH+}(K,t)$ and $\mathcal{U}^\Phi_{GH-}(K,t)$ be the set of all f in L^Φ_+ and L^Φ_-, respectively, which satisfy the following conditions:

$$\sup_{k\in-\mathbb{N}_0} \frac{N_\Phi(C^{k-n}_{\varphi^{-1}}(f))}{N_\Phi(C^k_{\varphi^{-1}}(f))} \leq Kt^n, \quad \forall n \in \mathbb{N} \tag{11}$$

and

$$\inf_{k\in\mathbb{N}_0} \frac{N_\Phi(C^k_{\varphi^{-1}}(f))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(f))} \geq K\frac{1}{t^n}, \quad \forall n \in \mathbb{N}. \tag{12}$$

From this point on, all the results are a reformulation of what has already been proven in [8] for L^p spaces. In particular, Theorems 1 and 2 are the Orlicz version of Theorems SS and SN from [8], while Corollaries 1 and 2 correspond to Corollaries SC and GH from [8], respectively.

Here, we recall a simple fact about the definition of upper and lower limits:

Proposition 3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, and let $t > 0$. Then, the following hold:*

1. *If $\overline{\lim}_{n\to\infty} a_n^{\frac{1}{n}} < t$, then there exists $K > 0$ such that $a_n \leq Kt^n$, for every $n \in \mathbb{N}$;*
2. *If $\underline{\lim}_{n\to\infty} a_n^{\frac{1}{n}} > t$, then there exists $K > 0$ such that $a_n \geq Kt^n$, for every $n \in \mathbb{N}$.*

Proposition 4. *The following hold:*

1. *Condition (6) holds if and only if $\chi_W \in \mathcal{U}^\Phi_C(K,t)$ for some $K > 0$ and $t < 1$;*
2. *Condition (7) holds if and only if $\chi_W \in \mathcal{U}^\Phi_D(K,t)$ for some $K > 0$ and $t > 1$;*

3. *Condition (2) holds if and only if there exist $K > 0$ and $t < 1$ such that $\chi_W \in \mathcal{U}^{\Phi}_{GH-}(K,t)$ and $C_\varphi(\chi_W) \in \mathcal{U}^{\Phi}_{GH+}(K,t)$.*

Proof. (1) Suppose that Condition (6) holds. Let $a_n = \sup_{k\in\mathbb{Z}} \frac{C^k_{\varphi^{-1}}(\chi_W)}{C^{k+n}_{\varphi^{-1}}(\chi_W)}$. Since $N_\Phi(C^k_{\varphi^{-1}}(\chi_W)) = \frac{1}{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}$, then by definition of Condition (6) and Proposition 3, we obtain that there exist $K > 0$ and $t < 1$ such that $\chi_W \in \mathcal{U}^{\Phi}_{C}(K,t)$.
For the converse, let $K > 0$ and $t < 1$ such that $\chi_W \in \mathcal{U}^{\Phi}_{C}(K,t)$, i.e.,

$$\sup_{k\in\mathbb{Z}} \frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})} = \sup_{k\in\mathbb{Z}} \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(\chi_W))} \leq Kt^n.$$

Hence,

$$\begin{aligned}\overline{\lim}_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}\right)^{\frac{1}{n}} &= \overline{\lim}_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(\chi_W))}\right)^{\frac{1}{n}} \\ &\leq \overline{\lim}_{n\to\infty} K^{\frac{1}{n}} t \\ &= t < 1.\end{aligned}$$

So, Condition (6) holds.

(2) By using the definition of Condition (7) and Proposition 3, similar to the proof of (1), we obtain the proof.

(3) Suppose that Condition (2) holds. We replace W by $\varphi^{-1}(W)$ in the first part of Condition (2). Then, by Proposition 3, there exist $K > 0$ and $0 < t < 1$ such that

$$\sup_{k\in-\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^k(\varphi^{-1}(W)))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k-n}(\varphi^{-1}(W)))})}\right) \leq Kt^n$$

and

$$\inf_{k\in\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}\right) \geq K\frac{1}{t^n},$$

for every $n \in \mathbb{N}$. Hence,

$$\sup_{k\in-\mathbb{N}_0} \frac{N_\Phi(C^{k-n}_{\varphi^{-1}}(\chi_{\varphi^{-1}(W)}))}{N_\Phi(C^k_{\varphi^{-1}}(\chi_{\varphi^{-1}(W)}))} = \sup_{k\in-\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^k(\varphi^{-1}(W)))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k-n}(\varphi^{-1}(W)))})}\right) \leq Kt^n,$$

and

$$\inf_{k\in\mathbb{N}_0} \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(\chi_W))} = \inf_{k\in\mathbb{N}_0} \left(\frac{\Phi^{-1}(\frac{1}{\mu(\varphi^k(W))})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{k+n}(W))})}\right) \geq K\frac{1}{t^n}.$$

So, we obtain that $\chi_W \in \mathcal{U}^{\Phi}_{GH-}(K,t)$ and $C_\varphi(\chi_W) \in \mathcal{U}^{\Phi}_{GH+}(K,t)$.

