C H A P T E R 3

A THEORETICAL APPROACH TO THE POROUS PIPE FLOW PROBLEM

3.1 Introduction

In the previous chapter various theories have been presented which attempt to solve the problem of flow through a porous pipe by means of the viscous Navier-Stokes equations.

In order to be able to reduce the flow equations to forms that could be solved by power series, similarity and several perturbation methods, numerous assumptions or approximations to, either the boundary conditions, or the flow properties had to be made. Each solution was limited to its own range of applicability and even within this range many of the solutions did not appear to agree with the experimental results available. These types of difficulties are frequently encountered with purely mathematical solutions to real flow problems.

The method used to model and examine the flow situation in this chapter is based on an incompressible potential flow technique.

The restriction imposed by incompressibility is thought not to be serious, due to the range of velocities considered. J L Hess and A M O Smith [26] found that the effects of compressibility could be neglected up to local Mach numbers of 0,5. The assumption of potential flow will best be evaluated by reference to comparisons with available solutions and experimental results.

The basic difference between this method and ones proposed previously, is the stage at which approximations and

assumptions are made. The difference may be illustrated as follows:

(i) Solutions given in Chapter 2

[Viscous, incompressible flow equations] + [Simplified and limited boundary conditions] + [Mathematical restrictions] ⇒ [Approximate solution of mathematical formulation]

(ii) Potential flow solution

[Non-viscous, incompressible potential flow equations] + [Unlimited boundary conditions] + [Unlimited geometry] ⇒ [Exact solution (exact implies 'to any prescribed accuracy')]

3.2 The Potential Flow Method

This potential flow method for calculating the incompressible flow about abritrary body shapes was developed by the Douglas Aircraft Company [26]. The company's main interest was in analysing the flow around aerofoils, engine fairings, intakes and fuselages. All these flows are characterised by the fact that they are external flows, (i e flow outside an object as opposed to flow within its boundaries). Mention is made in this work [26] that the method may be used for internal flows, e g flow inside a pipe. There is, however, no further explanation of this statement other than to indicate that there are restrictions imposed by continuity in internal flows.

The method utilises a distribution of sources and sinks over the body surface and the distribution is computed as the solution of an integral equation using a finite element approximation.

The strength of the technique lies in its generality and its flexibility in adapting to numerous boundary conditions for different body geometries. Due to the finite element method of representation the bodies are not restricted to analytical shapes, nor do the boundary conditions have to be analytic or continuous. Because the technique involves potential flow, superposition of different flow situations is possible.

3.3 Mathematical Formulation of the Problem

If non viscous, incompressible flow is assumed the steady flow Navier-Stokes equations (2.2.1) and (2.2.2) reduce to the Eulerian equation of motion:

$$(\underline{V}.\nabla)\underline{V} = -\frac{1}{\rho}\nabla p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.1)$$

and the continuity equation (2.2.3) becomes:

$$\nabla.\underline{V} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.2)$$

These two differential equations are solved using the normal component of fluid velocity on the body surface specified as a boundary condition. The normal velocity may be expressed mathematically as shown in the following expression:

$$\underline{V}.\underline{n}\big|_{\delta} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.3)$$

where δ denotes the body surface, $\underline{n}$ is the normal vector of the surface at a given point, and F is the specified boundary condition at the same point.

The assumption of potential flow implies that the velocity field is irrotational and may thus be written as the negative gradient of a potential function defined as ϕ:

$$\underline{V} = -\nabla\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.4)$$

Using equations (3.3.2) and (3.3.4) it can be shown that the potential, ϕ, satisfies Laplace's equation:

$$\nabla^2\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.5)$$

The boundary condition equation (3.3.3) written in terms of the potential function becomes:

$$-\nabla\phi.\underline{n}\Big|_S = F$$

or

$$\frac{\partial\phi}{\partial n}\Big|_S = -F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3.6)$$

The velocity field in potential flow is determined solely from the equation of continuity and the condition of irrotationality. The equation of motion (3.3.1) may be used to calculate pressures once the velocity field has been established.

The approach used in the present method is to represent the body, in this case a porous pipe, by a distribution of sources and sinks on its surface.

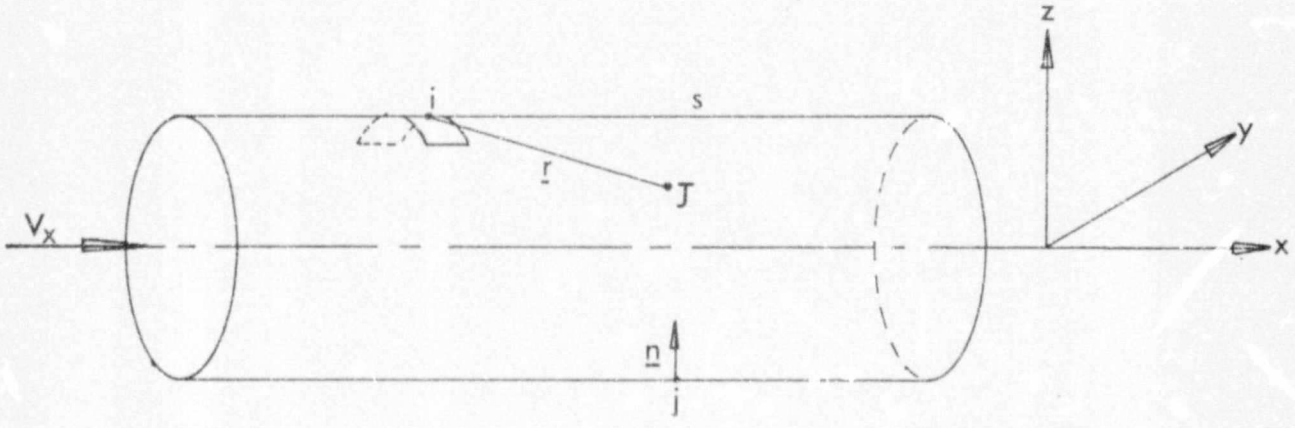

Fig 3.1 - Illustrates the notation used in the analysis with reference to a pipe.

