On polarity-based semantics for non-distributive modal logics

by

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Declaration

I declare that this thesis is my own, and that contributions from my supervisor were consistent with standard supervisory practices. This thesis is being submitted for the degree of *Master of Science* (MSc) at the University of the Witwatersrand, Johannesburg; and has not been submitted before for any degree or examination at any other University.

Signature of candidate

Date

Abstract

This masters study builds upon recent research in polarity-based semantics for non-distributive modal logics (NDMLs). Current formulations of polarity-based semantics for NDML impose compatibility requirements on additional relations of polarity-based frames, hindering applicability of the semantics, as arbitrary frames need not be compatible. In this study we develop a polarity-based semantics for NDML with modalites that are, in general, neither normal nor distributive and without the imposition of compatibility requirements. We provide a sound and complete axiomatization of this logic. The second half of the thesis focuses on a special class of enriched polarities, those who are in a sense liftings of Kripke frames. The compatibility of these liftings combined with the intuitive nature of the underlying Kripke frames makes for a useful case study in which to explore p-morphisms between enriched polarities, and enriched polarity-based models, from a relational perspective.

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Chapter 1

General Introduction

Non-classical propositional logic has been a major field of study in mathematical logic since the early 20th century. For the last several decades a number of modal logics built on various non-classical propositional bases have been developed including modal versions of intuitionistic [52], Łukasiewicz [34], and quantum logics. In recent years there has been growing interest in non-distributive modal logics (NDMLs) – modal logics built upon propositional bases in which the familiar classical distributive laws for conjunction and disjunction, $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \vdash \alpha \vee (\beta \wedge \gamma)$, need not necessarily hold. Several relational semantic frameworks have been thus far explored for NDML, most notably graph-based semantics (primarily of reflexive graph-based frames see for example [7] and [16]), and polarity-based semantics – the latter of which is the primary focus of this masters study.[1]

Polarities are triples $\mathbb{P} = (A, X, I)$ where A and X are non-empty sets, and $I \subseteq A \times X$ is a binary relation. Defined on $\mathbb{P}$ are two maps, $(\cdot)^{\uparrow} : \mathcal{P}(A) \to \mathcal{P}(X)$ and $(\cdot)^{\downarrow} : \mathcal{P}(X) \to \mathcal{P}(A)$, referred to as the *polars*[2] of I. If $B \subseteq A$ and $Y \subseteq X$ then the polar maps of I give

$$B^{\uparrow} = \{x \in X \mid (\forall a \in A)\,(a \in B \to aIx)\} \text{ and } Y^{\downarrow} = \{a \in A \mid (\forall x \in X)\,(x \in Y \to aIx)\}\,.$$

It is well known that these maps induce a Galois-connection [20, 156]. This Galois connection in turn induces associated closure operations, $(\cdot)^{\uparrow\downarrow} : \mathcal{P}(A) \to \mathcal{P}(A)$ and $(\cdot)^{\downarrow\uparrow} : \mathcal{P}(X) \to \mathcal{P}(X)$, under which subsets of A and X are deemed "stable" if they are equal to their closures (that is $B^{\uparrow\downarrow} = B$ or $Y^{\downarrow\uparrow} = Y$). It is to these Galois-stable subsets that propositional variables are evaluated in polarity-based semantics.

The presence of two underlying sets, A and X, in our polarity-based context, as opposed to a single underlying set W in Kripke frames has important ramifications for interpretations in polarity-based semantics. This *two-sortedness* ([26], [11]) of polarities leads to the encoding of interpretations by two relations, namely $\Vdash$ and $\succ$, read as, for example, "satisfaction" and "co-satisfaction" respectively.

Expanding non-distributive propositional logics to NDMLs requires the enrichment of polarities with additional relations, which correspond to modalities in the semantics. *Enriched polarities* or *polarity-based frames* are for our purposes tuples $\mathbb{F} = (\mathbb{P}, \{R_{\Box}, R_{\Diamond}, R_{\lhd}, R_{\rhd}\})$.[3]. Semantically, these additional relations can be given interpretations congruent to their respective modalities: $\Box$, $\Diamond$, $\lhd$, and $\rhd$. There have thus far been a number of interpretations of modalities in non-distributive modal logic including temporal [37], epistemic [11], and information theoretic [26] [39].

Current approaches such as [12], [16], [9], and [10] require that additional relations be *compatible* with the underlying polarity (see Chapter 4 for a formal statement). Stated informally, these requirements ensure that the semantic clauses for each of the modalities based on these relations send Galois stable sets to Galois stable sets. These compatibility requirements simplify the clauses for modalities in the

[1] Polarity-based semantics itself has been studied through the lens of several semantic frameworks. For a short overview see Section 1.1.3.

[2] See Chapter 3 for a more thorough description along with historical notes.

[3] Where $R_{\Box} \subseteq A \times X, R_{\Diamond} \subseteq X \times A, R_{\lhd} \subseteq A \times A$ and $R_{\rhd} \subseteq X \times X$ – see Definition 4.4. In general any logic algebraically captured by normal lattice expansions can be given a polarity based semantics; where every n-ary operation in the signature is assigned an $(n+1)$ relation of corresponding type in the semantic structures. See, for example, [16] and [17].

resulting semantics, as well as ensure that the modalities are normal, distributive, and have certain adjunction properties. The imposition of compatibility constraints is, however, not without cost. Crucially, compatibility requirements limit the applicability of the semantics, as arbitrary additional relations need not be compatible.[4]

A primary contribution of this study is the proposal and development of a polarity-based semantics for NDML that does not impose compatibility requirements on additional relations – a non-compatible polarity-based semantics for NDML. Our new semantics is adapted from a line of research developed in recent years such as [10], [11], [16] with compatibility requirements dropped. In their place we impose additional closures under $\uparrow$ and $\downarrow$ upon the semantic clauses associated with the relations: $R_\Box$, $R_\Diamond$, $R_\lhd$ and $R_\rhd$. This yields a semantics in which modalities are only mono/anti-tone, need not preserve meets/joins, or be normal. The axiomitizaion of this new semantics therefore only requires that the modalities be monotone (in the case of $\Box$ and $\Diamond$) and anti-tone (in the case of $\lhd$ and $\rhd$ – Section 4.2). We follow this with proofs of soundness and completeness (the subjects of Section 4.4.1 and Section 4.4.2 respectively).

Expressivity and definability are central topics in modern logical studies. For instance, in basic modal logic definability is encapsulated by the famous Goldblatt-Thomason theorem, which characterises classes of definable frames. The Goldblatt-Thomason theorem is discussed in more detail in Chapter 6. For compatible polarity-based semantics an analogous theorem to the Golblatt-Thomason theorem in basic modal logic has been formulated for a more general family of logics [17] – the so called *LE*-logics.[5]

In this masters thesis we initiate an analogous study of the expressivity of our non-compatible polarity-based semantics. Our study is constrained to a single truth preserving construction, p-morphisms – more specifically, p-morphisms for the subclass of polarity-based frames that are "liftings" of Kripke frames. Our notion of liftings of Kripke frames is adapted from recent attempts to apply polarity-based semantics in the context of rough set theory [9]. Restricting our focus to the special case of enriched polarities that are liftings of Kripke frames falls within the general scope of [17], as liftings of Kripke frames always produce compatible relations as is shown in Corollary 5.3. This preliminary study could form the basis for a future, more general, notion of p-morphism between enriched polarities in the general non-compatible setting. This could in turn form part of a broader study of truth preserving constructions on enriched polarities such as generated submodels, disjoint unions, and ultrafilter extensions. This could then form the basis of a potential Goldblatt-Thomason style theorem specialised to non-compatible polarity-based frames. This potential line of further research is addressed further in Chapter 7.

1.1 Historical overview and literature review

The historical roots of polarity-based semantics are branching and overlapping, passing through various fields and subfields over the last several decades. Our recounting of this history begins in the 19th century with the study of Boolean algebra in the early years of mathematical logic giving rise to lattice theory (Section 1.1.1). We then shift our attention to the founding of formal concept analysis as a subbranch of lattice theory in the 1980s (Section 1.1.2). Finally, in Section 1.1.3 we recount how polarity-based semantics in the form explored in this thesis emerged from the rapid development of substructural logic in the 1990s through to the 2010s.

1.1.1 From Boolean algebras to lattice theory

Lattice theory was born of the study of Boolean algebras in the 19th century – notably by Schröder, Boole, and Peirce. Dedekind also showed interest in lattices (what he called "dual groups") from an algebraic number theoretic perspective.

The study of abstract structures such as lattices was not in vogue in the late 19th century, and little, if any, further development of the theory took place until several decades later. In the late 1920s lattices of various kinds resurfaced in still further fields. K. Menger utilised modular lattices in his axiomatization

[4] See, for instance, Example 4.54.

[5] More generally, lattice expansions are general lattices with additional connectives of arbitrary finite arity that respect certain co-ordinate wise meet and join preservation requirements. For a full detailed exposition see for example [17]. Normal lattice expansions are often abbreviated to *LE*s.

of projective geometry. Soon after lattices appeared once more in formal logic through the research of F. Klein [1][51].

Lattices emerged as objects of study for algebraists in the 1930s and 1940s through the work of R. Remak, G. Birkhoff, and O. Ore. Birkhoff played a particularly substantial role in establishing lattice theory as a field of study in its own right – collating its applications, and grounding it firmly within the field of algebra. During this period Birkhoff coined the modern English terminology "lattice"[6] [1] [51]. Birkhoff saw lattice theory as [3]

> [A] proper vocabulary for discussing order, and especially systems which are in any sense hierachies."

And it is in this sense broadly that lattices are of interest to us in this study. Of particular relevance are two specific kinds of lattices – Boolean algebras, and concept lattices introduced in Section 2.3 and Chapter 3 respectively.

1.1.2 From lattice theory to formal concept analysis

By the mid-20th-century lattice theory had, to some extent, taken its place as a "basic tool of algebra" as Birkhoff had predicted decades prior, and growing into an algebraic subfield of its own. Increasing emphasis on the study of lattices as objects of interest in their own right, however, naturally widened the separation between lattice theory and the subfields of mathematics that had inspired its early development. Heightening abstraction and growing isolation came not without controversy, however. O Ore., one of the modern theory's founders is said to have remarked "I think lattice theory is played out" in a 1954 lecture, for example [1][46].

In the 1980s, R. Wille expressed similar feelings of disillusionment at the state of the field and sought to restructure lattice theory in a manner he hoped would "reinvigorate connections with our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory" [51]. In 1982 Wille published his foundational paper, *Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts.* This work, drawing from early works in formal logic which had formed the foundations of lattice theory a century or so prior, introduced an algebraic framework for reasoning about hierarchies of concepts. In this framework, that will be discussed in greater formal detail in Chapter 3, a concept is "reduced" to two constituent parts – its *extension* and *intention.* This reduction[7] is formalised through the introduction of *formal context* – a triple (G, M, I) mathematically identical to Birkhoff's polarities defined earlier and discussed in greater detail in Chapter 3. It is in this setting that Wille first used the terminology *formal concept* for ordered pairs, (A, B), of Galois-stable subsets of G and M respectively (Definition 3.19). Wille's approach in [51], along with subsequent publications throughout the decade, served as the foundation for formal concept analysis as an active subfield of applied lattice theory into the present day.

Wille's approach, whilst drawing from works in formal logic of the late 19th and early 20th centuries, did not aim to provide a logical framework for the treatment of formal concepts – a trend that largely persists in contemporary formal concept analysis. The logical study of hierarchies of concepts is not, however, an entirely recent pursuit. Notably, Hardegree published *An approach to the logic of natural kinds* in 1982 (several months after [51]) drawing from a 1969 paper by Thomason [49], develops a logic for polarities (what he calls *NK-structures*). In both [49] and [35], the non-distributive nature of the resulting logic is noted. Logics of the kind introduced by Hardegree – logics algebraically captured by complete lattices – are studied in far greater detail in the subsequent line of research explored in the next section.

1.1.3 A brief review of polarity-based semantics

Broadly, substructural logics are non-classical logics that are in some way weaker than classical logic, and have been a subfield of interest in the formal study of logic since the early 20th century (for a more detailed

[6] Inspired by the appearance of their Hasse diagrams.

[7] The reduction of concepts to their extension and intention is grounded in a long philosophical tradition which falls outside the scope of this masters thesis – for an overview of the philosophical history see [2].

history of substructural logic see [45], [44], and [42]) – notable examples include relevant logics logic, linear logic, Łukasiewicz systems of many-valued logic, and non-distributive logics. In the 1990s, there was a good deal of interest in the development of semantics for modal extensions of various substructural logics. This was, in several notable cases, accomplished through the development of representation theorems for the corresponding algebra.

Notably, in 1990 J Michael Dunn provided a uniform approach to relational semantics for a variety of substructural logics, including relevance logic and the Lambek calculus, through the development of his gaggle theory and representation theorems for the logics' corresponding algebras [21]. In [23], Dunn extends this approach for algebras whose corresponding logics are ones in which the distributive laws for conjunction and disjunction need not necessarily hold through the introduction of so-called partial gaggles. Several years later, in 2005, complete relational semantics were obtained in [22] for each of the logics treated in [23] via canonical extensions.

Later in 2005, Gehrke interpreted the semantics of [22] from a purely model theoretic perspective. Of primary interest to Gehrke in the study where polarity-based relational structures referred to as RS-frames. RS-frames are polarities satisfying four additional conditions, two referred to as *separating conditions* and the remaining two as *reduction conditions* (see [11, Definition 3] and [26]). The imposition of these RS-constraints results in a dual correspondence between the category of perfect lattices and the category of RS-frames [18]. In [26], Gehrke also proposed an information theoretic interpretation of this "two-sortedness" akin to be possible world interpretation of Kripke semantics. In this interpretation the elements of A are referred to as *worlds*in a similar fashion to in Kripke semantics, and the elements of X are referred to as *information quanta*, with aIx taken to mean x *is a part of* a. In this setting, satisfaction ($\Vdash$) is given a similar interpretation to classical Kripke semantics, but co-satisfaction ($\succ$) is taken to mean "a part of".

Following [26], polarity-based semantics have been proposed for a variety of substructural logics – such as linear logic [18], temporal logic [38]. More generally, alternative polarity-based semantics for substructural logics have been developed. In 2010 Suzuki developed his bi-approximation semantics for substructural logic, a polarity-based semantics for substructural logic in which additional relations are required to satisfy various "tightness" conditions for $\rightarrow$, $\leftarrow$, and $\circ$ [47][47]. A Sahlqvist theory for bi-approximation semantics was given in [48].

There have been several epistemic interpretations of polarity based semantics over the past several years. Notably, in [10] and [11] Conradie, Frittella, Palmigiano, Piazzai, Tzimoulis, and Wijnberg, develop a natural epistemic interpretation of the "two-sorted" semantics of [26]. This line of research models the epistemic attitudes of individual agents, $i \in \mathsf{Ag}$, as well as the interplay between the attitudes of the agents through the augmentation of the basic modal language with an additional normal box-type operator, C, referred to as a *common knowledge operator*. The resulting relational structures of interest in this setting are polarity based frames, $\mathbb{F} = (\mathbb{P}, \{R_i \mid i \in \mathsf{Ag}\})$ where $\Box_i$ are box-type modal operators. Possible interpretations of a formula such as $\Box_i\varphi$ could be "the category, φ, according to the agent, i" or "i knows φ" depending on the setting.

In [11] the RS requirements of [26] were maintained but were subsequently dropped [10], as well as in subsequent publications by this research group. The dropping of these requirements serves a similar purpose to the dropping of compatibility requirements in the formulation of our new polarity based semantics, in so far as arbitrary polarity based frames need not satisfy the RS requirements present in [26] and [11].

Using a modal expansion of Ploščica's representation of non-distributive lattices [43], reflexive graph-based frames [8] provide an alternative semantic framework for NDML – see for instance [7], [16], and [19]. Graph-based semantics has a wide variety of modelling applications; for instance the modelling of informational entropy [7], competing socio-political and scientific theories ([14] and [8] respectively), as well as epistemic applications – see for instance [16] and [7]. For a more thorough review of the graph-based semantics for NDML literature see [16] and [7].

[8] A reflexive graph is a ordered pair $\mathbb{X} = (Z, E)$ where Z is a non-empty set, and $E \subseteq Z \times Z$ is a reflexive relation. A graph based frame is a reflexive graph enriched with additional relations, analogous structure to our polarity based frames (see Definition 4.4).

1.2 Structure of this thesis

This thesis can be split roughly into two parts: the first half (Chapter 2–Chapter 4) focuses primarily on the introduction of our non-compatible semantics, whilst in the second (Chapter 5-Chapter 6), we explore truth preservation in the restricted context of p-morphisms between the liftings of Kripke frames.

We begin with a brief review of relevant mathematical and logical preliminaries in Chapter 2 before providing the necessary background in the theory of polarities/formal contexts in Chapter 3. The compatible semantics described in Section 1.1.3 are formally introduced in Section 4.5. The remainder of Chapter 4 is dedicated to our non-compatible semantics in the form of an axiomatization (Section 4.2), and proofs of soundness and completeness (Section 4.4.1 and Section 4.4.2).

The second half of this thesis, begins with Chapter 5, where liftings of Kripke frames in the style of [9] are introduced – this includes the statement and proof of several technical results used throughout Chapter 6. Chapter 6 concludes the body the thesis with a study of p-morphisms for liftings of Kripke frames.

Several areas for possible expansion and further research are summarised in Chapter 7. Finally, in Chapter 8, we briefly review the software developed during this masters study, which was used thoroughly in the generation of counter examples, examples, and the gathering of heuristic insights before the formal proof and statement of various results throughout this study.

1.3 Originality of results

In this thesis, unless otherwise stated, we will maintain the following convention as far as noting the originality of results and definitions is concerned. Numbered results and definitions in Chapter 2 and Chapter 3 are well known, and citations are given where necessary. In the remainder of the thesis, unless otherwise noted, both definitions and results should be taken as original if no citation is given.

Chapter 2

Preliminaries

In this chapter we briefly review some well-known definitions, results, and notation pertaining to sets and binary relations (Section 2.1), lattices (Section 2.2), Boolean algebras (Section 2.3), and modal logic (Section 2.4) relevant to the rest of the thesis. Our review is restricted to definitions and results used frequently throughout the main body of this thesis. The reader is referred to the cited references for more detailed explanations.

2.1 Sets and binary relations

Definition 2.1 (Image of a subset under a map f)**.** The *image* of a subset $U \subseteq W$ under a map $f : W \to Z$ is the set $f(U) = \{f(u) \mid u \in U\}$.

Definition 2.2 (Preimage of a subset under a map f)**.** The *preimage* of a subset $V \subseteq Z$ under a map $f : W \to Z$ is the set

$$f^{-1}(V) = \{w \in W \mid f(w) \in V\} = \{w \in W \mid \exists v \in V(f(w) = v)\}.$$

The notation $f^{-1}(v)$ is used for $f^{-1}(\{v\})$ where convenient.

Definition 2.3. For a binary relation, $R \subseteq W \times Z$, and subsets $U \subseteq W$ and $V \subseteq Z$ we use the notation

$$R[U] = \{z \in Z \mid \exists u \in U(uRz)\} \text{ and } R^{-1}[V] = \{w \in W \mid \exists v \in V(wRv)\}.$$

When $U = \{u\}$ (resp. $V = \{v\}$) is a singleton we will abuse notation by writing $R[\{u\}] = \{z \in Z \mid uRz\}$ as $R[u]$ (resp. $R^{-1}[\{v\}] = \{w \in W \mid wRv\}$ as $R^{-1}[v]$).

Definition 2.4 (The relations, Δ and Δ^c_W)**.** For a set, W, the relation, $\Delta_W \subseteq W \times W$, is $\Delta_W = \{(w, w) \mid w \in W\}$. Similarly, $\Delta^c_W = \{(w, w') \mid w \in W \text{ and } w \neq w'\}$.

2.2 Lattices

The logics studied in this thesis are algebraically captured by (normal) lattice expansions (LEs). In this section we review some well-known lattice theoretic definitions and results used at various points throughout the subsequent chapters of this thesis. For a brief overview of the development of lattice theory, see Section 1.1.1.

Definition 2.5 (Partial order [6])**.** A *partial order* on a set P is a binary relation, $\leq$, for which the following conditions hold for any $a, b \in P$:

(i) $a \leq a$ (reflexivity)

(ii) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)

(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$. (transitivity)

A non-empty set on which a partial ordering has been defined is a *partially ordered set.*

Definition 2.6 (Upper and lower bounds [20])**.** Let P be a partially ordered set, and $U \subseteq P$. An element $a \in P$ is an *upper bound* of U if $b \leq a$ for each $b \in U$. Similarly, $c \in P$ is a *lower bound* of A if $c \leq b$ for each $b \in U$.

Definition 2.7 (Supremum and infimum [6])**.** Let P be a partially ordered set and $U \subseteq P$. An element $a \in P$ is a *least upper bound* or *supremum* of U if a is an upper bound of U and $a \leq d$ for any upper bound, d, of U. Similarly, an element $c \in P$ is a *greatest lower bound* or *infimum* of U if b is a lower bound of U and $d \leq c$ for each lower bound, d, of U.

The supremum and infimum of U are denoted $\sup U$ and $\inf U$ respectively.

Lattices can be defined both order-theoretically (Definition 2.8) and algebraically (Definition 2.9). These two definitions are in a sense equivalent (Proposition 2.10), and both are used throughout this study depending on context [46].

Definition 2.8 (Lattice – order theoretic [6])**.** A partial order $\mathbb{L} = (L, \leq)$ is called a *lattice* if $\inf\{a, b\}$ and $\sup\{a, b\}$ exist for every $a, b \in L$.

Definition 2.9 (Lattice – algebraic [6])**.** A lattice is a triple $\mathbb{L} = (L, \vee, \wedge)$ where L is a non-empty set, and $\vee$ and $\wedge$ are binary operations on L satisfying the following conditions for every $a, b, c \in L$

L1. $a \vee b = b \vee a$ (Commutativity)
$a \wedge b = b \wedge a$

L2. $a \vee (b \vee c) = (a \vee b) \vee c$ (Associativity)
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

L3. $a \vee a = a$ (Idempotency)
$a \wedge a = a$

L4. $a = a \vee (a \wedge b)$ (Absorption laws)
$a = a \wedge (a \vee b)$

The operations $\vee$ and $\wedge$ are referred to as *join* and *meet* respectively.

Strictly speaking, the order-theoretic and algebraic lattice definitions describe different objects – an ordered pair and ordered triple respectively. From any lattice ordering in the sense of Definition 2.8, $\leq$, however, arises a pair of join and meet operations; and dually $\vee$ and $\wedge$ produce an ordering in a natural way, as detailed in Proposition 2.10.

Proposition 2.10 (Equivalence of algebraic and order theoretic lattice definitions [6])**.** If $\mathbb{L} = (L, \leq)$ is a lattice in the sense of Definition 2.8 then $(L, \vee, \wedge)$ is a lattice in the sense of Definition 2.9 where $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$. If $\mathbb{L} = (L, \vee, \wedge)$ is a lattice in the sense of Definition 2.9 then $(L, \leq)$ is a lattice in the sense of Definition 2.8, where $a \leq b$ if and only if $a = a \wedge b$.

Proposition 2.10 is well known fundamental result in lattice theory. For this reason a proof of Proposition 2.10 is omitted and we refer to texts such as [6] and [20].

Given the equivalence expressed in Proposition 2.10 will write $\bigwedge U$ for $\sup U$ and $\bigvee U$ for $\inf U$ when $\sup U$ and $\inf U$ exist.

Definition 2.11 (Atom[20])**.** An element x of a lattice $\mathbb{L}$ with least element 0 is an *atom* of $\mathbb{L}$ if $0 < x$ and $0 \leq a < x$ implies $a = 0$.

We denote the set of atoms of $\mathbb{L}$ by $\mathrm{At}(\mathbb{L})$.

Definition 2.12 (Join primality [41]). An element $a \in \mathbb{L}$ is *join prime* in $\mathbb{L}$ if $a \leq x \vee y$ implies $a \leq x$ or $a \leq y$ for every $x, y \in \mathbb{L}$.

If, for any $X \subseteq L$, we have

$$a \leq \bigvee X \text{ implies } a \leq x \text{ for some } x \in X;$$

then a is *completely join prime* in $\mathbb{L}$. Clearly complete join primality implies join primality.

Definition 2.13 (Atomic, complete, and perfect lattices). A lattice, $\mathbb{L} = (L, \vee, \wedge)$, is:

(a) *Atomic* ([20]) if for each $a \neq 0$ there is an $x \in \mathrm{At}(\mathbb{L})$ such that $x \leq a$.

(b) *Complete* ([50]) if $\bigvee U$ and $\bigwedge U$ exist for every $U \subseteq \mathbb{L}$.

(c) *Perfect* ([50]) if it is both complete and atomic.

Definition 2.14 (Bounded lattice [6]). A bounded lattice is a tuple $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ where $(L, \vee, \wedge)$ is a lattice, and $x \wedge \mathbf{0} = \mathbf{0}$ and $x \vee \mathbf{1} = \mathbf{1}$ for all $x \in L$.

Definition 2.15 (Distributive lattice [6]). A lattice, $\mathbb{L}$, is *distributive* if it satisfies the following additional properties for each $a, b, c \in \mathbb{L}$:

(D1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(D2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Proposition 2.16.

(a) Every finite lattice is complete.

(b) Every finite lattice is bounded.

(c) Every finite lattice is atomic.

(d) Every finite lattice is perfect.

Proof. Let $\mathbb{L}$ be a finite lattice.

Proposition 2.16(a) is a well known result – see for instance [6] or [50]. We include a sketch of the proof below. Let $\mathbb{L}$ be a finite lattice with the elements of L enumerated as in (b) – that is $L = \{a_1, \ldots, a_N\}$ for some $N \in \mathbb{N}$. Consider $U \subseteq L$ with $U = \{u_1, \ldots, u_M\}$ for some number $M \leq N$. Then define recursively, $u_i' \in L$ by: $u_1' = u_1$ and $u_i' = u_{i-1}' \vee u_i$ for $2 \leq i \leq M$. It is easily verifiable by induction that

$$\bigvee_{1 \leq j \leq M} u_j = \bigvee U = u_M'.$$

The proof that $\mathbb{L}$ respects finite meets is order dual.

For (b), as $\mathbb{L}$ is finite we can write $L = \{a_1, \ldots, a_N\}$ for some $N \in \mathbb{N}$. As $\mathbb{L}$ is a lattice it is closed under taking finite meets and joins. Thus $\bigwedge_{1 \leq i \leq N} a_i \leq a$ and $\bigvee_{1 \leq i \leq N} a_i \geq a$ for each $a \in \mathbb{L}$. Thus $\mathbb{L}$ is bounded below by $\bigwedge_{1 \leq i \leq N} a_i$ and above by $\bigvee_{1 \leq i \leq N} a_i$.

For (c), note that, by (b), $\mathbb{L}$ is bounded. Let $a \in \mathbb{L}$. If $a \in \mathrm{At}(\mathbb{L})$ we are done. If $a \notin \mathrm{At}(\mathbb{L})$ there exists an $x_0 \in \mathbb{L}$ such that $0 < x_0 < a$. Once more, if $x_0 \in \mathrm{At}(\mathbb{L})$ the result follows. If $x_0 \notin \mathrm{At}(\mathbb{L})$ then there is a x_1 such that $0 < x_1 < x_0$. As $\mathbb{L}$ is finite we may proceed in this way with the process terminating for some x_N (with $N \geq 1$) such that $0 < x_N < x_{N-1} < \ldots < x_0 < a$ where x_N is the desired atom. Proposition 2.16(d) follows directly from (a) and (c). $\dashv$

Definition 2.17 (Compliment of a lattice element [20]). For $a, b \in \mathbb{L}$ with $\mathbb{L}$ a bounded lattice we say that b is a *compliment of* a if $a \vee b = 1$ and $a \wedge b = 0$. If b is unique in this regard then we use the notation $a' = b$.

Definition 2.18 (Filter of a lattice [6]). A filter, F, of a lattice, $\mathbb{L}$, is a non-empty subset $F \subseteq L$ such that for any $a, b \in \mathbb{L}$:

(a) $a \in F$ and $a \leq b$ implies $b \in F$.

(b) $a, b \in F$ implies $a \wedge b \in F$.

Definition 2.19 (Ideal of a lattice [6]). An ideal, J, of a lattice, $\mathbb{L}$, is a non-empty subset $J \subseteq L$ such that for any $a, b \in L$:

(a) $a \in J$ and $b \leq a$ implies $b \in J$.

(b) $a, b \in J$ implies $a \vee b \in J$.

The sets of ideals and filters on $\mathbb{L}$ are denoted $J(\mathbb{L})$ and $F(\mathbb{L})$ respectively.

Definition 2.20 (Ideal and filter generated by a set [6]). For $X \subseteq L$ non-empty the *filter generated by* X is the least filter of $\mathbb{L}$ containing X. Similarly, the *ideal generated by* X is the least ideal of $\mathbb{L}$ containing X.

Definition 2.21 (Pricipal ideals and pricipal filters [6]). A principle ideal of a lattice, $\mathbb{L}$, is a subset of the form $\{u \in L \mid u \leq x\}$ for some $x \in L$. Similarly, a principle filter of L is a subset of the form $\{u \in L \mid x \leq u\} \subseteq L$ for some $x \in L$.

If $J_x = \{u \in L \mid u \leq x\}$ then J_x is said to be the *principal ideal generated by* x. Similarly, $F_x = \{u \in L \mid x \leq u\}$ is the *principal filter generated by* x.

Proposition 2.22. For any $x \in L$ the principal ideal (resp. filter) generated by x is indeed an ideal (resp. filter) of $\mathbb{L}$.

Proof. If $a \in J_x$ and $b \leq a$ then as $a \leq x$ from transitivity of $\leq$ it follows that $b \leq x$ and so $b \in J_x$. As $a, b \leq x$, x is an upper bound of $\{a, b\}$ and therefore $a \vee b = \sup\{a, b\} \leq x$ and so $a \vee b \in J_x$. The proof for F_x is order dual. $\dashv$

Proposition 2.23. If $f \in F(\mathbb{L})$ and $j \in J(\mathbb{L})$ then for any $u, v \in \mathbb{L}$:

(a) $u \in f$ and $v \in f$ if and only if $u \wedge v \in f$.

(b) $u \in j$ and $v \in j$ if and only if $u \vee v \in j$.

Proof. The forwards implication in (a) follows directly from Definition 2.18(b). For the converse, suppose that $u \wedge v \in f$. Then $u \wedge v \leq u$ and $u \wedge v \leq v$, and thus by Definition 2.18(a) it follows that $u \in f$ and $v \in f$.

Similarly for (b), the forwards implication follows directly from our assumption that j is an ideal and Definition 2.19(a). For the converse suppose that $u \vee v \in j$. Then as $u \leq u \vee v$ and $v \leq u \vee v$ it follows from Definition 2.19(a) that $u \in j$ and $v \in j$. $\dashv$

Definition 2.24 (Monotone and antitone operations). A *monotone* operation on a lattice, $\mathbb{L}$, is a unary operation operation $h : \mathbb{L} \to \mathbb{L}$ such that for any $u, v \in \mathbb{L}$:

$$u \leq v \text{ implies } h(u) \leq h(v).$$

Similarly, h is called *antitone* if

$$u \leq v \text{ implies } h(v) \leq h(u).$$

In general the operations in this definition need not be unary. As we concern ourselves exclusively with unary lattice operations, however, all monotone and antitone operations will be assumed unary unless stated otherwise.

Definition 2.25 (Galois connection)**.** Let $P_1 = (S_1, \leq_1)$ and $P_2 = (S_2, \leq_2)$ be partially ordered sets. Then maps $(\cdot)^{\vartriangle} : S_1 \to S_2$ and $(\cdot)^{\triangledown} : S_2 \to S_1$ define a *a Galois connection* if for each $a \in S_1$ and $b \in S_2$:[20]

$$a^{\vartriangle} \leq_2 b \text{ iff } a \leq_1 b^{\triangledown}.$$

Definition 2.26 (Lattice homomorphism [20])**.** A *lattice homomorphism* is a map $h : \mathbb{L}_1 \to \mathbb{L}_2$ where $\mathbb{L}_1 = \left(L_1, \vee^1, \wedge^1\right)$ and $\mathbb{L}_2 = \left(L_2, \vee^2, \wedge^2\right)$ such that for any $a, b \in L_1$

$$h\left(a \vee^1 b\right) = h(a) \vee^2 h(b) \text{ and } h\left(a \wedge^1 b\right) = h(a) \wedge^2 h(b).$$

That is, a lattice homomorphism is a map between lattices that preserves meets and joins.

Definition 2.27 (Complete lattice homomorphism [50])**.** A *complete lattice homomorphism* is a map $h : \mathbb{L}_1 \to \mathbb{L}_2$ between complete lattices $\mathbb{L}_1 = \left(L_1, \vee^1, \wedge^1\right)$ and $\mathbb{L}_2 = \left(L_2, \vee^2, \wedge^2\right)$ preserving arbitrary meets and joins. That is for any $U \subseteq L_1$:

$$h\left(\bigvee\nolimits^1_{a \in U} a\right) = \bigvee\nolimits^2_{a \in U} h(a) \text{ and } h\left(\bigwedge\nolimits^1_{a \in U} a\right) = \bigwedge\nolimits^2_{a \in U} h(a).$$

2.3 Boolean algebras

As noted in Section 1.1.1, the study of Boolean algebras in the late 19th century predates the founding of lattice theory as a field by several decades. In lattice theoretic terms, a Boolean algebra is a bounded (Definition 2.14), distributive (Definition 2.15) lattice, endowed with a complementation operation, satisfying four additional conditions as given in Definition 2.28 – two concerning complementation and two concerning the interaction of the top and bottom elements, 1 and 0, with $\wedge$ and $\vee$ respectively. Boolean algebras feature predominently in Chapter 6 of this thesis, where duality results for Boolean algebras with operators are leveraged to prove several results relating to truth preservation.

Definition 2.28 (Boolean algebra [6])**.** A *Boolean algebra*, $\mathbb{B}$, is a tuple $\langle B, \vee, \wedge, ', 0, 1\rangle$ where

(a) $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice;

(b) $a \wedge a' = 0$ and $a \vee a' = 1$ for each $a \in B$.

Throughout the remainder of this section, if left unspecified, $\mathbb{B}$ will refer to a Boolean algebra, $\mathbb{B} = \langle B, \vee, \wedge, ', 0, 1\rangle$ and $\mathbb{B}_i = \langle B_i, \vee_i, \wedge_i, '^i, 0_i, 1_i\rangle$ to indexed Boolean algebras with $i = 1, 2$.

Definition 2.29 (Complete/atomic Boolean algebra [50])**.** A Boolean algebra, $\mathbb{B}$, is complete/atomic if the underlying lattice $(B, \vee, \wedge)$ is complete/atomic. See Definition 2.13(b) and Definition 2.13(a) respectively.

Definition 2.30 (Operator [50])**.** A map $f : \mathbb{B} \to \mathbb{B}$ is an *operator* if for every $a, b \in B$

(a) $f(0) = 0$ (normality)

(b) $f(a \vee b) = f(a) \vee f(b)$ (additivity)

Definition 2.31 (Complete operator [50])**.** An operator, $f : \mathbb{B} \to \mathbb{B}$, is called a *complete operator* if it preserves arbitrary joins: $f\left(\bigvee U\right) = \bigvee_{u \in U} (f(u))$ for each $U \subseteq B$.

Definition 2.32 (Boolean algebra with operators (BAO) [5])**.** A *Boolean algebra with operators* or *BAO* is a tuple, $\mathcal{B} = \langle \mathbb{B}, \mathcal{F} \rangle$ where $\mathbb{B}$ is a Boolean algebra, and $\mathcal{F} = (f_i)_{i \in I}$ is a collection of operators.[1]

For the remainder of this section, $\mathcal{B}$ and $\mathcal{B}_i$ will refer to the BAOs $\mathcal{B} = \left(\mathbb{B}^1, \mathcal{F}^1\right)$ and $\mathcal{B}_i = \left(\mathbb{B}^i, \mathcal{F}^i\right)$ where $i = 1, 2$.

When referring to operators belonging to a BAO that is itself indexed (for example $\mathcal{B}_1$) we will use a superscript to refer to the BAO to which the operator belongs, and a subscript to denote its index – for example $f_i^1 \in \mathcal{F}_1$ in the case of operators belonging to a BAO, $\mathcal{B}_1 = \left(\mathbb{B}^1, \mathcal{F}^1\right)$.

Definition 2.33 (Q_f relations [50])**.** Let $\mathcal{B} = (\mathbb{B}, \mathcal{F})$ be an atomic Boolean algebra with operators, and $f \in \mathcal{F}$. Then the binary relation, $Q_f \subseteq \mathrm{At}(\mathbb{B}) \times \mathrm{At}(\mathbb{B})$, is defined as follows for $a, b \in \mathrm{At}(\mathbb{B})$

$$aQ_f b \text{ if and only if } a \leq f(b).$$

Definition 2.34. The *atom structure*, $\mathcal{B}_+$ of an atomic BAO, $\mathcal{B}$, is the relational structure

$$\mathcal{B}_+ = \langle \mathrm{At}(\mathbb{B}), \{Q_f \mid f \in \mathcal{F}\} \rangle .$$

Definition 2.35 (Perfect BAO [50])**.** A Boolean algebra with operators, $\mathcal{B} = (\mathbb{B}, \mathcal{F})$, is *perfect* if $\mathbb{B}$ is complete and atomic (Definition 2.29) and each $f \in \mathcal{F}$ is a complete operator (Definition 2.31).

Definition 2.36 (Boolean algebra homomorphism [40])**.** A map, $h : \mathbb{B}_1 \to \mathbb{B}_2$, is a *Boolean algebra homomorphism* if for each $a, b \in B_1$

(a) $h(a \vee^1 b) = h(a) \vee^2 h(b)$;

(b) $h(a \wedge^1 b) = h(a) \wedge^2 h(b)$;

(c) $h\left(a'^{1}\right) = h(a)'^{2}$.

If, additionally, h preserves arbitrary meets and joins (h is a complete homomorphism between the lattice reducts of $\mathbb{B}_1$ and $\mathbb{B}_2$) then h is a *complete Boolean homomorphism.*

Definition 2.37 (Modal homomorphism [50])**.** If $h : \mathcal{B}_1 \to \mathcal{B}_2$ respects operators then h is a *modal homomorphism.* That is, for all $a^1 \in B^1$

$$h\left(f_i^1\left(a^1\right)\right) = f_i^2\left(h\left(a^1\right)\right).$$

Definition 2.38 (BAO homomorphism [50])**.** A map $h : \mathcal{B}_1 \to \mathcal{B}_2$ is a *BAO homomorphism* if $h : \mathbb{B}_1 \to \mathbb{B}_2$ is a Boolean homomorphism, and a modal homomorphism.

Definition 2.39 (Complete BAO homomorphism [50])**.** A BAO homomorphism, $h : \mathcal{B}_1 \to \mathcal{B}_2$, is *complete* if it respects arbitrary meets and joins (is a complete lattice homomorphism between the lattice reducts of $\mathcal{B}_1$ and $\mathcal{B}_2$).

2.4 Modal Logic

In the subsection we briefly review some well known definitions from modal logic that will be used at various points throughout this thesis – particularly in Chapter 5 and Chapter 6. For further background see, for example, [5].

Definition 2.40 (Kripke frame [5])**.** A Kripke frame is a ordered pair $\mathsf{K} = (W, R)$ where W is a non-empty set and R is a (possibly empty) binary relation, $R \subseteq W \times W$.

[1]For our purposes $\mathcal{F}$ will be taken as a collection of unary operators, and the normality and additivity conditions are adapted accordingly.

We will refer to an element of $w \in W$ as a *state*, R as an *accessibility relation*, and W as a *set of states*.

Definition 2.41 (Kripke model [5])**.** A Kripke model is an ordered pair $\mathcal{M}_{\mathsf{K}} = (\mathsf{K}, V)$, where K is a Kripke frame, and V is a map $V : \mathsf{Prop} \to \mathcal{P}(W)$ referred to as a *valuation*.

Definition 2.42 (Power set algebra [20])**.** The *power set algebra* of a set W is the Boolean algebra $\mathcal{S}$ where

$$\mathcal{S}(W) = (\mathcal{P}(W), \cup, \cap, (\cdot)^c, \varnothing, W).$$

Relational information can be algebraically captured by expanding the power set algebra of the underlying set, $\mathcal{S}(W)$, with an additional operation, m_R.

Definition 2.43 (Complex algebra of a Kripke frame [5])**.** The *complex algebra* of a Kripke frame, $\mathsf{K} = (W, R)$, is the expansion of the power set algebra $\mathcal{S}(W)$ with the additional operation $m_R : \mathcal{P}(W) \to \mathcal{P}(W)$ where $R^{-1}[U]$.

We will use the notation K^+ to denote the complex algebra of K. Note that we will at times use the term "complex algebra" to refer to the complex algebra of an enriched polarity.

The term "complex algebra" arises from the once prevalent usage of the term "complex" to refer to subsets of groups [29].

Theorem 2.44 (Every complex algebra is a perfect BAO [50])**.** The complex algebra, K^+, of a Kripke frame, K, is a perfect BAO.

Proof. See [50] or [28, Theorem 1.2]. $\dashv$

Theorem 2.45 (A Kripke frame is isomorphic to the atom structure of its complex algebra.)**.** $\mathsf{K} \cong (\mathsf{K}^+)_+$

Proof. This is a well known result. See, for example, [50, Proposition 42]. $\dashv$

Chapter 3

Polarities

In this chapter we introduce polarities, and their associated lattices of formal concepts. Additionally, we state and prove several basic results used extensively throughout the rest of this masters thesis. All of the technical results in this chapter follow from well known properties of general Galois connections - see for example [20], [25], and [4].

Additionally, in Example 3.2, we provide a running working example that is briefly revisited in Example 3.21, Example 3.28, and again in Example 4.54.

Definition 3.1 (Polarity [4])**.** A polarity, $\mathbb{P}$, is a triple $\mathbb{P} = (A, X, I)$ where A and X are non-empty sets and $I \subseteq A \times X$. The sets A and X will be referred to as sets of *objects* and *features*, respectively. The binary relation I will be referred to as the *incidence relation* of $\mathbb{P}$.[1]

If $(a, x) \in I$ we will use the shorthand aIx: read "object a has feature x" or "a is I-related to x".

Example 3.2. Consider the polarity $\mathbb{P}_M = (A, X, I)$ where $A = \{\mathrm{IN}, \mathrm{DH}, \mathrm{KG}\}$ where IN, DH, KG stand for *Isaac Newton*, *David Hilbert*, and *Kurt Gödel*, respectively.[2] We let X be a set of descriptors $X = \{\mathrm{math}, \mathrm{phil}, \mathrm{phys}\}$ standing for *mathematician*, *philosopher*, and *physicist* respectively.

Define the incidence relation as follows: $(m, d) \in I \subseteq A \times X$ iff m is described using descriptor d in the introductory paragraph of m's Wikipedia page. Polarity $\mathbb{P}_M$ can be represented diagrammatically as presented in Figure 3.1.

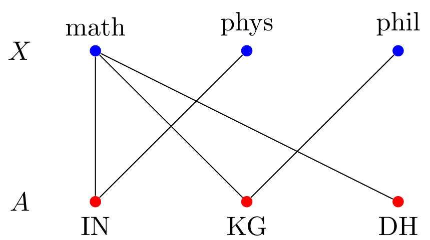


Figure 3.1: Polarity diagram for $\mathbb{P}_M$.

Definition 3.3 (Polar maps of $\mathbb{P}$ [4])**.** Let $\mathbb{P} = (A, X, I)$ be a polarity and $B \subseteq A$ and $Y \subseteq X$. Define maps $(\cdot)^{\uparrow} : \mathcal{P}(A) \to \mathcal{P}(X)$ and $(\cdot)^{\downarrow} : \mathcal{P}(X) \to \mathcal{P}(A)$ as follows

$$B^{\uparrow} = \{x \in X \mid (\forall a \in A)\,(a \in B \to aIx)\} \text{ and } Y^{\downarrow} = \{a \in A \mid (\forall x \in X)\,(x \in Y \to aIx)\}\,.$$

[1]Polarities are referred to as "formal contexts" in formal context analysis with G and M used in place of A and X – see [26],[20], and [25], for example. As this thesis is focused on polarity based semantics for NDML we will use the term "polarity" and associated notation as developed in the NDML literature – see [26],[10],[32], for example.

[2]This example is based on Wikipedia articles for the respective mathematicians as they were on 2022-07-02, 2022-07-07, and 2022-07-06 respectively. Snapshots of these pages can be found on the internet archive – `http://www.archive.org`.

The maps $(\cdot)^{\uparrow} : \mathcal{P}(A) \to \mathcal{P}(X)$ and $(\cdot)^{\downarrow} : \mathcal{P}(X) \to \mathcal{P}(A)$ are the *polars of I* or *polar maps of $\mathbb{P}$*.

