

**INTERMEDIATE PHASE MATHEMATICS
TEACHERS' REASONING ABOUT LEARNERS'
MATHEMATICAL THINKING**

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ABSTRACT

The focus of this study is to investigate the ways teachers use their content knowledge to understand and address the misconceptions which lie behind learners' errors (in their PCK). Misconceptions arise in mathematics and this phenomenon needs to be addressed by teachers. Misconceptions are instrumental for learning, but they are also instrumental in halting learners' progress in certain mathematical domains. The study highlights the relationship between how teachers hold content knowledge and use it to reason about learners' errors. Six teachers were interviewed following the 'think-aloud' method (REF) and they reasoned about learners' errors in five Grade 6 multiple-choice items. The findings show that different relationships between teachers' content knowledge and their pedagogical content knowledge emerge, with special reference to teachers' use of conceptual or procedural thinking, when thinking about the knowledge-base of the item. This distinction is used to further analyse the mode of teachers' proposed interventions, when they reflect on what would be best suited to address the misconceptions they identified.

DECLARATION

I declare that this Research Project is my own unaided work. It is submitted for the Master of Education by Coursework and Research Project in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination in any other university.

(Signature of candidate)

_____ day of _____ 2010.

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CHAPTER ONE : INTRODUCTION

Introduction

Since the 1980s, pedagogical content knowledge (PCK) has moved to the fore-front of research in teaching. Although content knowledge of the subject is and always will be important for teachers, authors are interrogating teachers' knowledge of learners' understanding and working with learners' thinking, (Ball 1990, Shulman 1986). PCK combines the notions of content and pedagogy; that is, subject specialized content domains in teacher knowledge are combined with knowledge of knowing how to mediate curriculum knowledge (Gees-Newsome, 1999). One of the key areas in PCK, which this study intends to foreground, is teachers' understanding and handling of misconceptions and errors in learners' thinking.

South African learners are generally weak in the mathematics field and perform well below most of the participating countries in international tests. Rather than attribute blame for this inadequate performance, a more constructive measure is to find a means of understanding learners' errors and misconceptions. Interventions that serve to improve teachers' knowledge for teaching are crucial and one of the means to develop this understanding is to look at the ways in which South African teachers interpret and address learners' errors and misconceptions.

Using the Schools International Assessment Test, the Gauteng Department of Education measured student performance in the Gauteng Province against International standards. The University of New South Wales – Educational Assessment Australia (EAA) which developed the test was requested to provide assessment materials to test and report on a sample of approximately 50 thousand students from Grade 3 to Grade 11 in Mathematics and English, in the Gauteng Province. This comprehensive sample was chosen so that the GDE could reliably benchmark learners' performance over a 3 year period (2006 – 2008). The test provided by EAA was part of a suite of international assessments conducted in approximately ten countries, mainly in the Asia Pacific region. The comparison between GDE learner performance and international achievement is, at face value, quite alarming. The Grade 6 results show a difference of 32% between the South African mean achievement and the mean of international countries. (Technical and Statistical Report 2006 - Gauteng Department of Education Schools International Assessment Task (SIAT) in Mathematics and English:4).

A cross-section of Gauteng schools was provided with a multiple choice test - "International Competitions and Assessments for Schools Mathematics 2006", henceforth ICAS 2006, to assess learners' content knowledge and benchmark South African performance. Learners from Grades 4, 5 and 6 (known as the Intermediate Phase) were among those who participated in the test. The Gauteng Department of Education approached the Wits School of Education to run a teacher development project and use the data that emerges from the ICAS tests to help teachers

understand South African learners' dismal performance in different mathematics domains, that is, Number, Measurement, Shape and Data handling. The Data-Informed Practice Improvement Project (DIPIP) uses small groups of teachers in Grades 3-9 who are tasked (inter alia) with analyzing why learners chose incorrect distractors for each of the 35 test items of the ICAS 2006 test. The teams, facilitated by a Witwatersrand postgraduate student or staff member, work together to develop lessons and tasks which can address the types of conceptual errors learners experience in certain topics. I am a team leader in one of the two Grade 6 groups and I was interested in working with my team of teachers on learners' errors. Each teacher shared their ideas about the reasons for the errors and misconceptions, and I was thereby able to increase my own knowledge about learners' errors and misconceptions through the experiences and reasoning of my teachers.

Aim of the study

My study aims to investigate teachers' pedagogical content knowledge by examining teachers' reasoning about learners' thinking. The study examines six Intermediate Phase teachers' thinking about reasons why learners chose incorrect answers. The quality of their explanations of the distractors, and their ideas of how to address them pedagogically may shed some light on what teachers know about learners' errors and misconceptions that result in these errors. This study aims to provide a sense of where these teachers are located pedagogically as far as learners' errors and misconceptions are concerned. More specifically, the study investigates the ways teachers use their content knowledge to understand and address the misconceptions which lie behind learners' errors (in their PCK). In this way this study aims to contribute to the field of research on PCK.

Rationale

I am perturbed about the state of mathematics teaching in South Africa as evidenced in the TIMMS 1995, 1999, 2003, ICAS 2006, HSRC 2008 and SAQMEQ 1999 assessment tests. The low performance by South African learners in all mathematical domains indicates that there is a nationwide crisis in mathematics achievement.

The data from the ICAS tests also evidences that our learners are underperforming. By means of working with teachers' mathematical knowledge I will endeavour to gain insight into the teachers' pedagogical content knowledge (PCK), with a specific focus on their thinking about learners' errors and the misconceptions underlying the errors, when they analyze assessment data.

As a teacher/lecturer who has taught students for close to four decades from primary school to high school and subsequently at a teacher training tertiary institution, I have come to realize that having good content knowledge does not improve pass rates, and therefore I have had to adapt my own PCK to address students' needs. One of my discoveries in this regard is that students

evidence errors and misconceptions in the classroom both formally and informally and are frequently classed as underperforming students. I have noted, with a measure of success, that when I change my strategies, representations of mathematical ideas and revisit their prior knowledge constructs in a mathematical domain, I am often surprised by the errors and misconceptions I discover, which I never thought existed in the thinking of my learners. If these kinds of experiences have occurred, I began to hypothesize how primary school teachers can both recognize and subsequently deal with errors and misconceptions as they arise and how conversant teachers in the field are, of the reasoning required to transform erroneous mathematical knowledge in their learners.

I currently engage with undergraduate mathematics teachers and my research will both enlighten me as to Intermediate Phase mathematics teachers' pedagogical content knowledge and their understanding of mathematical errors and misconceptions, and what further attention needs to be conveyed to students in a pre-service mathematics programme. My work with the DIPIP teachers has also highlighted that there is a need for teachers in the field to engage more extensively with learners' errors and misconceptions. In the beginning, my DIPIP group approached the error analysis superficially, that is, they could identify errors but were often unable to rigorously look at the misconceptions that lay behind the errors. As our work progressed, the teachers became more conscious of the role misconceptions play in learning.

The analysis in which the sample of teachers will engage through my research project, should provide me with an opportunity to articulate systematically some aspects of the pedagogical content knowledge that in-service teachers in the Intermediate phase (an under-researched group) have developed from their experiences. I believe that my research with teachers who are engaged with multiple – choice items in which misconceptions are embedded will impart so much more about their PCK compared to the performance data obtained in national and international assessment tests, in which South African learners have participated.

A shift from purely content based teaching that is transmitted by teacher to learner (which excludes learners as critical in the learning process), to teaching that incorporates PCK (which includes the learner as central in the learning process), started in the 1980's in countries such as the United States of America and the United Kingdom. These two areas are not mutually exclusive and the focus is on the *relationship* between a teacher's mathematical content knowledge and pedagogical content knowledge. Ball (2001), points out that research in pedagogical content knowledge incorporates knowledge about mathematical ideas and their representations, students' cognition and how learning with understanding takes place. A component of PCK includes a teacher's knowledge of misconceptions in different mathematical domains. Numerous studies on pedagogical content knowledge (Hill 2005, Ball 2003, Brodie 2001, Gees-Newsome 1999, Fennema 1992, Shulman 1986) shifted the emphasis from content knowledge to pedagogical content knowledge and it is presently acknowledged that both types of

knowledge enhance teaching. Less well researched, is teachers' understanding of learners' mathematical errors and misconceptions and in particular, how teachers think about learners' errors. In this study, I aim to investigate how teachers recognize and understand learners' errors and misconceptions. An important aspect of this is an attempt to probe erroneous thinking and find solutions to address mathematical flaws, both of which require pedagogical content knowledge.

In my observation as a mathematics teacher, I have found that in order to work with learners' errors, I needed to have a good understanding about the development of mathematical concepts and procedures, the connectedness of mathematical ideas within mathematical domains, the prior knowledge that learners bring into the classroom (from their understandings in previous years at school), and the constructions learners make in their thinking. This research, which focuses on teachers' reasoning about learners' performance, investigates ways in which teachers make the above connections.

I have therefore chosen to focus my research on the following:

Research Questions

1. What pedagogical content knowledge do Intermediate Phase teachers demonstrate when analyzing learners' performance on 5 multiple-choice test items?
 - 1.1 In what ways do teachers reason about the misconceptions that underlie learners' mathematical errors?
 - 1.2 What pedagogical suggestions do teachers offer to address these misconceptions?

It is my contention that until our teachers develop the capacity to change from procedural teaching approaches to approaches that embrace conceptual understanding by employing their content knowledge and mathematical knowledge for teaching, it is unlikely that teachers will recognize and understand errors and misconceptions in their learners. I am of the opinion that the contribution made by this study will assist researchers in South Africa to understand ways teachers reason about mathematical knowledge for teaching when they are reflecting on learners' errors.

Overview of chapters

Chapter 2 introduces the reader to the literature that theoretically frames my research. The focus of this chapter is to look at the relationship between the following constructs: "content knowledge", "conceptual" and "procedural" knowledge and "misconceptions".

Chapter 3 focuses on my methodology, my teacher sample and my interview structure.

Chapter 4 is a task analysis of the items used in the interview. Each item is analyzed according to the misconceptions embedded in the item distractors, and the analysis is supported by literature that relates to misconceptions in the different item domains.

Chapter 5 is theme based in terms of the teacher's content knowledge, their reasoning about misconceptions in the distractors and their proposed interventions to address these.

In Chapter 6, my analysis contrasts three teachers' PCK in more depth with a particular focus on their interventions to address misconceptions in three of the items.

Chapter 7 is a response to my research questions based on my findings, analysis and literature review. I consider the consistencies and inconsistencies between them and what I would like to envisage added to future teacher training programmes in mathematics education.

CHAPTER TWO : LITERATURE REVIEW

Theoretical framework and Literature review

A study of the nature of teachers' content and pedagogical content knowledge when analyzing errors and misconceptions in learners' thinking, focuses one's attention on certain key knowledge areas. The theoretical knowledge that informs my research points to:

- Content knowledge and pedagogical content knowledge
- The importance of the interconnections between conceptual and procedural knowledge
- The phenomenon of errors and misconceptions
- Handling misconceptions

Mathematics teachers in the field have content knowledge from their schooling, pre-service studies and acquired content knowledge while working in the classroom. The transfer of this knowledge to learners depends on individual pedagogical choices. The main issue is how new knowledge is to be transferred and whether cognizance is taken of learners as individuals, who are in the same learning community, but may have different cognitive levels and understandings. Teachers are responsible for assisting all learners in this community to acquire knowledge and therefore they need to bridge their content knowledge with good practice that benefits all. This means that teachers constantly have to make decisions about what actions to take in the representation and conceptual development of new topics, when and how to use procedures, how to probe their learners' reasoning about new content, how to investigate and address flaws or misconceptions involved with the content and what the origins of such flaws are. Working with learners' misconceptions demands reflection. It also requires thinking about the type of cognitive tasks needed in order to focus on misunderstandings and the errors they produce.

Content knowledge and pedagogical content knowledge

Modern day thinking about the complexities of PCK can be largely attributed to Lee Shulman and his colleagues who, in the 1980s, gave us a new understanding of the knowledge of practice and content. Shulman's views on teaching reformed past thinking about pedagogy and what teachers need to know for learners to learn. His views and vision paved the way for further research and development about good practice in the 1990s and into the new millennium. Shulman (1987) divides teacher knowledge into knowledge bases: content knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners, knowledge of educational contexts and knowledge of educational ends. The idea of types of teacher knowledge has led researchers, such as Zembal, Starr, Krajcik (1999) to examine knowledge bases for teaching. More specifically they examined pedagogical content knowledge and its implications for science in undergraduate teaching programmes.

According to Shulman (1986), a crucial characteristic of pedagogical content knowledge is the ability of a teacher to make multiple representations of the same mathematical content. Learners come to the mathematics class with prior knowledge that is constructed according to their own

comprehension. Some learners find aspects of mathematical topics easy to learn, while others find them difficult. In a reference to pedagogical content knowledge and misconceptions, Shulman states that:

An understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them...If those preconceptions are misconceptions, which they often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners. (1986:9-10)

The above idea that a teacher needs to be in a position to first understand learners' errors and misconceptions and then employ different strategies in an attempt to transform learners' mathematical knowledge structures that lie behind errors, is central to PCK. It suggests that a teacher needs to draw on a repertoire of representations which might be required when least expected in order to promote understanding.

The promotion of this understanding is enhanced by Adler (2005) and Kazima, Pillay, Adler (2008) who foreground the idea of a teacher's ability to "hold and use mathematics" – apart from being able to do the mathematics, a teacher needs to have clarity on the goals of a mathematics lesson, use approaches that transmit ideas and concepts, assess learners' responses and arguments, interpret their explanations, structure appropriate tasks, ask appropriate questions that promote thinking, and interpret curriculum resources. Adler (2005) makes a further point about the interpretations teachers make about learners' work on a specific task:

The teacher then needed to be able to judge the mathematical worth of learner productions which in turn would require being able to relate different responses to each other in relation to mathematics. (2005:4)

Adler contends that learners have a voice and before teachers make interpretations and judgements about the learners' knowledge, learners need to be able to reason and articulate their thinking, which in turn, allows teachers to analyse errors if they present themselves. Lampert (1991) was also concerned about the things teachers needed to do to promote learners' articulation of their understanding and misunderstandings. She looked at the type of tasks that learners need to communicate their comprehension of the mathematical content. The idea of a task-based understanding replaces the simplistic traditional notion that understanding can be measured against how well learners recall and apply rules. She states:

If the process of coming to know mathematics in the classroom is going to have some relationship to the process of coming to know mathematics in the discipline, then teachers will involve getting students to reveal and examine the assumptions they are making about mathematical structures, and it will involve presenting new material in a way that enables them to consider the reasonability of their own and teacher's assertions. (1991:125)

This entails choosing types of tasks or problems that will engage with mathematical thinking and learner constructions of mathematical knowledge. Lampert makes a clear distinction between pedagogical content knowledge, that is knowledge for teaching, and cognitive knowledge, which is knowledge about knowing and learning content. She believes that teachers need both types of knowledge.

Related to the area of learners' thinking and understanding, Ball and Bass (2000) emphasize the fact that teaching situations are unique and dynamic and no amount of experience can ever predict learner thinking at any given time. The most teachers can do is anticipate what learners may think and respond with a pedagogical action that is relevant to the unique situation. In other words like Lambert, Ball and Bass understand the importance of learners revealing and examining "the assumptions they are making about mathematical structures" and emphasize that teachers need to act on knowledge available in their repertoire at the time and draw on their own considerations of the learners to help learners restructure their thinking. This brings the notion of uncertainty. They say:

Knowing mathematics for teaching must take account of the regularities and uncertainties of practice, and must equip teachers to know in the contexts of the real problems they have to solve. (2000:90)

The implication here is that teachers' judgements about learners' thinking and the decisions they make about this thinking impacts on the kinds of pedagogic action that is used to assist learners to reflect and transform mathematical ideas and constructs. Whatever decisions are made by teachers, they can never be entirely certain that what they have chosen to do at any particular time is set in stone. Pedagogic decisions made one day may have to be refined or changed the next day. Embedded in this decision making is an evaluation of what will work best; what response does a particular learner need? Fennema and Franke (1992) explain further:

The knowledge a teacher has is responsible for the kinds of decisions they make, that is, they can reason, make judgements and reflect on actions taken in the past which may need to be modified for the future. (1992: 156)

Carpenter and Fennema (1991) studied programs which assist teachers to make instructional decisions based on learner thinking. They contend that teachers need to understand the developmental stages learners experience when acquiring new knowledge, and that teachers must be able to view this development reflected in learner solutions of problems. Their model of "Cognitively Guided Instruction" indicates a relationship between teacher decision making and learner thinking. The main idea here is that teachers can use learners' responses to tasks as a guide to assess what mental processes are occurring within a particular learner, and on the basis of this assessment they can then make decisions for interventions where appropriate.

Good decision making comes more with one's personal growth as a practicing teacher with good subject matter knowledge (Ball, 1988). Part of this growth is an ability to respond to learners' ideas and reasoning. Ball (2001) contends that responding to students' mathematical ideas is more complex than one may think. She says, "It requires being able to hear and interpret what the student is saying, and it includes being able to skillfully probe in cases where the student's idea is not clear" (2001:453).

In sum, the above researchers emphasize that teachers need to have knowledge of instructional strategies (this includes representations and activities for specific topics) and knowledge of topics, particularly the types of difficulties that learners experience with certain topics and the prior knowledge that learners may have on those topics (Zemba et al., 1999).

The nature of teachers' subject matter understanding concerns the depth of content specific domain knowledge (Ball, Lubienski, Mewborn, 2001). The importance of this knowledge cannot be over emphasized. In her analysis of Chinese mathematics teachers, Ma (1999) views the improvement in teachers' subject matter and students' mathematics education as being "interwoven and interdependent processes that must occur simultaneously" (1999:147). She claims that teachers can improve their subject matter knowledge while they are teaching. What matters here is that teachers need to feel confident enough to make decisions about their teaching and where their subject knowledge deficits are. New subject knowledge that enters the curriculum for the first time (as happened with curriculum 2005, for example, transformation geometry) will be taught at first apprehensively, but with judicious reflection on learners' thinking and on their own knowledge deficits, the second time should see growth in that subject knowledge and therefore improved practice in the classroom. Ma also looked at factors that hinder subject knowledge growth, particularly as it relates to topics that teachers mastered when they were at school. This being the case, Ma claims that teachers can quite easily slip into complacency and feel that they don't need to do any further study of the topic because it is so "basic". This deficiency is endorsed by Ball and Bass (2000) who state that:

Subject matter knowledge for teaching is often defined simply by the subject matter knowledge that students are to learn – that is, by the curricular goals for students. Put simply, most people assume that what teachers need to know is what they teach. (2000:86)

Embedded in the notion of mathematical knowledge for teaching is the mathematical knowledge that is needed to carry out the work of effective mathematics teaching (Ball et al., 2008). This embraces more than content knowledge and PCK – it includes assigning a meaning to the notion of 'effective teaching' in terms of the demands of tasks in which the teacher is continuously engaged. These tasks include the critical work of assessment, planning developmental units of work, interacting with parents, understanding the content of the curriculum, knowing what mathematical language to use in the classroom and what teachers do to respond to and manage these tasks, while always keeping the notion of mathematical proficiency foregrounded. Ball et

al. (2008) speak about specialized content knowledge (SCK) as that area of mathematical knowledge needed for teaching whereby a teacher is able, amongst other things, to make decisions about what suitable actions to take concerning learner errors and when other methodologies are needed to enhance learning – this can happen instinctively or as a result of reflection on one's own teaching. Alongside this SCK and equally important is what Ball et al. (2008) refer to as knowledge of content and students (KCS). The authors claim that teachers need to anticipate how learners are likely to think about what they may misunderstand. Teachers need to listen acutely and interpret learners' responses, so that they can discern where learners' thinking is incomplete, and be alert to the existence of the types of errors and misconceptions that can manifest in learners' knowledge constructs.

The research I have referred to on content knowledge, pedagogical content knowledge and mathematical knowledge for teaching evidences that when I probe teachers' understanding about learners' erroneous responses I will be foregrounding the ways teachers understand the thinking behind learners' (erroneous) responses to tasks, specifically the prior knowledge that the learners draw on and the assumptions they make about mathematical concepts. The analysis above suggests that a competent teacher's understanding of learners' thinking, demonstrates a teacher's ability to relate the structure and the sub-structures within the mathematical content domains as they pertain to tests' items, and to construct alternative methodologies to address learners' errors and misconceptions within these domains.

The importance of conceptual and procedural knowledge

Kilpatrick, Swafford and Findell (2001) developed the notion of mathematical proficiency which intertwines five strands – conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. They state, “One of the defining features of conceptual understanding is that knowledge must be connected so that it can be used intelligently. Teachers need to make connections within and among their knowledge of mathematics, students and pedagogy.” (2001:10). Therefore, a mathematical concept should never be treated in isolation. Ma (1999) makes the point that teachers at elementary level very often do not see the whole ‘knowledge package’ that pieces of mathematical knowledge belong to, for example, fractions, decimals and percentage are linked to the notion of a part of a whole. Once a new concept is grasped, learners can better understand the procedures used for calculations. This does not advocate that procedures cannot be used without conceptual understanding. The problem is, however, that many learners learn procedures through practice and drill without having any knowledge of the underlying structures behind the procedures (Kilpatrick et al., 2001).

Conceptual understanding occurs when mathematical ideas are grasped and are connected to what learners already know. These ideas can be represented in different ways and in different contexts. The principle idea is that knowledge that is understood can generate new knowledge

and this knowledge can be retrieved to solve problems (Kilpatrick et al., 2001). A key feature of this strand is that learners are in a position to make connections to related concepts and understand the mechanics of mathematical procedures and standard algorithms:

Conceptual understanding frequently results in students having less to learn because they can see the deeper similarities between superficially unrelated situations. Their understanding has been encapsulated into compact clusters of interrelated facts and principles. (2001:120)

In their discussion about the interrelationship between conceptual and procedural knowledge, Hiebert and Lefevre (1986) state that:

Building relationships between conceptual knowledge and the formal symbol system of mathematics is the process that gives meaning to symbols and building relationships between conceptual knowledge and the procedures of mathematics contributes to memory (storage and retrieval) of procedures and to their effective use. (1986:10)

Procedures such as the standard algorithm used to multiply two and three-digits numbers together (where one number is placed vertically beneath the other) can be retrieved quickly and successfully particularly in certain types of real life problems. For example, ‘A farmer plants 35 orange trees in 27 rows. How many trees does he plant altogether?’ The conceptual links underpinning the multiplication procedure in this case are that multiplication is an efficient way to do repeated addition, that the numbers are a composite of place value structures – hundreds, tens and units and that the distributive law construct enables all the parts of one number to be multiplied with all the parts of another in any order – the products are then added to provide a final answer. Conceptual knowledge can behave as a critic of a solution to a problem that has used a particular procedure. The reasonableness or inappropriateness of an answer can be measured against one’s conceptual understanding of the context involved. In the tree problem, a learner who makes conceptual links will know that the farmer has planted more than 600 (from 30×20) trees and judge his/her answer accordingly.

Long (2004) points out, “There is a positive correlation between children’s understanding of mathematical concepts and their ability to execute procedures” (62:2004) but she further adds that there is no hard and fast rule that conceptual understanding must always precede procedural learning. For example, the conceptual understanding that underpins the addition of fractions with different denominators should scaffold the procedure in order for the procedure to make sense. However, in the case of children learning to count, they first learn to say the numbers and learn the symbols before the cardinal value of each number means anything. Advancing Kilpatrick et al.’s notion of interwoven strands, Long alerts us to the fact that, “It is not always possible to distinguish concepts from procedures because understanding and doing are connected in complex ways” (64:2004).

A successful relationship between conceptual and procedural knowledge relies heavily on a teacher's pedagogy and the ability to promote this knowledge for effective use. Shulman's (1986) ideas for effective teaching embody the ability of teachers to represent content in powerful and different ways and to frame it comprehensibly to all learners in the same learning environment. Explanations need to be complete and sufficient enough to meet the needs of all learners.