A point source of unit magnitude is located on the surface of the pipe at a position i, (see Fig 3.1). The cartesian co-ordinates of this point are x_i, y_i, z_i. J is a point inside the pipe with co-ordinates x, y, z. The potential induced at J by the point source situated at i is given by:

$$\phi = \frac{1}{r(J,i)} \qquad \ldots\ldots\ldots\ldots \quad (3.3.7)$$

where $r(J,i)$ denotes the distance between points i and J and its magnitude is:

$$r(J,i) = ((x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2)^{\frac{1}{2}}$$

Equation (3.3.7) satisfies Laplace's equation (3.3.5) for any position of J which lies inside the pipe, except at the source, i. This is true for any number of sources and sinks, not necessarily of unit strength, distributed over the surface of the pipe. The total potential at J will be given by the summation of the individual potentials induced by each source. If i now becomes a general point on the surface and $\sigma(i)$ is the strength of the source located at this point, then the total potential at J is given by:

$$\phi = \iint_{\delta} \frac{\sigma(i)}{r(J,i)} \, d\delta \qquad \ldots\ldots\ldots\ldots \quad (3.3.8)$$

Since equation (3.3.7) satisfies Laplace's equation, expression (3.3.8) does also. The precise form of the function $\sigma(i)$ is calculated from the normal velocity boundary conditions given in equation (3.3.6).

In order to use equation (3.3.6) it is necessary to evaluate the spatial derivatives of equation (3.3.8) as the point J approaches a point j on the surface of the pipe (see Fig 3.1). These must be calculated with caution

as the derivatives of $1/r(j,i)$ become singular as the point i approaches the point j. Several potential flow works present rigorous evaluations of the integrand as the singularity is approached, e g Kellog [27]. It is found that the velocity in the normal direction approaches a function consisting of a Dirac delta function plus a function whose value is finite. Combining equations (3.3.6) and (3.3.8) gives:

$$\frac{\partial}{\partial n} \iint_{S} \frac{\sigma(i)}{r(j,i)} \, dS = -F_j$$

where F_j is the prescribed normal velocity at the point j. Applying the results of Kellog [27] to the derivative yields:

$$2\pi\sigma(j) - \iint_{S} \frac{\partial}{\partial n}\left(\frac{1}{r(j,i)}\right)\sigma(i)dS = F_j \quad . \quad . \quad . \quad . \quad (3.3.9)$$

The term $2\pi\sigma(j)$ is a consequence of the delta function. The kernel of the integral

$$-\frac{\partial}{\partial n}\left(\frac{1}{r(j,i)}\right)$$

is the inward normal velocity at the point j due to a unit point source at the point i.

The left hand side of equation (3.3.9) is purely a consequence of the geometry of the pipe and the boundary conditions only appear on the right hand side. This means that for a given pipe geometry several different flow situations may be calculated simultaneously.

The solution of equation (3.3.9) is the principal problem in the present theory. By using a finite element representation of the pipe the equation may be reduced to a set of linear equations in the unknown $\sigma(j)$.

3.4 Geometrical Representation of the Pipe

The pipe under consideration is axisymmetric and so the problem of integrating the left hand side of equation (3.3.9) is simplified. The circumferential integration may be performed immediately. For computational purposes the pipe is comprised of a series of elements (see Fig 3.2).

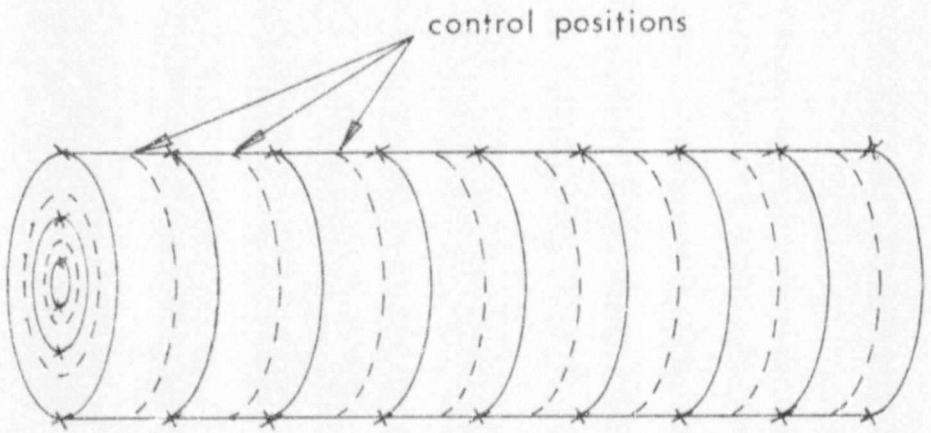

Fig 3.2 - Elemental representation of the porous pipe

The elements are either annular or cylindrical in shape. In the limit, on the pipe axis, the annular ones become circular. The source strength is assumed constant over an element. The term 'source' will be used to indicate source or sink, the resulting sign will indicate which singularity is present. Because of the axisymmetric nature of the problem the pipe may be thought of as having source rings distributed over its surface. Each element is made up of many source rings. Equation (3.3.7) can be adapted to give the potential due to a source ring instead of a point source, and equation (3.3.8) reduces to a single integral expression. Fig 3.3 shows a constant strength source ring of

radius R lying in the yz plane passing through the origin. The potential due to this source ring is calculated at the point J which has, for convenience, co-ordinates x, y, o.