Polarities in this form were discussed as objects of interest at least as early as Birkhoff in 1938 [3]. Birkhoff used the notation $(\cdot)^*$ and $(\cdot)^{\dagger}$ for the polar maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ respectively. The use of a single $(\cdot)'$ in place of $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ is common in the formal concept analysis literature (see [25], [51], [20] for example). The "up-down" notation is predominant in recent literature in polarity-based semantics for NDML (see for example [9], [16], [32]). We will at times omit bracketing when writing $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$.

The maps defined in Definition 3.4 are analogous to the polars maps of $\mathbb{P}$ generalised to an arbitrary binary relation $S \subseteq W \times Z$.

Definition 3.4 (The operations $(\cdot)^{(0)}$ and $(\cdot)^{(1)}$ [10])**.** For binary relation $S \subseteq W \times Z$ with $U \subseteq W$ and $V \subseteq Z$ define

$$S^{(1)}(U) := \{z \in Z \mid \forall w \in W(w \in U \to wSz)\} \text{ and } S^{(0)}(V) := \{w \in W \mid \forall z \in Z(z \in V \to wSz)\}.$$

For the remainder of this section, unless otherwise specified, we will fix $S \subseteq W \times Z$ and $U, U_1, U_2 \subseteq W$ and $V, V_1, V_2 \subseteq Z$. Additionally, we fix polarity $\mathbb{P} = (A, X, I)$ with $B, B_1, B_2 \subseteq A$ and $Y, Y_1, Y_2 \subseteq X$.

It will be useful to at times treat $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ as $I^{(1)}(\cdot)$ and $I^{(0)}(\cdot)$, respectively. Similarly, provided that W and Z are non-empty, $(\cdot)^{(1)}$ and $(\cdot)^{(0)}$ can be treated as the polar maps of a polarity (W, Z, S).

Lemma 3.5 (Antitonicity of $S^{(0)}$ and $S^{(1)}$ [10, Lemma 1.])**.**

(a) $V_1 \subseteq V_2$ implies $S^{(0)}(V_2) \subseteq S^{(0)}(V_1)$.

(b) $U_1 \subseteq U_2$ implies $S^{(1)}(U_2) \subseteq S^{(1)}(U_1)$

Proof. For (a), suppose $V_1 \subseteq V_2$. Then

$$\begin{aligned} w \in S^{(0)}(V_2) &\text{ iff } \forall z \in Z(z \in V_2 \to wSz) && \text{(Definition 3.4)}\\ &\text{implies } \forall z \in Z(z \in V_1 \to wSz) && \text{(as } V_1 \subseteq V_2.)\\ &\text{implies } w \subseteq S^{(0)}(V_1). && \text{(Definition 3.4)} \end{aligned}$$

Similarly, for (b), suppose $U_1 \subseteq U_2$. Then

$$\begin{aligned} z \in S^{(1)}(U_2) &\text{ iff } \forall w \in W(w \in U_2 \to wSz) && \text{(Definition 3.4)}\\ &\text{implies } \forall w \in W(w \in U_1 \to wSz) && \text{(as } U_1 \subseteq U_2)\\ &\text{implies } z \in S^{(1)}(U_1). && \text{(Definition 3.4)} \end{aligned}$$

Thus $U_1 \subseteq U_2$ implies $S^{(1)}(U_2) \subseteq S^{(1)}(U_1)$. ⊣

Corollary 3.6 (Antitonicity of $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ [20, Lemma 3.5])**.**

(a) $Y_1 \subseteq Y_2$ implies $Y_2^{\downarrow} \subseteq Y_1^{\downarrow}$

(b) $B_1 \subseteq B_2$ implies $B_2^{\uparrow} \subseteq B_1^{\uparrow}$

Proof. Noting that $(\cdot)^{\downarrow}$ and $(\cdot)^{\uparrow}$ is simply special notation for $I^{(0)}(\cdot)$ and $I^{(1)}(\cdot)$, substituting $(\cdot)^{\uparrow}$ for $S^{(1)}(\cdot)$ and $(\cdot)^{\downarrow}$ for $S^{(0)}(\cdot)$ in Lemma 3.5 produces Corollary 3.6. ⊣

Proposition 3.7 (Special cases for $(\cdot)^{(0)}$ and $(\cdot)^{(1)}$)**.**

(a) $S^{(1)}(U) = Z$ if $U = \varnothing$.

(b) $S^{(1)}(U) = \varnothing$ if $S = \varnothing$ and $U \neq \varnothing$.

(c) $S^{(0)}(V) = W$ if $V = \varnothing$.

(d) $S^{(0)}(V) = \varnothing$ if $S = \varnothing$ and $V \neq \varnothing$.

Proof. For (a)

$$\begin{aligned} S^{(1)}(\varnothing) &= \{z \in Z \mid \forall w \in W(w \in \varnothing \to (w,z) \in S)\} \\ &= \{z \in Z \mid \forall w \in W(w \notin \varnothing \text{ or } (w,z) \in S)\} \\ &= Z && \text{(vacuously)} \end{aligned}$$

For (b) suppose that $S = \varnothing$ and $U \neq \varnothing$

$$\begin{aligned} S^{(1)}(U) &= \{z \in Z \mid \forall w \in U \to (w,z) \in \varnothing\} \\ &= \{z \in Z \mid \forall w \in W(w \notin U \text{ or } (w,z) \in \varnothing\} \\ &= \varnothing. \end{aligned}$$

Similarly for (c)

$$\begin{aligned} S^{(0)}(\varnothing) &= \{w \in W \mid \forall z \in Z\,(z \in \varnothing \to (w,z) \in S)\} \\ &= \{w \in W \mid \forall z \in Z\,(z \notin \varnothing \text{ or } (w,z) \in S)\} \\ &= W. && \text{(vacuously)} \end{aligned}$$

For (d) suppose that $S = \varnothing$ and $V \neq \varnothing$

$$\begin{aligned} S^{(0)}(U) &= \{w \in W \mid \forall z \in Z\,(z \in V \to (w,z) \in S)\} \\ &= \{w \in W \mid \forall z \in Z\,((w,z) \in \varnothing \text{ or } z \notin V)\} \\ &= \varnothing. \end{aligned}$$ ⊣

In the same way that Corollary 3.6 follows directly from Lemma 3.5, we obtain Corollary 3.8 directly from Corollary 3.8.

Corollary 3.8 (Special cases for $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$)**.**

(a) $\varnothing^{\uparrow} = X$ always.

(b) $\varnothing^{\downarrow} = A$ always.

(c) $B^{\uparrow} = \varnothing$ if $B \neq \varnothing$ and $I = \varnothing$.

(d) $Y^{\downarrow} = \varnothing$ if $Y \neq \varnothing$ and $I = \varnothing$.

Proof. Follows directly from Corollary 3.8 with $S = I$. ⊣

Lemma 3.9 ([20, 7.22])**.** $U \subseteq S^{(0)}(V)$ if and only if $V \subseteq S^{(1)}(U)$.

Proof.

$$\begin{aligned} U \subseteq S^{(0)}(V) &\text{ iff } (\forall u \in U)\,(\forall v \in Z(v \in V \to uSv)) \\ &\text{ iff } (\forall u \in U)\,(\forall v \in V(uSv)) \\ &\text{ iff } (\forall u \in W)\,(\forall v \in V(u \in U \to uSv)) \\ &\text{ iff } (\forall v \in V)\,(\forall u \in W(u \in U \to uSv)) \\ &\text{ iff } V \subseteq S^{(1)}(U). \end{aligned}$$ ⊣

It should be noted that $S^{(0)}$ and $S^{(1)}$ do not, in general, have the additional property:

$$S^{(1)}(U) \subseteq V \text{ iff } S^{(0)}(V) \subseteq U.$$

This is illustrated by the following counter example.

Example 3.10. Consider the binary relation $S \subseteq W \times Z$ on the sets $W = \{1, 2, 3\}$ and $Z = \{a, b, c\}$ where $S = \{(1, a), (1, b), (1, c), (2, b), (2, c), (3, b)\}$. Then $S^{(0)}(\{a, b\}) = \{1\} \subseteq \{1, 2\}$, but $S^{(1)}(\{1, 2\}) = \{b, c\} \nsubseteq \{a, b\}$.

Corollary 3.11 ([20, Lemma 3.5])**.** $B \subseteq Y^{\downarrow}$ if and only if $Y \subseteq B^{\uparrow}$.

Proof. Noting that $Y^{\downarrow} = I^{(0)}(Y)$ and $B^{\uparrow} = I^{(1)}(B)$, we apply Lemma 3.9, which produces the above result as required. ⊣

Corollary 3.12 (Polar maps form a Galois connection [20, Example 7.24])**.** The polar maps of $\mathbb{P}$ form a Galois connection between $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(X), \subseteq)$.

Proof. Follows directly from Corollary 3.11. ⊣

Definition 3.13 (Galois stable subsets [11, Definition 1])**.** Subsets $U \subseteq A$ and $V \subseteq X$ are called *Galois-stable* if

(a) $U^{\uparrow\downarrow} = U$;

(b) $V^{\downarrow\uparrow} = V$.

We will use the adjectives *Galois-stable* and *stable* interchangeably throughout this thesis.

Proposition 3.14 ([20, Lemma 3.5])**.** For $B \subseteq A$ and $Y \subseteq X$

(a) $B \subseteq B^{\uparrow\downarrow}$ and $Y \subseteq Y^{\downarrow\uparrow}$

(b) $B^{\uparrow}$ and $Y^{\downarrow}$ are Galois Stable

Proof. For (a) let $b \in B$ we recall that

$$B^{\uparrow} = \{x \in X \mid \forall a \in A\,(a \in B \to aIx)\}$$

and

$$B^{\uparrow\downarrow} = \left\{a \in A \mid \left(\forall x \in B^{\uparrow}\right)(aIx)\right\}.$$

Then clearly bIx for each $x \in B^{\uparrow}$ by the membership condition for $B^{\uparrow}$ (that x is I-related to each $a \in B$). Thus $b \in B^{\uparrow\downarrow}$. The proof of $Y \subseteq Y^{\downarrow\uparrow}$ is symmetrical.

For (b) we need to show that $\left(B^{\uparrow}\right)^{\downarrow\uparrow} = B^{\uparrow}$ and $\left(Y^{\downarrow}\right)^{\uparrow\downarrow} = Y^{\downarrow}$. From (a) we have $B \subseteq B^{\uparrow\downarrow}$ and so applying Corollary 3.6 we have $\left(B^{\uparrow\downarrow}\right)^{\uparrow} \subseteq B^{\uparrow}$. Additionally, applying (b) to $B^{\uparrow}$ gives $B^{\uparrow} \subseteq \left(B^{\uparrow}\right)^{\uparrow\downarrow}$. Thus $\left(B^{\uparrow}\right)^{\downarrow\uparrow} = B^{\uparrow}$ as required. The proof for $Y^{\downarrow}$ is symmetrical. ⊣

Proposition 3.15 (Finite intersections of Galois-stable subsets are Galois-stable)**.** Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_i\}_{i\in I}$ be collections of Galois-stable subsets of A and X respectively. Then $\bigcap_{i\in I} U_i$ and $\bigcap_{i\in I} V_i$ are both Galois-stable.

Proof. This is a well known result in formal concept analysis. See for example [20, Proposition 7.2]. ⊣

Corollary 3.16. A and X are Galois stable.

Proof. As $A \subseteq A$ and $X \subseteq X$ we have $A \subseteq A^{\uparrow\downarrow} \subseteq A$ and $X \subseteq X^{\downarrow\uparrow} \subseteq X$ by Proposition 3.14(a). $\dashv$

Definition 3.17 (Galois closure operators [20, Example 7.24])**.** The maps $(\cdot)^{\uparrow\downarrow} : \mathcal{P}(A) \to \mathcal{P}(A)$ and $(\cdot)^{\downarrow\uparrow} : \mathcal{P}(X) \to \mathcal{P}(X)$ are the *Galois closure operators* or *closures* of $\mathbb{P}$.

In a similar fashion to Lemma 3.9, we do not in general have $B^{\uparrow} \subseteq Y$ if and only if $Y^{\downarrow} \subseteq B$. To see this we need only construct a polarity $\mathbb{P} = (W, Z, I)$ from Example 3.10 where $I = S$. This additional property, however, hold for Galois stable subsets of $\mathbb{P}$ as shown in Proposition 3.18

Proposition 3.18. If $B \subseteq A$ and $Y \subseteq X$ are Galois stable subsets of $\mathbb{P}$ then

$$B^{\uparrow} \subseteq Y \text{ if and only if } Y^{\downarrow} \subseteq B.$$

Proof. Suppose $B = B^{\uparrow\downarrow}$ and $Y = Y^{\downarrow\uparrow}$ then

$$B^{\uparrow} \subseteq Y \text{ iff } B^{\uparrow} \subseteq Y^{\downarrow\uparrow} \text{ iff } \left(B^{\uparrow}\right) \subseteq \left(Y^{\downarrow}\right)^{\uparrow}.$$

Thus by Corollary 3.11

$$B^{\uparrow} \subseteq Y \text{ iff } Y^{\downarrow} \subseteq \left(B^{\uparrow}\right)^{\downarrow} = B. \qquad \dashv$$

With the notions of Galois-stability within the context of polarities in place we can now introduce Wille's notion of a *formal concept* from [51].

Definition 3.19 (Formal concept)**.** Let $\mathbb{P} = (A, X, I)$ be a polarity. A *formal concept* is an ordered pair $\boldsymbol{c} = (\llbracket \boldsymbol{c} \rrbracket, (\!\![\boldsymbol{c}]\!\!)) \in \mathcal{P}(A) \times \mathcal{P}(X)$ such that $\llbracket \boldsymbol{c} \rrbracket^{\uparrow} = (\!\![\boldsymbol{c}]\!\!)$ and $(\!\![\boldsymbol{c}]\!\!)^{\downarrow} = \llbracket \boldsymbol{c} \rrbracket$. The sets $\llbracket \boldsymbol{c} \rrbracket$ and $(\!\![\boldsymbol{c}]\!\!)$ are referred to as the *extension* and *intention* of $\boldsymbol{c}$ respectively.

The *extension* of $\boldsymbol{c}$ is the subset of objects exemplifying the concept $\boldsymbol{c}$, and the intension is the set of attributes shared by all such objects [51][20].

We will sometimes refer to $(B)^{\uparrow}$ as *taking the intention of* B and $(Y)^{\downarrow}$ as *taking the extension of* Y, although "extension" and "intention" as nouns will be reserved for components of formal concepts.

As in [16], we will use the notation $\mathbb{L}(\mathbb{P})$ for the set of formal concepts of polarity $\mathbb{P}$. The terms "formal concept" and "concept" will be used interchangeably. Bold lowercase Roman letters such as $\boldsymbol{c}$ will be used for arbitrary concepts unless otherwise specified.

Following immediately from Definition 3.19 is the stability of $\llbracket \boldsymbol{c} \rrbracket$ and $(\!\![\boldsymbol{c}]\!\!)$.

Remark 3.20. As $\boldsymbol{c}$ is a formal concept $\llbracket \boldsymbol{c} \rrbracket = ((\!\![\boldsymbol{c}]\!\!))^{\downarrow} = \left(\left(\llbracket \boldsymbol{c} \rrbracket^{\uparrow}\right)\right)^{\downarrow} = \llbracket \boldsymbol{c} \rrbracket^{\uparrow\downarrow}$ and $(\!\![\boldsymbol{c}]\!\!) = (\llbracket \boldsymbol{c} \rrbracket)^{\uparrow} = \left((\!\![\boldsymbol{c}]\!\!)^{\downarrow}\right)^{\uparrow} = (\!\![\boldsymbol{c}]\!\!)^{\downarrow\uparrow}$.

Example 3.21. Continuing on from Example 3.2, consider the subsets $\{\mathrm{DH}, \mathrm{IN}\}$ and $\{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}$. Note that $\{\mathrm{DH}, \mathrm{IN}\}^{\uparrow} = \{\mathrm{math}\}$, but $\{\mathrm{math}\}^{\downarrow} = \{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\} \neq \{\mathrm{DH}, \mathrm{IN}\}$. Where as $\{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}^{\uparrow} = \{\mathrm{math}\}$ and $\{\mathrm{math}\}^{\downarrow} = \{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}$.

Intuitively, there's a sense in which $\{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}$ is a privileged subset of objects in the context of our polarity, where $\{\mathrm{DH}, \mathrm{IN}\}$ is not – $\{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}$ is equal to its Galois closure, where $\{\mathrm{DH}, \mathrm{IN}\}$ is not. That is $\{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\}$ is a Galois-stable subset in $\mathbb{P}_M$ in the sense of Definition 3.13 and $\{\mathrm{DH}, \mathrm{IN}\}$ is not.

Before defining our ordering on $\mathbb{L}(\mathbb{P})$ we show that set containment between the extensions of two concepts holds if and only if reverse inclusion for their intentions holds. An ordering on $\mathbb{L}(\mathbb{P})$ is then defined in terms of this characterisation (Definition 3.23).

Remark 3.22. For any $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{L}(\mathbb{P})$: $[\![\boldsymbol{c}]\!] \subseteq [\![\boldsymbol{d}]\!]$ iff $(\![\boldsymbol{d}]\!) \subseteq (\![\boldsymbol{c}]\!)$.

Proof. Suppose that $[\![\boldsymbol{c}]\!] \subseteq [\![\boldsymbol{d}]\!]$. Then $(\![\boldsymbol{c}]\!)^{\downarrow} \subseteq (\![\boldsymbol{d}]\!)^{\downarrow}$. Applying Corollary 3.6 gives $\left((\![\boldsymbol{d}]\!)^{\downarrow}\right)^{\uparrow} \subseteq \left((\![\boldsymbol{c}]\!)^{\downarrow}\right)^{\uparrow}$ and so by Remark 3.20 we have $(\![\boldsymbol{d}]\!) \subseteq (\![\boldsymbol{c}]\!)$ as required. Similarly, if $(\![\boldsymbol{d}]\!) \subseteq (\![\boldsymbol{c}]\!)$ then $[\![\boldsymbol{d}]\!]^{\uparrow} \subseteq [\![\boldsymbol{c}]\!]^{\uparrow}$ and so by Corollary 3.6 we have $[\![\boldsymbol{c}]\!] = [\![\boldsymbol{c}]\!]^{\uparrow\downarrow} \subseteq [\![\boldsymbol{d}]\!]^{\uparrow\downarrow} = [\![\boldsymbol{d}]\!]$. ⊣

Definition 3.23 (Ordering for $\mathbb{L}(\mathbb{P})$)**.** For any $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{L}(\mathbb{P})$ define the ordering $\leq$ as follows

$$\boldsymbol{c} \leq \boldsymbol{d} \text{ iff } [\![\boldsymbol{c}]\!] \subseteq [\![\boldsymbol{d}]\!] \text{ iff } (\![\boldsymbol{d}]\!) \subseteq (\![\boldsymbol{c}]\!).$$

Remark 3.24. A collection of subsets of any set U ordered by $\subseteq$ is a partially ordered set, which follows directly from the properties of set inclusion. Thus $(\mathbb{L}(\mathbb{P}), \leq)$ is a partially ordered set, as $\boldsymbol{c} \leq \boldsymbol{d}$ iff $[\![\boldsymbol{c}]\!] \subseteq [\![\boldsymbol{d}]\!]$ with $[\![\boldsymbol{c}]\!], [\![\boldsymbol{d}]\!] \subseteq A$.

Moreover, the subsets of U form a complete lattice with meet and join given by intersection and union respectively [3, pg. 7].

Proposition 3.25. Let $S \subseteq W \times Z$ and $\mathcal{U}$ and $\mathcal{V}$ be arbitrary collections of subsets of W and Z respectively. Then

(a) $S^{(1)}\left(\bigcup_{U\in\mathcal{U}} U\right) = \bigcap_{U\in\mathcal{U}} S^{(1)}(U)$.

(b) $S^{(0)}\left(\bigcup_{V\in\mathcal{V}} V\right) = \bigcap_{V\in\mathcal{V}} S^{(0)}(V)$.

For (a)

$$\begin{aligned}
z \in \bigcap_{U\in\mathcal{U}} S^{(1)}(U) &\text{ iff } (\forall U \in \mathcal{U})\,(\forall w \in W\,(w \in U \to wSz)) \\
&\text{iff } (\forall U \in \mathcal{U})\,(\forall w \in U\,(wSz)) \\
&\text{iff } \left(\forall w \in \bigcup_{U\in\mathcal{U}} U\right)(wSz) \\
&\text{iff } (\forall w \in W)\left(w \in \bigcup_{U\in\mathcal{U}} \to wSz\right) \\
&\text{iff } z \in S^{(1)}\left(\bigcup_{U\in\mathcal{U}} U\right).
\end{aligned}$$

For Proposition 3.25(b)

$$\begin{aligned}
w \in \bigcap_{V\in\mathcal{V}} S^{(0)}(V) &\text{ iff } (\forall V \in \mathcal{V})\,(\forall z \in Z\,(z \in V \to wSz)) \\
&\text{iff } (\forall V \in \mathcal{V})\,(\forall z \in V\,(wSz)) \\
&\text{iff } \left(\forall z \in \bigcup_{V\in\mathcal{V}} V\right)(wSz) \\
&\text{iff } (\forall z \in Z)\left(z \in \bigcup_{V\in\mathcal{V}} \to wSz\right) \\
&\text{iff } w \in S^{(1)}\left(\bigcup_{V\in\mathcal{V}} V\right).
\end{aligned}$$

Corollary 3.26. For any $\mathcal{U} \subseteq A$ and $\mathcal{V} \subseteq X$

(a) $\left(\bigcup_{U\in\mathcal{U}} U\right)^{\uparrow} = \bigcap_{U\in\mathcal{U}} U^{\uparrow}$

(b) $\left(\bigcup_{U\in\mathcal{U}} U\right)^{\downarrow} = \bigcap_{U\in\mathcal{U}} U^{\downarrow}$

This follows directly from application of Proposition 3.25 replacing notation $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ with $I^{(1)}(\cdot)$ and $I^{(0)}(\cdot)$ respectively.

Definition 3.27 (Concept lattice)**.** The *concept lattice* of $\mathbb{P}$ is the complete lattice $\mathbb{P}^{+} = (\mathbb{L}(\mathbb{P}), \leq)$ where $\wedge$ and $\vee$ are given as follows for $\mathcal{A} \subseteq \mathbb{L}(\mathbb{P})$

$$\bigwedge \mathcal{A} = \left(\bigcap_{\boldsymbol{a}\in\mathcal{A}} [\![\boldsymbol{a}]\!], \left(\bigcap_{\boldsymbol{a}\in\mathcal{A}} [\![\boldsymbol{a}]\!] \right)^{\uparrow} \right) \text{ and } \bigvee \mathcal{A} = \left(\left(\bigcap_{\boldsymbol{a}\in\mathcal{A}} (\![\boldsymbol{a}]\!) \right)^{\downarrow}, \bigcap_{\boldsymbol{a}\in\mathcal{A}} (\![\boldsymbol{a}]\!) \right).$$

Example 3.28 (Concept lattice of $\mathbb{P}_M$)**.** The concept lattice of $\mathbb{P}_M$ introduced in Example 3.2 is given in Figure 3.2.

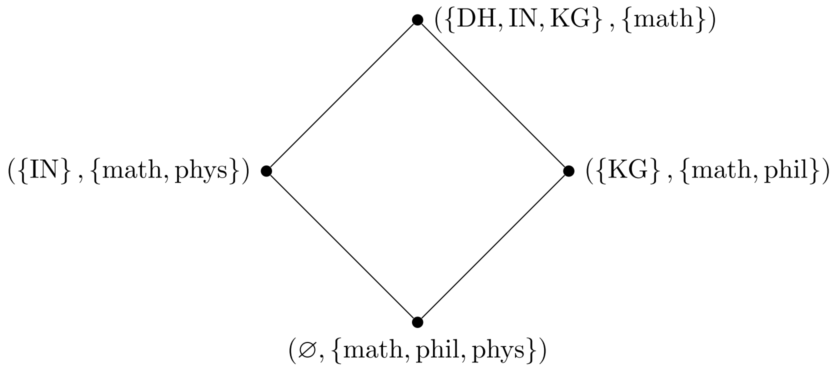


Figure 3.2: Concept lattice of $\mathbb{P}_M$.

Chapter 4

Polarity-based semantics

Polarity-based semantics have played a prominent role in non-distributive modal logic throughout the last two decades. During this time semantics have been developed for a number of signatures, for instance [47], [36], [10], [26], [18], and [38]. In this chapter we introduce a polarity based semantics in the signature of distributive modal logic (DML). The relational structures of interest in this setting are enriched polarity-based frames – polarities enriched with additional binary relations which correspond to unary modalities in the semantics. Current approaches in the NDML literature such as [16], [10], and [17] require these additional relations [1] to be compatible with the incidence relation of the underlying polarity – a detailed description of these compatibility requirements is given in Definition 4.49.

The primary contribution of this thesis is to provide a polarity-based semantics for NDML accommodating enriched polarities whose additional relations need not be compatible. Accommodating for enriched polarities with so called *non-compatible* relations significantly enhances the applicability of our resultant semantics – this is as the additional relations of an arbitrary enriched polarity may not satisfy the compatibility requirements. A simple example of such an arbitrary enriched polarity with non-compatible relations is given in Example 4.54. Whilst our non-compatible semantics is adapted from the compatible semantics of [16], given that our primary adaptation is in the removal of constraints on additional relations, the semantics presented in this chapter is in a sense a generalisation of its compatible counterpart in the restricted case where additional relations are binary. It is for this reason that the non-compatible semantics is presented before the more restricted compatible semantics in Section 4.5.

Following the presentation of the new semantics we give an axiomatization with proofs of soundness and completeness. A formal description of the compatible semantics in the DML signature is given in Section 4.5 before the presentation of examples of enriched polarities in our signature that fail to exhibit properties such as normality, distributivity, and various adjunction properties for the modalities $\Box, \Diamond, \lhd$, and $\rhd$ required by inference rules in the compatible axiomatization.

4.1 The DML signature

We begin by introducing the DML signature along with some associated notation and typing conventions. The order type conventions introduced here will also be used in Chapter 6.

Let $\mathcal{H}$ be a unary signature – a signature in which all connectives are of arity 1. We ascribe to each connective, $h \in \mathcal{H}$, a corresponding adjunction and order types $\alpha_h, \epsilon_h \in \{1, \partial\}$ respectively. Sets W_{ϵ_h} and W_{α_h} are assigned using the convention $W_1 = A$ and $W_\partial = X$. Similarly, w_1 will denote an element of A and w_∂ an element of X.

A binary relation, $R_h \subseteq W_{\epsilon_h} \times W_{\alpha_h}$ on $\mathbb{P}$ can then be assigned to each $h \in \mathcal{H}$. We use $\mathcal{R}_\mathcal{H}$ to denote a set of relations associated with the signature, $\mathcal{H}$. This notation was adapted from [13], where ϵ_h is an n_h-tuples (where n_h is the arity of h) and $\alpha_h \in \{1, \partial\}$. As we have explicitly restricted our considerations

[1] In [17] these additional relations can be of arbitrary arity.

to collections of unary connectives, our adjunction and order types both belong to the set $\{1, \partial\}$.[2]

Restricting our focus to unary signatures produces four possible (ϵ_h, α_h) pairs, as the order and adjunction types can either assume values 1 or ∂. These four cases are presented in Table 4.1, along with the domain and codomain of their associated binary relations on $\mathbb{P}$.

ϵ_h	α_h	$W_{\epsilon_h} \times W_{\alpha_h}$
1	∂	$A \times X$
∂	1	$X \times A$
1	1	$A \times A$
∂	∂	$X \times X$

Table 4.1: The four possible order/adjunction type values for unary connectives, h, along with the type of associated binary relation in each case.

We refer to the signature composed of four connectives, one of each kind given in Table 4.1, simply as DML – the signature of distributive modal logic, as introduced in [27]. Adopting the notation of [27], the DML signature consists of the connectives $\Box, \Diamond, \rhd$ and $\lhd$, where the order and adjunction types corresponding to each of the connectives are given in Table 4.2.

h	ϵ_h	α_h	$R_h \subseteq W_{\epsilon_h} \times W_{\alpha_h}$
$\Box$	1	∂	$R_\Box \subseteq A \times X$
$\Diamond$	∂	1	$R_\Diamond \subseteq X \times A$
$\rhd$	1	1	$R_\rhd \subseteq A \times A$
$\lhd$	∂	∂	$R_\lhd \subseteq X \times X$

Table 4.2: Unary connectives of DML, along with their corresponding order types and resultant binary relation R_h.

Although DML is a signature consisting of precisely one of each of these four connectives, we will use the $\Box, \Diamond, \rhd$, and $\lhd$ to refer to any connective of the corresponding order and adjunction types. For example, if we have a signature $\mathcal{H} = \{h\}$ where $\epsilon_h = 1$ and $\alpha_h = \partial$ we will write $\mathcal{H} = \{\Box\}$, even if this signature is, in a sense, a fragment of DML. Similarly, we will follow the DML notation style if our signature consists of more than one connective of a given order/adjunction type. For instance, if we had a signature consisting of connectives h_1, h_2 with $\epsilon_{h_1} = \epsilon_{h_2} = \alpha_{h_1} = \alpha_{h_2} = 1$ we would use $\{\rhd_1, \rhd_2\}$ when referring to the signature, as each of the connectives is "$\rhd$-type".

Definition 4.1 (Bitonal algebra)**.** A bitonal algebra is an algebra $\mathbb{B} = (\mathbb{L}, \{\Diamond, \Box, \lhd, \rhd\})$ in the signature of DML where $\mathbb{L}$ is a bounded lattice, and $\Diamond, \Box$ and $\lhd, \rhd$ are pairs of monotone and antitone operations respectively.

4.2 The basic $\mathcal{L}_P$-logic

Definition 4.2. The language, $\mathcal{L}_P$, of the basic non-distributive modal logic is defined as follows over a denumarable set of proposition letters, Prop:

$$\varphi ::= p \mid \bot \mid \top \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \Box\varphi \mid \Diamond\varphi \mid \rhd\varphi \mid \lhd\varphi.$$

Where $p \in \mathsf{Prop}$ and $\varphi_1, \varphi_2 \in \mathcal{L}_P$.

[2] This notation was adapted from [13], in which connectives of arbitrary arity are considered. In [13] and [17], if h is a connective of arity n_h, then the order-type, ϵ_h, is an n_h-tuple in $\{1, \partial\}^{n_h}$. The adjunction type, $\alpha_h \in \{1, \partial\}^{n_h}$, partitions the collection of connectives into sets $\mathcal{F} = \{f \in \mathcal{H} \mid \alpha_f = 1\}$ and $\mathcal{G} = \{g \in \mathcal{H} \mid \alpha_g = \partial\}$. In this framing f (resp. $g \in \mathcal{G}$) preserves finite joins (resp. meets) in its ith co-ordinate when $\epsilon_f(i) = 1$ (resp. $\epsilon_g = 1$) and turns finite meets (resp. joins) into finite joins (resp. meets) when $\epsilon_f(i) = \partial$ (resp. $\epsilon_g(i) = \partial$) [13]. Note that in our unary setting we happen to have $\alpha_h, \epsilon_h \in \{1, \partial\}$.

Conventionally, a formula $\varphi \in \mathcal{L}_P$ denotes the *category* or *concept* φ. The sequent $\varphi \vdash \psi$ is interpreted as "the concept φ is a sub-concept of the concept ψ", set-theoretically $[\![\varphi]\!] \subseteq [\![\psi]\!]$ and equivalently $(\![\psi]\!) \subseteq (\![\varphi]\!)$.[3]

The basic $\mathcal{L}_P$-logic is a set $\boldsymbol{L}_P$ of sequents $\varphi \vdash \psi$ with $\varphi, \psi \in \mathcal{L}_P$ with the axioms[4]:

$$
\begin{array}{llll}
p \vdash p \ \text{(Ax.)} & \bot \vdash p \ (\bot\text{L}) & p \vdash \top \ (\top\text{R}) & \\
p \vdash p \vee q \ (\vee\text{R1}) & q \vdash p \vee q \ (\vee\text{R2}) & p \wedge q \vdash p \ (\wedge\text{L1}) & p \wedge q \vdash q \ (\wedge\text{L2})
\end{array}
$$

and closed under the inference rules:

$$
\frac{\varphi \vdash \chi \qquad \chi \vdash \psi}{\varphi \vdash \psi}\ \text{(Cut)} \qquad \frac{\varphi \vdash \psi}{\varphi\,(\chi/p) \vdash \psi\,(\chi/p)}\ \text{(Subs.)} \qquad \frac{\chi \vdash \varphi \qquad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi}\ (\wedge\text{R}) \qquad \frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}\ (\vee\text{L})
$$

$$
\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}\ (\Box\text{M}) \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}\ (\Diamond\text{M}) \qquad \frac{\varphi \vdash \psi}{\rhd\psi \vdash \rhd\varphi}\ (\rhd\text{A}) \qquad \frac{\varphi \vdash \psi}{\lhd\psi \vdash \lhd\varphi}\ (\lhd\text{A})
$$

The stated axioms for propositional variables can be extended to arbitrary $\mathcal{L}_P$-formulae using uniform substitution, as given in Lemma 4.3 below.

Lemma 4.3. For $\varphi, \psi \in \mathcal{L}_P$, the following sequents are derivable in $\boldsymbol{L}_P$:

$$
\begin{array}{llll}
\varphi \vdash \varphi & \bot \vdash \varphi & \varphi \vdash \top & \\
\varphi \vdash \varphi \vee \psi & \psi \vdash \varphi \vee \psi & \varphi \wedge \psi \vdash \varphi & \varphi \wedge \psi \vdash \psi
\end{array}
$$

Proof. Each sequent follows immediately from the corresponding propositional sequents and uniform substitution. For example, $\psi \vdash \varphi \vee \psi$ follows directly from uniform substitution of ψ for q and φ for φ in $p \vdash p \vee q$. $\dashv$

From this point the names of the original propositional axioms will be used to refer to the corresponding sequents of Lemma 4.3, which will be taken as axioms. For example "($\wedge$L1)" will refer to $\varphi \wedge \psi \vdash \varphi$. This axiomatization is shown to be sound and complete in Section 4.4.

4.3 Non-compatible polarity-based semantics

Modalities are accommodated for in NDML via the enrichment of polarities with additional relations, which are then given modal interpretations via corresponding clauses in the semantics. As was referred to at the beginning of the chapter, these relations need not be binary in general, nor does the signature to which connectives belong have to be DML.

Definition 4.4 (Enriched polarity/polarity-based frame)**.** A *polarity-based frame* is a tuple $\mathbb{F} = (\mathbb{P}, \mathcal{R}_{\mathcal{H}})$ where $\mathbb{P}$ is a polarity, and $\mathcal{R}$ a collection of relations associated with a signature $\mathcal{H}$.

We will use the terms *polarity-based frame* and *enriched polarity* interchangeably. From an enriched polarity extends naturally the notion of a *polarity-based model*.

We specialize this definition of polarity-based frames to the signature of DML in the following definition.

Definition 4.5. A polarity-based frame in the signature of DML is a tuple $\mathbb{F} = (\mathbb{P}, \{R_\Box, R_\Diamond, R_\rhd, R_\lhd\})$, where the binary relations $R_\Box, R_\Diamond, R_\rhd, R_\lhd$ correspond to those given in Table 4.2.

We sometimes consider sub-languages of DML and reducts of the enriched polarities with additional relations corresponding to the modalities in the specific sub-language of DML. See, for example, Example 4.54 and Example 4.58(i).

[3] The $[\![\psi]\!]$ and $(\![\psi]\!)$ notation is defined in Definition 4.8.
[4] Suffixes L and R are used for left and right, and prefixes M and A are used for monotonicity and antitonicity respectively.

Analogous to the way in which a complex algebra is derived from a Kripke frame in classical modal logic (Definition 2.43), an enriched polarity gives rise to an associated complex algebra in our polarity-based setting. In the following definition we adapt the notion of complex algebras for compatible enriched polarities given in, for example, [9] and [16], to our non-compatible setting.

Definition 4.6 (Complex algebra of an enriched polarity)**.** For and enriched polarity $\mathbb{F} = (\mathbb{P}, \{R_\Box, R_\Diamond, R_\rhd, R_\lhd\})$ the complex algebra, $\mathbb{F}^+$, is the tuple

$$\mathbb{F}^+ = \left(\mathbb{P}^+, [R_\Box], \langle R_\Diamond\rangle, [R_\rhd\rangle, \langle R_\lhd]\right).$$

This is where for $\boldsymbol{c} \in \mathbb{P}^+$:

$$[R_\Box](\boldsymbol{c}) = \left(\left(R_\Box^{(0)}\left((\![c]\!)\right)\right)^{\uparrow\downarrow}, \left(R_\Box^{(0)}\left((\![c]\!)\right)\right)^{\uparrow}\right); \qquad \langle R_\Diamond\rangle(\boldsymbol{c}) = \left(\left(R_\Diamond^{(0)}\left([\![c]\!]\right)\right)^{\downarrow}, \left(R_\Diamond^{(0)}\left([\![c]\!]\right)\right)^{\downarrow\uparrow}\right)$$

and

$$[R_\rhd\rangle(\boldsymbol{c}) = \left(\left(R_\rhd^{(0)}\left([\![c]\!]\right)\right)^{\uparrow\downarrow}, \left(R_\rhd^{(0)}\left([\![c]\!]\right)\right)^{\uparrow}\right); \qquad \langle R_\lhd](\boldsymbol{c}) = \left(\left(R_\lhd^{(0)}\left((\![c]\!)\right)\right)^{\downarrow}, \left(R_\lhd^{(0)}\left((\![c]\!)\right)\right)^{\downarrow\uparrow}\right).$$

Definition 4.7 (Polarity-based Model)**.** A polarity-based model is an ordered pair $\mathcal{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a polarity-based frame and V is a map $V : \mathsf{Prop} \to \mathbb{F}^+$ referred to as a *valuation*, or *valuation on* $\mathbb{F}$.

Definition 4.8 (Satisfaction and co-satisfaction)**.** Let $\mathcal{M} = (\mathbb{F}, V)$ be a polarity-based model. The satisfaction and co-satisfaction relations $\Vdash\ \subseteq A \times \mathcal{L}_P$ and $\succ\ \subseteq X \times \mathcal{L}_P$, along with $[\![\psi]\!] = [\![\psi]\!]_\mathcal{M} = \{a \in A \mid \mathcal{M}, a \Vdash \psi\}$ and $(\![\psi]\!) = (\![\psi]\!)_\mathcal{M} = \{x \in X \mid \mathcal{M}, x \succ \psi\}$ are defined by the following simultaneous recursion:[5]

$$\begin{array}{llllll}
\mathcal{M}, a \Vdash p & \text{iff} & a \in [\![V(p)]\!] & \mathcal{M}, x \succ p & \text{iff} & x \in (\![V(p)]\!) \\
\mathcal{M}, a \Vdash \top & & \text{always} & \mathcal{M}, x \succ \top & \text{iff} & x \in A^{\uparrow} \\
\mathcal{M}, a \Vdash \bot & \text{iff} & a \in X^{\downarrow} & \mathcal{M}, x \succ \bot & & \text{always} \\
\mathcal{M}, a \Vdash \psi \wedge \varphi & \text{iff} & a \Vdash \psi \text{ and } a \Vdash \varphi & \mathcal{M}, x \succ \psi \wedge \varphi & \text{iff} & x \in ([\![\psi]\!] \cap [\![\varphi]\!])^{\uparrow} \\
\mathcal{M}, a \Vdash \psi \vee \varphi & \text{iff} & a \in ((\![\psi]\!) \cap (\![\varphi]\!))^{\downarrow} & \mathcal{M}, x \succ \psi \vee \varphi & \text{iff} & x \succ \psi \text{ and } x \succ \varphi \\
\mathcal{M}, a \Vdash \Box\psi & \text{iff} & a \in \left(R_\Box^{(0)}\left((\![\psi]\!)\right)\right)^{\uparrow\downarrow} & \mathcal{M}, x \succ \Box\psi & \text{iff} & x \in \left(R_\Box^{(0)}\left((\![\psi]\!)\right)\right)^{\uparrow} \\
\mathcal{M}, a \Vdash \Diamond\psi & \text{iff} & a \in \left(R_\Diamond^{(0)}\left([\![\psi]\!]\right)\right)^{\downarrow} & \mathcal{M}, x \succ \Diamond\psi & \text{iff} & x \in \left(R_\Diamond^{(0)}\left([\![\psi]\!]\right)\right)^{\downarrow\uparrow} \\
\mathcal{M}, a \Vdash \rhd\psi & \text{iff} & a \in \left(R_\rhd^{(0)}\left([\![\psi]\!]\right)\right)^{\uparrow\downarrow} & \mathcal{M}, x \succ \rhd\psi & \text{iff} & x \in \left(R_\rhd^{(0)}\left([\![\psi]\!]\right)\right)^{\uparrow} \\
\mathcal{M}, a \Vdash \lhd\psi & \text{iff} & a \in \left(R_\lhd^{(0)}\left((\![\psi]\!)\right)\right)^{\downarrow} & \mathcal{M}, x \succ \lhd\psi & \text{iff} & x \in \left(R_\lhd^{(0)}\left((\![\psi]\!)\right)\right)^{\downarrow\uparrow}.
\end{array}$$

If we interpret the polarity $\mathbb{P} = (A, X, I)$ as a structure consisting of a set of objects, A, and features, X, and incidence relation I pairing objects with their corresponding features then $a \Vdash \varphi$ and $x \succ \varphi$ can be read as "object a exemplifies concept φ" and "feature x describes concept φ" respectively.

Note that for each pair of clauses in Definition 4.8 there is one clause that is, in a sense, *primary* and one that is *secondary.* For instance, the satisfaction clause $\mathcal{M}, a \Vdash \top$ is primary and the co-satisfaction clause $\mathcal{M}, x \succ \top$ is secondary, as it can be derived from the primary clause and the relation $\mathcal{M}, x \succ \varphi$ iff $x \in (\![\varphi]\!) = ([\![\varphi]\!])^{\uparrow}$. Dually, there are instances when the co-satisfaction clause is primary and the satisfaction cause is secondary. For example, in the case of disjunction the co-satisfaction clause, $\mathcal{M}, x \succ \psi \vee \varphi$, is primary and the satisfaction clause, $\mathcal{M}, a \Vdash \psi \vee \varphi$, is secondary.

It can be useful at times to refer to expanded forms of the clauses in Definition 4.7. These expanded clauses are summarized in Table 4.3 below.

[5]Read as the "extension of ψ" and "intention of ψ" respectively.

Relation	Connective	Valuation	Satisfaction Clause	Co-Satisfaction Clause
$R_\Box \subseteq A \times X$	$\Box$	$\left(\left(R_\Box^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\uparrow\downarrow}, \left(R_\Box^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\uparrow}\right)$	$(\forall x \in X)\,(x \succ \Box\psi \to aIx)$	$(\forall a \in A)\,(\forall x' \in X)\,((x' \succ \psi \to aR_\Box x') \to aIx)$
$R_\Diamond \subseteq X \times A$	$\Diamond$	$\left(\left(R_\Diamond^{(0)}\llbracket\psi\rrbracket\right)^{\downarrow}, \left(R_\Diamond^{(0)}\llbracket\psi\rrbracket\right)^{\downarrow\uparrow}\right)$	$(\forall x \in X)\,(\forall a' \in A)\,((a' \Vdash \psi \to R_\Diamond x a') \to aIx)$	$(\forall a \in A)\,(a \Vdash \Diamond\psi \to aIx)$
$R_\rhd \subseteq A \times A$	$\rhd$	$\left(\left(R_\rhd^{(0)}\llbracket\psi\rrbracket\right)^{\uparrow\downarrow}, \left(R_\rhd^{(0)}\llbracket\psi\rrbracket\right)^{\uparrow}\right)$	$(\forall x \in X)\,(x \succ \rhd\psi \to aIx)$	$(\forall a \in A)\,(\forall a' \in A)\,((a' \Vdash \psi \to R_\rhd a' a) \to aIx)$
$R_\lhd \subseteq X \times X$	$\lhd$	$\left(\left(R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\downarrow}, \left(R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\downarrow\uparrow}\right)$	$(\forall x \in X)\,(\forall x' \in X)\,((x' \succ \psi \to R_\lhd x x') \to aIx)$	$(\forall a \in A)\,(a \Vdash \lhd\psi \to aIx)$

Table 4.3: Derivations corresponding to each row of the table are given in Proposition 4.9.

Proposition 4.9. The valuations in Table 4.3 correspond to their respective satisfaction and co-satisfaction clauses in adjacent columns of the table.

