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Kant's Account of Mathematical Cognition: The Role and Contribution of Intuition

In this paper I offer an interpretation of the role of intuition in mathematical cognition in Kant's philosophy of mathematics based on two novel and very recent interpretations of Kant's epistemology. I argue, with Lucy Allais, that the primary role of intuition in cognition is in presenting objects and I argue, with Karl Schafer, that the contribution of these intuitions is in providing the concepts involved in the cognition with real possibility and determinate content.[1] While it should be clear that I think both Allais' and Schafer's positions are fundamentally correct, the focus of this paper will not be to defend these positions against all its possible criticisms. Instead, the focus is on what can be inferred from these positions in the specific case of mathematical cognition.

The central concern of Kant's *Critique of Pure Reason* is explaining the possibility of synthetic *a priori* cognition (B19).[2] Kant never really entertains the idea that such cognition is impossible[3] – he claims that all the major judgements[4] of metaphysics, mathematics and pure physics are both synthetic and *a priori* (A10/B14-8). The mystery of synthetic *a priori* cognition is how it can simultaneously be both synthetic and *a priori*.

[1] Here I will draw on Allais' work in her "Kant, Non-Conceptual Content and the Representation of Space," *Journal of the History of Philosophy* 47 (2009): 383-413 and the 9th and 10th chapters of her manuscript on Kant's Transcendental Idealism. I will also draw on Schafer's "Kant's Conception of Cognition and Our Knowledge of Things-in-Themselves," *The Sensible and Intelligible Worlds* (Oxford: Oxford University Press, forthcoming).

[2] I will follow the convention of referencing passages from the *Critique* within the text with 'A' and 'B' referring to the first and second editions respectively. I have used Immanuel Kant, *Critique of Pure Reason*, trans. and ed. Paul Guyer and Allen Wood (Cambridge: Cambridge University Press, 1996). In referencing the *Prolegomena*, I will note only the volume and page numbers in text. Here, I have used Immanuel Kant, *Prolegomena to any future metaphysics that will be able to come forward as a science* in *Theoretical Philosophy After 1781*, trans. & eds. Henry Allison & Peter Heath, trans. Gary Hatfield (Cambridge: Cambridge University Press, 2002).

[3] It is disputed whether he merely claims this or also presents some argument for it. See Paul Guyer, *Kant* (London: Routledge, 2006), 48. Here, I will assume that Kant takes this existence as established in some way.

[4] For Kant, cognition always involves judgement. Kant defines cognition as "objective perception" – which for Kant is the same as objective representation with consciousness (A320/B376). For now, we can understand this as something akin to propositional knowledge, although there are important differences. Importantly, truth might not be a necessary condition for cognition, although it is for knowledge. See Stephen Engstrom, "Understanding and Sensibility," *Inquiry* 49 (2006): 21, note 2.

In his introduction Kant presents synthetic *a priori* judgements as a third way between judgements that are both analytic and *a priori* and those that are both synthetic and *a posteriori*. The distinction between analytic judgements and synthetic judgements is presented as the distinction between "judgements of clarification" (analytic judgements) and "judgements of amplification" (synthetic judgements). (A7/B11) Where analytic judgements are those that are concerned only with bringing to light the content of the concepts being analysed, synthetic judgements go beyond mere conceptual analysis. Furthermore, synthetic judgements can be denied without contradiction, while analytic judgements cannot (A151/B190-1). Kant defines synthesis as "the action of putting different representations together with each other and comprehending their manifoldness in one cognition" (A77/B103).

Kant focuses on two types of representations – those that are immediate and those that are mediate. Kant dubs the former intuitions and the latter concepts. For Kant, intuitions are singular and concepts general. The role of intuition is in giving objects. Furthermore, at least for us humans (perhaps only in the case of empirical intuition), the standard way objects are given is through affecting our senses (A19/B33).[5] The role of concepts is to enable thought about those objects via marks (features) that may be common to several objects (A320/B376). Kant claims further that all cognition requires both concepts and intuitions.

The problem of explaining the possibility of synthetic *a priori* cognition remains the problem of how we can go beyond mere conceptual analysis while ensuring the apriority of the resulting cognition. This problem is particularly perplexing because intuition, an integral component of cognition, only plays its role when it gives objects, which is standardly done through the senses. The problem of the possibility of synthetic *a priori* cognition is then the problem of the possibility of *a priori* intuition – i.e. intuition that gives objects, without this bringing in anything empirical.

[5] For those that find negative definitions useful: "an intuition for Kant is an immediate or non-descriptive, non-discursive or non-conceptual and non-propositional, essentially singular representation of some actual spatial or temporal object, or of the underlying spatial or temporal form of all such objects" Robert Hanna, "Mathematics for Humans: Kant's Philosophy of Arithmetic Revisited," *European Journal of Philosophy* 10 (2002): 331.

In this paper I will address this mystery in the case of mathematical cognition, something, according to Kant, quite different from philosophical cognition. The problem here is that while mathematical cognition is assumed to be necessary and universal, and so *a priori*, it also requires input from intuition in order to qualify as cognition. This suggests that *a priori* intuition must play an important role in such cognition and yet in a way that is different from philosophical cognition.

In section 1, I argue that the primary role Kant attributes to intuition in cognition is that it presents the objects that the cognition is of. This will involve showing that it is in a concept's inability to present objects that concepts are insufficient for full-blown cognition. From this it will follow that Kant's insistence that intuition plays a role in mathematical cognition follows from his claim that mathematical thought qualifies as cognition, and *not*, as is sometimes argued, from his claim that all mathematical judgements are synthetic.

In section 2, I investigate the connection between the intuitivity and syntheticity of mathematical judgements. Some argue that the intuitivity of mathematical judgements implies their syntheticity; others that their syntheticity implies their intuitivity. Here I argue against both claims. Against the latter claim I argue that the syntheticity of mathematical judgements implies a specific role for intuition, and not *that* intuition plays a role (which follows merely from mathematical judgements qualifying as cognition). Against the former claim I argue that the syntheticity of mathematical judgements follows from the content of mathematical concepts.

In section 3, I argue that the primary role of intuition in mathematical cognition is in defining mathematical concepts. The essential contribution of intuition here is in establishing the real possibility of these concepts – that is, that there are objects that fall under these concepts. A secondary, non-essential, contribution is in displaying marks that belong necessarily to these concepts, but are not contained within them, that is, providing the concepts with determinate content. This secondary role is in presenting marks that are necessarily and universally predicable to the concepts, resulting in judgements that are thereby also verified. This will involve drawing heavily on Kant's contrast between both philosophical and mathematical cognition and the philosophical and mathematical methods.

In section 4, I compare and contrast my position with the most prominent existing interpretations of the role of intuition in mathematical cognition. I will discuss both evidentialist interpretations and objectivist interpretations.

The primary purpose of the paper is to present an interpretation of one aspect of Kant's philosophy of mathematics and not an argument that his account of mathematical cognition is correct. While the latter is also an important endeavour, we must first determine which claims Kant argues for, before we can assess whether these claims are true.

Section 1: Concepts, Intuition & Cognition

Kant claims that cognition requires both intuition and concepts. According to Kant, concepts are general, mediate representations of objects that enable us to think about those objects. Concepts do not, and cannot, put us directly in touch with the objects that they represent. Intuitions, on the other hand, are singular, immediate representations that give us objects. Intuitions do not, and cannot, enable any thought about those objects.

> [N]either concepts without intuition corresponding to them in some way nor intuition without concepts can yield cognition. (A50/B74)

> It is thus just as necessary to make the mind's concepts sensible (i.e. to add an object to them in intuition) as it is to make its intuitions understandable (i.e. bring them under concepts). Further these two faculties or capacities cannot exchange their functions...Only from their unification can cognition arise. (A51/B75)

> The genus is **representation** in general (*reprasentatio*). Under it stands the representation with consciousness (*perceptio*). A **perception** that refers to the subject as a modification of its state is a **sensation** (*sensatio*); an objective perception is a **cognition** (*cognitio*). The latter is either an **intuition** or a **concept** (*intuitus vel conceptus*). The former is immediately related to the

> object and is singular; the latter is mediate, by means of a mark, which can be common to several things. (A320/B376-7)

One of the most difficult aspects of the final passage is Kant's claim that intuitions and concepts *on their own* qualify as cognition, contrary to the passages above it that stipulate that cognition arises only from their *coming together*. What is clear from this passage is that both intuitions and concepts are objective representations, that is, representations of objects. This might be classified as cognition in the sense used in the final passage.

The implication then is that cognition in the sense used in the prior passages must be something more than merely representing objects. Taken together with the preliminary definitions of concepts and intuitions, this more stringent sort of cognition can be characterized as thought about objects that are actually given.

> To **cognize** an object, it is required that I be able to prove its possibility (whether by the testimony of experience from its actuality or *a priori* through reason). But I can **think** whatever I like, as long as I do not contradict myself, i.e., as long as my concept is a possible thought, even if I cannot give any assurance whether or not there is a corresponding object somewhere within the sum total of all possibilities. But in order to ascribe objective validity to such a concept (real possibility, for the first sort of possibility was merely logical) something more is required (note, Bxxvi)

> If a cognition is to have objective reality, i.e., to be related to an object, and is to have significance and sense in that object, the object must be able to be given in some way. Without that the concepts are empty, and through them one has, to be sure, thought but not in fact cognized anything through this thinking, but rather merely played with representations. To give an object, if this is not meant only mediately, but it is rather to be exhibited immediately in intuition, is nothing other than to relate its representation to experience (whether this be actual or still possible). (A155-6/B194-5)

From such a characterization, it follows that the primary role of intuition in cognition is to give the objects that the cognition is of.

To see why Kant takes this position we need to consider these technical terms in more detail. In what follows, I discuss Kant's usage of 'concept', then 'intuition', and finally, 'cognition'.