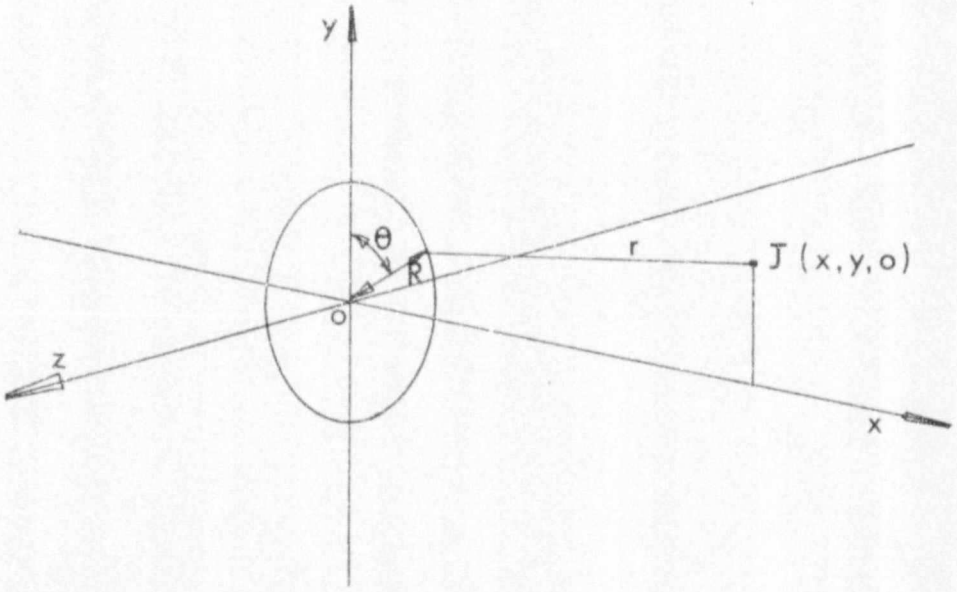

Fig 3.3 - A ring source of unit strength situated in the yz plane at the origin.

The potential at J is given by:

$$\phi = \int_{-\pi}^{\pi} \frac{R}{\pi} d\theta = 2R \int_{0}^{\pi} \frac{d\theta}{(x^2+y^2+R^2-2Ry\cos\theta)^{\frac{1}{2}}} \quad \ldots\ldots \quad (3.4.1)$$

where θ is the angle shown in Fig 3.3.

Velocities in the x and y directions at J may be calculated from the potential flow relationships:

$$V_x = \left(\frac{\partial \phi}{\partial x}\right)_J$$

and

$$V_y = \left(-\frac{\partial \phi}{\partial y}\right)_J$$

Differentiating equation (3.4.1) gives:

$$V_x = \left(\frac{\partial \phi}{\partial x}\right)_J = -2R\int_0^{\pi} \frac{x\, d\theta}{(x^2+y^2+R^2-2Ry\cos\theta)^{\frac{3}{2}}} \quad . \quad . \quad . \quad (3.4.2)$$

and

$$V_y = -\left(\frac{\partial \phi}{\partial y}\right)_J = +2R\int_0^{\pi} \frac{(y-R\cos\theta)\, d\theta}{(x^2+y^2+R^2-2Ry\cos\theta)^{\frac{3}{2}}} \quad . \quad . \quad . \quad (3.4.3)$$

Equations (3.4.2) and (3.4.3) involve elliptic integrals. They are reduced to standard forms by various substitutions and integrations. (See Appendix).

The equations become:

$$\left(\frac{\partial \phi}{\partial x}\right)_J = \frac{-4Rx\ E(k)}{((y+R)^2+x^2)^{\frac{1}{2}}((y-R)^2+x^2)} \quad . \quad . \quad . \quad . \quad . \quad (3.4.4)$$

and

$$-\left(\frac{\partial \phi}{\partial y}\right)_J = \frac{+\ 2R}{y((y+R)^2+x^2)^{\frac{1}{2}}}\left(K(k) + \frac{y^2-R^2-x^2}{(y-R)^2+x^2}E(k)\right) \quad . \quad (3.4.5)$$

where

$$k^2 = \frac{4Ry}{(y+R)^2+\ x^2}$$

and $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds respectively.

Equations (3.4.4) and (3.4.5) express the velocities induced at a point J, inside the pipe, due to a ring source, of unit magnitude, situated on the pipe surface.

Each element of the pipe's surface is comprised of many ring sources of the same strength. The total velocity at the point J due to an element is given by the integrated effect of all these ring sources. (See section 4.3).

The elements of the pipe are each assigned a number for reference purposes and the central ring of each element

is chosen as a control position (see Fig 3.2). The general point J may be situated on the control position of an element. In this case the equations (3.4.4) and (3.4.5) express the velocity induced by an element, i, at another element j's control position. When i = j, the equations express the velocity at an element's control position due to itself. The singularity problem is again encountered. It is overcome by integrating over the element in two parts, this this is explained in detail in Chapter 4.

If ϕ_{ij} denotes the potential at the jth element due to a unit source distribution over element i, then it is possible to calculate a set of such potentials Φ_{ij}, containing n^2 members, where n is the number of elements representing the pipe. Using equations (3.4.4) and (3.4.5) a set of velocities in the x and y directions may also be calculated.

Matrices A and B may now be defined as containing the normal and tangential influence coefficients. Matrix A contains n^2 elements of the type a_{ij}, where a_{ij} is the velocity induced by element i, at element j's control position, in a direction normal to element j. Similarly B consists of n^2 elements of the type b_{ij}, where b_{ij} is the velocity induced by element i, in a direction tangential to element j.

3.5 The Solution of Equation (3.3.9)

To obtain the normal velocities around the surface of the pipe the elements of matrix A must be multiplied by the actual values of the source strength σ.

The total normal velocity at an element j's control position, due to the other elements, is given by the

quantity:

$$\sum_{i=1}^{n} a_{ij} . \sigma_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5.1)$$

where σ_i is the unknown source strength distributed over element i.

It can be seen that expression (3.5.1) is the left hand side of equation (3.3.9) written in elemental form. Equation (3.3.9) may be rewritten as:

$$\sum_{i=1}^{n} a_{ij} . \sigma_i = F_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5.2)$$

where F_j is the prescribed normal velocity boundary condition at the element j's control position.

Equation (3.3.9) has thus been reduced to a set of linear equations which can be solved to give the unknown source strengths.