Proof. Let $\mathcal{M} = (\mathbb{F}, V)$ be a polarity-based model with $\mathbb{F} = (\mathbb{P}, \{R_\Box, R_\Diamond, R_\rhd, R_\lhd\})$.[6]

For $R_\Box \subseteq A \times X$:

$$\begin{aligned}
\mathcal{M}, x \succ \Box\psi \text{ iff } & x \in \left(R_\Box^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\uparrow} && \text{(Definition 4.8)}\\
\text{iff } & (\forall a \in A)\left(a \in R_\Box^{(0)}\llparenthesis\psi\rrparenthesis \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall a \in A)\,(\forall x' \in X)\,((x' \in \llparenthesis\psi\rrparenthesis \to aR_\Box x') \to aIx) && \text{(Definition 3.4)}\\
\text{iff } & (\forall a \in A)\,(\forall x' \in X)\,((x' \succ \psi \to aR_\Box x') \to aIx)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}, a \Vdash \Box\psi \text{ iff } & a \in \left(R_\Box^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\uparrow\downarrow} && \text{(Definition 4.8)}\\
\text{iff } & (\forall x \in X)\left(x \in \left(R_\Box^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\uparrow} \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall x \in X)\,(x \succ \Box\psi \to aIx)
\end{aligned}$$

For $R_\Diamond \subseteq X \times A$:

$$\begin{aligned}
\mathcal{M}, a \Vdash \Diamond\psi \text{ iff } & a \in \left(R_\Diamond^{(0)}\llbracket\psi\rrbracket\right)^{\downarrow} && \text{(Definition 4.8)}\\
\text{iff } & (\forall x \in X)\left(x \in R_\Diamond^{(0)}\llbracket\psi\rrbracket \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall x \in X)\,(\forall a' \in A)\,((a' \Vdash \psi \to xR_\Diamond a') \to aIx)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}, x \succ \Diamond\psi \text{ iff } & x \in \left(R_\Diamond^{(0)}\llbracket\psi\rrbracket\right)^{\downarrow\uparrow} && \text{(Definition 4.8)}\\
\text{iff } & (\forall a \in A)\left(a \in \left(R_\Diamond^{(0)}\llbracket\psi\rrbracket\right)^{\downarrow} \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall a \in A)\,(a \Vdash \Diamond\psi \to aIx)
\end{aligned}$$

For $R_\lhd \subseteq X \times X$:

$$\begin{aligned}
a \Vdash \lhd\psi \text{ iff } & a \in \left(R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\downarrow}\\
\text{iff } & (\forall x \in X)\left(x \in R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall x \in X)\,(\forall x' \in X)\,((x' \succ \psi \to xR_\lhd x') \to aIx) && \text{(Definition 3.4)}
\end{aligned}$$

and

$$\begin{aligned}
x \succ \lhd\psi \text{ iff } & x \in \left(R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\downarrow\uparrow}\\
\text{iff } & (\forall a \in A)\left(a \in \left(R_\lhd^{(0)}\llparenthesis\psi\rrparenthesis\right)^{\downarrow} \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall a \in A)\,(a \Vdash \lhd\psi \to aIx)
\end{aligned}$$

[6]We write $a \Vdash \psi$ for $\mathcal{M}, a \Vdash \psi$ and $x \succ \psi$ for $\mathcal{M}, x \succ \psi$ for the sake of readability.

For $R_\triangleright \subseteq A \times A$:

$$\begin{aligned}
x \succ \triangleright\psi \text{ iff } & x \in \left(R_\triangleright^{(0)}[\![\psi]\!]\right)^\uparrow && \\
\text{iff } & (\forall a \in A)\left(a \in R_\triangleright^{(0)}[\![\psi]\!] \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall a \in A)\,(\forall a' \in A)\,((a' \Vdash \psi \to a' R_\triangleright a) \to aIx) && \text{(Definition 3.4)}
\end{aligned}$$

and

$$\begin{aligned}
a \Vdash \triangleright\psi \text{ iff } & a \in \left(R_\triangleright^{(0)}[\![\psi]\!]\right)^{\uparrow\downarrow} && \\
\text{iff } & (\forall x \in X)\left(x \in \left(R^{(0)}[\![\psi]\!]\right)^\uparrow \to aIx\right) && \text{(Definition 3.4)}\\
\text{iff } & (\forall x \in X)\,(x \succ \triangleright\psi \to aIx)
\end{aligned}$$

⊣

Proposition 4.10. For any $\psi \in \mathcal{L}_P$ for subsets $[\![\psi]\!]$ and $(\![\psi]\!)$ are Galois stable.

Proof. We will proceed by a standard induction on the number of connectives in ψ.

BASE CASE: ψ has 0 connectives:

If ψ is a logical constant:

$\psi = \bot$ implies $[\![\psi]\!] = X^\downarrow$	Galois Stable by Proposition 3.14(b)
$(\![\psi]\!) = X$	Galois Stable by Corollary 3.16
$\psi = \top$ implies $[\![\psi]\!] = A$	Galois Stable by Corollary 3.16
$(\![\psi]\!) = A^\downarrow$	Galois Stable by Proposition 3.14(b)

If $\psi = p$ for $p \in \mathsf{Prop}$ then $[\![\psi]\!] = (\![p]\!)^\downarrow$ and $(\![\psi]\!) = [\![p]\!]^\uparrow$, both of which are Galois stable by Proposition 3.14(b).

INDUCTIVE HYPOTHESIS: Suppose that for $\psi \in \mathcal{L}_p$ containing at most n connectives $[\![\psi]\!]$ and $(\![\psi]\!)$ are Galois stable.

INDUCTIVE STEP: Let $\varphi, \gamma \in \mathcal{L}_p$ with φ and γ each have at most n connectives.

If $\psi = \varphi \wedge \gamma$ then $[\![\psi]\!] = [\![\varphi]\!] \cap [\![\gamma]\!]$ and $(\![\psi]\!) = ([\![\varphi]\!] \cap [\![\gamma]\!])^\uparrow$. By hypothesis $[\![\varphi]\!]$ and $[\![\gamma]\!]$ are Galois stable, and as the intersection of Galois stable sets is Galois stable, $[\![\psi]\!]$ is Galois stable. And by Proposition 3.14(b) $(\![\psi]\!)$ is Galois stable. The argument for $\psi = \varphi \vee \gamma$ is symmetric.

If $\psi = \circ\varphi$ where $\circ \in \{\Box, \Diamond, \lhd, \rhd\}$ by definition $(\![\psi]\!)$ and $[\![\psi]\!]$ are the up or down sets of some subset of A or X. Thus by Proposition 3.14(b) are Galois stable. ⊣

The interpretation of sequents in models is given in Definition 4.11 as in [10].

Definition 4.11. A sequent $\varphi \vdash \psi$ is *true in a model*, $\mathcal{M}$, (denoted $\mathcal{M} \Vdash \varphi \vdash \psi$) if $[\![\varphi]\!] \subseteq [\![\psi]\!]$ in $\mathcal{M}$. Equivalently

$$\mathcal{M} \Vdash \varphi \vdash \psi \text{ iff } \mathcal{M}, a \Vdash \varphi \text{ implies } \mathcal{M}, a \Vdash \psi \text{ for all } a \in A.$$

Remark 4.12. Definition 4.11 can be equivalently stated in terms of intentions as

$$\mathcal{M} \Vdash \varphi \vdash \psi \text{ iff } (\![\psi]\!) \subseteq (\![\varphi]\!).$$

This is as $[\![\psi]\!] \subseteq [\![\varphi]\!]$ if and only if $(\![\varphi]\!) \subseteq (\![\psi]\!)$, as $[\![\psi]\!]$ and $[\![\varphi]\!]$ are Galois stable subsets by Proposition 4.10.

As in classical Kripke semantics, the notion of truth in a model is generalised to frame validity (Definition 4.13) and then validity on classes of frames (Definition 4.14).

Definition 4.13 (Validity of a sequent on a frame)**.** A sequent $\varphi \vdash \psi$ is *valid on a polarity-based frame*, $\mathbb{F}$, (denoted $\mathbb{F} \Vdash \varphi \vdash \psi$) if $\mathcal{M} \Vdash \varphi \vdash \psi$ in every model $\mathcal{M} = (\mathbb{F}, V)$ based on $\mathbb{F}$.

Definition 4.14 (Validity of sequent on a class of frames)**.** A sequent $\varphi \vdash \psi$ is *valid on a class of frames*, F (denoted $\mathsf{F} \Vdash \varphi \vdash \psi$) if $\mathbb{F} \Vdash \varphi \vdash \psi$ in every $\mathbb{F} \in \mathsf{F}$.

We will sometimes use the shorthand $\Vdash \varphi \vdash \psi$ when $\varphi \vdash \psi$ is valid on the class of all polarity-based frames in the signature.

4.4 Soundness and completeness

In this section we prove that $\boldsymbol{L}_P$ is sound and complete with respect to the class of enriched polarity-based frames. We begin with several brief notes on our conventions for uniform substitution throughout the remainder of the thesis. In Section 4.4.1 we prove through a series of results (Lemma 4.20-Lemma 4.23) that $\boldsymbol{L}_P$ is sound with respect to the class of enriched non-compatible polarity-based frames: that every valid sequent is derivable. In Section 4.4.2 we prove, via a standard Lindenbaum-Tarski type construction, that $\boldsymbol{L}_P$ is complete with respect to the class of enriched non-compatible polarity-based frames: that every sequent valid on this class of frames is derivable in $\boldsymbol{L}_P$.

Definition 4.15. For $\varphi, \psi \in \mathcal{L}_P$ we use the notation $\mathsf{Var}(\varphi)$ to denote the set of proposition variables appearing in φ. Formally, we define $\mathsf{Var}(\varphi)$ recursively as follows

$$\begin{array}{lll}
\mathsf{Var}(p) = \{p\} & \mathsf{Var}(\bot) = \varnothing & \mathsf{Var}(\top) = \varnothing \\
\mathsf{Var}(\varphi \wedge \psi) = \mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) & \mathsf{Var}(\varphi \vee \psi) = \mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) & \mathsf{Var}(\Box\varphi) = \mathsf{Var}(\varphi) \\
\mathsf{Var}(\Diamond\varphi) = \mathsf{Var}(\varphi) & \mathsf{Var}(\lhd\varphi) = \mathsf{Var}(\varphi) & \mathsf{Var}(\rhd\varphi) = \mathsf{Var}(\varphi)
\end{array}$$

The single variable substitution rule, Subs., can be defined inductively as in Definition 4.16.

Definition 4.16 (Substitution)**.** For $\varphi, \chi \in \mathcal{L}_P$ with $p \in \mathsf{Var}(\varphi)$, the single variable substitution of χ for p in φ (denoted $\varphi(\chi/p)$) is defined as $\theta_{\chi/p}(\varphi)$ where $\theta_{\chi/p}$ is a map $\theta_{\chi/p} : \mathcal{L}_P \to \mathcal{L}_P$ defined inductively on $\mathcal{L}_P$ below.

Let $\psi, \gamma \in \boldsymbol{L}_P$ and $q \in \mathsf{Prop}$ be a proposition variable distinct from p

$$\begin{array}{lll}
\theta_{\chi/p}(p) = \chi & \theta_{\chi/p}(q) = q & \theta_{\chi/p}(\bot) = \bot \\
\theta_{\chi/p}(\top) = \top & \theta_{\chi/p}(\psi \wedge \gamma) = \theta_{\chi/p}(\psi) \wedge \theta_{\chi/p}(\gamma) & \theta_{\chi/p}(\psi \vee \gamma) = \theta_{\chi/p}(\psi) \vee \theta_{\chi/p}(\gamma) \\
\theta_{\chi/p}(\Box\psi) = \Box\theta_{\chi/p}(\psi) & \theta_{\chi/p}(\Diamond\psi) = \Diamond\theta_{\chi/p}(\psi) & \theta_{\chi/p}(\lhd\psi) = \lhd\theta_{\chi/p}(\psi) \\
\theta_{\chi/p}(\rhd\psi) = \rhd\theta_{\chi/p}(\psi) & &
\end{array}$$

Proposition 4.17 (Theorems of $\boldsymbol{L}_P$)**.** For $\varphi, \psi, \chi \in \mathcal{L}_P$ the following are theorems of $\boldsymbol{L}_P$:

$$\begin{array}{ll}
\varphi \wedge \psi \vdash \psi \wedge \varphi \ (\wedge\mathrm{C}) & \varphi \vee \psi \vdash \psi \vee \varphi \ (\vee\mathrm{C}) \\
(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi) \ (\wedge\mathrm{A1}) & \varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi \ (\wedge\mathrm{A2}) \\
(\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi) \ (\vee\mathrm{A1}) & \varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi \ (\vee\mathrm{A2})
\end{array}$$

Proof. For $(\wedge\mathrm{C})$:

$$\dfrac{\varphi \wedge \psi \ \vdash \ \psi \qquad \varphi \wedge \psi \ \vdash \ \varphi}{\varphi \wedge \psi \ \vdash \ \psi \wedge \varphi}\ (\wedge\mathrm{R})$$

For $(\vee\mathrm{C})$:

$$\dfrac{\psi \vdash \varphi \vee \psi \qquad \varphi \vdash \varphi \vee \psi}{\psi \vee \varphi \vdash \varphi \vee \psi}\,(\vee\mathrm{L})$$

For ($\wedge$A1):

$$\dfrac{\dfrac{\varphi \wedge (\psi \wedge \chi) \vdash \varphi \qquad \dfrac{\varphi \wedge (\psi \wedge \chi) \vdash (\psi \wedge \chi) \qquad \psi \wedge \chi \vdash \psi}{\varphi \wedge (\psi \wedge \chi) \vdash \psi}\,(\mathrm{Cut})}{\varphi \wedge (\psi \wedge \chi) \vdash \varphi \wedge \psi}\,(\wedge\mathrm{R}) \qquad \dfrac{\varphi \wedge (\psi \wedge \chi) \vdash \psi \wedge \chi \qquad \psi \wedge \chi \vdash \chi}{\varphi \wedge (\psi \wedge \chi) \vdash \chi}\,(\mathrm{Cut})}{\varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi}\,(\wedge\mathrm{R})$$

By a symmetric derivation we have ($\wedge$A2).

For ($\vee$A1):

$$\dfrac{\dfrac{\varphi \vdash \varphi \vee \psi \qquad (\varphi \vee \psi) \vdash (\varphi \vee \psi) \vee \chi}{\varphi \vdash (\varphi \vee \psi) \vee \chi}\,(\mathrm{Cut}) \qquad \dfrac{\dfrac{\psi \vdash \varphi \vee \psi \qquad \varphi \vee \psi \vdash (\varphi \vee \psi) \vee \chi}{\psi \vdash (\varphi \vee \psi) \vee \chi}\,(\mathrm{Cut}) \qquad \chi \vdash (\varphi \vee \psi) \vee \chi}{\psi \vee \chi \vdash (\varphi \vee \psi) \vee \chi}\,(\vee\mathrm{L})}{\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi}\,(\vee\mathrm{L})$$

By a symmetrical derivation we have ($\vee$A2). $\dashv$

Definition 4.18 (Simultaneous n-variable substitution)**.** The uniform substitution rule, (Subs.), is given for the sake of simplicity in terms of a single substitution χ/p. This single variable formulation, however, can be generalised to a uniform substitution of n formulae $\chi_1, \ldots, \chi_n$ for n propositional variables $p_1, \ldots, p_n$ in a formula φ (denoted $\varphi(\chi_1/p_1, \ldots, \chi_n/p_n)$) in the following way.

First we define substitution of n propositional variables $\mathsf{Var}\,(\varphi) = \{p_1, \ldots, p_n\}$ for a primed set of proposition variables $\{p'_1, \ldots, p'_n\}$ disjoint from $\{p_1, \ldots, p_n\}$ and all variables occurring in $\chi_1, \ldots, \chi_n$.

Let φ'_i be the formulae defined recursively by:

$$\varphi'_1 = \varphi\,(p'_1/p_1) \text{ and } \varphi'_i = \varphi'_{i-1}\,(p'_i/p_i) \text{ for } 2 \leq i \leq n.$$

Define γ_i recursively by:

$$\gamma_1 = \varphi'_1\,(\chi_1/p'_1) \text{ and } \gamma_i = \gamma_{i-1}\,(\chi_i/p'_i) \text{ for } 2 \leq i \leq n.$$

Then we use the notation $\varphi\,(\chi_1/p_1, \ldots, \chi_n/p_n)$ to refer to the formula

$$\varphi\,(\chi_1/p_1, \ldots, \chi_n/p_n) := \gamma_n.$$

The need for Definition 4.18 is illustrated by the following example.

Example 4.19. Let $\chi_1, \chi_2, \varphi \in \boldsymbol{L}_P$ with $\varphi = p_1 \vee \Diamond p_2$, $\chi_1 = p_2$, and $\chi_2 = p_1 \wedge p_2$.

Then performing the direct substitution of χ_1 for p_1 followed by χ_2 for p_2 is distinct from the formula obtained by direct substitution of χ_2 for p_2 followed by the substitution of χ_1 for p_1:

$$\varphi(\chi_1/p_1) = p_2 \vee \Diamond p_2 \text{ and } (\varphi\,(\chi_1/p_1))\,(\chi_2/p_2) = (p_1 \wedge p_2) \vee \Diamond\,(p_1 \wedge p_2)$$

but

$$\varphi\,(\chi_2/p_2) = p_1 \vee \Diamond\,(p_1 \wedge p_2) \text{ and } (\varphi\,(\chi_2/p_2))\,(\chi_1/p_1) = p_2 \vee \Diamond\,(p_2 \wedge p_2)\,.$$

Note, however, if we perform the procedure described in Definition 4.18 we obtain the formulae for the uniform 2-formulae substitution of χ_1/p_1 and χ_2/p_2:

$$\varphi\,(\chi_1/p_1, \chi_2/p_2) = p_2 \vee \Diamond\,(p_1 \wedge p_2) \text{ and } \varphi\,(\chi_2/p_2, \chi_1/p_1) = p_2 \vee \Diamond\,(p_1 \wedge p_2)\,.$$

The equivalence of such formulae is motivation for the inclusion of Definition 4.18.

Simultaneous n-variable substitution can be defined recursively in a manner similar to Definition 4.16, however, Example 4.19 and Definition 4.18 illustrate that simultaneous n-variable substitution can still be obtained via appropriate repeated uniform single variable substitutions.

4.4.1 Soundness

We begin by showing that all of our axioms are valid on the class of polarity-based frames.

Lemma 4.20. Let $\mathcal{M}$ be an arbitrary polarity-based model and $p, q \in \mathsf{Prop}$ then

(i) $\mathcal{M} \Vdash p \vdash p$

(ii) $\mathcal{M} \Vdash \bot \vdash p$

(iii) $\mathcal{M} \Vdash p \vdash \top$

(iv) $\mathcal{M} \Vdash p \vdash p \vee q$

(v) $\mathcal{M} \Vdash q \vdash p \vee q$

(vi) $\mathcal{M} \Vdash p \wedge q \vdash p$

(vii) $\mathcal{M} \Vdash p \wedge q \vdash q$

Proof.

$\mathcal{M} \Vdash p \vdash p$ follows immediately from $[\![p]\!] \subseteq [\![p]\!]$.

$\mathcal{M} \Vdash \bot \vdash p$ as $(\![p]\!) \subseteq X = (\![\bot]\!)$ always.

$\mathcal{M} \Vdash p \vdash \top$ as $[\![p]\!] \subseteq A = [\![\top]\!]$ always.

$\mathcal{M} \Vdash p \vdash p \vee q$ as $(\![p \vee q]\!) = (\![p]\!) \cap (\![q]\!) \subseteq (\![p]\!)$.

$\mathcal{M} \Vdash q \vdash p \vee q$ as $(\![p \vee q]\!) = (\![p]\!) \cap (\![q]\!) \subseteq (\![q]\!)$.

$\mathcal{M} \Vdash p \wedge q \vdash p$ as $[\![p \wedge q]\!] = [\![p]\!] \cap [\![q]\!] \subseteq [\![p]\!]$.

$\mathcal{M} \Vdash p \wedge q \vdash q$ as $[\![p \wedge q]\!] = [\![p]\!] \cap [\![q]\!] \subseteq [\![q]\!]$. ⊣

The following three lemmas (Lemma 4.21-4.23) show the soundness of the rules of inference on the class of polarity-based frames.

Lemma 4.21. Let $\mathcal{M}$ be an arbitrary polarity based model and $\varphi, \psi, \chi \in \mathcal{L}_P$. Then

(i) $\mathcal{M} \Vdash \varphi \vdash \chi$ and $\mathcal{M} \Vdash \chi \vdash \psi$ implies $\mathcal{M} \Vdash \varphi \vdash \psi$

(ii) $\mathcal{M} \Vdash \chi \vdash \varphi$ and $\mathcal{M} \Vdash \chi \vdash \psi$ implies $\mathcal{M} \Vdash \chi \vdash \varphi \wedge \psi$

(iii) $\mathcal{M} \Vdash \varphi \vdash \chi$ and $\mathcal{M} \Vdash \psi \vdash \chi$ implies $\mathcal{M} \Vdash \varphi \vee \psi \vdash \chi$

Proof. Inference (i) follows from transitivity of $\subseteq$ as $[\![\varphi]\!] \subseteq [\![\chi]\!]$ and $[\![\chi]\!] \subseteq [\![\psi]\!]$ implies that $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

For (ii)

$$\begin{aligned}\mathcal{M} \Vdash \chi \vdash \varphi \text{ and } \mathcal{M} \Vdash \chi \vdash \psi &\text{ implies } [\![\chi]\!] \subseteq [\![\varphi]\!] \text{ and } [\![\chi]\!] \subseteq [\![\psi]\!] \\ &\text{ implies } [\![\chi]\!] \subseteq [\![\varphi]\!] \cap [\![\psi]\!] \\ &\text{ implies } [\![\chi]\!] \subseteq [\![\varphi \wedge \psi]\!] \\ &\text{ implies } \mathcal{M} \Vdash \chi \vdash \varphi \wedge \psi.\end{aligned}$$

For (iii)

$$\begin{aligned}\mathcal{M} \Vdash \varphi \vdash \chi \text{ and } \mathcal{M} \Vdash \psi \vdash \chi &\text{ implies } [\![\varphi]\!] \subseteq [\![\chi]\!] \text{ and } [\![\psi]\!] \subseteq [\![\chi]\!] \\ &\text{ implies } (\![\chi]\!) \subseteq (\![\varphi]\!) \text{ and } (\![\chi]\!) \subseteq (\![\psi]\!) \\ &\text{ implies } (\![\chi]\!) \subseteq ((\![\varphi]\!) \cap (\![\psi]\!)) \\ &\text{ implies } ((\![\varphi]\!) \cap (\![\psi]\!))^{\downarrow} \subseteq [\![\chi]\!] \\ &\text{ implies } \mathcal{M} \Vdash \varphi \vee \psi \vdash \chi.\end{aligned}$$

⊣

Lemma 4.22. Let $\mathbb{F}$ be an arbitrary polarity based frame and $\varphi, \psi, \chi \in \mathcal{L}_P$:

$$\mathbb{F} \Vdash \varphi \vdash \psi \text{ implies } \mathbb{F} \Vdash \varphi\,(\chi/p) \vdash \psi\,(\chi/p)\,.$$

Proof. We show the contraposition

$$\mathbb{F} \nVdash \varphi\,(\chi/p) \vdash \psi\,(\chi/p) \;\text{ implies } \mathbb{F} \nVdash \varphi \vdash \psi.$$

Suppose that $\mathbb{F} \nVdash \varphi\,(\chi/p) \vdash \psi\,(\chi/p)$, and that $\mathcal{M} = (\mathbb{F}, V)$ is a polarity-based model in which $\mathcal{M} \nVdash \varphi\,(\chi/p) \nvdash \psi\,(\chi/p)$. Then

$$[\![\varphi\,(\chi/p)]\!]_{\mathcal{M}} \nsubseteq [\![\psi\,(\chi/p)]\!]_{\mathcal{M}}.$$

Let $\mathcal{M}' = (\mathbb{F}, V')$ where $V' : \mathsf{Prop} \to \mathbb{F}^+$ is the valuation

$$V'(q) = \begin{cases} ([\![\chi]\!]_{\mathcal{M}}, (\![\chi]\!)_{\mathcal{M}}) & q = p \\ V(q) & \text{otherwise.}\end{cases}$$

Then

$$[\![\varphi]\!]_{\mathcal{M}'} = [\![\varphi\,(\chi/p))]\!]_{\mathcal{M}} \nsubseteq [\![\psi\,(\chi/p)]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}'}.$$

Thus $\mathbb{F} \nVdash \varphi \vdash \psi$, as required. ⊣

Lemma 4.23. Let $\mathcal{M}$ be an arbitrary polarity based model and $\varphi, \psi \in \mathcal{L}_P$. Then

(i) $\mathcal{M} \Vdash \varphi \vdash \psi$ implies $\mathcal{M} \Vdash \Box\varphi \vdash \Box\psi$

(ii) $\mathcal{M} \Vdash \varphi \vdash \psi$ implies $\mathcal{M} \Vdash \Diamond\varphi \vdash \Diamond\psi$

(iii) $\mathcal{M} \Vdash \varphi \vdash \psi$ implies $\mathcal{M} \Vdash {\rhd}\psi \vdash {\rhd}\varphi$

(iv) $\mathcal{M} \Vdash \varphi \vdash \psi$ implies $\mathcal{M} \Vdash {\lhd}\psi \vdash {\lhd}\varphi$

Proof. For (i)

$$\begin{aligned}\mathcal{M} \Vdash \varphi \vdash \psi &\text{ implies } [\![\varphi]\!] \subseteq [\![\psi]\!] \\ &\text{ implies } (\![\psi]\!) \subseteq (\![\varphi]\!) \\ &\text{ implies } R_{\Box}^{(0)}\left((\![\varphi]\!)\right) \subseteq R_{\Box}^{(0)}\left((\![\psi]\!)\right) && \text{(Lemma 3.5(a))} \\ &\text{ implies } \left(R_{\Box}^{(0)}\left((\![\psi]\!)\right)\right)^{\uparrow} \subseteq \left(R_{\Box}^{(0)}\left((\![\varphi]\!)\right)\right)^{\uparrow} && \text{(Corollary 3.6(b))} \\ &\text{ implies } \left(R_{\Box}^{(0)}\left((\![\varphi]\!)\right)\right)^{\uparrow\downarrow} \subseteq \left(R_{\Box}^{(0)}\left((\![\psi]\!)\right)\right)^{\uparrow\downarrow} && \text{(Corollary 3.6(a))} \\ &\text{ implies } [\![\Box\varphi]\!] \subseteq [\![\Box\psi]\!] \\ &\text{ implies } \mathcal{M} \Vdash \Box\varphi \vdash \Box\psi\end{aligned}$$

For (ii)

$$
\begin{aligned}
\mathcal{M} \Vdash \varphi \vdash \psi &\text{ implies } [\![\varphi]\!] \subseteq [\![\psi]\!] \\
&\text{ implies } R_{\Diamond}^{(0)}\left([\![\psi]\!]\right) \subseteq R_{\Diamond}^{(0)}\left([\![\varphi]\!]\right) && \text{(Lemma 3.5(a))} \\
&\text{ implies } \left(R_{\Diamond}^{(0)}\left([\![\varphi]\!]\right)\right)^{\downarrow} \subseteq \left(R_{\Diamond}^{(0)}\left([\![\psi]\!]\right)\right)^{\downarrow} && \text{(Corollary 3.6(a))} \\
&\text{ implies } [\![\Diamond\varphi]\!] \subseteq [\![\Diamond\psi]\!] \\
&\text{ implies } \mathcal{M} \Vdash \Diamond\varphi \vdash \Diamond\psi
\end{aligned}
$$

For (iii)

$$
\begin{aligned}
\mathcal{M} \Vdash \varphi \vdash \psi &\text{ implies } [\![\varphi]\!] \subseteq [\![\psi]\!] \\
&\text{ implies } R_{\rhd}^{(0)}\left([\![\psi]\!]\right) \subseteq R_{\rhd}^{(0)}\left((\![\psi]\!)\right) && \text{(Lemma 3.5(a))} \\
&\text{ implies } \left(R_{\rhd}^{(0)}\left([\![\varphi]\!]\right)\right)^{\uparrow} \subseteq \left(R_{\rhd}^{(0)}\left((\![\psi]\!)\right)\right)^{\uparrow} && \text{(Corollary 3.6(b))} \\
&\text{ implies } (\![\rhd\varphi]\!) \subseteq (\![\rhd\psi]\!) \\
&\text{ implies } [\![\rhd\psi]\!] \subseteq [\![\rhd\varphi]\!] \\
&\text{ implies } \mathcal{M} \Vdash \rhd\psi \vdash \rhd\varphi.
\end{aligned}
$$

For (iv)

$$
\begin{aligned}
\mathcal{M} \Vdash \varphi \vdash \psi &\text{ implies } (\![\psi]\!) \subseteq (\![\varphi]\!) \\
&\text{ implies } R_{\lhd}^{(0)}\left((\![\varphi]\!)\right) \subseteq R_{\lhd}^{(0)}\left((\![\psi]\!)\right) && \text{(Lemma 3.5(a))} \\
&\text{ implies } \left(R_{\lhd}^{(0)}\left((\![\psi]\!)\right)\right)^{\downarrow} \subseteq \left(R_{\lhd}^{(0)}\left((\![\varphi]\!)\right)\right)^{\downarrow} && \text{(Corollary 3.6(a))} \\
&\text{ implies } [\![\lhd\psi]\!] \subseteq [\![\lhd\varphi]\!] \\
&\text{ implies } \mathcal{M} \Vdash \lhd\psi \vdash \lhd\varphi.
\end{aligned}
$$

$\dashv$

Therefore by Lemma 4.20-4.23 we have that $\boldsymbol{L}_P$ is sound with respect to the class of polarity-based frames.

4.4.2 Completeness

In this subsection we prove the completeness of $\boldsymbol{L}_P$ with respect to polarity-based frames – that is "if $\varphi \vdash \psi$ is valid in every enriched polarity, $\mathbb{F}$, then $\varphi \vdash \psi$ is derivable in $\boldsymbol{L}_P$". Our approach is similar to that of [10] and [7] in that we utilise a standard Lindenbaum-Tarski construction, from which an associated enriched polarity $\mathbb{F}_{\mathbb{A}_P}$ can be constructed. This enriched polarity is then used to show the contrapositive of the completeness condition: that if $\varphi \vdash \psi$ is not derivable in $\boldsymbol{L}_P$ then there exists a $\mathbb{F}$ such that $\mathbb{F} \nVdash \varphi \vdash \psi$.

Definition 4.24. Let $\varphi, \psi \in \mathcal{L}_P$ the relation $\sim \subseteq \mathcal{L}_P \times \mathcal{L}_P$ is defined by

$$\varphi \sim \psi \text{ iff } \varphi \vdash \psi \text{ and } \psi \vdash \varphi \text{ are both derivable in } \boldsymbol{L}_P.$$

Lemma 4.25 ($\sim$ is an equivalence relation)**.** Let $\varphi, \psi, \gamma \in \mathcal{L}_P$ then

(a) $\varphi \sim \varphi$ (reflexivity)

(b) $\varphi \sim \psi$ implies $\psi \sim \varphi$ (symmetry)

(c) $\varphi \sim \psi$ and $\psi \sim \gamma$ implies $\varphi \sim \psi$ (transitivity).

Proof. Reflexivity: follows directly from the definition of $\sim$ and $\varphi \vdash \varphi$ (Lemma 4.3).

Symmetry: follows directly from the definition of $\sim$. If $\varphi \sim \psi$ then $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable in $\boldsymbol{L}_P$ and so $\psi \sim \varphi$.

Transitivity: suppose that $\varphi \sim \psi$ and $\psi \sim \gamma$ then $\varphi \vdash \psi$ and $\psi \vdash \gamma$ and so by (Cut) we have $\varphi \vdash \gamma$. Similarly, as $\gamma \vdash \psi$ and $\psi \vdash \varphi$ it follows that $\gamma \vdash \varphi$ and so $\varphi \sim \gamma$. $\dashv$

Lemma 4.26. For $\varphi_0, \varphi_1, \psi_0, \psi_1 \in \mathcal{L}_P$

(a) $\varphi_0 \sim \varphi_1$ and $\psi_0 \sim \psi_1$ implies $\varphi_0 \wedge \psi_0 \sim \varphi_1 \wedge \psi_1$.

(b) $\varphi_0 \sim \varphi_1$ and $\psi_0 \sim \psi_1$ implies $\varphi_0 \vee \psi_0 \sim \varphi_1 \vee \psi_1$.

(c) $\varphi_0 \sim \varphi_1$ implies $h\varphi_0 \sim h\varphi_1$ for each $h \in \{\Diamond, \Box, \lhd, \rhd\}$.

Proof. For (a) and (b) suppose $\varphi_0 \sim \varphi_1$ and $\psi_0 \sim \psi_1$. Then $\varphi_0 \vdash \varphi_1$, $\varphi_1 \vdash \varphi_0$, $\psi_0 \vdash \psi_1$, and $\psi_1 \vdash \psi_0$.

Below it is shown that $\varphi_0 \wedge \psi_0 \vdash \varphi_1 \wedge \psi_1$. The proof tree for $\varphi_1 \wedge \psi_1 \vdash \varphi_0 \wedge \psi_0$ is identical apart from simultaneously replacing all 1s for 0s and 0s for 1s in our labelling.

$$\dfrac{\dfrac{\varphi_0 \wedge \psi_0 \ \vdash\ \varphi_0 \qquad \varphi_0 \ \vdash\ \varphi_1}{\varphi_0 \wedge \psi_0 \ \vdash\ \varphi_1}\,(\mathrm{Cut}) \qquad \dfrac{\varphi_0 \wedge \psi_0 \ \vdash\ \psi_0 \qquad \psi_0 \ \vdash\ \psi_1}{\varphi_0 \wedge \psi_0 \ \vdash\ \psi_1}\,(\mathrm{Cut})}{\varphi_0 \wedge \psi_0 \ \vdash\ \varphi_1 \wedge \psi_1}\,(\wedge\mathrm{R})$$

For (b) first note that $\varphi_0 \vdash \varphi_1$ implies that $\varphi_0 \vdash \varphi_1 \vee \psi_1$ from ($\vee$R1) and (Cut), and similarly $\psi_0 \vdash \psi_1$ implies that $\psi_0 \vdash \varphi_1 \vee \psi_1$ from ($\vee$R2) and (Cut). Then

$$\dfrac{\varphi_0 \ \vdash\ \varphi_1 \vee \psi_1 \qquad \psi_0 \ \vdash\ \varphi_1 \vee \psi_1}{\varphi_0 \vee \psi_0 \ \vdash\ \varphi_1 \vee \psi_1}\,(\vee\mathrm{L})$$

By a symmetric derivation we have $\varphi_1 \vee \psi_1 \vdash \varphi_0 \vee \psi_0$. Thus $\varphi_0 \vee \psi_0 \sim \varphi_1 \vee \psi_1$.

For (c) for h' either $\Box$ or $\Diamond$ and h'' either $\lhd$ or $\rhd$ we have:

$$\dfrac{\varphi_0 \ \vdash\ \varphi_1}{h'\varphi_0 \ \vdash\ h'\varphi_1}\,(h'\mathrm{M}) \qquad\qquad \dfrac{\varphi_0 \ \vdash\ \varphi_1}{h''\varphi_1 \ \vdash\ h''\varphi_1}\,(h''\mathrm{A}).$$

From symmetric derivations we have $h'\varphi_1 \vdash h'\varphi_0$ and $h''\varphi_0 \vdash h''\varphi_1$: replacing the 1 indices with 0 and 0 indices with 1 in the above derivation. Therefore $h\varphi_0 \sim h\varphi_1$. $\dashv$

Definition 4.27 (Equivalence classes of φ under $\sim$)**.** For $\varphi \in \mathcal{L}_P$ define $[\varphi] = \{\psi \in \mathcal{L}_P \mid \varphi \sim \psi\}$.

Definition 4.28 (Set of equivalence classes on $\mathcal{L}_P$)**.** The set of equivalence classes on $\mathcal{L}_P$ induced by $\sim$ is the set

$$\mathcal{L}_{P/\sim} := \{[\varphi] \mid \varphi \in \mathcal{L}_P\}.$$

Definition 4.29 (Lindenbaum-Tarski Algebra of $\boldsymbol{L}_P$)**.** The Lindenbaum-Tarski Algebra of $\mathcal{L}_P$, $\mathbb{A}_P$, is the algebra

$$\mathbb{A}_P := \left(\left(\mathcal{L}_{P/\sim}, \wedge^{\mathbb{A}_P}, \vee^{\mathbb{A}_P}\right), \left\{\Diamond^{\mathbb{A}_P}, \Box^{\mathbb{A}_P}, \lhd^{\mathbb{A}_P}, \rhd^{\mathbb{A}_P}\right\}\right).$$

This is where for $[\varphi], [\psi] \in \mathcal{L}_{P/\sim}$:

$$\begin{array}{lll} [\varphi] \wedge^{\mathbb{A}_P} [\psi] := [\varphi \wedge \psi] & \Diamond^{\mathbb{A}_P} [\varphi] := [\Diamond\varphi] & \lhd^{\mathbb{A}_P} [\varphi] := [\lhd\varphi] \\ [\varphi] \vee^{\mathbb{A}_P} [\psi] := [\varphi \vee \psi] & \Box^{\mathbb{A}_P} [\varphi] := [\Box\varphi] & \rhd^{\mathbb{A}_P} [\varphi] := [\rhd\varphi]. \end{array}$$

Proposition 4.30. The triple $(\mathcal{L}_{P/\sim}, \wedge^{\mathbb{A}_P}, \vee^{\mathbb{A}_P})$ is a lattice.

Proof. We will show that $\wedge^{\mathbb{A}_P}$ and $\vee^{\mathbb{A}_P}$ satisfy the conditions of Definition 2.9.

Commutativity: By Proposition 4.17, ($\vee$C) is derivable in $\boldsymbol{L}_P$. Thus $\varphi \vee \psi \vdash \psi \vee \varphi$ and $\psi \vee \varphi \vdash \varphi \vee \psi$ and so $\varphi \vee \psi \sim \psi \vee \varphi$. Therefore $[\varphi] \vee^{\mathbb{A}_P} [\psi] = [\psi] \vee^{\mathbb{A}_P} [\varphi]$. Similarly, as ($\wedge$C) is derivable in $\boldsymbol{L}_P$ we have $\varphi \wedge \psi \sim \psi \wedge \varphi$ and so $[\varphi] \wedge^{\mathbb{A}_P} [\psi] = [\psi] \wedge^{\mathbb{A}_P} [\varphi]$.

Associativity: By Proposition 4.17, ($\wedge$A1) and ($\wedge$A2) are derivable in $\boldsymbol{L}_P$. Therefore $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$ and $\varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi$ and so $\varphi \wedge (\psi \wedge \chi) \sim (\varphi \wedge \psi) \wedge \chi$. Thus

$$[\varphi] \wedge^{\mathbb{A}_P} ([\psi] \wedge^{\mathbb{A}_P} [\chi]) = ([\varphi] \wedge^{\mathbb{A}_P} [\psi]) \wedge^{\mathbb{A}_P} [\chi] .$$

Additionally, as ($\vee$A1) and ($\vee$A2) are theorems of $\boldsymbol{L}_P$ it follows that

$$\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi \text{ and } (\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi) .$$

Therefore $\varphi \vee (\psi \vee \chi) \sim (\varphi \vee \psi) \vee \chi$, and thus $([\varphi] \vee^{\mathbb{A}_P} [\psi]) \vee^{\mathbb{A}_P} [\chi] = [\varphi] \vee^{\mathbb{A}_P} \big([\psi] \vee^{\mathbb{A}_P} [\chi]\big)$.

Idempotency: The idempotency of $\vee^{\mathbb{A}_P}$ follows immediately from $\varphi \vdash \varphi \vee \varphi$ ($\vee$R1) and

$$\frac{\varphi \vdash \varphi \qquad \varphi \vdash \varphi}{\varphi \vee \varphi \vdash \varphi} (\vee\text{L}).$$

Thus $\varphi \vee \varphi \sim \varphi$ and so $[\varphi] \vee^{\mathbb{A}_P} [\varphi] = [\varphi]$.

For idempotency of $\wedge^{\mathbb{A}_P}$ we have $\varphi \wedge \varphi \vdash \varphi$ ($\wedge$L1), and

$$\frac{\varphi \vdash \varphi \qquad \varphi \vdash \varphi}{\varphi \vdash \varphi \wedge \varphi} (\wedge\text{R}).$$

So $[\varphi] \wedge^{\mathbb{A}_P} [\varphi] = [\varphi]$.

Absorption laws: First note that $\varphi \vdash \varphi \vee (\varphi \wedge \psi)$ ($\vee$R1) and $\varphi \wedge (\varphi \vee \psi) \vdash \varphi$ ($\wedge$L1), as well as the following derivations:

$$\frac{\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \varphi}{\varphi \vee (\varphi \wedge \psi) \vdash \varphi} (\vee\text{L}) \qquad\qquad \frac{\varphi \vdash \varphi \qquad \varphi \vdash \varphi \vee \psi}{\varphi \vdash \varphi \wedge (\varphi \vee \psi)} (\wedge\text{R}).$$

Thus $\varphi \vee (\varphi \wedge \psi) \sim \varphi$ and $\varphi \sim \varphi \wedge (\varphi \vee \psi)$. Therefore

$$[\varphi] = [\varphi] \wedge^{\mathbb{A}_P} \big([\varphi] \vee^{\mathbb{A}_P} [\psi]\big) \text{ and } [\varphi] \vee^{\mathbb{A}_P} \big([\varphi] \wedge^{\mathbb{A}_P} [\psi]\big) = [\varphi] .$$

As $\vee^{\mathbb{A}_P}$ and $\wedge^{\mathbb{A}_P}$ satisfy L1-L4 of Definition 2.9, $\big(\mathcal{L}_{P/\sim}, \wedge^{\mathbb{A}_P}, \vee^{\mathbb{A}_P}\big)$ is a lattice. ⊣

Definition 4.31. The lattice order, $\leq_{\mathbb{A}_P} \subseteq \mathcal{L}_{P/\sim} \times \mathcal{L}_{P/\sim}$, on $\big(\mathcal{L}_{P/\sim}, \wedge^{\mathbb{A}_P}, \vee^{\mathbb{A}_P}\big)$ is defined in the usual way, that is

$$[\varphi] \leq_{\mathbb{A}_P} [\psi] \text{ iff } [\varphi] = [\varphi] \wedge^{\mathbb{A}_P} [\psi] .$$

For the remainder of this section, when clear from context, we will write $\leq$ in place of $\leq_{\mathbb{A}_P}$ for the sake of readability.

Corollary 4.32. The relation $\leq_{\mathbb{A}_P}$ is a partial order on $\mathcal{L}_{P/\sim}$.

Proof. As $\mathbb{A}_P$ is a lattice (Proposition 4.30) the relation $\leq_{\mathbb{A}_P}$ is a partial order by Proposition 2.10. $\dashv$

Lemma 4.33. For any $\varphi, \psi \in \mathcal{L}_P$

$$[\varphi] \leq_{\mathbb{A}_P} [\psi] \text{ iff } \varphi \vdash \psi.$$

Proof. Suppose that $\varphi \vdash \psi$. Then

$$\frac{\varphi \vdash \varphi \qquad \varphi \vdash \psi}{\varphi \vdash \varphi \wedge \psi}\,(\wedge\mathrm{R})$$

Additionally, $\varphi \wedge \psi \vdash \varphi$ ($\wedge$L1). So $\varphi \wedge \psi \sim \varphi$ and therefore $[\varphi] \wedge^{\mathbb{A}_P} [\psi] = [\varphi]$. Thus $[\varphi] \leq_{\mathbb{A}_P} [\psi]$.

For the converse, suppose that $[\varphi] \leq_{\mathbb{A}_P} [\psi]$. Then $[\varphi] \wedge^{\mathbb{A}_P} [\psi] = [\varphi]$, and so $\varphi \wedge \psi \sim \varphi$. Thus $\varphi \vdash \varphi \wedge \psi$ and $\varphi \wedge \psi \vdash \varphi$ (note that we already have the second sequent from ($\wedge$L2)). Thus

$$\frac{\varphi \vdash \varphi \wedge \psi \qquad \varphi \wedge \psi \vdash \psi}{\varphi \vdash \psi}\,(\mathrm{Cut})$$

Therefore $[\varphi] \leq_{\mathbb{A}_P} [\psi]$ implies $\varphi \vdash \psi$. $\dashv$

Lemma 4.34. For any $\varphi \in \mathcal{L}_P$

(a) $[\bot] \leq_{\mathbb{A}_P} [\varphi]$

(b) $[\varphi] \leq_{\mathbb{A}_P} [\top]$

Proof. For (a) note that $\bot \wedge \varphi \sim \bot$, as $\bot \wedge \varphi \vdash \bot$ by ($\wedge$L1) and $\bot \vdash \bot \wedge \varphi$ by ($\bot L$). Therefore $[\varphi] \wedge^{\mathbb{A}_P} [\bot] = [\bot \wedge \varphi] = [\bot]$ by Definition 4.27. Thus $[\bot] \leq_{\mathbb{A}_P} [\varphi]$, as required.

Similarly, for (b) note that $\top \wedge \varphi \sim \varphi$ as $\top \wedge \varphi \vdash \varphi$ by ($\wedge$L2) and $\top \vdash \varphi \wedge \varphi$ from the following derivation

$$\frac{\varphi \vdash \top \qquad \varphi \vdash \varphi}{\varphi \vdash \top \wedge \varphi}\,(\wedge\mathrm{R}).$$

Thus $[\top] \wedge^{\mathbb{A}_P} [\varphi] = [\varphi]$ and so $[\varphi] \leq_{\mathbb{A}_P} [\top]$. $\dashv$

Proposition 4.35. $\mathbb{A}_P$ is a bitonal algebra.

