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Research Report

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A quasi-experimental case study involving teaching division to low attaining Grade 5 learners using variation theory.

I acknowledge that this research report is my own work and no part of it has been copied from another source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list.

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Abstract

This research was a quasi-experimental case study involving the teaching of division to low attaining Grade 5 learners using variation theory. This area of study interested me because as a primary school mathematics teacher I found that many of my Grade 5, 6 and 7 learners struggled to understand the concepts of division and successfully solve division problems. I hoped that through this research I would be able to identify the key areas of difficulty and find ways to assist learners in overcoming these difficulties and prevent them from occurring in future teaching and learning.

My research involved the entire Grade 5 group of learners at the school where I was teaching. However, the intervention was conducted with a small sample of six Grade 5 learners that were in my class at the time of the research.

This research was planned and conducted within the theoretical framework of variation theory. Variation theory is based on the premise that “we learn from discerning variation, and what varies in our experiences influences what we learn” (Rowland, 2008, p.153). Accordingly the focus of all teaching, within the intervention, including all materials and examples was on highlighting variation to promote teaching and learning.

My research involved a pre-test that was conducted with all Grade 5 learners, an intervention which involved six low attaining learners, an immediate post-test that was only conducted with the intervention group, and a delayed post-test that was again conducted with the whole grade. The pre-test was used to establish common errors across the entire grade and answer the first research question of “What are the specific features that learners struggle to understand within the concepts and procedures associated with division at a Grade 5 level?”. I identified a range of errors across the grade and within the intervention group. All errors corresponded to those highlighted in the literature on division.

The second research question was “How can variation theory be used to devise an intervention to improve learners’ understanding of the concepts and procedures of division?”. My research and intervention was based on my application of the principles of variation theory to key division concepts and procedures. Results suggested that I was able to highlight the variation between quotitive and partitive problems quite well. For example, I feel that by the end of the intervention learners were able to differentiate between quotitive and partitive problems, and for the most part solve them appropriately. However, I believe that I was unable to successfully highlight the variation in problems involving zero. This was illustrated when

learners were unable to work confidently with and solve problems involving zero. My recommendations include suggestions as to how this area of teaching could be improved in future.

My final research question, “What are the effects of the intervention on learner performance in this area?” required me to reflect on the intervention and evaluate its overall success. The results suggested that overall the intervention had allowed the six learners to close the ‘gap’ to a small degree between themselves and their peers. This was confirmed by the post-test results where all learners improved on their pre-test results by on average, 13% in comparison to the non-intervention group which improved by an average of 11.3%. However, I believe there were areas that could have been improved further within the intervention which might have allowed for a more extensive closing of the gap.

Chapter 1

Introduction

A quasi-experimental case study involving teaching division to low attaining Grade 5 learners using variation theory.

1.1. Introduction

In this research I intended to investigate and improve the teaching and learning of division through an intervention. As a researcher I chose to design this intervention based on a theoretical framework known as 'variation theory'. Also, as a mathematics teacher I wanted my learners to be able to find meaning and make sense of division and related mathematical concepts, perform meaningful mental operations and be able to abstract and generalize these to situations beyond those used in my intervention and the limited context of the classroom. In short, I wanted my students to learn mathematics. The following section will provide a brief description of what it means to 'learn' - in general and within the context of mathematics and explore the notion of the 'object of learning'. As variation theory is my chosen theoretical framework my description will be according to my perception of these concepts through the 'lens' of variation theory.

Marton et. al. (2004, p.4 - 5) define learning as 'the acquired knowledge of something' and the object of learning as 'a capability', thus, the process of learning can be seen as 'becoming capable of doing something as a result of having had certain experiences'. Watson and Mason (2005a, p.1) define learning mathematics as "becoming acquainted with generalizations of several types: concepts, techniques, classes of objects, properties, relationships and theorems." Accordingly, learning mathematics would involve the acquisition of a mathematical concept or skill, in other words, becoming capable of doing something mathematical through participating in a mathematical activity.

The capability has two distinct aspects, the first aspect is general which includes remembering, 'discerning' - which means noticing as a result of experience, not just being told (Marton et.al., 2004), interpreting, grasping or viewing the acts of learning. In variation theory the general aspect is referred to as the 'indirect object of learning' (Marton et.al., 2004, p.4-5). The specific aspect or 'direct object of learning' refers to the 'thing or subject on which these acts are carried out on' (Marton et.al., 2004, p.4-5). Within the mathematics classroom the thing or subject can be seen as a concept or skill. Similarly Watson and Mason (2006b, p.100 - 1) describe the object of learning as 'that which is the focus of attention'. This means that the object of learning is whatever the learner focuses and acts intelligently on. If division was the section being addressed in the Grade 5 mathematics classroom the object of learning would be the various division concepts being taught in the lessons. The learners would need to know and understand each concept before it can be said that they ‘know’ division. Within each concept dimensions of variation must be explored. The notion of dimension of variation can best be described as the different strategies or types of examples that make it possible for the learner to discern the critical feature/s (Marton et.al., 2004, p.15). It can be conceptualised as the different strategies to solve a division problem or the different numbers within a strategy.

The object of learning is defined by Marton et. al. (2004, p.22) as ‘critical features’ that must be discerned in order for learners to reach the desired meaning. For example, within the concepts of division the critical feature can be considered in terms of how the aspects of: divisor, dividend, quotient and remainder relate to each other. Consequently, within each division concept taught or capability developed the focus should be on how the aspects interact and relate to each other.

The problem that teachers sometimes experience with the object of learning is that they cannot control what becomes the object of learning (Watson and Mason, 2006b, p. 101). Thus, it is important that the teacher and learners are constantly engaged in a discourse that will enable the teacher to monitor and direct the learner’s focus appropriately. Accordingly, as a teacher and researcher, I needed to direct learners’ attention towards specific critical features that had to be discerned so that the object of learning – division- could be meaningfully acquired.

Through this research, I hoped to provide opportunities that promoted constructive learning experiences within a classroom. I proposed that this be done through the creation of an environment that was conducive for “common mathematical sense-making” (Watson and Mason, 2006b, p. 97) to take place where learners were able to make sense of mathematical concepts in a way that was meaningful and match the greater mathematical community.

Marton and Booth (1997) describe the starting point of sense making as the discernment of variation within a selection of examples of a concept. Liljestrand & Runesson (2006, p. 165) support this claim when they explain that learning takes place when an individual is able to discern the critical features of an object and that any critical feature only becomes evident if it varies. Marton et. al. (2004, p.11) describe a feature as an aspect or an attribute and critical features as the aspects or attributes necessary for defining an object (Marton et. al., 2004 p. 15).

Marton & Pang (2007, p.9) argue that what learners learn is dependant on the way in which the object of learning is dealt with i.e. the learning conditions, as well as the content. The manner in which variance and invariance is highlighted within the content, as well as classroom environment influences learners' ability make sense of the critical features. Thus, variation and variation theory – the manner in which a concept is dealt with within a learning environment, can be viewed as a scaffolding tool (Watson and Mason, 2006b, p. 97). I support this view as I believe that if learners are able to identify variants the learning process will be more effective. However, as Marton and Pang (2007) highlighted, learning requires more than just a pattern of variation or the communication of a concept, for learning to take place; the classroom environment also plays an important role in the learning process. Hence, if the teacher is aware of the learners' current knowledge he/she will be able to select exercises that are progressively more challenging and encourage the learners to engage in the accommodation process and thus promote learning. Learning and cognition will be described in greater detail in the literature review.

In summary, variation theory is about "how we perceive and experience the world around us" (Liljestrand & Runesson, 2006, p. 165). I decided to use variation theory as a theoretical framework to teach division in an intervention with a sample of low attaining Grade 5 learners. This decision was a result of previous teaching of division that had not developed their knowledge of division sufficiently to meet the expected curriculum requirements and there were large gaps in the knowledge that they had acquired. Further discussion on the potential benefits and limitations surrounding variation theory and how it can be used to promote learning are described in Chapter Three. The reason for the selection of division as my area of study is explained in the following section.

1.2. Background and aims of the research

I teach in an all girls, private, primary school. In my experience I have found that learners in Grade 4 to 7 struggle to understand the concepts and apply the procedures of division. Many of the Grade 6 and 7 learners that I teach struggle to understand some of the division concepts and procedures, including the effective use of the long division algorithm.

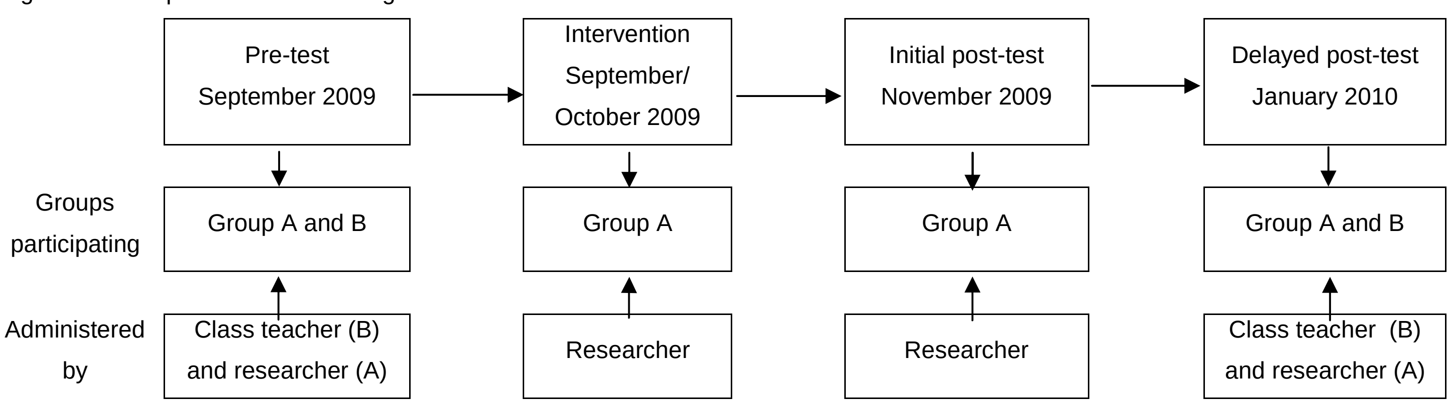
Hiebert and Lefevre (1986) distinguish between the two distinct, but related types of knowledge: procedural and conceptual. They describe procedural knowledge as "the rules or procedures for solving mathematical problems (Hiebert and Lefevre, 1986, p. 7). It includes the knowledge of the formal mathematical language, the algorithms and rules, as well as the strategies for solving problems and is sequential in nature (Hiebert and Lefevre, 1986). In mathematics, an algorithm can be viewed as: "a specific set of steps that, if executed accurately, will lead to consistent results" (Troutman and Lichtenberg, 2003, p. 241). Hiebert and Lefevre

(1986, p.4) describe conceptual knowledge as “the construction of relationships between pieces of information”. While there is a distinct difference between the two types of knowledge, each type requires the other for a complete understanding. Kilpatrick et.al. (2001) explain that procedural knowledge is embedded within conceptual knowledge. Thus, to enable a good conceptual knowledge, learners will need a sound understanding of procedural knowledge. I believe that neither of these knowledge types can be taught independently but have to be developed simultaneously. Furthermore, foundation concepts (pre-concepts) need to be stable and complete before new knowledge can be constructed using those concepts. For example, the gaps in learners’ knowledge and understanding of division often hinder their ability to develop other concepts such as fractions and decimals.

In my research, I wanted to explore the reasons for my sample of low attaining learners struggling with division and try to improve their knowledge and understanding of division concepts, including the ability to use the long division algorithm, as this forms part of the procedural knowledge base in Grade 5. I hoped to conduct an intervention with the sample to assist them in overcoming some of the difficulties.

Figure 1.1. presents a model of the case study, action research that was conducted for the purposes of this study. Group A refers to the sample of six Grade 5 learners that took part in the intervention and Group B refers to the remainder of the Grade 5 learners. The non-intervention group (Group B) consisted of 50 learners who took part in both the pre and delayed post-test. The learners were in Grade 6 when they wrote the delayed post-test.

Figure 1.1.: Proposed research design



I designed and conducted a pre-test on various key concepts and procedures of division. These were drawn from the literature and my experience of teaching division to Grade Five learners. The results were used to identify common misconceptions across the entire grade. I used the findings of this pre-test to assist me in designing and implementing an intervention addressing the key division concepts identified in the literature, including those highlighted as problematic in the pre-test. Finally, I investigated the success of the intervention through the analysis of a post-test. I conducted two post-tests. The first post-test was written by the sample of learners shortly after the intervention. A delayed post-test was written the following year (about 2 months after the intervention) by both Group A and B. Further discussion on the research methodology and details of the different parts of the research process have been provided in Chapter 4. The selection of the sample group of learners is explained in the following section.

1.3. Sample selection

I was teaching a small class of six Grade 5 learners at the time of this research. These learners had been identified as low attaining learners within mathematics and were pulled out of their mainstream classes to try and improve their mathematical understanding within a mathematics bridging class. The concept of a 'low attaining learner', the possible causes and suggestions as to how to address their difficulties to promote effective learning have been explained in Chapter 2, the literature review. I hoped that I would be able to help these learners close the attainment gap with their peers and bridge the gaps in their knowledge and understanding so that they were able cope with the academic demands of a mainstream class. Whilst my work with this group spanned a number of mathematical areas, the focus of this study was on the concepts and procedures associated with division at a Grade 5 level.

These six low attaining learners formed the sample for the intervention and are referred to as Group A. While there was little I could do to control the factors affecting the learners outside of my classroom, I hoped to design an intervention that would assist them in gaining the necessary skills and understanding to improve their mathematical performance in division problems so that they were able to meet the standards expected of a Grade 5 at my school. I believed that if I could improve the understanding and skills relating to division for these six learners I would be able to use the same strategies to assist other low-attaining Grade 6 and 7 learners. If this research proved successful I hoped to share my knowledge with colleagues to improve the overall teaching and learning of division within my school.

The selection of Group A was both purposive and opportunistic. It was not a representation of the grade as it only involved a selection of low attaining learners. The reason that I selected this small group was to gain a more in-depth understanding of the problems that these learners faced as I believed that this would provide insight into many of the difficulties other learners experience in this area. The aim of this case study concurs with any case study in that it focused on a specific instance that was designed to illustrate a more general principle (Opie, 2004, p. 74). With this sample of learners I had the unique opportunity to study and implement an intervention with an ‘extreme’ instance of mathematics low attainers (Denscombe, 2007 p. 40 & 41) of a common problem within the primary school – namely understanding division.

1.4. Rationale

Daniels and Anghileri (1995) identify the fundamental aim of teaching mathematics as equipping learners with the strategies, skills and knowledge that will enable them to solve real life problems with confidence. The South African Department of Education (2002) sees this as a basic human right in addition to equipping learners for lifelong learning which includes further studies in mathematics. Division is an important concept for both real life problems and further studies in mathematics. Hence, I believe that it is essential that learners master division along with many other foundational concepts at a primary school level, to allow them access to other areas of mathematics, in addition to enabling them to solve real life problems with ease.

The concept of division can be understood in a range of ways, for example, equal sharing and measurement are two key ways of understanding division. Across both interpretations the concept of division produces a relationship between two numbers – the dividend and the divisor. At a Grade Five level for division, this relationship is considered in terms of the quotient and remainder, although subsequently, learners will go onto fractional or decimal representations in which the remainder is subsumed in the quotient. In brief, the critical feature of division involves the relationship between the four parts. These parts are interconnected and if one is altered it affects all other parts. The parts and their relationship to each other can be expressed using the equation Dividend ÷ Divisor = Quotient + Remainder. In addition, to this critical feature it is also vital to recognise the relationship of division to multiplication, the inverse operation, as well as, addition and subtraction.

As stated previously, I have encountered many Grade 4-7 learners who have experienced difficulty understanding division and its algorithms. Middleton and Toluk (2004) recognize division concepts and procedures as being widely problematic. Booker et. al. (1992, p.164-5) claim that the reason children and teachers find division challenging can be linked to the use of inappropriate language and meaningless, rote learned procedures. I must confess that as a teacher I have frequently used both the inappropriate language and meaningless routine that they describe. I believe that although some of the language may be inappropriate it is sometimes directed at a short-term purpose. However, it is important that both the teachers and learners begin from accurate and appropriate conceptions with informal language working within these conceptions. For example, I always begin the section by working with the core concepts and when I introduce an algorithm I try to teach it through the concepts and make it meaningful to build on the previously learnt concepts. I find at times that I resort to drilling the meaningless routines to try and assist the learners in becoming fluent when using a procedure with little focus on the underlying concepts. While I am aware of the negative aspects of using inappropriate language and meaningless routines I find it difficult to avoid them and I must argue that it does have some benefit when it is developed during the learning process. Furthermore, division relies on learners having a sound understanding of the pre-requisite skills (Troutman and Lichtenberg, 2003, p.248). These skills have been explained in greater detail in Chapter 2 – the literature review. The difficulties learners experience and the possible reasons behind these difficulties are discussed in more detail in the literature review. I will also critique a variety of texts with regards to the different teaching methods they propose.

In my research, I hoped to expose some of the problems and misconceptions that my learners experienced and relate this to the literature. However, as my sample (Group A) was not representative of a Grade 5 class I expected the number and degree of difficulties experienced by my sample to be greater than that experienced by most Grade 5 children who are not low attainers.

I aimed to address Group A's difficulties through an intervention that was designed using the theoretical framework of variation theory. The importance of variation theory and the justification for its use as a theoretical framework is explained in Chapter Three. I believed that if the results from my case study intervention with Group A were good, my findings could be more broadly useful to primary mathematics teachers in other schools. I hoped that through this research I would be able to provide insight into why low attaining learners struggle to understand division and the long division algorithm. I anticipate that this research could provide a stepping-stone into further research and teacher development around teaching division and other concepts within a primary school.

1.5. Research Questions

Based on my aims, context and theoretical framework three critical questions framed my investigation. The questions I hoped to answer through my research were:

- What are the specific features that learners struggle to understand within the concepts and procedures associated with division at a Grade 5 level?
- How can variation theory be used to devise an intervention to improve learners' understanding of the concepts and procedures of division?
- What are the effects of the intervention on learner performance in this area?

In this chapter I have introduced my study as well as how and why I propose to use variation theory to teach division to a small sample of Grade 5 learners in a ‘quasi’ experimental, action research, case study. The following chapters have been structured in this way:

- Chapter 2 - explores three broad areas of literature. The first area relates to low attaining learners in mathematics. While this is not the focus of the study I felt it was important as my sample of learners fell in this category. The second area surrounds the concepts of division and teaching division. Within the review I explored the concepts of division that learners are required to understand and master at a primary school level, as well as misconceptions and errors that the literature suggests that learners may experience. I critiqued proposed teaching strategies and sequences suggested in the literature surrounding the teaching of division. Finally, I analysed examples commonly used to teach division.
- Chapter 3 – introduces the literature surrounding variation theory, which was my theoretical framework. I used my framework of variation theory and my knowledge of division to develop an intervention for the sample of girls.
- Chapter 4 – provides details surrounding the research methodology and the planning and administration of the pre-test, intervention and post-test. It also explains how the data will be analysed.
- Chapter 5 – contains the data gained throughout the research process, the analysis of the data and the findings.
- Chapter 6 – draws together all findings. It contains recommendations made and final conclusions.

Chapter 2
Literature review

I believe that before beginning any research it is important to understand key problems in the area to be researched in addition to having a good understanding of the theoretical framework that will guide the research. I will draw on two bodies of knowledge in my literature review – firstly, the issue of cognition and low attainment in mathematics and secondly, division, including - teaching strategies, errors and the selection of examples that address identified misconceptions and provide learners with the opportunities to learn new concepts and skills. The selection and sequencing of examples will be explained further with reference to variation theory in Chapter 3, in which I discuss my theoretical framework.

2.1. Cognition and learning

I believe that it is important to have some idea as to what it means to acquire knowledge as this forms the link between the theoretical framework and the object of learning because it discusses how the theory can be put into practice to assist learners in building and extending their knowledge base. Watson and Mason (2006b, p. 92) explain that the notion of learning includes the factual acquisition of concepts or conceptual development, conceptual re-organisation, schema development, alteration of a predisposition or perspective.

Piaget's theory of cognition provides one view of how concepts are acquired. His theories focus on the process of learning and not specifically what is learnt. I believe that Piaget's theory is an important foundational theory, but I hope to draw on various theories of cognition that focus specifically on the acquisition of mathematical concepts to build on my understanding of cognition. Variation theory (Runesson, 2006) is a theory about learning that emphasises the relationship between what is to be learnt and the process of learning. Accordingly, I felt that this would be a suitable theoretical framework for a study focused specifically on the learning of ideas related to division. To fully understand variation theory I felt that it was important that I look at Piaget's theory of cognition as it forms a general foundation to many other theories of cognition. However, I have used a 'lens' of variation theory to interpret his theory.

Piaget's theory of cognition involves three basic principles (Von Glasersfeld, 1997, p.8). The first principle of assimilation explains how the mind perceives and categorises the learning experience in terms of what it already knows. This was important in my study as I wanted to provide examples that would shape learners perceptions of a concept. However, in order to do this effectively, I needed to be aware of their current knowledge as this affected their perceptions and the way that they categorised the learning experience.

The second principle of accommodation is used when the learning experience does not fit with the current knowledge (Von Glasersfeld, 1997, p.8). This disturbance requires the learner to review the experience and re-organise their conceptual understanding and develop their schemata. Thus, their perspective is altered to accommodate and assimilate the new knowledge (Watson and Mason, 2006b, p. 92). The process of accommodation provides the opportunity for learning to take place and allows learners to ‘construct’ and make sense of the new knowledge. Watson and Mason (2006b, p. 92) explain that the ‘construction’ of new knowledge involves a “shift between attending to relationships within, and between, elements of current experience … and perceiving relationships as properties that might be applicable to other situations”. Watson and Mason (2006b, p. 92) focus their discussion on learning mathematics when they claim that “learning mathematics involved long-term conceptual development, advances in abstract understanding and improved applicability”. This claim extends beyond Piaget’s second principle of accommodation to his third principle of reflective abstraction (Von Glasersfeld, 1997, p.8).

Watson and Mason (2006b, p.94) describe abstraction as a shift from seeing a relationship as specific to a situation to seeing them as potential properties of similar situations. Piaget (Von Glasersfeld, 1997, p.8) describes reflective abstraction as the mind’s reflection on the mental operation, as well as, the abstraction and generalization of the operation. Watson and Mason (2006b, p.94) explain that the teacher needs to put in place special steps to assist learners in moving from the immediate doing to further engagement that allows abstraction and conceptualisation beyond the current problem or activity.

I recognise that variation theory does not always support and concur with the Piagetian orientation to learning, however, Piaget’s theories provide a well known and useful base to compare, and contrast, variation theory, highlighting development, strengths and weaknesses.

2.2. Low attaining learners

I considered my sample of learners as low attaining. In this section, I draw on literature that describes what the term ‘low attaining learner’ means and suggest possible causes of learners’ difficulties, as well as providing recommendations as to how teachers can assist learners in improving their understanding and attainment. I believed that this area of literature was important, as it provided insight into the possible reasons why my learners experienced difficulty in mathematics, and more specifically division. The suggestions assisted in the implementation of appropriate strategies to support the learners in overcoming these difficulties.

Barnes (2005, p.42) describes a low attaining mathematics learner as a child “who does not meet with the required standard of mathematics performance set out by the school”. At my school, learners’ performance was measured using continuous assessment of class activities and formal assessments. The learners that were part of my sample group did not meet the required standard of mathematics in the majority of the areas assessed in class in the previous grade. This research focused on these learners and their achievement in the area of division.

In her article Barnes (2005) identifies the general characteristics and causes of low attainment in learners. She makes reference to Denvir et. al. (1982) who separate the causes into three main categories: firstly, those which were controlled by the school, such as access to resources and teaching methods; secondly, those which were a result of environmental factors, for example, lack of food due to socio-economic conditions. The third category includes factors which were beyond the control of the school, for example, the prescribed curriculum.

Barnes (2005, p. 42 and 44) claims that low attainment in mathematics is something that can, with support by the teacher, and through the use of appropriate strategies, be addressed. I agree with this claim and believe that for the majority of learners, once the cause of the low attainment has been identified, the problem can, at least partially, be overcome through the implementation of appropriate strategies. I believe, for the majority of the children that I teach, the cause of their difficulties is related to the school environment or conditions relating to the school but not controlled by the school. As the curriculum is prescribed and can not be controlled, the only aspect that I could change in my intervention was the teaching strategies and the means through which the content was presented. For example the experiences and examples used, what and how attention was drawn to the object of learning, as well as, the classroom environment.

Barnes (2005) identifies a variety of strategies that she finds helpful for improving low attainders’ performance in mathematics, for example the importance of language development, the use of a calculator and using social interaction as a learning strategy. The focus of this research was on the development of understanding and the identification of “purposeful activities in meaningful contexts” (Haylock, 1991, p.5) to promote understanding of division – for my study. Barnes (2005, p. 44 & 45) stresses the importance of making learning relevant and meaningful encouraging all learners to participate actively. She explains that when learners participate actively in the learning process by engaging with the experiences and examples, teachers can observe the learners’ strengths and weaknesses and gain insight into their level of understanding and the extent of their knowledge (Barnes, 2005, p. 45). From this a teacher is able to adapt the work accordingly, and thus improve the learning experience.

2.2.1. Understanding and low attaining learners

As a teacher I strive for my students to understand what I have taught them. However, I found that some low attaining learners did not meet the required standard in class activities or formal assessments, while others were able to complete division tasks in class but forgot the concepts when it came to formal assessments. This lack of retention suggested that some of the learners did not have a sound understanding of the concept of division or the underlying concepts on which division relied, such as subtraction or multiplication. This supports Hiebert and Carpenter (1992) claim that understanding promotes remembering and the transferring of knowledge. Furthermore, Haylock (1991) state that understanding and knowing come from making connections by seeing how things fit together, relating classroom mathematics to real situations and identifying and describing patterns and relationships.

Skemp (1989) distinguishes between two different types of understanding, instrumental understanding and relational understanding. He describes instrumental understanding as “rules without reasons”, where learners know and are able to apply the rules/algorithms without actually understanding the concepts behind the rules and why the rule works and is used (Skemp, 1989). As many low attaining learners struggle to remember random facts the more algorithms they learn the more they forget. In addition, whilst they ‘know’ what to do in a specific situation they struggle to transfer this knowledge to other similar contexts where a slight variation occurs. This was evident in my class where learners who were able to apply an algorithm correctly in class were unable to recall these appropriately in a formal assessment. Skemp (1976, p.14-15) describes relational learning in mathematics as the “building up of a conceptual structure from which its possessor can produce an unlimited number of plans for getting from any starting point to any finishing point”. Accordingly drawing on this and other references made to relational understanding, my interpretation of relational understanding is that it involves the integration of new concepts, including the algorithm and critical ideas behind it, into an existing web of knowledge, thus improving the opportunities to improve and trigger memory of relevance to division. As explained in the next section, division relies heavily on learners’ understanding of many other concepts. Thus, for learners to apply division skills competently they need to develop a relational understanding of division. This was an important consideration when I designed my intervention.

Up to this point in the literature review I have provided a general theory on how knowledge is acquired, and I have looked at a broad description of the type of learners that formed my key sample. As the focus of my study is on the teaching and learning of division, the following part of this chapter looks at the different concepts Grade 5 learners are required to master, and suggestions from the literature as to the most effective way to teach the concepts and problems that are often encountered in the teaching and learning of division. The remainder of the chapter is dedicated to the introduction and discussion of various division concepts. I then outline literature found on common errors and error analysis. The writing on error analysis includes the identification and classification of errors and misconceptions, suggestions as to how these can be addressed and overcome, as well as prevented in future teaching.

2.3. The concepts of division

Troutman & Lichtenberg (2003, p.251) emphasise that division of whole numbers should be mastered at a primary school level. I agree with this as learners will not be able to further their mathematical studies if they have not mastered all of the core concepts taught at a primary school level. As division forms one of the concepts taught at a primary school level, it must be mastered before learners can continue their mathematical learning career. This section explores literature surrounding the key concepts of division in relation to the **critical feature of the relationship between the dividend, divisor, quotient and remainder**. In addition, it includes an analysis of the proposed teaching methods and tools that have been described in the literature. I begin by giving a general account of what teaching division entails.

Troutman & Lichtenberg (2003, p.250) recognise that children have difficulty calculating quotients and they emphasise that in the early grades the focus of the learning experiences should be on developing the meaning of division. Drawing from the literature, the meaning of division and key concepts can be broken into three key areas:

- division as ‘sharing’ (partitive)
- division as measurement (quotitive)
- division as the inverse of multiplication
    - division facts

Troutman & Lichtenberg (2003, p.251) propose that the writing of number sentences, mental arithmetic, and problem solving should form part of the development of the division concept. Troutman & Lichtenberg (2003), like many other authors, recommend that computational procedures be developed slowly and replicate the development of multiplication concepts. The link between multiplication and division is very important, not only for the development of the computational procedures, but also for the development of division concepts. Toluk and Middleton (2004) also stress the relationship between multiplication and division, when they state that an “understanding of the division concept beyond whole number partitioning should address the relationship between quotients as numbers and the operation of division as embodying a multiplicative relationship".

According to the literature there are two contextual situations or interpretations of division problems; namely the partitive and quotitive situations. Neuman (1999, p.103) describe the different interpretations as two different counting operations. Each situation type is described below.

2.3.1. Division as 'sharing' (partitive)

The first interpretation is when the number of groups is known, and the size of each group can be found by a process of sharing. Booker et. al. (1992, p. 166) refer to this process of sharing as partitioning. The word partitioning comes from the notion of division to find out the size or share of each part (Stern and Stern, 1949, p.269). Troutman and Lichtenberg (2003, p. 226, 227) describe this situation by saying that the product and one of the factors have been given. They name this type of question a distributive or partition type. Neuman (1999, p. 101 & 104) explain that the partitive interpretation is the only primitive form of division which concerns the relationship between two measurement variables and that the meaning of the divisor is the number of parts. The following question is an example of a partitive problem: *There are 35 sweets in a bag. 7 girls want to share them. How many sweets does each girl get?* In this question the dividend concerns sweets but the divisor concerns girls. According to Neuman's (1999, p. 103) interpretation it is not possible to use 7 girls as a unit to 'measure' 35 sweets. Thus, there are 2 different units of measurement.

Booker et.al. (1992, p.167) and Toluk and Middleton (2004) claim that the sharing approach provides the best basis for understanding division as it is the approach that young children are more familiar with and accordingly should be introduced to first. In addition, they also believe that it enables a consistent development of both the basic facts and the algorithm (Booker et.al., 1992, p.167). Haylock (2006) do not support this argument that sharing provides a good foundation for the development of division, as he claims that the idea of sharing only corresponds with division under certain conditions. In real life, children share a certain number of objects with friends or family. For example, Mary may share a packet of ten sweets with her sister. However, division in the classroom requires that a collection of objects be shared between a number of people, or into a number of groups, with little correlation to learners' experience. For example, a farmer packs 48 mangos into 12 boxes. Thus, Haylock (2006) claims that the artificial process may not be as familiar as we expect. He also highlights the differences in language. For example we often use "shared with" in real life and "shared between" in the mathematics classroom. This may not seem significant to those experienced in the division conventions, but it may be confusing to inexperienced or low attaining learners. Barnes (2005) who emphasises the importance of the development of language in Mathematics to promote meaningful learning for low attaining learners. Haylock (2006, p.78) explains that many of the sharing problems in division are specially contrived for mathematics and do not feel natural to learners, with the exception of money problems. Conversely, many authors recognised the benefit of these problems if appropriate language and context is used.

2.3.2. <u>Division as measurement (quotitive)</u>

The second interpretation of a division problem is the quotitive division situation. According to Neuman (1999, p. 103, 104) a quotitive problem has only one unit of measurement and can only be acquired with instruction as it is frequently an unfamiliar problem type. For example: *There are 32 marbles and some bags. 8 marbles will be placed in each bag. How many bags are needed?* In this case the dividend and divisor concerns marbles. Therefore, according to Neuman's (1999, p. 103) interpretation the 8 marbles can be used as the unit to 'measure' the 32 marbles. Another example is "how many 3's in 15?" Again there is only one unit of measurement, the 3 is the unit used to 'measure' the 15. Booker et.al. (1992, p. 166) state that the process used to solve a quotitive problem is 'essentially repeated subtraction'. This relationship with subtraction is why Troutman and Lichtenberg (2003, p. 226) called this type of division subtractive or measurement. Booker et. al. (1992, p.169) believe that this interpretation of division should only be taught once learners have internalised the concept of sharing. They claim that quotitive division requires that several objects be taken at a time and put into some special form of arrangement (Booker et. al., 1992) as opposed to being shared between a specified number of groups. For example, children are accustomed to sharing out objects such as sweets into a specified number of groups or between a specified number of children, where as quotitive questions requires learners put a specific amount of each object for example cupcakes into each group with the number of groups being the object they are trying to find out.

The relationship of ratio to division falls within this interpretation of division but as it is beyond the scope of Grade 5 it will be discussed in the section 'beyond Grade 5' (see page 35).

It is important that both the teachers and learners are aware of this special arrangement as it means that the 'divisor' has a different functions and interpretations across the quotitive and partitive conception of division. Accordingly, this brings into focus the critical feature of the relationship between the dividend, divisor and quotient as each conception requires a different way of thinking about the relationship. Neuman (1999, p. 101) questions whether children are able to differentiate between the meaning attached to the divisor in the quotitive and partitive division, and if they are able to experience the variation between the different problems. Booker et.al. (1992, p.167) state that in order for learners to gain a complete conception of division they must understand both the sharing (partitive) conception and the repeated subtraction (quotitive) conception. Based on the suggestions above, I began my intervention by introducing division through everyday problems, specifically with a partitive question first and then a quotitive. I also drew explicit attention to the similarities and differences between the two types of problems.

Booker et.al. (1992, p. 168, 178) claim that the 'sharing' language, for example 'trading' (also known as borrowing), or sharing with / between or what is left, that is developed through the teaching of the two conceptions can be replaced by formal, mathematical language such as 'divided by'. Thus, avoiding the inappropriate and problematic language described earlier that frequently causes the development of misconceptions.

The literature surrounding the discovery and learning of the two interpretations of division advocates the use of concrete objects and representational drawings. Troutman and Lichtenberg (2003, p. 228) believe that after a substantial amount of experience with both interpretations learners will be able to recognise a relationship between the two types of problems and their corresponding multiplication facts. Booker et.al. (1992, p.167) explain that "both forms relate to multiplication" and it is this relationship which provides the fundamental notation of division. This relationship can play an important role in the formal computations learners encounter at a later stage and is one of the prerequisite skills of division. The relationship between multiplication and division can be highlighted through the use of arrays which provide a simple pictorial representation of the relationship. Arrays will be discussed in section 2.5.2.

Troutman and Lichtenberg (2003, p. 227) assert that once learners make the connection between the division problems and the multiplication facts or repeated subtraction which they are familiar with, they will choose to work with the numbers as opposed to the concrete objects they initially used as they are quicker and easier to manipulate than concrete objects and representations. Troutman and Lichtenberg (2003, p. 227) named this as the point when learners had achieved maturity with whole numbers. In division this would be the point at which learners would use multiplication/division facts or algorithms to solve problems and not concrete or pictorial representations. The role of multiplication will be discussed in the following section.

2.3.3. <u>Division as the inverse of multiplication</u>

Weissglass (1979, p.70, 71) explains that division is the operation that 'undoes' multiplication and could be defined in terms of multiplication. Accordingly, multiplication should be used in assisting learners to further their understanding of division. Troutman & Lichtenberg (2003, p.226) suggest that as division is closely related to multiplication the development of division does not need to start from scratch but should rather draw on learners' experience of multiplication. The authors claim that division ideas are dependant on multiplication and that the transfer of knowledge from multiplication to division is easy for learners (Troutman & Lichtenberg, 2003, p. 226, 227). Booker et. al. (1992, p.165) also support the use of multiplication as an important tool for teaching division, but they recommend the use of everyday experience and sharing as the most meaningful introduction to division. As established earlier, multiplication is one of the prerequisite skills required for learners to be able to visualise the initial concepts of division, as well as, effectively find the answers for steps involved in the formal procedure (Booker et.al., 1992, p. 166). However, I believe that the link between multiplication and division should not be made formal or explicit until learners have a sound understanding of what division means. Furthermore, I agree with Troutman and Lichtenberg (2003, p. 230) when they recommend that when learners are introduced to division number sentences they should learn to associate it with the relevant multiplication facts and number sentences. I concur with their belief that this would promote the linking of concepts and the transfer of knowledge.

Haylock (2006, p.86) emphasises the benefit of learners being able to multiply and divide effectively using multiples of 10, 100 and 1000. He describes how these skills assist learners in performing mental calculations (Haylock, 2006, p.86). He believes that this aids learners in working with large numbers, as they are able to break up the numbers and multiply or divided using the multiples. Accordingly, this would be a useful tool for learning division.

Directly associated with the concept of division as the inverse of multiplication are the division facts. The following section will briefly explore the notion of division facts.

- <u>**Division facts**</u>

Booker et.al. (1992, p.170) believes that it is vital for learners to have ready access to the basic division facts for the smooth development of the division algorithm with larger numbers. They also claim that learners can only begin to master the basic division facts once they have acquired the concept of division and are able to write and correctly interpret division statements (Booker et.al., 1992, p. 170). This provides further support for the notion that it is important that learners understand the concept of division before the idea is formally related to multiplication. Accordingly, I planned for learners to have a sound understanding of division before I guided learners into 'noticing' the relationship to multiplication. Once they were aware of the relationship between multiplication and division I formally and explicitly introduced the concept to the learners.

Booker et.al. (1992, p.170) explains that division as sharing does not provide an efficient means of learning the basic facts because the sharing process requires learners to actively share objects one at a time and then count the number in each group. This process means that learners have to learn each division fact separately and in isolation. If learners use their knowledge of the number facts and other operations it is possible for them to learn the facts in related clusters. In addition, a sound knowledge of the multiplication facts is a useful tool for mastering the division facts, as division can be interpreted as knowing the product and one factor (Booker et.al, 1992, p. 171) – as explained earlier. For example, 32 divided by 8 is 4 because 4 eights are 32. This link between multiplication and division helps learners remember the division facts as they are related to known facts and they can be learnt in clusters. Booker et.al. (1992, p.171) suggest the use of drill and practice to promote the development of automatic recall. Furthermore, the authors highlight the need for learners to practice distinguishing between multiplication and division if they are to be able to use this knowledge efficiently and effectively (Booker et.al. 1992, p.171).

Booker et. al. (1992, p.171) recommend that learners be introduced to the division facts through a multiplication setting where they are able to recognise that one factor, and the product, can be used to find the other factor, as introduced in partitive sharing, and reinforced through the quotitive conception. This is then related to the division process and learners can be introduced to the division form of recording. Booker et.al. (1992, p.172) believe that once learners are able to find the missing factor, this thinking needs to be transferred to division situations explicitly. Learners need to practice matching the multiplication and the division facts. Through practicing this transition learners would then be able to automatically associate the multiplication and division facts and recognise clusters of division facts that match the clusters of multiplication facts (Booker et.al., 1992, p.172). This is an important consideration in the development of my intervention when planning the introduction of the division facts. Much of lesson three was based on these considerations along with the missing factor approach which is described below.

- <u>Missing factor</u>

Stern and Stern (1949, p.280, 281) discuss the division concept of finding the missing factor. It can be represented in two different ways: $3 \times \square = 12$ or $12 \div 3 = \square$. While they recognised that this interpretation helped learners compare and link the two operations of multiplication and division they claimed that this approach sometimes meant that learners had difficulty recognising that the two signs “set opposite operations in motion” (Stern and Stern, 1949, p.280, 281). The authors claimed that although the multiplication number sentence was mathematically correct it took away from the sense of dividing and did not highlight the unique structural characteristic of division (Stern and Stern, 1949).