For the converse, similar to the proof of (1), we can obtain the proof. □

Proposition 5. *The following statements are true.*

- *Let $\mathcal{U}^{\Phi}(K,t) \in \{\mathcal{U}^{\Phi}_{C}(K,t), \mathcal{U}^{\Phi}_{D}(K,t)\}$. If $f \in \mathcal{U}^{\Phi}(K,t)$, then $C^j_\varphi(f) \in \mathcal{U}^{\Phi}(K,t)$, for all $j \in \mathbb{Z}$;*
- *If $f \in \mathcal{U}^{\Phi}_{GH+}(K,t)$, then $C^j_\varphi(f) \in \mathcal{U}^{\Phi}_{GH+}(K,t)$, for all $j \geq 0$;*
- *If $f \in \mathcal{U}^{\Phi}_{GH-}(K,t)$, then $C^j_\varphi(f) \in \mathcal{U}^{\Phi}_{GH-}(K,t)$, for all $j \leq 0$.*

Proof. By a straightforward calculation, we can obtain all statements by definitions. □

Proposition 6. *Let $\mathcal{U}^{\Phi}(K,t) \in \{\mathcal{U}^{\Phi}_{C}(K,t), \mathcal{U}^{\Phi}_{D}(K,t), \mathcal{U}^{\Phi}_{GH+}(K,t), \mathcal{U}^{\Phi}_{GH-}(K,t)\}$. Then, the following are true.*

1. *If $f \in \mathcal{U}^{\Phi}(K,t)$ and $\alpha \in \mathbb{C} \setminus \{0\}$, then $\alpha.f \in \mathcal{U}^{\Phi}(K,t)$;*
2. *If $f_1, f_2 \in \mathcal{U}^{\Phi}(K,t)$ with disjoint supports, then $(f_1 + f_2) \in \mathcal{U}^{\Phi}(K,t)$.*

Proof. (1) It is clear by definition.

(2) Let $\mathcal{U}^{\Phi}(K,t) = \mathcal{U}^{\Phi}_{C}(K,t)$ and $f_1, f_2 \in \mathcal{U}^{\Phi}_{C}(K,t)$ be with disjoint supports. So, by definition we have

$$\sup_{k\in\mathbb{Z}} \frac{N_{\Phi}(C^{k}_{\varphi^{-1}}(f_i))}{N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_i))} \leq Kt^{n}, \quad \forall n \in \mathbb{N},$$

for $i = 1, 2$. Hence,

$$N_{\Phi}(C^{k}_{\varphi^{-1}}(f_i)) \leq Kt^{n} N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_i)),$$

for every $k \in \mathbb{Z}$ and $i = 1, 2$. Since f_1 and f_2 have disjoint supports, then by definition of $N_{\Phi}(.)$ it is clear that $N_{\Phi}(f_1) + N_{\Phi}(f_2) \leq 2N_{\Phi}(f_1 + f_2)$. Moreover, since f_1 and f_2 have disjoint supports, then for each $k \in \mathbb{Z}$, $f_1 \circ \varphi^{-k}$ and $f_2 \circ \varphi^{-k}$ have disjoint support too. By these observations for every $k \in \mathbb{Z}$, we have

$$\begin{aligned} N_{\Phi}(C^{k}_{\varphi^{-1}}(f_1 + f_2)) &= N_{\Phi}((f_1 + f_2) \circ \varphi^{-k}) \\ &= N_{\Phi}((f_1) \circ \varphi^{-k}) + N_{\Phi}((f_2) \circ \varphi^{-k}) \\ &\leq N_{\Phi}(C^{k}_{\varphi^{-1}}(f_1)) + N_{\Phi}(C^{k}_{\varphi^{-1}}(f_2)) \\ &\leq Kt^{n}(N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_1)) + N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_2))) \\ &\leq 2Kt^{n} N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_1 + f_2)). \end{aligned}$$

Therefore,

$$\sup_{k\in\mathbb{Z}} \frac{N_{\Phi}(C^{k}_{\varphi^{-1}}(f_1 + f_2))}{N_{\Phi}(C^{k+n}_{\varphi^{-1}}(f_1 + f_2))} \leq 2Kt^{n}, \quad \forall n \in \mathbb{N}.$$

This implies that $f_1 + f_2 \in \mathcal{U}^{\Phi}_{C}(K,t)$. For other cases, the proof is similar to the case $\mathcal{U}^{\Phi}_{C}(K,t)$. □