3.6 The Total Flow Field

An expression for the tangential velocities induced at the control positions due to the other elements can be expressed in the same form as expression (3.5.2), it is:

$$\sum_{i=1}^{n} b_{ij} \sigma_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6.1)$$

b_{ij} has been defined in section 3.4, σ_i is no longer an unknown so expression (3.6.1) can be evaluated. The potentials at the control positions may be evaluated similarly and the relevant expression is:

$$\sum_{i=1}^{n} \phi_{ij} \sigma_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.6.2)$$

The flow properties are now known at the element control positions. They can also be calculated at points inside the pipe. Equations (3.4.1), (3.4.2) and (3.4.3) were derived for a general point J which was not necessarily on the pipe surface. The flow properties at J may be calculated in exactly the same way as if J were on the pipe surface. However, instead of using unit source strengths on the elements their actual magnitudes may now be used.

3.7 Pressure Variations In the Pipe

Coefficients of pressure may be calculated at any positions inside, as well as on, the pipe. The coefficient is defined as:

$$Cp = \frac{(p-p_k)}{\frac{1}{2}\rho V_k^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.7.1)$$

where the subscript k denotes conditions on the axis at the entrance to the pipe.

Since potential flow has been assumed, pressures may be calculated using the equation of motion (3.3.1) (Bernoulli's equation)

$$p_k + \tfrac{1}{2}\rho V_k^2 = p + \tfrac{1}{2}\rho V^2 = p + \tfrac{1}{2}\rho(Vx^2+Vy^2)$$

Because the problem is axisymmetric Vr (radial velocity) may be substituted for Vy. The coefficient of pressure (expression 3.7.1) may be written in terms of velocities that have been previously evaluated:

$$Cp = 1 - \frac{(Vx^2+Vr^2)}{V_k^2}$$

This enables pressure gradients in the axial and radial directions to be studied and compared with experimental results.

3.8 Visualisation of the Flow Field

Using the calculated velocity components a stream function may be evaluated from the potential flow relations.

A stream function is defined as being constant along a stream line and the difference in value between two stream lines gives a measure of the fluid flux between them.

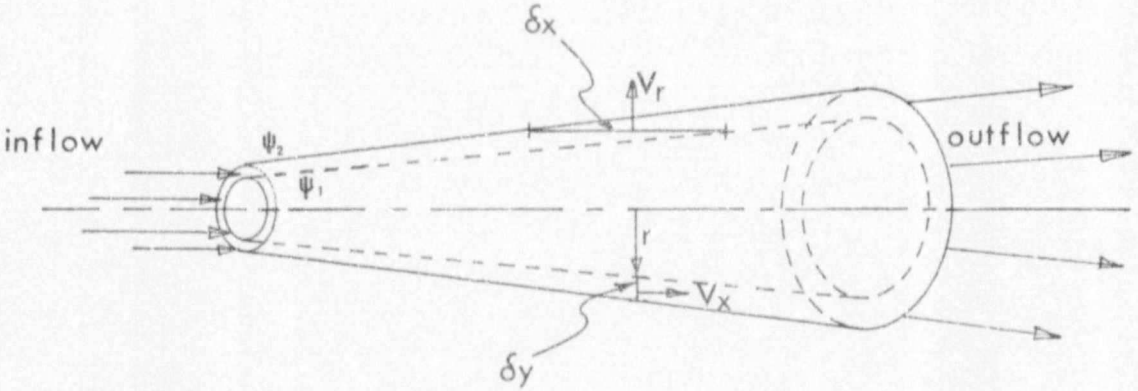

Fig 3.4 - Flow between two stream tubes

Because the problem is axisymmetric, it is more suitable to use stream tubes rather than stream lines (see Fig 3.4). The flux between the stream tubes, ψ_2 and ψ_1, may be calculated in the radial or the axial direction.

From the definition of a stream function:

$$\psi_2 - \psi_1 = \int_{\text{surface } \psi_1}^{\text{surface } \psi_2} V_x 2\pi r.dr = -\int_{\psi_1}^{\psi_2} V_y 2\pi r.dx \quad . \quad . \quad (3.8.1)$$

Expressions (3.8.1) enable stream functions to be evaluated at positions inside and on the pipe. These stream functions

are plotted by means of a contour plotting programme.

Lines of constant potential calculated from expression (3.6.3) are also plotted using the contouring program and prove to be orthogonal to the stream lines calculated using expressions (3.8.1). This provides a check on the accuracy of the integral calculations. The numerical methods given to evaluate the integrals are given in chapter 4.

3.9 Boundary Conditions

The solution of equation (3.3.9) requires a set of boundary conditions F_j. These specify the normal fluid velocities on the pipe surface. For a solid boundary, such as a non-porous pipe wall, these boundary conditions would be zero indicating no flow through the surface. The inlet and exit to the pipe may be thought of as fully porous surfaces, thus the inlet and exit velocity profiles are used as boundary conditions.

The porosity of the pipe may be simulated by allowing flow through the pipe surface. The magnitude and direction (i e either mass addition or extraction) of the flow through the surface is dictated by the values of F_j at the elements' control positions.

Since the boundary conditions need not be analytic or continuous the pipe may be only partially porous. This enables a porous 'test section' attached to lengths of non-porous piping to be simulated and studied. (See Fig 3.5).