2.4. <u>Curriculum demands for Grade 5</u>

The section briefly explores the core concepts highlighted, including the division that need to be addressed in South African schools in Grade 5, as well as pre-concepts that are necessary for the development of division concepts. The curriculum demands which can be related to division for Grade 5 include:

- solving problems in context (does not specify division problems),
- calculating problems that involve the division of at least whole 3-digit by 2-digit numbers,
- recognises, describes and uses:
    - the relationship between multiplication and division,
    - the commutative, associative and distributive properties with whole numbers (but does not necessarily know the names),
- writes a number sentence to describe a problem situation and
- solves or completes number sentences by inspection (Department of Education, 2002, p.41-47).

As indicated above the curriculum demands that learners work with both the division concepts and solve division problems relating to the real world. Accordingly my teaching of division integrated a conceptual approach with real world problems. The actual teaching of division and sequencing of concepts and other related aspects at a Grade 5 level will be discussed in the following section.

2.5. <u>Approaches to teaching division</u>

Neuman (1999, p.101) explains that "formal division, understood as related to everyday situations, only develops in interplay with informal knowledge". Many authors, researchers and teachers support Neuman's (1999) claim that the development of division should begin with everyday experiences and build on learners' informal knowledge.

The following sections will highlight further areas to be addressed and suggest how the various approaches can be implemented, as well as describing the core division concepts covered. This section in the literature review explores the following aspects of division:

- contexts
- notation
- representation
- procedures

all of which help learners extend the range of examples that they can work with and connect conceptions. As everyday experiences was the area I felt best to begin learning about division, my discussion on teaching division will begin here.

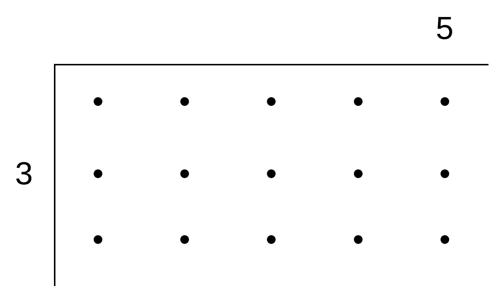
2.5.1. <u>Everyday experiences and interpretations of division</u>

Everyday experiences have been explored through problem sums where a real life situation is described to learners and they are required to solve the problem. Learners often use informal strategies to solve the problems and these problems form the foundation of the conceptual development process.

2.5.2. <u>Arrays</u>

Booker et.al. (1992, p.165) suggest that when the teaching and learning of division starts from informal sharing it is possible to assist learners to formalise their processes and promote a meaningful interpretation of division and its symbolism. Booker et. al. (1992, p.165) recommend that sharing be done in such a way as to form arrays rather than simply creating equal sized groups. Figure 2.1. demonstrates how an array can be used to represent the division sum 15 ÷ 3 = 5.

Figure 2.1. an example of an array for the question 15 ÷ 3 = 5



Booker et. al. (1992, p.165) believe that this type of representation assists learners to form the link between multiplication and division and it promotes the development of the concept of division as the inverse of multiplication. In addition, the use of arrays promotes further links to multiplication as it incorporates the part-part-whole conception, which builds on division’s relationship to multiplication. They argued that the use of materials to model problems with arrays leads to the development of language, which makes the recording of division meaningful and assists learners to differentiate between the number being shared (dividend) and the number it has been shared among (divisor). Furthermore, it provides a link between partitive and quotitive problems as in one you are looking for rows and the other for columns. The use of arrays provides the opportunity for a visual representation and I was alert to the development of this strategy in learners’ work.

2.5.3. <u>Division using base 10 blocks</u>

Booker et. al. (1992, p.165) suggests that the division of larger numbers should be modeled by the sharing of base 10 materials. They claim that this approach would lead into a natural language which would direct the final algorithm. In addition, the base ten materials promote the underlying place value which provides further meaning to the division process (Booker et.al., 1992). I supported the use of base ten materials (base 10 blocks) in the teaching of division as I felt that it provided a concrete representation of a problem which the learners were able to manipulate in order to find a solution and thus it formed an integral part of my intervention. Furthermore, I believed that this was easily transferred to a pictorial and a numerical representation. These concrete and pictorial representations promote the development of numerical representations and the mathematical sense making process. Further evidence supporting the use of base 10 blocks to promote relational understanding of division is provided in the section about the teaching of the division algorithm.

2.5.4. <u>Division using repeated subtraction</u>

Division as repeated subtraction involves learners repeatedly subtracting the divisor from the dividend to determine how many groups can be made and if there is a remainder. While division using repeated subtraction best supports the quotitive interpretation of division it can also be used to solve partitive problems. For example in the partitive problem of sharing 24 marbles between 6 boys, learners would have subtracted 6 each time to show that they had given one marble to each of the six boys. In the quotitive problem of packing 24 marbles into bags of 6 learners would subtract 6 each time to show that they had filled a bag. Thus, repeated subtraction can be useful in both cases as it helps learners cluster the marbles into appropriate groups to make the process of practical sharing quicker.

Division as repeated subtraction can be used to link arrays with division as the quotient can be found by counting how many rows of the divisor could be taken away from the dividend. Repeated subtraction also highlights the link between multiplication and division - if the number of times the divisor was subtracted from the dividend and the divisor are multiplied, the answer will be the dividend.

Booker et.al. (1992, p.165) advocate that the notion of division as repeated subtraction can only be introduced after learners have understood the meaning of division, and it should be built onto the concept of sharing. The authors argue that repeated subtraction is only useful to certain problem situations and is not always beneficial to the development of the concept of division. Booker et. al. (1992, p.165) claim that the link to multiplication is the most important idea for the meaningful development of division. I disagree with these claims, as I believe that the notion of repeated subtraction is as important to the notion of division as the inverse of multiplication and both concepts are vital in the development of division as a concept. I see each pre-concept as important, as they promote the understanding of different aspects of the division concepts. Repeated subtraction is built onto the notion of breaking a number or group of objects into smaller equal groups or clusters either to be shared out or made into groups of a specific size. The link to multiplication is also built on the concept of clustering into groups and develops the learners' ability to use their knowledge of multiplication to estimate or calculate the number of groups and aids in identification and remembering of the division facts. Thus, it is important that teachers be aware of the strengths of each conception and use them to assist learners in developing a more complete view of division.

Haylock (2006) proposed the development of repeated subtraction to be used to form an ad hoc, informal division method rather than building up to the traditional algorithm. However, none of the other literature I read encouraged the use of this method of calculation. Repeated subtraction is a lengthy process that may be easier initially and promotes the grouping of knowledge into chunks. It does not, however promote efficient division strategies that can be extended into later learning. I felt that the use of base 10 blocks which can also be used to promote 'chunking' and the methods suggested in the development of the division algorithm, as described later in this chapter, promoted a relational understanding and would be easier for learners to work with and of greater use for later learning in Mathematics. Thus, while I included repeated subtraction in the intervention I used it as a stepping stone in the development of division concepts rather than an end product in itself.

2.5.5. <u>Development of the division algorithm</u>

Troutman and Lichtenberg (2003, p. 241) stress that it is important that both the teachers and learners feel ownership of the algorithms and take part in the development of these procedures. The authors (Troutman & Lichtenberg, 2003, p.244) emphasise that children should invent their own arithmetic, rather than just learn the conventional algorithms, as actions or operations can not be instilled into learners but need to be developed by the learners themselves. They continue to explain that children should be encouraged to use their current mathematical schemes to solve unfamiliar problems (Troutman & Lichtenberg, 2003, p.244). Troutman & Lichtenberg (2003, p.244) suggests teachers ask learners to use their knowledge of place value to validate their thinking when solving unfamiliar problems. Most literature advocates the ultimate development of the algorithm as it provides stability (Troutman & Lichtenberg, 2003, p.249), however, it should be the final division concept taught or developed. Conversely, Haylock (2006, p. 101) feels that the algorithms are archaic, difficult to master and should not be taught anymore. I feel that learning the division algorithm was beneficial to learners and I was confident that with the new approaches I had found in the literature I would be able to create a positive and effective learning experience for my Grade 5 learners. In the following section I explain how different authors propose the division algorithm be developed so that the teachers and learners are able to claim ownership.

Troutman & Lichtenberg (2003, p.242) claim that the ability to multiply and divide by multiples of 10, a sound understanding of the properties of multiplication and its computational procedures are essential prerequisite skills for division. Furthermore, before learners begin to develop the computational procedures for division it is important that their teachers check that the following knowledge and skills are also in place: a sound understanding of the meaning of division and the relationship between multiplication and division (Troutman & Lichtenberg, 2003, p.248). Troutman & Lichtenberg (2003, p.242) emphasise that the development of all tools, skills and concepts must be done by first seeing and doing, then reflecting and understanding.

Booker et.al. (1992, p.175) believe that the development of the division algorithm is an extension of the sharing procedure used for the concepts and the division facts. This was supported by Troutman & Lichtenberg (2003, p.241) who said that the distributive or partitive interpretation was best for the development of the division algorithm. They advocate that if this interpretation is used in conjunction with place value concepts, it enables students to easily construct the algorithm for the division of whole numbers and this knowledge can be easily transferred to division with decimals.

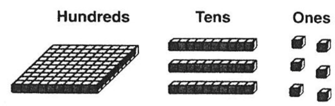
Booker et.al. (1992) and Troutman & Lichtenberg (2003) recommend the use of materials, such as base 10 blocks, to introduce the steps of the algorithm and to develop appropriate and meaningful language. The materials help learners work out what exactly is needed at every step of the algorithm, eventually forming the basis of the final symbolic form. Booker et.al. (1992, p176) cautioned that the use of concrete materials may encourage the learners to continue to share one at a time.

However, I felt that the base 10 blocks allowed learners to quickly progress to sharing in clusters according to place value. It is the teachers’ role to guide learners to the recognition that by beginning the sharing process with the largest place value digit then continuing to the smaller place value digits made the sharing process quicker. It is important that this is made explicit to learners as for all other operations the most common algorithms begin the process with the smallest place value digit. Booker et.al. (1992, p.177) emphasise that it is important that learners be provided with sufficient experience with materials to develop the concept and a new way of thinking before any recording is introduced. Furthermore, the recording process should match the actions with the concrete materials (Booker et.al., 1992, p.178).

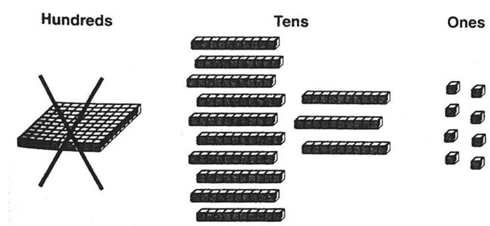
Booker et. al. (1992, p.177) stress that it is important that learners record all steps that are demonstrated when working with the base 10 blocks to ensure that the concrete procedures govern the many steps involved in the algorithm. For example they first need to represent the number using base 10 blocks. Troutman & Lichtenberg (2003, p.251) stress that special attention be given to situations where regrouping or trading is required. Troutman & Lichtenberg (2003, p.249) provide the following explanation as to how this can be done for the following computation.

$$6\overline{)136}$$

First represent 136 using base 10 blocks: hundreds, tens and ones (units).

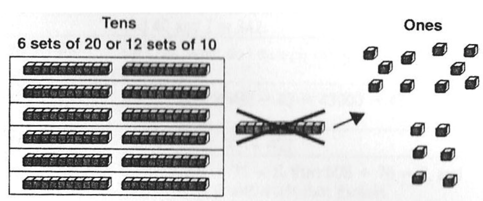


Then distribute the place value pieces into 6 sets each having the same number, if it is possible. Start with the largest piece. Make the sets as large as possible. The place value pieces that are 'left over' or cannot be divided into sets must be exchanged for the next largest pieces. In this case it is not possible to distribute the hundreds into six sets, so the hundreds must be exchanged for tens.



It can be seen that 136 has 13 tens.

The tens can then be distributed into 6 sets having the same number. The sets must be as large as possible. The 'left over' tens must be exchanged into ones. The number of tens must be recorded.



Computation:

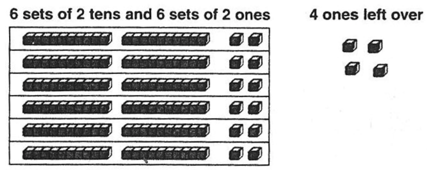
$$\begin{array}{r} 2 \\ 6\overline{)136} \\ -12 \\ \hline 16 \end{array}$$

2 ⟶ 2 in the tens place

−12 ⟶ 6 sets with 2 tens

16 ⟶ exchange 1 ten for 10 ones already had 6 ones

It can be seen that there are now 16 ones. These must be distributed into 6 sets. This must then be recorded and the number of left over must be indicated.



There are 0 hundreds, 2 tens and 2 ones in each set with 4 left over.

Computation:

$$\begin{array}{r l}
22 & \longrightarrow \text{2 in the ones place} \\
6\overline{)136} & \\
-12 & \longrightarrow \text{6 sets with 2 tens} \\
\hline
16 & \\
12 & \longrightarrow \text{6 sets with 2 ones} \\
\hline
4 & \longrightarrow \text{4 left over}
\end{array}$$

The number sentence can be recoded as follows.

(6 × 22) + 4 = 136

- 136 → total in original set
- 4 → a remainder of 4
- 22 → 2 tens and 2 ones in each set
- 6 → 6 sets

Only once the learners are confident in their understanding of the algorithm do Booker et.al. (1992, p.177) suggest that they dispense with some of the steps and move to short division. However, many of the authors reviewed did not promote the teaching of short division. Haylock (2006) claimed that the explanation becomes unwieldy and wordy.

Once learners have mastered the algorithm it is important that they build up to bigger dividends and divisors. It is essential that they become aware of the relationship between those two features and the quotient before they are introduced to the remainder. Toluk and Middleton (2004) claim that children find division with remainders challenging because most learning experiences fail to provide learners with opportunities to work with fractional quotients within a problem context. This was often included later in the learning experience. In the early grades I can understand that teachers may be concerned that if fractions and division are taught simultaneously it would confuse learners. However, at a Grade 5 level I feel that learners have had sufficient exposure to fractions and division. With further guidance, they will be able to understand the connection between fractions and division and it improves their understanding of remainders. In addition, an understanding of fractional remainders helps learners to understand the move to obtaining a decimal answer in future.

Although there are a number of steps that form part of the division algorithm, a good understanding of the underlying numerical concepts (Booker et.al. 1992, p.175) are viewed as supporting understanding of the algorithm. As mentioned previously, division requires learners to integrate a number of different concepts and operations when solving a division problem. For example, as the division algorithm is built up, so learners needed to 'trade' 1 ten for 10 ones. This process is used in subtraction and should not pose a problem for learners who have a sound understanding of the subtraction concept as the thinking required is no different (Booker et.al. 1992, p.175). The only 'new' concept is that of the remainder. The remainder is the part of the number that could not be shared using whole numbers and could be found once the division process is complete. Booker et. al. (1992, p.175) recommend that the concept of a remainder should not be introduced until learners have developed the concept of the algorithm and are secure in this knowledge. Learners are only introduced to continuing the division process into decimals to eliminate the remainder in Grade 7 and thus I did not include this area into the literature review. At a Grade 5 level learners are taught to represent the remainder as a whole number or as a fraction of the divisor. However, this can only be done in a meaningful way if learners understand the concept of a fraction.

Booker et. al. (1992, p.182) provide the following explanation to the recording and guidelines for appropriate language (see table 2.1).

Step	Language	Recording
Share the thousands	5387 divided by 6 What do you share first? (the thousands) Can you share 5 thousands among 6? No	6)5387
Share the hundreds	Trade for hundreds; 53 hundreds to be shared Can you share 53 hundreds among 6? Yes, 8 hundreds. Record with the hundreds. How many hundreds were shared? 48 How many hundreds remain to be shared? 5 Can you share 5 hundreds among 6? No	8 6)5387 48 5
Share the tens	Trade for tens; 58 tens to be shared Can you share 58 tens among 6? Yes, 9 tens each. Record with the tens How many tens were shared? 54 How many tens remain to be shared? 4 Can you share 4 tens among 6? No	89 6)5387 48 58 54 4
Share the ones **Record the remainder**	Trade for ones; 47 ones to be shared. Can you share 47 ones among 6? Yes, 7 ones each. Record with the ones. How many ones were shared? 42 How many ones remain? 5	897r5 6)5387 48 58 54 47 42 5
5387 divided by 6 is 897 remainder 5		

Table 2.1. Taken from Booker et. al. (1992, p.182)

2.6. <u>Special cases of division</u>

2.6.1. <u>Division with one and zero</u>

Learners do not find it difficult to understand that when you share out zero things each person gets zero, thus they are usually able to conceptualise that zero divided by any number will be zero. The sharing concept is helpful to learners trying to understand this instance. Conversely, the repeated subtraction conception does not necessarily promote learners' understanding of the concept of division of zero as learners would be confused by the fact that they are starting with nothing. Stronger learners may have recognised that when starting with 0 objects each group would receive 0. However, low attaining learners may find this a little more challenging. Division by zero is a source of much confusion for children and their teachers, and is one of the most challenging concepts to master (Booker et.al., 1992 and Troutman & Lichtenberg, 2003). Booker et. al. (1992, p.174) explain that the repeated subtraction conception suggests that zero could be taken away from the number infinitely many times and may give learners the view that division by zero gives infinity. This can be dealt with in one of two ways. Firstly we can simply say that division by zero is to be treated as 'undefined' with more explanation to come later. Alternatively, we can look at repeated subtraction using divisors that get closer and closer to zero – say 0.5, then 0.3, then 0.1, then 0.0001. This starts allowing learners to see that as they 'tend towards zero' repeated subtraction will give larger and larger answers and can lead into the notion that division by zero gives 'infinity' as an answer. Booker et.al. (1992, p.173) explain that because generalisations to further mathematics depend on learners understanding of division with zero it is important that this concept be developed so that learners are able to form a complete conception of division.

Learners find it difficult to separate dividing and multiplying by zero. They expect the answer to be zero when they divide by zero (Booker et.al., 1992, p.173) in the same way as the answer is zero when they multiply by zero. Booker et.al. (1992, p.173) explain that within the mathematical community a commonly accepted answer to a number divided by zero would be infinity. However, infinity is not a number and if you were dividing by zero the objects are to be shared by no-one. Consequently, sharing did not occur and accordingly dividing by zero is not possible. For this reason the answer to a question that involves a number being divided by zero is undefined. Anton (1995, p.2) confirms this explanation when he states that "in computations with real numbers division by zero is never allowed" and is said to be undefined. Furthermore, Booker et.al. (1992, 173) state that for division by zero to be possible it must also be possible to find a number that could be multiplied by zero to give a particular number as an answer. The repeated subtraction conceptual base incorporates the idea of 'stopping' at zero, thus conceptually $0 \div 7$ is valid. However, the use of repeated subtraction as a conceptual base for division posed some problems for learners (Booker et. al., 1992, p.174). For example, as mentioned earlier, in the question $0 \div 7$ learners try to repeatedly subtract 7 from 0. Booker et. al. (1992, p.174) continue to explain that according to the repeated subtraction conception when viewing the multiplication interpretation $7 \times 0 = 0$, $\therefore$ $0 \div 7 = 0$, does not pose a problem but when working with $7 \div 0$ learners are faced with a new challenge. Thus, the repeated subtraction conception is not helpful when trying to conceptualise division by zero. However, in the case of $0 \div 7$ it is more helpful to use the sharing conception to promote understanding and provide the learner with the appropriate interpretation. Accordingly when I introduced division involving 0 and 1 I used the sharing approach and began at a practical level – using base 10 blocks to assist learners in visualizing the context and providing the opportunity for them to see mathematical reasoning behind the facts for themselves.

2.6.2. Division with internal zeros

Division with internal zeros in the quotient is an important dimension of variation found in many of the different division concepts. It is evident when learners are unable to 'trade' or share part of what they have within a particular place value group. It is important that they learn to write the zero in the place where it is not possible to share. Omitting the zero is a common error that has been discussed later in the section on errors.

2.7. Beyond Grade 5 division

Troutman and Lichtenberg (2003, p. 232) suggest that once learners have a sound understanding of the division concepts that these are extended to, and integrated with, other topics in mathematics. For example as discussed earlier, the link between division and fractions or decimals can be explored.

2.7.1. Laws of division

Haylock (2006, p.86) explained the following laws of division, division is not commutative or associative but division is distributive over addition and subtraction. At a Grade 5 level it is not important that learners learn the names of these laws, as they would not be required to work with these properties. If they asked a question relating to these laws I would explain it to them, and for those learners who have mastered the necessary Grade 5 concepts and procedures, the distributive law would be informally introduced through the teaching of mental strategies. For example, when learners break up a number or divide the closest 'round' number then 'adjust' the answer. From this, the learners could be introduced to the written form of the strategy.

2.7.2. The ratio structure of division

Haylock (2006) introduces the ratio structure of division. He explains that the ratio structure is used to compare two quantities (Haylock, 2006, p.77). The process involves the inverse scaling structure of multiplication where the learner is trying to discover by what scale factor the one quantity must be increased to match the other (Haylock, 2006, p. 77). Haylock (2006, p.80) recognises that the answers to these problems can be difficult to interpret. While this problem ties in well with division as measurement it is beyond the scope of the Grade 5 curriculum demands and thus was not included earlier and I will not be addressing it in my intervention. This type of problem is usually only introduced at a Grade 7 level at my school.

This section in the literature review has explored the following core concepts of division:

- division as sharing – partitive
- division as measurement – quotitive
- division as the inverse of multiplication
    - division facts

While not explored as a specific concept all the above concepts explored the relationship between divisor, dividend, quotient and remainder and their meanings as critical features of division.

The teaching aspects that have been discussed in relation to how they could assist learners in extending the range of examples which they were able to work with were:

- contexts
- notation
- representations
- procedures

Most authors recommended starting with problems that enabled learners to use their existing knowledge and could be solved using concrete or representational methods although different authors focused on different approaches. For example: Troutman and Lichtenberg (2003) emphasised the relationship between division and multiplication, while Booker et. al. (1992) preferred the use of an array to represent the problem and writing it in the form of a division sum and accordingly emphasize the written representation of division. Each approach offered different insights to the learner and promoted the development of the specific concepts. The majority of the authors concluded their approach to teaching division with the development of the traditional algorithm. Similarly I hoped to draw on the different approaches to assist my learners to develop a sound understanding of division and the many dimensions of variation that fell under its broad umbrella before concluding the section with the development of the traditional algorithm. The following sections will briefly discuss the selection of examples before identifying the areas that the literature on division has identified as problematic and are possible causes of learners making the errors mentioned in this section.

2.8. Selecting examples for the intervention

I feel that the selection of examples is a vital part of the planning process, as this is the manner through which learners experience the majority of mathematical teaching and learning in Grades 4 – 7, including the manner through which I proposed to improve my sample of learners' understanding. Furthermore, in my encounters with variation theory, thus far, the greatest debates and key foci surrounded the selection of appropriate examples in the sense that it is through careful variation of questions that opportunities to discern key aspects of division were opened up.

Skemp (1989) describes the role of examples as activities that facilitate abstraction, which are unified through the formation of a concept and that subsequent examples can assimilate into the concept. Skemp's conception of examples ties in with his conception of relational understanding, which was described earlier. Rowland (2008, p. 150) distinguishes between two broad categories or types of examples that are used in teaching. The first type of examples are often called exercises. They are illustrative and practice-orientated (Rowland, 2008, p. 150). These 'exercises' are used once a learner has learnt a procedure. Learners practice the procedure over and over again until they achieve fluency. This type of example is the more common type and the selection of examples involves careful consideration to promote instrumental understanding but does not facilitate abstraction. The second type of example is inductive. This is when particular instances are given to demonstrate a general characteristic (Rowland, 2008, pg 150). It can be said that a general concept is taught through specific examples, thus facilitating abstraction. The general concept forms part of the knowledge base surrounding the concept and includes the appropriate strategies. Dockrell and McShane (1992) highlight the dynamic and important relationship between the knowledge base and the strategies. Based on the literature reviewed thus far and my experiences, I identified that one role of the teacher is to select relevant and meaningful activities and carefully structure a sequence of examples that promote a relational understanding and simultaneously develops both the concepts/knowledge and strategies. Accordingly, what students learn is dependant on the teacher's awareness of the possible opportunities for variation within that context (Watson and Mason, 2005a). While, inductive examples promote opportunities for highlighting variation and relational understanding, both types of examples have been highlighted as helpful in promoting learning for low attaining students.

Rowland (2008, p. 153) identifies four categories that examples highlight: variables, sequencing, representations, and learning objectives. An example may integrate more than one category. In his research Rowland (2008) analysed twelve different examples in terms of the four categories. Variation was highlighted in eight of the twelve examples, of which four of the examples were intentionally directed towards highlighting variation (Rowland, 2008, p. 153 - 161). What is evident in Rowland's article (2008) is that the selection of appropriate examples is challenging and that the selection of inappropriate examples can result in obscuring a concept that the teacher aims to highlight and confuse learners. What this means is that if teachers do not have a sound knowledge of the content that they are going to teach they may have difficulty selecting appropriate examples and as a result, may damage the learning experience.

Thus, Rowland (2008, p. 150) highlights, what I see as one possible problem or indicator of a problem, when he states

> *"a teacher's choice of examples for the purpose of abstraction will reflect his/her awareness of the nature of the concept ... what is considered paradigmatic rather than exceptional, and the dimensions of possible variation within that category."*

This means that teachers who do not have a good knowledge of the content may experience difficulty selecting appropriate examples as they are unaware of the critical features of the related abstract concept. In their study Rowland et.al. (2001) made a direct link between learner attainment and the teacher's content knowledge. This is a possible cause of the low learner achievement at my school, as the majority of teachers (pre Grade 6) are not trained to teach mathematics and have a limited mathematical background, and thus, lack three of Shulman's (1987) content related knowledge categories, namely: subject matter knowledge, pedagogical content knowledge and curricular knowledge. This leads me to the conclusion that if I want to see an improvement in mathematical attainment there will need to be in-service training regarding the concepts and selection of examples to assist those teachers who lack the subject content knowledge. However, this is beyond the scope of this study.

Troutman and Lichtenberg (2003, p. 232) emphasise what I see as another important consideration when selecting examples, they stress that it is important that the teacher should not allow examples to become predictable. For example, if the class has been working on division all week it is important that the follow up problems don't only involve division. If this happens the learners will simply be cued to identify the numbers and divide. However, if there is a mixture of different types of problems the learners will first need to recognise what the question is asking before they are able to identify what the required operation is. This was an important consideration in the planning of the pre and post-test questions for this research. Troutman and Lichtenberg (2003, p. 232) also suggest that teachers vary the output that learners must give. For example, learners can provide the computation, a pictorial representation, a number sentence or a written explanation or description. Further discussion surrounding examples and their selection will take place in Chapter 3.

2.9. Problems identified with the teaching and learning of division

Booker (1992 p.164) describe division "as one of the most difficult of the four operations to teach". This section will explore why teachers and learners find division difficult. It presents an overview of the literature surrounding the types of errors learners make when working with the division concepts discussed in the previous section. It then discusses some of the problems that have been identified in literature regarding the teaching and learning of division. Suggestions as to how these difficulties can be overcome and errors avoided will be included and these will be used in the development of the intervention.

2.9.1. Possible causes of difficulties

- Language

Booker et. al. (1992, p. 164) identify the two main reasons learners experience difficulty with division as the inappropriate language that is used, for example 'goes into' and 'how many … in …', and the meaningless rote learned procedures that dominate teaching. For example, when practicing the algorithm learners learn to divide, multiply, subtract and bring down with little to no focus on why they are performing the actions. In their analysis of the division algorithm and the associated teaching Booker et.al. (1992, p. 175) criticise the ill-conceived language that accompanies recording, the meaningless 'crutches' for example "Daddy, Mommy, Sister, Brother" to name the steps involved in long division, rote learning, the variety of recorded forms chosen in an ad hoc manner and the misleading names for, what they see as, one method. The authors state that the length of the procedure is governed by the size of the numbers and that it was only the amount of recording that varies. Thus they claim that there "is no such thing as short division" (Booker et.al. 1992, p.175). I support this claim as the same steps / procedures are used in what is known as long and short division. However, in short division some of the steps are not recorded and some of the information is recorded in a different place in each of the algorithm. Essentially, they are the same algorithm that was described in the section on developing the division algorithm. The 'short division' algorithm requires learners to mentally keep track of certain steps and thus demanded a higher level of thinking. For this reason, Booker et.al. (1992, p.175) claims that the 'short division' algorithm can only be considered short (easy) for those already skilled in division and able to process information in their head. Accordingly, the authors (Booker et.al., 1992) suggest that it is inappropriate to introduce such an approach to learners who are at the beginning of their development of division concepts as is commonly done in early schooling.

- Realistic contexts

Many authors claim that the best way to help learners find meaning in mathematics is to couch it in everyday problems. Gravemeijer (1997) contests this approach and draws attention to a problem with the 'reality' that teachers ascribed to word problems in that the problems do not match reality and learners are encouraged to refrain from realistic considerations and focuses on the mathematics or the problem. The author claims that word problems are stereotyped and do not require learners to reflect on their actions (Gravemeijer, 1997, p.389) but rather find the ‘trick’. Usually, the only ‘trick’ learners need to work out is what operation is required to solve the problem. As a result, learners “ignore relevant and plausibly familiar aspects of reality in answering word problems” (Gravemeijer, 1997, p.389) and consequently nullify the benefit of the problem solving. This view suggests that the classroom environment endorses a separation between school mathematics where the focus is on getting the right answers quickly as opposed to realistic everyday mathematics that draws on reasoning (Gravemeijer, 1997, p.389). This defeats the object of using everyday problems as they became yet another form of meaningless drill disguised in words and do not reap the benefit of learners being able to draw on their reasoning ability and informal knowledge. Gravemeijer (1997, p.390) recommends that teachers use a greater variety of word problems by varying the amount of data provided, asking for different outputs e.g. estimation, exact calculation or simply a representation of the problem, altering the number of steps required to solve the problems or even asking learners to formulate their own problems. This encourages the reasoning process and sense making within the mathematics classroom. I acknowledge the possible problems with everyday mathematics in the classroom. However, I believe that the benefits described by various authors earlier in the chapter regarding concepts of division and the recommendations made by Barnes (2005) in her work surrounding low attaining learners outweighs the possible problems and thus I have included the use of everyday examples in the intervention. What was taken from Gravemeijer’s (1997) argument was an awareness that careful attention needed to be paid to the selection of examples and the manner in which they were presented to avoid them becoming meaningless drill in ‘trick finding’, as was also emphasised in the previous section on examples.

- <u>Multiple steps and remainders</u>

Another problem that Booker et. al. (1992, p.165) identifies in the teaching and learning of division is that unlike the other operations where a basic fact associated with the operation is sufficient to give the result, division requires a multiplication fact to give a result close to the possible dividend, through estimation or a calculation done mentally or using pencil and paper techniques. Then an additional level of understanding and processing is required to determine the quantity that is unable to be shared (the remainder). Troutman and Lichtenberg (2003, p. 230) also recognise this problem when they acknowledge that for division, the answer is not always a whole number and the problems this poses for learners. Furthermore, the authors believe that some learners have difficulty representing division sums with remainders using formal notation (Troutman and Lichtenberg, 2003, p. 230). Remainders affect the interpretation of results across many concepts of division. For example they are found within the everyday problems, both interpretations of division, division as repeated subtraction etc. As a result, a poor understanding could impact negatively on the development of many concepts of division. Accordingly, I needed to continually make learners aware of the possibility of remainders within the results and how they affect the interpretation of a problem.

- **Direction of solution**

Booker et. al. (1992, p.169) explain that some of the problems that learners experience with division stem from the fact that it is necessary to read and solve division sums differently to the other operations. When solving addition, subtraction and multiplication sums using the traditional algorithm the tendency is to work from right to left starting with the lowest place value. In contrast, division sums are usually worked out from left to right starting with the highest place value. Furthermore, when translating from the division number sentence to the algorithm format the numbers are read in different directions. For example:

$42 \div 7$ is read from left to right but

$7\overline{)42}$ is read from right to left.

To overcome this difficulty Booker et.al. (1992, p. 169) recommend that learners be made explicitly aware of the different ways of reading and interpreting these problems. This is an area that had to be addressed clearly within the intervention.

- **Notation**

Booker et. al. (1992, p.165) recommend that the symbol '$\overline{)}$' be introduced right at the beginning as they believe that it promotes learners' understanding of the divisor and dividend because they feel that the symbol '÷' is not as effective. Booker et. al. (1992) argue that the '÷' symbol causes learners to believe that the bigger number must be divided by the smaller number. I did not agree with this claim as I think that the reason that learners draw the conclusion that the bigger number must be divided by the smaller number is that this is all they encountered in their initial learning experiences.

- **Links to fractions**

Even though learners begin working with fractions early in their primary school career they do not usually associate fractions and division. They would be able to explain that $\frac{1}{5}$ means a whole had been divided into five parts and they were working with one out of the five parts. But they seldom make the connection to $1 \div 5$ until Grade 7. In addition, in my experiences I found that it was not until learners reach Grade 7 and begin dividing with decimals that they realise that it is possible to divide a smaller number by a bigger number. I believe that the meaning attached to symbols and the relationship between the symbol and numbers is made meaningful as learners work with the concepts. In conclusion, I believe that the meaning is not gained from the symbol but rather learners attach meaning to a symbol and learn to interpret the symbol according to their experiences with it. Thus, teachers shape learners conceptions through the opportunities they provide for them to engage with the concepts. As working with examples is the manner through which learners engage with a concept, the selection of examples is key to the shaping of the learners experience. Accordingly it was an important consideration in the planning of the intervention. Examples, their role and selection have been explored in an earlier section and will be expanded on in the following chapter.

- Identifying the required operation as division

In my experience I found that many learners had difficulty identifying division as operation necessary to solve a problem. I have listed below the reasons Troutman and Lichtenberg (2003, p. 236) suggest as to why learners find this challenging. Learners

- do not know 'the meaning of multiplication and division'
- do not understand the different interpretations of division
- 'can't translate a written problem into a number sentence'
- have 'difficulty with basic facts'
- have 'difficulty reading problems when there are no aids or pictures'
- are unable to compute the given numbers

However, there are many other possible causes for this difficulty some of which have already been discussed. A number of the problems learners experience can be avoided if teachers are aware of learners' areas of weakness and try to pose questions in such a way as to minimize their impact. For example, if a child has poor comprehension skills the teacher could use a large font, simple language and provide supporting pictures to reduce the effect of the reading on learners' mathematical achievement.

Haylock (2006) suggest another approach for learners who have difficulty identifying which operation to use or how to write the appropriate number sentence. The author recommends that learners use a calculator or ask themselves "what would I put into the calculator to solve this problem?". He believes that this will help learners make the connection to division explicit and assist them in identification of the underlying mathematical structure (Haylock, 2006, p.76).

In this section the possible causes of the difficulties that learners experience was explored. I found that inappropriate language and the teaching of meaningless procedures that had to be learned by rote were the prime causes identified. Other areas of difficulty highlighted were the symbols, reading and interpreting number sentences, the many prerequisite skills and levels of thinking required to solve calculations.

2.10. Error analysis

I believe that errors, and error analysis, are an important part of the teaching and learning process. In this section I justify this claim and classify the different types of errors with particular focus on the errors that learners make while working with division. I identify possible causes of these errors and suggest how these might be overcome or avoided. Troutman & Lichtenberg (2003, p.287) explain that it is important that teachers are able to identify the possible causes of learners' difficulties and areas of weakness so that appropriate support can be provided. The authors advise that errors are an important signifier of a lack of understanding of a concept or concepts (Troutman & Lichtenberg, 2003, p.289). As a teacher and researcher planning an intervention it is important that I identified and classified the errors that my learners made as this helped me understand their difficulties and tailor an intervention that assisted them in overcoming any barriers to the development of division concepts. The following section classifies the errors making them easier to recognise and interpret.

Troutman & Lichtenberg (2003, p.289) identify three different areas relating to division that learners experience difficulties:

- place value and understanding the operation,
- difficulty with prerequisite skills and
- clerical difficulties.

Ryan and Williams (2007, p. 13) differentiate between six different types of errors - across mathematical working and not specific to division, which have been clustered into three broad groups:

- developmental errors, which included:
    - modeling
    - prototyping
    - overgeneralising
    - process-object linking
- errors with no obvious developmental or conceptual explanation – also referred to as slips
- uncertain diagnosis

These categories of errors have been compared and reviewed to provide a foundation for error analysis.

I feel that the possible difficulties learners experience with division as discussed earlier in the chapter can be classified and identified using the types of errors highlighted by Troutman & Lichtenberg's (2003) and Ryan and Williams (2007). The following paragraphs will explain how division errors can be classified within these general errors. Troutman & Lichtenberg's (2003, p.289) difficulty with prerequisite skills refers to learners who are unable to understand a concept because they did not acquire the necessary underlying concepts. For example, if a child does not understand subtraction and/or multiplication concepts they will not be able to comprehend the division concepts. A common division error or difficulty in this category would include the inability to use the basic multiplication facts to find related quotients or division facts (Troutman & Lichtenberg, 2003, p.292), as well as, incorrect computation of a number sentence, product or divisor. The omission of a zero can also be the result of a poor understanding of place value – a prerequisite skill. For example, if the answer was 240 the child might give an answer of 24 thus not recognizing the value of the zero. Ryan and Williams (2007) do not have a separate category for these types of errors, however, it is an underlying theme in their developmental errors.

Ryan and Williams (2007, p. 13) introduce the following four types of errors; 'modeling', 'prototyping', 'overgeneralising' and 'process-object' linking. These are described as developmental errors and the authors claim that they are a result of intelligent constructions (Ryan and Williams, 2007, p. 13). Ryan and Williams (2007, p.27) explain the concept of 'intelligent construction' as an indicator of the learner's state of knowledge and the current point in their concept formation process, it demonstrates connections or generalizations that the learner make, whether they are right or wrong. Modeling refers to when a child understands or represents a task context inappropriately or differently to the intended mathematical way. A common division error within this category is - difficulty identifying a division context (Troutman & Lichtenberg, 2003, p.292), and often results in the incorrect selection of an operation, incorrect writing of number sentences and the selection of inappropriate information to solve the problem or the inability to suggest appropriate missing information.

A prototype refers to a typical example of a concept. When a concept is taught it is most commonly done through the use of prototypical examples rather than defined mathematically. As a result, when some of the learners encounter a question that does not match the prototype, they are unable to solve the problem correctly (Ryan and Williams, 2007, p. 20). What usually happens is that the learners try to solve the atypical problem in the same way as they solved the prototypical problems without identifying that the problem does not match the prototype problem. A common error in this category occurs when working with some of the basic facts (Troutman & Lichtenberg, 2003, p.292) that do not match the prototype. For example, learners are unable to identify that when dividing by zero the answer is undefined as apposed to other typical division fact questions or when learners have difficulty regrouping and correctly aligning numbers when computing quotients of different sizes (Troutman & Lichtenberg, 2003, p.292).

Closely related to the prototypes are overgeneralisations. These errors most frequently occur when there is a variation in the theme, the rule used or the set of cases to which a rule can be applied is inappropriately extended. For example, when using short division learners may not have had trouble calculating questions that do not require regrouping/carrying over. When they move onto questions that require regrouping the learners simply omit the regrouping and do not carry over the remainder. The final type of developmental error refers to process object errors, these most frequently relate to structural conceptions. These errors occur when learners have not made the conceptual change of the process object reification. For example, when working on a repeated subtraction problem, learners may focus inappropriately on the process of subtraction instead of the object, which is how many times the divisor can be subtracted. As a result they may erroneously give an answer of 0 or whatever the remainder was instead of how many groups they subtracted. Another example of this occurred when I introduced the measurement concept of division to my intervention group and the girls assumed that the question was asking how many of the objects went into each group as was done in the sharing questions as opposed to how many groups were made. This error also highlights possible deficits within prerequisite skills as explained by Troutman & Lichtenberg's (2003). I believe that this type of error could also fall into Troutman & Lichtenberg's (2003, p.289) category of place value and understanding the operation which refers to an error that is a result of a lack of understanding of the focal concept itself or related concepts.