This study therefore focuses on mathematics teachers and how they reason about their own mathematical knowledge for teaching. Will teachers give me mainly procedural solutions for the handling of misconceptions, that is, to learn algorithms off by heart? Will they mainly refer to the (erroneous) mathematical procedure learners have employed, or will they refer to these in *relation to* the conceptual knowledge that underpins them, and suggest tasks that address the conceptual issues of structure and domain that are fundamental to 'knowing mathematics' for understanding? These are the kinds of questions which Kilpatrick's distinctions give rise to.

In conclusion to this section, it is important to note that if tasks comprise mechanical drill and practice of procedures and learners are not exposed to different ways of viewing concepts in different situations, that is, to different representations and contextual problems, it is unlikely that the mathematical proficiency Kilpatrick et al. speak about will ever develop. Learners will not be in a position to comprehend why their answers are incorrect, nor will they evidence an ability to justify and reason what mathematics to use in certain contexts. In my study I aim to determine the kinds of conceptual and /or procedural connections teachers make when they reflect on learners' errors in five ICAS test items.

The phenomenon of errors and misconceptions

Much has been written about errors and misconceptions, but the literature mostly deals with learners and very little research exists on teachers' understanding of learners' errors and their misconceptions. Ryan and McCrae (2005) have suggested that:

pre-service teachers who confront their own mathematical errors, misconceptions and strategies in order to reorganize their subject matter knowledge, have an opportunity to develop a rich pedagogical content knowledge. (2005:641)

Errors and misconceptions are not one and the same thing. According to Hansen and Drews (2005) errors can be:

the result of carelessness; misinterpretation of symbols or text; lack of relevant experience or knowledge related to that mathematical topic/learning objective/concept; a lack of awareness or inability to check the answer given; or the result of a misconception. (2005:14)

Hansen and Drews assert that the notion of 'misconception' is rooted in an underlying confusion about a concept or it evidences itself when learners over- or under-generalize mathematical

contexts without any conceptual understanding playing a role. Nesher (1987) also makes this point when she says that errors arise within conceptual frameworks and are based on previously acquired knowledge (1987:33). She advocates that teachers make time to look for not only single errors, but rather see if behind each single error, lies ‘clusters of errors’ (which she calls misconceptions) that are responsible for learners’ erroneous thinking. For example, learner A is shown $\frac{2}{5}$ and asked what it means. The response may be ‘5 5’ (two fives). There is clearly a misconception that learner A sees the two numbers in the fraction notation as two whole numbers. Embedded in this misconception is the notion that a learner does not have any concept of the fraction as representative of a part-whole relationship. It is evident that learner A is not able to make any conceptual links with the relationship between the numerator and denominator. Allied to this construct is the notion of division which is represented by the line in fraction notation, hence learner A ignores the line. A second example that shows evidence of a misconception is when Learner B is asked to provide the answer to 3^4 . The response is ‘12’. In this case learner B has a misconception about the function of the exponent and the base. Embedded in the error is that learner B thinks that any two numbers presented in this form is the action of multiplication and the exponent is treated as a whole number. The learner does not grasp the difference between 3×4 (derived from the repeated addition ($4 + 4 + 4$ or $3 + 3 + 3 + 3$)) and $3 \times 3 \times 3 \times 3$, which is presented as a power with a base of 3 and an exponent of 4, hence 3^4 .

The notion of erroneous thinking is further elucidated by Smith and Roschelle (1993). Smith and Roschelle alert us to the fact that when learners come to class, they bring with them preconceived conceptual ideas and beliefs which may conflict with the conceptual notions they subsequently experience in class. Smith and Roschelle point out that “errors are characteristic of initial phases of learning because students’ knowledge is inadequate and supports only partial understanding” (1993:123). The authors argue that if learners construct their own knowledge by being constantly in the process of interpreting what they experience both within and outside the classroom, it is not by accident that misconceptions will arise. A learner’s prior knowledge drives the process of interpretation and in learners who lack substantial prior knowledge, mathematical concepts take longer to master than in others whose prior knowledge in the domain is more sophisticated. However, the authors concede that in time learners are in a position to reorganize and transform their existing knowledge and initial misconceptions can be dispelled through construction as a result of new experiences. Although learners are diverse, many misconceptions are similar across these different individuals. Their preconceived notions may be strong and deeply rooted and they do not disappear when confronted with new knowledge, particularly if teachers are not aware of at least the common misconceptions that exist related to certain concepts that have to be learned.

Related to the notion of new knowledge being assimilated into existing structures, Olivier (1989) says:

Sometimes a new idea may be so different from any available schema that assimilation and accommodation is impossible. In such a case the learner creates a new 'box' and tries to memorize the idea. This is rote learning; because it is not linked to any previous knowledge it is not understood; it is isolated knowledge, therefore it is difficult to remember. Such rote learning is the cause of many mistakes in mathematics as pupils try to recall partially remembered and distorted rules. (1989:11)

One cannot expect flawed thinking to disappear in a short space of time (Smith and Roschelle 1993). Individuals will master the mathematical concepts in their own time according to their sense-making processes. Nesher proposes that when one has to teach new knowledge, "we must know how this knowledge is embedded in a larger meaning system that the child already holds and from which he derives his guiding principles" (1987:36).

The Data-Informed Practice Improvement Project alerts us to the fact that errors and misconceptions need not be viewed in a negative light – "Errors can have a positive effect for teachers, in that they can reveal incompleteness in learners' knowledge; and thus enable the teacher to contribute, or better still, guide the learner to realize for him or herself where s/he is going wrong" (Brodie, Shalem, Sapire, Manson and Sanni, 2008:2). DIPIP makes pertinent that teachers have to listen very carefully to what learners say, so that existing misconceptions can be identified and through directed questions and activities, attempt to restructure the learners' existing knowledge – all of which serve to inform one's own teaching. Answers are not merely correct or incorrect, one needs to probe why answers are wrong and use our findings to take corrective measures and re-evaluate the learners in order to transform their thinking.

The purpose of this study is to probe teachers' reasoning about the errors and misconceptions in the multiple-choice item distractors, and I aim to reflect on their conceptual ideas and beliefs. I am also interested to assess the pedagogic choices they will make to address these phenomena.

Handling errors and misconceptions

In order to address misconceptions, one must not only bear in mind what Kilpatrick et al. (2001), Hiebert and Lefevre (1986) have stated about conceptual understanding, but allied to their assertions is the notion that different non-routine tasks around the same mathematical domain ought to be structured, to probe the full extent of the learners' knowledge (Smith and Roschelle 1993). In addition, it is not sufficient to give learners tasks; it is important to let learners discuss how they think in the execution of such tasks. In view of this, Smith and Roschelle argue that it is important to give teachers an idea of the learners' level of comprehension and the role misconceptions are playing as obstacles to learning. Nesher also discusses teachers needing to "construct diagnostic items that disclose the specific nature of misconceptions" (1987:39). Nesher asserts that teachers will find it helpful to first be aware of what types of misconceptions exist in a domain before new knowledge is taught. She suggests that teachers need to alert

learners to possible misconceptions that can occur by emphasizing them as faulty ways of thinking.

On the question of misconceptions made public, Brodie (2005) points out “If learners have come to expect particular ways of working in a mathematics classroom, they will continue to make use of these expectations” (2005:180). That is why it is imperative to allow learners to discuss their mathematics and listen carefully to their constructions. Learners who have previously been unused to this approach from former teachers, will in all likelihood not publicize their thinking and therefore never open their misconceptions to interrogation by their teacher or peers. Smith (1993) emphasizes that discussion is far better than confrontation because we need to access the knowledge that students have if it is to be ‘refined’.

Another pertinent point that Nesher makes when handling misconceptions is that sometimes right answers can be a disguise for misconceptions. It is often a worthwhile consideration for a teacher to verify with a learner what understanding led to a particular answer.

As teachers think aloud during the interview I will observe the extent to which they are able to diagnose what the difficulties are within item domains, that is, be able to identify a structural origin of the misconceptions, talk about it and suggest what they consider to be the most fruitful way forward in terms of tasks they would give to eradicate learners’ flawed thinking.

The literature has evidenced that a teacher’s content knowledge is the basic platform from which to reason about learners’ thinking, particularly in terms of their errors and misconceptions. Procedural and conceptual knowledge are both acknowledged as types of knowledge that one draws on when ‘doing mathematics’. Teachers are not always privy to which type of knowledge a learner uses when doing a mathematical calculation or solving a problem. Not all learners get their answers right and this poses concerns about pedagogy. Teachers have to reflect whether their strategies and representations were meaningful and sensible to begin with. They have also to consider if the learners’ prior knowledge is flawed in one way or another and whether this flawed thinking lies behind learners’ misconceptions. Consequently, the teacher’s task is to initially recognize learners’ errors and misconceptions and subsequently attempt to intervene by transforming learners’ knowledge in order to eradicate flawed thinking if possible. Chapter 4 (task analysis) focuses on the items and their distractors and fleshes out the types of misconceptions (with referenced literature) related to the items used in my study.

CHAPTER THREE : METHODOLOGY

Methodology

My research aims to investigate the ways six Intermediate Phase teachers use their content knowledge to understand and address misconceptions that lie behind learners' errors (in their PCK) in five Grade 6 multiple choice items. Thereafter, I look at the relationship between their content knowledge, their reasoning about learners' errors and misconceptions and the ways they propose to address these through tasks that can serve to eradicate cognitive flaws. I was specifically interested in what teachers were reasoning about learners' thinking when mathematical misconceptions and errors emerged in learners' tasks. McMillan and Schumacher (2006) claim that:

Qualitative research is inquiry in which researchers collect data in face-to-face situations by interacting with selected persons in their settings (e.g. field research). Qualitative research describes and analyzes peoples' individual and collective social actions, beliefs, thoughts and perceptions. The researcher interprets phenomena in terms of the meanings people assign to them. (2006:315)

From what the above definition has to say it is appropriate that my study is classed as qualitative research. The empirical nature of the design included semi-structured interviews, which provided a selected number of teachers with an opportunity to express their perceptions, thoughts and beliefs about learners' mathematical errors and misconceptions. Teachers are unique individuals who have a wealth of professional experience and I was interested to explore their conceptions of their PCK with regard to what I have read in the literature, particularly the phenomenon of errors and misconceptions in learners both in the ways teachers reflect on these and their means of addressing them pedagogically.

The interviews provided me with insight into teachers' perspectives that I could interpret against my understanding of PCK in the area of errors and misconceptions. My interviews were grounded in the assumption that "people's perceptions are what they consider real and thus what directs their actions, thoughts and feelings" (McMillan and Schumacher 2006:315). The interpretation of the interviews rested upon the ability of the researcher to delve as deeply as possible and assign meaning to what is heard and said. I intended to remain objective at all times and not intervene in the reasoning of the participants.

Sample

Experienced teachers with mathematical content knowledge are better situated to furnish valuable perspectives to learners' errors and misconceptions and ways of addressing these. In order to generate a sample of teachers for this purpose, fifty teachers who were registered in the 2008 Witwatersrand ACE programme at the Wits School of Education completed an exercise containing 30 multiple-choice items selected from the Grades 5-6 Australian 2006 ICAS tests

(Refer to p1). The 30 items selected covered two different learning outcomes (Number and Measurement) and thus required a relatively wide range of mathematical understanding. The teachers were informed that the multiple-choice exercise was not for mark purposes and there was no implication for them in any way other than the exercise being used for my research. If they wished to receive their item responses they could do so after I had assessed them.

I chose the top six Intermediate Phase teachers all of whom achieved more than 60% in the exercise and were willing to participate in my interview. After consulting with these teachers, I arranged dates and times for their interview. Three of the teachers did not arrive for the interview as arranged and when I pursued the matter they declined to make another arrangement due to their time constraints. I was compelled to make up my complement of six teachers by contacting teachers who I had become acquainted with while my fourth year Intermediate Phase students from the School of Education at Wits were conducting their continuous practice. These teachers were responsible for individual students to whom I had been assigned. I was impressed by their mathematical content knowledge and the pedagogical advice they gave to my students. They consented to participate in my research interview.

The six teachers who I interviewed (Henceforth, Angie, Betty, Carla, Dawn, Ella and Fran) are all female and currently teaching mathematics at various schools in Gauteng. Five of them are teaching at public schools which range from disadvantaged and under-resourced socio-economic environments to medium and highly resourced schools. One teacher is teaching at a small private school. Their teaching experience ranges from fifteen to twenty seven years. They have all taught mathematics in various Intermediate Phase grades. One of the teachers spent a few years teaching in the Foundation Phase.

Data collection

As a basis for the audio – taped interview, I used five of the 30 multiple-choice exercise items given to the fifty ACE students. I selected to apply the interview guide approach as opposed to holding informal interviews where there is “no predetermination of question topics or phrasing” (McMillan and Schumacher 2006:315). I decided to interview the teachers because I wanted to hear how they reasoned and take cognizance of how they used mathematical language in their verbal explanations. The questions in the schedule (See Appendix A) allowed me to comprehensively investigate the ways the six teachers’ thought about learners’ errors and misconceptions by using the ‘think-aloud’ method (Young, 2005).

Young (2005) draws on the findings of Ericsson and Simon (1993) who worked extensively with think-aloud data. Ericsson and Simon claim that the think-aloud method captures what is held in short-term memory and that the sequence of thinking of the participants reflects what occurs cognitively while the participant is engaged in a specific activity. This notion is endorsed by Young who states that, “the think-aloud approach ensures specific focus is directed to the

participant's thoughts, which is useful in both minimizing distractions from a participant's sequence of thoughts and also aids the researcher in obtaining data that are most purposeful for their research goals" (2005:22). Young furthermore draws one's attention to the fact that the think-aloud approach has its advantages and disadvantages. One advantage is that the approach enables researchers to obtain evidence of depth of thinking. A disadvantage in using this approach is that not all thought is accessible at all times and issues of language and articulation can impede the mental processes of the participants from being accurately reported.

Young is of the opinion that think-aloud data is an under-utilized method of data collection:

I believe that going directly to the source of information (i.e. the students) and capturing what they verbalise, provides substantial information to both support and enhance that which we obtain using other common research methods. It also offers the opportunity for student voice to be heard, a voice often neglected in research. (2005:19)

The design of the think-aloud interview schedule commences with a broad background of the teachers' teaching experience and their opinions of teaching today. The second section of the interview deals with the teachers' understanding of curriculum alignment. It then proceeds to the domain contexts of the items and the questions then focus on the distractors for the purpose of discussing learners' errors and misconceptions. The interview ends with the teachers' reflections on interventions to address errors and misconceptions in the items.

I now intend to discuss the items I chose for the semi-structured interview:

Reasons for selecting the items

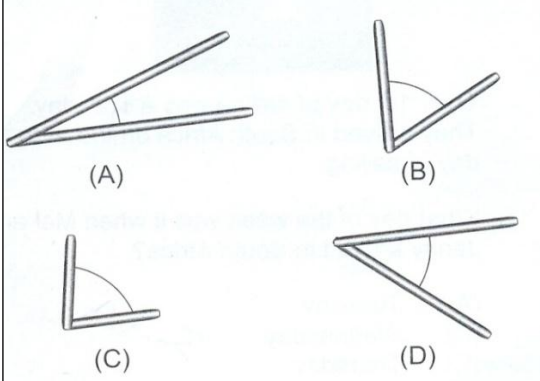
The 'Measurement' domain has a number of topics which are conceptually very different. I chose 'angle' (Item 1), 'SI unit conversions' (Item 3) and 'area' (Item 5) as key concepts in which learners tend to manifest errors and misconceptions in this domain. The items I chose for the 'Number' domain involve the concepts 'fraction' (Item 4) and 'subtraction of whole numbers' (Item 2). The reason I selected these concepts in particular in the Intermediate Phase level is the following:

- One of the earliest concepts in the learning of geometry is knowledge about angle (Barrett, Jones, Thornton, Dickson, 2003). Learners who struggle with angle are liable to struggle with geometry in higher grades and develop a negative disposition towards this branch of mathematics.
- I have encountered adults in society who have either no knowledge of or have forgotten about SI unit conversions and decimals – a life skill (Mitchell and Horne, 2008) that remains with us in every day contexts.
- Area is another life skill (Outhred and Mitchelmore, 1993) which should be accessible to all human beings.

- Many undergraduate students and adults in society have admitted to me on numerous occasions how they could never understand fraction concepts at school (Newstead and Murray, 1998) and ‘hated them’
- The skill of doing subtraction with whole numbers (Brodie et al., 2009) is useful throughout one’s life – very often this process has to be done mentally in certain circumstances and without pen and paper the borrowing procedure often becomes an obstacle in obtaining a correct mental answer.

Items used in the interview and an analysis of Gauteng learners’ (2006) responses to the 5 items selected

The following tables show the percentage of learners’ responses in the 2006 cohort (that wrote the ICAS test) per distractor per item. In all these items the percentage of learners that identified the correct answer is much lower than the total percentage that responded to the three distractors.

Item 1	Differentiate angle size from orientation and side length
	Which of these shows the largest angle?  (A) (B) (C) (D)
Grade	5
Content	Angle
Area	Measurement

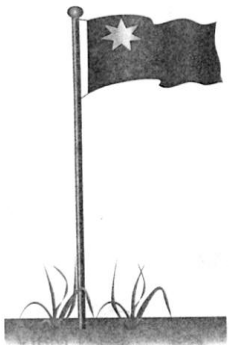
Analysis of Learner responses (%)

A	36
B	10
C	34
D	13
Blank	5
Correct answer	C

Item 2	Subtract three-digit numbers
$900 - 358 = ?$ (A) 542 (B) 552 (C) 642 (D) 658	
Grade	6
Content	Whole number subtraction
Area	Number

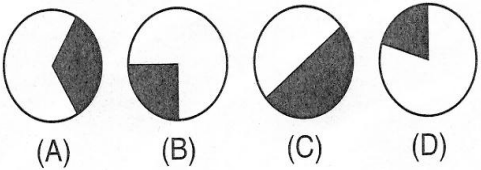
Analysis of Learner responses (%)

A	34
B	11
C	13
D	39
Blank	1
Correct answer	A

Item 3	Convert between units of length
 A flag pole has a height of 3.24 metres. Which one of these shows the height of the flag pole? (A) 3 m + 2 cm + 4 mm (B) 3 m + 20 cm + 4 mm (C) 3 m + 20 cm + 40 mm (D) 3 m + 200 cm + 40 mm	
Grade	6
Content	SI unit conversions
Area	Measurement

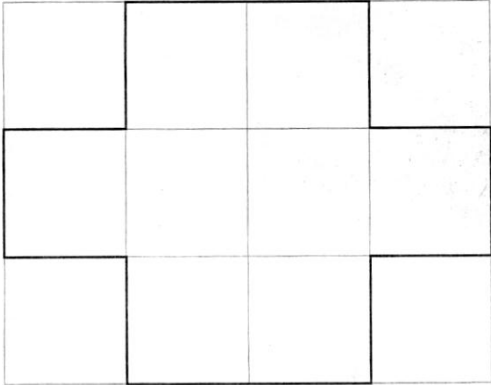
Analysis of Learner responses (%)

A	23
B	33
C	16
D	24
Blank	3
Correct answer	C

Item 4	Identify a circle with one third shaded
Which circle has one-third shaded?  (A) (B) (C) (D)	
Grade	6
Content	Fractions
Area	Number

Analysis of Learner responses (%)

A	31
B	17
C	28
D	20
Blank	2
Correct answer	A

Item 5	Calculate the area of a given portion based on squares
Holly drew this shape on 2 cm grid paper.  What is the area of Holly's shape? (A) 32 cm² (B) 28 cm² (C) 16 cm² (D) 8 cm²	
Grade	5
Content	Area
Area	Measurement

Analysis of Learner responses (%)

A	14
B	22
C	23
D	34
Blank	5
Correct answer	A

Interview Schedule (Refer to Appendix A)

Data Analysis

Before analyzing the transcripts of the teachers' interviews, a task analysis of each item was completed in terms of the mathematical knowledge required to obtain the correct answer to the item and the misconceptions (supported by literature) evident in the distractors (See Chapter 4). Responses provided by the teachers were evaluated against this analysis.

The 'think-aloud' interviews required that teachers reason about each item's content knowledge and the errors and misconceptions embedded in the item distractors. They then told me what interventions they would use to address errors and misconceptions in the items. I wanted to find out what kind of *relationship* existed between their content knowledge, their reasoning about the

errors and misconceptions and their interventions. Therefore, my findings were framed around four themes:

Theme One: Knowledge of mathematical content relevant to the items and use of procedural or conceptual thinking

My analysis for this theme focuses on the knowledge teachers have of the content, that is, do they show evidence that they recognize the mathematical concept(s) embedded in each item and are they in a position to draw on content knowledge required to produce the correct answer? This analysis suggests too, that by examining teachers' thinking about learners' thinking, one can access teachers' own content knowledge and conclude whether they think procedurally, conceptually or evidence a mix of the two.

Theme Two: Language use when explaining mathematical concepts

The focus of analysis for this theme is how teachers use mathematical language in the interview to convey meaning given the constraints that English is, in all probability, not be their home language. This theme was chosen because English is the official language of instruction in South African classrooms and many of the learners' first language could be one of the other ten official languages. The teachers in my sample come from different ethnic backgrounds and except for one teacher none of them has English as a first language.

Theme Three: Awareness of misconceptions and errors in the items

In this theme, I focus on how teachers reason about misconceptions in the item distractors by accessing their thinking through a direct question. I wanted to discover if they could identify one or all of the misconceptions, or merely construe the item distractors as choices learners make because they (the learners) do not have sufficient knowledge about the content.

Theme Four: Interventions to address perceived misconceptions

The interviews gave teachers an opportunity to both impart their content knowledge in the item sub-domains and to disseminate learner thinking about the incorrect distractors. The last section of the interview dealt with the kind of interventions they would make to address one or two of the misconceptions in the item distractors. In order to analyze and focus on the interventions, I decided to use a meta-structure against which to map the teachers' suggested interventions: Is the intervention mathematically correct? Does it address the misconception and is it age appropriate?

An analysis of the above four themes is conducted in Chapter 5. I initially check to see if teachers have mathematical content knowledge required for the correct answer in each item and then I examine the way they reason about errors and misconceptions embedded in the item distractors. I look for ambiguities in their mathematical language and lastly, I look at the means in which teachers address one or two errors and misconceptions. The themes allow me to gain insight into their PCK with a particular focus on their interventions. I intend to gain an overall

picture of their content knowledge, their perception of learner errors and misconceptions and their PCK. This is all in line with my aim, to investigate the ways these teachers use their content knowledge to understand and address misconceptions that lie behind learners' errors in five Grade 6 multiple-choice items (in their PCK).

In Chapter 6 three teachers are identified and contrasted to analyse their PCK in more depth. I selected these teachers because their conceptions of interventions highlight the differences in their thinking about what learners need to know. Their discussion on interventions enabled me to show the difference between teachers whose strategies demonstrate a conceptual/procedural and content/learner orientation in approach.

I cannot purport to know exactly what went on in the minds of the teachers in my sample at all times. I could only probe as far as time constraints would allow. The teachers did not have hours or days to reflect on my questions in the interview - I found this to be a positive rather than a negative factor, because the content of their responses is embedded in their own beliefs and understanding about their learners' thinking which are linked to their practice and experience. The teachers were constrained by time due to their own circumstances and lives and were only able to provide me with approximately two hours of their time for the interview. Their views and opinions are a product of their many years of experience in teaching mathematics to Intermediate Phase learners.

Limitations of the study

The answers furnished by a sample of six teachers cannot be used to generalize the thoughts about the teachers' understanding of learners' errors and misconceptions for the population of Intermediate phase teachers in South Africa. I had no knowledge of whether the teachers in my interviews had ever thought about their learners' errors and misconceptions before, but I was in a position to make an informed decision based on the analysis of their responses.

Ethical considerations

The teachers who participated in the interviews first gave me written permission for their participation. Before I presented the exercise to the ACE teachers I explained my reasons for my research and the nature of the interviews I wanted to conduct. I made it clear that their contribution would be part of a developmental process that aims to work towards a better education for South African learners. Those that decided not to participate had the freedom to refrain from taking part. They were also informed that their right to anonymity would be respected at all times.

