

Aspects of Electromagnetic Field Distributions in Multipath Conductive Structures

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Declaration

I declare that this thesis is my own, unaided work, other than where specifically acknowledged. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other university.

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Abstract

Conductive structures composed of multiple layers are found in a number of problem and applications. These multi-layer conductors normally present themselves in the form of flat conductive layers stacked on top of one another or cylindrical conductive layers centred on the same axis. Even though significant details vary between problems, the majority of their solutions are dependent on the determination of the current density, electric field and magnetic field distribution within these layers. Two groups of multi-layer conductors are identified which comprise the majority of multi-layer problems. The first group, predominantly concerned with eddy current testing and magnetic shielding, is shown to have received significant analytical investigation over the last fifty years. The second group, even though under consideration for the same amount of time, lacks any significant analytical development, with existing analysis limited to numerical simulations and experimental measurements.

This thesis addresses this second group's shortcomings, by focussing on the development of analytical solutions to the current density, electric field and magnetic field, along with their subsequent impedance. These focus points are selected as they form the fundamental building blocks of these multi-layer conductors, and will be invaluable in the development of current and future multi-layer applications.

This investigation starts by performing an in depth study of single layer equations, their existing solutions and factors governing them. A number of important findings are highlighted. It is firstly shown that the flat conductor approximations, which are required to allow for the single-layer flat conductor solution, produce negligible error under a given thickness-to-width ratio. Secondly, it is found that the well-known solution of Dowell has inherent limitations, while lastly, a previously proposed wave analyses method is shown to be identical to the existing single layer solution.

These findings are subsequently used in the development of a multi-layer methodology, where the methodology is based on the continuity of the tangential components of the electric field and magnetic field intensity. It enables the derivation of a solution to the current density, electric field and magnetic field intensity of a multi-layer conductor, with no limitation placed on the number of layers. The current density, electric field and magnetic field intensity solutions to the two- and three-layer flat and cylindrical conductors were subsequently derived, and verified through FEM. The advantages of these analytical solutions, over that of FEM, are clearly shown. The two most predominant being, near instantaneous results and insight into the factors governing the distributions through visual inspection. It is, however, found that a direct relation exists between the solution complexity and the number of layers, resulting in visual inspection becoming difficult for three-layers or more.

With the use of the distribution equations, impedance equations are derived. They are subsequently applied to three-layer differential conductor configurations and verified through experimental measurement. Measurement was achieved through the use of a precision LCR meter. The experimental verification serves two main purposes. Firstly, as FEM simulations are just an approximation and can suffer from convergence issues, it was used to verify the derived

impedance equations. Secondly, the verification of the impedance equations is used to infer the correctness of the distribution solutions, on which the impedance equations are based, which in turn infers the validity of the multi-layer methodology.

This thesis therefore provides a significant advancement in the understanding of multi-layer conductors, with one of the most important contributions being the development of the multi-layer methodology. This methodology serves as an enabling tool, allowing for the derivation of multi-layer conductor distributions equations, along with their subsequent impedance equations. With the use of these equations, freedom from numerical simulations methods (such as FEM) is achieved, while providing a means to understand the factors governing these conductors from more than just a qualitative perspective.

Opsomming

Geleidende strukture saamgestel uit veelvoudige lae word gevind in 'n aantal probleme en toepassings. Hierdie multi-laag geleiers is normaalweg in die vorm van plat geleidende lae wat bo-op mekaar gestapel is of silindriese geleiers wat om dieselfde as gesentraliseerd is. Alhoewel aansienlike detail tussen die probleme verskil, is die meerderheid oplossings afhanklik van die bepaling van die stroomdigtheid, elektriese veld en magnetiese veld verspreiding binne die lae. Die meerderheid multi-laag probleme val in een van twee groepe. Die eerste groep, wat oorheersend gemoeid is met werwelstroom toetsing en magnetiese afskerming, het aansienlike analitiese ondersoeke oor die laaste vyftig jaar ondervind. Die tweede groep, alhoewel net so lank ondersoek, het nog geen betekenisvolle analitiese ontwikkeling ondergaan nie, met die resultaat dat huidige analise tot numeriese simulaties en eksperimentele metings beperk is.

Hierdie tesis spreek hierdie tweede groep se tekortkominge aan, deur te fokus op die ontwikkeling van analitiese oplossings vir die stroomdigtheid, elektriese veld, magnetiese veld en hulle daaropvolgende impedansie. Hierdie punte is gekies aangesien hulle die fundamentele boublokke van multi-laag geleiers vorm, en sal van onskatbare waarde wees in die ontwikkeling van huidige en toekomstige multi-laag toepassings.

Hierdie ondersoek begin met 'n in diepte studie van die enkel laag vergelykings, hulle huidige oplossings en die faktore wat hulle beheer. 'n Aantal belangrike ontdekkings word uitgelig. Daar word eerstens gewys dat die plat geleier benadering, wat vereis word vir die enkel-laag plat geleier oplossing, 'n weglaatbare fout het onder 'n gegewe dikte-tot-wydte verhouding. Tweedens, word dit bevind dat die wel bekende oplossing van Dowell 'n inherente beperking het. Laastens, 'n vorige voorgestelde golf analise metode word uitgewys om identies te wees aan die huidige enkel-laag oplossing.

Hierdie bevindinge word daaropvolgend gebruik in die ontwikkeling van 'n multi-laag metodiek, met die metodiek gebaseer op die kontinuïteit van die tangens komponent van die elektriese veld en die magnetiese veld intensiteit. Dit stel die afleiding van die stroomdigtheid, elektriese veld en magnetiese veld van 'n multi-laag geleier in staat, met geen beperkinge op die aantal lae geplaas nie. Die stroomdigtheid, elektriese veld en magnetiese veld oplossing tot die twee- en drie-lae plat en silindriese geleiers word hierna afgelei, en bevestig deur eindige element analise. Die voordele van hierdie analitiese oplossing, in vergelyking met eindige element analise, word duidelik uitgewys. Die twee mees oorheersende is, die feitlik oombliklike resultate en die insig in die faktore wat die veldverspreidings beïnvloed deur direkte inspeksie van die vergelykings. Dit is wel gevind, dat daar 'n direkte verband tussen die ingewikkeldheid van die oplossing en die aantal lae is. Dit maak direkte inspeksie moeilik vir drie of meer lae.

Met die gebruik van die verspreidings vergelykings word impedansie vergelykings afgelei en daaropvolgend toegepas op drie-laag differentiele geleiers. Hierdie vergelykings word eksperimenteel bevestig deur gebruik te maak van 'n presisie LCR meter. Twee doele is bereik deur hierdie eksperimentele bevestiging. Eerstens, aangesien eindige element analysesimulasie net 'n benadering is en onderhewig is aan konvergensie probleme, is die praktiese meting gebruik om die afgeleide impedansie vergelykings te bevestig. Tweedens, die bevestiging van

die impedansie vergelykings word gebruik om die korrektheid van die verspreidings oplossings te bevestig, aangesien die impedansie vergelykings op dit gebaseer is. Aangesien die verspreidings oplossings gebaseer is op die multi-laag metodiek, impliseer dit die korrektheid van die multi-laag metodiek.

Gevolgtik verskaf hierdie tesis 'n aansienlike verbetering van die begrip van multi-laag geleiers, waar die multi-laag metodiek die belangrikste van die bydrae is. Hierdie metodiek stel in staat die afleiding van multi-laag geleier veldverspreidings vergelykings, asook die daaropvolgende berekening van impedansie vergelykings. Hierdie vergelykings verskaf vryheid van numeriese simulatie metodes (soos eindige element analise), terwyl dit toelaat vir die verstaan van die beherende faktore uit meer as net 'n kwalitatiewe perspektief.

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I would like to start with the most important person in my life, my loving wife Kerry. Thank you my darling for your undying support during the last three years. The late nights you sat with me, the patience you exerted, your willingness to listen to my explanations of my work, the financial support you provided for us for the last two years, your words of encouragement and the fact that you're always willing to help are true indications of how lucky I am to have you in my life. For a more loving, intelligent or beautiful wife I could never ask for! You really are the best thing that could have happened to me.

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Frequently used Symbols

Latin

a	Width of a turn	m
B	Magnetic flux density	T or Wb/m ²
b	Winding breadth	m
C	Capacitance	F
E	Electric field	V/m
E_{TEM}	Electric field component of a TEM wave	V/m
E_{TEM+}	Electric field component of the positive travelling TEM wave	V/m
E_{TEM-}	Electric field component of the negative travelling TEM wave	V/m
E_{TEM+_R}	Real component of E_{TEM+}	V/m
E_{TEM+_I}	Imaginary component of E_{TEM+}	V/m
E_{TEM+_M}	Absolute magnitude of E_{TEM+}	V/m
E_{TEM-_R}	Real component of E_{TEM-}	V/m
E_{TEM-_I}	Imaginary component of E_{TEM-}	V/m
E_{TEM-_M}	Absolute magnitude of E_{TEM-}	V/m
f	Frequency	Hz
H	Magnetic field intensity	A/m
H_{sn}	Magnetic field intensity at one of the boundary of layer n	A/m
h	Height	m
I	Current	A
I_T	Total current	A
I_n	Current flowing through layer n	A
I_{peak}	Peak current	A
I_{rms}	rms value of the current	A
I_{T_peak}	Total peak current	A
$I_v(\xi)$	Modified Bessel function of the first kind of order v and argument ξ	
J	Current density	A/m ²
J_p	Peak current density	A/m ²
J_{skin_p}	Peak current density of the skin-effect component	A/m ²
J_{prox_p}	Peak current density of the proximity-effect component	A/m ²
J_{rms}	rms value of the current density	A/m ²
J_x	Current density in the x-direction	A/m ²
j	Imaginary	
$K_v(\xi)$	Modified Bessel function of the second kind of order v and argument ξ	
k	Wavenumber	m ⁻¹
L	Inductance	H
l or ℓ	length	m
M	Subscript M indicates absolute magnitude	
N_1	Number of turns per layer	
P	Power	W
P_T	Total power	W
P_{avr}	Average power	W
P_{skin}	Skin-effect component of the power	W
P_{prox}	Proximity-effect component of the power	W
P	Subscript P normally indicates peak value	

R	Resistance	Ω
$\vec{S}$	Energy density flow vector	W/m^2
S	Surface/Area	m^2
S_n	Source for conductor n	
T	Period	S
t	time	s
W	Energy	J
w	Width of a flat conductor	m
$\hat{x}$	x-direction of the Cartesian coordinate system	
$\hat{y}$	y-direction of the Cartesian coordinate system	
$\hat{z}$	z-direction of the Cartesian coordinate system	

Greek

α	Attenuation coefficient	m^{-1}
β	Phase constant	rad/m
Γ	Reflection coefficient	
γ	Propagation constant	m^{-1}
ϵ	Permittivity	F/m
ϵ'	Real component of the complex permeability	F/m
ϵ''	Loss component of the complex permittivity	F/m
ϵ''_{NC}	Dielectric losses associated with the loss component of the complex permittivity	F/m
η_n	Impedance of medium η	Ω
η_{nR}	Real component of the impedance of medium η	Ω
η_{nI}	Imaginary component of the impedance of medium η	Ω
η	Conductor spacing factor/layer porosity/layer copper factor	
θ	Phase angle	rad
μ	Permeability	H/m
μ'	Real component of the complex permeability	H/m
μ''	Loss component of the complex permeability	H/m
ρ	Charge density	C/m^3
$\hat{\rho}$	Radial direction of the cylindrical coordinate system	
σ	Conductivity	S/m
$\hat{\phi}$	Azimuth direction of the cylindrical coordinate system	
Φ	Magnetic flux	Wb
ϕ	Phase angle	rad
ω	Angular frequency	rad/s

Chapter 1

Introduction

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1.1 Introduction

A multi-layer conductor refers to any conductor that is composed of multiple individual conductive layers that are connected together, with the boundaries of the layers parallel to one another. They are found in a number of different problems, with the conductor geometries and the orientation of the current, magnetic field, electric field and conductor differing between them. Their two most common geometries are, however, flat and cylindrical, as shown in Figure 1.1, where their most generic forms are given. As can be seen, each conductor consists of n individual layers, each with its own electric and magnetic properties. The value of n varies drastically between applications and the problem under consideration. Any current flowing within such a conductor will distribute itself among the layers, depending on the properties of the conductor. This is irrespective of whether the current originates from a source, or is induced via an external magnetic field.

Based on the applications of these multi-layer conductors, the two most prominent arrangements can be identified. An arrangement is defined, in this thesis, as a unique orientation of the conductive layers with regards to the field and current direction. The first arrangement is characterised by the magnetic field being perpendicular to the surface of the multi-layer conductor. Well known problems, such as eddy current testing and magnetic shielding, comprise the majority of this arrangement's applications and are well researched topics. The second arrangement is defined for the magnetic field parallel to the multi-layer conductor and includes applications such as dissipative filters and coaxial cables. This arrangement has, in contrast, received very little investigation and, as such, a large paucity of information exists.

This paucity will therefore form the basis of this thesis' objectives. However, in order to place them into perspective, an overview of both arrangements is required, and given in the next two sections. The most common problems with regards to each arrangement will be discussed, along with their existing solutions.

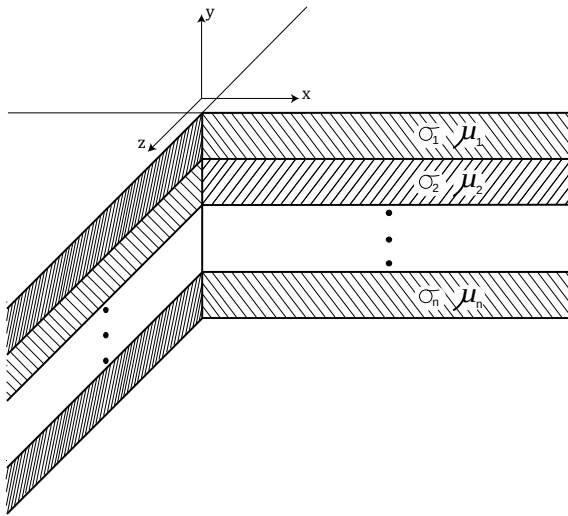


Figure 1.1a. Flat multi-layer conductor

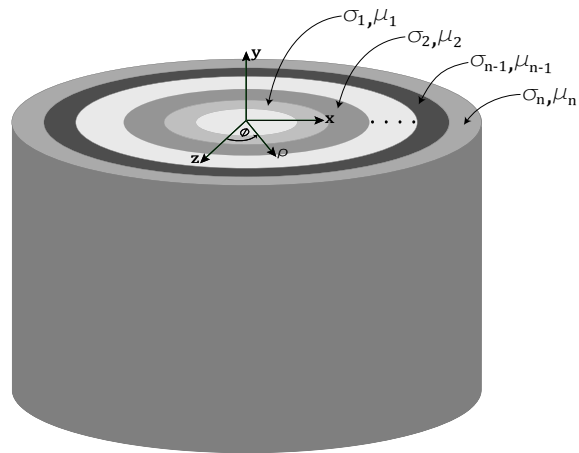


Figure 1.1b. Cylindrical multi-layer conductor

1.2 Arrangement 1: The magnetic field perpendicular to the multi-layer conductor

As was mentioned, this arrangement is defined for the magnetic field perpendicular to the surface of the conductor, with the current therefore confined to the plane in which the conductor lies. In both the flat and cylindrical multi-layer conductors of Figure 1.1 the magnetic field will be in the y -direction, with the current in the xz -plane. This arrangement is predominantly encountered in the problems of eddy current testing and magnetic shielding. Since significant details differ between these two problems, they will be discussed separately, starting with eddy current testing.

1.2.1 Eddy current testing

For this problem, the flat conductor of Figure 1.1a is taken to be infinitely long and wide (z - and x -directions). The thickness (y -direction), however, is taken to infinite for only half of the problem space. With regards to Figure 1.1a, all the space above the first layer can therefore be considered to be free space. The conductor itself will consist of a finite number of layers of finite thickness, with the last layer, however, stretching infinitely far in the negative y -direction. This is shown in Figure 1.2a, where the current carrying coil, which is responsible for the magnetic field (in the y -direction), can be seen. The cylindrical conductor under the eddy-current problem, is not nearly as complex, with only the length taken as infinite. This is also shown in Figure 1.2b, where the magnetic field creating coil can be seen to be situated around the cylindrical conductor.

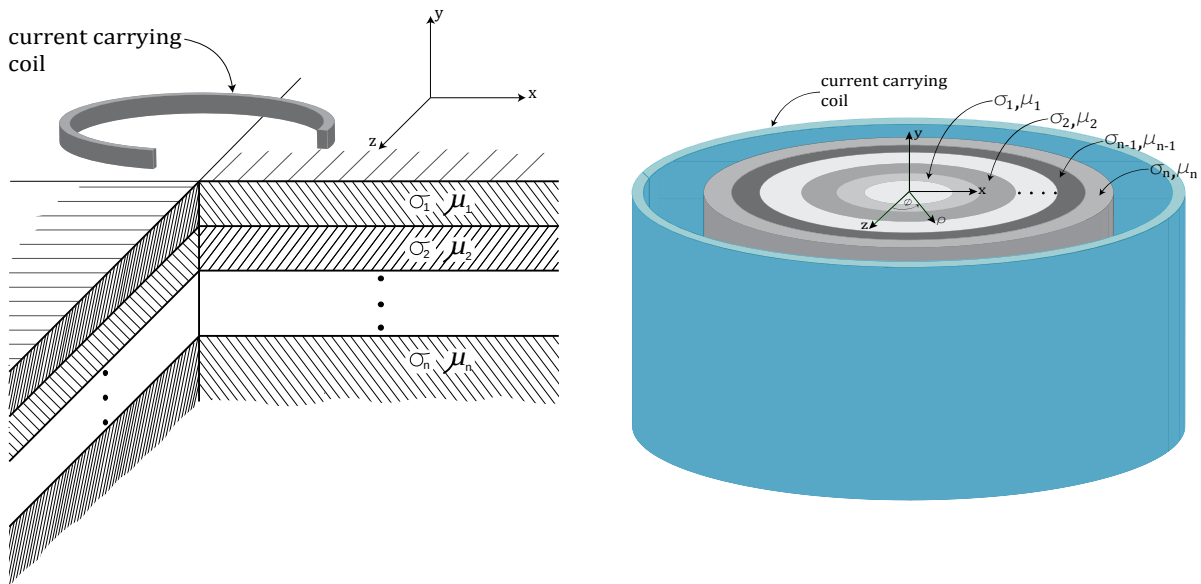


Figure 1.2

As the name suggests, these problems are concerned with the case where eddy currents are induced in conductors through the use of an excitation coil. Generally, the work has been concerned with the multi-layer conductors presented in Figure 1.2. However, more recent work has also considered the case where the conductivity varies continuously with distance (y -direction for Figure 1.2a and ρ -direction for Figure 1.2b). Since this second case is just the limit as an infinite number of layers' thicknesses approach zero, they will both be grouped under the same problem and discussion.

A large number of authors have developed methods to predict the resulting magnetic field, current density and reflected impedance (as seen by the coil) within these conductors ([1]-[9]). This is known as the “forward solution”. These methods consist, among others, of analytical equations ([1]-[4],[9]), with [2]’s solution being one of the more frequently used solutions, and is based on integrals of first-order Bessel functions. [8] expands on his work by replacing the integrals by a truncated region eigenfunction expansion (TREE) method. While [6]’s solution uses a matrix technique, where the multi-layer conductor is represented by stacked inductively coupled, concentric metal rings.

As eddy current testing forms part of electromagnetic nondestructive evaluation (ENDE) techniques, these solutions have been used and adapted, along with a large number of other techniques, to help predict material properties from given impedance measurements ([10]-[16]). This is known as the “inverse solution”. A common method is that of iteratively adjusting the theoretical predictions by use of the solutions discussed above, in order to match the measured results ([11][15]). A large number of more unconventional methods also exist, such as; neural networks ([12]), multi-frequency eddy current inversion technique ([13]), peak frequency detection ([14]), time-domain pulsed eddy-current technique ([10]) and a double coil method ([16]).

1.2.2 Magnetic shielding

The magnetic shielding is the second problem that falls under this arrangement. Both flat and cylindrical multi-layer shielding conductors exist. However, since the majority of the cylindrical shields do not adhere to the restrictions of this arrangement, but rather to that of arrangement 2, they will rather be discussed in the next section. For this problem, Figure 1.1a consists of a finite number of layers, with the electric and magnetic properties varying between them. Figure 1.3 shows this arrangement, along with the different regions comprising the shield. The length and width can be taken to be much larger than the region under consideration.

As is clear, the purpose of this conductor is to provide magnetic shielding from an external source incident on its surface. The shield normally consists of predominantly conductive and predominantly magnetic layers, interleaved to form a multi-layer conductor. It was shown by [17] and [18] that the interleaving of these layers provides improved effectiveness, as opposed to the alternative configuration of grouping all the conductive and magnetic layers separately and stacking the two groups on top of one another.

The predominant focus, to date, was to investigate the shielding effectiveness of these conductors. [19] presented a theoretical analysis in order to better understand what governs these shields, which produced explicit expressions for the magnetic field and power loss. A methodology for accurate calculation of shielding factors was given by [18], in which he used transfer relations for each layer, which were then spliced together. [20] continued on this work, by letting the transfer relations express the relationship between the spatial Fourier transforms of the tangential magnetic field intensity and the normal magnetic flux density at the surfaces of each layer. An alternative approach was taken by [21], in which a single non-ferrous layer of finite width and length was modelled by an array of distributed circuit components. [22]

undertook a comparison of this circuitual approach ([21]) to the Finite Formulation of Electromagnetic Fields (FFEF) of [23], and adapted by [24], in order to establish which is better suited to determine the induced current density and resultant magnetic flux density.

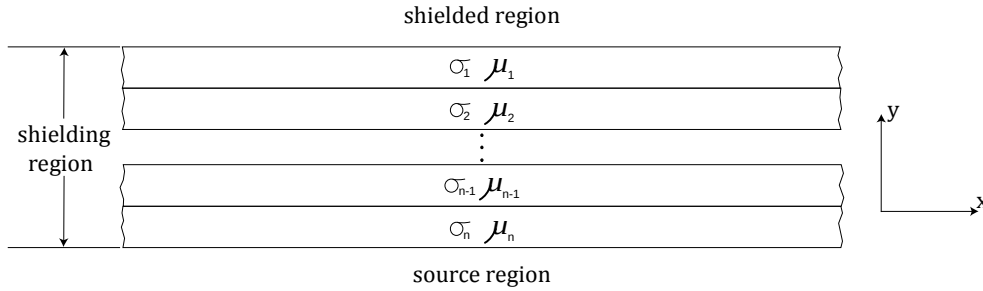


Figure 1.3. Multi-layer flat conductor used for magnetic shielding

1.3 Arrangement 2: The magnetic field parallel to the multi-layer conductor

This is the second predominant multi-layer conductor arrangement. The defining conditions of this arrangement are that both the magnetic field and current density are parallel to the surface of the conductor, perpendicular to each other, and is confined to one direction only. With regards to Figure 1.1a, the magnetic field will be in the z-direction and the current in the x-direction, while for Figure 1.1b the current will be in the y-direction and the magnetic field in the azimuth-direction. This is shown in Figure 1.4. For the cylindrical conductor the length ($\hat{y}$), and for the flat conductor the length and width ($\hat{x}$ and $\hat{z}$), will therefore have to be either infinite or approximated as such. This arrangement is encountered in a number of problems, such as dissipative filters, armoured or reinforced cables, coaxial cables, and magnetic shielding. Since no significant details differ between them, they will all be discussed within this section on a problem by problem basis.

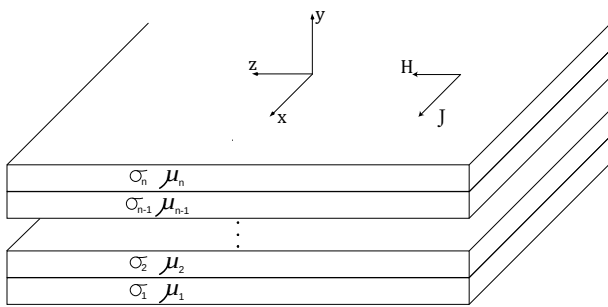


Figure 1.4a. Flat multi-layer conductor

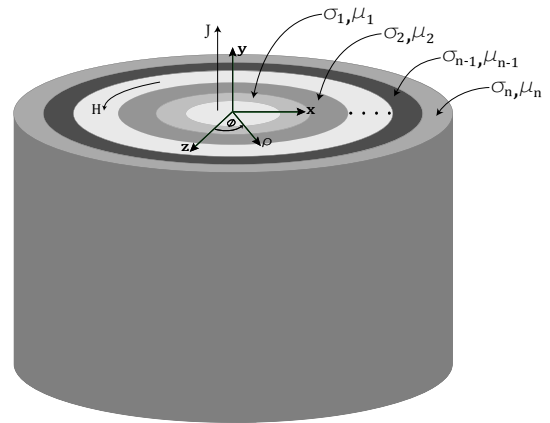


Figure 1.4b. Cylindrical multi-layer conductor

Dissipative filters refer to any filter in which unwanted energy is dissipated as heat. This is practically achieved through one of two means; hysteresis and/or dielectric losses or low conductivity regions, in conjunction with skin-and proximity-effects. The first is well known for its automotive application in the suppression of EMI due to the ignition system. This, however, does not fall under this arrangement, as the lossy hysteresis and dielectric materials are non-conductive, and will therefore not be considered. The second filtering method makes use of the

skin- and proximity-effects to force current into low conductivity layers in order to convert the unwanted electromagnetic energy into thermal energy. This concept was first realised in the mid 1960s, where it was applied to cylindrical conductors ([25][26]) and later applied in commercial aircrafts ([27]). Very little further development was made until this concept was reinvestigated as a means to increase the level of integration of power electronic filters, where they are better known as *frequency selective/low-pass/power electronic interconnects/filters*. A qualitative discussion and example on the workings of these filters can be found in [28]-[30].

Some of the first designs, such as that given in [29], were based on intuition and optimised via FEM simulations. Attenuation predictions were then made based on a lossy transmission-line model, where R , L , C , and G were determined from approximations based on parallel plane transmission lines. [28] followed a similar approach by simulating a large number of interconnects and extracting their resistance, inductance and capacitance values. Analytical parameter values were obtained through approximated equations. Both the simulated and analytical approximated parameters were then applied to a transmission line-model in order to obtain attenuation predictions and allow for a comparison. [31] attempted to obtain a better understanding of these filters through parametric studies. She extracted impedance parameters (through FEM) for varying layer thicknesses, conductivities and lengths and applied them to the same lossy transmission-line model to obtain impedance predictions. However, since none of these provide for a thorough fundamental understanding, [32] attempted to develop analytical equations through a wave-analysis approach. Even though this method was never fully developed, and hence could not be used, it did prove to be a step in the right direction.

With the extent of the dissipative filters' development now discussed, attention can be shifted to the remaining multi-layer conductors. As is clear, the armoured/reinforced cables and the coaxial cables both fall under the multi-layer cylindrical conductors. Since the attenuation of both these conductors is normally required to be as small as possible, they are designed to maintain their internal current density distribution as constant as possible for the frequencies under consideration. This is achieved by ensuring that the skin depth is significantly larger than the thickness of the cylindrical layers. The alternative to these conductors is the multi-layer cylindrical conductor filters discussed above and given in [25] and [26].

The last significant application that makes use of arrangement 2, is that of the magnetic shielding found in [33] and [34]. Figure 1.5 gives a top down view of the conductor used in [34], where the conductor consists of n cylindrical conductors centred around the same axis. As can be seen, the conductor consists of $2n+1$ regions, with the even regions making up the conductor layers, and the odd regions are free space. [33] is a precursor to [34] and is given for n equal to 2 (two conductive layers). In both these cases the conductive layers are composed of the same material, with σ and μ being the same for all layers. Unlike the filters discussed above, the conductive layers do not carry any resultant current. They are rather subjected to an eddy current inducing magnetic field, resulting from a current carrying conductor situated within region 1. [34] derives an analytical solution expressed in terms of the magnetic vector potential for each of the regions. It is important to note that from the constraints placed on arrangement 2, only the solution for the case where the current carrying conductor is situated at the centre of this multi-layer cylindrical conductor is of interest. This can easily be seen by realising that, for

any other placement, the magnetic field will be composed of an azimuth as well as a radial component.

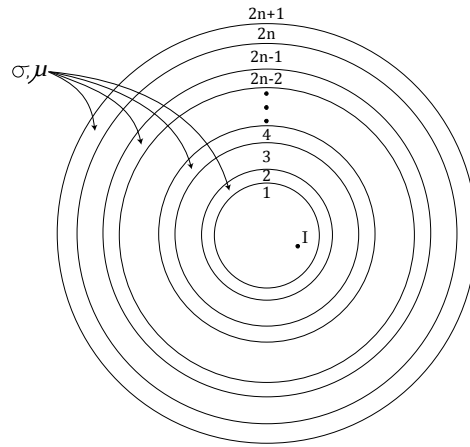


Figure 1.5. Top down view of the multi-layer cylindrical shield similar to that found in [34]

Even though these two arrangements cover a large section of multi-layer conductors, a number of other variations also exists, as was seen with the work of [33] and [34], along with some other undiscussed conductors. The purpose of this literature review was, however, to help put the objective of this thesis into perspective, which is discussed in the next section.

1.4 Problem description

From the previous investigation into multi-layer structures, it is clear that the conductors under the second arrangement are the most underdeveloped. Whereas arrangement 1 was subjected to numerous theoretical investigations leading to multiple solutions, arrangement 2 has only been scrutinised from first principle by [32], [33] and [34]. [32] however, did not deliver any tangible solution, whereas [33] and [34] only considered the cylindrical conductor where all the layers were composed of the same material and did not carry any resultant current. Furthermore, since arrangements 1 and 2 differ on a fundamental level, their solutions are not interchangeable.

It is therefore clear that the ad-hoc approach to arrangement 2 to date has delivered very little in the form of a proper solution. Currently, a solution to the electric field, magnetic field and current density within this multi-layer conductor, and hence its reflected impedance, is only possible through an iterative/numerical method, such as FEM. This, however, is not a viable solution, as it is slow, suffers from convergence errors and does not provide for any significant insight into the factors governing them.

One of the few solutions which will overcome these difficulties is an analytical one, as it does not suffer from the previously mentioned problems. It is also one of the only ways to advance the development of this arrangement and its applications.

1.4.1 Thesis objectives

From this problem description, the thesis objectives can be clearly defined as:

1. *Perform a thorough investigation of all existing equations and methods pertaining to the relevant behaviour of arrangement 2:*
 - o *Since the flat and cylindrical conductors formed the predominant geometries of this arrangement, attention will be focussed on them.*
 - o *This objective will concentrate on the single-layer conductors, as they are the only arrangement 2 conductors which have received significant analytical investigation to date.*
 - o *Investigation will be focussed on the electric field, magnetic field and current density within the single-layer conductors, as they form the basis for all future work and understanding.*
2. *Develop a multi-layer methodology for arrangement 2:*
 - o *This methodology must provide the steps required to obtain an analytical solution to the magnetic field, electric field and current density of any multi-layer conductor subjected to arrangement 2's constraints, with the methodology being independent of the number of layers.*
 - o *Apply this methodology to obtain the solutions to the flat and cylindrical geometries.*
3. *Develop impedance equations, based on this methodology and its subsequent equations, to assist in the future development of this arrangement's applications.*
4. *Design an experimental setup to confirm the validity of the multi-layer methodology and its impedance equations.*

1.5 Thesis layout

Chapters 2 and 3 will be aimed at fulfilling the first of the four objectives. Since the single-layer conductor is the simplest form of the multi-layer conductor, it follows that it is the most logical place to commence this investigation. Take note, for the remainder of this thesis, when reference is made to multi-layer conductors, it is with regards to the conductors of arrangement 2. Chapter 2 will therefore start with a thorough review of the existing equations, concepts and information pertinent to this single layer variant. This will primarily be focussed on the flat and cylindrical geometries, as they present the two predominant geometries of this arrangement.

Chapter 3 will be focussed on the wave analysis method of [32], in order to determine whether it has any future merit as a solution to the multi-layer conductors. Since [32] did not show the validity of this approach, attention will be directed to the establishment of its validity and how it compares to the methods found in chapter 2.

The multi-layer methodology is fully developed in chapter 4. It presents the steps required to obtain the desired solution, with no constraints on the number of layers or the geometry (given that the geometry is part of arrangement 2). These steps are applied to derive the solutions to two- and three-layer flat and cylindrical conductors in different scenarios and are verified through FEM simulations. This chapter will therefore satisfy the second objective.

Chapter 5 derives impedance equations for the multi-layer conductors based on the multi-layer methodology and equations of chapter 4. These equations, and chapter 4's methodology, are subsequently verified through experimental measurements. This, in turn, satisfies the last objective of this thesis.

Chapter 6 concludes this thesis by presenting recommendations for future work, while summarising the most important conclusions of this thesis.

A graphical outline of this thesis is given in Figure 1.6.

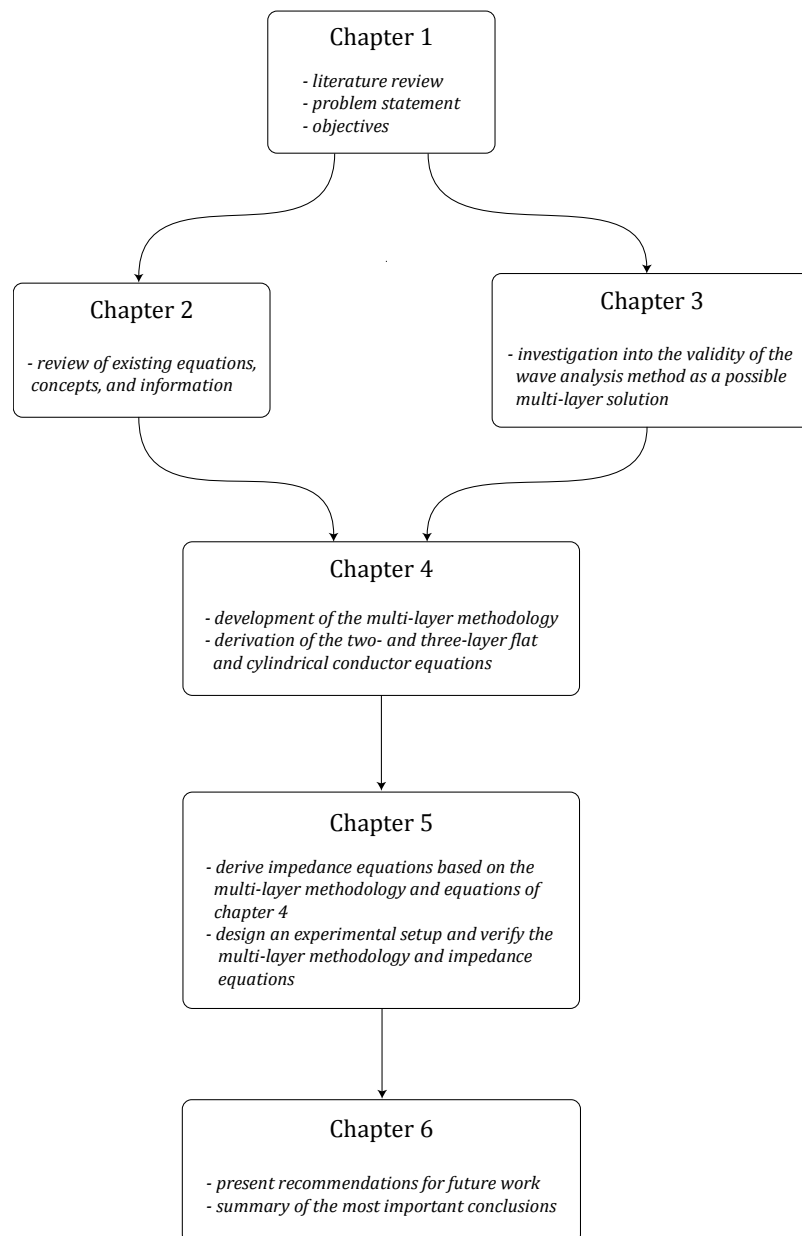


Figure 1.6. Thesis outline

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Chapter 2

Current State of the Equations

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2.1. Introduction

As identified in the previous chapter, the fundamental objective of this thesis is to develop a multi-layer methodology and determine the associated electric field, magnetic field and current density equations. This chapter will focus on the essential groundwork related to this, with focus on the single layer variants of the flat and cylindrical conductive geometries.

A thorough review of their existing equations will be undertaken in an effort to understand how they were obtained, under what conditions they hold and what similarities they share on an electromagnetic and mathematical level. This is necessary in order to identify a number of important and essential observations. These include the inadequacies associated with the existing cylindrical conductor equations, resulting in a rederivation, while the flat conductor's equations are rederived in order to be better suited for chapter 4. The constraints of the flat-conductor's simplification are furthermore scrutinised in order to determine under what conditions they hold, while Dowell's results will reveal their limited scope for adaptation, resulting in its rejection as a viable building block for the multi-layer conductors.

This chapter concludes with a thorough review of the orthogonality principle, which is essential for chapter 5, and will reveal the lack of a proper proof for cylindrical conductors. This groundwork, along with the associated findings, will prove to be essential in the subsequent chapters.

2.2. Semi-infinite single-layer flat conductor

Consider the semi-infinite conductive configuration given in Figure 2.1. As can be seen, it has a finite thickness in the y-direction and stretches infinitely far in the x- and z-directions. This is regularly used as an approximation to the finite flat conductor (which is the geometry under investigation) and proves to be very accurate under the correct conditions. The approximation is necessary to avoid edge effects and allows the problem to be dependent on one dimension only (a very much sought after condition), as two-dimensional equations are significantly more difficult to solve. This approximation is also known as the 1D approximation. The confirmation of the validity and the constraints under which this approximation holds are discussed in section 2.2.6. For the remainder of this investigation, all finite flat conductors will therefore be taken to comply with these constraints.

Even though the treatment of this problem ([1][2] and some adaptations [3][4]) have been around for a long time and appears in a number of texts, the exact origin is unknown. The most referenced version can be found in [3], where it is applied to transformer windings which are approximated as single layer sheets. [3] however, introduced limitations into his adaptation (as is thoroughly discussed in section 2.2.5) with the result that this investigation will be based on the derivations found in [1] and [2]. Some of their finer and more challenging points, not discussed in any of the references, will also be considered in this chapter.

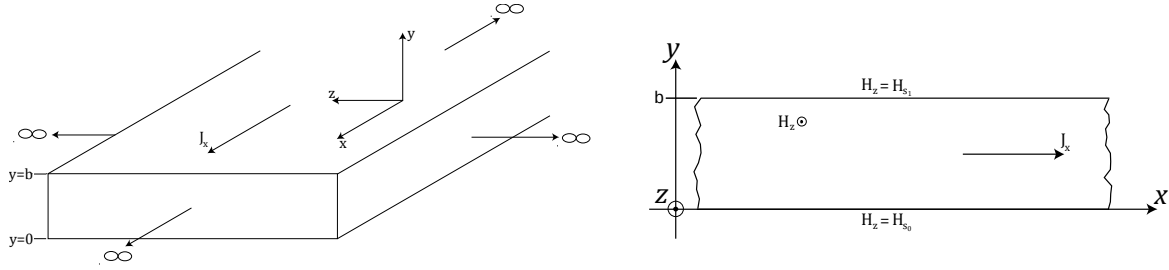


Figure 2.1. Semi-infinite single-layer flat conductor

2.2.1. General solution based on the semi-infinite approximation

The derivation is made under the assumption that any flat conductor is of the semi-infinite nature, with its finite thickness constrained to be in the y -direction and its unbounded dimensions in the x - and z -directions. All conductors are considered to be of a linear isotropic homogenous nature, with their magnetic permeability and conductivity therefore constant. Any current flowing in these conductors will be constrained to be in the x -direction, which, from the flat conductor assumption, forces any magnetic field to be in the z -direction only.

In general, more than one conductor may be present within the space under consideration (stacked in the y -direction). The following derivation will, however, only focus on one of these layers, with all constants kept completely generic. This allows for the subsequent solution to be applicable to any number of conductors (with each conductor having its own unique properties). Starting with Maxwell's equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (1)$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \quad (2)$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad (3)$$

$$\nabla \cdot \vec{B} = 0 \quad (4)$$

where

$$\vec{B} = \mu \vec{H} \quad (5)$$

and

$$\vec{J} = \sigma \vec{E} \quad (6)$$

Expanding the curl of the electric field,

$$\begin{aligned} \nabla \times \vec{E} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{j} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{k} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \end{aligned} \quad (7)$$

Applying this to the semi-infinite conductor (Figure 2.1) and recalling that the current, and hence the electric field, is only in the x -direction and only changes with y , produces,

$$\nabla \times \bar{E} = -\frac{\partial E_x}{\partial y} \hat{k} \quad (8)$$

Substituting this result into equation (1) and realising that the magnetic field is only in the z-direction,

$$-\hat{k} \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \hat{k} \quad (9)$$

Cancelling the vectors on either side and applying equations 6,

$$\frac{\partial J_x}{\partial y} = \sigma \mu \frac{\partial H_z}{\partial t} \quad (10)$$

Leaving this result for now and expanding equation (3), with the displacement current term neglected (substantiation of this is given in Chapter 3), produces,

$$\begin{aligned} \nabla \times \bar{H} &= \bar{J} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} &= \bar{J} \\ \hat{i} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \hat{j} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \hat{k} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) &= J_x \hat{i} + J_y \hat{j} + J_z \hat{k} \end{aligned} \quad (11)$$

Recalling that only the z-component and x-component of the magnetic field and current density are non-zero respectively, allows the above expansion to reduce to,

$$\frac{\partial H_z}{\partial y} = J_x \quad (12)$$

Differentiating both sides with respect to y and substituting into equation (10) produces,

$$\frac{\partial^2 H_z}{\partial y^2} = \mu \sigma \frac{\partial H_z}{\partial t} \quad (13)$$

Equation (13) is the well-known diffusion equation for the one-dimensional scenario and is equal to the Laplacian of the magnetic field intensity under the constraints of this configuration.

Since these conducting configurations are to be considered under sinusoidal excitation (as this is where they are most applicable), equation (13) can be moved into the frequency domain with the time derivative replaced by $j\omega$,

$$\frac{\partial^2 H_z}{\partial y^2} = j\omega\mu\sigma H_z \quad (14)$$

This allows the magnetic field to be written in complex notation,

$$H(y) = |H(y)|e^{j\phi(y)} = |H(y)| \angle \phi(y) = |H(y)| [\cos\phi(y) + j\sin\phi(y)] \quad (15)$$

and D^2 can be defined as,

$$D^2 = j\omega\mu\sigma \quad (16)$$

Taking the square root on both sides gives D as,

$$D = (j\omega\mu\sigma)^{\frac{1}{2}} = (j)^{\frac{1}{2}} (\omega\mu\sigma)^{\frac{1}{2}} \quad (17)$$

Using Euler's identity, the square root of j can be rewritten as,

$$j = e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} \quad (18)$$

$$\begin{aligned} (j)^{\frac{1}{2}} &= \left(e^{j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = \frac{1+j}{\sqrt{2}} \end{aligned} \quad (19)$$

which allows equation (17) to become,

$$D = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{(1+j)}{\delta} \quad (20)$$

where δ is the well-known skin depth defined as $\sqrt{2/\omega\mu\sigma}$. Equation (14) can now be rewritten as,

$$\frac{d^2 H_z}{dy^2} - D^2 H_z = 0 \quad (21)$$

where the partial derivatives have been replaced by ordinary derivatives. As can be seen, this is a linear second-order differential equation. The solution to equation (21) is therefore given by,

$$H_z(y) = K_1 e^{Dy} + K_2 e^{-Dy} \quad (22)$$

where K_1 and K_2 are dependent on the boundary values. A detailed discussion on this solution can be found in [5]. What is important to take note of is that the solution consists of two

exponential functions whose powers are dependent on y , conductor specific characteristics (σ and μ) and the frequency of excitation. It can therefore be understood that the exact solution will consist of two exponential functions (with scenario specific coefficients, as is discussed below) whose rate of decay is governed by the material's properties and the frequency. This is the form in which the generic equation is given by [1]. In this thesis, the $y=0$ point is chosen to be at the bottom of the conductor (Figure 2.2), in order to be better suited for chapter 4. This however is in contrast to [1], where the origin is in the middle of the conductor, and as such the results will have to be rederived. Equation (22) can also be written as,

$$H_z(y) = F \cosh(Dy) + G \sinh(Dy) \quad (23)$$

which will be used for the duration of this investigation. The equivalence between equations (22) and (23) is given in Appendix A. For the time being, it will be assumed that the boundary values are known, as shown in the practical generic setup of Figure 2.2, and given below,

$$H_z(0) = H_{s0} \quad (24)$$

$$H_z(b) = H_{s1} \quad (25)$$

H_{s0} and H_{s1} will be dependent on the scenario under investigation and will directly influence the values of the coefficients F and G in (23). As is self-evident, for the practical generic setup of Figure 2.2 to be applicable to these generic equations, the finite dimensions must be subjected to the semi-infinite approximation.

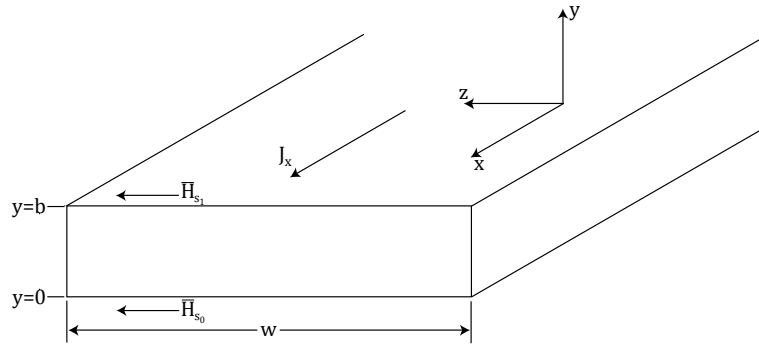


Figure 2.2. Practical generic conductor which can be approximated to the semi-infinite conductor

Setting H_{s0} equal to equation (23) with $y = 0$ and H_{s1} equal to (23) with $y = b$ produces two simultaneous equations. Solving these allows for the coefficients F and G to be written in terms of known constants,

$$F = H_{s0} \quad (26)$$

$$G = \frac{H_{s1} - H_{s0} \cosh(Db)}{\sinh(Db)} \quad (27)$$

The generic current density can be determined from equation (12),

$$J_x(y) = \frac{dH_z(y)}{dy}$$

$$J_x(y) = \frac{d}{dy} [F \cosh(Dy) + G \sinh(Dy)] = FD \sinh(Dy) + GD \cosh(Dy) \quad (28)$$

from which the electric field can be determined from equation (6),

$$E_x(y) = \frac{FD}{\sigma} \sinh(Dy) + \frac{GD}{\sigma} \cosh(Dy) \quad (29)$$

As these one-dimensional solutions are well known, it may seem unnecessary to devote so much attention to them. It will, however, be seen that the details regarding their origin and the generic solution to the electric and magnetic fields will form an integral part of this thesis and the eventual obtainment of a multi-layer solution.

Before moving on to the next section, two additional points should be mentioned. Firstly, the same generic solution can be obtained from the Laplacian of the electric field. This is briefly shown in Appendix B. Secondly, as the analysis will be carried out in the frequency domain, any discussion with regards to a constant current or magnetic field refers to a current or field that is sinusoidal in time and constant in peak value.

2.2.2. General solution applied to the case of zero net external field

In this scenario the conductor is carrying a known resultant current, I , with the previous constraint forcing it to be confined to the x-direction, as shown in Figure 2.3. The return conductor is taken to be infinitely far away such that the magnetic fields at the boundaries are solely due to the current carried within the conductor under consideration. This is also known as the skin-effect scenario, as only the skin effect will be present. Reference vectors $\bar{H}_{s0}$ and $\bar{H}_{s1}$, which were used to solve for coefficients F and G (equations (26) and (27) above), are also shown in Figure 2.3.

This configuration is, however, a contradiction in itself. For the hypothetical scenario, where the conductors are in fact infinitely wide and long (x- and z-directions), it follows that the return conductor can only be moved infinitely far away in the y-direction. The semi-infinite nature of these conductors, however, ensures that the return conductor's magnetic field will be constant for all distances (y-direction) outside the conductor. Therefore, irrespective of how far away the return conductor is moved, its magnetic field will always be present at the conductor under consideration. This is obviously only a theoretical concern, as all practical conductors are finite in dimensions and can only be approximated as semi-infinite. It is therefore clear that this configuration only holds for finite conductors which are mathematically approximated as semi-infinite.

The reason for investigating and applying the generic equations of the previous section to this configuration is twofold. Firstly, this isolated configuration is one of the two fundamental configurations that will make up any practical scenario. It is therefore important to have a thorough understanding of it before commencing with the multi-layer configurations of chapter 4. Secondly, it provides for an uncomplicated configuration to demonstrate how to determine the boundary values and apply the generic equations in order to obtain a configuration specific solution.

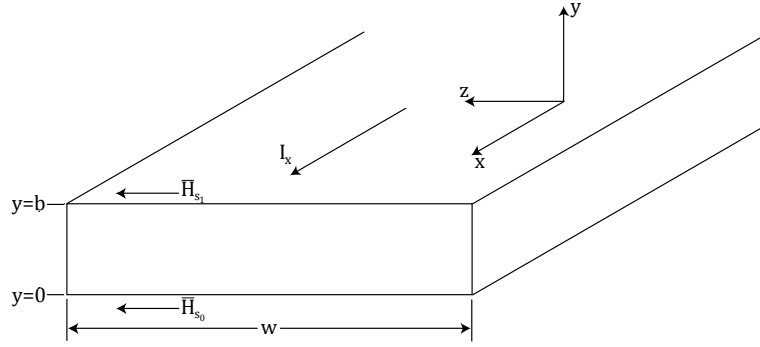


Figure 2.3. Configuration used for the derivation of the skin effect equations

In order to determine the solution specific to this scenario, F and G, and therefore $\bar{H}_{s0}$ and $\bar{H}_{s1}$ need to be determined under the constraints of this configuration. Ampere's law in integral form is therefore applied around the circumference of the conductor (path c in Figure 2.4), where it can be seen that four magnetic field intensity vectors are defined. $\bar{H}_0$ and $\bar{H}_1$ represent the magnitude and direction of the actual magnetic fields at the surface of the conductor due to the current I flowing within it (the directions can be seen to be correct from the right hand rule). $\bar{H}_{s0}$ and $\bar{H}_{s1}$ are (as now known) the reference vectors that define the positive reference directions and will be expressed in terms of the actual magnetic fields, $\bar{H}_0$ and $\bar{H}_1$ in the final solution. Since $b \ll w$, the thickness of the conductor will be omitted from integral.

$$\begin{aligned}
 \oint_c \bar{H} \cdot d\bar{l} &= \iint_S \bar{J} \cdot \bar{N} da = I \\
 \int_w H_{s1}(\hat{k}) \cdot d\bar{l}(\hat{k}) + \int_w H_{s0}(\hat{k}) \cdot d\bar{l}(-\hat{k}) &= I \\
 H_{s1}w - H_{s0}w &= I \\
 w(H_{s1} - H_{s0}) &= I
 \end{aligned} \tag{30}$$

It needs to be mentioned that since the current density, J, is still to be determined, it is not actually possible to solve its integral. However, as the integral is over the complete surface of the conductor, it follows that it will be equal to the total current flowing through it (which is known). Looking at the actual versus the reference magnetic field intensities, it is clear that,

$$\begin{aligned}
 \bar{H}_{s1} &= \bar{H}_1 \\
 H_{s1}(\hat{k}) &= H_1(\hat{k}) \\
 H_{s1} &= H_1
 \end{aligned}
 \tag{31}$$

and

$$\begin{aligned}
 \bar{H}_{s0} &= -\bar{H}_0 \\
 H_{s0}(\hat{k}) &= H_0(-\hat{k}) \\
 H_{s0} &= -H_0
 \end{aligned}
 \tag{32}$$

From symmetry it follows that the magnitude of the magnetic field is constant along the circumference of the conductor (defined as H), allowing for $H_1=H_0=H$. Substituting this into equations (31) and (32), and the subsequent results into equation (30),

$$\begin{aligned}
 w(H_{s1} - H_{s0}) &= I \\
 w(H_1 + H_0) &= I \\
 2wH &= I \\
 H &= \frac{I}{2w}
 \end{aligned}
 \tag{33}$$

With the magnitude of the magnetic field along the width of the conductor now known, it can be substituted back into equations (31) and (32), allowing for the reference vectors to be expressed in terms of parameters that are specific to this configuration.

$$H_{s1} = H = \frac{I}{2w} \tag{34}$$

$$H_{s0} = -H = -\frac{I}{2w} \tag{35}$$

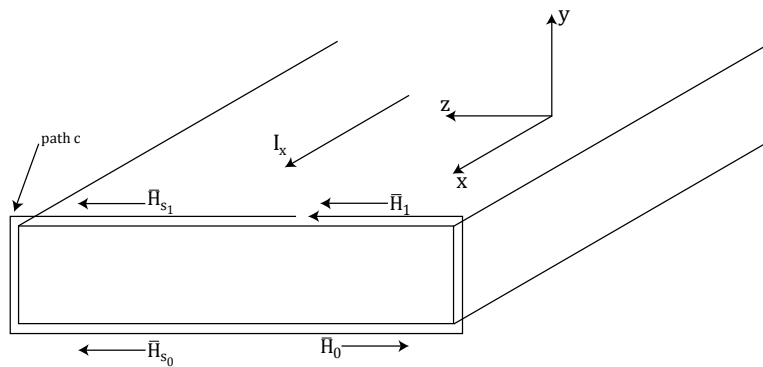


Figure 2.4. Skin-effect configuration with direction of integration indicated

Now that the boundary values have been obtained, the coefficients of the magnetic field equation can be solved. Substituting equations (34) and (35) into (26) and (27) produces,

$$F = -H = -\frac{I}{2w} \quad (36)$$

$$G = H \left[\frac{1 + \cosh(Db)}{\sinh(Db)} \right] = \frac{I}{2w} \left[\frac{1 + \cosh(Db)}{\sinh(Db)} \right] \quad (37)$$

Using these coefficients to solve for $H_z(y)$,

$$\begin{aligned} H_z(y) &= F \cosh(Dy) + G \sinh(Dy) \\ &= -\frac{I}{2w} \cosh(Dy) + \frac{I}{2w} \frac{1 + \cosh(Db)}{\sinh(Db)} \sinh(Dy) \\ H_z(y) \sinh(Db) &= \frac{I}{2w} \sinh(Dy) + \frac{I}{2w} [\sinh(Dy) \cosh(Db) - \cosh(Dy) \sinh(Db)] \end{aligned} \quad (38)$$

Using the identity of $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$ (found in [6]) equation (38) can be rewritten as,

$$\begin{aligned} H_z(y) \sinh(Db) &= \frac{I}{2w} \sinh(Dy) + \frac{I}{2w} \sinh(D(y-b)) \\ H_z(y) &= \frac{I}{2w} \frac{\sinh(Dy)}{\sinh(Db)} + \frac{I}{2w} \frac{\sinh(D(y-b))}{\sinh(Db)} \\ &= \frac{I}{2w} \left[\frac{\sinh(Dy)}{\sinh(Db)} + \frac{\sinh(D(y-b))}{\sinh(Db)} \right] \end{aligned} \quad (39)$$

Initial assessment of equation (39) may lead to the conclusion that the magnetic field intensity is an odd function of y , since $\sinh$ is odd. Even though this is the case, care must be taken, as the contribution of the other 'sub'-functions may result in the final function not being the same as its individual 'sub'-functions. This is encountered for the proximity effect of section 2.2.3. Since the magnetic field intensity is, however, odd, an integral taken from zero to b along the y -axis will be equal to zero. The real and imaginary parts of the magnetic field intensity are therefore odd images around the x -axis (at the $y = \frac{1}{2}b$ intersect). This is shown in Figure 2.5a and b for a non-magnetic conductor carrying a current of 8 A at 50 kHz and having a width of 1600 mm, thickness of 2 mm and conductivity of 58×10^6 S/m. As can be seen, the real part of the magnetic field intensity agree with equations (34) and (35) while the imaginary part is zero at the boundaries. This is to be expected, as the current is defined to be at zero phase and from equations (34) and (35) the boundary value magnetic fields are in phase and 180° out of phase with the resultant current for the top and bottom boundaries, respectively. This is further discussed in the next section.

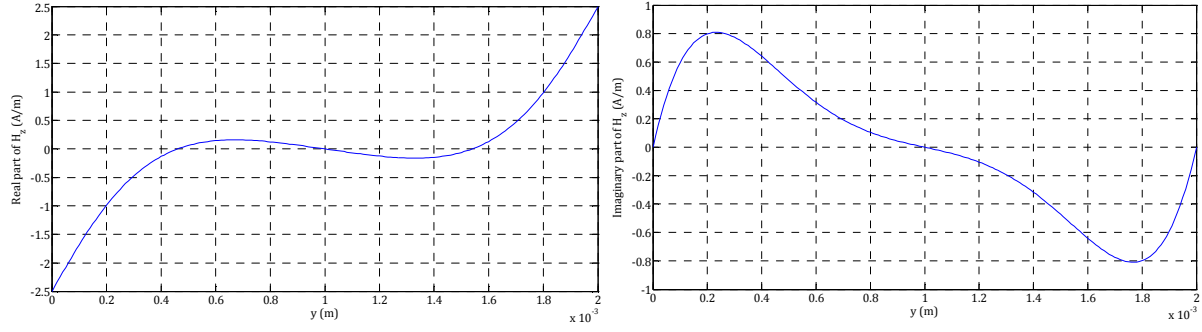


Figure 2.5a and b. Real and imaginary parts of the magnetic field intensity of Figure 2.4

The current density can now easily be obtained by either differentiating equation (39) with respect to y or using the already derived generic form of equation (28) and substituting the skin effect specific coefficients (which is done here). From equation (28),

$$\begin{aligned}
 J_x(y) &= FD \sinh(Dy) + GD \cosh(Dy) \\
 &= -HD \sinh(Dy) + HD \left[\frac{1 + \cosh(Db)}{\sinh(Db)} \right] \cosh(Dy) \\
 J_x(y) \sinh(Db) &= HD \cosh(Dy) + HD [\cosh(Db) \cosh(Dy) - \sinh(Db) \sinh(Dy)]
 \end{aligned} \tag{40}$$

Using the identity $\cosh(Dy - Db) = \cosh(Db) \cosh(Dy) - \sinh(Db) \sinh(Dy)$ from [6] the current density can be written as,

$$J_x(y) = \frac{I}{2w} \frac{D}{\sinh(Db)} [\cosh(Dy) + \cosh(D(y-b))] \tag{41}$$

which is an even function of y along the x -axis, as shown in Figure 2.6. Taking the integrals of the real and imaginary parts of $J_x(y)$ from zero to b will produce I (the total current flowing within the conductor) and zero, respectively. This confirms the initial condition that the conductor is carrying a resultant current, I , which is at zero phase.

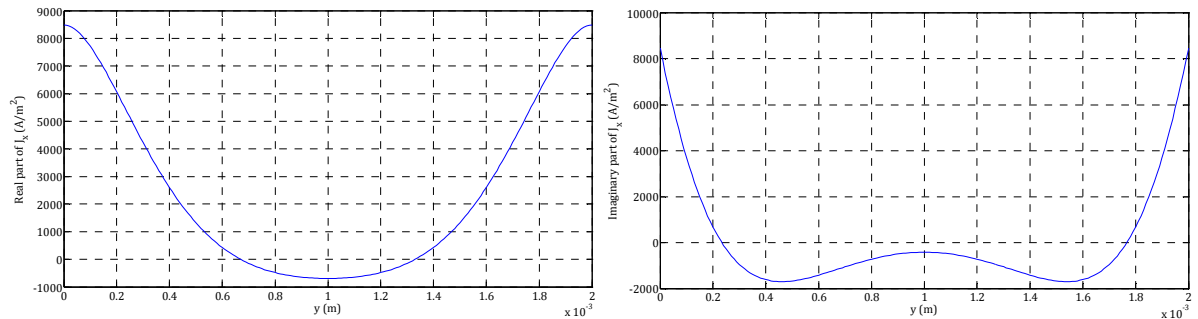


Figure 2.6a and b. Real and imaginary parts of the current density of Figure 2.4

The electric field is easily obtained by dividing the above result by the conductivity of the material and is given by,

$$E_x(y) = \frac{I}{\sigma 2w} \frac{D}{\sinh(Db)} [\cosh(Dy) + \cosh(D(y-b))] \quad (42)$$

As was mentioned at the beginning of this section, this isolated conductor is one of two fundamental scenarios that will be encountered. Section 2.2.4 will show that the superposition of these solutions, along with that of section 2.2.3, can provide a solution to any semi-infinite flat conductor scenario.

2.2.2.1. Magnetic field intensity and current density distribution in the time domains

Visualisation of the distributions within the conductor will help to provide additional insight into this configuration (skin-effect scenario), which is not easily achieved by investigation of the equations alone. This was already demonstrated by the visualisation of even and odd functions in Figure 2.5 and Figure 2.6. Visualisation in the frequency domain is, however, difficult (as compared to the time domain), as the simultaneous interpretation of the magnitude and phase angle is not very intuitive. As such, interpretation will rather be transferred to the time domain, which allows for a snapshot, or multiple snapshots, of the distributions in time.

Conversion from the frequency to the time domain is accomplished by multiplying by $e^{j\omega t}$ and taking the real part. Starting with the magnetic field intensity,

$$\begin{aligned} H(y, t) &= \text{Re} [H_z(y) e^{j\omega t}] \\ &= \text{Re} [|H_z(y)| e^{j(\omega t + \phi(y))}] \\ &= \text{Re} [|H_z(y)| [\cos(\omega t + \phi(y)) + j \sin(\omega t + \phi(y))]] \end{aligned} \quad (43)$$

The current density is obtained in the exact same fashion,

$$\begin{aligned} J(y, t) &= \text{Re} [J_x(y) e^{j\omega t}] \\ &= \text{Re} [|J_x(y)| e^{j(\omega t + \theta(y))}] \\ &= \text{Re} [|J_x(y)| [\cos(\omega t + \theta(y)) + j \sin(\omega t + \theta(y))]] \end{aligned} \quad (44)$$

The electric field will not be considered here, as it is identical to the current density except for the scaling by the constant conductivity. Applying equation (43) to the solved magnetic field intensity (equation (39)) produces a function that is dependent on both distance (y) and time (t).

Visualisation can be achieved by either varying the time in set increments, producing two-dimensional graphs (Figure 2.7), or by sweeping both the time and distance over set values, which results in a three-dimensional graph (Figure 2.8). In both these cases the time was varied over one period, which, for a fixed frequency, is equal to varying the phase angle between 0 to 2π rad (or as was done here between 0° and 360°). As is clear, the three-dimensional graph is equivalent to making the set increment of Figure 2.7 very small and stacking the graphs next to

one another. In both of these cases, the conductor was chosen to have a width of 2 m (which is necessary to determine the boundary values - equations (34) and (35)), conductivity of 59.6×10^6 S/m, permeability of free space, thickness of 20 mm and carry a current of 1.414 A (peak) at 1 kHz. (All currents and fields will be expressed in terms of peak values throughout this thesis). These values were not chosen to be representative of any practical scenario.

Figure 2.7 and Figure 2.8 substantiate the statement that was made about the magnetic field being an odd function of y , where it can be seen to hold for every point in time. Figure 2.8 has the added advantage of better demonstrating the periodic nature of the magnetic field in time, with it clearly shown that the magnetic field at the boundaries change their sign (and hence their direction) every 180° (due to the current changing sign every 180°).