Ryan and Williams (2007, p.27) stress the importance of developmental errors as they enable teachers to diagnose the "child's state of knowledge" as they are the natural outcome of intelligent, but partial mathematical development, and they signal a learning opportunity or zone. The authors stress the importance of identifying the learning opportunity or zone as, with targeted teaching, it becomes an area for potential development.

Clerical difficulties described by Troutman & Lichtenberg (2003, p.289) refer to the errors that occur when children work too fast, don't pay attention or are bored. Clerical errors are often called 'careless mistakes'. Ryan and Williams (2007, p. 13) refer to these errors as 'slips' and explain that they have no obvious developmental or conceptual explanation. This type of error often occurs when a child is rushing through a piece of work and copies a sum incorrectly or only answers part of a sum. Ryan and Williams (2007, p. 14) summarise that these errors are in contrast to the child's actual working practice and are often the result of poor motivation and high anxiety commonly associated with assessment conditions. These errors can also be attributed to a cognitive overload, learners jumping to conclusions, reading difficulties or the use of a misremembered fact (Ryan and Williams, 2007, p. 13).The authors explain that although addressing these errors by teaching the learners problem-solving strategies may improve assessment results they are of little interest to the teaching and learning process (Ryan and Williams, 2007, p. 14) as they are not a result of a misconception or lack of understanding.

Errors of uncertain diagnosis refer to those errors that cannot be diagnosed with the available evidence and required further engagement and discussion surrounding the reasoning behind the error.

Finally, Ryan and Williams (2007, p.27) explain that if an error is addressed at a superficial level by teaching a meaningless rule or process, it does not help the child in the long term, as the learner's underpinning knowledge, conceptions or misconceptions, have not been engaged with. Thus, one of the aims of my intervention was to engage with learners' current conceptions and misconceptions before teaching any new concept.

In this section the different types of errors were explored. In summary, I felt that the following categories, drawn from the two main authors reviewed in this section, best encapsulate the three common collections of errors:

- developmental errors,
- difficulty with prerequisite skills,
- clerical errors.

I explored the possible reasons behind the difficulties and errors and found that inappropriate language and the teaching of meaningless procedures that had to be learned by rote were the prime causes identified. Other areas of difficulty highlighted were the symbols, reading and interpreting number sentences, the many prerequisite skills and levels of thinking required to solve calculations.

This chapter discussed concepts relating to cognition and low attaining learners to provide a better understanding of the sample of learners. It then gave an overview of various concepts relating to division and the teaching of division, common areas of difficulty and associated errors. The concepts, relevant to Grade 5 highlighted in this chapter provided the core focus of the intervention as the aim of the intervention was to improve learners' knowledge of these concepts and procedures of division. The pre and post-test was used to establish which concepts and procedures the learners had mastered and which shoed evidence of errors. Accordingly, I was able to measure their progress during the intervention and the error analysis enabled me to diagnose and address misconceptions. In the following chapter I provide detail on the theoretical framework of variation theory, which underpinned the design of both the pre and post-test and the intervention in this study.

Chapter 3

Theoretical framework – Variation Theory

In this chapter I explain what variation theory is and justify why I chose it as my theoretical framework. Variation theory is a theory about learning. It differs from the Piagetian theory of cognition in that it does not simply focus on the process of learning but recognises that learning takes place in an environment and is the result of an experience or experiences which are perceived and interpreted by the learner. The lens through which the learner interprets the experience depends on what they have learnt or experienced previously, and what context they discerned it from (Marton & Booth, 1997, p.108). In this chapter I explain why I felt that variation theory provided an integrated and useful approach to learning by simultaneously taking into account the process of learning, the learner, the concepts to be learnt and the pedagogical approaches.

Liljestrand and Runesson (2006, p. 165) illustrate the relationship between learning and variation in variation theory when they explain that:

> *"in order to understand or see a phenomenon or situation in a particular way, one must discern all the critical aspects of the object in question simultaneously. Since an aspect is only noticeable if it varies, the experience of variation is a necessary condition for learning something in a specific way."*

Liljestrand and Runesson (2006, p. 181) describe the context in which learning takes place as the 'space of learning' and state that it is characterised by the joint interaction of the different features of the object that are focused on, and opened up, as dimensions of variation. They continue by saying that the features to which learners' attention is drawn to, determines what is *possible* for learners to learn (Liljestrand and Runesson, 2006, p. 181). Marton and Booth (1997) provide some insight into this discussion when they state that "we learn from discerning variation, and what varies in our experience influences what we learn." Watson and Mason (2006a, p.5) claim that the use of variation for the exposition and introduction of new procedures and techniques provides openings for learners to engage with hard mathematical thinking.

3.1. <u>Concepts within variation theory</u>

Various concepts relating to variation theory were introduced in the first chapter. I will begin this section with a brief overview of these concepts. The following concepts were introduced in Chapter 1:

- Learning: 'the acquired knowledge of something' (2004, p.4 - 5)

- Process of learning: ‘becoming capable of doing something as a result of having certain experiences’ (Marton et.al, 2004).
- Learning mathematics: “becoming acquainted with generalizations of several types: concepts, techniques, classes of objects, properties, relationships and theorems.” Watson and Mason (2005a, p.1). In other words the acquisition of a mathematical concept or skill / becoming capable of doing something mathematical.
- Object of learning: ‘a capability’ Marton et. al. (2004, p.4 - 5) or ‘that which is the focus of attention’ Watson and Mason (2006b, p.100 - 1).
    - Indirect object of learning: remembering, ‘discerning’ which means noticing as a result of experience, not just being told (Marton et.al., 2004), interpreting, grasping or viewing the acts of learning.
    - Direct object of learning: the ‘thing or subject on which these acts are carried out on’ (Marton et.al., 2004, p.4-5), within the mathematics classroom the thing or subject can be seen as a concept or skill.
- Critical feature: the object of learning is defined by ‘critical features’ that must be discerned in order for learners to reach the desired meaning (Marton et. al., 2004, p.22).
- Dimension of variation: the different strategies or types of examples that make it possible for the learner to discern the critical feature/s (Marton et.al., 2004, p.15).

I would like to expand upon two important concepts relating to variation theory – dimensions of variation and patterns of variation. Dimensions of variation refer to all the aspects that may vary within a single concept while promoting the development of the understanding of the critical feature/s. For example: in a single example learners may use counters, pictures, informal written methods, recall of division facts or formal written methods to find the solution. In this case, the strategy used to solve the problem has been varied and the critical feature of the relationship between the dividend and quotient was explored in various forms although the relationship / numbers have remained the same. This was just one example of the many possible dimensions of variation within a division question. Liljestrand and Runesson (2006, p. 166) argue that the possibilities of learning are defined by how the critical aspects of the objects of learning are brought out in the learning situation and that it is important that each critical aspect of learning must be experienced as a dimension of variation.

Within each dimension of variation there is a range of change which covers all the permitted changes within a single dimension of variation (Watson and Mason, 2006a, p.5). In other words, the range of change describes how the aspects within a dimension of variation can be varied (Watson and Mason, 2006b, p.98). An illustration of a common question in which this type of variation can be used at a grade 5 level is a sharing problem involving concrete objects such as sweets. When working with division the divisor and dividend are usually limited to three, two and one digit whole numbers. However, the range of change can be broadened, depending on the group's abilities and as division is extended into fractions and decimals the idea that a smaller number can be divided by a bigger number can be introduced. It is important that all mathematics students are aware of the extended range of change as it is helpful to learners if they are made aware that what they are learning is a small part of something much bigger, and something that they will be exploring in future years.

Discerning variation involves the identification, or noticing, of features that change or are different. These features are referred to as variants or aspects that vary. The identification of features that are common or do not change are referred to as invariants (Ling et.al., 2005, Watson and Mason, 2005b). Discerning variation enables learners to become aware of the dimensions of variation which are possible, and which constitute the scope and the extent of the concept being exemplified (Watson and Mason, 2005b, p. 3). In the incidences described by Watson and Mason (2006b) the activity of discerning variation occurs within a specified set of new examples or between a new example and existing knowledge, or simply through the drawing together of a collection of existing knowledge. It also occurs when learners are given the opportunity to attend to variation through observation, exploration or actively engaging with examples that highlight the variation (Watson and Mason, 2006b, p.101 and 102). However, the timing of the activities is important as the variation has to be detected from near simultaneous experiences. If the activities are too far apart the learners experience succession rather than variation (Watson and Mason, 2005b, p.3). Furthermore, the authors explain that even when dimensions of variation are used to highlight the variation it does not mean that the learners recognise the variation or dimensions of possible variation (Watson and Mason, 2005b, p.5). Watson and Mason (2006b, p.103) encourage teachers to ask learners questions such as;

> "What changes and what stays the same?"

- as this assists in highlighting the variables and invariants. Another useful way of identifying possible variation is to ask learners to construct mathematical objects that meet specific constraints (Watson and Mason, 2005b, p.6 and 7). The constraints specify invariance and indicate the dimensions of possible variation within the concept (Watson and Mason, 2005b, p.7).

Identifying features that vary, and those that remain the same, is not enough. Learners need to recognise that they are able to control the variation and it is important that they understand how and why some features vary and others do not, both within the definition of the object and when the component variables interacted (Watson and Mason, 2005b, p.5). Watson and Mason (2005b, p.5) highlight that it is unreasonable to expect learners to reconstruct every possible dimension of variation. Thus if the teacher is aware of the possible dimensions of variation for defining an object it is advantageous to their learners, as they are able to assist and guide them.

As mentioned earlier, the linking of old and new knowledge is important as it promotes relational understanding (Skemp, 1989). I believe that the links between new knowledge and existing knowledge can be developed, and enhanced, through the identification of critical features through variation, and invariance, patterns and contrasts between similar/related concepts. Thus, the use of variation theory can promote relational understanding and improved mathematical attainment.

The second concept I would like to expand on is patterns of variation. These patterns are built into the series of examples used to highlight the dimensions of variation. Liljestrand and Runesson (2006, p. 166) explain that a pattern of variation occurs when 'the principle is invariant, whereas the examples vary … constituting a pattern of variation and invariance'. Four distinct patterns of variation have been identified in the literature: contrast, generalization, separation and fusion (Marton et.al, 2004; Ling et.al. 2005). Each of the concepts will now be discussed briefly with an example to illustrate the concept.

Marton et.al. (2004, p. 16) explain that in order to understand what is possible to learn and what is not, it is important to 'pay close attention to what varies and what is invariant in a learning situation'. This is because invariance can only be recognised if something is changing, and change will only be recognised if something else is invariant (Watson and Mason, 2005b, p.2).

Contrast is described as experiencing something by comparing it to something else that is different but related (Marton et.al., 2004, p. 16). For example, in question 1 in the pre-test (see Appendix D) learners were asked what 72 ÷ 6 was. The intention of the question was to investigate the extent to which learners who have been exposed to the multiplication fact of 12 x 6, can contrast this information with the question involving the inverse operation. The question was intended to allow learners to see numbers that they were familiar with in a multiplicative relationship juxtaposed in a different relationship. Learners investigated how these related questions are inverse representations of the same number sentence.

Generalisation involves the 'experience of varying appearances' (Marton et.al., 2004, p. 16) or a sensing of the possible variation within a relationship (Watson and Mason, 2006b, p. 94). For example, a division question could be solved using long or short division. Alternatively, learners could be given the same problem that requires different representations or be asked in different forms i.e. numerically, pictorially or as a real world problem sum. Watson and Mason (2005a, p.1) claim that all learners generalise in all situations and they could therefore become frustrated when it is not clear how to do so mathematically. However, through the careful selection of examples learners can be guided into appropriate generalising.

Separation involves learners experiencing one aspect by separating it and varying that aspect while all other aspects remain constant (Marton et.al., 2004, p. 16). For example, if the aim of an exercise is to vary the value of the dividend all other aspects must be kept the same except for the increasing size of the dividend.

Finally, fusion entails several critical aspects being experienced simultaneously where learners are required to recognize and take into account several different dimensions of variation simultaneously (Marton et.al., 2004, p. 16). Ling et.al. (2005) support this explanation as they describe fusion as the combining of features discovered in previous experiences and that fusion requires learners to be aware of different features of differing value from different examples at the same time. The simultaneous drawing together of knowledge is called fusion (Ling et.al., 2005). In my experience, at a primary school level, learners are introduced to new dimensions of variation one at a time. The teacher usually waits for the learners to master a particular dimension before introducing the following dimension. Once learners master a second dimension of variation, a revision exercise is done in which both dimensions of variation are assessed within the same problem or sum, thus requiring fusion of several critical aspects. In division, learners are required to draw on their knowledge of problem representation and procedures associated with division, multiplication and subtraction in order to solve complex questions that go beyond the basic facts and require a calculation. Fusion relies on learners being aware of, and ‘noticing’, several critical aspects, including the relationship between them. If learners are able to draw the appropriate aspects into focus and allow irrelevant aspects to fade into the background the process of fusion occurs naturally. However, those children who were unable to identify the relevant features need guidance in order to experience fusion.

A danger I see in the use of variation theory is that if teachers are not careful, they could cause learners to compartmentalise their knowledge by separating the teaching of concepts as specific objects of learning and not promote the broader linking of knowledge and a relational understanding. I recognised the importance of relational understanding as I discovered that learners found it easier to learn concepts in smaller ‘chunks’ that build on the previous dimensions of variation. I believe that relational understanding comes from the careful development of patterns of variation where learners begin with known concepts and built up new dimensions of variation using carefully planned patterns of variation and ending with the fusion of concepts where learners are encouraged to connect the new knowledge in relevant and meaningful ways.

Liljestrand and Runesson (2006) distinguish between two main ways of highlighting variation. The first involves the selection of examples where critical features of the concept are contrasted with different examples that are related in different ways (Liljestrand and Runesson, 2006, p. 166). Therefore, according to the theory, if I am trying to get learners to conceptualise a quotitive division problem (Neuman, 1999, p. 103), which involves the measurement of one variable, I should contrast it with a partitive division problem (Neuman, 1999, p. 103), which involves two measurement variables. This is demonstrated in Worksheet 1 (see appendix F).

The second method involves learners being exposed to a variety of examples, which highlight a particular aspect of variation through a pattern of variation (Liljestrand and Runesson, 2006). For instance, if I am trying to highlight the role of place value as a critical feature of the relationship between the dividend, divisor and quotient within all concepts of division, I would use a series of examples that highlight this in a pattern of variation (Liljestrand and Runesson, 2006). In another example, the same method could be used to solve a problem while the dividend increases in size. This will highlight the effect of place value. An illustration of this is in Worksheet 3 (see appendix H) in the questions about division involving multiples of 10. In these questions the only aspect of variation is the increased size of the dividend. Both ways of highlighting variation are important, however, the selection of the strategy depend on the type of variation that needs to be highlighted.

3.2. <u>Selection of examples</u>

I will now discuss the implication of variation theory for my teaching and in the selection, structuring and sequencing of pre and post-test questions as well as examples in the intervention. If I want my children to understand a particular concept - the object of learning (Ling et.al., 2005, p. 51) - in this case, a division concept, I need to design my lessons in a specific way, making use of activities and questions to highlight the critical features that distinguishes the division concept from other similar or related concepts. Watson and Mason (2006a, p.3) explain that perceived variation and invariance generate expectations. When their expectations are confirmed the learners' confidence grows (Watson and Mason, 2006a, p.3) thus enhancing the learning experience. However, it is important that the learners actively make the observation and recognise the relationships as, the authors explain, if the learners think that the examples are random they will not be able to generate an expectation, receive the confirmation and feel the confidence (Watson and Mason, 2006a, p.3). Accordingly they will not make the mathematical connection and learning will tend to be random and inefficient. The importance of learners' participation in the learning process is emphasised by Barnes (2005) and has been discussed in detail in Chapter 2. As a result, Watson and Mason (2006a, p.3 and 2005a, p.9) recommend that teachers carefully consider the patterns and expectations that they would like learners to develop before selecting examples for learners to work and engage with, as it is their responsibility to organize the experience rather then simply exerting their mathematical authority. If learners are to experience clarity within significant mathematical variation and structure, develop well-founded expectations, make connections and generalizations and identify exceptions or ‘special cases’ they need to engage with a number of carefully selected and sequenced examples (Watson and Mason, 2006a, p.4). If there are too many patterns within the collection of examples it is unlikely that learners will generate appropriate, if any, expectations (Watson and Mason, 2006a, p.4). If there are too few opportunities to develop expectations then learners are forced to rely on the teacher to tell them what they should be seeing and point out the connections (Watson and Mason, 2006a, p.4). In effect, they will not be able to make sense of the concept themselves and will therefore not ‘own’ the concept.

Another important consideration is that not all learners experience the learning experience in the same way. Consequently, each learner may have discerned a different aspect of the object in question (Liljestrand and Runesson, 2006, p. 165). However, if learners are able to see an object in a certain way it is important that they are able to discern certain features of that object (Marton et. al. (2004, p.10). Liljestrand and Runesson (2006, p. 165 and 166) suggest that teachers assist learners by creating a learning situation that promotes the discernment of all the necessary aspects at the same time. Furthermore, I believe that it is up to the teacher to engage the learners in discourse that enables him/her to ensure that all learners’ attention is directed appropriately.

When planning a teaching or assessment sequence it is important that teachers begin with learners’ current or initial perceptions of the mathematical object (Watson and Mason, 2006b, p.108 and 109). From there, the authors recommend that teachers help learners analyse conventional concepts that they could encounter, as well as special cases or exceptions that will assist them in the generalization of concepts by selecting examples that highlight the variables and invariant aspects (Watson and Mason, 2006b, p.108 and 109), the range of permissible variation.

Watson and Mason (2005b, p.4) highlight the fact that the amount and nature of variation offered to illustrate a point is an important consideration when planning activities. There needs to be enough variation to make the invariance obvious and enough invariance to make the variance obvious (Watson and Mason, 2005b, p.4). What the authors are unable to clarify is what counts as enough. This is left up to the teacher to decide and unfortunately is an area that takes much practice and learning from experience to get to grips with. There were areas in this study where I believe I found the correct balance, and other areas I needed to rethink for future teaching, as there were too few or too many examples.

Variation theory provides a framework that I believe will promote learning within a mathematical context. It assisted me in indentifying which features were critical to the development and understanding of the concept of division. It also provided a lens through which examples relating to the critical feature could be analysed, evaluated, selected and presented. It gave me guidance as to how attention and awareness could be drawn to aspects that are essential to the learning and understanding of a division and finally how the development of a concept should take place.

In the next chapter I will introduce my research methodology and discuss the various aspects involved in the planning, implementation and analysis of the pre and post-test as well as the intervention.

Chapter 4

Research design/methods

My research was a case study, action research project using a ‘quasi’ experimental methodology. Figure 1.1. in Chapter 1 demonstrated the process that I followed and the teachers and learners that participated (with informed consent having been given).

Although my research followed an experimental format, with a pre-test, an intervention, an initial and delayed post-test, it cannot be considered true experimental research in the sense of maintaining controlled conditions, as there were many other factors that may have affected the learners performance.

Cohen and Manion (1980, p.174) define action research as a

> “small-scale intervention in the functioning of the real world, and a close examination of the effects of such intervention”.

Accordingly, my study was considered action research as it involved a small-scale intervention in a real classroom and the effects of the intervention were measured. Denscombe (2007, p.123) provides four defining characteristics of action research: practical, change, cyclical process, and participation. My study complied with all four of these characteristics. It was practical and dealt with a common problem encountered in teaching mathematics at a primary school level. I hoped to invoke change with the learners participating in the intervention. The cyclical process took place as I planned, implemented, assessed and revised future lessons and assessments in my intervention. Finally, the learners’ participation was as important as my own participation.

As a researcher I was a complete participant in the study. Opie (2004, p. 129) describes a complete participant as when the researcher is completely immersed in the participant role and I used this position to conduct the research.

In this chapter, I provide detail on the key data collection instruments and strategies used in this study. I outline how the instruments were administered and the analytical methods that were used to interpret the data gathered from their administration. A discussion of validity, reliability and ethical considerations have been included.

4.1. <u>The pre-test</u>

In my experience, Grade 5 learners find it difficult to explain their mathematical reasoning and understanding. Thus, questionnaires where learners were asked to explain their understanding of division were likely to be of little use. Interviews could be used to clarify findings in the pre-test and provide me with a more holistic view of learner's misconceptions, competencies and understandings. However, their value would depend on the learner's ability to articulate their reasoning and thus, I did not intend to incorporate interview instruments. The instrument I chose to use was a pre-test as this format was familiar to learners. Accordingly, the first step of my research involved the planning of a pre-test. The pre-test was designed using dimensions of variation that I initially viewed as critical to understanding division. Through the process of extending my reading of the literature surrounding division and variation theory, as well as the analysis of data, I became aware that I began my study with a more procedural orientation to division, as was evident in the pre-test. This was however broadened into a more conceptual orientation in the intervention and post-tests. The selection of examples for the pre-test included features drawn from literature on problems with division in mathematics learning. They incorporated the patterns of variation explained in Chapter 3: contrast, generalization, separation and fusion as suggested by Marton et. al. (2004, p. 16) and aimed to highlight the development of division and common misconceptions in division as suggested by Neuman (1999), Lichtenberg and Troutman (2003), Ryan and Williams (2007) and Cooper et.al. (1999) in their literature on teaching division and associated common misconceptions. Due to time constraints I could not test all possible dimensions of variation within division or all types of division examples, thus, my selection of questions excluded some dimensions of variation. The examples excluded were mainly those falling outside Grade 5 as specified by the curriculum, did not highlight a common misconception frequently experienced in Grade 5, or did not fit within the set of dimensions of variation that I initially identified as important for the learning of division in Grade 5.

Due to the fact that there was limited time available to administer the test I separated the pre-test into two smaller tests, both containing a similar number of 'easy' or short questions and more complex long questions, thus, ensuring the nature and level of the tests was balanced. Different dimensions of variation were included in each pre-test and between the two pre-tests I was able to assess all desired all dimensions of variation. Results indicated that the pre-tests were not as evenly balanced as I had expected. Details on this are provided in the analysis in Chapter 5. In each class half the children wrote the one test and the other half wrote the other test. The tests were evenly distributed throughout the class according to the first term's results. This way I was able to receive feedback on all questions from a cross section of the entire class and pick up common misconceptions across the grade and range of abilities as opposed to within one class or ability group. In Appendix D I have justified the selection of each specific division question across both pre-tests. The pre-tests presented to learners are shown in Appendix E. Questions relating to the 'other' three operations (addition, subtraction and multiplication) were included in both pre-tests to ensure that the learners did not guess that division was required and to see if they were able to identify the right operation to use. Problems 1 to 19 are computational, whereas problems 20 to 26 are situational or real life - based (Neuman, 1999, p. 102). Various concepts were assessed in the pre-test. For example, knowledge of the division facts, the relationship to multiplication and halving to name a few. However, as stated already, the focus of the pre-test was more on procedural knowledge. I used my experience of setting tests and class activities to structure the pre-test in such a way as to follow the progression of questions used in all my class tasks and assessments.

Finally the pre-test was designed in such a way as to answer my first research question:

> "What are the specific features that learners struggle to understand within the concept and procedures associated with division at a Grade 5 level?"

and to provide data that assisted in the planning of the intervention.

4.2. <u>Administering the pre-test</u>

I approached the three Grade 5 teachers that I worked with and asked them if they would be willing to allow me to use data from the books and assessments from learners in their class once the parents and learners had signed the informed consent letter (see appendix A). In addition, I asked if they would allow me to design the revision exercise for the term, which would be used as the pre-test for my study and provide me with the important data that I needed for my research. The format of the test and questions was drawn from the style of question which learners were familiar with and had been working with. According to learners' books and exercises the focus of teaching and learning was largely procedural. Thus, the format of the pre-test was largely of a procedural nature a format that I broadened within the subsequent intervention.

All learners wrote the pre-test as it formed part of the end of term revision done in class. End of term revision exercises are not always included in the mark schedule but form an important part of consolidation. They assist teachers with the planning of the following term's work, as they highlight areas that need to be revisited. The mathematics teachers administered the pre-test during one of the regular mathematics lessons. As stated previously the tests were handed out based on the previous term's performance with the top student in each class completing test one, the second completing test two, the third completing test one, and so on.

The pre-test provided me with some insight into the learner's procedural competency and conceptual understanding, as well as, highlighting misconceptions through the formal and informal methods the learners used in supporting calculations. While the focus of the questions may have been procedural the strategies that learners used to solve the problems provided insight into their level of conceptual understanding of division in relation to the context surrounding the question. For example, question 1 of the full pre-test could be solved in a variety of ways. If learners did not attempt the question it could demonstrate that they did not recognise the operation in question and thus did not have access to the knowledge necessary to attempt the question. If learners drew a pictorial representation of sharing 72 into 6 groups or into groups of 6 they would have had an understanding of the concept of division and it also demonstrated their preference for partitive or quotitive division and possibly a familiarity with the array-based representation. If the learners solved it using the multiplication fact of 6 x 12 = 72 they demonstrated an understanding of the relationship between multiplication and division. Finally, if learners solved the question using the division fact they had demonstrated a reified relationship between multiplication and division and the ability to simply draw on the division facts. If learners made errors, it was still possible for me to identify the level at which they conceptualized the problem if it was not a slip or careless error.

Another example would be the manner in which question 13 of pre-test 1 assessed the learners understanding of a remainder. Learners did not always recognise that the remainder was kilometers that still had to be cycled. This knowledge of the concept of a remainder proved to be a problematic area for the whole cohort.

I acknowledge that using a pre-test is not without its disadvantages. As the teachers administered the test, there were opportunities for bias. Based on Opie's (2004, p. 103) description of bias, I defined bias as a less than truthful representation of information or a 'skewing' of the results. One area where bias may have occurred was if teachers helped or provided support for learners during the assessment. To try to overcome this problem, I spoke to the teachers and emphasised the importance of them not assisting the learners before the pre-test was administered. I explained that in the same way that it is important that they assess the same class work, under the same conditions, for assessment and evaluation purposes to make the results a fair reflection of the learner's performance in comparison to the rest of the grade, it was important for my research that we minimized any possible factors that may affect the learners performance in the pre-test. I needed to evaluate common errors that learners made and if a teacher provided guidance it would mask any misconceptions that the learner may have had. However, if a learner was demonstrating signs of stress that would be detrimental to them or their learning, the teacher could guide the learner with the proviso that any support administered should be noted as the results would not be a true reflection of the learner's capabilities but indicate a possible area of inadequate understanding or misconception.

The information obtained from the pre-test served two purposes; firstly it assisted me in identifying the nature of the errors that learners in Grade 5 make. Through the analysis of these errors I was able to identify the misconceptions prevalent within both Group A and B learners. This aided in the development of my intervention as I was able to focus on addressing Group A's common errors and misconceptions. As my aim was to help the Group A learners reduce the gap between the Group A and B learners it was helpful to have a picture of the errors and misconceptions on questions (concepts and procedures) that Group A perhaps would not attempt as a low attaining learner. Furthermore, as Group A was very small and low attainers they had not had much experience with some of the dimensions of variation and were therefore not able to answer all the questions. Consequently, my analysis of the Group B responses to the pre-test assisted me in identifying common misconceptions that were not evident in Group A's responses. In addition, when I designed my intervention, I was able to address common misconceptions directly within the design. The second purpose of the pre-test was to identify the current level of the learners in Group A, in terms of their understanding and level of achievement in relation to their peers in Group B. I hoped that through the intervention I would be able to close the gap between the sample of learners and the mainstream Group B learners within the grade as was stated in the rationale. Learners' responses guided the dimensions of variation covered in both the intervention and the post-test.

Using the results from the pre-test, as well as literature on division and variation theory I designed an intervention that focused on the misconceptions / errors that were identified in the pre-test and informed by the broader mathematical community through the literature reviewed. I used the dimensions of variation to expose the critical features in relation to these to try and extend the learners and build on their current understanding of division. The design of the intervention context (as defined in the following section) was tentative and was adapted on the basis of learners' responses. I conducted the intervention with Group A. The planning and implementation of the subsequent intervention will be discussed in the following section.

4.3. <u>The intervention</u>

The intervention involved me teaching a series of lessons building on the dimensions of variation using the different patterns of variation in relation to the literature on division and evidence of errors from the pre-test. The intervention provided support for my second research question: "How can variation theory be used to design an intervention to improve learners understanding of the concepts and procedures of division?"

I estimated that the intervention would take approximately two weeks with eight 30 minutes periods each week. I recorded my discourse during the intervention so that I was able to analyse the different aspects of my teaching and identify their possible effect on the learning experience. For example, through the analysis of the language I used, I could evaluate the meaning that learners were able to make and then observe the effect on the learners' progress and development. It allowed me to accurately report on learner's responses using the same language that they used.

Throughout the intervention I arranged for the learners to have access to the following support materials, unless otherwise stated: Dienes blocks (base 10 blocks), times table grid and chart (see appendix S) and plastic counters. The Dienes blocks and plastic counters provided a concrete tool to those learners who required concrete objects to make sense of a mathematical problem. Furthermore, I planned to use the Dienes blocks to lead learners into long division in a way that I hoped would encourage sense making. Knowledge of the times tables was identified in the literature as a key tool to understanding division and solving problems. A lack of access to / fluency with the times tables created an unnecessary barrier to learning division. As many of my learners struggled to remember the times tables, the use of the times table chart helped to negate this barrier to learning division.

The following sections provide a brief outline of the concepts / objects of learning and pedagogical focus of each of the lessons within the intervention. This will be highlighted briefly at the start of each lesson description, as well as the critical feature addressed and dimensions varied. The order given below is the order in which they were introduced in the intervention. The following areas were included in the intervention:

- division as sharing – partitive
- division as repeated subtraction – quotitive
- symbols – reading and interpreting
- division as repeated subtraction
- division as the inverse of multiplication
- missing factor
- division facts
- division with 1 and 0
- representations - formal and informal including algorithms
- laws of division (only through ongoing development of mental strategies)

The focus of worksheets within each lesson was on the direct object of learning. However, other indirect objects of learning were included in each lesson. For example in lesson one we discussed the similarities and differences between quotitive and partitive problems. Furthermore in my attempt to broaden my teaching to include a more conceptual focus I introduced learners to different types of questioning. For example learners were given a number sentence and had to write their own questions.

4.3.1. <u>Lesson 1</u>

Object of learning: Quotitive and partitive division

Different parts of the division sum – divisor, dividend, quotient

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Wording of questions to give a quotitive or partitive sum

I began my intervention with practical, real-world problems in the first lesson. This enabled learners to use their informal knowledge and experience to solve both quotitive and partitive problems with the help of concrete materials, as suggested by Booker et.al. (1992, p. 166). The focus was on creating an awareness of the quotitive and partitive sums through highlighting the invariants and variables, as well as, highlighting the different parts of the division sum, the terminology and relationship between them. Finally, different representations including formal number sentences of the sum were explored to help learners become accustomed to these dimensions of variation. With this approach, I attempted to work across a range of different concepts and competencies relating to division rather than focusing separately on very specific objects within each lesson. During this lesson learners were able to use informal strategies to solve the problems. Informal strategies refer to the use of pictures and pictorial representations as well as written methods that do not include the traditional division algorithm. Murphy (2006, p.219) describes informal strategies as calculation strategies that children have invented that are based on the laws of arithmetic.

I varied the concrete materials by using real objects where possible and base 10 blocks to represent the concrete objects to be shared where it was not possible to use the actual objects - as suggested by Lichtenberg and Troutman (2003, p.242). I felt that this would assist learners in visualizing the context. Through a pattern of variation, the blocks were used for increasing sizes of dividends. As learners became accustomed to sharing the blocks, a written number sentence was introduced to represent the problem, thus generalizing the concept. Once learners became confident representing division using the number sentences the different patterns of variation were used to extend their experience to include the selected dimensions of variation within division. Their experience was extended through the use of increased dividends and asking learners to present different representations of the problems. The questions and a brief description of why each question was selected has been included on worksheet 1 (see appendix F). Following the worksheet, I engaged learners in a discussion about the similarities and differences between the two types of questions. We also revised the terminology and what the different words mean. Finally, I extended learners knowledge by exploring the relationship between the different parts of the sum. I estimated that this would take approximately two periods.

4.3.2. <u>Lesson 2</u>

Object of learning: Repeated subtraction

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Size of dividend

Forms of representation

The next concept to be introduced was division as repeated subtraction. I believed that this promoted the development of learners sharing out objects in chunks or larger groups. Although repeated subtraction has a better link to quotitive division it can also be used to consolidate the notion of partitive division or sharing. This was done using concrete materials before learners were encouraged to record their findings. At this stage, learners were encouraged to write number sentences using formal notation, however, calculations were largely informal. As explained both quotitive and partitive questions were used. The link between learners' informal methods and a more formalised form of recording the repeated subtraction was made. Once learners demonstrated that they understood the concept of repeated subtraction, I specified that they needed to use the more formalised notation introduced and not the pictorial representation or the Dienes blocks. In appendix G I have included a brief description of the reason behind the selection of each of the questions. I estimated this would take approximately two periods.

4.3.3. <u>Lesson 3</u>

Object of learning: Division facts

Relationship between multiplication and division

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Different types of questions

Different representations of answers

Once learners were familiar with the meaning of division, in terms of the above interpretations, and were able to represent their findings using the written form of the division the link was made between multiplication, their multiplication facts and their findings to the division problems. Hence, promoting a more structured approach to division. From there I formally introduced them to the division facts including division with zero, one and multiplies of 10. This was explored through a series of practical activities using base 10 blocks where learners needed to represent their findings in a number of ways, and a selection of examples (pattern of variation) used to highlight a pattern. Details regarding the selection of each example has been explained in appendix H. I estimated that this would take approximately two periods.

4.3.4. <u>Lesson 4</u>

Object of learning: Division algorithms

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Size of dividend and divisor – extension of algorithm procedure

Inclusion of remainders

Once learners became familiar with all the above concepts I introduced long division. I reverted back to initial simple division questions that fell within the division facts so that learners were able to contrast their knowledge of division thus far and the long division algorithm to identify the features that vary and those that were invariant. I used the base 10 blocks to guide and give meaning to the different steps in the algorithm as suggested by Booker et.al. (1992). Patterns of variation were used to assist learners to work through the dimensions of variation within long division. In appendix I, I explain the selection of examples used in these activities. I estimated that this would take approximately 5 periods. I devoted many periods to this concept as I wanted to begin at a concrete level and use the concrete procedures to develop the algorithm. I then wanted to allow time for the whole class to become familiar with the algorithm and both the formal and informal language associated with it. Research reviewed in Chapter 2 indicated that this was an area plagued with misconceptions and difficulties. According to the literature there were many different patterns of variation that needed to be explored and addressed to avoid learners developing misconceptions. I wanted to allow learners enough time to be able to calculate problems confidently using the algorithm.

4.3.5. <u>Lesson 5</u>

Object of learning: Division algorithms

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Sizes of dividends and divisors

Inclusion of remainders

I planned to introduce learners to short division in lesson 5, as I believed that they would have a sound understanding of the long division algorithm. Short division forms one of the algorithms that Grade 5 learners are expected to perform at my school. I hoped to guide them into what is commonly known as short division by investigating which steps can be left out of the recording process. I planned to begin with questions that involved division by a single digit with a two digit dividend. Appendix J justifies the selection of examples in worksheet 6. I estimated that this would take approximately three periods. However, following my analysis of lesson 4 I chose not to move onto short division but rather consolidate long division, the reasons for this will be explained in the following chapter and proved the literature correct. This was done with consent from both the principal and head of Mathematics in the Junior School.

4.3.6. Lesson 6

Object of learning: Fusion of different concepts and objects of learning from previous lessons

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Sizes of dividends and divisors
Inclusion of remainders
Different types of questions
Different representations of answers

The final two periods involved consolidation and application of all the concepts and skills from previous lessons. Providing learners the opportunity to experience fusion through the drawing together of all concepts learnt. If any of the learners had a good grasp on the concepts expected at a Grade 5 level I hoped to extend their application of division into other areas of the mathematics curriculum for example ratios as measurement and division involving fractions.

4.4. Post-test

Following the intervention I conducted an initial post-test with Group A. The post-test used the same format as the pre-test and included the same procedural questions. However, as explained earlier, following the further reading I engaged with, and the introduction and analysis of division concepts that were covered in the intervention through the lens of variation theory, I realised that I needed to assess learners conceptual knowledge in greater detail and through new patterns of variation that I had not identified in the pre-test. Furthermore, I wanted to directly assess specific objects of learning from the intervention. Thus, additional conceptual questions were added to the post-test. A description of the additional questions added to the post-test have been included in Appendix L. The post-test was used to establish if the misconceptions and errors that were evident in Group A's pre-test had been addressed and to determine if there was any development in Group A's knowledge. A delayed post-test that was identical to the initial post-test was conducted with the whole grade (both Group A and B in January 2010) to determine if the concepts covered with Group A had been retained. If a learner wrote pre-test 1 they then wrote post-test 1 so that I was able to compare their responses in the identical questions. Accordingly, the post-test was used to measure the development of Group B learners and established if the gap between Group A and B had been reduced, thus answering my third research question.

The following figure summarises the types of questions included in each test and which group wrote the tests.

Figure 4.1. Summary of the tests

	Pre-test	Initial post-test	Delayed post-test
Group A	Procedural questions	Procedural and conceptual questions	Procedural and conceptual questions
Group B	Procedural questions	X	Procedural and conceptual questions

4.5. Analytical methods

The analysis of the pre and post-tests in this study took place on two levels. Firstly, a broad analysis across the grade (approximately 50 learners) of learner performance in the pre and post-tests. Secondly, a more in-depth analysis of learners' responses in pre and post-tests, as well as the intervention. My data analysis included both qualitative and quantitative analysis. Breakwell (1995, p. 13) describes qualitative analysis as "a description of the processes occurring and details of differences in the character of these processes over time". The qualitative analysis included descriptions of errors and correct calculations that enabled me to identify misconceptions and critical features in relation to the categories provided by Troutman and Lichtenberg (2003) and Ryan and Williams (2007). The second type of analysis surrounded the actual intervention. In this chapter I have provided a brief outline of the lessons. This will be expanded on in the following chapter, along with details of the activities. All worksheets can be viewed in the appendices. In the following chapter I have provided a detailed analysis of learner responses to the different questions in the pre-test (5.1.), intervention (5.2.) and the post-test (5.3.), again, specific responses can be viewed in the appendices. Appendix N refers to the pre-test, Appendix O is the lesson analysis and appendix P refers to the post-tests (the initial and delayed post-tests were identical). Appendix Q and R summarises, compares and analyses pre and post-test results. I have linked the misconceptions highlighted in the learners' errors to the literature.

An example of my qualitative analysis of the pre-tests and the implications of the analysis for the intervention is summarised below. A learner in my sample was able to use the long division algorithm to answer question 17 in the pre-test. However, she was not able to correctly calculate the answer to question

19 where there is a zero in the quotient. I then described the process that the learner used and where the error occurred. In addition, I linked this error to the literature by Troutman and Lichtenberg (2003, p. 251) stating that in teaching division special attention should be given to questions involving a zero in the divisor, dividend or quotient as these situations pose the greatest difficulty for children. I would also relate it to Ryan and Williams' (2007, p. 206) description of a place value error. I would link the qualitative description to my findings in the quantitative data on error analysis- to examine the extent of this error.

Within the intervention, one focus would be on designing a sequence of examples within which this dimension of variation, place value and the role of 0, can be addressed. Following the intervention I then compared the learners' responses in the pre-test to their responses in the post-tests. If the learner no longer made the error it provided evidence supporting the use of variation theory in the teaching of division. However, if the learner still made the same error I would need to question the success of my intervention or explain why the error continued to occur following the intervention.