CHAPTER FOUR : TASK ANALYSIS

Task analysis of misconceptions and procedural errors embedded in the item distractors and curriculum mapping

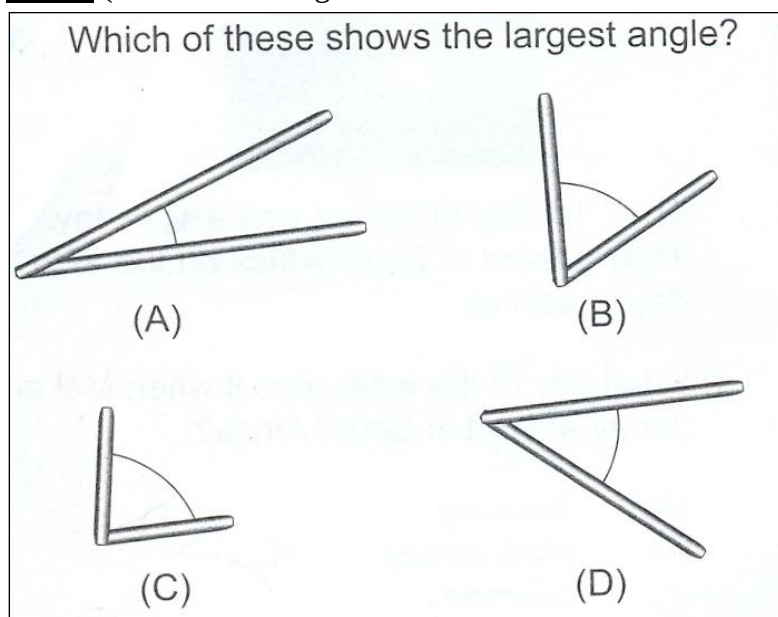
I have selected to investigate teachers' reasoning about learners' mathematical thinking. To this end five multiple-choice items are used as a basis for my investigation. Each item has one correct answer and three distractors. Before I was able to interview the teachers in my sample, I conducted a task analysis of the items. The literature and my own conceptual understanding of the content knowledge informed my analysis of the type of mathematical thinking that underpins each item. In this chapter I first focus on each item separately.

My aim for the first section of the task analysis is not to focus on all the methods and procedures one can use to obtain the correct answer, but rather, my approach is to establish what conceptual mathematical thinking lies within the specific sub-domain for each item. Mathematical domains are broad in content, for example, 'Number' items are linked to two of their sub-domains, for example, 'subtraction of three-digit numbers and fractions'. Having accomplished this, I turn my attention to the errors and misconceptions that arise when conceptual development has gone awry in the developmental process and the erroneous consequences that result. Studies have deduced that certain misconceptions are rooted in each sub-domain. The discussion about the different misconceptions is informed by some of these studies. Finally, I link the distractors given in each item and explain the misconception. This is given in table form at the end of each item.

The second section in this chapter deals with curriculum mapping of the item with the RNCS document. I want to establish how familiar the teachers are with the Grade 6 assessment standards and what assessment standards from previous grades they think are important for prior knowledge of the mathematics in the items. In sum, my frame of reference for an analysis of teachers' thinking about the items takes cognizance of the following categories:

- Content knowledge required for choosing the correct answer:
Choosing the correct answer requires prior knowledge of mathematical constructs within the sub-domain that connect with the mathematics evident in the item.
- Procedural and/or conceptual thinking that can be used to choose the correct answer:
The correct answer can be obtained by using mathematical constructs that are interconnected conceptually, or methods for doing the mathematics in certain items can be done procedurally by employing a step-by-step algorithm that may have been learnt as a recipe.
- What research says about the misconceptions embedded in the distractors:
The literature points out the types of misconceptions that occur in the construction of mathematical knowledge pertaining to the item distractors.
- Mapping the items to the Revised National Curriculum Statement

Item 1 (Differentiate angle size from orientation and side length)



The main conceptual base from which angle knowledge develops is the notion of turn. Embedded in the notion of turn, for example, turning of a door knob, opening a pair of scissors, opening a door attached to a hinge is the idea that there are two straight lines (although invisible), meeting at a given point and related to one another through an amount of turn as the one line moves away from the other around a fixed point. (Mitchelmore, 1998). A two-dimensional representation of an angle indicates the amount of turn by using a curve known as an arc from one line segment to the other. The arc can be placed close to or far from the vertex. Irrespective of its position it represents an amount of a single turning action. As the amount of turn increases, the angle increases in size. The length of the arms is irrelevant when the focus is on the amount of turn. A study of the four angles in the item indicates that angle C is the largest angle based on these constructs.

Misconceptions related to angle fall into three major categories (Barrett et al. 2003, Mitchelmore 1998, Magina and Hoyles 1997): 1) Learners identify the largest angle as the one with the longest arms, 2) the one with the biggest arc and 3) the one with the biggest area between the arms and bounded by the end points of the arms. Compounding these misconceptions is the orientation of the angle - if an angle is not orientated so that the one arm is horizontal and the other is turning anti-clockwise, learners find it difficult to discern and compare angle sizes. Barrett et al. (2003) go even further to say that the standard protractor used to measure angles tends to reinforce two of the misconceptions mentioned above, because the arms on the instrument are equal in size, the areas between the sub-division of the angles are equal in size. He prefers that learners measure angles with a geotriangle. This instrument shows learners that angle size is not dependent on arm length (because they are different lengths on the instrument) and area measures are visibly different even though the sub-divisions of angles are equal.

Misconceptions in the distractors

Distractor A	This angle has the longest line segments.
Distractor B	The orientation of the angle vertex is different from what the learner is used to seeing, that is, a horizontal arm and the angle looks ‘wider’ than the others from one endpoint of an arm to another. This gives the impression that the area between the arms is the biggest. The arm length is longer than C.
Distractor D	The line segments are long as well as being ‘wide apart’ and the angle looks as if it has the biggest area bounded by the endpoints of the line segments. Angle orientation may be new ie the turn is clockwise and this leads learners to think it is the largest angle.

Item 2 (Subtract three-digit numbers)

900 – 358 = ?

(A) 542

(B) 552

(C) 642

(D) 658

When primary school learners add two numbers together they cognitively experience two things: the digits of the numbers are situated in place value columns that are governed by grouping in tens (the decimal system) and it is inconsequential in what order the addends are written (commutative law). The sum is still the same. The commutative law does not hold for subtraction. None of the digits in the place value columns of the bigger number (which has a value greater than the number after the subtraction sign at primary level), can change places with one or more of the digits in the place value columns belonging to the smaller number. Such prior knowledge will predispose learners to the concept of borrowing in the standard algorithm or allow them to explore other methods of subtracting 358 from 900, such as adding up from the subtrahend until the minuend is reached. The parts added on are summed and the total is the difference.

Prior conceptual knowledge needed to understand the ‘borrowing’ concept is the notion that all numbers are made up of units which have been grouped into tens, hundreds, thousands etc. Taking a ‘group of ten’ from a larger place value column on the left and moving this group to the next place value column on the right is the process of writing the number in another way without losing its original value. This is known as decomposing the number. The procedural knowledge is the borrowing process. After a group of ten is moved to the next column (it may not skip a column), the digit in the column from which it was taken becomes one less. If there is a zero in

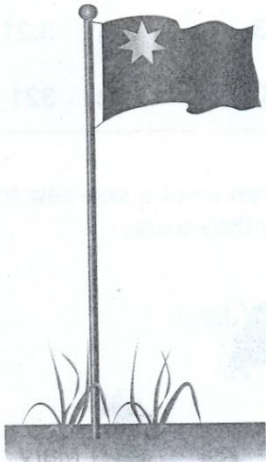
the larger number it is now replaced with a ten. This ten can then give a ‘group of ten’ to the next column if the digit present is smaller than the digit that has to be subtracted. The ten becomes one less and is replaced with a nine. Similarly, if there is a digit other than zero present, the ten is first added to the digit before subtraction can occur. The ‘trade’ in ‘groups of tens’ will result in the decomposition of the 900 in the item to $800 + 90 + 10$. The procedural thinking to choose A as the correct answer would be something similar to saying, “Zero can’t take away eight so I must go to the tens column. There is zero in the tens column so I must go to the hundreds column. I take one from the nine and change it to an eight. I give that one to the zero in the tens column and the zero changes to a ten. I take one from that ten and give it to the units column. I now have a nine left in the tens column and a ten in the units column. Now I can subtract the bottom digits from the top digits.” Brodie et al. (2009) assert, “Learners need to understand and be able to speak the language of place value in order to use the vertical subtraction algorithm with understanding (Brodie et al. 2009:12).

Research has shown that the most common misconception associated with the subtraction algorithm is the erroneous use of the borrowing procedure. It is very common amongst learners to subtract the smaller digit from the larger digit irrespective of their positions (Brodie et al. 2009, Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006). Co-incidental with this misconception are procedural errors arising out of “Borrow-from-zero and Borrow-across-zero” (Fernandez and Garcia 2008:232). For example, $540 - 297$. The zero in the units column may be subtracted from the digit in the subtrahend (because it is smaller) or in a three-digit number with two zeros, e.g. 600, the zero in the units column is changed to a ‘ten’ by means of borrowing but the zero in the tens column is ignored and either added to or subtracted from the digit underneath it. Fernandez furthermore asserts that through remediation, whereby the conceptual constructs involved with the borrowing process are made more meaningful, erroneous thinking can disappear.

Misconceptions in the distractors

Distractor B	Borrowing a group of ten begins in the hundreds column and the digit becomes an eight. The ‘borrowed’ ten is taken across the first zero in the tens column (first misconception) and given to the zero in units column. The second misconception of ‘smaller from bigger’ operates in the tens column.
Distractor C	The borrowing algorithm begins in the units column but because there is a zero in the tens column the learner is confused by the fact that zero cannot be made one less. The misconception of ‘smaller from bigger’ operates in the tens column and the digits are exchanged. The hundreds digits remain as they originally were and are subtracted.
Distractor D	The ‘smaller from bigger’ misconception is evident in all columns.

Item 3 (Convert between units of length)



A flag pole has a height of 3.24 metres.

Which one of these shows the height of the flag pole?

- (A) 3 m + 2 cm + 4 mm
- (B) 3 m + 20 cm + 4 mm
- (C) 3 m + 20 cm + 40 mm
- (D) 3 m + 200 cm + 40 mm

This item is testing the intersection of two domains – the one is decimal numbers and the other is measurement. Mitchell and Horne (2008) draw attention to the fact that when one measures a distance most answers have a ‘bit left over’, the answer to distance measurement is not always a measure of an exact number of whole numbers. “Rational numbers are necessary to describe leftovers” (2008:353).

In the given question, the first focus of attention should be on the 3.24 metres. The unit name given at the end of the decimal number informs the reader about the wholes that are being used. Whole metres are indicated before the decimal point. There are three metres. The ‘extra bit’ of a whole metre is given as .24. Learners need to know that the metre is bigger than the centimetre. There are 100 cm in 1 m. Similarly, the centimetre is bigger than the millimetre and there are 10 mm in 1 cm.

The second focus of attention should be on the fractional part of the number, that is, .24. Knowledge of the place value columns after the decimal point needs to be in place: the ‘2’ represents 2 tenths ($\frac{2}{10}$) and the ‘4’ represents 4 hundredths ($\frac{4}{100}$). The third focus of attention is how to intersect the two domains of knowledge to decompose 3.24 m. As has been stated the 3 is

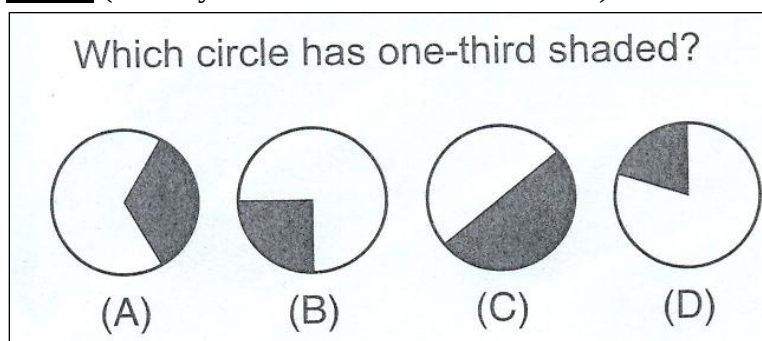
3 m. We know that .24 is the fractional part of a metre. $\frac{2}{10}$ of 1 m is the same as $\frac{2}{10}$ of 100 cm which is 20 cm. $\frac{4}{100}$ of 1 m is the same as $\frac{4}{100}$ of 100 cm which is 4 cm or 40 mm. Pooling all the answers together in an additive form decomposes the 3.24 m into 3 m + 20 cm + 40 mm which is the same as the answer given in C. Thus, not only do learners need prior knowledge of decimal numbers and their place value decomposition, they also need to intersect this knowledge in a given context such as length measurement.

Research has shown that when decimal numbers stand alone, that is, without a given context such as money or measurement, one of the major misconceptions that arises is the sense children give to the function of the decimal point, “Such a student considers a decimal number as two separate whole numbers separated by a dot” (Steinle, 2004:463). Steinle also points out that some errors learners cling to exist as incomplete knowledge rather than incorrect knowledge. The distractors are using the knowledge learners have of units of measurement and the conversion of these different units. The addition of units must ultimately be linked to the fractional parts of the given decimal.

Misconceptions in the distractors

Distractor A	Knowledge of the function of the decimal point is absent and therefore ignored. The number is read as 324. The 3 is the digit in the largest place value column (hundreds) and is linked with the largest unit of measure, that is, the metre. 3 m is added to the 2 which is in the second biggest place value column (tens) and linked with the second biggest unit of measure, the centimetre. The 4 is in the smallest place value column (units) and is linked with the smallest unit of measure, the millimetre. The addition signs have links with incomplete knowledge of expanded notation. ($324 = 300 + 20 + 4$).
Distractor B	The .24 is seen as a whole number separate from the ‘3’. The .24 is decomposed as expanded notation into $20 + 4$. The whole number misconception is dominant and this is the only distractor written in this form. The ‘cm’ and ‘mm’ are inconsequential.
Distractor D	Knowledge of conversions is used to assign a measurement value to the 2. Learners know that there are 100 cm in one metre. The word ‘metres’ is used in the question and the 2 is first assigned the metre as its unit. This ‘2 m’ is converted to the 200 cm and matches what is read in the deflector. The 4 is treated as 40 millimetres because the decimal number has a space at the end of it and this must be filled with a zero. The addition signs point to the use of expanded notation and therefore the ‘ $2 + 4$ ’ is read as $200 \text{ cm} + 40 \text{ mm}$. Seeing that 40 is ‘last’ it can only be written as mm because the other units of measurement have been used. An alternative way of thinking is that learners are influenced by the whole number misconception. .24 is read as $20 + 4$. They draw on prior knowledge when moving from one place value column to the next column on the right of it. They know that this involves multiplying by 10. The $20 + 4$ becomes $20 \times 10 + 4 \times 10$ and the units are assigned to the answers in size order.

Item 4 (Identify a circle with one third shaded)



This item is testing the conceptual part-whole relationship of “one third”. According to Amato (2005) difficulties occur because learners think that a part of a shape is a fraction and not a number. There are two sub-constructs involved in this relationship, “The fractional part and the unit and the idea that the fractional part is that quantity which can be iterated a certain number of times to produce the unit” (Fraser, Murray, Hayward, Erwin 2004:27). The sequence of teaching common fractions comes under scrutiny by Newstead and Murray (1998). They assert that early social and everyday life experiences of sharing should first be extended into the classroom, whereby learners are given problems in which they can devise their own strategies of sharing equally in situations before being provided with geometric diagrams in which parts are shaded, thereby using their informal knowledge to partition “fairly”.

In A, the iteration of the shaded part to make up the whole unit amounts to three. This matches with the ‘3’ in $\frac{1}{3}$ and the name ‘third’. It is assumed that learners have the knowledge that a ‘1’ in the numerator means ‘one part shaded out of three equal parts’ in this item. Prior knowledge must be embedded in the notion of equal sharing of the unit according to the name of the fraction and its denominator.

One of the misconceptions associated with the part-whole concept is that all parts are named as either halves or quarters because of the learners’ everyday exposure to halves and quarters (Newstead, 2000). The concept of a half is rooted in a number system. The representation of fractions as numbers can give rise to serious misconceptions if informal ideas are not monitored. A second misconception in the part-whole concept occurs when the fraction notation is misunderstood and read as two whole numbers with a line in between (Newstead and Murray, 1998). Instead of a fraction in notation form being conceptualized as one number it gets conceptualized as two separate whole numbers.

Misconceptions in the distractors

Distractor B	Learners may have correctly linked the name of one third to its number symbol notation of $\frac{1}{3}$ but the whole number misconception will generate the addition of the 1 and the 3 to produce an answer of 4. They then decide that the shaded region is one out of 4 equal parts. Their informal experience of a quarter (which they might socially know is smaller than a half), could also be a reason for this choice. There is confusion between a quarter and a third.
Distractor C	From everyday life experience of a half used continuously in rich contexts and no knowledge of linking a part with a name other than a half in the mind, this distractor will be a likely choice.
Distractor D	This choice could be made because a quarter is a part that is familiar to the children. They know it has a denominator of 4. The misconception that the denominator is the whole number 4 leads them to reason that 4 is bigger than the whole number 3. (They might know that a third is the same as $\frac{1}{3}$). Therefore the shaded part reflected in the diagram is viewed as smaller than a shaded quarter.

Item 5 (Calculate the area of a given portion based on squares)

Holly drew this shape on 2 cm grid paper.



What is the area of Holly's shape?

(A) 32 cm²
(B) 28 cm²
(C) 16 cm²
(D) 8 cm²

An ability to calculate the area of a 2-dimensional shape such as a rectangle is rooted in the development of both an understanding of the concepts of an array (or grid) of squares and linear measurement. Rows and columns of iterated squares in the array relate to the iterated unit of linear measurement along the straight borders of the enclosed region. Embedded in this connection is the meshing together of two operations - the addition of the squares covering the surface of the rectangle and the multiplication of its length to its width (Battista et al.1998, Outhred and Mitchelmore 1993). Allied to this relationship is the chosen unit of measurement for the dimension of the square used in the array and in the dimensions of the length and width of the rectangle. They have to be identical, for example, 1 cm. Each square of dimension 1 cm has its own area of 1 cm^2 . After learners experience counting iterated squares of any size in a rectangular array, the next step is to count the areas of 1 cm squares and give the answer in cm^2 . Once this construct has been established the relationship between the repeated addition of the squares in the rows and the equivalent idea of multiplying the length and the width (which are also made up of iterated 1cm units) will make more sense.

An understanding of the array does not come naturally to learners. It involves structuring and enumeration of the square units. Battista et al. (1998) assert that a grid consisting of rows with an equal number of congruent squares in each row is first experienced as repeated addition of the sum of squares in each row in the grid. The desired end product of the relationship between the enumeration of the squares in one row multiplied to the number of columns in the array, is the area formula for a rectangle.

Item 5 is an irregular shape composed of one rectangle and two adjoining squares. Approaches to the correct answer can vary. The length of the rectangle is 6 cm and the width is 4 cm. Its area is 24 cm^2 . Each identical adjoining square shape comprises an area of 4 cm^2 . The area of the irregular shape is $24 \text{ cm}^2 + 4 \text{ cm}^2 + 4 \text{ cm}^2 = 32 \text{ cm}^2$. Another approach is to count the squares and multiply the area of each square in the irregular shape by the number of squares that is, $8 \times 4 \text{ cm}^2 = 32 \text{ cm}^2$.

Studies have shown that learners who have not grasped the area concept confuse it with perimeter. They also rely heavily on the formula and sometimes use a non-multiplicative approach if they have not learnt it by rote (Cavanagh, 2007). A square larger than 1 cm used in an array or grid is not decomposed into its own array of 1 cm^2 units and hence there is no relationship made with the length and width of the rectangle that is measured in centimetres. The experience of counting squares in arrays without any further development will lead learners to the misconception that area is a matter of counting all the squares in a given array. Learners who do not experience finding the area of irregular shapes made up of a combination of squares and rectangles, think that the $A=L \times B$ formula must be used to find any area of any 2D shape (Cavanagh, 2007).

Misconceptions in the distractors

Distractor B	The learner confuses area and perimeter. The length of each square is given as 2 cm. The total distance around the shape is 28 cm. The 'cm ² ' in the deflector is ignored.
Distractor C	The learner does not have conceptual understanding of the area formula $A=L \times B$ but has learnt it by rote. The misconception that the area formula for a rectangle must be used for all area calculations is evident. The sense this learner has of the formula is that the L is always a bigger number than the B. The total number of squares is 8 and the length of one square is 2 cm. The 2 is substituted in the formula as the smaller number and the area is calculated as $8 \times 2 = 16$. The cm ² in the deflector is ignored.
Distractor D	The learner has counted all the squares inside the given shape because they all cover the surface of the enclosed region. The learner has no conceptual knowledge of the role of the square cm and the cm ² in the deflector is ignored.

Mapping the five items onto the Revised National Curriculum Statement (RNCS)

One of Shulman's (1987) teacher knowledge bases for teaching is knowledge of the curriculum. All South African teachers are guided by the RNCS on the content that has to be covered in a specific grade. The ICAS test items embed mathematical content knowledge specific to Grade 6, including prior content knowledge from previous grades that is required for choosing the correct answer.

For the purposes of this study each of the mappings in the table below is explained by stating the actual assessment standard provided in the Revised National Curriculum Statement, and the number of teachers in my sample who mapped the same assessment standards as I did. My own mapping was an exercise I performed without having a model of an 'ideal' or 'correct' mapping for each item in my possession. I introduced this task into my investigation, because I was interested to see how closely aligned my content knowledge mapping was with theirs. What they omitted from, or had in common with my mapping might provide me with additional information about their own mathematical content knowledge related to the items. The teachers were given the RNCS document and were asked to find the Learning Outcome and Assessment Standards in Grade 6 pertaining to each item. Grade 6 mathematics is built on prior mathematics constructs from previous grades. The teachers were all acquainted with the RNCS document. I used Item 1 as an example to clarify for the teachers what I wanted in determining the correct mathematical sub-domain and its associated assessment standards. I noted that I was compelled to do this with all six teachers and concluded this was the first time the teachers had engaged in such a task.

The researcher's curriculum mapping and teacher alignment

Item	Curriculum area	Assessment Standards for Grade 6 (researcher)	No of teachers	Assessment Standards in previous grades linked to the development of mathematical content knowledge in the item (researcher)	No. of teachers
1	LO4 measurement: Angles	Recognizes and describes angles in 2D shapes – right angles, angles smaller than right angles, angles greater than right angles	6	Recognizes and describes right angles	5
2	LO1 number: Subtraction of 3 digit whole numbers	Addition and subtraction of 3 – digit whole numbers	6	Addition and subtraction of 2-digit whole numbers	3
3	LO4 measurement: SI unit conversions	Solves problems involving calculating and converting between appropriate SI units	2	Lengths using millimetres, centimetres, metres and kilometres	1
Item	Curriculum area	Assessment Standards for Grade 6 (researcher)	No of teachers	Assessment Standards in previous grades linked to the development of mathematical content knowledge in the item (researcher)	No. of teachers
4	LO1 number: Proper fractions	Recognizes and presents common fractions with different denominators from halves, thirds, quarters to eighths. Common fractions in diagrammatic form	4	Common fractions including halves, quarters, thirds	3
5	LO4 measurement: Area	Area of polygons (using square grids and tiling) in order to develop understanding of square units Formula for the area of a rectangle	5	Area of 2D shapes using tiling	5

The data shows that Item 1 and Item 5 were the items that were mapped by the majority of teachers in alignment with my mapping choices. Three teachers were able to identify the prior content knowledge I had mapped for Item 2. The teachers that could not identify any prior knowledge assessment standard for this item, were not in a position to make the connection that learners need to first master procedural and conceptual knowledge for two - digit subtraction before moving onto three digit subtraction. There is evidence that Item 3 is the item in which

most teachers were unable to recognize the underlying content constructs required for the item, in terms of the intersection of decimal and length concepts. The problem may reside in the fact that decimals are taught in LO1 and length concepts are taught in LO4. Three of the teachers aligned their mapping with mine in Item 4. Two teachers mapped assessment standards for prior knowledge as the main assessment standard, and one teacher could not identify any prior knowledge assessment standard.