1.1. Concepts

Concepts differ from intuitions in that they relate to their objects only mediately and they do so by means of marks displayed by these objects. Marks are usually understood as something akin to properties of objects by which we can identify those objects as objects of a certain sort. Objects that display the marks contained in a concept fall under that concept.

The notion of such containment is notoriously difficult to capture. Despite common charges of obscurity against Kant[6], many philosophers interpret his conceptual containment as a version of genus-species hierarchies where the predicate-concept is a more general genus-concept and the subject-concept is a lower and more specific instance of the genus-concept.[78] For example, the genus-concept <unmarried>[910] can be divided into the species-concepts <bachelor> and <bachelorette>, where the former is the genus <unmarried> in conjunction with the differentia <man> and the latter with the differentia <woman>. Objects that display both the marks of <man> and <unmarried> are correctly subsumed under the concept <bachelor>.

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[6] Including Quine's very brief "[the] notion of containment…is left at a metaphorical level". William V. Quine, "Two Dogmas of Empiricism, " *From a Logical Point of View: 9 Logico-Philosophical Essays* (Cambridge, MA: Harvard University Press, 1959), 21.

[7] See, for example, Robert Hanna, "Analyticity within the Limits of Cognition Alone," *Kant and the Foundations of Analytic Philosophy* (Oxford Scholarship Online, 2004), accessed May 16, 2013, DOI: 10.1093/acprof:oso/9780199272044.001.0001, R. Lanier Anderson, "It Adds Up After All: Kant's Philosophy of Arithmetic in Light of the Traditional Logic," *Philosophy and Phenomological Research*, 69 (2004): 505-17, R. Lanier Anderson, "The Introduction to the Critique: Framing the Question," in *The Cambridge Companion to Kant's Critique of Pure Reason*, ed. Paul Guyer (Cambridge: Cambridge University Press, 2010), 86-8.

[8] Although presented in terms most appropriate for categorical judgements, I intend what follows to apply to other sorts of judgements too, particularly the hypothetical and disjunctive.

[9] I will follow the custom of indicating concepts by angle brackets (<,>). The words within the brackets do not have any special or essential relation to the particular English word used; rather the intent is to express the concept outside of any particular language.

[10] Bachelors being unmarried will be stressed many more times in this paper. Apologies to those that need no more convincing.

From this interpretation it should be clear that marks are most properly understood as the content of concepts and the objects that display instances of these marks, as the extension of those concepts. Thus, marks can be understood as properties displayed by objects, but also as concepts, or general representations themselves.[11] A bachelor falls under the concept <bachelor> and the concepts <man> and <unmarried>, because it displays instances of these marks contained within all three of the concepts. Yet, concepts are nothing more than a collection of sub-concepts, or marks. There is nothing more to <bachelor> than <unmarried> and <man>.[12]

Such an understanding of containment also helps to explain why Kant thinks that concepts are general representations. Most marks, or properties, can be shared by multiple objects, at least possibly. It may well be true at certain times that only one object displays an instance of the property of not being married, but nothing about the property of being unmarried excludes other objects from displaying an instance of the property as well. The possibility of multiple objects displaying instances of this property remains. Thus, from a concept's mediacy, its generality follows.

Yet, other properties are unitary. A common example is in the concept of a monotheistic god. It is contained within this concept that only one object can display the property of being a monotheistic god. Here being monotheistic is not a property that "can be common to several things". A potential solution to this counterexample to the generality criterion is similar to the solution that can be offered for other unitary properties, such as being a first-born child. A specific child might be the first-born child of a particular couple, but her twin could also have been the first-born, had circumstances been slightly different. Here only one object displays an instance of the particular property, but a *different* object could have displayed an instance of the property instead. As with the sole bachelor above, only one object happens to display the property, but the possibility of another object displaying it is not removed. That is, the concept leaves open which particular object falls under it.[13]

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[11] See Houston Smit, "Kant on Marks and the Immediacy of Intuition," *The Philosophical Review* 109 (2000): 235-266 for a thorough exposition of marks, both as the content of concepts (conceptual marks) and as property instances displayed by objects (intuitive marks, or tropes).

[12] Leaving aside, of course, the limits Kant places on the possibility of fully defining and analysing empirical concepts. I discuss definition further in 3.4.

[13] I do not think that the problem of unitary properties is entirely removed by this strategy. Kant is quite clear that marks can be *shared* by many objects, at least possibly. Such possibility is still removed in

One of Kant's most important claims about the nature of concepts is that no concept can ever guarantee the existence of any of the objects that would fall under it.[14] Certain concepts can guarantee the non-existence of the objects that would fall under them. These concepts would involve an outright contradiction. An object that is both square and non-square, is logically impossible and so, cannot exist. An object that is only square, and not also non-square is logically possible, because the concept of a square object does not contain any outright contradictions. But this is all a concept can do – we can establish logical possibility by means of concepts alone, but we cannot establish actuality. As we will discover later, Kant thinks that concepts alone cannot even establish what he calls real possibility.

A concept cannot guarantee the existence of the objects that fall under it because, as a mere collection of marks, it can entail only those marks contained within it. This is why Kant characterises analytic judgements as merely clarificatory. (A6/B10) Analytic judgements predicate one (or several) of the marks contained in the subject-concept of that subject-concept. Such a judgement is necessarily true precisely because the predicate-concept is entailed by the content of the subject-concept. If concepts entailed anything more than their content, some analytic judgements would be more than mere clarifications of their subject-concepts.

By use of a simple example we can extract an argument for Kant's claim that concepts cannot guarantee their objects from his definitions of analytic judgements and concepts.[15] We concede that concepts are merely collections of marks. We also concede that analytic judgements are necessarily true. In order to guarantee the actuality of its objects, one of the marks of a concept would have to be <existence>. The marks of <bachelor> would then include <unmarried>, <man> and <existence>.

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the case of unitary properties. This is sufficient to undermine Kant's claim that concepts are general. A further option might be to exclude unitary properties as proper marks. In such a case the only marks of a monotheist god would be properties like <omnipotence>, <omniscience> etc., but not <unique> or something similar that excludes the possibility of multiple objects falling under it. I exclude this solution from the main text, because it is not clear to me how one would present it in a way that is not *ad hoc*. Furthermore, because mathematical concepts do not contain unitary marks, the problem is not crucial to the main concerns of this paper. Yet, it should be clear that I do not think the problem should be set aside when trying to understand Kant's account of concepts.

[14] This is the force behind his objections to the ontological argument for the existence of God.

[15] Although this argument is not Kant's, it does follow from his definitions.

But then 'Bachelors exist' would be an analytic, and so, necessary truth. This cannot be. 'Bachelors exist' is quite clearly a contingent truth. Leaving this specific empirical example aside, if concepts could guarantee the existence of their objects, then *all* existential claims concerning objects would be analytic. This is clearly untenable. Thus, at least one of our starting assumptions is false.

1.2 Intuition

> An intuition is a representation of the sort which would depend immediately on the presence of an object. (4: 281)
>
> Everything that is to be given to us as objects must be given to us in intuition. (4:288)

Let us now turn to intuitions. Because Kant thinks that cognition is actual, and so possible, and because concepts cannot guarantee objects, there must be another kind of objective representation that does give us objects. Furthermore, it is because concepts are merely general and mediate representations that they cannot guarantee the objects that fall under them, and so, representations that do give us these objects cannot be general or mediate. Intuitions are thus characterised as singular immediate representations that give objects. The actuality of such representations follows from the actuality of cognition and the impossibility of cognition without such representations.

There has been much debate surrounding the singularity and immediacy criteria in the literature on intuition. Jaako Hintikka argues that the two criteria are equivalent whereas Charles Parson argues that, while immediacy entails singularity, singularity does not entail immediacy.[16] Neither pay much attention to the further criterion that intuitions give objects; rather the presence of an object is roughly assimilated into the immediacy criterion.[17] To see why this third criterion is essential, we must keep in

[16] Jaako Hintikka, "Kant on the Mathematical Method," and Charles Parsons, "Kant's Philosophy of Arithmetic" and "Postscript," in *Kant's Philosophy of Mathematics*, ed. Carl J. Posy (Dordrecht, Boston & London: Kluwer Academic Publishers, 1992), 21-79.
[17] See Parsons, "Kant's Philosophy of Arithmetic," 44.

mind both the nature of concepts explicated above, and where both Hintikka and Parsons' positions fall short.

Hintikka's interpretation of the singularity criterion is that an intuition is singular in that it is a particular instance of a concept. He further argues that all our intuitions are sensible and that Kant shows this in the Transcendental Aesthetic.[18] Kant argues there that the forms of all our intuitions are space and time and thus that all our intuitions would occur in space and time, which we can access only through outer and inner *sense.* Hintikka then concludes that any particular sensible instance of a concept could not but be immediately present. This follows because sensibility operates through the affectation of an object on the senses. Thus, on Hintikka's interpretation of intuition, it is because *our* intuitions are sensible that they involve the presence of their objects, and not that giving objects is an *essential* feature of intuition in general.

Parsons takes issue with Hintikka's interpretation by constructing an example of a singular representation, which is not immediately present, and so not an intuition, but a concept.[19] By doing so, Parsons aims to show that the immediacy criterion cannot be merely assimilated into the singularity criterion as Hintikka suggests. According to Parsons, the concept of a monotheistic god is singular[20], because it can have only one particular instantiation, and yet we can have no immediate access to such an object and so any representation of such a god cannot be an intuition. Thus, Parsons has shown that singularity is not sufficient to indicate intuitivity.