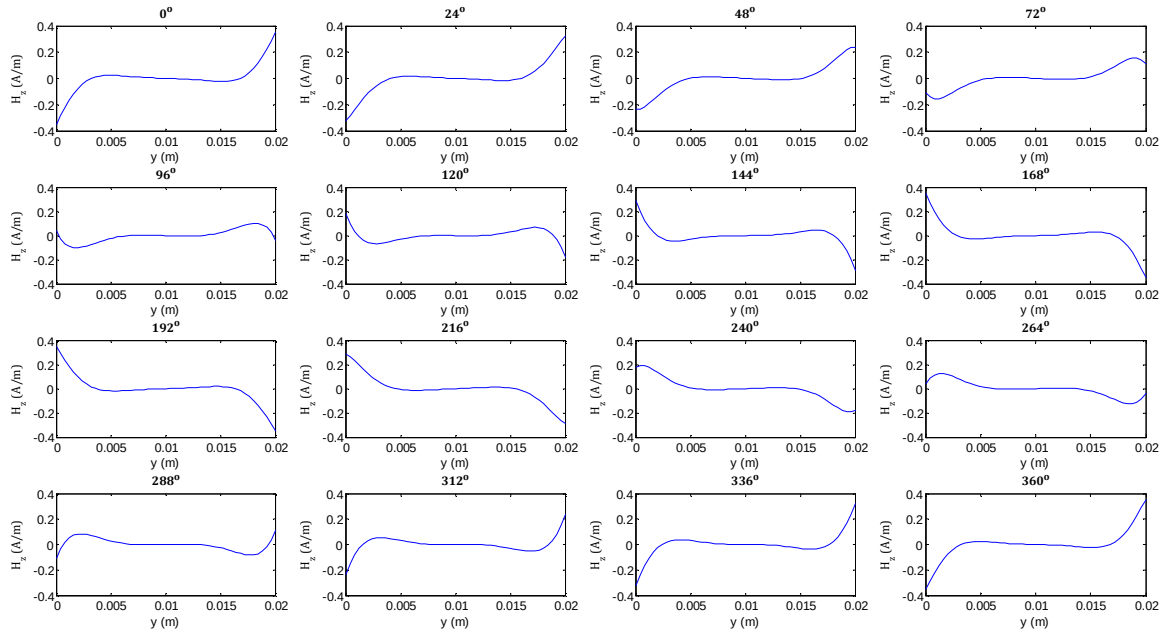


Figure 2.7. Magnetic field intensity as a function of distance for the phase angle changing from 0° to 360° in 24° increments

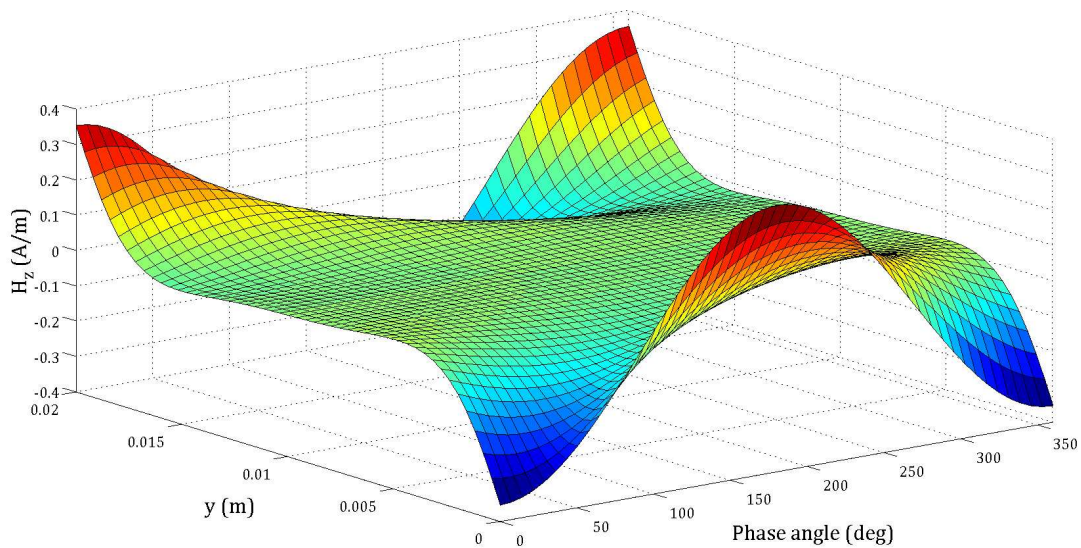


Figure 2.8. Magnetic field intensity as a function of distance and phase angle for the skin-effect scenario

Similar plots are given in Figure 2.9 and Figure 2.10 for the current density as a function of time. These plots again show the current density to be an even function of y , confirming that the conductor is carrying a resultant current (even and odd functions are discussed in more detail in section 2.4). Although the frequency dependence of the current density can be seen from the equations and the time-dependent plots, it is best illustrated by Figure 2.11, where it is plotted as a function of frequency and distance (y). As is expected, the current density is constant at zero hertz and migrates towards the boundaries at higher frequencies.

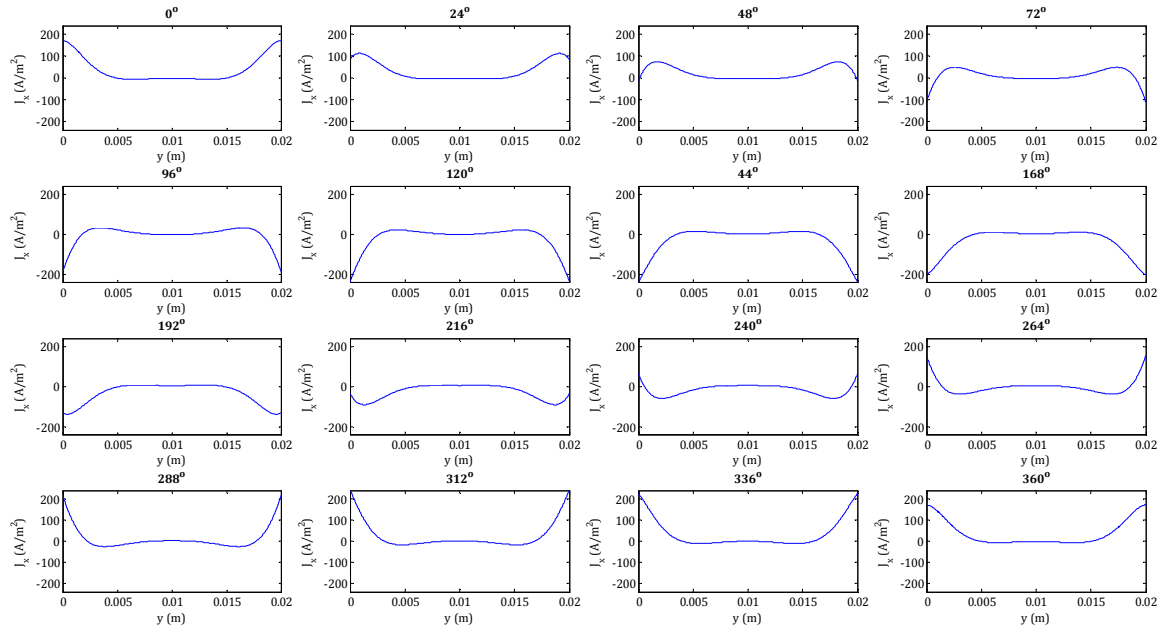


Figure 2.9. Current density as a function of distance for the phase angle changing from 0° to 360° in 24° increments

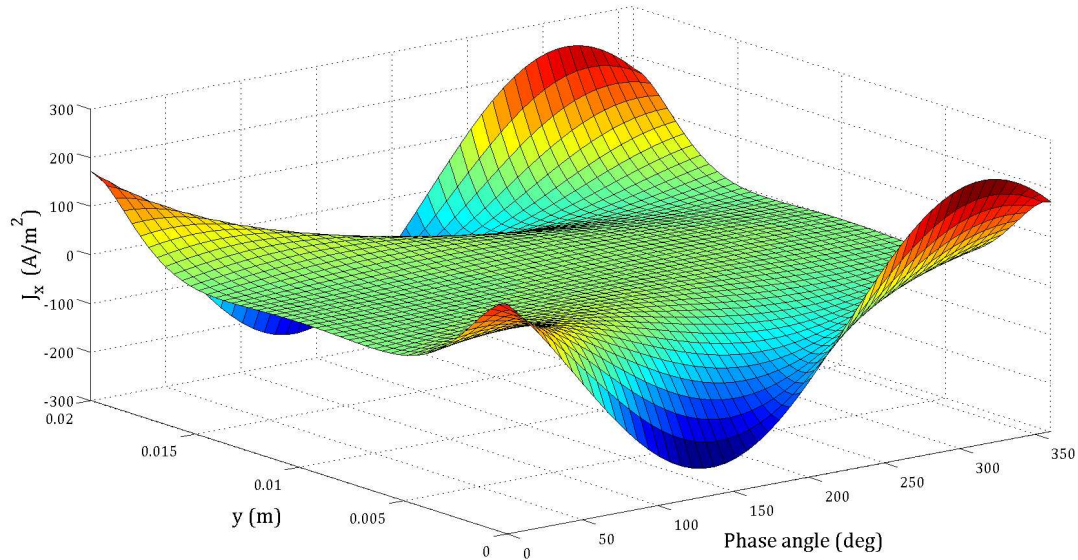


Figure 2.10. Current density as a function of distance and phase angle for the skin-effect scenario

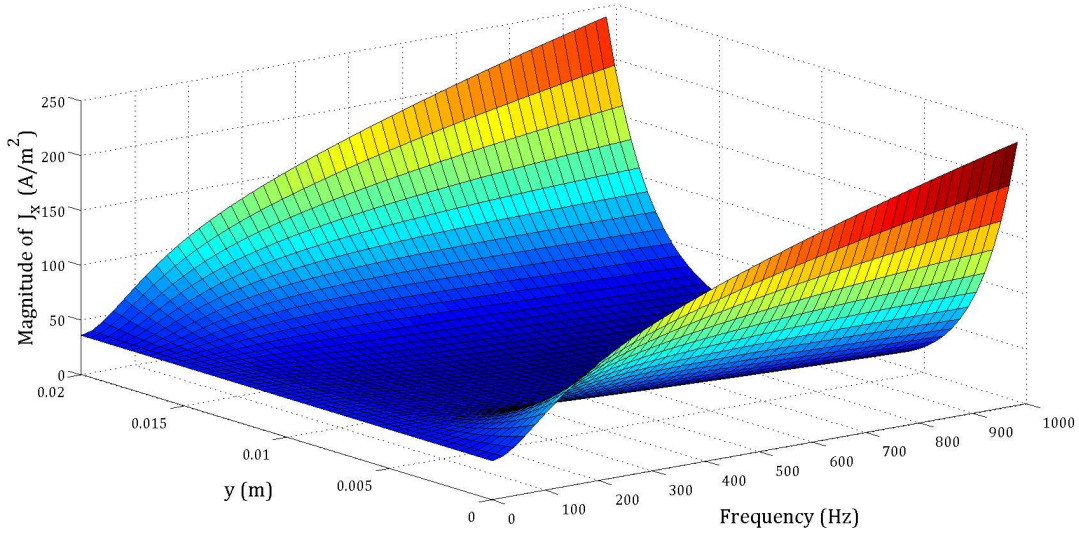


Figure 2.11. Magnitude of the current density as a function of distance and frequency for the skin-effect scenario

2.2.3. General solution applied to the case of non-zero external fields

As was mentioned before, any practical scenario will consist of a combination of two fundamental configurations. This section investigates the equations and distributions of the second configuration, shown in Figure 2.12. It depicts a semi-infinite conductor (or to be approximated as such) subjected to an external sinusoidal magnetic field in the z -direction. Although the origin of this field is of no importance here, it will be the result of a current-carrying conductor in close proximity (for the vast majority of practical cases), and is therefore also known as the proximity-effect scenario.

It should be noted that the conductor under consideration (Figure 2.12) does not carry any resultant current, but is subjected to internal circulating currents (eddy currents) due to the inbound field. The magnetic field inside the conductor will therefore be the superposition of the externally applied field and the ‘reacting’ field due to the circulating currents (which is what will be predicted by the final solution).

The field external to the conductor can be obtained by looking at the consequences of the semi-infinite nature of the conductor. Firstly, it ensures that the magnetic field outside a current-carrying conductor is constant for all y . Secondly, it forces the internal eddy currents to take on the form of differential currents (since $J_y = 0$). This is shown in Figure 2.14 and Figure 2.17 where the current density is an inverted image around the y -axis. Two constant opposing magnetic fields of equal magnitude are consequently present outside the conductor. These cancel each other out, with the field external to the conductor therefore only due to the externally applied source.

As a result, the reference fields, H_{s0} and H_{s1} , can be written as,

$$H_{s0} = H \quad (45)$$

$$H_{s1} = H \quad (46)$$

where H is the field due to the external source and will be known. The coefficients F and G (from equations (26) and (27)) reduce to,

$$F = H \quad (47)$$

$$G = \frac{H[1 - \cosh(Db)]}{\sinh(Db)} \quad (48)$$

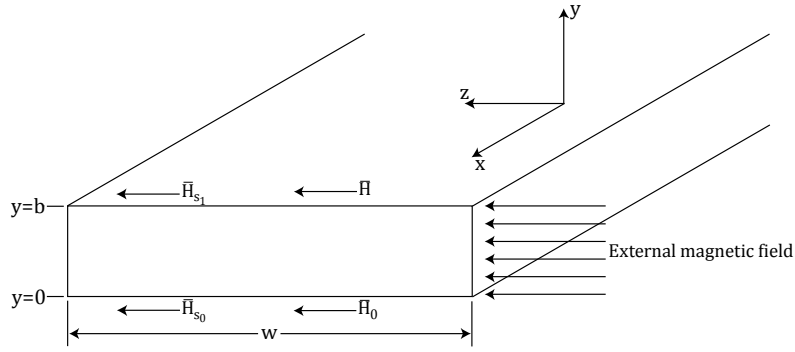


Figure 2.12. Configuration used for the derivation of proximity-effect equations

This allows the magnetic field intensity, current density and electric field equations of (23), (28) and (29) to be expressed in terms of known values,

$$H_z(y) = H \left[\cosh(Dy) + \left(\frac{1 - \cosh(Db)}{\sinh(Db)} \right) \sinh(Dy) \right] \quad (49)$$

$$J_x(y) = HD \left[\sinh(Dy) + \left(\frac{1 - \cosh(Db)}{\sinh(Db)} \right) \cosh(Dy) \right] \quad (50)$$

$$E_x(y) = \frac{HD}{\sigma} \left[\sinh(Dy) + \left(\frac{1 - \cosh(Db)}{\sinh(Db)} \right) \cosh(Dy) \right] \quad (51)$$

Using the identities given in (52) and (53) (from [6]), the magnetic field intensity and current density can be written in the form of equations (54) and (55).

$$\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y) \quad (52)$$

$$\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y) \quad (53)$$

$$H_z(y) = H \left[\frac{\sinh(Dy)}{\sinh(Db)} + \frac{\sinh(D(b - y))}{\sinh(Db)} \right] \quad (54)$$

$$J_x(y) = HD \left[\frac{\cosh(Dy)}{\sinh(Db)} - \frac{\cosh(D(b - y))}{\sinh(Db)} \right] \quad (55)$$

Comparing equation (49) to (54), and (50) to (55), reveals that no clear conclusion can be drawn on the nature of the equations' symmetry from visual inspection alone. This is due to the choice of the $y = 0$ point in the generic derivation of section 2.2.1. This obscurity can be avoided by rederiving the equation with the $y = 0$ point to be situated at half the thickness ($b/2$ in Figure 2.1). This was done by [1] and leads to significantly simpler equations for which the symmetry can be determined from inspection alone. The reason for choosing the $y=0$ point at the bottom of the conductor (Figure 2.1) will become evident in chapter 4, where it will be essential in the equations of the multi-layer conductors.

Returning now to equations (54) and (55) the plots of their real and imaginary components are given in Figure 2.13 and Figure 2.14. The conductor was chosen to be non-magnetic with a conductivity of 58×10^6 S/m, thickness of 2 mm and the externally applied magnetic field intensity has a peak value of 2.5 A/m. From these plots it is clear that the magnetic field intensity and current density are even and odd respectively, even though, as was stated before, initial inspection of the equations does not reveal this. This further substantiates the need for visualisation of the equations, whether in the time or frequency domain, in order to obtain a better understanding of their behaviour.

The time domain visualisation is given in the next section.

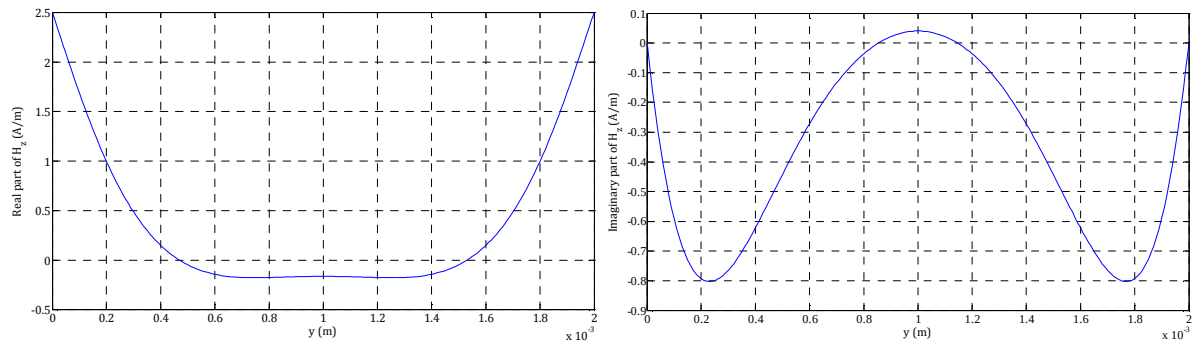


Figure 2.13a and b. Real and imaginary parts of the magnetic field intensity of Figure 2.12

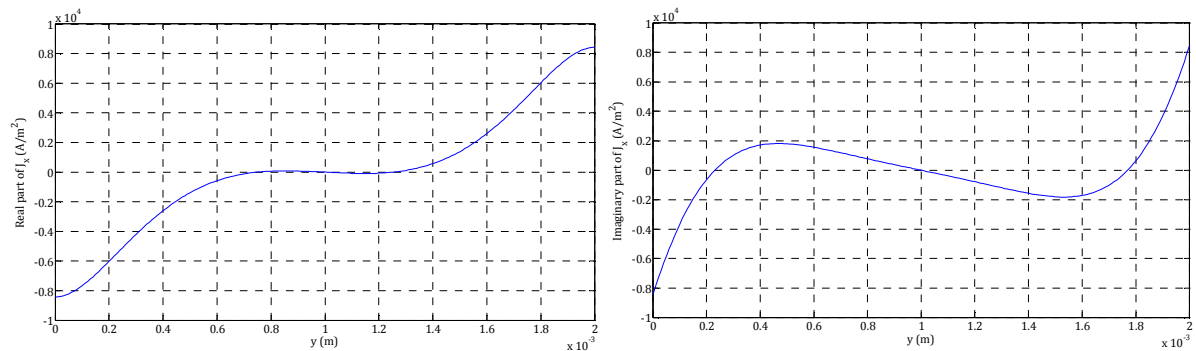


Figure 2.14a and b. Real and imaginary parts of the current density of Figure 2.12

2.2.3.1. Magnetic field intensity and current density distribution in the frequency and time domains

The conductor under consideration was chosen to have the same properties as that used in the visualisation of section 2.2.2.1. However, in this case, it carries no resultant current and an external magnetic field intensity of 2.828 A/m at a frequency of 1 kHz is incident on the conductor.

Figure 2.15 and Figure 2.16 again confirms the even nature of the magnetic field intensity, where it can be seen to be even around the centre of the conductor for all points in time. The periodic change of the magnetic field from positive to negative every half period (equal to 180°) is also clearly shown. Although it was mentioned before, it is important to realise that the magnetic field depicted in these plots is the resultant field present within the conductor, and is the superposition of the externally applied field and the internal field due to the reacting current. Further confirmation of these graphs can be obtained by considering the magnetic field intensity at the boundaries at zero phase ($t = 0$). Since the external field was defined to be real (zero phase), the magnetic field intensity at the boundaries should be in phase and equal to field (since the internal current is not contributing for $y < 0$ and $y > b$). Figure 2.15 and Figure 2.16 reveal this to be indeed the case, with the field at the boundaries equal to 2.828 A/m.

Consider now the current density distribution plotted in Figure 2.17 and Figure 2.18, where its odd symmetry is again evident. It is clear that at any point in time the resultant current within the conductor will be zero, resulting in only the external field being present outside the conductor (as was stated before). An obvious, but interesting, observation is that the current density's frequency is governed by the externally applied field, as it is this field which produces the emf and, therefore, this current density distribution internal to the conductor.

Similar to the skin-effect scenario, Figure 2.19 gives the absolute magnitude of the current density as a function of distance and frequency. Care must be taken here when interpreting this plot, as the phase angle information is not given. It can easily be misinterpreted and lead to the incorrect conclusion that the current density distribution is even. It must, however, be remembered that the current density distribution is odd for this case. If the actual distribution, at, say $t = 0$, is plotted against frequency, it will be an odd distribution whose magnitudes increase positive to the one side and negative to the other as the frequency increases.

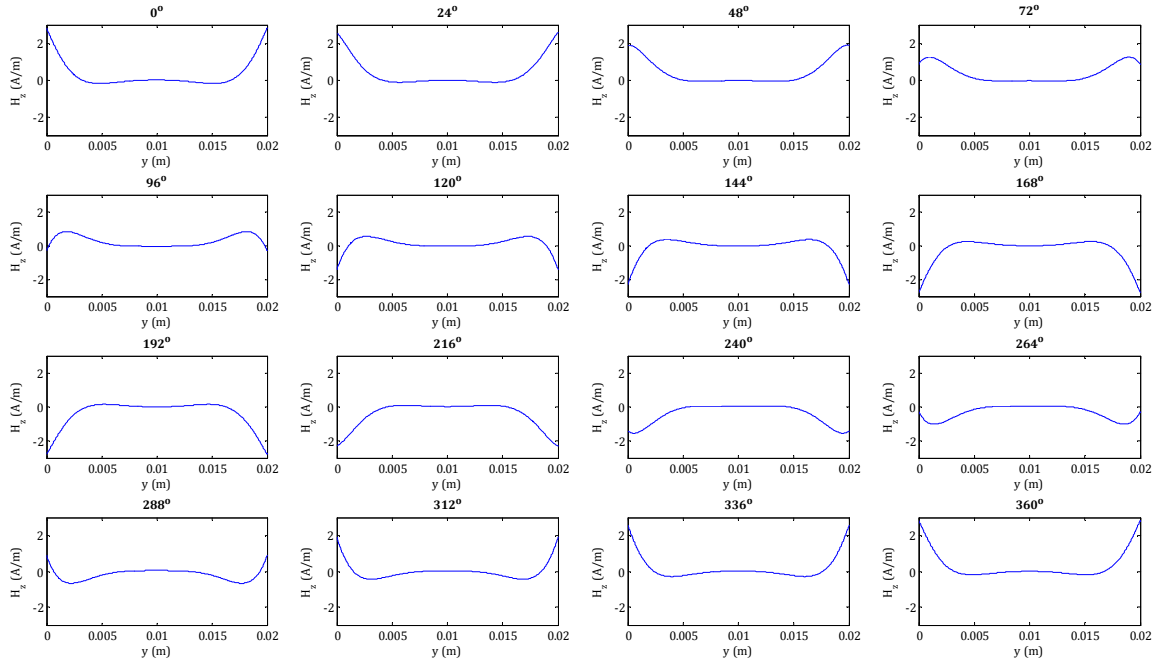


Figure 2.15. Magnetic field intensity as a function of distance for the phase angle changing from 0° to 360° in 24° increments

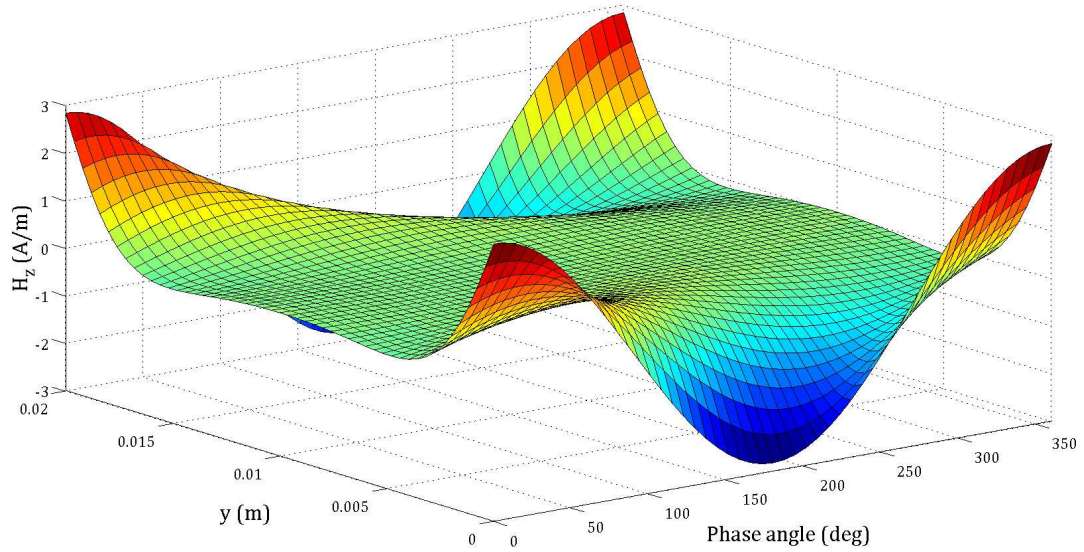


Figure 2.16. Magnetic field intensity as a function of distance and phase angle for the proximity-effect scenario

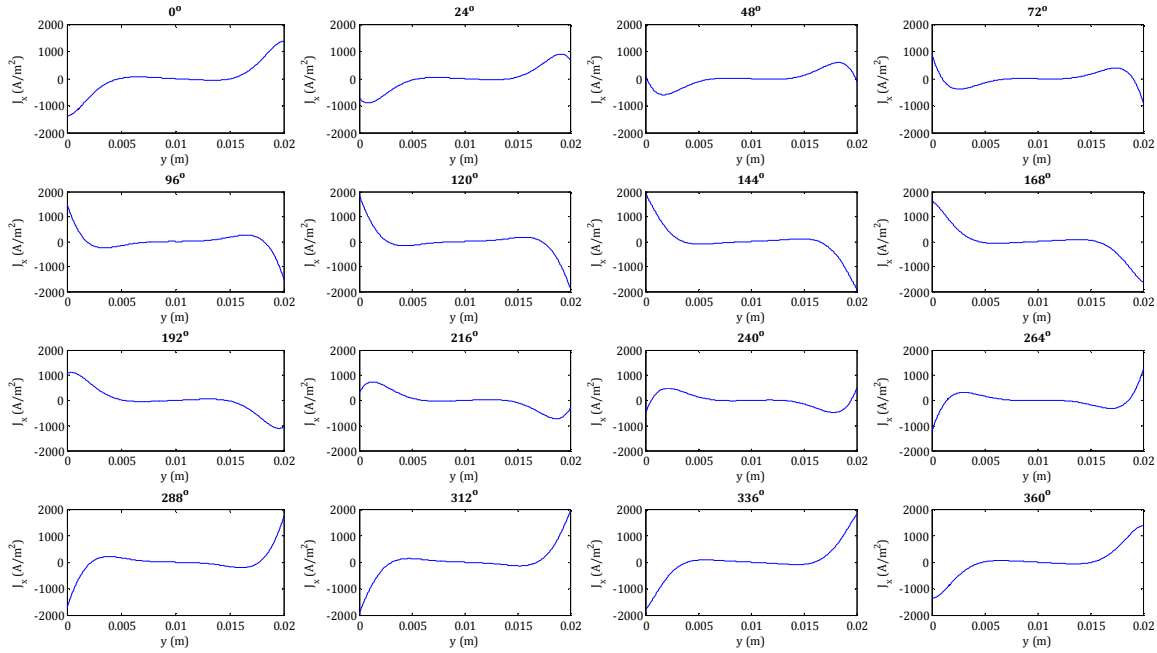
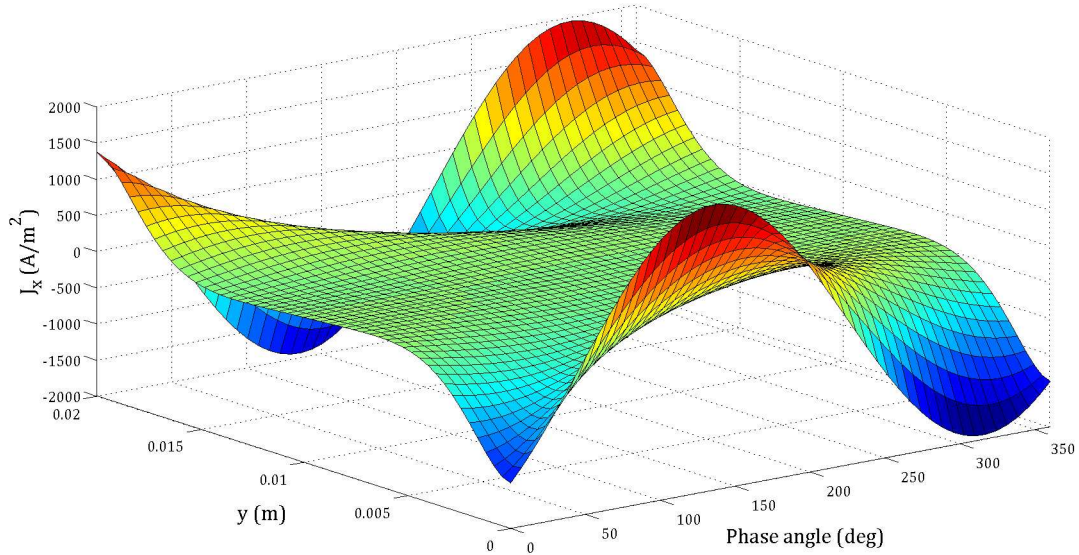
Figure 2.17. Current density distribution in the time domain as ωt changes from 0° to 360° in 24° increments

Figure 2.18. Current density as a function of distance and phase angle for the proximity-effect scenario

As a final note, the visualisation of the functions for the skin- and proximity-effect cases helped to illustrate one important, and in hindsight, obvious point. If the magnetic field is an odd function, then the current density will be an even function, and vice versa. For the skin-effect scenario this logically follows from the conductor carrying a resultant current (which is the even function). From the integral form of Ampere's law, the magnetic fields at the boundaries of the conductor will have opposite signs, as the magnetic field has to flow in a closed path (which is the odd function). In the case of the proximity effect, the magnetic field takes on the even function, as the field is of equal magnitude and sign at the boundaries of the conductor. The changing external field produces circulating emfs, which produce circulating currents within the conductor. As these have to flow in a closed path (integral form of Faraday's law), the current

will have equal magnitude and opposite signs at equal distances from the centre of the conductor (odd function).

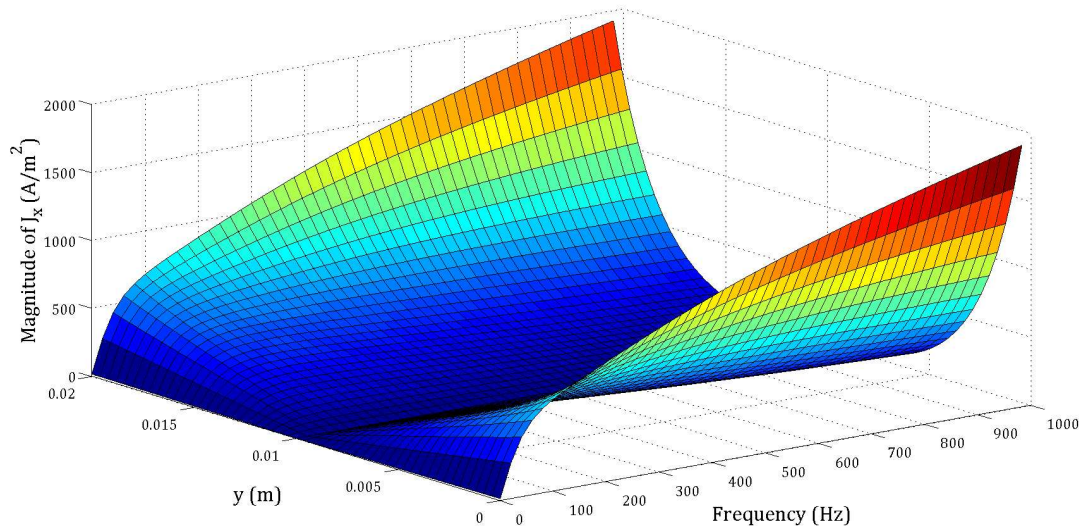


Figure 2.19. Magnitude of the current density as a function of distance and frequency for the proximity-effect scenario

2.2.4. Superposition of the skin and proximity effects

As is well known, superposition applies to current densities and magnetic field intensities as they are linear (unlike power which is non-linear and therefore superposition cannot generally be applied). This has been mathematically shown by [7] (pp.66-68) where Dowell's result ([3]) was manipulated to show that the skin- and proximity-effect currents are independent of each other. This is the main reason for studying the skin- and proximity-effect scenarios in the previous section, as they form the two fundamental and independent building blocks of any configuration, irrespective of its complexity.

Therefore, for any configuration subjected to both the skin- and proximity-effects (the geometry is unimportant) it follows that the final solution will be the superposition of the individual skin- and proximity-effect solutions. This, in turn, leads to the conclusion that the final solution can be obtained without the need to consider these effects separately, but by applying superposition to their boundary value magnetic fields and applying the resultant fields to the general solution.

2.2.5. Dowell's solution to the semi-infinite conductor

As was stated at the beginning of section 2.2 the one-dimensional derivation of Dowell ([3]) is one of the most referenced derivations of the semi-infinite conductor structure. As such, the details of the derivation, along with its self-imposed limitations, will be provided in order to justify the reasoning for selecting the derivation given in [1] and [2], as opposed to that of [3], for dealing with the semi-infinite conductors.

In order to bring the results of [3] into context with the work currently under investigation, this section will commence from the same place as that of Dowell's derivation. Dowell set out to quantify the losses in transformer windings brought about by the eddy currents associated with

the excitation frequency. As such, he approximated each round winding by a square winding of the same cross-section area, as is shown in Figure 2.20, where the transformer core has been neglected to maintain clarity. The details of the main paper are of no concern here, only the derivation of the magnetic field and current density, as given in the appendix of [3]. He starts by stating that equation (56) leads to (57),

$$\oint H dl = \sum i \quad (56)$$

$$H = \frac{\sum i}{b} = \frac{1}{b} [IN_1(p-1)] + \frac{1}{b} \int_0^x \eta b J dx \quad (57)$$

where it was stated (in [3]) that the magnetic path length was equal to the winding breadth, b , and η is given by,

$$\eta = \frac{N_1 a}{b} \quad (58)$$

with N_1 equal to the number of turns per layer and a the width of each turn. η is known as the ‘conductor spacing factor’, ‘layer porosity’ or the ‘layer copper factor’ ([8]) and, despite having been extensively used for many years, has been subjected to recent scrutiny. [8] and [9] state that η does not have any mathematical or physical justification and is only correct, as applied to the one-dimensional analysis, when it equals 1. Considering, however, the case of parallel windings to which Dowell’s derivation was originally applied, reveals that, depending on the spacing between the conductors, significant errors can be introduced in the one-dimensional equations due to two-dimensional effects. In such cases, η tends to compensate for these errors to a more or lesser extent. [8] and [9], however, stress that there is no theoretical link between the amount of compensation and the value of η . [9] goes on to say that the definition of η , equation (58), can therefore be ignored altogether and be replaced by a new definition if it produces better results. Given that semi-infinite flat conductors, and not parallel windings, are considered here, allows for the disregard of this matter (as η will be equal to one) and the continuation of this work.

As the steps leading from equation (56) to (57) were omitted in the appendix of [3], equation (57) may seem unsubstantiated. These missing steps are therefore given below, in order to obtain confidence in these results before further analysis is continued. Starting with the current carried in a single conductor of the transformer winding,

$$\begin{aligned} i &= \int_0^y \int_0^x J(x) dx dy + C \\ &= a \int_0^x J(x) dx + C \end{aligned} \quad (59)$$

where C is a constant of integration. As can be seen $J(x)$ is only a function of x as it is assumed that the layer is much wider than its thickness (i.e. the semi-infinite approximation). Take note that in this case the conductor’s finite dimension is in the x -direction. Dividing by the magnetic

path length, b (as stated by Dowell in [3]) and summing over each current carrying conductor in a single layer produces,

$$\begin{aligned}\frac{\sum i}{b} &= \frac{aN_1}{b} \int_0^x J(x) dx + \frac{N_1}{b} C \\ &= \frac{aN_1}{b} \int_0^x J(x) dx + K\end{aligned}\quad (60)$$

Substituting equation (58) into (60),

$$\begin{aligned}\frac{\sum i}{b} &= \int_0^x \eta J(x) dx + K \\ &= \frac{1}{b} \int_0^x \eta b J(x) dx + K\end{aligned}\quad (61)$$

Comparing this result to (57) reveals that K is equal to,

$$K = \frac{1}{b} [IN_1(p-1)] \quad (62)$$

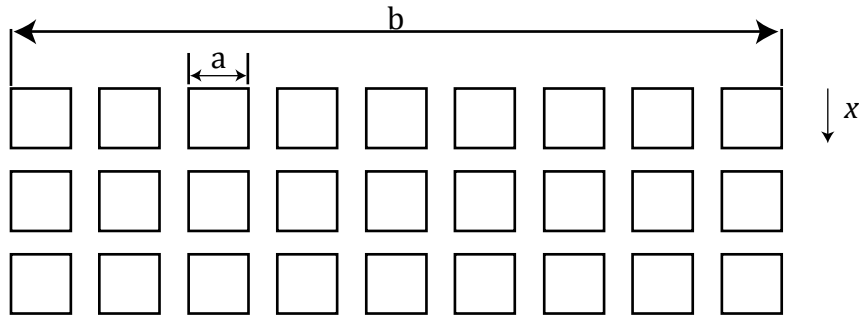


Figure 2.20. Approximation of the transformer windings by square conductors

This is where the derivation of [3] starts and continues from. It is, however, felt that the derivation found in [3] is substantially lacking in terms of proper explanation and reasoning, resulting in a derivation that could lead to misinterpretation and confusion.

In order to properly convey the reason for not using [3]'s result, a thorough rederivation of these results are needed, based purely on the semi-infinite conductor (i.e. neglecting the focus on transformer windings). Therefore, following in the path of [7], which gave a very brief explanation of [3]'s result, an extensive rederivation will be given along with the required explanation (none of which can be found in [3] or [7]).

Investigation will start with the main factor distinguishing the results of Dowell ([3]) from that of [1] and [2], the definitions of the magnetic fields at the boundaries. Taking Ampere's law around a conductor of width w (as can be found in Figure 2.21) with the thickness neglected,

$$\begin{aligned}
 \oint_c \bar{H} \cdot d\bar{s} &= \iint_s (\nabla \times \bar{H}) \cdot \bar{N} da = I \\
 &= \int_w H_{s1}(\hat{k}) \cdot ds(\hat{k}) + \int_w H_{s0}(\hat{k}) \cdot ds(-\hat{k}) = I \\
 &= H_{s1}w - H_{s0}w = I \\
 &= H_{s1} - H_{s0} = \frac{I}{w}
 \end{aligned} \tag{63}$$

where I is the total current carried by the conductor. Dowell makes the following distinguishing definitions with respect to the boundary magnetic field intensities,

$$H_{s0} = (k-1) \frac{I}{w} \tag{64}$$

$$H_{s1} = k \frac{I}{w} \tag{65}$$

Substitution of these defining equations into (63) reveals them to hold, thereby allowing these new definitions of H_{s0} and H_{s1} .

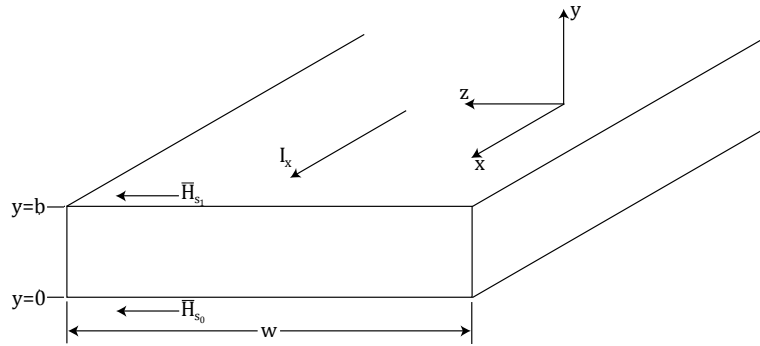


Figure 2.21. Semi-infinite approximation to the transformer layers as used in the derivation of Dowell [3]

Now that these boundary values are defined, the generic form of the magnetic field intensity can be determined. Starting with the point form of Ampere's law and applying it to the semi-infinite conductor of Figure 2.21,

$$\nabla \times \bar{H} = \bar{J} \tag{66}$$

$$\frac{dH_z(y)}{dy} = J_x(y) \tag{67}$$

Integrating both sides with respect to y ,

$$H_z(y) = \int_0^y J_x(y) dy + C \tag{68}$$

where C is the constant of integration. Using the boundary values to solve for C ,

$$H_z(0) = H_{s0} = (k-1) \frac{I}{w} = \int_0^0 J_x(y) dy + C \quad (69)$$

The integral goes to zero and C is given by,

$$C = (k-1) \frac{I}{w} \quad (70)$$

The magnetic field intensity is then given by,

$$H_z(y) = (k-1) \frac{I}{w} + \int_0^y J_x(y) dy \quad (71)$$

or equivalently,

$$H_z(y) = (k-1) \frac{I}{w} + \frac{1}{w} \int_0^y w J_x(y) dy \quad (72)$$

which is identical to equation (57) by realising that $b = w$, $p = k$, IN_1 of (57) is equal to I of (72) and the terms in the integral of both equations are identical ($N_1 a = w$). This can furthermore be confirmed by evaluating equations (57) and (72) at their boundaries and comparing the results. From $B = \mu H$ the magnetic flux density is given by,

$$B_z(y) = \mu(k-1) \frac{I}{w} + \frac{\mu}{w} \int_0^y w J_x(y) dy \quad (73)$$

Now that the magnetic field intensity and flux density have been solved, attention can be shifted to the current density. Starting with the diffusion equation written in terms of the electric field (derivation given in Appendix B),

$$\nabla^2 \bar{E} = j\omega\sigma\mu\bar{E} \quad (74)$$

Applying this to the semi-infinite conductor geometry (as shown in Figure 2.1),

$$\frac{d^2}{dy^2}(\bar{E}_x) = j\omega\sigma\mu\bar{E}_x \quad (75)$$

Using the definition $D^2 = j\omega\mu\sigma$ of equation (16), cancelling the vectors on either side and substituting equation (76) (below) produces a second- order homogenous differential equation,

$$\bar{J} = \sigma\bar{E} \quad (76)$$

$$\frac{d^2}{dy^2}(J_x) = j\omega\sigma\mu J_x \quad (77)$$

which has a solution of the form,

$$J_x(y) = A \cosh(Dy) + B \sinh(Dy) \quad (78)$$

In order to solve these two unknown coefficients, equation (73) along with Faraday's law and equation (76) will be utilised,

$$\begin{aligned} \nabla \times \bar{E} &= -\frac{\partial \bar{B}}{\partial t} \\ &= -j\omega \bar{B} \end{aligned} \quad (79)$$

$$-\frac{dJ_x(y)}{dy} = -j\omega \sigma B_z(y) \quad (80)$$

Substituting equation (73) into (80), and the generic form of the current density (78) into this result, produces,

$$\begin{aligned} \frac{dJ_x(y)}{dy} &= j\omega \sigma \mu \left[(k-1) \frac{I}{w} + \frac{1}{w} \int_0^y w J_x(y) dy \right] \\ DA \sinh(Dy) + DB \cosh(Dy) &= j\omega \sigma \mu \left[(k-1) \frac{I}{w} + \frac{A}{D} \sinh(Dy) + \frac{B}{D} (\cosh(Dy) - 1) \right] \end{aligned} \quad (81)$$

Substituting $D^2 = j\omega \sigma \mu$ (from equation (16)) and evaluating equation (81) at $y = 0$, allows for the determination of the unknown coefficient, B,

$$B = \frac{DI(k-1)}{w} \quad (82)$$

Replacing equation (82) for B in (78), and integrating over the cross-sectional area to obtain the total current flowing in the conductor,

$$\begin{aligned} I &= \int_0^w \int_0^b J_x(y) dy dz \\ &= \frac{wA}{D} \sinh(Db) + I(k-1) [\cosh(Db) - 1] \end{aligned} \quad (83)$$

Writing this in terms of A allows for the determination of the last unknown coefficient,

$$A = \frac{ID}{w} \left[\frac{1}{\sinh(Db)} - \frac{(k-1)(\cosh(Db) - 1)}{\sinh(Db)} \right] \quad (84)$$

Utilising the identities given below, allows A to be rewritten in the form of equation (87), which is the form given by Dowell ([3]) and [7].

$$\cosh(Db) - 1 = 2 \sinh^2\left(\frac{Db}{2}\right) \quad (85)$$

$$\sinh(Db) = 2 \sinh\left(\frac{Db}{2}\right) \cosh\left(\frac{Db}{2}\right) \quad (86)$$

$$A = \frac{ID}{w} \left[\frac{1}{\sinh(Db)} - (k-1) \tanh\left(\frac{Db}{2}\right) \right] \quad (87)$$

The final current density and magnetic field intensity are then,

$$\begin{aligned} J_x(y) &= \frac{DI}{w} \left[\frac{\cosh(Dy)}{\sinh(Db)} + (k-1) \left[\sinh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \cosh(Dy) \right] \right] \\ &= \frac{DI}{w} \left[\frac{\cosh(Dy)}{\sinh(Db)} + (k-1) \left[\sinh(Dy) - \tanh\left(\frac{Db}{2}\right) \cosh(Dy) \right] \right] \end{aligned} \quad (88)$$

$$\begin{aligned} H_z(y) &= (k-1) \frac{I}{w} + \int_0^y J_x(y) dy \\ &= \frac{I}{W} \left[\frac{\sinh(Dy)}{\sinh(Db)} + (k-1) \left[\cosh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \sinh(Dy) \right] \right] \\ &= \frac{I}{W} \left[\frac{\sinh(Dy)}{\sinh(Db)} + (k-1) \left[\cosh(Dy) - \tanh\left(\frac{Db}{2}\right) \sinh(Dy) \right] \right] \end{aligned} \quad (89)$$

with the electric field obtained by dividing $J_x(y)$ by σ .

2.2.5.1. Behaviour of the k-factor under different scenarios

Now that the derivation of Dowell is well understood, the exact reason for the extensive derivation of the previous section and the discarding of his result in favour of that given in [1] and [2] (which is what sections 2.2.1 to 2.2.4 are based on) will be explained.

The fundamental problem that was encountered with Dowell's result, is the definitions of the magnetic field intensities at the upper and lower boundaries (equations (64) and (65)). They are expressed in terms of the current flowing through the conductor, the width (w) and a constant (k). From these definitions and equations (88) and (89), it is clear that a solution is obtainable as long as a value of k can be found such that H_{s0} and H_{s1} is satisfied for that particular scenario. This is not normally a problem when considering the application for which Dowell's results were derived, i.e. transformer windings carrying a known current. However, a significant problem occurs when considering only the proximity-effect case where the conductor does not carry any resultant current and is subjected to an external magnetic field.

In order to demonstrate these limitations, the twin-busbar configuration, given in Figure 2.22 on p.42, will be considered under different current-carrying conditions. In all the configurations below, the magnetic field intensity and current density in the top conductor is of interest. The subscript A and B are, therefore, assigned to the top and bottom conductors respectively, in

order to avoid confusion as to which conductor reference is made (not to be confused with the coefficients used in the derivation of (88) and (89)).

$|I_A|=|I_B| \rightarrow I_A = -I_B = I$ (Figure 2.22a)

This is the well-known differential current scenario,

$$H_A = \frac{I}{2w} \quad (90)$$

$$H_B = \frac{I}{2w} \quad (91)$$

$$H_{s0} = (k-1) \frac{I}{w} = -H_A - H_B = -\frac{I}{w} \quad (92)$$

$$H_{s1} = k \frac{I}{w} = H_A - H_B = 0 \quad (93)$$

From this it is clear that a value of $k = 0$ will satisfy equations (92) and (93).

$|I_A|=|I_B| \rightarrow I_A = I_B = I$ (Figure 2.22b)

$$H_A = \frac{I}{2w} \quad (94)$$

$$H_B = \frac{I}{2w} \quad (95)$$

$$H_{s0} = (k-1) \frac{I}{w} = -H_A + H_B = 0 \quad (96)$$

$$H_{s1} = k \frac{I}{w} = H_A + H_B = \frac{I}{w} \quad (97)$$

The above criteria will hold when $k = 1$.

$|I_A|=\text{constant}, I_B = 0 \rightarrow I_A = I, I_B = 0$ (Figure 2.22c)

Take note that 'constant' in the above heading refers to constant in magnitude in the frequency domain. In this case, the bottom conductor is taken to be infinitely far away as not to have any influence on the top conductor.

$$H_A = \frac{I}{2w} \quad (98)$$

$$H_B = 0 \quad (99)$$

$$H_{s0} = (k-1) \frac{I}{w} = -H_A = -\frac{I}{2w} \quad (100)$$

$$H_{s1} = k \frac{I}{w} = H_A = \frac{I}{2w} \quad (101)$$

With the result that $k = \frac{1}{2}$ satisfies the equations.

$|I_A| > |I_B| \rightarrow I_A = I, I_B = \frac{1}{3}$ (Figure 2.22d)

$$H_A = \frac{I}{2w} \quad (102)$$

$$H_B = \frac{I}{6w} \quad (103)$$

$$H_{s0} = (k-1) \frac{I}{w} = H_B - H_A = \frac{I}{6w} - \frac{I}{2w} = -\frac{I}{3w} \quad (104)$$

$$H_{s1} = k \frac{I}{w} = H_A + H_B = \frac{I}{2w} + \frac{I}{6w} = \frac{2I}{3w} \quad (105)$$

In order to satisfy the above equations, k will have to be equal to $\frac{2}{3}$.

$|I_A| < |I_B| \rightarrow I_A = I, I_B = -5I$ (Figure 2.22e)

$$H_A = \frac{I}{2w} \quad (106)$$

$$H_B = \frac{5I}{2w} \quad (107)$$

$$H_{s0} = (k-1) \frac{I}{w} = -H_A - H_B = -\frac{I}{2w} - \frac{5I}{2w} = -\frac{3I}{w} \quad (108)$$

$$H_{s1} = k \frac{I}{w} = H_A - H_B = \frac{I}{2w} - \frac{5I}{2w} = -\frac{2I}{w} \quad (109)$$

resulting in k = -2.

$|I_A|=0, I_B = \text{constant} \rightarrow I_A = 0, I_B = I$ (Figure 2.22f)

$$H_A = 0 \quad (110)$$

$$H_B = \frac{I}{2w} \quad (111)$$

$$H_{s0} = (k-1) \frac{I_A}{w} = H_B = \frac{I_B}{2w} = \frac{I}{2w} \quad (112)$$

$$H_{s1} = k \frac{I_A}{w} = H_B = \frac{I_B}{2w} = \frac{I}{2w} \quad (113)$$

No value of k satisfies these conditions.

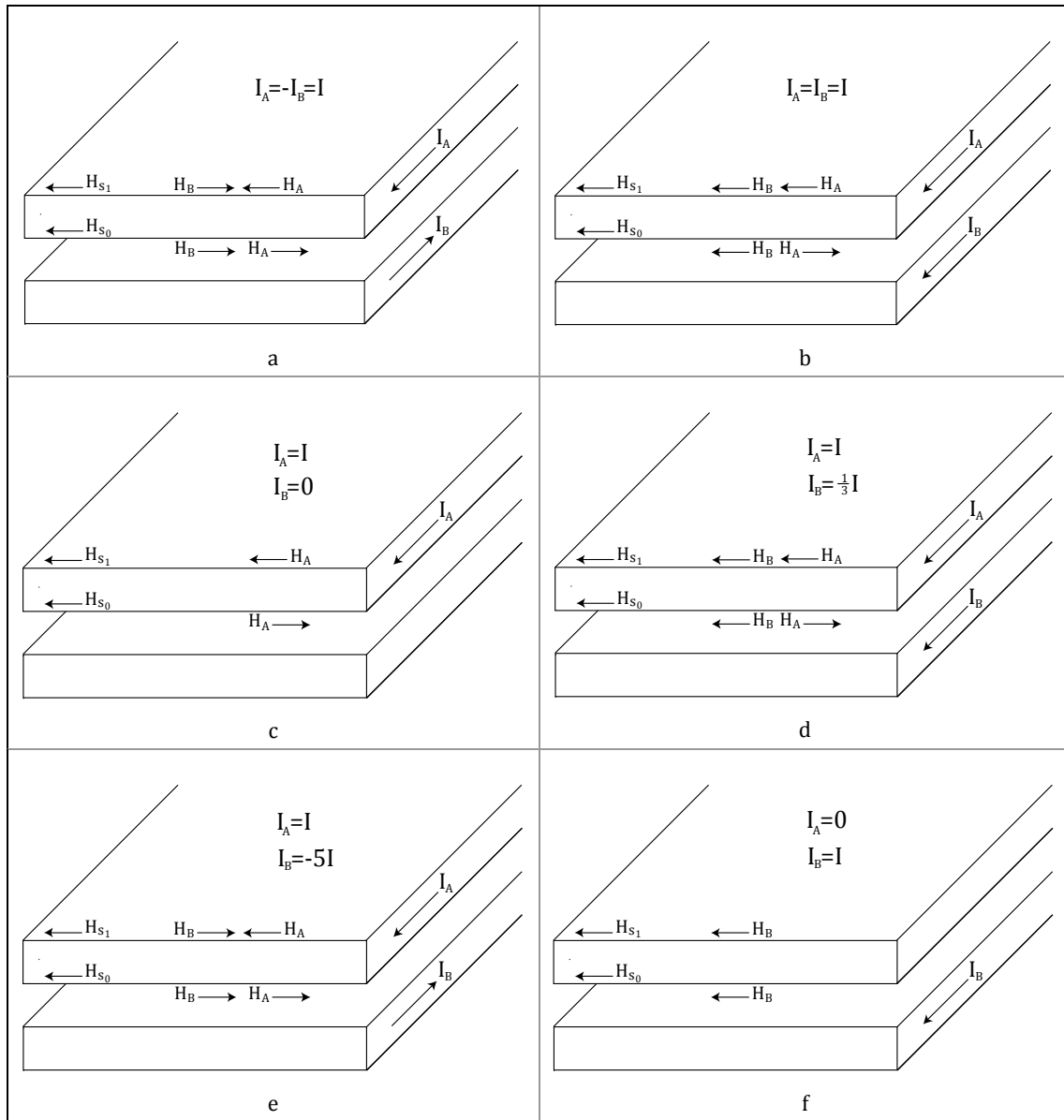


Figure 2.22. Twin-busbar configurations with different current-carrying conditions

2.2.5.2. Limitations of Dowell's results

The limitations of Dowell's definition of the boundary values (equations (64) and (65)) become apparent when trying to determine a value of k to satisfy the equations of the last example in the previous section (Figure 2.22f). The reason for this is that no relation between the current in the conductor under consideration, I_A , and the current in the neighbouring conductor, I_B , exists. This at first may seem to be only due to the fact that I_A is zero. This same problem will however be encountered for any scenario where the proximity-effect magnetic field cannot be written as a ratio of the current I_A . Consider, for example, the case where the main conductor is isolated (i.e. the bottom conductor does not exist), carries a current of 5A and is subjected to an externally applied field of 4 A/m, which is created by some unknown source. (This source will most probably be a current-carrying conductor but can be driven from a second independent source, with only the magnetic field intensity due to this current given).

The fundamental problem with these definitions can be understood by considering the magnetic field intensity and current density equations, rewritten in the forms of equations (114) and (115) below. Dowell's solutions were derived under the assumption that the conductor under consideration always carries a resultant current, which is correct for transformer windings, but not a necessary condition for single- or multi-layer conductors. As can be seen, they are both written in terms of a constant current I (which is the current carried by the conductor under consideration) and consist of two main terms, one being dependent of the value of k and the other not. (Consider for example equation (114), where it is clear that the first bracketed term (within the main brackets) is independent of k , while the second term is multiplied by the $2(k - \frac{1}{2})$ term).

The first k -independent term is identical to that which is obtained for the isolated conductor condition (i.e. skin effect only), where $k = \frac{1}{2}$. (For $k = \frac{1}{2}$ the second term reduces to zero). This can be verified by comparison to the previously derived results of section 2.2.2 (equations (39) and (41)). This can be thought of as an initial baseline. The second term is responsible for the adjustment of this baseline value (be it magnetic field intensity or current density equations) due to the presence of an additional, externally applied magnetic field, i.e. the proximity-effect. This adjustment is, therefore, achieved by changing the value of k from $\frac{1}{2}$ in order to account for this additional field.

With this in mind, it is clear that a severe limitation is encountered when the pure proximity-effect scenario is considered, as the conductor (under consideration) does not carry any current. Equations (114) and (115) show that it is not possible to obtain any magnetic field intensity or current density solution with I , the resultant current, equal to zero. Similar problems, as have been seen, are encountered with the determination of k for the case where the current, I , is non-zero, but the external field is not defined as a ratio of I (only given in A/m).

From these findings, it is clear that Dowell's results were derived under two assumptions. Firstly, that the conductor (under consideration) will always carry a resultant current, and secondly, that any external magnetic field will be defined in terms of a ratio of this current.

This outcome therefore clearly justifies the abandonment of Dowell's results in favour of [1] and [2] (on which sections 2.2.1 to 2.2.4 are based), as they provide for a more flexible framework to work from.

$$\begin{aligned}
 H_z(y) &= \frac{I}{W} \left[\frac{\sinh(Dy)}{\sinh(Db)} + (k-1) \left[\cosh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \sinh(Dy) \right] \right] \\
 &= \frac{I}{2W} \left[\frac{\sinh(Dy)}{\sinh(Db)} + \frac{\cosh(Db)}{\sinh(Db)} \sinh(Dy) - \cosh(Dy) \right] + \dots \\
 &\quad \left[2\left(k - \frac{1}{2}\right) \left[\cosh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \sinh(Dy) \right] \right]
 \end{aligned} \tag{114}$$

$$\begin{aligned}
J_x(y) &= \frac{DI}{w} \left[\frac{\cosh(Dy)}{\sinh(Db)} + (k-1) \left[\sinh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \cosh(Dy) \right] \right] \\
&= \frac{DI}{2w} \left[\left[\frac{\cosh(Dy)}{\sinh(Db)} - \sinh(Dy) + \frac{\cosh(Db)}{\sinh(Db)} \cosh(Dy) \right] + \dots \right. \\
&\quad \left. 2\left(k-\frac{1}{2}\right) \left[\sinh(Dy) - \frac{\cosh(Db)-1}{\sinh(Db)} \cosh(Dy) \right] \right] \quad (115)
\end{aligned}$$

2.2.6. Validity of the semi-infinite approximation

In the preceding section, the approximation of all finite flat conductors to semi-infinite conductors was taken to hold. The constraints which ensure the validity of this approximation need, however, to be scrutinised more closely, as significant concerns have been raised in the past as to the legitimacy of this approximation. These concerns were particularly focussed on discrepancies between the 1D impedance values, which were derived from the equations of the previous sections, and the actual values of finite configurations. (Remember, 1D refers to the results stemming from the semi-infinite approximation).

It was always assumed that the distributions predicted by the semi-infinite equations (1D equations) should coincide with the actual distributions of a finite, flat conductor (whose width is much larger than its thickness) for the vast majority of the conductor ([1]-[3]). This is the classical semi-infinite approximation used up to now. A large number of authors have, however, contradicted this assumption ([10]-[15]), stating that the edge-effect components contribute a significant percentage to the total impedance. As this edge-effect contribution is not accounted for in the 1D equations, discrepancies will occur between the 1D theoretical and actual impedances. (The edge-effect refers to magnetic field and current density which occur due to the presence of the boundaries resulting from a finite width).

In order to include these edge-effects the analysis will have to be approached from a two-dimensional perspective, as edge-effect fields and current density are two-dimensional in nature. [16], however, states that the inclusion of this second dimension complicates the analytical approach to the extent that no exact solution is possible, forcing the circumvention of this problem through a number of means.

[17] approached this problem from an empirical perspective by presenting a closed-form solution, obtained by adjustment of Dowell's solution to FEM-obtained 2-D results. [16] combined previous high and low frequency approximations (developed through use of conformal mapping and power-series, respectively) using asymptotic matching techniques in order to produce a solution applicable to a far larger frequency spectrum. While [14] proposed the use of a set of correction factors (for the 1D solutions), derived through conformal mapping techniques, in order to account for the resistance variation resulting from the edge-effect.

Although [10]-[15] raised these concerns, they were mostly focussed on round conductors approximated to flat conductors through Dowell's ([3]) method. Since [15], however, showed that the accuracy of the 1D equations are highly dependent on the exact configuration (error

ranges from 5% to 50%), the error contribution from the edge-effects may not be as significant as was initially expected. To this end, an independent comparison will be performed in order to establish the significance of the edge-effects in the flat conductor geometry and whether it is feasible to commence with the use of the 1D equations.

It may be questioned as to why the validity of the semi-infinite approximation (i.e. 1D approximation) is only dealt with now. The reason is that the 1D equations needed to be properly understood before this comparison can be performed.

2.2.6.1. Field and current density distribution

The implications of the edge-effect on the fields and current density distribution of a finite width conductor, in comparison to that predicted by the 1D equations, will be considered first. FEM simulations will be used as a representation of the practical, finite width conductor, as the determination of the actual distributions is not possible. To this end, Maxwell 2D Student Version will be used. This program is a two-dimensional FEM package in which the front view of the structure is drawn. The length (i.e. the third, uncontrollable dimension) is taken as infinite, with the current flowing in this direction and all results given on a per unit length basis.

Since [15] stated that the error is highly dependent on the configuration details, it follows that the choice of flat conductor configurations will influence this comparison. The differential-mode configuration of Figure 2.23 will therefore be selected, as it represents one of the most likely used practical configurations (chapter 5 will make extensive use of it). It also has the added advantage of encountering the least FEM convergence problems of all the flat conductor configurations. (The isolated flat conductor (skin-effect scenario) suffers from severe convergence problems).

Figure 2.23a is the FEM representation of the semi-infinite flat conductor (made possible by the use of the 'axes of even symmetry'-see Appendix C for more information) and its distributions will be used to establish confirmation of the validity of the 1D theoretical results. Once this has been obtained, the influence of the edge-effects will be considered and compared to the 1D results through the simulation of Figure 2.23b. The conductors are taken to have a conductivity of 58×10^6 S/m and permeability of free space, while the Nylon spacer has a relative permittivity of 5. The dimensions are as shown in Figure 2.23, with only the current carried and the width changing.

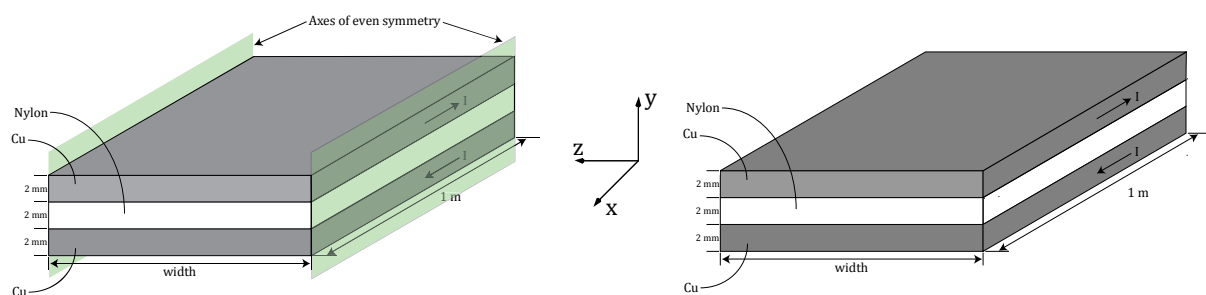


Figure 2.23a and b. FEM model of the differential-mode configuration with and without the presence of 'axes of even symmetry'

Figure 2.24 gives the theoretical and FEM results of the magnitude (absolute values) of the magnetic field intensity and current density of Figure 2.23a, where the conductors have a width of 150 mm, and carry a current of 5 A. As is clear, they are in near exact agreement, with the slight difference due to convergence problems of the FEM program. This confirms the correctness of the theoretical 1D equations, which should be no surprise as these are well known equations which have been thoroughly scrutinised over the last 50 years.