Proposition 7. *Let $(X, \mathcal{F}, \mu, \varphi)$ be with bounded distortion and H be the bounded distortion constant from Proposition 1. Then, for each $j \in \mathbb{Z}$, the following statements hold:*

1. *Let $\mathcal{U}^{\Phi}(K,t) \in \{\mathcal{U}^{\Phi}_{C}(K,t), \mathcal{U}^{\Phi}_{D}(K,t)\}$. If $C^{-j}_{\varphi}(\chi_W) \in \mathcal{U}^{\Phi}(K,t)$, then $C^{-j}_{\varphi}(\chi_F) \in \mathcal{U}^{\Phi}(HK,t)$, for all $F \subseteq W$ with $\mu(F) > 0$;*
2. *If $C^{-j}_{\varphi}(\chi_W) \in \mathcal{U}^{\Phi}_{GH-}(K,t)$, for $j \geq 0$, then $C^{-j}_{\varphi}(\chi_F) \in \mathcal{U}^{\Phi}_{GH-}(HK,t)$, for all $F \subseteq W$ with $\mu(F) > 0$;*
3. *If $C^{-j}_{\varphi}(\chi_W) \in \mathcal{U}^{\Phi}_{GH+}(K,t)$, for $j < 0$, then $C^{-j}_{\varphi}(\chi_F) \in \mathcal{U}^{\Phi}_{GH+}(HK,t)$, for all $F \subseteq W$ with $\mu(F) > 0$.*

Proof. (1) As in [20] (Proposition 4.1.8), let $\mathcal{U}^{\Phi}(K,t) = \mathcal{U}^{\Phi}_{C}(K,t)$, $C^{-j}_{\varphi}(\chi_W) \in \mathcal{U}^{\Phi}(K,t)$ and $F \subseteq W$ with $\mu(F) > 0$. We note that for each $k \in \mathbb{Z}$, $C^{k}_{\varphi^{-1}} = C^{-k}_{\varphi}$. By the Proposition 1, there exists $0 < H < \infty$ such that for all $t, s \in \mathbb{Z}$,

$$\frac{1}{H}\frac{N_{\Phi}(C^{t+s}_{\varphi}(\chi_W))}{N_{\Phi}(C^{s}_{\varphi}(\chi_W))} \leq \frac{N_{\Phi}(C^{t+s}_{\varphi}(\chi_F))}{N_{\Phi}(C^{s}_{\varphi}(\chi_F))} \leq H\frac{N_{\Phi}(C^{t+s}_{\varphi}(\chi_W))}{N_{\Phi}(C^{s}_{\varphi}(\chi_W))}.$$

So, for a fixed $n \in \mathbb{N}$, we have

$$\frac{N_\Phi(C_\varphi^{-(k+j)}(\chi_F))}{N_\Phi(C_\varphi^{-(k+j+n)}(\chi_F))} \leq H \frac{N_\Phi(C_\varphi^{-(k+j)}(\chi_W))}{N_\Phi(C_\varphi^{-(k+j+n)}(\chi_W))},$$

Hence,

$$\sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C_\varphi^{-(k+j)}(\chi_F))}{N_\Phi(C_\varphi^{-(k+j+n)}(\chi_F))} \right) \leq H \sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C_\varphi^{-(k+j)}(\chi_W))}{N_\Phi(C_\varphi^{-(k+j+n)}(\chi_W))} \right),$$

Moreover, since $C_\varphi^{-j}(\chi_W) \in \mathcal{U}^\Phi(K,t)$, then by definition we have

$$\sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C_{\varphi^{-1}}^{k+j}(\chi_W))}{N_\Phi(C_{\varphi^{-1}}^{k+j+n}(\chi_W))} \right) \leq Kt^n.$$

Therefore,

$$\sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C_{\varphi^{-1}}^{k}(\chi_{\varphi^j(F)}))}{N_\Phi(C_{\varphi^{-1}}^{k+n}(\chi_{\varphi^j(F)}))} \right) = \sup_{k\in\mathbb{Z}} \left(\frac{N_\Phi(C_{\varphi^{-1}}^{k+j}(\chi_F))}{N_\Phi(C_{\varphi^{-1}}^{k+j+n}(\chi_F))} \right) \leq HKt^n,$$

which means that $C_\varphi^{-j}(\chi_F) \in \mathcal{U}^\Phi(HK,t)$. The proofs of other cases are similar. □

Proposition 8. *If $\Phi \in \Delta_2$, then the following statements are true:*

1. *The set*

$$\{\sum_{i=0}^{n} a_i \chi_{F_i} : a_i \in \mathbb{C}, F_i \subseteq \varphi^{j_i}(W), j_i \in \mathbb{Z}, \mu(F_i) > 0, F_i \cap F_{i'} = \emptyset, i \neq i'\}$$

   is dense in $L^\Phi(\mu)$;
2. *The set*

$$\{\sum_{i=0}^{n} a_i \chi_{F_i} : a_i \in \mathbb{C}, F_i \subseteq \varphi^{j_i}(W), j_i < 0, \mu(F_i) > 0, F_i \cap F_{i'} = \emptyset, i \neq i'\}$$

   is dense in L_+^Φ;
3. *The set*

$$\{\sum_{i=0}^{n} a_i \chi_{F_i} : a_i \in \mathbb{C}, F_i \subseteq \varphi^{j_i}(W), j_i < 0, \mu(F_i) > 0, F_i \cap F_{i'} = \emptyset, i \neq i'\}$$

   is dense in L_-^Φ.