According to Breakwell (1995, p. 13) quantitative data "states what the processes are, how often they occur, and what differences in their magnitude can be measured over time". In the quantitative analysis I analysed how many of the learners made a particular error, how often they made that error and for the learners taking part in the intervention I used the quantitative analysis to measure the progress they made between the pre and post-test. I used my results to identify common errors across Group A and B and linked these findings to the literature on common misconceptions and errors of learners regarding division.

I used qualitative and qualitative analysis to track the progress of my sample of learners in relation to their peers. The learners that were not part of my sample had already completed the section on division and the teachers did not plan any further teaching time spent on it, thus their level of understanding and competence should not have been not directly improved through teaching. However, I was alerted by the improvements in many of the learners that all or some teachers had revisited division concepts and skills included in the pre-test in the final part of the year, and this resulted in the general improvement in the classes' results. In order to get a fair comparison, I therefore compared my intervention group's score changes with the changes of the Groups B's scores. The data gained in the pre and delayed post-test assisted me in answering my third research question about the effects of the intervention on my sample of learner's performance in division by indicating their overall improvement as well as the reduction of the gap between the Group A and B learners. Links between the qualitative and quantitative data were made explicit and all analysis was related to the literature to examine the ways in which my results linked with the research recommendations.

4.6. <u>Ethics</u>

According to Sikes (in Opie (ed), 2004, p. 32) as a researcher we must not harm anyone or do any moral wrong. As my research involved teachers and children, ethical consideration was of paramount importance. My research did not involve any covert research. All participants were aware of my intentions and details were made explicit in a letter attached to the informed consent form (see appendix A, B and C). Any learner that did not return a signed informed consent letter did not form part of the study. Learners and teachers anonymity was maintained at all times and only relevant findings were revealed while still maintaining anonymity for those participating in the study.

My research was experimental with one group of learners partaking in an intervention and another group not taking part in the intervention. My research was not detrimental to either group. All learners had already been taught the division work. Group A was identified as low achieving and my intention was to try and help them improve their results. Thus, the aim was to benefit the Group A without detriment of the Group B learners.

In my experience, I found that many teachers feel threatened when other people look at their books or evaluate their learners. They often feel that they are being judged if their learners make a mistake. I explained that I was not judging their teaching but looking for common errors that Grade 5 learners make when solving division problems. I stressed that the results would not be analysed to reflect their classes' performance in relation to the other classes but rather the whole grade would be used to identify common misconceptions. I hoped that by setting their minds at ease in this regard would promote their co-operation in administering a fair test.

4.7. <u>Validity</u>

"Validity concerns the relationship between the claim and the accompanying process of data gathering" (Scaife, in Opie (ed), 2004, p.69). My pre and post-tests were designed to test for errors and misconceptions in division. As they had been designed with the intention of assessing particular dimensions of variation within division based on literature surrounding division, variation and common errors and misconceptions I believe that they will both be valid and accordingly the results will be valid.

I was aware that there would be some tension between my role as a teacher and as a researcher. I acknowledged that it would be difficult to avoid a subjective view on the teaching and learning that takes place in my classroom. However, I hoped that by recording my discourse I would be able to accurately report on my teaching and that I would gain insight into my teaching. At the same time there were some benefits to being both the teacher and researcher. I hoped to use my existing good relationship with the focal student group to my advantage in terms of getting ‘honest’ accounts of learning and experiences of learning from the students involved in the intervention group. I was also to accurately report on the teacher’s perspective and provide an account of my difficulties and the advantages of the new strategies I tried.

An additional problem that was predicted was that the weaker learners may not be able to complete the pre-test within the allocated time. If this was the case, teachers were instructed to allow the learners additional time to complete the activity if they had the time available. However, if there were time constraints, those learners would be excluded from the quantitative data on the questions they were unable to attempt but the qualitative analysis of the questions that were not attempted could provide important clues indicating a lack of understanding of those areas.

While additional questions were included in the post-test I do not feel that this affected the validity of the test in any way. Furthermore I do not feel that the procedural nature of the pre or post-test affected the validity of the tests.

The advantage of using a pre-test that was administered by the class teacher in the mathematics lesson was that it was in the same format as learners are accustomed to being assessed. This meant that I was able to minimise distortion in the results caused by the stress of unfamiliar environments, researchers and assessment formats which can occur with interviews or questionnaires.

4.8. Reliability

Scaife (in Opie (ed), 2004, p.68) describes reliability as the extent to which a data gathering process produces similar results in similar conditions. My study was very specific, and is therefore not likely to be repeatable. However, I do believe that my findings will be useful to other teachers and researchers for the reasons mentioned earlier. The reliability of this study is likely to be achieved by locating the data on specific misconceptions seen in the sample within the broader literature.

This chapter has discussed how I planned to conduct my study from the designing, administration and analysis of the pre-test, to the content and analytical strategies of the actual lessons to the designing, administration and analysis of the post-test. The following chapter will report on the findings in the pre-test, intervention and post-test as well as the analysis of these.

Chapter 5

Results and analysis

In this chapter I present my analysis of learners' understandings of division - as gleaned from the pre-test, written test scripts in the first instance, but then more in-depth understandings from the oral and written responses they gave during the course of the intervention lessons and finally the initial and delayed post-test. The planning for each lesson and worksheets were discussed in the previous chapter and a breakdown of the questions is included in the appendices. In this chapter, I present my observations of each lesson and following the analysis of my findings, I draw together all common findings from the lessons and tests in a summary. I also present an analysis of learner performance in the post-tests, and use this to consider evidence of growth of procedural and conceptual knowledge. I have identified the nature of their approaches, any misconceptions, as well as, their ability to use, and apply, the procedures associated with division and I draw on this throughout the different areas of analysis. Through this process, I acknowledge that it was difficult to gain an understanding of their knowledge of division until I worked extensively with the learners and listened to their interpretation of questions and how they solved problems.

My analysis of learner responses is arranged around the various ways of understanding learner thinking that were detailed in Chapter 2. I have therefore focused on error categories drawn from the literature I presented in Chapter Two, as well as the types of questions and strategies used by learners in their attempts to solve various problems. From this I hope to gain insight into the type of understanding (relational or instrumental) that learners are working with. In the initial part of the chapter I have not made explicit reference to the literature, however, the error categories and question clusters draw on the knowledge I gained through the literature review. The latter part of the chapter does make more explicit reference to the literature and incorporates various aspects of the literature review. I have tried to incorporate a reflexive and critical focus on how the pedagogical approaches that I have used may have impacted on learner responses. When conducting my grounded analysis I found that errors were highlighted within three different areas of the problem solving process, namely – interpretation of the question, strategies used to solve the problem and knowledge of underlying mathematical structure. Accordingly these were included in my areas of analysis. These categories have been summarised below.

Key areas of analysis will be:

- the identification of errors in:
    - interpretation,

- procedural strategies for solving the problem, and
    - understanding of mathematical structure.
- The error analysis will be done in terms of Troutman & Lichtenberg's (2003, p.289) identification of three key areas of errors in relation to division:
    - place value and understanding the operation
    - difficulty with prerequisite skills and
    - clerical difficulties

  and Ryan and Williams' (2007, p. 13) six different types of errors which they cluster into three broad groups:
    - developmental errors
        - modeling
        - prototyping
        - overgeneralising
        - process-object linking
    - errors with no obvious developmental or conceptual explanation
        - slips
    - uncertain diagnosis.
- the actual strategies used to solve the problem
- type and level of understanding and
- pedagogical approaches and their effect

The following paragraph provides a brief summary of key findings presented in this chapter. In this chapter I begin by showing how the intervention group (Group A: n = 6) as a whole, performed poorly in the division questions in comparison to their peers in the pre-test. I then discuss how the development of various division concepts through variation theory assisted learners in improving their understanding and knowledge of division at a conceptual and procedural level. Learners' progress was continually tracked culminating in them writing a post-test immediately after the intervention. This took place in the last month of the school year. When learners returned the following year a delayed post-test was written by all learners (Group A and B). What became evident was that the learners involved in the intervention had not retained all the new knowledge that they had been exposed to during the previous year. However, there had been an improvement in their performance in comparison to the Group B (n = 50) learners, some of whom had received further teaching and consolidation of division concepts by their mathematics teachers following the pre-test that they had written. Thus, although the gap had not been eliminated it had been reduced.

The following abbreviations and pseudonyms have been used throughout this chapter:

LD – long division

SD – short division

Q – question

M/DF - Multiplication or Division Facts

NWS – no working shown

Drew – a pictorial representation was used to find the solution

Group A learners (pseudonyms)

C – Catherine

J - Jenny

K – Kim

M – Mel

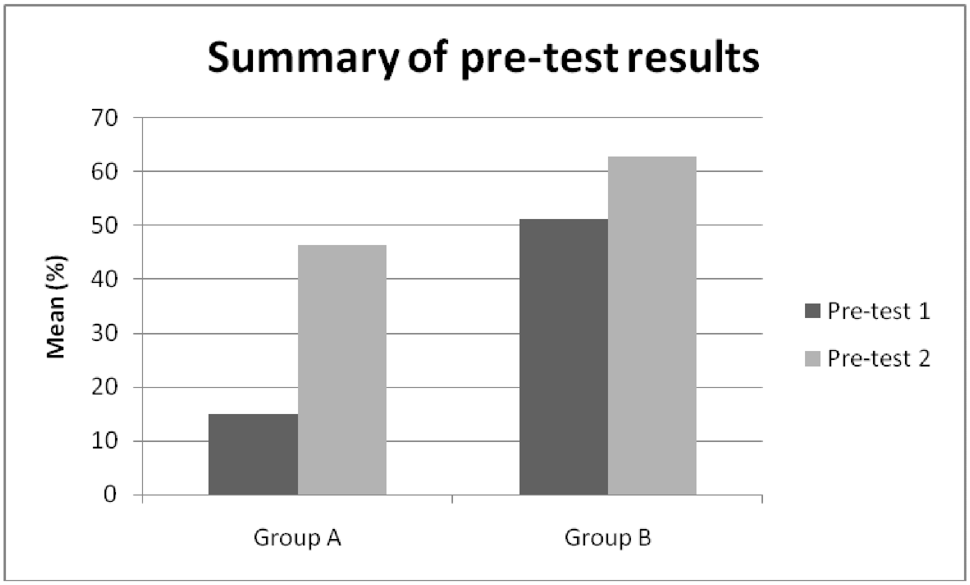
R – Rosie

S – Salina

Where a block in a table has been left open the learner did not attempt that question or problem.

5.1. Pre-test results

Appendix N contains a question-by-question breakdown of Group A learners' responses to the division questions, with details of strategies used, as well as, errors made. Appendix Q contains the full analysis of Group B's pre-test results. Graph 5.1 provides a summary of the mean pre-test score for Group A (those partaking in the intervention) and Group B (those Grade 5 learners that were not part of the intervention group). Table 5.1. then breaks down the Group A results into individual performance in each of the pre-tests. In Group A Mel, Catherine and Kim wrote both pre-tests. Salina and Jenny only wrote pre-test 1 and Rosie only wrote pre-test 2. In Group B, as explained in the previous chapter half of each class wrote one test and the other half wrote the other test. Tests were assigned so that a cross section of Group B learners wrote each pre-test.



Graph 5.1. Summary of pre-test results

Pre-test analysis	Pre-test 1 (%)	Pre-test 2 (%)
Salina *	0	Did not write
Mel	12.5	28.6
Catherine	37.5	85.7
Jenny *	0	Did not write
Kim	25	42.9
Rosie *	Did not write	28.6
Mean	15	46.45

Table 5.1. Summary of Group A pre-test results

* Rosie did not write pre-test 1 and Salina and Jenny did not write pre-test 2

As mentioned in the previous chapter, the dimensions and patterns of variation assessed in the questions were distributed as evenly as possible between the two tests so that across the two tests, I was able to analyse the full range of dimensions and patterns of variation. However, the graph indicates that learners in Group A and B found pre-test 2 easier. Although the questions were distributed as equally as possible, pre-test two included one procedural question that drew on a multiplication fact and the more challenging word sum only involved division by a single digit. In contrast pre-test one did not have a straight procedural question that could be solved by simply drawing on a multiplication fact. I believe that another reason for this was the fact that pre-test 1 had two more questions that fell into the category that relied on a knowledge of the division facts and the link between multiplication and division. In addition, there were two less questions that involved questions that fell into the category that required carrying over and learners to work with a remainder. Furthermore, the more challenging word sum in pre-test one required division by a double digit number and working with a remainder. However, I believe that the differences between the tests did not affect the validity of the claims as learners' performance in the post-test was measured against their performance in the same questions they answered in the pre-test.

The results in the graph indicate learners' ability to select and apply the appropriate procedures to solve division problems. It did not provide evidence of learners' conceptual understanding. Details surrounding learners' conceptual understanding has been provided in the qualitative analysis provided at a later stage. It is evident in the graph that there is a significant gap between the performance of the Group A and B learners. Group B learners performed better in both pre-test one and two. All learners participating in the intervention produced results that were below Group B's average in pre-test one. Catherine was the only learner participating in the intervention who achieved a result that was above Group B's mean result for pre-test two.

5.1.1. <u>Group A: pre-test</u>

The table containing the results for Group A's pre-test results can be found in the table below and appendix N. In this section I have summarised the results and analysed my findings according to the criteria stated at the beginning of the chapter.

Table 5.2. Group A question by question pre-test analysis (the non division questions have been omitted from my analysis)

Group A pre-test 1

Name	Q 1	Q 3	Q 5	Q 7	Q 9	Q 11	Q 13 a & b	Other notes and error analysis
Salina	X NWS	X NWS	X NWS	X NWS	X NWS	X NWS		Cannot classify errors as no working was shown.
Mel	X SD	√ SD	X SD			X NWS		Did not attempt 3 of the division questions. Q 1 & 5 - Did not carry over remainder in calculation. Errors are developmental- understanding of the operation and prototypical.
Catherine	√ NWS	√ NWS		X NWS	X NWS	√ NWS	X NWS	Did not attempt Q 5. Cannot analyse errors as no working was shown.
Jenny	X Break up	X LD.				X NWS	X	Only attempted 4 questions. Developmental errors. Q1 - tried to break up and divide 4÷2 = 2, 6÷3 = 3 64÷4 = 22 - Overgeneralisation of addition and does not understand the operation. Q3 - First two steps correct but didn't complete the sum - doesn't understand the operation, process object error – has not completed the reification of foundational division concepts e.g. sharing and division facts thus having to focus on processes instead of using them as objects to solve long division sums. Q 13 - multiplied instead of divided - doesn't understand operation, modeling error.
Rosie								Did not have time to finish both pretests
Kim		√ NWS				√ NWS		Only attempted two division sums.

Pre-test 2 results

Name	Q 1	Q 3	Q 5	Q 7	Q 9	Q 10	Q 12	Other notes and error analysis
Salina								She did not have time to finish both tests.
Mel	√ NWS					√ NWS	X	Q 12 Incorrect operation used, - instead of ÷.Developmental error, modeling and doesn't understand the operation.
Catherine	√ NWS	√ NWS	√ NWS	√ NWS		√ NWS	√ NWS	
Jenny								Did not attempt any of the division questions.
Rosie	√	X	X	X	X	X	√	Q1 - number sentence given as working. Developmental errors Q 3, 10, 12 - format of addition or subtraction used. Overgeneralisation, and doesn't understand the operation. Q 5, 7, 9 - multiplied instead of divided - modeling error and doesn't understand the operation.
Kim	√ Facts					√ NWS	X	Only attempted three division sums. Developmental errors Q 1 - division fact. Incorrect setting out - set out like +,- and x sums. Must have used multiplication facts to solve. Modeling error and doesn't understand the operation. Q 12 - Incorrect operation done. Multiplied instead of divided. Overgeneralisation, and doesn't understand the operation.

To summarise, in pre-test one, all learners omitted questions. Salina and Catherine attempted the most questions (6/7) in this test; however, all Salina's answers were incorrect while Catherine had 3 correct answers. Kim attempted the least number of questions (2/7) but both answers were correct. Mel and Jenny each omitted 3 questions. Mel answered one of her questions correctly while Jenny was unable to reach any correct solutions. Rosie did not do this pre-test due to time constraints.

In pre-test 2 Rosie was the only learner to attempt all the questions. Question 12 was set out in the format she uses for multiplication and she must therefore have solved the problem mentally to have reached the correct solution. Jenny did not attempt any of the division questions in this pre-test. Catherine attempted, and correctly answered, 6 of the 7 division questions. Mel and Kim each attempted 3 division questions and correctly answered 2 of these.

- <u>Strategies used (in instances where working was shown)</u>

This paragraph provides a summary of the strategies used to solve the division questions in the pre-test. One learner attempted long division in pre-test 1 for one of the questions but she was unable to reach the correct solution. The same learner attempted to break up and divide one of the problems but again was unable to reach the correct solution. One of the learners attempted short division three times. She reached the correct solution once, however, no carrying over was required in this sum. This learner was unable to calculate the more advanced calculation that required a number to be carried over. In conclusion, the majority of the learners did not show any working when they were trying to solve the division questions. Those learners that were able to draw on their knowledge of the multiplication/division facts were able to correctly solve some of the problems without a calculation. However, in the questions that required a calculation learners' lack of attempt appeared indicative of a lack of knowledge surrounding the procedures and tools used for division. Two of the six learners attempted to use an algorithm to solve the calculations, but only one of these was able to correctly execute the calculation, indicating learner lack of knowledge of the procedures of division.

- <u>Types of errors (in instances where working was shown)</u>

I will now summarise the types of errors learners made in their division calculations. If no working was done the errors could not be assessed. All five errors in pre-test 1 found within calculations appeared to be related to learners' understanding of the division concept. However, some of the errors fell within two categories. For example; in question 12 of pre-test 2 Kim overgeneralised and demonstrated a lack of understanding of the operation and in question 5, 7 and 9 Rosie's errors indicated a deficit within her understanding of a division concept, as well as the error being a modeling error. Of the ten errors in pre-test 2, 5 were a result of the incorrect operation being used and 4 for attempting to solve using the format of a different algorithm, such as, when the sum was set out as if it were an addition, subtraction or multiplication sum. These errors are all developmental in Ryan and Williams' (2007) terms and indicate some serious misconceptions. The errors made by the Group A learners indicated extensive gaps in their knowledge of the concepts of division, as well as the pre-concepts and sense making capabilities, as they were unable to decode the problem sufficiently to establish the appropriate operation or eliminate the inappropriate operations. Accordingly, my intervention took learners back to the initial sharing concepts so that I was able to provide a solid foundation on which more advanced concepts could be developed.

5.1.2. <u>Analysis of Group B pre-test results and comparison to Group A pre-test results</u>

In this section I analyse the overall achievement of the Group B learners in the two pre-tests and compare these to the achievement of the Group A learners. I then break down the questions into four broad categories according to dimensions of variation. As the grounded research progressed four categories emerged in the data from learners responses to particular types of questions.

- The first cluster involves questions that involve division facts and thus can be solved using multiplication/division facts.
- The second involves numbers that are beyond the division facts and are subsequently best solved using a calculation, but no carrying over is necessary in the calculation.

- The third required a calculation but carrying over is necessary within the calculation.
- The final cluster involved questions that have a remainder.

While the clusters discussed above suggest specific procedures be used learners were free to choose how to solve the problems. Thus, whilst my procedural dimensions of variation were mathematically grounded in procedural complexity, the empirical data suggests that these levels are interchangeable for students that have procedural fluency. While the categories are based on procedural strategies, each group demands that learners work with a variety of concepts. In each category I state both Group A and B learners results, list the strategies learners used – as this provides some indication of how the learners conceptualised the problem and then investigate the errors, classifying them according the groupings stated at the beginning of the chapter. From this classification findings are presented.

As was highlighted in Graph 5.1. the average for Group B in pre-test 1 was 51.25%. This was 36.25% higher than the attainment of the Group A learners for the same pre-test. Pre-test 2 saw the Group B learners achieving an average of 62.9%. This was 16.45% higher than the average for the Group A learners. It is important to note that both Group A and B learners did better in the second pre-test.

- <u>Questions relying on division facts</u>

The first cluster of questions relied on division facts (refer to appendix D for associated questions). The following table summarises learner attainment and provides details of learner responses. Correct and incorrect responses are written as a fraction of those that wrote the test. Those that did not attempt a question have been included in the total for incorrect responses.

Table 5.3. Pre-test questions relying on division facts (A and B refer to the groups. With Group B, 26 learners wrote pre-test 1 and 24 learners wrote pre-test 2)

Pre-test	Question	Correct responses	Incorrect responses	Strategies used	Errors made (excluding those who did not attempt the calculation)
1	1	A: 1/5 B: 24/26	A: 4/5 B: 2/26	SD: B -14 M/DF: B - 5 Drew: B – 2 NWS: A – 4, B - 5 Did not attempt: A-1	A – cannot evaluate NWS B – Both errors relate to an incorrect division fact used
2	1	A: 4/5 B: 24/24	A: 1/5	SD: B – 16 LD: B – 4	

				M/DF: B – 3 NWS: A – 2, B – 1 Number sentence: A - 1 Other: A – 1 Did not attempt: A – 1	
2	10	A: 3/5 B: 20/24	A: 2/5 B: 4/24	SD: B – 10 LD: B – 2 M/DF: B – 8 Drew: B - 1 NWS: A – 3 Other: A – 1 Did not attempt: A – 1, B - 3	A: Calculation carried out as if it were subtraction. B: did not carry over in short division

Error analysis

In Group A, none of the learners did any working for Q1 pre-test 1, thus it is not possible to diagnose the errors made. Rosie was the only Group A learner who answered Q10 incorrectly and she used an inappropriate format in her attempt to solve this problem indicating a deficit in her understanding of the operation and a developmental error.

There were three Group B learners who did not attempt Q10 pre-test 2, indicating that it contained a concept they had not mastered. The only Group B error that occurred in Q 10 involved a short division calculation, in which the child forgot to carry over the remaining tens to their units.

Strategies analysis

The majority of the Group A learners did not show any working for the problems in this section. Rosie wrote a number sentence to represent the problem and Kim must have relied on her knowledge of division facts, as she set out the sum as if it were an addition, subtraction or multiplication sum.

In conclusion, the learners in both Group A and B excelled in this area of questioning, namely those questions involving division facts. The majority of Group B learners favoured short division to solve these questions, even though they formed part of the known division facts. Group A learners had access to multiplication charts and grids and it would appear from the lack of any kind of calculation that they used these to assist them in reaching the correct solution. The Group B learner who forgot to carryover in the calculation and those that did not attempt Q10 in pre-test 2 suggest developmental difficulties, which may indicate a lack of understanding of the operation or inadequate pre-requisite skills. In Group A, Kim and Rosie's errors both relate to procedural practices and indicate developmental difficulties.

- <u>Questions beyond the division facts</u>

The second cluster of questions were beyond the division facts and thus favoured the use of a calculation. However, because the questions did not require any carrying over – the divisor could be divided into each digit of the dividend without leaving a remainder. This meant that informal and mental strategies could be used to reach the correct solution.

Table 5.4. Pre-test questions beyond the division facts (A and B refer to the groups)

Pre-test	Question	Correct responses	Incorrect responses	Strategies used	Summary of errors made (excluding those who did not attempt the calculation)
1	3	A: 3/5 B: 21/26	A: 2/5 B: 4/26	SD: A – 1, B – 13 LD: A- 1, B - 5 NWS: A – 3 , B – 4 Halved: B – 2 Other: B – 1 Did not attempt: B - 1	A - 1 did not complete the algorithm, 1 NWS B - 2 did not complete LD algorithm, SD Copied the sum incorrectly, B - Subtracted format but divided
2	12	A: 2/5 B: 12/24	A: 3/5 B: 12/24	SD: B – 12 NWS: A – 1 Other: A – 3 B - 7 Did not attempt: A -1, B - 5	A: 2 incorrect operation B - 6 performed incorrect operation, 1 set out in incorrect format

<u>Error analysis</u>

In Q3 pre-test 1, the 3 learners from Group A and B (including Jenny) who used long division in incorrect ways did not complete the algorithm and as a result did not finish. I considered this a developmental error as they did not understand the operation and the algorithm fully although they demonstrated some instrumental understanding. One of the Group B learners did short division and copied the sum incorrectly – I viewed this as a slip and not developmental.

The learners who did not attempt Q3, pretest 1, using formal or informal strategies did not demonstrate any understanding of the concept or prerequisite skills. Salina did not do any working, accordingly her errors cannot be diagnosed. It is important to note that in Q12, pre-test 2 – a contextual problem, there is little difference in the proportion of errors between the Group A and B learners. This question required learners to understand the relationship between multiplication and division, as well as the problem context, in order to recognise that the required operation was division. Five of the Group B learners and one Group A learner did not attempt this question, it is possible that they did not know how to interpret the question, or that they did not have the necessary tools to find the solution. As informal strategies could have been used to solve the problems I believe it is possible that these learners could not identify the operation required and perhaps did not have a sound understanding of the prerequisite skills. Eight learners over both groups performed the incorrect operation. This demonstrated that they did not understand the context and thus did not fully understand the operation; this is viewed as a developmental error. Rosie found the correct answer, however her calculation was set out as if it were using the subtraction algorithm. The fact that she found the correct answer demonstrated that she has some understanding of the concept of division as she was able to solve the problem using an informal method, however, it would appear she did not understand the traditional procedures and algorithms of division. Of the 12 learners who recognised the context, and could identify division as the required operation, all were able to solve it correctly.

The errors in this group of questions were largely developmental, I believe that there was only one slip. Many of the learners appeared unable to accurately interpret the context of question 12, and thus could not understand the operation. The difference in the performance of Group A and B learners in this question was less. This can also be viewed as a modeling error as learners were not able to identify or represent the context appropriately. While I cannot accurately diagnose the difficulty of the learners who did not attempt the questions I can infer that they too did not have the necessary knowledge to decode or solve the problem. In this section there were many learners who did not correctly execute the algorithms they had selected. This also forms part of the developmental errors category.

It is interesting to note that Group B appear to be stronger procedurally and as a result they tend to use the formal procedures to solve the division questions. Furthermore, the Group B learners are more willing to attempt and show working for the division questions.

- <u>Questions requiring Carrying over with no remainder</u>

The third section of questions could not be solved effectively using informal strategies, as it was beyond the division facts and accordingly, required learners to perform a calculation to find the solution. Carrying over was required within the calculation but there was no remainder.

Table 5.5. Pre-test questions beyond the division facts that required a carrying over in the calculation but had no remainder (A and B refer to the groups)

Pre-test	Question	Correct responses	Incorrect responses	Strategies used	Summary of errors made (excluding those who did not attempt the calculation)
1	11	A: 2/5 B: 22/26	A: 3/5 B: 4/26	SD: B – 13 LD: B - 2 M/DF: B - 6 Drew: B – 1 NWS: A – 4, B - 2 Did not attempt: B – 1 Other : A - 1	A: 3 NWS B: All errors included the use of incorrect division facts
2	3	A: 1/5 B: 15/24	A: 4/5 B: 9/24	SD: B – 12 LD: B – 8 NWS: A – 1, B – 1 Number sentence: A - 2 Other: A – 1 Did not attempt: A - 3, B - 3	A: format of subtraction used B: 1 incorrect carrying over SD, 1 working right to left, 1 incorrect multiplication, 1 incorrect algorithm, 1 incorrect fact, 1 correct calculation but incorrect interpretation of answer
2	5	A: 1/5 B: 18/24	A : 4/5 B: 6/24	SD: B – 1 LD: B – 17 NWS: A – 1 Other: A – 1, B - 2 Did not attempt: A- 3, B - 4	A: 1 multiplied instead of divided B: 1 answer written in incorrect place value column LD, 1 incorrect use of algorithm
2	7	A: 1/5 B: 11/24	A: 4/5 B: 13/24	SD: B – 1 LD: B – 16 NWS: A – 1 Other: A – 1, B- 1 Did not attempt: A – 3, B - 6	A: 1 incorrect operation B: 1 incorrect algorithm, incorrect fact , 1 Incorrect operation, 3 incorrect multiplication, 1 correct calculation but incorrect interpretation of answer, 1 incorrect subtraction

Error analysis

All group B errors in Q 1 in pre-test one related to multiplication or division facts. It would appear that these errors either related to a deficiency within the prerequisite skills which indicated a developmental error, or they were clerical errors, in which case they were not developmental. In Group A only one out of the five learners was able to answer Q1. Catherine solved Q1 with no working, Jenny attempted to break up the number and divide it but she did not recognise the full value (based on place value) of each digit and only worked with the digits. She did not use the appropriate divisor for each part of the sum. She attempted to generalise a strategy, but clearly did not understand the strategy or the importance of place value. This error indicates several difficulties – not understanding the operation or context and inadequate pre-requisite skills. Mel used short division but did not carry over in her working for Q1. One Group B learner did not do any working for Q3 and thus the error could not be diagnosed.

Division algorithms

The results for question 3, 5 and 7 of pre-test 2 indicate learners' problems when working with the algorithms. In group B the results were as follows: 15/24, 18/24 and 11/24. Long division was used in 41 of the responses with short division being used in only 14 of the responses. Three of the learners who used short division did not apply the algorithm correctly. One learner forgot to carry over, one worked from right to left and the third did not complete the steps correctly. One of learners who used long division calculated the solution correctly, but wrote the answer above the incorrect place value. 13 learners did not attempt this – indicating an area that the learners have little to no knowledge of and a developmental error, with 3 using the incorrect algorithm or operation indicating a modeling – developmental error. Catherine was the only Group A learner to answer Q3, 5 and 7 correctly, although she did not show any working. Rosie was the only other learner to attempt these questions and performed the incorrect operation for two of the sums and was unable to use the algorithm. Her errors indicate developmental problems and that she does not understand the operation. In this group of questions the majority of errors related to a lack of understanding of the algorithm, as well as the operation of division.

It was interesting to note that this group of questions highlighted the strength of Group B's procedural fluency in comparison to Group A. Furthermore, Group B were better at identifying the correct operation to use to solve these problems even if they were not able to reach the correct solution.

- Questions with carrying over and a remainder

The final cluster of questions required learners to perform a calculation that involved carrying over and the solution involved a remainder.

Table 5.6. Pre-test questions beyond the division facts that required carrying over in the calculation and had a remainder (A and B refer to the groups)

Pre-test	Question	Correct responses	Incorrect responses	Strategies used	Errors made (excluding those who did not attempt the calculation)
1	5	A: 0/5 B: 14/26	A: 5/5 B: 12/26	SD: A – 1, B – 17 LD: B - 5 NWS: A - 1B – 1 Did not attempt: A – 3, B - 3	A: 1 did not carry over in SD B: 5 incorrect facts, 2 no carrying, 1 incorrect algorithm, 1 incorrect multiplication
1	7	A: 0/5 B: 9/26	A: 5/5 B: 17/26	SD: B – 3 LD: B – 15 NWS: A – 2, B – 1 Did not attempt: A – 3 B - 7	A: 2 NWS B: 2 incorrect algorithm, 2 incorrect copying, 2 didn't bring down, 2 incorrect subtraction, 1 incorrect multiplication, 1 incorrect fact
1	9	A: 0/5 B: 11/26	A: 5/5 B: 15/26	SD: B – 5 LD: B – 14 NWS: A – 2 Did not attempt: A – 3, B - 6	A: 2 NWS B: 1 incorrect algorithm, 2 omitted 0, 1 didn't complete last step, 1 muddled place value, 1 incorrect multiplication, 1 NWS, 1 incorrect carrying over
1	13 a and b	A: 0/5 B: 0/26	A: 5/5 B: 26/26	SD: B – 1 LD: B – 12 M/DF: B – 1 NWS: A – 1, B - 3 Other: A – 1 B - 1 Did not attempt: A - 3 B - 8	A: 1 incorrect operation B: 8 correct calculations- but did not get to correct answer, 1 didn't complete last step, 1 added instead of subtracting, 1 incorrect multiplication, 3 NWS, 1 incorrect divisor, 1 multiplied instead of divided, 2 incorrect fact
2	9	A: 0/5 B: 8/24	A: 5/5 B: 16/24	SD: B – 2 LD: B – 14 M/DF: B – 1 Other: A-1 Did not attempt: A: 4, B - 5	A: Multiplied instead of divided B: 1 incorrect copying, 1 omitted 0, 3 incorrect subtraction, 2 incorrect multiplication, 1 added remainder into total, 1 incorrect carrying over, 1 incorrect operation, 1 incorrect multiples

These questions highlighted learners' difficulty with the algorithms and the lack of understanding of the remainder. The results for Q 5, 7, 9 and 13 - pre-test 1 and Q9 - pre-test 2 were as follows, 14/26, 9/26, 11/26, 0/26 and 8/24. Group A learners struggled with the questions involving carrying over and working with a remainder, with no correct answers by any of the learners.

Strategies used

It is significant to note that 60 of the responses used long division with 28 using short division all of which came from Group B. None of the Group A learners attempted to use either algorithm to find a solution. 45 were not attempted across Group A and B.

Error analysis

Although no learners gave the correct answer for question 13 of pre-test one, 8 learners had completed the calculation correctly but they were unable to interpret their answers. Some errors could possibly be classified as slips, such as when learners used the incorrect divisor or multiplication fact, or copied the sum incorrectly. However, the majority of errors appeared to be developmental and indicated a lack of understanding of the operation, for example a place value error that occurred four times when learners omitted the zero in the quotient could be classified as the learner not understanding the operation. 4 learners did not complete all the steps of the algorithm and one did not carry over in the short division. This indicates an incomplete understanding of the process and operation.

5.1.3. Summary of pre-test results

While it is not possible to analyse learners' knowledge of division from the unanswered questions, it is clear that there are gaps in learners' knowledge.

- Strategies

Group A learners found the correct solution for some of the questions with no working, I assume that they solved these using their knowledge of multiplication and division facts. However, they did not answer many of the questions - they did not draw on any of the prerequisite skills and knowledge to develop informal strategies to attempt to solve these problems and often made mistakes in their attempts to use the formal strategies. Group B seemed to prefer the use of formal algorithms, and demonstrated broad familiarity with these algorithms.

In general, all learners appeared to understand the relationship between multiplication and division. This was evident in learners' ability to use the multiplication and division facts to solve some of the problems especially those that fell within the multiplication facts. Group B learners demonstrated an increased procedural fluency and tended to use an algorithm to solve a problem rather than draw on the division facts. Although there was no written evidence, the learners in Group A that did not draw pictures may have used their times table grids or knowledge of the multiplication facts when finding solutions. Learners were all able to perform division calculations that involved division facts with no remainders.

- <u>Error analysis</u>

Some of the Group A learners attempted to solve division sums using the format for the addition, subtraction and multiplication algorithms. This was an indication of an overgeneralisation of the multiplication algorithm. Learners assumed that in the same way as the subtraction and addition algorithm could be linked, so could the multiplication and division algorithms. This overgeneralisation demonstrated that learners did not have a sound understanding of the concepts of division.

If no working was shown, no certain diagnosis of the underlying cause of the error could be made. The majority of errors in the pre-tests, in both groups, could be considered developmental. The errors fell into a variety of categories including modeling, prototypical, overgeneralisation and process/object. All errors were related to inadequate/incomplete prerequisite skills or a partial/lack of understanding of the division concepts.

Following this analysis I decided to begin my intervention with the teaching of everyday sharing / partitive division concepts, as this has been recognized as providing the best basis for understanding division (Booker et. al. 1992, p. 167). From there I contrasted this with quotitive problems to provide learners with the opportunity to distinguish variation, identify critical features and consolidate all previous knowledge. This type of teaching directed learners towards more abstract contexts where concepts could be generalised and fused (Marton et. al. 2004, p. 16).

5.2. <u>Intervention analysis</u>

This section provides the introduction to the analysis of the intervention. In each lesson set questions were answered by the Group A learners. Responses to these questions, field notes and video footage formed the basis of my analysis. The first section provides a brief overview of each of the lessons and highlights the object of learning, critical features and dimensions varied of each lesson as done in Chapter 4. I then move onto a more detailed analysis of the activities and finally I document learners' progress and understanding within the various concepts of division. This section documents and analyses learner responses during the intervention. Following this the post-test analysis will provide insight into learners' development.

5.2.1. Lesson overview

Copies and justification for question selection can be found in Appendix F – K.

- Lesson 1 analysis

Object of learning: Quotitive and partitive division

Different parts of the division sum – divisor, dividend, quotient

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Wording of question to give a quotitive or partitive sum

As discussed in the previous chapter the aim of the first lesson was to introduce learners to division through word sums and provide the opportunity to create and use number sentences. At the same time, the intention was to create an awareness of the quotitive and partitive sums through highlighting the invariants and variables. In addition, I focused on learners becoming aware of the different terminology and parts of a division sum. As with the introduction of all concepts I began by assessing learners' current knowledge. In this case I was surprised that one of the learners was able to identify, and name, the quotient and another learner identified the divisor before we had gone over it as a class. These terms had not been introduced in the intervention, thus the learners were able to recall this from previous experiences. I was then able to draw on this and extend their knowledge into my direct object of learning for this lesson.

As mentioned in Chapter 4 indirect objects of learning were integrated into discussions surrounding the problems that addressed the direct objects of learning. For example, to extend the two learners that had a sound understanding of division I employed variation to explore the relationship between the different parts of the sum. They were able to tell me what would happen to the other parts of the sum if I made either the divisor or dividend bigger or smaller. They were also able to tell me what I would need to do if I wanted to get a bigger quotient. One of the other learners participated in this discussion and was able to tell me the effect of my actions, but was not able to tell me what to do if I wanted a larger or smaller quotient. The remaining three learners did not participate in this discussion. The remainder of the section will discuss different concepts that were taught through these questions.

- Lesson 2 analysis

Object of learning: Repeated subtraction

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Size of dividend

Forms of representation

The focus of lesson two was on the development of repeated subtraction. Question one was used as revision of the previous lesson and learners were able to use any method to solve the problem. From question two onwards all learners used repeated subtraction to solve the problems as was the instruction.

- Lesson 3 analysis

Object of learning: Division facts

Relationship between multiplication and division

Critical feature: The relationship between the divisor, divided and quotient

Dimensions varied: Different types of questions

Different representations of answers

Lesson three saw the formal introduction of the link between the multiplication and division facts. However, many of the learners had already made this connection and the transition appeared effortless. All of them began using their times table charts as we had done when working with fractions earlier in the year. However, those that had a better knowledge of the multiplication facts quickly abandoned the charts and relied on their memorised knowledge.

- Lesson 4 analysis

Object of learning: Division algorithms

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Size of dividend and divisor – extension of algorithm procedure

Inclusion of remainders

Lesson 4 involved the introduction of the algorithms and a more procedural approach. Support was given to all learners in this lesson. Catherine, Mel and Kim were able to complete the sums once initial teaching of the concept was complete. However, Salina, Jenny and Rosie required continuous support.

- Lesson 5 analysis

Object of learning: Division algorithms

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Sizes of dividends and divisors

Inclusion of remainders

Worksheet 5 was used as homework and reinforcement of the long division method of solving division calculations. Thus all calculations were done without my support. However, support may have been provided at home. All sums were corrected in class.

- Lesson 6 analysis

Object of learning: Fusion of different concepts and objects of learning from previous lessons

Critical feature: The relationship between the divisor, divided, quotient and remainder

Dimensions varied: Sizes of dividends and divisors

Inclusion of remainders

Different types of questions

Different representations of answers

This lesson was revision of all concepts and skills taught in class. Furthermore, it required the fusion of all division concepts and provided learners the opportunity to consolidate all knowledge covered in class.