Proof. By Proposition 4.30, $\mathbb{L} = \left(\mathcal{L}_{P/\sim}, \wedge^{\mathbb{A}_P}, \vee^{\mathbb{A}_P}\right)$ is a lattice. Additionally, $[\bot] \leq_{\mathbb{A}_P} [\varphi]$ and $[\varphi] \leq_{\mathbb{A}_P} [\top]$ for any $\varphi \in \boldsymbol{L}_P$ by Lemma 4.34. So $\mathbb{L}$ is bounded. What remains is to show that $\Diamond^{\mathbb{A}_P}$ and $\Box^{\mathbb{A}_P}$ are monotone and that $\lhd^{\mathbb{A}_P}$ and $\rhd^{\mathbb{A}_P}$ are antitone.

Suppose that $[\varphi] \leq_{\mathbb{A}_P} [\psi]$. For $\Diamond$:

$$\begin{aligned}
[\varphi] \leq_{\mathbb{A}_P} [\psi] &\text{ iff } \varphi \vdash \psi && \text{(Lemma 4.33)}\\
&\text{ implies } \Diamond\varphi \vdash \Diamond\psi && ((\Diamond\mathrm{M}))\\
&\text{ iff } [\Diamond\varphi] \leq_{\mathbb{A}_P} [\Diamond\psi] && \text{(Lemma 4.33)}
\end{aligned}$$

The argument for $\Box$ is identical – replacing $\Diamond$ with $\Box$ in the above chain of implications.

Proceeding similarly in showing the anti-tonicity of $\lhd$:

$$\begin{aligned}
[\varphi] \leq_{\mathbb{A}_P} [\psi] & \text{ iff } \varphi \vdash \psi && \text{(Lemma 4.33)} \\
& \text{implies } \lhd \psi \vdash \lhd \varphi && ((\lhd\text{A})) \\
& \text{iff } [\lhd\psi] \leq_{\mathbb{A}_P} [\lhd\varphi] && \text{(Lemma 4.33)}
\end{aligned}$$

The argument for $\rhd$ is identical – replacing $\lhd$ with $\rhd$ in our chain of implications above.

Thus $\Box^{\mathbb{A}_P}$ and $\Diamond^{\mathbb{A}_P}$ are monotone and $\lhd^{\mathbb{A}_P}$ and $\rhd^{\mathbb{A}_P}$ are anti-tone, and $\mathbb{A}_P$ is a bitonal algebra. $\dashv$

Definition 4.36 (Validity in $\mathbb{A}_P$)**.** A sequent $\varphi \vdash \psi$ is *valid in* $\mathbb{A}_P$ if for any assignment $\overline{v} : \boldsymbol{L}_P \to \mathbb{A}_P$:

$$\mathbb{A}_P \Vdash \varphi \vdash \psi \text{ iff } \overline{v}\,(\varphi) \leq_{\mathbb{A}_P} \overline{v}\,(\psi)\,.$$

Proposition 4.37. For any $\varphi, \psi \in \mathcal{L}_P$

$$\mathbb{A}_P \Vdash \varphi \vdash \psi \text{ iff } \varphi \vdash \psi \text{ is derivable in } \boldsymbol{L}_P.$$

Proof. For the forwards implication suppose that $\mathbb{A}_P \Vdash \varphi \vdash \psi$. Then for any assignment, $\overline{v}$, we have $\overline{v}(\varphi) \leq_{\mathbb{A}_P} \overline{v}\,(\psi)$. In particular, taking $\overline{v}$ as the natural assignment $\overline{\eta}\,(\varphi) = [\varphi]$:

$$\begin{aligned}
\overline{\eta}\,(\varphi) \leq_{\mathbb{A}_P} \overline{\eta}\,(\psi) & \text{ implies } [\varphi] \leq_{\mathbb{A}_P} [\psi] \\
& \text{ iff } \varphi \vdash \psi && \text{(Lemma 4.33)}
\end{aligned}$$

Thus $\varphi \vdash \psi$ is derivable in $\boldsymbol{L}_P$.

For the converse, suppose that $\varphi \vdash \psi$ is derivable in $\boldsymbol{L}_P$, and that $\mathsf{Var}\,(\varphi) \cup \mathsf{Var}\,(\psi) = \{p_1, \ldots, p_n\}$. Let $\overline{v}$ be an arbitrary assignment on $\mathbb{A}_P$, and let $\overline{v}\,(p_i) = [\gamma_i]$ with $\gamma_i \in \boldsymbol{L}_P$. Then

$$\overline{v}\,(\varphi) = \overline{\eta}\,(\varphi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)) = [\varphi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)]$$

and

$$\overline{v}\,(\psi) = \overline{\eta}\,(\psi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)) = [\psi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)]\,.$$

Thus

$$\frac{\varphi \;\vdash\; \psi}{\varphi(\gamma_1/p_1, \ldots, \gamma_n/p_n) \;\vdash\; \psi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)}\,\text{(Subs.)}$$

But

$$\begin{aligned}
& \varphi(\gamma_1/p_1, \ldots, \gamma_n/p_n) \vdash \psi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n) \\
\text{iff } & [\varphi(\gamma_1/p_1, \ldots, \gamma_n/p_n)] \leq_{\mathbb{A}_P} [\psi\,(\gamma_1/p_1, \ldots, \gamma_n/p_n)] && \text{(Lemma 4.33)}
\end{aligned}$$

That is, $\overline{v}\,(\varphi) \leq \overline{v}\,(\psi)$. Hence, $\varphi \vdash \psi$ implies $\mathbb{A}_P \Vdash \varphi \vdash \psi$. $\dashv$

The Lindebaum-Tarski Algebra, $\mathbb{A}_P$, will be associated with a corresponding enriched polarity, $\mathbb{F}_{\mathbb{A}_P}$. For the sake of clarity, however, this is first done for an arbitrary bitonal algebra. Denote this arbitrary bitonal algebra by $\mathbb{L} = \left((L, \wedge, \vee), \left\{\Box^{\mathbb{L}}, \Diamond^{\mathbb{L}}, \lhd^{\mathbb{L}}, \rhd^{\mathbb{L}}\right\}\right)$. This construction is adapted from [10, Appendix 4.].

Definition 4.38. For a lattice, $\mathbb{L}$, let the polarity $\mathbb{P}_{\mathbb{L}} = (F(\mathbb{L}), J(\mathbb{L}), I_{\mathbb{L}})$ where $F\,(\mathbb{L})$ and $J\,(\mathbb{L})$ are the sets of filters and ideals of $\mathbb{L}$ respectively, and for $f \in F\,(\mathbb{L})$ and $j \in J\,(\mathbb{L})$, let $f I_{\mathbb{L}} j$ iff $f \cap j \neq \varnothing$.

Lemma 4.39. In the polarity $\mathbb{P}_{\mathbb{L}}$ for a particular $u \in \mathbb{L}$:

(a) $\{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow} = \{j \in J(\mathbb{L}) \mid u \in j\}$.

(b) $\{j \in J(\mathbb{L}) \mid u \in j\}^{\downarrow} = \{f \in F(\mathbb{L}) \mid u \in f\}$.

Proof. For (a) note that

$$\begin{aligned}\{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow} &= \{j \in J(\mathbb{L}) \mid \forall f \in F(\mathbb{L})\,(u \in f \to f \cap j \neq \varnothing)\} \\ &= \{j \in J(\mathbb{L}) \mid \forall f \in F(\mathbb{L})\,(u \notin f \text{ or } f \cap j \neq \varnothing)\}\end{aligned}$$

For the inclusion $\{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow} \supseteq \{j \in J(\mathbb{L}) \mid u \in j\}$ consider a $j \in J(\mathbb{L})$ such that $u \in j$. Then for any $f \in F(\mathbb{L})$ where $u \in f$ clearly $\{u\} \subseteq f \cap j \neq \varnothing$ and so $j \in \{j \in J(\mathbb{L}) \mid \forall f \in F(\mathbb{L})\,(u \in f \to f \cap j \neq \varnothing)\}$.

For the inclusion $\{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow} \subseteq \{j \in J(\mathbb{L}) \mid u \in j\}$ suppose that $j \in \{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow}$. Let $F_u = \{x \in L \mid u \leq x\}$ be the principal filter generated by u. Then clearly $u \in F_u \in F(\mathbb{L})$ as $u \leq u$. Then as $f \cap j \neq \varnothing$ for every $f \in F(\mathbb{L})$ such that $u \in f$ we have that $F_u \cap j \neq \varnothing$. Let $u_0 \in F_u \cap j$ then $u \leq u_0$, and as j is an ideal of J (and thus downwards closed) we must have $u \in j$. Therefore $j \in \{j \in J(\mathbb{L}) \mid u \in j\}$ and the equality in (a) follows.

The proof of (b) is order dual. ⊣

Remark 4.40. The equalities in Lemma 4.39(a) and Lemma 4.39(b) can be rewritten as biconditional statements in the following way for arbitrary $f \in F(\mathbb{L})$ and $j \in J(\mathbb{L})$:

$$f \in \{j \in J(\mathbb{L}) \mid u \in j\}^{\downarrow} \text{ iff } u \in f$$

and

$$j \in \{f \in F(\mathbb{L}) \mid u \in f\}^{\uparrow} \text{ iff } u \in j.$$

This observation guarantees that certain subsets of $F(\mathbb{L})$ and $J(\mathbb{L})$ will be Galois-stable in the polarity associated with $\mathbb{L}$, $\mathbb{P}_{\mathbb{L}}$.

Corollary 4.41. Subsets of the form $\{f \in F(\mathbb{L}) \mid u \in f\} \subseteq F(\mathbb{L})$ and $\{j \in J(\mathbb{L}) \mid u \in j\} \subseteq J(\mathbb{L})$ are stable in $\mathbb{P}_{\mathbb{L}}$ in the sense of Definition 3.13.

Proof. This result follows directly from Remark 4.40, as

$$\begin{aligned}(\{f \in F(\mathbb{L}) \mid u \in f\})^{\uparrow\downarrow} &= (\{j \in J(\mathbb{L}) \mid u \in j\})^{\downarrow} && \text{(Remark 4.40)} \\ &= (\{f \in F(\mathbb{L}) \mid u \in j\}) && \text{(Remark 4.40)}\end{aligned}$$

and

$$\begin{aligned}(\{j \in J(\mathbb{L}) \mid u \in j\})^{\uparrow\downarrow} &= (\{f \in F(\mathbb{L}) \mid u \in f\})^{\uparrow} && \text{(Remark 4.40)} \\ &= \{j \in J(\mathbb{L}) \mid u \in j\} && \text{(Remark 4.40)}\end{aligned}$$

⊣

Definition 4.42. Unary operations in the signature $\left\{\Box^{\mathbb{L}}, \Diamond^{\mathbb{L}}, \lhd^{\mathbb{L}}, \rhd^{\mathbb{L}}\right\}$ can be associated with binary relations on $\mathbb{P}_{\mathbb{L}}$ in the following ways:

(a) $f R_{\Box^{\mathbb{L}}} j$ iff $\Box u \in f$ for some $u \in j$.

(b) $j R_{\Diamond^{\mathbb{L}}} f$ iff $\Diamond u \in j$ for some $u \in f$.

(c) $f_0 R_{\rhd^{\mathbb{L}}} f_1$ iff $\rhd u \in f_0$ for some $u \in f_1$.

(d) $j_0 R_{\lhd^{\mathbb{L}}} j_1$ iff $\lhd u \in j_0$ for some $u \in j_1$.

An enriched polarity, $\mathbb{F}_{\mathbb{L}}$, can thus be associated with bitonal algebra, $\mathbb{B}$, in the following way:

Definition 4.43. For a bitonal algebra, $\mathbb{B}$, the enriched polarity, $\mathbb{F}_{\mathbb{B}}$, is given by $\mathbb{F}_{\mathbb{B}} = (\mathbb{P}_{\mathbb{B}}, \{R_{\Box^{\mathbb{B}}}, R_{\Diamond^{\mathbb{B}}}, R_{\lhd^{\mathbb{B}}}, R_{\rhd^{\mathbb{B}}}\})$.

In particular, an enriched polarity, $\mathbb{F}_{\mathbb{A}_P} = (\mathbb{P}_{\mathbb{A}_P}, \{R_{\Box^{\mathbb{A}_P}}, R_{\Diamond^{\mathbb{A}_P}}, R_{\lhd^{\mathbb{A}_P}}, R_{\rhd^{\mathbb{A}_P}}\})$, can be associated with the Lindebaum-Tarski Algebra, $\mathbb{A}_P$. From Lemma 4.39, the canonical assignment

$$V(p) = (\{f \in F(\mathbb{L}) \mid [p] \in f\}, \{j \in J(\mathbb{L}) \mid [p] \in j\})$$

is well defined, as $[\![V(p)]\!]^{\uparrow} = (\![V(p)]\!)$ by Lemma 4.39.

Definition 4.44 (The canonical polarity-based model)**.** The canonical polarity-based model is the model $\mathcal{M}_{\mathbb{A}_P} = (\mathbb{F}_{\mathbb{A}_P}, V_{\mathbb{A}_P})$ where $V_{\mathbb{A}_P}$ is the canonical assignment, $V_{\mathbb{A}_P} : \mathsf{Prop} \to \mathbb{F}^{+}_{\mathbb{A}_P}$, given by $V_{\mathbb{A}_P}(p) = (\{f \in F(\mathbb{L}) \mid [p] \in f\}, \{j \in J(\mathbb{L}) \mid [p] \in j\})$.

Lemma 4.45. For $j \in J(\mathbb{A}_P)$ and $f \in F(\mathbb{A}_P)$:

(a) $[\top] \in f$.

(b) $[\bot] \in j$.

Proof. Suppose $f \in F(\mathbb{A}_P)$. As f is a filter, it is non-empty. Consider an $[\varphi] \in f$. By Lemma 4.34(b) we have that $[\varphi] \leq_{\mathbb{A}_P} [\top]$, and as f is a filter and $[\varphi] \in f$ we have that $[\top] \in f$.

The proof of (b) is order dual. ⊣

For the sake of readability of the Truth Lemma (Lemma 4.47), we will defer the body of the argument for modalities to the following lemma, which will then be referenced for the relevant cases in the induction proof of Lemma 4.47.

Lemma 4.46. For $\psi \in \mathcal{L}_P$, and $f \in F(\mathbb{A}_P)$ and $j \in J(\mathbb{A}_P)$, with $R_{h^{\mathbb{A}_P}}$ and $h \in \{\Diamond, \Box, \lhd, \rhd\}$ defined as in Definition 4.42

(a) $f \in R^{(0)}_{\Box^{\mathbb{A}_P}}(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\})$ iff $[\Box\psi] \in f$.

(b) $j \in R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})$ iff $[\Diamond\psi] \in j$.

(c) $f \in R^{(0)}_{\rhd^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})$ iff $[\rhd\psi] \in f$.

(d) $j \in R^{(0)}_{\lhd^{\mathbb{A}_P}}(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\})$ iff $[\lhd\psi] \in j$.

Proof. For (a):

$$\begin{aligned}
& f \in R^{(0)}_{\Box^{\mathbb{A}_P}}(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\}) \\
\text{iff } & (\forall j \in J(\mathbb{A}_P))([\psi] \in j \to f R_{\Box^{\mathbb{A}_P}} j) && \text{(Definition 3.4)} \\
\text{iff } & (\forall j \in J(\mathbb{A}_P))\big([\psi] \in j \to (\exists [\chi] \in \mathbb{A}_P)\big([\chi] \in j \text{ and } \Box^{\mathbb{A}_P}[\chi] \in f\big)\big) && \text{(Definition 4.42(a))} \\
\text{iff } & (\forall j \in J(\mathbb{A}_P))([\psi] \in j \to (\exists [\chi] \in \mathbb{A}_P)([\chi] \in j \text{ and } [\Box\chi] \in f)). && \text{(Definition 4.29)}
\end{aligned}$$

Consider $j_{[\psi]} = \{[\chi] \in \mathbb{A}_P \mid [\chi] \leq [\psi]\}$. Clearly $[\psi] \in j_{[\psi]}$. So instantiating $j_{[\psi]}$ for j in the universal quantification in the last line in our chain of biconditional statements above:[7]

$$\begin{aligned}
& \big([\psi] \in j_{[\psi]} \to (\exists [\chi] \in \mathbb{A}_P)\big([\chi] \in j_{[\psi]} \text{ and } [\Box\chi] \in f\big)\big) \\
\text{implies } & (\exists [\chi] \in \mathbb{A}_P)\big([\chi] \in j_{[\psi]} \text{ and } [\Box\chi] \in f\big). && \text{(As } [\psi] \in j_{[\psi]})
\end{aligned}$$

Then $[\chi] \leq [\psi]$, as $j_{[\psi]}$ is the principal ideal generated by $[\psi]$. Thus from the monotonicity of $\Box^{\mathbb{A}_P}$:

$$\begin{aligned}
[\chi] \leq [\psi] & \text{ implies } \Box^{\mathbb{A}_P}[\chi] \leq \Box^{\mathbb{A}_P}[\psi] \\
& \text{ implies } [\Box\chi] \leq [\Box\psi].
\end{aligned}$$

Therefore, as f is a filter (and thus upwards closed) and $[\Box\chi] \in f$, we have $[\Box\psi] \in f$ as required.

For the converse we will show that

$$[\Box\psi] \in f \text{ implies } (\forall j \in J(\mathbb{A}_P))\,([\psi] \in j \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,([\chi] \in j \text{ and } [\Box\chi] \in f)).$$

Suppose that $[\Box\psi] \in f$ and $[\psi] \in j$. Then for the instantiation of $[\psi]$ for $[\chi]$ in the existential quantification above we clearly have ($[\psi] \in j$ and $[\Box\psi] \in f$). Thus the converse holds and

$$f \in R^{(0)}_{\Box^{\mathbb{A}_P}}\,(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\}) \text{ iff } \Box\psi \in f.$$

For (b) we proceed in a similar manner to (a):

$$\begin{aligned}
&j \in R^{(0)}_{\Diamond^{\mathbb{A}_P}}\,(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\}) && \\
\text{iff } &(\forall f \in F(\mathbb{A}_P))\,([\psi] \in f \rightarrow jR_{\Diamond^{\mathbb{A}_P}} f) && \text{(Definition 3.4)}\\
\text{iff } &(\forall f \in F(\mathbb{A}_P))\,\big([\psi] \in f \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f \text{ and } \Diamond^{\mathbb{A}_P}[\chi] \in j\big)\big) && \text{(Definition 4.42(b))}\\
\text{iff } &(\forall f \in F(\mathbb{A}_P))\,([\psi] \in f \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,([\chi] \in f \text{ and } [\Diamond\chi] \in j)) && \text{(Definition 4.29)}
\end{aligned}$$

Consider $f_{[\psi]} = \{[\chi] \in \mathbb{A}_P \mid [\psi] \leq [\chi]\}$. Clearly $[\psi] \in f_{[\psi]}$. So instantiating $f_{[\psi]}$ for f in the universal quantification on the last line of the chain of biconditional statements:

$$\begin{aligned}
&\big([\psi] \in f_{[\psi]} \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f_{[\psi]} \text{ and } [\Diamond\chi] \in j\big)\big) &&\\
\text{implies } &(\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f_{[\psi]} \text{ and } [\Diamond\chi] \in j\big). && (\text{As } [\psi] \in f_{[\psi]})
\end{aligned}$$

Then $[\psi] \leq [\chi]$, as $f_{[\psi]}$ is the principal filter generated by $[\psi]$. Thus from monotonicity of $\Diamond^{\mathbb{A}_P}$:

$$\begin{aligned}
[\psi] \leq [\chi] &\text{ implies } \Diamond^{\mathbb{A}_P}[\psi] \leq \Diamond^{\mathbb{A}_P}[\chi]\\
&\text{ implies } [\Diamond\psi] \leq [\Diamond\chi].
\end{aligned}$$

Therefore, as j is an ideal (and thus downwards closed) and $[\Diamond\chi] \in j$, we have $[\Diamond\psi] \in j$ as required.

For the converse we will show

$$[\Diamond\psi] \in j \text{ implies } (\forall f \in F(\mathbb{A}_P))\,([\psi] \in f \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,([\chi] \in f \text{ and } [\Diamond\chi] \in j)).$$

Suppose that $[\Diamond\psi] \in j$ and $[\psi] \in f$. Then instantiating $[\psi]$ for $[\chi]$ in the existential quantification above, we clearly have $[\psi] \in f$ and $[\Diamond\psi]$. Thus the converse holds and

$$j \in R^{(0)}_{\Diamond^{\mathbb{A}_P}}\,(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\}) \text{ iff } [\Diamond\psi] \in j.$$

For (c):

$$\begin{aligned}
&f \in R^{(0)}_{\rhd^{\mathbb{A}_P}}\,(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\}) && \\
\text{iff } &(\forall f' \in F(\mathbb{A}_P))\,([\psi] \in f' \rightarrow fR_{\rhd^{\mathbb{A}_P}} f') && \text{(Definition 3.4)}\\
\text{iff } &(\forall f' \in F(\mathbb{A}_P))\,\big([\psi] \in f' \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f' \text{ and } \rhd^{\mathbb{A}_P}[\chi] \in f\big)\big) && \text{(Definition 4.42(c))}\\
\text{iff } &(\forall f' \in F(\mathbb{A}_P))\,([\psi] \in f' \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,([\chi] \in f' \text{ and } [\rhd\chi] \in f)) && \text{(Definition 4.29)}
\end{aligned}$$

As in (b), consider $f_{[\psi]} = \{[\chi] \in \mathbb{A}_P \mid [\psi] \leq [\chi]\}$. Clearly $[\psi] \in f_{[\psi]}$. So instantiating $f_{[\psi]}$ for f in the existential quantification on the last line of our chain of biconditionals above:

$$\begin{aligned}
&\big([\psi] \in f_{[\psi]} \rightarrow (\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f_{[\psi]} \text{ and } [\rhd\chi] \in f\big)\big) &&\\
\text{implies } &(\exists\,[\chi] \in \mathbb{A}_P)\,\big([\chi] \in f_{[\psi]} \text{ and } [\rhd\chi] \in f\big). && (\text{As } [\psi] \in f_{[\psi]})
\end{aligned}$$

Then $[\psi] \leq [\chi]$, as $f_{[\psi]}$ is the principal filter generated by $[\psi]$. Thus from the antitonicity of $\rhd^{\mathbb{A}_P}$:

$$\begin{aligned}
[\psi] \leq [\chi] &\text{ implies } \rhd^{\mathbb{A}_P}[\chi] \leq \rhd^{\mathbb{A}_P}[\psi]\\
&\text{ implies } [\rhd\chi] \leq [\rhd\psi].
\end{aligned}$$

Therefore, as f is a filter (and therefore upwards closed) and $[\rhd\chi] \in f$, we have $[\rhd\psi] \in f$ as required.

For the converse we will show that:

$$[\rhd\psi] \in f \text{ implies } (\forall f' \in F(\mathbb{A}_P))\,([\psi] \in f' \to (\exists\,[\chi] \in \mathbb{A}_P)\,([\chi] \in f' \text{ and } [\rhd\chi] \in f))\,.$$

Suppose that $[\rhd\psi] \in f$ and $[\psi] \in f'$. Then, in the instantiation of $[\psi]$ for $[\chi]$ in the existential quantification above, we clearly have $([\psi] \in f' \text{ and } [\rhd\psi] \in f)$. Thus the converse holds and

$$f \in R^{(0)}_{\rhd^{\mathbb{A}_P}}\,(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\}) \text{ iff } [\rhd\psi] \in f.$$

The proof of (d) is order dual to (c). ⊣

Lemma 4.47 (Truth lemma)**.** For every $\varphi \in \mathcal{L}_P$

(a) $\mathcal{M}_{\mathbb{A}_P}, f \Vdash \varphi$ iff $[\varphi] \in f$.

(b) $\mathcal{M}_{\mathbb{A}_P}, j \succ \varphi$ iff $[\varphi] \in j$.

Proof. We proceed by induction on the number of connectives in φ.

Suppose that $\varphi = p \in \mathsf{Prop}$ then

$$\begin{aligned} \mathcal{M}_{\mathbb{A}_P}, f \Vdash p &\text{ iff } f \in [\![V_{\mathbb{A}_P}(p)]\!] \\ &\text{ iff } f \in \{f \in F(\mathbb{A}_P) \mid [p] \in f\} \\ &\text{ iff } [p] \in f \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\mathbb{A}_P}, j \succ p &\text{ iff } j \in (\![V_{\mathbb{A}_P}(p)]\!) \\ &\text{ iff } j \in \{j \in J(\mathbb{A}_P) \mid [p] \in j\} \\ &\text{ iff } [p] \in j. \end{aligned}$$

Suppose $\varphi = \top$. For satisfaction: $[\top] \in f$ always from Lemma 4.45(a), and so trivially $\mathcal{M}_{\mathbb{A}_P}, f \Vdash \top$ implies $[\top] \in f$. The converse follows directly from the definition of $\Vdash$ in Definition 4.8 – $\mathcal{M}_{\mathbb{A}_P}, f \Vdash \top$ always. Thus $\mathcal{M}_{\mathbb{A}_P}, f \Vdash \top$ iff $[\top] \in f$, as required.

For co-satisfaction:

$$\begin{aligned} \mathcal{M}_{\mathbb{A}_P}, j \succ \top &\text{ iff } j \in (F(\mathbb{A}_P))^{\uparrow} \\ &\text{ iff } j \in (\{f \in F(\mathbb{A}_P) \mid [\top] \in f\})^{\uparrow} && \text{(Lemma 4.45(a))} \\ &\text{ iff } [\top] \in j. && \text{(Remark 4.40)} \end{aligned}$$

Similarly, for $\varphi = \bot$ we have $\mathcal{M}_{\mathbb{A}_P}, j \succ \bot$ implies $[\bot] \in j$, as $[\bot] \in j$ for any $j \in J(\mathbb{A}_P)$ by Lemma 4.45(b). The converse follows directly from the definition of $\succ$ in Definition 4.8.

For satisfaction:

$$\begin{aligned} \mathcal{M}_{\mathbb{A}_P}, f \Vdash \bot &\text{ iff } f \in (J(\mathbb{A}_P))^{\downarrow} \\ &\text{ iff } f \in (\{j \in J(\mathbb{A}_P) \mid [\bot] \in j\})^{\downarrow} && \text{(Lemma 4.45(b))} \\ &\text{ iff } [\bot] \in f. && \text{(Remark 4.40)} \end{aligned}$$

Induction hypothesis: Suppose that (a) and (b) hold for all φ with at most N connectives, and suppose ψ and χ have at most N connectives.

Suppose that $\varphi = \psi \wedge \chi$:

$$\begin{aligned} \mathcal{M}_{\mathbb{A}_P}, f \Vdash \psi \wedge \chi &\text{ iff } \mathcal{M}_{\mathbb{A}_P}, f \Vdash \psi \text{ and } \mathcal{M}_{\mathbb{A}_P}, f \Vdash \chi \\ &\text{ iff } [\psi] \in f \text{ and } [\chi] \in f && \text{(induction hypothesis)} \\ &\text{ iff } [\psi] \wedge^{\mathbb{A}_P} [\chi] \in f && \text{(Proposition 2.23(a))} \\ &\text{ iff } [\psi \wedge \chi] \in f && \text{(Definition 4.29)} \end{aligned}$$

For co-satisfaction:

$$
\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \psi \wedge \chi \text{ iff } & j \in (\llbracket \psi \rrbracket \cap \llbracket \chi \rrbracket)^{\uparrow} \\
\text{iff } & j \in (\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\} \cap \{f \in F(\mathbb{A}_P) \mid [\chi] \in f\})^{\uparrow} && \text{(Induction hypothesis)} \\
\text{iff } & j \in \{f \in F(\mathbb{A}_P) \mid [\psi] \in f \text{ and } [\chi] \in f\}^{\uparrow} \\
\text{iff } & j \in \left\{f \in F(\mathbb{A}_P) \mid [\psi] \wedge^{\mathbb{A}_P} [\chi] \in f\right\}^{\uparrow} && \text{(Proposition 2.23(a))} \\
\text{iff } & j \in \{f \in F(\mathbb{A}_P) \mid [\psi \wedge \chi] \in f\}^{\uparrow} && \text{(Definition 4.29)} \\
\text{iff } & j \in \{j \in J(\mathbb{A}_P) \mid [\psi \wedge \chi] \in j\} && \text{(Remark 4.40)} \\
\text{iff } & [\psi \wedge \chi] \in j.
\end{aligned}
$$

Suppose $\varphi = \psi \vee \chi$. In this case we first address co-satisfaction, as for disjunction co-satisfaction is primary.

$$
\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \psi \vee \chi \text{ iff } & \mathcal{M}_{\mathbb{A}_P}, j \succ \psi \text{ and } \mathcal{M}_{\mathbb{A}_P}, j \succ \chi && \text{(Definition 4.8)} \\
\text{iff } & [\psi] \in j \text{ and } [\chi] \in j && \text{(Induction hypothesis)} \\
\text{iff } & [\psi] \vee^{\mathbb{A}_P} [\chi] \in j && \text{(Proposition 2.23(b))} \\
\text{iff } & [\psi \vee \chi] \in j && \text{(Definition 4.29)}
\end{aligned}
$$

For satisfaction:

$$
\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, f \Vdash \psi \vee \chi \text{ iff } & f \in (⦇\psi⦈ \cap ⦇\chi⦈)^{\downarrow} \\
\text{iff } & f \in (\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\} \cap \{j \in J(\mathbb{A}_P) \mid [\chi] \in j\})^{\downarrow} && \text{(Induction hypothesis)} \\
\text{iff } & f \in \{j \in J(\mathbb{A}_P) \mid [\psi] \in j \text{ and } [\chi] \in j\}^{\downarrow} \\
\text{iff } & f \in \left\{j \in J(\mathbb{A}_P) \mid [\psi] \wedge^{\mathbb{A}_P} [\chi] \in j\right\}^{\downarrow} && \text{(Proposition 2.23(b))} \\
\text{iff } & f \in \{j \in J(\mathbb{A}_P) \mid [\psi \wedge \chi] \in j\}^{\downarrow} && \text{(Definition 4.29)} \\
\text{iff } & f \in \{f \in F(\mathbb{A}_P) \mid [\psi \wedge \chi] \in f\} && \text{(Remark 4.40)} \\
\text{iff } & [\psi \wedge \chi] \in f.
\end{aligned}
$$

Suppose that $\varphi = \Box\psi$. Then

$$
\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, f \Vdash \Box\psi \text{ iff } & f \in \left(R^{(0)}_{\Box^{\mathbb{A}_P}}\left(⦇\psi⦈\right)\right)^{\uparrow\downarrow} && \text{(Definition 4.8)} \\
\text{iff } & f \in \left(R^{(0)}_{\Box^{\mathbb{A}_P}}\left(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\}\right)\right)^{\uparrow\downarrow} && \text{(Induction hypothesis)} \\
\text{iff } & f \in (\{f \in F(\mathbb{A}_P) \mid [\Box\psi] \in f\})^{\uparrow\downarrow} && \text{(Lemma 4.46(a))} \\
\text{iff } & f \in \{f \in F(\mathbb{A}_P) \mid [\Box\psi] \in f\} && \text{(Corollary 4.41)} \\
\text{iff } & [\Box\psi] \in f.
\end{aligned}
$$

Note that the above chain of biconditionals also show that the set $R^{(0)}_{\Box^{\mathbb{A}_P}}(⦇\psi⦈)$ is Galois-stable. This is as $[\Box\psi] \in f$ if and only if $f \in R^{(0)}_{\Box^{\mathbb{A}_P}}(⦇\psi⦈)$ by Lemma 4.46(a), and $f \in \left(R^{(0)}_{\Box^{\mathbb{A}_P}}(⦇\psi⦈)\right)^{\uparrow\downarrow}$ if and only if $[\Box\psi] \in f$ by the chain of biconditionals. Analogous observations can be made regarding the Galois-stability of $R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)$, $R^{(0)}_{\triangleright^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)$, and $R^{(0)}_{\triangleleft^{\mathbb{A}_P}}(⦇\psi⦈)$ using corresponding chains of biconditionals below.

For co-satisfaction:

$$
\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \Box\psi \text{ iff } & j \in \left(R^{(0)}_{\Box^{\mathbb{A}_P}}\left(⦇\psi⦈\right)\right)^{\uparrow} && \text{(Definition 4.8)} \\
\text{iff } & j \in \left(R^{(0)}_{\Box^{\mathbb{A}_P}}\left(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\}\right)\right)^{\uparrow} && \text{(Induction hypothesis)} \\
\text{iff } & j \in (\{f \in F(\mathbb{A}_P) \mid [\Box\psi] \in f\})^{\uparrow} && \text{(Lemma 4.46(a))} \\
\text{iff } & j \in (\{j \in J(\mathbb{A}_P) \mid [\Box\psi] \in j\}) && \text{(Lemma 4.39(a))} \\
\text{iff } & [\Box\psi] \in j.
\end{aligned}
$$

Suppose that $\varphi = \Diamond\psi$. Then

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \Diamond\psi \text{ iff } & j \in \left(R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)\right)^{\downarrow\uparrow} && \text{(Definition 4.8)}\\
\text{iff } & j \in \left(R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})\right)^{\downarrow\uparrow} && \text{(Induction hypothesis)}\\
\text{iff } & j \in (\{j \in J(\mathbb{A}_P) \mid [\Diamond\psi] \in j\})^{\downarrow\uparrow} && \text{(Lemma 4.46(b))}\\
\text{iff } & j \in \{j \in J(\mathbb{A}_P) \mid [\Diamond\psi] \in j\} && \text{(Corollary 4.41)}\\
\text{iff } & [\Diamond\psi] \in j.
\end{aligned}$$

For satisfaction

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, f \Vdash \Diamond\psi \text{ iff } & f \in \left(R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)\right)^{\downarrow} && \text{(Definition 4.8)}\\
\text{iff } & f \in \left(R^{(0)}_{\Diamond^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})\right)^{\downarrow} && \text{(Induction hypothesis)}\\
\text{iff } & f \in (\{j \in J(\mathbb{A}_P) \mid [\Diamond\psi] \in j\})^{\downarrow} && \text{(Lemma 4.46(b))}\\
\text{iff } & f \in \{f \in F(\mathbb{A}_P) \mid [\Diamond\psi] \in f\} && \text{(Lemma 4.39(b))}\\
\text{iff } & [\Diamond\psi] \in f.
\end{aligned}$$

Suppose that $\varphi = \rhd\psi$. Then for the satisfaction clause

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, f \Vdash \rhd\psi \text{ iff } & f \in \left(R^{(0)}_{\rhd^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)\right)^{\uparrow\downarrow} && \text{(Definition 4.8)}\\
\text{iff } & f \in \left(R^{(0)}_{\rhd^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})\right)^{\uparrow\downarrow} && \text{(Induction hypothesis)}\\
\text{iff } & f \in (\{f \in F(\mathbb{A}_P) \mid [\rhd\psi] \in f\})^{\uparrow\downarrow} && \text{(Lemma 4.46(c))}\\
\text{iff } & f \in \{f \in F(\mathbb{A}_P) \mid [\rhd\psi] \in f\} && \text{(Corollary 4.41)}\\
\text{iff } & [\rhd\psi] \in f. && \text{(Lemma 4.46(c))}
\end{aligned}$$

For co-satisfaction

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \rhd\psi \text{ iff } & j \in \left(R^{(0)}_{\rhd^{\mathbb{A}_P}}(\llbracket\psi\rrbracket)\right)^{\uparrow} && \text{(Definition 4.8)}\\
\text{iff } & j \in \left(R^{(0)}_{\rhd^{\mathbb{A}_P}}(\{f \in F(\mathbb{A}_P) \mid [\psi] \in f\})\right) && \text{(Induction hypothesis)}\\
\text{iff } & j \in (\{f \in F(\mathbb{A}_P) \mid [\rhd\psi] \in f\})^{\uparrow} && \text{(Lemma 4.46(c))}\\
\text{iff } & j \in \{j \in J(\mathbb{A}_P) \mid [\rhd\psi] \in j\}. && \text{(Lemma 4.39(a))}
\end{aligned}$$

Finally, suppose that $\varphi = \lhd\psi$. Then for co-satisfaction

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, j \succ \lhd\psi \text{ iff } & j \in \left(R^{(0)}_{\lhd^{\mathbb{A}_P}}((\!|\psi|\!))\right)^{\downarrow\uparrow} && \text{(Definition 4.8)}\\
\text{iff } & j \in \left(R^{(0)}_{\lhd^{\mathbb{A}_P}}(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\})\right)^{\downarrow\uparrow} && \text{(Induction hypothesis)}\\
\text{iff } & j \in (\{j \in J(\mathbb{A}_P) \mid [\lhd\psi] \in j\})^{\downarrow\uparrow} && \text{(Lemma 4.46(d))}\\
\text{iff } & j \in \{j \in J(\mathbb{A}_P) \mid [\lhd\psi] \in j\} && \text{(Corollary 4.41)}\\
\text{iff } & j \in [\lhd\psi].
\end{aligned}$$

For satisfaction

$$\begin{aligned}
\mathcal{M}_{\mathbb{A}_P}, f \Vdash \lhd\psi \text{ iff } & f \in \left(R^{(0)}_{\lhd^{\mathbb{A}_P}}((\!|\psi|\!))\right)^{\downarrow} && \text{(Definition 4.8)}\\
\text{iff } & f \in \left(R^{(0)}_{\lhd^{\mathbb{A}_P}}(\{j \in J(\mathbb{A}_P) \mid [\psi] \in j\})\right)^{\downarrow} && \text{(Induction hypothesis)}\\
\text{iff } & f \in (\{j \in J(\mathbb{A}_P) \mid j \in [\lhd\psi]\})^{\downarrow} && \text{(Lemma 4.46(d))}\\
\text{iff } & f \in \{f \in F(\mathbb{A}_P) \mid f \in [\lhd\psi]\} && \text{(Lemma 4.39(b))}\\
\text{iff } & f \in [\lhd\psi].
\end{aligned}$$

Thus the induction hypothesis holds, and the result follows. ⊣

Theorem 4.48 (Completeness)**.** If $\Vdash \varphi \vdash \psi$ then $\varphi \vdash \psi$ is derivable in $\boldsymbol{L}_P$.

Proof. We will proceed by showing the contrapositive as stated at the beginning of this subsection: if $\varphi \vdash \psi$ is not derivable in $\boldsymbol{L}_P$ then there exists an $\mathbb{F}$ such that $\mathbb{F} \nVdash \varphi \vdash \psi$. In particular, we will show that if $\varphi \vdash \psi$ is not derivable then $\mathcal{M}_{\mathbb{A}_P}, f_{[\varphi]} \Vdash \varphi$ and $\mathcal{M}_{\mathbb{A}_P}, f_{[\varphi]} \nVdash \psi$.

Suppose that $\varphi \vdash \psi$ is not derivable in $\boldsymbol{L}_P$. Then $[\varphi] \in f_{[\varphi]}$ as $\varphi \vdash \varphi$, and $[\psi] \notin f_{[\varphi]}$ as $\varphi \nvdash \psi$. Applying the Truth Lemma (Lemma 4.47):

$$[\varphi] \in f_{[\varphi]} \text{ iff } \mathcal{M}, f_{[\varphi]} \Vdash \varphi \qquad \text{(Lemma 4.47)}$$

and

$$[\psi] \notin f_{[\varphi]} \text{ iff } \mathcal{M}, f_{[\varphi]} \nVdash \psi \qquad \text{(Lemma 4.47)}$$

Thus, $\mathcal{M} \nVdash \varphi \vdash \psi$ as for specific $f_{[\varphi]} \in F(\mathbb{A}_P)$ we have $\mathcal{M}, f_{[\varphi]} \Vdash \varphi$ and $\mathcal{M}, f_{[\varphi]} \nVdash \psi$. Therefore $\mathcal{M} \nVdash \varphi \vdash \psi$ (Definition 4.11), and thus $\mathbb{F}_{\mathbb{A}_P} \nVdash \varphi \vdash \psi$. $\dashv$

4.5 Comparison with compatible semantics

The non-compatible semantics introduced earlier in this chapter are an adaptation of polarity-based semantics for non-distributive modal logic developed over the past several years (for a more detailed discussion see Section 1.1.3). Our new semantics is, in a sense, a generalization of the compatible semantics – in so far as compatibility requirements on additional relations are dropped, and the semantic clauses for the corresponding modalities are adapted to ensure stability. These semantic clauses do, however, become equivalent to those of the compatible semantics when compatibility conditions are imposed.

In the section we present the compatible semantics of [16], before providing several examples of non-compatible polarity-based frames illustrating the failure of normality, distributivity of modalities over conjunction and disjunction, as well as the failure of various adjunction properties that otherwise hold in the compatible setting.

Definition 4.49 (Compatible relation [16],[10])**.** A relation, $R_h \subseteq W_{\epsilon_h} \times W_{\alpha_h}$, is *I-compatible* or simply *compatible* if $R_h^{(0)}(w^{\epsilon_h})$ and $R_h^{(1)}(w^{\alpha_h})$ are Galois-stable in $\mathbb{P} = (A, X, I)$. This is where $w^1 = a \in A$ and $w^\partial = x \in X$, and the ϵ_h and α_h conditions are as in Table 4.1 and Table 4.2.

Compatible relations are defined in terms of singletons in order to maintain first-order definability. Notably, as shown in Proposition 4.50 below, compatibility over singletons implies compatibility over subsets. The proof of Proposition 4.50 is adapted from [10, Lemma 4].

Proposition 4.50. Let $R_h \subseteq W_{\epsilon_h} \times W_{\alpha_h}$ be an I-compatible relation on $\mathbb{P}$, and $U \subseteq W_{\epsilon_h}$ and $V \subseteq W_{\alpha_h}$. Then

(a) $R_h^{(0)}(V)$ is Galois-stable.

(b) $R_h^{(1)}(U)$ is Galois-stable.

Proof. For (a)

$$\begin{aligned} R_h^{(0)}(V) &= R_h^{(0)}\left(\bigcup_{v \in V} \{v\}\right) \\ &= \bigcap_{v \in V} R_h^{(0)}(\{v\}) && \text{(Proposition 3.25(b))} \end{aligned}$$

Then, as $R_h^{(0)}(\{v\})$ is Galois-stable, and the intersection of Galois-stable sets is Galois-stable by Proposition 3.15, we have that $R_h^{(0)}(V)$ is itself Galois-stable as required. Similarly, for (b):

$$\begin{aligned} R_h^{(1)}(U) &= R_h^{(1)}\left(\bigcup_{u\in U}\{u\}\right) \\ &= \bigcap_{u\in U} R_h^{(1)}(\{u\}) && \text{(Proposition 3.25(a))} \end{aligned}$$

Then, as $R_h^{(1)}(\{u\})$ is Galois-stable, and the intersection of Galois-stable sets is Galois-stable by Proposition 3.15, $R_h^{(1)}(V)$ is itself Galois-stable as required. ⊣

From this point the stability of compatible relations over subsets may be assumed without direct reference to Proposition 4.50.

4.5.1 Compatible polarity-based semantics

The compatible polarity-based semantics discussed in this section share an underlying language, $\mathcal{L}_P$, with the non-compatible semantics introduced earlier in this subsection. Notationally, we will distinguish between the respective logics of the two languages by the introduction of a prime in the writing of the compatible logic, $\boldsymbol{L}_{P'}$, as well as in the writing of the satisfaction and co-satisfaction symbols when discussing non-compatible models – $\Vdash'$ and $\succ'$ respectively. This additional notation is largely introduced for the benefit of Chapter 5 and Chapter 6 – primarily to avoid confusion when proving results pertaining to the agreement of the two semantics when compared over the class of enriched polarities that are lifting of Kripke frames.

Definition 4.51 (Compatible enriched polarity)**.** An enriched polarity, $\mathbb{F} = (\mathbb{P}, \mathcal{R})$, is compatible if each relation, $R_h \in \mathcal{R}$, is compatible in the sense of Definition 4.49.

Similarly, we adapt our definition of polarity-based models in the following definition.

Definition 4.52. A polarity-based model, $\mathcal{M} = (\mathbb{F}, V)$, is compatible if the enriched polarity, $\mathbb{F}$, is compatible.