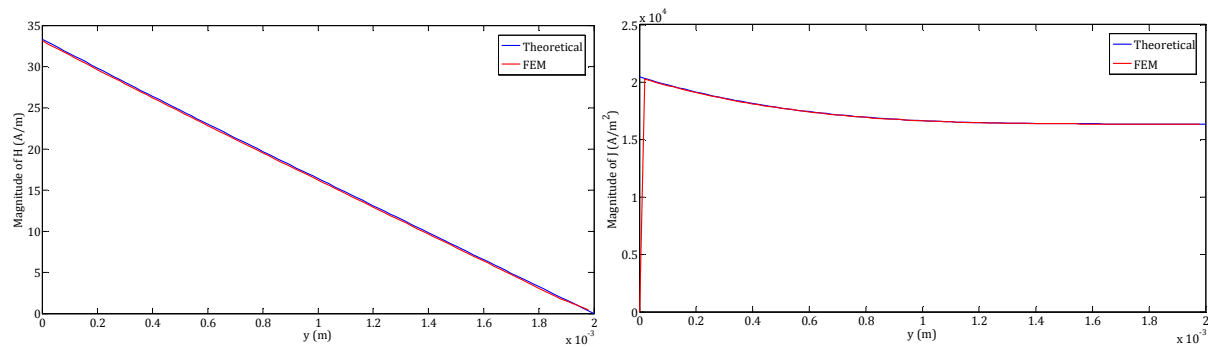


Figure 2.24a and b. Theoretical and FEM obtained magnitude values of the magnetic field intensity and current density of Figure 2.23a

The case of the finite width (Figure 2.23b) is given in Figure 2.25, which plots the absolute magnitude of the magnetic field intensity for multiple points along the width of the conductor. The conductors again have a width of 150 mm and carry a current of 1/3 A. The zero point is taken to be in the centre of the conductor, with the boundaries at plus and minus 75 mm. As the fields and current density are mirror images around the centre of the conductor (vertical axis), only the distribution for the one side is given (i.e. $x=0$ mm to $x=75$ mm). The theoretical result is also given for comparison. As is clear, the FEM results are in reasonably close agreement with the theoretical result for the majority of the conductors' width, with only the distribution at the ends deviating completely. This is undoubtedly the result of the edge-effects.

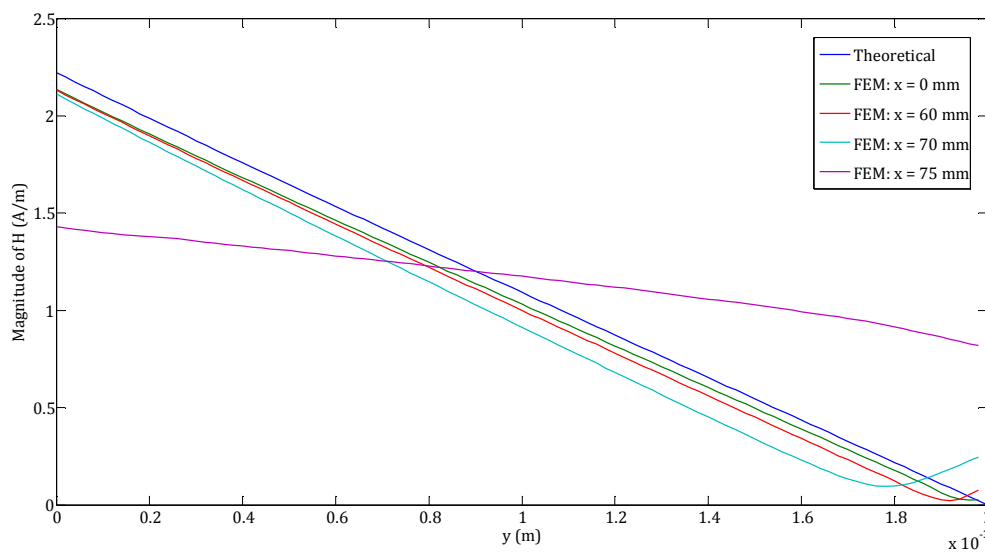


Figure 2.25. Absolute values of the magnetic field intensity at multiple points along the width

The overall agreement between the simulated and theoretical 1D distribution will, therefore, depend on the width to thickness ratio, with an increase in agreement expected for a larger ratio. This leads to the next obvious (and important) question, of what is the minimum ratio between the thickness and finite width in order to still allow for the use of the 1D equations. This is discussed in the next section.

2.2.6.2. Resistance, inductance and capacitance

As will be seen, chapter 5 focuses on the verification of the multi-layer methodology described in chapter 4, through comparison of theoretical and physical impedance values. This same approach will be used here to determine the minimum allowable width-to-thickness ratio, as it provides for quantifiable values which can easily be compared. The impedance equations used here can be found in chapter 5.

Table 2.1 gives the theoretically- and FEM-determined resistance, inductance and capacitance values of Figure 2.23a at a frequency of 1 kHz. This is just to confirm the agreement between the 1D theoretical and FEM results for the semi-infinite case. As is clear, near identical results are obtained. The minute discrepancies shown in the percentage column (with the analytical results taken to be correct) can be attributed to convergence errors of the FEM program.

Table 2.1. Theoretical and FEM produces impedance values for Figure 2.23a (per unit length)

Width (mm)	Resistance			Inductance			Capacitance		
	Analytical ($\mu\Omega$)	FEM ($\mu\Omega$)	% error	Analytical (nH)	FEM (nH)	% error	Analytical (nF)	FEM (nF)	% error
50	369.75	368.98	0.21	82.417	82.272	0.18	1.1068	1.1077	0.08
150	123.25	122.72	0.43	27.472	27.501	0.11	3.3203	3.322	0.05

Table 2.2 gives the 1D analytical and FEM results obtained for the finite-width conductors of Figure 2.23b. It is clearly shown that the variation brought about by the edge-effect is negligible, with the largest error being 4.85% (where the FEM result is taken to be correct for the error determination). This error will be taken to be the largest allowable error, limiting all future finite conductors to a minimum thickness to width ratio of 1:25 (6 mm:150 mm).

Table 2.2. Theoretical and FEM produces impedance values for Figure 2.23b (per unit length)

Width (mm)	Resistance			Inductance			Capacitance		
	Analytical ($\mu\Omega$)	FEM ($\mu\Omega$)	% error	Analytical (nH)	FEM (nH)	% error	Analytical (nF)	FEM (nF)	% error
150	123.25	122.72	0.43	27.472	26.2	4.85	3.32	3.35	0.9
300	61.62	61.45	0.28	13.736	13.41	2.43	6.64	6.67	0.45
800	23.11	23.06	0.22	5.151	5.11	0.8	17.71	17.74	0.17
1600	11.55	11.53	0.17	2.576	2.56	0.63	35.42	35.47	0.14

This is clearly not an extensive investigation into the ideal ratio, as factors such as percentage error versus frequency have not been considered. Since the focus is to obtain a reasonable estimation of the limits of the 1D equations, the ratio given above will be taken as adequate for this thesis.

One last factor which is not accounted for in the 1D equations, is the adjustment of the distributions due to the end-effects. This refers to the presence of the boundaries resulting from a finite length in a practical configuration. By selecting the length significantly larger than the thickness, this effect can easily be avoided (as was done above for the edge effects) and will, therefore, not be considered to be a factor of concern in the remainder of this work.

2.3. Single-layer cylindrical conductors

As was stated in chapter one, this investigation will not be limited to flat conductors but will be extended to cylindrical conductors as well. Focus will, however, be limited to coaxial and multi-axial configurations, as these make up a significant portion of cylindrical conductor applications. (Multi-axial conductors refer to two or more cylindrical conductors centred on the same axis). Alternative cylindrical configurations, which are not limited to the multi-axial constraints, are discussed in chapter 6 under the ‘recommendations for future work’ section.

Consider, therefore, Figure 2.26, which is the basic building block of these multi-axial configurations. The length is taken to be infinite in comparison to its diameter, thereby allowing the end-effects to be ignored. A basic derivation of the magnetic field and current density within cylindrical conductors can be found in [1] and [2], but is limited to the solid variety only (i.e. not hollow). As a far more comprehensive and flexible solution is required, a complete derivation is therefore needed. Applying the same steps as that used in the beginning of section 2.2.1 to cylindrical conductors (without any simplifications) leads to the Laplacian of the magnetic field intensity given below,

$$\nabla^2 \bar{H} = \sigma \mu \frac{\partial \bar{H}}{\partial t} \quad (116)$$

Expansion of this equation in the Cartesian coordinate-system is relatively straightforward and leads to equation (13) of section 2.2.1 under the semi-infinite approximation. This expansion is, however, far more complex when done in the cylindrical coordinate system, with the result that the full expansion is given in Appendix D. For the work at hand the final result is given in equations (117) to (119) on a component basis,

$$(\nabla^2 \bar{H})_{\rho} = \frac{\partial^2 H_{\rho}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_{\rho}}{\partial \rho} - \frac{H_{\rho}}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 H_{\rho}}{\partial \phi^2} - \frac{2}{\rho^2} \frac{\partial H_{\phi}}{\partial \phi} + \frac{\partial^2 H_{\rho}}{\partial z^2} \quad (117)$$

$$(\nabla^2 \bar{H})_{\phi} = \frac{\partial^2 H_{\phi}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_{\phi}}{\partial \rho} - \frac{H_{\phi}}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 H_{\phi}}{\partial \phi^2} + \frac{2}{\rho^2} \frac{\partial H_{\rho}}{\partial \phi} + \frac{\partial^2 H_{\phi}}{\partial z^2} \quad (118)$$

$$(\nabla^2 \bar{H})_z = \frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} \quad (119)$$

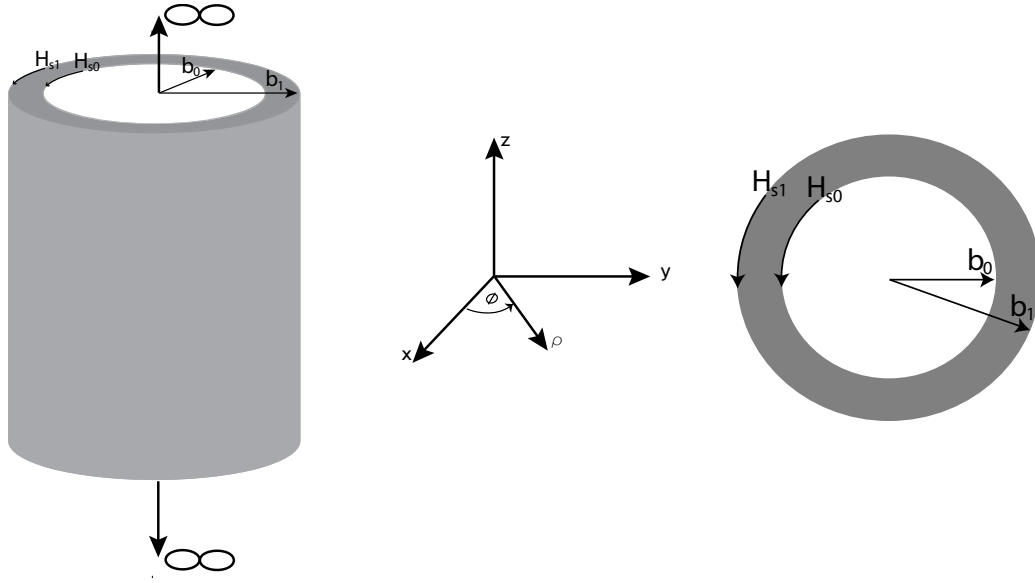


Figure 2.26. Single layer cylindrical conductor

Based on the constraints of these cylindrical conductors, the Laplacian of the magnetic field intensity can be simplified to a solvable state. The current and magnetic field intensity will be limited to the $\hat{z}$ - and $\hat{\phi}$ -directions, respectively, and only dependent on the radial $\hat{\rho}$ -component. Equations (117) and (119) consequently become zero and (118) simplifies to,

$$\nabla^2 \bar{H} = \left[\frac{\partial^2 H_\phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_\phi}{\partial \rho} - \frac{H_\phi}{\rho^2} \right] \hat{\phi} = \sigma \mu \frac{\partial \bar{H}}{\partial t} \quad (120)$$

Cancelling the unit vectors on either side and replacing the time derivative of H with $j\omega$,

$$\frac{\partial^2 H_\phi(\rho)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_\phi(\rho)}{\partial \rho} - \frac{H_\phi(\rho)}{\rho^2} = j\omega \sigma \mu H_\phi(\rho) \quad (121)$$

Letting $D^2 = j\omega \sigma \mu$ (as was done previously) allows the above equation to be rewritten as,

$$\frac{\partial^2 H_\phi(\rho)}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial H_\phi(\rho)}{\partial \rho} - \left(D^2 + \frac{1}{\rho^2} \right) H_\phi(\rho) = 0 \quad (122)$$

From [18] and [5] it can be seen that equation (122) is a parametric modified Bessel differential equation. From here on forward, equation (122) will be referred to as a modified Bessel function only. As is discussed in [18] and [5] the above equation will have a solution of the form,

$$H_\phi(\rho) = A I_1(D\rho) + B K_1(D\rho) \quad (123)$$

where I_1 and K_1 are first order modified Bessel functions (subscript 1) of the first and second kind, respectively (I and K). A and B are arbitrary coefficients that need to be determined by

means of boundary values. The generic boundary values are shown in Figure 2.26 with their exact values dependent on the scenario in which they are to be determined (as was done for the flat semi-infinite conductor scenarios). Applying the boundary values to equation (123),

$$H_\phi(b_0) = H_{s0} = AI_1(Db_0) + BK_1(Db_0) \quad (124)$$

$$H_\phi(b_1) = H_{s1} = AI_1(Db_1) + BK_1(Db_1) \quad (125)$$

From equation (124),

$$A = \frac{H_{s0} - BK_1(Db_0)}{I_1(Db_0)} \quad (126)$$

Substituting this result into (125) and writing the result in terms of B,

$$B = \frac{H_{s1}I_1(Db_0) - H_{s0}I_1(Db_1)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \quad (127)$$

Substituting B into equation (126) allows A to be expressed in terms of known constants and functions,

$$A = \frac{1}{I_1(Db_0)} \left[H_{s0} - K_1(Db_0) \left[\frac{H_{s1}I_1(Db_0) - H_{s0}I_1(Db_1)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \right] \quad (128)$$

Equations (127) and (128) are therefore the generic solutions to the coefficients of the single-layer cylindrical conductor. The current density can now be obtained from the point form of Ampere's law (neglecting the displacement current term) and the now solved magnetic field intensity. The curl of $\vec{H}$ written in terms of cylindrical coordinates is given below,

$$\nabla \times \vec{H} = \left[\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right] \hat{\rho} + \left[\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right] \hat{\phi} + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{\partial H_\rho}{\partial \phi} \right] \hat{z} \quad (129)$$

Since the magnetic field is solely in the azimuth direction ($\hat{\phi}$) and only dependent on the radial direction ($\hat{\rho}$) the curl can be reduced to,

$$\nabla \times \vec{H} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi(\rho)) \hat{z} = J_z \hat{z} \quad (130)$$

$$\begin{aligned}
J_z &= \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi(\rho)) \\
&= \frac{dH_\phi(\rho)}{d\rho} + \frac{H_\phi(\rho)}{\rho}
\end{aligned} \tag{131}$$

Substituting the solved for magnetic field intensity,

$$J_z(\rho) = \frac{A}{\rho} I_1(D\rho) + \frac{B}{\rho} K_1(D\rho) + A \frac{dI_1(D\rho)}{d\rho} + B \frac{dK_1(D\rho)}{d\rho} \tag{132}$$

Applying the chain rule allows this equation to be written as,

$$J_z(\rho) = \frac{A}{\rho} I_1(D\rho) + \frac{B}{\rho} K_1(D\rho) + ADI_1'(D\rho) + BDK_1'(D\rho) \tag{133}$$

Utilising the identities of the modified Bessel functions of the first and second kind (equations 165 and 209 in [18]) given below,

$$D\rho I_v'(D\rho) + vI_v(D\rho) = D\rho I_{v-1}(D\rho) \tag{134}$$

$$D\rho K_v'(D\rho) + vK_v(D\rho) = -D\rho K_{v-1}(D\rho) \tag{135}$$

allows the current density to be rewritten in the following form,

$$\begin{aligned}
J_z(\rho) &= \left[A \left[DI_1'(D\rho) + \frac{1}{\rho} I_1(D\rho) \right] + B \left[DK_1'(D\rho) + \frac{1}{\rho} K_1(D\rho) \right] \right] \cdot \frac{\rho}{\rho} \\
&= \frac{1}{\rho} A \left[D\rho I_1'(D\rho) + I_1(D\rho) \right] + \frac{1}{\rho} B \left[D\rho K_1'(D\rho) + K_1(D\rho) \right]
\end{aligned} \tag{136}$$

Applying the identities of equations (134) and (135) with $v=1$ to the above equation results in the solved for current density,

$$J_z(\rho) = ADI_0(D\rho) - BDK_0(D\rho) \tag{137}$$

The electric field can be acquired by dividing the current density by the conductivity of the medium,

$$E_z(\rho) = \frac{1}{\sigma} J_z(\rho) = \frac{1}{\sigma} [ADI_0(D\rho) - BDK_0(D\rho)] \tag{138}$$

2.3.1. General solution applied to the case of zero net external field

Similar to the skin-effect scenario of the semi-infinite flat conductor (section 2.2.2), the cylindrical conductor will carry a resultant current I , as shown in Figure 2.27. The boundary values are determined by applying Ampere's law in integral form around the boundaries of the conductor,

$$\oint_c \vec{H} \cdot d\vec{s} = \iint_s (\nabla \times \vec{H}) \cdot \vec{N} da = \iint_s \vec{J} \cdot \vec{N} da \quad (139)$$

As is clear when the inner boundary is substituted into the above equation, J will be zero, as no current flows for $\rho < b_0$, which reduces H_{s0} to zero as well. Substituting the outer boundary, b_1 , into the above equation results in,

$$\begin{aligned} \int_0^{2\pi} H_{s1} b_1 d\phi &= I \\ H_{s1} b_1 2\pi &= I \\ H_{s1} &= \frac{I}{b_1 2\pi} \end{aligned} \quad (140)$$

The skin-effect equations are subsequently obtained by substitution of these boundary values into the above generic equations.

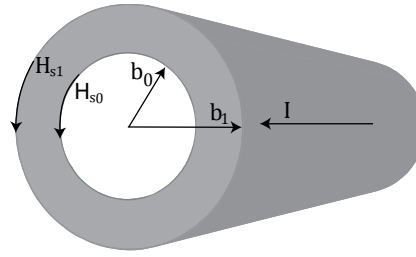
$$A = -\frac{I}{2\pi b_1} \frac{K_1(Db_0)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \quad (141)$$

$$B = \frac{I}{2\pi b_1} \frac{I_1(Db_0)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \quad (142)$$

$$H_\phi(\rho) = \frac{I}{2\pi b_1} \left[\frac{I_1(Db_0)K_1(D\rho) - K_1(Db_0)I_1(D\rho)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \quad (143)$$

$$J_z(\rho) = -\frac{ID}{2\pi b_1} \left[\frac{K_1(Db_0)I_0(D\rho) + I_1(Db_0)K_0(D\rho)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \quad (144)$$

$$E_z(\rho) = -\frac{ID}{\sigma 2\pi b_1} \left[\frac{K_1(Db_0)I_0(D\rho) + I_1(Db_0)K_0(D\rho)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \quad (145)$$

Figure 2.27. Cylindrical conductor carrying current I

Similar to the flat-conductor scenarios a better understanding of the field and current density distribution can be achieved through visualisation. Choosing a typical example of $b_0=2$ mm, $b_1=3$ mm, $\sigma=59.6 \times 10^6$ S/m, $\mu_r = 1$, $f = 20$ kHz and $I = 1.414$ A (peak) the following graphs can be obtained.

Figure 2.28 to Figure 2.33 give the absolute value (magnitudes) and time variation of the magnetic field intensity and current density within the conductor. Unlike the semi-infinite visualisations, these time-domain plots do not directly emphasise the even or odd nature of a distribution. This is because the centre point is not $\rho = (b_1 - b_0)/2$, as was the case for the flat conductors (where b_0 was equal to zero), but is $\rho = 0$. Therefore, in order to obtain a better visualisation, the plots should extend from $-b_1$ to $+b_1$. This however is not as straight forward as the equations (which were used to draw these graphs), are discontinuous below b_0 and will not be modified to accommodate for this (as it is not required for the multi-layer solution). The nature of the current density distribution is extensively discussed in section 2.4, where it is used in the investigation of the orthogonality of the skin-and proximity-effect powers. These findings help to emphasises the importance of proper interpretation of visualised information, as it may easily lead to incorrect conclusions.

Figure 2.34 gives the distribution of the absolute value of the current density as a function of radius and frequency. As is expected the current density tends to a constant value as the frequency tends to zero, and increases at the circumference (and decreases at the centre) for an increase in frequency.

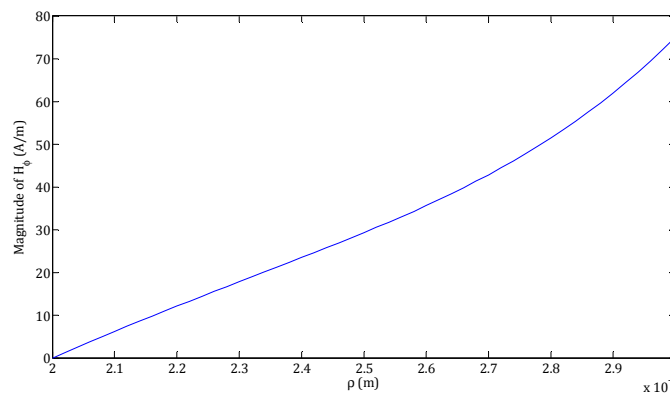


Figure 2.28. Magnitude of the magnetic field intensity along the radius of the conductor for the skin-effect scenario

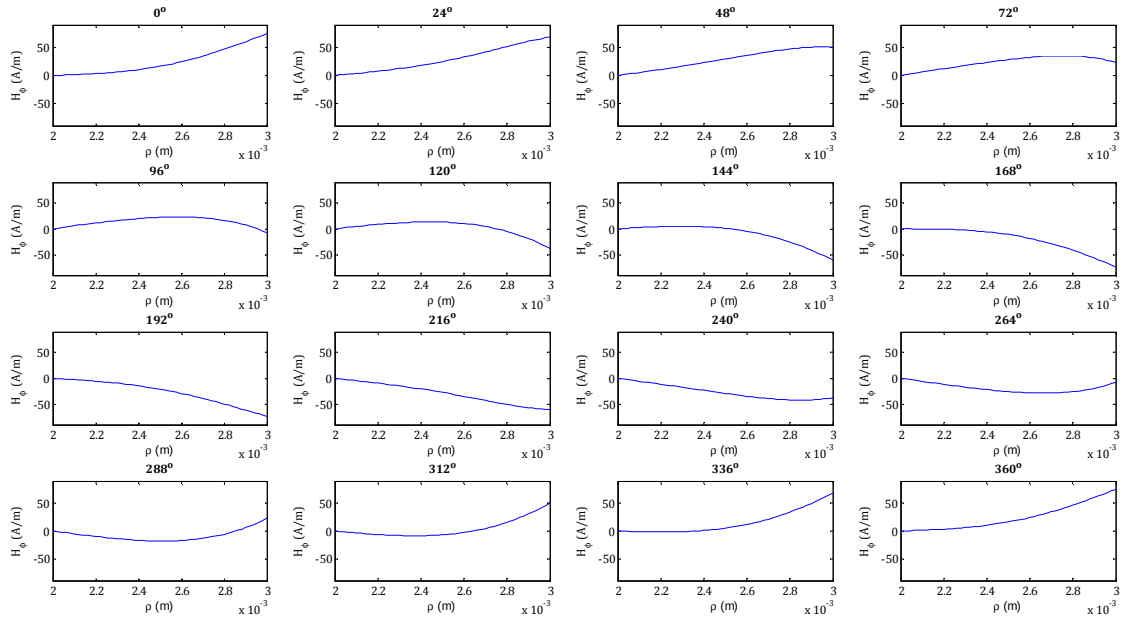


Figure 2.29 Magnetic field intensity in the time domain as ωt changes from 0° to 360° in 24° increments for the skin-effect scenario

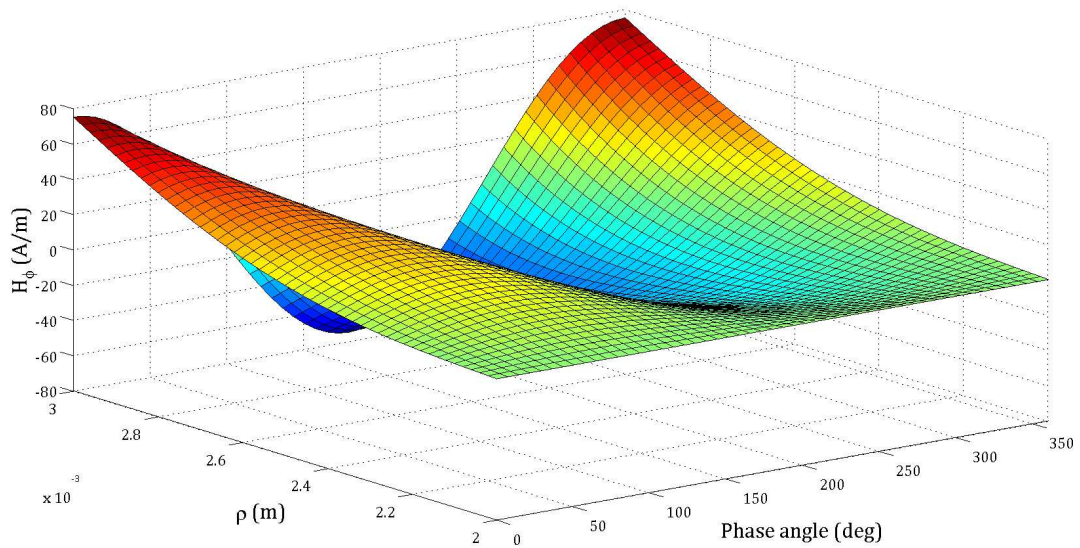


Figure 2.30. Magnetic field intensity as a function of distance and phase angle for the skin-effect scenario

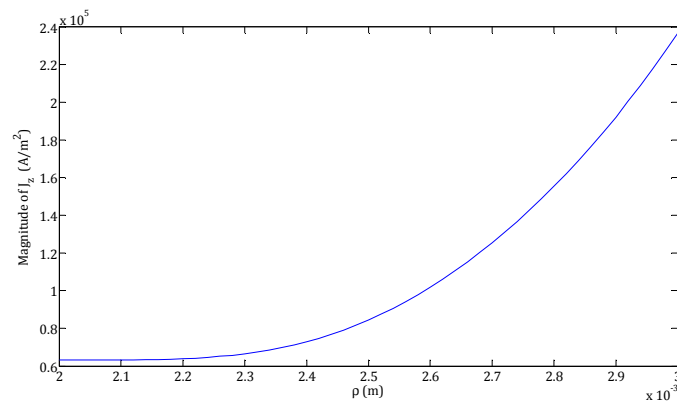


Figure 2.31. Magnitude of the current density along the radius of the conductor for the skin-effect scenario

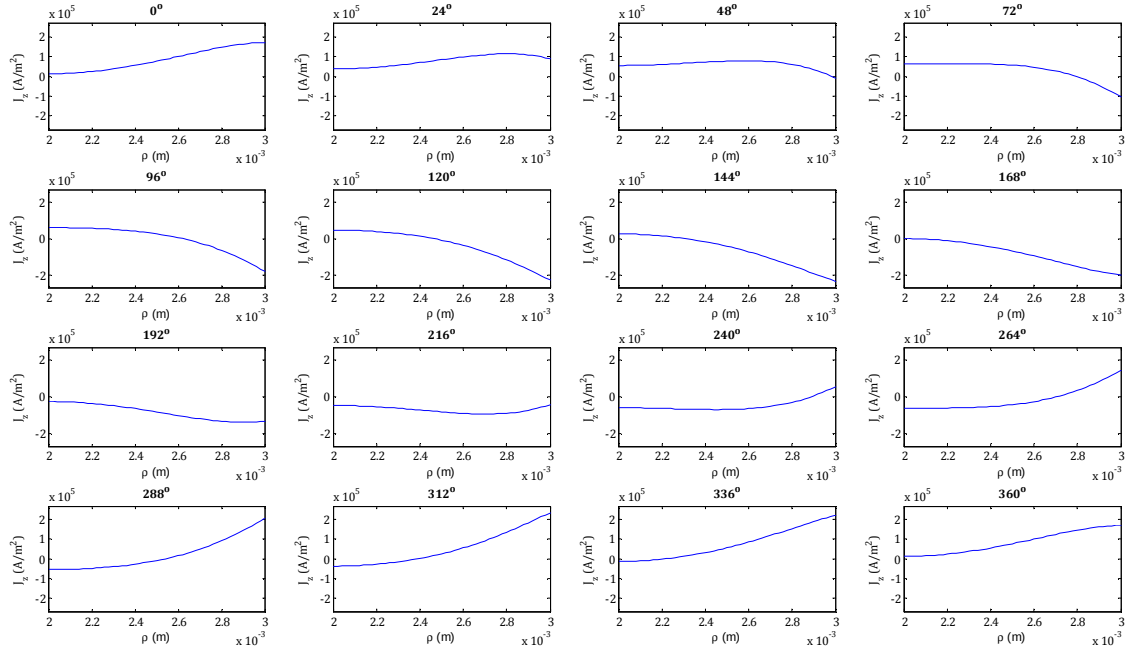


Figure 2.32. Current density in the time domain as ωt changes from 0° to 360° in 24° increments for the skin-effect scenario

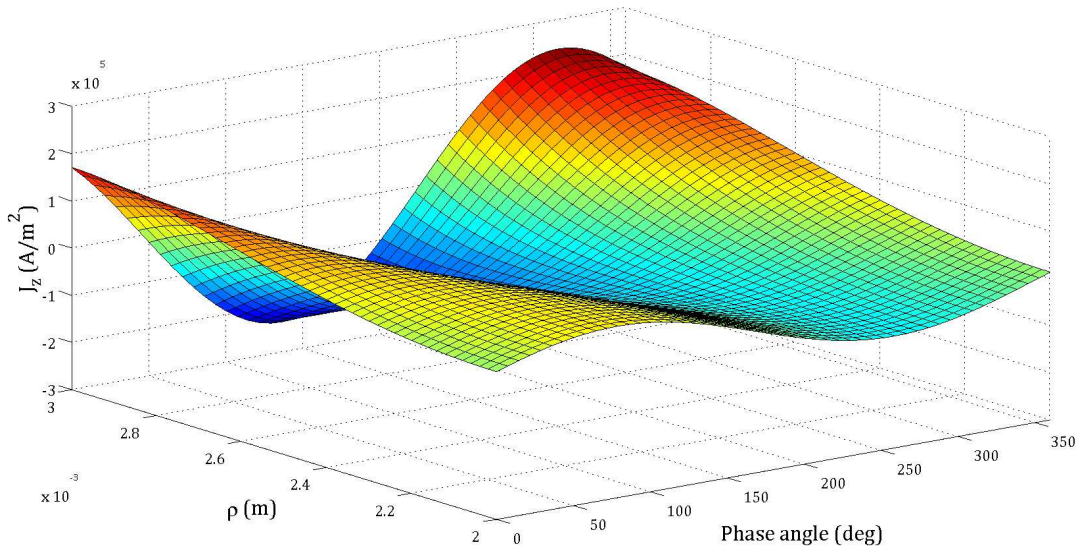


Figure 2.33. Current density as a function of distance and phase angle for the skin-effect scenario

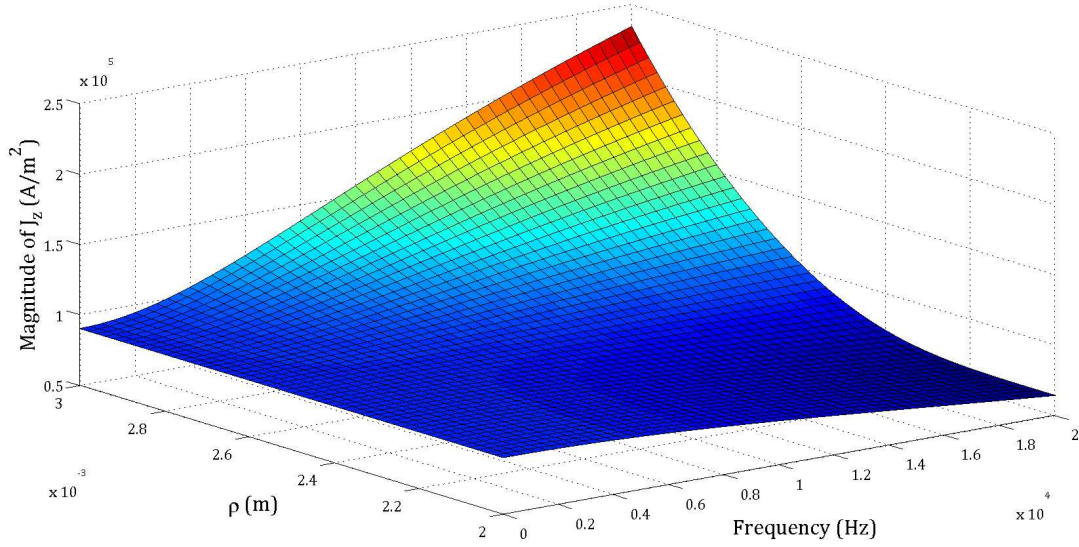


Figure 2.34. Magnitude of the current density as a function of distance and frequency for the skin-effect scenario

2.3.2. General solution applied to the case of non-zero external fields

As was previously mentioned, only coaxial configurations will be considered where all the conductors are centred on the same axis. For this cylindrical proximity-effect scenario, the conductor under consideration will not carry any resultant current, while being subjected to a magnetic field created by current carrying conductors centred within it. This can best be explained by Figure 2.35, where the inner, current carrying conductor, is responsible for a magnetic field in the azimuth direction. As can be understood, this changing magnetic field results in eddy currents within the outer conductor (which is the conductor of interest). The boundary values are determined in an identical fashion to that of the skin effect, where, I , is the current carried by the inner conductor.

$$\oint_c \vec{H} \cdot d\vec{s} = \iint_s \vec{J} \cdot \vec{N} da \quad (146)$$

$$\int_0^{2\pi} H_{s0} b_0 d\phi = I \quad (147)$$

$$H_{s0} = \frac{I}{b_0 2\pi}$$

$$\int_0^{2\pi} H_{s1} b_1 d\phi = I \quad (148)$$

$$H_{s1} = \frac{I}{b_1 2\pi}$$

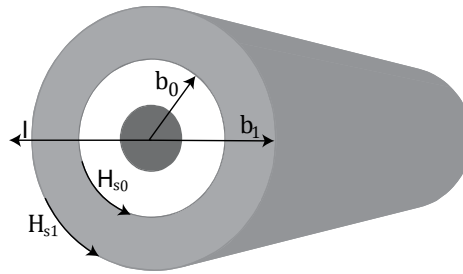


Figure 2.35. Cylindrical conductor subjected to an external magnetic field

From these results, it can be seen that the difference between the cylindrical skin- and proximity-effect scenarios is that a magnetic field intensity is present at the inner boundary. Applying these boundary values to the generic equations of the coefficients A and B ((128) and (127)) results in,

$$A = \frac{I}{2\pi I_1(Db_0)} \left[\frac{1}{b_0} - K_1(Db_0) \right] \left[\frac{\frac{1}{b_1} I_1(Db_0) - \frac{1}{b_0} I_1(Db_1)}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \right] \quad (149)$$

$$B = \frac{I}{2\pi} \frac{\frac{1}{b_1} I_1(Db_0) - \frac{1}{b_0} I_1(Db_1)}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \quad (150)$$

Substituting these results into the magnetic field and current density equations ((123) and (137)), and dividing the current density by the conductivity to obtain the electric field,

$$H_\phi(\rho) = \frac{I}{2\pi} \left[\frac{I_1(D\rho)}{b_0 I_1(Db_0)} + \left[K_1(D\rho) - \frac{I_1(D\rho) K_1(Db_0)}{I_1(Db_0)} \right] \left[\frac{\frac{I_1(Db_0)}{b_1} - \frac{I_1(Db_1)}{b_0}}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \right] \right] \quad (151)$$

$$J_z(\rho) = \frac{ID}{2\pi} \left[\frac{I_0(D\rho)}{b_0 I_1(Db_0)} - \left[\frac{I_0(D\rho) K_1(Db_0)}{I_1(Db_0)} + K_0(D\rho) \right] \left[\frac{\frac{I_1(Db_0)}{b_1} - \frac{I_1(Db_1)}{b_0}}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \right] \right] \quad (152)$$

$$E_z(\rho) = \frac{ID}{\sigma 2\pi} \left[\frac{I_0(D\rho)}{b_0 I_1(Db_0)} - \left[\frac{I_0(D\rho) K_1(Db_0)}{I_1(Db_0)} + K_0(D\rho) \right] \left[\frac{\frac{I_1(Db_0)}{b_1} - \frac{I_1(Db_1)}{b_0}}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \right] \right] \quad (153)$$

Applying these derived equations to the configuration of Figure 2.35 with $b_0 = 10 \mu\text{m}$, $b_1 = 40 \mu\text{m}$, $\sigma = 14.5 \times 10^6$, $\mu_r = 600$, $f = 150 \text{ kHz}$, $I = 1.41 \text{ mA}$ peak results in Figure 2.36 to Figure 2.38.

Similar to the skin-effect scenario, not much information can be gleaned from these graphs alone about the symmetry of these functions. They do, however, show the periodic nature of the distributions as they increase and decrease towards the inner and outer radius respectively, opposite to that of the skin-effect scenario. This opposing trend between the skin- and proximity-effects was also displayed by the semi-infinite flat conductors. This should come as no surprise, as it is the same factors which govern and influence their distributions (even though their geometries differ).

However, unlike the skin-effect case, the analysis of these functions, and their associated graphs, is significantly more complex. The symmetry of these functions is briefly addressed in the next section where they are applied in the investigation of the orthogonality between the skin- and proximity effect power components.

Figure 2.38 depicts the magnitude of the current density as a function of frequency and distance. As is expected the current density tends to a constant value for low frequencies and increases towards the inner radius for higher frequencies.

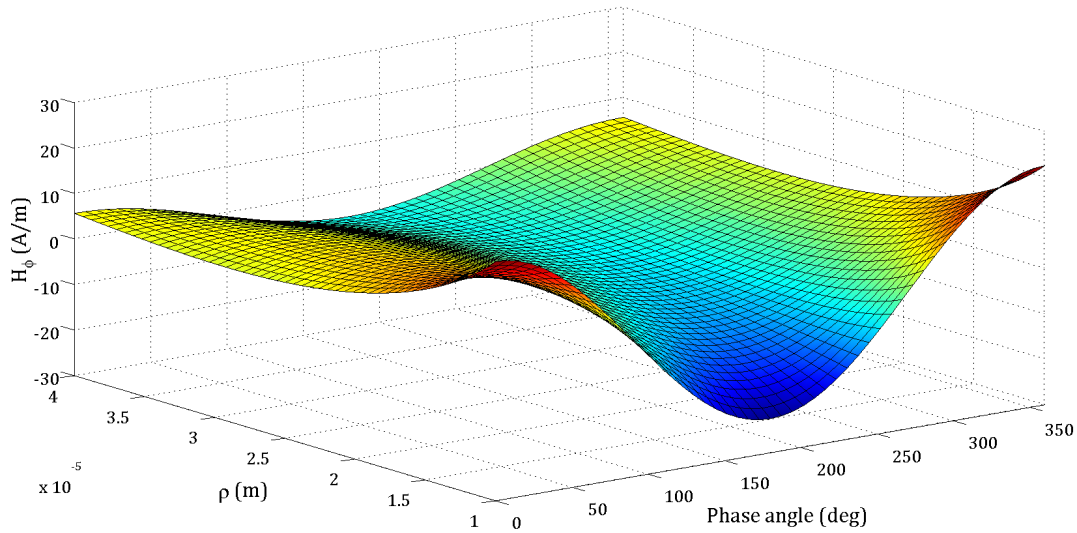


Figure 2.36. Magnetic field intensity as a function of distance and phase angle for the proximity-effect scenario

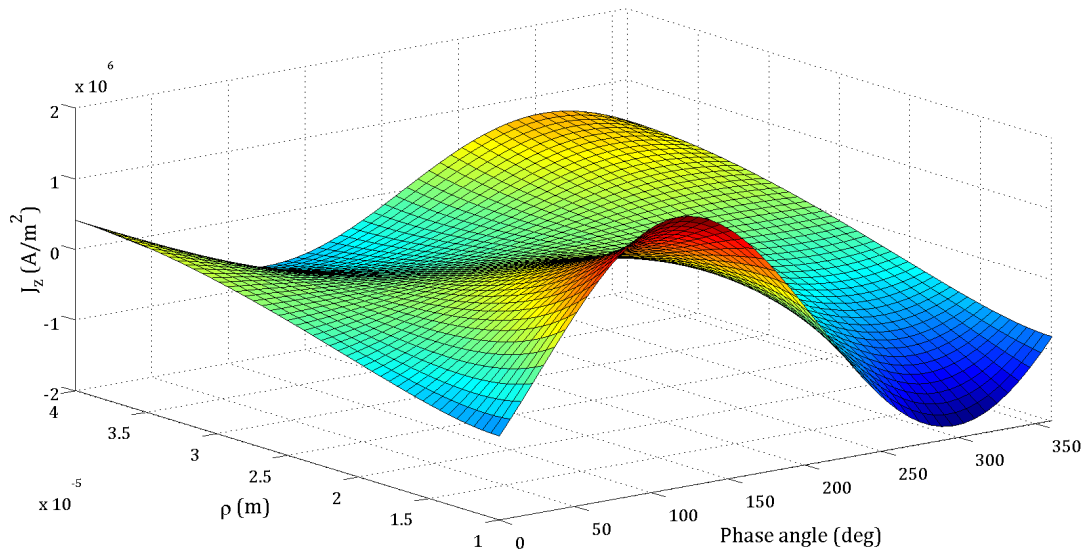


Figure 2.37. Current density as a function of distance and phase angle for the proximity-effect scenario

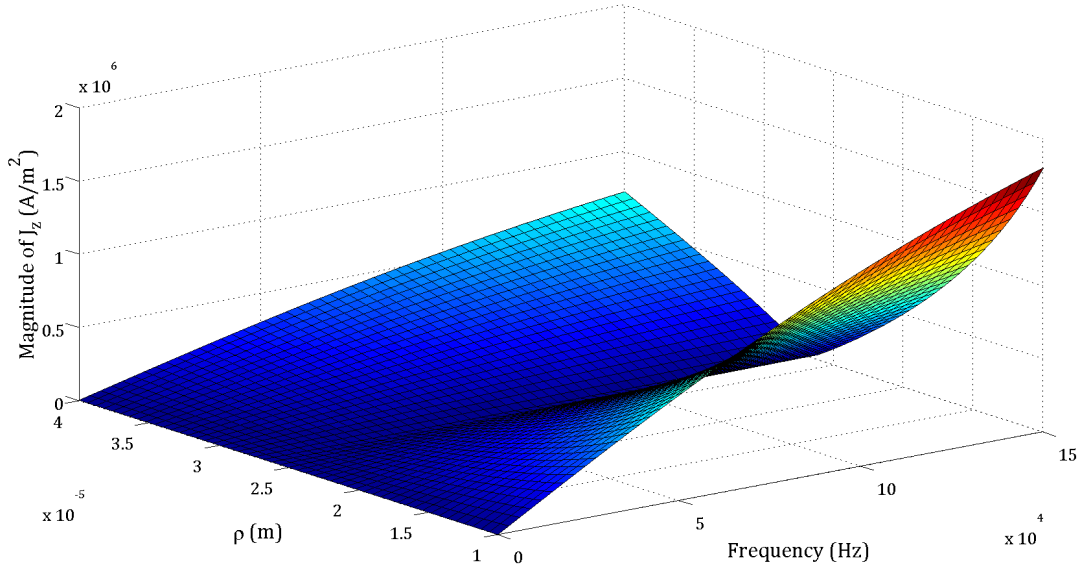


Figure 2.38. Magnitude of the current density as a function of distance and frequency for the proximity-effect scenario

2.4. Orthogonality between the skin- and proximity-effect power components

Orthogonality between the skin- and proximity-effects refers to the two effects being independent of one another. This may seem to be a reasonable and logical conclusion, as it is known (and discussed in 2.2.4) that the fields and current densities are linear and subjected to superposition. This does, however, not generally extend to the dissipated power within current carrying conductors, due to its non-linear I^2R nature, with the result that certain criteria have to be met before orthogonality can be enforced on the power components. This concept, and its associated criteria, needs to be well understood, as it will be applied in chapter 5, where the resistance of multi-layer, semi-infinite and cylindrical conductors will be determined. A thorough investigation is therefore needed.

The orthogonality principle (between skin- and proximity-effect powers) can be found in [1] (pp. 19-20) and [19] and is given below with the inclusion of a number of additional expansions. As the current density distribution is known, the power derivation will start with equation (154) below. ψ is used to represent a general dimension (direction) in order to keep the derivation applicable to any coordinate system. This is the time domain representation of the current density and can be obtained by applying the results from the previous sections to equation (44). Even though both the time and frequency domain representations will reduce to the same result, it was decided to choose the more difficult of the two approaches as it tends to provide more insight into the problem.

$$J(\psi, t) = |J_p(\psi)| \cos(\omega t + \phi(\psi)) \quad (154)$$

$J_p(\psi)$ is the peak value of the phasor. The voltage over a unit length conductor of infinitesimal cross-sectional area, da , is given by,

$$\begin{aligned}\frac{V(\psi, t)}{l} &= J(\psi, t) da \cdot \frac{R}{l} \\ &= J(\psi, t) da \cdot \frac{1}{\sigma da}\end{aligned}\quad (155)$$

where R/l , the resistance per unit length of the infinitesimal area is equal to $1/\sigma da$. This can be seen to be correct as V/l on the left hand side is the electric field (see Appendix E for a detailed discussion on potential difference and the polarity of the sign) while the right hand side is J/σ , which is also equal to the electric field. The instantaneous power per unit length dissipated over this area element is,

$$\frac{dP(\psi, t)}{l} = \frac{V(\psi, t)}{l} \cdot J(\psi, t) da \quad (156)$$

Substituting (155) into (156) and expanding,

$$\frac{dP(\psi, t)}{l} = |J_p(\psi)|^2 \cos^2(\omega t + \phi(\psi)) \frac{da}{\sigma} \quad (157)$$

The total instantaneous dissipated power (per unit length) is therefore obtained by integrating both sides,

$$\frac{P(t)}{l} = \frac{1}{\sigma} \iint_S |J_p(\psi)|^2 \cos^2(\omega t + \phi(\psi)) da \quad (158)$$

where S represents the area over which the integral is to be taken. The average dissipated power is therefore,

$$\frac{P_{avr}}{l} = \frac{1}{\sigma} \frac{1}{T} \int \iint_S |J_p(\psi)|^2 \cos^2(\omega t + \phi(\psi)) da dt \quad (159)$$

Using the following identities from [6],

$$\cos^2(u) = \frac{1 + \cos(2u)}{2} \quad (160)$$

$$\cos(u + v) = \cos(u)\cos(v) - \sin(u)\sin(v) \quad (161)$$

and applying them to (159), produces equation (162) below, where the order of integration has been interchanged,

$$\begin{aligned}\frac{P_{avr}}{l} &= \frac{1}{\sigma} \frac{1}{T} \iint_S \int_T |J_p(\psi)|^2 \left[\frac{1}{2} + \frac{1}{2} \cos(2\omega t) \cos(2\phi(\psi)) - \frac{1}{2} \sin(2\omega t) \sin(2\phi(\psi)) \right] dt da \\ &= \frac{1}{\sigma} \iint_S \frac{|J_p(\psi)|^2}{2} da\end{aligned}\quad (162)$$

As can be seen, the average power is not a function of time (as is to be expected). Equation (162) can be written in the form of (163) when considering this problem in the frequency domain.

$$\frac{P_{avr}}{l} = \frac{1}{2\sigma} \iint_S |J_p^2(\psi)| da = \frac{1}{2\sigma} \iint_S J_p(\psi) \cdot J_p^*(\psi) da \quad (163)$$

with $J_p(\psi)$ given by,

$$J_p(\psi) = |J_p(\psi)| \underline{\phi(\psi)} \quad (164)$$

As was previously discussed, superposition applies without any constraints to current density, and, as such, $J_p(\psi)$ can be broken up into its skin-and proximity-effect components (for the case where both cases are present). Equation (163) therefore becomes,

$$\frac{P_{avr}}{l} = \frac{1}{2\sigma} \iint_S (J_{skin_p}(\psi) + J_{prox_p}(\psi)) \cdot (J_{skin_p}^*(\psi) + J_{prox_p}^*(\psi)) da \quad (165)$$

Multiplying out,

$$\frac{P_{avr}}{l} = \frac{1}{2\sigma} \iint_S [J_{skin_p}(\psi) J_{skin_p}^*(\psi) + J_{prox_p}(\psi) J_{prox_p}^*(\psi) + J_{skin_p}(\psi) J_{prox_p}^*(\psi) + J_{prox_p}(\psi) J_{skin_p}^*(\psi)] da \quad (166)$$

Equation (166) shows that it is possible to separate the skin- and proximity-effect power components for the case where $J_{skin_p}(\psi) J_{prox_p}^*(\psi) + J_{prox_p}(\psi) J_{skin_p}^*(\psi)$ is zero. The required criteria under which these products reduce to zero form, therefore, the main part of this discussion.

2.4.1. Semi-infinite flat conductors

The easier, and far more investigated, semi-infinite flat conductor case will be considered first. Given below is a list of identities associated with even and odd functions:

- The function $f(y)$ is even if $f(-y) = f(y)$ (a)
- The function $f(y)$ is odd if $f(-y) = -f(y)$ (b)
- The product of two even functions is an even function (c)
- The product of two odd functions is an even function (d)
- The product of an even and an odd function is an odd function (e)
- The integral of an odd function from $-A$ to $+A$ is zero (f)
(where A is finite and the function has no vertical asymptotes between $-A$ and A)
- The integral of an even function from $-A$ to $+A$ is twice the integral from 0 to $+A$ (g)
(where A is finite, and the function has no vertical asymptotes between $-A$ and $+A$)

As can be seen all the above identities are associated with functions centred around the $y = 0$ axis. Since the semi-infinite flat conductors of section 2.2 are centred around $y = \frac{1}{2}b$, caution will have to be taken when applying identities (a) and (b). Even though the time domain plots of the current density clearly show the even and odd nature of the skin-and proximity-effect components, a more rigorous mathematical approach needs to be followed. Identities (f) and (g) will, therefore, first be applied to the skin- and proximity-effect equations ((41) and (51)) (respectively). This is done to obtain initial confirmation of their even and odd nature.

Starting with the skin-effect current density (equation (41)) and identity (g) the integral will be taken from zero to b and compared against the same integral taken from $\frac{1}{2}b$ to b .

$$\begin{aligned}\int_0^b J_x(y) dy &= \int_0^b \frac{I}{2w} \frac{D}{\sinh(Db)} [\cosh(Dy) + \cosh(D(y-b))] dy \\ &= \frac{I}{2w} \frac{D}{\sinh(Db)} \left[\frac{\sinh(Dy)}{D} + \frac{\sinh(D(y-b))}{D} \right]_0^b \\ &= \frac{I}{w}\end{aligned}\tag{167}$$

$$\begin{aligned}\int_{\frac{1}{2}b}^b J_x(y) dy &= \int_{\frac{1}{2}b}^b \frac{I}{2w} \frac{D}{\sinh(Db)} [\cosh(Dy) + \cosh(D(y-b))] dy \\ &= \frac{I}{2w} \frac{D}{\sinh(Db)} \left[\frac{\sinh(Dy)}{D} + \frac{\sinh(D(y-b))}{D} \right]_{\frac{1}{2}b}^b \\ &= \frac{I}{2w}\end{aligned}\tag{168}$$

As can be seen equation (168) is half of equation (167), which from identity (g) further substantiates the even nature of the skin-effect current density equation. Turning attention to the proximity-effect current density and identity (f) the following integral results,

$$\begin{aligned}\int_0^b J_x(y) dy &= \int_0^b HD \left[\sinh(Dy) + \frac{[1 - \cosh(Db)]}{\sinh(Db)} \cosh(Dy) \right] dy \\ &= HD \left[\frac{\cosh(Dy)}{D} + \frac{[1 - \cosh(Db)]}{\sinh(Db)} \frac{\sinh(Dy)}{D} \right]_0^b \\ &= 0\end{aligned}\tag{169}$$

which supports the odd symmetry of the proximity-effect current density.

As these previous results are necessary, but not sufficient, to prove the nature of these equations, they will have to be subjected to identities (a) and (b) as well. As is already known, the current densities ((41) and (51)), were derived for the flat conductor's centre point situated at $y = \frac{1}{2}b$. This makes the direct application of identities (a) and (b) not possible. Instead of adjusting the argument of the functions to accommodate for this shift, it will be easier to

rederive the skin- and proximity-effect equations for the conductor centred at $y=0$, with the top and bottom boundaries at $y = \frac{1}{2}b$ and $y = -\frac{1}{2}b$. (Even though this may seem irrational, the proof for the proximity-effect equation holding under identity (b) is much more difficult with the centre point at $y=\frac{1}{2}b$). The equations of Stoll ([1]) were also derived under these ‘easier’ conditions. The current densities are given as,

$$J_{skin} = \frac{I}{2w} \frac{\cosh(Dy)}{\sinh(\frac{1}{2}Db)} \quad (170)$$

$$J_{prox}(y) = DH \frac{\sinh(Dy)}{\cosh(\frac{1}{2}Db)} \quad (171)$$

From [6] it is known that cosh and sinh are even and odd functions, respectively. This proves without any doubt that the skin- and proximity-effect currents are even and odd functions themselves (respectively). As was mentioned previously, the origin in the derivation of section 2.2 was chosen to be at the bottom of the conductor, in order to facilitate with the multi-layer equations of chapter 4.

Returning to equation (166) with this newly obtained knowledge and identities (c), (d), (e) and (f) allows it to be written as,

$$\begin{aligned} \frac{P_{avr}}{l} &= \frac{1}{2\sigma} \iint_S \left[J_{skin_p}(\psi) J_{skin_p}^*(\psi) + J_{prox_p}(\psi) J_{prox_p}^*(\psi) \right] da \\ &= \frac{1}{2\sigma} \iint_S J_{skin_p}(\psi) J_{skin_p}^*(\psi) da + \frac{1}{2\sigma} \iint_S J_{prox_p}(\psi) J_{prox_p}^*(\psi) da \\ &= P_{skin} + P_{prox} \end{aligned} \quad (172)$$

where the skin and proximity power components are separated and proven to be orthogonal, i.e. independent of each other.

2.4.2. Cylindrical conductors

Obtaining a vigorous proof in the same vein for the cylindrical conductors is, however, far more difficult. Although [19] and [20] investigated round conductors, it was with regards to their application in transformer and inductor windings, where a large number of them were placed parallel next to one another. This allows for Dowell’s semi-infinite flat-conductor simplification to be applied, which in turn allows the above orthogonality proof to hold. [20] does, however, extend this orthogonality concept further, but only for the case where the external proximity causing magnetic field is transverse onto certain sections of the round conductor. An example of this is when two current carrying round conductors are parallel to one, resulting in their magnetic fields being transverse on one another.

A vigorous mathematical proof does, therefore, not seem to exist for the case where the magnetic field is in the azimuth direction and parallel to the boundaries. As the formulation of a

proof is outside the scope of this thesis it will not be done here. What will, however, be considered, is a qualitative discussion which will provide enough evidence towards the validity of the orthogonality principle, as applied to the cylindrical case, in order for it to be accepted as correct.

The skin effect will be considered first, where it will be shown to be an even function, one of the two requirements of the orthogonality principle. Equation (174) below, gives the current density distribution within a solid cylindrical conductor, where only the skin-effect is present. This is the most basic of the cylindrical conductor scenarios and can be obtained from the hollow, cylindrical skin-effect equation ((144) and reproduced as (173) below) by letting b_0 tend to zero, which forces $I_1(Db_0)$ to zero as well.

$$J_z(\rho) = -\frac{ID}{2\pi b_1} \left[\frac{K_1(Db_0)I_0(D\rho) + I_1(Db_0)K_0(D\rho)}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \quad (173)$$

$$J_z(\rho) = \frac{ID}{2\pi b_1} \frac{I_0(D\rho)}{I_1(Db_1)} \quad (174)$$

From equation 160, p. 201 in [18] it can be seen that the zeroth order modified Bessel function of the first kind ($I_0(D\rho)$) is an even function. This then results in the current density being an even function as well. Expanding this concept mathematically to the hollow cylindrical case is significantly more complicated and will not be done here. It can however be hypothesised that since it holds for a solid, cylindrical conductor, it will hold for the case where the inside radius is not zero.

The second criteria for orthogonality, the odd symmetry of the proximity-effect, can now be considered. Demonstrating this is not as easily done as the skin-effect scenario. From the time domain representation of Figure 2.37, it may at first glance seem that the distribution is even. This is, however, not the case and can be demonstrated by considering identity (f), which states that the integral of an odd function from $-A$ to $+A$ is zero. This cannot be directly applied as equation (152) (the proximity-effect current density and given in (175) for clarity) is only defined from b_0 to b_1 .

$$J_z(\rho) = \frac{ID}{2\pi} \left[\frac{I_0(D\rho)}{b_0 I_1(Db_0)} - \left[\frac{I_0(D\rho)K_1(Db_0)}{I_1(Db_0)} + K_0(D\rho) \right] \left[\frac{\frac{I_1(Db_0)}{b_1} - \frac{I_1(Db_1)}{b_0}}{K_1(Db_1)I_1(Db_0) - K_1(Db_0)I_1(Db_1)} \right] \right] \quad (175)$$

This identity can, however, be interpreted in a different way. It states that for every positive point there is a corresponding negative point, such that their sum is equal to zero. Integrating (or summing over an infinite amount of infinitesimal points) over the range of the function will therefore equate to zero. This can be directly applied to the current density equations when considering a symmetrical geometry (such as the semi-infinite flat conductors). However, in the

cylindrical case, the change in area increases as the radius increases. Consequently, in order to apply identity (f) to equation (175) the integral needs to be taken over the entire area of the conductor (and not just from b_0 to b_1). This is done in (176) below,

$$\iint_s J_z(\rho) dz = \iint_s \frac{ID}{2\pi} \left[\frac{I_0(D\rho)}{b_0 I_1(Db_0)} - \left[\frac{I_0(D\rho) K_1(Db_0)}{I_1(Db_0)} + K_0(D\rho) \right] \left[\frac{\frac{I_1(Db_0)}{b_1} - \frac{I_1(Db_1)}{b_0}}{K_1(Db_1) I_1(Db_0) - K_1(Db_0) I_1(Db_1)} \right] \right] dz$$

$$= 0$$

(176)

which shows that the proximity-effect current density satisfies identity (f). Even though the orthogonality principle for the cylindrical case has not been cast in stone by the above qualitative analysis, it does provide enough support for it to be taken as correct.

2.5. Conclusion

This chapter focussed on the review of existing equations, concepts and information that will be essential in the development of a multi-layer methodology in chapter 4, and its subsequent application in chapter 5. It started out by identifying two of the main building blocks used in the majority of conductive configurations, the flat- and cylindrical conductive geometries. The flat conductor was approximated by a semi-infinite conductor, while the cylindrical conductor was confined for use in multi-axial configurations. This allowed the mathematics governing their behaviour to be reduced to a solvable state. As the flat conductor is severely approximated, a thorough investigation was done to identify the conditions under which this approximation holds.

The existing equations for these conductive geometries were reviewed, which led to the rederivation of the flat conductor's equations, as they were not suited for direct use in chapter 4. The only cylindrical equations that were found were for solid, cylindrical conductors. As these were inadequate, a rederivation for hollow, cylindrical conductors followed. Both of these geometries' generic equations were applied to their respective skin- and proximity-variants, as these are the fundamental building blocks that comprise any scenario, and a thorough understanding of these scenarios is required.

Since Dowell's derivation of the 1D equations is so frequently referenced, a thorough investigation was warranted. It was found that the self-imposed limitations, brought about in the derivation, limits their use to current carrying conductors only, thereby eliminating their possible use in the multi-layer derivation.

Lastly, attention was focussed on the orthogonality principle, as this is an important concept for the resistance determination in chapter 5. The orthogonality between the skin- and proximity-effect for the semi-infinite geometry is quite well known, with the proof relatively easily obtained. The orthogonality of the cylindrical geometry, while undoubtedly true, was found to

be non-existent. Since this proof is outside the scope of this thesis, it was reduced to providing substantiating evidence to justify the use of this principle on cylindrical geometries.

The information and results of this chapter will be essential to chapter 4. However, before continuing with their application, chapter 3 will consider a different perspective on the 1D-equations.

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Chapter 3

The TEM Plane Wave Approach

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3.1. Introduction

As was seen, one of chapter 2's main focus points was the study of existing equations predicting the behaviour of single layer flat and cylindrical conductors, in preparation of chapter 4's multiplayer methodology. These equations resulted from first principle application of Maxwell's equation, leading to the well-known diffusion equation, where they were applied to the two fundamental scenarios, the skin- and proximity-effects.

A different approach has, however, been proposed to chapter 4's multi-layer problem by [1], in which the derivation is based on a TEM plane wave incident on the surface of one or more conductors. The basic concept, when applied for example to the single-layer semi-infinite flat conductor of Figure 3.1, is that two TEM plane waves are incident on each side of the conductor. The waves originate at the boundaries (surfaces) and propagate into the conductor. The magnetic field, electric field and density distribution within the conductor is then the result of the interaction between the waves and the conductor's properties.

Even though the single layer wave approach has been suggested before ([2][3][4]), it has not yet been fully developed for single- or multi-layer ([1]) conductor use. Therefore, before any further consideration can be made, it needs to be established whether it is scientifically sound with regards to this application.

A logical argument dictates that only a solution based on Maxwell's equations will produce the correct results. As the TEM plane wave is derived from these equations, it may be argued that this method may be applicable to this single- and multi-layer problem. This however needs to be mathematically established before further consideration can be given. A proof will therefore be undertaken and will be based on a purely generic semi-infinite flat material.

3.2. The TEM plane wave methodology

Figure 3.1 shows the semi-infinite structure on which this derivation will be based. Even though this method is considered for multi-layer conductors, the proof will be focussed on the simpler single-layer variant. Similar to the flat conductors of chapter 2, the medium will be taken to be semi-infinite along the length and width in order to reduce the problem to one-dimension only.

As was stated above, the medium will be kept completely generic in order to obtain an all-encompassing solution which can then be tailored to the specific scenario at hand. The medium will therefore be taken to have conductive, magnetic and dielectric properties. From this the propagation constant and wave number are defined as [4],

$$\gamma = jk = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} = j\omega\sqrt{(\mu' - j\mu'')(\epsilon' - j\epsilon'')} \quad (1)$$

where

$$\epsilon = \epsilon' - j\epsilon'' \quad (2)$$

$$\mu = \mu' - j\mu'' \quad (3)$$

$$\epsilon'' = \frac{\sigma}{\omega} + \epsilon''_{NC} \quad (4)$$

As is clear, the permittivity and permeability are complex, with the real components associated with lossless materials, while the imaginary components are the loss-representing factors incurred through electric and magnetic interaction with non-ideal/loss incurring materials. The imaginary permittivity component consists again of two separate components, ϵ''_{NC} , which refers to the dielectric losses and the conductivity, σ , which is associated with resistive losses [4].

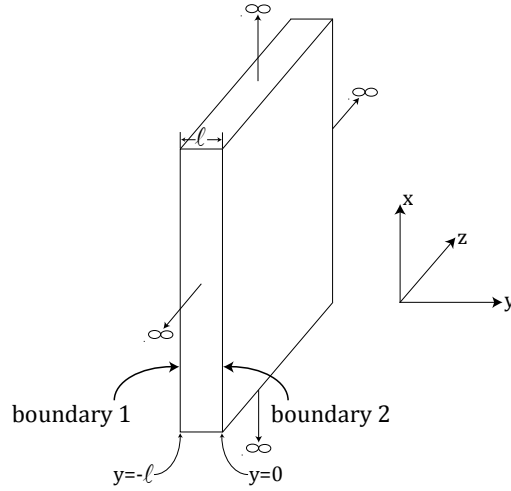


Figure 3.1. Semi-infinite generic structure on which the TEM plane wave

The TEM plane waves will be represented by their electric fields (defined to be in the $\hat{x}$ -direction), although the magnetic fields could just as easily have been used ($\hat{z}$ -direction). Figure 3.2 shows a cross-sectional view of Figure 3.1. From the above discussion two sources will be present, one at boundary 1 (source 1) and the other at boundary 2 (source 2), both transmitting waves into the medium that are sinusoidal and continuous in time.