5.2.2. Analysis by question type

Whilst the intervention did broaden into a more conceptual orientation it was necessary to develop learners' procedural competencies. During the intervention learners were able to solve these problems using any strategy that they chose with the exception of Q4 to 7 in lesson 2 in which they were required to use repeated subtraction. The questions ranged across the concepts and dimensions of variation presented in Chapter 2, such as: the different interpretations of division – quotitive and partitive, the relationship between multiplication and division and the related facts, the symbols and terminology of division, division repeated subtraction

In this section I have clustered the questions into the four overarching clusters, as was explained earlier in the chapter, based on procedural dimensions of variation used in the pre-test analysis. The clusters were grouped as follows:

- questions that involve division facts
- questions that involve numbers that are beyond the division facts
- questions that require a calculation and carrying over is necessary within the calculation
- questions that have a remainder.

It is important to note that other than a small number of questions in lesson two where I specified that learners must use repeated subtraction to solve the questions learners were free to use any strategy- formal or informal to solve the problems. Within each cluster of questions I will highlight the dimensions of variation, learners' performance, strategies used, common errors and difficulties. This will lead into an analysis of the concepts taught throughout the intervention.

- <u>Questions involving division facts</u>

There were 114 questions that learners attempted without support in this area. All questions and activities examined within this section fell within the known multiplication and division facts. As the table below indicates 87.7% of the responses to these questions in the intervention were correct.

Table 5.7. Questions involving division facts

Lesson	Question	Number of responses	Unassisted correct responses
1	1, 2, 3, 4, 7, 8, 9	42	34
2	1, 2, 4, 7a	24	19
3	1-5	30	30
6	1, 2, 5	18	17
Total:		114	100
%			87.7%

The learners did not like working with the repeated subtraction as they felt it was laborious and they said that they knew that there were more efficient ways of dealing with division problems. Once learners were formally introduced to the relationship between multiplication and division, and accordingly the division facts they used these to solve the numerical number sentences. This accounts for the five questions in lesson three where all learners used the division facts to solve the problems and accounts for 30 of the responses for this strategy. Although these questions were limited to the division facts, learners used a range of strategies to solve the problems. Graph 5.2. indicates the learners selection of strategies within this collection of questions. It is interesting to note that although arrays were not formally taught, Mel set out her pictorial representations in this format.

The graph below summarises the primary strategies used by learners to solve problems that involved the division facts. Strategies were clustered according to suggestions made in the literature. I broke the strategies into 4 main groups.

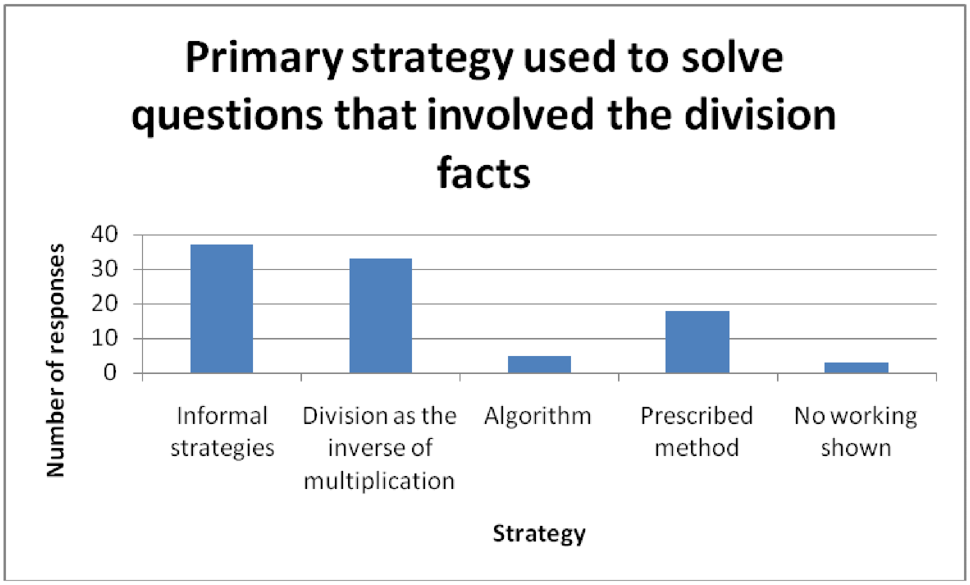
- Informal strategies (n=37) which included

- Drawing (n=22)
- Dienes blocks (n=8)
- Counting in groups (n=7)

- Division as the inverse of multiplication (n=33)
    - Division/multiplication fact (n=31)
    - Times table chart (n=2)
- Algorithm
    - Short division (n=1)
    - Long division (n=4)
- Prescribed method
    - Repeated subtraction (n=18)

There were 3 children that did not show any working.

Graph 5.2. Primary strategy used to solve questions that involved the division facts.



The graph indicates learners' preference for using informal strategies. It is also significant to note that many questions were solved using pictorial representations - including the array format, which was not formally taught, when they could have all been solved using multiplication/division facts.

However, the most commonly used formal strategy was the use of multiplication and division facts for solving problems. This is in contrast to the methods used by Group B in the pre-test, Group B learners favoured the algorithms for all division problems.

Each Group A learner tended to favour a different method. Catherine favoured the pictorial representation, initially Jenny preferred using blocks, however, in later lesson used pictorial representations with increasing frequency. Kim moved from drawing pictures to counting in groups, Rosie used a range of strategies - she began by using her times table chart, then progressed to counting in groups and used short division to solve the final questions and Salina favoured working with blocks. The analysis of learners' preferences highlights that although many of them changed strategies, most learners tended to remain within the same broad grouping of strategy. As there were so few errors within this category of question I have not conducted an extensive error analysis.

- <u>Questions beyond division facts with no carrying over or remainder</u>

Questions in this category focused on extending division facts to work with multiples of 10 and the introduction of the long division algorithm. Learners did not have much trouble in this area with 93.7% of the questions answered correctly. 52/63 calculations were solved using long division, 7 were solved without any working being shown, 2 used division facts and 1 used a fraction method.

It is significant to note that as learners' fluency and confidence in using the algorithms developed the range of strategies that they used to solve problems narrowed. Many of the learners abandoned their informal preferences demonstrated earlier in favour of the algorithms.

- <u>Questions beyond division facts that require carrying over</u>

The majority of questions were answered using long division in this section with 72/84 questions being answered using long division. Rosie continued to favour short division and Salina occasionally reverted to pictorial representations. 70.2% of the questions were answered correctly. This is significantly lower than the previous category of questions. This indicates the increased complexity of the calculations in this category. Some learners had difficulty with regrouping/carrying over occurred in Q6 of lesson 6 which asked learners to calculate 1000 ÷ 4. This question highlights a second difficulty, that of internal zero's. This error was described as a place value error in Chapter 2 as the learners are unsure of what to do in the places where the quotient was 0.

- <u>Questions that have a remainder</u>

This section was the most challenging as learners not only had to calculate the quotient but they were faced with a remainder. 4 of the learners continued to use long division to solve these problems with Rosie using short division and Salina favouring concrete and pictorial representations. In the purely numerical questions learners fared better achieving 61.9% correct in comparison to 19.4% achieved in the problems where they had to use and interpret the remainder. What is interesting to note is that in the questions that required learners to interpret the remainder, a further 27.7% calculated the sum correctly but were unable to interpret the answer correctly. Overall, learners achieved 52.4% correct for these problems.

It is significant to note that the difficulty Group A learners had making sense of these problems in the pre-test persist throughout the intervention and post-tests. Similarly the Group B learners' responses in the pre and post-test indicated that this area was problematic. It would appear that both the procedures and interpretations of a remainder were problematic and not sufficiently addressed in the intervention.

In conclusion as learners' proficiency with the algorithm developed their willingness to attempt questions improved as well as their ability to find the correct solution. Furthermore as their procedural competency improved so the range of strategies decreased. This was evident in the Group B's responses to the pre and post-test questions.

5.2.3. Analysis of concept development and strategies for solving division problems

I have analysed learners' development according to the question types. Each question type tended to favour the development of particular strategies. The following section will explore in more depth the strategies used by learners. I do this in order to present detail on how the increasing procedural fluency mentioned in the last section linked with both the conceptual understanding of the problem situation and learners willingness to adopt strategies. A further distinction that was salient was differences in responses to the more familiar problems and those that were less familiar (i.e. relating to understanding / creation of a problem situation). Learners' development according to the development of strategies for solving division problems can be tracked using the following strategies:

- Informal strategies and interpretations of division
- Formal representations
    - Division as repeated subtraction
    - The relationship between multiplication and division and the division facts
    - Long division

However, concepts of division cannot be viewed in isolation thus where links occurred between concepts I included them. At the end of this section I will show how all the concepts worked together as learners were able to fuse the dimensions of variation and concepts.

- Using informal strategies to solve problems

I found that the girls used informal strategies when we worked with the everyday problems and were exploring the different interpretations of division. Thus this section will focus on these two areas of the intervention. What became evident is the many ways in which the girls used informal strategies in their sense making experiences. It is important to notice that their strategies are not always appropriate to the problem and question, although they usually reached the correct answer even if their strategy was not appropriate.

Everyday problems and the different interpretations of division

As there was so much evidence supporting the use of everyday experience to develop the concept of division and thus introduce the critical feature of the relationship between the divisor, dividend, quotient and remainder, I chose to begin with this concept and its dimensions of variation. Furthermore, the use of everyday problems simultaneously provided the opportunity for learners to become aware of the two different interpretations of division, namely quotitive and partitive.

I found that the girls did not find the partitive questions (Q1 and 3) on the first worksheet in the first lesson difficult. They approached the problems with confidence as they were familiar and they made use of the opportunity to draw on a variety of support tools. This supported the claim that Booker et.al. (1992, p. 167) made when he said that learners are more familiar with partitive problems, as well as, the assertion of Toluk and Middleton (2004) that learners find this interpretation (partitive or sharing) of division easier.

The second, fourth and tenth question of the first lesson were quotitive division sums. Booker et. al. (1992) identified the increased complexity of quotitive problems when they explained that these problems require learners to take several objects and put them into a special form of arrangement. This was evident in the explanation learners gave when explaining their working to the rest of the class.

The following extract is from Salina's description of the second question of how she found her answer to question two using dienes blocks. It is important to note has she has equated the mathematics of the second problem which she is answering to the first problem which she solved appropriately and correctly.

> *Q2: A boy had 12 marbles he wanted to put the marble into bags. If he put 3 marbles into each bag how many bags could he make?*

S: I took three containers and I put them out.

T: Why did you take three containers?

S: Because it says there three bags?

T: Does it say three bags?

S: It says here “A boy has 12 marbles. If he put three marbles in each bag how many bags could he make?” So then what I did was […] I held them in my hand and I put one in each [Salina places the dienes block in each of the three containers one at a time]. Then I got my answer. It was 4.

T: Ok so you could make four bags?

S: No three bags.

What was evident in Salina’s description as with four of the learners was that they were able to correctly identify the numbers and operations necessary to solve the problem. They understood that it was possible to solve the problem using the same number sentence and method as they did in the first (partitive) calculation thus identifying the equivalence of the underling mathematical structure. However, it was clear from the description that she had not identified the variation and her strategy did not model ‘real-life’ but it ‘worked’. She was unable to distinguish that the first question had specified the number of groups and the second question specified the size of each group. The first question wanted to find out how many were in each group as opposed to how many groups in the second question. Thus, the learners who were not able to make sense of the problem and were not able to use an appropriate strategy to solve the problem. In opposition to Salina’s partitive / sharing approach described above that was not appropriate, Mel’s strategy for the second question described below was appropriate for this quotitive problem and indicated an explicit awareness of the underlying mathematical structure.

M: Well I worked it out […] in my head […] three times something equals what? So I said that equals four. So I know there are four bags and there are three in each bag.

When Mel drew her picture she drew three in the first bag then three in the second bag … creating an array of four columns with three marbles in each.

Mel’s insightful response drew on the relevant times table, indicating a prior knowledge of this concept and the notion that multiplication as the inverse operation, and a pictorial representation that formed an array as described by (Booker et.al., 1992). The array is a more advanced form of a pictorial representation than the sharing format used by Jenny as it is intentionally organized into appropriate groupings demonstrating that learner’s understanding of the organization of division problems has developed and is not just a collection of shared objects in a group. This representation may have assisted in providing the initial link to the multiplication fact. I interpreted Mel’s understanding of these questions as more relational (Skemp, 1989) as she was able to understand the subtle difference or variation between the two different questions, integrate several different strategies and draw on a web of knowledge to find and make meaning. She demonstrated the ability to integrate the concepts of multiplication facts, division number sentences, the relationship between multiplication and division and breaking up objects into specified sized groups to make sense of a problem and use appropriate strategies to solve it. On the other hand, the remaining 5 learners' understanding was described as instrumental (Skemp, 1989) as they were able to apply a rule, but did not demonstrate an understanding of the context. The fact that they all used an inappropriate strategy and explanation for the quotitive sharing indicates that were not able to make sense of the question. Some of the learners did not recognise that they had been told how much to put into each container and had to work out the number of containers that they were only able to identify the numbers. This was made clear when one learner answered by saying that there were four marbles rather than 4 bags were needed. I felt that although Salina and those who solved the problem in the same way she had were able to solve the problem procedurally the approach that they used indicated a poor, or incomplete, understanding of the context and underlying mathematical concept. Accordingly, the learners who selected the inappropriate strategy were experiencing a developmental error (Ryan and Williams, 2007) and did not fully understand the operation (Troutman and Lichtenberg, 2003). This error demonstrated their current level of understanding as well as the area that I, as a teacher, needed to continue addressing as it was a learning opportunity (Ryan and Williams, 2003) that learners have not completed.

However, the fact that all learners were able to identify a mathematical equivalence between the two concepts and three of the learners were able to recognise the relationship between multiplication and division indicated that they did have a partial understanding of division and some of the procedures used to solve problems. This relationship will be explored further in the section on multiplication and division.

As misconceptions arose I addressed them, assisting learners to make sense of the problems and use appropriate informal strategies to solve them. We then compared the two types of questions, the information given, what the answer was looking for, and how to solve the problem. The learners did not have any trouble identifying similarities between questions and were therefore able to identify the invariants once they had understood the context. They could identify that there was something different about the quotitive and partitive questions but only one learner was able to accurately describe the difference and thus had achieved a complete relational understanding of the two types of problems. Two learners were able to provide a partially correct description and thus had a partially relational understanding of the two types of questions. The remaining three learners still did not have a relational understanding and were still using an instrumental understanding to solve the problems.

In conclusion, learners were able to compute the everyday problems with ease. However, they generally represented their answers using a single method. As their repertoire developed they were able to represent a single problem using more than one representation. The learners were able to solve the partitive problems using appropriate sharing strategies with ease and they were able to articulate their findings. Five of the learners initially struggled with the quotitive problem. By the end of lesson one, two of the learners were able to solve quotitive problems accurately using appropriate grouping strategies, however only one of these learners could write a quotitive problem. I believe that one of the learners had a full relational understanding which was demonstrated by her insightful descriptions and the integration of her knowledge of different problem solving strategies and representations, one had a partial relational understanding and the remaining four are still developing their understanding of the different types of division word problems. These areas were being continually addressed during the teaching of other division concepts. By the end of lesson six I believe that three learners had a relational understanding of the partitive problems and they all had a better understanding of the quotitive problems. However, whilst I do not think that any of the learners had a fully relational understanding of this interpretation. For example, they could all solve a partitive and quotitive problem accurately and appropriately, only three learners could write a partitive problem and no learners felt comfortable writing a quotitive problem without support.

Creating appropriate problem situations

Although the content of these questions was no more complex than that used in the questions learners solved using the informal methods or with multiplication or division fact the way in which they were asked was new to learners. These questions were used to develop learners' conceptual understanding of division and thus the focus was not on procedures but rather on how learners interpreted the question, what strategies were used to solve them and how they represented their finings. As conceptual development has not been the focus of previous mathematical learning experiences these questions did not match the prototypical examples learners had encountered previously.

Learners found Q3 of lesson six (38 ÷ 2 =) challenging as they were required to write their own question for the sum, extending the scope of variation. Four of the learners were able to do this. Mel, Salina and Catherine provided partitive problems the following example was the question Salina wrote:

> "You have 38 chocolates and 2 of your friends want how many will each get?"

The fact that 3 out of the four learners who wrote appropriate problems chose to write partitive problems I believe demonstrated that they were more comfortable and had a better understanding of this type of interpretation as opposed to the quotitive problems. Kate provided the following question – this type of question had not been directly addressed during the intervention and demonstrated her knowledge of the relationship between multiplication and division

"Find out what you have to times 2 by to get 38?"

This was a new dimension of variation that was introduced during the intervention. As this was not a typical example that the learners had encountered previously, and did not match the prototype examples that the learners were accustomed to, the errors relating to this question form part of the prototypical examples and indicate that this is an area that needs to be developed further. Q4 was another example that was not prototypical. Those learners that reached the correct answer did so with support. The learners had difficulty understanding that they did not need to rewrite the information in a more appropriate order, or work out the answer but simply use the information to write a number sentence to represent the information. The prototypical problems indicate areas that learners are not familiar with and need to be revisited in later teaching. While these examples explored learners understanding of the interpretations of division it was presented in a format that was unfamiliar. It also required learners to move away from informal notation towards more formal and traditional notation. This type of questioning can be viewed as separate dimensions of variation of the broader umbrella of problems involving everyday experiences and interpretations of division. The variation was that the question did not ask learners to solve the problem, but rather represent the question in a different manner, thus exploring learners understanding of the concept without using a procedural approach. Thus, I attempted to integrate the teaching of this dimension of variation regarding questioning approach into the everyday examples and interpretations of division. It is possible that I provided insufficient patterns of variation and examples within this dimension as my focus was on a different dimension and had not provided enough variation to make the invariance obvious with enough invariance to make the variance obvious, as highlighted by Watson and Mason in Chapter 2 (2005b, p.4). Asking learners to formulate questions and represent problems using different formats and identifying missing information are important skills and should be addressed in more depth. Conversely, Q5 and 6 of lesson 6 were all variations of typical examples that were used throughout the intervention and in previous years of schooling. All learners were able to indentify appropriate strategies to solve the problems. However 4 of the learners required some support when solving Q6 (lesson 6)

These dimensions of variation highlighted the importance of example selection and the many possibilities. Some of the difficulties that the learners experienced within the dimensions of variation discussed in this section stem from a lack of experience within a specific frame of reference and as a result learners were unsure of which strategies would be helpful to solving these problems.

- Moving towards formal representations

Initially learners favoured informal methods however as their knowledge of division developed they moved away from these towards more formal representations such as number sentences and algorithms. The following sections will explore the formal and more traditional approaches to solving division problems.

Using division as repeated subtraction to solve problems

Repeated subtraction forms an important conceptual link between subtraction and division but according to Booker et.al. (1992) repeated subtraction should only be developed after the concept of division is in place and should be an extension of the partitive approach. I agree that repeated subtraction fits best within the partitive approach but I felt that it was important that I showed the learners how repeated subtraction could be also be used to solve quotitive questions. This provided the opportunity to consolidate the similarities and differences between the two interpretations. Thus, although division as repeated subtraction was a separate and distinct dimension of variation, it promoted the generalisation of the interpretations covered in an earlier section. As explained in Chapter 3, Marton et. al. (2004) described generalisation as the “experience of varying appearance”, while the format and type of questions used within this concept were the same as those used in the previous section the manner and approach to solving appeared different while the conclusion reached was the same. The remainder of this section will detail learners’ development of the repeated subtraction conception of division.

In lesson 2, I went through question 2 - 4 with the girls. I drew a picture showing how the groupings or sharing was done and made the link to repeated subtraction for the learners. I found that the girls grasped the connection to repeated subtraction easily, however, some of them were not sure how to find the correct answer once they had completed the calculation correctly. This was evident when they gave the remainder that they had found at the end as their final answer. Mel overcame the problem by keeping a tally of the number of subtractions she had made down the right hand side of the sum. Salina marked of each subtraction with showing what she subtracted on the right e.g. 1 flower, at the end she counted the number of subtractions she made.

According to Troutman and Lichtenberg (2003), the errors learners made during repeated subtraction were due to difficulty with prerequisite skills e.g. Salina did not remember how to subtract if there was a 0 in the top number and Rosie simply subtracted the smaller digit from the bigger digit regardless of which was on top. According to the authors (Troutman and Lichtenberg, 2003) description, these learners would be unable to understand the new concept of division as repeated subtraction, as they have not acquired the necessary prerequisite skill of subtraction. To overcome this I needed to address this deficit before the learners could achieve a relational understanding. As time was limited this would have to be done outside of school hours in the extra lessons offered at school.

Some of the children were confused when they reached question 8 as there was no word sum, only a number sentence. Once I explained that it was the same as all the other sums they were able to answer the questions without much difficulty. It would appear that all the learners had a good instrumental understanding of division as repeated subtraction. Some of the learners began moving towards a more relational understanding when they were able to integrate their calculation, knowledge of division, interpretation of the question to correctly answer the question.

Learners were not familiar with this interpretation of division and they did not choose to continue with it as they found it laborious, especially those learners who had some knowledge of division and its algorithms knew that there were more efficient ways to solve division problems.

Using the relationship between multiplication and division and the division facts to solve division problems

All the girls found it easy to make the link between the multiplication and division facts, some of them demonstrated this understanding during the development of previous concepts and the fusion of knowledge across different dimensions of variation. Further, all of them began using their times table charts as we had done when working with fractions earlier in the year. However, those who appeared to have a better knowledge of the multiplication facts quickly abandoned the charts and relied on their knowledge. Any errors made were quickly corrected once identified either by the learners themselves or with support from me. These errors were clerical (Troutman and Lichtenberg, 2003) and not developmental (Ryan and Williams, 2007). They were as a result of the girls rushing through the sums that they had found easy. They were in contrast to the learners’ other answers and working practice in this section (Ryan and Williams, 2007).

Learners did not have any difficulty solving the division by one and the division of zero questions. They referred to the times table facts as their justification. The learners did not identify the variation in the division of one and division by zero. They assumed that these questions were the same as the previous questions and cited the times tables as the justification. This developmental error was the result of an intelligent overgeneralization (Ryan and Williams, 2007). Booker et. al. described this difficulty in their article and recommended the use of sharing as the most effective manner in helping learners make sense of this concept. Thus, I encouraged learners to go back to concrete or representational pictures to help them use their knowledge of sharing to help them develop within this learning opportunity. When learners couched the problem in an everyday context they quickly realised their mistakes, thus supporting Barnes’ (2005) claim that every day problems are the best way to assist learners in making meaning. Learners came to the correct conclusion when I had guided them and provided the context. However, as later evidence gained in the post-tests suggested, the learners did not make sense of the concept for themselves and thus ‘own’ it, but rather they relied on me to point out the connections. A possible cause of this was that I did not provide enough examples within this dimension of variation for learners to have the opportunity to develop expectations and make sense of the concept through working within a pattern of variation. This possible problem was commented on by Watson and Mason (2006a) and highlighted in the literature review.

Learners found it easy to divide by ten and understood the role of place value in this situation as they had encountered it in the multiplication of 10 and mental calculation exercises as suggested by Haylock (2006). They also found it helpful to integrate their knowledge of simplifying fractions when dividing by multiples of 10.

The importance of these two dimensions of variation, the relationship of multiplication and division and the division facts, cannot be diminished. These dimensions of variation become prerequisite skills for later dimensions of variation. Jenny continued to find it challenging to identify the correct division facts, throughout the intervention as the reification of these concepts had not taken place. Evidence of this difficulty for Jenny occurred continually throughout the activities. The effect of this difficulty became more evident in the post-tests.

Using long division to solve division problems

Initially, the plan was to introduce long division in lesson four and progress to short division in lesson 5. However, as learners demonstrated some difficulty in working out problems mentally, I decided to rather consolidate the long division. Support was given to all learners in lesson 4. Catherine, Mel and Kim were able to complete the sums once initial teaching of the algorithm was complete. However, Salina, Jenny and Rosie required continuous support.

As explained in the previous chapter I used dienes blocks to guide learners from working with concrete materials and division facts into the algorithm. We worked through question 1 of lesson 4, together for the division by a single digit. In Question 2 of lesson 4, learners were given the opportunity to try to solve the problem before we worked through it as a class. From question 3 in lesson 4 onwards Catherine, Mel and Kim were able to solve the problems involving division by a single digit without further support, while the other four learners continued to receive support. Learners completed one or two sums on their own and then as a class we would go through the calculation. This assisted learners, as well as me as the teacher and researcher, to identify errors and provide the opportunity to clarify any misconceptions that become evident when learners explained their working.

We then moved onto division by a double-digit number. Learners found it challenging to identify the multiples. However, once I showed them how to work out the multiples for each number they were able to solve the problems with little support.

Rosie's error in Q 8 (lesson 4) of division by a single digit and Jenny's error in Q5 division by a double digit would appear to be slips in this activity as they were in contrast to all other sums. However, as support was available and provided throughout the activity an accurate measure of learners knowledge could only be obtained when learners had completed a piece of work unaided, such as the post-test.

When learners were calculating the sums I found that they were unable to find a quotient and hold that sum in their memory to be used again when they were doing the multiplication/check part of the sum. Each time they would have to rework the operation. An example, based on observations made from the video footage, of the thought process that took place at each step of the long division sum was as follows

> If a learner needed to find out what 65 divided by 9 was, the learner was then able to use their times tables grids to find out that the quotient was 7. However, when multiplying 9 and 7 in the second part of the step the learners needed to go back to their times table chart to find the answer. They could not recall this fact that they had looked up in the division step.

Worksheet 5 provided an opportunity for learners to consolidate their knowledge of long division and showcase their new skills. Worksheet 5 was used as a homework assignment and reinforcement of the long division method of solving division calculations. Thus all calculations were done without my support. However, support may have been provided at home. All sums were corrected in class. Five of the learners used long division to find the solution to the lesson questions, while one learner chose to use short division. All learners except Rosie and Salina attempted all the sums.

Catherine made one subtraction error. This error was contrasted with her standard working practice and was classified as a slip and not a developmental error. Jenny made two errors, both involved the use of an incorrect number fact. While these appear to be slips, it is important to recognize that she had support and access to a multiplication grid throughout the intervention. In addition, she made this same error in numerous previous activities. Thus, her errors were considered developmental and were a result of a difficulty with prerequisite skills. This was viewed as a process object error as Jenny did not view the number facts as a reified object that could be used in the process of long division but had to be found as a process and thus resulted in errors. Accordingly, this influenced the accuracy of the division process. Although Salina explained that she had solved the sums using long division on a scrap piece of paper her answers with decimals indicate that she had used a calculator as division into decimals is only introduced in Grade 7. While Salina did not demonstrate her procedural competency she did demonstrate that she knew the required operation was division and which numbers represented the divisor and dividend.

Salina required support with Q7 and 8 in lesson 6. Rosie did not ask for support but was unable to reach the correct solution. Her error in Q7 (lesson 6) can be considered developmental and was the result of a process object problem and partial understanding of the concept of division and an incomplete understanding of the algorithm.

It was clear that although the learners did not always reach the correct solution there had been development in their knowledge of the division algorithm. I hoped that through the consolidation of the initial division concepts and dimensions of variation at the beginning of the intervention, and the fact that I had guided learners from these concepts into the development of the division algorithm, I had created a meaningful algorithm. Through this approach, of both the separation and subsequent fusion of concepts, I was able to provide the opportunity to use generalisation as the tool through which the algorithm was developed. Consequently, I hoped that learners would develop a relational understanding and a sound knowledge. This was measured in the delayed post-test with the initial post-test providing evidence of more consolidated development.

- <u>Fusion of different dimensions of variation</u>

The fusion of concepts and dimensions of variation is a difficult task for any learner, but even more so for low attaining learners. This section will demonstrate that although the majority of learners mastered the separate concepts, they found it very challenging to fuse them. Question 1, 2, 9 and 10 of lesson 6 provided the opportunity for learners to fuse several dimensions of variation together.

Learners showed that they understood separate dimensions of variation in question one and two (Lesson 6) as they were able to reach the correct answer for the problem. However, only four of the learners showed that they could represent the problem in more than one way and none of the learners attempted to demonstrate more than two methods.

Three learners were able to correctly identify the correct operation for Q9 – lesson 6 and two learners for Q10 – lesson 6. However, none of these learners were able to correctly complete the full sum. This problem was exacerbated by learners' poor understanding of the concept of a remainder as highlighted earlier. Examples of the difficulties in this area were highlighted in Question 9 of lesson 6. The question required learners to first work out the distance that each person had to drive. To do this they had to recognise that there would be 4 driving sessions and the total distance had to be divided by 4. Learners had to use their knowledge of division, the partitive interpretation, the algorithm and the meaning of the dividend, divisor and quotient, together with their knowledge of addition/subtraction to calculate this problem. Catherine and Mel correctly calculated this, however, they then had to establish at which distances the drivers needed to change over – thus adding on the calculated distance to the previous answer each time. Learners were unable to interpret their answer as the distance that each person had to drive and add this onto the total. This indicated that they did not have a full understanding of the context and thus the error was a modeling error and an area that required further development. The complexity of this type of problem was further increased in question 10 of lesson 6 where learners found a remainder and had to interpret this to make sense of the problem.

This dimension of variation required learners to fuse at least two different operations together with various dimensions of variation within division to solve the problem. Although the other operations were not addressed in the intervention all prerequisite skills were covered prior to the intervention. Accordingly, learners had the necessary skills to solve the problem, however, they would have had to fuse these concepts in order to successfully solve this problem.

5.3. Post-test analysis

The initial post-test was written by Group A approximately 2 weeks after the intervention was completed. The entire grade wrote the delayed post-test approximately 2 months after the intervention. The initial and delayed post-test contained identical questions. The delayed post-test was written at the beginning of the following school year. No support was provided by the teacher although learners still had access to all the support tools. Both post-tests contained additional conceptual questions to fill the gaps that were found when analysing the pre-test. Thus, analysis involving the post-tests took place at two levels. I begin by analysing and comparing the performance of Groups A and B in the pre and delayed post-test, therefore highlighting the knowledge gained during the intervention and the areas that remained problematic. Comparison of responses of Group A and B learners in the additional conceptual questions in the delayed post-test provided insight into learners' proficiency with the more procedural based questions as compared to the more conceptual based questions. I also evaluated the change in results from the initial to the delayed post-test and the number of questions attempted by Group A learners to provide an indication of the retention of this knowledge and learners' confidence in working with the division concepts.

Graphs 5.3. and 5.4. compare the number of correct responses Group A and B learners gave in the pre and delayed post-test. Group A and B's results are displayed in terms of the mean performance and change. Group A's results have also been represented on an individual basis in the table below each graph.

Graph 5.3. Comparison of results in pre-test 1 and delayed post-test 1

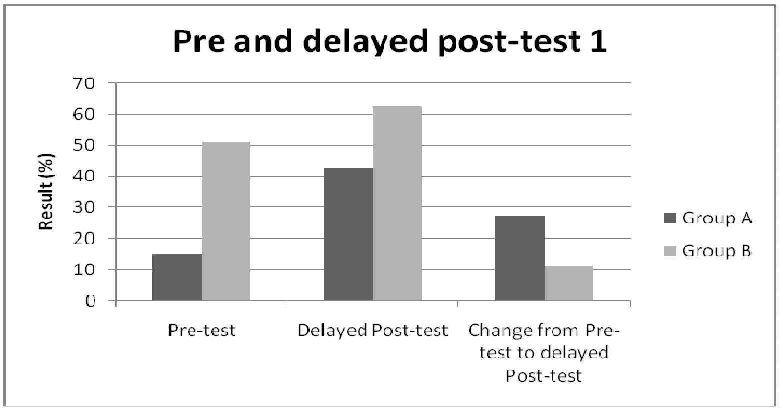


Table 5.8. Group A results in pre and delayed post-test 1

Test 1	Pre-test	Delayed Post-test	Change from Pre-test to delayed Post-test
Salina	0	25	25
Mel	12.5	37.5	25
Catherine	37.5	87.5	50
Jenny	0	25	25
Kim	25	37.5	12.5
Mean	15	42.5	27.5

Graph 5.4. Comparison of results in pre-test 2 and delayed post-test 2

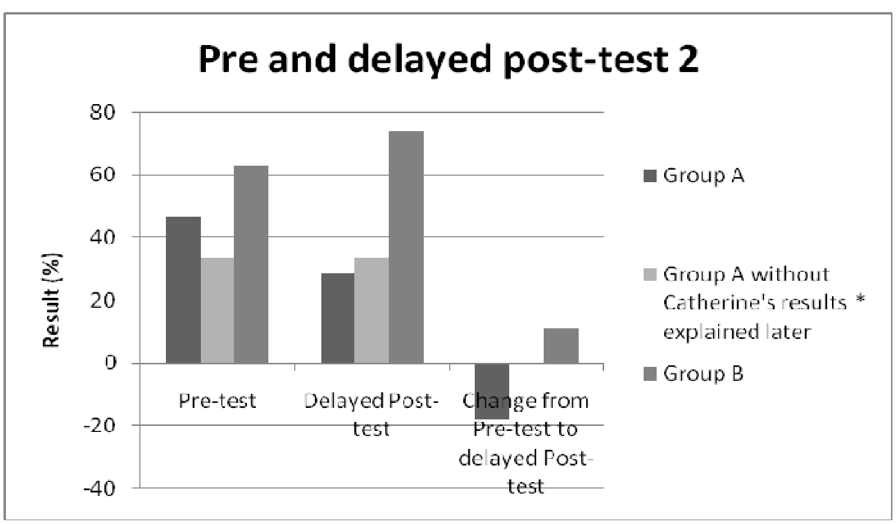


Table 5.9. Group A results in pre and delayed post-test 2

	Pre-test	Delayed Post-test	Change from Pre-test to delayed Post-test
Mel	28.6	28.6	0
Catherine	85.7	14.2	-72.8
Rosie	28.6	28.6	0
Kim	42.9	42.9	0
Mean	46.45	28.575	-17.875
Mean without Catherine's results	33.367	33.367	0

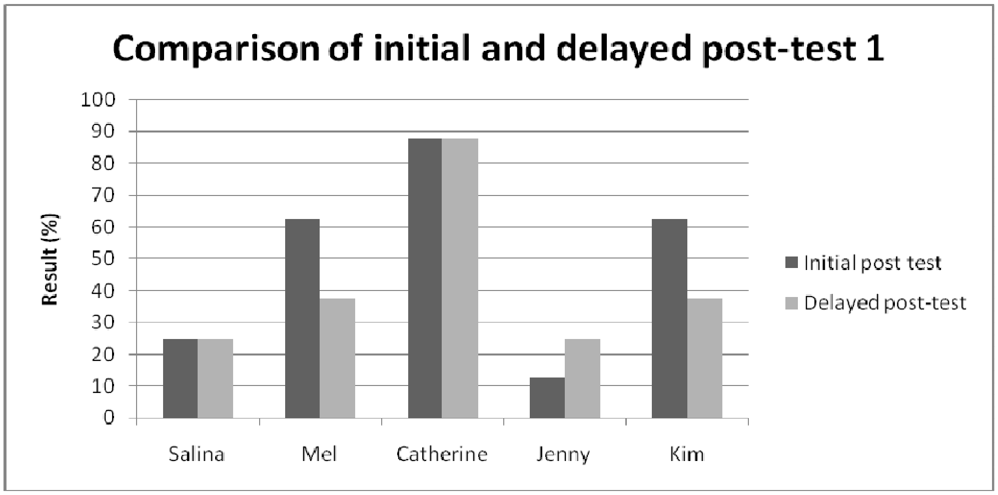
It is important to note that Catherine, when writing post-test 2 first, said that she could not remember how to divide but in writing of post-test 1 immediately following post-test 2 (within the same lesson) she then remembered. Thus post-test 2 is perhaps not a fair reflection of her development. Furthermore without Catherine's results there was no change in the Group A results. It must be noted that although division did not form part of the planned teaching for the Group B learners from the time of the pre-test, all teachers revised and consolidated division concepts during the intervention and post intervention time.

It is evident from graph 5.3. that all learners improved in post-test 1 when the results were compared to the same questions from the pre-test results and the improvement was greater for Group A then the improvement in Group B learners. As pre-test 1 drew more on learners' knowledge of the relationship between multiplication and division, and the division facts, it would appear that learners' knowledge in this area had improved during the course of the intervention. Graph 5.4. indicated that none of Group A learners improved on their pre-test results, while Group B learners demonstrated a small improvement. Group A's progress was disappointing and I hoped that through my analysis I could identify the reason for this lack of progress in the hope that I could improve future teaching and learning.

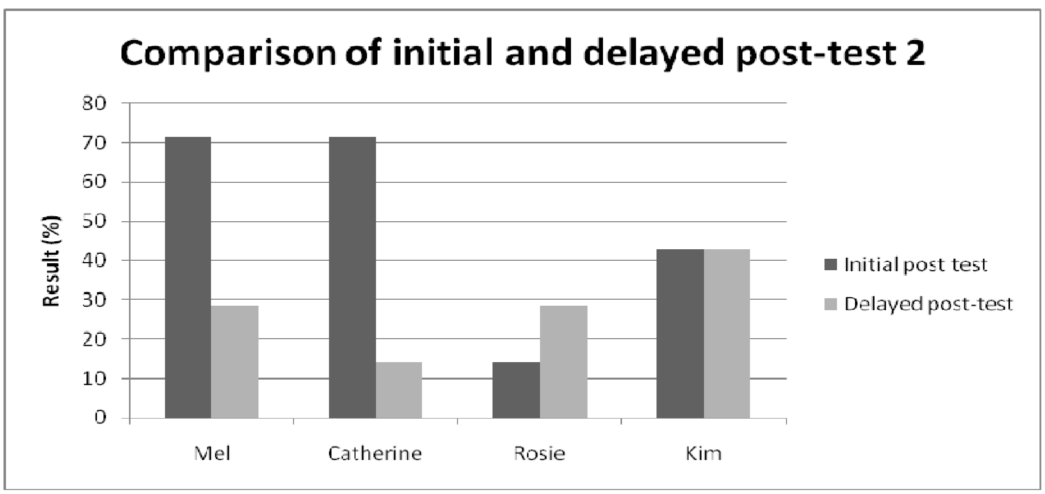
Post-test 2 demanded learners work in greater depth with calculations that required carrying over and remainders. These concepts are more complex and required learners to work with more advanced procedures. It was not my intention to make the one test more challenging than the other and it did affect the learners' performance. However, as explained earlier, learners' performance was only measured within the same test and thus was still considered valid. Graph 5.4 emphasises the fact of Group A learners' ongoing difficulty within the carrying over and working with the remainder. This was discussed in detail in the section which broke the questions down into clusters earlier in the chapter.

Group A wrote an initial post-test straight after the intervention and then another post-test a few months later. This measured how many of the concepts and procedures they were able to retain over an extended period. This analysis incorporated both the more procedural questions that formed part of the pre-test and the additional conceptual questions that were only part of the post-tests. A separate analysis will be done comparing learners' performance in the procedural and conceptual questions. I expected some knowledge to be lost over the extended period, as the intervention analysis indicated that several learners had not achieved a relational understanding of the concepts.

Graph 5.5. Comparison of Group A's initial and delayed post-test 1 results



Graph 5.6. Comparison of Group A's initial and delayed post-test 2 results

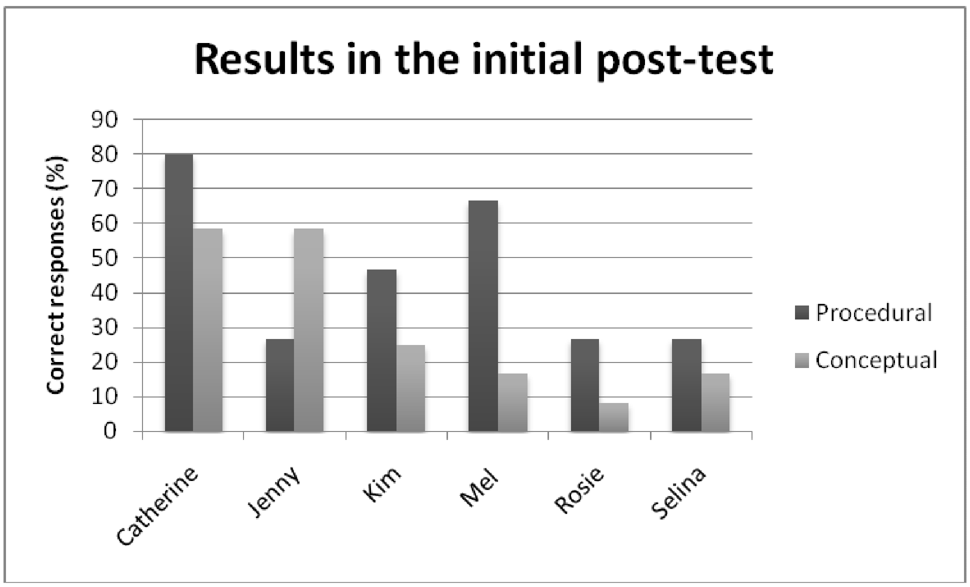


Graphs 5.5. and 5.6. above indicate the difference between the results that learners obtained during the initial post-test and the delayed post-test. Some learners appeared to have retained all concepts and procedures (3 learners in post-test 1 and 2 learners in post-test 2) while the other learners results dropped.