Is immediacy sufficient for intuitivity? If Hintikka is correct in arguing that immediacy follows from the sensibility of intuitions, then immediacy might not be sufficient either. As Hintikka correctly points out, Kant does *not* claim that sensibility is an essential feature of intuition in general.[21] The sensibility of our intuition follows from the nature of our finite faculties. *Our* intuitions are sensible, but this does not preclude non-sensory intuition in other beings – beings whose intuitions are limited by something other than space and time, or not limited at all. But then if sensibility is what entails the immediacy of our intuitions, and intuition is not essentially sensible,

[18] Hintikka, "Kant on the Mathematical Method," 23.
[19] Parsons, "Kant's Philosophy of Arithmetic, " 45.
[20] The singular concept rears its head again. See the previous sub-section, and particularly, note 13
[21] Hintikka, "Kant on the Mathematical Method," 23.

then neither are they essentially immediate. Thus, we retain the possibility of non-immediate intuitions.

Furthermore, while we concentrate almost exclusively on the mediacy of concepts, it is important to note that for concepts to be mediate representations of objects, they must relate immediately to those representations that act as mediators.[22] Granted, concepts must represent their objects mediately; but in order to do so, it seems they must relate to their marks immediately. Further, if it is only through intuitions that concepts can relate to their objects, then at least some of the concept's marks, themselves concepts, must relate immediately to those intuitions.

Thus, we have seen that neither singularity nor immediacy is a sufficient indicator of intuitivity. Certain concepts can be necessarily singularly referring, and all concepts relate immediately to the mediators between the concepts and their objects. All that remains is the feature that intuitions are essentially representations that give objects.

Lucy Allais argues that the primary purpose of intuition is to give objects and that understanding intuition in such a way helps us make sense of the immediacy and singularity criteria in a way that is more loyal to what Kant says about these criteria in the *Critique*.[23] On her account, intuition giving objects should be interpreted as the presentation of objects to consciousness – a sort of direct acquaintance with an object rather than mere sensory affectation by an object. Immediacy should be interpreted as such acquaintance and singularity as being presented with a particular, but unclassified, object.

Understanding intuition in such a way helps to explain why intuition cannot enable thought. For Kant, thought comes in the form of judgements where one concept is predicated of another. Intuitions cannot replace one of these concepts because, on Allais' account, intuitions are unclassified. It is only via concepts that we can describe objects as of a certain kind. We do so by bringing an intuition under a concept via

[22] Smit, "Kant on Marks," 262, note 45
[23] Lucy Allais, "Concept and Intuitions," MS

instances of the marks contained within the concept displayed by the object presented.[24]

On a view such as Hintikka's or Parsons' intuitions can be assimilated with singular *terms*, already classified.[25] In this context, singular terms like 'God', or more damning, definite descriptions and demonstratives like 'the cat in the corner' or 'this', can be used in judgements. This is problematic because the line between intuition and concept is blurred.[26] This runs explicitly against Kant's insistence that intuitions and concepts cannot play one another's role.

A potential problem for Allais' view is that not all intuitions seem to present objects. While Kant refers to space and time as the *a priori* forms of intuition, he also claims that they are themselves intuitions. If all intuitions present their objects, then we need to interpret Kant here as claiming that space and time are themselves objects as well as operating as the forms of intuition in general. This will turn quite heavily on how broad Kant intended the use of 'object'.

> There is therefore only one way possible for my intuition to precede the actuality of the object and occur as an *a priori* cognition, *namely if it contains nothing else except the form of sensibility, which in me as subject precedes all actual impressions through which I am affected by objects*. For I can know *a priori* that the objects of the senses can be intuited only in accordance with this form sensibility. From this it follows: that propositions which relate merely to this form of sensory intuition will be possible and valid for the objects of the senses; also, conversely, that intuitions which are possible *a priori* can never relate to things other than objects of our senses. (4: 282)

Here Kant is concerned with how *a priori* cognition can relate to objects without being grounded on those objects, but it also illuminates one way in which Kant

[24] See Smit "Kant on Marks," 235-266 for an account of how intuitions display instances of marks.
[25] This follows from a conceptualist interpretation of intuition where intuition only becomes intuition once a sensory mush is brought under a concept, and so classified. Allais' account departs from this view in being non-conceptualist. She argues that intuitions are intuitions prior to being brought under concepts.
[26] See Manley Thompson, "Singular Terms and Intuitions in Kant's Epistemology," in *Kant's Philosophy of Mathematics*, ed. Carl J. Posy (Dordrecht, Boston and London: Kluwer Academic Publishers, 1992), 95 on why intuitions cannot have linguistic counterparts of any kind and Kirk Dallas Wilson, "Kant on Intuition," *The Philosophical Quarterly* 25 (1975): 251-2.

thought space and time are given as objects themselves (at least in a broad sense). While space and time are not themselves objects like a cat or a piano, which are bounded, they are necessary aspects of objects in that they present the boundaries. Crucially, just as the object of an empirical intuition can be divided both spatially and temporally, space and time themselves can be divided spatially and temporally.[27] It is precisely because the latter holds that the former does. This is also why whatever holds for space and time, holds for the objects in space and time. To this extent then, space and time are objects just as much as the objects represented in empirical intuition.[28]

1.3 Cognition

So far we have glossed cognition as something more robust than, but essentially, objective representation. In his excellent exposition of 'cognition', Karl Schafer provides two quite specific criteria for cognition: real possibility and determinate identity.[29][30] The former encapsulates the idea that cognition is objective representation of objects of possible experience and the latter encapsulates the idea that cognition is objective representation that is of objects as classified as objects of specific kinds and not of other kinds.

Schafer distinguishes between concepts and intuitions as objective representations, and cognition when they come together as follows:

> [T]he sort of cognition involved in intuitions and concepts - i.e. cognition of an object or objects - is impossible without the sort of cognition involved in judgements - i.e. cognition that these objects possess certain features.[31]

Cognition in the first sense is objective representation in that it is representation of an object, but nothing more. Cognition in the second more robust sense is representation

[27] Wilson, "Kant on Intuition," 256.
[28] See also Smit "Kant on Marks," 240-2 on objects.
[29] Schafer, "Kant's Conception of Cognition," 12-6.
[30] Houston Smit expresses a similar understanding of 'cognition'. See his "Kant on Marks," 242-3.
[31] Schafer, "Kant's Conception of Cognition," 5.

of objects, which includes *that* the objects exist and that they display instances of specific marks, and not others.

Cognition in the first sense need not be constrained by real possibility or determinate identity. On their own, the concepts cannot have real possibility because concepts cannot guarantee any of their objects. On their own, the objects of intuitions in Allais' sense have no determinate identity, because their objects are unclassified. However, when concepts and intuition come together it results in cognition of objects that have both reality and determinate identity – thereby providing the concepts with real possibility and determinate content. The concepts provide the classification via marks and intuition provide the objects displaying instances of these marks thereby guaranteeing the reality of the object the cognition is of.

In order to fully explicate Kant's notion of real possibility we would require a far-reaching detour into both the Transcendental Aesthetic and the Deduction. For our purposes here it is sufficient to note that only objects constrained by space and time, the categories, have real possibility. While the concept <God> contains no contradiction of marks, and so has logical possibility, it has no real possibility because its object would neither be constrained by space and time, nor enter into normal relations of cause and substance. <God> has no real possibility for us[32], because its object is not a possible object of our experience. In Kant's terms, if the object of cognition has reality, the robust cognition of that object would have objective validity. Further, it is only through the intuition that the concepts that apply to the intuitions have real possibility.

Schafer explicates his determinate identity criterion in terms of numerical identity/diversity and qualitative identity/diversity. For an object falling under a concept to have determinate identity we must be able to distinguish it both as an object of one sort rather than another (it must be classified as an object of a certain kind), but we must also be able to distinguish it as a particular object different from another object both of which would fall under the same concept. Furthermore, a

[32] Here I intend 'real' to apply to that which is possible for us to experience. Kant, of course, explores non-theoretical ways in which we can know that God has real possibility in his practical philosophy. Such alternative routes will be left aside in my considerations of real possibility in this paper.

concept would have determinate content if it contains enough marks by which we can include some objects as falling under it and exclude others.

For our purposes here, it is crucial to note that intuition factors in both constraints of cognition. According to Schafer the two constraints are explicated as follows:

> **Real Possibility**: In order to cognize X we must be able to become conscious of X as a real possibility.
> **Determinate Content:** We can only cognize X to the degree that we are able to become conscious of X's determinate identity.[33]

In both cases, cognition requires the consciousness of X, the object. This can only be supplied by intuition. In the first case, intuition provides us with the actuality of the object, and so the real possibility of the concept that represents objects of such a kind, and in the second case, intuition displays instances of marks that enable us to bring the objects under certain concepts, and not others. Furthermore, it is only intuition that can enable us to distinguish between two qualitatively identical objects, that is, detect numerical diversity.

In Schafer's words:

> [I]ntuition is central for Kant's account of theoretical cognition because exhibiting the object of a concept in intuition simultaneously accomplishes two tasks. First, it allows one to establish the real possibility of this object. And, second, it does so in a manner that gives the concept in question the determinacy of content that true cognition of an object requires.[34]

1.4 Summary

This first section can be summarised as follows. For Kant, concepts are essentially mediate representations of objects. From such mediacy, it follows that most concepts are general representations and that concepts cannot give the objects that they refer to. Because cognition is essentially object-directed, and concepts cannot give us those

[33] Schafer, "Kant's Conception of Cognition," 17.
[34] Schafer, "Kant's Conception of Cognition," 19.

objects, cognition requires a further objective representation that does give us the objects that the cognition is of. From this it follows that intuitions essentially give objects. Furthermore, since it is because concepts are mediate that they cannot give objects, intuition, which does give objects, cannot be mediate. Yet, because intuition merely presents its objects unclassified, it cannot enable thought. Thus we find that cognition requires both representations that give us the objects of cognition, and representations that enable us to think about those objects in the form of judgements. This provides the judgement with objective validity – the concepts have real possibility and determinate content. The one role excludes the other, and so, we need both kinds of representation in order to have cognition.