The basic compatible $\mathcal{L}_P$-logic is a set $\boldsymbol{L}_{P'}$ of sequents $\varphi \vdash \psi$ with $\varphi, \psi \in \mathcal{L}_P$ with the propositional axioms[8]:

$$\begin{array}{llll} p \vdash p \ \text{(Ax.)} & \bot \vdash p \ (\bot\text{L}) & p \vdash \top \ (\top\text{R}) & \\ p \vdash p \vee q \ (\vee\text{R1}) & q \vdash p \vee q \ (\vee\text{R2}) & p \wedge q \vdash p \ (\wedge\text{L1}) & p \wedge q \vdash q \ (\wedge\text{L2}) \end{array}$$

and modal axioms:

$$\begin{array}{llll} \top \vdash \Box\top \ (\text{Norm-}\Box) & \Diamond\bot \vdash \bot \ (\text{Norm-}\Diamond) & \Box p \wedge \Box q \vdash \Box(p \wedge q) \ (\text{Dist-}\Box) & \Diamond(p \vee q) \vdash \Diamond p \vee \Diamond q \ (\text{Dist-}\Diamond) \\ \top \vdash \rhd\bot \ (\text{Norm-}\rhd) & \lhd\top \vdash \bot \ (\text{Norm-}\lhd) & \rhd p \wedge \rhd q \vdash \rhd(p \vee q) \ (\text{Dist-}\rhd) & \lhd(p \wedge q) \vdash \lhd p \vee \lhd q \ (\text{Dist-}\lhd) \end{array}$$

and closed under the inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi}\ \text{(Cut)} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)}\ \text{(Subs.)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi}\ (\wedge\text{R}) \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}\ (\vee\text{L})$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi}\ (\Box\text{M}) \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}\ (\Diamond\text{M}) \qquad \frac{\varphi \vdash \psi}{\rhd\psi \vdash \rhd\varphi}\ (\rhd\text{A}) \qquad \frac{\varphi \vdash \psi}{\lhd\psi \vdash \lhd\varphi}\ (\lhd\text{A})$$

This axiomatization of the compatible semantics features four additional modal axioms when compared to our axiomatization of the non-compatible semantics in Section 4.3. As will be shown in Example 4.57-Example 4.57, these axioms do not hold in the general setting of non-compatible enriched polarities.

[8]Suffixes L and R are used for left and right, and prefixes M and A are used for monotonicity and antitonicity respectively.

Definition 4.53 (Compatible polarity-based model)**.** Let $\mathcal{M} = (\mathbb{F}, V)$ be a compatible polarity-based model. Then the satisfaction and co-satisfaction relations $\Vdash' \subseteq A \times \mathcal{L}_P$ and $\succ' \subseteq X \times \mathcal{L}_P$, as well as the sets $[\![\psi]\!] = [\![\psi]\!]_{\mathcal{M}} = \left\{a \in A \mid \mathcal{M}, a \Vdash' \psi\right\}$ and $(\![\psi]\!) = (\![\psi]\!)_{\mathcal{M}} = \left\{x \in X \mid \mathcal{M}, x \succ' \psi\right\}$ are defined by the following simultaneous recursion:

$\mathcal{M}, a \Vdash' p$	iff	$a \in [\![V(p)]\!]$	$\mathcal{M}, x \succ' p$	iff	$x \in (\![V(p)]\!)$
$\mathcal{M}, a \Vdash' \top$		always	$\mathcal{M}, x \succ' \top$	iff	$x \in A^{\uparrow}$
$\mathcal{M}, a \Vdash' \bot$	iff	$a \in X^{\downarrow}$	$\mathcal{M}, x \succ' \bot$		always
$\mathcal{M}, a \Vdash' \psi \wedge \varphi$	iff	$a \Vdash' \psi$ and $a \Vdash' \varphi$	$\mathcal{M}, x \succ' \psi \wedge \varphi$	iff	$x \in ([\![\psi]\!] \cap [\![\varphi]\!])^{\uparrow}$
$\mathcal{M}, a \Vdash' \psi \vee \varphi$	iff	$a \in ((\![\psi]\!) \cap (\![\varphi]\!))^{\downarrow}$	$\mathcal{M}, x \succ' \psi \vee \varphi$	iff	$x \succ' \psi$ and $x \succ' \varphi$
$\mathcal{M}, a \Vdash' \Box\psi$	iff	$a \in R_{\Box}^{(0)}((\![\psi]\!))$	$\mathcal{M}, x \succ' \Box\psi$	iff	$x \in \left(R_{\Box}^{(0)}((\![\psi]\!))\right)^{\uparrow}$
$\mathcal{M}, a \Vdash' \Diamond\psi$	iff	$a \in \left(R_{\Diamond}^{(0)}([\![\psi]\!])\right)^{\downarrow}$	$\mathcal{M}, x \succ' \Diamond\psi$	iff	$x \in R_{\Diamond}^{(0)}([\![\psi]\!])$
$\mathcal{M}, a \Vdash' \rhd\psi$	iff	$a \in R_{\rhd}^{(0)}([\![\psi]\!])$	$\mathcal{M}, x \succ' \rhd\psi$	iff	$x \in \left(R_{\rhd}^{(0)}([\![\psi]\!])\right)^{\uparrow}$
$\mathcal{M}, a \Vdash' \lhd\psi$	iff	$a \in \left(R_{\lhd}^{(0)}((\![\psi]\!))\right)^{\downarrow}$	$\mathcal{M}, x \succ' \lhd\psi$	iff	$x \in R_{\lhd}^{(0)}((\![\psi]\!))$

Notably, as the additional relations on our frame are assumed to be compatible, taking closures – as we did when defining satisfaction and co-satisfaction for our non-compatible models – would be redundant, as the subsets that we would be taking the closures of are already stable by assumption.

As discussed at the beginning of this chapter as well as in Section 1.1.3, whilst the imposition of these compatibility requirements serves to simplify the semantic clauses as well as insure various desirable properties such as normality and distributivity of the modalities, it does come at a significant cost to the applicability of the resultant semantic framework. In Example 4.54 we provide an elementary example of an enriched polarity with a non-compatible additional relation, intended to illustrate how simple relations arising naturally in a modelling context may very well be incompatible in the sense of Definition 4.49.

Example 4.54. Returning to our example from Example 3.2, consider the polarity $\mathbb{P}_M = (A, X, I)$ where A, X, and I are as defined in Example 3.2. Suppose that we enrich $\mathbb{P}_M$ with an additional $\rhd$-type relation, $R_{\rhd}$, where $aR_{\rhd}b$ iff a is mentioned on b's Wikipedia page – denote the resultant enriched polarity by $\mathbb{F}_M = (\mathbb{P}_M, \{R_{\rhd}\})$.

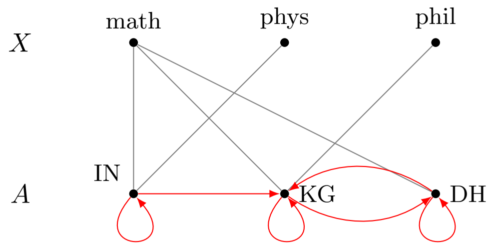


Figure 4.1: The enriched polarity, $\mathbb{F}_M = (\mathbb{P}_M, \{R_{\rhd}\})$, with incidence relation, I_{Δ^c}, in gray and the additional relation, $R_{\rhd}$, in red.

Note that in this example $\{\mathrm{IN}\}$ is stable as $\{\mathrm{IN}\}^{\uparrow\downarrow} = \{\mathrm{math}, \mathrm{phys}\}^{\downarrow} = \{\mathrm{IN}\}$. However, $R_{\rhd}^{(0)}(\{\mathrm{IN}\}) = \{\mathrm{IN}, \mathrm{KG}\}$ and $\{\mathrm{IN}, \mathrm{KG}\}$ is not stable as $\{\mathrm{IN}, \mathrm{KG}\}^{\uparrow\downarrow} = \{\mathrm{math}\}^{\downarrow} = \{\mathrm{DH}, \mathrm{IN}, \mathrm{KG}\} \neq \{\mathrm{IN}\}$. Therefore, $R_{\rhd}$ is not compatible in the sense of Definition 4.49.

Proposition 4.55. Let $\mathcal{M} = (\mathbb{F}, V)$ with $\mathbb{F} = (\mathbb{P}, \{R_{\Box}, R_{\Diamond}, R_{\lhd}, R_{\rhd}\})$ a compatible enriched polarity. Then for any $\varphi \in \mathcal{L}_P$, $a \in A$, and $x \in X$

(a) $\mathcal{M}, a \Vdash' \varphi$ iff $\mathcal{M}, a \Vdash \varphi$

(b) $\mathcal{M}, x \succ' \varphi$ iff $\mathcal{M}, x \succ \varphi$

Proof. We proceed by induction on φ. Satisfaction and co-satisfaction clauses in the compatible and non-compatible semantics are identical for logical constants, $\top$ and $\bot$, as well as conjunction and disjunction.

For (a) consider first the modalities that have primary satisfaction clauses: $\Box$ and $\rhd$. As $\mathbb{F}$ is compatible, for each $U \subseteq A$ and $V \subseteq X$

$$R_{\Box}^{(0)}(V) = \left(R_{\Box}^{(0)}(V)\right)^{\uparrow\downarrow} \text{ and } R_{\rhd}^{(0)}(U) = \left(R_{\rhd}^{(0)}(U)\right)^{\uparrow\downarrow}.$$

Therefore

$$\mathcal{M}, a \Vdash' \Box\varphi \text{ iff } a \in R_{\Box}^{(0)}(([\varphi])) \text{ iff } a \in \left(R_{\Box}^{(0)}(([\varphi]))\right)^{\uparrow\downarrow} \text{ iff } \mathcal{M}, a \Vdash \Box\varphi$$

and

$$\mathcal{M}, a \Vdash' \rhd\varphi \text{ iff } a \in R_{\rhd}^{(0)}([\![\varphi]\!]) \text{ iff } a \in \left(R_{\rhd}^{(0)}([\![\varphi]\!])\right)^{\downarrow\uparrow} \text{ iff } \mathcal{M}, a \Vdash \rhd\varphi.$$

For $\Diamond$ and $\lhd$ co-satisfaction is primary, and so we address the co-satisfaction clauses first. For each $U \subseteq A$ and $V \subseteq X$

$$R_{\Diamond}^{(0)}(U) = \left(R_{\Diamond}^{(0)}(U)\right)^{\downarrow\uparrow} \text{ and } R_{\lhd}^{(0)}(V) = \left(R_{\lhd}^{(0)}(V)\right)^{\downarrow\uparrow}.$$

Therefore

$$\mathcal{M}, x \succ' \Diamond\varphi \text{ iff } x \in R_{\Diamond}^{(0)}([\![\varphi]\!]) \text{ iff } a \in \left(R_{\Diamond}^{(0)}([\![\varphi]\!])\right)^{\downarrow\uparrow} \text{ iff } \mathcal{M}, x \succ \Diamond\varphi$$

and

$$\mathcal{M}, x \succ' \lhd\varphi \text{ iff } x \in R_{\lhd}^{(0)}(([\varphi])) \text{ iff } x \in \left(R_{\lhd}^{(0)}(([\varphi]))\right)^{\uparrow\downarrow} \text{ iff } \mathcal{M}, x \succ \lhd\varphi.$$

The remaining steps in the induction are to show that (a) holds for $\Diamond$ and $\lhd$, and (b) for $\Box$ and $\rhd$.

For $\Box$ and $\rhd$, co-satisfaction clauses for $\succ'$ and $\succ$ are identical and expressed in terms of the primary $\Vdash'$ and $\Vdash$ clauses, which were shown to be equivalent. Therefore $\mathcal{M}, x \succ' \varphi$ iff $\mathcal{M}, x \succ \varphi$ as required for (b).

Similarly, for $\Diamond$ and $\lhd$ the clauses for $\Vdash'$ and $\Vdash$ are identical, and expressed in terms of the primary $\succ'$ and $\succ$ clauses, which were shown to be equivalent. Thus $\mathcal{M}, a \Vdash' \varphi$ iff $\mathcal{M}, a \Vdash \varphi$ as required for (a). $\dashv$

Remark 4.56. Notably, the modalities $\Box/\Diamond$ and $\lhd/\rhd$ are not distinguishable algebraically in our semantics. In this way our non-compatible semantics is dissimilar from the compatible polarity-based semantics reviewed in Section 4.5.1 where $\Box$ is meet preserving, $\Diamond$ is join preserving, $\lhd$ is meet reversing, and $\rhd$ is join reversing [16].

The modalities $\Box/\Diamond$ and $\lhd/\rhd$ are distinguishable in the non-compatible semantics, as for $\Diamond/\lhd$ satisfaction clauses are primary and for $\Box/\rhd$ co-satisfaction clauses are primary. For this reason, as well as maintaining an analogue with the compatible semantics – especially in the special case of liftings of Kripke frames, which are the focus of Chapter 5 – we retain all four modalities in the signature.

4.5.2 Failure of normality and distributivity in the non-compatible setting

Absent from the axiomatization of the non-compatible semantics in Section 4.2 are modal axioms of the kind seen in the axiomatization of their compatible counterparts – namely distributivity of modal operators over conjunction or disjunction, and normality. Below we present various examples of enriched polarities with non-compatible additional relations in which these properties fail.

Example 4.57 (Failure of normality.)**.** The axioms for normality in the compatible semantics ((i)-(iv) below) are not sound in the non-compatible setting. Presented below are two polarity based frames, $\mathbb{F}_1$ and $\mathbb{F}_2$, demonstrating the failure of (i)-(ii) and (iii)-(iv) respectively.

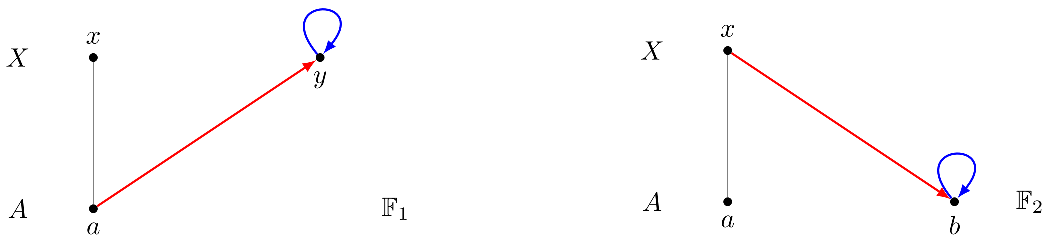
(i) $\top \vdash \Box\top$

(ii) $\lhd\top \vdash \bot$

(iii) $\Diamond\bot \vdash \bot$

(iv) $\top \vdash \rhd\bot$

Let $\mathbb{F}_1 = ((\{a\}, \{x, y\}, \{(a, x)\}), \{R_\Box, R_\lhd\})$ where $R_\Box = \{a, y\}$ and $R_\lhd = \{(y, y)\}$. And let $\mathbb{F}_2 = ((\{a, b\}, \{x\}, \{(a, x)\}), \{R_\Diamond, R_\rhd\})$ where $R_\Diamond = \{(x, b)\}$ and $R_\rhd = \{(b, b)\}$. Enriched polarities $\mathbb{F}_1$ and $\mathbb{F}_2$ are depicted in Figure 4.2.



(a) (b)

Figure 4.2: Illustrations of $\mathbb{F}_1$ (4.2(a)) and $\mathbb{F}_2$, polarity-based frames in which (i)-(ii) and (iii)-(iv) are invalid respectively. Incidence relations are illustrated by solid grey lines, $R_\Box$ and $R_\Diamond$ as depicted with red arrows, and $R_\lhd$ and $R_\rhd$ with blue.

In the case of (i) note that

$$\begin{aligned}
\top \vdash \Box\top &\text{ iff } \llparenthesis \Box\top \rrparenthesis \subseteq \llparenthesis \top \rrparenthesis \\
&\text{ iff } \left(R_\Box^{(0)} \llparenthesis \top \rrparenthesis\right)^{\uparrow} \subseteq (A)^{\uparrow} \\
&\text{ iff } \left(R_\Box^{(0)} \left(A^{\uparrow}\right)\right)^{\uparrow} \subseteq (A)^{\uparrow} \\
&\text{ iff } \left(R_\Box^{(0)} \left(A^{\uparrow}\right)\right)^{\uparrow} = (A)^{\uparrow} \qquad \left(\text{as } \left(R_\Box^{(0)} \left(A^{\uparrow}\right)\right)^{\uparrow\downarrow} \subseteq A \text{ always.}\right)
\end{aligned}$$

But $\mathbb{F}_1$ is such that $A^{\uparrow} = \{a\}^{\uparrow} = \{x\}$ and so $\left(R_\Box^{(0)} \left(A^{\uparrow}\right)\right)^{\uparrow\downarrow} = \left(R_\Box^{(0)} (\{x\})\right)^{\uparrow\downarrow} = \varnothing^{\uparrow\downarrow} = \varnothing \neq A$. Thus $\top \nvdash \Box\top$ in $\mathbb{F}_1$.

For (ii) note that

$$\begin{aligned}
\lhd\top \vdash \bot &\text{ iff } \llparenthesis \bot \rrparenthesis \subseteq \llparenthesis \lhd\top \rrparenthesis \\
&\text{ iff } X \subseteq \left(R_\lhd^{(0)} \llparenthesis \top \rrparenthesis\right)^{\downarrow\uparrow} \\
&\text{ iff } X \subseteq \left(R_\lhd^{(0)} \left(A^{\uparrow}\right)\right)^{\downarrow\uparrow}.
\end{aligned}$$

In $\mathbb{F}_1$ we have $A^{\uparrow} = \{a\}^{\uparrow} = \{x\}$. Therefore

$$\begin{aligned}
\left(R_\lhd^{(0)} \left(A^{\uparrow}\right)\right)^{\downarrow\uparrow} &= \left(R_\lhd^{(0)} (\{x\})\right)^{\downarrow\uparrow} \\
&= \varnothing^{\downarrow\uparrow} = \varnothing.
\end{aligned}$$

Thus $\llparenthesis \bot \rrparenthesis = X \nsubseteq \left(R_\lhd^{(0)} \left(A^{\uparrow}\right)\right)^{\downarrow\uparrow} = \varnothing$, and $\lhd\top \nvdash \bot$ in $\mathbb{F}_1$.

For (iii) note that

$$\begin{aligned}
\Diamond\bot \vdash \bot &\text{ iff } \llparenthesis \Diamond\bot \rrparenthesis \supseteq \llparenthesis \bot \rrparenthesis \\
&\text{ iff } \left(R_\Diamond^{(0)} [\![\bot]\!]\right)^{\downarrow\uparrow} \supseteq X \\
&\text{ iff } \left(R_\Diamond^{(0)} \left(X^{\downarrow}\right)\right)^{\downarrow\uparrow} \supseteq X.
\end{aligned}$$

But in $\mathbb{F}_2$,

$$\begin{aligned}\left(R_{\Diamond}^{(0)}\left(X^{\downarrow}\right)\right)^{\downarrow\uparrow} &= \left(R_{\Diamond}^{(0)}\left(\{x\}^{\downarrow}\right)\right)^{\downarrow\uparrow}\\ &= \left(R_{\Diamond}^{(0)}\left(\{a\}\right)\right)^{\downarrow\uparrow}\\ &= (\varnothing)^{\downarrow\uparrow} = \varnothing \not\supseteq X.\end{aligned}$$

Thus $\Diamond\bot \nvdash \bot$ in $\mathbb{F}_2$.

For (iv) we proceed similarly. Note that

$$\begin{aligned}\top \vdash \rhd\bot \text{ iff } [\![\top]\!] &\subseteq [\![\rhd\bot]\!]\\ \text{iff } A &\subseteq \left(R_{\rhd}^{(0)}\left([\![\bot]\!]\right)\right)^{\uparrow\downarrow}\\ \text{iff } A &= \left(R_{\rhd}^{(0)}\left(X^{\downarrow}\right)\right)^{\uparrow\downarrow}.\end{aligned}$$

But

$$\begin{aligned}\left(R_{\rhd}^{(0)}\left(X^{\downarrow}\right)\right)^{\uparrow\downarrow} &= \left(R_{\rhd}^{(0)}\left(\{a\}\right)\right)^{\uparrow\downarrow}\\ &= (\varnothing)^{\uparrow\downarrow} = \varnothing \neq A.\end{aligned}$$

Thus $\top \nvdash \rhd\bot$ in $\mathbb{F}_2$.

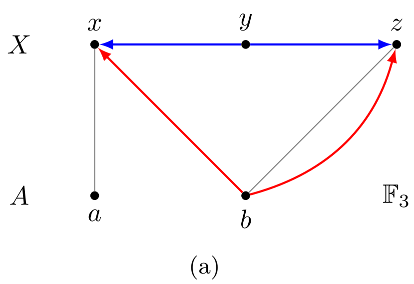
Example 4.58 (Failure of distributivity.)**.** The modal operators of our non-compatible semantics fail to distribute over disjunction and conjunction in the manner seen in distributive modal logic ([27]), and compatible non-distributive modal logic (see for instance the soundness results of [16]). In this example we present two basic polarity based models, $\mathcal{M}_1$ and $\mathcal{M}_2$, in which each of the modal operators of our language fail to distribute in the manner expected in the compatible setting ((i)-(iv)).

(i) $\Box p \wedge \Box q \vdash \Box(p \wedge q)$

(ii) $\lhd(p \wedge q) \vdash \lhd p \vee \lhd q$

(iii) $\Diamond(p \vee q) \vdash \Diamond p \vee \Diamond q$

(iv) $\rhd p \wedge \rhd q \vdash \rhd(p \vee q)$



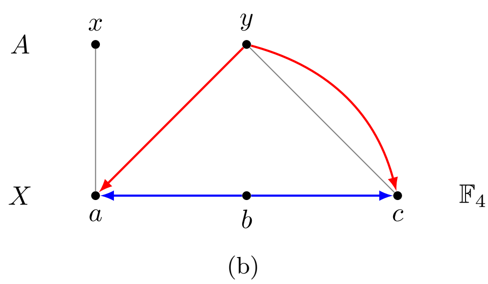


Figure 4.3: $\mathbb{F}_3$ (left) and $\mathbb{F}_4$ (right) with additional relations depicted using red ($R_\Box$ and $R_\Diamond$) and blue for ($R_\lhd$ and $R_\rhd$). Incidence relations are in grey.

As depicted in Figure 4.3(a), let $\mathcal{M}_1 = (\mathbb{F}_3, V)$ where $\mathbb{F}_3 = ((\{a,b\}, \{x,y,z\}, \{(a,x),(b,z)\}), \{R_\Box, R_\lhd\})$ where $R_\Box = \{(b,x),(b,z)\}$, $R_\lhd = \{(y,z),(y,x)\}$, $[\![V(p)]\!] = \{a\}$, and $[\![V(q)]\!] = \{b\}$. Note that

For (i) note that $\Box p \wedge \Box q \vdash \Box\,(p \wedge q)$ iff $\llbracket \Box p \wedge \Box q \rrbracket \subseteq \llbracket \Box\,(p \wedge q) \rrbracket$ where

$$\begin{aligned}
\llbracket \Box p \wedge \Box q \rrbracket &= \left(R_{\Box}^{(0)} \llparenthesis p \rrparenthesis\right)^{\uparrow\downarrow} \cap \left(R_{\Box}^{(0)} \llparenthesis q \rrparenthesis\right)^{\uparrow\downarrow} \\
&= \left(R_{\Box}^{(0)} \left(\{a\}^{\uparrow}\right)\right)^{\uparrow\downarrow} \cap \left(R_{\Box}^{(0)} \left(\{b\}^{\uparrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_{\Box}^{(0)} \left(\{x\}\right)\right)^{\uparrow\downarrow} \cap \left(R_{\Box}^{(0)} \left(\{z\}\right)\right)^{\uparrow\downarrow} \\
&= \{b\}^{\uparrow\downarrow} \cap \{b\}^{\uparrow\downarrow} = \{b\}
\end{aligned}$$

and

$$\begin{aligned}
\llbracket \Box\,(p \wedge q) \rrbracket &= \left(R_{\Box}^{(0)} \llparenthesis p \wedge q \rrparenthesis\right)^{\uparrow\downarrow} \\
&= \left(R_{\Box}^{(0)} \left((\llbracket p \rrbracket \cap \llbracket q \rrbracket)^{\uparrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_{\Box}^{(0)} \left((\varnothing)^{\uparrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_{\Box}^{(0)} \left(\{x, y, z\}\right)\right)^{\uparrow\downarrow} = \varnothing^{\uparrow\downarrow} = \varnothing.
\end{aligned}$$

Clearly $\{b\} \nsubseteq \varnothing$. Thus $\llbracket \Box p \wedge \Box q \rrbracket \nsupseteq \llbracket \Box\,(p \wedge q) \rrbracket$, and so $\Box p \wedge \Box q \nvdash \Box\,(p \wedge q)$ in $\mathcal{M}_1$.

For (ii) note that $\lhd\,(p \wedge q) \vdash \lhd p \vee \lhd q$ iff $\llparenthesis \lhd\,(p \wedge q) \rrparenthesis \supseteq \llparenthesis \lhd p \vee \lhd q \rrparenthesis$ where

$$\begin{aligned}
\llparenthesis \lhd\,(p \wedge q) \rrparenthesis &= \left(R_{\lhd}^{(0)} \llparenthesis p \wedge q \rrparenthesis\right)^{\downarrow\uparrow} \\
&= \left(R_{\lhd}^{(0)} \left((\llbracket p \rrbracket \cap \llbracket q \rrbracket)^{\uparrow}\right)\right)^{\downarrow\uparrow} \\
&= \left(R_{\lhd}^{(0)} \left((\varnothing)^{\uparrow}\right)\right)^{\downarrow\uparrow} \\
&= \left(R_{\lhd}^{(0)} \left(\{x, y, z\}\right)\right)^{\downarrow\uparrow} = \varnothing^{\downarrow\uparrow} = \varnothing
\end{aligned}$$

and

$$\begin{aligned}
\llparenthesis \lhd p \vee \lhd q \rrparenthesis &= \llparenthesis \lhd p \rrparenthesis \cap \llparenthesis \lhd q \rrparenthesis \\
&= \left(R_{\lhd}^{(0)} \llparenthesis p \rrparenthesis\right)^{\downarrow\uparrow} \cap \left(R_{\lhd}^{(0)} \llparenthesis q \rrparenthesis\right)^{\downarrow\uparrow} \\
&= \left(R_{\lhd}^{(0)} \left(\{x\}\right)\right)^{\downarrow\uparrow} \cap \left(R_{\lhd}^{(0)} \left(\{z\}\right)\right)^{\downarrow\uparrow} \\
&= \{y\}^{\downarrow\uparrow} \cap \{y\}^{\downarrow\uparrow} = \{x, y, z\}\,.
\end{aligned}$$

Then as $\varnothing \nsupseteq \{x, y, z\}$ we have $\llparenthesis \lhd\,(p \wedge q) \rrparenthesis \nsupseteq \llparenthesis \lhd p \vee \lhd q \rrparenthesis$ and thus $\lhd\,(p \wedge q) \nvdash \lhd p \vee \lhd q$ in $\mathcal{M}_1$.

Let $\mathcal{M}_2 = (\mathbb{F}_4, V)$ where $\mathbb{F}_4 = \left(\left(\{a, b, c\}, \{x, y\}, \{(a, x), (c, y)\}\right), \{R_{\diamond}, R_{\rhd}\}\right)$ where $R_{\diamond} = \{y, a, y, c\}$, $R_{\rhd} = \{b, a, b, c\}$, $\llbracket p \rrbracket = \{a\}$, and $\llbracket q \rrbracket = \{c\}$. This model is depicted in Figure 4.3(b).

For (iii) note that $\diamond\,(p \vee q) \vdash \diamond p \vee \diamond q$ iff $\llparenthesis \diamond\,(p \vee q) \rrparenthesis \supseteq \llparenthesis \diamond p \vee \diamond q \rrparenthesis$ where:

$$\begin{aligned}
\llparenthesis \diamond\,(p \vee q) \rrparenthesis &= \left(R_{\diamond}^{(0)} \llbracket p \vee q \rrbracket\right)^{\downarrow\uparrow} \\
&= \left(R_{\diamond}^{(0)} \left((\llparenthesis p \rrparenthesis \cap \llparenthesis q \rrparenthesis)^{\downarrow}\right)\right)^{\downarrow\uparrow} \\
&= \left(R_{\diamond}^{(0)} \left((\{x\} \cap \{y\})^{\downarrow}\right)\right)^{\downarrow\uparrow} \\
&= \left(R_{\diamond}^{(0)} \left((\varnothing)^{\downarrow}\right)\right)^{\downarrow\uparrow} \\
&= \left(R_{\diamond}^{(0)} \left(\{a, b, c\}\right)\right)^{\downarrow\uparrow} \\
&= \varnothing^{\downarrow\uparrow} = \varnothing
\end{aligned}$$

and

$$
\begin{aligned}
(\!|\Diamond p \vee \Diamond q|\!) &= \left(R_\Diamond^{(0)} [\![p]\!]\right)^{\downarrow\uparrow} \cap \left(R_\Diamond^{(0)} [\![q]\!]\right)^{\downarrow\uparrow} \\
&= \left(R_\Diamond^{(0)} \{a\}\right)^{\downarrow\uparrow} \cap \left(R_\Diamond^{(0)} \{c\}\right)^{\downarrow\uparrow} \\
&= (\{y\})^{\downarrow\uparrow} \cap (\{y\})^{\downarrow\uparrow} = \{y\}\,.
\end{aligned}
$$

Thus $\{y\} \nsubseteq \varnothing$, and so $(\!|\Diamond(p \vee q)|\!) \nsupseteq (\!|\Diamond p \vee \Diamond q|\!)$ and $\Diamond(p \vee q) \nVdash \Diamond p \vee \Diamond q$ in $\mathcal{M}_2$.

Finally, for (iv) we note that $\rhd p \wedge \rhd q \vdash \rhd(p \vee q)$ iff $[\![\rhd p \wedge \rhd q]\!] \subseteq [\![\rhd(p \vee q)]\!]$ where

$$
\begin{aligned}
[\![\rhd p \wedge \rhd q]\!] &= \left(R_\rhd^{(0)} ([\![p]\!])\right)^{\uparrow\downarrow} \cap \left(R_\rhd^{(0)} ([\![q]\!])\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} (\{a\})\right)^{\uparrow\downarrow} \cap \left(R_\rhd^{(0)} (\{c\})\right)^{\uparrow\downarrow} \\
&= \{b\}^{\uparrow\downarrow} \cap \{b\}^{\uparrow\downarrow} \\
&= \varnothing^{\downarrow} \\
&= \{a, b, c\}
\end{aligned}
$$

and

$$
\begin{aligned}
[\![\rhd(p \vee q)]\!] &= \left(R_\rhd^{(0)} ([\![p \vee q]\!])\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} \left(((\!|p|\!) \cap (\!|q|\!))^{\downarrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} \left(\left(\{a\}^{\uparrow} \cap \{c\}^{\uparrow}\right)^{\downarrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} \left((\{x\} \cap \{y\})^{\downarrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} \left((\varnothing)^{\downarrow}\right)\right)^{\uparrow\downarrow} \\
&= \left(R_\rhd^{(0)} (\{a, b, c\})\right)^{\uparrow\downarrow} \\
&= \varnothing^{\uparrow\downarrow} = \varnothing.
\end{aligned}
$$

And as $\{a, b, z\} \nsubseteq \varnothing$ we have that $\rhd p \wedge \rhd q \nvdash \rhd(p \vee q)$ in $\mathcal{M}_2$.

Remark 4.59 (Loss of adjunction properties in the non-compatible setting)**.** As distributivity is guaranteed in the compatible semantics, one can define for each of the four modalities $\Box, \Diamond, \lhd$ and $\rhd$ a corresponding adjoint $\blacklozenge, \blacksquare, \blacktriangleleft$ and $\blacktriangleright$ respectively. For instance, in [12] and [11] the distributivity of $\Box$ over conjunction implies the existence of a left adjoint, $\blacklozenge$, such that $\varphi \leq \Box\psi$ if and only if $\blacklozenge\varphi \leq \psi$ and satisfaction and co-satisfaction of $\blacklozenge$ are defined as:

$$
\mathcal{M}, a \vdash \blacklozenge\varphi \text{ iff } (\forall x \in X)\,(\mathcal{M}, x \succ \blacklozenge\varphi \to aIx)
$$

and

$$
\mathcal{M}, x \succ \blacklozenge\varphi \text{ iff } (\forall a \in A)\,(\mathcal{M}, a \vdash \varphi \to aR_\Box x)\,.
$$

On first glance, these adjoints could be given analogous definitions in our non-compatible semantics by taking the Galois-closures of the relevant terms, as we did when obtaining semantic clauses for each of the four modalities in Table 4.3. For instance, $\blacklozenge$ we would naturally have:

$$
\begin{aligned}
\mathcal{M}, a \vdash \blacklozenge\varphi \text{ iff } \boldsymbol{a} &\in \left(\left(R_\Box^{(1)} [\![\psi]\!]\right)^{\downarrow}, \left(R_\Box^{(1)} [\![\psi]\!]\right)^{\downarrow\uparrow}\right) \\
&\text{iff } (\forall x \in X)\,(\forall a' \in A)\,((a' \Vdash \psi \to a' R_\Box x) \to aIx)
\end{aligned}
$$

We can verify using the valuation $[\![V(p)]\!] = [\![V(q)]\!] = \{a\}$ that frame $\mathbb{F}_3$ in Example 4.58 does not exhibit the adjunction property: $p \leq \Box q$ if and only if $\blacklozenge p \leq q$. Thus the intuitive candidate obtained by taking Galois-closures does not, in general, have the adjunction property.

More generally, however, as non-compatible modalities are in general non-distributive over conjunction and disjunction in the way described in the previous example, left adjoints in the cases of $\Box$ and $\lhd$, and right adjoints in the cases of $\Diamond$ and $\rhd$, do not exist [24].

Chapter 5

Kripke frames as enriched polarities

Possible world semantics has seen use across a diversity of disciplines since its development in the 1970s. Sub-structural logic by comparison, whilst seeing great technical strides over the past several decades, has not seen the same degree of cross-disciplinary proliferation. Mai Gehrke acknowledges the difficulty in achieving the same level of applicability in the development of relational semantics for substructural logics in the introduction of [26]:

> The immense success of relational semantics in the setting of modal logics has prompted a sprawling literature that attempts to apply the methods in the setting of substructural logics as well, but the level of unity and power of these methods in the setting of substructural logic is not the same.

This chapter elaborates upon known connections between Kripke semantics and polarity-based semantics, with Kripke frames seen as a specific subclass of compatible enriched polarities. The familiarity of Kripke semantics offers potentially important ground upon which to build intuitive understanding for the relational semantics of NDML. This is the broad goal of the remaining two chapters of this thesis: to introduce Kripke frames as a special class of enriched polarities using the methods of [9] in this chapter, and then to examine truth preservation in the context of liftings using primarily relational methods in Chapter 6.

5.1 Lifting sets and Kripke frames

The definitions of set liftings (Chapter 5) and lifting of Kripke frames (Definition 5.7) are adapted from [9]. Working from these definitions we introduce our notion of liftings of classical Kripke models in (Definition 5.13) in the subsequent section.

Unless otherwise specified, throughout the remainder of this chapter we let $\mathsf{K} = (W, R)$ be a Kripke frame and $\mathcal{M} = (\mathsf{K}, V)$ a model based on K.

Many of the constructions used in this chapter rely on the following typing convention in producing two typed copies of the underlying set, W. For a set U, the notation U_A and U_X will refer to typed copies of U given by $\{u_A \mid u \in U\}$ and $\{u_X \mid u \in U\}$ respectively. That is, U_A and U_X are typed copies of U with each $u \in U$ subscripted with an A or X respectively.

5.1.1 Sets as polarities

Definition 5.1 (Liftings of sets [9].)**.** The *lifting* a non-empty set, W, is the polarity $\mathbb{P}_W = (W_A, W_X, I_{\Delta^c})$ where $I_{\Delta^c} = \{(u_A, v_X) \in W_A \times W_X \mid u \neq v\}$. The notation $\mathbb{P}_W = (W_A, W_X, I_{\Delta^c})$ will be used for the lifting of W where convenient.

Proposition 5.2 (Properties of I_{Δ^c})**.** For $\mathbb{P}_W = (W_A, W_X, I_{\Delta^c})$ and $U, V \subseteq W$

(a) $u_X \in (U_A)^{\uparrow}$ if and only if $u \in U^c$.

(b) $v_A \in (V_X)^{\downarrow}$ if and only if $v \in V^c$.

(c) $(U_A)^{\uparrow} = (U^c)_X$.

(d) $(U_X)^{\downarrow} = (U^c)_A$.

Proof. For (a), suppose that $u_X \in (U_A)^{\uparrow}$. Then $b_A I_{\Delta^c} u_X$ for each $b_A \in U_A$. So, $u \neq b$ for each $b \in U$, and thus $u \in U^c$. For the converse, suppose that $u \in U^c$, then $u \neq b$ for each $b \in U$. Thus $b_A I_{\Delta^c} u_X$ for each $b \in U_A$, and so $u_X \in U_A^{\uparrow}$.

For (b), suppose that $v_A \in V_X^{\downarrow}$, then $v_A I_{\Delta^c} b_X$ for each $b_X \in V_X$. Thus, $v \neq b$ for each $b \in V$, and $v \in V^c$. If $v \in V^c$ then $v \neq b$ for each $b \in V$, and so $v_A I_{\Delta^c} b_X$ for each $b_X \in V_X$, and so $v_A \in V_X^{\downarrow}$.

The (c) and (d) equivalences follow directly from (a) and (b) respectively. This is as $u_X \in (U_A)^{\uparrow}$ if and only if $u \in U^c$ if and only if $u_X \in (U^c)_X$, as $(U^c)_X$ is simply an X-subscripted copy of U^c. Similarly, setting $U = V$ in (b) gives $v_A \in (U_X)^{\downarrow}$ if and only if $v \in U^c$ if and only if $v_A \in (U^c)_A$. ⊣

Corollary 5.3. All subsets $B_A \subseteq W_A$ and $Y_X \subseteq W_X$ are I_{Δ^c}-stable in the polarity $\mathbb{P}_W$. That is $B_A^{\uparrow\downarrow} = B_A$ and $Y_X^{\downarrow\uparrow} = Y_X$.

Proof. Let $a_A \in B_A \subseteq W_A$ and $x_X \in Y_X \subseteq W_X$. Then

$$\begin{aligned} a_A \in B_A^{\uparrow\downarrow} &\text{ iff } a_A \in \left((B^c)_X\right)^{\downarrow} && \text{(Proposition 5.2(c))} \\ &\text{ iff } a_A \in \left((B^c)^c\right)_A && \text{(Proposition 5.2(d))} \\ &\text{ iff } a_A \in B_A. \end{aligned}$$

Thus $B^{\uparrow\downarrow} = B$, and B is I_{Δ^c}-stable. Similarly for $Y_X \subseteq W_X$:

$$\begin{aligned} x_X \in Y^{\downarrow\uparrow} &\text{ iff } x_X \in \left((Y^c)_A\right)^{\uparrow} && \text{(Proposition 5.2(d))} \\ &\text{ iff } x_X \in \left((Y^c)^c\right)_X && \text{(Proposition 5.2(c))} \\ &\text{ iff } x_X \in Y_X. \end{aligned}$$ ⊣

The choice of I_{Δ^c} as incidence relation over I_{Δ}[1] may appear counter intuitive on first glance. The complex algebra of (S_A, S_X, I_{Δ}) is, however, not in general isomorphic to the powerset algebra of S. See [9, Example 3.5] for an example of a set whose set algebra is not isomorphic to the complex algebra of its resultant polarity when lifted in this way. The choice of I_{Δ^c}, however, ensures that the complex algebra of $\mathbb{P}_S$ is isomorphic to the powerset algebra of S, as is shown in Remark 5.4.

Remark 5.4. For any set W, the powerset algebra, $\mathcal{S}(W)$, is isomorphic to $\mathbb{P}_W^+$.

Proof. This is a well known result – see for instance [9] and [16].

We will provide a brief sketch of the proof here. Strictly speaking, the structures $\mathcal{S}(W)$ and $\mathbb{P}_W^+$ are not comparable as $\mathcal{S}(W)$ has the additional complementation operator, $(\cdot)^c$. The complex algebra, $\mathbb{P}_W^+$, does however accommodate for an additional operation that respects complementations under isomorphism.

Consider the map $\theta : \mathcal{S}(W) \to \mathbb{P}_W^+$ where for $U \in \mathcal{P}(W)$:

$$\theta(U) := \left(U_A, (U^c)_X\right).$$

It follows from the fact $U_A^{\uparrow} = (U^c)_X$ (Proposition 5.2(c)) that θ is well defined. Clearly, θ is injective as $\theta(U) = \theta(V)$ implies that $U_A = V_A$, and thus $U = V$ for any $U, V \in \mathcal{P}(W)$. Similarly, for any $\boldsymbol{c} \in \mathbb{P}_W^+$

[1] Where $I_{\Delta} = \{(u_A, v_X) \in S_A \times S_X \mid u = v\}$

we have $[\![\boldsymbol{c}]\!] = U_A$ for some $U \in \mathcal{P}(W)$ and so by Proposition 5.2(c) it follows that $\theta(U) = \boldsymbol{c}$. Thus θ is bijective.

Additionally, noting that $(U \cap V)_A = (U_A \cap V_A)$, we have

$$\begin{aligned}
\theta(U \wedge V) &= \theta(U \cap V) && \text{(Definition of } \theta\text{)} \\
&= \left((U \cap V)_A, (U \cap V)^c_X\right) \\
&= \left((U_A \cap V_A), ((U \cap V)^c_X)\right) \\
&= \left(U_A \cap V_A, (U_A \cap V_A)^{\uparrow}\right) \\
&= \left([\![\theta(U)]\!] \cap [\![\theta(V)]\!], ([\![\theta(U)]\!] \cap [\![\theta(V)]\!])^{\uparrow}\right) \\
&= \theta(U) \wedge \theta(V).
\end{aligned}$$

Thus θ respects meets. Verifying that θ respects joins is similar.

Additionally, $\theta(\varnothing) = (\varnothing_A, W_X)$ and $\theta(W) = (W_A, \varnothing_X)$, the respective top and bottom elements of $\mathbb{P}^+_W$ by Definition 3.23. Complementation can be accommodated for via the addition of a complementation operation $(\cdot)' : \mathbb{P}^+_W \to \mathbb{P}^+_W$ given by

$$(U_A, (U^c)_X)' := ((U^c)_A, U_X). \qquad \dashv$$

We will use the notation $\boldsymbol{U}$ for the element $\boldsymbol{U} = (U_A, (U^c)_X) \in \mathbb{P}^+_W$ where convenient.

5.1.2 Kripke frames as enriched polarities

In order to accommodate for each of the modalities in our signature, $\Box, \Diamond, \lhd$ and $\rhd$, four binary relations (one of each type given in Table 4.2) are defined in terms of the accessibility relation of the underlying Kripke frame, R – $I_{R^c}, J_{R^c}, H_{R^c}$ and K_{R^c} respectively.

Definition 5.5 (I_R, J_R, H_R, K_R [9])**.** For binary relation $R \subseteq W \times W$ define:

(a) $I_R = \{(u_A, v_X) \in W_A \times_X \mid (u, v) \in R\}$

(b) $J_R = \{(v_X, v_A) \in W_X \times W_A \mid (u, v) \in R\}$

(c) $H_R = \{(u_A, v_A) \in W_A \times W_A \mid (u, v) \in R\}$

(d) $K_R = \{(u_X, v_X) \in W_X \times W_X \mid (u, v) \in R\}$

As $I_{R^c}, J_{R^c}, H_{R^c}$ and K_{R^c} serve as additional relations in the enriched polarity associated with a Kripke frame, for convenience their definitions are provided in full in the following remark.

Remark 5.6.

(a) $I_{R^c} = (I_R)^c = \{(u_A, v_X) \in S_A \times S_X \mid (u, v) \notin R\}$

(b) $J_{R^c} = (J_R)^c = \{(v_X, v_A) \in S_X \times S_A \mid (u, v) \notin R\}$

(c) $H_{R^c} = (H_R)^c = \{(u_A, v_A) \in S_A \times S_A \mid (u, v) \notin R\}$

(d) $K_{R^c} = (K_R)^c = \{(u_X, v_X) \in S_X \times S_X \mid (u, v) \notin R\}$

As these sets have identical membership requirements, they are equivalent in the sense that:

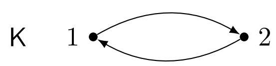
$$(u_A, v_X) \in I_{R^c} \text{ iff } (v_X, u_A) \in J_{R^c} \text{ iff } (u_A, v_A) \in H_{R^c} \text{ iff } (u_X, v_X) \in K_{R^c}.$$

Definition 5.7 (Lifting of Kripke frames [9])**.** The *lifting* of a Kripke frame, $\mathsf{K} = (W, R)$, is the enriched polarity, $\mathsf{P}_\mathsf{K} = (\mathbb{P}_\mathsf{K}, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$, where $\mathbb{P}_\mathsf{K} = \mathbb{P}_W = (W_A, W_X, I_{\Delta^c})$.

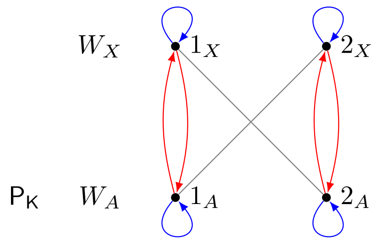
At certain points $I_{R^c}, J_{R^c}, H_{R^c}$, and K_{R^c} will be referred to as $R_\Box, R_\Diamond, R_\triangleright$, and $R_\triangleleft$ respectively.

Below we present several examples of Kripke frames and their liftings. For clarity, we will use the following color convention: red lines for relations I_{R^c} and J_{R^c}, and blue lines for H_{R^c} and K_{R^c}. Gray solid lines will be used for the incidence relations, I_{Δ^c}.