Following one wave from source 1 (ignoring source 2 for now) created at a specific time, it can be seen to travel to boundary 2, where it will be partially reflected (the difference is transmitted into medium three, which is not of interest here). This reflected component will travel back to boundary 1 where it will again be partially reflected. This reflected wave will then superimpose with a new wave created by source 1 at boundary 1 at the time of reflection. This superimposed wave will then travel to boundary 2 where it will undergo the same process of partial reflection at boundary 2, partial reflection at boundary 1 and superposition with a new wave. While this process is underway with a given wave, it must be remembered that new waves are created continuously by source 1 and undergo this same process. This allows all the positive and negative travelling waves, which were created by source 1, to be combined, resulting in two waves, E_{1i} and E_{1r} . The subscript 1 refers to the wave created by the source at boundary 1, while the i and r refers to incident and reflected, respectively. These two waves are shown in Figure 3.2.

Also shown are the incident and reflected waves created in an identical fashion by the source at boundary 2, E_{2i} and E_{2r} . The subscript 2 obviously refers to the waves created by source 2, while i and r again refer to the incident and reflected waves.

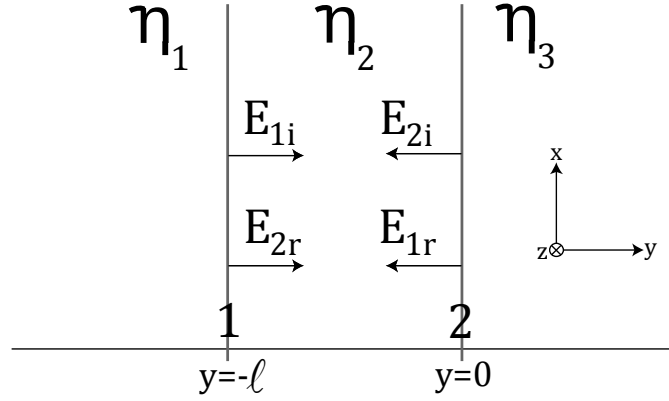


Figure 3.2. Side-view of the generic material

Looking now at the superimposed resultant waves due to the source at boundary 1, they can be represented by equations (5) and (6), which show their time domain and frequency domain forms.

$$E_{1i}(t, y) = |E_{1iM}| e^{j(\alpha t - k(\ell + y))} \text{ in phasor form } \Rightarrow E_{1i}(y) = |E_{1iM}| e^{-jk(\ell + y)} \quad (5)$$

$$E_{1r}(t, y) = |E_{1rM}| e^{j(\alpha t + ky + \theta)} \text{ in phasor form } \Rightarrow E_{1r}(y) = |E_{1rM}| e^{j(ky + \theta)} \quad (6)$$

The subscript M refers to the absolute magnitude of the electric field. $E_{1i}(y)$ is taken to be at zero phase at boundary 1 ($y = -\ell$), hence the $(\ell + y)$ term in the exponent of the e term (equation (5)). θ is the phase shift of the reflected wave at interface 2 with respect to the incident wave at 1. The reflection coefficient at boundary 2 is then given by,

$$\Gamma_1 = \frac{E_{1r}(0)}{E_{1i}(0)} = \frac{|E_{1rM}| e^{j\theta}}{|E_{1iM}| e^{-jk\ell}} = \frac{|E_{1rM}| e^{j\theta}}{|E_{1iM}| e^{-\alpha\ell} e^{-j\beta\ell}} = |\Gamma_1| e^{j\phi} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \quad (7)$$

Where $\phi = \theta + \beta\ell$ and $|\Gamma_1| = |E_{1rM}| / |E_{1iM}| e^{-\alpha\ell}$. The subscript 1 refers to all the waves originating from source 1. Therefore, from (7) $|E_{1rM}| = |\Gamma_1| |E_{1iM}| e^{-\alpha\ell}$, which allows for the total electric field due to source 1 to be written in terms of the incident field, E_{1i} ,

$$\begin{aligned} E_1(y) &= E_{1i}(y) + E_{1r}(y) \\ &= |E_{1iM}| e^{-jk(y+\ell)} + |\Gamma_1| |E_{1iM}| e^{-\alpha\ell} e^{j(ky+\theta)} \end{aligned} \quad (8)$$

Consider now the superimposed resultant waves due to the source 2 at boundary 2,

$$E_{2i}(y) = |E_{2iM}| e^{j(ky+R)} \quad (9)$$

$$E_{2r}(y) = |E_{2rM}| e^{j(-k(\ell+y)+\psi+R)} \quad (10)$$

where M again refers to the absolute magnitude. The incident wave, $E_{2i}(y)$, is defined with respect to the incident wave, $E_{1i}(y)$ (which was taken to be at zero phase), and R therefore represents the phase shift between them. Ψ is the phase shift of the reflected wave, $E_{2r}(y)$, with respect to the incident wave at boundary 2, $E_{2i}(y)$ and will therefore have a total phase shift of $\Psi+R$. The reflection coefficient at boundary 1 can therefore be written as,

$$\Gamma_2 = \frac{E_{2r}(-\ell)}{E_{2i}(-\ell)} = \frac{|E_{2rM}|e^{j\psi}e^{jR}}{|E_{2iM}|e^{-jk\ell}e^{jR}} = \frac{|E_{2rM}|e^{j\psi}e^{jR}}{|E_{2iM}|e^{-\alpha\ell}e^{-j\beta\ell}e^{jR}} = |\Gamma_2|e^{j\delta} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \quad (11)$$

where $\delta=\psi+\beta\ell$. The total electric field due to the source 2 can now be written as,

$$\begin{aligned} E_2(y) &= E_{2i}(y) + E_{2r}(y) \\ &= |E_{2iM}|e^{j(ky+R)} + |\Gamma_2||E_{2iM}|e^{-\alpha\ell}e^{-j(k(y+\ell)+\psi+R)} \end{aligned} \quad (12)$$

As is clear, there will now be four waves present within medium two, two in the positive and two in the negative y-direction. Adding all the waves and grouping the positive and negative travelling waves produces,

$$\begin{aligned} E_{TEM}(y) &= e^{-jk(y+\ell)} \left[|E_{1iM}| + |\Gamma_2||E_{2iM}|e^{-\alpha\ell}e^{j(y+R)} \right] + e^{jky} \left[|E_{2iM}|e^{jR} + |\Gamma_1||E_{1iM}|e^{-\alpha\ell}e^{j\theta} \right] \\ &= E_{TEM+}(y) + E_{TEM-}(y) \end{aligned} \quad (13)$$

As can be seen, this reduces to one resultant positive and one resultant negative travelling wave. The subscript 'TEM' is used to denote their association with the TEM plane wave approach and is used to distinguish them from the electric fields used in the next section. Using Euler's identity, the resultant positive and negative travelling waves can be rewritten in terms of magnitude and phase shift,

$$\begin{aligned} E_{TEM+}(y) &= e^{-jk(y+\ell)} \left[(|E_{1iM}| + |\Gamma_2||E_{2iM}|e^{-\alpha\ell} \cos(\psi+R)) + j(|\Gamma_2||E_{2iM}|e^{-\alpha\ell} \sin(\psi+R)) \right] \\ &= e^{-jk(y+\ell)} \left[E_{TEM+R} + jE_{TEM+I} \right] \\ &= e^{-jk(y+\ell)} |E_{TEM+M}| e^{j \tan^{-1} \left(\frac{E_{+I}}{E_{+R}} \right)} \\ &= e^{-jk(y+\ell)} |E_{TEM+M}| e^{j\chi} \end{aligned} \quad (14)$$

where

$$E_{TEM+R} = (|E_{1iM}| + |\Gamma_2||E_{2iM}|e^{-\alpha\ell} \cos(\psi+R)) \quad (15)$$

$$E_{TEM+I} = |\Gamma_2||E_{2iM}|e^{-\alpha\ell} \sin(\psi+R) \quad (16)$$

$$|E_{TEM+M}| = \sqrt{E_{TEM+R}^2 + E_{TEM+I}^2} \quad (17)$$

$$\chi = \tan^{-1} \left(\frac{E_{+I}}{E_{+R}} \right) \quad (18)$$

and

$$\begin{aligned}
 E_{TEM-}(y) &= e^{jky} \left[\left(|E_{2iM}| \cos R + |\Gamma_1| |E_{1iM}| e^{-\alpha \ell} \cos \theta \right) + j \left(|E_{2iM}| \sin R + |\Gamma_1| |E_{1iM}| e^{-\alpha \ell} \sin \theta \right) \right] \\
 &= e^{jky} \left[E_{TEM-R} + j E_{TEM-I} \right] \\
 &= e^{jky} |E_{TEM-M}| e^{j \tan^{-1} \left(\frac{E_{-I}}{E_{-R}} \right)} = e^{jky} |E_{TEM-M}| e^{j\tau}
 \end{aligned} \tag{19}$$

where

$$E_{TEM-R} = |E_{2iM}| \cos R + |\Gamma_1| |E_{1iM}| e^{-\alpha \ell} \cos \theta \tag{20}$$

$$E_{TEM-I} = |E_{2iM}| \sin R + |\Gamma_1| |E_{1iM}| e^{-\alpha \ell} \sin \theta \tag{21}$$

$$|E_{TEM-M}| = \sqrt{E_{TEM-R}^2 + j E_{TEM-I}^2} \tag{22}$$

$$\tau = \tan^{-1} \left(\frac{E_{-I}}{E_{-R}} \right) \tag{23}$$

χ and τ are the resultant phase angles of the positive and negative travelling resultant waves ($E_{TEM+}(y)$ and $E_{TEM-}(y)$) with respect to the original incident wave at boundary 1 ($E_{1i}(y)$). Inspecting the positive travelling wave of equation (14) it can be written as,

$$\begin{aligned}
 E_{TEM+}(y) &= e^{-jk(y+\ell)} |E_{TEM+M}| e^{j\chi} \\
 &= e^{-jky} e^{-\alpha \ell} e^{-j\beta \ell} |E_{TEM+M}| e^{j\chi} \\
 &= e^{-jky} e^{-\alpha \ell} |E_{TEM+M}| e^{j(\chi - \beta \ell)}
 \end{aligned} \tag{24}$$

By redefining the magnitude and phase as,

$$|E_{TEM+M}| = e^{-\alpha \ell} |E_{TEM+M}| \tag{25}$$

$$\xi = \chi - \beta \ell \tag{26}$$

the positive travelling becomes,

$$E_{TEM+}(y) = e^{-jky} |E_{TEM+M}| e^{j\xi} \tag{27}$$

Defining the phasors,

$$E_{TEM+} = |E_{TEM+M}| e^{j\xi} \tag{28}$$

$$E_{TEM-} = |E_{TEM-M}| e^{j\tau} \tag{29}$$

which are the complex magnitudes of the positive and negative travelling waves respectively. The solution to the electric field within the conductor can now be written as,

$$\begin{aligned} E_{TEM}(y) &= E_{TEM+}(y) + E_{TEM-}(y) \\ &= E_{TEM+} e^{-jky} + E_{TEM-} e^{jky} \end{aligned} \quad (30)$$

The magnetic field representation of these waves can be obtained by dividing equation (30) by the impedance, η_2 , and adjusting the sign as to satisfy the Poynting vector,

$$\begin{aligned} H_{TEM}(y) &= -e^{-jky} \frac{E_{TEM+}}{\eta_2} + e^{jky} \frac{E_{TEM-}}{\eta_2} e^{j\tau} \\ &= -e^{-jky} \frac{|E_{TEM+}|}{\eta_2} e^{j\xi} + e^{jky} \frac{|E_{TEM-}|}{\eta_2} e^{j\tau} \end{aligned} \quad (31)$$

If η_2 is complex, which it will be for a dissipative medium, the magnetic field intensities will be phase shifted with respect to their electric fields. The impedance can be written as,

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \eta_{2R} + j\eta_{2I} = |\eta_2| e^{j\delta} \quad (32)$$

The magnetic field intensity can therefore be written as,

$$H_{TEM}(y) = H_{TEM+} e^{-jky} + H_{TEM-} e^{jky} \quad (33)$$

where

$$H_{TEM+} = \frac{|E_{TEM+}|}{|\eta_2|} e^{j(\xi - \delta + 180^\circ)} \quad (34)$$

$$H_{TEM-} = \frac{|E_{TEM-}|}{|\eta_2|} e^{j(\tau - \delta)} \quad (35)$$

Comparison of equation (33) to equation (22) of chapter 2 (given below for ease of reference) reveals them to have near identical forms. Since the unknown coefficients of both equations will be dependent on the scenario under investigation, the only difference is the exponents of the exponential terms (given in equations (1) and (37)).

$$H_z(y) = K_1 e^{Dy} + K_2 e^{-Dy} \quad (36)$$

$$D = (1 + j) \sqrt{\frac{\omega \mu \sigma}{2}} \quad (37)$$

It must, however, be remembered that D of equation (37) was limited to conductive structures while jk of equation (1) was derived under complete generic conditions. Therefore, if it can be shown that D and jk are equivalent (under the same material conditions) and the ratio of the electric to the magnetic fields are the same, then the TEM plane wave approach and the original

one-dimensional derivation are equivalent on a physics level. The difference in interpretation will be considered once the mathematical proof has been obtained.

3.3. Complete generic frequency response of the original one dimensional derivation

As was seen in chapter 2 the general one dimensional derivation (chapter 2, section 2.2.1) was applied to conductors under the assumption that the displacement current term is negligible (low frequency approximation). This, however, prevents for a comparison to the previously derived TEM plane wave result. Instead of simplifying the TEM plane wave result to the frequency limited conductive scenario, the original one dimensional derivation will be redone without any limitations on frequency or material properties in order to allow for a comparison. This is to ensure that the two approaches are equivalent on an unconstrained level, i.e. equivalent for all possible scenarios. Starting with Faraday and Ampere's law under sinusoidal excitation,

$$\nabla \times \bar{E} = -j\omega\bar{H} \quad (38)$$

$$\nabla \times \bar{H} = (\sigma + j\omega\epsilon)\bar{E} \quad (39)$$

with

$$\mu = \mu' - j\mu'' \quad (40)$$

$$\epsilon = \epsilon' - j\epsilon''_{NC} \quad (41)$$

where the imaginary terms denote the losses and phase shift associated with the magnetic and dielectric properties of the generic material. It is important to note that the imaginary component of the permittivity in equation (41) consists only of the dissipative component of the dielectric part of the material. This is because the conductivity in equation (39) is dealt with separately from ϵ , whereas the conductivity was grouped under the dissipative, ϵ'' , term (which formed part of ϵ) in equation (4).

Irrespective of where the conductivity is taken into account, the end result will still be the same. This can easily be seen by taking Ampere's law for the two approaches. From [4] (p.410) and the definition given in equation (4) the point form of Ampere's law is given in equation (42) for the TEM plane wave approach.

$$\begin{aligned} \nabla \times \bar{H} &= j\omega\epsilon\bar{E} \\ &= j\omega(\epsilon' - j\epsilon'')\bar{E} \\ &= j\omega\left(\epsilon' - j\left(\frac{\sigma}{\omega} + \epsilon''_{NC}\right)\right)\bar{E} \\ &= (j\omega\epsilon' + \sigma + \omega\epsilon''_{NC})\bar{E} \end{aligned} \quad (42)$$

This can be compared to the more general definition of Ampere's law (which the original one dimensional derivation is based on) along with the definition given in equation (41),

$$\begin{aligned}
\nabla \times \vec{H} &= (\sigma + j\omega\epsilon) \vec{E} \\
&= (\sigma + j\omega(\epsilon' - j\epsilon''_{NC})) \vec{E} \\
&= (\sigma + j\omega\epsilon' + \omega\epsilon''_{NC}) \vec{E}
\end{aligned} \tag{43}$$

The obvious question that arises is why the difference in the definitions of ϵ . Looking first at the definitions of this section (equations (38) to (41)), it can be seen that they follow the more 'logical' approach, where the conductivity forms part of the current and is separate from the permittivity, which only deals with the dielectric effects. In order to understand the reason for the definitions of the TEM plane wave method ((1) to (4)), the original derivation of the TEM plane wave will have to be considered (p. 397 in [4]). The original derivation was done under the assumption that the wave is propagating through empty space, which led to the wave equations being derived under zero conductivity conditions. When adjusting these equations to deal with materials, it is easy enough to take the non-conductive effects of the permeability and permittivity into account by adjusting the already existing permeability and permittivity constants, while making them complex to deal with non-conductive losses. Since the conductivity component has, however, been removed from the original wave equations, it is easier to add it to dielectric component than to rederive the equations, hence the form of equation (4). This can clearly be seen in the first line of equation (42), where the conductivity component has been removed due to it being zero in the original conditions of the wave equations' derivation.

With this equivalence now confirmed and discussed, attention can be returned to the one-dimensional derivation of the generic material of Figure 3.1. As was the case for chapter 2, the electric field and current density will be solely in the x-direction, thereby confining the magnetic field to the z-direction. Simplifying equations (38) and (39) under these constraints results in,

$$\frac{\partial E_x}{\partial y} = j\omega\mu H_z \tag{44}$$

$$\frac{\partial H_z}{\partial y} = (\sigma + j\omega\epsilon) E_x \tag{45}$$

where the displacement current term is obviously kept a non-zero term. Rewriting equation (45) in terms of E and substituting into (44),

$$\frac{\partial^2 H_z}{\partial y^2} = j(\sigma + j\omega\epsilon)\omega\mu H_z \tag{46}$$

Let $Q^2 = j(\sigma + j\omega\epsilon)\omega\mu$,

$$\frac{\partial^2 H_z}{\partial y^2} - Q^2 H_z = 0 \tag{47}$$

This is again a second-order linear differential equation. When solved, the complete frequency response of the magnetic field intensity is given by,

$$H_{orig}(y) = H_z(y) = H_{orig1}e^{-Qy} + H_{orig2}e^{Qy} \quad (48)$$

where Q equals,

$$Q = \sqrt{j(\sigma + j\omega\epsilon)\mu\omega} \quad (49)$$

The subscript 'orig' is used to denote the results obtained from the original derivation (expanded to accommodate a completely generic material) in order to avoid confusion with the results obtained in the previous section. Expanding equation (49) with the definitions of (40) and (41) results in,

$$\begin{aligned} Q &= \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \\ &= \sqrt{j\omega(\mu' - j\mu'')(\sigma + \omega\epsilon_{NC}' + j\omega\epsilon')} \end{aligned} \quad (50)$$

The electric field is obtained next by applying Ampere's law to equation (48),

$$\frac{\partial H_{orig}(y)}{\partial y} = QH_{orig2}e^{Qy} - QH_{orig1}e^{-Qy} \quad (51)$$

The electric field is therefore given by,

$$\begin{aligned} E_x(y) &= \frac{QH_{orig2}e^{Qy} - QH_{orig1}e^{-Qy}}{(\sigma + j\omega\epsilon)} \\ &= E_{orig1}e^{-Qy} + E_{orig2}e^{Qy} \end{aligned} \quad (52)$$

where

$$E_{orig1} = \frac{-QH_{orig1}}{(\sigma + j\omega\epsilon)} \quad (53)$$

$$E_{orig2} = \frac{QH_{orig2}}{(\sigma + j\omega\epsilon)} \quad (54)$$

3.4. Equivalence of the TEM plane wave and original one-dimensional approach

Now that complete responses of both the TEM plane wave and original one-dimensional methods have been derived, a comparison between the two methods can be made. Before comparing their resultant electric and magnetic field intensities, it first needs to be shown that jk and Q are equivalent. Starting with the completely expanded form of jk from equation (1),

$$\begin{aligned}
 jk &= j\omega\sqrt{\mu\epsilon} \\
 &= j\omega\sqrt{(\mu' - j\mu'')(\epsilon' - j\epsilon'')} \\
 &= j\omega\sqrt{(\mu' - j\mu'')(\epsilon' - j(\sigma/\omega + \epsilon''_{NC}))} \\
 &= \sqrt{j^2\omega^2(\mu' - j\mu'')(\epsilon' - j(\sigma/\omega + \epsilon''_{NC}))} \\
 &= \sqrt{j\omega(\mu' - j\mu'')(\sigma + \omega\epsilon''_{NC} + j\omega\epsilon')}
 \end{aligned} \tag{55}$$

Comparing this to the expanded form of Q given in equation (50) and rewritten below for clarity,

$$\begin{aligned}
 Q &= \sqrt{j\omega(\mu' - j\mu'')(\sigma + j\omega(\epsilon' - j\epsilon''_{NC}))} \\
 &= \sqrt{j\omega(\mu' - j\mu'')(\sigma + \omega\epsilon''_{NC} + j\omega\epsilon')}
 \end{aligned} \tag{56}$$

reveals that they are identical. All that remains is to compare the magnetic to electric field ratios of these two methods. The easiest way to do this is to write the magnetic field intensities of both these methods in terms of their electric fields. Starting with the TEM plane wave,

$$\begin{aligned}
 H_{TEM}(y) &= H_{TEM+}e^{-jky} + H_{TEM-}e^{jky} \\
 &= -\frac{E_{TEM+}e^{-jky}}{\eta_2} + \frac{E_{TEM-}e^{jky}}{\eta_2}
 \end{aligned} \tag{57}$$

It must be remembered that the definitions of ϵ for the two methods differ.

$$\eta_2 = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu' - j\mu''}{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}} \tag{58}$$

Therefore,

$$H_{TEM}(y) = -\sqrt{\frac{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}{\mu' - j\mu''}}E_{TEM+}e^{-jky} + \sqrt{\frac{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}{\mu' - j\mu''}}E_{TEM-}e^{jky} \tag{59}$$

Consider now the magnetic field of the original one-dimensional method,

$$\begin{aligned}
 H_{orig}(y) &= H_{orig1}e^{-Qy} + H_{orig2}e^{Qy} \\
 &= \frac{H_{orig1}Q(\sigma + j\omega\epsilon)}{Q(\sigma + j\omega\epsilon)}e^{-Qy} + \frac{H_{orig2}Q(\sigma + j\omega\epsilon)}{Q(\sigma + j\omega\epsilon)}e^{Qy} \\
 &= \left[\frac{H_{orig1}Q}{(\sigma + j\omega\epsilon)} \right] \frac{(\sigma + j\omega\epsilon)}{Q} e^{-Qy} + \left[\frac{H_{orig2}Q}{(\sigma + j\omega\epsilon)} \right] \frac{(\sigma + j\omega\epsilon)}{Q} e^{Qy} \\
 &= -E_{orig1} \frac{(\sigma + j\omega\epsilon)}{Q} e^{-Qy} + E_{orig2} \frac{(\sigma + j\omega\epsilon)}{Q} e^{Qy}
 \end{aligned} \tag{60}$$

Using equations (40) and (41) allows the $(\sigma + j\omega\epsilon)/Q$ term to be expanded as,

$$\begin{aligned}
 \frac{(\sigma + j\omega\epsilon)}{Q} &= \frac{\sigma + j\omega\epsilon}{\sqrt{j(\sigma + j\omega\epsilon)\omega\mu}} \\
 &= \sqrt{\frac{(\sigma + j\omega\epsilon)}{j\omega\mu}} \\
 &= \sqrt{\frac{\epsilon - j\sigma/\omega}{\mu}} \\
 &= \sqrt{\frac{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}{\mu' - j\mu''}}
 \end{aligned} \tag{61}$$

which allows equation (60) to be written as,

$$H_{orig}(y) = -\sqrt{\frac{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}{\mu' - j\mu''}} E_{orig_1} e^{-Qy} + \sqrt{\frac{\epsilon' - j(\sigma/\omega + \epsilon''_{NC})}{\mu' - j\mu''}} E_{orig_2} e^{Qy} \tag{62}$$

As can be seen, the ratio of electric field to magnetic field is identical for both methods (which is to be expected when considered from a logical perspective). As the same boundary values will be used to determine the unknown coefficients for both these methods, it follows that their coefficients will therefore also be equal. This in turn proves, on a physics level, that these two methods are identical.

3.5. Energy flow

Since it is now known that these two methods are mathematically equivalent, it follows that this equivalence should also apply when considered from an energy flow perspective. Poynting's theorem (and the corresponding equation) will therefore be employed, which gives the flow of energy associated with electric and magnetic fields,

$$\bar{S} = \bar{E} \times \bar{H} \quad (W/m^2) \tag{63}$$

Considering first the TEM plane wave method, to which the Poynting theorem is more regularly applied, reveals that the energy is flowing into the material in the direction of wave propagation. Even though this is not as regularly done, Poynting's theorem can be applied to any scenario where electric and magnetic fields exist at the same point in space. It can therefore be applied to the results of the original one-dimensional derivation which shows that, identical to the TEM plane wave, energy is flowing into the material. At this stage this should come as no surprise since, as the electric and magnetic fields are equivalent, their cross-product must also produce the same result.

3.6. Subsequent interpretation of the two methods

Since equivalence has now been proven from all possible perspectives, the only advantage one may pose over the other is in the way they are interpreted. In the TEM plane wave scenario the source is considered to be at the boundaries of the material, with the fields and current density distribution being a result of the interaction between the wave and the material. I.e. the current density is a result of the electric field of the TEM wave interacting with the free and bound charges within the material. As the electric field attenuates with propagation into the material, for example due to ohmic losses, the magnetic field attenuates as well, with the ratio of electric to magnetic field governed by the impedance of the material. The original one-dimensional derivation, however, considers this from the more 'practical' perspective, where a sinusoidal voltage source is present over the ends of the material which creates an electric field within it. The electric field interacts with the free and bound charges within the material which results in a current, which in turn produces a magnetic field parallel to the surface of the conductor.

Irrespective of which interpretation is preferred, the end result is still governed by the same mathematical equations, with Poynting's theorem stating that energy is flowing into the conductor from some external source.

3.7. Application to a conductive material

Now that equivalence is no longer a question, attention will only be focussed on one equation with the origin open to the choice of the two methods discussed. The above obtained results were for a completely generic structure with no limitation on the applied frequency. These results should, however, reduce to that obtained in chapter 2 for the case where the material is a good conductor and the frequency is low enough such that the displacement current term can be neglected. This is a general simplification which is made by a number of authors [3][5]. The limitation of this simplification is, however, unknown and as such it is felt important to obtain a rough estimate for what materials and frequencies this simplification holds. Starting with Q from equation (50),

$$Q = \sqrt{j\omega(\mu' - j\mu'')(\sigma + \omega\epsilon''_{NC} + j\omega\epsilon')} \quad (64)$$

The following characteristics are selected to represent an average conductive material:

$$\begin{aligned} \mu' &= \mu = \text{constant value} \\ \mu'' &= \epsilon''_{NC} = 0 \\ \epsilon' &= \epsilon_0 \\ \sigma &= \text{constant value} \end{aligned}$$

Q will therefore reduce to,

$$\begin{aligned} Q &= \sqrt{j\omega\mu(\sigma + j\omega\epsilon_0)} \\ &= (1+j)\sqrt{\frac{\omega\mu(\sigma + j\omega\epsilon_0)}{2}} \end{aligned} \quad (65)$$

As can be seen (65) will reduce to the (37) (which is used for conductive structures in chapter 2) under the condition that $\sigma \gg \omega\epsilon_0$. The ratio of σ to $\omega\epsilon_0$ at which the $\omega\epsilon_0$ term can be neglected, is very much dependent on the desired accuracy. In order to obtain a worst-case estimate a very poor conductor will be selected (Graphite with $\sigma = 7 \times 10^4$ S/m) and σ must be two orders of magnitude larger than $\omega\epsilon_0$ for the simplification to hold.

$$\begin{aligned} 7 \times 10^4 &\gg \omega \cdot 8.854 \times 10^{-12} \\ \therefore \omega &\leq 79 \times 10^{12} \\ \therefore f &\leq 12.57 \times 10^{12} \end{aligned}$$

From this it can clearly be seen that for even a very poor conductor the displacement current term is still negligible for the frequency range under consideration. Returning now to equation (65) it can be simplified to,

$$Q = (1 + j) \sqrt{\frac{\omega \mu \sigma}{2}} \quad (66)$$

which is what was used in chapter two.

3.8. Conclusion

As is now clear, regardless of the perspective from which the problem is approached, the mathematical solution will remain the same. In hindsight this is to be expected. Since the fields and current distributions are governed by Maxwell's equations, any method or perspective which is mathematically sound and governed by these equations will lead to the same solution. The only advantage one method may therefore pose over the other, is in the way the problem is considered and the possible available tools which have already been developed. These tools may include additionally derived equations (based on Maxwell's equations) along with software tailored to a specific perspective.

Extending this conclusion to include different geometries reveals this TEM plane wave perspective to be independent of geometry, and can therefore just as easily be applied to cylindrical conductors. Without deriving the cylindrical conductor solution (from a TEM plane wave perspective) the geometry independence can easily be seen by considering a hollow cylindrical conductor for which the radius is large compared to the thickness of the cylinder wall. If only a small section is considered, such that the difference in the top and bottom circumference is negligible, it can be approximated by a semi-infinite flat conductor, for which the TEM plane wave already has a solution.

From the above information it can therefore be concluded that, with regards to the TEM plane wave method's use towards a multi-layer solution, it poses no advantage over that presented in chapter 2. It follows that the same mathematical difficulties will be encountered, irrespective of the perspective from which the problem is approached. Therefore, without neglecting the insight gained from this TEM perspective, the perspective and approach presented in chapter 2 will still be used as it provides a more 'fundamental' approach to this problem.

3.9. References

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Chapter 4

Multi-layer Methodology

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4.1 Introduction

The objective of a multi-layer conductor solution was first outlined in chapter 1, with chapters 2 and 3 providing a detailed investigation into existing equations and methods to be used towards this objective. This multi-layer topology can be seen in Figure 4.1, where it takes on the forms of multi-layer flat and multi-axial cylindrical conductors based on the flat and cylindrical geometries of chapter 2. Similar to the single-layer variants of chapter 2, these multi-layer conductors are either considered to be semi-infinite or can be approximated as such.

Similar multi-layer problems have been encountered in the past in the fields of eddy current testing ([1][2] and the papers that follow from them [3]-[9]), and multi-layered shielding ([10][11] and the papers that follow from them [12]-[17]). None of their solutions are, however, directly applicable to this problem and as such, only a single, incomplete method ([18]) could be found to date. Chapter 3 showed that this method of [18] is in fact identical to the single-layer solution presented in chapter 2, which led to the conclusion that all single-layer solutions are identical from a fundamental perspective. This fundamental equivalence helps to suggest that a multi-layer solution can be approached from one of two fundamental perspectives. Either start by using the single-layer equation and make the conductivity a function of distance, or apply a single-layer equation to each of the layers in the conductor. The first method, briefly attempted in Appendix F, resulted in a differential equation in which the coefficients are also functions of distance. This unfortunately increased the solution difficulty to the point where it was abandoned in favour of the second approach.

The second approach, based on the application of single-layer equations, forms therefore the basis of the solution presented in this chapter.

4.2 Multi-layer methodology

In the derivation towards a multi-layer solution the multi-layer semi-infinite flat and cylindrical conductors of Figure 4.1 will be kept in their generic form. Only once a general solution has been obtained will it be applied to the different configurations. Before continuing, some definitions and information need to be given. Firstly, during the remainder of this thesis these structures will be referred to as either multi-layer conductors or structures, or just conductors or structures, depending on the context. Secondly, configuration refers to the scenario in which the multi-layer structure is used. For example, the scenario where two multi-layer flat conductors are placed parallel to one another, with each carrying a current of equal magnitude and opposite direction, may be referred to as the differential-mode configuration. Finally, a reasonable and practical assumption that will be made is that either the total current flowing through the complete multi-layer conductor (skin-effect case) and/or the externally applied magnetic field will be known (proximity-effect case).

As is known from chapter 2, the solution of the single-layer equations is possible given that the boundary values of that layer are known. In the case, however, where a single-layer solution is applied to each of the layers of a multi-layer conductor a significant problem will arise.

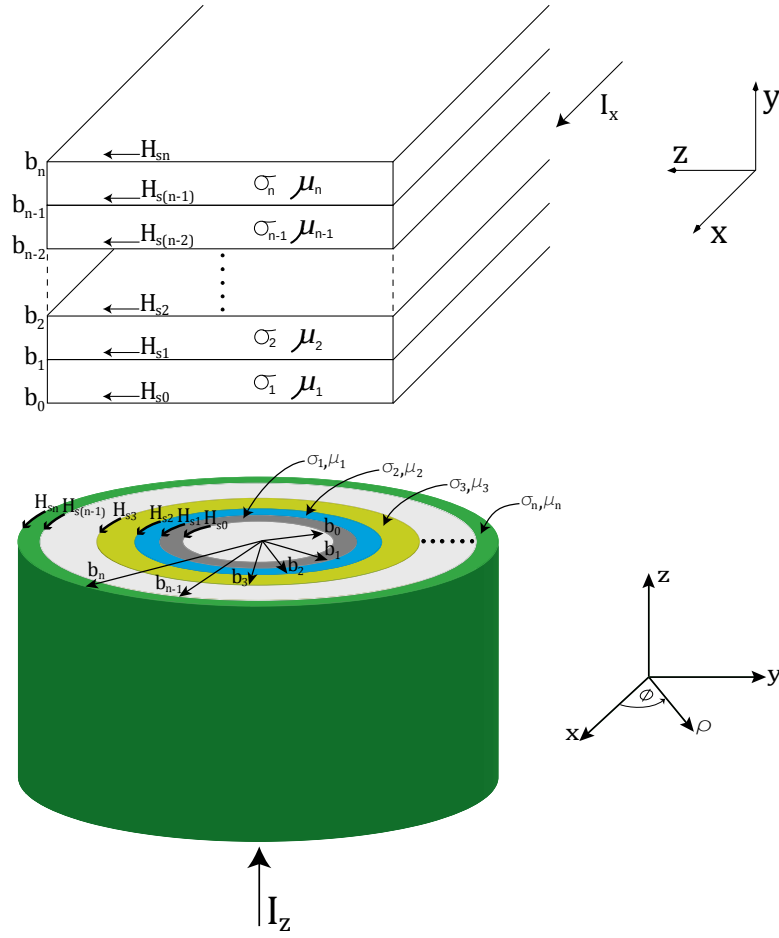


Figure 4.1. Multi-layer flat and cylindrical conductors

Consider the flat n -layer conductor of Figure 4.1 subjected to the two separate cases of the skin- and proximity-effects. For the skin-effect case the total current flowing through the multi-layer conductor is given as, I_T , while for the proximity-effect case the externally applied magnetic field is known. Application of the integral form of Ampere's law shows that for both these cases only the external boundaries, H_{s0} and H_{sn} , can be determined. For a solution using the single-layer equations the remaining boundary values internal to the conductor (H_{s1} and $H_{s(n-1)}$) are, therefore, still unknown and required. This forms the crux of this problem and its solution one of the main contribution of this thesis.

Equations (1) and (2) (below) are the single-layer flat and cylindrical generic solutions from chapter 2 and will be applied to each of the layers of Figure 4.1. i refers to the layer number and ranges from 0 to n for an n -layer structure. (The generic solutions written in terms of the electric field (derived in Appendix B) could just as easily have been made use of). There will therefore be n equations, resulting in $2*n$ unknown coefficients (F_1 to F_n and G_1 to G_n for the flat and A_1 to A_n and B_1 to B_n for the cylindrical conductors) which need to be solved for. Since each of these functions is only defined and correct within its respective domain, simultaneous equations are not possible while still being dependent on y or ρ , as substitution of one function into another will lead to incorrect results. Therefore, in order to obtain a solution $2*n$ simultaneous equations (not functions) are required, with each equation written in terms of known and unknown constants while being independent of distance (i.e. not functions).

$$H_{z_i}(y) = F_i \cosh(D_i y) + G_i \sinh(D_i y) \quad (1)$$

$$b_{i-1} \leq y \leq b_i$$

$$H_{\phi}(\rho) = A_i I_1(D_i \rho) + B_i K_1(D_i \rho) \quad (2)$$

$$b_{i-1} \leq \rho \leq b_i$$

where

$$D_i = (1+j) \sqrt{\frac{\omega \mu_i \sigma_i}{2}} = \frac{(1+j)}{\delta_i} \quad (3)$$

As these equations ((1) and (2)) are confined to their respective domains, a way to relate them to one another is required. The solution lies in the application of Maxwell's equations to the boundaries between the layers. Application of Faraday's law to a small boundary section reveals that the tangential component of the electric field must be continuous across this boundary ([19]-[21]). The same can be found for the tangential component of the magnetic field intensity by applying Ampere's law in a similar manner ([19]-[21]). It must be remembered that, under the constraints of arrangement 2, the electric field and magnetic field intensity are always parallel to the surfaces of the conductor, with the result that only tangential components exist at the boundaries (i.e. no normal components). Therefore, for the remainder of this thesis, all references to either the electric field or magnetic field intensity is with regards to the fields being parallel to the surfaces of the conductor. These tangential continuity criteria are frequently used for TEM plane waves incident on a boundary ([19]). Similar criteria were used in the solutions of the eddy current and shielding multi-layer problems of [1][2] and [11], respectively. However, in these cases they were also concerned with the behaviour of the normal component of the magnetic field. Even though these problems ([1][2] and [11]) are to some extent similar, their boundary conditions; fields; direction of energy propagation and constraints, such as being confined to carry no resultant current, are completely different from the problem at hand. This prevents the use of their solutions and forces the development of a new multi-layer solution.

Utilising the continuity of the electric field and magnetic field intensity, the steps required to obtain the desired $2 \times n$ simultaneous equations are given below (with notation as in Figure 4.1):

1. Determine the magnetic field intensity at the two exterior boundaries (b_0 and b_n) and use these values to obtain two equations written in terms of constants:

These boundary values are heavily dependent on the configuration in which the multi-layer conductor is to be considered, with their determination discussed in sections 4.3.1.1 and 4.3.2.1. Irrespective of the configuration, the magnetic fields at the two exterior boundaries are determinable and allows for two equations written in terms of constants.

2. Enforce the continuity of the electric field across the boundaries:

As was stated, Faraday's law dictates that the tangential electric field must be continuous across the boundaries. Therefore, for each two layers neighbouring one another, the condition that their electric fields must be equal at their common boundary needs to be enforced. Applying this condition to each common boundary produces $n-1$ equations, with all of them written in terms of constants.

3. Enforce the continuity of the magnetic field intensity across the boundaries:

Also discussed above was the continuity of the tangential magnetic field intensity across the boundaries (Ampere's law). Enforcement of this condition at each boundary common to two layers will produce $n-1$ equations written in terms of constants.

4. Solving of the simultaneous equations:

As was seen, steps one to three produces $2*n$ equations ($2 + n-1 + n-1$), which allows for the creation of simultaneous equations and the solving of the unknown coefficients.

In steps 1 to 4 above, both the electric field and magnetic field intensity are made use of. Conversion between the two is therefore essential and will be dependent on which field the initial generic solution is based (electric field-derived in Appendix B or magnetic field intensity-derived in chapter 2, section 2.2.1). Conversion from electric to magnetic field is therefore accomplished by Faraday's law and magnetic to electric field by Ampere's law. It is important to realise that, irrespective from which field the general solution was derived, the method outline in 1 to 4 above remains the same. This should come as no surprise as the electric and magnetic fields are related through Maxwell's equations and are just different perspectives of the same problem.

As is now known, the central point of this methodology is the continuity of the tangential electric field and the tangential magnetic field intensity across the boundaries. Since the continuity is independent of geometry, it follows that this methodology (steps 1 to 4) can be applied to any multi-layer geometry subjected to arrangement 2's constraints, given that its single-layer solution is known.

4.3 Flat and cylindrical multi-layer conductors

From the above discussion, it is clear that this methodology imposes no limit on the number of layers contained within a multi-layer structure. Even though this allows for an unlimited number of layers to be applied to all possible configurations, each one solvable by use of this method, attention will only be focussed on two- and three-layer conductors. The reason for this stems from the verification of this methodology.

As direct verification of a methodology is not possible, the resulting solutions will rather be used and verified against the distributions predicted by FEM (practical verification of the distribution is near-impossible). Since the fundamental building block of this method is the continuity of the E and H fields across the boundaries, a conductor with at least a single internal boundary is needed. In order to obtain complete confidence in this method, both two- and three-layer conductors will therefore be considered. As a final note, limiting the conductors to three layers is also necessary as the rate of convergence (for FEM) is partially dependent on the amount of layers, with the solution time becoming impractical for larger numbers.

Considering steps 1 to 4 above, it is clear that only the values obtained in step 1 are configuration dependent. Generic solutions can therefore be obtained by denoting the magnetic fields at the external boundaries as H_{s0} and H_{sn} (as was done in Figure 4.1). A configuration-specific solution then follows by substituting the correct values for these placeholders (i.e. H_{s0} and H_{sn}), thereby saving from going through a complete derivation for each configuration. This will be done below for both the two- and three-layer variants of the flat and cylindrical multi-layer conductors.

4.3.1 Flat multi-layer conductors

As is clear, different configurations will result in different boundary values which adjust the distributions internal to these multi-layer conductors. The general (or generic) configuration, from which all other flat conductor configurations stem, will therefore first be investigated in order to determine how to obtain the required boundary values. This will then be reduced to a finite configuration (in order to assist with the validation of section 4.4) and will be followed by the derivation of the flat conductor's generic two- and three-layer equations.

4.3.1.1 General configuration: Multi-conductor, multi-layer flat configuration

Shown in Figure 4.2 is the m-conductor, multi-layer flat configuration, from which all other multi-layer flat configurations originate. Each of the individual conductors carries a known resultant current and potentially consists of multiple individual layers (such as that given in Figure 4.1). Conductor k, being one of the m-conductors, is the conductor under consideration. (The whole structure obviously complies with the semi-infinite approximation).

It is important to note that if the return component of a current is not included in this configuration, i.e. one of the other conductors is not carrying a conductor's return current, then the conductor that is carrying this current is considered to be infinitely far away. The configuration of Figure 4.2 obviously includes the possibility of either the skin- or proximity-effects, or a combination of both. For the skin-effect only the conductor under consideration will carry a resultant current. While for the proximity-effect, the conductor under consideration will carry no resultant current, with the currents in the neighbouring conductors responsible for the external magnetic field.

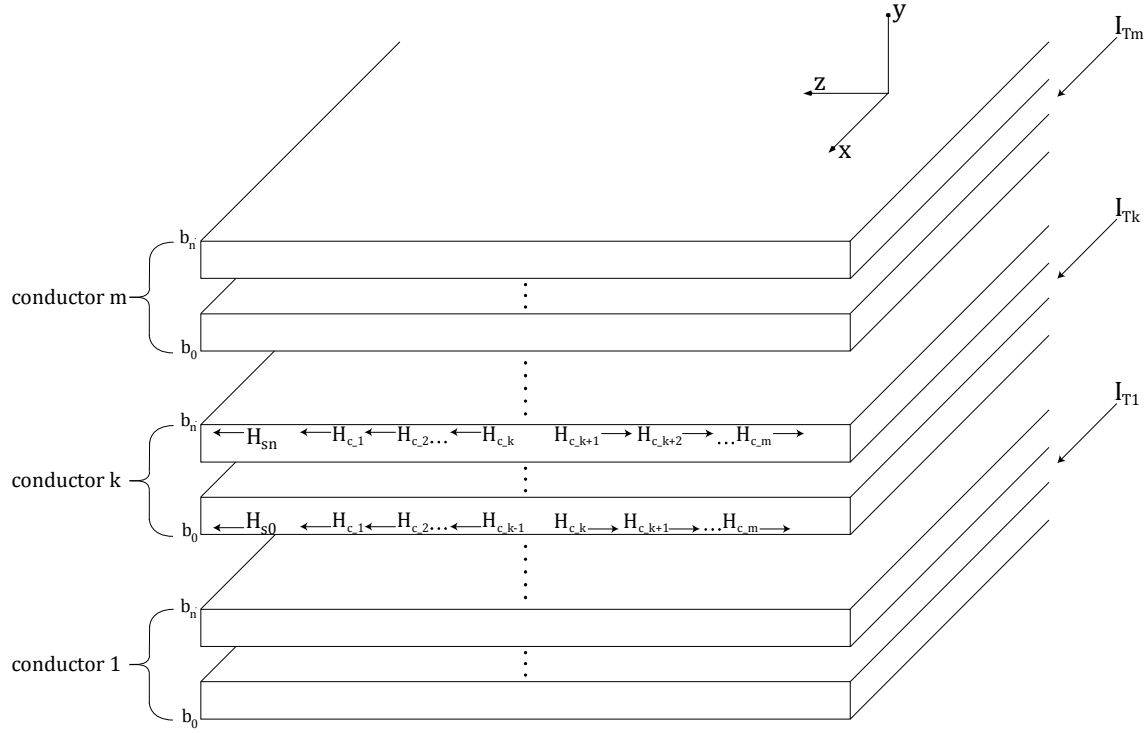


Figure 4.2. Multi-layer conductor forming part of an m-conductor flat-configuration

The outer boundary values, H_{s0} and H_{sn} , are configuration specific and will depend on the magnetic field created at these boundaries by the currents in the layers. The general solutions of these boundaries are therefore,

$$H_{s0}(\hat{k}) = \left[\sum_{i=1}^{k-1} H_{c,i} - \sum_{i=k}^m H_{c,i} \right] \hat{k} \quad (4)$$

$$H_{s0} = \sum_{i=1}^{k-1} H_{c,i} - \sum_{i=k}^m H_{c,i}$$

$$H_{sn}(\hat{k}) = \left[\sum_{i=1}^k H_{c,i} - \sum_{i=k+1}^m H_{c,i} \right] \hat{k} \quad (5)$$

$$H_{sn} = \sum_{i=1}^k H_{c,i} - \sum_{i=k+1}^m H_{c,i}$$

where c is used to denote that the field, $H_{c,i}$, is created by a resultant current (in order to avoid confusion with the notation use in chapter 2), while i refers to the number of the multi-layer conductor in which this current flows. This field can be determined from Ampere's law (as is done in chapter 2, section 2.2.2) and is given by,

$$H_{c,i} = \frac{I_{Ti}}{2w} \quad (6)$$

In the summation of the resultant currents (equations (4) to (6)) it is vital to remember that the currents, and therefore their subsequent magnetic fields, are phasors and that their phase angles need to be taken into account.

It is clear from the above discussion that this general conductor is representative of all possible configurations. In order to assist in the validation of the multi-layer methodology, it will be reduced to the configuration discussed below.

Two-conductor, multi-layer flat configuration

The general configuration of Figure 4.2 reduces to Figure 4.3 when m and k are both made equal to 2, with the top conductor being the conductor under consideration. Equations (4) and (5) therefore reduce to,

$$H_{s0} = H_{c_{-1}} - H_{c_{-2}} \quad (7)$$

$$H_{sn} = H_{c_{-1}} + H_{c_{-2}} \quad (8)$$

Substituting equation (6) into (7) and (8) allows the boundary magnetic fields to be expressed in terms of the currents that create them,

$$H_{s0} = \frac{I_{T1}}{2w} - \frac{I_{T2}}{2w} \quad (9)$$

$$H_{sn} = \frac{I_{T1}}{2w} + \frac{I_{T2}}{2w} \quad (10)$$

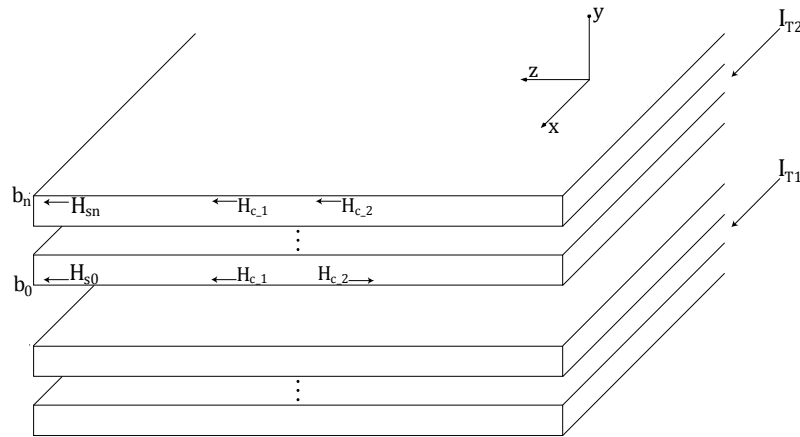


Figure 4.3. Two-conductor multi-layer flat configuration

It is now easy to see that the above result will reduce to the separate skin- and proximity-effect scenarios when I_{T1} and I_{T2} are made zero, respectively. Take first the case of the skin-effect. I_{T1} is therefore zero and equations (9) and (10) reduce to,

$$H_{s0} = -\frac{I_{T2}}{2w} \quad (11)$$

$$H_{sn} = \frac{I_{T2}}{2w} \quad (12)$$

For the proximity-effect case I_{T2} is zero and equations (9) and (10) become,

$$H_{s0} = \frac{I_{T1}}{2w} \quad (13)$$

$$H_{sn} = \frac{I_{T1}}{2w} \quad (14)$$

This configuration will be applied in section 4.4.3 for the validation of the methodology.

4.3.1.2 Multi-layer methodology: General solution to the two-layer semi-infinite flat conductor

Even though the generic solution expressed in terms of the magnetic field intensity is the more frequently used form, the electric field variant will be used here in order to demonstrate that both these approaches are equally valid. The general solution of layer, i , which forms part of a generic multi-layer structure is therefore,

$$E_{xi}(y) = L_i \cosh(D_i(y - b_{i-1})) + M_i \sinh(D_i(y - b_{i-1})) \quad (15)$$

$$b_{i-1} \leq y \leq b_i$$

where L_i and M_i are the unknown coefficients that need to be solved. The magnetic field intensity is obtained by applying Faraday's law to equation (15),

$$H_{zi}(y) = \frac{L_i D_i}{j\omega\mu_i} \sinh(D_i(y - b_{i-1})) + \frac{M_i D_i}{j\omega\mu_i} \cosh(D_i(y - b_{i-1})) \quad (16)$$

$$b_{i-1} \leq y \leq b_i$$

The current density is obtained by either taking the curl of the magnetic field intensity, or by multiplying the electric field by σ_i ,

$$J_{xi}(y) = \sigma_i L_i \cosh(D_i(y - b_{i-1})) + \sigma_i M_i \sinh(D_i(y - b_{i-1})) \quad (17)$$

$$b_{i-1} \leq y \leq b_i$$

where

$$D_i = (1+j) \sqrt{\frac{\omega\mu_i\sigma_i}{2}} = \frac{(1+j)}{\delta_i} \quad (18)$$

The arguments of the functions have been shifted such that the $(y-b_{i-1})$ term becomes zero when the bottom of each layer is considered. For example, looking at the bottom of the second layer

($i=2$) the argument ($y-b_1$) reduces to zero for $y=b_1$. This convention was chosen to simplify the subsequent algebra, but is however not necessary. If the argument was chosen to be (Dy) for all i a solution will still be obtainable. This will be seen when cylindrical conductors are considered in the next section, where it was decided to use the solution derived from the Laplacian of the magnetic field intensity with its argument kept constant for all i . As can be understood, the exact solution to the coefficients will depend on whether a shift in origin was decided upon or not. This dependence is required to ensure a correct solution to the problem.

Attention will now be focussed on obtaining a general solution to the two-layer semi-infinite flat conductor, depicted in Figure 4.4. The bottom boundary, b_0 , will be taken to be at $y = 0$. From equations (15) and (16) the general solutions of the fields and current density for each layer are,

Layer 1:

$$E_{x1}(y) = L_1 \cosh(D_1 y) + M_1 \sinh(D_1 y) \quad (19)$$

$$H_{z1}(y) = \frac{L_1 D_1}{j\omega\mu_1} \sinh(D_1 y) + \frac{M_1 D_1}{j\omega\mu_1} \cosh(D_1 y) \quad (20)$$

$$J_{x1}(y) = \sigma_1 L_1 \cosh(D_1 y) + \sigma_1 M_1 \sinh(D_1 y) \quad (21)$$

$$D_1 = (1+j) \sqrt{\frac{\omega\mu_1\sigma_1}{2}} \quad (22)$$

$$0 \leq y \leq b_1$$

Layer 2:

$$E_{x2}(y) = L_2 \cosh(D_2 (y - b_1)) + M_2 \sinh(D_2 (y - b_1)) \quad (23)$$

$$H_{z2}(y) = \frac{L_2 D_2}{j\omega\mu_2} \sinh(D_2 (y - b_1)) + \frac{M_2 D_2}{j\omega\mu_2} \cosh(D_2 (y - b_1)) \quad (24)$$

$$J_{x2}(y) = \sigma_2 L_2 \cosh(D_2 (y - b_1)) + \sigma_2 M_2 \sinh(D_2 (y - b_1)) \quad (25)$$

$$D_2 = (1+j) \sqrt{\frac{\omega\mu_2\sigma_2}{2}} \quad (26)$$

$$b_1 \leq y \leq b_2$$

with L_1 , M_1 , L_2 and M_2 being the unknown coefficients which need to be solved.

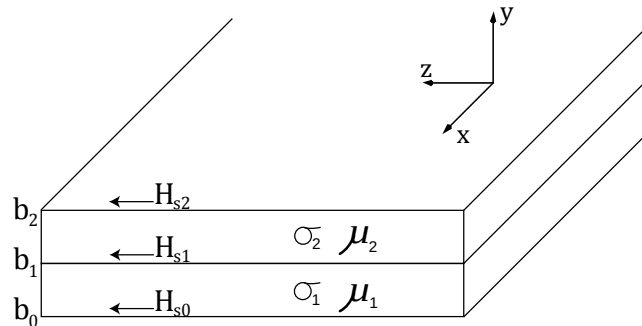


Figure 4.4. Two-layer flat conductor

Step 1

As a general solution is to be obtained, the external boundary value magnetic field intensities will be kept in their generic forms, H_{s0} and H_{s2} . They will be given explicit values in section 4.4, where they will be applied to specific configurations for the validation of this methodology. From equations (20) and (24),

$$H_{z1}(0) = H_{s0} = \frac{M_1 D_1}{j\omega\mu_1} \quad (27)$$

$$H_{z2}(b_2) = H_{s2} = \frac{L_2 D_2}{j\omega\mu_2} \sinh(D_2(b_2 - b_1)) + \frac{M_2 D_2}{j\omega\mu_2} \cosh(D_2(b_2 - b_1)) \quad (28)$$

Step 2

From Faraday's law, the tangential electric field must be continuous across the boundary. Equating, therefore, E_{x1} and E_{x2} (equations (19) and (23)) at their common boundary, b_1 ,

$$\begin{aligned} E_{x1}(b_1) &= E_{x2}(b_1) \\ L_1 \cosh(D_1 b_1) + M_1 \sinh(D_1 b_1) &= L_2 \end{aligned} \quad (29)$$

Step 3

Ampere's law imposes the continuity of the tangential magnetic field intensity across the boundary, forcing equations (20) and (24) to be equal at $y = b_1$,

$$\begin{aligned} H_{z1}(b_1) &= H_{z2}(b_1) \\ \frac{L_1 D_1}{j\omega\mu_1} \sinh(D_1 b_1) + \frac{M_1 D_1}{j\omega\mu_1} \cosh(D_1 b_1) &= \frac{M_2 D_2}{j\omega\mu_2} \end{aligned} \quad (30)$$

Step 4

The simultaneous equations are therefore,

$$H_{s0} = \frac{M_1 D_1}{j\omega\mu_1} \quad (31)$$

$$H_{s2} = \frac{L_2 D_2}{j\omega\mu_2} \sinh(D_2(b_2 - b_1)) + \frac{M_2 D_2}{j\omega\mu_2} \cosh(D_2(b_2 - b_1)) \quad (32)$$

$$L_1 \cosh(D_1 b_1) + M_1 \sinh(D_1 b_1) = L_2 \quad (33)$$

$$\frac{L_1 D_1}{j\omega\mu_1} \sinh(D_1 b_1) + \frac{M_1 D_1}{j\omega\mu_1} \cosh(D_1 b_1) = \frac{M_2 D_2}{j\omega\mu_2} \quad (34)$$

which are equations (27) to (30) from steps one to four. The advantage behind the shift in origin (equation (15) and (16)) can clearly be seen from (31), where it already produced a solution for M_1 . All the remaining unknowns will therefore be written in terms of L_1 , which is chosen as the

next unknown to be solved. Rewriting equation (31) in terms of M_1 and substituting into (33) allows L_2 to be written in terms of L_1 only,

$$L_2 = L_1 \cosh(D_1 b_1) + j\omega\mu_1 \frac{H_{s0}}{D_1} \sinh(D_1 b_1) \quad (35)$$

Similarly, substitution of (31) into (34) permits M_2 to be expressed solely in terms of L_1 ,

$$M_2 = \frac{\mu_2}{\mu_1} \left[L_1 \frac{D_1}{D_2} \sinh(D_1 b_1) + j\omega\mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \quad (36)$$

L_1 can now be solved by substituting these two new equations ((35) and (36)) into (32). Grouping terms,

$$\left[\begin{array}{l} j\omega\mu_2 H_{s2} - j\omega\mu_1 H_{s0} \frac{D_2}{D_1} \sinh(D_1 b_1) \cdot \sinh(D_2 (b_2 - b_1)) - \dots \\ j\omega\mu_2 H_{s0} \cosh(D_1 b_1) \cdot \cosh(D_2 (b_2 - b_1)) \end{array} \right] = \left[\begin{array}{l} L_1 D_2 \cosh(D_1 b_1) \cdot \sinh(D_2 (b_2 - b_1)) + \dots \\ \frac{\mu_2}{\mu_1} L_1 D_1 \sinh(D_1 b_1) \cdot \cosh(D_2 (b_2 - b_1)) \end{array} \right] \quad (37)$$

L_1 is therefore given by,

$$L_1 = \frac{\left[\begin{array}{l} j\omega\mu_2 H_{s2} - j\omega\mu_1 H_{s0} \frac{D_2}{D_1} \sinh(D_1 b_1) \cdot \sinh(D_2 (b_2 - b_1)) - \dots \\ j\omega\mu_2 H_{s0} \cosh(D_1 b_1) \cdot \cosh(D_2 (b_2 - b_1)) \end{array} \right]}{\left[\begin{array}{l} D_2 \cosh(D_1 b_1) \cdot \sinh(D_2 (b_2 - b_1)) + \dots \\ \frac{\mu_2}{\mu_1} D_1 \sinh(D_1 b_1) \cdot \cosh(D_2 (b_2 - b_1)) \end{array} \right]} \quad (38)$$

Since L_2 and M_2 are written in terms of the now solved L_1 ((35) and (36)), it follows that all the coefficients have been solved.