Section 2: Intuitivity and Syntheticity

Two of Kant's most controversial claims concerning mathematical cognition are that "**Mathematical judgements are all synthetic**" (A10/B14) and that mathematical cognition involves the construction of concepts, where construction means, "to exhibit *a priori* the intuition corresponding to [the concept]". (A713/B741) These two claims are also two of the most central of Kant's philosophy of mathematics. Often this leads us to read these two claims as intimately connected and perhaps even as implying one another. In this section, I investigate the relationship between the intuitivity and syntheticity of mathematical propositions. I will start with the claim that the latter implies the former, before turning to the claim that the former implies the later.

2.1. Does Syntheticity Imply Intuitivity?

Where analytic judgements are clarificatory, synthetic judgements are ampliative. (A/B) As explicated above in 1.1, the former means that predicate-concepts are contained as conceptual marks in subject-concepts, and as such analytic judgements merely make explicit in a judgement what is implicit in the subject-concept on its own. Synthetic judgements on the other hand, do not clarify their subject concepts. In this case the predicate concept is not one of the implicit conceptual marks contained in the subject concept. The synthetic concept is amplified, added to, by means of the predicate concept.

> Either the predicate *B* belongs to the subject *A* as something that is (covertly contained in this concept *A*; or *B* lies entirely outside the concept *A*, though to be sure it stands in connection with it. In the first case, I call the judgement analytic, in the second synthetic. Analytic judgements (affirmative ones) are thus those in which the connection of the predicate is thought through identity, but those in which this connection is thought without identity are to be called synthetic judgements. One could also call the former judgements of clarification and the latter judgements of amplification, since through the predicate the former do not add anything to the concept of the subject, but only break it up by means of analysis into its component concepts, which were already thought in it (though confusedly); while the latter, on the contrary, add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis (A6-7/B10-11)

Strictly speaking then the distinction between analytic and synthetic judgements is about the relationship between the subject- and predicate-concepts, and nothing more. However, the distinction is usually taken as also involving truth – the verification of the judgement. Analytic truths are true in virtue of merely the content of the concepts. The verification of synthetic judgements, however, requires something more. As Kant is fond of saying, we must "go beyond the concepts" (4: 272, B15, A25/B40, A718/B746), the implication[35] being that synthetic judgements require intuition for their verification.

> 1) that through analytic judgements our cognition is not amplified at all, but rather the concept, which I already have, is set out, and made intelligible to me; 2) that in synthetic judgements I must have in addition to the concept of the subject something else (*X*) on which the understanding depends in cognizing a predicate that does not lie in that concepts as nevertheless belonging to it (A7-8/B11)

So, if a judgement is synthetic (and true), then it involves intuition. Yet, does intuitivity really follow from syntheticity? If the synthetic judgement is true (really

[35] Along with the claim that the intuition/concept distinction is exhaustive.

true, of real objects in the world), then the judgement qualifies as cognition. As has already been established, if a judgement qualifies as cognition, then both concepts and intuitions must be involved; otherwise, it would not be cognition. Intuitivity seems to follow directly from the judgement being true of objects in the world, regardless of the relationship between the concepts of the judgement. Thus, even though intuitions are involved in synthetic judgements elevated to the level of cognition, the syntheticity of such judgements does not imply *that* intuitions are involved, their being cognition does.

Why then does Kant insist that we must go beyond the mere concepts specifically in synthetic judgements? My suggestion is that intuitions play the same role in analytic and synthetic judgements that qualify as cognition in that, in both cases, the intuition provides us with the objects the judgements apply to. Thus, as already argued above, intuition provides real possibility and determinate identity. Intuition provides the objective validity of *both* analytic and synthetic judgements.

The implication of this suggestion is controversial. I am, in effect, suggesting the reality of analytic judgements being elevated to the level of cognition without the judgements thereby becoming synthetic. The motivation behind making such a claim is in considering the differences between two analytic judgements such as 'All monads are simple' and 'All bachelors are unmarried'. The first major difference is of course that <bachelor> is an empirical concept while <monad> is, arguably, a non-empirical concept.[36] The second major difference is that we can have cognition of bachelors, but not of monads. This follows from intuition presenting us with bachelor-objects, but not with monad-objects.

This is the point at which I diverge from a perhaps more standard way of reading Kant. I claim that even an analytic judgement about bachelors can be elevated to the level of cognition. Following Schafer, cognition has two requirements – that the concepts have real possibility and determinate content, which is provided by intuition by displaying the reality and features of the objects. <bachelor> has real possibility

[36] Although it is neither a category nor a mathematical concept. Yet, this might be one of those concepts, like <god>, that pure reason directs us towards when not kept in line with the boundaries of cognition.

because bachelors are objects of possible intuitions; monads are not. However, the concepts of cognition have determinate content only to the extent that their objects have determinate identity. This means that we need marks by which to distinguish objects that fall under the concept from objects that do not. <unmarried> is one of these marks. Via a display of an instance of this mark, we can start to distinguish between a bachelor and a non-bachelor. Thus, we have the beginnings of a concept with determinate content, even if it were the only mark that had been derived from the analysis of <bachelor>.[37] While intuition has not provided the mark itself, it remains crucial in enabling the differentiation between different objects of the same sort and objects of different sorts. Furthermore, while we can do the same thing with <monad> and <simple>, an analytic judgement involving these two predicates would fail to qualify as cognition because it does not meet the first criterion of cognition.[38] An analytic judgement involving <bachelor> and <unmarried> however does.

Yet, this does not imply that I think the role and contribution of intuition in cognition is entirely the same when it comes to analytic and synthetic judgements. The difference, I suggest, is in the forming of the judgement in the first place. In the case of analytic judgements, the judgement can be formed merely by analyzing the subject concept. Intuition comes in only after the judgement has been formed in order to establish the judgement's objective validity. In synthetic judgements however, the subject concept is not analyzed, rather it is put together with another concept – the subject and predicate concepts are synthesized. The special role of intuition in synthetic judgements is in *enabling* such synthesis. It does so by presenting an object that displays instances of marks contained in two unconnected concepts.

Thus, an analytic judgement's truth can be assessed prior to and independently of its objective validity, but a synthetic judgement's truth and objective validity go hand in hand. In assessing the one, we are assessing the other. The truth of analytic judgements depends on their concepts; their objective validity on the objects they apply to. Both the truth of synthetic judgements and their objective validity depend on the objects they apply to.

[37] See Schafer, "Kant's Conception of Cognition," 24-8 on incomplete analysis and approaching full determination of the content of concepts.
[38] This suggests, as will be expanded upon later, that demonstrating real possibility is a more essential role for intuition, than presenting instances of marks.

2.2 Does Intuitivity Imply Syntheticity?

It should already be clear from the previous sub-section, that this implication is also too simplistic a formulation. That intuition plays a role in a judgement does not imply that the judgement is synthetic. However, if that intuition plays a role in bringing the two concepts of the judgement together in the first place and the truth of the judgement can be assessed *only* via the presentation of some of the objects it applies to, *then* we infer the syntheticity of the judgement. In general, however, it is much simpler to infer the syntheticity of a judgement by considering whether the predicate-concept is contained in the subject-concept.

2.3. Taking Stock

This is what we have established so far. The primary role of intuition in cognition is to present objects. This contribution amounts to providing the concepts involved in the judgement with real possibility and determinate content. That is, intuition is what allows a judgement to be objectively valid. In the specific case of synthetic judgements, objects presented by intuition enable previously unconnected concepts to be connected while simultaneously providing the verification of the resulting judgement.

Section 3 – Mathematical Cognition

Let us then finally turn to the specific case of mathematical cognition. When Kant writes about mathematical cognition, and the mathematical method, it is usually used as a mere illustrative contrast with philosophy's cognition and method. Although he posits various similarities between mathematics and philosophy, his focus is on separating philosophy from mathematics. These are the highlights: Philosophical and mathematical cognition are both *a priori*, their judgements are synthetic and finally, their methods are analytic. The crucial difference is that where philosophical concepts are given, mathematical concepts are made and only mathematics makes use of axioms and demonstrations.

3.1. Apriority

Kant distinguishes between two ways in which a judgement may be said to be *a priori*. In the first case, the concepts of the judgement are *a priori* (or pure) *and* the judgement itself is *a priori*:

> Among *a priori* cognitions, however, those are called **pure** with which nothing empirical is intermixed (B3)

In the second case, the concepts are empirical, but the judgement is still *a priori*. An example of the latter is, once again, 'All bachelors are unmarried'. This judgement is *a priori* because the justification of the judgement does not require any particular experience. The formation of the concept <bachelor>, however requires experience. Putting together <man> and <unmarried> is grounded on empirical objects that display instances of both these marks. Such formation is mostly a pragmatic affair. We have no need for the concept of an unmarried object that is purple, although if we had a sufficient number of objects that we needed to predicate something of, we could create such a concept. In that case too, a purachelor (= <unmarried> + <purple>) would be an empirical concept even though 'All purachelors are unmarried' would be an *a priori* judgement. Notice that the formation of the concept would require objects, but the justification of the judgement does not require any object. Furthermore, the cognition (constituted by the judgement) would require intuition in order to confer objective validity on the judgement and so, real possibility and determinate identity on the object(s) that the judgement applies to, that is, what the cognition is of.

For Kant, the only concepts that are properly *a priori* are the categories and all mathematical concepts. All other concepts are grounded on (possibly) experienced objects. While these pure concepts are not derived from experience, it is crucial that they apply to objects of possible experience. In the case of the categories, they operate as the conditions of all possible experience, and in the case of mathematical concepts, they apply to the objects of possible experience by describing the nature of the forms of our sensibility – space and time.

Kant thinks that mathematical judgements are *a priori* because they are both necessary and universal – two conclusive indicators of apriority.