Example 5.8. For the Kripke frame, $\mathsf{K} = (W, R) = (\{1, 2\}, \{(1, 2), (2, 1)\})$, we set $W_A = \{1_A, 2_A\}$, $W_X = \{1_X, 2_X\}$. Then $\mathsf{P}_\mathsf{K} = (\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ with $I_{R^c} = \{(1_A, 1_X), (2_A, 2_X)\}$, $J_{R^c} = \{(1_X, 1_A), (2_X, 2_A)\}$, $H_{R^c} = \{(1_A, 1_A), (2_A, 2_A)\}$, and $K_{R^c} = \{(1_X, 1_X), (2_X, 2_X)\}$.

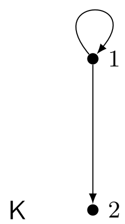


(a)

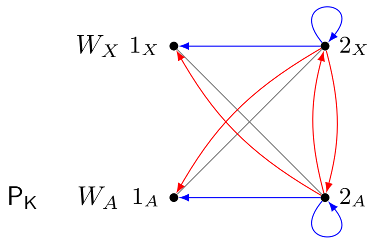


(b)

Example 5.9. Let $\mathsf{K} = (W, R) = (\{1, 2\}, \{(1, 1), (1, 2)\})$. Setting $W_A = \{1_A, 2_A\}$ and $W_X = \{1_X, 2_X\}$ gives $\mathsf{P}_\mathsf{K} = (\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ with $I_{R^c} = \{(2_A, 2_X), (2_A, 1_X)\}$, $J_{R^c} = \{(2_X, 2_A), (1_X, 2_A)\}$, $H_{R^c} = \{(2_A, 2_A), (1_A, 2_A)\}$, and $K_{R^c} = \{(2_X, 2_X), (1_X, 2_X)\}$.



(a)



(b)

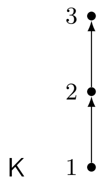
Lifting diagrams can grow fairly complex following with the addition of states to the underlying Kripke frame. For the sake of readability, we thus omit the gray incidents relation lines in Example 5.10. Instead, we draw I_{R^c} in red, J_{R^c} in blue, and H_{R^c} and K_{R^c} in gray.

Example 5.10. Let $\mathsf{K} = (W, R) = (\{1, 2, 3\}, \{(1, 2), (2, 3)\})$. Then setting $W_A = \{1_A, 2_A, 3_A\}$ and $W_X = \{1_X, 2_X, 3_X\}$ we have $\mathsf{P}_\mathsf{K} = \{\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\}\}$ where

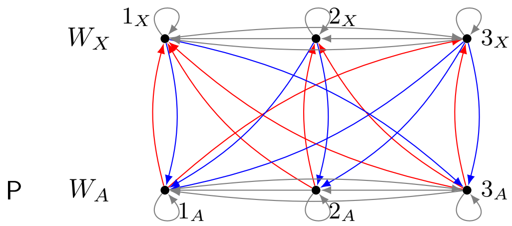
$$I_{R^c} = \{(1_A, 1_X), (1_A, 3_X), (2_A, 1_X), (2_A, 2_X), (3_A, 1_X), (3_A, 2_X), (3_A, 3_X)\};$$
$$J_{R^c} = \{(1_X, 1_A), (1_X, 3_A), (2_X, 1_A), (2_X, 2_A), (3_X, 1_A), (3_X, 2_A), (3_X, 3_A)\};$$
$$H_{R^c} = \{(1_A, 1_A), (1_A, 3_A), (2_A, 1_A), (2_A, 2_A), (3_A, 1_A), (3_A, 2_A), (3_A, 3_A)\};$$

and

$$K_{R^c} = \{(1_X, 1_X), (1_X, 3_X), (2_X, 1_X), (2_X, 2_X), (3_X, 1_X), (3_X, 2_X), (3_X, 3_X)\}.$$



(a)



(b)

Additional examples of lifted Kripke frames can be found in Chapter 8.

Remark 5.11 (Liftings of Kripke frames are compatible)**.** The lifting, P_K, of a Kripke frame, $\mathbb{P}_\mathsf{K}$, is always compatible in the sense of Definition 4.51. This follows directly from Corollary 5.3, as every typed subset $B_A \subseteq W_A$ and $Y \subseteq W_X$ is stable.

Notably, there can exist polarities distinct from $\mathbb{P}_S$ whose complex algebras are also isomorphic to $\mathcal{S}(S)$ – polarities that satisfy the isomorphism in Remark 5.4. Two additional methods of lifting sets are discussed in [9, Remark 3.6]. These additional methods are of interest in the context of many-valued semantics, but don't provide any additional utility in the Boolean case, which is the focus of this study.

5.2 Equivalence of compatible and non-compatible semantics on liftings of Kripke frames

The remainder of this chapter will focus on the comparison of our compatible and non-compatible semantics on lifting of Kripke frames. In addition to the compatible and non-compatible semantics introduced in Chapter 4, this section also refers to satisfaction on Kripke models. Our primary focus, however, lies in the interplay between the two polarity-based semantic frameworks in this special context. For this reason, we refrain from defining the familiar basic modal language, and instead interpret the language of DML for polarities, $\mathcal{L}_P$, through the definition of a third satisfaction relation, $\Vdash_\mathsf{K}$, in Definition 5.12. With the addition of classical negation, $\mathcal{L}_P$ becomes expressively equivalent to the basic modal language under this interpretation. Consequently, upon the introduction of classical negation, the modalities $\Box, \Diamond, \rhd$, and $\lhd$ become interdefinable – for example $\Diamond\varphi = \neg\Box\neg\varphi$, $\rhd\varphi = \neg\Diamond\varphi$, and $\lhd\varphi = \Diamond\neg\varphi$.

Definition 5.12. The satisfaction relation, $\Vdash_\mathsf{K} \subseteq W \times \mathcal{L}_P$, on a model, $\mathcal{M}_\mathsf{K} = (\mathsf{K}, V)$, based on Kripke frame,$\mathsf{K} = (W, R)$, is defined recurseively as follows:

$$\begin{array}{lll}
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} p & \text{iff} & w \in V(p) \text{ for any } p \in \mathsf{Prop} \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \wedge \varphi_2 & \text{iff} & \mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \text{ and } \mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \vee \varphi_2 & \text{iff} & \mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \text{ or } \mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \varphi_1 \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \Box\varphi & \text{iff} & \forall v \in W(wRv \to \mathcal{M}_\mathsf{K}, v \Vdash_\mathsf{K} \varphi) \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \Diamond\varphi & \text{iff} & \exists v \in W(wRv \text{ and } \mathcal{M}_\mathsf{K}, v \Vdash_\mathsf{K} \varphi) \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \rhd\varphi & \text{iff} & \forall v \in W(wRv \to \mathcal{M}_\mathsf{K}, v \nVdash \varphi) \\
\mathcal{M}_\mathsf{K}, w \Vdash_\mathsf{K} \lhd\varphi & \text{iff} & \exists v \in W(wRv \text{ and } \mathcal{M}_\mathsf{K}, v \nVdash \varphi).
\end{array}$$

As $\Vdash_\mathsf{K}$ is the only satisfaction relation that we have defined on Kripke models, and thus the only satisfaction relation to the left of which will appear pointed Kripke models, it will often be clear from context that $\Vdash$ refers to $\Vdash_\mathsf{K}$. The satisfaction relations may also be at times disambiguated by the presence of typing, with states in the Kripke model always going untyped. In such cases, the subscript K may be omitted for the sake of readability.

We now extend the notion of set or Kripke frame lifting to a Kripke model in the intuitive way detailed in Definition 5.13 below.

Definition 5.13 (Lifting of a Kripke model)**.** The polarity-based model, $\mathcal{N} = (\mathsf{P}, V')$, is a *lifting* of a

Kripke model, $\mathcal{M}_{\mathsf{K}} = (\mathsf{K}, V_1)$, if $\mathsf{P} = (\mathbb{P}_{\mathsf{K}}, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ is a lifting of K in the sense of Definition 5.7 and $a \in V_1(p)$ iff $a_A \in [\![V'(p)]\!]$ for all $p \in \mathsf{Prop}$.

If left unspecified, the notation $\mathcal{N}$ refers to the lifting of a corresponding Kripke model, $\mathcal{M}_{\mathsf{K}}$ or $\mathcal{M}$.

As our focus lies with lifting of Kripke frames for the remainder of this thesis, we restate, for convenience, the equivalency of compatible and non-compatible satisfaction and co-satisfaction on compatible models given in Proposition 4.55 for the special case of models that are lifting of Kripke models. For any $a \in W$

Remark 5.14.

(a) $\mathcal{N}, a_A \Vdash' \varphi$ iff $\mathcal{N}, a_A \Vdash \varphi$

(b) $\mathcal{N}, a_X \succ' \varphi$ iff $\mathcal{N}, a_X \succ \varphi$

Proof. Follows directly from Proposition 4.55, as lifting of Kripke frames are always compatible (Remark 5.11). $\dashv$

Proposition 5.15. If $\mathcal{N}$ is a lifting of $\mathcal{M}_{\mathsf{K}}$, then, for any $\psi \in \mathcal{L}_P$ and $a \in W$,

$$\mathcal{N}, a_A \Vdash \psi \text{ if and only if } \mathcal{N}, a_X \not\succ \psi.$$

Proof. Let $\psi \in \mathcal{L}_P$, and $a_A \in W_A$. Then

$$\begin{aligned} a_A \Vdash \psi \text{ iff } & a_A \in [\![\psi]\!] \\ \text{iff } & a_A \in (\![\psi]\!)^{\downarrow} \\ \text{iff } & (\forall x_X \in W_X)\,(x_X \succ \psi \to a_A I_{\Delta^c} x_X) \\ \text{iff } & (\forall x_X \in W_X)\,(x_X \succ \psi \to a \neq x \text{ in } \mathsf{K}) \\ \text{iff } & (\forall x_X \in W_X)\,(a = x \text{ in } \mathsf{K} \to x_X \not\succ \psi) \\ \text{iff } & a_X \not\succ \psi. \end{aligned}$$ $\dashv$

Proposition 5.16. If $\mathcal{N} = (\mathbb{F}, V_2)$ is a lifting of model $\mathcal{M} = (\mathsf{K}, V_1)$, then, for any $\psi \in \mathcal{L}_P$,

$$\mathcal{M}, w \Vdash_{\mathsf{K}} \psi \text{ if and only if } \mathcal{N}, w_A \Vdash \psi.$$

Proof. We proceed by induction on the number of connectives, n, in ψ.

Consider $n = 0$. If $\psi \in \mathsf{Prop}$ then by Definition 5.13, $\mathcal{M}, a \Vdash_{\mathsf{K}} p$ if and only if $\mathcal{N}, a_A \Vdash p$. If $\psi = \top$ both $\mathcal{N}, a_A \Vdash \top$ always and $\mathcal{M}, a \Vdash_{\mathsf{K}} \top$ always. Similarly for $\psi = \bot$, we have $\mathcal{M}, a \Vdash_{\mathsf{K}} \bot$ never, and as $\mathcal{N}, a_X \succ \bot$ always, by Proposition 5.15 $\mathcal{N}, a_A \Vdash \bot$ never.

Suppose the result holds for ψ with at most N connectives.

Suppose $\psi = \varphi \wedge \gamma$ with φ and γ each having at most N connectives.

$$\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi \wedge \gamma \text{ iff } \mathcal{M}, a \Vdash_{\mathsf{K}} \varphi \text{ and } \mathcal{M}, a \Vdash_{\mathsf{K}} \gamma$$

and

$$\mathcal{N}, a_A \Vdash \varphi \wedge \gamma \text{ iff } \mathcal{N}, a_A \Vdash \varphi \text{ and } \mathcal{N}, a_A \Vdash \gamma.$$

And by the inductive hypothesis, $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi$ [resp. γ] if and only if $\mathcal{N}, a_A \Vdash \varphi$ [resp. γ].

Similarly, suppose $\psi = \varphi \vee \gamma$ with φ and γ each having at most N connectives. Suppose $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi \vee \gamma$. We have $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi \vee \gamma$ if and only if $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi$ or $\mathcal{M}, a \Vdash_{\mathsf{K}} \gamma$. Without loss of generality suppose that $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi$ then

$$\begin{aligned} \mathcal{M}, a \Vdash_{\mathsf{K}} \varphi \text{ iff } & \mathcal{N}, a_A \Vdash \varphi && \text{(hypothesis)} \\ \text{iff } & \mathcal{N}, a_X \not\succ \varphi && \text{(Proposition 5.15)} \end{aligned}$$

and

$$\mathcal{N}, a_X \succ \varphi \vee \gamma \text{ iff } \mathcal{N}, a_X \succ \varphi \text{ and } \mathcal{N}, a_X \succ \gamma.$$

But, $\mathcal{N}, a_X \not\succ \varphi$ and so $\mathcal{N}, a_X \not\succ \varphi \vee \gamma$. And so $\mathcal{N}, a_A \Vdash \varphi \vee \gamma$ by Proposition 5.15.

For the converse we show the contrapositive – $\mathcal{M}, a \nVdash_{\mathsf{K}} \varphi \vee \gamma$ implies $\mathcal{N}, a_A \nVdash \varphi \vee \gamma$.

$$\begin{aligned} \mathcal{M}, a \nVdash_{\mathsf{K}} \varphi \vee \gamma \text{ iff } & \mathcal{M}, a \nVdash_{\mathsf{K}} \varphi \text{ and } \mathcal{M}, a \nVdash_{\mathsf{K}} \gamma \\ \text{iff } & \mathcal{N}, a_A \nVdash \varphi \text{ and } \mathcal{N}, a_A \nVdash \gamma && \text{(hypothesis)} \\ \text{iff } & \mathcal{N}, a_X \succ \varphi \text{ and } \mathcal{N}, a_X \succ \gamma && \text{(Proposition 5.15)} \\ \text{iff } & \mathcal{N}, a_X \succ \varphi \vee \gamma \\ \text{iff } & \mathcal{N}, a_A \nVdash \varphi \vee \gamma. && \text{(Proposition 5.15)} \end{aligned}$$

Suppose $\psi = \Box\varphi$. Then

$$\begin{aligned} \mathcal{N}, a_A \Vdash \Box\psi \text{ iff } & a_A \in I^{(0)}_{R^c}\left(\llparenthesis\varphi\rrparenthesis\right) \\ \text{iff } & (\forall b_X \in W_X)\,(b_X \succ \varphi \to a_A I_{R^c} b_X) \\ \text{iff } & (\forall b_A \in W_A)\,(b_A \nVdash \varphi \to a_A I_{R^c} b_X) && \text{(Proposition 5.15)} \\ \text{iff } & (\forall b_A \in W_A)\,(aRb \text{ in } \mathsf{K} \to b_A \Vdash \varphi) && \text{(contraposition)} \\ \text{iff } & (\forall b \in W)\,(aRb \text{ in } \mathsf{K} \to b \Vdash_{\mathsf{K}} \varphi) && \text{(hypothesis)} \\ \text{iff } & \mathcal{M}, a \Vdash_{\mathsf{K}} \Box\varphi. \end{aligned}$$

Suppose $\psi = \Diamond\varphi$. Then

$$\begin{aligned} \mathcal{N}, a_A \Vdash \Diamond\varphi \text{ iff } & \mathcal{N}, a_X \not\succ \Diamond\varphi && \text{(Proposition 5.15)} \\ \text{iff } & \neg((\forall b_A \in W_A)(b_A \Vdash \varphi \to a_X J_{R^c} b_A)) \\ \text{iff } & (\exists b_A \in W_A)(b_A \Vdash \varphi \text{ and } \neg a_X J_{R^c} b_A) \\ \text{iff } & (\exists b \in W)(b \Vdash_{\mathsf{K}} \varphi \text{ and } aRb) && \text{(hypothesis)} \\ \text{iff } & \mathcal{M}, a \Vdash_{\mathsf{K}} \Diamond\varphi \end{aligned}$$

Suppose $\psi = \rhd\varphi$. Then

$$\begin{aligned} \mathcal{N}, a_A \Vdash \rhd\varphi \text{ iff } & a \in H^{(0)}_{R^c}\left(\llbracket\varphi\rrbracket\right) \\ \text{iff } & (\forall b_A \in W_A)\,(b_A \Vdash\!\to a_A H_{R^c} b_A) \\ \text{iff } & (\forall b_A \in W_A)\,(b_A \Vdash \varphi \to \neg aRb \text{ in } \mathsf{K}) \\ \text{iff } & (\forall b \in W)\,(b \Vdash_{\mathsf{K}} \varphi \to \neg aRb \text{ in } \mathsf{K}) && \text{(hypothesis)} \\ \text{iff } & (\forall b \in W)\,(aRb \to b \nVdash_{\mathsf{K}} \varphi) && \text{(contraposition)} \\ \text{iff } & \mathcal{M}, a \Vdash_{\mathsf{K}} \rhd\varphi. \end{aligned}$$

Suppose $\psi = \lhd\varphi$. Then

$$\begin{aligned} \mathcal{N}, a_A \Vdash \lhd\varphi \text{ iff } & \mathcal{N}, a_X \not\succ \lhd\varphi && \text{(Proposition 5.15)} \\ \text{iff } & \neg\,(\forall b_X \in W_X)\,(b_X \succ \varphi \to a_X K_{R^c} b_X)) \\ \text{iff } & (\exists b_X \in W_X)\,(b_X \succ \varphi \text{ and } \neg a_X K_{R^c} b_X) \\ \text{iff } & (\exists b_X \in W_X)\,(b_X \succ \varphi \text{ and } aRb \text{ in } \mathsf{K}) \\ \text{iff } & (\exists b_A \in W_A)\,(b_A \nVdash \varphi \text{ and } aRb \text{ in } \mathsf{K}) && \text{(Proposition 5.15)} \\ \text{iff } & (\exists b \in W)\,(b \nVdash_{\mathsf{K}} \varphi \text{ and } aRb) && \text{(hypothesis)} \\ \text{iff } & \mathcal{M}, a \Vdash_{\mathsf{K}} \lhd\varphi. \end{aligned}$$

Thus the induction holds, and the result follows. ⊣

Corollary 5.17. $\mathcal{M}, a \Vdash_{\mathsf{K}} \varphi$ if and only if $\mathcal{N}, a_X \not\succ \varphi$.

Proof. Follows directly from Proposition 5.15 and Proposition 5.16. $\dashv$

Chapter 6

p-morphisms for liftings of Kripke frames

Bounded morphisms, or p-morphisms, are well known truth preserving maps in classical modal logic – that is modal satisfaction of formulae is preserved between a state and its image under a p-morphism in their corresponding Kripke models (for a formal statement of this result see Theorem 6.10). In this way p-morphisms are one of four truth preserving operations in classical modal logic, along with generated subframes, disjoint unions, and ultrafilter extensions, coalesced in the famous Goldblatt-Thomason theorem:

> A first-order definable class of frames, K, is modally definable if and only if it is closed under taking p-morphic images, generated subframes, and disjoint unions, and additionally reflects ultrafilter extensions [31][5].

Given the utility of the Goldblatt-Thomason theorem in understanding expressivity in classical modal logic, analogs of the theorem have been developed for many classical and non-classical logics since the publication of Goldblatt and Thomason's original result ([31]) in 1975.[1]

Goldblatt-Thomason style theorems have been developed for NDML – see [30] and [17]. In [17], such a result is developed for logics algebraically captured by normal lattice expansions (normal LE-logics) for arbitrary signatures. These results, whilst applicable to a variety of logics, were developed by dualising from the algebraic context. Consequently, these formulations are, in a sense, less intuitive than their counterparts from classical modal logic. Additionally, the Goldblatt-Thomason theorem of [17] assumes normality, which our logic, $\boldsymbol{L_P}$, in general lacks (see Example 4.57 as well as the axiomatization of $\boldsymbol{L_P}$ in Section 4.2). As detailed in Chapter 5, liftings of Kripke frames are always compatible, and thus maintain normality.[2] The context of liftings thus provides an hospitable environment to explore relational implications of the algebraic formulation of p-morphisms for polarity-based frames.

In Section 6.1, we specialise the definition of p-morphisms for enriched polarity-based frames in the setting of general LE-logics given in [17] (Definition 6.1) to models[3] in the signature of DML. Restricting our scope to the class of enriched polarity-based frames that are liftings of Kripke frames, we state and prove relational results for p-morphisms between liftings.

In an analogous fashion to the manner in which liftings of sets, frames, and models were defined in Chapter 5, a map $f : \mathsf{K}_1 \to \mathsf{K}_2$ can be lifted in order to obtain a pair of binary relations, (S_f, T_f) (see Section 6.2.2). In Section 6.2.2 map liftings are defined in addition to the statement and proof of some technical results used throughout the chapter. The rest of the chapter concerns two collections of results for liftings of Kripke frames, outlined below.

[1]For a brief review of some of these developments see [17].

[2]See the axiomatization of compatible semantics in Section 4.5.1, which has axioms for normality (Norm-$\Box$, Norm-$\Diamond$, Norm-$\lhd$, and Norm-$\rhd$), and thus makes the normality of $\Box, \Diamond, \lhd$, and $\rhd$ explicit. This axiomatization is known to be sound with respect to the class of compatible polarity-based frames, as noted in [12].

[3]The definition in [17] is for frames.

First, Section 6.2.3 establishes that lifting a p-morphisms between Kripke models results in a p-morphism between the corresponding lifted models (Theorem 6.19); and that, additionally, the procedure of lifting preserves surjectivity (Proposition 6.20) and injectivity Proposition 6.21.

The second collection of results concerns the converse of Theorem 6.19 and is the topic of Section 6.3. The converse of Theorem 6.19, broadly stated, is that p-morphisms between liftings of Kripke frames induce a p-morphism between their underlying Kripke frames (Theorem 6.44). The proof of Theorem 6.44 requires the definition of "untyping" sets and relations in Section 6.3.1.

6.1 p-morphisms for LE-Logics

In [17], a definition of p-morphisms for general LE-logics is given. In this section, we begin by specializing this general definition of p-morphisms for LE-logics given in [17] to our polarity-based setting in Definition 6.1. Clauses (p1)–(p5) are adapted directly from [17]. As the focus of this chapter will be on truth preservation between liftings of Kripke frames, all enriched polarities and associated models will be assumed compatible unless otherwise indicated.

In this section, if left unspecified, $\mathcal{M}$, will refer to an arbitrary compatible enriched polarity-based model, $\mathcal{M} = ((\mathbb{P}, \{R_\Box, R_\Diamond, R_\lhd, R_\rhd\}), V)$ where $\mathbb{P} = (A_1, X, I)$. We will also refer to indexed versions of polarity-based frames and models using the conventions: $\mathcal{M}_i = ((\mathbb{P}_i, \mathcal{R}^i_{\mathcal{H}}), V_i)$ where $\mathbb{P}_i = \left(A^i, X^i, I^i\right)$ and $i = 1, 2$.

Our considerations of truth in addition to validity requires the introduction of a clause similar in nature to the local harmony clause – clause (p6). As [17]'s approach is frame-centric, it does not include a clause of this kind. As will be shown later in this chapter, the addition of (p6) is sufficient to ensure the preservation of satisfaction between polarity-based models that are liftings of Kripke frames (see Corollary 6.51).

Definition 6.1 (p-morphism between polarity-based models)**.** A p-morphism between compatible polarity-based models, $\mathcal{M}_1 = ((\mathbb{P}_1, \mathcal{R}^1_{\mathcal{H}}), V_1)$ and $\mathcal{M}_2 = ((\mathbb{P}_2, \mathcal{R}^2_{\mathcal{H}}), V_2))$ is a pair of binary relations $(S, T) : \mathcal{M}_1 \to \mathcal{M}_2$[4] such that[5]:

p1. $S \subseteq A^1 \times X^2$ and $T \subseteq X^1 \times A^2$.

p2. $S^{(1)}(a^1), T^{(1)}(x^1), S^{(0)}(x^2)$ and $T^{(0)}(a^2)$ are Galois stable for all $(a^1, a^2, x^1, x^2) \in A^1 \times A^2 \times X^1 \times X^2$.

p3. $\left(T^{(0)}(a^2)\right)^{\downarrow} \subseteq S^{(0)}\left((a^2)^{\uparrow}\right)$ for each $a^2 \in A^2$

p4. $T^{(0)}\left[\left(S^{(1)}(a^1)\right)^{\downarrow}\right] \subseteq (a^1)^{\uparrow}$ for each $a^1 \in A^1$.

p5. $(S^{\epsilon_h})^{(0)}\left[\left((R^2_h)^{(0)}(w^2_{\alpha_h})\right)^{\uparrow^{\epsilon_h}}\right] = \left(R^1_h\right)^{(0)}\left[\left((T^{\alpha_h})^{(0)}(w^2_{\alpha_h})\right)^{\downarrow^{\alpha_h}}\right]$ for each $w^2_{\epsilon_h} \in W^2_{\epsilon_h}$ and $R^1_h \in \mathcal{R}^1_{\mathcal{H}}$ and $R^2_h \in \mathcal{R}^2_{\mathcal{H}}$.

p6. The (p6) clause, analogous to local harmony in the classical context, composed of the following sub-clauses:

p6a. $(\![p]\!)_{\mathcal{M}_2} \subseteq S^{(1)}([\![p]\!]_{\mathcal{M}_1})$ for every $p \in \mathsf{Prop}$.

p6b. $(\![p]\!)_{\mathcal{M}_1} \subseteq T^{(0)}\left([\![p]\!]_{\mathcal{M}_2}\right)$ for every $p \in \mathsf{Prop}$.

Where ϵ_h and α_h are the order and adjunction types discussed in Section 4.1. We adopt the conventions: $T = T^1 = S^\partial$ and $S = S^1 = T^\partial$, as well as $\uparrow=\uparrow^1=\downarrow^\partial$ and $\downarrow=\downarrow^1=\uparrow^\partial$ where $\partial = \partial^1 = 1^\partial$ and $1 = 1^1 = \partial^\partial$.

[4] We adopt the abuse of notation used in [17] notation, using the $\to$ arrow notation to illustrate that the binary relations, (S, T), play a similar role to p-morphisms in the classical context, despite being a pair of binary relations and not maps from $\mathcal{M}_1$ to $\mathcal{M}_2$ as the notation would suggest.

[5] Where $\mathcal{R}_{\mathcal{H}} = \{R_h \subseteq W_{\epsilon_h} \times W_{\alpha_h} \mid h \in \mathcal{H}\}$ is a collection of binary relations associated with the signature $\mathcal{H}$.

We now specialise (p5) further to the signature of DML.

Remark 6.2. Clause (p5) can be unpacked into subclauses (p5.1)–(p5.4), which are instantiated with the four types of relations corresponding to the connectives given in Table 4.1.

p5.1. $S^{(0)}\left[\left((R^2_\Box)^{(0)}(x^2)\right)^\uparrow\right] = \left(R^1_\Box\right)^{(0)}\left[\left(S^{(0)}\left(x^2\right)\right)^\uparrow\right]$ for each $x^2 \in X^2$ and $R^1_\Box \subseteq A^1 \times X^1$ and $R^2_\Box \subseteq A^2 \times X^2$.

p5.2. $T^{(0)}\left[\left((R^2_\Diamond)^{(0)}(a^2)\right)^\downarrow\right] = \left(R^1_\Diamond\right)^{(0)}\left[\left(T^{(0)}(a^2)\right)^\downarrow\right]$ for each $a^2 \in A^2$ and $R^1_\Diamond \subseteq X^1 \times A^1$ and $R^2_\Diamond \subseteq X^2 \times A^2$.

p5.3. $S^{(0)}\left[\left((R^2_\triangleright)^{(0)}(a^2)\right)^\uparrow\right] = \left(R^1_\triangleright\right)^{(0)}\left[\left(T^{(0)}\left(a^2\right)\right)^\downarrow\right]$ for each $a^2 \in A^2$ and $R^1_\triangleright \subseteq A^1 \times A^1$ and $R^2_\triangleright \subseteq A^2 \times A^2$.

p5.4. $T^{(0)}\left[\left((R^2_\triangleleft)^{(0)}(x^2)\right)^\downarrow\right] = \left(R^1_\triangleleft\right)^{(0)}\left[\left(S^{(0)}(x^2)\right)^\uparrow\right]$ for each $x^2 \in X^2$ and $R^1_\triangleleft \subseteq X^1 \times X^1$ and $R^2_\triangleleft \subseteq X^2 \times X^2$.

Definition 6.3 (p-morphism between compatible enriched polarities [17])**.** A p-morphism between compatible enriched polarities $\mathbb{F}_1 = (\mathbb{P}_1, \mathcal{R}_1)$ and $\mathbb{F}_2 = (\mathbb{P}_2, \mathcal{R}_2)$ is a pair of binary relations, $(S,T) : \mathbb{F}_1 \to \mathbb{F}_2$, satisfying (p1)–(p5).

Similarly, a pair of binary relations, $(S,T) : \mathcal{M}_1 \to \mathcal{M}_2$, satisfying (p1)–(p6) is a p-morphism between polarity-based models.

Remark 6.4. As $\mathcal{M}_1$ and $\mathcal{M}_2$ are assumed compatible, clause (p2) implies that $S^{(1)}(U^1), T^{(1)}(V^1), S^{(0)}(V^2)$ and $T^{(0)}(U^2)$ are Galois stable for all $(U^1, U^2, V^1, V^2) \subseteq A^1 \times A^2 \times X^1 \times X^2$ by Proposition 4.50.

Definition 6.5 (Surjective p-morphism between compatible enriched polarities [17])**.** A p-morphism, $(S,T) : \mathbb{F}_1 \to \mathbb{F}_2$, is *surjective* if for each $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{P}_2^+$ we have $\boldsymbol{a} \neq \boldsymbol{b}$ implies $S^{(0)}(([\boldsymbol{a}])) \neq S^{(0)}(([\boldsymbol{b}]))$ (or equivalently $T^{(0)}([\![\boldsymbol{a}]\!]) \neq T^{(0)}([\![\boldsymbol{b}]\!])$).

If (S,T) is a surjection we write $(S,T) : \mathbb{F}_1 \twoheadrightarrow \mathbb{F}_2$, and refer to $\mathbb{F}_2$ as the *p-morphic image* of $\mathbb{F}_1$.

Definition 6.6 (Injective p-morphism between compatible enriched polarities [17])**.** A p-morphism,$(S,T) : \mathbb{F}_1 \to \mathbb{F}_2$, is *injective* if for every concept $\boldsymbol{a} \in \mathbb{P}^+$ there exists a $\boldsymbol{b} \in \mathbb{P}_2^+$ such that $S^{(0)}\left(([\boldsymbol{b}])\right) = [\![\boldsymbol{a}]\!]$ (or equivalently $T^{(0)}\left([\![\boldsymbol{b}]\!]\right) = ([\boldsymbol{a}])$).

Similarly, a surjective/injective p-morphism between polarity-based models is a pair of binary relations, $(S,T) : \mathcal{M}_1 \to \mathcal{M}_2$, satisfying (p1)–(p6) as well as Definition 6.6 in the case of injectivity and Definition 6.5 for surjectivity.

It may appear counter intuitive that p-morphisms of the kind defined in Definition 6.5 are referred to as "surjective" and those in Definition 6.6 as "injective" when they seem to more closely resemble the definitions of injectivity and surjectivity for maps respectively. This discrepancy arises from duality considerations discussed in further detail in [17].

Lemma 6.7. For $\boldsymbol{a} \in (\mathbb{F})^+$ we have $\left(T^{(0)}\left([\![\boldsymbol{a}]\!]\right)\right)^\downarrow = S^{(0)}\left(([\boldsymbol{a}])\right)$.

Proof. See [17, Lemma 21]. ⊣

A specialised corollary to Lemma 6.7 for the liftings of Kripke frames is given in Lemma 6.25.

6.2 p-morphisms for liftings of Kripke frames

The remainder of the chapter focuses on p-morphisms between liftings of Kripke frames. Unless otherwise specified, we let $\mathsf{K} = (W, R)$ be an arbitrary Kripke frame, $\mathcal{M} = (\mathsf{K}, V)$ a model based on K with valuation V, and following the conventions of Chapter 5, $\mathcal{N}$ will refer to the lifting of Kripke model $\mathcal{M}$ in the sense of Definition 5.13. Additionally, we will often refer to indexed versions of these objects in the following ways:

$$\mathsf{K}_i = \left(W^i, R_i\right) \text{ and } \mathcal{M}_i = (\mathsf{K}_i, V_i) \text{ for } i = 1, 2.$$

Polarities corresponding to the liftings of W^1 and W^2 are $\mathbb{P}_i = \left(W_A^i, W_X^i, I_{\Delta_i^c}\right)$ along with enriched polarities and polarity-based models (see Definition 5.7 and Definition 5.13 respectively)

$$\mathsf{P}_i = \left(\mathbb{P}_i, \left\{I_{R_i^c}, J_{R_i^c}, H_{R_i^c}, K_{R_i^c}\right\}\right) \text{ and } \mathcal{N}_i = (\mathsf{P}_i, V_i') \text{ for } i = 1, 2.$$

6.2.1 p-morphisms between Kripke frames

For convenience, we recall some well known truth preservation results and definitions from classical modal logic. Note that any references to modal formulae in Definition 6.8 Definition 6.12 will refer to formulae in $\mathcal{L}_P$ in our case, but refer to the basic modal language in the sources referenced.

Definition 6.8 (p-morphism between Kripke models [5])**.** A mapping $f : \mathcal{M}_1 \to \mathcal{M}_1$ is a *p-morphism*[6] if it satisfies the following conditions.

(b1) The same proposition letters are satisfied by w and $f(w)$.

(b2) If wR_1v then $f(w)R_2f(v)$ for each $w, v \in W^1$.

(b3) If $f(w)R_2v_2$ then there exists a $v \in W^1$ such that wR_1v and $f(v) = v_2$.

A p-morphism between Kripke frames is a map satisfying both of the relational clauses of Definition 6.8.

Definition 6.9 (p-morphism between Kripke frames)**.** A map, $f : \mathsf{K}_1 \to \mathsf{K}_2$, is a p-morphism if it satisfies (b2) and (b3) of Definition 6.8.

The next theorem may be proved by induction on formula, as in [5].

Theorem 6.10 (Modal satisfaction is invariant under p-morphisms [5])**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Kripke models and $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a p-morphism between them. Then for any $w \in \mathcal{M}_1$ and modal formula φ:

$$\mathcal{M}_1, w \Vdash \varphi \text{ iff } \mathcal{M}_2, f(w) \Vdash \varphi.$$

Definition 6.11 (p-morphic image of Kripke models [5])**.** If a p-morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ is surjective (for each $v_2 \in W^2$ there is a $v_1 \in W^1$ such that $v_2 = f(v_1)$) then $\mathcal{M}_2$ is said to be the *p-morphic image* of $\mathcal{M}_1$, denoted $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$.

Definition 6.12 (Injective p-morphism between Kripke models [5])**.** If a p-morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ is such that $f\left(w^1\right) = f\left(w^2\right)$ implies $w_1 = w_2$ for every $w^1, w^2 \in W^1$ then f is an *injective p-morphism between $\mathcal{M}_1$ and $\mathcal{M}_2$*.

6.2.2 Lifting maps between Kripke frames

In this section we define liftings of maps between Kripke frames (Definition 6.13), and, in Example 6.14, present an example in which the lifting of a p-morphism between two Kripke models is a p-morphism between their respective liftings. In the next section we will show that this result holds in general: That

[6] The terms "p-morphism" and "bounded morphism" refer to the same object. As noted in [5], bounded morphism is a more modern term. We, however, use the term p-morphism for the sake of consistency with recent publications in NDML.

a lifting of a p-morphism between Kripke models is always a p-morphism between their respective lifted models (Theorem 6.19). In the remainder of this subsection we state and prove several technical results, which will be used in the proof of Theorem 6.19, and related results.

Definition 6.13 (Liftings of maps between Kripke frames)**.** The *lifting of a map* $f : W^1 \to W^2$ is a pair of relations $S_f \subseteq W^1_A \times W^2_X$ and $T_f \subseteq W^1_X \times W^2_A$ given by

$$S_f = \left\{(a^1_A, x^2_X) \in W^1_A \times W^2_X \mid f\left(a^1\right) \neq x^2\right\} \text{ and } T_f = \left\{(x^1_X, a^2_A) \in W^1_X \times W^2_A \mid f\left(x^1\right) \neq a^2\right\}.$$

That is

$$a^1_A S_f x^2_X \text{ iff } f\left(a^1\right) \neq x^2 \text{ and } a^1_X T_f x^2_A \text{ iff } f\left(a^1\right) \neq x^2.$$

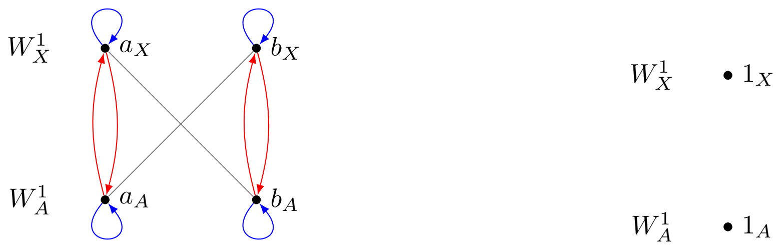
Note that S_f and T_f can be written in the equivalent forms $S_f = I_{Z^c}$ and $T_f = J_{Z^c}$ where $Z = \left\{(w, f(w)) \mid w \in W^2\right\}$.

Lifting a map in this way preserves several properties of the original map, $f : \mathcal{M}_1 \to \mathcal{M}_2$, when f is a p-morphism between Kripke models. Crucially, if f is a p-morphism then (S_f, T_f) is a p-morphism in the sense of Definition 6.1 (Theorem 6.19). Additionally lifting preserves surjectivity and injectivity (Theorem 6.19 and Proposition 6.21).

Example 6.14 (Lifting of a p-morphism between Kripke frames)**.** In this example we consider a classical p-morphism between two Kripke models, $\mathcal{M}_1$ and $\mathcal{M}_2$, from Section 5.1.2 and Chapter 8 – Example 5.8 and Example 8.1 respectively. The Kripke frames from these examples are augmented with the valuations $V_1(p) = \{a, b\}$ and $V_2(p) = \{1\}$ respectively. The labels of states in these frames have been altered from Example 5.8 and Example 8.1 for the sake of clarity.

(a) $\mathcal{M}_1 = ((W^1, R_1), V_1)$

(b) $\mathcal{M}_2 = ((W^2, R_2), V_2)$



(a) $\mathcal{N}_1 = (((W^1_A, W^1_X, I_{\Delta^c_1}), I_{R^c_1}, J_{R^c_1}), V'_1)$

(b) $\mathcal{N}_2 = (((W^2_A, W^2_X, I_{\Delta^c_2}), I_{R^c_2}, J_{R^c_2}), V'_2)$

The mapping $f : \mathcal{M}_1 \to \mathcal{M}_2$ where $f(w) = 1$ for all $w \in W^1$ is a p-morphism between K_1 and K_2 (when a, b and 1 satisfy the same proposition letters).

Let $Z \subseteq W^1 \times W^2$ with $Z = \left\{(w, f(w)) \mid w \in W^2\right\} = \{(a, 1), (b, 1)\}$. We verify that relations $S \subseteq W^1_A \times W^2_X$ and $T \subseteq W^1_X \times W^2_A$ given by $S = I_{Z^c} = \varnothing$ and $T = J_{Z^c} = \varnothing$ constitute a p-morphism, $(S, T) : \mathcal{N}_1 \to \mathcal{N}_2$ in the sense of Definition 6.1.

(p1) is satisfied by definition of S and T.

(p2) is satisfied as $S^{(1)}(a_A) = S^{(1)}(b_A) = S^{(0)}(1_x) = T^{(0)}(1_A) = T^{(1)}(a_x) = T^{(1)}(b_x) = \varnothing$, which is Galois stable.

(p3) is satisfied as $\left(T^{(0)}(1_A)\right)^{\downarrow} = \left(W^1_X\right)^{\downarrow} = \varnothing \subseteq W^1_A = S^{(0)}(\varnothing) = S^{(0)}(1^{\uparrow}_A)$.

(p4) requires that $T^{(0)}\left[\left(S^{(1)}(a_1)\right)^{\downarrow}\right] \subseteq a_1^{\uparrow}$ for each $a_1 \in W^1_A$. Note that $S^{(1)}(a_A) = S^{(1)}(b_A) = \varnothing$. Thus

$$T^{(0)}\left[\left(S^{(1)}(a_A)\right)^{\downarrow}\right] = T^{(0)}\left[\left(S^{(1)}(b_A)\right)^{\downarrow}\right] = T^{(0)}\left[(\varnothing)^{\downarrow}\right] = T^{(0)}\left[\{1_A\}\right] = \varnothing.$$

And as $\varnothing \subseteq a_A^{\uparrow} = \{b_X\}$ and $\varnothing \subseteq b_A^{\uparrow} = \{a_X\}$, (S,T) has (p4).

For (p5) it is sufficient to show that (p5.1) is satisfied – see Remark 6.31. Noting that $R^2_{\Box} = R^2_{\Diamond} = R^2_{\lhd} = R^2_{\rhd} = \varnothing$:

For (p5.1)

$$\begin{aligned} S^{(0)}\left[\left((R^2_{\Box})^{(0)}(1_X)\right)^{\uparrow}\right] &= S^{(0)}\left[(\varnothing)^{\uparrow}\right] \\ &= S^{(0)}\left[1_X\right] \\ &= \varnothing \end{aligned}$$

and

$$\begin{aligned} \left(R^1_{\Box}\right)^{(0)}\left[\left(S^{(0)}\left(1_X\right)\right)^{\uparrow}\right] &= \left(R^1_{\Box}\right)^{(0)}\left[(\varnothing)^{\uparrow}\right] \\ &= \left(R^1_{\Box}\right)^{(0)}\left[\{a_X, b_X\}\right] \\ &= \varnothing \end{aligned}$$

For (p6a), $V_1(p) = \{a, b\}$ and $V_2(p) = \{1\}$. By Definition 5.13:

$$[\![V_1'(p)]\!] = [\![p]\!]_{\mathcal{M}_{\mathsf{P}_1}} = \{a_A, b_A\} \text{ and } [\![V_2'(p)]\!] = [\![p]\!]_{\mathcal{M}_{\mathsf{P}_2}} = \{1_A\}.$$

Thus $(\!|p|\!)_{\mathcal{M}_{\mathsf{P}_2}} = \{1_A\}^{\uparrow} = \varnothing = S^{(1)}(\{a_A, b_A\}) = S^{(1)}([\![p]\!]_{\mathcal{M}_{\mathsf{P}_1}})$, and (p6a) is satisfied.

For (p6b) we have $(\!|p|\!)_{\mathcal{N}_1} = \{a_A, b_A\}^{\uparrow} = \varnothing$, and so $(\!|p|\!)_{\mathcal{N}_1} \subseteq T^{(0)}\left([\![p]\!]_{\mathcal{N}_2}\right)$ vacuously.

Thus (p6) is satisfied, and $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ a p-morphism in the sense of Definition 6.1.

The remainder of this section is devoted to the statement and proof of several technical results for liftings of maps between sets. The three equalities listed in Lemma 6.15 are leveraged at several points throughout this chapter.

Lemma 6.15. Let $S_f \subseteq W^1_A \times W^2_X$ and $T_f \subseteq W^1_X \times W^2_A$ be liftings of a map $f : W^1 \to W^2$ and $U_X \subseteq W^2_X$, $U_A \subseteq W^2_A$, $V_A \subseteq W^1_A$, and $V_X \subseteq W^1_X$ then:

(a) $S_f^{(0)}(U_X) = \left(\left(f^{-1}\left(U\right)\right)^c\right)_A$

(b) $T_f^{(0)}(U_A) = \left(\left(f^{-1}\left(U\right)\right)^c\right)_X$

(c) $S_f^{(1)}(V_A) = \left(\left(f\left(V\right)\right)^c\right)_X$

(d) $T_f^{(1)}(V_X) = \left(\left(f\left(V\right)\right)^c\right)_A$

Proof. For (a)

$$\begin{aligned} S_f^{(0)}(U_X) &= \left\{a^1_A \in W^1_A \mid \forall x^2_X \in W^2_X \left(x^2_x \in U_X \to a^1_A S_f x^2_X\right)\right\} \\ &= \left\{a^1_A \in W^1_A \mid \forall x^2_X \in U_X \left(a^1_A S_f x^2_X\right)\right\} \\ &= \left\{a^1_A \in W^1_A \mid \forall x^2 \in U\left(f(a) \neq x\right)\right\} \\ &= \left(\left\{a^1 \in W^1 \mid \forall x^2 \in U\left(f(a) \neq x\right)\right\}\right)_A \\ &= \left(f^{-1}\left(U^c\right)\right)_A \\ &= \left(\left(f^{-1}\left(U\right)\right)^c\right)_A. \end{aligned}$$

For (b)

$$\begin{aligned}
T^{(0)}(U_A) &= \left\{x^1_X \in W^1_X \mid \forall a^2_A \in U_A(x^1_X T_f a^2_X)\right\} \\
&= \left\{x^1_X \in W^1_X \mid \forall a^2 \in U(f(x) \neq a^2)\right\} \\
&= \left(\left\{x^1 \in W^1 \mid \forall a^2 \in U(f(x) \neq a^2)\right\}\right)_X \\
&= \left(\left(f^{-1}(U)\right)^c\right)_X .
\end{aligned}$$

For (c)

$$\begin{aligned}
S^{(1)}_f(V_A) &= \left\{x^2_X \in W^2_X \mid \forall a^1_A \in V_A \left(a^1_A S_f x^2_X\right)\right\} \\
&= \left\{x^2_X \in W^2_X \mid \forall a^1_A \in V_A \left(f(a) \neq x\right)\right\} \\
&= \left(\left\{x^2 \in W^2 \mid \forall a^1 \in V \left(f(a) \neq x\right)\right\}\right)_X \\
&= \left(\left(f\left(V\right)\right)^c\right)_X
\end{aligned}$$

For (d)

$$\begin{aligned}
T^{(1)}_f(V_X) &= \left\{a^2_A \in W^2_A \mid \forall x^1_X \in V_X \left(x^1_X T_f a^2_A\right)\right\} \\
&= \left\{a^2_A \in W^2_A \mid \forall x^1_X \in V_X \left(f(x) \neq a\right)\right\} \\
&= \left(\left\{a^2 \in W^2 \mid \forall x^1 \in V \left(f(x) \neq a\right)\right\}\right)_A \\
&= \left(\left(f\left(V\right)\right)^c\right)_A
\end{aligned}$$

⊣

Lemma 6.16(a) and Lemma 6.16(b) imply immediately that liftings of p-morphisms satisfy (p3) and (p4) of Definition 6.1.