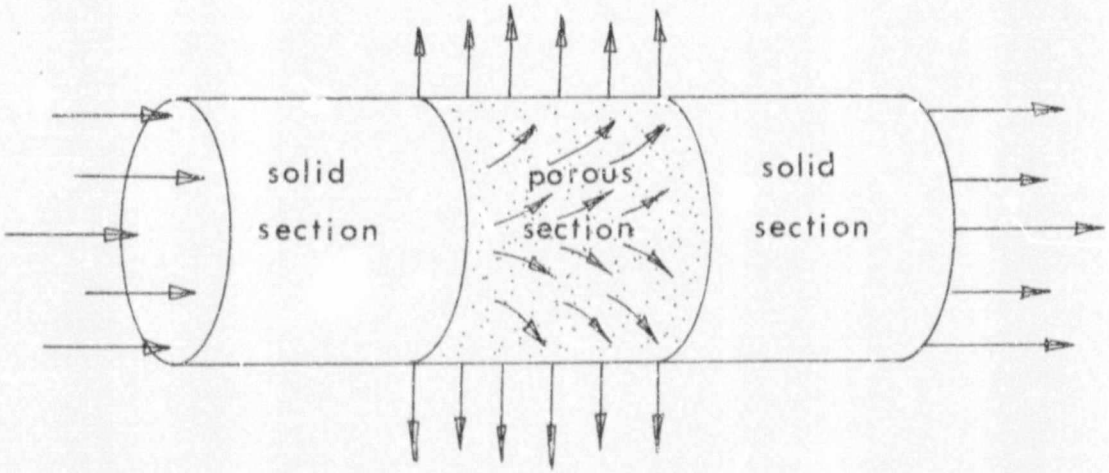

Fig 3.5 - A typical model that can be simulated, showing possible discontinuities in boundary conditions.

A detailed description of solutions obtained for various boundary conditions is given in chapter 4.

3.10. The Effects of Axial Pressure Gradients

In section 3.9 it was mentioned that any distribution of suction or blowing could be applied to the pipe. In the laboratory it is extremely difficult to establish uniform mass transfer along the length of a porous pipe. Constant transfer is approximated in the limit as the porosity tends to zero, or when the axial pressure gradient is negligible compared with that across the porous surface. These restrictions would limit the flexibility of any solution to the porous pipe flow problem, and so it is necessary to be able to establish the actual mass transfer across the surface without these assumptions.

A prerequisite for the solution of the potential flow equation (3.3.9), is that the normal velocities are specified on the surface of the pipe and at its inlet and outlet. Inlet and outlet velocity profiles may assume several forms but basically they are well defined. Problems arise when a mass transfer distribution, along the porous surface, has to be assumed. In practice the mass transfer across a porous surface is controlled by the pressure difference across it (Darcy's law). In the case under consideration the local pressure, inside the pipe, will effect how much flows in (or out) at that position. If p_a denotes the local pressure outside the porous pipe then according to Darcy's law:

$$V_n = k(p-p_a)^n \qquad \ldots\ldots\ldots\ldots \quad (3.10.1)$$

where k is a constant incorporating the porosity of the material.

From the manufacturer's data sheets for the porous material used in the experimental work, it was found that for a large range of velocities the pressure drop through the material versus velocity graphs were linear. It was decided to use this data for the computer program, thus equation (3.10.1) becomes

$$V = k(p-p_a). \qquad \ldots\ldots\ldots\ldots \quad (3.10.2)$$

It would be possible to represent the manufacturer's data by a polynomial fit and incorporate this into the program, but it was thought that the small increase in accuracy that could be obtained did not warrant the extra complications involved.

If equation (3.10.2) is examined closely it can be seen where problems arise. The radial velocity through the porous surface, Vr, is used as a boundary condition in the solution of the flow equation (3.3.9). To calculate V_r it is necessary to know the local pressure, p, inside the pipe. This is, however, only available after equation (3.3.9) has been solved. An iterative method has to be employed.

A mass transfer distribution is assumed, for example a uniform one, indicating constant velocity through the surface along the entire length of porous pipe. The solution to the flow equation is found using the assumed boundary conditions, and coefficients of pressure are calculated inside the pipe as described in section 3.7. The value of V_r at all points on the porous surface may now be revised and a new solution to equation 3.3.9 evaluated. This procedure is repeated until equilibrium is reached, i e until the radial velocities and local pressures cease to change.

One restriction is placed on the porous surface boundary conditions. It must be remembered that, although the radial velocity is changed due to the axial pressure gradient, the total mass flowing through the surface must remain constant, because continuity must be obeyed.

CHAPTER 4

REPRESENTATION OF THE POTENTIAL FLOW METHOD IN A COMPUTER PROGRAMME

4.1 Introduction

Chapter 3 described the outline of the potential flow method used in the solution of the porous pipe flow problem. The basic equation (3.3.9) was reduced to a set of linear equations to be solved for the unknown source strength distribution over the pipe surface, by representing the porous pipe in elemental form. From this source distribution the whole flow field can be determined.

This chapter discusses the finer points of converting the technique into a computer programme. Special attention is given to the difficulties encountered when dealing with the singularities described in section 3.3.

The problem is solved for various pipe flows, ranging from high suction rates (mass extraction) through the porous wall to high rates of mass addition. The flow properties such as pressure and velocity are presented, for a variety of boundary conditions, in the form of graphs as shown in Fig 4.11, Chapter 6 and the Appendix.

Streamlines and lines of constant potential are calculated and drawn by means of a standard I B M contour plotting programme and the resulting contours are illustrated in section 4.10.

Comparisons have been made with existing theories which were discussed in Chapter 2, and these are shown in Chapter 6.

4.2 The Size and Quantity of Elements

The effectiveness of the elemental representation of the pipe depends on how accurately the actual source strength density over the surface of the pipe may be approximated to by uniform source densities over the elements. It seems reasonable to assume that the more elements there are the closer the approximate elemental distribution will approach the exact continuous one.

It was mentioned in section 3.9 that the boundary conditions for the pipe flow are satisfied at the element's control position, this may mean that at positions on the element other than at the control point there may be slight 'leakage' through the surface. These effects may also be reduced by increasing the number of elements used to describe the surface. 'Leakage' is more marked in regions of rapidly changing potential, e g at corners, and in these areas it was found necessary to crowd the elements.

The obvious conclusion drawn from the arguments given above is that as many elements as possible should be used to approximate the body surface. This soon becomes impractical as the time taken for the computer programme to run is not linearly proportional to the number of elements used, but varies almost as the square of the number, and the small increases in accuracy achieved may not warrant the vast increases in running time. Also as more elements are introduced into the programme, accumulative errors, due to truncation and round off errors in the computer, will increase because of an increased amount of calculations.