4.3.1.3 Multi-layer methodology: General solution to the three-layer semi-infinite flat conductor

The three-layer conductor of Figure 4.5 will now be considered, where it will be shown that the complexity of the simultaneous equations, and the subsequent solutions of the coefficients, drastically increase with an increase in layer numbers. The general solutions of each layer's electric field and magnetic field intensity and current density are,

Layer 1:

$$E_{x1}(y) = L_1 \cosh(D_1 y) + M_1 \sinh(D_1 y) \quad (39)$$

$$H_{z1}(y) = \frac{L_1 D_1}{j\omega\mu_1} \sinh(D_1 y) + \frac{M_1 D_1}{j\omega\mu_1} \cosh(D_1 y) \quad (40)$$

$$J_{x1}(y) = \sigma_1 L_1 \cosh(D_1 y) + \sigma_1 M_1 \sinh(D_1 y) \quad (41)$$

$$D_1 = (1+j) \sqrt{\frac{\omega\mu_1\sigma_1}{2}} \quad (42)$$

$$0 \leq y \leq b_1$$

Layer 2:

$$E_{x2}(y) = L_2 \cosh(D_2(y - b_1)) + M_2 \sinh(D_2(y - b_2)) \quad (43)$$

$$H_{z2}(y) = \frac{L_2 D_2}{j\omega\mu_2} \sinh(D_2(y - b_1)) + \frac{M_2 D_2}{j\omega\mu_2} \cosh(D_2(y - b_1)) \quad (44)$$

$$J_{x2}(y) = \sigma_2 L_2 \cosh(D_2(y - b_1)) + \sigma_2 M_2 \sinh(D_2(y - b_2)) \quad (45)$$

$$D_2 = (1+j) \sqrt{\frac{\omega\mu_2\sigma_2}{2}} \quad (46)$$

$$b_1 \leq y \leq b_2$$

Layer 3:

$$E_{x3}(y) = L_3 \cosh(D_3(y - b_2)) + M_3 \sinh(D_3(y - b_2)) \quad (47)$$

$$H_{z3}(y) = \frac{L_3 D_3}{j\omega\mu_3} \sinh(D_3(y - b_2)) + \frac{M_3 D_3}{j\omega\mu_3} \cosh(D_3(y - b_2)) \quad (48)$$

$$J_{x3}(y) = \sigma_3 L_3 \cosh(D_3(y - b_2)) + \sigma_3 M_3 \sinh(D_3(y - b_2)) \quad (49)$$

$$D_3 = (1+j) \sqrt{\frac{\omega\mu_3\sigma_3}{2}} \quad (50)$$

$$b_2 \leq y \leq b_3$$

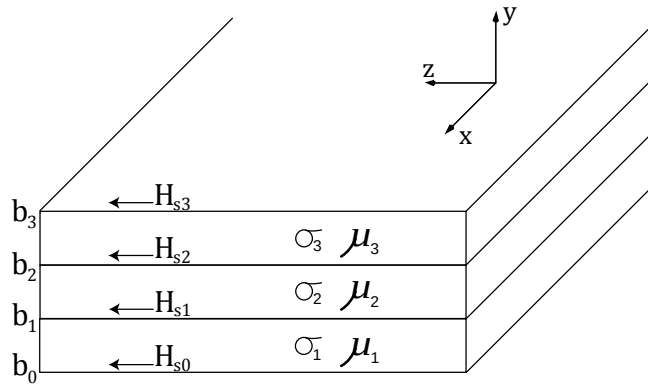


Figure 4.5. Three-layer flat conductor

Step 1

Similar to the previous two-layer case, the magnetic fields at the external boundaries will be kept in their generic form in order to allow for the final solution to be applied to any configuration. Therefore,

$$H_{z1}(0) = H_{s0} = \frac{M_1 D_1}{j\omega\mu_1} \quad (51)$$

$$H_{z3}(b_3) = H_{s3} = \frac{L_3 D_3}{j\omega\mu_3} \sinh(D_3(b_3 - b_2)) + \frac{M_3 D_3}{j\omega\mu_3} \cosh(D_3(b_3 - b_2)) \quad (52)$$

Step 2

The tangential electric field must be continuous across each of the two boundaries. Equating E_{x1} and E_{x2} at boundary b_1 and E_{x2} and E_{x3} at b_2 ,

$$\begin{aligned} E_{x2}(b_1) &= E_{x1}(b_1) \\ L_2 &= L_1 \cosh(D_1 b_1) + M_1 \sinh(D_1 b_1) \end{aligned} \quad (53)$$

$$\begin{aligned} E_{x3}(b_2) &= E_{x2}(b_2) \\ L_3 &= L_2 \cosh(D_2(b_2 - b_1)) + M_2 \sinh(D_2(b_2 - b_1)) \end{aligned} \quad (54)$$

Step 3

Ampere's law requires that the tangential magnetic field intensity must be continuous across each boundary as well,

$$\begin{aligned} H_{z1}(b_1) &= H_{z2}(b_1) \\ \frac{L_1 D_1}{j\omega\mu_1} \sinh(D_1 b_1) + \frac{M_1 D_1}{j\omega\mu_1} \cosh(D_1 b_1) &= \frac{M_2 D_2}{j\omega\mu_2} \end{aligned} \quad (55)$$

$$\begin{aligned} H_{z2}(b_2) &= H_{z3}(b_2) \\ \frac{L_2 D_2}{j\omega\mu_2} \sinh(D_2(b_2 - b_1)) + \frac{M_2 D_2}{j\omega\mu_2} \cosh(D_2(b_2 - b_1)) &= \frac{M_3 D_3}{j\omega\mu_3} \end{aligned} \quad (56)$$

Step 4

The six simultaneous equations required to solve the six unknown coefficients are therefore given by equations (51) to (56). Similar to the previous two-layer scenario the shift in origin (i.e. the $(y-b_{i-1})$ term) helped to simplify the equations, with the result that M_1 is already solved (equation (51)). L_1 will again be chosen to be solved first with all the other coefficients written in terms of it. Rewriting (51) in terms of M_1 and substituting into (53) and (55) produces,

$$L_2 = L_1 \cosh(D_1 b_1) + \frac{H_{s0} j \omega \mu_1}{D_1} \sinh(D_1 b_1) \quad (57)$$

$$M_2 = \frac{\mu_2}{\mu_1} \left[\frac{D_1}{D_2} L_1 \sinh(D_1 b_1) + j \omega \mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \quad (58)$$

Substituting these new expressions for L_2 and M_2 into (54) and (56) allows L_3 and M_3 to be written in terms of L_1 ,

$$L_3 = \left[L_1 \cosh(D_1 b_1) + \frac{H_{s0} j \omega \mu_1}{D_1} \sinh(D_1 b_1) \right] \cosh(D_2 (b_2 - b_1)) + \dots \quad (59)$$

$$\frac{\mu_2}{\mu_1} \left[\frac{D_1}{D_2} L_1 \sinh(D_1 b_1) + j \omega \mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \sinh(D_2 (b_2 - b_1))$$

$$M_3 = \frac{\mu_3}{\mu_2} \left[\frac{D_2}{D_3} \left[L_1 \cosh(D_1 b_1) + \frac{H_{s0} j \omega \mu_1}{D_1} \sinh(D_1 b_1) \right] \sinh(D_2 (b_2 - b_1)) + \dots \right. \quad (60)$$

$$\left. \frac{D_2}{D_3} \frac{\mu_2}{\mu_1} \left[\frac{D_1}{D_2} L_1 \sinh(D_1 b_1) + j \omega \mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \cosh(D_2 (b_2 - b_1)) \right]$$

Substitution of (59) and (60) into (52) produces an equation with only one unknown, L_1 .

$$j \omega \mu_3 H_{s3} = \left[\left[L_1 \cosh(D_1 b_1) + \frac{H_{s0} j \omega \mu_1}{D_1} \sinh(D_1 b_1) \right] \cosh(D_2 (b_2 - b_1)) + \dots \right. \quad (61)$$

$$\left. \frac{\mu_2}{\mu_1} \left[\frac{D_1}{D_2} L_1 \sinh(D_1 b_1) + j \omega \mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \sinh(D_2 (b_2 - b_1)) \right] D_3 \sinh(D_3 (b_3 - b_2)) + \dots$$

$$\left[\frac{\mu_3}{\mu_2} \left[\frac{D_2}{D_3} \left[L_1 \cosh(D_1 b_1) + \frac{H_{s0} j \omega \mu_1}{D_1} \sinh(D_1 b_1) \right] \sinh(D_2 (b_2 - b_1)) + \dots \right. \right. \quad (61)$$

$$\left. \left. \frac{D_2}{D_3} \frac{\mu_2}{\mu_1} \left[\frac{D_1}{D_2} L_1 \sinh(D_1 b_1) + j \omega \mu_1 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \right] \cosh(D_2 (b_2 - b_1)) \right] \right] D_3 \cosh(D_3 (b_3 - b_2))$$

Grouping terms and simplifying,

$$\begin{aligned}
& L_1 \begin{bmatrix} \cosh(D_1 b_1) \cosh(D_2(b_2 - b_1)) + \dots \\ \frac{\mu_2 D_1}{\mu_1 D_2} \sinh(D_1 b_1) \sinh(D_2(b_2 - b_1)) \\ \frac{\mu_3 D_2}{\mu_2 D_3} \cosh(D_1 b_1) \sinh(D_2(b_2 - b_1)) + \dots \\ \frac{\mu_3 D_1}{\mu_1 D_3} \sinh(D_1 b_1) \cosh(D_2(b_2 - b_1)) \end{bmatrix} \begin{bmatrix} D_3 \sinh(D_3(b_3 - b_2)) + \dots \\ D_3 \cosh(D_3(b_3 - b_2)) \end{bmatrix} \\
& = \begin{bmatrix} j\omega\mu_3 H_{s3} - \left[\frac{H_{s0} j\omega\mu_1}{D_1} \sinh(D_1 b_1) \cosh(D_2(b_2 - b_1)) + \dots \right. \\ \left. j\omega\mu_2 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \sinh(D_2(b_2 - b_1)) \right] \\ \frac{D_2 \mu_3}{D_3 \mu_2} \frac{H_{s0} j\omega\mu_1}{D_1} \sinh(D_1 b_1) \sinh(D_2(b_2 - b_1)) + \dots \\ j\omega\mu_3 \frac{H_{s0}}{D_3} \cosh(D_1 b_1) \cosh(D_2(b_2 - b_1)) \end{bmatrix} \begin{bmatrix} D_3 \sinh(D_3(b_3 - b_2)) - \dots \\ D_3 \cosh(D_3(b_3 - b_2)) \end{bmatrix} \quad (62)
\end{aligned}$$

L_1 is therefore given by,

$$\begin{aligned}
& L_1 = \begin{bmatrix} j\omega\mu_3 H_{s3} - \left[\frac{H_{s0} j\omega\mu_1}{D_1} \sinh(D_1 b_1) \cosh(D_2(b_2 - b_1)) + \dots \right. \\ \left. j\omega\mu_2 \frac{H_{s0}}{D_2} \cosh(D_1 b_1) \sinh(D_2(b_2 - b_1)) \right] \\ \frac{D_2 \mu_3}{D_3 \mu_2} \frac{H_{s0} j\omega\mu_1}{D_1} \sinh(D_1 b_1) \sinh(D_2(b_2 - b_1)) + \dots \\ j\omega\mu_3 \frac{H_{s0}}{D_3} \cosh(D_1 b_1) \cosh(D_2(b_2 - b_1)) \end{bmatrix} \begin{bmatrix} D_3 \sinh(D_3(b_3 - b_2)) - \dots \\ D_3 \cosh(D_3(b_3 - b_2)) \end{bmatrix} \\
& \quad \begin{bmatrix} \cosh(D_1 b_1) \cosh(D_2(b_2 - b_1)) + \dots \\ \frac{\mu_2 D_1}{\mu_1 D_2} \sinh(D_1 b_1) \sinh(D_2(b_2 - b_1)) \\ \frac{\mu_3 D_2}{\mu_2 D_3} \cosh(D_1 b_1) \sinh(D_2(b_2 - b_1)) + \dots \\ \frac{\mu_3 D_1}{\mu_1 D_3} \sinh(D_1 b_1) \cosh(D_2(b_2 - b_1)) \end{bmatrix} \begin{bmatrix} D_3 \sinh(D_3(b_3 - b_2)) + \dots \\ D_3 \cosh(D_3(b_3 - b_2)) \end{bmatrix} \quad (63)
\end{aligned}$$

Since all the other unknown coefficients have been expressed in terms of L_1 , it is clear that a complete solution has been obtained. The experimental verification of these equations is given in section 4.4.

4.3.1.4 Multi-layer methodology: Discussion of the obtainment of an n-layer conductor solution

Comparison of two- and three-layer solutions presented above showed that the complexity of the steps, along with the final results, drastically increases with layer numbers. Solution by hand, therefore, becomes impractical for larger layer numbers, with the result that a more

automated solution process is required. Mathematical programs, such as Mathematica, is therefore ideally suited to such a problem. In its simplest form, the simultaneous equations will still have to be created by the user, with its solution then left to the program. More advanced algorithms can, however, be created in which the generic equations of (15) and (16), along with the boundary conditions and layer numbers, are used to create the simultaneous equations and subsequently solve them.

As the purpose of this chapter is the creation and subsequent proof of the multi-layer methodology, this complete automation process will be relinquished to future work. The three-layer cylindrical conductor does, however, take a step in this direction, with its simultaneous equations solved via Mathematica in section 4.3.2.3.

4.3.2 Cylindrical multi-layer conductors

Similar to the flat conductor case, this section will start with the general cylindrical configuration and the determination of the boundary values. The finite configurations, to be used in the verification of the multi-layer methodology (section 4.4), will then be considered and will be followed by the derivation of the cylindrical two- and three-layer generic equations.

4.3.2.1 General configuration: The multi-layer conductor, multi-layer cylindrical-configuration

The generalised multi-layer multi-conductor cylindrical configuration is given in Figure 4.6. This general configuration consists of multiple cylindrical conductors aligned on the same axis, with each conductor consisting of multiple layers. Figure 4.6 therefore consists of m -cylindrical conductors, with each conductor carrying a resultant current and being made up of multiple layers (m can be any number). n represents the amount of layers of each conductor and may differ between conductors. Conductor k , which can be any of the m -multi-layer conductors, is to be considered.

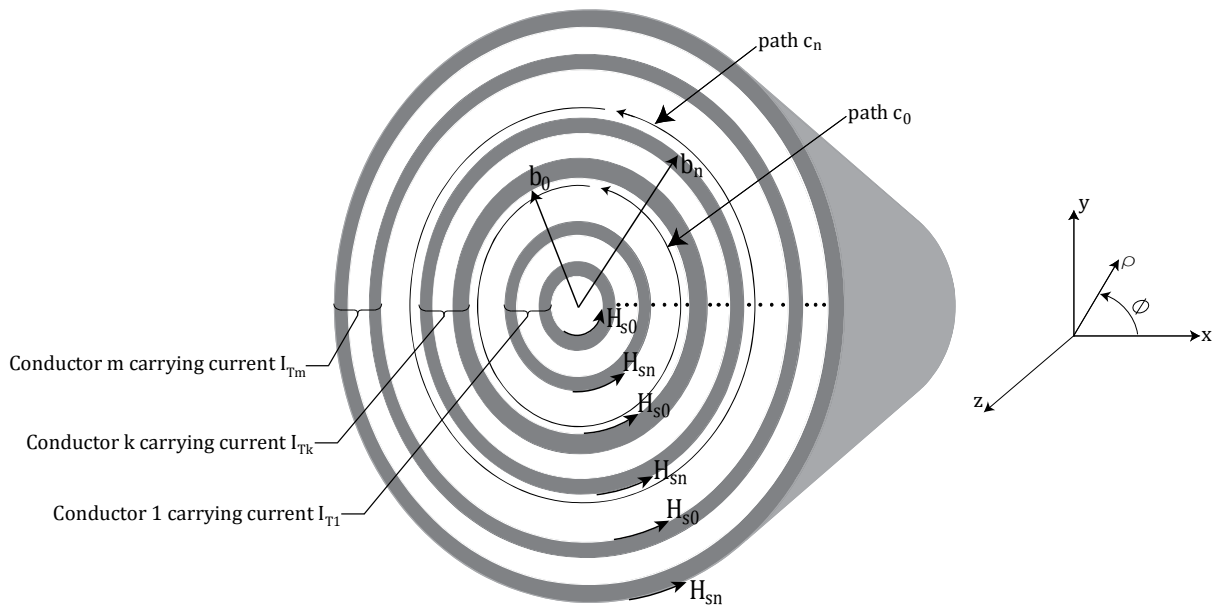


Figure 4.6. Multi-layer conductor, multi-layer cylindrical configuration

The magnetic field intensity at the inner boundary, b_0 , will first be determined. Starting with Ampere's law in integral form, where the path of integration is c_0 ,

$$\oint_{c_0} \bar{H}_{s0} \cdot d\bar{s} = \iint_S \bar{J} \cdot \bar{N} da$$

$$\int_0^{2\pi} H_{s0} b_0 d\phi = \int_0^{b_0} \int_0^{2\pi} J d\phi \rho d\rho$$
(64)

The area of integration on the right-hand side is therefore over all the current carrying conductors whose outer radius is smaller than b_0 . This integral reduces to,

$$H_{s0} = \frac{\sum_{i=1}^{k-1} I_{Ti}}{2\pi b_0}$$
(65)

where it must be remembered to include the phase of the currents in the summation. As is clear, this is the superposition of the magnetic field intensities at b_0 from all the conductors' currents whose radii are smaller than b_0 . It follows that the magnetic field intensity at the outer boundary ($\rho=b_n$) will then be the superposition of the magnetic field intensities due to all the currents flowing in the conductors whose radii are smaller or equal to b_n .

$$H_{sn} = \frac{\sum_{i=1}^k I_{Ti}}{2\pi b_n}$$
(66)

This configuration is obviously representative of all possible cylindrical configurations. In order to assist in the verification of section 4.4, it will be reduced to the configuration given below.

Two-conductor, multi-layer cylindrical configuration

Similar to the flat conductor case, Figure 4.6 will be reduced to the two-conductor multi-layer configuration shown in Figure 4.7 ($m = 2$). Both conductors will, however, be considered, since the corresponding boundary values of the two conductors differ from each other (due to the corresponding radii differing, i.e. $b_{1,0} \neq b_{2,0}$ and $b_{1,n} \neq b_{2,n}$). The subscript 1_ and 2_ will therefore be used to distinguish between the two conductors. Starting with the focus on the inner conductor (i.e. $k = 1$), the boundary values can be obtained from equations (65) and (66),

$$H_{1,s0} = 0$$
(67)

$$H_{1,sn} = \frac{I_{T1}}{2\pi b_{1,n}}$$
(68)

This is to be expected as the current within any cylindrical conductor does not produce a magnetic field for $\rho < b_0$. The magnetic field at $b_{1,0}$ is therefore zero as there is no current

carrying conductor present for $\rho < b_{1,0}$ and the only field at $b_{n,0}$ is due to the current within conductor 1.

It follows that the distribution within the inner conductor is independent of the outer conductor, meaning that whether the outer conductor is carrying the return current of I_{T1} (i.e. $I_{T2} = -I_{T1}$), some other independent current, or is not present at all, the distribution will remain unaffected. This is the familiar skin-effect scenario and allows for the inner conductor to be situated anywhere within the outer conductor (even though the restriction was made that all the conductors are situated on the same axis).

Take note, however, that this independence (from the outer conductor and position) only applies to the distributions within the inner conductor. The impedance equations investigated in the next chapter will be found to be dependent on the presence of the outer conductor, while the freedom of position will no longer be applicable.

Moving now to the outer conductor ($k=2$) the boundary values are given by,

$$H_{2,s0} = \frac{I_{T1}}{2\pi b_{2,0}} \quad (69)$$

$$H_{2,sn} = \frac{I_{T1} + I_{T2}}{2\pi b_{2,n}} \quad (70)$$

The distributions within the outer conductor will clearly be influenced by the magnetic field of the inner conductor's current. By making I_{T1} zero, the boundary values clearly reduce to that of the skin-effect case, while, if I_{T2} is zeroed, then the configuration simplifies to the proximity-effect scenario.

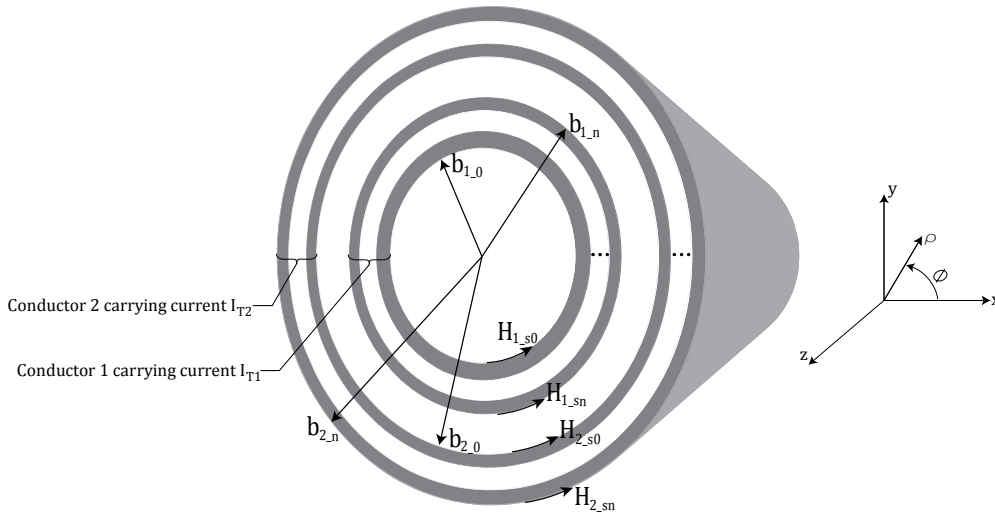


Figure 4.7. Two conductor multi-layer cylindrical configuration

4.3.2.2 Multi-layer methodology: General solution to the two-layer cylindrical conductor

With the possible configurations now discussed, attention can be focussed on the derivation of the generic equations. As mentioned in section 4.3, the derivation will focus on the two- and three-layer conductor variants. The solution will, however, be derived from the single-layer solution based on the magnetic field intensity, as opposed to the electric field variant used for the flat conductors. From chapter 2, the general solution is given by equation (71), where i refers to a specific layer in a multi-layer conductor. The electric field and current density are obtained from equation (71) and Ampere's law, while A_i and B_i are the unknown coefficients that need to be solved.

Comparison of these equations to that used for the flat conductors (equations (15) and (16)), reveals that the arguments of these functions have not been shifted with respect to the layer's inner radius. This was purposefully done in order to demonstrate that, even though the shift used for the flat conductors helped with the simplification of the algebra, it is not a requirement and that a solution can be obtained without it.

$$H_{\phi i}(\rho) = A_i I_1(D_i \rho) + B_i K_1(D_i \rho) \quad (71)$$

$$E_{zi}(\rho) = \frac{1}{\sigma_i} [A_i D_i I_0(D_i \rho) - B_i D_i K_0(D_i \rho)] \quad (72)$$

$$J_{zi}(\rho) = A_i D_i I_0(D_i \rho) - B_i D_i K_0(D_i \rho) \quad (73)$$

$$b_{i-1} \leq \rho \leq b_i$$

where

$$D_i = (1+j) \sqrt{\frac{\omega \mu_i \sigma_i}{2}} = \frac{(1+j)}{\delta_i} \quad (74)$$

Consider the generic two-layer structure given in Figure 4.8. From equations (71) and (72) the general single-layer solutions to the magnetic field intensity and electric field of the two layers are,

Layer 1:

$$H_{\phi 1}(\rho) = A_1 I_1(D_1 \rho) + B_1 K_1(D_1 \rho) \quad (75)$$

$$E_{z1}(\rho) = \frac{1}{\sigma_1} [A_1 D_1 I_0(D_1 \rho) - B_1 D_1 K_0(D_1 \rho)] \quad (76)$$

$$J_{z1}(\rho) = A_1 D_1 I_0(D_1 \rho) - B_1 D_1 K_0(D_1 \rho) \quad (77)$$

$$D_1 = (1+j) \sqrt{\frac{\omega \mu_1 \sigma_1}{2}} \quad (78)$$

$$b_0 \leq \rho \leq b_1$$

Layer 2:

$$H_{\phi 2}(\rho) = A_2 I_1(D_2 \rho) + B_2 K_1(D_2 \rho) \quad (79)$$

$$E_{z 2}(\rho) = \frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 \rho) - B_2 D_2 K_0(D_2 \rho)] \quad (80)$$

$$J_{z 2}(\rho) = A_2 D_2 I_0(D_2 \rho) - B_2 D_2 K_0(D_2 \rho) \quad (81)$$

$$D_2 = (1 + j) \sqrt{\frac{\omega \mu_2 \sigma_2}{2}} \quad (82)$$

$$b_1 \leq \rho \leq b_2$$

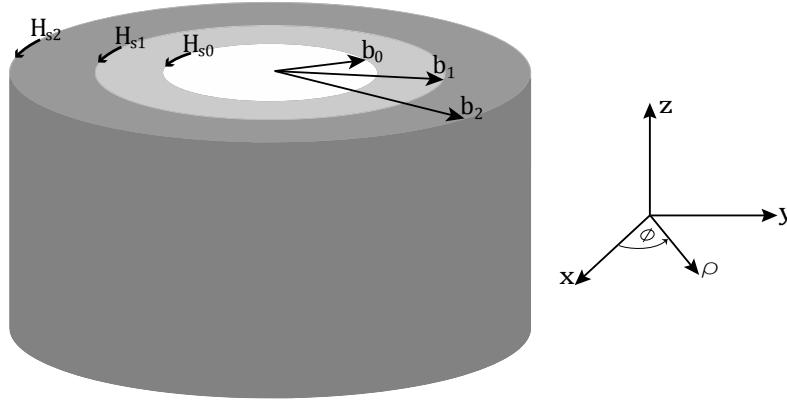


Figure 4.8. two-layer cylindrical conductor

As is now known, four simultaneous equations will be required to solve the four unknowns (A_1 , B_1 , A_2 , B_2). Following the steps laid out in section 4.2:

Step 1

As was stated before, the boundary values will be kept in their generic forms (H_{s0} and H_{s2}) until the final solution is applied to the previously discussed configurations. Equations (75) and (79) therefore become,

$$H_{\phi 1}(b_0) = A_1 I_1(D_1 b_0) + B_1 K_1(D_1 b_0) = H_{s0} \quad (83)$$

$$H_{\phi 2}(b_2) = A_2 I_1(D_2 b_2) + B_2 K_1(D_2 b_2) = H_{s2} \quad (84)$$

Step 2

Faraday's law dictates that the tangential electric field must be continuous across the boundary. Equating the single-layer general electric field solutions (to layers one and two) with $\rho = b_1$,

$$E_{z1}(b_1) = E_{z2}(b_1)$$

$$\frac{1}{\sigma_1} [A_1 D_1 I_0(D_1 b_1) - B_1 D_1 K_0(D_1 b_1)] = \frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 b_1) - B_2 D_2 K_0(D_2 b_1)] \quad (85)$$

Step 3

From Ampere's law the tangential magnetic field intensity must be continuous across the boundary,

$$\begin{aligned} H_{\phi_1}(b_1) &= H_{\phi_2}(b_1) \\ A_1 I_1(D_1 b_1) + B_1 K_1(D_1 b_1) &= A_2 I_1(D_2 b_1) + B_2 K_1(D_2 b_1) \end{aligned} \quad (86)$$

Step 4

The simultaneous equations are therefore given by,

$$H_{\phi_1}(b_0) = A_1 I_1(D_1 b_0) + B_1 K_1(D_1 b_0) = H_{s_0} \quad (87)$$

$$H_{\phi_2}(b_2) = A_2 I_1(D_2 b_2) + B_2 K_1(D_2 b_2) = H_{s_2} \quad (88)$$

$$\frac{1}{\sigma_1} [A_1 D_1 I_0(D_1 b_1) - B_1 D_1 K_0(D_1 b_1)] = \frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 b_1) - B_2 D_2 K_0(D_2 b_1)] \quad (89)$$

$$A_1 I_1(D_1 b_1) + B_1 K_1(D_1 b_1) = A_2 I_1(D_2 b_1) + B_2 K_1(D_2 b_1) \quad (90)$$

A_1 is selected to be solved first, with all the other unknowns written in terms of it. Equations (87) and (88) can therefore be rewritten as,

$$B_1 = \frac{H_{s_0} - A_1 I_1(D_1 b_0)}{K_1(D_1 b_0)} \quad (91)$$

$$B_2 = \frac{H_{s_2} - A_2 I_1(D_2 b_2)}{K_1(D_2 b_2)} \quad (92)$$

B_1 and B_2 can now be substituted into (90),

$$A_1 I_1(D_1 b_1) + \frac{H_{s_0} - A_1 I_1(D_1 b_0)}{K_1(D_1 b_0)} K_1(D_1 b_1) = A_2 I_1(D_2 b_1) + \frac{H_{s_2} - A_2 I_1(D_2 b_2)}{K_1(D_2 b_2)} K_1(D_2 b_1) \quad (93)$$

Grouping terms and writing A_2 in terms of A_1 ,

$$A_2 = \frac{A_1 \left[I_1(D_1 b_1) - \frac{I_1(D_1 b_0)}{K_1(D_1 b_0)} K_1(D_1 b_1) \right] + \frac{H_{s_0}}{K_1(D_1 b_0)} K_1(D_1 b_1) - \frac{H_{s_2}}{K_1(D_2 b_2)} K_1(D_2 b_1)}{\left[I_1(D_2 b_1) - \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} K_1(D_2 b_1) \right]} \quad (94)$$

Substituting this result into (92),

$$B_2 = \frac{H_{s2} - \left[A_1 \left[I_1(D_1 b_1) - \frac{I_1(D_1 b_0)}{K_1(D_1 b_0)} K_1(D_1 b_1) \right] + \frac{H_{s0}}{K_1(D_1 b_0)} K_1(D_1 b_1) - \frac{H_{s2}}{K_1(D_2 b_2)} K_1(D_2 b_1) \right]}{\left[I_1(D_2 b_1) - \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} K_1(D_2 b_1) \right]} I_1(D_2 b_2) \quad (95)$$

As can now be seen, all the other unknown constants are written in terms of A_1 , which can be solved by substituting (91), (94) and (95) into (90),

$$A_1 = \frac{\frac{H_{s0}}{K_1(D_1 b_0)} K_1(D_1 b_1) - \frac{H_{s2}}{K_1(D_2 b_2)} K_1(D_2 b_1)}{\left[I_1(D_2 b_1) - \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} K_1(D_2 b_1) \right]} \left[\frac{1}{\sigma_2} D_2 I_0(D_2 b_1) + \frac{1}{\sigma_2} \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} D_2 K_0(D_2 b_1) \right] + \dots$$

$$A_1 = \frac{\frac{1}{\sigma_1} \frac{H_{s0}}{K_1(D_1 b_0)} D_1 K_0(D_1 b_1) - \frac{1}{\sigma_2} \frac{H_{s2}}{K_1(D_2 b_2)} D_2 K_0(D_2 b_1)}{\left[\frac{1}{\sigma_1} D_1 I_0(D_1 b_1) + \frac{1}{\sigma_1} \frac{I_1(D_1 b_0)}{K_1(D_1 b_0)} D_1 K_0(D_1 b_1) - \dots \right]} \left[\frac{I_1(D_1 b_1) - \frac{I_1(D_1 b_0)}{K_1(D_1 b_0)} K_1(D_1 b_1)}{\left[I_1(D_2 b_1) - \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} K_1(D_2 b_1) \right]} \left[\frac{1}{\sigma_2} D_2 I_0(D_2 b_1) + \frac{1}{\sigma_2} \frac{I_1(D_2 b_2)}{K_1(D_2 b_2)} D_2 K_0(D_2 b_1) \right] \right] \quad (96)$$

Now that A_1 has been solved, the other coefficients can be obtained by substituting (96) into (91), (94) and (95). Comparing the complexity of this solution (A_1) to the solution of the two-layer flat conductor (equation (38)) highlights the advantage of the shift in origin undertaken in the single-layer equations of the multi-layer flat conductors.

4.3.2.3 Multi-layer methodology: General solution to the three-layer cylindrical conductor

The exact same approach will be taken to obtain the general solution to a three-layer cylindrical conductor (Figure 4.9). The equations of the magnetic field intensity and electric field for the three layers are,

Layer 1:

$$H_{\phi 1}(\rho) = A_1 I_1(D_1 \rho) + B_1 K_1(D_1 \rho) \quad (97)$$

$$E_{z1}(\rho) = \frac{1}{\sigma_1} \left[A_1 D_1 I_0(D_1 \rho) - B_1 D_1 K_0(D_1 \rho) \right] \quad (98)$$

$$J_{z1}(\rho) = A_1 D_1 I_0(D_1 \rho) - B_1 D_1 K_0(D_1 \rho) \quad (99)$$

$$D_1 = (1+j) \sqrt{\frac{\omega \mu_1 \sigma_1}{2}} \quad (100)$$

$$b_0 \leq \rho \leq b_1$$

Layer 2:

$$H_{\phi 2}(\rho) = A_2 I_1(D_2 \rho) + B_2 K_1(D_2 \rho) \quad (101)$$

$$E_{z 2}(\rho) = \frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 \rho) - B_2 D_2 K_0(D_2 \rho)] \quad (102)$$

$$J_{z 2}(\rho) = A_2 D_2 I_0(D_2 \rho) - B_2 D_2 K_0(D_2 \rho) \quad (103)$$

$$D_2 = (1+j) \sqrt{\frac{\omega \mu_2 \sigma_2}{2}} \quad (104)$$

$$b_1 \leq \rho \leq b_2$$

Layer 3:

$$H_{\phi 3}(\rho) = A_3 I_1(D_3 \rho) + B_3 K_1(D_3 \rho) \quad (105)$$

$$E_{z 3}(\rho) = \frac{1}{\sigma_3} [A_3 D_3 I_0(D_3 \rho) - B_3 D_3 K_0(D_3 \rho)] \quad (106)$$

$$J_{z 3}(\rho) = A_3 D_3 I_0(D_3 \rho) - B_3 D_3 K_0(D_3 \rho) \quad (107)$$

$$D_3 = (1+j) \sqrt{\frac{\omega \mu_3 \sigma_3}{2}} \quad (108)$$

$$b_2 \leq \rho \leq b_3$$

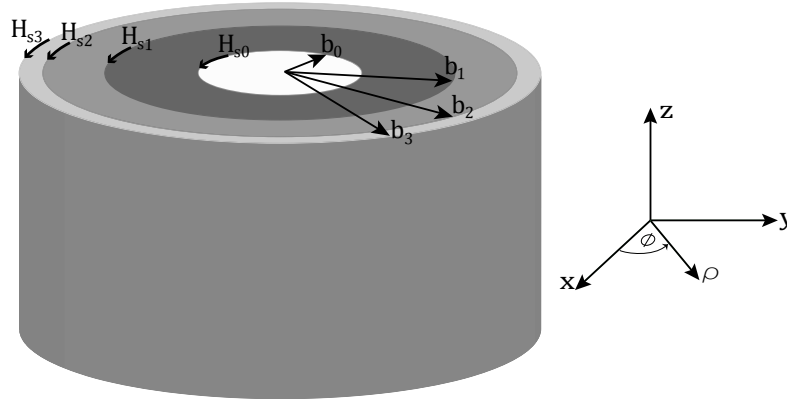


Figure 4.9. Three-layer cylindrical conductor

Step 1

The values of the magnetic field intensity at the inner and outer boundaries ($\rho=b_0$ and $\rho=b_1$) will again remain in their generic form to allow for the obtainment of a general solution. Equations (97) and (105) therefore become,

$$H_{\phi 1}(b_0) = A_1 I_1(D_1 b_0) + B_1 K_1(D_1 b_0) = H_{s0} \quad (109)$$

$$H_{\phi 3}(b_3) = A_3 I_1(D_3 b_3) + B_3 K_1(D_3 b_3) = H_{s3} \quad (110)$$

Step 2

Enforcing the continuity of the tangential electric field across the first ($\rho=b_1$) and second boundary ($\rho=b_2$).

$$E_{z1}(b_1) = E_{z2}(b_1)$$

$$\frac{1}{\sigma_1} [A_1 D_1 I_0(D_1 b_1) - B_1 D_1 K_0(D_1 b_1)] = \frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 b_1) - B_2 D_2 K_0(D_2 b_1)] \quad (111)$$

$$E_{z2}(b_2) = E_{z3}(b_2)$$

$$\frac{1}{\sigma_2} [A_2 D_2 I_0(D_2 b_2) - B_2 D_2 K_0(D_2 b_2)] = \frac{1}{\sigma_3} [A_3 D_3 I_0(D_3 b_2) - B_3 D_3 K_0(D_3 b_2)] \quad (112)$$

Step 3

Equating the magnetic fields at the first boundary ($\rho=b_1$),

$$H_{\phi 1}(b_1) = H_{\phi 2}(b_1)$$

$$A_1 I_1(D_1 b_1) + B_1 K_1(D_1 b_1) = A_2 I_1(D_2 b_1) + B_2 K_1(D_2 b_1) \quad (113)$$

and the second boundary ($\rho=b_2$),

$$H_{\phi 2}(b_2) = H_{\phi 3}(b_2)$$

$$A_2 I_1(D_2 b_2) + B_2 K_1(D_2 b_2) = A_3 I_1(D_3 b_2) + B_3 K_1(D_3 b_2) \quad (114)$$

Step 4

Judging from the complexity of the two-layer cylindrical conductor's simultaneous equations, it was decided to employ Mathematica to solve these six simultaneous equations in order to avoid calculation mistakes. The results for these six coefficients are given in Appendix G due to their complexity and size.

4.4 Simulation and theoretical results and discussion

In order for this thesis' main contribution, the multi-layer methodology, to be accepted as correct, it needs to undergo a thorough verification process. Since the direct verification of a methodology is not possible, its validity will be inferred through comparison of the distributions predicted by its subsequent equations against simulation obtained distributions. FEM simulations will be used to determine the distributions as they cannot be directly obtained through experimental measurement. (The distributions can only be inferred through impedance measurements, as is done in the next chapter).

The analytical and FEM obtained results of both the flat and cylindrical multi-layer configurations will first be given, before commencing with the discussion of these results.

4.4.1 Calculation and graphical software

Extreme care should be taken in the choice of software used to render graphs and determine values from the derived equations. It was found that under certain conditions, some of the constants adopt very small values. This leads to round-off errors, as these numbers are too small to be adequately represented using the normal storage scheme employed by most software programs. Significant errors therefore occur when these small incorrect values are multiplied by very large numbers, which frequently occur for the semi-infinite flat configuration equations. Mathematica was therefore employed, as significant round-off error can be avoided by adjusting the precision of the constants. i.e. Mathematica, unlike other programs (such as Matlab), allows the user to select the amount of decimal points (precision) which are to be accurately represented. The values of the graphs in the following sections were therefore determined with Mathematica, with the results exported to Matlab for final plotting.

4.4.2 FEM software

Maxwell 2D Student Version (discussed in chapter 2, section 2.2.6.1) will be used for the validation of the magnetic field intensity and current density predicted by these newly derived equations. Since the validity of the semi-infinite approximation has already been shown in chapter 2 (section 2.2.6) the use of 'even symmetry axis' will be used in the simulation of the flat conductors in order to reduce the convergence time of the FEM simulations (see Appendix C for more information on the use of 'even symmetry axis').

4.4.3 Two- and three-layer semi-infinite flat conductors

The two- and three-layer conductor equations will be applied to the configuration of Figure 4.3 (shown in Figure 4.10 for ease of reference) in order to verify the equations and the methodology from which it stems. Table 4.1 gives the properties of these conductors, where conductor one is the conductor under consideration (the values were not chosen to represent any practical configurations). As can be seen both conductors were given the same amount of layers and properties and were made to carry equal and opposite currents (as in a differential configuration). This is due to no limitation of the equations, as all that is required of the second conductor is the resultant current it carries, but rather to the limitations of the FEM program. As was briefly mentioned in chapter 2 (section 2.2.6.1) the FEM program encounters convergence problems for certain flat configurations (specifically those that are unsymmetrical). The symmetrical differential configuration is therefore chosen in order to reduce the convergence time and error, thereby allowing for a more accurate comparison.

Take note that layer one of conductor one is at its top, with layers following sequentially downwards. Table 4.2 gives the equations used to obtain the theoretical distributions shown in Figure 4.11 and Figure 4.12. Since the FEM program does not provide simulated results of the electric field, it was decided to divide the FEM obtained current density by the conductivity of the layer through which it flows. This is plotted against the theoretically obtained electric field in Figure 4.12 for comparison. (This was only done for this first case (flat-case 1) in order to show that the validation of the current density infers validation of the electric field as well). The vertical lines indicate the boundaries between the different layers of the conductors.

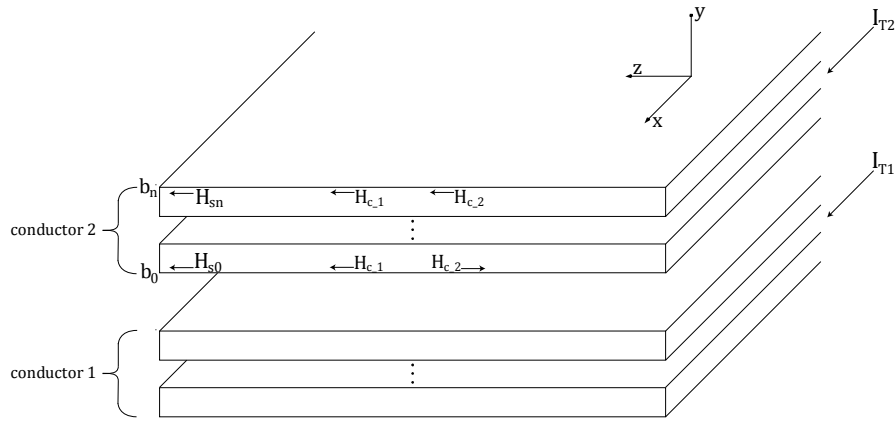


Figure 4.10. Two-conductor multi-layer flat configuration

Table 4.1. Properties of the two- and three-layer conductors as applied to the configuration of Figure 4.10

2-layer conductor (flat-case 1)				3-layer conductor (flat-case 2)			
f	10 MHz			f	100 kHz		
Conductor 1		Conductor 2		Conductor 1		Conductor 2	
n	2	n	2	n	3	n	3
I_{T1}/w	2500 A/m	I_{T2}/w	-2500 A/m	I_{T1}/w	10 A/m	I_{T2}/w	-10 A/m
b_1	100 μm	b_1	100 μm	b_1	200 μm	b_1	200 μm
b_2	150 μm	b_2	150 μm	b_2	1000 μm	b_2	1000 μm
σ_1	$13.3 \times 10^6 \text{ S/m}$	σ_1	$13.3 \times 10^6 \text{ S/m}$	b_3	2500 μm	b_3	2500 μm
μ_{r1}	1	μ_{r1}	1	σ_1	$2 \times 10^5 \text{ S/m}$	σ_1	$2 \times 10^5 \text{ S/m}$
σ_2	$38 \times 10^6 \text{ S/m}$	σ_2	$38 \times 10^6 \text{ S/m}$	μ_{r1}	1	μ_{r1}	1
μ_{r2}	10	μ_{r2}	10	σ_2	$8 \times 10^5 \text{ S/m}$	σ_2	$8 \times 10^5 \text{ S/m}$
				μ_{r2}	20	μ_{r2}	20
				σ_3	$1 \times 10^7 \text{ S/m}$	σ_3	$1 \times 10^7 \text{ S/m}$
				μ_{r3}	1	μ_{r3}	1

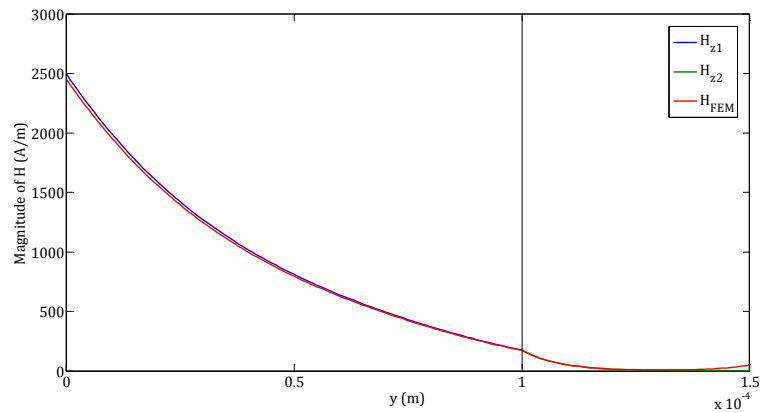


Figure 4.11. Flat case 1: Magnetic field intensity of the FEM and analytical results

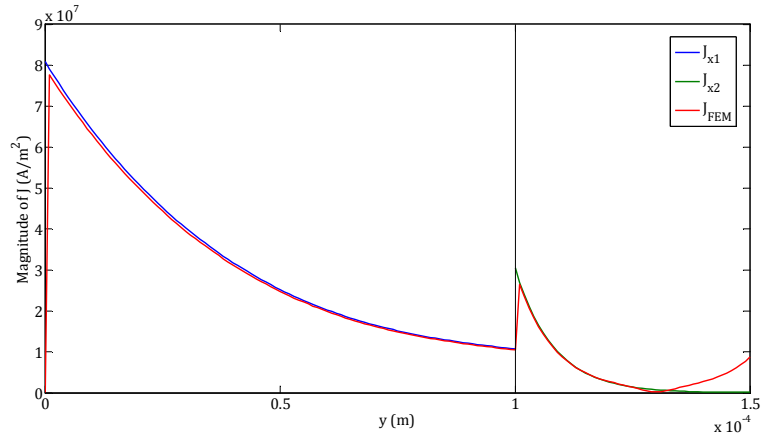


Figure 4.12. Flat case 1: Current density of the FEM and analytical results

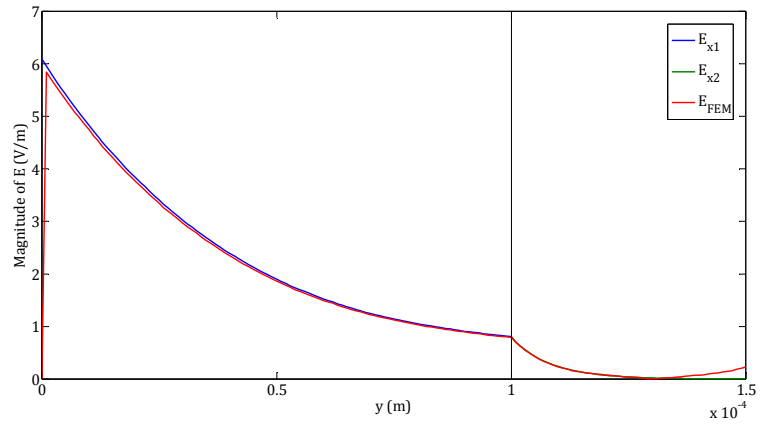


Figure 4.13. Flat case 1: Electric field of the FEM and analytical results

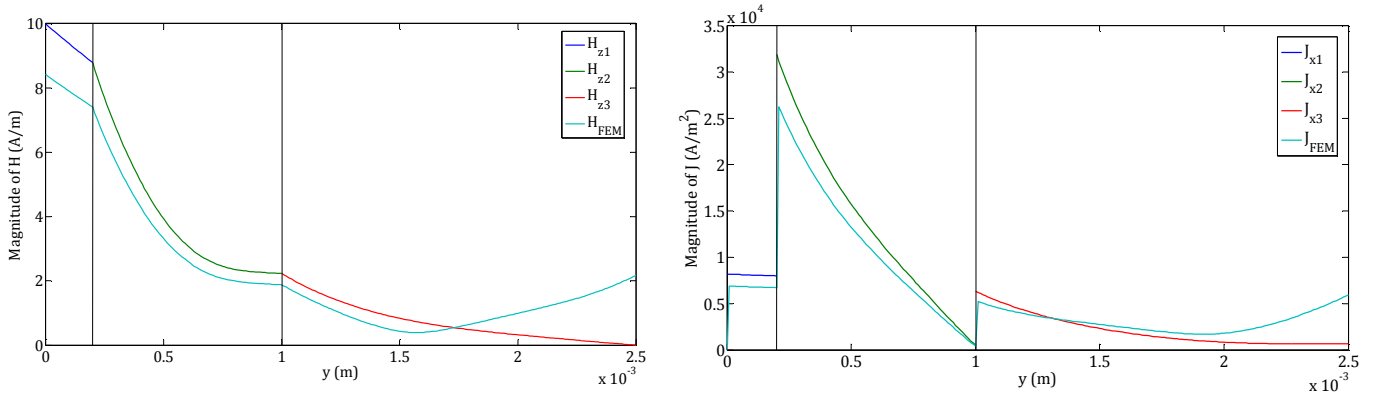


Figure 4.14. Flat case 2: Magnetic field intensity and current density of the FEM and analytical results

Table 4.2. Equations used to obtain the magnetic field intensity, electric field and current density equations

2-layer conductor (flat-case 1)		3-layer conductor (flat-case 2)	
H_{s0}	(9)	H_{s0}	(9)
H_{s2}	(10)	H_{s3}	(10)
D_1	(22)	D_1	(42)
D_2	(26)	D_2	(46)
L_1	H_{s0}, H_{s2}, D_1 and D_2 into (38)	D_3	(50)
L_2	H_{s0}, D_1 and L_1 into (35)	L_1	H_{s0}, H_{s3}, D_1, D_2 and D_3 into (63)
M_1	H_{s0} and D_1 into (31)	L_2	H_{s0}, D_1 and L_1 into (57)
M_2	H_{s0}, L_1, D_1 and D_2 into (36)	L_3	H_{s0}, D_1, D_2, L_1 into (59)
$E_{x1}(y)$	L_1, M_1 and D_1 into (19)	M_1	H_{s0} and D_1 into (51)
$H_{z1}(y)$	L_1, M_1 and D_1 into (20)	M_2	H_{s0}, D_1, D_2 and L_1 into (58)
$J_{x1}(y)$	L_1, M_1 and D_1 into (21)	M_3	$H_{s0}, D_1, D_2, D_3,$ and L_1 into (60)
$E_{x2}(y)$	L_2, M_2 and D_2 into (23)	$E_{x1}(y)$	D_1, L_1 and M_1 into (39)
$H_{z2}(y)$	L_2, M_2 and D_2 into (24)	$H_{z1}(y)$	D_1, L_1 and M_1 into (40)
$J_{x2}(y)$	L_2, M_2 and D_2 into (25)	$J_{x1}(y)$	D_1, L_1 and M_1 into (41)
		$E_{x2}(y)$	D_2, L_2 and M_2 into (43)
		$H_{z2}(y)$	D_2, L_2 and M_2 into (44)
		$J_{x2}(y)$	D_2, L_2 and M_2 into (45)
		$E_{x3}(y)$	D_3, L_3 and M_3 into (47)
		$H_{z3}(y)$	D_3, L_3 and M_3 into (48)
		$J_{x3}(y)$	D_3, L_3 and M_3 into (49)

4.4.4 Two- and three-layer cylindrical conductors

The generic two and three-layer cylindrical conductor equations will be implemented in the 2-conductor multi-layer configuration of Figure 4.7 (reproduced in Figure 4.15 for ease of reference). As was mentioned earlier, since the radii of the corresponding boundaries differ, the boundary values will also differ, resulting in both the inner and outer conductors (i.e. conductors 1 and 2) being considered.

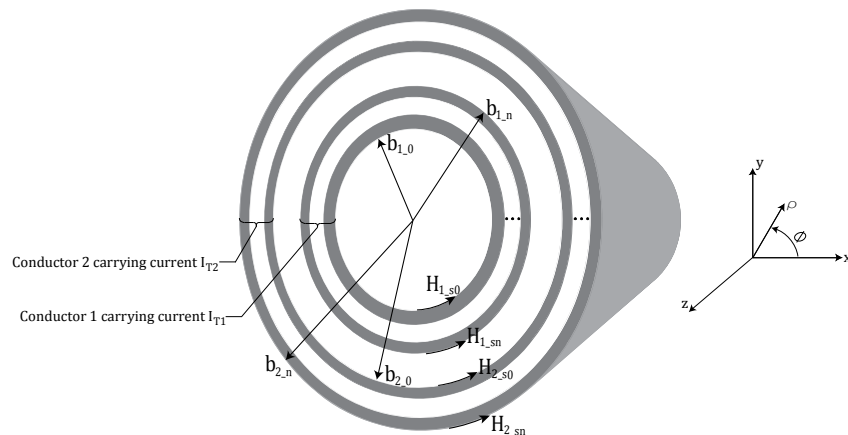


Figure 4.15. Two conductor multi-layer configuration

As will be seen, more two- and three-layer cylindrical scenarios will be considered (compared to the flat conductor scenarios of the previous section) as converged FEM solutions are far easier obtained for cylindrical than flat conductors. Table 4.5 gives the equations used to obtain the analytical distributions of these scenarios. The electric field will not be plotted for these scenarios as the previous section showed that the current density is representative of the electric field.

Scenario 1

The skin-effect scenario will first be considered, where conductor 1 (inner conductor) is under consideration. As was mentioned, the outer conductor will have no influence on the inner conductor and, in order to emphasise this, two separate cases are considered. For both cases the inner conductor will be a two-layer conductor with identical properties (given in Table 4.3), with the first case (cyl-case A) having no outer conductor and the second case's (cyl-case B) outer conductor properties given in the right hand side of Table 4.3. The outer conductor of the second case is taken to be single layer.

The plots of the magnetic field intensity and current density are given in Figure 4.16 and Figure 4.17. In some of the plots it may seem that only one graph is present. This, however, is not the case, as the FEM and analytical results coincide so well that they have near identical results.

Table 4.3. Properties of the two-layer conductors as applied to the configuration of Figure 4.15

cyl-case A and cyl-case B				cyl-case B			
f	100 Hz						
2-layer inner conductor				Single-layer outer conductor			
I _{T1}	3 A	σ _{1,1}	580×10 ⁵ S/m	I _{T2}	-3 A	σ _{2,1}	430×10 ⁵ S/m
b _{1,0}	2×10 ⁻³ m	μ _{r1,1}	600	b _{2,0}	7×10 ⁻³ m	μ _{r2,1}	300
b _{1,1}	3×10 ⁻³ m	σ _{1,2}	145×10 ⁵ S/m	b _{2,1}	10×10 ⁻³ m		
b _{1,2}	4×10 ⁻³ m	μ _{r1,2}	50				

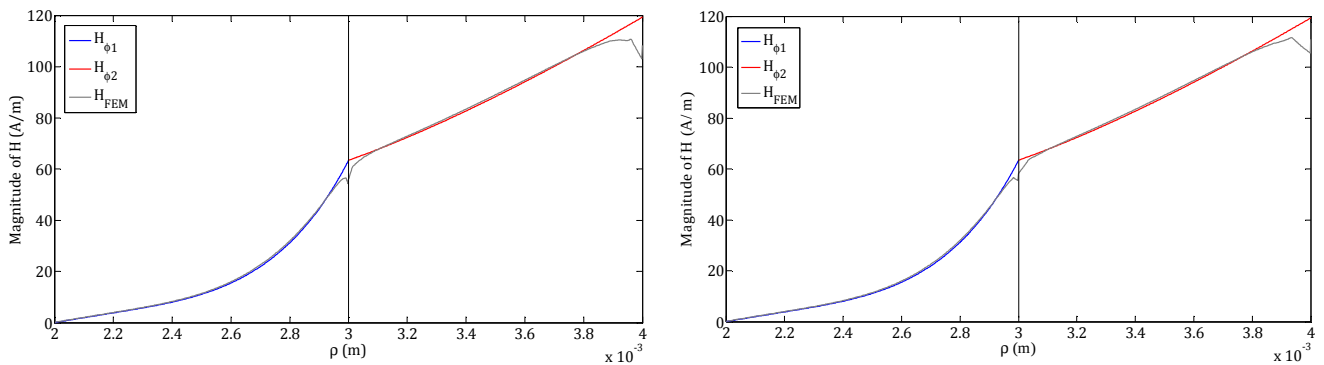


Figure 4.16. cyl-case A and cyl-case B: Magnetic field intensity of the FEM and analytical results

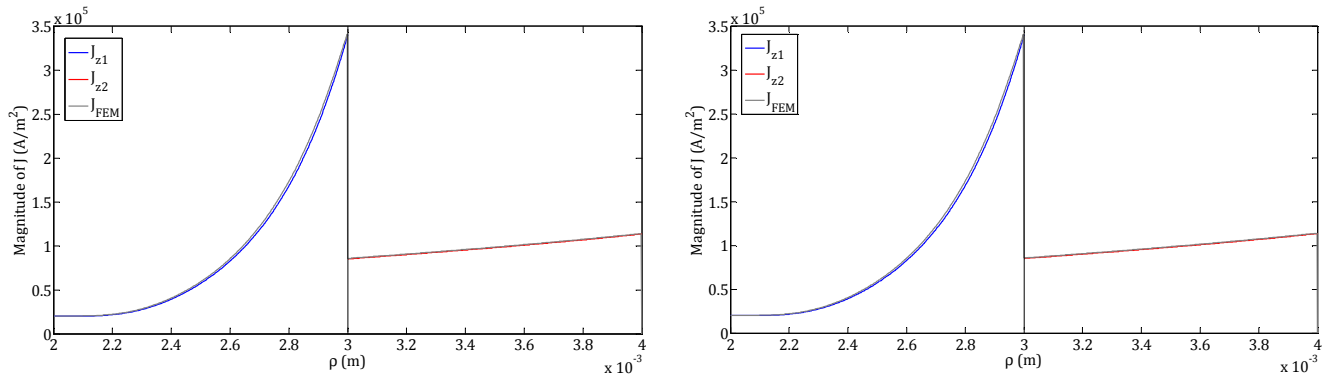


Figure 4.17. cyl-case A and cyl-case B: Current density of the FEM and analytical results

Scenario 2

The outer conductor (conductor 2 of Figure 4.15) is the conductor under consideration in this scenario. Although the inner conductor (conductor 1) is not layer limited, it was chosen to consist of a single layer. The properties of these two conductors are given in Table 4.4 with the magnetic field intensity and current density given in Figure 4.18 and Figure 4.19.

Table 4.4 Properties of the two- and three-layer conductors as applied to the configuration of Figure 4.15

cyl-case C				cyl-case D			
f	400 Hz			f	300 Hz		
2-layer outer conductor				3-layer outer conductor			
I _{T1}	1 A	σ _{2,1}	175x10 ⁵ S/m	I _{T1}	1 A	σ _{2,1}	145x10 ⁵ S/m
b _{2,0}	15x10 ⁻³ m	μ _{r2,1}	1	b _{2,0}	2x10 ⁻³ m	μ _{r2,1}	50
b _{2,1}	20x10 ⁻³ m	σ _{2,2}	850 x10 ⁵ S/m	b _{2,1}	2.5x10 ⁻³ m	σ _{2,2}	580 x10 ⁵ S/m
b _{2,2}	22x10 ⁻³ m	μ _{r2,2}	1200	b _{2,2}	3x10 ⁻³ m	μ _{r2,2}	300
				b _{2,3}	4x10 ⁻³ m	σ _{2,3}	900 x10 ⁵ S/m
						μ _{r2,3}	600
Single-layer inner conductor				Single-layer inner conductor			
I _{T2}	-1 A	σ _{1,1}	600 x10 ⁵ S/m	I _{T2}	-3 A	σ _{1,1}	600 x10 ⁵ S/m
b _{1,0}	0 m	μ _{r1,1}	420	b _{1,0}	0 m	μ _{r1,1}	150
b _{1,1}	10x10 ⁻³ m			b _{1,1}	1x10 ⁻³ m		

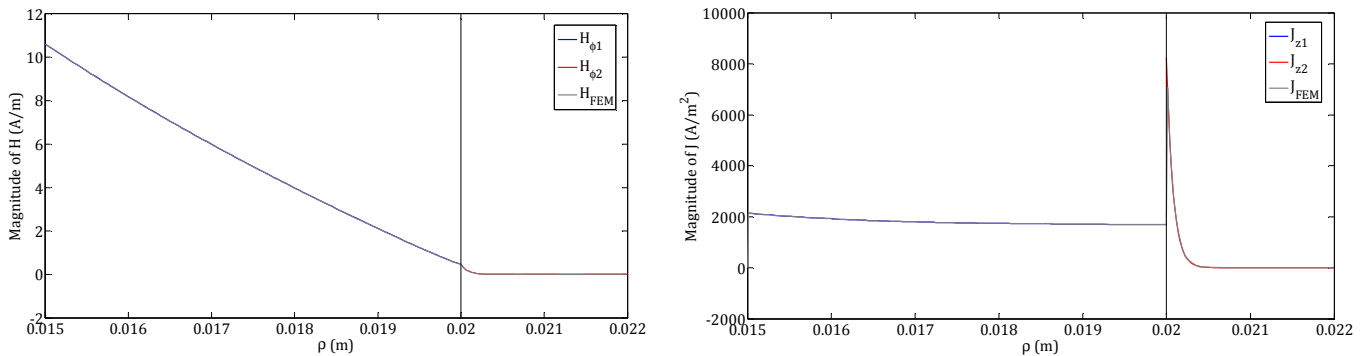


Figure 4.18. cyl-case C: Magnetic field intensity and current density of the FEM and analytical results

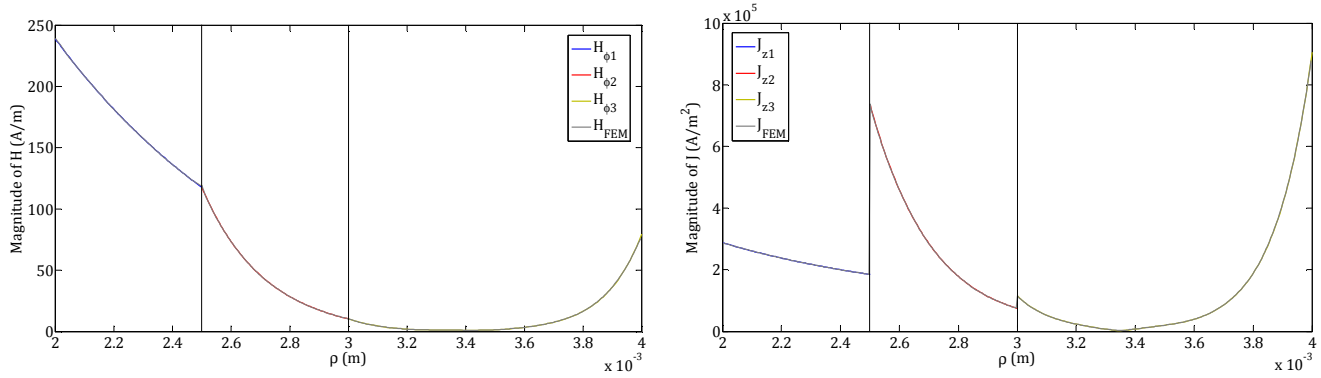


Figure 4.19. cyl-case D: Magnetic field intensity and current density of the FEM and analytical results

Table 4.5. Equations used to obtain the magnetic field intensity, electric field and current density equations

2-layer conductor		3-layer conductor	
H_{s0}	(67) for scenario 1 and (69) for scenario 2	H_{s0}	(69)
H_{s2}	(68) for scenario 1 and (70) for scenario 2	H_{s3}	(70)
D_1	(78)	D_1	(100)
D_2	(82)	D_2	(104)
A_1	H_{s0} , H_{s2} , D_1 and D_2 into (96)	D_3	(108)
A_2	H_{s0} , H_{s2} , D_1 , D_2 and A_1 into (94)	A_1	G.(1) – Appendix G
B_1	H_{s0} , D_1 and A_1 into (91)	A_2	G.(3) – Appendix G
B_2	H_{s2} , D_2 and A_2 into (92)	A_3	G.(5) – Appendix G
$E_{z1}(\rho)$	D_1 , A_1 and B_1 into (76)	B_1	G.(2) – Appendix G
$H_{\phi1}(\rho)$	D_1 , A_1 and B_1 into (75)	B_2	G.(4) – Appendix G
$J_{z1}(\rho)$	D_1 , A_1 and B_1 into (77)	B_3	G.(6) – Appendix G
$E_{z2}(\rho)$	D_2 , A_2 and B_2 into (80)	$E_{z1}(\rho)$	D_1 , A_1 and B_1 into (98)
$H_{\phi2}(\rho)$	D_2 , A_2 and B_2 into (79)	$H_{\phi1}(\rho)$	D_1 , A_1 and B_1 into (97)
$J_{z2}(\rho)$	D_2 , A_2 and B_2 into (81)	$J_{z1}(\rho)$	D_1 , A_1 and B_1 into (99)
		$E_{z2}(\rho)$	D_2 , A_2 and B_2 into (102)
		$H_{\phi2}(\rho)$	D_2 , A_2 and B_2 into (101)
		$J_{z2}(\rho)$	D_2 , A_2 and B_2 into (103)
		$E_{z3}(\rho)$	D_3 , A_3 and B_3 into (106)
		$H_{\phi3}(\rho)$	D_3 , A_3 and B_3 into (105)
		$J_{z3}(\rho)$	D_3 , A_3 and B_3 into (107)

4.4.5 Discussion

From these results the validity of this methodology is without question. Starting with the analysis of the flat conductor plots, near perfect agreement was reached for the 2-layer conductors of Figure 4.11 to Figure 4.13. The 3-layer results of Figure 4.14 are, however, less ideal and may at first be considered to be due to unaccounted for edge effects in the simulation. This however is not the case, as the use of the ‘axis of even symmetry’ converts the simulated

finite conductors to semi-infinite, thereby preventing this problem. Under these conditions, Ampere's law states that no magnetic field should be present outside this differential configuration (i.e. for $y > b_3$). Figure 4.14 shows that the analytical results are in perfect agreement with this, unlike the simulation result which has a non-zero magnetic field at $y = b_3$. This leads to the conclusion that this discrepancy can only be due to convergence problems on the part of the simulation program, even though differential configurations were used to mitigate this problem.

Consider now the cylindrical conductors, where virtually flawless agreement was reached for all two- and three-layer conductors (Figure 4.16 to Figure 4.19). The minute disagreements that were found at the boundaries of Figure 4.16 will again be attributed to convergence problems experienced by the FEM simulation.

From section 4.2 it can be remembered that a central requirement to this methodology is the continuity of the tangential electric field and the tangential magnetic field intensity. This can be seen to hold for all the plots, where the electric field is obtained by dividing the current density by the conductivity of the layer through which it is flowing. From Maxwell's equations it follows that if adequate agreement is found for either the magnetic field intensity, electric field or current density, then this agreement will also hold for the remaining two.

4.5 Implications of this multi-layer methodology

With the above sections taken into consideration, the significance of this multi-layer methodology is clear and can be given as:

- The resulting analytical equations allow for nearly instantaneous results, as compared to the FEM simulations which take significantly longer and may experience convergence problems. These problems are related to the complexity of the conductor, with more layers producing increasing convergence problems. This is also one of the reasons only two- and three-layer conductors were considered, as convergence difficulties already started to arise with three layers (Figure 4.14).
- The analytical equations allow for complex parametric studies, something which is far more difficult to obtain when utilising the normal FEM approach.
- Investigation of the equations allows for insight and understanding as to what factors influence the distributions, something which is not possible with FEM.
- It allows for the accurate determination of the impedance a multi-layer configuration will present to a source. This forms one of the main contributions of the next chapter.

4.6 Conclusion

The multi-layer methodology of section 4.2 formed one of the main contributions of this thesis and was defined without any geometric constraints. Since the continuity of the tangential electric field and tangential magnetic field intensity is geometry independent it can clearly be applied to all multi-layer geometries (of arrangement 2), given that the single-layer solution is known.

With the flat and cylindrical geometries selected, their two- and three- layer equations were derived and applied to chosen multi-layer configurations. Comparison against FEM-obtained results confirmed the methodology's validity, while accentuating a number of advantages it may pose over the generally used FEM simulations. These include: near instantaneous results, which can be used in complex parametric studies; insight into the factors governing the distributions and the determination of a multi-layer conductor's impedance (done in the next chapter).

The two- and three-layer equations have, however, shown an increase in the solution complexity with an increase in layer numbers. This, however, is unavoidable, as a significant interdependency exists between the layers, with more layers resulting in a higher interdependency and more complex solutions. The complexity can, however, be limited by shifting the argument of the functions. This is clearly visible when considering the difference between the two-layer flat and cylindrical conductors. Since they both stem from second-order linear differential equations, their complexities should be the same, given that they use the same argument (neglecting the difference between y and ρ). However, with their arguments shifted (with regards to one another), a clear increase in complexity can be seen for the cylindrical solutions which did not make use of this.

These results will be used in chapter 5, along with derived multi-layer impedance equations, to provide additional validation of this methodology through experimental measurements. The determination of a multi-layer conductor's impedance will prove to be central to the future application of this work.

4.7 References

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Chapter 5

Frequency Dependent Impedance

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5.1 Introduction

Chapters 2 and 3 provided the groundwork needed for the development of the much sought after multi-layer methodology of chapter 4. As was shown, this methodology allows for the prediction of the fields and current density distributions in multi-layer structures. Although this is an extremely important aspect, it also functions as a crucial step towards the practical application of these structures. One of the most important results that can come from this methodology, is the ability to predict the impedances of these multi-layer structures. Up to now this was determined either by simulation or physical measurement, both being very slow and un insightful processes.

This chapter will therefore focus on the derivation of the multi-layer impedance equations, in order to predict what impedance, such a structure, will present to a source. The equations will subsequently be applied to a number of multi-layer structures, and the theoretical results compared against physical measurements. This will serve two purposes. Firstly, it will provide a means to confirm the impedance equations. Secondly, physical verification will provide irrefutable proof to the validity of the multi-layer methodology. This is because chapter 4 only inferred the validity through FEM simulations, which in itself is only an approximation. While physical confirmation of the impedance equations infers the validity of the distribution equations (from which they are derived), this in turn provides unquestionable confirmation of the multi-layer methodology.