All the teachers were able to identify the Learning Outcomes in the Mathematics curriculum for each of the items and state the relevant mathematical sub-domain. Although the teachers were able to map some of the assessment standards it is evident that they are not aware of the importance of all the prior content knowledge constructs that are needed for answering the items successfully. An example of this is item two. In two digit subtraction the learner has to know how to subtract from a number ending in a zero. If learners cannot master the mathematical process behind this operation they will experience difficulty subtracting from a number ending in two zeros.

Summary:

An analysis of the items and their associated misconceptions suggests that a teacher's knowledge embraces more than knowing how to do the mathematics. Misconceptions in the distractors clearly indicate that learners do evidence misunderstandings in mathematical domains and these are not alluded to in the RNCS document. By omission, it appears that misconceptions are not expected. Also, the assessment standards do not give teachers manifold representations of the kinds of tasks that learners are expected to do, in which the assessment standards are embedded (Lampert 1991). A teacher's PCK linked to his or her content knowledge is vitally important, without this knowledge they may not be able to identify when a learner has made a careless error, or if there are deeper misconceptions that are demonstrated in a learner's mathematical behaviour.

CHAPTER 5: FINDINGS

Introduction

My research aims to investigate Intermediate Phase teachers' PCK by examining their reasoning about learners' errors and misconceptions and the ways they proposed to address these through tasks that can serve to eradicate learners' misunderstandings. Before I interviewed the teacher on the specific tasks, I asked them to reflect on how their teaching has changed over the years and they expressed a change in their pedagogy.

In their reflections (See Appendix B) regarding their early days in the classroom, the teachers used phrases and words such as 'recipes', 'rote learning', 'drill', 'talk and chalk', 'regurgitate' and 'repetition'. For example, Betty states "Now we are doing more practical stuff where we help learners to have a more hands on approach to maths rather than using rote learning or expecting children to learn recipes and talk and chalk". Dawn adds, "Before it was a case of showing children everything and they had to regurgitate everything". Today they all acknowledge a paradigm shift in their pedagogy – learning is enhanced by other resources besides a textbook, that is, peer teaching, an 'openness' to the notion that the learners' thinking and opinions count in the learning process. Conceptual growth demands more attention and a variety of different methodologies to promote understanding. The slower learner is recognized and not overlooked, and the idea that learning mathematics can be fun is more prominent in how teachers view their teaching today. For example, Angie says, "I allow them to play mathematics games and they don't realize they are learning through these games". The over-all impression one gets is that the teachers are treating their learners as a community in which individual differences are recognized and accepted, and it is evident that learning mathematics has developed into a combination of teacher and learner contributions and a wider use of resources.

The teachers claim that because of time constraints due to departmental policy pressures (excessive paperwork demands and numerous workshops), curriculum pressures, school activities and big classes, they do not always have the time to discuss common mathematical problems experienced by learners in the same grade. One teacher said that she sits with teachers in the same grade as herself and talks about why learners are not attaining mathematical grade levels, and they look at what was achieved in previous grades to try and find the source of this problem. The teachers individually try to remediate and help learners in their classrooms with mathematical problems that surface in tests and written work. They do not receive outside inputs from more experienced authorities on mathematical errors and misconceptions, unless they read more about them and this is not encouraged by subject heads. Only one teacher said she gains more knowledge about errors and misconceptions in area meetings and from reading.

In the interviews, I probed their reasoning about learner thinking in terms of the errors and misconceptions embedded in the given distractors (see Chapter 4 on Item Analysis) and thereafter I asked the teachers for the kind of instructional interventions they believed would address their learners' erroneous thinking. In order to answer my research questions, I decided to frame my analysis of the teachers' responses around four themes of teachers' knowledge that I consider to be pertinent to my study. (The themes are all equally important and therefore are not ranked).

Theme One: Knowledge of mathematical content relevant to the items and use of procedural or conceptual thinking.

Theme Two: Language use when explaining mathematical concepts.

Theme Three: Awareness of errors and misconceptions in the items.

Theme Four: Interventions to address perceived misconceptions.

The interviews I conducted with my sample of teachers show evidence that the four themes deserve consideration. In my own teaching experience, I have established that these themes have affected my own practice over the course of considerable years. My research into what other authors such as Shulman 1987, Lampert 1991, Carpenter and Fennema 1991, Fennema and Franke 1992, Ball 1998, 2001, 2007, Ma 1999, Ball and Bass 2000, Adler 2005, Kazima 2008 specify on content knowledge, PCK and mathematical knowledge for teaching, suggests that these themes can contribute to an analysis of teachers' reasoning of learners' errors and the misconceptions that underlie them.

In addition, I want to indicate that not all the themes will be considered for each of the five items. I chose to look at the data across the five items and discuss themes which I consider to be applicable in terms of the teachers' reasoning for this study.

Themes

Theme One: Knowledge of mathematical content relevant to the items and the use of procedural or conceptual thinking

This theme was covered by asking the following questions:

- In order for a learner to answer this item correctly, what prior mathematical knowledge needs to be present?
- What mathematical understanding is required to choose the correct answer?

Evidence for the scope and depth of the teachers' content knowledge in each item's sub-domain is the teachers' understanding of the prior knowledge required for that particular content. It must be noted that sometimes teachers answered my questions according to their *own* knowledge of the content and explained what they thought the learners were thinking about the items. Such

evidence can be read when teachers use the word “they” (meaning the learners) in their responses. Whether the teachers answered me directly or indirectly through what they considered to be learner thinking, I was able to acquire insight into their content knowledge.

Item 1 (Differentiate angle size from orientation and side length)

It must be noted that this item involves conceptual thinking only. The arc was foregrounded by three of the teachers as representative of the size of the angle and can be used as a means for comparison with other angles. For example, “Know that the size of the angle is indicated by an arc” (Dawn), “A wider arc means it is big” (Ella) and “They need to understand that little arc. What the meaning of it is” (Angie). The concept of angle as representing the notion of an amount of turn or rotation was mentioned by three of the teachers. Betty explained, “Rotation means a circular movement, clockwise and anti-clockwise. This is how an angle is formed.” Carla said, “The arms are moving in a clockwise direction which is a rotation and we are measuring the amount of turn” and Ella added, “Angles show an amount of turn (the arc). The smaller it is, the smaller the angle.” Fran and Carla also mentioned the idea that an angle is formed where two lines meet at a vertex. Angie added furthermore, “I would say sorting but that is the classification of angles”.

In order to choose angle C as the biggest, Carla, Angie and Betty used the right angle as a benchmark to estimate which angle is closest to it in size. Carla said, “If they are able to see a right angle they are able to see a perfect L and the one arm that is nearest to the L will tell them it is bigger in size”. Betty stated, “It looks like a right angle. A lot of them have been taught to identify angles and to look at the rotation. This angle is closer to 90°”. Angie explained, “They could have looked at one that is closest to 90° to decide which one is the biggest.” A useful strategy of using the right angle as a benchmark for choosing the largest angle is clearly evidenced in these statements. Dawn, Ella and Fran noted that the arc of angle C is the biggest.

The item shows four acute angles in different orientations. The angles also have different arm lengths. This can pose problems if learners have misconceptions in these two areas (Barrett 2003, Mitchelmore 1998, Magina 1997). The notion of angle orientation was mentioned by Angie only, “It doesn’t matter in which direction you place an angle. That does not determine the size of the angle”. Apart from the arc as representative of the amount of turn and hence size of an angle, arm length and angle orientation are also connected to content knowledge of angles.

The teachers’ content knowledge relevant to the item is accurate but incomplete. The angle with the longest arms was chosen as the biggest angle by the majority of the learners in the ICAS test. As evidenced, the teachers did not view it as important to mention that learners need to be taught about the irrelevance of arm length as indicative of size (a misconception). I consider this observation to be important because the arms are two straight lines that catch the attention of the learners. The teachers’ emphasis on the arc is important when comparing angle sizes and their

responses showed evidence of this. What they all missed concerning the arc construct is that it is irrelevant where it is placed in the angle, it can be close to or far from the vertex. Some learners may look at the size of the arc and perceive that the angle is bigger if the arc is bigger than the other arcs given in a group of angles (a misconception). Teachers preferred to use the right angle as a benchmark for comparing angle sizes but this is not a key issue mentioned in the literature. Angle orientation was not considered important by the majority of the teachers even though different angle orientations can lead to errors and misconceptions concerning angle size. In light of the fact that the teachers recognized certain misconceptions in the distractors and omitted others, I argue that they are not cognizant of all the errors and misconceptions associated with the angle concept.

Item 2 (Subtract three-digit numbers)

Algorithms are procedures used for doing a mathematical calculation. For example, the standard algorithm that is used in the subtraction operation is commonly known as the ‘borrowing method’. Other algorithms use the notion of partitioning a number into its place value components (expanded notation) and the ‘same change’ or ‘change and compensate’ algorithm adds to or subtracts from both given numbers to make the subtraction process easier.

Four of the teachers, Carla, Dawn, Ella and Fran stated in their responses that the prior knowledge needed for this item is that learners need to know how to borrow. For example, Carla says, “In this case, you are having a double zero in a number and so you need to understand the concept of borrowing” and Fran states, “They must know how to borrow from the zeros from one number”. Dawn and Ella did not mention the zeros but they linked borrowing to increasing the number at the top to enable the subtraction process. Dawn said, “Always subtract from the top down and if the larger digits are at the bottom you are going to use the borrowing method.” Ella explained, “Borrowing helps us to increase numbers that we have to take away from because we always take away from a bigger number.” All the teachers described in detail what procedure is used to subtract the two three- digit numbers. Ella described the borrowing procedure as,

900 is a big number. I don’t have a unit and I don’t have a ten. I have to borrow one from the hundreds. But I can’t take it and run to the units. I have to go back the way I came. So I take that 1 and write it before zero. Now I have the value of 10. But I haven’t reached my destination. Then from this 10 I borrow one and am left with 9. I take the one to the units. then I have 10 units. Now I can subtract.

Each of the four teachers explained the borrowing method correctly but it is interesting to note that Dawn and Fran introduced the notion of expanded notation when they talked about splitting the borrowed 100 into $90 + 10$. For example, Fran said,

They knew that the larger number is above the smaller number. They said $0 - 8$ you can’t do so they had to borrow from the hundreds. They must know that they have to share the 100 between two. Know what to give to the units and what to give to the tens. They give

10 to the units which leaves the tens with 90. They then subtract each column.

Dawn explained,

He can't subtract 8 from 0 so he has to go to the tens to borrow. The tens also has a 0 so he has to go to the hundreds. He will borrow a 100 from the hundreds and there is less remaining so now there is only 800 left. He takes the 100 to the tens and he needs another ten to go to the units. When he takes a 10 from the tens 90 will remain the '9'. He now has the 10 to subtract the 8 from to get 2 and the 5 from the 9 gives 4 and the 3 from the 8 gives 5.

Angie and Betty used the idea of 'breaking up numbers' or expanded notation which is linked to an understanding of place value and is different from the standard algorithm. Angie said, "The child must be able to break up numbers and build them up again" and Betty stated, "They also need place value and expanded notation which is breaking down of numbers". She later qualified what she meant – use expanded notation and subtract $900 - 300 = 600$. $600 - 50 = 550$. $550 - 8 = 542$.

Angie introduced another procedure for subtracting three digit numbers. By rounding the subtrahend she used another algorithm called 'change and compensate' or 'same change'. She explained,

Mentally change the 358 to 360 and it's easier to subtract that from the 900. You take 60 from 100 and there is 40 left over. There is 800 left. 300 from 800 is equal to 500 and the two that you added to 58 to get it to 60 you add to your answer to get to 542. I teach them to write it out in expanded notation because the "borrow" I was taught at school does not teach them the concept behind it. They still don't understand what they are doing. That is a recipe they follow.

Betty preferred to expand the subtrahend into the sum of its place value parts and subtract each part from the minuend separately.

Research has indicated (Brodie et al. 2009, Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006) that the borrowing method is responsible for errors and misconceptions in terms of 'borrowing from one or across two zeros' which is the mathematical procedure required for this item. It is evident from the responses that application of the standard algorithm is the most popular method used by the teachers for the subtraction of three digit numbers. The fact that other algorithms are not used by the majority of teachers is not incorrect.

Angie and Carla also mentioned knowing about place value, but this knowledge has little bearing on the borrowing process because a group of ten is taken from the column to the left irrespective of its place value column name. Many of the learners were confused about the two zeros and chose to subtract the smaller digit from the bigger digit in the columns. Two teachers, Angie and Betty, were able to use another algorithm for subtraction. The borrowing method is not the only

method that learners need to know in order to subtract three digit numbers. From what the literature says (Brodie et al. 2009, Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006), if the borrowing method is taught as a ‘fait accompli’, the errors and misconceptions that may arise derive from an inability to make sense of each of the steps.

Item 3 (Convert between units of length)

This item requires conceptual knowledge of unit conversions and the role of the decimal point when working with different SI units. Common ideas that emerged in all the responses are firstly, that one needs to have knowledge of metres, centimetres and millimetres and secondly, that one needs to know how to convert from one unit to another. Angie further noted (with reference to place value) that, “They would also need to know where the zero point is and the difference between the m, cm, and mm and the relationship is 10, 10, 10.” Betty believed that prior knowledge of decimals and place value is important for this item and she mentioned the intersection of a decimal number with a unit of length – “the 2 is 2 tenths of a metre and the 4 is 4 hundredths of a metre.” Without this latter construct, the item’s digits cannot be decomposed into their associated length unit values and this is what the item is ultimately testing. Angie and Betty stated that learners need to be able to envisage that the number before the point is associated with the unit given in the question – “The 3 before the point is 3 metres,” (Angie) and Betty adds,

3 stands as the whole number and is 3 metres. When we teach place value they need to know that whatever lies after the comma that is the fractional part of the metre because the height is represented as metres in the question 3.24 metres. They need to know what a metre is.

Ella and Fran spoke about working with a common unit - “There must be a common unit to work with so all different units have to be converted.” (Ella). She explained, “They converted first because they couldn’t add the different units together. They converted all the units to millimetres and thereafter they divided by 1000. They have to take it back to metres because the answer is in metres. Fran chose centimetres – “They are converting the 40 mm to 4 cm and they are adding that to the 20 cm which makes it 24 cm. The 3 m is 300 cm which makes it 3.24. The height has to be in m.” Although this approach is correct it does not depict how the decimal can be analyzed in terms of its length components.

Dawn used a process of elimination by trial and error – knowledge of conversions is used to prove that the given units in the distractor are correct – Dawn explained, “It is 20 centimetres because they know that it can’t be 200 centimetres because 200 centimetres equals 2 metres. It can’t be 2 centimetres because that would be 2 spaces away from the decimal. In that case it must be 20cm”.

Five of the teachers in the sample did not connect the place values after the point in the decimal number with their corresponding SI units. Carla attempted to do this when she spoke about the

place value connection after the point with centimetres and millimetres, but she too, did not make the correct connection. She equated the first place after the point with centimetres and the second place with millimetres. The first place after the point given the whole as metres is decimetres. The second place is centimetres and the third place is millimetres. From what Carla has evidenced, confusion and misconceptions can develop because decimal knowledge is learnt separately from SI units and conversion knowledge in the curriculum, and learners need to experience a combination of both. The teachers were unable to explain why $1 \text{ mm} = 0,001 \text{ m}$ or why $1 \text{ cm} = 0,01 \text{ m}$. Without this prior knowledge and experience, learners might ignore the point in 3.24 m completely and treat the digits as whole numbers. Although Dawn and Ella arrived at the correct answer by converting to a single SI unit, the question remains, what would they have reasoned if they were given 3.24 m (without the distractors) and asked to decompose the fractional section of the number into its SI unit components?

Item 4 (Identify a circle with one third shaded)

This item is involved with conceptual knowledge only. All the teachers said that learners need to have conceptual knowledge of the part-whole relationship, that is, the idea that the whole is divided into smaller parts. For example, Fran said, “The piece is smaller than a whole number.” Allied to this understanding is that all the parts must be equal (stated by all the teachers). Betty, Carla and Ellen mentioned the notion of equal sharing, “Dividing into equal parts and sharing.” (Betty), “A third means three equal parts” (Carla) and “Know how to share equally amongst the number of people” (Ella). Dawn and Ella included knowledge of the function of the numerator and denominator as prior knowledge required for this item:

The denominator is the bottom number and shows you into how many parts the whole has been divided. The bigger the denominator the smaller the fraction is (Dawn). The top number tells you the number of shaded parts and the bottom number tells you into how many parts the whole is divided. (Ella)

Carla and Angie acknowledged that the naming of fractions and estimating fractions as iterated units in a whole is important – Carla explained, “They need to visualize the third. Look at the shape of the shaded area and does the other side (unshaded) look like two thirds and estimate and see if this fits into the circle.” Angie said “In the case of a third, three sections. They must see that two more of one of the sections will fill up the whole circle.” The statements show a bias toward knowing about dividing a continuous whole equally as an aspect of the content knowledge of fractions. For this item, such knowledge is pertinent given that a whole circle is used in the item. This does not mean that the knowledge of the teachers is limited to continuous wholes and does not incorporate discrete wholes.

The Grade 6 teachers in my sample were unaware how fractions are developed in the Foundation Phase at their respective schools. All the teachers except for Dawn identified the connection between the name ‘third’ and the three in the denominator of a fraction. Once again this does not indicate that Dawn has no knowledge of this connection. In order to choose A as the correct

answer a learner needs to be able to estimate whether the shaded part shown in the circle is one of three equal parts. Apart from Ella, the other teachers argued that learners would have to distinguish which circle contains the one-third piece – “Need to visualize the third. Look at the shape of the shaded area and does the other side (unshaded) look like two thirds and estimate and see if this fits into the circle.” (Carla) and “They need to be able to break the whole up into smaller sections and equal sections. In the case of a third, three sections. That two more of one of the sections will fill up the whole circle.” (Angie). Betty and Dawn used prior knowledge of a half and a quarter as benchmarks to estimate if the size of the shaded part lies between the two. They concluded that A shows one third. Betty explained, “Knowing that one half is greater than one third and one quarter is smaller than one third” and Dawn added, “They will see that the half shows exactly two equal parts. Cut both halves in two again and it will give you four equal parts. They see a size shaded smaller than a half but looks bigger than a quarter.”

In general, the teachers’ responses show that their knowledge of the fraction concept is in place in terms of the part-whole relationship and the notion of equal sharing. Betty, Dawn Ella and Fran did not mention visualizing a given fraction as that part of the whole which can be iterated to make up the whole. This knowledge is extremely important for the identification of fractions and key to choosing the correct distractor in this item (Fraser, 2004). Angie, Betty, Carla and Fran did not mention the relationship between the numerator and denominator as part of knowledge required for the fraction concept. In the Foundation Phase, knowledge of a half and a quarter is made explicit. None of the teachers stated that knowledge and experience of sharing a whole into three equal parts in contexts other than diagrammatic form (such as the given circle) is important. Without these experiences, learners fail to see the difference between a third and a half or a third and a quarter, which are meaningful everyday experiences for learners (Newstead and Murray, 1998).

Item 5 (Calculate the area of a given portion based on squares)

Carla, and Ella connected the area concept with the notion of covering a surface. For example, “Area is a surface that has to be covered with squares that have to be tessellated” (Carla), “The area is the amount of space that needs to be covered” (Ella). Dawn explained that area is the action of using the area formula – “They must multiply two different sides of a square or a rectangle. Your answer is going to be in square metres or square centimetres.” At the end of her response Ella added, “Formula is important”. It is evident that the teachers made two distinctly different connections with the area concept – covering a surface and use of the area formula. The former connection is closest to the area concept while the latter connection is a procedural vehicle for finding out the area and is not used in Grade 6. The formula is introduced in Grade 7.

Each response used a different strategy to arrive at the correct answer. A common feature in all the responses evidenced knowledge of a common mathematical construct – the area of one 2cm by 2cm tile needed to be known, that is, 4 square centimetres. Betty, Dawn and Fran used a

procedure that involved counting the tiles and thereafter multiplying the total by 4, “Each block has been chopped into 4 square cm. 8 blocks x 4 sq cm = 32 sq cm.” (Betty). Fran said, “They counted 8 tiles. They said 4×8 is 32 because length times breadth is 4, that is 2×2 because a tile is a square.” A different strategy used a subtractive approach with the formula, “If you multiply 6 by 8 you are going to get 48 and then from the 48 you take away 2, 4. It is 4 times 4 which is 16. 48 take away 16 is 32 square centimetres.” (Ella). An additive strategy was used by Angie who explained,

Imagine the two little blocks on the side are not there. Each square is 2cm and so each square is 4 square cm. 6×4 is 24 plus the 8 on the sides gives you 32. They must be able to multiply the length by the width to get the area. They need to know the formula or understanding how to get there if you don’t know the formula.

The responses reflect the assertions of Outhred (1993) and Battista (1998), that area connects two operations together – addition and multiplication. Addition embraces the notion of counting all the square tiles and multiplication embraces the notion of multiplying the length to the breadth of one tile.

All six teachers were able to apply their content knowledge to thinking about strategies they could use to arrive at the correct answer. None of the teachers said that area is a measure for the surface of a 2 - dimensional shape. Similarly, they did not evidence explicit knowledge required to further understand the area concept - area measurement uses tessellated congruent shapes such as squares (a more convenient shape because the length of a square unit can be lined up against the iterated units of the length and breadth of a rectangle). The iteration of the squares forms a grid and are counted within the grid. Knowledge of the area formula for a rectangle is then developed in Grade 7 once the area concept is embedded. I suggest that without conceptual knowledge of area, learners in Grade 7 will come to rely on the area formula (in Grade 7) for a rectangle (learnt by rote), which is a calculation using linear dimensions. This may be the root cause of the misconception that perimeter (which also uses linear dimensions) and area is one and the same thing in some of the learners’ thinking. The evidence suggests that teachers have content knowledge to arrive at the correct answer but some rely on the formula for the calculation as opposed to using the area concepts embedded in the Grade 6 assessment standards.

Teachers’ thinking about how area is mathematically conceptualized is evident in their responses. They all mentioned square units that cover the surface of a 2-dimensional shape as a key knowledge construct for determining its area. For example – “You can count the squares if the side of the square is equal to one square centimetre. Each of the bigger squares on the grid paper contains four smaller squares” (Angie), “Square units are units for area and each tile can be cut up into four square units (Betty) and “The surface of the diagram has to have squares” (Carla). Added to this construct, Angie, Carla, and Ella stated that learners need to know the

formula for the area of a rectangle – “The formula is the length times the width” (Angie), “Know the formula $l \times b$ ” (Carla) and “Area is length times breadth and uses squares” (Ella).

Angie, Carla and Ella mentioned that the rectangle formula is knowledge that is needed for the area of the shape in the item, but they did not elaborate on any conceptual connections with the grid of squares. The formula is derived from counting the number of squares alongside the linear dimensions (length and breadth) of the rectangle then multiplying these together. This is made possible because the grid is composed of iterated rows and columns with the same number of iterated square units in each. Their responses suggest that learners should know the formula by rote which then becomes a procedure to be used for area. Without embedded knowledge of this construct, the formula for a rectangle is meaningless and is used procedurally for area calculation. This can lead to misconceptions between calculations for area and perimeter (Cavanagh 2007). Only one teacher, Dawn, was concerned about the confusion learners evidence between the area and perimeter concepts but her explanation of these concepts is mechanical – “The answer for perimeter will be in centimetres and not square centimetres”. This statement does not refer to the fundamental difference between the two, that is, in perimeter, iterated length units are counted around the border of the shape and for area, iterated areas of square units are counted in the array inside the shape. Therefore the perimeter answer is given in centimetres and the area answer is given in square centimetres.