Proof. Since $\Phi \in \Delta_2$, then the set of simple functions is dense in $L^\Phi(\mu)$ and we easily obtain the proof. □

Proposition 9. *Suppose that $\Phi \in \Delta_2$ and $(X, \mathcal{F}, \mu, \varphi)$ has bounded distortion.*

1. *Let $\mathcal{U}^\Phi(K,t) \in \{\mathcal{U}_C^\Phi(K,t), \mathcal{U}_D^\Phi(K,t)\}$. If $\chi_W \in \mathcal{U}^\Phi(K,t)$, then $\mathcal{U}^\Phi(HK,t) = L^\Phi(\mu)$;*
2. *If $\chi_W \in \mathcal{U}_{GH-}^\Phi(K,t)$, then $\mathcal{U}_{GH-}^\Phi(HK,t) = L_-^\Phi$;*
3. *If $C_\varphi(\chi_W) \in \mathcal{U}_{GH+}^\Phi(K,t)$, then $\mathcal{U}_{GH+}^\Phi(HK,t) = L_+^\Phi$.*

Proof. (1) This proof follows as in [5] (Proposition 4.1.10) and, for the convenience of the reader, it is provided here with the proper modifications. Let $\mathcal{U}^\Phi(K,t) \in \{\mathcal{U}_C^\Phi(K,t), \mathcal{U}_D^\Phi(K,t)\}$. As we have in Proposition 7 part (1), if $\chi_W \in \mathcal{U}^\Phi(K,t)$, then $\chi_F \in \mathcal{U}^\Phi(HK,t)$, for all $F \subseteq W$

with $\mu(F) > 0$. Also, by Proposition 5, the first part, $\chi_f \in \mathcal{U}^{\Phi}(HK, t)$, for all $F \subseteq \varphi^i(W)$, with $\mu(F) > 0$. Since by Proposition 6 $\mathcal{U}^{\Phi}(HK, t)$ is a linear space, then we have

$$\{\sum_{i=0}^{n} a_i \chi_{F_i} : a_i \in \mathbb{C}, F_i \subseteq \varphi^{j_i}(W), j_i \in \mathbb{Z}, \mu(F_i) > 0, F_i \cap F_{i'} = \emptyset, i \neq i'\} \subseteq \mathcal{U}^{\Phi}(HK, t).$$

By the fact that simple functions are dense in Orlicz space $L^{\Phi}(\mu)$, in the case $\Phi \in \Delta_2$, the assumption $X = \cup_{j=-\infty}^{+\infty} \varphi^j(W)$ and by Proposition 8, part (1), we obtain the result.

(2) By Proposition 7 part (2), if $\chi_W \in \mathcal{U}^{\Phi}_{GH-}(K, t)$, then $\chi_F \in \mathcal{U}^{\Phi}_{GH-}(HK, t)$, for all $F \subseteq W$ with $\mu(F) > 0$. Also, by Proposition 5, the first part, $\chi_f \in \mathcal{U}^{\Phi}_{GH-}(HK, t)$, for all $F \subseteq \varphi^i(W)$, with $\mu(F) > 0$ and $i \geq 0$. Similar to part (1), by the linearity of $\mathcal{U}^{\Phi}_{GH-}(K, t)$ (Proposition 6), we obtain that

$$\{\sum_{i=0}^{n} a_i \chi_{F_i} : a_i \in \mathbb{C}, F_i \subseteq \varphi^{j_i}(W), j_i < 0, \mu(F_i) > 0, F_i \cap F_{i'} = \emptyset, i \neq i'\} \subseteq \mathcal{U}^{\Phi}_{GH-}(HK, t)$$

Hence, by Proposition 8, part (3), we obtain the result.

(3) The proof is similar to the proof of part (2). □

Theorem 1. *If $(X, \mathcal{F}, \mu, \varphi)$ is a dissipative system of bounded distortion generated by W, then the following hold:*

1. *If Condition (6) is satisfied, then C_φ is a proper contraction under an equivalent norm, i.e., $r(C_\varphi) < 1$;*
2. *If Condition (7) holds, then C_φ is a proper dilation under an equivalent norm, i.e., $r(C_\varphi^{-1}) < 1$;*
3. *If Condition (2) is satisfied, then C_φ is a generalized hyperbolic operator.*

So, C_φ has the shadowing property in all three cases.