Lemma 6.16. If $S_f \subseteq W^1_A \times W^2_X$ and $T_f \subseteq W^1_X \times W^2_A$ are liftings of a map $f : W^1 \to W^2$ then

(a) $\left(T^{(0)}_f\left(a^2_A\right)\right)^{\downarrow} = S^{(0)}_f\left({a^2_A}^{\uparrow}\right)$ for any $a^2_A \in W^2_A$.

(b) $T^{(0)}_f\left[\left(S^{(1)}_f(a_1)\right)^{\downarrow}\right] \subseteq \left\{a^1_A\right\}^{\uparrow}$ for any $a^1_A \in W^1_A$.

Proof. For (a) let $a^2_A \in W^2_A$ then

$$\begin{aligned}
\left(T^{(0)}_f\left(a^2_A\right)\right)^{\downarrow} &= \left(\left(\left(f^{-1}\left(\left\{a^2\right\}\right)\right)^c\right)_X\right)^{\downarrow} && \text{(Lemma 6.15(b))} \\
&= \left(\left(\left(f^{-1}\left(\left\{a^2\right\}\right)\right)^c\right)^c\right)_A && \text{(Proposition 5.2(d))} \\
&= \left(f^{-1}\left(\left\{a^2\right\}\right)\right)_A
\end{aligned}$$

and

$$\begin{aligned}
S^{(0)}_f\left({a^2_A}^{\uparrow}\right) &= S^{(0)}_f\left(\left(\left\{a^2\right\}^c\right)_X\right) && \text{(Proposition 5.2(c))} \\
&= \left(\left(f^{-1}\left(\left\{a^2\right\}^c\right)\right)^c\right)_A && \text{(Lemma 6.15(a))} \\
&= \left(\left(f^{-1}\left(\left\{a^2\right\}\right)^c\right)^c\right)_A && \text{(as } f^{-1}\left(V^c\right) = \left(f^{-1}\left(V\right)\right)^c \text{ always)} \\
&= \left(f^{-1}\left(\left\{a^2\right\}\right)\right)_A \\
&= \left(T^{(0)}_f\left(a^2_A\right)\right)^{\downarrow}. && \text{(Proposition 5.2(d) and Lemma 6.15(b))}
\end{aligned}$$

For (b)

$$\begin{aligned}
T_f^{(0)}\left[\left(S_f^{(1)}(a_1)\right)^{\downarrow}\right] &= T_f^{(0)}\left[\left(\left(\left(f\left(\{a^1\}\right)\right)^c\right)_X\right)^{\downarrow}\right] && \text{(Lemma 6.15(c))}\\
&= T_f^{(0)}\left[\left(\left(\left(f\left(\{a^1\}\right)\right)^c\right)^c\right)_A\right] && \text{(Proposition 5.2(d))}\\
&= T_f^{(0)}\left[\left(f\left(\{a^1\}\right)\right)_A\right]\\
&= \left(\left(f^{-1}\left(f\left(\{a^1\}\right)\right)\right)^c\right)_X && \text{(Lemma 6.15(b))}\\
&= \left(\left(f^{-1}\left(f\left(\{a^1\}\right)\right)\right)_A\right)^{\uparrow} && \text{(Proposition 5.2(c))}
\end{aligned}$$

Noting that for an arbitrary $f : X \to Y$ and $U \subseteq X$ that $U \subseteq f^{-1}(f(U))$, we have $\left\{a^1\right\} \subseteq f^{-1}\left(f\left(\{a^1\}\right)\right)$. Thus $\left\{a_A^1\right\} \subseteq f^{-1}\left(f\left(\{a^1\}\right)\right)_A$. Therefore

$$T_f^{(0)}\left[\left(S_f^{(1)}(a_1)\right)^{\downarrow}\right] = \left(\left(f^{-1}\left(f\left(\{a^1\}\right)\right)\right)_A\right)^{\uparrow} \subseteq \left\{a_A^1\right\}^{\uparrow}. \qquad \text{(Corollary 3.6(b))}$$

⊣

Remark 6.17. If $\mathbb{F}$ is a lifting of Kripke frame $\mathsf{K} = (W, R)$ then $(R_\Box)^{(0)}(x_X) = \left(R^{-1}[x]\right)^c_A$ where $R_\Box = I_{R^c}$.

Proof. First noting that $R^{-1}[x] = \{w \in W \mid wRx\}$ we have

$$\begin{aligned}
(R)^{(0)}(x_X) &= \{a_A \in W_A \mid a_A R_\Box x_X\}\\
&= \{a_A \in W_A \mid a_A I_{R^c} x_X\}\\
&= \{a_A \in W_A \mid \neg aRx\}\\
&= \left(\left(R^{-1}[x]\right)^c\right)_A
\end{aligned}$$

⊣

6.2.3 Lifted p-morphisms are p-morphisms

In this subsection we show that the lifting, $(S_f, T_f) : \mathcal{N}_1 \to \mathcal{N}_2$, of a p-morphism between Kripke frames, $f : \mathcal{M}_1 \to \mathcal{M}_2$, is a p-morphism in the sense of Definition 6.1. Moreover, in Proposition 6.20 and Proposition 6.21 the operation of lifting p-morphisms between underlying Kripke frames is shown to preserves injectivity and surjectivity respectively; that is if $\mathcal{M}_2$ is the p-morphic image of $\mathcal{M}_1$ under f in the sense of Definition 6.11) then $\mathcal{N}_2$ is the p-morphic image of $\mathcal{N}_1$ under (S_f, T_f) in the sense of Definition 6.6, and if f is injective in the sense of Definition 6.12 then (S_f, T_f) is an injective p-morphism in the sense of Definition 6.6.

If left unspecified, $f : \mathsf{K}_1 \to \mathsf{K}_2$ will refer to an arbitrary p-morphism between Kripke frames, and (S_f, T_f) to the lifting of f in the sense of Definition 6.13.

Lemma 6.18. If $f : \mathsf{K}_1 \to \mathsf{K}_2$ is a p-morphism then $f^{-1}\left(R_2^{-1}[w^2]\right) = R_1^{-1}\left[f^{-1}(w^2)\right]$ for each $w^2 \in W^2$.

Proof. Note that expanding $f^{-1}\left(R_2^{-1}[w^2]\right)$ and $R_1^{-1}\left[f^{-1}(w^2)\right]$ as in Definition 2.3 gives

$$\begin{aligned}
f^{-1}(R_2^{-1}\left[w^2\right]) &= f^{-1}\left(\left\{v^2 \in W^2 \mid v^2 R_2 w^2\right\}\right)\\
&= \left\{w^1 \in W^1 \mid f\left(w^1\right) \in \left\{v^2 \in W^2 \mid v^2 R_2 w^2\right\}\right\}\\
&= \left\{w^1 \in W^1 \mid f\left(w^1\right) R_2 w^2\right\}
\end{aligned}$$

and

$$\begin{aligned}
R_1^{-1}\left[f^{-1}\left(w^2\right)\right] &= R_1^{-1}\left[\left\{y^1 \in W^1 \mid f\left(y^1\right) = w^2\right\}\right]\\
&= \left\{w^1 \in W^1 \mid \left(\exists v^1 \in f^{-1}\left(w^1\right)\right)\left(w^1 R_1 v^1\right)\right\}\\
&= \left\{w^1 \in W^1 \mid \exists v^1 \in W^1 (w^1 R_1 v^1 \text{ and } v^1 \in f^{-1}\left(w^2\right)\right\}\\
&= \left\{w^1 \in W^1 \mid \exists v^1 \in W^1 (w^1 R_1 v^1 \text{ and } f(v^1) = w^2\right\}.
\end{aligned}$$

First, suppose that $w^1 \in f^{-1}\left(R_2^{-1}[w^2]\right)$. Then $f\left(w^1\right) R_2 w^2$. By (b3) of Definition 6.8, there exists a $v^1 \in W^1$such that $w^1 R_1 v^1$ and $f\left(v^1\right) = w^2$, and thus $w^1 \in R_1^{-1}\left[f^{-1}\left(w^2\right)\right]$.

For the remaining inclusion suppose that $w^1 \in R_1^{-1}\left[f^{-1}\left(w^2\right)\right]$, then there exists a $v^1 \in W^1$ such that $w^1 R_1 v^1$ and $f\left(v^1\right) = w^2$. By (b2) of Definition 6.8 we have $f\left(w^1\right) R_2 f\left(v^1\right)$ and as $f\left(v^1\right) = w^2$ we have $f\left(w^1\right) R_2 w^2$ and $w^1 \in f^{-1}\left(R_2^{-1}[w^2]\right)$. Thus

$$w^1 \in f^{-1}\left(R_2^{-1}[w^2]\right) \text{ if and only if } w^1 \in R_1^{-1}\left[f^{-1}\left(w^2\right)\right].$$

And so $f^{-1}\left(R_2^{-1}[w^2]\right) = R_1^{-1}\left[f^{-1}(w^2)\right]$. ⊣

Lemma 6.18 is a result leveraged at several points throughout the remainder of this chapter.

Theorem 6.19 (Liftings of p-morphisms are p-morphisms)**.** If $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism in the sense of Definition 6.8 then $(S_f, T_f) : \mathcal{N}_1 \to \mathcal{N}_2$ is a p-morphism in the sense of Definition 6.1.

Proof. Suppose $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism in the sense of Definition 6.8. We show that $(S_f, T_f) : \mathcal{N}_1 \to \mathcal{N}_2$ constitutes a p-morphism in the sense of Definition 6.1 by verifying (p1)–(p6).

Condition (p1) is satisfied by definition of S_f and T_f.

Condition (p2) is satisfied as all typed subsets of W^1 and W^2 are $I_{\Delta_1^c}$-stable and $I_{\Delta_2^c}$-stable respectively by Corollary 5.3.

Condition (p3) is satisfied by the equality in Lemma 6.16(a).

Condition (p4) is satisfied by Lemma 6.16(b).

For (p5) we show that (p5.1) is satisfied, and note that the remaining sub-clauses, (p5.2)–(p5.4), follow similar arguments. For (p5.1) let $x_X^2 \in W_X^2$. Then

$$\begin{aligned}
\left(R_\Box^1\right)^{(0)}\left[\left(S_f^{(0)}\left(x_X^2\right)\right)^{\uparrow}\right] &= \left(R_\Box^1\right)^{(0)}\left[\left(\left(\left(f^{-1}\left(x^2\right)\right)^c\right)_A\right)^{\uparrow}\right] && \text{(Lemma 6.15(a))}\\
&= \left(R_\Box^1\right)^{(0)}\left[\left(\left(\left(f^{-1}\left(x^2\right)\right)^c\right)^c\right)_X\right] && \text{(Proposition 5.2(c))}\\
&= \left(\left(R_1^{-1}\left[f^{-1}\left(x^2\right)\right]\right)^c\right)_A && \text{(Lemma 6.18)}\\
&= \left(\left(f^{-1}\left(R_2^{-1}\left[x^2\right]\right)\right)^c\right)_A && \text{(Lemma 6.18)}\\
&= S_f^{(0)}\left[\left(\left(\left(R_2^{-1}\left[x^2\right]\right)^c\right)^c\right)_X\right] && \text{(Lemma 6.15(a))}\\
&= S_f^{(0)}\left[\left(\left(\left(R_2^{-1}\left[x^2\right]\right)^c\right)_A\right)^{\uparrow}\right] && \text{(Proposition 5.2(c))}\\
&= S_f^{(0)}\left[\left((R_\Box^2)^{(0)}(x_X^2)\right)^{\uparrow}\right]. && \text{(Remark 6.17)}
\end{aligned}$$

Thus (S_f, T_f) satisfies (p5.1). The arguments showing that (S_f, T_f) satisfies (p5.2)–(p5.4) are similar.

For (p6a) let $p \in \mathsf{Prop}$. Let $b_X \in (\!|p|\!)_{\mathcal{N}_2}$, then $b_A \notin [\![p]\!]_{\mathcal{N}_2}$ by Definition 5.13, and so $b \notin V_2(p)$ by Corollary 5.17. Therefore by (b1) of Definition 6.8, $f(a) \neq b$ for each $a \in V_1(p)$, and thus, $a_A S_f b_X$ for each $a_A \in [\![p]\!]_{\mathcal{N}_1}$ by Definition 6.13. Therefore $b_X \in S_f^{(1)}((\!|p|\!)_{\mathcal{N}_1})$ for arbitrary $b_X \in (\!|p|\!)_{\mathcal{N}_2}$, and thus $(\!|p|\!)_{\mathcal{N}_2} \subseteq S_f^{(1)}([\![p]\!]_{\mathcal{N}_1})$.

Finally, for (p6b), suppose that $a_X^1 \in (\!|p|\!)_{\mathcal{N}_1}$. Then $\mathcal{N}_1, a_X^1 \succ p$, and so $\mathcal{N}_1, a_A^1 \nVdash p$ by Proposition 5.15. Equivalently, $a^1 \notin V_1(p)$ by Definition 5.13; and as f is a p-morphism, and thus satisfies (b1) of Definition 6.8, we have $f(a^1) \notin V_2(p)$ – that is $f\left(a^1\right) \neq y^2$ for each $y^2 \in V_2(p)$. Therefore $a_X^1 \in T_f^{(0)}\left([\![p]\!]_{\mathcal{N}_2}\right)$, and thus $(\!|p|\!)_{\mathcal{N}_1} \subseteq T^{(0)}\left([\![p]\!]\right)_{\mathcal{N}_2}$. ⊣

Preservation of surjectivity (Proposition 6.20) and injectivity (Proposition 6.21) are two natural corollaries of Theorem 6.19.

Proposition 6.20 (Liftings of surjective p-morphisms are surjective)**.** If $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a surjective p-morphism in the sense of Definition 6.11 then $(S, T) : \mathcal{N}_1 \to \mathcal{N}_2$ is a surjective p-morphism in the sense of Definition 6.5

Proof. Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be an onto p-morphism, and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{P}_2^+$. Note that $(\![\boldsymbol{a}]\!) = U_X$ and $(\![\boldsymbol{b}]\!) = V_X$ for some $U, V \subseteq W^2$. Suppose that $S_f^{(0)}((\![\boldsymbol{a}]\!)) = S_f^{(0)}((\![\boldsymbol{b}]\!))$. We need to show that $\boldsymbol{a} = \boldsymbol{b}$ – equivalently $U = V$. Note that

$$\begin{aligned} S_f^{(0)}((\![a]\!)) = S_f^{(0)}((\![b]\!)) &\text{ iff } \left(\left(f^{-1}(U)\right)^c\right)_A = \left(\left(f^{-1}(V)\right)^c\right)_A && \text{(Lemma 6.15(a))} \\ &\text{iff } \left(f^{-1}(U)\right)^c = \left(f^{-1}(V)\right)^c \\ &\text{iff } f^{-1}(U) = f^{-1}(V) \\ &\text{iff } \left\{w \in W^2 \mid f(w) \in U\right\} = \left\{w \in W^2 \mid f(w) \in V\right\}. \end{aligned}$$

Consider $u \in U$. As f is onto there exists a $w_u \in W^1$ such that $f(w_u) = u$. So $w_u \in f^{-1}(U)$. However, $f^{-1}(U) = f^{-1}(V)$, and so $w_u \in f^{-1}(V)$. Thus $f(w) = u \in V$, and so $U \subseteq V$. A similar argument shows that $V \subseteq U$.

Thus $V = U$, and so $U_X = (\![\boldsymbol{a}]\!) = (\![\boldsymbol{b}]\!) = V_X$. Therefore $\boldsymbol{a} = \boldsymbol{b}$. ⊣

Proposition 6.21 (The lifting of an injective p-morphism is injective)**.** If $f : \mathcal{M}_1 \to \mathcal{M}_2$ is an injective p-morphism in the sense of Definition 6.12 then $(S_f, T_f) : \mathcal{N}_1 \to \mathcal{N}_2$ is an injective p-morphism in the sense of Definition 6.6.

Proof. As typed subsets are stable in liftings of Kripke frames by Corollary 5.3, we need to show that for arbitrary $U_A \subseteq W_A^1$ there exists a $V_X \subseteq W_X^2$ such that $S^{(0)}[V_X] = U_A$.

Let $V_X = ((f(U))^c)_X$ then

$$\begin{aligned} S^{(0)}[V_X] &= S^{(0)}[((f(U))^c)_X] \\ &= \left(\left(f^{-1}((f(U))^c)\right)^c\right)_A && \text{(Lemma 6.15(a))} \\ &= \left(\left(\left(f^{-1}(f(U))\right)^c\right)^c\right)_A && \text{(as } f^{-1}(X^c) = \left(f^{-1}(X)\right)^c \text{ always.)} \\ &= U_A && \text{(as } f^{-1}(f((U))) = U \text{ for any injective } f\text{.)} \end{aligned}$$

Thus (S_f, T_f) is injective. ⊣

6.3 From p-morphisms between liftings to p-morphisms between underlying Kripke frames

The converse of Theorem 6.19 holds – that is if there is a p-morphism between liftings of Kripke Frames $\mathcal{N}_1$ and $\mathcal{N}_2$ (where $\mathcal{N}_1$ is the lifting of frame $\mathcal{M}_1$ and $\mathcal{N}_2$ the lifting of $\mathcal{M}_2$) then there exists a p-morphism between $\mathcal{M}_1$ and $\mathcal{M}_2$. The statement and proof of this result (Theorem 6.44), as well as the establishment of necessary notation and requisite technical results, are the focuses of the remainder of this chapter.

6.3.1 Untyping relations and the equivalence of S^* and T^*

As liftings are "typed" via the $(\cdot)_A$ and $(\cdot)_X$ notation of Section 5.1.1, it is first necessary to establish an operation that "untypes" typed elements of W_A and W_X in order to construct maps between liftings of Kripke frames in the typed setting and Kripke frames and their p-morphisms in the untyped setting. Formally shown in this section is an important intuitive property of $(S, T) : \mathcal{N}_1 \to \mathcal{N}_2$ – that S^* and T^* are identical when ignoring typing (Corollary 6.30).

Definition 6.22 (S^* and T^*)**.** For $S \subseteq W^1_A \times W^2_X$ and $T \subseteq W^1_X \times W^2_A$ define $S^* \subseteq W^1 \times W^2$ and $T^* \subseteq W^1 \times W^2$ as $S^* = \left\{\left(w^1, v^2\right) \in W^1 \times W^2 \mid w^1_A S v^2_X\right\}$ and $T^* = \left\{\left(w^1, v^2\right) \in W^1 \times W^2 \mid w^1_X T v^2_A\right\}$.

Remark 6.23. The relations S and S^* are referred to as *typed* and *untyped* relations respectively. Note that $w^1 S^* w^2$ if and only if $w^1_A S w^2_X$ and $w^1 T^* w^2$ if and only if $w^1_X T w^2_A$.

Remark 6.24. If $S \subseteq W^1_A \times W^2_X$ then $S^{(0)}\left(U^2_X\right) = \left(S^{*(0)}\left(U^2\right)\right)_A$ and $S^{(1)}\left(U^1_A\right) = \left(S^{*(1)}\left(U^1\right)\right)_X$.

Similarly, if $T \subseteq W^1_X \times W^2_A$ then $T^{(0)}\left(U^2_A\right) = \left(T^{*(0)}\left(U^2\right)\right)_X$ and $T^{(1)}\left(U^1_X\right) = \left(T^{*(1)}\left(U^1\right)\right)_A$.

Lemma 6.25. If $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then for any $U \subseteq W^2$:

$$S^{*(0)}\left(U^c\right) = \left(T^{*(0)}(U)\right)^c \text{ and } T^{*(0)}\left(U^c\right) = \left(S^{*(0)}(U)\right)^c.$$

Proof. By Lemma 6.7 $\left(T^{(0)}\left([\![\boldsymbol{a}]\!]\right)\right)^{\downarrow} = S^{(0)}\left(([\boldsymbol{a}])\right)$ for any $\boldsymbol{a}_2 \in (\mathbb{P}_2)^+$. Noting that the liftings of arbitrary subsets of W^2 are $I_{\Delta^c_2}$-stable (Corollary 5.3), set $\boldsymbol{a}_2 = \left(U_A, (U^c)_X\right)$. Then

$$\begin{aligned}
\left(\left(\left(T^{*(0)}\left(U\right)\right)_X\right)^c\right)_A &= \left(\left(T^{*(0)}\left(U\right)\right)_X\right)^{\downarrow} && \text{(Proposition 5.2(d))}\\
&= \left(T^{(0)}\left(U_A\right)\right)^{\downarrow} && \text{(Remark 6.23)}\\
&= \left(T^{(0)}\left([\![\boldsymbol{a}]\!]\right)\right)^{\downarrow} &&\\
&= S^{(0)}\left(([\boldsymbol{a}])\right) && \text{(Lemma 6.7)}\\
&= S^{(0)}\left((U^c)_X\right) &&\\
&= \left(S^{*(0)}\left((U^c)\right)\right)_A && \text{(Remark 6.23)}
\end{aligned}$$

Therefore $\left(\left(\left(T^{*(0)}\left(U\right)\right)_X\right)^c\right)_A = \left(S^{*(0)}\left((U^c)\right)\right)_A$ and so $\left(\left(T^{*(0)}\left(U\right)\right)_X\right)^c = S^{*(0)}\left((U^c)\right)$.

Using the equality $\left(\left(T^{*(0)}\left(V\right)\right)_X\right)^c = S^{*(0)}\left((V^c)\right)$ and substituting U^c for U before taking compliments gives the second equality. ⊣

Lemma 6.26. If $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then:

(a) For any $a^2 \in W^2$ we have $(\forall w^1 \in W^1)((w^1, a^2) \notin T^* \to \left(\forall b^2 \in W^2\right)(b^2 \neq a^2 \to \left(w^1, b^2\right) \in S^*)$.

(b) For any $a^1 \in W^1$ there is precisely one $y^2 \in W^2$ such that $(a^1, y^2) \notin S^*$ and $(a^1, y^2) \notin T^*$.

Proof. Let (S, T) be a p-morphism. For (a) let $a^2 \in W^2$. Then p3 becomes

$$\begin{aligned}
&\left(T^{(0)}\left(a^2_A\right)\right)^{\downarrow} \subseteq S^{(0)}\left(\left\{a^2_A\right\}^{\uparrow}\right) &&\\
\text{iff } &\left(\left(T^{(0)}\left(a^2_A\right)\right)^c\right)_A \subseteq S^{(0)}\left(\left(\left\{a^2\right\}^c\right)_X\right) && \text{(Proposition 5.2(c) and (d))}\\
\text{iff } &\left(\left(T^{*(0)}\left\{a^2\right\}\right)^c\right) \subseteq S^{*(0)}\left(\left\{a^2\right\}^c\right) && \text{(Remark 6.24)}\\
\text{iff } &(\forall w^1 \in W^1)(w^1 \notin T^{*(0)}\left(a^2\right) \to w^1 \in \left(S^{*(0)}\left(\left\{a^2\right\}^c\right)\right) && \text{(Definition 3.4)}\\
\text{iff } &(\forall w^1 \in W^1)\left((w^1, a^2) \notin T^* \to \left(\forall b^2 \in W^2\right)\left(b^2 \in \left\{a^2\right\}^c \to w^1 S^* b^2\right)\right). && \text{(Definition 3.4)}\\
\text{iff } &(\forall w^1 \in W^1)((w^1, a^2) \notin T^* \to \left(\forall b^2 \in W^2\right)(b^2 \neq a^2 \to (w^1, b^2) \in S^*). && \text{(Definition 3.4)}
\end{aligned}$$

For (b) let $a^1 \in W^1$. Then (p4) becomes

$$
\begin{aligned}
& T^{(0)}\left[\left(S^{(1)}\left(a_A^1\right)\right)^{\downarrow}\right] \subseteq \left\{a_A^1\right\}^{\uparrow} \\
& \text{iff } T^{*(0)}\left[\left(S^{*(1)}\left(a^1\right)\right)^{c}\right] \subseteq \left\{a^1\right\}^{c} && \text{(Proposition 5.2(c)-(d) and Remark 6.24)} \\
& \text{iff } \left(\forall w^1 \in W^1\right)\left(w^1 \in T^{*(0)}\left[\left(S^{*(1)}\left(a^1\right)\right)^{c}\right] \to w^1 \neq a^1\right) \\
& \text{iff } \left(\forall w^1 \in W^1\right)\left(\left(\forall y^2 \in W^2\right)\left(y^2 \in \left(S^{*(1)}\left(a^1\right)\right)^{c} \to w^1 T^* y^2\right) \to w^1 \neq a^1\right) && \text{(Definition 3.4)} \\
& \text{iff } \left(\forall w^1 \in W^1\right)\left(\left(\forall y^2 \in W^2\right)\left(\left(a^1, y^2\right) \notin S^* \to w^1 T^* y^2\right) \to w^1 \neq a^1\right) \\
& \text{iff } \left(\forall w^1 \in W^1\right)\left(\left(\forall y^2 \in W^2\right)\left(\left(a^1, y^2\right) \in S^* \text{ or } w^1 T^* y^2\right) \to w^1 \neq a^1\right) \\
& \text{iff } \left(\forall w^1 \in W^1\right)\left(w^1 = a^1 \to \left(\exists y^2 \in W^2\right)\left(\left(a^1, y^2\right) \notin S^* \text{ and } \left(w^1, y^2\right) \notin T^*\right)\right) && \text{(contraposition)} \\
& \text{iff } \left(\exists y^2 \in W^2\right)\left(\left(a^1, y^2\right) \notin S^* \text{ and } \left(a^1, y^2\right) \notin T^*\right)
\end{aligned}
$$

Thus there is at least one $y^2 \in W^2$ such that $\left(a^1, y^2\right) \notin S^*$ and $\left(a^1, y^2\right) \notin T^*$.

What remains to be shown is that y^2 is the only such element of W^2. Suppose the contrary: that there is a $z^2 \in W^2$ such that $z^2 \neq y^2$ and $\left(a^1, z^2\right) \notin S^*$ and $\left(a^1, z^2\right) \notin T^*$. As $\left(a^1, y^2\right) \notin T^*$ we have from Lemma 6.26(a) that $\left(a^1, z^2\right) \in S^*$ – a contradiction. Thus there is only one $y^2 \in W^2$ such that $\left(a^1, y^2\right) \notin S^*$ and $\left(a^1, y^2\right) \notin T^*$. ⊣

Lemma 6.26(b) guarantees the existence of a “privileged element”, $y^2 \in W^2$ for each $w^1 \in W^1$ such that y^2 is neither S^*-related nor T^*-related to w^1. At several points throughout the remainder of this section it will be helpful to refer to the privileged $y^2 \in W^2$ given in Lemma 6.26(b) corresponding to a particular $w^1 \in W^1$. For this reason we introduce a function $\theta_{(S,T)} : \mathsf{P}_1 \to \mathsf{P}_2$ in the following definition.

Definition 6.27. If $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then define the map $\theta_{(S,T)} : W^1 \to W^2$ by

$$\theta_{(S,T)}\left(w^1\right) = y^2 \text{ if and only if } \left(w^1, y^2\right) \notin S^* \text{ and } \left(w^1, y^2\right) \notin T^*.$$

The existence of precisely one such $y^2 \in W^2$ (Lemma 6.26(b)) ensures that $\theta_{(S,T)}$ is well-defined as a function. We will write θ instead of $\theta_{(S,T)}$ when the specific (S, T) is clear from context.

Through the series of corollaries (Corollary 6.28–Corollary 6.30) it is shown formally what may appear as an intuitive property of (S, T) throughout this chapter – when untyped, the relations comprising a p-morphism between lifted Kripke models, S and T, are identical (Corollary 6.30).

Corollary 6.28. If $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $S^{*(0)}\left(W^2\right) = T^{*(0)}\left(W^2\right) = \varnothing$.

Proof. We will proceed by showing that $a^1 \notin S^{*(0)}\left(W^2\right)$ for arbitrary $a^1 \in W^1$. As in Definition 6.27 let $\theta\left(a^1\right)$ be the unique element of W^2 such that $a^1 \notin S^{*(0)}\left(\left\{\theta\left(a^1\right)\right\}\right)$. So

$$
\begin{aligned}
S^{*(0)}\left(W^2\right) &= S^{*(0)}\left(\bigcup_{y \in W^2} \{y\}\right) \\
&= S^{*(0)}\left(\left\{\theta\left(a^1\right)\right\} \cup \left(\bigcup_{y \in \{\theta(a^1)\}^c} \{y\}\right)\right) \\
&= S^{*(0)}\left(\left\{\theta\left(a^1\right)\right\}\right) \cap \left(\bigcap_{y \in \{\theta(a^1)\}^c} S^{*(0)}(\{y\})\right). && \text{(Proposition 3.25(b))}
\end{aligned}
$$

Then $a^1 \notin S^{*(0)}\left(\left\{\theta\left(a^1\right)\right\}\right) \cap \left(\bigcap_{y \in \{\theta(a^1)\}^c} S^{*(0)}(\{y\})\right)$ as $a \notin S^{*(0)}\left(\left\{\theta\left(a^1\right)\right\}\right)$. Thus for arbitrary $a^1 \in W^1$ the set $S^{*(0)}\left(W^2\right) \cap W^1$ is empty, and as $S^{*(0)}\left(W^2\right) \subseteq W^1$ we have $S^{*(0)}\left(W^2\right) = \varnothing$.

The argument showing that $T^{*(0)}\left(W^2\right) = \varnothing$ is similar. $\dashv$

Corollary 6.29. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $S^{*(0)}(U) = T^{*(0)}(U)$ for any $U \subseteq W^2$.

Proof. Let $U \subseteq W^2$. For the first inclusion, $S^{*(0)}(U) \subseteq T^{*(0)}(U)$, assume the contrary – that $S^{*(0)}(U) \nsubseteq T^{*(0)}(U)$. Then there is a $w^1 \in W^1$ such that $w^1 \in S^{*(0)}(U)$ and $w^1 \in \left(T^{*(0)}(U)\right)^c$. Therefore, by Lemma 6.25, we have $w^1 \in S^{*(0)}\left(U^c\right)$. But then $w^1 \in S^{*(0)}\left(U^c\right) \cap S^{*(0)}(U)$; and so by Proposition 3.25(b), $w^1 \in S^{*(0)}\left(U \cup U^c\right) = S^{*(0)}\left(W^2\right)$. However, $S^{*(0)}\left(W^2\right)$ is empty by Corollary 6.28 – a contradiction. Therefore $S^{*(0)}(U) \subseteq T^{*(0)}(U)$.

Similarly, for the remaining $T^{*(0)}(U) \subseteq S^{*(0)}(U)$ inclusion suppose the contrary – that $T^{*(0)}(U) \subseteq S^{*(0)}(U)$. Then there is a $w^1 \in W^1$ such that $w^1 \in T^{*(0)}(U)$ and $w^1 \in \left(S^{*(0)}(U)\right)^c$. Therefore, by Lemma 6.25, we have $w^1 \in T^{*(0)}\left(U^c\right)$. But then $w^1 \in T^{*(0)}\left(U^c\right) \cap T^{*(0)}(U)$; and so by Proposition 3.25(b), $w^1 \in T^{*(0)}\left(U \cup U^c\right) = T^{*(0)}\left(W^2\right)$. However, $T^{*(0)}\left(W^2\right)$ is empty by Corollary 6.28 – a contradiction. Therefore $T^{*(0)}(U) \subseteq T^{*(0)}(U)$. $\dashv$

Corollary 6.30. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $S^* = T^*$.

Proof. Suppose that $\left(w^1, w^2\right) \in S^*$ then

$$\begin{aligned} w^1 \in S^{*(0)}\left(\{w^2\}\right) &\text{ iff } w^1 \in T^{*(0)}\left(\{w^2\}\right) && \text{(Corollary 6.29)} \\ &\text{implies } w^1 T^* w^2 \\ &\text{implies } \left(w^1, w^2\right) \in T^*. \end{aligned}$$

Similarly if $\left(w^1, w^2\right) \in T^*$ then

$$\begin{aligned} w^1 \in T^{*(0)}\left(\{w^2\}\right) &\text{ iff } w^1 \in S^{*(0)}\left(\{w^2\}\right) && \text{(Corollary 6.29)} \\ &\text{implies } w^1 S^* w^2 \\ &\text{implies } \left(w^1, w^2\right) \in S^* \end{aligned}$$

Thus $S^* = T^*$. $\dashv$

Using the equality $S^* = T^*$, several of the earlier results of this section can be simplified, as in Corollary 6.33 for Lemma 6.26(a) or Corollary 6.32 for Lemma 6.25.

We restate the observations of Remark 5.6 in terms of liftings of maps for convenience, as the equivalence of (p5.1)–(p5.4) will be used throughout this chapter.

Remark 6.31. If $S_f \subseteq W^1_A \times W^2_X$ and $T_f \subseteq W^1_X \times W^2_A$ are liftings of a map between the Kripke frames, K_1 and K_2, then (p5.1)-(p5.4) are equivalent.

This follows directly from Remark 5.6 and Corollary 6.30.

Corollary 6.32. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $S^{*(0)}\left(U^c\right) = \left(S^{*(0)}(U)\right)^c$ and $T^{*(0)}\left(U^c\right) = \left(T^{*(0)}(U)\right)^c$

Proof. Using Lemma 6.25 and Corollary 6.29

$$S^{*(0)}\left(U^c\right) = \left(T^{*(0)}(U)\right)^c = \left(S^{*(0)}(U)\right)^c = T^{*(0)}\left(U^c\right). \qquad \dashv$$

Corollary 6.33. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then for any $w^1 \in W^1$ and $U \subseteq W^2$ precisely one of the following inclusions holds: $w^1 \in S^{*(0)}(U)$ or $w^1 \in S^{*(0)}(U^c)$.

Proof. Let $w^1 \in W^1$ and $U \subseteq W^2$. First, w^1 is in at least one of $S^{*(0)}(U)$ and $S^{*(0)}(U^c)$ as $w^1 \in S^{*(0)}(U)$ or $w^1 \in \left(S^{*(0)}(U)\right)^c = S^{*(0)}(U^c)$ by Corollary 6.32.

To show that w^1 is in at most one of $S^{*(0)}(U)$ and $S^{*(0)}(U^c)$ suppose the contrary. Then $w^1 \in S^{*(0)}(U) \cap S^{*(0)}(U^c) = S^{*(0)}(U \cup U^c) = S^{*(0)}\left(W^2\right) = \varnothing$ – a contradiction by Corollary 6.28. ⊣

An analogous result to Corollary 6.33 for T^* is obtained by substituting T^* for S^* using Corollary 6.30.

The following remark provides a simplification of Lemma 6.26(a) in light of the equality between S^* and T^* for liftings established in Corollary 6.30.

Remark 6.34. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then for any $a^2 \in W^2$

$$(\forall w^1 \in W^1)((w^1, a^2) \notin S^* \to \left(\forall b^2 \in W^2\right)(b^2 \neq a^2 \to S^*(w^1, b^2)).$$

Proof. Follows directly from the use of Corollary 6.30 to substitute S^* for T^* in Lemma 6.26(a). ⊣

6.3.2 A Kripke frame p-morphism from (S,T)

As remarked at the beginning of this section, the converse of Theorem 6.19 holds in that the existence of a p-morphism between liftings of Kripke frames implies the existence of a p-morphism between the underlying Kripke frames themselves. In order to show this we leverage several results from the more general setting of [17] and [33], as well as duality theoretic results for perfect Boolean algebras with operators in [50] and [28]. To this end we first establish that the lifting of a Kripke frame, P, is a perfect Boolean algebra with operators (Lemma 6.37). In Definition 6.38 a map $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ is defined as in [17, Definition 22] specialised to the setting of liftings of Kripke frames. Then after noting that $h_{(S,T)}$ is a complete and modal BAO homomorphism (Lemma 6.39) we appeal to Theorem 6.41 to show that a map, $H_{(S,T)}$, defined in terms of $h_{(S,T)}$ (Definition 6.43) is indeed a p-morphism between the underlying Kripke frames, K_2 and K_1.

At several points throughout this section we will reference results from [17] and [33], which both refer to "$\mathcal{L}$-frames" and "$\mathcal{L}$-algebras".[7] In the setting of the language considered in this thesis, an $\mathcal{L}$-frame is a compatible enriched polarity $\mathbb{F} = (\mathbb{P}, \{R_\Box, R_\Diamond, R_\rhd, R_\lhd\})$ and a $\mathcal{L}$-algebra its corresponding complex algebra

$$\mathbb{F}^+ = \left(\mathbb{P}^+, \{[R_\Box], \langle R_\Diamond\rangle, [R_\rhd\rangle, \langle R_\lhd]\}\right).$$

These are compatible polarity-based frames and their corresponding complex algebras respectively in our setting.

In this section we will use the notation $\boldsymbol{U}$ to refer to the element, $\boldsymbol{U} = \left(U_A, (U_A)^\uparrow\right)$, of the concept lattice, $(\mathbb{P}_W)^+$ of the lifting of the underlying set, W, of a Kripke frame, $\mathsf{K} = (W,R)$. As every typed subset, $U_A \subseteq W_A$ and $V_X \subseteq W_X$ is I_{Δ^c}-stable (Corollary 5.3), any $U \subseteq W$ has a corresponding concept lattice element, $\boldsymbol{U}$. Similarly, for any $\boldsymbol{U} \in \left(\mathbb{P}_W^+\right)$, we will use U to refer to the untyped version of $[\![\boldsymbol{U}]\!]$. Note this notation will be used not only for sets denoted by capital roman letters, but for the concept lattice element corresponding to any arbitrary subset of W^1, such as in Lemma 6.39(c).

Remark 6.35. The concept lattice of a lifted Kripke frame, $(\mathsf{P})^+$, has greatest element $\boldsymbol{W} = (W_A, \varnothing_X)$ and least element $\boldsymbol{\varnothing} = (\varnothing_A, W_X)$. This follows directly from Corollary 5.3 and Remark 5.4, and the inclusions $\varnothing \subseteq U$ and $U \subseteq W$ for all $U \subseteq W$.

[7]Definitions 6 and 10 of [17] respectively.

Later in this section repeated use is made of the observation that the atoms of the complex algebra of the lifting of a Kripke frame are precisely the concepts corresponding to liftings of singletons (Lemma 6.36).

Lemma 6.36. The atoms of $(\mathsf{P})^+$ are precisely the elements of the form $\boldsymbol{w} = (\{w\}_A, (\{w\}^c)_X)$.

Proof. First, $\boldsymbol{w}$ is atomic, as for any $\boldsymbol{U} \in (\mathsf{P})^+$

$$\boldsymbol{\varnothing} \leq \boldsymbol{U} < \boldsymbol{w} \text{ iff } \varnothing_A \subseteq U_A \subsetneq \{w\}_A .$$

Therefore, as $\varnothing$ is the only proper subset of $\{w\}$, we have $U_A = \varnothing$. Thus $\boldsymbol{U} = \boldsymbol{\varnothing}$, and $\boldsymbol{w}$ is an atom of $(\mathsf{P})^+$.

Next suppose that $\boldsymbol{\varnothing} < \boldsymbol{U}$ is an atom of $(\mathsf{P})^+$. Then for any $\boldsymbol{V} \in (\mathsf{P})^+$

$$\boldsymbol{\varnothing} \leq \boldsymbol{V} < \boldsymbol{U} \text{ implies } \boldsymbol{V} = \boldsymbol{\varnothing}.$$

Suppose for the purposes of constructing a contradiction that U has at least two distinct elements, $x, y \in U$. Then $\varnothing \subsetneq \{x\} \subsetneq \{x, y\} \subseteq U$. Therefore, there is a $\boldsymbol{x} = \boldsymbol{V} \in (F)^+$ such that $\boldsymbol{\varnothing} \leq \boldsymbol{x} < \boldsymbol{U}$ and $\boldsymbol{x} \neq \boldsymbol{\varnothing}$ and so $\boldsymbol{U}$ is not an atom – a contradiction. $\dashv$

The set of atoms of $(\mathsf{P})^+$, $\mathrm{At}\left((\mathsf{P})^+\right)$, is thus precisely the set of concepts that are liftings of singletons. That is: $\mathrm{At}\left((\mathsf{P})^+\right) = \left\{\boldsymbol{x} \in (\mathbb{P}_W)^+ \mid x \in W\right\}$.

In order to utilise the duality results of [50] and [28] (Theorem 6.41) it must first be established that the complex algebras of liftings of Kripke frames are perfect BAOs.

Lemma 6.37.

(a) $(\mathsf{P})^+$ is a complete BAO.

(b) $(\mathsf{P})^+$ is an atomic BAO.

(c) $(\mathsf{P})^+$ is a perfect BAO.

Proof. By [33, Proposition 21], $(\mathsf{P})^+$ is a complete $\mathcal{L}$-algebra. Therefore, the lattice reduct of $(\mathsf{P})^+$ is a complete lattice and the operators $[R_\Box]$, $\langle R_\Diamond\rangle$, $[R_\rhd\rangle$, and $\langle R_\lhd]$ are complete operators. Additionally, as discussed in Remark 5.4, $(\mathsf{P})^+$, accommodates for complementation. Therefore, $(\mathsf{P})^+$ is a complete BAO.

For (b) we need to show that for any $\boldsymbol{U} \in (\mathsf{P})^+$ such that $\boldsymbol{\varnothing} < \boldsymbol{U}$ implies that there exists a $\boldsymbol{a} \in \mathrm{At}\left((\mathsf{P})^+\right)$ such that $\boldsymbol{a} \leq \boldsymbol{U}$. As $\boldsymbol{\varnothing} < \boldsymbol{U}$ we have that U is non-empty, and therefore there exists a $x \in U$, and so $\boldsymbol{x} \leq \boldsymbol{U}$ (Definition 3.23). By Lemma 6.36, $\boldsymbol{x} \in \mathrm{At}\left((\mathsf{P})^+\right)$. Thus $(\mathsf{P})^+$ is atomic.

Therefore, as $(\mathsf{P})^+$ is complete and atomic ((a) and (b)) it is a perfect BAO ((c)). $\dashv$

As in [17, Definition 22], a natural map $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ is defined in terms of $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$.

Definition 6.38. Let $(S, T) : \mathsf{P}_1 \to \mathsf{P}_2$ be a p-morphism. Define $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ where $h_{(S,T)}(\boldsymbol{a}) := \left(S^{(0)}\left[(\!\![\boldsymbol{a}]\!\!)\right], T^{(0)}\left[[\![\boldsymbol{a}]\!]\right]\right)$ for $\boldsymbol{a} \in (\mathsf{P}_2)^+$.[8]

The map, $h_{(S,T)}$, is a complete BAO homomorphism, as shown in Lemma 6.39 below.

Lemma 6.39. Let $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ be the map defined in Definition 6.38 then

[8]This definition is adapted to our setting from [17, Definition 22].

(a) $h_{(S,T)}$ is a complete lattice homomorphism between the lattice reducts of $(\mathsf{P}_2)^+$ and $(\mathsf{P}_1)^+$.

(b) $h_{(S,T)}$ is a complete modal homomorphism.

(c) $h_{(S,T)}(\boldsymbol{U}) = \boldsymbol{S^{*(0)}}(\boldsymbol{U^c})$.[9]

(d) $h_{(S,T)}(\boldsymbol{U}') = \left(h_{(S,T)}(\boldsymbol{U})\right)'$.[10]

(e) $h_{(S,T)}$ is a complete BAO homomorphism.