Several attempts were made at optimising the number of elements required for a pipe of given dimensions. For

a pipe having a length to diameter ratio of 10:1, insufficient accuracy was obtained with less than approximately forty elements. This was shown by large areas of leakage at the corners of the pipe and by the fact that the results continued to change as more elements were included. For more than forty elements the changes in results for increased numbers of elements diminished considerably. The computing time for fifty elements was twenty minutes and this increased to thirty minutes for sixty elements and to forty-five minutes for seventy. It was decided that the number of elements to be used would be in the region of sixty and experience would indicate whether it was necessary to increase or decrease this number for a given case.

For a pipe of given dimensions, once the number of elements has been decided upon, the size of the individual elements may be determined. If several small elements are in the vicinity of a large one, the accuracy obtained in that region is that associated with the large element. This is because the accuracy of the calculations in a certain area, due to the elemental representation of the pipe is proportional to the size of the elements in the region under consideration. If some of the elements are made smaller whilst a few remain large the inaccuracy of the calculations will still be dominated by the large ones. Thus distributions like this are inefficient since no additional accuracy results from the inclusion of the small elements. It was decided that the dimensions of an element should be no more than fifty per cent greater than those of adjacent elements. In this way the size of the elements may be made to change gradually from large to small around the pipe's surface.

Elements are input into the programme in x and y co-ordinates. As mentioned previously the problem is axisymmetric and so

a point in the x-y plane represents a ring in the three dimensional transform.

The elements are defined in a clockwise manner as shown in Fig 4.1.

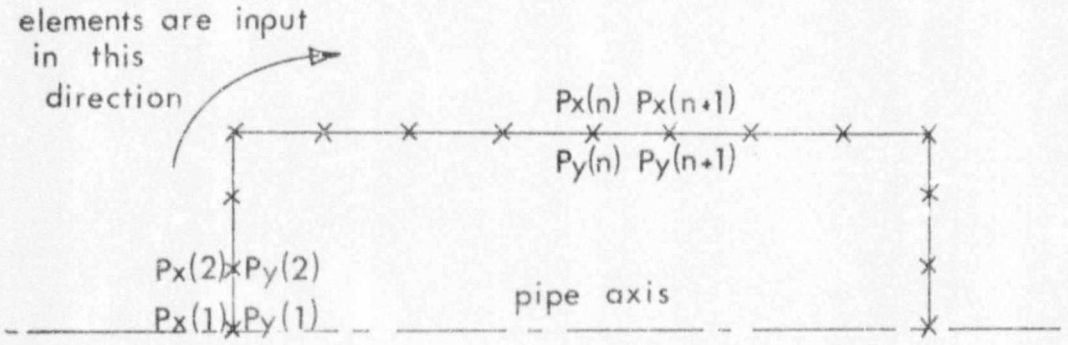

Fig 4.1 - Shows the procedure adopted for inputting the elements into the programme.

It must be remembered that although element P(n) - P(n+1) appears to describe a straight line it does in fact represent a cylinder.

The most critical features of the pipe geometry are the sharp corners. As was mentioned above it is necessary to have a concentration of elements in this region. It is, however, impossible to eliminate 'leakage' at the corners entirely but it may be minimised by a suitable choice of smaller elements. Another possible solution to the 'leakage' problem would be to round the corner slightly. This was attempted but the boundary conditions became awkward to define, see Fig 4.2.

At the rounded corner the normal velocity boundary condition was evaluated such that its component in the axial direction gave the axial input velocity at that position.

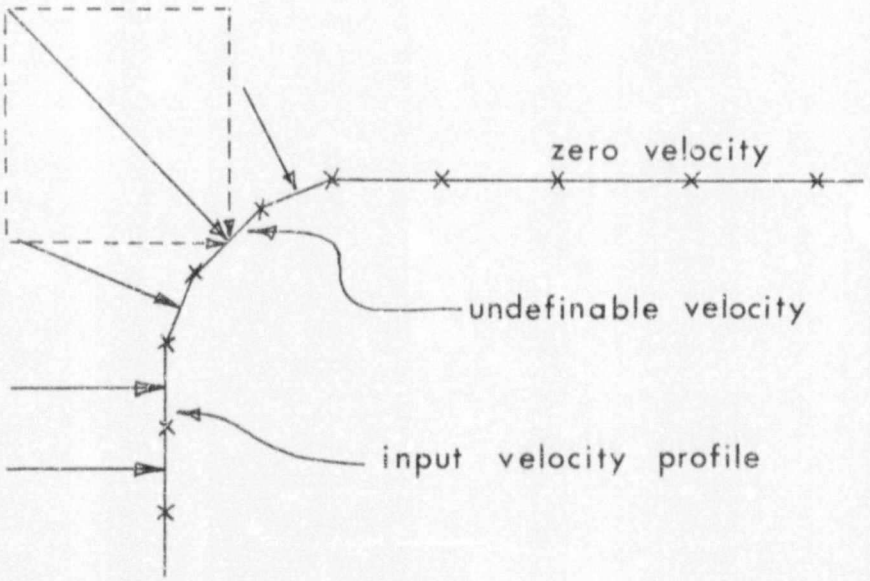

Fig 4.2 - Shows the difficulties encountered when rounding the pipe corners.

In introducing this boundary condition, however, a radial velocity component was also introduced, which effectively was the same as introducing leakage into the pipe's surface.