> [P]roperly mathematical propositions are always *a priori* and not empirical judgements, because they carry necessity with them, which cannot be taken from experience. (4:268)

> Necessity and strict universality are therefore secure indications of an *a priori* cognition, and also belong together inseparably. (B4)

Especially since Saul Kripke's exploration of necessity, Kant's claim that these two criteria are indicators of apriority has become dubious at best. That gold has atomic number 79 is both a necessary and universal truth about the universe, and yet seems an *a posteriori* truth, precisely because it is justified by experience and not from the concept <gold> alone. However, it should be noted that this merely follows from our concept of gold. If <gold> included the marks of <atomic number 79> the seemingly *a posteriori* claim would turn out analytic, and so, *a priori*. In this case, <gold> would remain an empirical concept, but the judgement would be *a priori*.[39]

Mathematical judgements are *a priori* in the first sense. Their justification is independent of any particular experience and the concepts are *a priori*. On its own, the possibility of cognition constituted by such judgements is not particularly problematic. Indeed, the apriority of (non-applied) mathematical judgements and concepts is taken as a common-sense basic assumption of mathematics. The possibility of mathematics becomes mysterious once it is conceded that mathematical judgements and the formation of their concepts are synthetic. It is these two syntheticity claims that will occupy us for the remainder of the section.

3.2. Syntheticity

[39] This is not meant to justify Kant's characterisation of apriority as equivalent to necessity and universality. To establish all of Kant's assumptions is simply too large a project for a paper of any reasonable length. Here, I intend merely to suggest a way in which we can set the problem aside as it pertains to my aims in this paper.

That mathematical judgements are synthetic is a surprisingly easy claim to establish. According to Kant, mathematical concepts are quite simple compared to empirical concepts because they contain very few marks. The concept of a triangle contains at most <shape>, <three> and <straight line>, or <shape>, <three> and <angle>. That the sum of its internal angles is equal to 180° (or two-right angles) is not included. Thus from the content of mathematical concepts and the definition of 'synthetic' it follows that any judgement that predicates the sum of its internal angles being equal to 180° to triangles is synthetic. This follows for all mathematical judgements. <the sum of 5 and 7> contains only <5>, <7> and <addition>, and not also <12>.

The only problematic claim Kant makes concerning the syntheticity of mathematical judgements is that seemingly analytic judgements like 'a = a', the whole is equal to itself, and '(a + b) > a', the whole is greater than the part, are also, in a way, synthetic. Here it is crucial to note that Kant is concerned almost exclusively with mathematical cognition, and not judgements used in mathematics in general. On their own, these two seemingly analytic judgements are not classified as cognition, because without any corresponding intuitions, they do not have any objective validity.

Yet, as established above, even when intuition comes in to confer such objective validity, the judgement does not thereby become synthetic, no more than 'All bachelors are unmarried' becomes synthetic once we have intuitions of bachelors.[40] In order to be synthetic, intuition is what must enable the synthesis of the predicate concept and the subject concept in the first place and what verifies the truth of the resulting judgement. 'a = a' is verified from mere concepts alone, as is '(a + b) > a', because <a> is contained in both subject-concepts. That <a> applies to any objects requires intuition, but the truth of the judgement does not depend on the intuition.

Let us then reconsider what Kant says about these judgements.

> [A] few principles that the geometers presuppose are **actually analytic** and rest on the principle of contradiction; but they also only serve, as identical propositions, for the chain of method, and not as principles, e.g., *a* = *a*, the

[40] Thus we have a textual, albeit indirect, justification for my claim that even analytic judgements can be elevated to the level of cognition without thereby becoming synthetic.

> whole is equal to itself, or $(a + b) > a$, i.e., the whole is greater than the part. And yet even these, **although they are valid in accordance with mere concepts, are admitted in mathematics only because they can be exhibited in intuition**. (B16-7, emphasis my own)

Here it is clear that Kant thought the judgements themselves are analytic, but that they are of use in mathematics only because they are of use in construction (exhibition in intuition). By being exhibited in intuition the subject concept is shown to have both real possibility and determinate content – that is, (in the case of 'a = a') it displays the reality of an object falling under the concept <a>, and it displays the marks of being equal to itself thereby providing the object with determinate identity. Thus, these analytic judgements are true by virtue of their concepts alone, but they are objectively valid only insofar as they are exhibited in intuition. This is required because Kant thinks of mathematics as objectively valid cognition (that is, *of objects*), and not merely a collection of true judgements.

Furthermore, this passage indicates that Kant did not consider these judgements as mathematical; rather they are merely of use in establishing properly mathematical judgements. They do not qualify as mathematical principles. That is, while these judgements can constitute cognition, they are not part of the body of mathematical cognition. It is only properly mathematical judgements that Kant claims are synthetic.

3.3. Method

Before we turn to the construction of mathematical concepts let us consider one way in which Kant did not think mathematics is synthetic. Although he is not always consistent on this point, I will interpret Kant as claiming that the method of mathematical inference is indeed analytic, that is, in mathematics we investigate the conditions under which mathematical judgements are true.

> The analytic method, insofar as it is opposed to the synthetic, is something completely different from a collection of analytic propositions; it signifies only that one proceeds from that which is sought as if it were given, and ascends to the conditions under which alone it is possible. In this method one

> often uses nothing but synthetic propositions, **as mathematical analysis exemplifies**, and it might better be called the *regressive* method to distinguish it from the synthetic or *progressive* method. (note, 4: 276, my emphasis)

> [*P*]*ure mathematics* and *pure natural science*; for only these can present objects to us in intuition, and consequently, if they happen to contain an *a priori* cognition, can show its truth or correspondence with the object *in concerto*, i.e., *its actuality*, **from which one could then proceed along the analytic path to the ground of its possibility**. (4: 279, my emphasis)

Here it is clear that Kant thinks that mathematical inference involves demonstrating the possibility of mathematical concepts in intuition and then that we investigate the conditions under which these concepts are possible – that is, under which conditions these judgements qualify as cognition.

However, Kant also claims that mathematical judgements arise synthetically, and not through the analysis of concepts.

> Because pure mathematical cognition, in its propositions, must therefore go beyond the concept to that which is contained in the intuition corresponding to it, its propositions can and must never arise through the analysis of concepts, i.e. analytically, and so are one and all synthetic. (4: 272)

In order to reconcile these seemingly contradictory passages, we must note that Kant thinks we initiate the process of mathematical inference synthetically, that is through the definition of mathematical concepts, but then that the method of inference is analytic; that is, we investigate the conditions under which the concepts have real possibility. It is crucial to note that both the judgements that define mathematical concepts and the judgements resulting from mathematical inference are synthetic. Yet, this leaves open the possibility that the inference itself is analytic in its method. So, the formation of the judgements is never analytic in the sense of analyzing the relevant concepts, but the method is analytic, in the sense that we take the concepts, once made, as given and then investigate the conditions under which these concepts have real possibility.

Kant never claims that the analytic/synthetic distinction as it applies to judgements is the same as the distinction as it applies to methods. The former applies to the relation between subject and predicate concepts. The latter applies to the direction in which the investigation proceeds. The synthetic method proceeds from conditions to consequences; the analytic from consequences to conditions.

It is important also to note that Kant often takes the time to ensure that we do not confuse the analyticity of the mathematical method with the syntheticity of its judgements, as at (4: 272) above, and again at (4:283):

> Now space and time are the intuitions upon which pure mathematics bases all its cognitions and judgements, which come forward as at once apodictic and necessary; for mathematics must first exhibit all its concepts in intuition – and pure mathematics in pure intuition – i.e., it must first construct them, failing which (since **mathematics cannot proceed analytically, namely through the analysis of concepts, but only synthetically**) it is impossible for it to advance a single step, that is, as long as it lacks pure intuition, in which alone the material for synthetic judgements *a priori* can be given. (4: 283, my emphasis)

3.4. Construction

Let us then finally turn to what has been implicit in much of the sections above – the definition of mathematical concepts. We have already established that mathematical judgements are synthetic only on the condition that the predicate concepts are not contained in the subject concepts and that intuition is involved in the very formation of mathematical concepts as well as crucial to the verification of those judgements. All this turns on what Kant claims about the definition of mathematical concepts.

> [A]ll mathematical cognition has this distinguishing feature, that it must present its concept **beforehand** *in intuition* and indeed *a priori*, consequently in an intuition that is not empirical but pure, **without which means it cannot take a single step**; therefore its judgements are always *intuitive*, in the place of which philosophy can content itself with *discursive* judgements *from mere*

> *concepts*, and can indeed exemplify its apodictic teaching through intuition but can never **derive them from it**. (4: 281)

> [T]hat philosophical definitions come about only as expositions of given concepts, but mathematical ones as constructions of concepts that are originally made, thus the former come about only analytically through analysis (the completeness of which is never apodictically certain), while the latter come about synthetically, and therefore **make** the concept itself, while the former only **explain** it. (A730/A758)

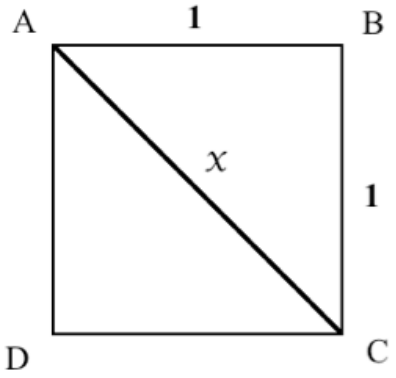
From these passages, it is clear that Kant thinks intuition is involved from the very beginning of mathematical inference and that the judgements of mathematics are derived from intuition. Together with the concession that mathematical inference must start with concepts, before the conditions of the real possibility of those concepts can be established, this passage gestures towards the idea that the very definition of mathematical concepts depends on intuition. It is my suggestion that the first, and primary, role of construction, that is, exhibition in intuition, in mathematical cognition is in the definition of mathematical concepts. The second, and indeed secondary, role is in providing new predicates that are not contained in the mathematical concept, but that do belong to it necessarily.