Summary:

The main aim of this theme was to investigate how aware Grade 6 teachers were about the embedded prior content knowledge constructs a learner has to have in place before the learners can attend to the mathematics in the specific items. I found that the teachers evidenced a mix of procedural and conceptual knowledge but most of their responses indicate that they tend to think procedurally. Here I allude to their reliance on a formula for area and the borrowing method for subtraction. They were able to make conceptual links with area but their conceptual reasoning lacked depth. The same can be said of their angle responses. They concentrated on the arc as key to conceptual understanding of angle but this was confined to a 2D diagram instead of broadening the notion of turn to real life examples.

With regard to the items themselves, the reasoning of the teachers integrated PCK with content knowledge in their explanations for choosing the correct answer. They were able to link their own knowledge constructs to mathematically viable strategies when they explained the correct answer in the items. The only item in which I detected some discomfort was Item 3, and this may have been prompted by the fact that they may have insufficient experience with the synthesis of different SI units into one decimal number.

Theme Two: Language use when explaining mathematical concepts

Mathematics classes are taught through the medium of English and I argue that some teachers struggle with mathematical explanations, that is, they may have knowledge of the mathematical

content or concepts, but the language they use to explain mathematical ideas may be incorrect. As a consequence, their learners inadvertently receive a mathematical message that can lead to misunderstanding and confusion. In support of my argument I have chosen to identify words, phrases or sentences used by the teachers in two of the items (Item 1 and Item 5). In my analysis of each of these items I first state the context of the item, then what the teacher said and thereafter, why I consider that such utterances could lead to confusion, particularly where conceptual knowledge is at stake. It must be noted that not all the teachers used inappropriate language in their explanations in all of the items. I have identified only those statements that may result in confusion or errors and misconceptions and are in conflict with language that learners may have learnt previously (from other teachers, socially, or from textbooks). I am conscious of the fact that the teachers were speaking to the researcher as someone who possessed knowledge. Nevertheless, consider that the language they used in the interview can point to some advantageous findings about teachers' knowledge.

Item 1 (Differentiate angle size from orientation and side length)

The teachers explain the significance of the arc and angle orientation:

Betty said, "An arc is the *area* from one point on a line to the other line, the width". Dawn explained, "It shows from where to where the angle is *stretched*." She later added, "The *space* is all that is between the legs." Fran said, "The arc indicates how *wide* an angle is" and Angie stated, "That learners need to look at the *length* of the arc in between those two legs." She also made reference to the different orientations of the angles in the item - "Even though B is twisted and stands on its head it is smaller than C."

The item shows four angles each having two straight lines which meet at a common point. Each angle has its own arc which is a mathematical symbol used to show an amount of rotation. The arc, in the above statements, is linked to the words 'area', 'space', 'wide', 'length' and 'stretched'. The question arises – does an angle have an area? Area is the measure of the interior of a bounded surface. If the two straight lines of an angle are connected with a third line to form a triangle, the answer to the question is in the affirmative. The interior region of this triangle can be measured in square units. This is the area concept that is taught in the curriculum. The teacher has inadvertently used one concept (area) to describe another (the amount of turn). If one considers the notion of 'space', it could be argued that space is all around us in a three-dimensional world. The arm of an angle rotates through this space and the arc indicates how much turn is made. This amount of turn can be measured in degrees in a two-dimensional plane. There is 'space' between the two straight lines but all the space between the lines from vertex to endpoints of the line segments is not what is represented by the arc. One can connect the notion of space measurement with area (2D) or volume (3D) but not with an amount of turn. The words 'wide' and 'length' are associated with linear measurement in which iterated units are used to measure the distance from one endpoint to another. If one were to connect the endpoint of one straight line of the angle with the endpoint of the other straight line this new line would

have a length (or width). As one moves closer to the vertex and joins points in the middle of each straight line to one another, the length (or width) becomes less. Teachers who connect these ideas to the arc may inadvertently, through their classroom language, exacerbate the misconception that the length of the arc (which is a curve that joins one end point to another) indicates the size of an angle. If the arc is drawn connecting the endpoints of the line segments it will definitely be longer than if it were drawn closer to the vertex. The latter would produce a shorter linear measure and lead learners to the misconception that the bigger the arc, the bigger the angle. The word ‘stretched’ is also associated with the concept of length. Implicit in the meaning of ‘stretch’ is the notion that a line is ‘getting longer’ (as with a piece of elastic that is stretched). Learners who are familiar with this word and its implicit meaning may assume that the longer the arms of the angle the bigger it is in size (a popular misconception).

Item 5 (Calculate the area of a given portion based on squares)

The teachers explain the meaning of area:

The inappropriate use of the words ‘area’ and ‘space’ has been discussed in the way language was used by Betty and Dawn in Item 1. In Item 5, teachers were asked what is meant by area. Ella explained, “The area is the amount of *space* that needs to be covered” and Fran said, “Area is the measurement of *space* inside a shape”. I previously made mention in item 1 that area is the measure of a surface. Informally people often refer to a 2-dimensional bounded region as a ‘space’, but teachers need to speak accurately when they are making sense of a concept such as area. ‘Space’ is connected with a 3-dimensional notion whereas ‘surface’ is connected to a 2-dimensional notion.

Fran’s language can confuse learners with the concept of volume. We live in a 3-dimensional world, the social meaning that learners attach to the phrase ‘inside a shape’ may well conjure up an image of a 3-dimensional object that has an outside (the surface) and an inside, which can be hollow or solid. This understanding is linked to the notion of volume, which is the measure of the space occupied by an object and it is likely that Fran’s language could, in the learners’ future, lead to a misconception between area and volume (Barrett et al., 2003).

Summary:

The teachers in my sample have had numerous years of experience teaching mathematics through the medium of English. In my analysis of their language, it has come to the fore, that in their struggle to convey mathematical meaning in English, misconceptions can inadvertently develop irrespective of the quality of their content knowledge. Had I not witnessed the teachers’ content knowledge being correctly applied to those items, their general struggle with language about the meaning of area might have lead me to conclude incorrectly, that their conceptual content knowledge is shaky.

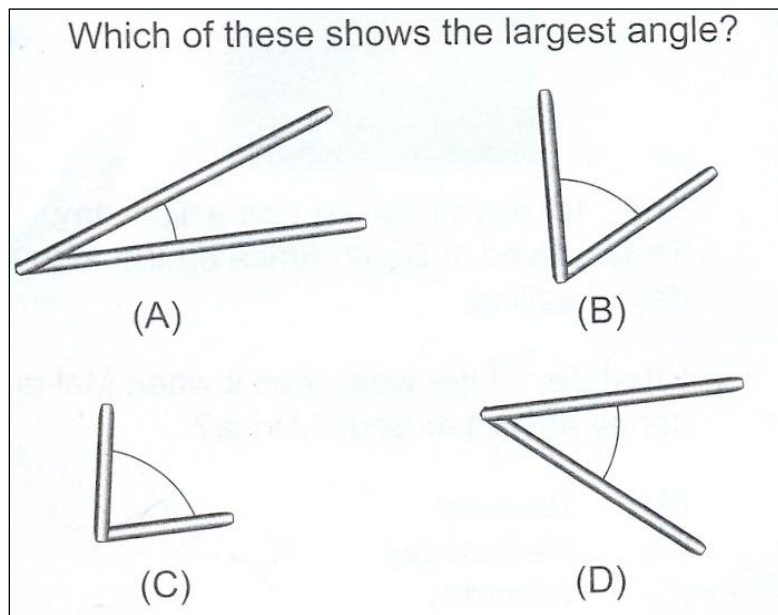
Theme Three: Awareness of errors and misconceptions in the items

The teachers' perceptions of misconceptions and errors in the items were elicited by asking them two questions:

- What are the incorrect distractors testing?
- Why was distractor (I name the distractor) chosen by the majority of the learners?

The teachers were furnished with percentages from the ICAS test results of learners who chose each incorrect distractor. They had to attempt to ascertain what the learners were thinking and reasoning.

Item 1 (Differentiate angle size from orientation and side length)



Angie, Betty and Fran expressed the notion that the orientation of angle B made it 'look bigger' than the other angles. Angie said, "It's not in the normal position we normally teach". Betty stated, "B looks similar to C because it is pointing in a different direction" and Fran said, "It looks very similar to C except that it's in a different position." Carla and Dawn offered an alternative suggestion for choosing angle B – "Their estimate of the space looks bigger at the end points" (Carla) and "He looks at the space at the end of the lines and sees it as the biggest" (Dawn).

According to Angie, angle D was chosen because, "The angle is below. It looks as if the angle is hanging down from the line and that could have confused the child because they are used to seeing the arc above the horizontal position. Dawn offered a different reason for angle D – "He looks at the size of the lines and sees the whole drawing as being bigger (as a totality). A child that is right eye dominant will see D as the bigger one immediately."

All the teachers, except Carla, decided that angle A was chosen as the largest angle because it has the longest arms. For example, Ella echoes what the teachers thought about angle A – “The longer the arms the bigger the angle is”. A further point was made by Angie and Betty – “The children looked at the angle on the outside (without the arc) and that is by far the biggest angle” (Angie) and “They might have looked at the other side of the angle and seen the biggest angle” (Betty). Dawn qualified what she meant by right eye dominance when she spoke about angle A – “the size of the legs influence the decision. A right eye dominant child might choose this as the biggest.”

An interesting observation made by Angie and Betty is that the learners put an arc on the ‘other side’ of an angle and include it in the group of angle to be compared. The group is given as a set of acute angles and if the ‘missing arcs’ on the other side of the vertex is mentally drawn, the group becomes a set of four acute angles and four reflex angles. If learners do include the reflex angles in the group it is an error rather than a misconception, which can easily be rectified in the classroom, by acknowledging that there are two angles involved. The learners should focus on the angle with the given arc. According to Dawn, a right eye dominant learner will look at the angles in the item and the right eye will sweep to the right and note that angle A has the longest line segments and angle D has the largest ‘area’.

The teachers appeared to be confident about what learners think if they choose the distractors with misconceptions. It is evident that they need to become familiar with all the main errors and misconceptions and deepen their understanding about the problems learners have with the angle concept. They were aware of the misconception that the angle with the longest arms in the biggest angle, but none of them mentioned the misconception that the closer the arc is to the vertex, the smaller the angle looks. The misconception linked to angle orientation was not acknowledged by all the teachers in the group.

Item 2 (Subtract three-digit numbers)

$900 - 358 =$		?
(A)	542	
(B)	552	
(C)	642	
(D)	658	

With reference to distractor B, teachers were able to establish that learners struggle to borrow across two zeros - the learners are able to borrow once but not twice in the same calculation, for example, “They knew how to borrow the 10 from the 100, and they gave it to the units. $10 - 8$ is

2. After that they forgot to give the rest over. They forgot to give that 90 to the tens and said $0 - 5$ is 5. They changed the 900 to 800.” (Fran), Carla said, “They borrowed from the tens and didn’t reduce it. They borrowed from the 9 and reduced it and forgot the middle one.”

Responses given for choosing distractor C were varied. “The 9 was not changed to an 8 and they turned the other zeros to a 10. They changed the digit in the tens position to a 90. That’s why they got 642” (Angie), “They still borrowed. They gave the units and they gave the tens but they forgot that they borrowed and said $9 - 3$ is 6” (Fran) and “ $58 + 42$ gives me a double zero. They used an inverse operation for the double zero and only subtracted with the 9.” (Carla). Betty explained, “The item is testing borrowing and carrying. They did remember part of it. They might have said $0 - 8 = 8$. Perhaps they added also. They don’t understand that you can subtract a whole number from nought. They just said $9 - 6$ is 3. $0 - 5$ is 5.”

The teachers explained why learners chose distractor D. Angie said, “In earlier grades they were taught that you cannot subtract more from less so they just swapped the two numbers around and zero from 8 is 8.” Carla stated, “They are doing it backwards. In Grade 1 they are taught the big number must subtract the smaller number and never that the smaller number must subtract the bigger number.” Dawn added, “They saw the opportunity to subtract but did not use the borrowing method and subtracted the top digit from the bottom ones because there is nothing on top. The basic knowledge is that we always subtract the small number from the big one.”

When learners subtract, the borrowing algorithm can be linked to misconceptions which result in learners making procedural errors such as taking the smaller digit at the top from the bigger digit at the bottom. A number that has one zero at the end causes confusion and they may add the digits in the column. Alternatively, if the bigger number at the top contains two zeros learners don’t know how to borrow across them and they may resort to adding the digit at the bottom with the digit at the top (Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006).

Fran, Ella, Betty, Carla and Dawn recognized the errors learners make when borrowing across two zeros. They were able to reason that the learners knew they had to borrow from the 9 in the hundreds column in order for the zero in the units column to change to a ten. Their explanation as to why the zero in the tens column is not changed to a 9 is inadequate. According to Carla and Fran, the learners “may have forgotten to do this”. ‘Forgetting’ puts the blame squarely on the learners’ shoulders. The statement has come from the teachers and indicates that the teachers have missed a misconception in the borrowing procedure. Only Ella felt that the error is a result of the learner ‘missing the borrowing concept’ but she gave no further explanation to mathematically justify this statement in terms of why that zero changes to a 9.

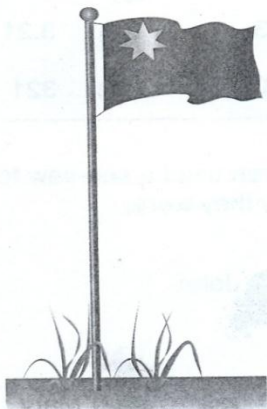
The teachers were able to identify the mathematical error in distractor C. They all said that the learners took 3 away from 9. Except for Carla, they acknowledged that the learners ‘borrowed’

in order to subtract from the zeros. None of the teachers suggested that the learners subtracted from left to right and when they came to the zero in the tens column, they may have exercised procedural thinking whereby ‘a zero always changes to a ten’ in order to subtract. Thereafter the units received a ten from the tens column which left 9 tens at the top. Carla’s thinking about “working backwards to the double zero” by adding the digits in the answer to the minuend is procedurally correct, but she did not elaborate why the learners only worked with the tens and units digits, then reverted to subtracting the hundreds digits. Although Angie, Fran, Carla and Betty were able to express what they thought the learners did mathematically to obtain the answer in distractor C, none of them made explicit that two zeros at the end of the minuend causes a lot of the confusion and learners look for procedures that they think enable the zeros to be worked into their calculation. Therefore, what is missing in their responses is the stated recognition that learners have problems when borrowing across two zeros. Unless they are able to unpack the mathematics behind the double borrowing process for their learners, the misconception may be viewed as a careless mistake or even ignored.

The general trend underscoring the responses given by the teachers for distractor D foregrounds the misconception that larger digits subtract smaller digits. Dawn, Angie and Carla have seen that when faced with the problem of the smaller digit placed above a larger digit (in this case the zeros) learners may resort to subtracting the zeros from the larger digit underneath. The learners continued to do this in the hundreds column.

The teachers were able to successfully unpack what the learners were thinking when probed about the incorrect distractors. They saw that borrowing across two zeros is a problem for many learners because of the misconception that learners may reason it is correct to borrow once but not twice in the same calculation. None of the teachers elaborated what they thought is the root cause of the problem – that the value of the digits in each place value column is decomposed into groups of ten and a group of ten is transferred to the column on the right to aid the process of subtraction when a smaller digit is above a larger digit in the subtrahend.

Item 3 (decimals and SI unit conversions)



A flag pole has a height of 3.24 metres.

Which one of these shows the height of the flag pole?

- (A) 3 m + 2 cm + 4 mm
- (B) 3 m + 20 cm + 4 mm
- (C) 3 m + 20 cm + 40 mm
- (D) 3 m + 200 cm + 40 mm

The reasons given by four of the teachers for learners choosing distractor A focused on whole number schemes. The decimal point and addition signs were ignored. Angie said, “They read it as three hundred and twenty four”, Carla added, “They read the number in the order of the digits as 3+2+4 and didn’t look at the decimal. They put m, cm and mm in the in the order of largest to smallest to match the order of the numbers”. Ella stated, “They didn’t know anything about conversions. They think its units, tens and hundreds. They see the number as 324 and the point just came there automatically” and Fran added, “They don’t understand the conversions and ignored the units.”

A common idea used by the teachers for distractor B is that the decimal point separates two whole numbers. The second whole number is expanded and the SI units are ignored – “There is two digits after the comma so they were looking at breaking up a whole number after the point into 20 + 4” (Carla), “They didn’t see that the 4 represents 4 mm. They used expanded notation with 24” (Betty), “They read it as 24. They see 3m + 24” (Angie) and “They saw two digits after the comma and put 20 and 4. They do not have enough basic knowledge of decimals” (Dawn).

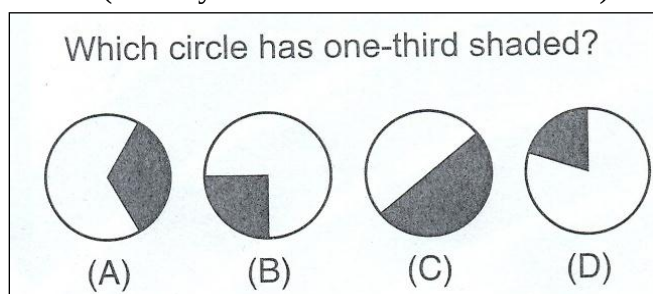
Suggestions posited for distractor D were more complex and varied. Fran said, “They linked 200cm to 2m. I don’t know why they did this. They then changed the 40mm to cm. There was a muddle up with conversions.” Dawn explained, “Metres are bigger than centimetres and millimetres. He knows that the bigger number comes before the decimal. He sees the 200 cm as

10 times more than the 40 mm because their basic knowledge tells them that cm is 10 times a millimetre”. Angie, Betty, Carla and Ella struggled to make sense of what the distractor is testing.

The height of the flagpole is a distance measured from the ground to its topmost point. It seldom happens that the units used (in this case, metres) do not have an extra bit to include in the final measure. The extra bit is the fractional part of the unit used and the whole measure is then expressed as a decimal number with the name of the unit written after the number (Mitchell 2008). A major misconception learners evidence when they see a decimal number is the meaning they give to the point. Learners think its function is to separate two whole numbers. Their knowledge of decimals is incomplete (Steinle, 2004).

There is a general consensus amongst the teachers that learners who chose distractor A ignored the point and read the number as a whole number (a misconception) but none of the teachers offered a suggestion as to why the point was ignored. For distractor B, Carla, Betty, Angie and Dawn noted that the learners were expressing 3.24 as two whole numbers and expanded the ‘24’ to match the expansion in the distractor. Fran’s thinking about distractor D was justified in that the ‘m’ given in the question influenced the learners’ thinking that the ‘2’ in the decimal must be 2 m which is equivalent to 200 cm. Angie thought in the same vein, “The child is seeing the ‘2’ as metres and knows that 200 cm = 1 m. Carla added another insightful idea – “The learners have been taught that there are always three places after the point in a decimal with metres. The learner read his section as a whole number made up of hundreds, tens and units as in ‘240’ and therefore the 200 is acceptable”. It is evident that the teachers were able to connect the whole number misconception with the distractors. Distractor D proved to be the most challenging but Angie, Fran and Carla made a connection with the influence of the given unit (m). The teachers were unable to voice the deeper problem learners evidence and that is, they did not deduce that learners struggle to make sense of the mathematical constructs in the decimal sub-domain, as well as decomposing the decimal into its different SI units.

Item 4 (Identify a circle with one third shaded)



Three teachers provided similar reasons as to why learners chose distractor B and distractor C. They said that learners' social experience and classroom experiences of a half and a quarter dominates their thinking – “They find quarters far easier. A half is any fraction, they are used to halves” (Angie), “They see quarters more often than the third from everyday life experience. The child is going with what he sees every day, a half.” (Carla) and Betty added, “They have worked with too many halves and not enough of anything else.” Fran offered an insight connected to distractor B, “It had one piece and they don't have the knowledge that a quarter is smaller than one third.” Betty also contributed an insight, “They have no understanding of equal sharing and what one third means. They are unfamiliar with dividing a circle into thirds. Shapes worked with are usually rectangular such as the fraction wall.”

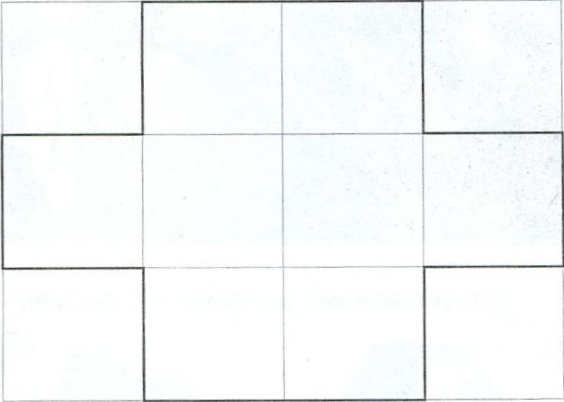
Distractor D was more challenging and Angie expressed an interesting idea linked to the number of sides in a triangle and the meaning of three embedded in the word ‘third’, “They saw a triangular shape and decided it was a third.” Ella explained, “They don't understand sharing and the denominator” and Dawn added, “Most children can't see the differences and they can't judge size. They have a slight idea that a third is smaller than a half.”

With reference to this item, the literature points out that errors and misconceptions linked to fractions are rooted in too much exposure to the half and quarter in and out of the classroom and too little exposure to a variety of other fractions such as thirds, fifths etc. (Yoshida, 2004). Learners who experience the notion of fraction by shading in pre-partitioned geometric shapes without first sharing objects equally may not find the sharing of a geometric shape into iterated parts such as a third easy to do (Newstead and Murray 1998). Every part of a whole is a number that has a relationship between the numerator (given as a number symbol) and the denominator (given as a number symbol). Learners must have this construct embedded in their conceptual knowledge of a fraction. The denominator tells us about the number of iterated parts in the whole (Amato 2005, Fraser et al. 2004).

Carla, Angie, Betty and Fran have connected the idea that learners have both formal and informal knowledge of a half and a quarter. They recognize that these constructs play a role in some learners thinking that all fractions are halves or quarters (a misconception). They also said that learners misunderstand the meaning of a third, but they did not elaborate further. Teachers pointed out that the learners don't have sufficient knowledge of equal sharing (Betty, Ella and Dawn). Dawn thought that too much concentration on the rectangle and no experience of other geometric shapes contributed to erroneous thinking in this item. None of the teachers expressed the idea that learners should first experience informal sharing themselves (in order to embed what fair sharing means) as a possible reason why learners who chose the incorrect distractors were unable to share the circle into three equal iterated parts. Only Ella mentioned that learners don't understand the denominator. This is an important statement because if learners cannot relate the word ‘third’ with a denominator of three, they may guess the answer or rely on the knowledge they do have, which is a half and a quarter.

Item 5 (Calculate the area of a given portion based on squares)

Holly drew this shape on 2 cm grid paper.



What is the area of Holly's shape?

(A) 32 cm²
(B) 28 cm²
(C) 16 cm²
(D) 8 cm²

With reference to what distractor B is testing Betty said, “They counted the centimetres around the shape” and Fran stated, “They just counted the centimetres.” Carla had a different idea – “The rectangle gives 24 and they added one square to make 28.” Responses for distractor C were either connected to the area formula for a rectangle or multiplying the number of tiles by 2 – “They were using the area formula” (Angie), “8 blocks x 2 cm = 16. This is connected with the formula of a rectangle” (Betty), “They read 2 cm in the question and counted the blocks and multiplied by 8. They didn’t know the concept of length and breadth” (Carla) and “They had a bit of knowledge that you have to times by 2. So they counted the squares and said 2 x 8 = 16 cm (Fran). The teachers all recognized that learners counted the number of tiles for the area of the shape in distractor D. Dawn explained, “They have no basic knowledge of area so they counted the amount of squares in the shape” and Angie reasoned, “They just counted the square tiles. They ignored the fact that each tile is 2 cm and gave each tile a measurement of 1 cm.”