Proof. (1) If Condition (6) holds, then by Proposition 4, we have $\chi_w \in \mathcal{U}^{\Phi}_{C}(K, t)$, for some $K > 0$ and $0 < t < 1$. On the other hand, by Proposition 8, $L^{\Phi}(\mu) = \mathcal{U}^{\Phi}_{C}(HK, t)$. Hence, by definition, for every $f \in L^{\Phi}(\mu)$, and $\forall n \in \mathbb{N}$

$$\sup_{k \in \mathbb{Z}} \frac{N_\Phi(C^k_{\varphi^{-1}} f)}{N_\Phi(C^{k+n}_{\varphi^{-1}}(f))} \leq HKt^n.$$

Especially for all $n \in \mathbb{N}$, we have $\sup_{k \in \mathbb{Z}} \frac{N_\Phi(C^n_\varphi f)}{N_\Phi(f)} \leq HKt^n$. Hence,

$$\left(\frac{N_\Phi(C^n_\varphi f)}{N_\Phi(f)}\right)^{\frac{1}{n}} \leq (HK)^{\frac{1}{n}} t, \quad \forall n \in \mathbb{N}$$

and so $r(C_\varphi) = \lim_{n \to \infty} \|C^n_\varphi\|^{\frac{1}{n}} \leq t < 1$. Therefore, we obtain the result. Similar to part (1), we can prove (2).

(3) If Condition (2) holds, then by Proposition 4, we have $\chi_w \in \mathcal{U}^{\Phi}_{GH-}(K, t)$ and $C_\varphi(\chi_w) \in \mathcal{U}^{\Phi}_{GH+}(K, t)$, for some $K > 0$ and $0 < t < 1$. On the other hand, by Proposition 8, $\mathcal{U}^{\Phi}_{GH+}(HK, t) = L^{\Phi}_{+}$ and $\mathcal{U}^{\Phi}_{GH-}(HK, t) = L^{\Phi}_{-}$. So, by the definitions of $\mathcal{U}^{\Phi}_{GH+}(HK, t)$ and $\mathcal{U}^{\Phi}_{GH-}(HK, t)$, we have for every $f \in L^{\Phi}_{+}$,

$$\sup_{k \in -\mathbb{N}_0} \frac{N_\Phi(C^{k-n}_{\varphi^{-1}}(f))}{N_\Phi(C^k_{\varphi^{-1}}(f))} \leq HKt^n, \quad \forall n \in \mathbb{N}$$

and for every $g \in L^{\Phi}_{-}$

$$\inf_{k\in\mathbb{N}_0} \frac{N_\Phi(C^k_{\varphi^{-1}}(g))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(g))} \geq HK\frac{1}{t^n}, \quad \forall n \in \mathbb{N}.$$

So, for every $n \in \mathbb{N}$ and $k \in -\mathbb{N}_0$

$$\frac{N_\Phi(C^{k-n}_{\varphi^{-1}}(f))}{N_\Phi(C^{k}_{\varphi^{-1}}(f))} \leq HKt^n$$

and for all $k \in \mathbb{N}_0$

$$\frac{N_\Phi(C^k_{\varphi^{-1}}(g))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(g))} \geq HK\frac{1}{t^n} \Rightarrow \sup_{k\in\mathbb{N}_0} \frac{N_\Phi(C^{k+n}_{\varphi^{-1}}(g))}{N_\Phi(C^{k}_{\varphi^{-1}}(g))} \leq \frac{1}{HK}t^n$$

Hence, by taking $k = 0$ in both cases, we obtain that for all $n \in \mathbb{N}$,

$$\frac{N_\Phi(C^n_{\varphi}(f))}{N_\Phi(f)} \leq HKt^n \quad \text{and} \quad \frac{N_\Phi(C^n_{\varphi^{-1}}(g))}{N_\Phi(g)} \leq \frac{1}{HK}t^n.$$

By these observations, we obtain that $r(C_\varphi|_{L^{\Phi}_{+}}) \leq t < 1$ and $r(C_{\varphi^{-1}}|_{L^{\Phi}_{-}}) \leq t < 1$. This completes the proof. □

Here, we recall the definitions of a factor map [8].