3.4.1. Definition

The concept of a triangle is first formed by putting together the concepts <three>, <shape> and <straight line> (or <angle>). Such a concept is logically possible, because it contains no contradictions, but so is the concept of a uniangle (<one>, <shape> and <straight line> (or <angle>)). The real possibility of this concept is demonstrated only once it is exhibited in intuition. Because we cannot demonstrate a uniangle in intuition, it is not admitted as a concept of mathematical cognition (see A47-8/B65).

So too with numerical concepts such as <five>. We can say that the definition of this concept is <1+4>, expanded to <1+1+1+1+1>, that is, <one>, <one>, <one>, <one>, <one> and <addition>. While such a concept is logically possible, its real possibility

is demonstrated only in intuition. Even the real possibility of concepts such as <√2> can only be demonstrated by drawing the diagonal of a unit square (see Fig.1 below). Similarly, the real possibility of <pi> can be demonstrated by drawing the relationship between the lengths of the circumference and diagonal of a circle.



(Fig. 1)

If both AB and BC are one unit long, we can establish, from the Pythagorean Theorem, the length of AC, x, as √2 units.

√(AB + BC) = √AC (*Pythag*)
√(AB + BC)
= √(1+1)
= √2
∴ √AC = √2

The concepts of non-real numbers such as √(-2), however cannot be demonstrated in intuition. Because of this inability, we can neither demonstrate its real possibility nor its determinate content. In the first case, we can neither find nor construct an object that would fall under the concept. In the second case, we cannot establish which concepts we could predicate of this really impossible concept. We cannot judge whether it is even or odd, whether it is negative or positive, and so on. All that we can say about this number is that it is non-real.

Such definition should not be read as a way in which Kant becomes a realist about mathematical objects, in the Platonic sense. By constructing a triangle in intuition, we demonstrate the real possibility of the concept <triangle>, yet the particular triangle in intuition is not an ideal triangle. It contains all sorts of contingent marks that have nothing necessarily to do with <triangle>. All that we do in demonstrating real possibility is demonstrating that <triangle> applies to objects in the world. No further ideal object is posited, created or recollected.

3.4.2. Additional Predicates

The idea that exhibition in intuition provides predicates for mathematical concepts not contained within those concepts is difficult. It is however clear that Kant thought that this is part of what intuition provides in construction:

> [F]or just as empirical intuition makes possible for us, without difficulty, **to amplify (synthetically in experience) the concept** we form of an object of intuition through **new predicates that are presented by the intuition itself**, so too will pure intuition do the same only with this difference: that in the latter case the synthetic judgement will be *a priori* certain and apodictic, but in the former only *a posteriori* and empirically certain, because the former contains only what is met with in contingent empirical intuition, while the latter contains what is necessarily must be met with in pure intuition[.] (4:281, my emphasis)

The first thing to note is that this secondary role of construction – providing predicates – occurs only after the specific concept has been defined. Once the concept has been constructed further predicates can then be read off the constructed intuition. A simple example is that of a triangle, which can be constructed from the concepts <three>, <shape> and <straight line>. A further predicate that can be found in the intuition is that a triangle also has three angles, that is, the intuition also displays an instance of the mark <angle>. A more complicated predicate that can be read of is that the internal angles add up to 180°. These predicates cannot be inferred from the concept alone, because it has already been stipulated that the content of the initial concept is exhausted by <three>, <straight line> and <shape>.

In the case of arithmetical sums, the concept of <5 + 7> can be constructed by putting together an intuition of <5> with <7>. The resulting intuition then also displays the mark <12>. Further, more complicated, predicates that can be discovered are <even> and 3 instances of 4, or <3 x 4>. Here, the concept of <5 + 7> has already been stipulated as <7>, <5> and <addition>, so these further predicates cannot be inferred from this collection of marks alone.

As Kant puts it:

> Since the concept is first given through the definition, it contains just that which the definition would think through it (A731/B759)

Put in another way, the concepts <7 + 5>, <12> and <3 x 4> all have the same reference (or extension), but not the same sense (or intension).

The second thing to note is the emphasis Kant places in the first passage on the difference between the predicates that empirical intuitions provide and those that *a priori* intuition provides. In the former case, intuitions display instances of marks not contained in the concepts that the object falls under, but this does not necessitate that all objects under that concept must display these marks. Even if all observed bachelors were taller than married men, the opposite possibility remains. In the case of *a priori* intuition, Kant claims that the instances of the marks displayed by particular intuitions of mathematical concepts must be displayed by all other intuitions that fall under that concept as well. Not only particular triangles, but *all* triangles are three-angled, and their internal angles all add up to 180°.

The easiest way to make sense of this difference is by way of Kant's distinction between *a priori* and empirical intuitions. The latter are contingent. The empirical intuition of a tall bachelor is possible, but need not ever have occurred. *A priori* intuitions, however, carry necessity and universality with them both because they are *a priori*, and because they constitute the necessary and universal structure of our sensibility. Regardless of any intuition's particular content, it will necessarily and universally be structured by the limits of space and time. Because mathematical cognition is *a priori* and because it requires intuition, the only intuitions that can play a role are *a priori*.

This seems dangerously close to a circular argument. Mathematical cognition is *a priori* (necessary and universal), therefore it relies on *a priori* intuition. Mathematical cognition relies on *a priori* intuition, therefore the cognition itself is *a priori*. Yet, this is a misrepresentation. Kant's concern is in explaining the possibility of mathematical cognition which he assumes is necessary and universal, and so, *a priori*. The only possible explanation for this possibility is *a priori* intuition. Thus, while the first half

of the circle holds, the second does not. Kant never argues for the apriority, necessity and universality of mathematical cognition; he assumes it.

Yet this gets us no closer to understanding how marks, not contained in the constructed concept, can be inferred from the intuition in a way that ensures that those marks hold for all objects that would fall under that concept. In an effort to do this we need to turn to Kant's notion of schemata.

> For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. Now this is not possible in any way but by determining my object in accordance with the conditions of either empirical or pure intuition...I put together in pure intuition, just as in an empirical one, the manifold that belongs to the schema of a triangle in general and thus to its concept, through which general propositions must be constructed. (A718/B746)

> Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept. (A140/B179-80)

> [I]t is not images of objects but schemata that ground our pure sensible concepts. No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc. but would always be limited to one part of this sphere. **The schema of the triangle can never exist anywhere except in thought, and signifies a rule of the synthesis of the imagination with regard to pure shapes in space**. (A140-1/B180, my emphasis)

In these passages Kant makes clear that it is not the *particular* intuition from which we can derive predicates that belong to the constructed concept, but from the schema of this concept combined with the *a priori* forms. While our concepts are not in themselves limited by space and time, the objects that fall under them are. This is one way in which predicates can belong necessarily to concepts without being contained

within them: namely spatial and temporal marks[41] that are not contained within the concepts and yet the object(s) that fall under these concepts necessarily display instance of these marks.

Schemata are most generally characterised as a third thing between concepts and intuitions that enable these two sorts of representations to come together. (A138/B177) In particular, a schema is a "representation of a general procedure of the imagination for providing a concept with its image". (A140/B179-80) The schema of a concept is derived from the content of that concept and so guides the production of the intuition of the object that falls under the concept, but, because intuition is limited by the pure forms of space and time, the schema too is limited by these constraints. So, in bringing together the conceptual and the intuitive representations of an object, the schema is constrained by both the pure concepts (the categories and mathematical concepts) and the pure intuitions of space and time.

So, during the construction of a mathematical concept the construction is guided by the concept's schema, but limited by the pure forms of space and time. This suggests that, in order to discover predicates that necessarily belong to concepts not contained in the concepts, we must turn to the schemata, rather than the particular intuitions (which would contain many empirical, and so, contingent, marks).

While Kant is clear that the additional predicates are discovered in the construct*ing* rather than the construct*ed*, it is not clear how this discovery occurs. A delicate balance must be struck between discovering predicates, or marks, outside of the concept, while inferring "only what we ourselves have put into" the objects under those concepts (Bxviii). According to Kant, space and time are the only non-conceptual content that we put into objects ourselves. This is why objects are only appearances rather than things in themselves as well as why neither space nor time are empirical, contingent, marks of objects, but necessary and universal.

[41] Here it is important to recall that Kant never claims that space and time are only *a priori* intuitions. Space and time are also represented by concepts, and so <space> and <time> can constitute marks contained in other concepts. The argument of the Transcendental Aesthetic is that it is merely our *initial* representations of space and time that are *a priori* intuitions.

From space and time certain marks can be inferred. That both triangles and squares enclose areas is evident from <shape>, contained in both <triangle> and <square>, but that the sum of the internal angles of the former is half that of the latter is inferred from their both being limited by the same spatial constraints. In general, objects falling under specific mathematical concepts, must all display certain spatial and temporal properties – they must be infinitely divisible (continuous), they must have both spatial and temporal boundaries, they must be related to one another both spatially and temporally, and they must be spatially and temporally located.

While objects, which are given in intuition, are necessarily constrained by space and time, intuition displays instances of these marks, whereas concepts, not constrained by space and time, need not contain such marks. Because space and time are pure intuitions, they do not display any marks other than these marks necessarily displayed by all objects given by intuition. This is why the predicates provided in intuition belong to the constructed mathematical concepts necessarily and universally. The particular empirical objects that are drawn will display all sorts of instances of other contingent marks, but those displayed by pure space and time are the only ones that we take notice of in the constructing, because those are the only limits placed on the schemata and so, the constructing. So, in being constrained by space and time, but guided by the conceptual content, schemata provide us with additional predicates not contained in the concepts as marks, but exclude the instances of the contingent marks displayed in the constructed objects.[42]

Furthermore, precisely because these discovered predicates are necessary and universal properties of all objects falling under the constructed concept, the judgements including these predicates present the conditions under which it is possible for objects to fall under that concept. That is, despite being synthetic judgements, the method of inference that lead to these judgements is analytic (*not* in the sense of analysing concepts) rather than synthetic.