One of the major misconceptions learners experience is the confusion between area and perimeter (Cavanagh 2007). Aligned with this misconception is an absence of knowledge of the area concept – a construct that is based on an array or grid of square units that are counted. Perimeter is the enumerated linear units around the outside of a shape whereas the area formula for a rectangle (length x breadth) is derived from an array of squares covering the surface of a shape. Outhred and Mitchelmore (2000) and Battista et al. (1998), point out that an array (grid) of squares does not come naturally to learners. Deficits in these constructs may render the formula of the area of a rectangle meaningless and

something to be learnt by rote. A consequence of this is that learners may misuse the formula when asked to find the area (Cavanagh, 2007).

Betty and Fran were the only two teachers who recognized that the learners' thoughts were connected to perimeter when thinking about distractor B. The other teachers were unable to see that the confusion between area and perimeter was being tested. Angie, Carla, Betty and Fran noted that learners were applying the area of a rectangle formula in distractor C. They were of the opinion that learners knew that two numbers are multiplied when using this formula, but the teachers did not consider in depth about why the learners used the 2 and the 8.

Distractor D is about enumerating the number of 2 cm x 2 cm square tiles. Dawn talked about learners having no 'basic knowledge' of area but she does not qualify what she means by this. Angie was the only teacher who seemed to connect the misconception of counting any size iterated square units sufficient for area measure. She connects the learners' conceptual knowledge with the squares to be counted, that is, each square must be 1 cm x 1 cm in size, but the 2 cm tile length confuses the learners and they therefore superimpose this '1 cm x 1 cm' knowledge on each tile.

The teachers quoted for this item were able to make connections with learner thinking in the distractors, but they did not furnish sufficient underlying reasons for why learners think this way. The misconception that area is the same as perimeter was not foregrounded by the majority of the teachers. Time constraints could have been responsible for this outcome.

Summary:

The teachers' reflections on the distractors show an awareness of how the learners can think erroneously. The teachers were able to detect the errors and explain why they occur but they did not always verbalise the misconceptions underlying the errors. Item 3 lacked sufficient depth in this regard. Although, as previously stated, the teachers' content knowledge is satisfactory, I suggest that their PCK is weaker than their content knowledge, in terms of knowing all the errors and misconceptions that exist in the item sub-domains and the embedded erroneous mathematical constructs that lie behind them.

Theme Four: Interventions to address perceived misconceptions

Items 1 (angle), 4 (fraction) and 5 (area) were chosen for this analysis because the interventions proposed were varied and interesting with regard to the teachers' PCK. It should be noted that the teachers either told me what they would say to a learner or they imagined I was the learner in the intervention.

Item 1 (Differentiate angle size from orientation and side length)

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
Angie	“It’s not the length of the arms that determines the size of the angles. The angle is the space between the two arms. This space could be the area between. The area is the movement from one arm.”	Learners have to focus on the notion of angle as that which involves the movement (turn) from one of the arms as opposed to the length of the arms. Angie speaks about the area between the arms and this has nothing to do with angle size and in this instance is a misconception.	Angie does not forge a link between her spoken words with a representation that uses a manipulative. Learners with this misconception need to see a demonstration of the ‘movement’ Angie is talking about so learners can see how one arm turns away from the other around a common vertex.	Yes, in Grade 6 learners are introduced to angles that are smaller and greater than a right angle. The item uses four acute angles.
Betty	“I would use geostrips and show the learner how the arms turn. I would open them wide and close them so they can see the size. The wider the arms the bigger it is and that as the arms move closer to each other the smaller the angle becomes. It’s the turn and it has nothing to do with the length of the arms but rather it’s that portion in the centre.”	The knowledge learners receive in this intervention serves to draw their attention to the notion of angle as an amount of turn. The teacher opens and closes the geostrips while aligning the concepts of ‘bigger’ and ‘smaller’ with the opening and closing of the arms. The teacher’s move is mathematically correct.	The visual experience that learners receive from the geostrip demonstration may serve to transform the learner’s thinking that the length of the arms dictates angle size. The intervention focuses learners’ attention on the rotation of one arm while it is connected to a common point with the other arm. The learner is told that the lengths of the arms do not matter. It would have been more appropriate if Betty had mentioned that she would use two geostrips of different lengths for her demonstration.	Yes, in Grade 6 learners are introduced to angles that are smaller and greater than a right angle. The item uses four acute angles.
Carla	“Go back to the concept of the right angle where you have a perfect L. Look at the L which gives you 90 degrees. See where the one line of the angle looks like a	Carla is showing learners how to use the right angle as a benchmark to gauge which is the biggest angle. She superimposes the right angle onto		In Grade 5 the learner has learnt about right angles and by using the right angle as a benchmark for angles less than 90° is appropriate for Grade

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
	bisecting line. A bisecting line is 45 degrees. Is it half way or is it less than that? Is the bisecting line nearer to the L line or nearer to the base line of the L? If it is nearer to the base line it is smaller.”	each angle in the item and wants the learners to focus on the arm that is closest to the perfect ‘L’ shape. This is visually correct. She introduces another construct which is the angle bisector of a right angle which draws the learner’s attention away from the right angle as a benchmark.	By using the right angle as a benchmark against which to compare the four acute angles the intervention can address the notion of the largest acute angle if degrees are not given. However, this is a static notion of turn. Carla has not linked the notion of turn as a movement of the angle arms to this task. The angle bisector is not visually drawn and this can serve to confuse the learner.	6. Angle bisector knowledge is introduced to learners in higher grades and is therefore inappropriate as a tool to be used for comparing angle sizes in Grade 6
Dawn	“Take A4 paper and fold it and say it is a chair and ask where he will sit and will he be comfortable. Bend an A3 paper and fold it but make the angle smaller. He must see the two different lengths of the legs of the papers. Ask him if this is a bigger chair. The backrest (of the A3 paper) is bent over towards the seat. Ask if he will still be comfortable when he sits on it. Then I will show the two different papers and show where the angle is and try to make him understand that the legs don’t matter for the angle size.”	Dawn has tried to give the learner a sense of angle size by linking it with the learner’s real life experience of sitting in a chair that is either comfortable or uncomfortable depending on how far back the ‘back of the chair’ is. The A4 paper is used for the more comfortable (bigger angle) chair to represent an angle with shorter arms and the A3 paper represents the uncomfortable chair (smaller angle) with longer arms. What the learner sees are two paper angles and has to visualize them as chairs. The notion of turn is not demonstrated for the learner in this context and this is key to an understanding of angle size.	The misconception learners have about the length of the arms dictating angle size is addressed. In this case, it may have been more appropriate if the two papers were aligned in angle size where the learner is able to clearly see that the turn is the same but the lengths of the arms are different.	Yes, the angles used in the intervention are either less than or 90 degrees which is aligned with the grade 6 assessment standards. Also learners at this age are able to visualize a chairs that have different backrests and they can ‘see’ themselves sitting on the two different chairs mentioned in the intervention. They also understand the notion of bodily comfort.

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
Ella	“Ask the learner if he remembers we said an angle is an amount of turn between two arms. Look at A and C. Which one is open wider? Get him to then explain why he made A his first choice and not C if C is wider than A. I would make an instrument out of cardboard and move the arm. Maybe I can use the arms of a clock.”	The teacher correctly draws the learner’s attention to the notion of turn that has been mentioned in class on a previous occasion and gets the learner to compare A (the misconception) with C (the correct answer). The learner has to focus on which angle has the bigger turn but the word ‘wide’ is used inappropriately and can cause confusion. Ella re-inforces the concept of turn with a cardboard manipulative or by using the hands of a clock. This intervention combines a 2D discussion with a 3D representation and it should transform learners’ understanding about angle size. She does not say why she would use a clock in terms of the hands being different lengths.	The misconception that the length of the arms of an angle dictate its size can be addressed by isolating angles A and C from the rest of the angles in the group and getting the learner to think aloud while justifying his choice of A over C if angle C has a bigger turn. The hands of a clock can also serve to focus attention on the amount of turn because the hour and minute hands are of different lengths yet they both rotate around a fixed point.	Yes, in Grade 6 learners are introduced to angles that are smaller and greater than a right angle. The item uses four acute angles.
Fran	“I would get you to start drawing a right angle with a protractor and then smaller size angles in order to see what the degrees are. I would tell you that it doesn’t matter how long the arms are, it’s the reading of where the two meet. The arms are moving wider as the degrees change.”	Fran uses the protractor and degree measure to embed the notion of angle size by getting learners to construct a right angle then construct smaller acute angles. This idea does not serve to re-inforce the mathematical notion of turn and its representation with an arc as used in the item. She does draw their attention to the fact that the arms are moving away from each other as the degrees change which embeds the notion of turn.	The misconception that arm length dictates the size of angles is not addressed by using a standard protractor. The line segments on a protractor are all the same length and the protractor does not demonstrate the notion of turn.	Use of the protractor is introduced in Grade 7 and therefore Fran’s intervention is inappropriate for a Grade 6 learner.

Item 4 (Identify a circle with one third shaded)

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
Angie	“I would ask the learner to write one third in mathematical terms. If this is written as one over two I would correct it. Folding double would show them one of two equal slices. I would explain the function of the numerator and the denominator. The denominator tells you into how many equal slices the whole was cut. The three tells us that the whole has been cut into 3 equal slices.”	Angie expects that learners may write one half instead of the required one third. The notion of a half is re-inforced by paper folding. This is a correct construct to use for a half. Angie then switches the learners’ focus to the role of the numerator and denominator and aligns this with mathematically correct knowledge by explaining the three in the denominator. Knowledge of the one in the numerator is ignored.	The misconception that a third is different from a half is not fully addressed. By not giving learners a manipulative that shows iterated thirds and iterated halves from the same whole an opportunity is missed which may transform learners’ knowledge that a half and a third are different in size. It is insufficient to talk about the three in the denominator without practical experience.	In Grade 4 learners have to learn about fractions smaller than a half (one third) and a quarter (one fifth etc). If this knowledge is absent by Grade 6 Angie needs to pay attention to paper folding with different shapes using the spectrum of Grade 4 fractions. This intervention is not age appropriate.
Betty	“One third means $1 \div 3$. I would show the learners how to divide a circle into two equal pieces for one half, then 3 equal pieces for one third and four equal pieces for one quarter. This will show practically through comparison that a third is smaller than a half.”	Betty’s practical demonstration of showing the learners how the same circle is divided into iterated halves, thirds and quarters gives learners a correct visual experience of how the sizes of the parts change depending on the number used in the denominator. Learner also receive conceptual knowledge that fraction notation is another representation of division. Both these strategies are mathematically correct.	Yes, learners have the opportunity to use the same shape as was used in the item in order to transform knowledge that a half or a quarter is totally different in meaning from a third. The denominator represents the whole circle divided by 3 and therefore three equal parts is re-inforced.	Yes, the teacher is remediating the learner’s incorrect constructs of a third learnt in Grade 4.
Carla	“Cut out a template of one third. Teach the word ‘third’ that it means 3 parts. How many templates would make a jigsaw? Each piece must be	Equal iterated parts that are used to fill a whole is mathematically correct (in this case the template that is used to represent one third). The template also allows learners to		Yes, learners have already become acquainted with fractions other than halves and quarters in Grade 4 of which the third is one such

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
	equal. Use templates for a half and a quarter.”	have practical experience of putting thirds together to make up a whole. The notion of a jigsaw combined with the fitting together of equal parts is a useful strategy to re-inforce the part-whole concept. Learners perform the same task using a half template and a quarter template for the same whole.	The three in the denominator is not linked to the naming of the fraction of a third. By using the third template to fill a whole embeds the notion of the denominator as three equal parts. The activity allows learners to visually link the three that is seen in the denominator with the final product obtained after using the template. Learners can compare halves with quarters and thirds by repeating the same procedure.	fraction.
Dawn	“Use a fraction wall where the child needs to see how many bricks they need to build on the whole starting with halves. The bricks become more and the child needs to tell me if the bricks become smaller than the previous one and how bricks are used for the thirds compared to the halves.”	Yes partially. A fraction wall represents different fractions within the same whole and visually learners can see at glance the difference in size between a half, a third and a quarter. The conceptual mathematical knowledge missing from the intervention is the connection between the numbers represented as words in the numerator and the denominator.	If learners can see the fraction wall they will be able to compare the sizes of a half, third and a quarter easily because the three rows are consecutive. This can help to address the misconception that a third can be called a half or a quarter. The wall shows iterated fractions that are different in size making up the same whole. The item uses 4 circles of the same size, three of which represent a half, third and a quarter. The wall does not address the numerical notation of one third.	The strategy of using the fraction wall is appropriate for Grade 6 learners who need to compare fraction sizes. Fractions represented on the wall are learnt in Grade 4.
Fran	“I would get the learner to cut paper into halves and quarters. Then I would say that those are not the only fractions we get. The smaller the number at the bottom the bigger the portion is so if it is a four there will be more portions than a third. If you have a twelfth it is very	Fran has made mathematically correct statements to learners in terms of how portions get smaller as the number in the denominator increases but she does not follow what she has explained to include a third. She mentioned that she will cut a cake but does not say how this activity	Paper folding into halves and quarters serves to re-inforce their dominance in learners’ thinking and may lead them to think that all fractions are called halves and quarters. The visual experience of the comparison between a half and a third is lacking. The intervention does not serve to address	Paper folding into halves and quarters are part of the assessment standards in the Foundation Phase. It would be more age appropriate to get learners in Grade 6 to look at different models in which Grade 4 fractions, for example, a third, are

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
	much smaller. I would use a cake and cut this cake.”	would connect with a third. She mentions the notion of portion but does not state that each portion in the whole must be equal to the next portion.	the misconceptions evidenced in the part-whole relationship for fractions such as a third, fifth etc. Paper folding can address the misconception that as the number in the denominator gets bigger the portion gets smaller but paper folding into thirds is doable but more difficult.	represented. Cutting a cake would serve this purpose better.

Item 5 (Calculate the area of a given portion based on squares)

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
Angie	“I could cut off the two squares on the side and put them on the top to make a larger rectangle. I would count the cm for the width which is 4 and I would count the cm as the length which is 8 and multiply them together to get 32. I could cut off the two squares on the side and put them on the top to make a larger rectangle. I would count the cm for the width which is 4 and I would count the cm as the length which is 8 and multiply them together to get 32.”	The strategy used to convert the given shape in the item into a rectangle by moving tiles around is mathematically correct. This is possible because the tile units are congruent. Counting the number of centimetres along the width and multiplying this answer to the number of centimeters along the length will give a correct area measure.	By making use of the rectangle formula Angie employs a procedure that is often learnt without any conceptual understanding. Angie’s intervention works in terms of building an array of squares but she does not address the misconception that exists in learner thinking concerning their confusion between area and perimeter.	Although Angie did not explicitly use the word ‘formula’ her procedure for finding the answer used the formula method. Her intervention was more teacher orientated. Using the formula for the area of a rectangle is inappropriate for a Grade 6 learner. This formula is taught in Grade 7.
Betty	“Each tile is used to cover the surface and area is the number of tiles used to fit onto a place. In this case 8. Each tile does not represent 1 square cm	The strategy used by Betty is mathematically correct. She is re-inforcing the notion of area as square centimetre units (rather than larger square tiles)	Learners who do not have the area concept embedded correctly are inclined to count the number of tiles in the diagram. These learners need to	Yes. In Grades 5 and 6 the area concept is developed by covering a surface with iterated squares units that are counted.

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
	but 4 sq cm. Use the grid paper that is given in the diagram and trace the shape onto sq cm paper so that the learner can see the sq units that have to be counted in each tile.”	that cover a surface. She gets learners to trace the given tiles onto square centimetre grid paper and then counting the square centimetres bounded by the original diagram.	transform their knowledge from the act of counting squares of any size to the mathematical construct that iterated square units are counted. In this case the diagram uses square centimetres. Betty’s intervention can serve to address this misconception.	
Carla	“Each big block is 2cm by 2cm which is 4. Count the number of big blocks and multiply by eight.”	If the area of one tile is known (which can be found by using the formula side x side) then the tiles can be counted and multiplied to the area of one tile. The strategy is mathematically correct.	The area concept is developed by enumerating the number of square units covering a surface. The multiplicative aspect is a further development of this enumeration and if learners do not have the former embedded, multiplication of the area of one tile to the number of tiles will have no conceptual meaning. The teacher has not addressed any misconceptions linked to area and perimeter.	Using the formula to find the area of one tile is not appropriate for Grade 6. The formula for the area of a square shape is learnt in Grade 7.
Ella	“Each square is 2cm and 2cm x 2cm = 4 square cm. Now count all the squares and times by 8.”			
Dawn	“Tell the child to paste little squares of paper 1cm on the given shape. How many squares did he use (not 28). He covered the area of the shape. Area means surface of the shape. Measure the perimeter and show him that he has to work out the area of each square and count how many squares he used to cover the surface of the shape.”	The intervention is mathematically correct. Learners have to experience the practical aspect of pasting square centimetres onto the given shape before counting can begin. Learners also have to measure the linear dimensions of the border of the shape in centimetres and compare the two answers.	Dawn has employed an intervention whereby learners have to work out the area of the shape by practically counting all the pasted square centimetres. Learners have to measure the perimeter in centimetres as a second task. Both tasks will produce different answers. This strategy can address the misconceptions learners have with area and perimeter.	Yes. The tasks are within the boundaries as set out in the assessment standards for area and perimeter in Grades 5 and 6.

Teacher	Intervention	Is the mathematics correct?	Is the misconception addressed?	Is it age appropriate?
Fran	“Draw a square 2cm by 2cm. I would get you to count the squares in the grid and do other shapes with grids. Then I would get them to physically tile shapes.”	This intervention focuses attention on the area of one tile. Although not explicitly stated, the tile is divided into a grid containing 4 square centimetres. Mathematically this is correct. Nothing is mentioned about the tiles in the diagram given in the item.	Many learners count iterated tiles of any size which is the misconception that Fran wants to address. By getting learners to sub-divide one tile into a grid of 4 square centimetres, their attention is focused on the square centimetre as the counting unit for area rather than the tile.	Yes. Counting square units is as a means to calculate area is a Grade 5 and 6 assessment standard.

Summary:

Having mapped the teachers’ interventions against my meta-structure, I have noted the following trends:

Bigger ideas and mathematical structure

In many cases, the teachers’ focus extended beyond the requirements for a task that serves to address a misconception. Besides focusing on the item in question, learners are exposed to broader notions and structures that underpin knowledge required for the sub-domains. For example, in Item 1, Angie, Betty, Carla, Dawn and Ella pay attention to the meaning of angle as it relates to the notion of turn and two arms that meet at a common vertex. In Item 4 attention is paid to the notion that a proper fraction is part of a whole and that the parts must be equal in size. Reference is made to the role of the denominator - it tells one how many equal parts make up the whole. The notion of sharing equally as a division operation is indirectly re-inforced. In Item 5 all teachers conceptualized area as the number of iterated square units that tessellate to cover a surface. Area measure can subsequently be calculated by counting the number of squares in a shape.

Experiential learning

Active engagement as opposed to being factually informed can enhance the transformation of knowledge. In Item 1, Betty does not assume that all learners know what the notion of ‘turn’ means as it relates to the angle concept and chooses to provide learners with a visual experience to embed this knowledge. In item 4, Carla associates the notion of equal fraction pieces that make up a whole, with learners actively building a jigsaw puzzle, where the template is used as one of the fraction pieces. In item 5, Dawn gets the learners to experience the concept of area by allowing them to paste equal square units onto a surface until the surface is covered.

Content knowledge of the teacher

Some teachers impose their own content knowledge on the learners. This was evident in Item 5 where Angie, Carla and Ella required learners to count the linear measures of the length and breadth of a rectangle and multiply them together. This formula is derived in Grade 7 after

learners have experienced the notion of covering a surface with iterated square units and counting them. This lays the foundation for the formula to be developed.

Representations

In nearly all cases, teachers chose to use one representation in their intervention strategies. The representations used to address erroneous thinking were mathematically correct. The literature does point out that multiple representations are more effective - this incorporates the use of different strategies to promote understanding (Shulman 1986, Lampert 1991, Ball 2000, Adler 2005). In some instances one representation can be limiting. For example, in Item 4 Fran wants learners to practically cut paper into halves and quarters but she does not extend this activity into the range of fractions that are used in Grade 5. Paper folding into thirds is more difficult. An emphasis on halves and quarters leads learners to erroneously think that all diagrams showing any proper fraction is either a half or a quarter.

Manipulatives

Apparatus that is used appropriately can enhance learning but frequently the apparatus on the market is expensive and therefore teachers have to devise creative ideas to avoid these costs. The aids and manipulatives used in the interventions are effective and accessible to all teachers. They give learners visual and practical knowledge to embed concepts, for example, cardboard strips to make an angle, a fraction wall poster to compare fraction sizes, square centimetres made from paper to tile a surface and cardboard to make a fraction template.

Teachers' focus on misconceptions

Some teachers indicate that they are, for the most part, more aware of misconceptions after thinking about and discussing the errors and misconceptions in the item distractors during the interview. Their interventions are focused on one or two of these misconceptions. For example, in Item1, Angie, Betty, Dawn, Ella and Fran chose to focus their interventions on the misconception learners have about the length of the arms dictating the size of the angle. In Item 4, Angie, Betty and Carla focused more on the misconception that all proper fractions represented in a diagram are called halves and quarters. In Item 5, Angie, Betty, Dawn and Fran focused on area as a concept, that is, the notion of the measure of covering a surface with squares and counting the number of congruent square units which is different from the perimeter concept which counts the number of equal iterated units around a shape.

My analysis examined teachers' content knowledge, language, awareness of errors and misconceptions and interventions to address misconceptions. The teachers all evidence an understanding of the mathematical concepts used in the items and were able to explain (either conceptually or procedurally) how they would choose the correct answer. Their language was easy to comprehend in the interview but some mathematical words could prove to be ambiguous if used in the classroom. Their knowledge of errors and misconceptions is often limited in the different item sub-domains but in all instances they were able to foreground and reason about one or two of the main misconceptions that learners evidence. Their interventions took cognizance of the broader mathematical structures associated with the misconceptions while they were engaged in

the transformation of knowledge. In most cases the teachers were in a position to think creatively and employ useful strategies to enhance learning.

CHAPTER 6 : ANALYSIS

Introduction

The ability to reason about learners' errors and misconceptions is underscored by a teacher's mathematical content knowledge (Ball 2001, Ma 1999, Ball 1988). I argue that content knowledge and its related procedural and conceptual knowledge (Long 2005, Kilpatrick et al. 2001, Hiebert and Lefevre 1986) in the different mathematical domains are important for the recognition and understanding of learners' errors and misconceptions. Research has shown that there is a vast difference between the careless mistakes inadvertently made by learners and the misconceptions they bring to class which indicate conceptual confusions embedded in their prior knowledge constructions (Hansen and Drews 2005, Smith and Roschelle 1993, Nesher 1987).

In the previous chapter I focused on a sample of six teachers' content knowledge in five multiple-choice items, and I then analyzed their reasoning about misconceptions embedded in the item distractors. All of the teachers demonstrated that they possessed sufficient mathematical content to proceed with an investigation into their reasoning about learners' thinking, with a specific focus on misconceptions. I endeavoured to deepen my understanding of their PCK by asking them to describe an intervention they would use to address a misconception in three of the multiple-choice items. I wanted to ascertain if teachers would provide me with mainly procedural solutions for the handling of misconceptions in which learners are required to learn algorithms off by heart. Would they mainly refer to the (erroneous) mathematical procedure learners employed for a task or would they refer to the errors and misconceptions in *relation* to the conceptual knowledge that underpins them, and suggest tasks that address the conceptual issues of structure and domain that are fundamental to 'knowing mathematics' for understanding? I also noted their use (or misuse) of language in their mathematical explanations. An intervention to address mathematical errors and misconceptions includes giving learners different representations of the subject matter in question (Lampert 1991). I assumed that in turn, these representations are linked to strategies that endeavour to transform erroneous mathematical constructs.