Catherine's comment suggested that her delayed post-test 2 (as explained already) was not a fair reflection of her knowledge and understanding. A drop in the results could indicate one of two things; either the learner has forgotten some of the concepts over the holiday period and once they begin working again they will remember or their understanding was instrumental and they did not have a complete or relational understanding of the concepts. Rosie and Jenny's improved results are contrary to expectation and suggest that relational understanding had begun to develop in terms of retention and improvement of results.

5.3.1 <u>Conceptual questions added to the post-test</u>

As explained earlier, I found that the pre-test did not adequately assess learners' conceptual understanding and thus I added in a few questions into the post-test. The results were varied in the new conceptual questions that were added. The following graph indicates the breakdown of learner attainment in the procedural vs. the new conceptual questions in the initial post-test.

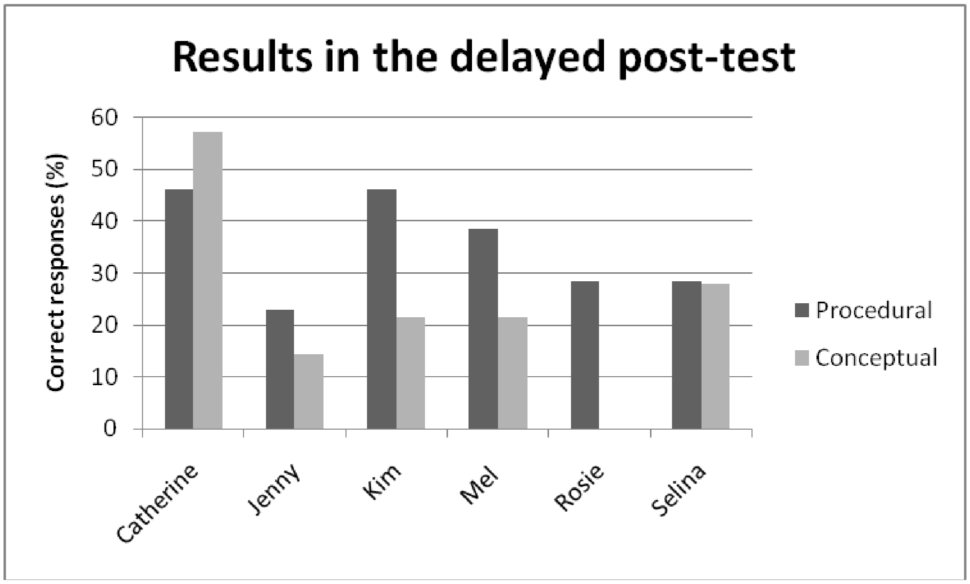


Graph 5.7. Results for the conceptual and procedural questions in the initial post-test

Five out of the six learners scored a higher percentage correct in the procedural based questions which were found in both in the pre and post-test and lower in the more conceptual based questions which were added to the post-tests following the intervention. This could perhaps be attributed to the fact that fusion was required and that they required higher order thinking skills and the need to identify and perform different strategies to find a solution such as the ability to substitute, then solve a problem in order to compare it with the question sum such as was the case in question 31 of the post-test.

Learners used generally selected one of three methods to solve the problems – pictorial representations, multiplication/division facts or algorithms. They all favoured the use of division facts in the questions that required a number that formed part of the known facts or could be found on their times table grid. Two of the learners chose to use pictures to calculate the problem if the dividend was smaller then 72. Five of the girls chose to use long division for questions that were not division facts and one of the girl's favoured short division – which was not taught during the intervention.

Graph 5.8. demonstrated learners performance in the procedural and conceptual questions in the delayed post-test. While learners' performance and changes have been discussed extensively it is important to note that learners procedural competency was significantly higher in the delayed post-test in comparison to their conceptual knowledge.



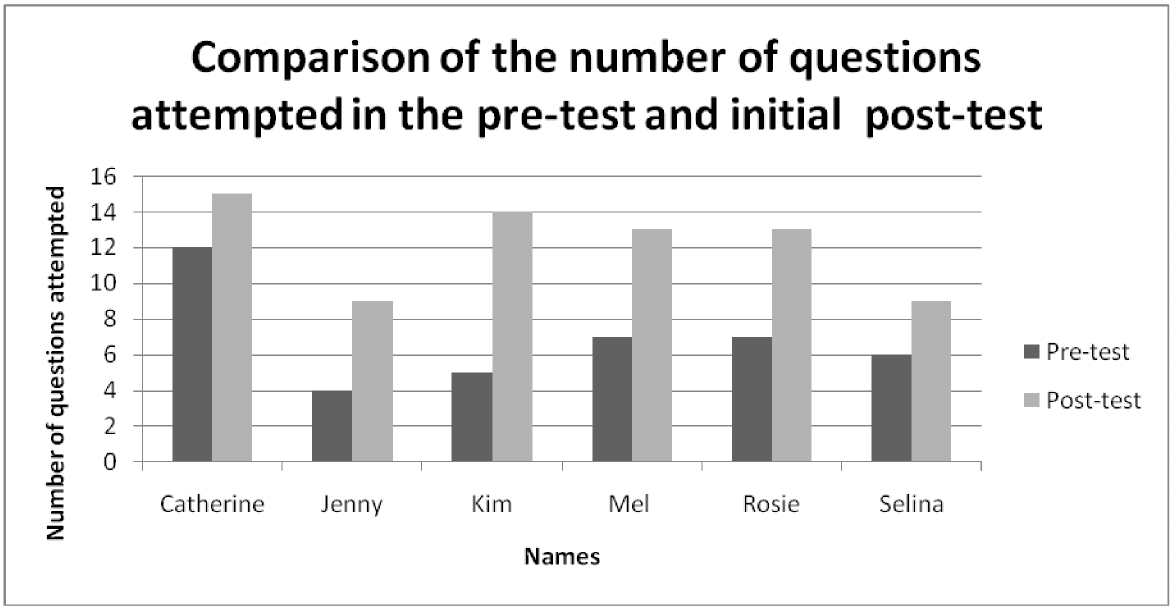
Graph 5.8. Results for the conceptual and procedural questions in the delayed post-test

According to the results, there was an overall improvement in all the learners' knowledge and understanding of division. I will now analyse each learner's attitudinal progress and their willingness to attempt questions before discussing their overall progress and the errors that occurred during the post-tests.

5.3.2. <u>Group A attitudinal improvement and willingness to attempt questions</u>

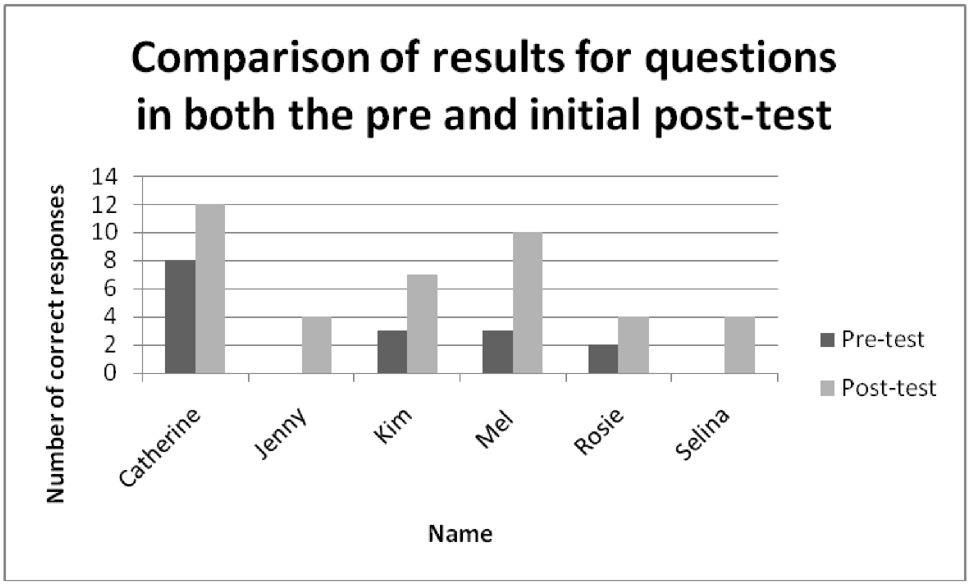
A further factor worth noting related to the number of questions learners attempted. If learners did not attempt a question it could indicate a lack of understanding – that they did not know where to start or how to solve the problem or they took to long to answer some of the questions. All Group A learners attempted more questions in the initial post-test in comparison to the pre-test. Furthermore, all learners got more answers correct in the post-test questions. However, three of the learners, Jenny, Kim and Rosie also got more answers incorrect in the post-test than they did in the pre-test but it must be noted that these girls attempted more questions in the post-test than in the pre-test. Of these learners, Jenny did not get any answers in the post-test incorrect that were correct in the pre-test, Kim omitted one sum in the post-test that she got correct in the pretest and she got one incorrect in the post-test that was correct in the pre-test and Rosie got two sums incorrect in the post-test that were correct in the pre-test.

The following graph indicates how many of the questions learners attempted in the pre and initial post test. The comparison was across both test 1 and 2. For example, Catherine attempted 12 division questions altogether in pre-tests 1 and 2 and she attempted 15 questions altogether in the initial post-tests 1 and 2.



Graph 5.9. Comparison of the number of questions attempted in the pre-test and initial post-test

All learners attempted to answer more questions in the post-test than they had in the pre-test. This indicates that the learners' new knowledge and understanding of the concept of division enabled them to attempt more questions following the intervention. Furthermore, five out of the six learners answered four or more questions correctly in the post-tests that were left out or incorrect in the pre-tests. This is indicated in the graph below.



Graph 5.10. Comparison of learners overall attainment in questions attempted in both the pre and initial post-test.

However, Catherine and Rosie got 2 questions wrong in the post-test that were correct in the pre-test, Kim answered one question wrong in the post-test that she answered correctly in the pre-test and left out one question in the post-tests that she had answered correctly in the pretests. However, the overall attainment improved in the initial post-test. Unfortunately, as discussed earlier some of these gains were only temporary and learners did not retain all knowledge gained.

5.4. Summary of change for each Group A learner

There was evidence of a significant improvement in Catherine's knowledge, the errors she made in the initial post-test were mainly non-developmental and indicated that she has a more complete understanding of division. Q3 and 27 as indicated in the full post-test in appendix M would appear to be slips as they were out of her standard working practice as evident throughout the intervention. In Q3 she subtracted incorrectly and in Q27 it would appear that she misread the sum. Q30 (in appendix M) was evidence of an area that requires some development as it would appear that she did not fully understand the context and only partially answered the sum.

Jenny, Rosie and Salina's errors in both post-tests tended to be developmental as they occurred frequently throughout the intervention. They are indicative of an area that the girls' understanding is incomplete and requires development. However, all three girls attempted far more questions in the post-test than in the pre-test and the results in the delayed post-tests (see table 5.8. and 5.9.) indicated an improvement in their knowledge at the same time. The results also indicated numerous areas in which there was opportunity for teaching and development.

Kim’s results indicated an improvement as she attempted nine more questions and she got more correct than in the pre-test. Her errors appeared to be developmental and related to underdeveloped prerequisite skills, for example: finding the appropriate division fact required a calculation process rather than simply being a recalled fact. This difficulty resulted in frequent errors due to her inability to recall her multiplication and division facts instantaneously. Essentially, given that she could not work with the division facts as reified objects when solving a division problem, she had to go though the process of working out answers while solving the division problem. This is likely to slow her openings for progress significantly, and suggest that before reification of the calculation process into a division fact is a key area to focus on.

Mel showed the greatest improvement. She attempted six more questions in the initial post-test and she achieved 7 more correct answers than in the pre-test. Furthermore she got one less incorrect answer. Her errors appeared to be slips and for the most part not developmental. However, some of the new concepts introduced in the intervention, for example the relationship between the critical features, required some development as she did not know how to answer these questions. An illustration of this was that she did not know what to expect if the dividend remained the same but the divisor increased in size. It must be noted that Mel’s poor performance in the delayed post-test was indicative of some fragility of the knowledge gained during the intervention. However, I believe that with a little more support and consolidation this knowledge can become solidified.

In conclusion, all learners improved as a result of the intervention. While all of the learners require further teaching in order to develop the areas in which they have only developed a partial understanding, some learners appeared to have gained a more complete knowledge of some of the concepts taught. In the sections where sufficient patterns of variance and invariance were used; for example the questions relating to everyday contexts, the two interpretations of division (partitive and quotitive), the relationship to multiplication and division and division without internal zeros learners tended to show greater progress. In the dimensions of variation where insufficient variance and invariance (Watson and Mason 2005b) was provided learners did not make the progress that I had hoped for, for example division involving 0 and 1 and remainders. This was discussed extensively in Chapter 3.

In this chapter a significant amount of analysis surrounded clusters of questions that were grouped according to procedural dimensions of variation. While these indicated a favoured procedural orientation learners were not limited to this. It was interesting to note that the recommended procedural hierarchy that was advocated throughout the literature in my review and that I came to accept was not developed in such a straightforward manner by the learners. What became clear in the pre-test, through the intervention and in the post-test was that questions that required a similar level of ‘sense making’ were addressed in different ways by the learners across Group A and B. Group B learners appeared far more willing to use formal division algorithms and apply them accurately. Group A learners favoured the informal methods and found it difficult to apply the algorithms with consistent accuracy.

The following chapter will summarise the analysis and draw together the findings highlighting strengths and weaknesses of the intervention. Based on the analysis recommendations will be made for future teaching.

Chapter 6

Conclusion and recommendations

In this research I investigated a sample of Grade five learners' knowledge of division. I conducted an intervention using the theoretical framework of variation theory to develop the learners' knowledge of the division concepts and to assist them in making sense of the mathematical contexts relating to division concepts. I recognise that many of the choices regarding the selections I made relating to concepts, terminology and teaching strategies from the literature to a large degree reflected my preferences towards teaching practices that have been developed in my teaching experience and studies. Through my findings I hoped to contribute the body of knowledge surrounding the teaching and learning of division.

6.1. <u>Answers to Research Questions</u>

Through my research I hoped to answer three key questions. Discussions surrounding my findings were done extensively in the previous chapter, but the following section will summarise and draw together all the research gathered to answer the questions.

> "What are the specific features that learners struggle to understand within the concepts and procedures associated with division at a Grade 5 level?"

The literature on error analysis proved very useful in this area of research. Ryan and Williams (2007) differentiated between six types of errors in three board categories:

- developmental errors which included:
    - modeling,
    - prototyping,
    - overgeneralising,
    - process-object linking,
- slips which held no obvious developmental or conceptual explanation
- errors that could not be diagnosed.

Troutman and Lichtenberg (2003) provided three mathematical areas where learners may experience difficulty, namely:

- place value and understanding the operation
- difficulty with the prerequisite skills
- clerical difficulties.

When linking the research on division to the categorization of errors, I was able to identify the areas of difficulty which learners experienced.

The pre-test, intervention and initial and delayed post-test took place after the teaching of the section on division. The Group B learners appeared to have a good understanding of relevant Grade 5 division concepts. They were able to answer the majority of the division questions accurately in both the pre and post-test. However, the girls in Group B appeared to struggle with the questions that required more than one operation. The intervention group, which has been described as low attaining, did not attempt to answer the majority of the division questions in the pre-test and many of those that they did attempt, they got wrong. Through an analysis of the errors, many of the difficulties related to inadequate prerequisite skills such as a knowledge of the times tables and the ability to subtract or multiply accurately, as well as a general lack of knowledge surrounding the actual division concepts. A few of the errors appeared to be slips or careless mistakes. Group A's results indicated weak levels of procedural fluency over and above the issues identified within Group B.

The errors that were highlighted in the pre-test and the difficulties that the learners experienced in the intervention were in-line with those highlighted in the literature on division. For example, the learners found the partitive problems easier to work with than the quotitive as noted by Booker et.al. (1992), Toluk and Middleton (2004) and Troutman and Lichtenberg (2003). The difficulties with division involving 0 and 1 were emphasised by Booker et.al. (1992) and Troutman and Lichtenberg (2003).

> "How can variation theory be used to devise an intervention to improve learners' understanding of the concepts and procedures of division?"

Much of the literature aided in the development of my understanding of variation theory. Variation theory assisted me in planning the intervention, as it helped me to identify some of the critical features that learners need to know in order to access all the division concepts. I then considered how these critical features related to each other and the learners current knowledge and therefore worked to highlight the variation and the invariants in my teaching. The use of a new theoretical framework made me question my current teaching strategies and engage at a deeper level with the concepts, as well as, with a variety of different strategies to find what I thought would be the most effective strategy for re-teaching these concepts. For example; through the integration of Marton et. al's. (2004) concepts of contrast, generalisation separation and fusion and Liljestrand and Runesson (2006) notion of pattern of variation I was able to improve the structure of my intervention planning. Ling et.al.'s (2005) discussion on the object of learning alongside Watson and Mason's (2005 & 2006) literature surrounding examples assisted in the selection of specific examples within the lessons. However, it was through the integration of variation theory, alongside strategies suggested in the literature surrounding division I was able to plan and implement the intervention lessons. Nevertheless the implementation did not always go as expected and as a result I had to re-look at what I taught, evaluate it and try to find more appropriate methods to use in the following lesson. For example, Catherine 'went blank' in the one post-test but did extremely well in the other one which was written directly afterwards. Learning is not always tidy and sequential especially in the case of my intervention. Learners who had been exposed to many division concepts, but had not got to grips with some of the simple concepts such as sharing, while they were able to implement some of the more complex procedures.

The most beneficial aspect of variation theory was the manner in which it taught me to select examples based on the dimensions of variation and the way to highlight the variation within these. I feel that this aspect of variation theory had the greatest impact and what I suspect will be a long-lasting effect on my teaching. The more I worked with division using the 'lens' of variation theory the more I became aware of the potential and possibilities that variation theory brought to the fore. So although I consider this study to have been a success, I acknowledge the vast room there is for improvement in both the selection of examples, patterns of variation used within each dimension of variation, the way in which the concepts were presented and the manner in which they were linked to the object of learning.

One problematic area that I foresee is that variation theory requires that the teacher has an extensive and in-depth knowledge of the concepts taught, as well as, the prerequisite skills and concepts that it leads onto. At a primary school level the majority of teachers are not subject specialists and their conceptual knowledge is often limited. Additionally, whilst I have a relatively strong mathematical background, I noticed that my selections of variation types tended to foreground procedural variation more readily than variation at a more conceptual level. Both of these factors are barriers to effectively using variation theory as they represent limitations in my ability as a teacher, and for teachers in general, to identify the critical features and may find it challenging to select examples that highlight the variation and invariant features.

"What are the effects of the intervention on learner performance in this area?"

The intervention can be considered a success. As indicated in the comparison of the pre and post-test results in Chapter 5 (discounting Catherine's performance in the delayed post-test 2) all Group A learners' knowledge surrounding division improved. In some areas learners' progress was more noticeable than in others. For example, the majority of learners appeared to have a relational understanding of the relationship between multiplication and division and the division facts. Conversely many of the children appeared to have a partial/limited knowledge and instrumental understanding of the remainder and division with 0 and 1. I felt that there was a direct link to the amount and quality of variation, I used in the intervention to the development of learners knowledge and understanding of concepts. A further indication of the effectiveness of variation in the teaching of mathematics concepts was when some learners found it useful to draw on the variation principles that I had introduced into the classroom to assist them in making meaning. For example; the learners used a wide range of strategies to solve the everyday problems and learners felt comfortable working with the division facts in many different contexts, but they were also happy to use work with the facts by identifying the relevant times table fact.

In the concepts where I used an extensive pattern of variation to highlight the variance and invariance, for example in the teaching of the division facts, learners found it easier to make mathematical sense of the concept for themselves and retained this knowledge as was highlighted in the post-test. For example; the majority of learners developed expectations and an understanding that enabled them to differentiate and correctly solve partitive and quotitive questions. However, where insufficient examples were given learners did not make sense of the concept for themselves. Accordingly they did not form links to their existing web of knowledge and could not recall the relevant skills or information relating to the concept. For example; division of one and division by zero, different question types and working meaningfully with a remainder. The reasons for this appeared to relate to the arguments made in the literature review chapter.

In conclusion the learners demonstrated clear procedural gains in relation to the use of multiplication and division facts. Some learners demonstrated greater fluency in working with the long division algorithm while others demonstrated a partial understanding of the algorithm. Results in the post-test indicated that difficulties with understanding and interpreting remainders persisted. While gains in conceptual understanding can not be measured evidence suggests that learners understanding of the concepts relating to division had improved.

Recommendations

Initially, I found it very challenging to work with variation theory and adapt my current teaching practices. I found it difficult to identify the critical features and general concepts of division as I had always viewed division in a procedural manner as apposed to a conceptual. However, having worked within the theoretical framework of variation theory and evaluated my previous teaching of division in order to plan and implement the intervention using some new teaching strategies, I do not think that I used variation theory to its maximum potential. With further study and practice I expect that I will be able to improve on my teaching, and as a result, improve the learning that takes place in my classroom. I believe that variation theory has much to offer teachers in the primary school and with practice will prove beneficial. However, teachers would need extensive training within variation theory and mathematical content and ongoing support to be able to use variation theory within mathematics to its full potential.

In conclusion, I believe that this research - a quasi-experimental, case study involving teaching division to Grade 5 learners using variation theory, resulted in an improvement in learner's knowledge of division. This is supported by the findings in the post-tests. Secondly, I believe that variation theory helped me to think about division and my teaching strategies in new ways which contributed to more effective teaching strategies and accordingly the learners' progress. I plan to use my findings in this research to further improve my own teaching and through the sharing of my knowledge I hope to improve the quality of teaching surrounding division at my school and in the schools in the surrounding community.

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Appendices

<u>Appendix A: Informed consent letter for Group B learners</u>

1 April 2009

Dear Parent or Guardian

<u>Masters Research Project – University of the Witwatersrand, Johannesburg</u>

I am currently doing some research in Mathematics Education at the University of the Witwatersrand. The study's focus is on teaching Mathematics using Variation Theory at a primary school level. I am interested in looking at how learners answer questions in their class work and assessments. I would like to use data gathered from the learner's class activities and assessments for my research. I believe that through my research, I can make a meaningful contribution to mathematics education, and gain an understanding of the ways in which Variation Theory impacts mathematics teaching and students' experiences of learning mathematics in school.

The focus of my research will be on the answers learners give to different examples within class tasks and assessments. Only my supervisor, Prof Hamsa Venkat, and myself will have access to the data. The school will be anonymous and all names in the transcript will be pseudonyms. When reporting my findings, it is my intention to illuminate the critical features within learner's responses. Useful anonymised data may be used for teacher development and broader research at WITS. In this regard I undertake to ensure that no untoward references are made about the pupils or the teacher.

I must stress that participation is voluntary. Your child is under no obligation to participate and there are no consequences should you or she choose not to. All participants also have the right to withdraw from the study at any future point. I would be very grateful for this opportunity however, and if you are agreeable to this process please read and complete the attached consent form and return it to school.

If you have any questions or concerns or would like to discuss the aims of my research in more detail, please do not hesitate to contact me on 011 531 1880. Should you wish to, you can also contact my supervisor, Prof Hamsa Venkat on (011) 717 3742.

Yours sincerely
Kerry Samuel

Consent form for participation in a research project.

(Please delete clearly where applicable)

I have read the above and **give consent / do not give consent** for my child to participate in the research project of Kerry Samuel subject to the conditions laid out in the accompanying letter. These include the use of the data from learner's class work or assessments research purposes and in articles for publication in academic journals on condition that the school is anonymous and all participants are referred to by pseudonyms.

Name of learner: …………………………………………………………………….

Signature of learner:……………………………………………………………………

Name of parent or guardian: ……………………………………………………………

Signature of parent or guardian: ………………………………………………………..

Date: ……………………………………………………………………………………….

Appendix B: Informed consent letter for the Principal

1 April 2009

Dear Principal

Masters Research Project – University of the Witwatersrand, Johannesburg

I am currently doing some research in Mathematics Education at the University of the Witwatersrand. The study's focus is on teaching Mathematics using Variation Theory at a primary school level. I am interested in looking at how learners answer questions in their class work and assessments. I would like to use data gathered from the learner's class activities and assessments for my research. I believe that through my research, I can make a meaningful contribution to mathematics education, and gain an understanding of the ways in which Variation Theory impacts mathematics teaching and students' experiences of learning mathematics in school.

My research would involve intervention in which I teach during the mathematics lessons. It will involve approximately eight lessons (two weeks) and will incorporate all the curriculum demands for Grade 5. I would like to use a tape recorder to assist me in recording the language that I use when teaching the division concepts. The focus of my research will be on the answers learners give to different examples within class tasks and assessments. Only my supervisor, Prof Hamsa Venkat, and myself will have access to the data. The school will be anonymous and all names in the transcript will be pseudonyms. When reporting my findings, it is my intention to illuminate the critical features within learner's responses. Useful anonymised data may be used for teacher development and broader research at WITS. In this regard I undertake to ensure that no untoward references are made about the pupils or the teacher.

I must stress that participation is voluntary. Your child is under no obligation to participate and there are no consequences should you or she choose not to. All participants also have the right to withdraw from the study at any future point. I would be very grateful for this opportunity however, and if you are agreeable to this process please read and complete the attached consent form and return it to school.

If you have any questions or concerns or would like to discuss the aims of my research in more detail, please do not hesitate to contact me on 011 531 1880. Should you wish to, you can also contact my supervisor, Prof Hamsa Venkat on (011) 717 3742.

Yours sincerely

Kerry Samuel

Consent form for school’s participation in a research project.

(Please delete clearly where applicable)

I have read the above and **give consent / do not give consent** for the learners at my school to participate in the research project of Kerry Samuel subject to the conditions laid out in the accompanying letter. These include the participation of my learners in the intervention and the use of the data from learner’s class work or assessments research purposes and in articles for publication in academic journals on condition that the school is anonymous and all participants are referred to by pseudonyms.

Name of school: ……………………………………………………………………….

Name of principal: ……………………………………………………………

Signature of principal: …………………………………………………………..

Date: ……………………………………………………………………………………..

<u>Appendix C: Informed consent letter learners participating in the intervention</u>

1 April 2009

Dear Learner and Parent or Guardian

<u>Masters Research Project – University of the Witwatersrand, Johannesburg</u>

I am currently doing some research in Mathematics Education at the University of the Witwatersrand. The study's focus is on teaching Mathematics using Variation Theory at a primary school level. I am interested in looking at how learners answer questions in their class work and assessments. I would like to use data gathered from the learner's class activities and assessments for my research. I believe that through my research, I can make a meaningful contribution to mathematics education, and gain an understanding of the ways in which Variation Theory impacts mathematics teaching and students' experiences of learning mathematics in school.

My research would involve an intervention in which I teach division during the mathematics lessons. It will involve approximately eight lessons (two weeks) and will incorporate all the curriculum demands for Grade 5. I would like to use a tape recorder to assist me in recording the language that I use when teaching the division concepts. The focus of my research will be on the answers learners give to different examples within class tasks and assessments. Only my supervisor, Prof Hamsa Venkat, and myself will have access to the data. The school will be anonymous and all names in the transcript will be pseudonyms. When reporting my findings, it is my intention to illuminate the critical features within learner's responses. Useful anonymised data may be used for teacher development and broader research at WITS. In this regard I undertake to ensure that no untoward references are made about the pupils or the teacher.

I must stress that participation is voluntary. Your child is under no obligation to participate and there are no consequences should you or she choose not to. All participants also have the right to withdraw from the study at any future point. I would be very grateful for this opportunity however, and if you are agreeable to this process please read and complete the attached consent form and return it to school.

If you have any questions or concerns or would like to discuss the aims of my research in more detail, please do not hesitate to contact me on 011 531 1880. Should you wish to, you can also contact my supervisor, Prof Hamsa Venkat on (011) 717 3742.

Yours sincerely

Kerry Samuel

Consent form for participation in a research project.

(Please delete clearly where applicable)

I have read the above and **give consent / do not give consent** for my child to participate in the research project of Kerry Samuel subject to the conditions laid out in the accompanying letter. These include the participation in the intervention and use of the data from learner's class work or assessments research purposes and in articles for publication in academic journals on condition that the school is anonymous and all participants are referred to by pseudonyms.

Name of learner: …………………………………………………………………….

Signature of learner:…………………………………………………………………

Name of parent or guardian: ……………………………………………………………

Signature of parent or guardian: ………………………………………………………..

Date: …………………………………………………………………………………….

Appendix D: The full pre-test

Pre test number	Question number	Question	Dimension of variation explained	Additional reasons and notes
2	1	72 ÷ 6 = $\square$		A known basic fact (Cooper, Heirdsfield, Irons, and Mullinggan, 1999). Reversed multiplication (Neuman, 1999, pg 109). Contrast to multiplication (Marton, Runesson, Tsui, 2004, pg16).
2	2	628 + 21 = $\square$		
1	1	64 ÷ 4 = $\square$	Extension of known facts (Cooper, Heirdsfield, Irons, and Mullinggan, 1999) beyond the tables.	Learners are able to use counting strategies to solve this problem (Cooper, Heirdsfield, Irons, and Mullinggan, 1999). They are also able to work out with short division, by drawing /dealing (Neuman, 1999, pg 112) or extending tables. Four levels of reasoning may be used: situational, referential, general or formal (Gravemeijer in Gravemeijer, 1997 pg 395).
1	2	672 - 59 = $\square$		
1	3	482 ÷ 2 = $\square$	Increasing size of dividend. Divisor remains a single digit.	No carrying within sum and no remainder.
1/2	4&4	475 x 6 = $\square$		
2	3	834 ÷ 3 = $\square$	Carry over within the sum.	Has no remainder. Will need to work out using calculation. Separation by keeping all factors the same only varying the

				number. (Marton, Runesson, Tsui, 2004, pg16)
1	6	1674 + 325 = $\boxed{}$		
1	5	9612 ÷ 7 = $\boxed{}$	Variation by increasing size of dividend and having remainder. (This example contains two new dimensions of variation.)	The divisor has been kept to one digit. Generalisation of short division and extension to bigger numbers. Requires fusion of several critical aspects – multiples, short division, carrying over and remainder. (Marton, Runesson, Tsui, 2004, pg16)
2	5	462 ÷ 21 = $\boxed{}$	A variation on question 5 and 7 – the divisor is a double-digit number the dividend is still a three-digit number.	Learners will tend to use long division for this sum. Some may solve by chunking (Neuman, 1999, pg 109). There is no borrowing in the subtraction. No remainder.
2	6	8718 + 687 = $\boxed{}$		
2	7	512 ÷ 16 = $\boxed{}$	Borrowing is now required in this example	Division by double-digit number. No remainder. Some may solve by chunking (Neuman, 1999, pg 109). Requires fusion of several critical aspects – multiples, long division, and carrying over. (Marton, Runesson, Tsui, 2004, pg16)
1	8	3497 + 2214 = $\boxed{}$		

1	7	1497 ÷ 24 = ☐	Increasing size of the dividend.	No remainder. Requires fusion of several critical aspects – multiples, long division, and carrying over. (Marton, Runesson, Tsui, 2004, pg16)
2	8	684 x 67 = ☐		
2	9	7594 ÷ 32 = ☐	Division with remainder.	Requires fusion of several critical aspects – multiples, long division, carrying over, borrowing in subtraction and remainders. (Marton, Runesson, Tsui, pg16)
1	9	2658 ÷ 26 = ☐	A 0 found in the middle of the quotient.	Place value error (Ryan and Williams, 2007, pg 206). Remainder. Requires fusion of several critical aspects – multiples, long division, carrying over and remainders. (Marton, Runesson, Tsui, 2004, pg16)
1	11	Jamey has 32 oranges if she puts 4 oranges into each bag. How many bags will she be able to make?	Problem solving / word sum variation on question 1.	Learners can solve problem without calculation, as it is a known multiplication fact (Cooper, Heirdsfield, Irons, and Mullinggan, 1999). Reversed multiplication (Neuman, 1999, pg 109). 1 variable, which means it is a quotitive problem (Neuman, 1999, pg 103). Learners will need to select appropriate operation to solve the problem.
1	10	If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?		
2	10	There are 6 boys at a party. If there are 72 sweets how many sweets did each boy get?	Increased number of variables to 2.	Learners can solve without calculation. It is a known multiplication fact (Cooper, Heirdsfield, Irons, and Mullinggan, 1999). Reversed multiplication (Neuman, 1999, pg 109). 2 variables, which means it is a partitive problem (Neuman, 1999, pg 103). Selects appropriate

				operation to solve the problem.
1	12	Sue walked 214 m and ran 187 m on the way to school. How much further did she walk than he ran?		
2	12	Craig delivers 3 times as many papers as Jack. If Craig delivers 369 papers how many does Jack deliver?	Increased size of dividend within problem.	Will need to calculate using an algorithm. Select correct operation for problem in a missing factor form. (Troutman and Lichtenberg, 2003, pg 292)
2	13	Sue earns R 25 a day. How much does she earn if she works for 12 days?		
1	13	David is cycling a mountain trail. He is able to cycle 31 km a day. The trail is 543 km. a. How many days will it take him to finish the trail? b. How far will he have to cycle on the last day?	Increase size in divisor.	Problem solving with long division.

Appendix E: The actual pre-tests

Pre-test 1

Grade 5

Revision exercise 1

Name: ____________________

Maths class: ________________

Instructions

1. Answer the questions on the paper provided.
2. You may use any method to answer the questions.
3. Please show all working that you do.

1. 64 ÷ 4 = $\square$
2. 672 - 59 = $\square$
3. 482 ÷ 2 = $\square$
4. 475 x 6 = $\square$
5. 9612 ÷ 7 = $\square$
6. 1674 +325 = $\square$
7. 1497 ÷ 24 = $\square$
8. 3497 + 2214 = $\square$
9. 2658 ÷ 26 = $\square$
10. If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?
11. Jamey has 32 oranges. If she puts 4 oranges into each bag. How many bags will she be able to make?
12. Sue walked 214 m and ran 187 m on the way to school. How much further did she walk than she ran?
13. David is cycling a mountain trail. He is able to cycle 31 km a day. The trail is 543 km.
    a. How many days will it take him to finish the trail?
    b. How far will he have to cycle on the last day?

Pre-test 2

Grade 5

Revision exercise 2

Name: ___________________

Maths class: _______________

Instructions

1. Answer the questions on the paper provided.
2. You may use any method to answer the questions.
3. Please show all working that you do.

1. $72 \div 6 = \boxed{}$
2. $628 + 21 = \boxed{}$
3. $834 \div 3 = \boxed{}$
4. $475 \times 6 = \boxed{}$
5. $462 \div 21 = \boxed{}$
6. $8718 + 687 = \boxed{}$
7. $512 \div 16 = \boxed{}$
8. $684 \times 67 = \boxed{}$
9. $7594 \div 32 = \boxed{}$
10. There are 6 boys at a party. If there are 72 sweets, how many sweets did each boy get?
11. If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?
12. Craig delivers 3 times as many papers as Jack. If Craig delivers 369 papers, how many does Jack deliver?
13. Sue earns R 25 a day. How much does she earn if she works for 12 days?

Appendix F: Worksheet 1 justification of questions

Question	Reason for selection and focus
1. A boy has 12 marbles. He shares his marbles with his brother and sister so that they can play a game. How many marbles does each of the children get? a. What was the answer? b. How did you find your answer? c. If you can, write a number sentence for the problem? d. On the number sentence label what each of the part means?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Small numbers Familiar situation Dimension of variation – different representations
2. A boy had 12 marbles he wanted to put the marbles in bags. If he put 3 marbles in each bag. How many bags could he make? a. What was the answer? b. How did you find your answer? c. If you can, write a number sentence for the problem? d. On the number sentence label what each of the part means?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation - same numbers as question 1 but the type of sum is different
3. Six girls share 18 smarties. How many smarties does each girl get? a. What was the answer? b. How did you find your answer? c. If you can, write a number sentence for the problem? d. Can you show the answer in a different way?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – same sum as question1 but different numbers for the dividend, divisor and quotient. Different representations required.
4. 18 cup cakes are packed onto small plates to sell in a shop. Each plate holds six cupcakes. How many trays did they make? a. What was the answer? b. How did you find your answer?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation - Same numbers as question 3 but type of sum different – same type of sum as question2 but different numbers. Different

c. If you can, write a number sentence for the problem? d. How would you tell someone else to work it out?	representations required.
5. How are question three and four similar?	To highlight the invariant elements.
6. How are question three and four different?	To highlight the variables
7. Write two of your own questions for the sum: 24 ÷ 6 = 4 using the two different kinds of questions shown above.	To assess if learners have understood the different dimensions of variation possible within this sum. Dividend increasing in size.
8. Something is missing from the question. What information do you need? The girls had a picnic. They shared a packet of 35 sweets. How many sweets did each girl get?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) To assess if learners are able to identify the critical features of a division sum. Dimension of variation – different output required. Dividend increasing in size.
9. Use dienes blocks to solve the following problem. Draw a sketch to show your answer. After market day one group made R48 profit. If there were 4 girls in the group how much money did each girl receive?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – two different strategies and tools specified for solving and representing the problem. Different output required. Dividend increasing in size.
10. There were 43 eggs to be packed into boxes. Each box can hold 6 eggs. a. How many boxes could be filled? b. Were there any eggs left over?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – remainder
11. What does it mean to divide?	To assess learners understanding of the concept of division.
12. What are the different parts of a division sum?	To assess learner's knowledge of the terminology: dividend, divisor, quotient, and remainder.
13. What do the different parts of the sum mean?	To assess learners understanding of the different parts of a division sum. Attention will be paid to the relationship between the different parts in an oral discussion.

Appendix G: Worksheet 2 justification of questions

Question	Reason for selection and focus
1. 12 friends went on a treasure hunt and found 36 pieces of treasure. They shared the treasure equally. How much treasure did each friend take home?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Revision of sharing strategies and writing number sentences Dimension of variation – Same as previous lessons questions
2. Tammy has R 66 she wants to buy packets of coloured pens if each packet costs R11 how many can she buy?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – using repeated subtraction
3. Mary and three friends had to share 52 sweets. How many sweets does each child receive?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – using repeated subtraction to solve a partitive question
4. Three Grade 6 girls are making flower arrangements for the Grannies tea at school. If they need to make 42 arrangements how many must each girl make?	Partitive (Stern and Stern, 1949, Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – same as question 3, only changed divisor and dividend. Less support given to learners.
5. There are 22 masks for the school play. At the end of each performance they are packed into boxes. Each box can hold 5 masks. a. How many boxes can they fill? b. How many masks are left over? c. How many boxes do they need?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – remainder
6. Sarah's class has 33 children in it. She is brining cupcakes that her mom baked for her birthday. Her containers will only hold 7 cupcakes each. How many containers will she need?	Quotitive (Booker et.al., 1992, Neuman, 1999 and Troutman and Lichtenberg, 2003) Dimension of variation – larger dividend and divisor

7. Can you solve these using repeated subtraction a. 84 ÷ 12 = X b. 67 ÷ 8 = X c. 105 ÷ 13 = X	Dimension of variation: Only a number sentence given. a. Is a division fact. b. Is a division fact with a remainder. c. Is beyond the division facts with a remainder.
8. What does it mean to use repeated subtraction to solve a division sum?	To assess if learners are able to verbalise their understanding of repeated subtraction

Appendix H: Worksheet 3 justification of questions

The focus of this lesson was on the development of the division facts from explicit links made to multiplication facts.