Definition 10. *Let (X, S) and (Y, T) be two linear dynamical systems. We say that T is a factor of S if there exists a linear, continuous, and surjective map $\Pi : X \to Y$ such that $\Pi \circ S = T \circ \Pi$. The map Π is called the factor map. Moreover, we say Π admits a bounded selector if there exists $L > 0$ such that*

$$\forall y \in Y, \ \exists x \in \Pi^{-1}(\{y\}), \quad \text{with} \quad \|x\| \leq L\|y\|.$$

Lemma 1. *Suppose that $(X, \mathcal{F}, \mu, \varphi)$ has bounded distortion, $\Phi \in \Delta'$ and $h_k = \frac{d\mu\circ\varphi^{-k}}{d\mu}$, the Radon–Nikodym derivative of $d\mu \circ \varphi^{-k}$ with respect to μ. Let B_w be the backward shift on Orlicz sequence space $l^\Phi(\mathbb{Z})$ with weights*

$$w_k = \frac{N_\Phi(C^{k-1}_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^{k}_{\varphi^{-1}}(\chi_W))}.$$

Then, B_w is a factor of the map C_φ by a factor map Π admitting a bounded selector.

Proof. By the invertibility of C_φ, we have $0 < \inf_{n\in\mathbb{Z}} |w_n| \leq \sup_{n\in\mathbb{Z}} |w_n| < \infty$, and consequently, we obtain that B_w is invertible. Now, we define the map

$$\Pi : L^\Phi(\mu) \to l^\Phi(\mathbb{Z}), \quad \Pi(f) = \mathbf{x} = \{x_k\}_{k\in\mathbb{Z}},$$

for all $f \in L^\Phi(\mu)$, where $x_k = \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{N_\Phi(\chi_W)} \int_W C^k_\varphi(f)d\mu$. It is obvious that Π is linear and $B_w \circ \Pi = \Pi \circ C_\varphi$. By our assumptions, we have

$$\|h_k|_W\|_\infty \leq \sup_{B\subseteq W, \mu(B)\neq 0} \frac{\mu(\varphi^{-k}(B))}{\mu(B)}.$$

So, there exists $B \subseteq W$ with $\mu(B) \neq 0$ such that $\frac{\mu(\varphi^{-k}(B))}{\mu(B)} \geq \|h_k|_W\|_\infty$. Since $\Phi \in \Delta'$, then we can find $N > 0$ such that

$$\|h_k|_W\|_\infty \leq N \frac{\Phi^{-1}(\frac{1}{\mu(B)})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{-k}(B))})} \leq N \sup_{B\subseteq W, \mu(B)\neq 0} \frac{\Phi^{-1}(\frac{1}{\mu(B)})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{-k}(B))})}, \quad \forall k \in \mathbb{Z},$$

and therefore,

$$\|h_k|_W\|_\infty \leq N \sup_{B\subseteq W, \mu(B)\neq 0} \frac{\Phi^{-1}(\frac{1}{\mu(B)})}{\Phi^{-1}(\frac{1}{\mu(\varphi^{-k}(B))})} = N \sup_{B\subseteq W, \mu(B)\neq 0} \frac{N_\Phi(C^k_\varphi(\chi_B))}{N_\Phi(\chi_B)}, \quad \forall k \in \mathbb{Z}.$$

Moreover, we recall the Jensen's inequality that is $\Phi(\int_X g d\mu) \leq \int_X \Phi(g) d\mu$, for every measurable and integrable function g on X. Now, let $f \in L^\Phi(\mu)$. Then,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \Phi\left(\frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(f).N_\Phi(\chi_W)} \int_W C^k_\varphi(f) d\mu\right) &= \sum_{k\in\mathbb{Z}} \Phi\left(\int_W \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(f).N_\Phi(\chi_W)} C^k_\varphi(f) d\mu\right) \\
&\leq \sum_{k\in\mathbb{Z}} \int_W \Phi\left(\frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(f).N_\Phi(\chi_W)} f\circ\varphi^k\right) d\mu \\
&= \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \Phi\left(\frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(f).N_\Phi(\chi_W)} f\right) h_k d\mu \\
&\leq \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(\chi_W)} \Phi\left(\frac{f}{N_\Phi(f)}\right) \|h_k|_{\varphi^k(W)}\|_\infty d\mu
\end{aligned}$$

Since

$$\|h_k|_W\|_\infty \leq N \sup_{\varphi^k(B), B\subseteq W} \frac{N_\Phi(\chi_B)}{N_\Phi(C^k_{\varphi^{-1}(\chi_B)})},$$

then by using Proposition 1, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \Phi\left(\frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{NH.N_\Phi(f).N_\Phi(\chi_W)} \int_W C^k_\varphi(f) d\mu\right) &\leq \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{H.N_\Phi(\chi_W)} \frac{HN_\Phi(\chi_W)}{N_\Phi(C^k_{\varphi^{-1}(\chi_W)})} \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \\
&= \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \\
&= \int_X \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \leq 1.
\end{aligned}$$