My analysis of their PCK gives me an opportunity to align teachers' reasoning about errors and misconceptions with the kind of intervention they believed would best address learners' problems. I wanted to find out if teachers could firstly articulate the misconceptions and secondly, address the misconceptions with an age appropriate intervention that seeks to transform knowledge, thereby giving the learner meaningful mathematical constructs to replace misunderstanding and confusion. In my view, misconceptions are best served by a reconstruction of knowledge through learners' active engagement - what learners hear, see and do should be linked to age appropriate explanation, demonstration and practical activity that *make mathematical sense*. I intend to deepen my understanding of how teachers in the field reason about learners' errors and misconceptions and I use this chapter to profile three teachers, Betty,

Carla and Ella. My analysis is focused on content knowledge (which I identify as procedural or conceptual or both), focusing on a specific aspect. I highlight teachers' ability to recognize one or all of the misconceptions in five of the items and comment on whether their intervention for one misconception in three items is mathematically meaningful for the reconstruction of learners' knowledge. The lens through which I analyze their interventions is *the relationship between their content knowledge and reasoning about learner error and whether the intervention is age appropriate*. This relationship informs my claim that the way teachers espouse their interventions, reflects whether their knowledge is dominated by conceptual or procedural understanding of learners' errors. I show that in the case of Betty's interventions, she uses conceptual content knowledge when she reasons about learners' errors and misconceptions. I show that Ella uses procedural content knowledge to reason about learners' errors. Carla's reasoning about learners' errors and misconceptions is a mixture of both.

My teacher profiles are structured into three parts – I initially look at the teachers' mathematical content knowledge followed by their reasoning about learners' errors and misconceptions in all of the items. These are given in item order (1,2,3,4,5). In order to obtain a coherent story, I have chosen their interventions in three items, that is, Items 1, 2 and 5. Although Item 1 and Item 5 are both from the Measurement domain, they are conceptually very different. Misconceptions with the angle concept may have a serious impact on learners' conceptual understanding of geometry throughout their schooling. Early misconceptions in area measurement can have negative consequences for future use in daily life and in advanced mathematics. Item 2 was chosen because the operation of subtraction is a life skill, particularly when a subtraction calculation is done without a calculator. I chose these items because the connection between teachers' content knowledge, awareness of errors and misconceptions and interventions are significant in terms of how their reasoning informs their PCK and whether their PCK is learner orientated/conceptual, or learner orientated/conceptual/procedural, or content orientated/procedural.

The following table contrasts different modes in the way Betty, Ella and Carla articulate the content of the items, explain learners' errors and misconceptions and focus their attention in the interventions:

(It must be noted that Item 1, Item 3 and Item 4 are conceptual constructs and do not provide an option for procedural thinking whereas Item 2 and Item 5 do).

Dimensions of PCK

	Betty	Ella	Carla
Content of the items	Conceptual	Procedural and conceptual	Conceptual and procedural
Learners' errors	<u>Conceptual</u> – Item 1, Item 2, Item 3, Item 4, Item 5	<u>Conceptual</u> – Item 1, Item 3, Item 4 <u>Procedural</u> – Item 2, Item 5	<u>Conceptual</u> – Item 1, Item 3, Item 4 <u>Procedural</u> – Item 2, Item 5
Interventions	<u>Learner orientated/Conceptual</u> – Item 1, Item 2, Item 5	<u>Learner orientated/Conceptual</u> – Item 1 <u>Content orientated/Procedural</u> – Item 2, Item 5	<u>Learner orientated/Conceptual</u> – Item 1 <u>Learner orientated/Conceptual/Procedural</u> – Item 2 <u>Content orientated/Procedural</u> – Item 5

Teacher Profiles:

Betty

My overall view of Betty is that she is consistently learner orientated and in her discussion of the content in the items, she always mentioned *prior knowledge content and conceptual constructs that learners require for doing the mathematics in a specific item*. For example, in Item 1, she stated that having knowledge of a right angle can be used as a benchmark against which to compare the angle sizes in the item. She also referred to the concept of rotation as the main angle construct. For Item 2, she suggested that ways of doing subtraction include a vertical and horizontal subtraction, subtracting from a number with two zeros, place value and expanded notation. Betty was also able to demonstrate a depth of content knowledge required for Item 3, that is, the intersection of the measurement and decimal domains, SI unit conversions, the identification of the digits in the place value columns after the decimal point as fractions which have to be multiplied with the given whole of a metre to calculate their corresponding SI unit amounts. (This reasoning is core to choosing the correct answer for the item). In Item 4, Betty used learners' prior experience of a half and a quarter to note that the shaded part (one third) is smaller than a half but bigger than a quarter. She then proceeded to identify the number in the denominator as a cue to the size of the shaded part in the whole. Betty used the area concept as a basis for area calculation in Item 5. She reasoned that each big block (tile) is made up of 4 cm^2 and proceeded to count the number of tiles before multiplying the sum by 4. It is evident from this procedure that learners need to have knowledge of the area concept as the number of square centimetres that cover the surface of a shape.

Based on Betty's ideas for what mathematical content learners need for the items, she was able to recognize many of the errors and misconceptions evident in the item distractors. I have applied her ideas and suggestions to show the breadth of her reasoning about learner thinking:

Betty is aware that learners think the angle with the longest arms is the biggest angle (Item 1). They look at angles in different orientations other than the 'horizontal' position and surmise they are larger when this is not the case (Barrett et al. 2003, Mitchelmore 1998, Magina and Hoyles 1997).

"They might have been looking at the length of the arms which are longer than C. They might have looked at the point where the two points meet and B looks similar to C because it is pointing in a different direction".

Betty suggests that learners become confused when they have to borrow across two zeros (Item 2). She noted that when learners have borrowed once they are unable to borrow twice in the same calculation, and resort to subtracting the zero left behind in the tens column from the digit in the subtrahend or adding the zero to the digit underneath it. In so doing, she was of the opinion that some learners have a partial grasp of the borrowing procedure (Brodie et al. 2009, Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006).

"Understanding of place value might be a problem. They said $10 - 8$ is 2 and forgot that they should say $9 - 5$. That concept of borrowing has not been grasped because they then said $10 - 5$. They don't understand that you can't subtract a whole number from 0".

Betty reasoned that learners treat the numbers on both sides of the comma as whole number schemes (Item 3). They ignore the unit mentioned after the decimal number and therefore do not link it with the whole number before the point. Consequently, errors arise because learners are unable to link the place value fractions with the given unit after the decimal (Mitchell and Horne 2008).

"They read 3 for 3 m, 2 for 2 cm, 4 is 4 mm read the number from left to right at random. They didn't see that the 4 represents 4 cm. They used expanded notation with 24".

Betty asserts that learners think that all shaded parts in a whole are either called a 'half' or a 'quarter' due to an emphasis on halves and quarters in their real life and classroom experiences (Item 4). She added that a learner's inexperience of equal sharing in different shapes (other than the rectangle) with different proper fractions will lead to a misconception about the size of a shaded part of a whole. Betty also pointed out that learners who have no understanding of fraction notation and the role of the denominator will make errors when deciding on the size of the shaded part given in the item (Newstead, 2000).

"They misread the shaded part and decided it looks like a third. They grapple with equal sharing. They have worked with too many halves and not enough of anything else. No understanding of equal sharing or what one third means. They are unfamiliar with dividing a circle into thirds. Shapes worked with are usually rectangular, such as the fraction wall. Misunderstood the value/size of a third compared to a half."

Betty suggests that learners use the procedure for calculating perimeter when they have to calculate area (Item 5). Betty stated that learners who have been drilled with the formula for a rectangle will seek to use it in any irregular shape in which they can identify a 'width' (small number) and a 'length' (large number) even if these numbers are not the dimensions of a rectangle. She also stated that learners have misconceptions about the notion of counting congruent square shapes irrespective of their size whilst not being cognizant of counting square centimetres (which is one of the square units used for area calculation) (Cavanagh 2007, Battista et al. 1998, Outhred and Mitchelmore 1993).

"They counted the centimetres around the shape. They just counted the squares or used the formula $L \times B$ and said $4 \text{ cm} \times 2 \text{ cm} = 8 \text{ cm}$."

It is evident from Betty's suggestions in the light of the conceptual emphasis she demonstrates about her own content knowledge, that she knows the key mathematical ideas and is aware of the confusion in learner thinking about the items. Her ideas take cognizance of the 'gaps' in learner's knowledge and she was able to explain many of the misconceptions (arm length and orientation of an angle, borrowing incorrectly when two zeros are at the end of the minuend, using whole number schemes in a decimal number calling any shaded fraction of a whole a half or a quarter, confusing area with perimeter) that arise in learner thinking if their mathematical content knowledge is confused.

Betty's interventions

Item 1 (Differentiate angle size from orientation and side length)

In her angle discussion, Betty focuses on the angle concept as a notion of turn or rotation. She requires learners to understand that, "The size of the angle is determined by the turn of the angle". The approach she uses to develop this mathematical understanding is both visual and practical, and she also takes into account the learners' misconception that the angle with the longest arms is the largest angle:

I would use geostrips and show them how the arms turn. I would open them wide and close them so they can see the size. The wider the arms the bigger it is and that as the arms move closer to each other the smaller the angle becomes. It's the turn and it has nothing to do with the length of the arms but rather its that portion in the centre.

Betty's manipulative takes the learners' focus away from the board and she gives them practical knowledge of the notion of 'turn'. Her choice of manipulative is meaningful, because the vertex of the strips can be oriented in any direction (this addresses the misconception about angle orientation) and both arms can open and close to produce large and small angles. At the same time she points out that the length of the arms is irrelevant. This is the key issue that Betty wants to address and she brings it to her learners' attention. During the demonstration her learners can focus on the rotation of the arms whilst seeing that the 'portion at the vertex' is becoming larger and smaller. I assert that when Betty's learners subsequently look at a two-dimensional representation of an angle, the construct of an angle as the amount of turn should be firmly

embedded. I suggest that Betty has used a conceptual representation in this intervention. She has used her conceptually based content knowledge and reasoning about learner error to inform her PCK.

Item 2 (Subtract three-digit numbers)

In her discussion on the content knowledge needed for subtraction, Betty made mention that learners need to have mathematical knowledge of place value and expanded notation. She has reasoned that the borrowing algorithm is often partially grasped by learners, and therefore she prefers to focus learners' attention away from borrowing by using a strategy for the subtraction operation that will make more sense to them. In so doing, Betty wants to dispel the misconception that smaller numbers must always be subtracted from bigger numbers. She uses a different strategy other than the borrowing procedure to transform her learner's thinking:

Add 658 to 358 with Dienes' blocks to see if it comes to 900. Then I would tell him that what you did was say $8 - 0 = 8$ and $5 - 0 = 5$. Take 900 and break it up into hundreds. Practically subtract 50 from 100 with Dienes' blocks and he would be left with 50. Then subtract 8 from 50 and he would be left with 42. He would add the 800 to the 42.

Betty commences with the learner's error. She first wants to prove to the learner why they think incorrectly by getting them to use the construct that if ' $a - b = c$, then $a = b + c$ '. By making this move, she allows learners to mathematically experience why they are incorrect rather than simply telling them they are wrong. Her next step is to use a well known manipulative (Dienes' blocks) which allows learners to do practical work with place value and expanded notation. The algorithm expands the subtrahend into its place value parts, and the parts are then separately subtracted from the minuend as a step-by-step procedure. Learners who struggle with the borrowing procedure can then use this algorithm with confidence and the confusion about borrowing across two zeros is avoided. Betty's intervention forges a relationship between what she considers as content knowledge necessary for learners and her reasoning about their misconceptions with the borrowing method.

Item 5 (Calculate the area of a given portion based on squares)

Betty wants her learners to grasp her knowledge of the area concept by assisting them to understand that square SI units are units used for covering the surface of a region. Betty is careful to keep learners' attention focused on what area is. She wants to make her learners aware that a 'tile' can be partitioned into square SI units irrespective of its size:

Each tile is used to cover the surface and area is the number of tiles used to fit onto a place. In this case 8. Each tile does not represent 1 square centimetre but 4 square centimetres. Use the grid paper given in the diagram and trace the given shape onto square centimetre paper so that the learner can see the square units that have to be counted in each tile.

Betty introduces square centimetre paper onto which the tiles given in the shape are to be traced. What learners practically see for themselves, is that each 'tile' is comprised of 4 single square centimetres. This is key in removing the notion that tiles of any size can be counted to provide an area measure. They may only count the square centimetres. Betty's approach takes cognizance that the learners will be better placed to embed the area concept if they do a practical activity in which they are engaged in counting square centimetres. This intervention combines a practical experience of counting squares covering a surface with the area concept rather than counting squares centimeters on the boundary of a shape, the latter embeds the perimeter concept.

Comment

I view Betty's practice as consistently conceptually orientated. Betty's reflections suggest that she believes that learners need to be taught conceptually, particularly if they have misconceptions. This means that procedures learnt in the classroom may have to be replaced by other meaningful procedures that provide learners with a better understanding of mathematical constructs. Betty utilizes educationally sound manipulatives to achieve her goals (geostrips, Dienes' blocks, square centimetre paper) thereby enabling the learners to experience mathematics in more profound ways which build constructs sensibly. From her espoused pedagogy, Betty foregrounds learners' attention with regard to a practical demonstration or practical work. Consistent with the conceptual way she frames the content of the items, Betty chooses modes of interventions to enable her *to adapt her pedagogy to interrupt her learners' way of thinking*. This perspective has been demonstrated throughout Betty's interventions. She is clear about what she wants to achieve conceptually and she is consistent about fore-grounding her learners' conceptual needs. I have also noted that her espoused pedagogy is aligned with her statements about how she thinks her teaching has changed over the years (See Appendix B).

Ella

My overall view of Ella is that she is *content and procedurally orientated in terms of the constructs that learners require for doing the mathematics in a specific item*. Ella is limited in what she sees as important for learners to know. Her interventions demonstrate that there is a relationship between what 'works' for her and what is directly transmitted to her learners. Ella noted that in Item 1, an angle is a measure of turn and the arc is indicative of the amount of turn, hence the arc shows the size of an angle. For Item 2, she is of the opinion that the borrowing procedure helps one to subtract a larger digit from a smaller digit in a number. She mentioned that in Item 3 one needs to know how to do SI unit conversions but added that these conversions are necessary, because we need to change to one common unit when working with different units in the same calculation. (The notion of decimals and their intersection with measurement units was omitted). The constructs Ella mentioned for Item 4 are that one must be able to share equally, know the meaning of the numerator and denominator in fraction notation and know that 'third' means three equal parts. For Item 5, she explained that one must be aware that area measures an amount of surface and know how to use the formula for the area of a rectangle.

Ella was able to recognize some of the errors and misconceptions evident in the item distractors. I have used her ideas and suggestions to show her reasoning about learner thinking:

According to Ella, learners think the angle with the longest arms is the biggest (Item 1) (Barrett et al. 2003, Mitchelmore 1998, Magina and Hoyles 1997).

“The longer the arms the bigger the angle is”.

Ella suggests that learners experience problems with the borrowing procedure because they do not think that they can subtract unless the larger digit is in the minuend (Item 2) (Brodie et al. 2009 Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006).

“He took one from 9 to the tens column and to the units. He never cancelled. He is missing the borrowing concept. He lacks foundation in the use of the minus sign. He doesn’t know what to do when there is a small number taking away a big number. He is not aware that he can break a number and he thinks that the big number must always be on top and the small numbers below.”

Ella asserts that learners ignore the decimal point and work with a whole number scheme thereby reading the decimal as a number with hundreds, tens and units place values only. Learners also see a two digit decimal as needing a zero at the end and if the zero is absent, learners think the last digit of the two digit decimal represents millimetres (Item 3) (Mitchell and Horne 2008, Steinle 2004).

“They don’t know anything about conversions. To them it is units tens hundreds. They see the number as 324 and the point just came there automatically. They were distracted by the zero that is supposed to be at the end and because there is no zero at the end they said it is 4 mm.”

Ella thinks that learners call any fraction portion of a whole a ‘quarter’ because they are unfamiliar with equal sharing into thirds and fifths (Item 4) (Newstead, 2000).

“They can’t divide into equal parts and learners see a quarter and not a third or a fifth in an object. They don’t understand the denominator and equal sharing.”

According to Ella, learners think they have to count all the congruent squares in a shape irrespective of their linear dimensions (Item 5) (Battista et al., 1998).

“They just counted the squares in the shape.”

It is evident that Ella’s reasoning about learner errors and misconceptions *is influenced by a procedural emphasis in the way she articulates the mathematical content related to the items.* She does not seem to venture out of these boundaries and she made no mention of many other misconceptions, for example, learners’ confusion between area and perimeter or that area means to count SI unit squares as opposed to counting square units that have different linear dimensions. In her reasoning, there is evidence of conceptual thinking related to a decimal number. Ella recognizes that learners do not understand the role of the decimal point and work

with whole number schemes instead. Ella is more focused on procedure and the correct application of the borrowing algorithm in subtraction. Within the confines of the procedural emphasis in her knowledge she was able to ascertain, why learners are confused.

Ella's interventions

Item 1 (Differentiate angle size from orientation and side length)

Ella's intervention is focused on dispelling the misconception that the length of the arms of an angle indicates its size. Ella wants learners to focus on the angle concept as a notion of turn or rotation. She directs learners' attention to the item and expects learners to explain why they chose a particular angle as the largest (other than the correct one) after they are reminded that an angle is a measure of turn:

Ask the learner if he remembers we said an angle is an amount of turn between two arms. Look at A and C. Which one is open wider? Get him to then explain why he made A his first choice and not C, if C is wider than A. I would make an instrument out of cardboard and move the arm. Maybe I can use the arms of a clock.

The order of Ella's intervention steps is problematic. Learners who have not conceptualized angle as a notion of turn will not be able to justify why one angle is 'wider' than the other. Learners need to first gain an understanding for this justification by watching how the cardboard arms or the hands of a clock move. After learners have had the opportunity to internalize this construct, learners' attention can then be focused on a two dimensional comparison between angles. Ella does not refer to the role of the arc in her intervention – this is mentioned in her content knowledge and she may assume that learners know about the arc.

Item 2 (Subtract three-digit numbers)

The intervention is informed by Ella's own knowledge and procedural reasoning for doing a subtraction calculation. Ella attempts to drill the procedure that *she uses* rather than paying attention to a level of conceptual understanding required for remediating learners' errors and misconceptions with the borrowing algorithm:

What do you understand by subtraction? Small from big. Look at 900 and 358. Which one is big and which is small? If we have a number with zeros you must know that you have to borrow from the digit in the number with zeros in order to increase the number you have to take away from. You can't take 8 units from 0 units. You must go where you are able to borrow. Next door is another zero. Go further and borrow one. You are left with 8. Take one from the tens and you are left with 9 tens. Take one to the units and you now have 10 units.

Although Ella has noted that learners have difficulties with the borrowing procedure, she has reasoned that what the learners need is more practice and a reminder in terms of the steps used for this procedure. She knows how the method works, even when there are two zeros at the end of the minuend, and what she offers to her learners is a repetition of how she understands the

calculation can be successfully achieved. Ella's intervention demonstrates more of the same procedure rather than giving learners an alternative that attempts to give sense to a procedure.

Item 5 (Calculate the area of a given portion based on squares)

Ella wants her learners to recognize that area uses squares to cover a surface and that the formula is length times breadth. It must be noted that the formula is not required for Grade 6, but I have chosen to use Ella's intervention to highlight that Ella is merely transmitting to her learners how she uses her own content knowledge to arrive at the answer correct as opposed to attending to learners' errors and misconceptions:

Each square is 2 cm and so $2\text{ cm} \times 2\text{ cm} = 4\text{ square cm}$. Now count all the squares and times by 8.

Ella's learners are provided with no explanation as to why they need to establish that each tile is 4 square centimetres which she has calculated by using the formula for area. This explanation is a key conceptual construct missing in her intervention. Each tile is composed of four congruent square units. Her next step is to get them to count all the squares. Ella has not stated which squares need to be counted, and coupled with this is the assumption that learners can count more quickly if they count the tiles and multiply by 4.

Comment

I have shown that a greater emphasis on procedural knowledge reflected in Ella's content knowledge influences the way she expects to address learners' errors and misconceptions. Secondly, her pedagogy illustrates that *her content knowledge is fore-grounded and her learners' needs are back-grounded*. The procedure Ella uses for doing calculations (such as addition) with different SI units in the same calculation, by first converting to the same unit, is a method that makes sense. She does not consider that strategies which use a more conceptual approach would better serve to address learners' errors and misconceptions. There is an absence of practical work where the learners are engaged in some aspect of concept development through the use of manipulatives, or working with different representations, in order to transform their knowledge. Unlike Betty, who provides learners more meaningful mathematical experiences, Ella does not engage her learners in any type of 'hands on' conceptual learning in her three interventions.

Carla

My overall impression of Carla is that she evidences both *procedural and conceptual thinking in the way she uses her content knowledge to explain and address what learners require for doing the mathematics in the items*. There is a strong relationship between her content knowledge and what she believes learners need to know. For example, in Item 1, she reasoned that an angle measures the amount of turn as the arm rotates, and that the arms of an angle meet at a common vertex. She uses knowledge of a right angle to act as a benchmark in order to choose the largest angle out of a given group of angles. For Item 2, she believes that one needs to know how to

perform vertical and horizontal subtraction and be in a position to use the borrowing algorithm when a number ends in two zeros (procedural). In Item 3, she spoke about knowing decimal place values and how these place values are connected to centimetres and millimetres (conceptual). In Item 4, she mentioned that one needs to know that the fraction concept is based on the notion of equal sharing and the name of a fraction, for example, one third, tells us there are three equal parts in a given whole. The notion of equal sharing enables one to visualize and estimate the size of a fraction in a given whole, when only one piece is shaded and the rest of the whole not partitioned. For Item 5 she mentioned that area is the amount of surface that is covered with tessellated squares (conceptual). Carla also believes that one needs to know that the formula 'length x breadth' is important for the area concept (procedural).

In her reasoning about errors and misconceptions in the items, Carla suggested the following:

Carla thinks that learners look at the distance between the endpoints of the arms to determine angle size (Item 1) (Barrett et al. 2003, Mitchelmore 1998, Magina and Hoyle 1997).

"Their estimate of the space looks bigger at the endpoints."

Carla suggests that learners can borrow once from the column on the left of the smaller digit and reduce the digit in the column they borrow from (Item 2). They can borrow a second time in the same calculation but do not reduce the digit they borrow from. The two zeros at the end of the number are problematic. Learners think that the smaller digit must be subtracted from a larger digit and they swap the digits in the minuend and subtrahend in order to do so, or they resort to addition when there is a zero present (Brodie et al. 2009, Fernandez and Garcia 2008, Sadi 2007, Baker and Chick 2006).

"They borrowed from the tens but didn't reduce it. They borrowed from the 9 and they reduced it but forgot the middle one. $00 - 58 = 58$ They haven't been taught that you can say $0 - 8$. They are doing it backwards. In Grade 1 they are taught the big number must subtract the smaller number and never that the smaller number must subtract the bigger number."

Carla reasons that learners read the decimal as two separate whole numbers and ignore the decimal point (Item 3). They expand the two whole numbers they see and attach SI units in order of their sizes to the digits in the whole numbers from left to right (Mitchell and Horne 2008, Steinle 2004).

"They didn't look at the metres concept. They read the number in the order of the digits as $3 + 2 + 4$ and didn't look at the decimal. They put m, cm and mm in the order of largest to smallest to match the order of the numbers. There are 3 places in the number. The 2 must have 200 and the 4 must be 40. There are two digits after the comma so they were looking at breaking up a whole number after the point into $20 + 4$."

Carla asserts that learners are influenced by their knowledge of a half and a quarter from everyday life and call any shaded part in a whole by these names (Item 4) (Newstead, 2000). “They just see it as a piece which is smaller than a half. They don’t understand the concept of a third. They see quarters more often than the third from everyday life experience.”

Carla states that learners use the formula for a rectangle but they have problems with establishing the dimensions of the length and breadth (Item 5). The dimension of a tile given in the question is called the breadth and the dimension for the length is obtained by counting the number of tiles in the rectangle (Outhred and Mitchelmore 2000, Battista et al. 1998).

“They read 2cm in the question and counted the blocks and multiplied by 8 to make 16. They didn’t know the concept of length and breadth. They just counted the number of blocks because they were told that area is the number of blocks.”