Question	Reason for selection and focus
1. 2 x 3 = ___ so 6 ÷ 2 = ___ and 6 ÷ 3 = ___	Recall of multiplication and explicit link made to division. The division facts are introduced by contrasting (Marton et. al., 2004) them with the corresponding multiplication fact and the second division fact associated with the multiplication fact. Learners must have ready access to the division facts to be able to develop division algorithms with larger numbers (Booker et.al., 1992). Important to explicitly link multiplication and division facts so that division facts can be developed and integrated into the web of existing knowledge of multiplication facts.
2. 5 x 6 = ___ so 30 ÷ 5 = ___ and 30 ÷ 6 = ___	Dimension of variation: the numbers have changed – variation but concept / principle is invariant: the relationship between the multiplication facts and division facts. (Pattern of variation – Liljestrand and Runesson, 2006)
	Learners are asked to make up two of their own questions. Provide learners the opportunity to generalize the dimension of variation covered in previous questions (Marton et.al. 2004).
5. 24 ÷ 6 = ___ 6. 15 ÷ 3 = ___ 7. 56 ÷ 7 = ___ 8. 45 ÷ 9 = ___ 9. 20 ÷ 10 = ___	Learners were asked to give the answers to the following and then state the corresponding multiplication table. Dimension of variation remains constant (Principle) the way in which the question has been asked and the numbers have varied.
10. What do you know about multiplication and	Provide learners the opportunity to generalize the

division?	dimension of variation covered in previous questions (Marton et.al. 2004).
<u>Division with 1</u> 1. What do you think the answer is to: 12 ÷ 1 = 2. Explain how you found your answer. 3. Can you think of a rule when dividing by 1?	Division by one. (Division laws) Dimension of variation: division fact that does conform to previous rules.
4. What do you think the answer is to: 1 ÷ 12 = 5. Explain how you found your answer. 6. Can you think of a rule when dividing one by a number?	Division of one. (Division laws) Dimension of variation: division fact that does not conform to rules that have been established and does not fall within the relationship established between multiplication and division. Contrasted with the division by one question.
<u>Division with 0</u> 1. What do you think the answer is to: 0 ÷ 3 = 2. Explain how you found your answer. 3. Can you think of a rule when dividing 0 by a number?	Division of zero. (Division laws) Dimension of variation: division fact that does conform to previous rules.
4. What do you think the answer is to: 3 ÷ 0 = 5. Explain how you found your answer. 6. Can you think of a rule when dividing by zero?	Division by zero. (Division laws) Dimension of variation: division fact that does not conform to rules that have been established and does not fall within the relationship established between multiplication and division. Contrasted with the division of zero question.
<u>Division with multiples of 10</u> 1. 30 ÷ 10 = 30 ÷ 3 = 30 ÷ 30 = 300 ÷ 10 = 300 ÷ 3 = 300 ÷ 30 =	Division with multiples of 10 Troutman and Lichtenberg (2003) emphasise the importance of being able to divide by multiples of ten and of multiples of ten. Dimension of variation: division by multiples of ten. Use of same base number and build up to multiples of 10.
70 ÷ 10 = 70 ÷ 7 = 70 ÷ 70 = 700 ÷ 10 =	Only the numbers vary.

700 ÷ 7 = 700 ÷ 70 =	
40 ÷ 10 = 40 ÷ 2 = 40 ÷ 20 = 400 ÷ 10 = 400 ÷ 2 = 400 ÷ 20 =	Only the numbers vary.
2. Did you see a pattern? 3. If you did describe it.	Give learners the opportunity to identify and generalise the pattern.
4. Explain how you would solve 600 ÷ 3 =	Dimension of variation: Principle the same, only the appearance / numbers vary. Extend pattern explored in previous questions.
5. Can you use the same way to solve 240 ÷ 60 =? 6. How would you solve this problem?	Dimension of variation: Principle the same, only the appearance / numbers vary. Extend pattern explored in previous questions.

Appendix I: Worksheet 4 justification of questions

Question	Reason for selection and focus
Division by one digit Can you solve this problem using base 10 blocks? Show your working using the pictures of the base 10 blocks. 1. 84 ÷ 4 = x 2. 96 ÷ 3 = x 3. 396 ÷ 3 = x 4. 864 ÷ 2 = x 5. 3693 ÷ 2 = x	Dimension of variation: division by single digit number – no regrouping required and no remainder Pattern of variation: increasing size of dividend.
Can you try these? 6. 75 ÷ 5 = x 7. 858 ÷ 6 = x 8. 2555 ÷ 7 = x	Dimension of variation: division by single digit number –regrouping required but no remainder. Pattern of variation: increasing size of dividend.
What about these? 9. 76 ÷ 4 = x 10. 649 ÷ 9 = x 11. 8642 ÷ 8 = x 12. 645 ÷ 6 = x	Dimension of variation: division by single digit number –regrouping required and remainder. Pattern of variation: increasing size of dividend.
Division by two digits Can you solve this problem using base 10 blocks? Show your working using the pictures of the base 10 blocks. 1. 338 ÷ 13 = x 2. 490 ÷ 14 = x 3. 6804 ÷ 21 = x	Dimension of variation: division by double digit number –regrouping required but no remainder. Pattern of variation: increasing size of dividend.
Can you try these? 4. 847 ÷ 15 = x 5. 4873 ÷ 23 = x	Dimension of variation: division by double digit number –regrouping required and remainder. Pattern of variation: increasing size of dividend.
What about these? 6. 1728 ÷ 16 = x 7. 3149 ÷ 31 = x	Dimension of variation: division by double digit number –regrouping required and remainder. Pattern of variation: increasing size of dividend.

Appendix J: Worksheet 5 justification of questions

The intention of this worksheet was to teach short division. However, it was actually used as consolidation and reinforcement.

Question	Reason for selection and focus
Can you find a quicker way of working these out? 1. 84 ÷ 4 = x 2. 693 ÷ 3 = x 3. 2486 ÷ 2 = x	Dimension of variation: division by single digit number – no regrouping required and no remainder Pattern of variation: increasing size of dividend.
Can you try these? 4. 84 ÷ 6 = x 5. 344 ÷ 8 = x 6. 1557 ÷ 9 = x	Dimension of variation: division by single digit number –regrouping required but no remainder. Pattern of variation: increasing size of dividend.
What about these? 7. 92 ÷ 7 = x 8. 837 ÷ 5 = x 9. 6532 ÷ 4 = x 10. 7228 ÷ 3 = x	Dimension of variation: division by single digit number –regrouping required and remainder. Pattern of variation: increasing size of dividend.
11. What is similar between long and short division? 12. What is different between long and short division?	This was included to assist learners to contrast long and short division.

Appendix K: Worksheet 6 justification of questions

Question	Reason for selection and focus
Represent the following sums using as many different ways as you can think of. 1. There was a jar with 24 sparkles in it. If Sam ate 4 each day, how many days would the sparkles last?	Quotitive problem. A division fact. Dimension of variation – focus on different forms of representation.
2. Five friends shared a bag of forty marbles. How many marbles did each of the friends receive?	Partitive problem. Division fact. Dimension of variation – focus on different forms of representation.
Write your own question for the following sum. 3. 38 ÷ 2 = x	Can learners use the information given to write a question? Can they identify the different parts of the division sum?
The following information has been jumbled up. Can you sort it out and write a number sentence. 4. There were 4 glasses left over. The glasses were packed in boxes of 6 at the glass factory. There were 12 boxes of glasses. There were 76 glasses.	Can learners identify relevant information and structure it in a more logical manner.
Solve the following problems remember to show all workings. 5. Three friends went on a picnic. They took 15 biscuits. How many biscuits did each of them have to eat?	Partitive problem. Division fact.
6. Four friends had to do a ribbon dance for Drama. They wanted to make their own ribbons. They bought 1000 cm of ribbon. If they shared the ribbon equally. How long was each girl's ribbon?	Partitive problem. Beyond division fact.
7. 8736 ÷ 8 = x 8. 5362 ÷ 53 = x	
9. The Sandy family decided to go to Cape Town on holiday. The distance to Cape Town is 1472. Mom and Dad decide to each take two turns driving. At	Problem requires more than one operation be performed. Requires the fusion of several skills.

what distances should they swop? a. Dad – 0 km b. Mom - _______________ c. Dad - _______________ d. Mom - _______________	
10. Seven friends decided to collect stationary for new Grade Ones at an underprivileged school. There are 28 children. Each child needs twelve pencils for the year. The friends collected 343 pencils. a. Did they collect enough pencils? b. How many spare pencils or how many pencils were they short? c. How many pencils did each of the friends collect?	Problem requires more than one operation be performed. Requires the fusion of several skills.

Appendix L: The full post-test analysis

I have only included a justification of those questions not included in the pre-test.

Grade 5 Revision Exercise

1. For each of these questions write the answer and explain how you found your answer. a. 4 ÷ 1 = ☐ b. 1 ÷ 4 = ☐ c. 5 ÷ 0 = ☐ d. 0 ÷ 5 = ☐	This question was to assess if learners had understood the concept of division with 0 and 1.
2. 14 ÷ 4 = 3 rem2 a. Write a story that this sum could be finding the answer to. b. Explain what each part of the sum means in your story. c. Name the different parts of the sum.	This question was to assess if learners understood the role and meaning of each part of the sum.
3. 72 ÷ 6 = ☐	
4. 628 + 21 = ☐	
5. If 40240 ÷ 4 =10060 What would 40240 ÷ 40 equal?	This was to test if learners were able to generalise their knowledge of division by multiples of 10. Explore the concept of place value. Dimension of variation – divisor increse by powers of 10

6. 64 ÷ 4 = ☐
7. 672 - 59 = ☐
8. 482 ÷ 2 = ☐
9. 475 x 6 = ☐
10. 834 ÷ 3 = ☐
11. 1674 + 325 = ☐
12. 9612 ÷ 7 = ☐
13. 672 - 59 = ☐
14. 462 ÷ 21 = ☐
15. 219 x 25 = ☐
16. 512 ÷ 16 = ☐
17. 3497 + 2214 = ☐
18. 1497 ÷ 24 = ☐

19. 684 x 67 = $\boxed{}$
20. 7594 ÷ 32 = $\boxed{}$
21. 8718 + 687 = $\boxed{}$
22. 2658 ÷ 26 = $\boxed{}$
23. Jamey has 32 oranges if she puts 4 oranges into each bag. How many bags will she be able to make?
24. If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?
25. There are 6 boys at a party. If there are 72 sweets how many sweets did each boy get?
26. Sue walked 214 m and ran 187 m on the way to school. How much further did she walk than he ran?
27. Craig delivers 3 times as many papers as Jack. If Craig delivers 369 papers how many does Jack deliver?
28. Sue earns R 25 a day. How much does she earn if she works for 12 days?
29. David is cycling a mountain trail. He is able to cycle 31 km a day. The trail is 543 km.
    a. How many days will it take him to finish the trail?
    b. How far will he have to cycle on the last day?

Work with rounding up of remainders was done in worksheet 2. Dimension of variation – rounding up and down. Dimension of variation – contextual sensitivity.

30. The following question was asked in a Grade 5 problem solving group work exercise.
    Danni has a bag with 68 sweets in. She wants to give each person 11 sweets. How many people can she give the sweets to? How many will be left over?

    Sam worked out the answer this way:

This was to assess learners' understanding of repeated subtraction.

68 – 11 – 11 – 11 - ….

What would Sam get if she carried on with her working?

Jane disagreed with Sam's method and said that there must be another way.

What would you tell Jane to do?

31. 12 ÷ 4 = 3
    a. What would happen to the answer if the 12 was changed to
        i. a bigger number?
        ii. a smaller number?
    b. What would happen to the answer if the 4 was changed to
        i. a bigger number?
        ii. a smaller number?

This was to assess if learners could identify and explain the relationship between the divisor, dividend, quotient and remainder.

Appendix M: The actual post-test

Post-test 1

Revision exercise 1

Name: ___________________ Maths class: _______________

Instructions

1. You may use any method to answer the questions.
2. Please show all working that you do.

1. 64 ÷ 4 = □
2. 672 - 59 = □
3. 482 ÷ 2 = □
4. 475 x 6 = □
5. 9612 ÷ 7 = □
6. 1674 +325 = □
7. 1497 ÷ 24 = □
8. 3497 + 2214 = □
9. 2658 ÷ 26 = □
10. If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?
11. Jamey has 32 oranges. If she puts 4 oranges into each bag. How many bags will she be able to make?
12. Sue walked 214 m and ran 187 m on the way to school. How much further did she walk than she ran?
13. David is cycling a mountain trail. He is able to cycle 31 km a day. The trail is 543 km.
    a. How many days will it take him to finish the trail?
    b. How far will he have to cycle on the last day?
14. The following question was asked in a Grade 5 problem solving group work exercise.

    Danni has a bag with 68 sweets in. She wants to give each person 11 sweets. How many people can she give the sweets to? How many will be left over?

    Sam worked out the answer this way: 68 – 11 – 11 – 11 - ….

    a. What would Sam get if she carried on with her working?

    Jane disagreed with Sam's method and said that there must be another way.

    b. What would you tell Jane to do?
15. If 40240 ÷ 4 =10060. What would 40240 ÷ 40 equal?
16. For each of these questions write the answer and explain how you found your answer.
    a. 4 ÷ 1 = □
    b. 1 ÷ 4 = □
    c. 5 ÷ 0 = □
    d. 0 ÷ 5 = □

Post-test 2

Revision exercise 2

Name: ___________________ Maths class: _______________

Instructions

1. You may use any method to answer the questions.
2. Please show all working that you do.

1. 72 ÷ 6 = $\boxed{}$
2. 628 + 21 = $\boxed{}$
3. 834 ÷ 3 = $\boxed{}$
4. 475 x 6 = $\boxed{}$
5. 462 ÷ 21 = $\boxed{}$
6. 8718 + 687 = $\boxed{}$
7. 512 ÷ 16 = $\boxed{}$
8. 684 x 67 = $\boxed{}$
9. 7594 ÷ 32 = $\boxed{}$
10. There are 6 boys at a party. If there are 72 sweets, how many sweets did each boy get?
11. If Tom has 267 marbles and Mary has 167 marbles. How many marbles do they have altogether?
12. Craig delivers 3 times as many papers as Jack. If Craig delivers 369 papers, how many does Jack deliver?
13. Sue earns R 25 a day. How much does she earn if she works for 12 days?
14. 12 ÷ 4 = 3
    a. What would happen to the answer if the 12 was changed to
        i. a bigger number?
        ii. a smaller number?
    b. What would happen to the answer if the 4 was changed to
        i. a bigger number?
        ii. a smaller number?
15. 14 ÷ 4 = 3 rem2
    a. Write a story that this sum could be finding the answer to.
    b. Explain what each part of the sum means in your story.
    c. Name the different parts of the sum.

Appendix N: Group A pre-test analysis

(Note that non-division questions have been omitted from analysis)

Group A pre-test 1

Name	Q 1	Q 3	Q 5	Q 7	Q 9	Q 11	Q 13 a & b	Other notes and error analysis
Salina	X NWS	X NWS	X NWS	X NWS	X NWS	X NWS		Cannot classify errors as no working was shown
Mel	X SD	√ SD	X SD			X NWS		Did not attempt 3 of the division questions. Q 1 & 5 Did not carry over remainder in calculation. Errors are developmental - understanding of the operation and prototypical
Catherine	√ NWS	√ NWS		X NWS	X NWS	√ NWS	X NWS	Did not attempt Q 5. Cannot analyse errors as no working was shown.
Jenny	X Break up	X LD.				X NWS	X	Only attempted 4 questions. Developmental errors. Q1 tried to break up and divide 4÷2 = 2, 6÷3 = 3 64÷4 = 22 - Overgeneralisation of addition and does not understand the operation. Q3 First two steps correct but didn't complete the sum - Doesn't understand the operation, process object error – has not completed the reification of foundational division concepts e.g. sharing and division facts thus having to focus on processes instead of using them as objects to solve long division sums. Q 13 multiplied instead of divided - Doesn't understand operation, modeling error.
Rosie								Did not have time to finish both pretests
Kim		√ NWS				√ NWS.		Only attempted two division sums

Pre-test 2 results

Name	Q 1	Q 3	Q 5	Q 7	Q 9	Q 10	Q 12	Other notes and error analysis
Salina								She did not have time to finish both tests
Mel	√ NWS					√ NWS	X	Q 12 Incorrect operation used, - instead of ÷.Developmental error, modeling and doesn't understand the operation
Catherine	√ NWS	√ NWS	√ NWS	√ NWS		√ NWS	√ NWS	
Jenny								Did not attempt any of the division questions.
Rosie	√	X	X	X	X	X	√	Q1 number sentence given as working Developmental errors Q 3, 10, 12 Format of addition or subtraction used Overgeneralisation, and doesn't understand the operation. Q 5, 7, 9 Multiplied instead of divided Modeling error and doesn't understand the operation.
Kim	√ Facts					√ NWS	X	Only attempted three division sums. Developmental errors Q 1 Division fact. Incorrect setting out - set out like +,- and x sums. Must have used multiplication facts to solve. Modeling error and doesn't understand the operation. Q 12 Incorrect operation done. Multiplied instead of divided. Overgeneralisation, and doesn't understand the operation.

Appendix O: Lesson analysis

Lesson 1 Question	Correct responses	Strategies / tools	Analysis
1. A boy has 12 marbles. He shares his marbles with his brother and sister so that they can play a game. How many marbles does each of the children get?	6	C: Drawing - sharing J: Correct used blocks K: Drawing - Sharing M:Drawing a picture (array) R: Used her fingers and a times table chart but represented using short division. S: Correct answer used dienes blocks	Children were able to solve this partitive question without any difficulty. Two of the learners used concrete tool and three relied on a pictorial representation. One learner relied on a times table chary and checked by counting in threes.
2. A boy had 12 marbles he wanted to put the marbles in bags. If he put 3 marbles in each bag. How many bags could he make?	6 – correct numerical answer 2 – incorrect units 1 – no units given 3 – inappropriate (partitive) method in explanation	C: Used same sharing method as for partitive problem but gave correct answer J: Correct numerical answer no units given, correct multiplication fact incorrect division number sentence – used times table chart – inappropriate method described K: Counted in 3's M: Drawing a picture (array) R: Used her fingers but represented using short division. S: Correct answer but when she explained she solved it by sharing 12 between 3 and could not identify the error in her working.	All learners arrived at the correct 'numerical' answer. Two of the girls had the incorrect units. When I asked them to explain their answer they described the answer as the number of marbles put into each bag instead of the number of bags and one of the learners did not ascribe a unit and thus did not demonstrate an understanding of the question. All learners were able to make the link between the equivalence of the numerical form of two sums. Three of the girls used the multiplication fact to find their answer or guide their working.
3. Six girls share 18 smarties. How many smarties	6	C: Drew picture sharing out J: Sharing picture K: Correct answer counting in	Learners all answered this partitive question correctly using an appropriate strategy.

does each girl get?		groups – said 3's but I think she meant 6 correct number sentence M: Drawing a picture (array). Able to write the multiplication fact R:Times table chart but represented using short division S: Correct answer used dienes blocks	
4. 18 cup cakes are packed onto small plates to sell in a shop. Each plate holds six cupcakes. How many trays did they make?	6 correct 2 - Appropriate method (one was given support) 4 – used a partitive method initially to solve (one gave correct answer and units) 1 – used appropriate method but initially gave wrong units	C: Drew picture but created array with 3 cupcakes allocated to each tray but in explanation used description as if it were a partitive problem J: Initial answer found using partitive method. Support provided and then able to explain answer using quotitive grouping K: Counted in 6's M: Drawing a picture (array). Able to write the multiplication fact R: Correct but rubbed out first answer which had the incorrect units explanation was partitive S: Correct answer, used dienes blocks	Some of the learners are still struggling to recognise the quotitive nature of the sum. However, once it was pointed out to learners that this was a quotitive question the learners were able to rework the question appropriately
5. How are question three and four similar?	4 able to identify similarities 1 did not understand the question and gave an invalid answer 1 did not	C: "It is the same sum and answer" J: Repeated last question K: Identified that they have the same sum, answer and number sentence M: ID numbers are the same. R: No answer S: "Same answer, same number	Although some of the learners could not write down an appropriate answer they were al able to join into the oral discussion and provide appropriate answers to be recorded on the board. Some of the learners gave an incorrect answer however with support of peers and the teacher they were

	answer 6 were able to provide oral feedback.	sentence"	able to self correct.
6. How are question three and four different?	1 correct 2 partially correct 3 incorrect	C: "In number 3 it tells you what to divide and in number 4 you have to work out what to divide" J:Repeated part of question 4 K:Different questions and examples M: "they ask different things they used the same numbers but they said it differently" R: No answer S: No answer	The learners found this more challenging and some of them just gave an example or referred concrete differences e.g. smarties and cupcakes, in the example rather than explain how the nature and grouping of the questions were different. 3 of the learners were unable to articulate the difference at all. 2 were able to partially articulate the answer 1 appeared to have a good understanding of the difference.
7. Write two of your own questions for the sum: 24 ÷ 6 = 4 using the two different kinds of questions shown above.	1 correct response for both 1 both incorrect 4 able to write partitive but unable to write quotitive support	C: Correctly wrote a partitive and quotitive sum. Mirrored quotitive on question 2. J: Able to write partitive questions but needed help writing quotitive K: Wrote quotitive question with support M: able to write partitive without support. Needed help writing quotitive. R: Questions did not make sense. Trying to share smaller number of whole things such as	1 learner with a relational understanding of the two types of questions 4 with a partial understanding unable to transfer knowledge of how to answer a question into writing own 1 with no understanding of how to write a division question

		uniform between big group of people. Correct numbers and number sentence. S: Wrote two partitive questions	
8. Something is missing from the question. What information do you need? The girls had a picnic. They shared a packet of 35 sweets. How many sweets did each girl get?	3 correct after it was explained what kind of answer was required 2 required further support to find the correct answer 1 only able to give oral answer	C: "How many packet and how many girls" later crossed out "how many packet". J: Correct K: No written answer M: Rubbed out first answer. Able to find answer with a little support. R: Correct S: First wrote 12 as an answer, then with support gave the correct answer	All the learners had trouble understanding how to answer this question. This format was unfamiliar and is not used in senior primary mathematics at the school where the intervention took place.
9. Use dienes blocks to solve the following problem. Draw a sketch to show your answer. After market day one group made R48 profit. If there were 4 girls in the group how much money did each girl receive?	6	C: Correct picture and answer J: Correct but drew picture to solve K: Correct picture no written answer M: Correct answer, drew picture showing correct sharing of dienes blocks R: Correct S: Correct answer	All girls able to find this answer easily.
10. There were 43 eggs to be packed into boxes. Each box can hold 6 eggs. a. How many boxes	2 correct answer no support 1 incorrect answer but following	C: Had 7 squares (boxes) with six circles (eggs) in each and one box with only one egg. Correct answers J: Correct answer – no working shown	2 learners able to find correct answer using appropriate strategy without support 4 learners still don't have a relational understanding need support to use correct grouping

could be filled? b. Were there any eggs left over?	support able to explain correct answer 3 correct answer but support given.	K: Incorrect used addition/subtraction format M: Showed eggs packed in boxes of 6 drew one box at a time. Able to identify and name remainder. R: Correct used short division S: Correct answer. Drew boxes with six in each counting up to 42 but then gave an answer of 43 eggs. Incorrect answer but correct working.	strategy to find answer.
11. What does it mean to divide?	6	C: "Divide means share out" J: To break up something K: Division means putting things into smaller parts M: "make it smaller" R: "to separate that number into smaller parts" S: "Divide means to find out how many times a number goes into another number.	
12. What are the different parts of a division sum?	6	C: Correct terms – no remainder J: Correct terms – no remainder K: Correct terms – no remainder M: Correct naming – did not include remainder R: Correct naming for divisor and quotient– did not include remainder called the dividend the denominator S: Correct naming – did not include remainder	Although learners did not initially identify the remainder as a part of the sum they all knew what it was and where it came from when I questioned them. They have not had much opportunity to work with remainders yet.
13. What do the different parts of	6	C: Correct meaning for each term.	

the sum mean?		J: Correct meaning for each term. K: Able to correctly explain different terms M: Able to correctly explain different terms R: “The denominator has to be divided into the divisor to get the quotient S: Able to correctly name the different parts of the sum	

Lesson 2 Question	Correct responses	Strategies / tools	Analysis
1. 12 friends went on a treasure hunt and found 36 pieces of treasure. They shared the treasure equally. How much treasure did each friend take home?	6	C: Correct answer – drew picture J: Correct - picture K: Correct NWS M: Correct NWS – used counters R: No answer given but picture shows correct interpretation and working S: Correct answer. NWS	Learners did not have any trouble solving this partitive problem.
2. Tammy has R 66 she wants to buy packets of coloured pens if each packet costs R11 how many can she buy?	5 1 subtraction error	C: Correct J: Correct K: Correct M: Correct. R: Subtraction error in the last part of the repeated subtraction S: Correct	The concept of repeated subtraction was taught in this question. The initial part of the sum was done as a class and then learners had the opportunity to finish the sum before we went through it together.
3. Mary and three friends had to share 52 sweets. How many sweets does	3 – using repeated subtraction 1 – using	C: Correct J: Drew picture to work out correct answer K: Correct repeated subtraction	Of the correct answers 1 did not write the final answer but her calculation was correct. 5 learners knew what procedure to

each child receive?	pictures 2 – incorrect 1 of these a subtraction error	but no final written answer given M: Correct R: Did not reach final answer S: Needed support – had difficulty subtracting when there was a 0 in the units or the top number was smaller than the bottom number	follow – two of the learners made subtraction errors. One of the learners was still not comfortable to begin with a calculation and preferred to draw before she attempted to calculate the answer.
4. Three Grade 6 girls are making flower arrangements for the Grannies tea at school. If they need to make 42 arrangements how many must each girl make?	6 – two of the learners required support	C: Correct J: Needed support with subtraction K: Correct M: Correct R: Correctly performed repeated subtraction but did not give final answer. S: Correct	Five of the learners were able to make the link from the picture representation to how repeated subtraction could be used to solve the problem. One of the learners did the correct calculation but did not write the final answer.
5. There are 22 masks for the school play. At the end of each performance they are packed into boxes. Each box can hold 5 masks. a. How many boxes can they fill? b. How many masks are left over? c. How many boxes do they need?	4 correct 2 correct with support 3 did not write up final correct answer	C: Correct - able to correctly identify remainder and identify that an extra box was needed for the remaining masks J: Able to do subtraction but needed help writing up answer K: Correct - able to correctly identify remainder and identify that an extra box was needed for the remaining masks M: Correct – able to correctly identify remainder and identify that an extra box was needed for the remaining masks R: Correct but needed support to correctly give final answer	All able to perform repeated subtraction two required some support with the subtraction. Three focused on the actual calculation but did not actually answer the question.

		and complete repeated subtraction S: Correct repeated subtraction first drew a picture showing how many masks were in each box. Did not actually answer questions.	
6. Sarah's class has 33 children in it. She is brining cupcakes that her mom baked for her birthday. Her containers will only hold 7 cupcakes each. How many containers will she need?	5 correct calculation 1 incorrect subtraction 4 of the correct calculation did not reach the final answer The fifth correct answer had support	C: Correct - did not allocate a spare box for the left over cupcakes J: Able to do repeated subtraction no final answer written K: Incorrect subtraction M: Correct repeated subtraction – able to correctly identify remainder and identify that an extra box was needed for the remaining cup cakes R: Correct with support S: Correct repeated subtraction but did not actually answer final question.	While four of the learners were able to correctly perform the calculation their lack of final answer suggests that they were not able to interpret their answer in terms of the question. Thus it suggests that their understanding of these problems is instrumental as they are able to identify the information necessary to work out the question, and perform the calculation. However, they are not able to use their findings to correctly answer the question.
7. Can you solve these using repeated subtraction a. $84 \div 12 = X$ b. $67 \div 8 = X$ c. $105 \div 13 = X$	A - 4 correct without support B – 2 correct, 1 correct calculation but gave remainder as answer	C: Correct for a and b, c correct repeated subtraction but gave remainder as answer instead of number of 13's subtracted. J: a - correct repeated subtraction, b - kept swapping number subtracting, c - subtracted wrong number K: Correct answer for a. b and c – correct repeated	Some of the learners were unsure what to do with these sums so I had to explain that these were the same as the word problems but the questions were not asked in a story. Those that got an incorrect answer was due to a subtraction error not an error in understanding.

	C – 1 correct, 2 correct calculation but gave remainder as answer	subtraction but gave remainder as answer instead of number of times subtracted M: Performed repeated subtraction with ease. Keeps a record of number of subtraction next to each operation. For one sum she tried to subtract from remainder. R: Incorrect repeated subtraction of a. Did not finish. S: Only had time for first question needed support with subtraction.	
8. What does it mean to use repeated subtraction to solve a division sum?		C: J: K: M: "You are subtracting to find out your division sum" R: S:	Only one learner had time to finish this question. The girls struggle to summarise their knowledge into a sentence explaining the concept.

Lesson 3 Question	Correct responses	Strategies / tools	Analysis
1 – 5 division fact as a question multiplication fact as a proof.	6	C: All correct J: All correct K: All correct M: All correct – corrected some clerical errors R: number sentence incorrect in one of own examples S: All correct	Learners found it easy to consolidate the link between multiplication and division as we have made this link earlier when simplifying and finding equivalent fractions.
Division by 1			We worked through this together. All girls could answer correctly

Division of 1			We worked through this together. All girls initially gave the dividend as the answer. When I drew a picture for them they were able to make the link to fractions.
Division of 0			We worked through this together. All girls could answer correctly and used the multiplication table as their reason.
Division by 0			We worked through this together. All girls initially gave the 0 as the answer. When I used concrete materials to share they could see that it was not possible to share into 0 groups and thus were happy to accept undefined as the answer.
Division with multiples of 10		C: All correct but could not describe the pattern J: K: All correct – able to identify and describe pattern when dividing by 10 M: All division by 10 correct. 2/6 correct when dividing multiple of 10 by a single digit. 1/6 correct when dividing by multiple of 10. R: All correct – able to identify and describe pattern S:	Not all the girls were able to identify the pattern on their own. Some of the learners needed me to explain the pattern making it explicit.
600 ÷ 3 =		C: Correct her reason is 6 ÷ 3 = 2 J: K: Correct reason was that she divided, could not fully	All the girls were able to find the correct answer on their own. Some used the pattern they found in the previous section to help them. Some of them divided relying on the times tables and

		articulate her method M: Absent R:Correct – no working or reason S: Correct	drawing on strategies taught in fractions.
240 ÷ 60		C: correct her reason is 24 ÷ 6 = 4 J: K: Correct reason was that she divided, could not fully articulate her method M: Absent R: Correct solved using fraction style S: Correct	All the girls were able to find the correct answer on their own. Some used the pattern they found in the previous section to help them. Some of them divided relying on the times tables and drawing on strategies taught in fractions.

Lesson 4 Question	Correct responses	Strategies / tools	Analysis
1 – 3 division by single digit number without carrying no remainder 4 and 5 given as homework	1. 6 2. 6 3. 6 4. 5 5. 5	C: All correct J: All correct – prefers using blocks first. 4, 5 incorrect. K: All correct M: All correct R: All correct S: All correct – preferred working with blocks first	We worked through 1 together. Some of the learners preferred to first solve the problem using dienes blocks, some of the learners preferred to work with the numbers. A few of the girls introduced the rest to the rhyme daddy, mommy, sister, brother. However, not all the girls used this. Some preferred to use the dienes blocks to make sense of the steps. 4, 5 given as homework.
6 – 8 division by single digit number with carrying no remainder	6. 6 7. 6 8. 5	C: All correct J: All correct K: All correct M: All correct R: All correct except 8 wrote 0 in last 2 quotient places	

		but the rest of the long division sum was correct. S: Correct – preferred working with blocks first.	
9 – 12 division by single digit number with carrying and remainder	9. 6 10. 6 11. 6 12. 5	C: All correct J: All correct. 12 incorrect division in the middle of the sum. K:All correct M: All correct R:All correct with support S: All correct	
1 – 3 division by double digit number with carrying no remainder	6 correct for all questions	C: All correct J: All correct K: All correct M: All correct R: All correct with support – had some trouble including the remainder in the same area as the quotient S: All correct	
4 – 7 division by double digit number with carrying and remainder	4. 6 5. 5 6. 6 7. 6	C: All correct J: 5 copied down from example left out heading. 6 correct. 7 correct. K: All correct – needed some support M: All correct R: All correct with support S: All correct	Jenny's error was clerical and does not indicate a lack of understanding.

Lesson 5 Question	Correct responses	Strategies / tools	Analysis
1-3 division by one digit,	1. 6	C: 1-2 correct – absent when 3 was	

no carrying or remainder	2. 5 3. 4/5	given J: All correct K: All correct M: All correct R: All correct – short division S: 1 and 3 correct, 2 incorrect	
4-6 division by one digit with carrying, no remainder	4. 6 5. 5 6. 6	C: 4 and 6 correct 5 incorrect (subtraction error) J: 4 and 5 incorrect – multiplication and division (prerequisite skills) 6 correct K: All correct M: All correct R: All correct – short division S: All correct	
7-10 division by one digit with carrying and remainder.	7. 6 8. 6 9. 4 10. 5	C: All correct J: All correct K: All correct M: All correct except 9 incorrect copying of sum R: Didn't finish S: 7 and 8 correct – has decimal and only answer written – must have used a calculator. Incomplete	
11 - 12			Not done – as short division was not introduced

Lesson 6 Question	Correct responses	Strategies / tools	Analysis
1. There was a jar with 24 sparkles in it. If Sam ate 4 each day, how many days would the	6	C:Correct – used long division J:Correct – used long division, and a picture K:Correct – division number sentence and picture M:Correct – picture and long division	Had to represent in as many ways as possible.

sparkles last?		R:Correct – used blocks started with 24 subtracted groups of 4 until was left with 0 then counted the groups. S:Correct – used pictures and blocks	
2. Five friends shared a bag of forty marbles. How many marbles did each of the friends receive?	5	C:Correct used a picture to show working J:Used a picture K:Correct – division and multiplication number sentence M:Correct – appropriate picture and long division. In the long division she did 40 ÷ 8 = 5 R:Incorrect – drew a picture which had 4 groups and 11 in each. S:Correct used pictures and blocks	Had to represent in as many ways as possible.
3. Write your own question for the following sum or picture. 38 ÷ 2 = x	2	C:Wrote a partitive problem. J:Just worked out the answer. K: "Find out what you have to times 2 by to get 38" M: Appropriate partitive problem. Then calculated the sum as follows 30 ÷ 2 = 15 8 ÷ 2 = 4 15 + 4 = 19 Check: 19 + 19 = 38 R:Could not do this on her own. S:Correct	Learners find it challenging to write a sum for a given problem. (This was a new concept that was introduced in the intervention)
4. The following information has been jumbled up. Can you sort it out and write a number sentence.	4	C:just rewrote the information in a more appropriate order. J: Wrote a number sentence first adding all the numbers. With support then able to construct a number sentence. K:Appropriate number sentence.	One of the learners required support when writing the number sentence.

There were 4 glasses left over. The glasses were packed in boxes of 6 at the glass factory. There were 12 boxes of glasses. There were 76 glasses.		Picture showing 6 people with 12 circles under each. M: Appropriate number sentence R:Incorrect tried to divide 4 by 12 then multiply 12 by 76. S:Appropriate number sentence	
5. Three friends went on a picnic. They took 15 biscuits. How many biscuits did each of them have to eat?	6	C:Correct – drew picture J:Drew a picture to work out. K:Correct – Picture, multiplication and division number sentence. M:Correct – long division R:Correct – short division S: Correct used pictures	
6. Four friends had to do a ribbon dance for Drama. They wanted to make their own ribbons. They bought 1000 cm of ribbon. If they shared the ribbon equally. How long was each girl's ribbon?	6	C:Correct – long division J:with support able to calculate using long division. K:Correct – long division required some support M: Correct Long division – did not require any support R:Correct short division – needed some help as she forgot to carry over. She reworked sum after I pointed out the error. S:Needed help – used long division	
7. 8736 ÷ 8 = x	5	C:Correct – long division J:Correct	

		K:Correct M: Correct R:Incorrect forgot to carry over when internal zero occurred in quotient and then did not carry over any further. S:Correct needed help, used long division	
8. 5362 ÷ 53 = x	4	C:Correct – long division J:Correct – needed support to work out multiples. K:Correct M: Incorrect – incorrect subtraction R:Correct S:Incorrect	
9. The Sandy family decided to go to Cape Town on holiday. The distance to Cape Town is 1472. Mom and Dad decide to each take two turns driving. At what distances should they swop? Dad – 0 km Mom - Dad - Mom -	0	C:Correct division but didn't finish the sum. J:Needed support to find out what needed to be worked out. K:Incomplete. M: Correct division but did not complete the sum R:Incorrect multiplied 28x343 S:Division sum correct with support – did not finish	No learners were able to complete the entire sum correctly and answer each part.
10. Seven friends decided to collect stationary		C: correct but incorrect multiplication in first part of the sum I had to correct this.	

for new Grade Ones at an underprivileged school. There are 28 children. Each child needs twelve pencils for the year. The friends collected 343 pencils. Did they collect enough pencils? How many spare pencils or how many pencils were they short? How many pencils did each of the friends collect?		J:Able to work out with support. K:Incomplete. M: Incorrect calculated 343 ÷ 28 = x did not complete the sum. R:Ran out of time S:Ran out of time.	