It was decided to reduce the 'leakage' due to the sharp corner as much as possible by merely crowding the elements in this regio[illegible]

4.3 Evaluation of the Influence Coefficient Matrices

Once the pipe has been defined and the number and size of elements has been determined the next step is to evaluate the influence coefficient matrices A, B and Φ as described in section 3.4. Expressions (3.4.4) and (3.4.5) are used to evaluate the velocities induced at an element j's control position, due to a ring source, of unit magnitude situated on an element i. To find the total effect of an element i at element j's control position it is necessary

to sum the individual effects of all the ring sources which are distributed over element i. This is done by a numerical integration method described in section 4.4. To complete the matrices A and B the diagonal elements must be calculated. This is where extra care is required. The diagonals of the matrices represent the effects an element induces on its own control position, i e when i = j. The singularity problem, which has been explained in sections 3.3 and 3.4, has to be overcome and so a special numerical integration method is used for the singular elements and is described in section 4.5.

4.4 Integration over a Non-Singular element

Consider a line segment element as shown in Fig 4.3. To simplify calculations and programming techniques the x origin of the system is moved to the centre of the element, whereas the y co-ordinate remains unchanged, thus enabling expressions for each element to be identical. For complete generality the slope of the segment with respect to the positive x - axis is denoted by β.

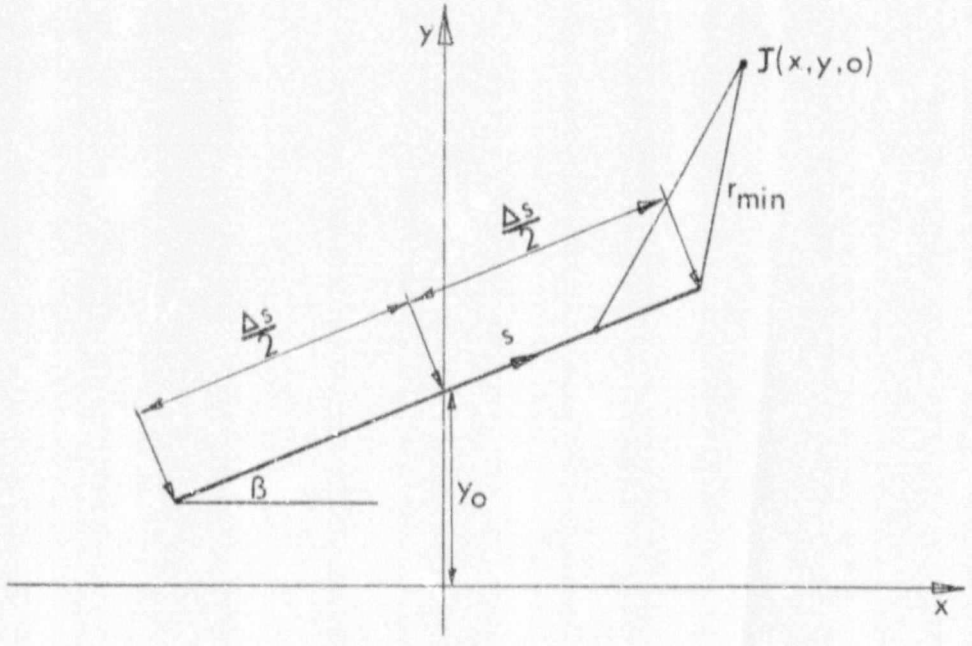

Fig 4.3 - Integration Over a Line Segment

δ is the distance measured along the line from the mid point. Δs is the total length of the segment. r_{min} is the shortest distance from one end of the segment to the external point J.

Expressions (3.4.4) and (3.4.5) are given in terms of the source radius R. From Fig 4.3 it is easily seen that:

$$R = y_o + \delta \sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4.1)$$

The expressions are also only valid for a ring source situated at the origin, but this does not present any severe difficulties as the x co-ordinate may be decremented by an amount b, where:

$$b = \delta \cos\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4.2)$$

Thus for a ring source situated anywhere on the line segment shown in Fig 4.3, the expressions (3.4.4) and (3.4.5) may be evaluated by suitable inclusions of equations (4.4.1) and (4.4.2) for each position, i e the expressions now become:

$$\left(\frac{\partial\phi}{\partial x}\right)_J = \frac{-4R(x-b)E(k)}{((y+R)^2+(x-b)^2)^{\frac{1}{2}}((y-R)^2+(x-b)^2)} \quad . \quad . \quad . \quad . \quad (4.4.3)$$

$$-\left(\frac{\partial\phi}{\partial y}\right)_J = \frac{2R}{y((y+R)^2+(x-b)^2)^{\frac{1}{2}}}\left(K(k) + \frac{y^2-R^2-(x-b)^2}{(y-R)^2+(x-b)^2}E(k)\right) \quad (4.4.4)$$

and

$$k^2 = \frac{4Ry}{(y+R)^2+(x-b)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4.5)$$

The methods of actually evaluating $E(k)$ and $K(k)$, the elliptic integrals, is given in section (4.6)

The required integrations are performed numerically by the

trapezoidal rule. The line segment is divided by equally spaced points into a number of subsegments. The number of subelements is taken as:

$$16\Delta s/r_{min}$$

rounded to the nearest integer. This means that the further the point J lies from the element, the fewer the number of subelements used in the calculation. The number 16 is purely arbitrary and any other number could certainly be used, however 16 appears to give good results.

The above calculations have enabled the elements in the influence matrices to be calculated other than the ones which lie on the diagonals, i e the singular ones.

4.5 Integration over a Singular Element

Expressions (4.4.3) and (4.4.4) for the velocity components Vx and Vy are singular for $x = b$ and $y = R$. In the transformed co-ordinate system the singularity occurs at $s = 0$. The evaluation of integrals involving the singularities requires the calculation of principal values. J L Hess and A M O Smith [26] suggested a method for dealing with these types of element and their method is adopted here.