5.2 Impedance

As is expected, the factors which will govern each of the impedance components will be the frequency, geometry, configuration and material properties. Therefore, in order to derive the equations that will be compared to the experimental measurements, a configuration needs to be decided upon. Since inductance is only defined for a closed current path, the differential configurations for the flat and cylindrical conductors will be selected, as it is also one of the most widely found configurations in practical applications.

Since the resistance and inductance will be determined from the current density and magnetic field distributions, it follows that the constraints under which they were derived need to be maintained. The two most important factors in this case being that the current is confined to one direction and that the current entering and leaving the multi-layer conductor remains the same. The presence of a displacement current through the capacitor, formed between the two multi-layer conductors, will however violate both these conditions. As such, the capacitance will be derived to ensure that its reactance remains large enough for the frequencies under consideration, that it can be neglected.

By determining the individual impedance components, the configuration can be reduced to the transmission-line model shown in Figure 5.1. The conductance, G , will be taken as zero for the dielectric material used in the next section. By ensuring that the capacitance is small enough, such that its reactance can be taken as infinite, the circuit is allowed to reduce to a series RL model. This will be used to model the measured impedance in the experimental verification section.

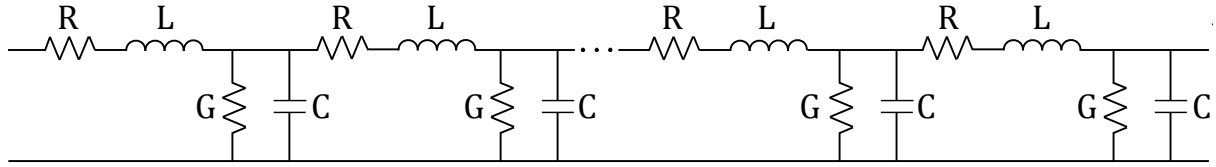


Figure 5.1. Circuit representation of transmission-line structure

5.2.1 Resistance

5.2.1.1 General solution

In order to determine the resistance provided by a single-or multi-layer configuration, the total power delivered by the source, and converted to heat in the conductor, is firstly required. From chapters 2 and 4 the current density is determined on a per-layer basis. The power dissipated within each layer will therefore first be determined, with the total power given by the summation over all the layers. Starting with the well-known power equation under sinusoidal conditions,

$$P = |I_{rms}|^2 R \quad (1)$$

Take note, that this current must be expressed in terms of rms values, as opposed to peak values which were used for the majority of this thesis. As the current density is not constant over the cross-sectional area of the conductor, this equation cannot be directly applied. Looking at the power dissipated by an infinitesimal current filament flowing through an infinitesimal area,

$$\begin{aligned} dP &= |dI_{rms}|^2 R \\ &= |J_{rms} da|^2 R \\ &= |J_{rms}|^2 |da|^2 R \\ &= |J_{rms}|^2 da^2 \frac{l}{\sigma da} \end{aligned} \quad (2)$$

where $l/(\sigma da)$ is the resistance experienced by the current element dI . In this case the current density must be expressed in terms of rms values. The total power dissipated per unit length is then,

$$\frac{P_k}{l} = \frac{1}{\sigma} \iint_S |J_{k_rms}|^2 da \quad (3)$$

or equivalently,

$$\frac{P_k}{l} = \frac{1}{2\sigma} \iint_S |J_{k_peak}|^2 da \quad (4)$$

where P_k is the power dissipated and J_{k_rms} or J_{k_peak} the rms and peak current density distribution within layer k (respectively). As peak values have been used to date, equation (4)

will generally be favoured in this thesis. This general power solution can now be applied to the semi-infinite flat and cylindrical conductors. For the flat conductors (4) becomes,

$$\frac{P_k}{l} = \frac{w}{2\sigma} \int_{b_{k-1}}^{b_k} |J_k(y)|^2 dy \quad (5)$$

As can be understood, P_k is the power dissipated per unit length for a layer of width, w . For the cylindrical conductors the power per layer becomes,

$$\frac{P_k}{l} = \frac{\pi}{\sigma} \int_{b_{k-1}}^{b_k} |J_k(\rho)|^2 \rho d\rho \quad (6)$$

Letting P_T be the total power supplied to all the layers, the resistance per unit length (as seen by the source) is obtained from equation (1),

$$\frac{R}{l} = \frac{P_T}{l} \frac{1}{|I_{T_rms}|^2} = \frac{P_T}{l} \frac{2}{|I_{T_peak}|^2} \quad (7)$$

where I_T is the total current flowing through the source and the conductor. The resistance is obviously unaffected by whether peak or rms values are used. Equation (7) is straight forward for isolated conductors, but becomes more complicated in the presence of neighbouring conductors, where the orthogonality principle (as described in chapter 2) needs to be applied. This is demonstrated in more detail in the following example.

5.2.1.2 Example

The power delivered by the source, and hence the resistance as seen by it, is largely dependent on the presence of neighbouring conductors. In order to clarify this, the resistance of the multi-layer configuration depicted in Figure 5.2 will be determined. Conductor 1 will be taken to have n_1 layers and carry current, I_1 , while conductor 2 will have n_2 layers and carries current, I_2 . It is important to take note that I_1 and I_2 are supplied by two independent sources, S_1 and S_2 , with no dependence between them. I_1 and I_2 are defined in terms of peak values.

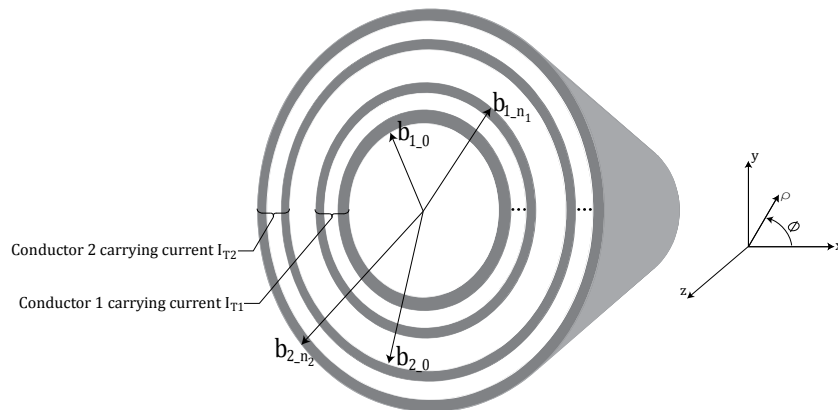


Figure 5.2. Multi-layer cylindrical conductor used in the resistance determination

Power determination and allocation

In order to determine the power delivered by each source, the current density (in each layer) needs to be broken up into its principle components (if it consists of more than one) and allocated to the source which created it. This will be clarified as the example progresses. Starting with the current densities in the layers of the inner conductor, it is clear that they are solely due to S_1 , as no magnetic field due to I_2 will be present here (as can be shown from Ampere's law). It therefore follows that S_1 is the sole provider of the inner conductor's losses. These losses can be determined on a per-layer basis and are given by,

$$\frac{P_{skin_cond1_layerk}}{l} = \frac{\pi}{\sigma_{cond1_layerk}} \int_{b_{1,k-1}}^{b_{1,k}} |J_{skin_cond1_layerk}(\rho)|^2 \rho d\rho \quad (8)$$

where the subscript 'skin' refers to the current density being due to the skin effect, 'cond1' to the layer being part of conductor 1, and 'layerk' being the layer number k of the n_1 layers. These current densities will be expressed in terms of peak values, since I_{inner} (which is used in their derivation) is defined as a peak value.

Shifting attention to conductor 2, it can be realised that the current density will consist of two superimposed components. It will firstly consist of the skin-effect component, which is the current density in the absence of conductor 1's magnetic field and will be designated by the subscript, 'skin_cond2'. This current and its associated losses are obviously supplied by source, S_2 . The second component will be due to the proximity effect, resulting from conductor 1's magnetic field. This will be designated with the subscript 'prox_cond2'. Since I_1 is responsible for this second component, it follows that its associated losses are supplied by S_1 . This separation of power components is allowed from the orthogonality principle, as discussed in chapter 2. The individual power components in the outer conductor are therefore,

$$\frac{P_{prox_cond2_layerk}}{l} = \frac{\pi}{\sigma} \int_{b_{2,k-1}}^{b_{2,k}} |J_{prox_cond2_layerk}(\rho)|^2 \rho d\rho \quad (9)$$

$$\frac{P_{skin_cond2_layerk}}{l} = \frac{\pi}{\sigma_{cond2_layerk}} \int_{b_{2,k-1}}^{b_{2,k}} |J_{skin_cond2_layerk}(\rho)|^2 \rho d\rho \quad (10)$$

The total power supplied by source, S_1 , is consequently the summation of all the skin-effect and proximity-effect losses associated with it,

$$\frac{P_{S_1}}{l} = \sum_{k=1}^{n_1} P_{skin_cond1_layerk} + \sum_{k=1}^{n_2} P_{prox_cond2_layerk} \quad (11)$$

Since conductor 2's current only resulted in skin-effect losses, the total power supplied by S_2 is,

$$\frac{P_{S_2}}{l} = \sum_{k=1}^{n_2} P_{skin_cond2_layerk} \quad (12)$$

Resistance as seen by source 1

With the power delivered by each source now known, the resistance (as seen by the source) can be determined. Utilising equation (7) the resistance (per unit length) is given by,

$$\frac{R_{S_1}}{l} = \frac{P_{S_1}}{l} \frac{2}{|I_1|^2} \quad (13)$$

where I_1 is again the total peak current flowing through conductor 1 and its source.

Resistance as seen by source 2

The resistance, as seen by source, S_2 , is obtained in a similar manner,

$$\frac{R_{S_2}}{l} = \frac{P_{S_2}}{l} \frac{2}{|I_2|^2} \quad (14)$$

where I_2 is the total peak current flowing through conductor 2 and its source. It is important to realise that proximity-effect current densities (such as $J_{\text{prox_cond2_layerk}}$) will not produce any resultant currents (as discussed in chapter 2) and will therefore not adjust the total current flowing within a conductor.

5.2.2 Inductance

As is known, the total inductance of any conductive configuration is comprised of an internal and external component. Under most circumstances, the internal component is neglected in the face of the larger external inductance. However, considering the dimension involved in the flat and cylindrical differential configurations, the internal inductance will constitute a significant portion of the total inductance, and can therefore not be neglected. As the predicted distributions are only defined within the conductor, it follows that the internal and external inductances will be determined separately and added at the end.

5.2.2.1 Internal Inductance

Considering the complexity of the multi-layer conductors and the dependence of the current density and magnetic field on distance, it was decided to determine the internal inductance from an energy perspective. This energy is given by equation (15), where the magnetic field is determined from chapter 4. Since only the internal inductance is under consideration, the integral needs to be confined to the volume of the conductor. Take note that this equation only holds for linear magnetic materials in which the magnetic flux density and magnetic field intensity are related by a constant μ . Equation (16) shows how this energy is related to the inductance of a current carrying conductor.

$$W = \frac{1}{2} \iiint_{\text{vol}} \vec{B} \cdot \vec{H} dv \quad (15)$$

$$W = \frac{1}{2}LI^2 \quad (16)$$

It must be emphasised that equation (15) only holds in the time domain. This can be better understood by realising that the energy is given by the distribution of the magnetic flux density and magnetic field intensity over a given volume. If either of their distributions or intensities varies with time, it follows that the energy contained in the field will also be time dependent. A similar argument holds for the current, I , in equation (16), in the sense that the energy contained in the inductor will vary as I varies.

Since the magnetic field is defined on a per-layer basis (for a multi-layer conductor), the energy will have to be determined on a per-layer basis as well, with the total energy given by the eventual summation. From chapters 2 and 4, the distribution was derived in the frequency domain, with the time domain representation given by the real part of the product of its phasor and $e^{j\omega t}$. With the frequency domain representation therefore given by $H(\xi)$, where ξ can either be y or ρ for the flat or cylindrical conductors, the magnetic field intensity can be written as a function of time,

$$\begin{aligned} H(\xi, t) &= \text{Re}[H(\xi)e^{j\omega t}] \\ &= \text{Re}[|H(\xi)|e^{j\theta(\xi)}e^{j\omega t}] \\ &= |H(\xi)|\cos(\omega t + \theta(\xi)) \end{aligned} \quad (17)$$

where

$$|H(\xi)| = \sqrt{(\text{Re}[H(\xi)])^2 + (\text{Im}[H(\xi)])^2} \quad (18)$$

and

$$\theta(y) = \tan^{-1} \frac{\text{Im}[H(\xi)]}{\text{Re}[H(\xi)]} \quad (19)$$

In order for equation (17) to hold, $H(\xi)$ must obviously be defined in terms of peak value. As can be seen, the magnetic field is phase shifted along the conductor and significantly complicates this derivation. From the linearity constraint, $\vec{B} = \mu\vec{H}$, equation (15) can be written as,

$$W_k(t) = \frac{\mu}{2} \iiint_{vol} H_k(\xi, t)^2 dv \quad (20)$$

where the subscript k refers to the energy and magnetic field intensity of layer k , with the integral taken over the volume of that layer. Substituting (17) into the above equation,

$$\begin{aligned}
 W_k(t) &= \frac{\mu_k}{2} \iiint_{vol} [H_k(\xi) \cos(\omega t + \theta_k(\xi))]^2 dv \\
 &= \frac{\mu_k}{2} \iiint_{vol} |H_k(\xi)|^2 \cos^2(\omega t + \theta_k(\xi)) dv
 \end{aligned} \tag{21}$$

The above equation now gives the energy stored in a single layer's magnetic field at a given time. The total energy stored within a multi-layer conductor is therefore just the summation of the energy within each layer. This total internal energy will be designated by $W_{int_T}(t)$. In order to determine the inductance from $W_{int_T}(t)$, it follows that (16) needs to be written as a function of time as well. With the current taken to be at reference phase (which is the condition under which $H(\xi)$ is derived), equation (16) becomes,

$$W_{int_T}(t) = \frac{1}{2} L I(t)^2 \tag{22}$$

where

$$I(t) = I_{peak} \cos(\omega t) \tag{23}$$

Since the current flowing through the single- or multi-layer conductor is known, I_{peak} and therefore $I(t)$ will also be known. Since equation (21) is the generic equation for the energy contained in each layer, it follows that each layer will have a similar form, with only μ_k , $H_k(\xi)$ and $\theta_k(\xi)$ differing between layers. Inspection of equations (21) and (23) reveals that one of the easiest ways to determine the inductance is to evaluate the equations at $t=0$. The integral term of (21) reduces to $[|H(\xi)| \cos(\theta(\xi))]^2$, which is the square of the real part of the phasor $H(\xi)$ (i.e. $\text{Re}[H(\xi)]^2$), while $I(t)$ reduces to I_{peak} . The inductance is then given by,

$$L = \frac{\sum_{k=1}^{k=n} \mu_k \iiint_{vol} [\text{Re}[H_k(\xi)]]^2 dv}{I_{peak}^2} \tag{24}$$

where the summation is taken over all n layers comprising the multi-layer conductor. It is important to realise that equation (24) will still hold even if the current is not taken to be at zero phase. In such a case, the denominator of (24) will change to $(I_{peak} \cos(\phi))^2$. Since the magnetic field of each layer will then be derived with this phase shifted current as one of its parameters, their respective phases will also adjust, resulting in (24) still being valid.

5.2.2.2 External Inductance

External inductance can either be determined from an energy perspective (similar to what was done above) or from the fundamental definition, $L = \Phi/I$. Irrespective of which one is used, the magnetic flux density or magnetic field intensity (which are just related by a constant) need to be known as a function of distance. Since the geometries of flat and cylindrical conductors obviously differ, it follows that their external magnetic field functions will also differ. The differential configurations given in Figure 5.3 and Figure 5.4 will be used in the determination of these functions.

Semi-infinite flat differential conductor configuration

The semi-infinite flat differential configuration of Figure 5.3 has been under significant investigation in both chapters 2 and 4 (both the single and multi-layer version). As is known, Ampere's law in integral form predicts that the magnetic field between the two conductors is constant and given by equation (25), while the field outside the conductors is zero.

$$H = \frac{I}{w} \quad (25)$$

where w is the width of the section under consideration and I the total current flowing within it. (I.e. since it is semi-infinite it can be stated to have a current to width ratio of I/w for the whole conductor). The derivation of the inductance will be approached from an energy perspective. In order to show equivalence, Appendix H obtains the same result when derived from a flux density perspective. Starting with the fundamental definition of energy contained in the magnetic field of a linear material,

$$W = \frac{1}{2} \iiint_{vol} \vec{B} \cdot \vec{H} dv \quad (26)$$

The derivation of the internal inductance (previous section) was complicated by the magnetic field being phase shifted along the conductor (where the current was taken to be at zero phase). This forced the derivation to be meticulously done in the time domain for each layer, taking care to take the phase shifts into account. The external inductance, fortunately, does not suffer from this problem. As can be seen from equation (25), the external magnetic field is completely in phase with the current which created it. This allows for a much easier derivation. Even though H and I are time dependent, this dependence will not be emphasised, since the same result can be obtained with H and I being constant in time. For a linear, magnetic material (26) can be written as,

$$W = \frac{\mu}{2} \iiint_{vol} H^2 dv \quad (27)$$

Substituting (25) into (27),

$$\begin{aligned} W &= \frac{\mu}{2} \int_0^w \int_0^h \int_0^l \left(\frac{I}{w} \right)^2 dx dy dz \\ W &= \frac{\mu}{2} \frac{lhI^2}{w} \end{aligned} \quad (28)$$

Equating this energy to that contained in an inductor,

$$\begin{aligned}\frac{1}{2}LI^2 &= \frac{\mu}{2} \frac{hlI^2}{w} \\ \therefore L &= \frac{\mu lh}{w}\end{aligned}\quad (29)$$

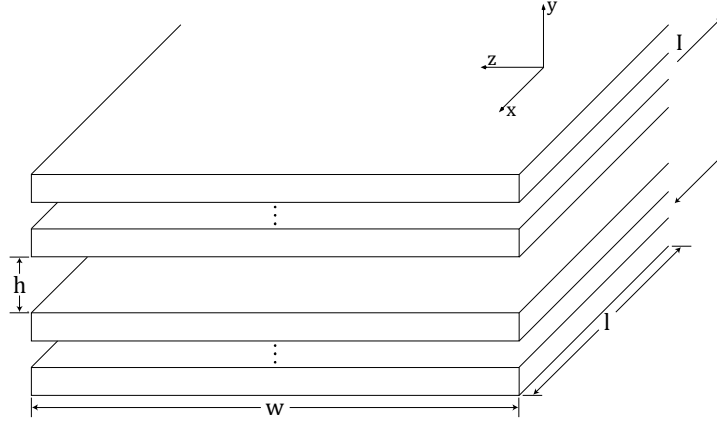


Figure 5.3. Multi-layer flat differential configuration

Differential cylindrical configuration

The derivation of the external inductance will be based on the cylindrical differential configuration given in Figure 5.4. This configuration has also been investigated in the previous chapters, with Ampere's law revealing that no magnetic field will be present for $\rho > b_{2,n2}$ and that the magnetic field intensity responsible for the external inductance is given by,

$$H = \frac{I}{2\pi\rho} \quad (30)$$

for

$$b_{1,n1} \leq \rho \leq b_{2,0}$$

Where I is the differential current carried by the inner and outer conductors. The inductance will again be determined from an energy perspective, with Appendix H giving the derivation from a flux perspective. For a linear material, the energy contained in the magnetic field can be written as,

$$W = \frac{\mu}{2} \iiint_{vol} H^2 dv \quad (31)$$

Substitution of equation (30) into (31) produces,

$$\begin{aligned}W &= \frac{\mu}{2} \int_{b_{1,n1}}^{b_{2,0}} \int_0^l \int_0^{2\pi} \left(\frac{I}{2\pi\rho} \right)^2 \rho d\phi dl d\rho \\ &= \frac{l\mu I^2}{4\pi} \ln \left| \frac{b_{2,0}}{b_{1,n1}} \right|\end{aligned}\quad (32)$$

Equating this energy to that contained in an inductor,

$$\begin{aligned} \frac{1}{2}LI^2 &= \frac{l\mu I^2}{4\pi} \ln \left| \frac{b_{2,0}}{b_{1,n_1}} \right| \\ \therefore L &= \frac{l\mu}{2\pi} \ln \left| \frac{b_{2,0}}{b_{1,n_1}} \right| \end{aligned} \quad (33)$$

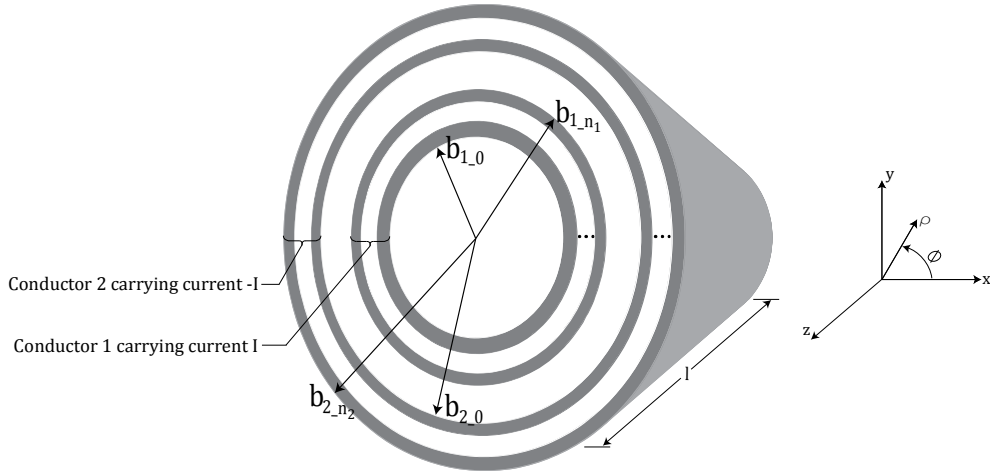


Figure 5.4. Multi-layer cylindrical differential configuration

5.2.3 Capacitance

As was previously discussed, the capacitance is only determined to ensure that its value is sufficiently small such that its reactance can be taken as infinite. Therefore, for a given capacitance, a maximum frequency can be determined that is sufficiently below the resonant point in order to maintain the 1D conditions. The experimental verification of the next section is limited to this condition.

The capacitance is significantly easier to determine than the inductance of the previous sections and can be derived from Gauss' law and can be found in [1] and [2]. With reference to Figure 5.3, the capacitance for the semi-infinite flat conductor configuration is given by,

$$C = \frac{\epsilon w l}{h} \quad (34)$$

where ϵ is the permittivity of the dielectric spacer separating the two multi-layer conductors. The capacitance of the coaxial configuration depicted in Figure 5.4 is,

$$C = \frac{\epsilon 2\pi l}{\ln \left| \frac{b_{0,Out}}{b_{2,In}} \right|} \quad (35)$$

It is important to note that capacitance will exist between conductors which are at different potentials. Therefore, for a hypothetical configuration that contains multiple differential currents, fed from multiple sources, and running through multiple multi-layer conductors, the capacitance needs to be determined between each pair of conductors at different potentials.

5.3 Experimental verification

As was mentioned before, experimental verification of the impedance equations is done for two reasons. Firstly, to confirm the correctness of the impedance equations for future use, and secondly, the indirect physical verification of the multi-layer methodology from which they stem. It follows that, since both the flat- and cylindrical multi-layer equations stem from the same multi-layer methodology, adequate verification of the one will automatically infer the verification of the other. Therefore, as constraints on the availability of cylindrical conductors were experienced, it was decided to focus the verification solely on the flat multi-layer conductors.

5.3.1 Experimental setup

Before commencing with the measurements, some information with regards to the equipment, calibration and material selection needs to be given:

1. Impedance measurement by use of an LCR meter:

The impedance measurement of the differential configurations will be performed by use of an Agilent E4980A Precision LCR Meter (shown in Figure 5.5). This instrument allows for the measurement of the complex impedance between 20 Hz and 2 MHz. The exact accuracy of the device is dependent on the frequency under consideration [3], but it was found that no reasonable measurements can be taken for impedances below 0.5 m Ω to 1 m Ω . Furthermore, for the configurations under consideration (discussed in the next section), it was found that for very low frequencies, where the load is mostly resistive and the phase angle very small, the reactance (and subsequent inductance) readings were very sporadic. This was contributed to the inherent angle discrimination limitation of the LCR meter, as a small change in angle will have a profound influence on an already small reactance. The configurations to be measured were, therefore, designed to have an expected impedance well above this minimum value.

As a final note, the LCR meter is able to store 201 points during one frequency sweep, with the points either linearly or logarithmically distributed between 20 Hz and 2 MHz. For the linear sweep, this results in the measurement points being separated by 9999.9 Hz, which gives an even distribution of information over the frequency range. However, if more low-frequency information is required, a logarithmic sweep would be better suited, as the measurement frequencies are close together at low and far apart at high frequencies. If neither of these will suffice, the option also exists to select a frequency section and allocate the measurement points to desired frequencies. As the linear and logarithmic sweeps were sufficient, this was not done here.



Figure 5.5. Agilent A4980A Precision LCR Meter

2. Perform calibration:

The Agilent E4980A Precision LCR Meter is equipped with a calibration option. This allows the impedance of the connection conductors, which are used to connect the device under test (DUT) to the LCR meter, to be measured and stored for the range of frequencies under consideration (shown in Figure 5.6). This is used as the baseline for the actual measurement of the test samples (DUT).

Particular care must be exercised when doing this measurement, as the contact resistance between the connecting conductors and the input clamps of the LCR meter is very sensitive to any adjustments. The most consistent calibration method was found to comprise of inserting the connecting conductors into the LCR meter's input clamps and then shorting out the DUT at the end of its connecting conductors (by use of a conductive strip and spring loaded clips). This ensures the calibration measurement of the connecting conductors only. Upon completion of the calibration measurement (before the removal of the shorting conductor), it was found that the LCR meter measures approximately $0.5\text{ m}\Omega$ to $1\text{ m}\Omega$ (the exact value varies between measurements and also depends on the frequency under consideration). As the ideal value would be $0\text{ }\Omega$, it follows that the smallest measurable impedance is therefore between $0.5\text{ m}\Omega$ to $1\text{ m}\Omega$ (as was stated in point 1 above). The shorting conductor is then finally removed, taking care not to adjust the connection between the connecting conductors and the input clamps.

Even though this calibration procedure proved to be the most consistent and allowed for a repeatable experimental setup, some unavoidable errors still remain. Firstly, the contact resistance between the connecting conductors and the input clamps of the LCR meter is very sensitive to any movement. In the removal of the shorting strip at the end of the calibration process, some unavoidable movement occurs, which tends to adjust the impedance from that stated above. As the shorting conductor is removed in this process, it is not possible to measure this adjusted impedance. Secondly, even though the shorting strip is placed as close to the DUT as possible, the entire connecting conductor is not shorted out. This leads to some unwanted, and unaccounted for,

impedance in the final measurement. This impedance will be designated as unaccounted for contact impedance.

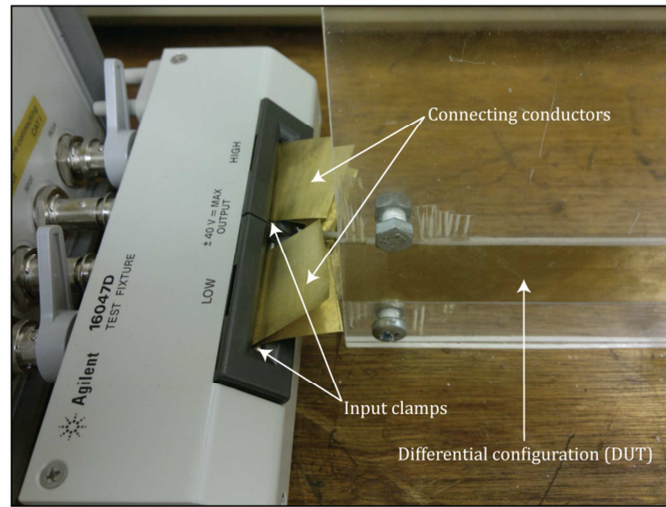


Figure 5.6. Connecting conductors and input clamps

3. Equivalent circuit representation:

As was discussed in section 5.2, the differential mode configuration can be simplified to the transmission-line model of Figure 5.1. For this experimental setup the conductance and capacitance are small enough such that their impedances to ground are significantly larger than that of the RL components (for the frequencies under consideration). This permits for the disregard of the conductance and capacitance and allows the model to reduce to a series resistance and inductance. The measured impedance (by use of the LCR meter) will therefore be modelled upon this.

4. Determine the conductivity of the conductive media:

Since all the available conductors were bought off the shelf, their exact conductivities vary from their ideal values and need to be determined experimentally. This can either be obtained through the use of the LCR meter, which confines the conductors to a differential mode configuration and prevents the characterisation of individual sections, or through a four terminal measurement, which is chosen here.

The four terminal method consists of applying a known DC current through the conductor and measuring the voltage difference over a given length. Care must be taken to ensure that the current density distribution is constant through the length over which the voltage is measured. The conductivity can then be obtained from equation (36), where l is the length, A the cross-sectional area and R the DC resistance.

$$\sigma = \frac{l}{RA} \quad (36)$$

5. Selection of conductive media:

Conductive sheets of varying thicknesses of copper, brass and stainless steel were initially available for the construction of the multi-layer conductors. It was, however, found that due to the copper's high conductivity and the LCR meter's minimum impedance measurement limitation, that any conductor constructed from this material presented too small a resistance to be measured successfully. Even though the stainless steel's conductivity is well within the measureable range, it was found to exhibit frequency dependent magnetic properties, which also complicates the measurements significantly. The only conductor devoid of both these problems is the brass, and was therefore chosen for the construction of the multi-layer conductors. As will be seen in sections 5.3.2.1 to 5.3.2.3, brass layers of varying thicknesses, along with dielectric layers, will be used to construct a reference differential and two three-layer differential configurations.

6. Experimental setup:

As was discussed in section 5.2, the differential configuration will be used for the verification of the impedance equations. The experimental implementation is shown in Figure 5.7, where it can be seen to consist of the connecting conductors, used to connect the configuration to the LCR meter, and the differential configuration itself. The differential configuration is made up of two multi-layer conductors (the amount of layers is discussed in the next section), separated by a dielectric spacer, which are shorted together at their ends. Two Perspex layers are situated outside the configuration and are used, along with bolts spaced 10 cm apart, to keep the conductors aligned and the space between the conductors as constant as possible.

Even though utmost care was taken in the construction, a number of unavoidable factors are, however, present that may influence the measurements:

- The brass layers used in the construction were found to exhibit some tolerances over their lengths. The values given in Table 5.1 are therefore average values.
- As the brass layers were cut by hand, slight variations in the width occur over their length.
- From the tendency of the Perspex to bend under the pressure of the applied bolts, the space between the two conductors may therefore increase above the thickness of the dielectric spacer. In addition, this space may not be constant, as the bolts are finite distances apart. This space between the two conductors, which will consist of the dielectric material and a possible airgap (depending on the deformation of the Perspex), will in future be referred to as the dielectric spacer.

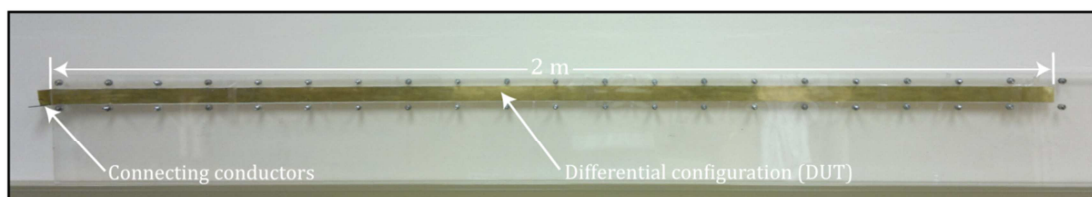


Figure 5.7. Experimental setup

7. Perform reference measurement:

Before commencing with the measurement of multi-layer conductors, a reference measurement should be performed. This reference setup will consist of the simplest possible, theoretically predictable configuration and will be used to confirm initial agreement between the measured and theoretical results. This is crucial to establish whether any unforeseen effects may plague this experimental setup and hence affect the multi-layer measurements.

5.3.2 Flat multi-layer conductor configurations and measurement results

From the previous section it is known that, due to the lack of a larger range of conductive materials and the inherent limitations of the LCR meter, construction of the multi-layer conductors will be limited to the materials given in Table 5.1. A reference differential-mode configuration will first be constructed and measured, which will be followed by the construction and measurement of two three-layer differential-mode configurations. Upon the completion of each measurement, a thorough comparison will be made against the theoretical results, highlighting and discussing any discrepancies that might occur.

As will be seen, the exact placement and thickness of each layer has a profound influence on the impedance-versus-frequency characteristics. The multi-layer conductors' compositions were therefore carefully chosen to emphasise a change with frequency in order to assist with the verification.

Table 5.1. Selected layers for the construction of the flat multi-layer differential configurations

Material	Brass (B1)	Brass (B2)	Nomex
Conductivity	13.1×10^6	14.01×10^6	0
Relative Permittivity	1	1	2.7 at 60 Hz
Thickness	100 μm	55 μm	250 μm
Width	30 mm	30 mm	30 mm
Length	2 m	2 m	2 m

5.3.2.1 Single-layer differential-mode configuration: Reference configuration

The reference configuration is depicted in Figure 5.8. As can be seen, it consists of two 200 μm brass conductors (made up of two 100 μm brass layers each), each of which forms a single conductor of the differential-mode configuration. As is obvious, the current is sinusoidal in time, with the arrows indicating the relative orientation of the current in each conductor.

The reason for using two 200 μm brass conductors, instead of two 100 μm brass conductors, is to help accentuate the change in impedance with frequency. It is important to realise that the impedance will only start to experience a significant change with frequency as soon as the current density distribution (and associated with that, the magnetic field) experiences a significant change from its DC distribution. Due to the skin-depth being an inverse exponential

function of frequency (chapters 2, 3, 4), the current density is reasonably constant with an increase in frequency, up to the point where the skin-depth equals the thickness of the conductors (from now on known as the skin-depth frequency). The skin-depth of 100 μm brass is reached at 1.93 MHz, with the result that the impedance will be reasonably constant for most of the LCR meter's measureable range. The thickness was therefore increased to 200 μm , which is equal to the skin depth at 483 kHz, resulting in a far more frequency dependent impedance (for the measureable range).

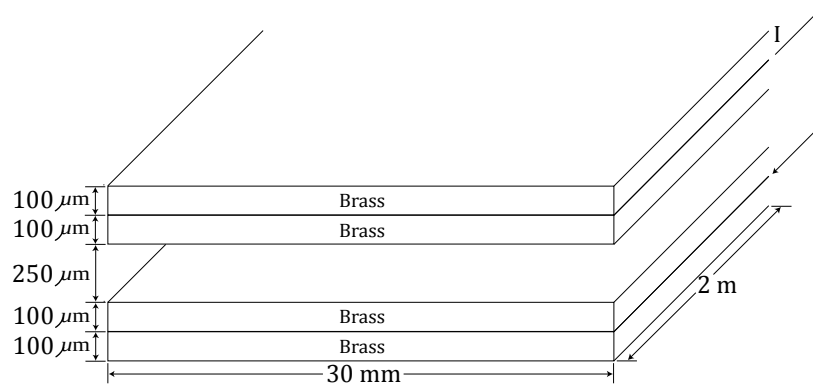


Figure 5.8. Reference configuration

From point 4 in the previous section, the assumption was made that the capacitive reactance to ground was large enough, in comparison to the resistive and inductive impedances, to be taken as infinite. Using equation (34) the capacitance is calculated to be 5.74 nF, which translates to 13.86 Ω at 2 MHz. As can be seen from Figure 5.9 and Figure 5.11 the theoretical resistive and inductive impedance values are 0.099 Ω and 0.397 Ω at 2 MHz respectively, which are significantly smaller than that of the capacitive reactance.

A final criterion that needs to be satisfied is that of the semi-infinite approximation. From chapter 2, the minimum thickness to width ratio was chosen to be 1:25. The ratio of this configuration is 1:46, well above the minimum ratio, and should therefore prevent the edge-effects from having a significant influence on the measured results.

Consider now the measured and theoretically determined resistance and inductance plots of Figure 5.9 and Figure 5.11. As can be seen, each figure consists of a linear and logarithmic plot. Both these plot types are included as they help to highlight different aspects of the same data, with both needed to obtain a clear and comprehensive understanding. As will become clear, the linear plot is more suited at emphasising high frequency discrepancies, while the logarithmic plot accentuates the frequency characteristics and gives a far more detailed representation of the low-frequency data.

Focussing first on the resistance results of Figure 5.9, it is clear that very good agreement is reached between measured and theoretical results, with only slight discrepancies at the low and high frequencies. Figure 5.10 plots the measured and theoretical resistance values for conductor thicknesses of plus and minus 5%. As can be seen, for the 5% increase in thickness, near perfect agreement is reached at the lower frequencies, while the 5% decrease results in excellent

agreement at the higher frequencies. With this information at hand, it can be argued that the actual thickness of the experimental conductor is closer to the minus 5% variant. This is because at low frequencies, the LCR meter's measurement limitation and unaccounted for contact impedance, is large enough, in comparison to the actual impedance, to induce a measurement discrepancy. While at high frequencies, the reactive inductance increases the impedance to such an extent that the contribution of the LCR meter's limitation and unaccounted for contact impedance becomes negligible. From these results and arguments, it can be stated that the single layer theoretical resistance equations are indeed correct.

As a final note, looking at Figure 5.9b, a corner frequency can be seen to exist at between 400 and 500 kHz, which marks the beginning of a steep increase of resistance with frequency. This change in resistance occurs due the skin-depth decreasing to below that of the conductor thickness, as was explained at the beginning of this section. This can be verified by calculating the skin-depth of 200 μm to be reached at 483 kHz, which is in agreement with Figure 5.9b. The corner frequency is therefore clearly dependent on the conductor's thickness, and is reached as soon as the skin-depth decreases below the conductor's thickness.

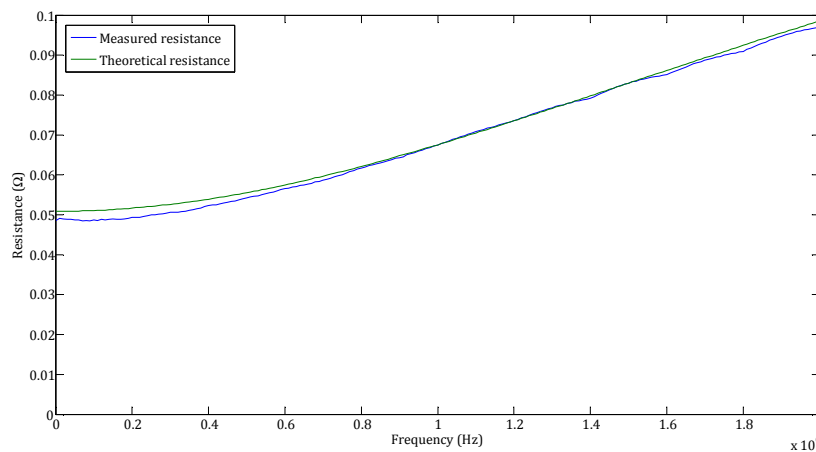


Figure 5.9a. Linear plot: Measured and theoretical resistance of the reference configuration of Figure 5.8

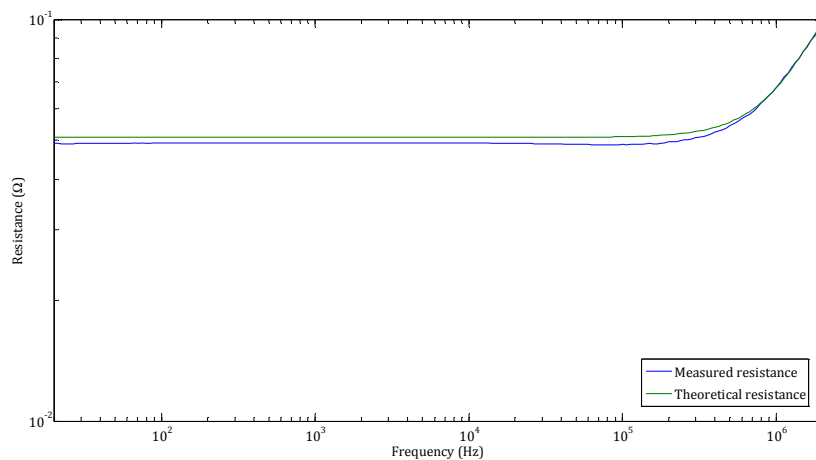


Figure 5.9b. Logarithmic plot: Measured and theoretical resistance of the reference configuration of Figure 5.8

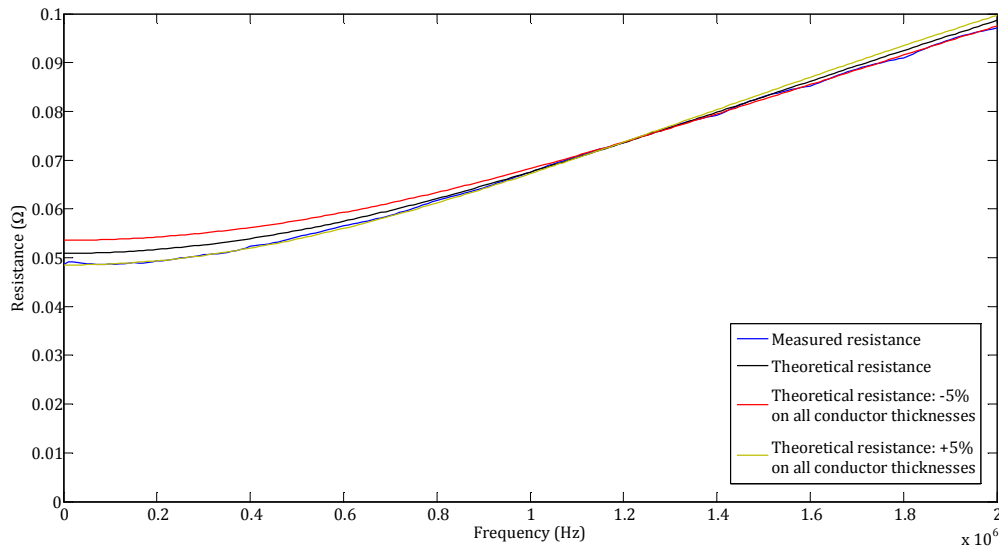


Figure 5.10. Linear plot: Measured and theoretical resistance, with the inclusion of theoretical tolerances, of the reference configuration of Figure 5.8

Moving on to the inductance, the measured and theoretical values are given in Figure 5.11a and b. The discrepancies between the measured and theoretical inductances can be broken up into three parts. Firstly, as can be seen, for the vast majority of both plots, the measured and theoretical inductances are out by a semi-constant. Secondly, below 50 kHz, both plots show a rapid decrease of inductance, while thirdly, sporadic behaviour is noted on the logarithmic plot for frequencies below 1 kHz.

Considering the third discrepancy first, it must be remembered from point 1 in the previous section, that the LCR meter has an inherent minimum angle measurement limitation, which is the reason for the sporadic inductance measurement values clearly shown in the logarithmic plot (below 1 kHz). The logarithmic plot highlights this shortcoming, in contrast to the linear plot, as it has a large number of sample points below 1 kHz, whereas the linear plot only has one (at 20 Hz).

It is, therefore, very tempting to solely allocate the sharp decrease of the measured inductance (below 50 kHz) to the minimum angle measurement limitation of the LCR meter (part 2 above). This is, however, not completely responsible for this variation, as at 50 kHz the impedance angle is already above 10° , well within the measureable range of the LCR meter. It is strongly suspected that this is due to a combination of the LCR meter's angle measurement limitation, along with unaccounted for contact impedance and the minimum measurable impedance of the LCR meter.

With these discrepancies now understood, attention can be shifted to the first disagreement, vis-à-vis the semi-constant difference between the measured and theoretical results. Figure 5.12 gives these results, along with theoretical variations on the conductor and dielectric spacer thicknesses. Consider first the discrepancies between the measured and ideal theoretical results. At a reasonably low frequency of 200 kHz, the difference between the measured and theoretical results is 2.92 nH, which translates to a difference in reactance of 3.66 mΩ, and a difference in impedance of 2.36 mΩ, where the resistance was taken to be 50 mΩ (from Figure

5.9). This impedance variation can easily be accounted for by the measurement limitations of the LCR meter's measurement accuracy and unaccounted for contact impedance. This, however, is not enough to account for the higher frequency variation, as at 1.64 MHz the reactance difference is 40.27 m Ω (even though the inductances only differ by 3.91 nH), resulting in an impedance difference of 38.77 m Ω (where the resistance was taken as 87.43 m Ω from Figure 5.9). From this it is clear that a fundamental difference in inductance exists between the measured and theoretical configurations. Since the measured and theoretical resistance had very good agreement for the entire frequency band (Figure 5.9) and since the inductance values differ by a constant value (which only slightly varies with frequency), it follows that the majority of this difference cannot be due to the internal inductance (which is very frequency dependant), but can only be due to the constant external inductance. This can be confirmed by the 10% and 20% increase in dielectric spacer plots of Figure 5.12, where the increase in spacer thickness shifts the entire theoretical plot by a constant value. The reason for this spacer variation is given in point 6 of the previous section. Even though this accounts for the vast majority of the discrepancy, it can be seen that even better agreement is reached by including slight variations of the conductor thickness as well (the black semi-dotted line in Figure 5.12). After taking these two variations into account, the last remaining discrepancies can be attributed to the LCR meter's measurement accuracy limitation and the unaccounted for contact impedance.

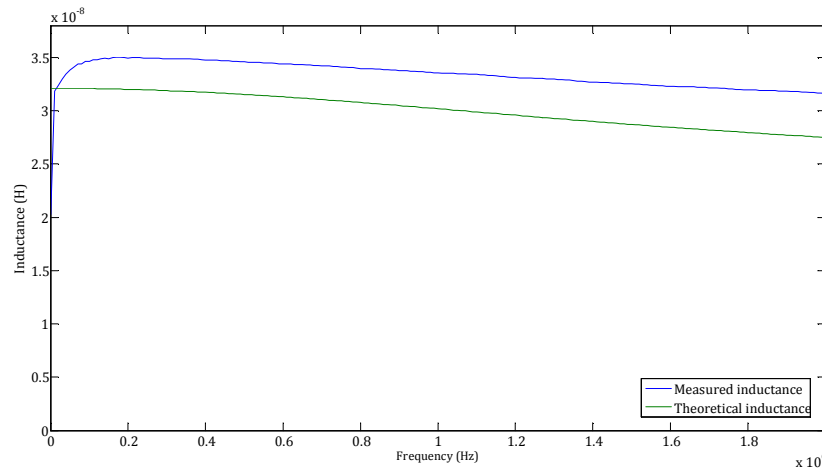


Figure 5.11a. Linear plot: Measured and theoretical inductance of the reference configuration of Figure 5.8

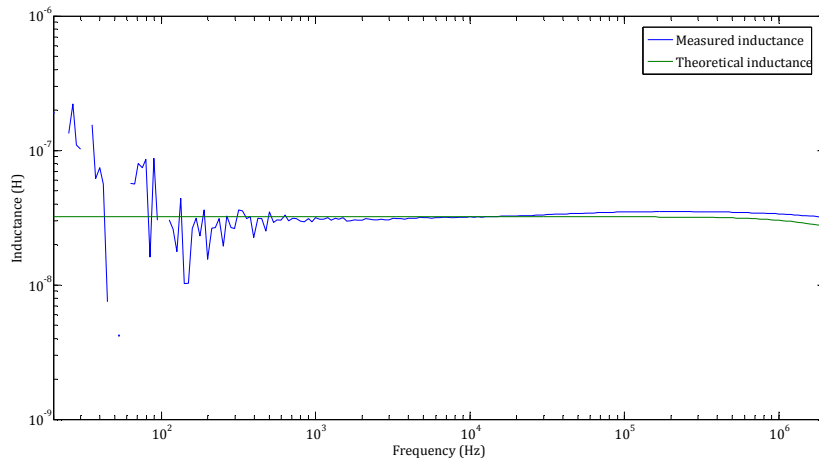


Figure 5.11b. Logarithmic plot: Measured and theoretical inductance of the the reference configuration of Figure 5.8

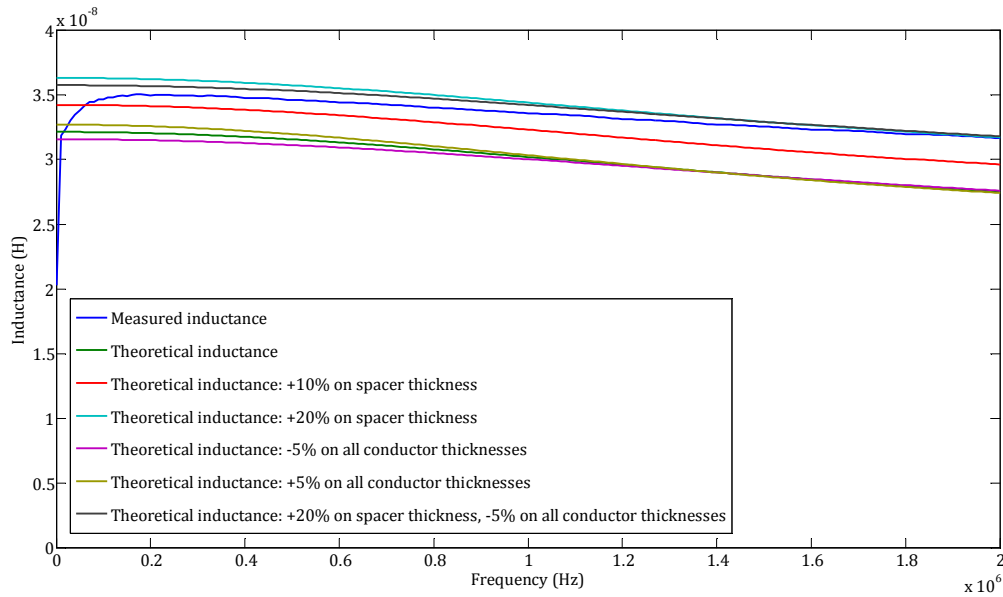


Figure 5.12. Linear plot: Measured and theoretical inductance, with the inclusion of theoretical tolerances, of the reference configuration of Figure 5.8

The linear and logarithmic impedance plots are given in Figure 5.13a and b. Even though the same conclusions as that reached earlier should be possible from a thorough scrutinisation of these plots, it is significantly easier to subject the impedance to the series RL model and approach the analysis from there (as was done above). This, however, does not mean that the impedance plots should not be considered at all. On the contrary, these plots can provide confirmation of what has already been obtained, while highlighting some aspects that are not that clear from the resistance and inductance plots alone.

The linear impedance magnitude plot confirms the effect of the semi-constant inductance discrepancy, which causes the divergence at higher frequencies as the frequency dependent reactance forms a larger component of the total impedance. This same discrepancy causes the measured impedance angle to be consistently larger than the theoretical angle. The logarithmic plots of Figure 5.13b tend to shift the focus from the smaller discrepancies to the overall frequency trends of the impedances, which can be seen to be identical. On the other hand, the logarithmic angle plot highlights the angle measurement limitation of the LCR meter for very small angles, which was first seen in the logarithmic inductance plot of Figure 5.11b.

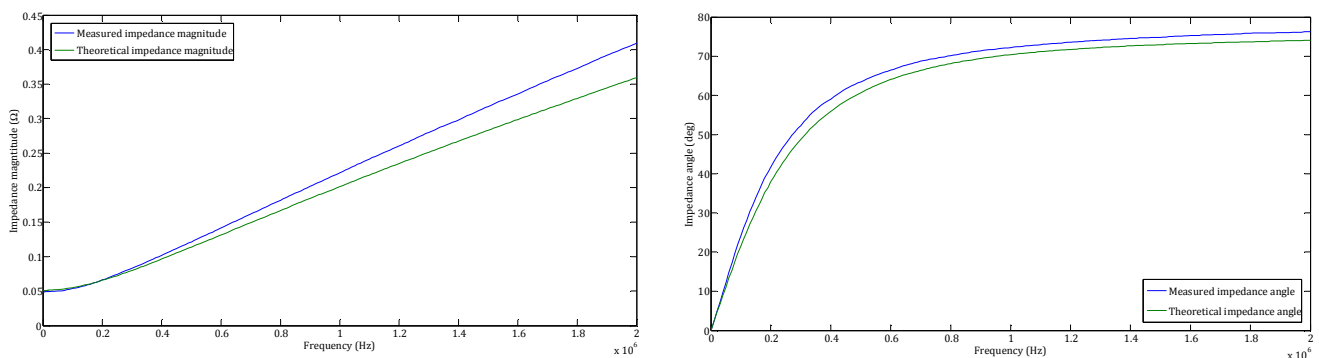


Figure 5.13a. Linear plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.8

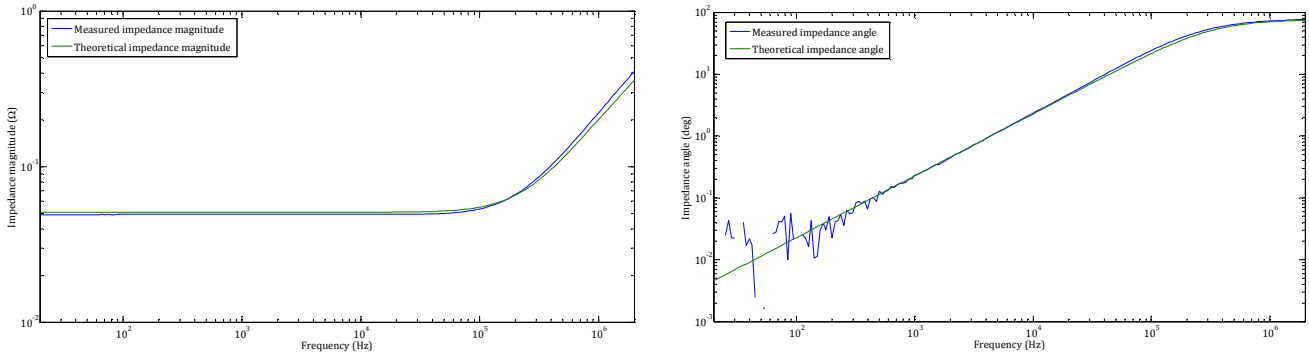


Figure 5.13b. Logarithmic plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.8

The agreement of the measured and theoretical results should, however, come as no surprise, as single-layer structures have been studied for a very long time. The verification of this reference configuration was, however, necessary, as it provided for a well verified baseline which allowed for the identification of all experimental discrepancies and their associated causes.

5.3.2.2 Three-layer differential-mode configuration: Configuration 1

The three-layer differential mode configuration, known as configuration 1, is given in Figure 5.14. Each three-layer conductor consists of an inner 100 μm brass conductor, a 250 μm Nomex dielectric and a 55 μm outer brass conductor. At first glance this may not seem like a three-layer conductor, but closer inspection reveals that all three layers carry current in the same direction, resulting in the necessity of the three-layer equations to solve the fields and current density distributions. The middle Nomex layer can therefore be thought of as a very low conductivity conductor. Two of these three-layer conductors are then used to form a differential-mode configuration, with the two three-layer conductors separated by a 250 μm Nomex dielectric spacer. Since this spacer is identical to that used in the previous reference configuration, it follows that the capacitance will be identical as well, allowing the capacitive reactance to be taken as infinite.

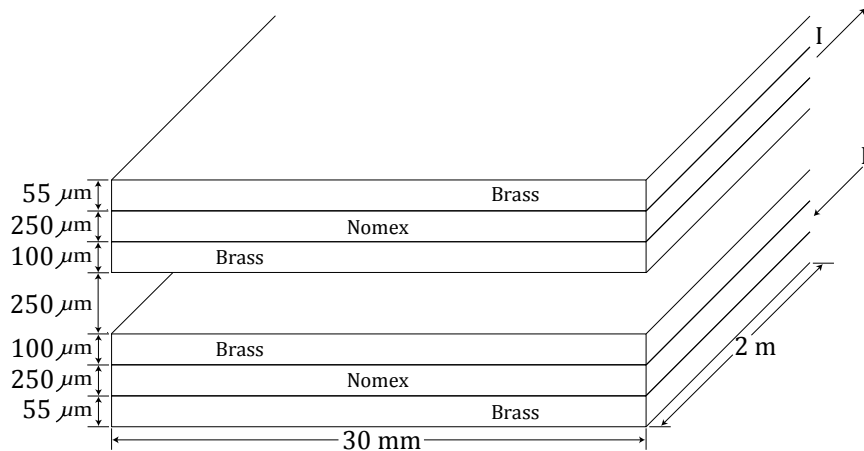


Figure 5.14. Configuration 1

Starting with the resistance given in Figure 5.15, it is clear that excellent agreement is reached for almost the entire frequency spectrum (20 Hz to 2 MHz), with only slight variation at the high frequencies. At 20 Hz the difference is less than 1 m Ω , while at 2 MHz the difference is 3.6 m Ω . With such a small general variation, it can be concluded that the majority of this difference is due to the measurement limitation of the LCR meter (which varies with frequency), with additional variations brought about by contact impedance and conductor and dielectric variations.

The logarithmic plot of Figure 5.15b helps to highlight the excellent agreement of the general trends of these two plots, with a prominent corner frequency at 650 kHz. It is interesting to note that the skin-depth, and hence the corner frequency, of an isolated 55 μm brass layer is calculated to be reached at 5.98 MHz, while for an isolated 100 μm brass layer it is reached at 1.93 MHz. If the middle Nomex layer was removed and each conductor was reduced to a 155 μm brass conductor, i.e. comprised of a 100 μm and 55 μm brass layers, the skin-depth frequency would become 785 kHz. This points to the ability to adjust the corner frequency through the correct selection of configuration, material and geometric properties.

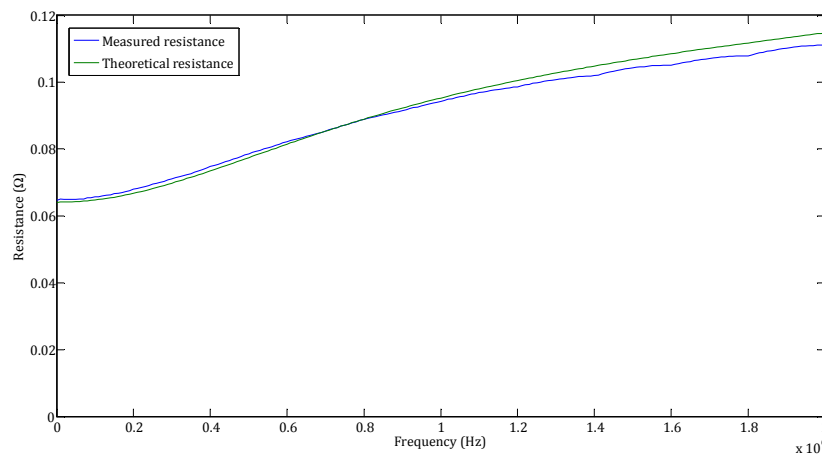


Figure 5.15a. Linear plot: Measured and theoretical resistance of the three-layer configuration of Figure 5.14

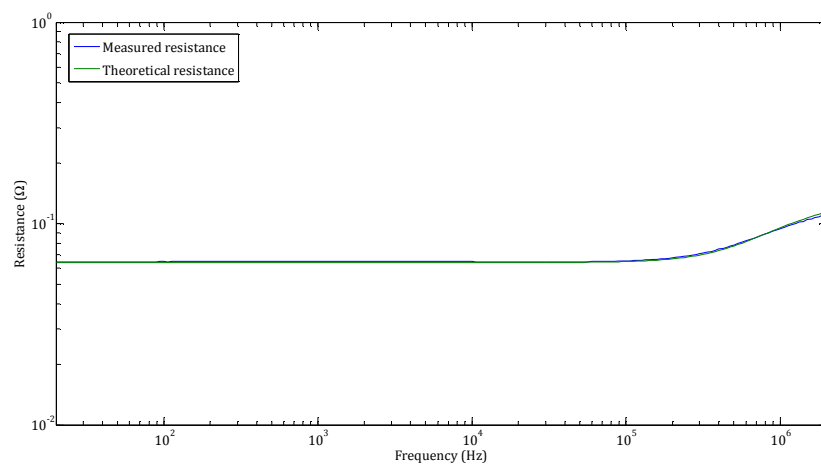


Figure 5.15b. Logarithmic plot: Measured and theoretical resistance of the three-layer configuration of Figure 5.14

Moving on to the inductance plots of Figure 5.16, it is clear that this configuration suffers from similar problems to the reference configuration. The logarithmic plot clearly highlights the minimum angle measurement limitation of the LCR meter, which is responsible for the sporadic inductance values below 1 kHz. This is also visible as the spike in the linear plot and is echoed in the impedance plots of Figure 5.18.

In both the linear and logarithmic plots, a sharp increase in inductance can clearly be seen between 1 kHz and 70 kHz. At 30 kHz, the measured inductance is 38.22 nH, which translates to a reactance of 7.2 m Ω and a total impedance of 65.3 m Ω (R was measured at 64.91 m Ω). With the inductance forming such a small part of the total impedance, it is clear that a small variation in measured angle (due to the minimum angle discrimination limitation of the LCR meter), along with unaccounted for contact impedance, can easily account for this discrepancy. This can be further substantiated by the fact that the measured inductance increases with frequency up to approximately 70 kHz, where it 'stabilises'. As the frequency increases, so does the reactance and impedance, with the result that the error inducing components become a smaller fraction of the total reactance and impedance.

Considering the frequencies above 70 kHz, it is clear that a near-constant difference exists between the measured and theoretical inductances. Similar to the reference configuration, the majority of this difference is due to variation on the main dielectric spacer separating the two three-layer structures. This is clearly shown in Figure 5.17, where the theoretical inductance is given for variations of the main dielectric spacer and conductors. As can be seen, near perfect agreement is reached for the case where the main spacer's thickness is increased by 25%, along with a thickness decrease of 5% for all the brass conductors. This increase in dielectric spacer thickness will, however, not affect the conclusion reached on the agreement of the resistances. This is because, from the semi-infinite approximation, the magnetic field between the two conductors is constant, irrespective of the spacer's thickness. The current density equations, and thus the resistance equations, are therefore not affected by this change.

It is important to realise that, for this three-layer case, a significantly larger deviation of the impedance is possible due to variations on each layer, as compared to the much simpler case of the reference configuration. i.e. due to tolerances on each layer, with the tolerances being independent of each other, a far larger number of variations from the ideal impedance are possible, as compared to the much simpler reference configuration. Therefore, even though it is the most likely, the excellent agreement of the last plot in Figure 5.17 is not necessarily the only possible explanation to the deviation of the measured from the ideal results.