[42] Thus, we find a general solution to some of the problems that have been levied against Kant. I have in mind here, in particular, Philip Kitcher's argument that Kant's account is circular in that we need to be able to distinguish between spatial and accidental properties in order to distinguish between spatial and accidental properties. Kitcher also argues that Kant's position is circular in other ways. See his "Kant and the Foundations of Mathematics," in *Kant's Philosophy of Mathematics*, ed. Carl J. Posy (Dordrecht, Boston and London: Kluwer Academic Publishers, 1992), 123-9 and his Philip Kitcher, *The Nature of Mathematical Knowledge* (Oxford: Oxford University Press, 1984), 51-7.

3.4.3. Definition and Predicates

One concern with distinguishing the definition phase from the discovering predicate phase is that, once marks are discovered, they can be included in new definitions. Once we have constructed <triangle> and discovered that its internal angles are equal to 180°, we can include it as a conceptual mark of <triangle>. Now, the judgement 'All triangles have internal angles with the sum of 180°' could turn out analytic. The crucial thing to note here is that it is only during the exhibition of the triangle in intuition that this predicate was discovered. Exhibition in intuition was both involved in the formation of the new concept <triangle> and thereby was also required to verify the supposedly now analytic judgement in the first place. Yet once formed, it can be verified from concepts alone. Construction would still be required for its first role, conferring real possibility on the concept, but it is no longer required for its second role, discovering new predicates that apply to the concept necessarily and universally.[43]

This is why I claim that demonstrating the real possibility of defined concepts is the primary, indeed essential, contribution of construction, and that facilitating the presenting of new predicates is merely a secondary, non-essential, contribution. This is in line with the primary role of intuition above – intuitions give objects, and only thereby present instances of marks, not all of which we need to take note of.

3.5. Axioms and Demonstrations

Of course, Kant presents his contrast between philosophy and mathematics not merely in terms of definition, but also in terms of axioms and demonstrations, both of which Kant claims are features of mathematics, but not philosophy.

According to Kant, axioms are synthetic *a priori* intuitive principles in the sense that the concepts are immediately connected, *not* via some further representation. (A732/B760-1) While discursive principles are also synthetic and *a priori*, they are

[43] Of course, there remain *other* concepts that are predicable to the redefined <triangle> for which intuition would be required to play its secondary role once more.

not immediate, because they require a further representation through which the principle can be formed. Kant claims that axioms are possible in mathematics because the concepts are immediately connected by being constructed in intuition and yet not via the intuition. Thus we have a further indication that what matters in construction is not the particular intuition, but merely the content of the concepts and the limitations that are placed on objects that fall under these concepts. We do not connect the subject and predicate concepts via an intuition that displays instances of marks contained in both (as is the case in synthetic *a posteriori* judgement and my general account of syntheticity above in 2.1. and 2.2.), we connect the two concepts immediately. The intuition serves merely to confer real possibility on the synthesis of these two concepts. Furthermore, we broaden the determinate identity of the object falling under the subject concept by having access to another conceptual mark by which to identify that object as an object of a certain sort.

A demonstration, which Kant characterises as "an apodictic proof, insofar as it is intuitive" (A734/B762), is possible only in mathematics precisely because the way in which we turn to intuition does not thereby remove the resulting judgement's necessity and universality. Thus, a demonstration in this sense is just a construction of a mathematical concept as explicated above.

3.6. Summary

Mathematics is synthetic in its definitions and resulting judgements. The mathematical concepts themselves do not need to be analyzed as in philosophy, because they are made, and so given as fully explicated. Mathematics is analytic in that it investigates the conditions under which these concepts are really possible. In doing so we provide the concepts with determinate content – that is, predicates by which we can distinguish objects that nevertheless necessarily, yet merely, belong to the concepts, but are not contained within them as originally defined.

The role of intuition is to present objects that fall under these made concepts, without thereby deriving anything from the particular intuition. We discover additional predicates that belong necessarily, and yet merely synthetically, to these concepts by taking note of the limitations that are placed on objects that fall under the concept.

Unlike synthetic *a posteriori* judgements, the additional predicates are not derived from the particular intuition of the object. In the case of mathematical cognition, the contribution of intuition is presenting an object that provides the concept with real possibility, but it also provides specific limitations on any real possible object that falls under the concept, which can be interpreted as additional concepts that must be predicated of these concepts without necessarily being contained in those concepts.

Section 4 – Other Interpretations of Kant's Philosophy of Mathematics

Before concluding I would like to contrast my position with the most prominent current positions on the role intuition plays in mathematical cognition. My aims here are to highlight how my position differs from these others and note where my position is preferable. Through such juxtaposition, my hope is also that this section serves to clarify aspects of my position.

Following Gordon Brittan we can distinguish broadly between two lines of interpretation of the role Kant attributes to intuition in mathematical construction.[44] The first, which Brittan calls the 'evidentialist' line, can further be divided between those interpretations that place the role of intuition at the level of mathematical axioms, principles and basic propositions (I will call it the 'premise position'[45]) and those that place the role of intuition at the level of mathematical inference, focussing specifically on the proof procedures (I will call it the 'proof position'). In both these evidentialist cases intuition furnishes us with crucial content that concepts alone cannot provide. These positions are thus more in line with the secondary role I attribute to intuition in construction.

Brittan also presents his own, non-evidentialist, position that places the role of intuition at the level of reference (I shall call it the 'objectivist position'). Here intuition furnishes us with the objects that mathematical propositions are about, without which mathematical cognition would fail to be objectively valid. This

[44] Gordon Brittan, "Kant's Philosophy of Mathematics," in *A Companion to Kant*, ed. Graham Bird (Malden, MA, Oxford & Victoria: Blackwell Publishing Ltd, 2006), 222.
[45] Michael Friedman refers to this position as the 'model position'. Not much hinges on this difference in labels. See Michael Freidman, *Kant and the Exact Sciences* (Cambridge & London: Harvard University Press, 1992), 81.

interpretation is thus in line with the primary role I attribute to intuition in construction, and ultimately the position that my position has most in common with.

4.1. The Evidentialist Interpretation:

That intuition plays an evidentialist, or verifying, role is by-and-large the standard way to read Kant when it comes to intuition, construction and mathematical cognition. In the introduction to his seminal collection on Kant's philosophy of mathematics, Carl Posy even lets this interpretation guide his investigation of the notion of intuition in the first place:

> [Kant] must define "intuition" in a way that includes the evidence for mathematical judgements.[46]

This sort of position is suggested by reading the notion of synthetic *a priori* cognition as a contrast to analytic judgements. While analytic judgements are verified by concepts alone, synthetic judgements must be verified in some other way, namely via intuition. To this extent, I agree with the evidentialist interpretation. Where I disagree with the position, is that this is *all* that intuition does. Furthermore, I disagree that verification is the *primary* thing that intuition does.

4.1.1. The Premise Position

The premise position takes as its starting point Kant's exposition of the syntheticity of mathematics at B14:

> For since one found that the inferences of the mathematician all proceed in accordance with the principle of contradiction (which is required by the nature of any apodictic certainty), one was persuaded that the principles could also be cognized from the principle of contradiction, in which, however, they erred: for a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only

[46] Carl J. Posy, introduction to *Kant's Philosophy of Mathematics*, ed. Carl J. Posy (Dordrecht, Boston and London: Kluwer Academic Publishers, 1992), 3.

> insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself. (B14)

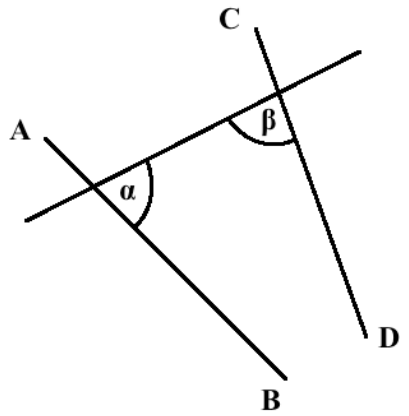
Those, such as Russell, who take the premise position, read this passage as claiming that the method of mathematics is analytic in the sense that it can be derived by logic along with definitions alone, but that some of the starting premises of such derivation are synthetic and thus that the derived mathematical judgements must also be synthetic. This position takes these starting premises as some of the most basic of mathematics. The simplest way to understand this is to think of these basic premises as models, all of which are internally coherent (that is, not self-contradictory), but not all of them true.

A common example utilised in this interpretation is the internal logical consistency of both Euclidean and non-Euclidean geometries. Geometries that include Euclid's fifth postulate are internally coherent, but so are those that exclude it and perhaps replace it with others.

Euclid's fifth postulate states that

> [I]f a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).[47]

[47] Richard Fitzpatrick, trans. and ed. *Euclid's Elements of Geometry* (lulu.com, 2007) available at farside.ph.utexas.edu/Euclid/elements.pdf (accessed 31 July 2013) 7.



(Fig. 2)

Thus, in this figure (See Fig. 2), if $\alpha + \beta < 180°$, then AB and CD will eventually intersect, if sufficiently extended. However, if $\alpha + \beta = 180°$, then AB and CD will remain equidistant regardless of how far we extend them. Furthermore, if $\alpha + \beta > 180°$, then AB and CD will never intersect, but they will also become further and further apart the further they are extended.

Taken at the conceptual level, neither Euclidean geometry nor non-Euclidean geometries have *prima facie* legitimacy – i.e. no logically consistent geometry has priority on the basis of mere analysis of the concepts involved. Yet, once intuition is involved, some sort of *evidence* for the correct model is provided – in this case, intuition is what verifies Euclid's fifth postulate. In this case, it is the features of the spatial constraints on the world. Thus, it is the *a priori* (forms of) intuitions of space and time that furnish us with the *a priori* evidence that we necessarily represent our universe as Euclidean.[48]

One concern for this position is in what sense intuition can furnish us with a model for arithmetic. One suggestion is that only intuition can provide us with the evidence that there are a certain number of objects. That is, just as 'parallel lines never meet' is only verified by an intuition of parallel lines in the world, '2 + 2 = 4' is only verified by an intuition of 4 objects in the world. In a world where there are less than four objects, '2 + 2 = 4' would turn out false (or at least without truth value).