This means that $N_\Phi(\Pi(f)) \leq NHN_\Phi(f)$ and so Π is a bounded linear map. Here, we prove that Π admits a bounded selector. For $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}} \in l^\Phi(\mathbb{Z})$, we show that $N_\Phi(f) \leq N_\Phi(\mathbf{x})$, in which

$$f = \sum_{k\in\mathbb{Z}} \frac{x_k}{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))} C^k_{\varphi^{-1}}(\chi_W).$$

Since $\Phi \in \Delta'$, then we have

$$\begin{aligned}\int_X \Phi(\frac{f}{N_\Phi(\mathbf{x})})d\mu &= \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \Phi(\frac{x_k C^k_{\varphi^{-1}}(\chi_W)}{N_\Phi(\mathbf{x}).N_\Phi(C^k_{\varphi^{-1}}(\chi_W))})d\mu \\ &\leq \sum_{k\in\mathbb{Z}} \int_{\varphi^k(W)} \Phi(\frac{x_k}{N_\Phi(\mathbf{x})})\Phi(\frac{C^k_{\varphi^{-1}}(\chi_W)}{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))})d\mu \\ &= \sum_{k\in\mathbb{Z}} \Phi(\frac{x_k}{N_\Phi(\mathbf{x})}) \int_{\varphi^k(W)} \Phi(\frac{C^k_{\varphi^{-1}}(\chi_W)}{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))})d\mu \\ &\leq \sum_{k\in\mathbb{Z}} \Phi(\frac{x_k}{N_\Phi(\mathbf{x})}) \leq 1.\end{aligned}$$

This implies that $N_\Phi(f) \leq N_\Phi(\mathbf{x})$. It is easy to see that for every $k \in \mathbb{Z}$, $(\Pi(f))_k = x_k$. Therefore, we obtain that $\Pi(f) = \mathbf{x}$. This completes the proof. □

As is known, Orlicz spaces $L^\Phi(\mu)$ are an interpolation of the spaces $L^1(\mu)$ and $L^\infty(\mu)$. Hence, the concepts that are well defined and valid on $L^1(\mu)$ and $L^\infty(\mu)$, are well defined on Orlicz spaces too.

Theorem 2. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system of bounded distortion generated by W and $\Phi \in \Delta'$. If the composition operator C_φ has the shadowing property, then either Condition (2) or (6) and (7) holds.*

Proof. By our assumptions and by Lemma 1, we have that $B_w : l^\Phi(\mathbb{Z}) \to l^\Phi(\mathbb{Z})$ is a factor of C_φ, in which

$$w_k = \frac{N_\Phi(C^{k-1}_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}.$$

Also, as we saw in Lemma 1, the factor map Π admits a bounded selector. Since C_φ has the shadowing property, then by Lemma 4.2.2 of [8], B_w has the shadowing property too. It is easy to see that Theorem 18 of [20] holds for Orlicz sequence spaces $l^\Phi(\mathbb{Z})$. So, we have that at least one of the conditions A, B, C of Theorem 18 of [20] holds. Since

$$w_k w_{k+1} ... w_{k+n} = w_k \frac{N_\Phi(C^k_{\varphi^{-1}}(\chi_W))}{N_\Phi(C^{k+n}_{\varphi^{-1}}(\chi_W))}$$

and also $0 < \inf|w_k| \leq \sup|w_k| < \infty$ (it comes from invertibility of C_φ and $0 < \mu(W) < \infty$), then we easily obtain that the conditions A, B, C of Theorem 18 of [20] imply Conditions (2), (6) and (7), respectively. □

By Theorems 1 and 2, we have the following characterization:

Corollary 1. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system of bounded distortion generated by W. Then, the following are equivalent:*

1. *The composition operator C_φ has the shadowing property;*
2. *One of Conditions (2) or (6) and (7) holds.*

As is known, every generalized hyperbolic operator has the shadowing property. Then, by applying Theorems 1 and 2, we have the following characterization:

Corollary 2. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system of bounded distortion. Then, the following are equivalent:*

1. *The composition operator C_φ is generalized hyperbolic;*
2. *The composition operator C_φ has the shadowing property.*

Finally, we provide an equivalent condition for the composition operator C_φ to have the shadowing property based on Radon–Nikodym derivatives.