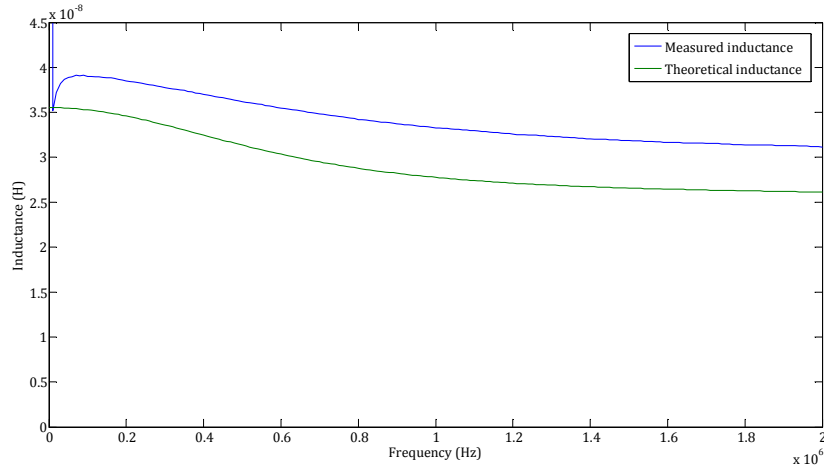


Figure 5.16a. Linear plot: Measured and theoretical inductance of the three-layer configuration of Figure 5.14

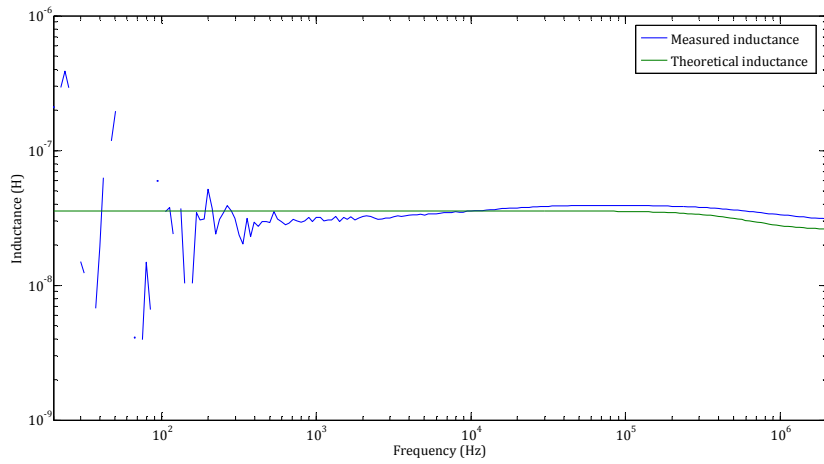


Figure 5.16b. Logarithmic plot: Measured and theoretical inductance of the three-layer configuration of Figure 5.14

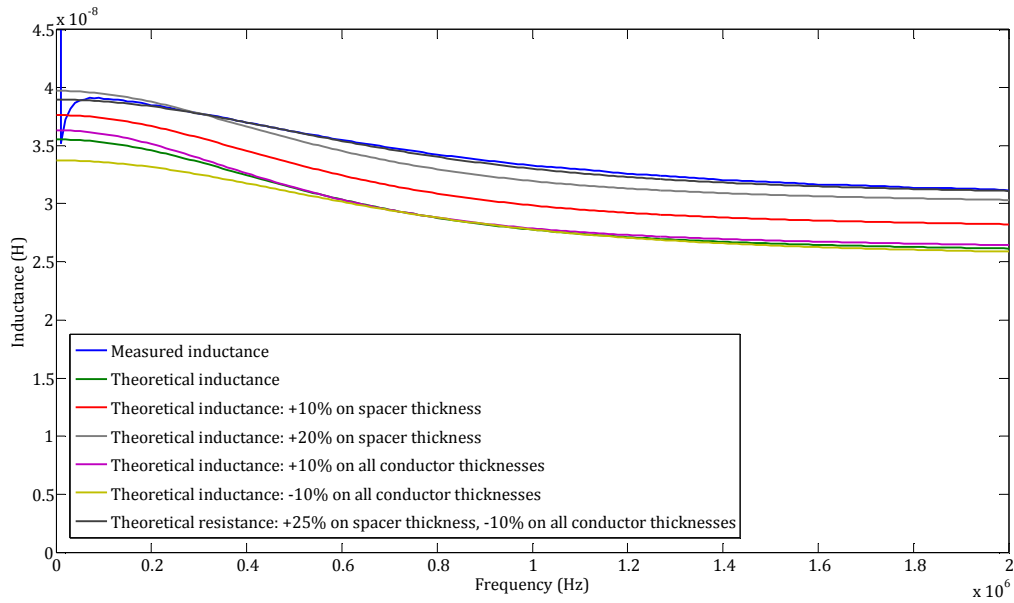


Figure 5.17. Linear plot: Measured and theoretical inductance, with the inclusion of theoretical tolerances, of the reference configuration of Figure 5.14

The impedance plots of Figure 5.18 echo the previous conclusions. The impedance magnitude plot shows how the constant difference between the measured and theoretical inductance results in the divergence of the impedance at higher frequencies, which also accounts for the difference between the measured and theoretical angles (best visible in the linear impedance angle plot). Lastly, the logarithmic impedance angle plot shows, again, the minimum angle measurement limitation of the LCR meter.

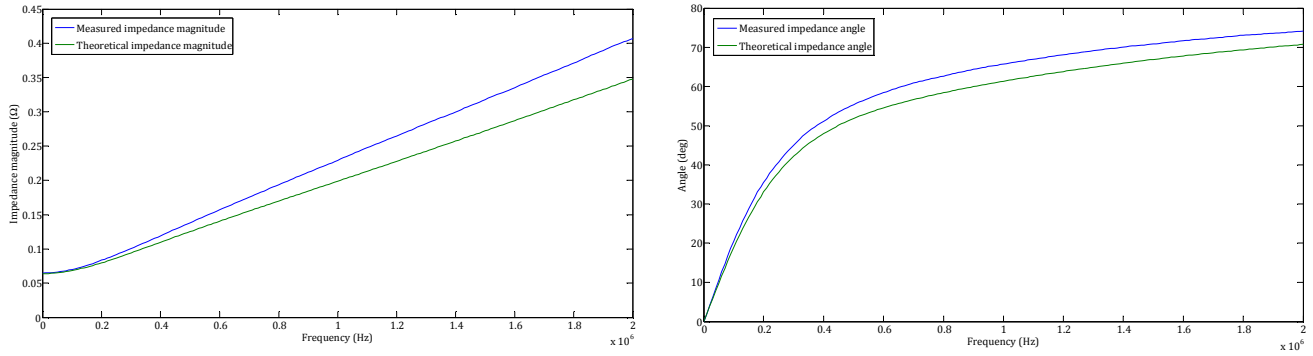


Figure 5.18a. Linear plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.14

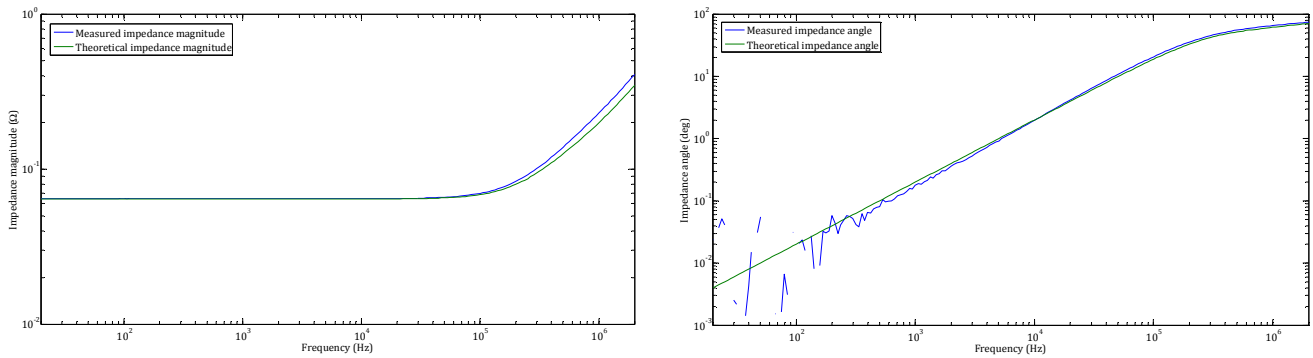


Figure 5.18b. Logarithmic plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.14

From these experimental results, it can be stated with significant confidence that the multi-layer impedance equations, and therefore the multi-layer distribution equations and the multi-layer methodology, are indeed correct. This will be further substantiated in the next section, where the impedance characteristics of a second three-layer differential mode configuration are experimentally verified.

5.3.2.3 Three-layer differential-mode configuration: Configuration 2

The second three-layer conductor is given in Figure 5.19. As can be seen, each three-layer conductor consists of an inner 55 μm brass conductor, a 250 μm Nomex dielectric and an outer 100 μm brass conductor. As before, the two three-layer conductors are separated by a 250 μm Nomex dielectric spacer.

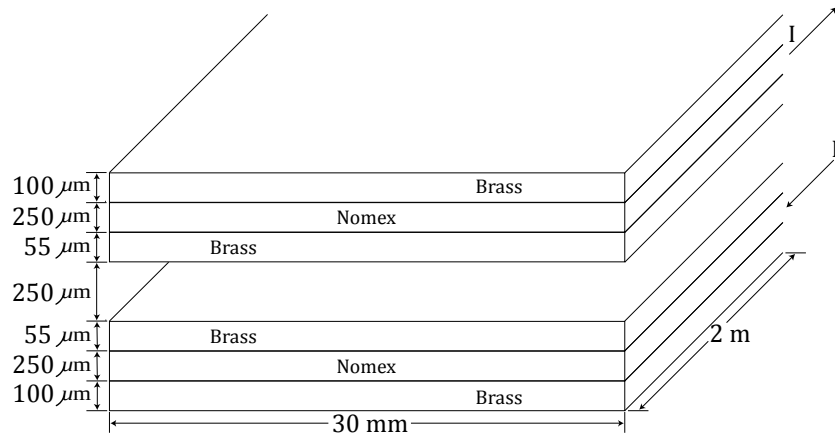


Figure 5.19. Configuration 2

Starting again with the resistance (Figure 5.20), it is clear that excellent agreement is reached across the majority of the measurable range, with only a slight discrepancy at the higher frequencies. In this case, the error will not be due to the minimum measurable angle or unaccounted for contact impedance. This is because the impedance at these higher frequencies is dominated by the inductive reactance which increases the impedance far above the sub 10 mΩ range (in which these error components lie). Figure 5.21 shows the resistance for a 5% variation (positive and negative) of the brass layers. In this case, all the conductors' thicknesses were either increased or decreased by 5%. As can be seen, they create an envelope in which the measured and ideal theoretical results lie. (Even though not shown here, a variation on the second dielectric layer will also have an influence on the impedance). From this it is clear that this discrepancy is due to some combination of variations on the layers' thicknesses.

From Figure 5.20b, the corner frequency can be seen to be situated at approximately 180 kHz. Closer inspection, however, suggests that a second 'corner frequency' exists at 1.2 MHz. This will be analysed and discussed in more detail in section 5.4.

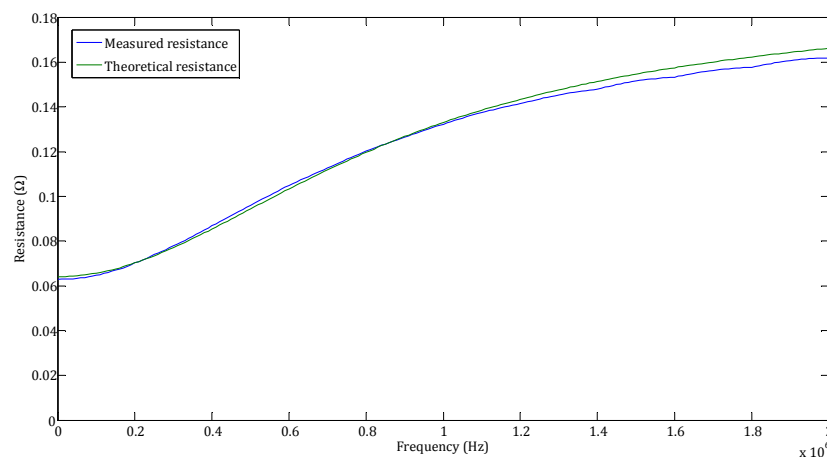


Figure 5.20a. Linear plot: Measured and theoretical resistance of the three-layer configuration of Figure 5.19

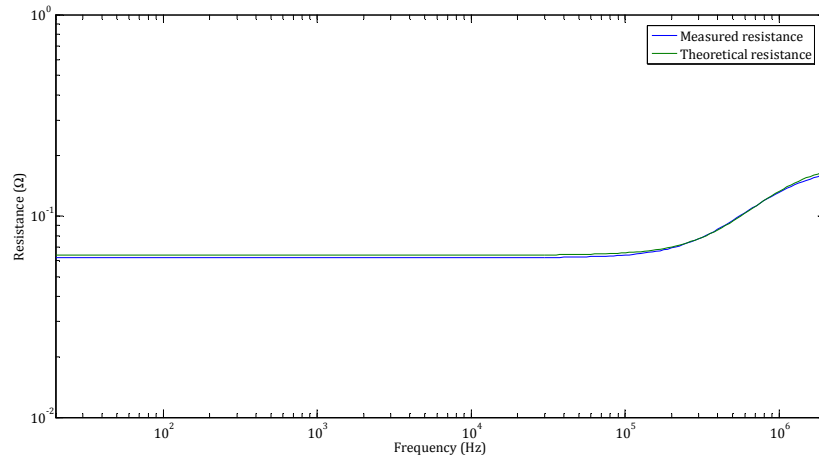


Figure 5.20b. Logarithmic plot: Measured and theoretical resistance of the three-layer configuration of Figure 5.19

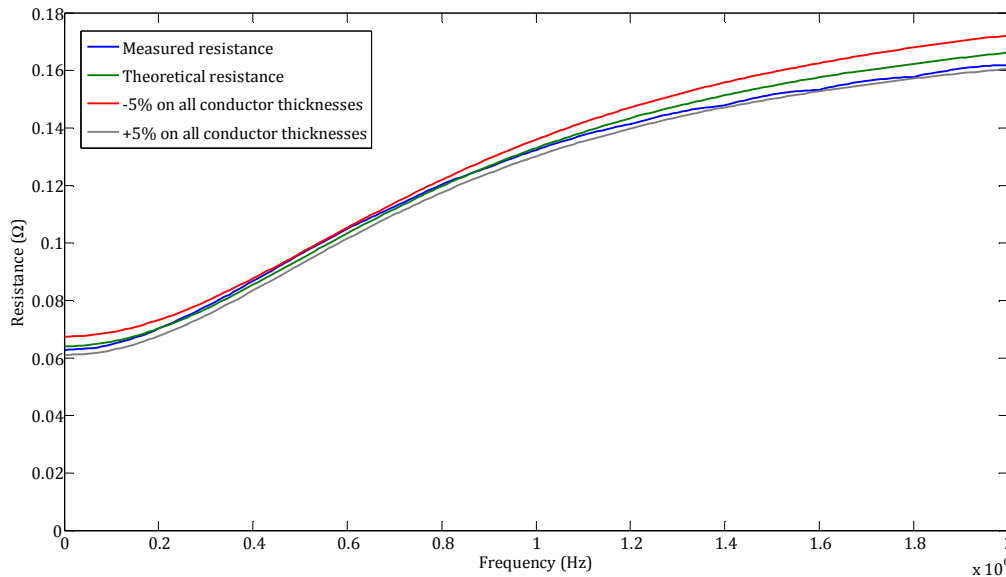


Figure 5.21. Linear plot: Measured and theoretical resistance, with the inclusion of theoretical tolerances, of the reference configuration of Figure 5.19

With the agreement of the measured and theoretical resistances now confirmed, attention can be shifted to the inductance plots of Figure 5.22. Again, the measured inductance seems to be suffering from the same problems that led to the discrepancies of configuration 1. The minimum angle measurement limitation of the LCR meter is again dominant for frequencies below 1 kHz (also shown by the impedance plots of Figure 5.24). The sharp increase in inductance between 0 and 70 kHz can also be attributed to a combination of the minimum angle discrimination limitation of the LCR meter and the unaccounted for contact impedance, since the inductive reactance at this frequency is significantly smaller than the resistance. Further substantiation can again be obtained by the increase of the inductance up to approximately 70 kHz. It again 'stabilises', as the inductive reactance starts to become a significant component of the total impedance and the error inducing components become negligible.

For frequencies above 70 kHz, a semi-constant difference between the measured and theoretical inductances exist. The majority of this difference is again due to variation on the main dielectric spacer separating the two three-layer structures, with the result that excellent agreement is

reached when the thickness of the spacer is increased by 25% and all the conductors undergo a 10% thickness decrease. The reason for the main spacer variation can again be attributed to the explanation given in point 6 of section 5.3.1.

Similar to the previous 3-layer configuration, this one possible variation of the thicknesses from their ideal values may not be the only possible explanation for the difference between the measured and ideal results. i.e. even though this 25% spacer increase and 10% conductor decrease resulted in good agreement with the measured results, it may not necessarily be the only combination of variations that lead to such an agreement.

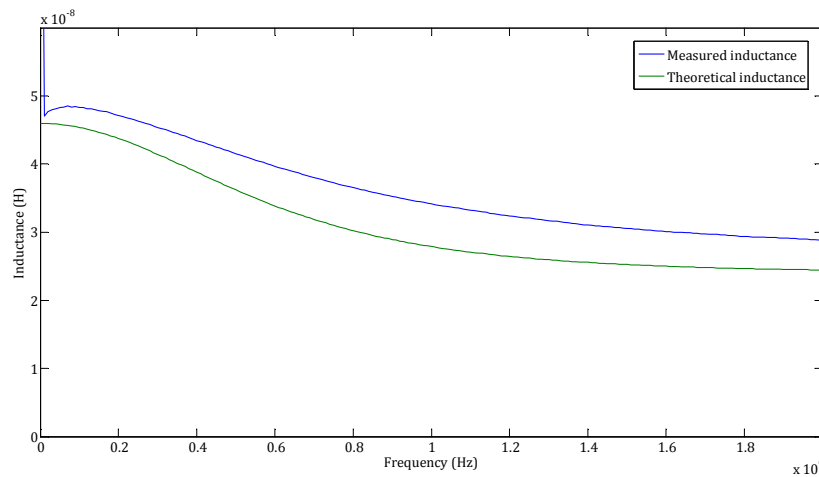


Figure 5.22a. Linear plot: Measured and theoretical inductance of the three-layer configuration of Figure 5.19

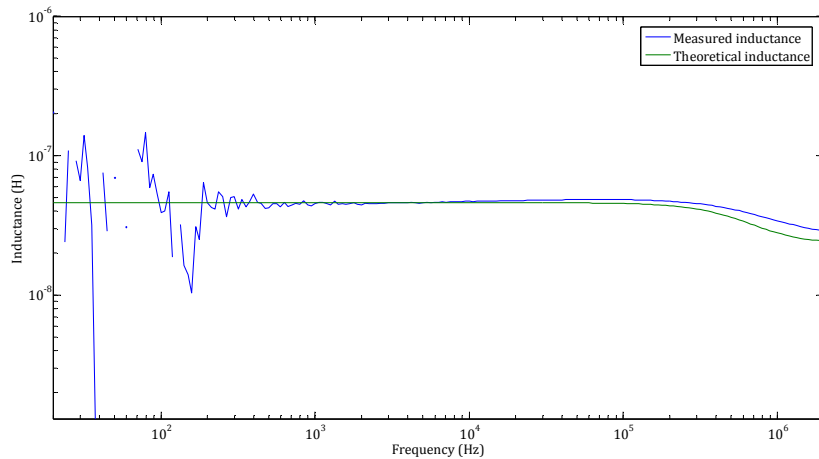


Figure 5.22b. Logarithmic plot: Measured and theoretical inductance of the three-layer configuration of Figure 5.19

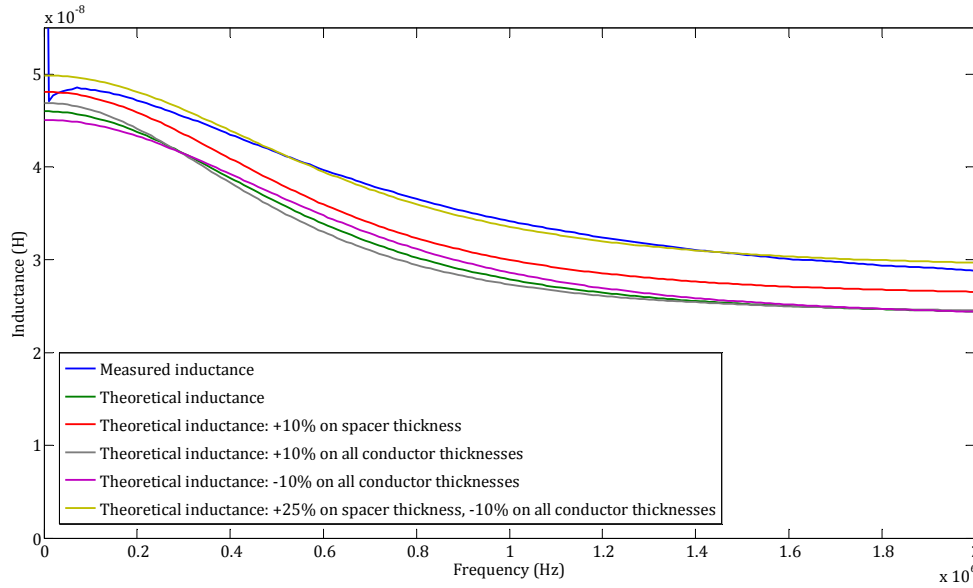


Figure 5.23. Linear plot: Measured and theoretical inductance, with the inclusion of theoretical tolerances, of the reference configuration of Figure 5.19

The impedance plots, given in Figure 5.24, can again be used to confirm the previous results and conclusions. From the linear impedance plots, it is again evident that the near-constant difference between the measured and theoretical inductances leads to the divergence of the impedance magnitudes and angles. While the logarithmic plots show that the trends of the measured and theoretical results are identical. Lastly, the angle measurement limitation of the LCR meter is again evident in Figure 5.24b.

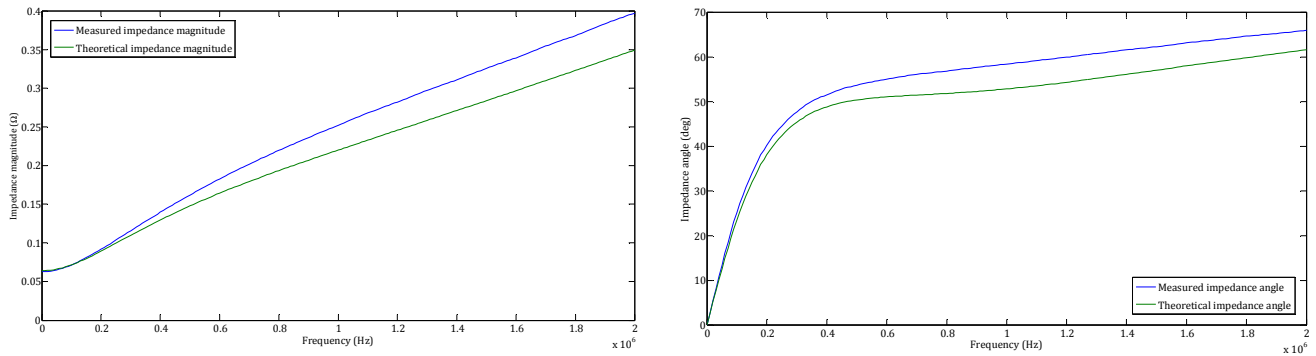


Figure 5.24a. Linear plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.19

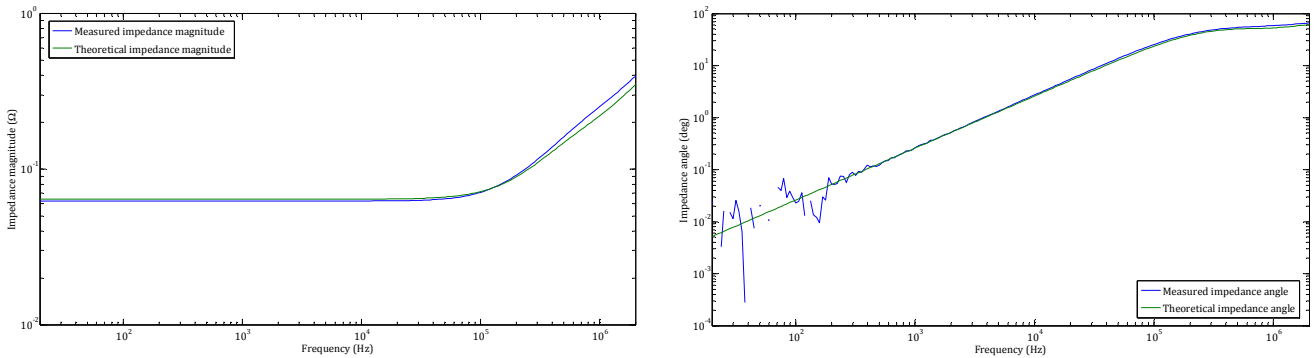


Figure 5.24b. Logarithmic plot: Measured and theoretical impedance magnitude and angle of the reference configuration of Figure 5.19

As is clear, these results echo the conclusion made at the end of the previous configuration. The impedance equations and the multi-layer distribution equations from which they were derived, are indeed correct, thereby confirming the validity of the multi-layer methodology.

5.4 Observations

From the configurations investigated in the previous section, a number of interesting, and potentially useful, observations were made. As they were not pertinent to that section, they will be discussed and analysed here. Since it is now known that the multi-layer methodology (and its subsequent equations) is correct, any additional data or graphs will be obtained via the theoretical impedance equations.

1. Impedance variation due to layer placement:

Even though configurations 1 and 2 were constructed from identical components, with only the brass layer placements differing (Figure 5.14 and Figure 5.19), they exhibited significantly different impedance characteristics. This is shown in the resistance and inductance plots of Figure 5.25, where the sensitivity to layer placement can clearly be seen.

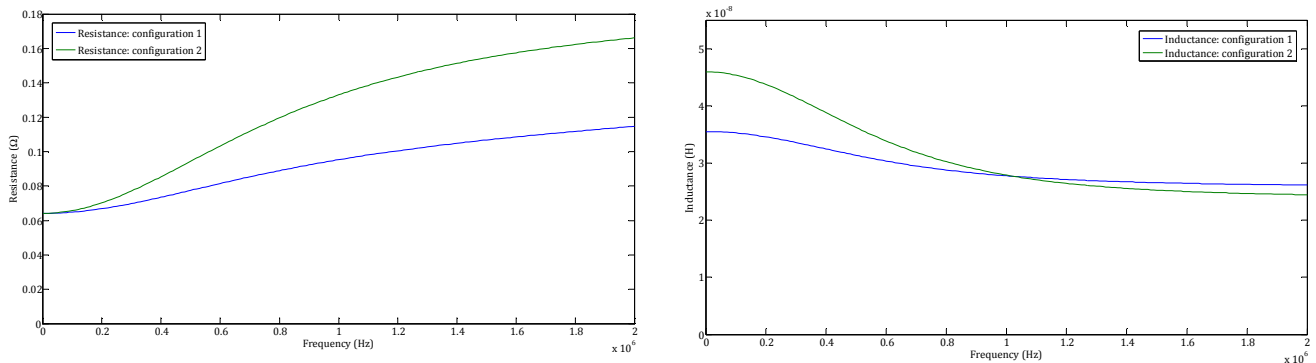
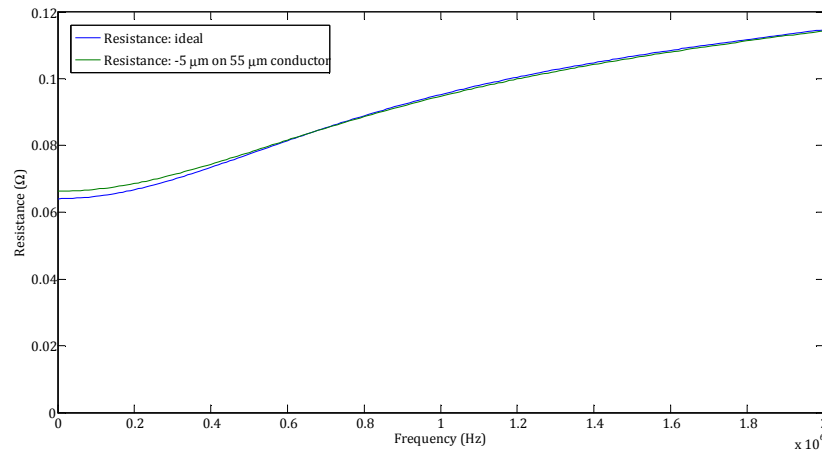
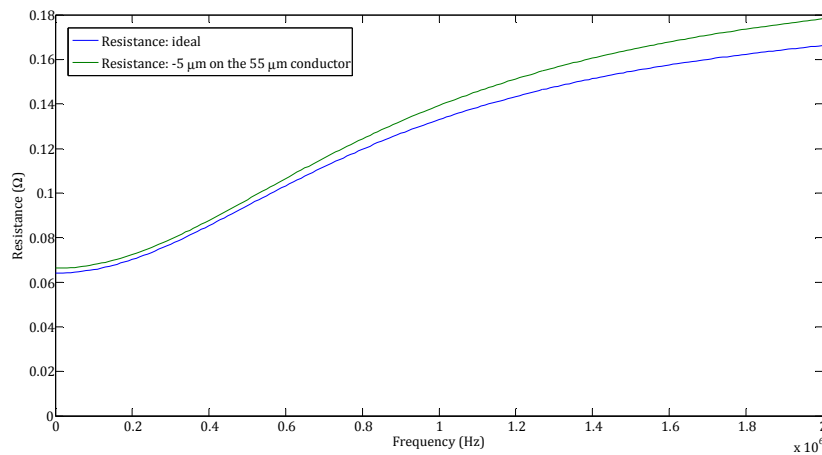


Figure 5.25. Linear plot: Resistance and inductance of configurations 1 and 2

2. Impedance sensitivity to layer tolerances:

It was found that the impedance's sensitivity to thickness tolerances was highly dependent on the exact configuration under consideration. This is demonstrated in Figure 5.26, where the 55 μm conductor was reduced to 50 μm (i.e. a 10% tolerance). As is clear, configuration 2 is far more sensitive to a variation of the inner 55 μm conductor than configuration 1. This can be explained from a qualitative perspective by realising that, at high frequencies, the larger part of the current flows through the inner conductor. If this conductor is decreased in thickness, as is the case for the 10% decrease in configuration 2, a smaller cross sectional area is available for the current, resulting in a larger resistance seen by the source. This sensitivity also helps to explain and appreciate the difficulty of obtaining a repeatable and accurate measurement.

Figure 5.26a. Linear plot: Resistance variation of configuration 1 due to a 5 μm reduction of the 55 μm layerFigure 5.26b. Linear plot: Resistance variation of configuration 1 due to a 5 μm reduction of the 55 μm layer

3. Multiple corner frequencies:

The previous section hinted at the possibility of multiple corner frequencies at higher frequencies. Figure 5.27 below shows the logarithmic plots of the resistance and inductance for frequencies up to 100 MHz. As can be seen, multiple corner frequencies do in fact exist.

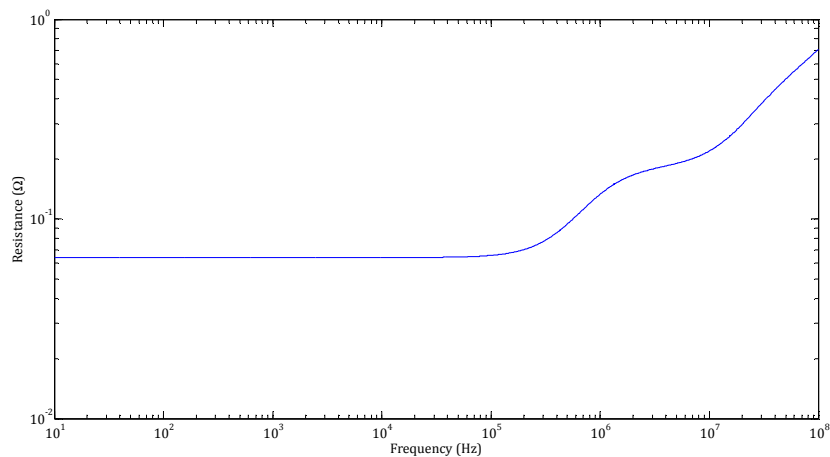


Figure 5.27a. Logarithmic plot: Resistance of configuration 2 up to 100 MHz

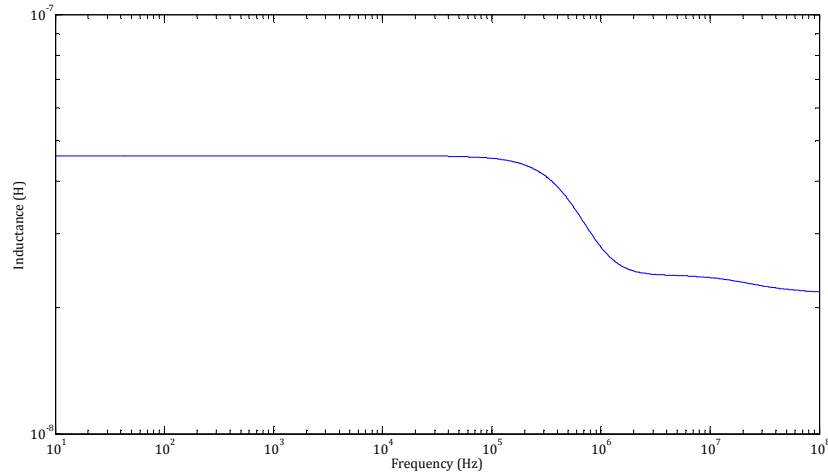


Figure 5.27b. Logarithmic plot: Inductance of configuration 2 up to 100 MHz

As is evident from these observations, the impedance and its characteristics are very dependent on the exact details of the configuration under investigation. Since the observations that were given here were discovered as a by-product of the impedance validation process, which was only focussed on a handful of configurations, it can be stated with reasonable confidence that numerous more undiscovered ones still exist. This in itself may warrant further study of these configurations.

5.5 Conclusion

As is now clear, the validity of the multi-layer methodology is without question. This was first shown through FEM verification in chapter 4 and was experimentally inferred through the verification of the multi-layer impedance equations in this chapter.

The development that led to this conclusion brought about a number of advancements and important findings, which would not have been possible otherwise. It firstly allowed for the derivation of multi-layer differential impedance equations, which will prove invaluable in future applications. It was found that in order to avoid violation of these conductors' constraints (arrangement 2 of chapter 1) and allow for the differential-mode configuration to be represented by a series R L model, the capacitive current needs to be negligible in comparison to the total current. The capacitance, therefore, needs to be small enough such that its reactance can be taken as infinite in comparison to the series resistance and inductance of the configuration.

The experimental work helped to emphasise the difficulties of constructing a configuration which allows for accurate and repeatable measurements. Particular problems were encountered with measurement limitations of the LCR meter, where it was found to produce errors for very small impedances and very small impedance angles. Unrelated to this, unaccounted for experimental errors were also identified and compensated for in order to verify the impedance equations.

All these factors were essential in providing the experimental verification of the multi-layer methodology, thereby verifying the obtainment of one of this thesis' main objectives.

5.6 References

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Chapter 6

Conclusions and Recommendations

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6.1. Introduction

All of the objectives laid out in chapter 1 were concerned with the attainment of a multi-layer methodology, the derivation of multi-layer field and impedance equations and their subsequent verification. This methodology needed to allow for the derivation of the electric field, magnetic field and current density within multi-layer conductors, while enabling the derivation of their impedances. The objectives are reiterated below, where, from chapter 1, arrangement 2 was defined as all multi-layer conductors for which the magnetic field and current are both parallel to the surface of the conductor, perpendicular to each other and confined to one direction only.

1. *Perform a thorough investigation of all existing equations and methods pertaining to the relevant behaviour of arrangement 2:*
 - o *Since the flat and cylindrical conductors formed the predominant geometries of this arrangement, attention will be focussed on them.*
 - o *This objective will concentrate on the single-layer conductors, as they are the only arrangement 2 conductors which have received significant analytical investigation to date.*
 - o *Investigation will be focussed on the electric field, magnetic field and current density within the single-layer conductors, as they form the basis for all future work and understanding.*
2. *Develop a multi-layer methodology for arrangement 2:*
 - o *This methodology must provide the steps required to obtain an analytical solution to the magnetic field, electric field and current density of any multi-layer conductor subjected to arrangement 2's constraints, with the methodology being independent of the number of layers.*
 - o *Apply this methodology to obtain the solutions to the flat and cylindrical geometries.*
3. *Develop impedance equations, based on this methodology and its subsequent equations, to assist in the future development of this arrangement's applications.*
4. *Design an experimental setup to confirm the validity of the multi-layer methodology and its impedance equations.*

The conclusions drawn throughout this thesis will be discussed under each of these objectives, while making reference to the chapters in which they were made.

6.2. Investigation of existing equations and methods

Both chapters 2 and 3 were concerned with establishing a thorough understanding of existing equations and methods which could be used towards the development of a multi-layer methodology.

Chapter 2 focussed on the review of existing equations, concepts and information, specifically those pertaining to flat and cylindrical conductors. The approximations involved in these geometries were scrutinised and it was found that the application of the semi-infinite approximation to finite flat conductors depends on the allowable error. A minimum thickness to width ratio of 1:25 was subsequently established.

The review of the existing equations led to the rederivation of the flat conductor's equations to be better suited for the multi-layer solution, while it was discovered that the existing equations for the cylindrical conductors were only for the solid variant. A derivation of the hollow cylindrical conductor, therefore, followed. These equations were then applied to the skin- and proximity-effects, which were identified as the fundamental building blocks of any scenario, in order to obtain a better understanding of their behaviour.

The orthogonality principle, which refers to the independence of the skin- and proximity-effect power components, was identified as a crucial concept in the resistance determination of objective three. It was shown that, in contrast to the flat conductor, the cylindrical conductor lacks a rigorous proof of this principle. A subsequent qualitative analysis, however, provided enough substantiation to accept its validity under the cylindrical geometry's constraints, which is reflected in the resistance verification of chapter 5.

In addition, two alternative methods to the obtainment of a multi-layer solution were investigated. The first method was the solution of Dowell, considered in chapter 2. It was shown to possess self-imposed limitations, which were brought about in its original derivation. These limitations restrict this solution to current carrying conductors only, thereby preventing the solution of the isolated proximity-effect scenario. This restriction therefore led to its elimination as a viable step towards an eventual multi-layer solution. The second method was investigated in chapter 3 and is based on a wave-analysis approach. Since its validity was under question, it was applied to the single-layer flat conductor and compared to the already well known solutions investigated in chapter 2. It was consequently shown that these two methods are identical from a fundamental perspective. The only advantage one method may pose over the other is in the perspective from which the problem is considered and the available tools that have already been developed. As such, the single-layer solutions investigated in chapter 2 were chosen for use in the development of the multi-layer solution.

6.3. Development of a multi-layer methodology

The multi-layer methodology was developed in chapter 4 under two conditions. Firstly, it must be possible to determine the magnetic field at the external boundaries of the multi-layer conductor. This is accomplished by knowing the total current flowing through the conductor, along with the dimensions of the conductor and any externally applied magnetic fields. While secondly, a generic solution to the single layer conductor must exist. This second criteria was fulfilled by the investigation done in chapter 2. These single-layer solutions are then related to one another through the application of Maxwell's equations to each of the boundaries between layers. This allows the multi-layer methodology to be defined in the four steps summarised below for a conductor composed of n layers:

1. Determine the magnetic field intensity at the external boundaries and use these values to obtain two equations.
2. Enforce the continuity of the tangential electric field across the boundaries and use this to obtain $n-1$ equations.

3. Enforce the continuity of the tangential magnetic field intensity across the boundaries and use this to obtain $n-1$ equations.
4. Solve the $2*n$ simultaneous equations.

This methodology allows for the derivation of a solution for any multi-layer conductor (subjected to arrangement 2's constraints), independent of its geometry or amount of layers, given that it fulfils the two criteria discussed previously. The solution, therefore, consists of n individual solutions, one for each layer. From this methodology the derivation of two- and three-layer flat and cylindrical conductors followed, where they were applied to a number of different scenarios. These were compared against FEM simulations in order to obtain initial confirmation of this methodology.

A number of important aspects came to light during these derivations. It was firstly found that, because of the interdependency that exists between the layers, more layers will result in an increased complexity of the individual layers' solutions. This complexity can, however, be minimized by shifting the argument of the single-layer solutions. In addition, the solution of an n layer conductor can be kept in its generic form and applied to a specific scenario by replacing the constants with their actual values. This avoids the need to rederive the solution for different scenarios.

Lastly, the verification through FEM emphasised the expected advantages of this methodology and its equations. Firstly, they allow for near instantaneous and always correct results, as opposed to the simulations which can take days and suffer from convergence problems. While secondly, they provide insight into the factors governing the distributions through visual inspection of the equations. However, because of the direct relation between solution complexity and the number of layers, visual inspection becomes difficult for three layers and more. In such cases the solutions can be used in parametric studies to determine the influence of the various factors. This is done in chapter 5 where the impedance components are plotted for various frequency sweeps.

6.4. Develop impedance equations based on this methodology and its subsequent equations

Since impedance equations are configuration dependent, the differential-mode configuration was chosen (chapter 5). This is representative of a large number of practical problems, thereby satisfying the objective that these equations must assist in the future development of multi-layer conductor applications. When viewed from a circuit perspective, the differential configuration can be modelled as a transmission-line structure, consisting of cascaded cells made up of series R L and parallel G C components. The conductance (G) component is taken to be negligible.

The resistance was shown to be dependent on two components. The first is the distribution of the current density within each layer, while the second is the presence of surrounding conductors. Eddy currents, and therefore additional losses, are induced within neighbouring conductors and are reflected as an increase in the total resistance.

The inductance was separated into its two primary components, internal and external, and derived individually. Both components were derived from an energy perspective in the time domain, where it was shown that the internal inductance is dependent on the distribution of the magnetic field within the layers.

In order to avoid violation of arrangement 2's constraints, the capacitive current needs to be negligible in comparison to the total current. The capacitance was, therefore, determined in order to ensure that its reactance can be taken as infinite in comparison to the impedance of the series resistance and inductance. Under these conditions the differential-mode configuration can be represented by a series R L model.

6.5. Design an experimental setup to confirm the validity of the multi-layer methodology and its impedance equations

The experimental verification performed in chapter 5 was required to confirm the impedance equations, which infer the validity of the multi-layer distributions equations and, in turn, the multi-layer methodology. The experimental setup consisted of three differential-mode configurations whose impedances were measured by use of an Agilent E4980A Precision LCR meter and modelled as a series resistance and inductance. The first differential-mode configuration was comprised of two single-layer conductors and was used to establish what unaccounted-for effects are present within the experimental setup. The remaining two both consisted of two three-layer conductors, which formed the differential conductors.

Physical measurements resulted in overall excellent agreement between the measured and theoretical results. Slight resistance discrepancies were found, but were accounted to thickness variation of the conductive layers. Marginally larger inductance differences were encountered, which were attributed to the space between the two differential conductors being larger than what was initially measured. All these discrepancies were, however, well within allowable experimental error and could be accounted for. The experimental verification therefore confirms the validity of the multi-layer methodology, along with its subsequent distribution and impedance equations.

A number of important observations were made during this experimental verification, which, either confirm already known characteristics, or may prove of value in future applications. It was firstly found that the impedance values are highly dependent on layer placement. Secondly, depending on layer placement, the impedance may exhibit a high sensitivity to layer tolerances. Lastly, multiple corner frequencies were identified to exist, with their placement highly dependent on the details of the differential configuration.

6.6. Thesis contribution

The main contributions of this thesis can be summarised as follows:

1. Development of a verified multi-layer methodology that enables the derivation of the analytical solutions to the magnetic field, electric field and current density within any multi-layer conductor subjected to arrangement 2's constraints.

2. The derivation of the generic multi-layer equations of the two- and three-layer flat and cylindrical conductors. Their generic forms allow for their application to any configurations in which the magnetic field intensity at their external boundaries can be determined.
3. Derivation of multi-layer resistance and inductance equations. Up to now it was not possible to analytically determine the impedance that a multi-layer conductor (arrangement 2) will present to a source, which was a severe limiting factor in the design and application of these multi-layer conductors. These equations will prove instrumental in the further development of multi-layer conductor applications.
4. An improved understanding of the factors governing the behaviour of the multi-layer conductors of arrangement 2.
5. Freedom from numerical simulations methods (such as FEM). Up to now, the determination of the internal field and current density distributions and impedance of these conductors, was only possible through a numerical simulation approach.

6.7. Recommendations for future work

The following is a list of topics which were not addressed in this thesis, but may benefit from further research:

- From chapter 2 the orthogonality between the skin- and proximity-effect power components for the cylindrical conductor was shown to be true from a qualitative discussion. The development of a mathematical proof of the orthogonality principle for cylindrical conductors is therefore recommended.
- As was seen in chapter 5, the solutions of three- or more layered conductors become very complex and visual inspection of the equations become difficult. Investigate methods to simplify these solutions.
- Develop fitting/search algorithms based on the derived equations to determine a material, geometry and dimension combination, if one exists, that would be required to obtain a given filter versus frequency characteristic. A tolerance or allowable deviation can be built into the algorithm for the cases (which would be the majority) where the desired response cannot be exactly obtained.
- Signal distortion arises from the frequency dependence of the impedance components. Investigate the viability of using these results and equations to construct a multi-layer conductor that will minimize signal distortion over a larger frequency spectrum. This ties in with the previous point on the development of a fitting/search algorithm, where in this case, the desired impedance components versus frequency characteristics would be a flat response.
- Investigate the observations made at the end of chapter 5, which are, impedance variation due to layer placement, impedance sensitivity to layer tolerances and existence of multiple corner frequencies, in order to determine if they can be used to improve or create new multi-layer conductor applications.
- As was discussed in chapter 1, the design of multi-layer dissipative filters has been based to date on intuition, FEM simulations and approximated equations. The multi-

layer methodology, distribution and impedance equations should, therefore, prove invaluable in the design, application and improvement of these filters.

- Extend the derived solution of the flat and cylindrical conductors to account for magnetic losses. This should be possible by replacing the permeability in the single layer equations by its complex counterpart.
- The lengths of both the flat and cylindrical multi-layer conductors were taken as infinite (or approximated as such) throughout this thesis. Investigate the implications on the methodology and equations when a multi-layer conductor of finite length starts to radiate significant electromagnetic energy, i.e. takes on antenna-like characteristics. The length-to-thickness ratio will still be such that the end-effects do not contribute a significant factor to the investigation.
- Investigate the implications, and if needed, the adjustment of the methodology, when the capacitive current component of the differential-mode configuration becomes significant (i.e. the capacitive reactance cannot be approximated as infinite).
- Investigate the possibility of adapting the multi-layer methodology and equations to the case where the cylindrical layers are parallel to one another, but not centred on the same axis.

Appendix A

Equivalence of the Two Forms of the 1D Solution

A.1. Introduction

This appendix shows the equivalence of the two forms of the one-dimensional solution encountered in chapter 2.

A.2. Equivalence

From chapter 2 the differential equation of the flat conductor reduced to,

$$\frac{d^2 H_z}{dy^2} - D^2 H_z = 0 \quad (1)$$

where

$$D = (1+j) \sqrt{\frac{\omega \mu \sigma}{2}} = \frac{(1+j)}{\delta} \quad (2)$$

The solution is given by,

$$H_z(y) = K_1 e^{Dy} + K_2 e^{-Dy} \quad (3)$$

where K_1 and K_2 are dependent on the boundary values. This equation can also be written in the following form,

$$H_z(y) = F \cosh(Dy) + G \sinh(Dy) \quad (4)$$

In order to see the equivalence the $\cosh(Dy)$ and $\sinh(Dy)$ terms need to be expanded into their exponential forms,

$$H_z(y) = F \left[\frac{e^{Dy} + e^{-Dy}}{2} \right] + G \left[\frac{e^{Dy} - e^{-Dy}}{2} \right] \quad (5)$$

Grouping e^{Dy} and e^{-Dy} terms,

$$H_z(y) = \left[\frac{F}{2} + \frac{G}{2} \right] e^{Dy} + \left[\frac{F}{2} - \frac{G}{2} \right] e^{-Dy} \quad (6)$$

Defining,

$$K_1 = \left[\frac{F}{2} + \frac{G}{2} \right] \quad (7)$$

and

$$K_2 = \left[\frac{F}{2} - \frac{G}{2} \right] \quad (8)$$

allows equation (6) to be written in the form of equation (3).

Appendix B

Laplacian of the Electric Field

B.1. Introduction

This appendix gives the derivation of the Laplacian of the electric field, along with its subsequent solutions under the constraints of the flat and cylindrical conductors.

B.2. Laplacian of the electric field

Starting with Faraday's law (equation (1)) and taking the curl on both sides,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (1)$$

$$\nabla \times (\nabla \times \bar{E}) = -\frac{\partial}{\partial t} (\nabla \times \mu \bar{H}) \quad (2)$$

where $\bar{B} = \mu \bar{H}$ for a linear material. From the identity given in [1] equation (2) can be written as,

$$\nabla \times (\nabla \times \bar{E}) = -\frac{\partial}{\partial t} (\nabla \times \mu \bar{H}) = \nabla (\nabla \cdot \bar{E}) - \nabla^2 \bar{E} \quad (3)$$

Since there are no localised charge densities within a conductor, equation (3) reduces to,

$$\nabla^2 \bar{E} = \frac{\partial}{\partial t} (\nabla \times \mu \bar{H}) \quad (4)$$

From Ampere's law,

$$\begin{aligned} \nabla \times \bar{H} &= \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \\ &= \sigma \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t} \end{aligned} \quad (5)$$

For the frequencies and materials under consideration, the displacement current term can be neglected. Equation (3) therefore becomes,

$$\nabla^2 \bar{E} = \mu \sigma \frac{\partial \bar{E}}{\partial t} \quad (6)$$

Under sinusoidal excitation this reduces to,

$$\nabla^2 \bar{E} = j\omega \mu \sigma \bar{E} \quad (7)$$

B.3. Solution under the constraints of the semi-infinite flat conductor

Under the semi-infinite flat conductor approximation only E_x is non-zero and is a function of y . The Laplacian of the electric field under sinusoidal excitation (equation (7)), therefore, becomes,

$$\frac{d^2 E_x}{dy^2} = j\omega\mu\sigma E_x \quad (8)$$

Writing this as a homogenous equation reveals that it is a second-order linear ordinary differential equation. The solution is given by,

$$E_x(y) = L \cosh(Dy) + M \sinh(Dy) \quad (9)$$

L and M are unknown coefficients that need to be solved for. Chapter 4, which makes use of this solution, introduced an additional shift in the argument of the hyperbolic functions. This will not affect the validity of equation (9) or of the solution used in chapter 4. All that the shift will result in is a change in the values of L and M , with the final electric field solution still being the same.

B.4. Solution under the constraints of the cylindrical conductor

As was mentioned in chapter 2, the expansion of the vector Laplacian in cylindrical coordinates is significantly more difficult than in the Cartesian coordinates. From Appendix D the components of the vector Laplacian of the electric field is,

$$(\nabla^2 \bar{E})_\rho = \frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial \rho} - \frac{E_\rho}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 E_\rho}{\partial \phi^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial^2 E_\rho}{\partial z^2} \quad (10)$$

$$(\nabla^2 \bar{E})_\phi = \frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \rho} - \frac{E_\phi}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} + \frac{\partial^2 E_\phi}{\partial z^2} \quad (11)$$

$$(\nabla^2 \bar{E})_z = \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} \quad (12)$$

From the constraints of the cylindrical geometry only the z -component of the electric field will exist and will only be dependent on the radial direction. Equations (10) to (12) therefore simplifies to,

$$\nabla^2 \bar{E} = \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} \quad (13)$$

From equation (7) this becomes,

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} = j\omega\mu\sigma E_z \quad (14)$$

Letting $D^2 = j\omega\mu\sigma$ allows (14) to be written as,

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} - D^2 E_z = 0 \quad (15)$$

Multiply by ρ^2 throughout,

$$\rho^2 \frac{\partial^2 E_z}{\partial \rho^2} + \rho \frac{\partial E_z}{\partial \rho} - \rho^2 D^2 E_z = 0 \quad (16)$$

The standard format of the modified Bessel function is,

$$\rho^2 \frac{\partial^2 E_z}{\partial \rho^2} + \rho \frac{\partial E_z}{\partial \rho} - (\rho^2 D^2 + v^2) E_z = 0 \quad (17)$$

which has a solution of the form,

$$E_z(\rho) = P I_v(D\rho) + Q K_v(D\rho) \quad (18)$$

Since v is equal to zero, the solution of equation (16) is,

$$E_z(\rho) = P I_0(D\rho) + Q K_0(D\rho) \quad (19)$$

where I_0 and K_0 are zeroth order (subscript 0) modified Bessel functions of the first (I) and second kind (K).

B.5. References

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Appendix C

Maxwell: Even Symmetry Axes in the Creation of an Infinite Dimension

C.1. Introduction

This appendix investigates the modelling of single- and multi-layer semi-infinite flat conductors in Maxwell, a finite element simulation program. In particular, the details regarding the creation and simulation of an infinite dimension will be considered.

C.2. Creation of an infinite dimension

In order for any configuration to be accurately simulated the inherent constraints of the FEM program's modelling space need to be well understood. Maxwell is a two-dimensional program in which two of the dimensions are controllable and finite, with the third taken to be infinite. A picture of the modelling interface is given in Figure C.1, where the $\hat{U}$ and $\hat{V}$ directions, aligned with the horizontal and vertical axes, are the two controllable dimensions. The third direction is therefore at right angles to both $\hat{U}$ and $\hat{V}$ and will be assigned $\hat{W}$, for ease of reference. Maxwell requires that all dimensions in the $\hat{U}$ and $\hat{V}$ directions be finite, while the dimension in the $\hat{W}$ direction is infinite. All currents are defined to flow along $\hat{W}$, which forces the resulting magnetic fields to be in the $\hat{U}$ and $\hat{V}$ directions.

Consider now the single layer (single- or multi-layer does not matter) semi-infinite flat conductor, which is to be modelled in Maxwell (Figure C.2). From the constraints that the current and magnetic field are confined to the $\hat{x}$ - and $\hat{z}$ -directions, respectively, it logically follows that the $\hat{x}$ -direction need to be aligned with the $\hat{W}$ -direction of the FEM program. The finite thickness will therefore be in the $\hat{V}$ -direction, which leaves the problem of modelling the infinite width as a finite dimension in the $\hat{U}$ -direction.

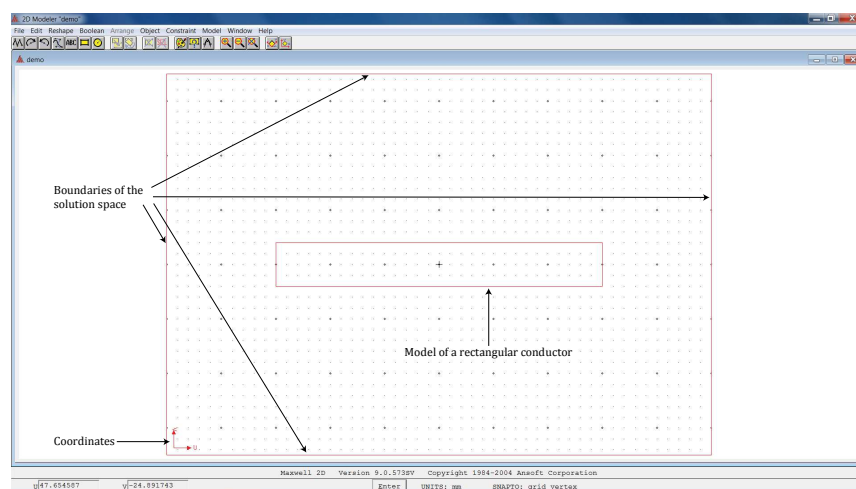


Figure C.1. Modelling interface of Maxwell

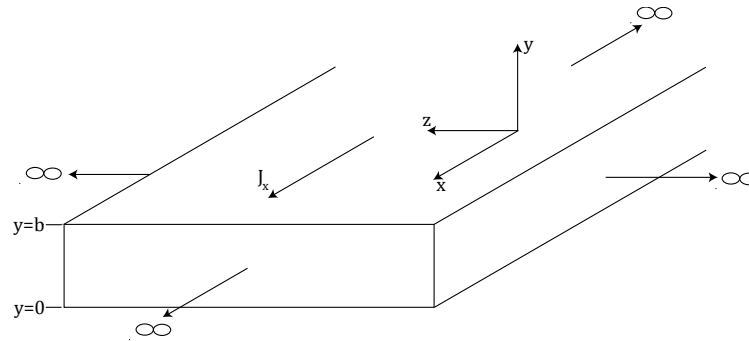


Figure C.2. Semi-infinite flat conductor to be modelled in Maxwell

In an attempt to circumvent this problem, attention is focussed on Maxwell's ability to define boundaries (of the modelling space) as 'even symmetry' axes. An 'even symmetry' axis acts as an axis of reflection, with everything to its side identically mirrored to its other side. For example, if the left hand boundary in Figure C.1 is defined as an 'even symmetry' axis, Maxwell will interpret the actual model as consisting of two rectangular conductors. The obvious advantage of this is that the problem becomes symmetrical, allowing for a shorter convergence time.

What is however unknown, is how Maxwell will interpret two 'even symmetry' axes opposite one another, as shown in Figure C.3. One of two possible interpretations is expected. The first, and what is hoped for, is that the left 'even symmetry' axis will reflect everything to its right, including the right 'even symmetry' axis, while the right 'even symmetry' axis will reflect everything to its left, including the left 'even symmetry' axis. As these two 'even symmetry' axes reflect each other an infinite amount of reflections occur, resulting in an infinite width. This is illustrated in Figure C.4. The second possible scenario is that the two 'even symmetry' axes remain unaware of each other and that just two reflections occur (one due to the left and the other due to the right 'even symmetry' axes). This is depicted in Figure C.5.

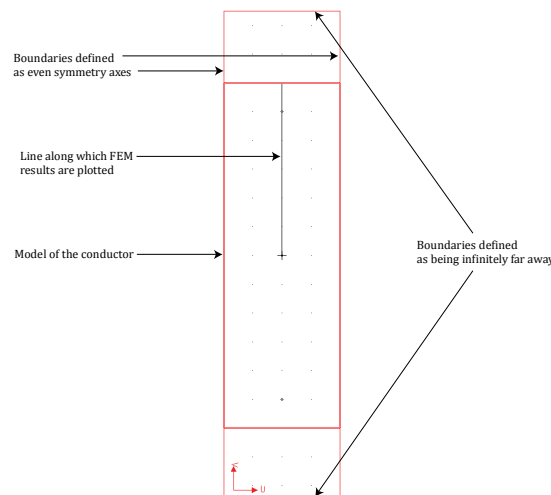


Figure C.3. Model of a conductor with two 'even symmetry' axes'

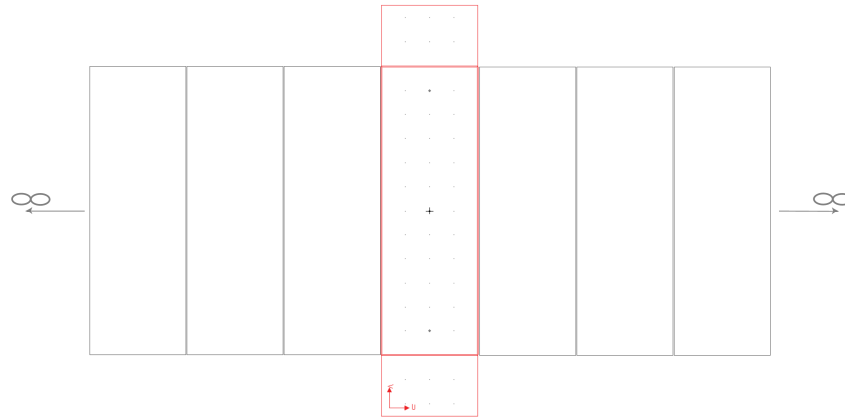


Figure C.4. Possible scenario: Infinite amount of reflections occurring due to the ‘even symmetry axes’

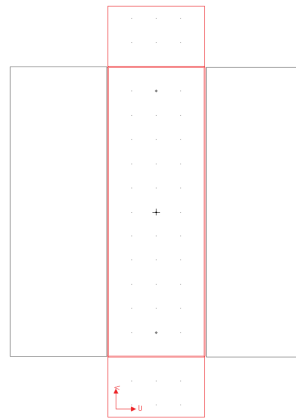


Figure C.5. Possible scenario: Two reflections occurring due to the ‘even symmetry axes’

In order to determine which of these two interpretations is true the following simulations will be performed. The width ($\hat{U}$ -direction) of the conductor in Figure C.3 will be chosen to be 8 mm wide, while the thickness ($\hat{V}$) will be selected as 24 mm. Depending on the interpretation the actual width (as seen by the program) will therefore be either infinite or 24 mm (3×8 mm). For comparison the one-dimensional semi-infinite analytical results of chapter two will be used to represent the infinite scenario. While the configuration of Figure C.6, with a thickness and width of 24 mm each, will be simulated and used to represent the finite reflections scenario.

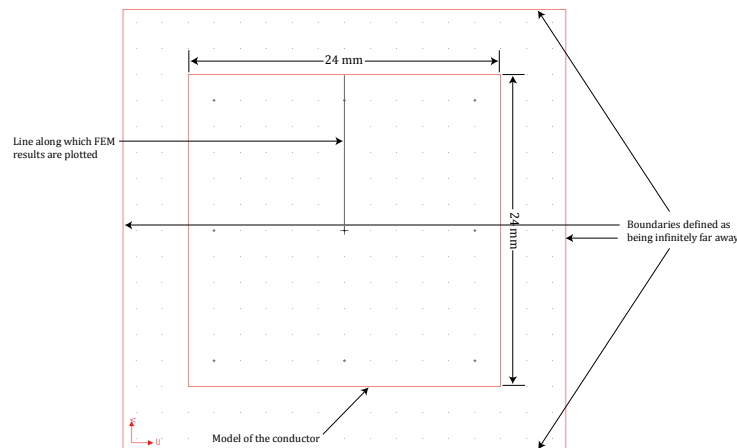


Figure C.6. Resultant model if only two reflections occur due to the ‘even symmetry axes’ in Figure C.3

Figure C.7 and Figure C.8 give the FEM produced current density distribution over the surface of the two conductors of Figure C.3 and Figure C.6. As is clear, these two distributions share no similarities, with the even boundaries of Figure C.7 producing a symmetrical current density distribution around the horizontal axis, while the lack of even boundaries in Figure C.8 result in excessive current crowding at the edges. This strongly suggests that the presence of the even boundaries result in an infinite width, which is confirmed by the plots depicted in Figure C.9 to Figure C.12. H and J of the analytical results is obtained from the equations in chapter 2, while the FEM values are those obtained along the lines shown in Figure C.3 and Figure C.6.