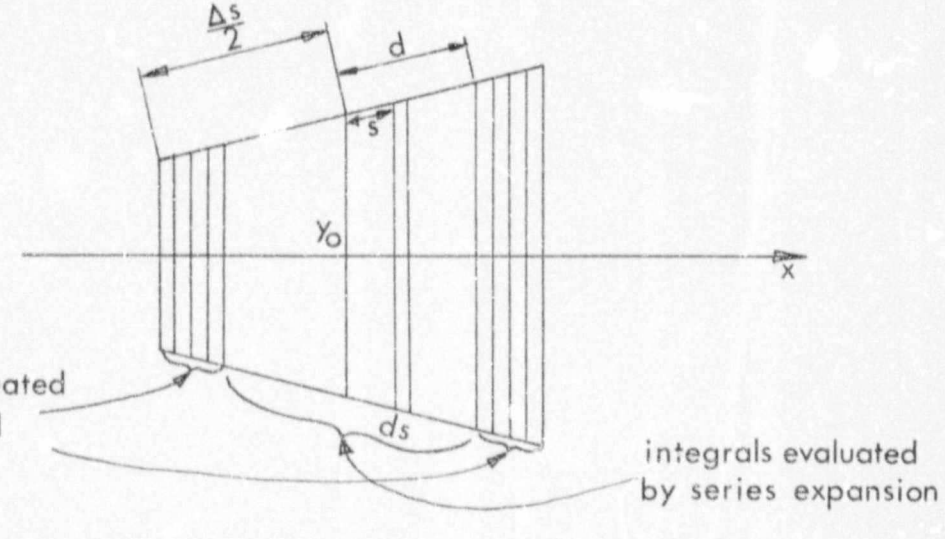

Fig 4.4
The Singular Subelement

Fig 4.4 illustrates how the singular element is divided:

(i) Two extreme regions over which the integral may be evaluated by the trapezoidal ruie in the ordinary way.

(ii) The singular subelement of width 2d over which the integral is evaluated by series expansion.

The choice of the magnitude of d is discussed later.

The expressions (4.4.3) and (4.4. are expanded in terms of δ/y_o (see Appendix), and the resulting expressions are integrated between $\delta = -d$ to $\delta = +d$. The contributions of the singular subelement to the velocity at the mid point are:

$$V_x = -\sin 2\beta\left(\frac{d}{y_o}\right)\left(1 + \frac{1}{144}\left(\frac{d}{y_o}\right)^2\left[13+6\sin^2\beta+6\ell n\left(\frac{1}{8}\frac{d}{y_o}\right)\right]+ \ldots.\right) \quad . \quad . \quad (4.5.1)$$

$$V_y = \cdot 2\left(\frac{d}{y_o}\right)\left(\left[\sin^2\beta+\ell n\left(\frac{1}{8}\frac{d}{y_o}\right)\right] - \frac{1}{48}\left(\frac{d}{y_o}\right)^2\left[3\cos^2\beta-2\sin^4\beta + 3\ell n\left(\frac{1}{8}\frac{d}{y_o}\right)\right] + \ldots.\right) \quad . \quad . \quad . \quad . \quad (4.5.2)$$

The same may be done for the potential induced at the mid point. (See Appendix). The resulting expression is:

$$\phi = 4y_o\left(\frac{d}{y_o}\right)\left(\left(1-\ell n\left(\frac{1}{8}\frac{d}{y_o}\right)\right) - \frac{1}{44}\left(\frac{d}{y_o}\right)^2\left(2-2\sin^2\beta + 3(1+2\sin^2\beta)\ell n\left(\frac{1}{8}\frac{d}{y_o}\right)\right) + \ldots.\right) \quad . \quad . \quad . \quad (4.5.3)$$

The selection of the distance d depends on two things. It is desirable to have d as small as possible to minimise

truncation errors introduced by the series (4.5.1) to (4.5.3), but it is also desirable to have d as large as possible to reduce errors occuring from the use of numerical integration techniques near a singularity. The truncation errors depend on the magnitude of d/y_o and the integration errors are proportional to $\frac{1}{d}$.

It is found that the most difficult elements to deal with are those with small values of y_o, that is elements near the axis of the pipe, at the inlet and outlet.

The rule for calculating d, which was developed by J L Hess and A M O Smith [26] by trial and error, is used in the programme.

$$d = 0{,}08\ y_o \quad \text{if} \quad 0{,}08\ y_o < \Delta s/2$$

$$d = \Delta s/2 \quad \text{if} \quad 0{,}08\ y_o > \Delta s/2$$

So far any contribution to the velocity that depends on the limiting process of approaching the surface is not included and must be added separately. In Chapter 3 it was mentioned that the limiting process of approaching the surface gives rise to a velocity of magnitude 2π, for unit value of source density in a direction normal to the element.

The velocity components induced by an element at its own mid point consist of the sum of three contributions:

(i) the numerical integration of the ends of the element
(ii) the series integration over the middle portion
(iii) the components arising from the limiting process of approaching the surface.

The potential induced by an element at its own mid point consists of contributions from the first two of these three.

4.6 Calculation of the Elliptic Integrals

The evaluation of expressions (4.4.3) and (4.4.4) at various points on an element require the evaluation of elliptic integrals of the first and second kinds.

The computer programme calculates these by using a series representation of the integrals. The first step performed is the calculation of the argument k, using expression (4.4.5). The complete integrals in series form are:

$$K(k) = 1.0 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{2.4}\right)^2 k^4 + \left(\frac{3.5}{2.4.6}\right)^2 k^6 + \ldots . \quad . \quad . \quad (4.6.1)$$

and

$$E(k) = 1.0 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{3}{2.4}\right)^2 \frac{k^4}{3} - \left(\frac{3.5}{2.4.6}\right)^2 \frac{k^6}{5} - \ldots . \quad . \quad . \quad (4.6.2)$$

These are evaluated by means of a 'DO LOOP' in the programme and are terminated as subsequent terms become negligible.

The previous three sections have described how the elements in the influence coefficient matrices A and B are calculated in the computer programme.

4.7 The Boundary Conditions

The calculations so far have only depended on the geometry of the pipe, i e the length to diameter ratio. It is now necessary to discuss the types of boundary conditions that may be used to solve the entire problem. As was mentioned in section 3.9, it is possible to have a partially porous pipe. This enables the situation studied in the laboratory to be simulated exactly. A length of non-porous pipe may precede the porous test section. The differences in porosity are simply realised as changes in boundary