Appendix P: Group A In depth analysis of initial and delayed post test

Question	Catherine	Jenny	Kim	Mel	Rosie	Salina
1a. 4 ÷ 1 = b. 1 ÷ 4 = c. 5 ÷ 0 = d. 0 ÷ 5 =	a. Correct b. Incorrect gave undefined as answer c. Incorrect gave o as answer d. Incorrect gave undefined as answer	a. Correct b. Incorrect gave 1 as answer. c. Incorrect gave 5 as answer. d. Incorrect gave 1 as answer.	a. Correct b. Incorrect gave 1 as answer. c. Incorrect gave 5 as answer. d. Correct	a. Incorrect gave 1 as answer. b. Incorrect gave 4 as answer. c. Incorrect gave 5 as answer d. Correct	a. Incorrect gave 1 as answer. b. Incorrect gave 4 as answer. c. Incorrect gave 0 as answer d. Incorrect gave 5 as answer.	a. Incorrect gave 1 as answer. b. Incorrect gave 1 as answer. c. Incorrect gave 5 as answer d. Incorrect gave 5 as answer.
2. 14 ÷ 4 = 3 rem 2	a. Correct b. Correct c. Correct gave a very detailed answer	a. Correct b. Did not explain what each part means but drew an appropriate picture. c. Named the parts but did not link to the parts.	a. Correctly used dividend in sentence but left out divisor. b. Did not explain numbers in words. But drew an appropriate picture. c. Named divisor and dividend but did not match to parts of the sum.	a. Correct b. Did not answer c. Did not answer	a. Inappropriate question. Did not demonstrate an understanding of how to write a question. b. explained 14 – how many c. Named divisor, quotient and remainder	a. Correct b. Correct meaning given for 14 and 4. c. Used fraction terms correctly did not use dividend or divisor
3. 72 ÷ 6	Incorrect – LD – did not subtract	Correct – LD but did not use	Correct – no working(Correct in pre-test)	Correct – LD (Correct in pre-test)	Incorrect – SD did not divide 6 into 12	Correct – LD (Did not answer this

	correctly (Correct in pre-test)	conventional method (Did not attempt in pre-test)			correctly (Correct in pre-test)	pre-test)
5. If 40240 ÷ 4 =10060 What would 40240 ÷ 40 equal?	Correct – just gave answer	Did not attempt	Incorrect. Used LD to work out but incorrect division.	Incorrect – LD incorrect subtraction	Incorrect left out a 0 gave answer of 106 instead of 1006	Attempted to divide.
6. 64 ÷ 4	Correct – LD (Correct in pre-test)	Incorrect – divided incorrectly, selected incorrect multiple (Incorrect in pre-test tried to break up and divide)	Correct – LD (Did not attempt in pre-test)	Correct – LD (Incorrect in pre-test used short division)	Incorrect – multiplied instead of divided (Did not answer this pretest)	Correct – LD (Incorrect in pre-test)
8. 482 ÷ 2	Correct – LD (Correct in pre-test)	Correct – LD (incorrect in pre-test)	Left out (Correct in Pre test)	Correct – LD (Correct in pre-test used SD)	Correct answer – used incorrect format set out as a multiplication sum (Did not answer this pre-test)	Correct – LD (Incorrect in pre-test)
10. 834 ÷ 3	Correct - LD (Correct in pretest)	Correct – LD (Did not attempt in pre-test)	Correct – LD (Did not attempt in pre test	Correct – LD (Did not attempt in pre-test)	Incorrect – SD incorrect carrying (Incorrect in pre-test used multiplication	Correct – LD (Did not answer this pre-test)

					format)	
12. 9612 ÷ 7	Correct – LD (Did not attempt in pre-test)	Did not attempt (Did not attempt in pre-test)	Correct – LD (Did not attempt in pre test)	Correct – LD (Incorrect in pre-test)	Correct – SD (Did not answer in pre-test)	Incorrect – LD incorrect multiplication (Incorrect in pre-test)
14. 462 ÷ 21	Correct – LD (Correct in pre-test, but NWS)	Did not attempt (Did not attempt in pre-test)	Correct – LD (Did not attempt in pre test	Correct – LD (Did not attempt in pre-test)	Incorrect – SD incorrect selection of multiple for second part of division (Incorrect in pre-test)	I assisted with this question used LD (Did not answer this pre-test)
16. 512 ÷ 16	Correct – LD (correct in pre-test but NWS)	Did not attempt(Did not attempt in pre-test)	Incorrect – LD, used a higher multiple (Did not attempt in pretest)	Correct – LD (Did not attempt in pre-test)	Incorrect – SD didn’t carry over (Incorrect in pre-test)	Assisted with second step in long division (Did not answer this pre-test)
18.1497 ÷ 24	Correct – LD (Incorrect in pre-test but NWS)	Did not attempt (Did not attempt in pre-test)	Incorrect – LD incorrect division and selection of multiple (Did not attempt in pretest)	Incorrect – LD incorrect subtraction (Did not attempt in pre-test)	Incorrect – SD incorrect selection of multiple (Did not answer this pre-test)	Incorrect – LD incorrect multiples calculated (Incorrect in pre-test)
20. 7594 ÷ 32	Correct –LD (Did not attempt in pre-test)	Did not attempt (Did not attempt in pre-test)	Incorrect - Correct quotient but found no remainder – incorrect calculation of	Incorrect – LD incorrect selection of multiple (Did not attempt in pre-test)	Incorrect – SD Incorrect selection of multiples (Incorrect in pre-test)	Incorrect – LD Incorrect multiples (Did not answer this pre-test)

			multiples– LD (Did not attempt in pretest)			
22. 2658 ÷ 26	Correct – LD (Incorrect NWS)	Did not attempt (Did not attempt in pre-test)	Correct – LD (Did not attempt in pretest)	Correct – LD (Did not attempt in pre-test)	Incorrect - SD incorrect selection of multiple (Did not answer this pre-test)	Did not finish (incorrect in pre-test)
23. Word sum 32 ÷ 4 (quotitive)	Correct – LD (Incorrect in pretest – NWS)	Incorrect – gave 10 as answer drew a picture (incorrect in pre-test)	Correct – drew picture (Correct in pretest NWS)	Correct – no working (Incorrect in pre-test)	Correct - no working (Did not answer this in the pre-test)	Did not finish (incorrect in pre-test)
25. Word sum 72 ÷ 6 (partitive)	Correct – LD (Correct in pre-test NWS)	Correct – drew a picture (Did not attempt in pre-test)	Incorrect – 12 rem 2. Drew picture. (Correct in pre-test NWS)	Correct – no working (Correct in pre-test)	Correct – SD (incorrect in pre-test)	Did not finish (Did not answer this pre-test)
27. 369 ÷ 3	Incorrect – multiplied 369 x 3 (Correct in pre-test NWS)	Incorrect subtracted instead of divided (Did not attempt in pre-test)	Correct division and found appropriate answer but copied sum incorrectly (Incorrect in pre-test multiplied instead of divided)	Incorrect – no working (Must have subtracted gave 366) (Incorrect in pre-test)	Incorrect – multiplied instead of divided (correct in pre-test – but in wrong format)	Did not finish (Did not answer this pre-test)
29a. 543 ÷ 31	a. correct division but	a. Incorrect gave 1 as	a. Incorrect division and	a. Did not answer	Did not finish the post-test	Did not finish (Did not

b. remainder	did not give answer b. incorrect did 31 – 16 (remainder) (Incorrect – multiplied instead of divided)	answer did not show working b. Incorrect gave 1 as answer (Incorrect in pre-test – multiplied instead of divided for a)	gave inappropriate answer b. Incorrect – Subtracted 31 from total distance (Did not attempt in pre-test)	b. Did not answer (Did not answer in pre-test)		answer this question)
30a. 68 ÷ 11 using repeated subtraction b. shorter method	a. correct b. did long division, but did not explain in words	a. Incorrect gave 24 as answer (68 – 44) b. Explained method used above	a. Incorrect - Did long division but placed answer in incorrect place value position. b. Did not explain	a. Incorrect gave answer as 62 sweets left over b. Did not answer	Did not finish the post-test	Did not finish
31a. Changing dividend b. Changing divisor	a. Gave answers the other way around b. Gave answers the other way around	Correct but I read and explained what question was asking for all questions	a. Did not finish.	a. Circled one of the questions – did not demonstrate an understanding of what the question is asking	Did not finish the post-test	Did not finish
Number attempted in pre-test	12 /15	4/15	5/15	7 /15	7/15	6/15
Pre-test	Correct: 8/15	Correct: 0/15	Correct: 3/15	Correct: 3/15	Correct:2/15	Correct: 0/15

	Incorrect: 4/15 Not attempted: 3/15	Incorrect: 4/15 Not attempted:11 /15	Incorrect: 2/15 Not attempted: 10/15	Incorrect: 4/15 Not attempted:8 /15	Incorrect: 5/15 Not attempted:8 /15	Incorrect: 6/15 Not attempted:9 /15
Number attempted in post test	27/27	20/27	24/27	21/27	21/27	17/27
Questions in post-test that were also in the pre-test	Correct: 12/15 Incorrect:3/15 Not attempted: 0/15	Correct: 4/15 Incorrect:5/15 Not attempted: 6/15	Correct: 7/15 Incorrect:7/15 Not attempted: 1/15	Correct: 10/15 Incorrect:3/15 Not attempted: 2/15	Correct: 4/15 Incorrect:9/15 Not attempted:2 /15	Correct: 4/15 Incorrect:3/15 Not attempted: 6/15 Had assistance: 2/15
Change in results	Correct: 4 more Incorrect: 1 less Attempted: 3 more	Correct: 3 more Incorrect: 1 more Attempted: 5 more	Correct: 4 more Incorrect: 5 more Attempted: 9 more	Correct: 7 more Incorrect: 1 less Attempted: 6 more	Correct: 2 more Incorrect: 4 more Attempted: 6 more	Correct: 4 more Incorrect: 3 less Attempted: 7 more
New conceptual questions	Correct:7/12 Incorrect: 5/12	Correct:7/12 Incorrect:4 /12 Not attempted:1 /12	Correct:3/12 Incorrect:7/12 Not attempted: 2/12	Correct:2/12 Incorrect:6 /12 Not attempted:4 /12	Correct:1/12 Incorrect:7 /12 Not attempted:4 /12	Correct:2/12 Incorrect:6 /12 Not attempted:4 /12
Total post-test	Correct: 19/27 Incorrect: 8/27 Not attempted: 0/27	Correct:11 /27 Incorrect: 9/27 Not attempted: 6/27	Correct: 10/27 Incorrect: 14/27 Not attempted: 3/27	Correct: 12/27 Incorrect: 9/27 Not attempted: 6/27	Correct:5 /27 Incorrect:16 /27 Not attempted: 6/27	Correct: 6/27 Incorrect: 9/27 Not attempted: 10/27 Assisted: 2/27

Appendix Q: Group B pre and post-test analysis

Class 1: Pre and post-test 1

Name	Term 1 mark	Position in class	Concent returned	Question 1	Description	Question 3	Description	Question 5	Description	Question 7	Description	Question 9	Description	Question 11	Description
1	74%	19	Yes	√ √	SD SD	√ √	SD SD	√ √	SD SD	√ √	LD LD	√ √	LD SD	√ √	SD SD
2	94%	1	Yes	√ X	SD SD incorrect selection of multiplication fact	√ √	SD SD	√ √	SD SD	√ √	LD LD	√ X	LD SD did not carry over	√ √	SD Stated division fact
3	81%	15	Yes	√ √	LD SD	√ √	LD SD	X √	LD - Inncorrect division / multiplication fact SD	√ √	LD LD	√ √	LD SD	√ √	SD SD
4	93%	3	Yes	√ √	SD SD	√ √	SD SD	X √	SD - forgot to carry remainder to next number SD	X √	LD - Incorrect multiplication fact LD	√ X	LD Added instead of divided	√ √	SD SD
5	80%	17	No	√ √	SD SD	√ √	SD SD	√ √	SD SD	√ √	LD SD	X X	LD - Did not complete the last step SD ommited the 0	√ √	SD SD
6	87%	13	No	√ √	SD LD	√ √	SD SD	√ X	SD LD could not find correct multiple	X √	LD - Did not bring down the last digit LD	√ √	SD LD	√ √	SD SD
7	88%	11	Yes	√ √	SD SD	√ √	SD SD	√ √	SD SD	X √	LD - incorrect subtraction - incorrect remainder LD	√ √	LD LD	√ √	Used multiplication fact SD
8	92%	5	Yes	√ √	SD SD	√ X	SD SD incorrect division fact	X √	SD - Incorrect table fact SD	X √	LD - Did not bring down the last digit LD	X √	LD - Incorrect multiplication SD	√ √	SD SD
9	91%	7	Yes	√ √	SD Stated fact	X √	Incorrect copying of sum SD	√ √	SD SD	√ √	LD LD	√ √	LD LD	√ √	SD SD
10	90%	9	Yes	√ √	SD SD	√ √	SD SD	√ √	SD SD	√ √	LD LD	√ √	LD LD	√ √	SD SD
Total correct responses				10/10, 9/10		9/10, 9/10		7/10, 9/10		6/10, 10/10		8/10, 7/10		10/10, 10/10	
				1		3		5		7		9		11	

Name	Question 13 a	Description	Question 13 b	Description	Q14 A	Q14 B	Q 15	Q16 A	Q16 B	Q 16 C	Q 16 D	Total correct		Other notes
1	X X	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km Correct calculation but chose remainder as answer.	X X	Added answer and remainder. Incorrect interpretation of answer. Added answer and remainder	X	√	√ NWS	√	√	X NWS	√	6/8, 6/8, 11/15		
2	X √	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km Correct answer but not writen in a mathematically correct number sentence	X X	Added maximum km per day and remainder Did not attempt	√	√	√	√	X NWS	X NWS	√	6/8, 5/8, 10/15		
3	X X	LD - Stopped working half way LD Incorrect selection of multiple	X	Did not attempt as did not finish the previous question. Did not attempt	√	described representation did not suggest division	√	√	√	X NWS	X NWS	5/8, 6/8, 10/15		
4	X X	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km LD incorrect selection of multiple	√ X	Added remainder and quotient	√	√	√	√	X NWS	X NWS	√	5/8, 5/8, 10/15		
5	X X	LD - Added instead of subtracted in calculation Correct calc but did not answer the question	X	Used right answer from the previous sum - although it was incorrect Did not attempt	√	√	√	√	√	X NWS	X NWS	5/8, 5/8, 10/15		
6	X X	LD - Incorrect multiplication LD Correct calculation but did not answer the question		Did not attempt. Did not attempt.	√	√						5/8, 5/8, 7/15		Did not finish
7	X X	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km Multiplied instead of dividing	XX	Added remaining km to km per day	X Correct calc but did not give appropriate answer	X stated division but wrote the numbers the wrong way around	√ LD	√ NWS	X NWS	X NWS	X NWS	5/8, 6/8, 8/15		
8	X X	LD - Incorrect divisor Correct calculation but did not answer the question	X √	? Cannot understand reasoning	√	√	√	√	X NWS	X NWS	X NWS	3/8, 6/8, 10/15		
9	X X	LD - Correct calculation but did not answer the question Correct calculation but did not answer the question	X	Did not attempt Stated the max per day	√	X	√	√	√	X NWS	√	5/8, 6/8, 11/15		
10	X X	LD - Correct calculation but did not answer the Q LD correct calc but didn't answer the Q	X X	Added remaining km to km per day Did not attempt	√	X	√ LD	√	X NWS	Did not attempt	Did not attempt	6/8, 6/8, 9/15		
Total correct responses	0/10, 1/10		1/10, 1/10		8/10,	6/10,	9/10,	9/10,	4/10,	0/10,	4/10,			
	13a		13 b		14 a	14 b	15	16a	16b	16 c	16d			

Class 1: Pre and post-test 2

Name	Term mark	Position in class	Concent returned	Q 1	Description	Q 3	Description	Q 5	Description	Q 7	Description	Q 9	Description	Q 10	Description	Q 12	Description	Q 14 A	Q 14 B	Q 15 A	Q 15 B	Q 15 C	Total correct		Other notes
11	91%	8	No	√ √	SD NWS	√ √	SD SD	√ √	LD SD	√ √	LD SD	X X	LD - Added a 6 into the answer SD - incorrect calculation of remainder	√ √	SD Stated division fact	√ √	SD SD	X	X	√	√	√	6/7, 6/7, 9/12		Marked off L and S on question to indicate method to use
12	91%	6	Yes	√√	SD NWS	√ √	SD SD	√√	LD LD	√ X	LD - Did not complete last step LD - Incorrect selection of multiple	X √	LD - Copied sum incorrectly LD	√ √	SD NWS	√ X	SD LD	√	√	√	√	√	6/7, 5/7, 10/12		
13	88%	12	No	√ √	SD NWS	X X	SD - incorrect carry over	√ √	LD SD	√ √	LD LD	X X	LD - Incorrect multiplication LD Copied sum incorrectly	√ √	SD SD	√ √	SD SD	√	X	√	√	Correct words but did not match to the parts of the sum	5/7, 5/7, 8/12		
14	81%	16	Yes	√ √	SD SD	√ √	SD SD	√ √	LD LD	√ √	LD LD	√ X	LD LD - Incorrect selection of multiple	√ √	SD SD	√ √	SD SD	X	X	√	√	√	7/7, 6/7, 9/12		
15	92%	4	Yes	√ √	SD SD	√ √	SD SD	√ √	LD SD	√ √	LD SD	X √	LD - Left out zero in subtraction answer - incorrect remainder SD	√ √	SD SD	√ √	SD SD	X	X	X	X	X	6/7, 7/7, 7/12		
16	93%	2	Yes	√ √	SD SD	√ √	SD SD	√ √	LD SD	√ √	LD SD	√ √	LD LD	√ √	SD SD	√ X	SD Multiplied instead of divided	√	√	X	X	√	7/7, 6/7, 9/12		
17	90%	10	No	√ √	SD NWS	√ √	SD SD	√ √	LD SD	√ √	LD SD	X X	LD - Incorrect subtraction and remainder LD - incorrect multiplication	√ √	SD Stated division fact	√ √	SD NWS	X	X	X	X	X	6/7, 6/7, 6/12		
18	84%	14	Yes	√ √	SD SD	√ √	SD SD	√ √	LD SD	X √	LD - Incorrect multiplication and remainder SD	√ X	LD SD	√ √	SD SD	X √	Incorrect operation used (multiplied) SD	√	√	X	X	Partially correct called the ÷quotient	5/7, 6/7, 8/12		
19	78%	18	No	√ √	SD NWS	√ √	SD LD	√ X	LD - answer not in correct place value place LD - Copied sum incorrectly	√ √	LD LD	√ √	LD LD	√ √	SD LD	√ √	SD SD	√	√	√	X	X	7/7, 6/7, 9/12		
Total correct				9/9, 9/9		8/9, 8/9		9/9, 8/9		8/9, 8/9		4/9, 4/9		9/9, 9/9		8/9, 7/9		5/9,	4/9,	5/9,	4/9,	4/9,			

Class 2: Pre and post-test 1

Name	Term 1 mark	Position in class	Concent returned	Q 1	Description	Q 3	Description	Q 5	Description	Q 7	Description	Q 9	Description	Q 11	Description	Q 13 a	Description	Q 13 b	Description	Q14 A	Q14 B	Q 15	Q16 A	Q16 B	Q 16 C	Q 16 D	Total correct		Other notes
20	80%	5	Yes	X √	SD - Incorrect application of algorithm SD	√ √	SD - Incorrect application of algorithm SD	X √	SD - Incorrect application of algorithm SD	X √	SD - Incorrect application of algorithm SD	X X	SD - Incorrect application of algorithm SD - ommited 0	√ √	Division fact SD	X	Did not attempt SD		Did not attempt Did not attempt	√	√	X SD	√	X	X	X	2/8, 5/8, 8/15		Writes divisor and dividend in the wrong place when using the algorithm
21	94%	1	Yes	√ √	Division fact LD	√ √	SD LD	√ √	SD LD		Did not attempt Did not attempt	√	Did not attempt LD	√ √	Drew groups with four in each until 32 objects had been drawn SD	X X	No working shown Correct calculation, incorrect answer	√	Did not attempt	X	X	√	√	X	X		4/8, 6/8, 8/15		
22	73%	9	Yes	X √	Incorrect division fact SD	√ √	No working done SD	√ X	SD SD incorrect selection of multiple	X	Did not do LD did not follow steps	√ X	SD LD incorrect techniques	√ √	SD NWS	X	Did not attempt	X	Did not attempt	X	X	√	X	X	X	√	4/8, 3/8, 5/15		
23	87%	3	No	√ √	Drew working - sharing SD	√ √	SD SD	√ √	SD SD	√	Did not attempt SD	√	Did not attempt SD	√ √	Drew working - sharing SD	X	Did not attempt Correct calculation	X	Did not attempt	√	√	X Added a 0	√	X	X	√	4/8, 6/8, 10/15		
24	66%	13	Yes	√ X	Division fact SD	√ X	NWS	X	Did not attempt	X	Did not attempt	X	Did not attempt	√ √	No working done	X	Did not attempt		Did not attempt	X	X	X	X	X	X	X	3/8, 1/8, 1/15		
25	72%	11	Yes	√ √	Multiplication fact SD	√ √	Addition checksum done SD	X	Did not attempt LD	√	Did not attempt LD	√	Did not attempt LD	√ √	Multiplication fact SD	√	Did not attempt Did multiplication as working	X	Did not attempt	√	X	√	√	X	X	√	3/8, 6/8, 10/15		
26	58%	15	Yes	√	Did not attempt SD	√	Did not attempt SD	X	Did not attempt SD	X	Did not attempt SD	√	Did not attempt SD	√ √	Division fact written in SD form NWS	X X	Multiplied instead of divided		Did not attempt Did not attempt	X	X	X	√	X	X	X	1/8, 4/8, 5/15		
27	76%	7	No	√	Division fact	√	NWS	X	No working done	X	NWS	X	NWS	X	Incorrect division fact	X	No working shown	X	No working shown										
Total correct				5/8, 6/7		7/8, 6/7		3/8, 3/7		0/8, 3/7		1/8, 4/7		7/8, 7/7		0/8, 1/7		0/8, 1/7		3/7,	2/7,	3/7,	5/7,	0/7,	0/7,	3/7,			

Class 2: Pre and post-test 2

Name	Term mark	Position in class	Concent returned	Question 1	Description	Question 3	Description	Question 5	Description	Question 7	Description	Question 9	Description	Question 10	Description	Question 12	Description	Q 14 A	Q 14 B	Q 15 A	Q 15 B	Q 15 C	Total correct		Other notes
28	73%	10	No	√ √	SD Mental	√ √	SD SD	√ X	SD SD	√ √	SD SD	X X	SD - Incorrect carrying SD	√ √	Division fact Mental	√ √	SD SD	√	√				6/7, 5/7, 7/12		Does not write answer in correct
29	88%	2	Yes	√ √	SD SD	√	Did not attempt SD	√	Did not attempt SD	√	Did not attempt SD	X	Did not attempt SD	√ √	Division fact SD	√	Did not attempt SD	√	√	X	√	√	2/7, 6/7, 10/12		
30	69%	12	No	√ √	SD	√	Did not attempt LD	√	Did not attempt SD	X	Did not attempt LD	X	Did not attempt LD	X √	SD - did not carry LD	X √	Multiplied instead of divided SD	X	X				1/7, 5/7, 5/12		
31	63%	14	No	√ √	Division fact	X	Did not attempt SD	X X	Used multiplication form SD	X	Did not attempt SD	X	Did not attempt SD	√ √	Division fact	√	Did not attempt SD	√	X	√	√	X Did not match to parts of the sum	2/7, 3/7, 6/12		Division sums written in the same format as multiplication
32	75%	8	No	√ √	Division fact NWS	X X	No working SD	X X	Looks like she attempted to multiply SD	X X	Looks like attempted to multiply SD	X X	Looks like attempted to multiply SD	√ √	Division fact Multiplication fact	X X	Multiplied instead of divided Multiplied instead of divided						2/7, 2/7, 2/12		
33	54%	16	Yes	√ X	SD SD	X √	SD - worked out from left to right SD	√	Did not attempt SD	X	Did not attempt SD	X	Did not attempt SD	√ √	Drew working - sharing SD	X √	Multiplied instead of divided SD	X	X				2/7, 4/7, 4/12		
34	80%	6	No	√ √	SD checked with multiplication sum	X √	SD - incorrect multiple used LD	X	Did not attempt LD	√	Did not attempt LD	√	Did not attempt LD	√ √	Division fact	X X	Multiplied instead of divided	√	X	√	X	X	2/7, 5/7, 7/12		
Total correct				7/7, 6/7		1/7, 5/7		1/7, 3/7		1/7, 4/7		0/7, 1/7		6/7, 7/7		1/7, 5/7		4/7,	2/7,	2/7,	2/7,	1/7,			

Class 3: Pre and post-test 1

Name	Term 1 mark	Position in class	Concent returned	Question 1	Description	Question 3	Description	Question 5	Description	Question 7	Description	Question 9	Description	Question 11	Description	Question 13 a	Description	Question 13 b	Description	Q14 A	Q14 B	Q 15	Q16 A	Q16 B	Q 16 C	Q 16 D	Total correct		Other notes
35	81%	5	No	√√	SD LD	√√	SD LD	√X	SD LD - did not carry down the last number	√X	LD Set out division sum but did not do any of it	√X	LD LD incorrect multiplication	√√	SD NWS	X	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km Did not attempt		Did not attempt Did not attempt				√	X	X	X	6/8, 3/8, 4/15		Did not attempt blank blocks
36	78%	7	No	√√	SD SD	√√	LD SD	XX	SD - Incorrect calculation no carrying recorded - incorrect carrying SD incorrect selection of multiples	X√	LD - Added to find multiple - incorrect multiple used LD	√√	LD LD	√√	LD SD	XX	Used repeated addition to find nearest multiple - incorrect selection of multiple LD correct calculation incorrect answer	X√	No working used	√	X	√					4/8, 6/8, 8/15		
37	68%	11	No	√√	Written out multiples LD	√√	Set out like a multiplication sum - divided each of the top digits by 2 LD	√√	SD LD used comma instead of writing remainder	X√	SD - Incorrect selection of multiple and carrying LD	X√	SD - Cannot follow working and place value error LD used comma instead of writing remainder	√√	NWS NWS	X	SD - written out mulitples incorrectly Correct calc but did not answer the question	XX	No working used Subtracted answer from 31	X Divided to find answer but gave incorrect answer	√ desribed division						4/8, 6/8, 7/8		Did not finish
38	72%	9	No	√√	No working SD	√√	No working SD	XX	SD - Incorrect multiplication fact SD	X √	SD - Incorrect remainder - incorrect subtraction LD	X√	SD - Omitted zero - place value error SD	√√	NWS SD	X	No answer - working rubbed out Correct calclation but incorrect answer	X	Did not attempt 31 ÷ 17 +16	X Correct calculation incorrect answer	X tried to divide dividend by answer	√	√	X	X	√	3/8, 5/8, 7/8		All incorrect sums reworked correctly - look like a teachers handwriting
39	81%	3	No	√√	LD LD	X√	LD - Does not follow correct method - does not divide into last digit LD	√√	LD LD used point instead of writing remainder	√X	LD LD Incorrect calculation of multiples	√X	LD LD Incorrect calculation of multiples	√√	Wrote out multiples SD	XX	LD - Correct calculartion Incorrect answer - did not add an extra day for the left over km Did not attempt		Did not attempt Did not attempt	√	X	X Added a 0	X	X	X	√	5/8, 4/8, 6/8		
40	56%	13	No	X	LD - Incorrect multiplication fact	X	LD - Did not finish last step	X	LD - Incorrect multiplication facts	X	LD - Copied sum incorrectly	X	LD - Skipped steps - did not complete sum																Divided all sums and did not finish
41	89%	1	No	√√	SD - Correct calculation but unable to read own answer to copy LD	√√	SD LD	√X	LD LD	XX	LD - divided by 12 even though wrote 24 in sum LD	X√	LD - Ommited zero - place value error LD	X√	LD - Correct calculation but did not write answer LD	X	Did not attempt LD incorrect remainder			X	X						3/8, 4/8, 4/8		
42	64%	11	No	√√	No working SD	√√	Halved each digit - informal notes SD	XX	LD - Incorrect multiple selected and did not complete the sum SD did not carry over	√	SD	√	SD	√	SD	X	SD incorrect calculation of remainder	X	NWS	X	√	√	X	X	X	X	2/8, 5/8, 7/8		Did not finish
Total correct responses				7/8, 7/7		6/8, 7/7		4/8, 2/7		3/8, 3/7		2/8, 4/7		5/8, 7/7		0/8, 0/7		0/8, 1/7		2/7,	2/7,	3/7,	2/7,	0/7,	0/7,	2/7,			
				1		3		5		7		9		11		13a		13b		14a	14b	15	16a	16b	16c	16d			

Class 3: Pre and post-test 2																									
Name	Term mark	Position in class	Concent returned	Question 1	Description	Question 3	Description	Question 5	Description	Question 7	Description	Question 9	Description	Question 10	Description	Question 12	Description	Q 14 A	Q 14 B	Q 15 A	Q 15 B	Q 15 C	Total correct		Other notes
43	81%	4	No	√ √	SD Stated division fact	X √	LD - incorrect multiple /multiplication LD	√ √	LD LD	√	LD	X	LD - could not identify multiples	√	Stated division fact	√	SD	√	√	X	X	X	2/7, 6/7, 8/12		Did not finish
44	53%	14	No	√ √	LD Stated division fact	√ X	LD - Correct calculation but answer gave incorrect remainder in answer sentance LD incorrect selection of multiple	√ √	LD - Correct calculation but answer gave incorrect remainder in answer sentence LD	√ √	LD - Correct calculation but answer gave incorrect remainder in answer sentence LD	√	LD	X	stated dividend	X	multiplied instead of divided	X	X	X	X	X	4/7, 4/7, 4/12		Did not finish
45	71%	10	No	√ √	SD NWS	√ √	LD LD	√ √	LD LD	X √	LD - Incorrect multiplication LD	X X	LD - Incorrect multiplication LD incorrect multiplication	√ √	LD LD	√ X	LD NWS	X	√	X	X	X	5/7, 5/7, 6/12		
46	76%	8	No	√ √	LD NWS	X √	LD - Incorrect division - Cant make any sense of answer - did not see that she could divide into first digit LD	√ √	LD LD	√ √	LD LD	X √	LD - Incorrect multiplication LD	√	NWS	√	SD	√	√	X	X	X	3/7, 7/7, 9/12		Did not finish
47	64%	12	No	√ √	LD SD	√ √	LD LD	√ √	LD LD	X √	LD - Incorrect multiplication LD	X X	LD - Incorrect subtraction LD incorrect selection of multiple	√ √	LD Stated division fact	√ √	SD NWS	X	X				5/7, 6/7, 6/12		
48	88%	2	Yes	√ √	LD Stated division fact	√ √	LD SD	√ √	LD LD	X √	LD - Incorrect subtraction LD	√ X	LD LD incorrect selection of multiple	√ √	Division fact SD	√ √	LD SD	X	X	X	X	Correct terms but did not match to the different parts of the sum	6/7, 6/7, 6/12		
49	80%	6	Yes	√ √	Division Fact nws	√	LD	√	LD	X	LD - Did not apply algorithm correctly and stopped half way	X	LD - Incorrect subtraction	√ √	Multiplication fact NWS	X X	Added instead of subtraced Multiplied instead of divided	X	X				4/7, 2/7, 2/12		
50	66%	12	No	√ √	NWS NWS	√ √	LD LD	√ √	LD LD	X √	LD - Incorrect multiplication LD	√ √	LD LD	√ √	Multiplication fact Stated division fact	X √	Multiplied instead of divided SD	√	X	√	X	√	5/7, 7/7, 10/12		
				8/8, 8/8		6/8, 6/8		8/8, 7/8		2/8, 7/7		2/8, 3/8		5/8, 7/8		3/8, 5/8		3/8,	3/8,	1/8,	2/8,	1/8,			
				1		3		5		7		9		10		12		14a	14b	15a	15b	15c			

Appendix R: Analysis of results tables

Pre-test analysis	Pre-test 1	Pre-test 2
Group A	15	46.45
Group B	51.25	62.9
Salina	0	NA
Mel	12.5	28.6
Catherine	37.5	85.7
Jenny	0	NA
Kim	25	42.9
Rosie	NA	28.6

Comparison of pre and post test averages

Test 1

	Pre-test	Delayed Post-test	Change from Pre-test to delayed Post-test	Initial post test
Group B	51.25	62.5	11.25	
Salina	0	25	25	25
Mel	12.5	37.5	25	62.5
Catherine	37.5	87.5	50	87.5
Jenny	0	25	25	12.5
Kim	25	37.5	12.5	62.5

Test 2

	Pre-test	Delayed Post-test	Change from Pre-test to delayed Post-test	Initial post test
Group B	62.9	74.3	11.4	
Mel	28.6	28.6	0	71.4
Catherine	85.7	14.2	-72.8	71.4
Rosie	28.6	28.6	0	14.2
Kim	42.9	42.9	0	42.9

Comparison of initial and delayed post-test results

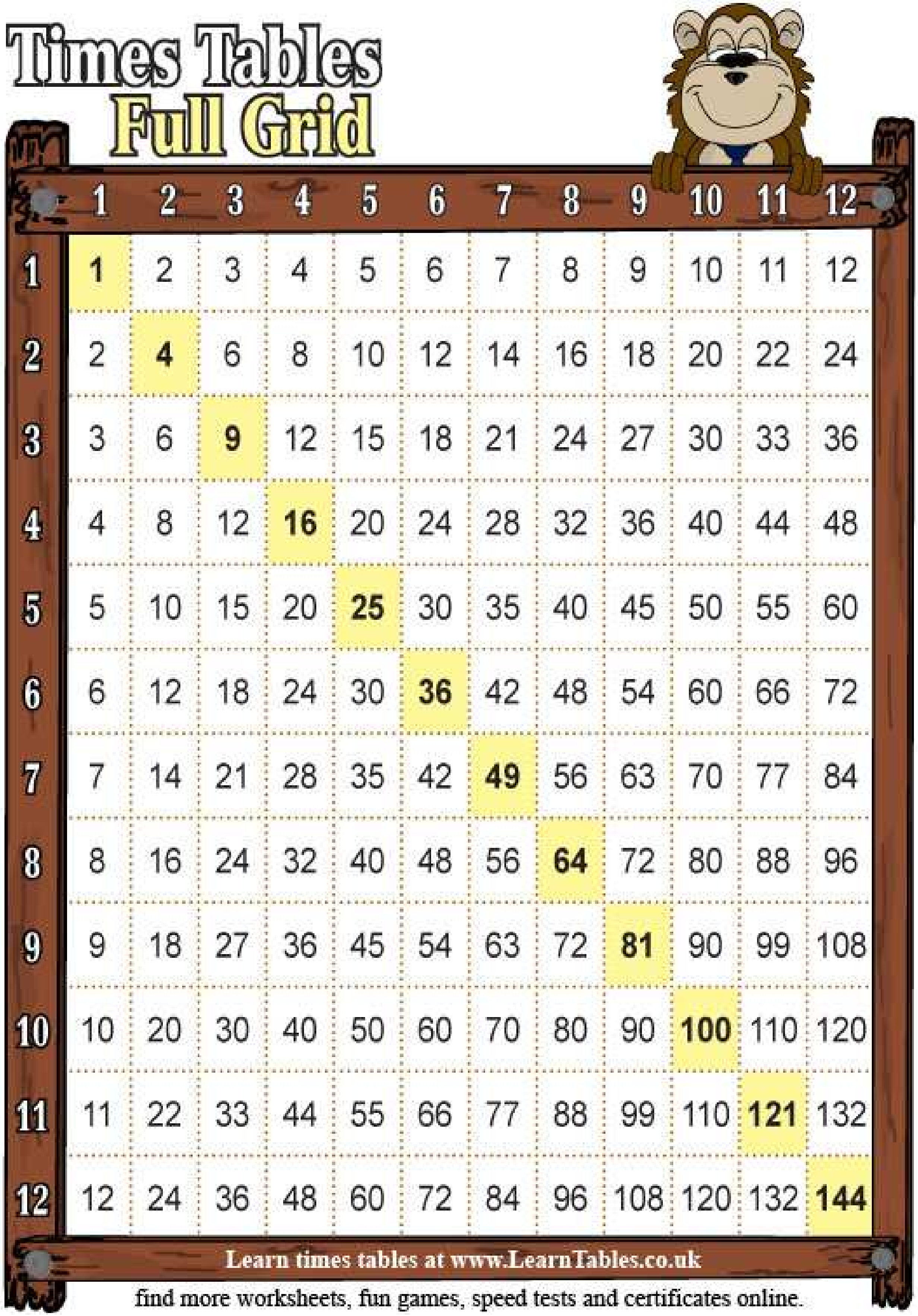
Test 1

	Initial post test	Delayed post-test	Change from initial post-test to delayed Post-test
Salina	25	25	0
Mel	62.5	37.5	-25
Catherine	87.5	87.5	0
Jenny	12.5	25	12.5
Kim	62.5	37.5	-25

Test 2

	Initial post test	Delayed post-test	Change from initial post-test to delayed post-test
Mel	71.4	28.6	-42.8
Catherine	71.4	14.2	-57.2
Rosie	14.2	28.6	14.4
Kim	42.9	42.9	0

Appendix S: Times table grid and chart



	1	2	3	4	5	6	7	8	9	10	11	12
1	**1**	2	3	4	5	6	7	8	9	10	11	12
2	2	**4**	6	8	10	12	14	16	18	20	22	24
3	3	6	**9**	12	15	18	21	24	27	30	33	36
4	4	8	12	**16**	20	24	28	32	36	40	44	48
5	5	10	15	20	**25**	30	35	40	45	50	55	60
6	6	12	18	24	30	**36**	42	48	54	60	66	72
7	7	14	21	28	35	42	**49**	56	63	70	77	84
8	8	16	24	32	40	48	56	**64**	72	80	88	96
9	9	18	27	36	45	54	63	72	**81**	90	99	108
10	10	20	30	40	50	60	70	80	90	**100**	110	120
11	11	22	33	44	55	66	77	88	99	110	**121**	132
12	12	24	36	48	60	72	84	96	108	120	132	**144**

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1 x 1 =	1	2 x 1 =	2	3 x 1 =	3	4 x 1 =	4
1 x 2 =	2	2 x 2 =	4	3 x 2 =	6	4 x 2 =	8
1 x 3 =	3	2 x 3 =	6	3 x 3 =	9	4 x 3 =	12
1 x 4 =	4	2 x 4 =	8	3 x 4 =	12	4 x 4 =	16
1 x 5 =	5	2 x 5 =	10	3 x 5 =	15	4 x 5 =	20
1 x 6 =	6	2 x 6 =	12	3 x 6 =	18	4 x 6 =	24
1 x 7 =	7	2 x 7 =	14	3 x 7 =	21	4 x 7 =	28
1 x 8 =	8	2 x 8 =	16	3 x 8 =	24	4 x 8 =	32
1 x 9 =	9	2 x 9 =	18	3 x 9 =	27	4 x 9 =	36
1 x 10 =	10	2 x 10 =	20	3 x 10 =	30	4 x 10 =	40
1 x 11 =	11	2 x 11 =	22	3 x 11 =	33	4 x 11 =	44
1 x 12 =	12	2 x 12 =	24	3 x 12 =	36	4 x 12 =	48
5 x 1 =	5	6 x 1 =	6	7 x 1 =	7	8 x 1 =	8
5 x 2 =	10	6 x 2 =	12	7 x 2 =	14	8 x 2 =	16
5 x 3 =	15	6 x 3 =	18	7 x 3 =	21	8 x 3 =	24
5 x 4 =	20	6 x 4 =	24	7 x 4 =	28	8 x 4 =	32
5 x 5 =	25	6 x 5 =	30	7 x 5 =	35	8 x 5 =	40
5 x 6 =	30	6 x 6 =	36	7 x 6 =	42	8 x 6 =	48
5 x 7 =	35	6 x 7 =	42	7 x 7 =	49	8 x 7 =	56
5 x 8 =	40	6 x 8 =	48	7 x 8 =	56	8 x 8 =	64
5 x 9 =	45	6 x 9 =	54	7 x 9 =	63	8 x 9 =	72
5 x 10 =	50	6 x 10 =	60	7 x 10 =	70	8 x 10 =	80
5 x 11 =	55	6 x 11 =	66	7 x 11 =	77	8 x 11 =	88
5 x 12 =	60	6 x 12 =	72	7 x 12 =	84	8 x 12 =	96
9 x 1 =	9	10 x 1 =	10	11 x 1 =	11	12 x 1 =	12
9 x 2 =	18	10 x 2 =	20	11 x 2 =	22	12 x 2 =	24
9 x 3 =	27	10 x 3 =	30	11 x 3 =	33	12 x 3 =	36
9 x 4 =	36	10 x 4 =	40	11 x 4 =	44	12 x 4 =	48
9 x 5 =	45	10 x 5 =	50	11 x 5 =	55	12 x 5 =	60
9 x 6 =	54	10 x 6 =	60	11 x 6 =	66	12 x 6 =	72
9 x 7 =	63	10 x 7 =	70	11 x 7 =	77	12 x 7 =	84
9 x 8 =	72	10 x 8 =	80	11 x 8 =	88	12 x 8 =	96
9 x 9 =	81	10 x 9 =	90	11 x 9 =	99	12 x 9 =	108
9 x 10 =	90	10 x 10 =	100	11 x 10 =	110	12 x 10 =	120
9 x 11 =	99	10 x 11 =	110	11 x 11 =	121	12 x 11 =	132
9 x 12 =	108	10 x 12 =	120	11 x 12 =	132	12 x 12 =	144