Carla is aware of some of the ways learners think which leads to their making important errors and misconceptions. Except for Item 5 in which her content knowledge is influenced by procedural and conceptual thinking about area, she evidences content knowledge that is conceptual in all items – for example, she acknowledges that in Item 1, learners misunderstand the angle concept as a notion of turn and look at the distance between the endpoints of the arms. In Item 4 learners are able to borrow from one zero at the end of a number but they become confused when two zeros are present at the end. She has noted that some learners lack conceptual thinking about the role of the decimal point in Item 3, and that the part-whole notion in terms of different fraction sizes is absent in learners’ understanding.

Carla’s interventions

Item 1 (Differentiate angle size from orientation and side length)

In this intervention Carla wants learners to conceptualize how to choose the largest angle. She draws their attention away from the distance between the end-points of the arms and instead, uses the right angle as a tool to achieve her end:

Go back to the concept of the right angle where you have a perfect L. Look at the L which gives you 90 degrees. See where the one line of the angle looks like a bisecting line. A bisecting line is 45 degrees. Is it half way or is it less than that? Is the bisecting line nearer to the L line or nearer to the base line of the L? If it is nearer to the base line it is smaller.

Carla uses a right angle. She informs the learner that the moving arm (which she incorrectly calls the bisecting arm but is relying on a learner’s knowledge of the angle bisector as the extra arm in between), is between the vertical part of the ‘L’ shape and the horizontal part of the ‘L’ shape. She does not specifically state that the moving arm is making larger and smaller ‘acute’ angles. The assumption is that the learner is comparing acute angles only. Depending on the moving arm’s position versus the vertical and horizontal portions of the L shape, the learner can compare

larger and smaller angles. This method is only useful for angles with one arm lying in a horizontal position. If the angle is orientated differently the 'L' becomes more difficult for learners to see, and this can cause confusion. Carla has not considered that learners may need to have more experience with the notion of a turning arm by giving them a three-dimensional experience.

Item 2 (Subtract three-digit numbers)

Carla intends to dispel the notion learners have that a smaller digit cannot subtract a larger digit and therefore learners have to understand the borrowing procedure:

I have 14 counters and I want to take away 9 counters. My answer will be 5. Now without using counters I will say you cannot do $4 - 9$ so you will use the concept of place value and say $14 - 9$. 14 is one ten + 4. $24 - 19$ is 5 again. Use the concept of the neighbour. Can I borrow one ten from my neighbour which has 20? Yes, 20 is going to lend me one ten and 4 becomes 14.

Carla develops learners' understanding from the mathematics they can do, that is $14 - 9$ and she then links the 14 to other constructs with which her learners are familiar, expanded notation and place value. Thereafter the learners are confronted with the problem of $4 - 9$ which promotes conflict in their thinking and the learners believe that they cannot subtract a bigger digit from a smaller digit. She keeps the digits 4 and 9 the same which is an important conceptual move. Learners are once more asked to think ' $24 - 19$ ' and she uses expanded notation with the 24. Her use of the word 'neighbour' is effective because this is linked to learners' real life experience. What is also important is that she has not said 'tens column'. The act of borrowing in subtraction does not use place value headings and therefore what learners are constructing in their mind is that the 'neighbour' always means the column on the left. Carla likewise, through what she says, re-inforces the notion that what is borrowed from the column on the left is one ten and not just 'one'. The activity she proposed addresses conceptual understanding behind the borrowing procedure, and learners are able to see mathematically, with the aid of taking a group of ten from the column on the left, a bigger digit can be subtracted from a smaller digit in any place value column. What has been omitted in this intervention is that her learners have not learned what step to take if there is a zero present in the column from which a ten is taken.

Item 5 (Calculate the area of a given portion based on squares)

Carla's intervention is the same as that of Ella. She relies heavily on how she knows area is calculated with a formula and informs her learners of her procedure, even though it is not age appropriate to Grade 6:

Area is L and B. Each block is 2 cm by 2 cm which is 4. Underlying concept is the formula. You say 2×2 which is area. Area is $l \times b$ and that is the space occupied by the block. Then you say $4 \times 8 = 32$. You can also see the rectangle and two squares and say $4 \times 6 = 24$ and add the $4 + 4$ to the 24.

In this intervention Carla omits to differentiate between the square tile (block) and the tessellated squares units that are used to cover the surface of a shape. In her explanation she imposes the correct procedure for obtaining the correct answer by making use of the rectangle formula. Her learners are unclear as to why they must first multiply two by two. The explanation is devoid of conceptual understanding required for the area concept because she equates length x breadth with the space occupied by one tile. Therefore her learners are left with the notion that area is always equal to length x breadth. This notion is inappropriate as an intervention for learners who have area misconceptions in Grade 6.

Comment

Carla's interventions are interesting and although her content knowledge of the items is predominantly conceptual, this is not followed consistently in her espoused interventions. Her interventions demonstrate that she can be more cognizant of learners' conceptual needs by drawing on their prior mathematical knowledge to address their confusions, for example, using the right angle as a benchmark, area is the 'space occupied by a block (square)'. It is interesting to note that although her intervention for borrowing correctly in Item 2 is procedural, she wants to make the procedure more meaningful by interrupting learners' thinking with a conceptual explanation (expanded notation to explain why we borrow one group of ten). On the other hand, Carla demonstrates how her own understanding of the content in the 'area' item impels her to impose itself on her learners without taking into consideration what is at the root of their misunderstanding – that area is the measure of the number of iterated square SI units that cover the surface of a shape. She is more concerned with giving them a correct procedure to follow in this particular case. Two of her interventions (angle – using an angle bisector and area – using a formula) are inappropriate for Grade 6 learners, because these are concepts learned in Grade 7. I would argue that her PCK is concept and procedure driven, depending on her content knowledge. For example, she knows that area uses tessellated square units to cover a surface, but her intervention uses a formula to find the area of a rectangle. She has made mention that learners need to know how to borrow from 'two zeros' at the end of a number, but her intervention is driven by borrowing from one zero at the end and not two. Conversely, Ella thinks procedurally about 'borrowing correctly' irrespective of the number of zeros at the end of a number, and she imposes the notion of 'borrowing correctly' in her intervention.

Summary of the profiles

A Report of the Primary Mathematics Research Project points out that:

'Good' teachers use all kinds of methods and approaches, irrespective of 'traditional' or 'progressive' stereotypes, according to the nature and content of the topic being dealt with, the level of prior knowledge and comprehension of the children being taught.

(Schollar et al. 2004:41)

According to Schollar et al., there is nothing implicitly incorrect if learners are taught procedurally. However, handling learners' errors and misconceptions requires more than simply reteaching

content knowledge to learners who misunderstand mathematics. Their errors are not due to carelessness (Hansen and Drews 2005) and therefore they need the content to be reconstructed and represented in ways that is conceptually meaningful and sensible. My analysis drew distinctions between three teachers' and the relationships evidenced between their content knowledge, PCK and espoused pedagogy (interventions) when dealing with learners' errors and misconceptions.

I noted how Betty in particular, pays attention to learners who are struggling with mathematics by using conceptual representations in her interventions. Betty knows how to perform the mathematics in question, but she is also cognizant that learners construct knowledge which can lead to misconceptions. Therefore Betty has considered other strategies and remedies which she reasons might serve to address learners' confusions in more conceptual ways. Ella, on the other hand, reasons that learners need to 'get' the knowledge that she has by drilling methods with which she is conversant, and that have served her best (for example how to borrow twice in the same subtraction calculation). I was interested to note that her reflection about how her teaching has changed over the years is inconsistent with her demonstrated pedagogy in her interventions (See Appendix B). Unlike Betty, she does not always intervene with an age appropriate strategy. For example, her area intervention is beyond the level of understanding of Grade 6 learners. Carla has shown that she is capable of using a mixed strategy (conceptual and procedural) approach in her interventions. She is generally cognizant of what learners need to know but even on a conceptual level she uses representations that fall short of what learners require for conceptual growth and understanding. For example, her borrowing procedure stopped at the 'tens' column and yet what her learners need is to know how to borrow twice in the same calculation (borrowing from the tens column and then borrowing from the hundreds column). When she does use a different conceptual strategy, for example in her angle intervention, she does not consider that learners need to initially understand the fundamental meaning of angle. Like Ella, she too uses an age inappropriate intervention for area. The angle bisector is not Grade 6 content. Neither Ella nor Carla begin their area intervention with any form of prior knowledge about the area construct. Betty keeps her intervention on area within the learners' conceptual development by returning them to working with square centimetre units covering a shape. Carla seems limited with regard to the building blocks learners need to transform their knowledge, and she does not employ strategies that are different and more conceptually meaningful. Instead, she tends to use strategies that are governed by and lie within the boundaries of her own content knowledge and she does not always consider whether her strategies are age appropriate.

CHAPTER 7: CONCLUSION

Conclusion

My study aimed to investigate the ways teachers use their content knowledge to understand and address misconceptions that lie behind learners' errors in five multiple-choice items (ie. their PCK). In order to achieve this aim, I chose to use a key component of teachers' PCK, namely, that teachers need to be able to recognize errors and misconceptions embedded in learners' thinking, and devise strategies that endeavour to address learners' incorrect perceptions (Brodie et al. 2008, Adler 2005, Ball and Bass 2000, Smith et al. 1993, Nesher 1987, Shulman 1986). To this end, I framed the following research questions:

1. What pedagogical content knowledge do Intermediate Phase teachers demonstrate when analyzing learners' performance on 5 multiple-choice test items?
 - 1.1 In what ways do teachers reason about the misconceptions that underlie learners' mathematical errors?
 - 1.2 What pedagogical suggestions do teachers offer to address these misconceptions?

The methodology I chose was to interview six Intermediate Phase teachers in the field and probe their reasoning about the distractors in five multiple choice ICAS items and the interventions they suggested to transform misconceptions. My interview used a "think-aloud" method (Young, 2005) in which teachers answered questions I asked pertaining to the items. I initially probed their content knowledge by getting them to explain what mathematics is required to arrive at the correct answer. Thereafter teachers had to identify misconceptions in the item distractors and reason why some learners chose the distractors. They were then asked to explain how they would address one of the misconceptions in the item distractors by means of an intervention with learners.

I administered a task analysis (See Chapter 4) of each item as a frame of reference for my findings. I looked at the appropriate mathematical content required for the correct answer and used the literature to provide me with typical misconceptions evident in the item distractors. Each distractor was then analysed for an embedded misconception.

The data obtained from my six interviews was analysed (See Chapter 5) by using four themes – knowledge of mathematical content relevant to the items and use of procedural or conceptual thinking, language use when reasoning about learners' errors, awareness of errors and misconceptions in the items, interventions to address perceived misconceptions. I chose to conduct a deeper analysis on three teachers (See Chapter 6), whose pedagogical actions in their interventions were significant in terms of their own PCK.

Ball (2000) argues that mathematical content knowledge on its own is insufficient for effective teaching. The content needs to be unpacked in ways learners can understand – for example,

choosing what representations to use for new ideas, modifying or changing strategies, choosing tasks that are best suited to learners, and listening to learners' thinking. For most people, content knowledge of mathematics is obtained through procedural and /or conceptual learning (Long 2004, Kilpatrick et al. 2001). Hiebert and Lefevre (1986) argue that the relationship between conceptual knowledge and mathematical procedures assists in the storage and retrieval of knowledge for future applications. I assert that teachers of mathematics require both kinds of knowledge but more importantly, teachers need to have knowledge of what their learners may find difficult, and they need to use representations that are meaningful and useful (Ball 2000). Without conceptual knowledge of a mathematical idea and knowing ways of developing this idea, I argue that an intervention with learners who have misconceptions can prove to be meaningless. Armed with these two knowledge bases, teachers are more likely to detect if learners' erroneous thinking is due to carelessness or whether learner errors arise as a result of an incorrect conceptual underpinning of mathematical constructs (Hansen and Drews 2005, Nesher 1987). Knowledge of misconceptions requires that teachers alter their own strategies in relation to the content to address learners' confusions (Brodie et al. 2008, Smith et al. 1993).

One of the biggest tasks facing mathematics teachers in current education, is what decisions to make that will best serve the learners' needs in the reconstruction of knowledge (Ball 2000). This is important because unless cognitive structures in the building of mathematical concepts are transformed, misconceptions may continue throughout learners' lifetimes. This means that consideration must be given to the kind of pedagogical action necessary for knowledge transformation to be effective. Lampert (1991) is concerned that tasks given to learners are meaningful and meet their needs, Shulman (1986) talks about a teacher's ability to make multiple representations as a key aspect of PCK. Ball and Bass (2000) speak about teachers always anticipating what learners may think and respond with appropriate pedagogical action.

My analysis of the interviews evidenced that the teachers in my sample were not fully conversant with all the misconceptions embedded in the item distractors. At times they could identify more obvious misconceptions and ignored others which are equally important. For example, in Item 1 they could identify the common misconception that learners think that the angle with the longest arms is the largest, but misconceptions about angle orientation and the position of the arc in an angle were not mentioned. I noted that *none of the teachers mentioned all of the misconceptions* evidenced in all of the items' distractors, but those misconceptions that were identified by the teachers were consistent with the literature.

I also took cognizance of how my sample of teachers used mathematical language during the interview (see Theme 2 in Chapter 5). Although the teachers were aware that I knew what they were talking about, I was concerned that the same language may be used in the classroom, which could cause confusion in the learners' minds. For example, teachers spoke about the 'length' of

the arc instead of the ‘amount of turn’ of the arc. Such ambiguities in mathematical meaning can also play a role in the development of misconceptions.

With reference to my questions, I have found evidence that provides me with insight into the ways in which the teachers reason about mathematical content. They are able to explain the mathematical constructs and thinking processes they would use for finding the correct answer in five multiple-choice items. The teachers are able to explain why the misconceptions they identified in different domains occur but they are not fully conversant with all the misconceptions that emerge in each sub-domain (See Chapter 4). For example, they could establish that learners become confused with the borrowing algorithm when subtracting two numbers, but their reasoning stopped at the notion that learners think that a smaller digit is always subtracted from a larger digit. They were unable to identify that learners also struggle with the borrowing procedure when the larger number has two zeros at its end. In such a case, borrowing occurs twice in the same calculation and the process can lead to misconceptions if learners are only used to doing calculations in which they have to borrow once.

Teachers’ pedagogical suggestions point to evidence of *two types of relationships* between content knowledge and the way they perceive interventions to address learners’ misconceptions. The first type of relationship is concerned with teachers attending to the transformation of knowledge in ways that *learners’ needs are fore-grounded*. They are able to suggest creative ideas and strategies that will promote a greater understanding of a concept by reconstructing knowledge through sensible and meaningful tasks. The second type of relationship is about teachers who *impose their own understanding of a concept and foreground the content* instead of learners’ (mis)understanding. Their strategies (conceptual or procedural) foreground their thinking about the content and there is insufficient evidence to show that strategies and representations need to be changed in order to address specific learners’ errors and misconceptions.

My analysis indicates that one of the teachers (Betty) is more learner orientated whereby she makes a conscious decision to transform knowledge through the effective use of manipulatives and practical tasks. In my view, these types of conceptual experiences may serve to enhance learners’ understanding, as opposed to being drilled in a process of learning that is only procedural and content orientated. I found evidence that the second teacher’s (Ella) pedagogic actions are consistent with how she understands the content of the items, and this knowledge is reflected in her interventions – if she explains the content procedurally then her intervention is focused on mastering the correct procedure. There is no evidence that she is able to step outside of the boundaries of what she knows. In addition, she does not give learners tasks or representations that reconstruct their incorrect conceptual knowledge. Ella wants learners to ‘get’ the mathematics as she understands it, whether it is by procedural or conceptual means. The third teacher, Carla, is more aware of learners’ needs for conceptual modes of intervention. I found

evidence in her angle intervention that she is able to promote conceptual understanding by making use of a different representation that is meaningful, but in the sub-domain of area, she only requires that learners reflect her own procedural knowledge. Therefore I describe Carla's proposed interventions as both learner and content orientated. In sum, my analysis of the three teachers shows that the ways in which teachers reason about the mathematical content required by the items and about the erroneous reasoning of the learners, informs their PCK, particularly when a concept can be taught both procedurally and conceptually. It also shows that investigating teachers' reasoning of learners' errors can tell us about the form in which they hold their content knowledge.

I also noted that Betty is aware of the importance of making learning meaningful in the development of mathematical concepts, particularly when misconceptions have to be addressed. The strategies that she proposed for her interventions were consistent with the development of concepts, by giving learners mathematical experiences that are not focused on board work or telling alone. Betty's learners are exposed to her 'doing' something concrete with a manipulative (using geostrips) as part of a demonstration, or the learners are engaged with a 'hands on' activity to enhance concept development (using Dienes' blocks, counting square centimetre units within a shape on grid paper). I contend that Ella's learners may remember the procedure she drilled, but her pedagogy derives from her own understanding of the content, and she may think that learners need only to be reminded of how she perceives the content. Although procedures can be learnt before conceptual understanding occurs (Long 2004), I consider that learners with misconceptions need interventions that are grounded in a variety of sense making activities. The literature states that teachers must be able to make judgements and decisions about the work that is required to restructure learning (Adler 2005, Ball and Bass 2000, Fennema and Franke 1992). My research has further indicated that when handling misconceptions, some teachers may think conceptually about the content and this reflects in the conceptual decisions they make for pedagogic action. A second group of teachers may reason about the content conceptually but their pedagogic action is more procedural. A third group of teachers may think about the content procedurally and this thinking is reflected in their pedagogic decisions.

Recommendations for further research

Research on a larger group of teachers in the field could show whether the three groups I have identified in relation to content reasoning, reasoning about errors and misconceptions and pedagogy are prevalent in equal measure, or whether one group predominates over the other two. This study focused on two mathematical domains, Number and Measurement. Further research on teachers' reasoning about learner errors and misconceptions can be conducted in the domains of Space and Shape, Data Handling and Number Patterns. Research on teachers' reasoning about errors reflected in learners' work in open-ended questions, could provide further insight in this field of investigation.

Recommendations for Mathematics Teacher Education

All teacher training institutions need to develop student teachers' mathematics PCK as much as possible *before* they enter teaching. Such mathematics courses must focus primarily on knowledge of the mathematics curriculum in all primary school grades, irrespective of the phase that a student teacher is registered for. Teachers need to be aware of what prior content knowledge a learner should have been exposed to in any mathematical domain. A teacher also needs to be well versed in the assessment standards for the learner's current grade. This knowledge can be made available during a teacher's undergraduate mathematics courses. Any course devoted to pedagogy *must include work on mathematical errors and misconceptions*. This entails paying attention to what the literature specifies and thereafter providing students with assignments to create strategies with accompanying activities that attempt to transform erroneous knowledge in the mathematical domains, with a focus on learners rather than on content.

Prior to my conducting this research I experimented with a small group of undergraduate Intermediate Phase teachers who were enrolled in a mathematics course to develop their classroom pedagogy. The students' content knowledge was in place and the course sought to give them tools and principles from the literature, to help them make choices that enable teaching to be effective. One of the aspects of the course addressed the notion of mathematical errors and misconceptions. The students worked with multiple-choice items from the ICAS tests. They were constantly surprised to find that there is a wealth of misconceptions in mathematical domains with which they were not familiar. They were interested in this exposure to errors and misconceptions and were grateful to have had the opportunity to analyze the item distractors with their embedded misconceptions, and create their own pedagogical actions to address them.

Limitations

I am aware that this study reveals its own strengths and weaknesses. I was fortunate to work with a group of teachers who felt comfortable with me, and were in a position to speak to me openly and spontaneously. They appeared to be keen to share their thoughts and did not find my questions difficult to answer. However, the interview was conducted on a colleague-to-colleague basis and they may have assumed that I 'understood' their explanations, and therefore some important details may have been disregarded in their responses. Due to time constraints I was unable to probe their thinking prodigiously and my interview schedule had to be limited. Therefore I cannot purport to know how they would reason in mathematical domains, for example, 'Space and Shape' which did not form part of this study. Had I been in a position to include other domains in this study, the teacher profiles may appear different from the ones I have constructed. A study of six teachers may be too few to determine whether the two relationships mentioned above are the only relationships that exist between teachers' content and pedagogy.

Before conducting this study with experienced Intermediate Phase teachers, I was unaware of the various ways teachers view their learners' errors. Some teachers have the ability to 'go behind the scenes' and take a hard look at the way learners are thinking and act pedagogically to address confusions. Other teachers are not aware that learners, who have misunderstandings about the content, may need to experience completely different strategies for the reconstruction of mathematical knowledge.

Appendix A

Interview schedule

The schedule comprises four sections in which questions are asked. If a teacher does not grasp the meaning of the questions put to them in the interview I will rephrase them as the need arises.

Section A

In this section teachers share their teaching experience in the Intermediate Phase in South Africa today:

1. What grade(s) are you currently teaching?
2. How long have you been teaching mathematics?
3. How have your teaching approaches changed from when you first started teaching?
4. Do you as a mathematics teacher ever discuss learners' misconceptions and errors with your colleagues in your mathematics department?

Section B

This section aims to investigate the teachers' knowledge of the curriculum through their analysis of the items in terms of curriculum alignment. The teachers will be given all the Assessment Standards stated in the National Curriculum Statement for Grades 1-6.

This section focuses on the errors and misconceptions in the six items:

1. What area of the mathematics is the item testing?
2. What is its Learning Outcome?
3. What assessment standards from the RNCS document can be mapped onto the item?
4. Can you find any assessment standards that are required as prior knowledge for the item?

Section C

This section aims to investigate the teachers' knowledge of the curriculum through their Assessment Standards stated in the National Curriculum Statement for Grades 1-6:

1. In order for a learner to answer this item correctly what prior mathematical knowledge needs to be present?
2. What mathematical understanding is required to choose the correct answer?
3. What are the incorrect (I name two of the distractors) distractors testing?
4. Why was distractor (I name the distractor) chosen by the majority of the learners?
5. What mathematical understanding has to replace the misconception(s) that the learner has?
6. What feedback would you give to a learner who chose one of the wrong answers?

Section D

This section will show if the teacher's thinking about teaching the mathematical area in the item has been impacted after being probed about the distractors in Section C:

Choose any item and explain what avenue you would take to avoid misconceptions developing in the mathematical area concerned.

Appendix B

Angie:	As an inexperienced teacher, I used to drill and push it out of the child. My approach has changed and I go for more hands on practical work. I have come to the conclusion that I'm not going to get the children to be better at maths but I can change their attitudes. They used to hate maths but now they can't wait to get to class. The change in attitude is also due to the different methods I use to teach maths. For example, I allow them to play maths games and they don't realize they are learning through these games. They enjoy the hands on approach to concepts
Betty:	Now there's more focus on doing more practical stuff in the classroom. A lot of schools where I taught in the past there were just text books and a lot of the stuff used to be rote learning. Now we are doing more practical stuff where we help learners to have a more hands on approach to maths rather than using rote learning or expecting children to learning recipes and talk and chalk.
Carla:	Today I use different experimental methodologies. If one concept doesn't work in one way I teach it in another way because the children are very weak, whereas in the past we used to teach using one method only. When I started in 1983 there were hardly any resources, maybe just a text book and there is much more available today to help concepts. You would adjust your methods to the children based on your experience. Today there are lots of resources available.
Dawn:	Before it was a case of showing children everything and I gave them methods and they had to regurgitate everything. They gave me answers and then I could see if they understood. Now I use a multiple method way of teaching where I listen to what the children feel what would be easy for them to use. I still bring a little bit of uniformity to the whole process to make teaching easier. I throw things open to children and get their opinion of a method that all of them can use to help them all.
Ella:	Now we use new approaches and we have to monitor slow ones. Learners can go to the board and explain a sum. Learners are sometimes scared of the teacher and they learn from peers. If they do peer teaching there is a great difference and you can come in and assist where they get stuck. There are no particular methods followed and we use mixed approaches. You have to facilitate now and give individual learners more attention if they struggle. If I prepare a lesson for group work and it doesn't work I can go to pair work and I can also open it to the class.
Fran:	In the old days there was a lot of rote learning, repetition and individual work. Today there is more group work and not so much rote learning. I use a lot more resources. I still tend to rely on some rote learning in my teaching

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