Proof. Results (a) and (b) follow from the observation that $h_{(S,T)}$ is a complete $\mathcal{L}$ homomorphism as shown in [17, Proposition 23]. In order for $h_{(S,T)}$ to be a complete BAO homomorphism, however, it must preserve complementation as defined in Remark 5.4; that is, for (d) we require that for any $\boldsymbol{U} \in (\mathsf{P}_2)^+$

$$h_{(S,T)}\left((\boldsymbol{U})'\right) = \left(h_{(S,T)}(\boldsymbol{U})\right)'.$$

We will first show (c), and then use (c) to show (d). Let $\boldsymbol{U} = (U_A, (U^c)_X) \in (\mathsf{P}_2)^+$. Note that $\boldsymbol{U}' = ((U^c)_A, U_X) \in (\mathsf{P}_2)^+$ and $\boldsymbol{U}' = \boldsymbol{U^c}$. So

$$\begin{aligned}
h_{(S,T)}(\boldsymbol{U}) &= \left(S^{(0)}\left((U^c)_X\right), T^{(0)}(U_A)\right) && \text{(Definition 6.38)}\\
&= \left(\left(S^{*(0)}(U^c)\right)_A, \left(T^{*(0)}(U)\right)_X\right) && \text{(Remark 6.24)}\\
&= \left(\left(S^{*(0)}(U^c)\right)_A, \left(S^{*(0)}(U)\right)_X\right) && \text{(Corollary 6.30)}\\
&= \left(\left(S^{*(0)}(U^c)\right)_A, \left(\left(S^{*(0)}(U^c)\right)^c\right)_X\right) && \text{(Corollary 6.32)}\\
&= \boldsymbol{S^{*(0)}}(\boldsymbol{U^c})
\end{aligned}$$

Thus, (c) holds. Substituting U^c for U in (c) we also find that $h_{(S,T)}(\boldsymbol{U^c}) = \boldsymbol{S^{*(0)}}(\boldsymbol{U})$. But

$$\left(h_{(S,T)}(\boldsymbol{U}')\right) = \left(h_{(S,T)}(\boldsymbol{U^c})\right) = \boldsymbol{S^{*(0)}}(\boldsymbol{U}) \qquad \text{((c))}$$

and

$$\begin{aligned}
\left(h_{(S,T)}(\boldsymbol{U})\right)' &= \left(\boldsymbol{S^{*(0)}}(\boldsymbol{U^c})\right)' && \text{((c))}\\
&= \left(\left(\boldsymbol{S^{*(0)}}(\boldsymbol{U})\right)^{\boldsymbol{c}}\right)' && \text{(Corollary 6.32)}\\
&= \left(\boldsymbol{S^{*(0)}}(\boldsymbol{U})\right) && \text{(as } \boldsymbol{U}' = \boldsymbol{U}^c\text{)}
\end{aligned}$$

Therefore, $h_{(S,T)}$ respects complementation ((d)). As $h_{(S,T)}$ respects complementation, is a complete lattice homomorphism ((a)), and a complete modal homomorphism ((b)) it follows that $h_{(S,T)}$ is a complete BAO homomorphism. ⊣

Let $\mathcal{A}$ and $\mathcal{A}'$ be perfect Boolean algebras with operators.

Definition 6.40. For a complete BAO homomorphism $\eta : \mathcal{A} \to \mathcal{A}'$ define the map $\eta_+ : \mathrm{At}(\mathcal{A}') \to \mathcal{A}$ by[11]

$$\eta_+(p') := \bigwedge \{a \in A \mid p' \leq \eta(a)\}.$$

Finally, Theorem 6.41 gives that $\eta_+ : \mathrm{At}(\mathcal{A}) \to \mathcal{A}'$ is a p-morphism.

Theorem 6.41 (η_+ is a p-morphism)**.** If $\eta : \mathcal{A} \to \mathcal{A}'$ is a complete and modal homomorphism between perfect BAOs $\mathcal{A}$ and $\mathcal{A}'$, then $\eta_+ : \mathrm{At}(\mathcal{A}) \to \mathcal{A}'$ is a p-morphism between the atom structures $\mathcal{A}_+$ and $\mathcal{A}'_+$ [50].[12]

[9] By $\boldsymbol{S^{*(0)}}(\boldsymbol{U^c})$, we are referring to the element $\left(\left(S^{*(0)}(U^c)\right)_A, \left(\left(S^{*(0)}(U^c)\right)^c\right)_X\right)$ of $(\mathsf{P}_1)^+$.

[10] Where complementation is as defined in Remark 5.4.

[11] Adapted from [50, Definition 44] and [28, Theorem 1.10].

[12] See Definition 2.34 for definition of atom structure.

Proof. See [50, Proposition 46] for an sketch of a proof that η_+ is a p-morphism, and [28, Theorem 1.10] for a more detailed proof in the general context of relational structures. $\dashv$

As $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ is a complete BAO homomorphism, we may apply Definition 6.40 and Theorem 6.41 to define the map $h_{(S,T)+} : \mathrm{At}((\mathsf{P}_1)^+ \to (\mathsf{P}_2)^+$ given by

$$h_+(\boldsymbol{x}) := \bigwedge \left\{ \boldsymbol{a} \in (\mathsf{P}_2)^+ \mid \boldsymbol{x} \leq (S^{(0)}([\![a]\!]), T^{(0)}([\![a]\!])) \right\}.$$

By Theorem 6.41, h_+ is a p-morphism between the atom structures $\left(\mathsf{P}_1^+\right)_+$ and $\left(\mathsf{P}_2^+\right)_+$. By Lemma 6.36, atoms in the concept lattice of liftings of Kripke Frames are precisely the formal concepts corresponding to singleton subsets in the powerset algebra of the underlying Kripke frame – concepts of the form $\boldsymbol{x} = (\{x\}_A, ((\{x\}^c)_X))$.

In Remark 5.4, it was noted that the powerset algebra, $\mathcal{S}(W)$ of a set, W, is isomorphic to the concept lattice of the lifting of the set, $\mathbb{P}_W^+$. This result can be extended to show that the full complex algebra K (K^+) and the complex algebra of P (P^+) are isomorphic, as in the below remark.

Remark 6.42. $\mathsf{K}^+ \cong \mathsf{P}^+$

Proof. See [9, Proposition 3.7]. $\dashv$

With Theorem 6.41 established, a map $H_{(S,T)}$ between the original underlying sets W^1 and W^2 can be defined in terms of $h_{(S,T)+}$, which we will write h_+ when the meaning is clear from context.

Definition 6.43. If $h_+ : \left(\mathsf{P}_1^+\right)_+ \to \left(\mathsf{P}_2^+\right)_+$ is the map given above, define $H_{(S,T)} : W^1 \to W^2$ as follows

$$H_{(S,T)}\left(w^1\right) = w^2 \text{ iff } h_+\left(\boldsymbol{w^1}\right) = \left(\left\{w^2\right\}_A, \left(\left\{w^2\right\}^c\right)_X\right) = \boldsymbol{w^2}.$$

Using Theorem 6.41 with $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ in place of $\eta : \mathcal{A} \to \mathcal{A}'$, together with Remark 6.42 provides Theorem 6.44, the converse to Theorem 6.19 in the sense discussed earlier in this section.

Theorem 6.44. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism in the sense of Definition 6.3 then $H_{(S,T)} : \mathsf{K}_1 \to \mathsf{K}_2$ is a p-morphism in the sense of Definition 6.9.

Proof. Let $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ be a p-morphism in the sense of Definition 6.1. Then $(\mathsf{P}_1)^+$ and $(\mathsf{P}_2)^+$ are perfect BAOs and $h_{(S,T)} : (\mathsf{P}_2)^+ \to (\mathsf{P}_1)^+$ is a complete BAO homomorphism by Lemma 6.37(c) and Lemma 6.39(e) respectively. Moreover, by Theorem 6.41, $h_{(S,T)+} : \left(\mathsf{P}_1^+\right)_+ \to \left(\mathsf{P}_2^+\right)_+$ is a p-morphism between the atom structures of P_1^+ and P_2^+. As h_+ maps atoms in $(\mathsf{P}_1)^+$ to atoms in $(\mathsf{P}_2)^+$, the map $H_{(S,T)}$ is well defined, as $h_+\left(\boldsymbol{w^2}\right)$ is always an atom $\boldsymbol{w^1}$ in $(\mathsf{P}_1)^+$ by Lemma 6.36, and can thus be easily associated with precisely one element $w^1 \in W^1$. Additionally, by Remark 6.42 we have that $\mathsf{K}_1^+ \cong \mathsf{P}_1^+$ and $\mathsf{K}_2^+ \cong \mathsf{P}_2^+$; and by Theorem 2.45 we have $\mathsf{K}_1 \cong \left(\mathsf{K}_1^+\right)_+$ and $\mathsf{K}_2 \cong \left(\mathsf{K}_2^+\right)_+$.

These constructions are summarised in Figure 6.3. Using Figure 6.3, we oberve that $H_{(S,T)}$ is the composition of isomorphisms and p-morphisms (starting in the bottom left corner of the diagram, and moving up to the top left corner, horizontally to the top right corner and then straight down to the bottom right), and is thus itself a p-morphism. $\dashv$

Additionally, if $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ is a p-morphism between liftings of models then $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism between Kripke models, as shown in Corollary 6.48 Using Theorem 6.44 in conjunction with Theorem 6.10 provides the truth preservation result, Corollary 6.51 at the end of the chapter.

$$
\begin{tikzcd}
(\mathsf{K}_1^+)_+ \arrow[rr, "\text{iso.}"] & & (\mathsf{P}_1^+)_+ \arrow[r, "h_{(S,T)+}"] & (\mathsf{P}_2^+)_+ \arrow[rr, "\text{iso.}"] & & (\mathsf{K}_2^+)_+ \arrow[dd, "\text{iso.}"] \\
& \mathsf{K}_1^+ \arrow[ul, "(\cdot)_+"'] & \mathsf{P}_1^+ \arrow[l, "\text{iso.}"'] \arrow[u, "(\cdot)_+"'] & \mathsf{P}_2^+ \arrow[l, "h_{(S,T)}"'] \arrow[r, "\text{iso.}"] \arrow[u, "(\cdot)_+"'] & \mathsf{K}_2^+ \arrow[ur, "(\cdot)_+"] & \\
\mathsf{K}_1 \arrow[uu, "\text{iso.}"] \arrow[ur, "(\cdot)^+"] \arrow[rr, "\text{lifting}"] \arrow[rrrrr, bend right, "H_{(S,T)}"'] & & \mathsf{P}_1 \arrow[u, "(\cdot)^+"'] \arrow[r, "{(S,T)}"] & \mathsf{P}_2 \arrow[u, "(\cdot)^+"'] & & \mathsf{K}_2 \arrow[ul, "(\cdot)^+"'] \arrow[ll, "\text{lifting}"']
\end{tikzcd}
$$

Figure 6.3: In this diagram we refrain from naming the isomorphisms for the sake of readability. Note that $(\cdot)^+$ refers to different operations when moving between Kripke Frame and their complex algebras in the one case, and polariy-based frames and their complex algebras in the other – see Definition 2.43 and Definition 4.6 for clarification.

Lemma 6.45. $w^1 \in S^{*(0)}(U^c)$ if and only if $\theta_{(S,T)}\left(w^1\right) \in U$.

Proof. For the forward implication suppose $w^1 \in S^{(0)}(U^c)$ for some $U \subseteq W^2$. Either $\theta\left(w^1\right) \in U$ or $\theta\left(w^1\right) \in U^c$. If $\theta\left(w^1\right) \in U^c$ then $\left(w^1, \theta\left(w^1\right)\right) \in S^*$, which is false by definition of θ (Definition 6.27). Therefore $\theta\left(w^1\right) \in U$.

For the converse suppose $\theta\left(w^1\right) \in U$. Then $w^1 \notin S^{*(0)}(U)$, as $\left(w^1, \theta\left(w^1\right)\right) \notin S^*$. Thus $w^1 \in S^{*(0)}(U^c)$ by Corollary 6.33. ⊣

Theorem 6.46. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $(S_{H_{(S,T)}}, T_{H_{(S,T)}}) = (S,T)$.

Proof. We first show that $S_{H_{(S,T)}} = S$.

Suppose $\left(w_A^1, w_X^2\right) \in S_{H_{(S,T)}}$. Now

$$
\begin{aligned}
\left(w_A^1, w_X^2\right) \in S_{H_{(S,T)}} &\text{ iff } H_{(S,T)}\left(w^1\right) \neq w^2 && \text{(Definition 6.13)}\\
&\text{ iff } h_+\left(\boldsymbol{w^1}\right) \neq \boldsymbol{w^2} && \text{(Definition 6.43)}\\
&\text{ iff } \bigwedge\left\{\boldsymbol{U} \in (\mathsf{P}_2)^+ \mid \boldsymbol{w^1} \leq (S^{(0)}(U_X^c), T^{(0)}(U_A)\right\} \neq \left(\left\{w^2\right\}_A, \left(\left\{w^2\right\}^c\right)_X\right) \\
& && \text{(Definition 6.43)}\\
&\text{ iff } \bigwedge\left\{\boldsymbol{U} \in (\mathsf{P}_2)^+ \mid \left\{w^1\right\} \subseteq S^{*(0)}(U^c)\right\} \neq \left(\left\{w^2\right\}_A, \left(\left\{w^2\right\}^c\right)_X\right)\\
&\text{ iff } \bigcap\left\{U \in \mathcal{P}(W^2) \mid w^1 \in S^{*(0)}(U^c)\right\} \neq \left\{w^2\right\}. && \text{(Remark 5.4)}
\end{aligned}
$$

As $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism, a map $\theta_{(S,T)} : W^1 \to W^2$ can be defined as in Definition 6.27. Using Lemma 6.45 and observing that $\theta\left(w^1\right) \in \left\{\theta\left(w^1\right)\right\} \in \mathcal{P}\left(W^2\right)$ we have:

$$
\bigcap\left\{U \in \mathcal{P}(W^2) \mid w^1 \in S^{*(0)}(U^c)\right\} = \bigcap\left\{U \in \mathcal{P}(W^2) \mid \theta\left(w^1\right) \in U\right\} = \left\{\theta\left(w^1\right)\right\}.
$$

Thus our initial chain of biconditional statements above gives

$$\begin{aligned}
\left(w^1_A, w^2_X\right) \in S_{H_{(S,T)}} &\text{ iff } \bigcap\left\{U \in \mathcal{P}(W^2) \mid w^1 \in S^{*(0)}\left(U^c\right)\right\} \neq \left\{w^2\right\} \\
&\text{ iff } \left\{\theta\left(w^1\right)\right\} \neq \left\{w^2\right\} \\
&\text{ iff } \theta\left(w^1\right) \neq w^2 \\
&\text{ iff } \theta\left(w^1\right) \in \left\{w^2\right\}^c \\
&\text{ iff } w^1 \in S^*\left(\left(\left\{w^2\right\}^c\right)^c\right) && \text{(Lemma 6.45)} \\
&\text{ iff } w^1 S^* w^2 \\
&\text{ iff } \left(w^1_A, w^2_X\right) \in S && \text{(Remark 6.23)}
\end{aligned}$$

Thus $S_{H_{(S,T)}} = S$.

What remains to be shown is that $T_{H_{(S,T)}} = T$. By Theorem 6.44, the map $H_{(S,T)} : \mathsf{K}_1 \to \mathsf{K}_2$ is a p-morphism. Therefore by Theorem 6.19 the lifting $(S_{H_{(S,T)}}, T_{H_{(S,T)}}) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism. Thus ${S_{H_{(S,T)}}}^* = {T_{H(S,T)}}^*$ by Corollary 6.30. As shown in the body of this proof, $\left(w^1_A, w^2_X\right) \in S_{H_{(S,T)}}$ if and only if $\left(w^1_A, w^2_X\right) \in S$. Chaining these equivalences together we have

$$\begin{aligned}
\left(w^1_X, w^2_A\right) \in T_{H_{(S,T)}} &\text{ iff } \left(w^1_A, w^2_X\right) \in S_{H_{(S,T)}} && \text{(Theorem 6.19 and Corollary 6.30)} \\
&\text{ iff } \left(w^1_A, w^2_X\right) \in S && (\text{as } S = S_{H_{(S,T)}}) \\
&\text{ iff } \left(w^1, w^2\right) \in S^* && \text{(Remark 6.23)} \\
&\text{ iff } \left(w^1, w^2\right) \in T^* && \text{(Corollary 6.30)} \\
&\text{ iff } \left(w^1_X, w^2_A\right) \in T. && \text{(Remark 6.23)}
\end{aligned}$$

Thus $\left(w^1_X, w^2_A\right) \in T_{H_{(S,T)}}$ if and only if $\left(w^1_X, w^2_A\right) \in T$, and so $T_{H_{(S,T)}} = T$ as required. ⊣

The following corollary of Theorem 6.46 shows that the privileged element, $\theta_{(S,T)}(w^1)$ is indeed $H_{(S,T)}\left(w^1\right)$, as we would expect intuitively.

Corollary 6.47. If $(S,T) : \mathsf{P}_1 \to \mathsf{P}_2$ is a p-morphism then $\theta_{(S,T)}\left(w^1\right) = H_{(S,T)}\left(w^1\right)$ for any $w^1 \in W^1$.

Proof. Suppose that $\theta_{(S,T)}\left(w^1\right) = w^2$ for some $w^2 \in W^2$ then

$$\begin{aligned}
\theta_{(S,T)}\left(w^1\right) = w^2 &\text{ iff } \left(w^1_A, w^2_X\right) \notin S && \text{(Definition 6.27 and Corollary 6.30)} \\
&\text{ iff } \left(w^1_A, w^2_X\right) \notin S_{H_{(S,T)}} && \text{(Theorem 6.46)} \\
&\text{ iff } H_{(S,T)}\left(w^1\right) = w^2. && \text{(Definition 6.13)}
\end{aligned}$$

Thus $H_{(S,T)}\left(w^1\right) = \theta_{(S,T)}\left(w^2\right)$. ⊣

Corollary 6.49 and Corollary 6.50 are informal converse statements of Proposition 6.21 and Proposition 6.20 respectively.

With Corollary 6.47 established, we can now show two important corollaries of Theorem 6.44, namely that $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ is a model p-morphism (Corollary 6.48), and that a typed version of $H_{(S,T)}$ preserves modal satisfaction between lifting of Kripke models (Corollary 6.51).

Corollary 6.48. If $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ is a p-morphism in the sense of Definition 6.1 then $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ is a model p-morphism in the sense of Definition 6.8.

Proof. Let $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ be a p-morphism in the sense of Definition 6.1. By Theorem 6.44, $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism between underlying Kripke frames, K_1 and K_2, and thus satisfies (b2) and (b3) of Definition 6.8. What remains to be shown is that $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ satisfies (b1) of Definition 6.8. That is, we require for any $w^1 \in W^1$ and $p \in \mathsf{Prop}$ that

$$\mathcal{M}_1, w^1 \Vdash_{\mathsf{K}} p \text{ iff } \mathcal{M}_2, H_{(S,T)}\left(w^1\right) \Vdash_{\mathsf{K}} p.$$

This is equivalent to

$$w^1_A \in [\![p]\!]_{\mathcal{N}_1} \text{ iff } \left(H_{(S,T)}\left(w^1\right)\right)_A \in [\![p]\!]_{\mathcal{N}_2}.$$

For the forwards implication, suppose that $\mathcal{M}_1, w^1 \Vdash_{\mathsf{K}} p$. By Corollary 6.47, we have $H_{(S,T)}\left(w^1\right) = \theta_{(S,T)}\left(w^1\right) = y^2$ where $y^2 \in W^2$ is the unique element of W^2 such that $\left(w^1_A, y^2_X\right) \notin S$. As (S,T) is a p-morphism, and thus has (p6a)

$$(\!|p|\!)_{\mathcal{N}_2} \subseteq S^{(1)}\left([\![p]\!]_{\mathcal{N}_1}\right).$$

As $\left(w^1_A, y^2_X\right) \notin S$ and $w^2_A \in [\![p]\!]_{\mathcal{N}_1}$ we have that $y^2_A \notin (\!|p|\!)_{\mathcal{N}_2}$ and therefore $\mathcal{N}_2, y^2_A \not\succ p$. Additionally, by Proposition 5.15

$$y^2_X \notin (\!|p|\!)_{\mathcal{N}_2} \text{ iff } y^2_A \in [\![p]\!]_{\mathcal{N}_2}.$$

For the converse suppose that $\mathcal{M}_2, H_{(S,T)}\left(w^1\right) \Vdash_{\mathsf{K}} p$. Let $v^2 := H_{(S,T)}\left(w^1\right) \in W^2$. By Corollary 6.47, v^2 is the unique element of W^2 such that $\left(w^1, v^2\right) \notin S^*$ and $\left(w^1, v^2\right) \notin T^*$, or equivalently in the typed/lifted setting $\left(w^1_A, v^2_X\right) \notin S$ and $\left(w^1_X, v^2_A\right) \notin T$. Since (S,T) is a p-morphism it has (p6b)

$$(\!|p|\!)_{\mathcal{N}_1} \subseteq T^{(0)}\left([\![p]\!]_{\mathcal{N}_2}\right).$$

As $\mathcal{M}_2, v^2 \Vdash_{\mathsf{K}} p$, by Proposition 5.16 and Corollary 5.17 we have that $v^2_A \in [\![p]\!]_{\mathcal{N}_2}$ and $v^2_X \notin (\!|p|\!)_{\mathcal{N}_2}$ respectively. But, $\left(w^1_X, v^2_A\right) \notin T$ and $v^2_A \in [\![p]\!]_{\mathcal{N}_2}$. So $w^1_X \notin T^{(0)}\left([\![p]\!]_{\mathcal{N}_1}\right)$, and thus $w^1_X \notin (\!|p|\!)_{\mathcal{N}_1}$. Then, by Proposition 5.15, $w^1_A \in [\![p]\!]_{\mathcal{N}_1}$, and therefore $\mathcal{M}_1, w^1 \Vdash_{\mathsf{K}} p$. ⊣

Corollary 6.49. If $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ is an injective p-morphism in the sense of Definition 6.6 then $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ is an injective p-morphism in the sense of Definition 6.12.

Proof. By Corollary 6.47, it need only be shown that $\theta_{(S,T)}$ is injective. We proceed by contradiction. Suppose that

(1) (S,T) is injective;
and

(2) $\theta_{(S,T)}$ is not injective.

As elements of the concept lattice of liftings of Kripke frames correspond to subsets of the underlying set (by Corollary 5.3), the definition of injectivity in Definition 6.6 gives

$$\begin{aligned} &\left(\forall \boldsymbol{U^1} \in (\mathsf{P}_1)^+\right)\left(\exists \boldsymbol{V^2} \in (\mathsf{P}_2)^+\left(S^{*(0)}\left(\left(V^2\right)^c\right) = U^1\right)\right) \\ \text{iff } &\left(\forall U^1 \subseteq W^1\right)\left(\exists V^2 \subseteq W^2\left(S^{*(0)}\left(\left(V^2\right)^c\right) = U^1\right)\right). \end{aligned} \tag{1}$$

Additionally, as $\theta_{(S,T)}$ assumed non-injective

$$\left(\exists x^1, y^1 \in W^1\right)\left(x^1 \neq y^1 \text{ and } \theta_{(S,T)}\left(x^1\right) = \theta_{(S,T)}\left(y^1\right)\right) \tag{2}$$

Let $z^2 = \theta_{(S,T)}\left(x^1\right) = \theta_{(S,T)}\left(y^1\right)$. From (1) we have for particular $U^1 = \left\{x^1\right\}^c$ that there exists $V^2 \subseteq W^2$ such that $S^{*(0)}\left(\left(V^2\right)^c\right) = \left\{x^1\right\}^c$. Either $z^2 \in V^2$ or $z^2 \notin V^2$.

First suppose $z^2 \in V^2$. As $\left(x^1, z^2\right) \notin S^*$ (by (2) and Definition 6.27) we have from Remark 6.34 that $\left(x^1, b^2\right) \in S^*$ for every $b^2 \in \left\{z^2\right\}^c$. So $x^1 \in S^{*(0)}\left(\left\{z^2\right\}^c\right)$. But $\left(V^2\right)^c \subseteq \left\{z^2\right\}^c$ and so by the antitonicity of $S^{*(0)}$ (Lemma 3.5(a)) we have $S^{*(0)}\left(\left\{z^2\right\}^c\right) \subseteq S^{*(0)}\left(\left(V^2\right)^c\right)$. Therefore $x^1 \in S^{*(0)}\left(\left(V^2\right)^c\right) = \left\{x^1\right\}^c$ – a contradiction.

Suppose then that $z^2 \notin V^2$. As $z^2 \in \left(V^2\right)^c$, the inclusions $y^1 \in \left\{x^1\right\}^c = S^{*(0)}\left(\left(V^2\right)^c\right)$ imply $\left(y^1, z^2\right) \in S^*$. This contradicts our assignment of $\theta_{(S,T)}\left(y^1\right) = z^2$ in (2).

Therefore (1) implies the negation of (2) and our result follows. ⊣

Corollary 6.50. If $\mathcal{N}_1 \twoheadrightarrow \mathcal{N}_2$ in the sense of Definition 6.5 then $\mathcal{M}_1 \twoheadrightarrow \mathcal{M}_2$ in the sense of Definition 6.11.

Proof. Suppose $\mathcal{N}_1 \twoheadrightarrow \mathcal{N}_2$ and let $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ be a surjective p-morphism. By Theorem 6.44, $H_{(S,T)}$ is a p-morphism and is equivalent to $\theta_{(S,T)}$ by Corollary 6.47. Thus it is sufficient to show that $\theta_{(S,T)}$ is surjective: that for any $y^2 \in W^2$ there is a $x^1 \in W^1$ such that $\theta_{(S,T)}\left(x^1\right) = y^2$.

As (S,T) is surjective, by Corollary 5.3 and the definition of surjectivity for p-morphisms (Definition 6.11) for any $U, V \subseteq W^2$:

$$\begin{aligned} &S^{(0)}(\boldsymbol{U}) = S^{(0)}(\boldsymbol{V}) \text{ implies } \boldsymbol{U} = \boldsymbol{V} \\ \text{iff } &S^{(0)}(U^c) = S^{(0)}(V^c) \text{ implies } U = V \\ \text{iff } &S^{*(0)}(U) = S^{*(0)}(V) \text{ implies } U = V. && \text{(Corollary 6.32)} \end{aligned}$$

For the purposes of constructing a contradiction suppose that there is a $y^2 \in W^2$ such that $\theta_{(S,T)}(x^1) \neq y^2$ for every $x^1 \in W^1$ – that is $\left(\exists y^2 \in W^2\right)\left(\forall x^1 \in W^1\left(y^2 \neq \theta_{(S,T)}\left(x^1\right)\right)\right)$. Thus by definition of $\theta_{(S,T)}$ (Definition 6.27)

$$\left(\exists y^2 \in W^2\right)\left(\forall x^1 \in W^1\left(\left(x^1, y^2\right) \in S^*\right)\right).$$

Therefore $S^{*(0)}\left(\left\{y^2\right\}\right) = W^1$. But $S^{*(0)}(\varnothing) = W^1$ by Proposition 3.7(c). Thus from the chain of biconditional statements for the surjectivity of S above:

$$S^{*(0)}\left(\left\{y^2\right\}\right) = S^{*(0)}(\varnothing) \text{ implies } \left\{y^2\right\} = \varnothing.$$

This is an absurdity, as $\left\{y^2\right\} \neq \varnothing$. Thus $\theta_{(S,T)}$ is surjective. ⊣

Corollary 6.51. If $(S,T) : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism between liftings of Kripke models then, for any $\varphi \in \mathcal{L}_P$ and $w \in W$,

$$\mathcal{N}_1, w_A \Vdash \varphi \text{ iff } \mathcal{N}_2, \left(H_{(S,T)}(w)\right)_A \Vdash \varphi.$$

Proof. This result follows from application of Corollary 6.48, Theorem 6.10, and Proposition 5.16.

Let $(S,T) : \mathcal{N}_1 \to \mathcal{N}_2$ be a p-morphism between liftings of Kripke models. Then, by Theorem 6.44, $H_{(S,T)} : \mathcal{M}_1 \to \mathcal{M}_2$ is a p-morphism between the Kripke models, $\mathcal{M}_1$ and $\mathcal{M}_2$. By Theorem 6.10

$$\mathcal{M}_1, w \Vdash_{\mathsf{K}} \varphi \text{ iff } \mathcal{M}_2, H_{(S,T)}(w) \Vdash_{\mathsf{K}} \varphi.$$

Then by Proposition 5.16

$$\mathcal{M}_1, w \Vdash_{\mathsf{K}} \varphi \text{ iff } \mathcal{N}_1, w_A \Vdash \varphi$$

and

$$\mathcal{M}_2, H_{(S,T)}(w) \Vdash_{\mathsf{K}} \varphi \text{ iff } \mathcal{N}_2, \left(H_{(S,T)}\right)_A \Vdash \varphi.$$

Therefore

$$\mathcal{N}_1, w_A \Vdash \varphi \text{ iff } \mathcal{N}_2, \left(H_{(S,T)}\right)_A \Vdash \varphi.$$

⊣

Chapter 7

Conclusions

The last several decades have seen growing research interest in non-distributive modal logics built upon various non-classical propositional bases. As discussed in further detail in Section 1.1.3, non-distributive modal logics have been interpreted on several classes of relational structures, with reflexive graphs and enriched polarities being two notable examples – the latter of which was the focus of this study. Polarity-based semantics have been developed for a number of contexts, as elaborated upon in our historical overview in Section 1.1. Current approaches only interpret logical languages on enriched polarities whose additional relations satisfy compatibility requirements that ensure semantic clauses corresponding to modalities always take Galois-stable sets to Galois-stable sets. The imposition of compatibility requirements hinder the applicability of the resultant semantics, as the additional relations of arbitrary enriched polarities need not send Galois-stable sets to Galois-stable sets.

The presence of these constraints, and the limits that they place on applicability, lead to several natural questions, for instance: What possible form could a potential non-compatible polarity-based semantics take? What logical properties would the resultant semantic framework have? Is it possible to produce a sound and complete axiomatization of the corresponding logic? What logical properties of the compatible semantics do we lose by dropping the requirements on additional relations? What can be said of the truth preserving constructions between non-compatible enriched polarities? These were some of the central questions motivating this thesis.

Following a review of well-known results for polarities in Chapter 3, in Chapter 4 we introduced a non-compatible polarity-based semantics for non-distributive model logic in the signature of DML, along with an axiomatization. The new semantics was adapted from [16] via the removal of the compatibility requirements placed on additional relations, and adapting instead the semantic interpretation of the modalities by taking the Galois-closure of their corresponding semantic clauses to ensure that relevant terms still send Galois-stable sets to Galois-stable sets, as discussed in greater technical detail in Section 4.3. This axiomatization was shown to be sound and complete with respect to the class of non-compatible polarity-based frames in Section 4.4.1 and Section 4.4.2 respectively.

Notably, this axiomatization lacks axioms for normality and distributivity of modalities present in its compatible counterpart. This is because an axiomatization with such axioms would not be sound with respect to the class of non-compatible polarity-based frames, as demonstrated through the examples of Section 4.5.2.

The second half of the thesis, Chapter 5 and Chapter 6, focused on a special class of enriched polarities, those who are in a sense *liftings of Kripke frames* (Definition 5.7). The compatibility of these liftings (Remark 5.11) combined with the intuitive nature of the underlying Kripke frames made for a useful case study in which to explore p-morphisms between enriched polarities, and enriched polarity-based models, from a mostly relational perspective rather than an algebraic one (Section 6.2).

Further research

In this final section we outline some possible avenues for further research.

Non-compatible semantics

Our non-compatible polarity-based semantics were formulated for the unary DML signature. Polarity-based semantics in the compatible setting have been generalized to signatures of arbitrary arity, and number of relations as in [16] and [17]. Conceivably, the approach taken in this thesis could be extended using indexing and typing conventions similar to [16, Appendix 2]. As discussed in the closing paragraphs of Section 4.1, our approach can easily be adapted to accommodate for enriched polarities with any number of relations.

Rough set theory and formal concept analysis offer important and complimentary tools in modern data analysis with the unification of the two approaches being a topic of interest (for a concise overview of these efforts see [53]). A logical framework for "rough concepts" have already been developed in [9] and [32], suggested as a means through which to unify the two approaches. These approaches do, however, impose the familiar I-compatibility requirements on the additional relations of enriched polarities seen in the crisp case. The development of non-compatible semantics in this rough setting could be of particular interest given the position of its compatible counterpart at the intersection of two theories both with high levels of applicability in data sciences.

In addition to polarity-based semantics for rough concepts, recent years have also seen the growth of many-valued polarity-based modal logics – see for instance [15] and [9]. Once more, such extensions could potentially be extended using say a broadly similar methodology to the one adopted in this thesis for developing an non-compatible version of many-valued polarity-based modal logics.

As mentioned in Chapter 1, reflexive graph-based frames offer an alternative relational framework for NDML. Reflexive graphs and polarities are closely related in that every reflexive graph induces a corresponding polarity, and every polarity induces a corresponding reflexive graph.[1] Current approaches such as [7], [16], and [19] impose compatibility requirements on the additional relations of graph-based frames in ways analogous to the polarity-based setting. There thus may be similar potential in the development of non-compatible graph-based semantics through the dropping of analogous compatibility requirements.

Truth preservation

There is room for further development of truth preserving constructions between non-compatible enriched polarity-based frames, ultimately building towards the formulation of a Goldblatt-Thomason style theorem in the non-compatible setting (see Chapter 6). More specifically, this would involve the formulation of polarity-based analogues to the familiar p-morphisms, generated subframes, and disjoint unions of basic modal logic in the non-compatible polarity-based setting. In [17], a Goldblatt-Thomason style theorem is provided for a large class of logics captured algebraically by normal lattice expansions, LE-logics. As [17] assumes normality, however, its results do not apply in the general non-compatible setting. Additionally, the definitions of truth preserving constructions in [17], obtained predominantly through algebraic means, lack the intuitive strength of their analogues in classical Kripke semantics. The counter intuitive nature of various aspects of polarity-based semantics, such as the "two-sortedness" of satisfaction and co-satisfaction, have been well noted (see for example [26] and [37]); and so the extent to which the counter intuitive nature of these constructions is intrinsic to polarity-based semantics regardless of approach (algebraic or relational) is unclear.

Definition 6.1 is a possible candidate definition for p-morphisms between non-compatible enriched polarity-based frames and models, although it may be insufficient to insure preservation of satisfaction in the non-compatible setting. A possible first approach in strengthening Definition 6.1 could be the extension of an axiom such as p2 to arbitrary subsets of $A_1 \times A_2 \times X_1 \times X_2$ rather than elements of $A_1 \times A_2 \times X_1 \times X_2$, as non-compatible frames do not have a property analogous to Proposition 4.50. Strengthening the definition in this way would make the verification of maps as p-morphisms significantly more computationally intensive.[2]

Some additional questions related to truth preservation: In the specific context of liftings of Kripke

[1] See the note following the statement and proof of Birkhoff's representation theorem in [16].
[2] As the number on verifications required would be of the order $2^{|A_1 \times A_2 \times X_1 \times X_2|}$ rather than $|A_1 \times A_2 \times X_1 \times X_2|$.

frames could Definition 6.1 be weakened in some way, whilst still retaining the truth preserving results of Section 6.2? On which other special classes of compatible enriched polarities could a weaker definition suffice? The latter question was addressed partially in Chapter 6 for liftings of Kripke frames.

Further questions

As noted in previous subsections, the imposition of compatibility requirements on additional relations reduces the applicability of the resultant semantics, as arbitrary additional relations need not send Galois-stable sets to Galois-stable sets. Developing a semantics for non-distributive modal logics based on non-compatible polarities, therefore creates several natural directions for applications. One possible direction could be the development of an epistemic logic of categories in the style of [10] based on the non-compatible semantics introduced in this thesis.

Could non-compatible semantics be established using general Galois connections?

There is significant room for the exploration of the applicability of the semantics by way of in depth case studies in various contexts – epistemic and data theoretic for example. The development of case studies could be facilitated by computational tools such as the software library developed to verify various computational aspects of this thesis (see Chapter 8). Computational tools could themselves be greatly enhanced through the integration of established proof assistant software.

Dedication

I am immensely grateful to my supervisor, Prof. Willem Conradie, for his consistent time, dedication, and understanding throughout the past several years, and through many delays and setbacks including those caused by a global pandemic.

Words are wholly insufficient to express my gratitude for everything my mother has done for me, both during the this masters study, and in the two decades prior. I will never forget how you scribed emacs commands for me when I struggled to type.

Thank you to my grandfather for his support, love, and understanding – even when it meant repeated delays in get-togethers.

To my friends and comrades for their love and support throughout this process; we made it!

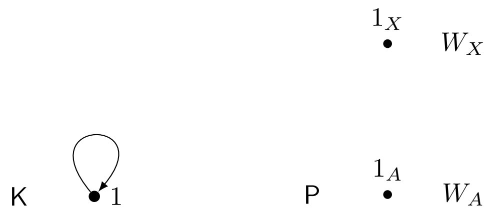
Chapter 8

Appendix

Additional example liftings of Kripke frames

The number of possible ordered pairs with first and second coordinates that are elements of a set, W, is $|W|^2$. In the context of lifting of Kripke frames, the typed versions of states, $u, v \in W$, are related by the corresponding lifted version of $R \subseteq W \times W$ if they are not R-related in the original Kripke frame. Thus, informally speaking, the number of edges encoding relational information in a lifting diagram is proportional to $|W|^2 - |R|$.

Example 8.1. Consider the Kripke frame comprising a single reflexive point, $\mathsf{K} = (W, R)$, where $W = \{1\}$, $R = \{(1,1)\}$. Then setting $W_A = \{1_A\}$ and $W_X = \{1_X\}$ the lifting of W is $\mathbb{P}_W = (\{1_A\}, \{1_X\}, \varnothing)$. Additionally, $\mathsf{P}_\mathsf{K} = (\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ where $I_{R^c} = J_{R^c} = H_{R^c} = K_{R^c} = \varnothing$. The Kripke frame, K, and it's lifting, P_K are presented diagrammatically below.



(a) (b)

Figure 8.1

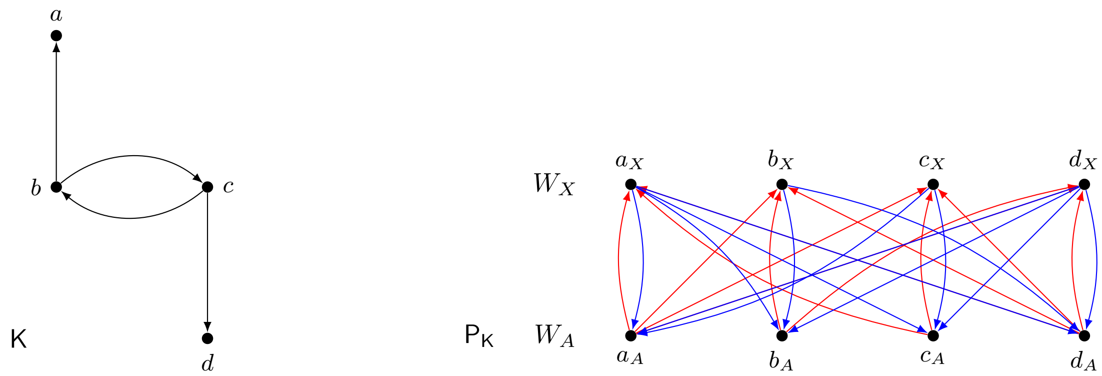
Due to the size of our remaining two examples, and the consequent number of lines on the diagrams, we omit H_{R^c} and K_{R^c} from the remaining examples along with the incidence relations for the underlying polarities.

Example 8.2. Consider the Kripke frame $\mathsf{K} = (\{a, b, c, d\}, \{(b, a), (b, c), (c, b), (c, d)\})$. Then $\mathsf{P}_\mathsf{K} = (\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ with:

$$\begin{aligned}
I_{R^c} = &\{(a_A, b_X), (a_A, c_X), (a_A, d_X), (a_A, a_X), (b_A, d_X), (b_A, b_X),\\
&(c_A, a_X), (c_A, c_X), (d_A, a_X)(d_A, b_X), (d_A, c_X), (d_A, d_X)\}\\
J_{R^c} = &\{(a_X, b_A), (a_X, c_A), (a_X, d_A), (a_X, a_A), (b_X, d_A), (b_X, b_A),\\
&(c_X, a_A), (c_X, c_A), (d_X, a_A)(d_X, b_A), (d_X, c_A), (d_X, d_A)\};\\
H_{R^c} = &\{(a_A, b_A), (a_A, c_A), (a_A, d_A), (a_A, a_A), (b_A, d_A), (b_A, b_A),\\
&(c_A, a_A), (c_A, c_A), (d_A, a_A)(d_A, b_A), (d_A, c_A), (d_A, d_A)\};
\end{aligned}$$

and

$$K_{R^c} = \begin{array}{l}\{(a_X, b_X), (a_X, c_X), (a_X, d_X), (a_X, a_X), (b_X, d_X), (b_X, b_X), \\ \quad (c_X, a_X), (c_X, c_X), (d_X, a_X)(d_X, b_X), (d_X, c_X), (d_X, d_X)\}.\end{array}$$



(a) (b)

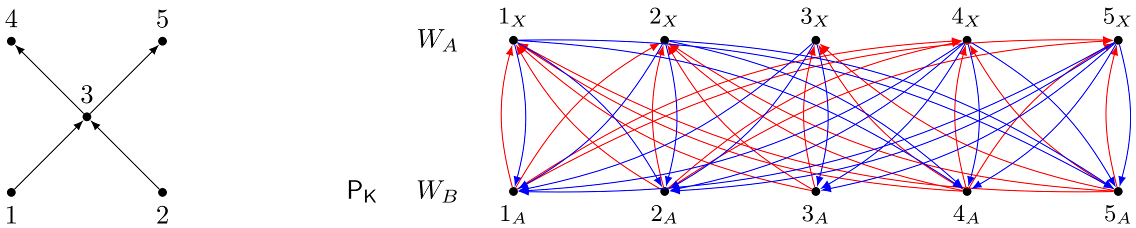
Figure 8.2

Example 8.3. Consider $\mathsf{K} = (W, R) = (\{1, 2, 3, 4, 5\}, \{(1, 3), (2, 3), (3, 4), (3, 5)\})$. Then $\mathsf{P_K} = (\mathbb{P}_W, \{I_{R^c}, J_{R^c}, H_{R^c}, K_{R^c}\})$ with:

$$\begin{aligned}
I_{R^c} &= \begin{array}{l}\{(1_A, 1_X), (1_A, 2_X), (1_A, 4_X), (1_A, 5_X), (2_A, 1_X), (2_A, 2_X), (2_A, 4_X), (2_A, 5_X), \\ \ (3_A, 1_X), (3_A, 2_X), (3_A, 3_X), (4_A, 1_X), (4_A, 2_X), (4_A, 3_X), (4_A, 4_X), (4_A, 5_X), \\ \ (5_A, 1_X), (5_A, 2_X), (5_A, 3_X), (5_A, 4_X), (5_A, 5_X)\},\end{array} \\
J_{R^c} &= \begin{array}{l}\{(1_X, 1_A), (1_X, 2_A), (1_X, 4_A), (1_X, 5_A), (2_X, 1_A), (2_X, 2_A), (2_X, 4_A), (2_X, 5_A), \\ \ (3_X, 1_A), (3_X, 2_A), (3_X, 3_A), (4_X, 1_A), (4_X, 2_A), (4_X, 3_A), (4_X, 4_A), (4_X, 5_A), \\ \ (5_X, 1_A), (5_X, 2_A), (5_X, 3_A), (5_X, 4_A), (5_X, 5_A)\},\end{array} \\
H_{R^c} &= \begin{array}{l}\{(1_A, 1_A), (1_A, 2_A), (1_A, 4_A), (1_A, 5_A), (2_A, 1_A), (2_A, 2_A), (2_A, 4_A), (2_A, 5_A), \\ \ (3_A, 1_A), (3_A, 2_A), (3_A, 3_A), (4_A, 1_A), (4_A, 2_A), (4_A, 3_A), (4_A, 4_A), (4_A, 5_A), \\ \ (5_A, 1_A), (5_A, 2_A), (5_A, 3_A), (5_A, 4_A), (5_A, 5_A)\},\end{array}
\end{aligned}$$

and

$$K_{R^c} = \begin{array}{l}\{(1_X, 1_X), (1_X, 2_X), (1_X, 4_X), (1_X, 5_X), (2_X, 1_X), (2_X, 2_X), (2_X, 4_X), (2_X, 5_X), \\ \ (3_X, 1_X), (3_X, 2_X), (3_X, 3_X), (4_X, 1_X), (4_X, 2_X), (4_X, 3_X), (4_X, 4_X), (4_X, 5_X), \\ \ (5_X, 1_X), (5_X, 2_X), (5_X, 3_X), (5_X, 4_X), (5_X, 5_X)\}.\end{array}$$



(a) (b)

Figure 8.3

Notes on available software

To assist in the generation of examples and counter examples, as well as checking conjectures over a large number of enriched polarities we sought the assistance of software packages. Most of the available software, however, was specialised for formal concept analysis, and not necessarily NDML. These packages, thus lacked support for enriching polarities with additional relations, $(\cdot)^{(0)}$ and $(\cdot)^{(1)}$ operations, and so on. For this reason we developed a Python package to perform basic operations on enriched polarity based frames such as: finding extensions and intentions, $(\cdot)^{(0)}$ and $(\cdot)^{(1)}$ of specific subsets, check stability, lift Kripke frames, and so on. Whilst this package, at the time of writing, lacks support for logical operations, it faciliated the checking of lists of conditions over hundreds of thousands of possible enriched polarities. Searches of this kind facilitated in the gathering of heuristic insights, which informed the formulation of various results and definitions throughout the thesis. The python package was also used extensively in generating and checking most of the examples that appear in this document.

The package can be found at the public repository `https://codeberg.org/rclingman`.

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