Theorem 3. *Let $(X, \mathcal{F}, \mu, \varphi)$ be a dissipative system generated by W, $h_k = \frac{d\mu\circ\varphi^{-1}}{d\mu}$, $m_k = ess\,inf_{x\in W} h_k(x)$, and $M_k = ess\,sup_{x\in W} h_k(x)$. If the sequence $\{\frac{M_k}{m_k}\}_{k\in\mathbb{Z}}$ is bounded, then the following are equivalent:*

1. *The composition operator C_φ has the shadowing property;*
2. *One of the following properties holds:*

$$\overline{\lim}_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{M_k}{m_{k+n}}\right)^{\frac{1}{n}} < 1 \tag{13}$$

$$\underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{Z}} \left(\frac{M_k}{m_{k+n}}\right)^{\frac{1}{n}} > 1 \tag{14}$$

$$\overline{\lim}_{n\to\infty} \sup_{k\in-\mathbb{N}_0} \left(\frac{M_{k-n}}{m_k}\right)^{\frac{1}{n}} < 1 \quad \text{and} \quad \underline{\lim}_{n\to\infty} \inf_{k\in\mathbb{N}_0} \left(\frac{M_k}{m_{k+n}}\right)^{\frac{1}{n}} > 1. \tag{15}$$

Furthermore, Conditions (13) and (14) imply that, under an equivalent norm, C_φ is a proper contraction or a proper dilation, respectively. Condition (15) implies that C_φ is generalized hyperbolic.

Proof. This proof follows Theorem RN from reference [20]. □

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References

1. Abdenur, F.; Díaz, L.J. Pseudo-orbit shadowing in the C1 topology. *Discret. Contin. Dyn. Syst.* **2007**, *17*, 223–245. [CrossRef]
2. Eisenberg, M.; Hedlund, J.H. Expansive automorphisms of Banach spaces. *Pac. J. Math.* **1970**, *34*, 647–656. [CrossRef]
3. Hedlund, J.H. Expansive automorphisms of Banach spaces. II. *Pac. J. Math.* **1971**, *36*, 671–675. [CrossRef]
4. Mazur, M. Hyperbolicity, expansivity and shadowing for the class of normal operators. *Funct. Differ. Equ.* **2000**, *7*, 147–156.
5. Singh, R.K.; Manhas, J.S. *Composition Operators on Function Spaces, North-Holland Mathematics Studies*; North-Holland Publishing Co.: Amsterdam, The Netherlands, 1993; Volume 179.
6. Walters, P. On the pseudo-orbit tracing property and its relationship to stability. In *The Structure of Attractors in Dynamical Systems*; Springer: Berlin/Heidelberg, Germany, 1978; Volume 668, pp. 231–244.
7. Bernardes, N.C., Jr.; Cirilo, P.R.; Darji, U.B.; Messaoudi, A.; Pujals, E.R. Expansivity and shadowing in linear dynamics. *J. Math. Anal. Appl.* **2018**, *461*, 796–816. [CrossRef]
8. D'Aniello, E.; Darji, U.B.; Maiuriello, M. Generalized hyperbolicity and shadowing in L^p spaces. *J. Difer. Equ.* **2021**, *298*, 68–94. [CrossRef]
9. Krasnosel'skii, M.A.; Rutickii, Y.B. *Convex Functions and Orlicz Spaces*; Noordhoff: Groningen, The Netherlands, 1961.
10. Rao, M.M.; Ren, Z.D. *Theory of Orlicz Spaces*; Marcel Dekker: New York, NY, USA, 1991.
11. Akbarbaglu, I.; Azimi, M.R. Universal family of translations on weighted Orlicz spaces. *Positivity* **2022**, *26*, 1. [CrossRef]
12. Azimi, M.R.; Farmani, M. Subspace-supercyclicity of conditional weighted type translations on $L^p(G)$. *Adv. Oper. Theory* **2023**, *8*, 37. [CrossRef]
13. Azimi, M.R.; Jabbarzadeh, M.R. Hypercyclicity of Weighted Composition Operators on L^p-Spaces. *Mediterr. J. Math.* **2022**, *19*, 164. [CrossRef]

14. Cui, Y.; Hudzik, H.; Kumar, R.; Maligranda, L. Composition operators in Olicz spaces. *J. Aust. Math. Soc.* **2004**, *76*, 189–206. [CrossRef]
15. Jabbarzadeh, M.R. The essential norm of a composition operator on Orlicz spaces. *Turk. J. Math.* **2010**, *34*, 537–542. [CrossRef]
16. Jabbarzadeh, M.R.; Haghighatjoo, S. Equivalent metrics on normal composition operators. *Rocky Mt. J. Math.* **2020**, *50*, 989–999. [CrossRef]
17. Bongiorno, D.; D'Aniello, E.; Darji, U.B.; Piazza, L.D. Linear dynamics induced by odometers. *Proc. Am. Math. Soc.* **2022**, *150*, 2823–2837. [CrossRef]
18. Gilmore, C. Linear dynamical systems. *Irish Math. Soc. Bull.* **2020**, *86*, 47–77. [CrossRef]
19. Grosse-Erdmann, K.G.; Peris, A. *Linear Chaos*; Universitext; Springer: London, UK, 2011.
20. Bernardes, N.; Messaoudi, A. Shadowing and structural stability for operators. *Ergod. Theory Dyn. Syst.* **2021**, *41*, 961–980. [CrossRef]

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