As Friedman points out, however, such a reading has the significant problem that such verification sounds dangerously like *a posteriori* evidence.[49] In this interpretation, unlike mine set out above, it is the intuition itself that provides the evidence, not the *a*

[48] According to Kant, at least.
[49] Freidman, *Kant and the Exact Sciences*, 100-1. See also Brittan, "Kant's Philosophy of Mathematics," 226-8 for other serious problems for this interpretation.

priori constraints on all intuition. In the geometric case, we need an intuition of two lines not meeting, and in the arithmetic, we need an intuition of four objects. What verifies the corresponding judgements is not general features of space and time, but specific instances of marks displayed by the objects represented by the intuition.

On my position above, the intuitions themselves do not provide any evidence; the *a priori* forms of intuition merely limit the ways in which the concepts can be constructed. That is, objects falling under the concepts are constrained by more than the content of the concepts they fall under; they are also constrained by space and time.

Of course, on my position, the constructed intuition does provide the constructed concept with real possibility. But, the premise position does more. It reads specific marks off the object represented in intuition. On my position, we have to take note of what we necessarily think into an object that can be represented in intuition. On the premise position, we have to take note of the intuition of that object, because we are focussing on what are necessary features of the world. While we necessarily represent the world as spatial and temporal, and so these are necessary features of all empirical reality, the two positions do not boil down to the same idea. On the proof position we are looking for these features outside in the world, *a posteriori*. On mine, we are investigating them from within, *a priori*.

4.1.2. The Proof Position

In this version of the evidentialist interpretation, intuition comes in during the formation of specific mathematical judgements, which are established through mathematical inference. According to this position, certain marks predicable of mathematical concepts *cannot* be found within those concepts, and so intuition must provide this content during the process of inferring the judgement.

In his version of this position, Michael Friedman concentrates on infinity. The notion of infinity is crucial for all mathematical cognition. For example, both arithmetic and algebra rely on the infinity of the number line and geometric construction relies on the possibility and legitimacy of infinitely extending geometric lines and the infinite

divisibility of these lines. Friedman explicates the requirement of infinity by focussing on the differences between the logic contemporary to Kant and the logic available to us now – the former being monadic and the latter, polyadic.[50] Monadic logic cannot generate an infinity of objects, whereas full polyadic logic can.[51] For Kant, general logic constitutes the form of all conceptual thought (A54/B78). So if logic alone cannot provide a representation of infinity, we need an intuitive input over and above conceptual analysis for the notion. Here we must turn to the intuitions of space and time.

In the Transcendental Aesthetic, in arguing that these representations are intuitive, Kant claims that both space and time are given as infinite and that any part of space or time is necessarily represented a part of the bigger (infinite) whole (A24-5/B39-40, A31-2/B47-8). Here Kant claims that it is precisely because of this representation of infinity that space and time cannot be given initially as concepts. As it pertains to the representation of space Kant argues:

> [E]very concept [is] a representation that is contained in an infinite set of different possible representations (as their common mark), which thus contains these **under itself**; but no concept, as such, can be thought as if it contained an infinite set of representations **within itself**…Therefore the original representation of space is an ***a priori*** **intuition**, not **a concept**. (A25/B40).

For Kant concepts can only *apply* to an infinity of objects (at least possibly) via the marks contained within the concept, but never *contain* an infinite amount of objects or marks. In analysing concepts we can discover which marks belong to the concept i.e. the content *within* that concept. However, implicit in the assumption that such analysis can be completed lies the necessity of a *finite* set of marks, but concepts need not be applicable to a finite set of objects.

This is not the way Kant would stress the point. For Kant, the finitude of conceptual content is a function of the way concepts are divided into a hierarchical structure (see 1.1. above). It is possible for any concept to apply to an infinity of objects and we

[50] For a full discussion of these ideas see Freidman, *Kant and the Exact Sciences*, 58-66.
[51] Details of why this is the case need not concern us here.

could also, in principle, continue to divide concepts infinitely into narrower sub-concepts.[52] But this still leaves us with only an infinity under concepts, never *within* them. Thus the only way in which we can access a representation of infinity, so crucial in mathematics, is through the *a priori* intuitions of space and time. This reading is also suggested in the Prolegomena, where Kant claims:

> [T]hat we can require that a line should be drawn to infinity (*in indefinitum*), or that a series of changes (e.g. spaces traversed through motion) should be continued to infinity, presupposes a representation of space and of time that can **only** inhere in intuition[.] (4: 285, my emphasis)

The proof position is preferable to the premise position in that it no longer turns to empirical intuition for verification. Instances of infinity are displayed in all intuition, but more importantly, infinity is one of the necessary and universal features that all objects in intuition are constrained by. The continuity and divisibility of lines, for example, follows from the continuity and divisibility of space and time.

The crux of Friedman's position, however, is that certain features *cannot* be expressed conceptually, and that intuition is the *only* way in which we have access to crucial content, such as infinity. Yet, this is untenable in conjunction with the contrast between intuition and concepts offered in this paper. Just by describing what is lacking in the conceptual content of mathematical cognition as 'infinity', we have already indicated our ability to form a concept, and so marks, of infinity. That is, we have access to <infinity>. We are capable of forming the judgement 'All lines are continuous', where one of the marks of <continuous> is <infinity>.

Perhaps we should read Friedman's position as claiming that our *initial* representation of infinity is intuitive, via which we form the concept <infinity>. Yet, then we leave behind the claim that intuition plays a crucial role in mathematical cognition. As I expressed above (in 3.4.3), once we have access to a mark that necessarily belongs to a concept, we can define that concept as including that mark. Intuition is no longer required in the case of that concept. Of course, intuition would still be required to provide the new concept with real possibility, but this is not the role Friedman

[52] Although it is not clear that there would be any point to such an exercise.

attributes to intuition. Furthermore, we cannot claim that intuition enables the synthesis of the two concepts, because, as Kant's claims about axioms suggest, the concepts must be connected immediately, and not via a third representation. On Friedman's account then, intuition is not a crucial component in Kant's account of mathematical cognition, a highly dubious consequence.

4.2. The Objectivist Interpretation

This third interpretation[53] moves away from intuition as verifying, and focuses instead on intuition as giving the objects that mathematical cognition is about. This position is often presented along the same lines as my position with the focus on the real possibility of concepts and the objective validity of the resulting judgements. The crucial difference, however, is that intuition is not also attributed the role of presenting new predicates. This omission is due to a shared rejection of the verification position.

While I share their rejection, I think there is a way in which we can make sense of the discovery of new predicates without turning to empirical intuition, or even *a priori* intuition as a mediating factor between the two concepts. By focussing on the ways in which the content (marks) of concepts and the *a priori* forms of our intuition limit the appearance of objects and the constructability of mathematical concepts, I have presented a way in which intuition presents new predicates without forcing the intuition to be empirical or a mediation between the two concepts. As discussed above, we manage this by focussing on the schemata of the concepts.

While I think this is a merely secondary role that the *a priori* forms of intuition plays, and indeed, not always necessary (as in the case of redefined concepts), intuition can also play this role. Omitting this role would do an injustice to Kant's text and his thought about mathematical cognition. Thus, my position is objectivist with a sort of verificationist amendment.

[53] Here I have in mind, in particular, the interpretations offered by Brittan in his "Kants Philosophy of Mathematics" and Anja Jauernig in her "The Synthetic Nature of Geometry, and the Role of Construction in Intuition," in *Proceedings of the XIth International Kant Congress 2010 in Pisa*, eds. Stefano Basin *et al*, Berlin: De Gruyter, forthcoming.

Conclusion

In this paper, I aimed to present an interpretation of the role Kant attributes to intuition in the case of mathematical cognition. I did this by drawing from two novel interpretations of Kant's epistemology – Allais' interpretation of intuition as a representation that presents its object and Schafer's interpretation of cognition as constrained by both a real possibility criterion and a determinate content criterion. As a result, I argued that the role of intuition in mathematical cognition is to present the objects that the cognition is of. I further argued that this contribution amounted to conferring real possibility and determinate content to the mathematical concept involved.

In doing so I also considered Kant's claim that mathematics is synthetic and yet *a priori*. My interpretation here was that Kant claimed that the definition of mathematical concepts is synthetic in that previously unconnected concepts are connected *a priori*. The real possibility of these concepts is then demonstrated in constructed intuition. The concepts are also thereby provided with determinate content, because the now-synthesised concept contains several marks by which we can determine whether certain objects fall under that concept or not. Intuition also enables us to distinguish between two objects falling under the same concept.

I also considered mathematical judgements as synthetic by showing that the initial definition of these concepts does not contain all the marks that necessarily belong to these concepts. By turning to construction in intuition, we can discover these predicates and thereby redefine the concepts. Any resulting judgement here would then turn out as analytic. Yet, intuition would still be required to confer real possibility on these concepts. Thus, even if redefinition is legitimate[54], intuition would still play a role in mathematical cognition.

While I think Allais and Schafer are correct in their reinterpretation of Kant's epistemology, I did not defend their positions outright. However, in presenting a

[54] It might not be. See Lewis White Beck, "Can Kant's Synthetic Judgements be made Analytic?," Ruth E. Chadwick, & Clive Cazeaux, *Immanuel Kant: Critical Assessments, Volume 2: Kant's Critique of Pure Reason* (London and New York: Routledge, 1992), 347-60.

coherent consequence of their positions in the case of mathematical cognition, I hope to have at least suggested that these positions ought to be serious contenders in scholarship on Kant's epistemology.

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