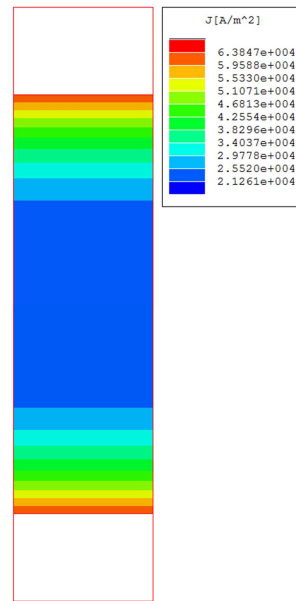


Figure C.7. Surface current density plot of Figure C.3

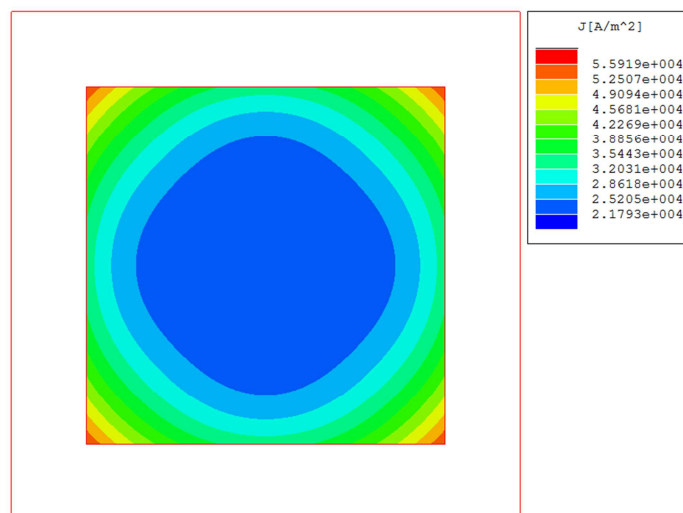


Figure C.8. Surface current density plot of Figure C.6

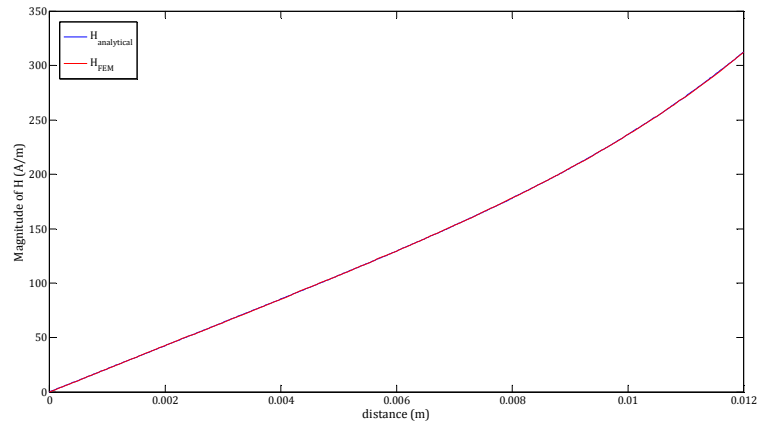


Figure C.9. Magnetic field intensity of the analytical and FEM results (with the presence of even boundaries)

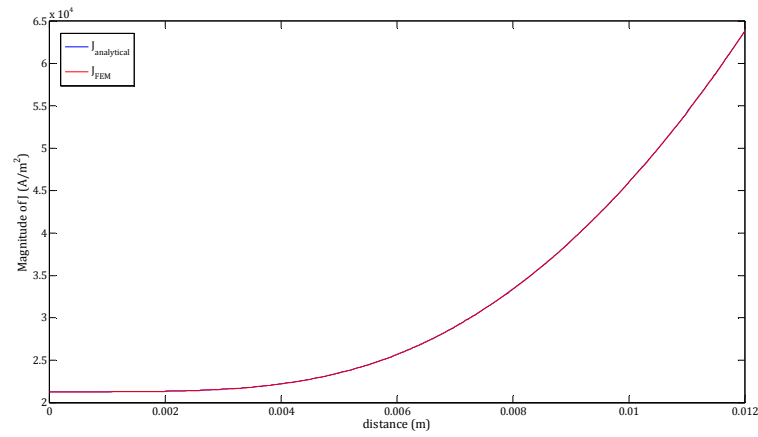


Figure C.10. Current density of the analytical and FEM results (with the presence of even boundaries)

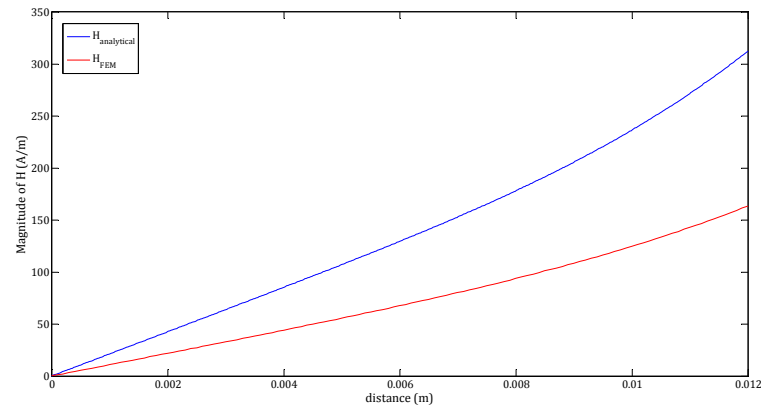


Figure C.11. Magnetic field intensity of the analytical and FEM results (without the presence of even boundaries)

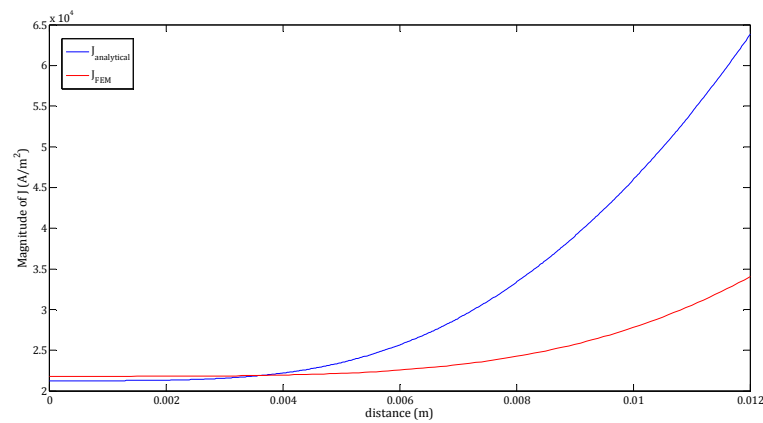


Figure C.12. Current density of the analytical and FEM results (without the presence of even boundaries)

C.3. Conclusion

This experiment and its subsequent results showed that through the use correctly placed 'even symmetry' axes it is possible to convert a finite dimension into an infinite one. This is a very powerful tool, especially when the effects of the boundaries need to be neglected on purpose (as is the case with the semi-infinite flat conductors). The use of 'even symmetry' axes can, therefore, be used with absolute confidence in the creation of infinite simulation dimensions, as is done in chapters 2 and 4.

Appendix D

Expansion of the Vector Laplacian in Cylindrical Coordinates

D.1. Introduction

The expanded form of the vector Laplacian is best known in its Cartesian coordinate form. This appendix gives its expanded form in the cylindrical coordinates system, as it is required by chapter 2 and could not be found in standard reference texts.

D.2. Laplacian

The fundamental definition of the Laplacian is given by equation (1), where R can be a scalar or vector.

$$\nabla \cdot (\nabla R) = \nabla^2 R \quad (1)$$

Under the condition of R being a scalar it reduces to the well-known forms given in [1], for the Cartesian, cylindrical and spherical coordinates systems. For R being a vector, equation (1) is given by the fundamental definition,

$$\nabla^2 \bar{R} = \nabla (\nabla \cdot \bar{R}) - \nabla \times (\nabla \times \bar{R}) \quad (2)$$

which holds for all coordinate systems. Under the Cartesian system it reduces to,

$$\bar{R} = R_x \hat{i} + R_y \hat{j} + R_z \hat{k} \quad (3)$$

$$\nabla^2 \bar{R} = \nabla^2 R_x \hat{i} + \nabla^2 R_y \hat{j} + \nabla^2 R_z \hat{k} \quad (4)$$

The Laplacian of the scalar quantities, R_x , R_y and R_z is then given by,

$$\nabla^2 R_{x/y/z} = \frac{\partial^2}{\partial x^2} (R_{x/y/z}) + \frac{\partial^2}{\partial y^2} (R_{x/y/z}) + \frac{\partial^2}{\partial z^2} (R_{x/y/z}) \quad (5)$$

As can be seen, the Laplacian of the vector, $\bar{R}$, in Cartesian coordinates is equal to a vector whose components are Laplacians of the scalar components R_x , R_y and R_z . This ease does not, however, extend to other coordinate systems and is one of the reasons Cartesian coordinates is generally preferred. In order to obtain the Laplacian of $\bar{R}$ in cylindrical coordinates the expansion will have to start from the fundamental definition of equation (2). Redefining $\bar{R}$ in cylindrical coordinates,

$$\bar{R} = R_\rho \hat{\rho} + R_\phi \hat{\phi} + R_z \hat{z} \quad (6)$$

Expanding the curl in the last bracket of (2),

$$\begin{aligned}
\nabla \times \bar{R} &= \left[\frac{1}{\rho} \frac{\partial R_z}{\partial \phi} - \frac{\partial R_\phi}{\partial z} \right] \hat{\rho} + \left[\frac{\partial R_\rho}{\partial z} - \frac{\partial R_z}{\partial \rho} \right] \hat{\phi} + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho R_\phi) - \frac{\partial R_\rho}{\partial \phi} \right] \hat{z} \\
&= \left[\frac{1}{\rho} \frac{\partial R_z}{\partial \phi} - \frac{\partial R_\phi}{\partial z} \right] \hat{\rho} + \left[\frac{\partial R_\rho}{\partial z} - \frac{\partial R_z}{\partial \rho} \right] \hat{\phi} + \frac{1}{\rho} \left[R_\phi + \rho \frac{\partial R_\phi}{\partial \rho} - \frac{\partial R_\rho}{\partial \phi} \right] \hat{z}
\end{aligned} \tag{7}$$

Taking the curl of this equation,

$$\begin{aligned}
\nabla \times (\nabla \times \bar{R}) &= \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} (\nabla \times \bar{R})_z - \frac{\partial}{\partial z} (\nabla \times \bar{R})_\phi \right] \hat{\rho} + \left[\frac{\partial}{\partial z} (\nabla \times \bar{R})_\rho - \frac{\partial}{\partial \rho} (\nabla \times \bar{R})_z \right] \hat{\phi} \\
&\quad + \frac{1}{\rho} \left[(\nabla \times \bar{R})_\phi + \rho \frac{\partial}{\partial \rho} (\nabla \times \bar{R})_\phi - \frac{\partial}{\partial \phi} (\nabla \times \bar{R})_\rho \right] \hat{z}
\end{aligned} \tag{8}$$

Substituting the $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$ components from (7) into (8),

$$\begin{aligned}
\nabla \times (\nabla \times \bar{R}) &= \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} (\nabla \times \bar{R})_z - \frac{\partial}{\partial z} (\nabla \times \bar{R})_\phi \right] \hat{\rho} + \left[\frac{\partial}{\partial z} (\nabla \times \bar{R})_\rho - \frac{\partial}{\partial \rho} (\nabla \times \bar{R})_z \right] \hat{\phi} \\
&\quad + \frac{1}{\rho} \left[(\nabla \times \bar{R})_\phi + \rho \frac{\partial}{\partial \rho} (\nabla \times \bar{R})_\phi - \frac{\partial}{\partial \phi} (\nabla \times \bar{R})_\rho \right] \hat{z} \\
&= \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} \left(\frac{1}{\rho} R_\phi + \frac{\partial R_\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial R_\rho}{\partial \phi} \right) - \frac{\partial}{\partial z} \left(\frac{\partial R_\rho}{\partial z} - \frac{\partial R_z}{\partial \rho} \right) \right] \hat{\rho} \\
&\quad + \left[\frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial R_z}{\partial \phi} - \frac{\partial R_\phi}{\partial z} \right) - \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} R_\phi + \frac{\partial R_\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial R_\rho}{\partial \phi} \right) \right] \hat{\phi} \\
&\quad + \frac{1}{\rho} \left[\frac{\partial R_\rho}{\partial z} - \frac{\partial R_z}{\partial \rho} + \rho \frac{\partial}{\partial \rho} \left(\frac{\partial R_\rho}{\partial z} - \frac{\partial R_z}{\partial \rho} \right) - \frac{\partial}{\partial \phi} \left(\frac{1}{\rho} \frac{\partial R_z}{\partial \phi} - \frac{\partial R_\phi}{\partial z} \right) \right] \hat{z} \\
&= \left[\frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^{-1}) R_\phi + \frac{1}{\rho^2} \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \phi \partial \rho} - \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^{-1}) \frac{\partial R_\rho}{\partial \phi} - \frac{1}{\rho^2} \frac{\partial^2 R_\rho}{\partial \phi^2} - \frac{\partial^2 R_\rho}{\partial z^2} + \frac{\partial^2 R_z}{\partial z \partial \rho} \right] \hat{\rho} \\
&\quad + \left[\frac{\partial}{\partial z} (\rho^{-1}) \frac{\partial R_z}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_z}{\partial z \partial \phi} - \frac{\partial^2 R_\phi}{\partial z^2} - \frac{\partial}{\partial \rho} (\rho^{-1}) R_\phi \right. \\
&\quad \left. - \frac{1}{\rho} \frac{\partial R_\phi}{\partial \rho} - \frac{\partial^2 R_\phi}{\partial \rho^2} + \frac{\partial}{\partial \rho} (\rho^{-1}) \frac{\partial R_\rho}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\rho}{\partial \rho \partial \phi} \right] \hat{\phi} \\
&\quad + \left[\frac{1}{\rho} \frac{\partial R_\rho}{\partial z} - \frac{1}{\rho} \frac{\partial R_z}{\partial \rho} + \frac{\partial^2 R_\rho}{\partial \rho \partial z} - \frac{\partial^2 R_z}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^{-1}) \frac{\partial R_z}{\partial \phi} - \frac{1}{\rho^2} \frac{\partial^2 R_z}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \phi \partial z} \right] \hat{z}
\end{aligned} \tag{9}$$

Realising that the differential of ρ and ρ^{-1} with respect to all coordinates, except itself, is zero reduces this equation to,

$$\begin{aligned}
\nabla \times (\nabla \times \bar{R}) = & \left[\frac{1}{\rho^2} \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \phi \partial \rho} - \frac{1}{\rho^2} \frac{\partial^2 R_\rho}{\partial \phi^2} - \frac{\partial^2 R_\rho}{\partial z^2} + \frac{\partial^2 R_z}{\partial z \partial \rho} \right] \hat{\rho} \\
& + \left[\frac{1}{\rho} \frac{\partial^2 R_z}{\partial z \partial \phi} - \frac{\partial^2 R_\phi}{\partial z^2} + \frac{R_\phi}{\rho^2} - \frac{1}{\rho} \frac{\partial R_\phi}{\partial \rho} - \frac{\partial^2 R_\phi}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial R_\rho}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\rho}{\partial \rho \partial \phi} \right] \hat{\phi} \\
& + \left[\frac{1}{\rho} \frac{\partial R_\rho}{\partial z} - \frac{1}{\rho} \frac{\partial R_z}{\partial \rho} + \frac{\partial^2 R_\rho}{\partial \rho \partial z} - \frac{\partial^2 R_z}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2 R_z}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \phi \partial z} \right] \hat{z}
\end{aligned} \tag{10}$$

Returning now to equation (2) and taking the divergence of $\bar{R}$,

$$\begin{aligned}
\nabla \cdot \bar{R} = & \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho R_\rho) + \frac{1}{\rho} \frac{\partial R_\phi}{\partial \phi} + \frac{\partial R_z}{\partial z} \\
= & \frac{\partial R_\rho}{\partial \rho} + \frac{R_\rho}{\rho} + \frac{1}{\rho} \frac{\partial R_\phi}{\partial \phi} + \frac{\partial R_z}{\partial z}
\end{aligned} \tag{11}$$

Taking the gradient of this,

$$\begin{aligned}
\nabla (\nabla \cdot \bar{R}) = & \frac{\partial}{\partial \rho} (\nabla \cdot \bar{R}) \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\nabla \cdot \bar{R}) \hat{\phi} + \frac{\partial}{\partial z} (\nabla \cdot \bar{R}) \hat{z} \\
= & \frac{\partial}{\partial \rho} \left[\frac{\partial R_\rho}{\partial \rho} + \frac{R_\rho}{\rho} + \frac{1}{\rho} \frac{\partial R_\phi}{\partial \phi} + \frac{\partial R_z}{\partial z} \right] \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left[\frac{\partial R_\rho}{\partial \rho} + \frac{R_\rho}{\rho} + \frac{1}{\rho} \frac{\partial R_\phi}{\partial \phi} + \frac{\partial R_z}{\partial z} \right] \hat{\phi} + \\
& + \frac{\partial}{\partial z} \left[\frac{\partial R_\rho}{\partial \rho} + \frac{R_\rho}{\rho} + \frac{1}{\rho} \frac{\partial R_\phi}{\partial \phi} + \frac{\partial R_z}{\partial z} \right] \hat{z} \\
= & \left[\frac{\partial^2 R_\rho}{\partial \rho^2} - \frac{R_\rho}{\rho^2} + \frac{1}{\rho} \frac{\partial R_\rho}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \rho \partial \phi} + \frac{\partial^2 R_z}{\partial \rho \partial z} \right] \hat{\rho} \\
& + \left[\frac{1}{\rho} \frac{\partial^2 R_\rho}{\partial \phi \partial \rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^{-1}) R_\rho + \frac{1}{\rho^2} \frac{\partial R_\rho}{\partial \phi} + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^{-1}) \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 R_\phi}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_z}{\partial \phi \partial z} \right] \hat{\phi} \\
& + \left[\frac{\partial^2 R_\rho}{\partial z \partial \rho} + \frac{\partial}{\partial z} (\rho^{-1}) R_\rho + \rho^{-1} \frac{\partial R_\rho}{\partial z} + \frac{\partial}{\partial z} (\rho^{-1}) \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial z \partial \phi} + \frac{\partial^2 R_z}{\partial z^2} \right] \hat{z} \\
= & \left[\frac{\partial^2 R_\rho}{\partial \rho^2} - \frac{R_\rho}{\rho^2} + \frac{1}{\rho} \frac{\partial R_\rho}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial R_\phi}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial \rho \partial \phi} + \frac{\partial^2 R_z}{\partial \rho \partial z} \right] \hat{\rho} \\
& + \left[\frac{1}{\rho} \frac{\partial^2 R_\rho}{\partial \phi \partial \rho} + \frac{1}{\rho^2} \frac{\partial R_\rho}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 R_\phi}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_z}{\partial \phi \partial z} \right] \hat{\phi} \\
& + \left[\frac{\partial^2 R_\rho}{\partial z \partial \rho} + \rho^{-1} \frac{\partial R_\rho}{\partial z} + \frac{1}{\rho} \frac{\partial^2 R_\phi}{\partial z \partial \phi} + \frac{\partial^2 R_z}{\partial z^2} \right] \hat{z}
\end{aligned} \tag{12}$$

By subtracting (10) from (12) the expanded vector Laplacian is given on a component basis,

$$\begin{aligned}
(\nabla^2 \bar{R})_{\hat{\rho}} &= [\nabla(\nabla \cdot \bar{R})]_{\hat{\rho}} - [\nabla \times (\nabla \times \bar{R})]_{\hat{\rho}} \\
&= \left[\frac{\partial^2 R_{\rho}}{\partial \rho^2} - \frac{R_{\rho}}{\rho^2} + \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial R_{\phi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \rho \partial \phi} + \frac{\partial^2 R_z}{\partial \rho \partial z} \right] \\
&\quad - \left[\frac{1}{\rho^2} \frac{\partial R_{\phi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \phi \partial \rho} - \frac{1}{\rho^2} \frac{\partial^2 R_{\rho}}{\partial \phi^2} - \frac{\partial^2 R_{\rho}}{\partial z^2} + \frac{\partial^2 R_z}{\partial z \partial \rho} \right] \\
&= \frac{\partial^2 R_{\rho}}{\partial \rho^2} - \frac{R_{\rho}}{\rho^2} + \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial R_{\phi}}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \rho \partial \phi} + \frac{\partial^2 R_z}{\partial \rho \partial z} \\
&\quad - \frac{1}{\rho^2} \frac{\partial R_{\phi}}{\partial \phi} - \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \phi \partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 R_{\rho}}{\partial \phi^2} + \frac{\partial^2 R_{\rho}}{\partial z^2} - \frac{\partial^2 R_z}{\partial z \partial \rho} \\
&= \frac{\partial^2 R_{\rho}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial \rho} - \frac{R_{\rho}}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 R_{\rho}}{\partial \phi^2} - \frac{2}{\rho^2} \frac{\partial R_{\phi}}{\partial \phi} + \frac{\partial^2 R_{\rho}}{\partial z^2}
\end{aligned} \tag{13}$$

$$\begin{aligned}
(\nabla^2 \bar{R})_{\hat{\phi}} &= [\nabla(\nabla \cdot \bar{R})]_{\hat{\phi}} - [\nabla \times (\nabla \times \bar{R})]_{\hat{\phi}} \\
&= \left[\frac{1}{\rho} \frac{\partial^2 R_{\rho}}{\partial \phi \partial \rho} + \frac{1}{\rho^2} \frac{\partial R_{\rho}}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 R_{\phi}}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_z}{\partial \phi \partial z} \right] \\
&\quad - \left[\frac{1}{\rho} \frac{\partial^2 R_z}{\partial z \partial \phi} - \frac{\partial^2 R_{\phi}}{\partial z^2} + \frac{R_{\phi}}{\rho^2} - \frac{1}{\rho} \frac{\partial R_{\phi}}{\partial \rho} - \frac{\partial^2 R_{\phi}}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial R_{\rho}}{\partial \phi} + \frac{1}{\rho} \frac{\partial^2 R_{\rho}}{\partial \rho \partial \phi} \right] \\
&= \frac{1}{\rho} \frac{\partial^2 R_{\rho}}{\partial \phi \partial \rho} + \frac{1}{\rho^2} \frac{\partial R_{\rho}}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 R_{\phi}}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_z}{\partial \phi \partial z} \\
&\quad - \frac{1}{\rho} \frac{\partial^2 R_z}{\partial z \partial \phi} + \frac{\partial^2 R_{\phi}}{\partial z^2} - \frac{R_{\phi}}{\rho^2} + \frac{1}{\rho} \frac{\partial R_{\phi}}{\partial \rho} + \frac{\partial^2 R_{\phi}}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial R_{\rho}}{\partial \phi} - \frac{1}{\rho} \frac{\partial^2 R_{\rho}}{\partial \rho \partial \phi} \\
&= \frac{\partial^2 R_{\phi}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R_{\phi}}{\partial \rho} - \frac{R_{\phi}}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 R_{\phi}}{\partial \phi^2} + \frac{2}{\rho^2} \frac{\partial R_{\rho}}{\partial \phi} + \frac{\partial^2 R_{\phi}}{\partial z^2}
\end{aligned} \tag{14}$$

$$\begin{aligned}
(\nabla^2 \bar{R})_{\hat{z}} &= [\nabla(\nabla \cdot \bar{R})]_{\hat{z}} - [\nabla \times (\nabla \times \bar{R})]_{\hat{z}} \\
&= \left[\frac{\partial^2 R_{\rho}}{\partial z \partial \rho} + \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial z} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial z \partial \phi} + \frac{\partial^2 R_z}{\partial z^2} \right] \\
&\quad - \left[\frac{1}{\rho} \frac{\partial R_{\rho}}{\partial z} - \frac{1}{\rho} \frac{\partial R_z}{\partial \rho} + \frac{\partial^2 R_{\rho}}{\partial \rho \partial z} - \frac{\partial^2 R_z}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2 R_z}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \phi \partial z} \right] \\
&= \frac{\partial^2 R_{\rho}}{\partial z \partial \rho} + \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial z} + \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial z \partial \phi} + \frac{\partial^2 R_z}{\partial z^2} \\
&\quad - \frac{1}{\rho} \frac{\partial R_{\rho}}{\partial z} + \frac{1}{\rho} \frac{\partial R_z}{\partial \rho} - \frac{\partial^2 R_{\rho}}{\partial \rho \partial z} + \frac{\partial^2 R_z}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 R_z}{\partial \phi^2} - \frac{1}{\rho} \frac{\partial^2 R_{\phi}}{\partial \phi \partial z} \\
&= \frac{\partial^2 R_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 R_z}{\partial \phi^2} + \frac{\partial^2 R_z}{\partial z^2}
\end{aligned} \tag{15}$$

D.3. References

- [1] P. Lorrain, D.P. Corson, F. Lorrain, *Electromagnetic fields and waves*, 3rd Edition, W.H. Freeman and Company, 1988.

Appendix E

Different Perspectives on the Potential Difference

E.1. Introduction

The determination of the potential difference (voltage) over a circuit component (for example a resistor) produces a difference in sign, when considered from a fundamental as opposed to a circuit perspective. This appendix investigates and clarifies this difference.

E.2. The potential difference from a fundamental perspective

The unit of potential difference is the volt, which is the reason it is also referred to as the voltage. They refer however to the same quantity, and will be interchanged without discretion throughout this investigation. The formal definition of potential difference is given in equation (1) and will be explained with regards to Figure E.1. Consider the two charged plates of Figure E.1 which produces an electric field, with a positive unit test charge situated inside this field. The energy of this field-charge system is measured in terms of potential energy and any conversion of the potential energy to another form is considered a loss to the system. Given that the charge is stationary the total energy of this field-charge system is comprised of the potential energy stored in the electric field and the potential energy of the charge due to its position within the field. Take note that the unit charge is small enough in order not to influence the field distribution (created by the external charges in the conductors). The potential or voltage is then defined as the change of energy of this field-charge system due to movement of the unit test charge within this field.

If the unit charge is released it will accelerate and move a distance d . This will reduce the potential energy of the system (as the potential energy of the charge was converted to kinetic energy) and the equation (1) will have a negative value. By doing the opposite and having an external source move the test charge from the negative plate against the electric field to the positive plate will increase the potential energy of the system and equation (1) will be positive.

$$V = -\int \vec{E} \cdot d\vec{s} \quad (1)$$

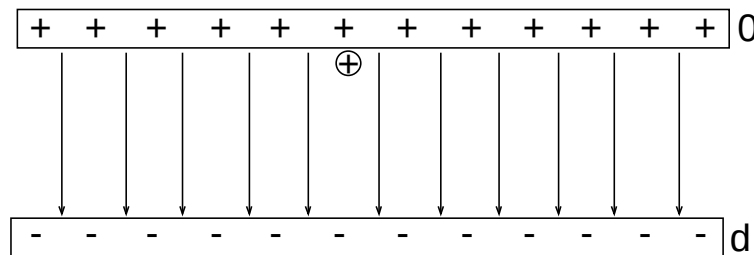


Figure E.1. Field-charge system used in the definition of the potential difference

Given below in Figure E.2, is a basic circuit consisting of a voltage source and a resistive load. The source and load are expanded to allow for an explanation from a fundamental electromagnetics perspective, as opposed to a circuit perspective.

The voltage source is generic and can be any source (such as a battery or a dc motor). The connection between the source and the load can be considered ideal and is of no importance. The flow of current will be represented by two positive unit test charges. They start at their respective positions (as indicated) and move simultaneously through the load and the source.

The resistive load and its test charge will be analysed first. It is represented by two ideal conductive plates having opposite charge, which creates a resultant field over a resistive material, with a test charge situated at the positive plate (identical to Figure E.1). Considering now the change in potential energy of this field-charge setup,

$$V_R = -\int_0^{d_1} \vec{E}_R \cdot d\vec{s}_R \quad (2)$$

where the subscript R is associated with the resistive load.

$$\begin{aligned} \vec{E}_R &= E_R \hat{i} \\ d\vec{s}_R &= ds_R \hat{i} \end{aligned} \quad (3)$$

Since the electric field and the path of integration are parallel and in the same direction, equation (2) reduces to,

$$V_R = -\int_0^{d_1} E_R ds_R = -E_R d_1 \quad (4)$$

Since V_R is negative it can be seen that the field-charge setup of the resistive load has lost potential energy due to the movement of the test charge. This energy was converted to kinetic energy which in turn was converted into thermal energy (I^2R).

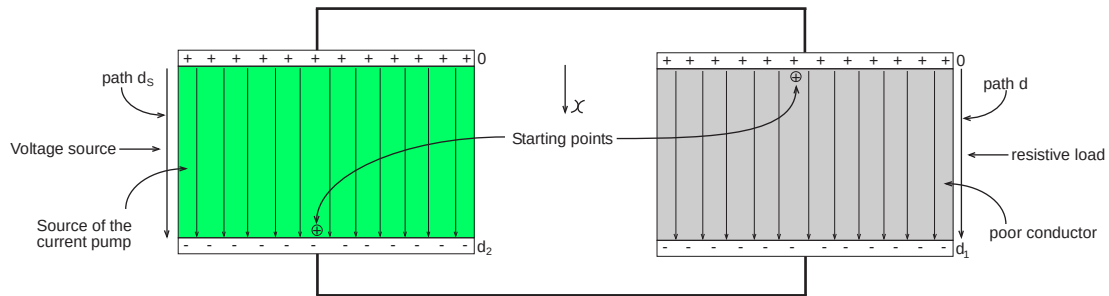


Figure E.2. Series connected voltage source and resistor expanded to allow for an electromagnetic perspective

As the charge of the resistive load moves through the resistor a similar charge will move through the voltage source. The internal workings of the voltage source can be modelled as a positive and a negative plate with a charge pump situated between them. The charge pump applies a force to the positive test charge against the opposing force from the electric field (created by the charged plates), that moves the charge from the negative plate (or terminal) to the positive plate. The internal workings of this charge pump is obviously dependent on the type of voltage source under consideration, but as the same model applies to all of them is not of significant importance to this discussion. The change in potential of the field-charge setup of the voltage source can now be determined,

$$V_s = -\int_{d_2}^0 \bar{E}_s \cdot d\bar{s}_s \quad (5)$$

where the subscript S is associated with the voltage source. The following is a quote from Hayt ([1]) and should be taken into consideration when evaluating line integrals, “One of the pitfalls in evaluating line integrals is a tendency to use too many minus signs when a charge is moved in the direction of a *decreasing* coordinate value. This is taken care of completely by the limits of the integral, and no misguided attempt should be made to change the sign of $d\bar{s}$ ”.

Therefore, in order to evaluate the line integral from d_2 to 0, the limits of the integral should be changes around (as can be seen in equation (5)), but the sign of the differential vector element $d\bar{s}$ should not be changed from its counterpart in equation (3). Therefore,

$$\begin{aligned} \bar{E}_s &= E_s \hat{i} \\ d\bar{s}_s &= ds_s \hat{i} \end{aligned} \quad (6)$$

Equation (5) therefore becomes,

$$\begin{aligned} V_s &= -\int_{d_2}^0 \bar{E}_s \cdot d\bar{s}_s \\ &= -\int_{d_2}^0 E_s \cdot ds_s \\ &= -E_s [0 - d_2] \\ &= E_s d_2 \end{aligned} \quad (7)$$

As can be seen the change in potential is positive, which is to be expected, as the potential energy of the field-charge setup of the voltage source has gained energy. This gained energy is in turn supplied by the charge pump and by whichever means it functions. When considering this from an energy perspective it is clear that the current pump supplies energy to the field-charge setup of the voltage source. This energy is in turn transferred to the resistive load, where it first gets transferred to kinetic and then to thermal energy.

E.3. The potential difference from a circuit perspective

The setup in Figure E.2 can now be considered from a circuit perspective, shown in Figure E.3. Taking KVL around the circuit results in,

$$-V_S + V_R = 0 \quad (8)$$

Multiplying by the current in order to obtain the power,

$$-P_{S_C} + P_{R_C} = 0 \quad (9)$$

where the subscript C refers to the circuit perspective.

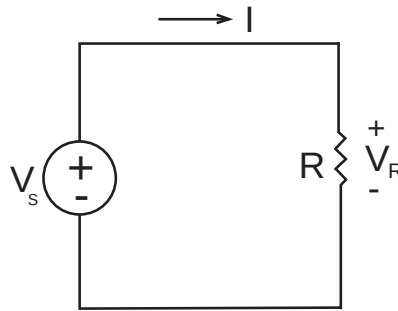


Figure E.3. Circuit perspective of a series connected voltage source and resistive load

The power delivered and lost can now be determined from the fundamental perspective. By multiplying the change in potential of the resistive load and voltage source (equations (4) and (7)) by the actual values of the charges which moved through their electric fields, produces the change in their potential energies. Dividing these results by the time it took for the charges to move through them, produces the power delivered and lost, by the voltage source and resistive load, respectively. Stated another way, if the change in potential of the resistive load and voltage source (equations (4) and (7)) are multiplied by the current flowing through them (charge per time) the power of the load and source can be determined. Adding these power,

$$P_{S_F} - P_{R_F} = 0 \quad (10)$$

where the subscript F designates that these values have been obtained from the fundamental perspective. As can be seen the circuit perspective and the fundamental perspective have opposing signs. I.e. the source powers of the fundamental and circuit perspectives have opposing signs. Similarly, the resistor powers of the fundamental and circuit perspectives also have opposite signs.

This can be understood by considering the different conventions used in these two analyses. From a circuit perspective a resistor absorbs power, with the sign of that power being positive, while the fundamental approach states that if power is lost by a system its potential energy per time is decreased and the power therefore has a negative sign. From this is clear that the

reference points of these two approaches are opposite, with the result that if voltages are calculated from a fundamental perspective and needs to be converted to a circuit perspective (which is used most of the time) the sign of the result needs to be changed.

This can be thought of in the sense that the circuit perspective uses the negative as a reference point, whereas the fundamental perspective uses the positive as a reference point. Therefore, if the direction of integration of equation (1) is turned around, i.e. integrating from the negative to the positive conductor, or if it is assumed that the electric field flows from the negative to the positive (i.e. opposite to its actual direction) the same result as the circuit perspective can be obtained.

E.4. References

- [1] W.H. Hayt, Jr., J.A. Buck, *Engineering Electromagnetics*, 7th edition, p.86, 2006.

Appendix F

Multi-layer Conductor Analysis for the Case where the Conductivity is a Function of Distance

F.1. Introduction

This appendix investigates the feasibility of a different approach to the solution of a multi-layer conductor. The multi-layer derivation will be considered from the perspective where the conductivity of a single conductor varies as a function of distance. As working in Cartesian coordinates generally proves easier than cylindrical coordinates, the semi-infinite flat conductor will be considered.

F.2. Rederivation of the 1D solution for σ being a function of y

The single-layer equation and solution of chapter 2 was derived under to condition that the conductivity is constant for the thickness of the layer. For the approach taken in this appendix the equation will have to be rederived in order to take the variation of the conductivity into account. Starting with Faraday's law in point form and assuming that the total conductor structure is infinitely long and wide,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (1)$$

$$\nabla \times \bar{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial E_x}{\partial y} \quad (2)$$

$$\frac{\partial E_x}{\partial y} = \mu(y) \frac{\partial \bar{H}_z}{\partial t} \quad (3)$$

Under sinusoidal excitation the time derivative can be replaced by $j\omega$ and from Ohm's law in point form (equation (4) below) equation (3) can be written as,

$$\bar{J} = \sigma \bar{E} \rightarrow \bar{E} = \frac{\bar{J}}{\sigma} \quad (4)$$

$$\frac{d}{dy} \left[\frac{J_x(y)}{\sigma(y)} \right] = j\omega \mu(y) H_z(y) \quad (5)$$

Taking the derivative on the left hand side and applying the product rule,

$$\begin{aligned}\frac{d}{dy}[J_x(y)\sigma^{-1}(y)] &= \frac{d}{dy}[J_x(y)]\sigma^{-1}(y) + \frac{d}{dy}[\sigma^{-1}(y)]J_x(y) \\ &= \frac{d}{dy}[J_x(y)]\sigma^{-1}(y) - \sigma^{-2}(y)\frac{d}{dy}[\sigma(y)]J_x(y)\end{aligned}\quad (6)$$

Multiply by $\sigma(y)/\sigma(y)$ on the right hand side,

$$\begin{aligned}\frac{d}{dy}[J_x(y)\sigma^{-1}(y)] &= \frac{d}{dy}[J_x(y)]\sigma^{-1}(y) - \sigma^{-2}(y)\frac{d}{dy}[\sigma(y)]J_x(y) \\ &= \frac{\frac{d}{dy}[J_x(y)]\sigma(y) - \frac{d}{dy}[\sigma(y)]J_x(y)}{\sigma^2(y)}\end{aligned}\quad (7)$$

Substitute this result back into equation (5) and move the $\sigma^2(y)$ term to the right hand side,

$$\frac{d}{dy}[J_x(y)]\sigma(y) - \frac{d}{dy}[\sigma(y)]J_x(y) = j\omega\mu(y)\sigma^2(y)H_z(y)\quad (8)$$

From Ampere's law in point form the following can be obtained,

$$\begin{aligned}\nabla \times \vec{H} &= \vec{J} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} &= J_x\hat{i} + J_y\hat{j} + J_z\hat{k}\end{aligned}\quad (9)$$

Under the semi-infinite approximation of the flat conductor equation (9) reduces to,

$$\frac{dH_z(y)}{dy} = J_x(y)\quad (10)$$

Substituting this into equation (8) allows it to be written in terms of the magnetic field intensity,

$$\frac{d^2H_z(y)}{dy^2}\sigma(y) - \frac{d}{dy}[\sigma(y)]\frac{dH_z(y)}{dy} = j\omega\mu(y)\sigma^2(y)H_z(y)\quad (11)$$

Dividing throughout by $\sigma(y)$ and converting equation (11) into a homogenous equation,

$$\frac{d^2H_z(y)}{dy^2} - \sigma^{-1}(y)\sigma'(y)\frac{dH_z(y)}{dy} - j\omega\mu(y)\sigma(y)H_z(y) = 0\quad (12)$$

This is the equation that needs to be solved in order to obtain the solution of a conductor whose conductivity varies along its thickness.

F.3. Second-order linear ordinary differential equation with variable coefficients

Before investigating the possibility of a solution, some clarification needs to be given. Equation (12) is a second-order linear ordinary differential equation (also just known as a second-order linear differential equation). The second-order part refers to the highest order derivative present within this equation, while the linear description refers to two factors. Firstly, the argument of the derivative is of the first degree, i.e. to the power of one, while, secondly, each coefficient depends at most on the independent variable, which in this case is y . Lastly, the ordinary part refers to all the derivatives being ordinary derivatives, i.e. only one independent variable (y in this case) is present and all other variables are dependent on it.

The majority of the second-order linear ordinary differential equations that are encountered have constant coefficients, resulting in the well-known solution expressed in terms of elementary functions ([1]). This is the solution that was encountered and used in chapters 2 and 4, for each of the individual layers of the multi-layer conductor. However, in the case of equation (12) each of the coefficients are dependent on y , thereby preventing the solution by elementary functions (section 6.1 of [1]), as this will result in an incorrect answer. Equations of this type are normally solved by use of an infinite series, such as the power series.

F.4. Problems encountered with the conductivity function

The method for solving a second-order linear ordinary differential equation with variable coefficients is given in section 6.1 of [1]. This solution method is, however, not the problem when considering equation (12). It is rather the function $\sigma(y)$ will have to take to be representative of a practical multi-layer conductor. This can be clarified by realising that for $\sigma(y)$ to represent a multi-layer conductor, it should contain step function for each transition to a different layer. Considering a three-layer conductor as an example, the function of the conductivity as a function of y should look like that given in Figure F.1, where the values of σ_1 to σ_3 and b_1 to b_3 were arbitrarily chosen.

This function, however, proves to be a significant problem, as the derivative of the conductivity in equation (12) will produce an infinite result. It therefore follows, that in order for equation (12) to be solved, an analytical approximation to the step function is required. Even though approximations to this function exist, such as the tanh function, the rate of change of the conductivity at the transition between layers will have to be limited in order to avoid an infinite result. This in turn will produce errors in the final solution, as $\sigma(y)$ will not be representative of the actual conductor. As no obvious solution to this problem exists, the solution of equation (12) will be abandoned in favour of the solution presented in chapter 4.

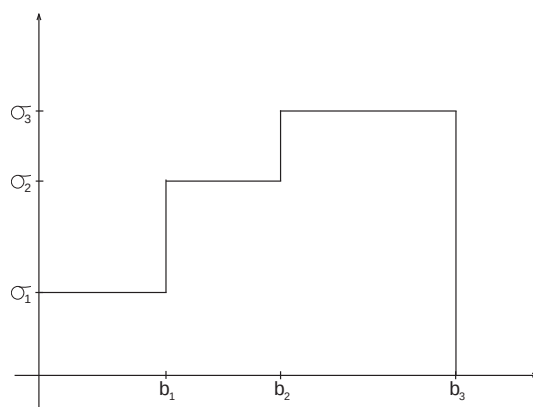


Figure F.1. Conductivity of a three-layer conductor

F.5. References

- [1] D.G. Zill, M.R. Cullen, *Differential Equations with Boundary-Value Problems*, 5th edition, Brooks/Cole, 2001.

Appendix G

Coefficients of the three-layer Cylindrical Conductor

G.1. Introduction

This appendix gives the six coefficients of the three-layer cylindrical conductor, as determined by use of Mathematica. Since chapter 4 and Mathematica's notation differ slightly, a lookup table is given below.

Table G.1. Look-up table

Mathematica notation	Chapter 4 notation
A1	A_1
B1	B_1
A2	A_2
B2	B_2
A3	A_3
B3	B_3
Hs0	H_{s0}
Hs3	H_{s3}
b0	b_0
b1	b_1
cond1	σ_1
cond2	σ_2
cond3	σ_3
D1	D_1
D2	D_2
D3	D_3
BesselI[u,v]	Modified Bessel function of the first kind of order u and argument v
BesselK[u,v]	Modified Bessel function of the second kind of order u and argument v

$$\begin{aligned}
A1 = & \{ (cond2 D3 Hs0 Bessel[1, b2 D2] (cond1 D2 Bessel[0, b1 D2] Bessel[1, b1 D1] - cond2 D1 Bessel[0, b1 D1] Bessel[1, b1 D2]) (Bessel[1, b3 D3] Bessel[0, b2 D3] + Bessel[0, b2 D3] Bessel[1, b3 D3]) + \\
& cond2 Bessel[1, b1 D2] (D1 Hs0 Bessel[1, b3 D3] Bessel[0, b1 D1] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + \\
& Bessel[1, b2 D3] (-cond1 D2 D3 Hs3 Bessel[0, b1 D2] Bessel[0, b2 D3] Bessel[1, b0 D1] + cond3 D1 D2 Hs0 Bessel[0, b1 D1] Bessel[0, b2 D2] Bessel[1, b3 D3]) + \\
& D3 Bessel[0, b2 D3] (-cond1 D2 Hs3 Bessel[0, b1 D2] Bessel[1, b0 D1] Bessel[1, b2 D3] + cond2 D1 Hs0 Bessel[0, b1 D1] Bessel[1, b2 D2] Bessel[1, b3 D3]) + \\
& D2 (cond3 Hs0 Bessel[0, b2 D2] (cond1 D2 Bessel[0, b1 D2] Bessel[1, b1 D1] - cond2 D1 Bessel[0, b1 D1] Bessel[1, b1 D2]) (Bessel[1, b3 D3] Bessel[1, b2 D3] - Bessel[1, b2 D3] Bessel[1, b3 D3]) + \\
& cond1 Bessel[0, b1 D2] (Hs0 Bessel[1, b3 D3] Bessel[1, b1 D1] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + \\
& Bessel[1, b2 D3] (-cond2 D3 Hs3 Bessel[0, b2 D3] Bessel[1, b0 D1] Bessel[1, b1 D2] + cond3 D2 Hs0 Bessel[0, b2 D2] Bessel[1, b1 D1] Bessel[1, b3 D3]) + \\
& cond2 D3 Bessel[0, b2 D3] (-Hs3 Bessel[1, b0 D1] Bessel[1, b1 D2] Bessel[1, b2 D2] Bessel[1, b3 D3] + Hs0 Bessel[1, b1 D1] Bessel[1, b2 D2] Bessel[1, b3 D3])) / \\
& (-cond1 D2 Bessel[1, b1 D1] Bessel[1, b0 D1] (cond2 D3 Bessel[1, b2 D2] Bessel[0, b1 D2] (Bessel[1, b3 D3] Bessel[0, b2 D3] + Bessel[0, b2 D3] Bessel[1, b3 D3]) + \\
& cond3 D2 Bessel[0, b2 D2] Bessel[0, b1 D2] (Bessel[1, b3 D3] Bessel[1, b2 D3] - Bessel[1, b2 D3] Bessel[1, b3 D3]) + \\
& Bessel[0, b1 D2] (Bessel[1, b3 D3] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + (cond3 D2 Bessel[1, b2 D3] Bessel[0, b2 D2] + cond2 D3 Bessel[0, b2 D2] Bessel[1, b3 D3]) + \\
& cond2 D1 Bessel[0, b1 D1] Bessel[1, b0 D1] (-cond2 D3 Bessel[1, b1 D2] (Bessel[1, b3 D3] Bessel[0, b2 D3] + Bessel[0, b2 D3] Bessel[1, b3 D3]) + \\
& cond3 D2 Bessel[0, b2 D2] Bessel[1, b1 D2] (-Bessel[1, b3 D3] Bessel[1, b2 D3] + Bessel[1, b2 D3] Bessel[1, b3 D3]) + \\
& Bessel[1, b1 D2] (Bessel[1, b3 D3] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + (cond3 D2 Bessel[1, b2 D3] Bessel[0, b2 D2] + cond2 D3 Bessel[0, b2 D2] Bessel[1, b3 D3]) + \\
& Bessel[1, b0 D1] (cond2 D3 Bessel[1, b2 D2] (cond1 D2 Bessel[0, b1 D2] Bessel[1, b1 D1] - cond2 D1 Bessel[0, b1 D1] Bessel[1, b1 D2]) (Bessel[1, b3 D3] Bessel[0, b2 D3] + Bessel[0, b2 D3] Bessel[1, b3 D3]) + \\
& cond2 D1 Bessel[1, b1 D2] Bessel[0, b1 D1] (Bessel[1, b3 D3] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + \\
& (cond3 D2 Bessel[1, b2 D3] Bessel[0, b2 D2] + cond2 D3 Bessel[0, b2 D2] Bessel[1, b2 D2]) Bessel[1, b3 D3]) + \\
& D2 (cond3 Bessel[0, b2 D2] (cond1 D2 Bessel[0, b1 D2] Bessel[1, b1 D1] - cond2 D1 Bessel[0, b1 D1] Bessel[1, b1 D2]) (Bessel[1, b3 D3] Bessel[1, b2 D3] - Bessel[1, b2 D3] Bessel[1, b3 D3]) + \\
& cond1 Bessel[0, b1 D2] Bessel[1, b1 D1] (Bessel[1, b3 D3] (cond2 D3 Bessel[0, b2 D3] Bessel[1, b2 D2] - cond3 D2 Bessel[0, b2 D2] Bessel[1, b2 D3]) + \\
& (cond3 D2 Bessel[1, b2 D3] Bessel[0, b2 D2] + cond2 D3 Bessel[0, b2 D2] Bessel[1, b2 D2]) Bessel[1, b3 D3])) \}
\end{aligned}$$

G.(1)

$$\begin{aligned}
B1 = & \{ (cond1 \ cond2 \ D2 \ D3 \ Hs3 \ Bessel[1, b0 \ D1] (Bessel[1, b1 \ D2] \ BesselK[0, b1 \ D2] - Bessel[0, b1 \ D2] \ BesselK[1, b1 \ D2]) (Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b2 \ D3]) - \\
& Hs0 \ (cond1 \ D2 \ Bessel[1, b1 \ D1] (cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[0, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[0, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad Bessel[0, b1 \ D2] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3]) + \\
& \quad cond2 \ D1 \ Bessel[0, b1 \ D1] (cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[1, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[1, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) - \\
& \quad \quad Bessel[1, b1 \ D2] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad \quad (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3])) / \\
& (-cond1 \ D2 \ Bessel[1, b1 \ D1] \ BesselK[1, b0 \ D1] (cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[0, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[0, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad Bessel[0, b1 \ D2] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad \quad + cond2 \ D1 \ Bessel[0, b1 \ D1] \ BesselK[1, b0 \ D1] (-cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[1, b1 \ D2] (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad \quad cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[1, b1 \ D2] (-Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad \quad Bessel[1, b1 \ D2] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad \quad \quad + Bessel[1, b0 \ D1] (cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[0, b1 \ D2] \ BesselK[1, b1 \ D1] - cond2 \ D1 \ BesselK[0, b1 \ D1] \ BesselK[1, b1 \ D2]) (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad \quad cond2 \ D1 \ Bessel[1, b1 \ D2] \ BesselK[0, b1 \ D1] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad \quad \quad (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3]) + \\
& \quad \quad \quad D2 (cond3 \ Bessel[0, b2 \ D2] (cond1 \ D2 \ BesselK[0, b1 \ D2] \ BesselK[1, b1 \ D1] - cond2 \ D1 \ BesselK[0, b1 \ D1] \ BesselK[1, b1 \ D2]) (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& \quad \quad \quad cond1 \ Bessel[0, b1 \ D2] \ BesselK[1, b1 \ D1] (Bessel[1, b3 \ D3] (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& \quad \quad \quad \quad (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3])))
\end{aligned}$$

$$\begin{aligned}
A^2 = & \{ (cond2 \ D3 \ Hs3 \ Bessel[1, b0 \ D1] \ (cond1 \ D2 \ BesselK[0, b1 \ D2] \ BesselK[1, b1 \ D1] - cond2 \ D1 \ BesselK[0, b1 \ D1] \ BesselK[1, b1 \ D2]) \ (Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b2 \ D3]) + \\
& Bessel[1, b1 \ D1] \ (D1 \ Hs0 \ Bessel[1, b3 \ D3] \ BesselK[0, b1 \ D1] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& Bessel[1, b2 \ D3] \ (-cond1 \ D2 \ D3 \ Hs3 \ BesselK[0, b1 \ D2] \ BesselK[0, b2 \ D3] \ BesselK[1, b0 \ D1] + cond3 \ D1 \ D2 \ Hs0 \ BesselK[0, b1 \ D1] \ BesselK[0, b2 \ D2] \ BesselK[1, b3 \ D3]) + \\
& D3 \ Bessel[0, b2 \ D3] \ (-cond1 \ D2 \ Hs3 \ BesselK[0, b1 \ D2] \ BesselK[1, b0 \ D1] \ BesselK[1, b2 \ D3] + cond2 \ D1 \ Hs0 \ BesselK[0, b1 \ D1] \ BesselK[1, b2 \ D2] \ BesselK[1, b3 \ D3])) + \\
& D1 \ Bessel[0, b1 \ D1] \ (Hs0 \ Bessel[1, b3 \ D3] \ BesselK[1, b1 \ D1] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& Bessel[1, b2 \ D3] \ (-cond2 \ D3 \ Hs3 \ BesselK[0, b2 \ D3] \ BesselK[1, b0 \ D1] \ BesselK[1, b1 \ D2] + cond3 \ D2 \ Hs0 \ BesselK[0, b2 \ D2] \ BesselK[1, b1 \ D1] \ BesselK[1, b3 \ D3]) + \\
& cond2 \ D3 \ Bessel[0, b2 \ D3] \ (-Hs3 \ BesselK[1, b0 \ D1] \ BesselK[1, b1 \ D2] \ BesselK[1, b2 \ D3] + Hs0 \ BesselK[1, b1 \ D1] \ BesselK[1, b2 \ D2] \ BesselK[1, b3 \ D3])) / \\
& (-cond1 \ D2 \ Bessel[1, b1 \ D1] \ BesselK[1, b0 \ D1] \ (cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& Bessel[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D2] \ BesselK[1, b3 \ D3])) \\
& + cond2 \ D1 \ Bessel[0, b1 \ D1] \ BesselK[1, b0 \ D1] \ (-cond2 \ D3 \ Bessel[1, b2 \ D2] \ BesselK[1, b1 \ D2] \ (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& cond3 \ D2 \ Bessel[0, b2 \ D2] \ BesselK[1, b1 \ D2] \ (-Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] + Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& Bessel[1, b1 \ D2] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D2] \ BesselK[1, b3 \ D3])) \\
& + Bessel[1, b0 \ D1] \ (cond2 \ D3 \ Bessel[1, b2 \ D2] \ (cond1 \ D2 \ BesselK[0, b1 \ D1] - cond2 \ D1 \ BesselK[0, b1 \ D1] \ BesselK[1, b1 \ D2]) \ (Bessel[1, b3 \ D3] \ BesselK[0, b2 \ D3] + Bessel[0, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& cond2 \ D1 \ Bessel[1, b1 \ D2] \ BesselK[0, b1 \ D1] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D2] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3]) + \\
& D2 \ (cond3 \ Bessel[0, b2 \ D2] \ (cond1 \ D2 \ BesselK[0, b1 \ D2] \ BesselK[1, b1 \ D1] - cond2 \ D1 \ BesselK[0, b1 \ D1] \ BesselK[1, b1 \ D2]) \ (Bessel[1, b3 \ D3] \ BesselK[1, b2 \ D3] - Bessel[1, b2 \ D3] \ BesselK[1, b3 \ D3]) + \\
& cond1 \ Bessel[0, b1 \ D2] \ BesselK[1, b1 \ D1] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ BesselK[0, b2 \ D3] \ BesselK[1, b2 \ D2] - cond3 \ D2 \ BesselK[0, b2 \ D2] \ BesselK[1, b2 \ D3]) + \\
& (cond3 \ D2 \ Bessel[1, b2 \ D3] \ BesselK[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D3] \ BesselK[1, b2 \ D2]) \ BesselK[1, b3 \ D3]))))
\end{aligned}$$

$$\begin{aligned}
b2 = & \{ (cond2 \ D3 \ Hs3 \ Bessel[1, b0 \ D1] \ (cond2 \ D1 \ Bessel[1, b1 \ D2] \ Bessel[0, b1 \ D1] + cond1 \ D2 \ Bessel[0, b1 \ D2] \ Bessel[1, b1 \ D1]) \ (Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D3]) - \\
& D1 \ Bessel[0, b1 \ D1] \ (-cond2 \ D3 \ Hs3 \ Bessel[1, b1 \ D2] \ Bessel[1, b0 \ D1] \ (Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D3]) + \\
& Hs0 \ Bessel[1, b1 \ D1] \ (cond2 \ D3 \ Bessel[1, b2 \ D2] \ (Bessel[1, b3 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b3 \ D3]) + cond3 \ D2 \ Bessel[0, b2 \ D2] \ (Bessel[1, b3 \ D3] \ Bessel[1, b2 \ D3] - Bessel[1, b2 \ D3] \ Bessel[1, b3 \ D3])) - \\
& Bessel[1, b1 \ D1] \ (cond2 \ D1 \ D3 \ Hs0 \ Bessel[1, b2 \ D2] \ Bessel[0, b1 \ D1] \ (Bessel[1, b3 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& D2 \ (cond1 \ D3 \ Hs3 \ Bessel[0, b1 \ D2] \ Bessel[1, b0 \ D1] \ (Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D3]) + \\
& cond3 \ D1 \ Hs0 \ Bessel[0, b2 \ D2] \ Bessel[0, b1 \ D1] \ (Bessel[1, b3 \ D3] \ Bessel[1, b2 \ D3] - Bessel[1, b2 \ D3] \ Bessel[1, b3 \ D3])))) / \\
& (-cond1 \ D2 \ Bessel[1, b1 \ D1] \ Bessel[1, b0 \ D1] \ (cond2 \ D3 \ Bessel[1, b2 \ D2] \ Bessel[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& cond3 \ D2 \ Bessel[0, b2 \ D2] \ Bessel[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ Bessel[1, b2 \ D3] - Bessel[1, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& Bessel[0, b1 \ D2] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2] - cond3 \ D2 \ Bessel[1, b2 \ D2] \ Bessel[1, b2 \ D3]) + (cond3 \ D2 \ Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D2] \ Bessel[1, b2 \ D2]) \ Bessel[1, b3 \ D3])) \\
& + cond2 \ D1 \ Bessel[0, b1 \ D1] \ Bessel[1, b0 \ D1] \ (-cond2 \ D3 \ Bessel[1, b2 \ D2] \ Bessel[1, b1 \ D2] \ (Bessel[1, b3 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& cond3 \ D2 \ Bessel[0, b2 \ D2] \ Bessel[1, b1 \ D2] \ (-Bessel[1, b3 \ D3] \ Bessel[1, b2 \ D3] + Bessel[1, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& Bessel[1, b1 \ D2] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2] - cond3 \ D2 \ Bessel[1, b2 \ D2] \ Bessel[1, b2 \ D3]) + (cond3 \ D2 \ Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D2] \ Bessel[1, b2 \ D2]) \ Bessel[1, b3 \ D3])) \\
& + Bessel[1, b0 \ D1] \ (cond2 \ D3 \ Bessel[1, b2 \ D2] \ (cond1 \ D2 \ Bessel[0, b1 \ D2] \ Bessel[1, b1 \ D1] - cond2 \ D1 \ Bessel[0, b1 \ D1] \ Bessel[1, b1 \ D2]) \ (Bessel[1, b3 \ D3] \ Bessel[0, b2 \ D3] + Bessel[0, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& cond2 \ D1 \ Bessel[1, b1 \ D2] \ Bessel[0, b1 \ D1] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2] - cond3 \ D2 \ Bessel[1, b2 \ D2] \ Bessel[1, b2 \ D3]) + \\
& (cond3 \ D2 \ Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2]) \ Bessel[1, b3 \ D3]) + \\
& D2 \ (cond3 \ Bessel[0, b2 \ D2] \ (cond1 \ D2 \ Bessel[0, b1 \ D2] \ Bessel[1, b1 \ D1] - cond2 \ D1 \ Bessel[0, b1 \ D1] \ Bessel[1, b1 \ D2]) \ (Bessel[1, b3 \ D3] \ Bessel[1, b2 \ D3] - Bessel[1, b2 \ D3] \ Bessel[1, b3 \ D3]) + \\
& cond1 \ Bessel[0, b1 \ D2] \ Bessel[1, b1 \ D1] \ (Bessel[1, b3 \ D3] \ (cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2] - cond3 \ D2 \ Bessel[0, b2 \ D2] \ Bessel[1, b2 \ D3]) + \\
& (cond3 \ D2 \ Bessel[1, b2 \ D3] \ Bessel[0, b2 \ D2] + cond2 \ D3 \ Bessel[0, b2 \ D3] \ Bessel[1, b2 \ D2]) \ Bessel[1, b3 \ D3]))))
\end{aligned}$$

G.(4)

$$\begin{aligned}
A3 = & \{ \{ Hs3 \text{ Bessel}[1, b0 D1] \text{ (cond2 D3 Bessel}[1, b2 D2] \text{ Bessel}[0, b2 D3] \text{ (cond1 D1 Bessel}[0, b1 D1] + \text{cond2 D1 Bessel}[0, b1 D1] \text{ Bessel}[1, b1 D2]) \text{ Bessel}[1, b1 D2])} \\
& \text{cond2 D1 Bessel}[1, b1 D2] \text{ Bessel}[0, b1 D1] \text{ (-cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] + \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \\
& \text{D2 (cond3 Bessel}[0, b2 D2] \text{ (-cond1 D2 Bessel}[0, b1 D2] \text{ Bessel}[1, b1 D1] + \text{cond2 D1 Bessel}[0, b1 D1] \text{ Bessel}[1, b1 D2]) \text{ Bessel}[1, b1 D2])} \\
& \text{cond1 Bessel}[0, b1 D2] \text{ Bessel}[1, b1 D1] \text{ (-cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] + \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} \} \\
& \text{D2 Bessel}[1, b1 D1] \text{ (cond1 Hs3 Bessel}[0, b1 D2] \text{ Bessel}[1, b0 D1] \text{ (cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] + \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \\
& \text{cond2 Bessel}[1, b2 D2] \text{ (cond1 D3 Hs3 Bessel}[0, b1 D2] \text{ Bessel}[1, b0 D1] - \text{cond3 D1 Hs0 Bessel}[0, b1 D1] \text{ Bessel}[0, b2 D2] \text{ Bessel}[1, b3 D3])} + \\
& \text{cond3 Bessel}[0, b2 D2] \text{ (cond1 D2 Hs3 Bessel}[0, b1 D2] \text{ Bessel}[1, b0 D1] \text{ Bessel}[1, b2 D3] - \text{cond2 D1 Hs0 Bessel}[0, b1 D1] \text{ Bessel}[1, b2 D2] \text{ Bessel}[1, b3 D3])} \} \\
& \text{cond2 D1 Bessel}[0, b1 D1] \text{ (Hs3 Bessel}[1, b1 D2] \text{ Bessel}[1, b0 D1] \text{ (-cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] + \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \\
& \text{Bessel}[1, b2 D2] \text{ (cond2 D3 Hs3 Bessel}[0, b2 D3] \text{ Bessel}[1, b0 D1] \text{ Bessel}[1, b1 D2] - \text{cond3 D2 Hs0 Bessel}[0, b2 D2] \text{ Bessel}[1, b1 D1] \text{ Bessel}[1, b3 D3])} + \\
& \text{cond3 D2 Bessel}[0, b2 D2] \text{ (Hs3 Bessel}[1, b0 D1] \text{ Bessel}[1, b1 D2] \text{ Bessel}[1, b2 D3] - \text{Hs0 Bessel}[1, b1 D1] \text{ Bessel}[1, b2 D2] \text{ Bessel}[1, b3 D3])} \} / \\
& \text{(cond1 D2 Bessel}[1, b1 D1] \text{ Bessel}[1, b0 D1] \text{ (cond2 D3 Bessel}[1, b2 D2] \text{ Bessel}[0, b1 D2] \text{ (Bessel}[1, b3 D3] \text{ Bessel}[0, b2 D3] + \text{Bessel}[0, b2 D3] \text{ Bessel}[1, b3 D3])} + \\
& \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[0, b1 D2] \text{ (Bessel}[1, b3 D3] \text{ Bessel}[1, b2 D3] - \text{Bessel}[1, b2 D3] \text{ Bessel}[1, b3 D3])} + \\
& \text{Bessel}[0, b1 D2] \text{ (Bessel}[1, b3 D3] \text{ (cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] - \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \text{cond3 D2 Bessel}[1, b2 D3] \text{ Bessel}[1, b2 D2])} \\
& - \text{cond2 D1 Bessel}[0, b1 D1] \text{ Bessel}[1, b0 D1] \text{ (-cond2 D3 Bessel}[1, b2 D2] \text{ Bessel}[1, b1 D2] \text{ (Bessel}[1, b3 D3] \text{ Bessel}[0, b2 D3] + \text{Bessel}[0, b2 D3] \text{ Bessel}[1, b3 D3])} + \\
& \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b1 D2] \text{ (-Bessel}[1, b3 D3] \text{ Bessel}[1, b2 D3] + \text{Bessel}[1, b2 D3] \text{ Bessel}[1, b3 D3])} + \\
& \text{Bessel}[1, b1 D2] \text{ (Bessel}[1, b3 D3] \text{ (cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] - \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \text{cond3 D2 Bessel}[1, b2 D3] \text{ Bessel}[1, b2 D2])} \\
& - \text{Bessel}[1, b0 D1] \text{ (cond2 D3 Bessel}[1, b2 D2] \text{ (cond1 D2 Bessel}[0, b1 D2] \text{ Bessel}[1, b1 D1] - \text{cond2 D1 Bessel}[0, b1 D1] \text{ Bessel}[1, b1 D2]) \text{ (Bessel}[1, b3 D3] \text{ Bessel}[1, b3 D2])} \\
& \text{cond2 D1 Bessel}[1, b1 D2] \text{ Bessel}[0, b1 D1] \text{ (Bessel}[1, b3 D3] \text{ (cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] - \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \\
& \text{(cond3 D2 Bessel}[1, b2 D3] \text{ Bessel}[0, b2 D2] + \text{cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2]) \text{ Bessel}[1, b3 D3])} + \\
& \text{D2 (cond3 Bessel}[0, b2 D2] \text{ (cond1 D2 Bessel}[0, b1 D2] \text{ Bessel}[1, b1 D1] - \text{cond2 D1 Bessel}[0, b1 D1] \text{ Bessel}[1, b1 D2]) \text{ (Bessel}[1, b3 D3] \text{ Bessel}[1, b2 D3] - \text{Bessel}[1, b2 D3])} \\
& \text{cond1 Bessel}[0, b1 D2] \text{ Bessel}[1, b1 D1] \text{ (Bessel}[1, b3 D3] \text{ (cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2] - \text{cond3 D2 Bessel}[0, b2 D2] \text{ Bessel}[1, b2 D3])} + \\
& \text{(cond3 D2 Bessel}[1, b2 D3] \text{ Bessel}[0, b2 D2] + \text{cond2 D3 Bessel}[0, b2 D3] \text{ Bessel}[1, b2 D2]) \text{ Bessel}[1, b3 D3])} \} \}
\end{aligned}$$

$$\begin{aligned}
B3 = & \{D2 \text{ Bessel}[1, b1 D1] \{cond2 \text{ Bessel}[1, b2 D2] \{cond3 D1 \text{ Hs0 Bessel}[1, b3 D3] \text{ BesselK}[0, b2 D2] + cond1 D3 \text{ Hs3 Bessel}[0, b2 D2] \text{ BesselK}[1, b0 D1]\} + \\
& cond1 \text{ Hs3 Bessel}[0, b1 D2] \text{ BesselK}[1, b0 D1] \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[0, b2 D2] + cond2 D3 \text{ Bessel}[0, b2 D2] \text{ BesselK}[1, b2 D2]\} + \\
& \text{Bessel}[0, b2 D2] \{-cond1 cond3 D2 \text{ Hs3 Bessel}[1, b2 D3] \text{ BesselK}[0, b1 D2] \text{ BesselK}[1, b0 D1] + cond2 cond3 D1 \text{ Hs0 Bessel}[1, b3 D3] \text{ BesselK}[0, b1 D1] \text{ BesselK}[1, b2 D2]\} - \\
& \text{Hs3 Bessel}[1, b0 D1] \{cond3 D2 \text{ Bessel}[1, b2 D3] \{cond1 D2 \text{ Bessel}[0, b2 D2] \text{ BesselK}[1, b1 D1] + \text{Bessel}[0, b2 D2] \{-cond1 D2 \text{ BesselK}[0, b1 D2] \text{ BesselK}[1, b1 D1] + cond2 D1 \text{ BesselK}[0, b1 D1] \text{ BesselK}[1, b1 D2]\} + \\
& cond2 D1 \text{ Bessel}[1, b1 D2] \text{ BesselK}[0, b1 D1] \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[0, b2 D2] + cond2 D3 \text{ Bessel}[0, b2 D2] \text{ BesselK}[1, b2 D2]\} + \\
& cond2 D3 \text{ Bessel}[0, b2 D3] \{ \text{Bessel}[1, b2 D2] \{cond1 D2 \text{ BesselK}[0, b1 D2] \text{ BesselK}[1, b1 D1] - cond2 D1 \text{ BesselK}[0, b1 D1] \text{ BesselK}[1, b1 D2] \text{ BesselK}[1, b2 D2]\} + \\
& cond2 D1 \text{ Bessel}[0, b1 D1] \{ \text{Bessel}[1, b2 D2] \{cond3 D2 \text{ Hs0 Bessel}[1, b3 D3] \text{ BesselK}[0, b2 D2] \text{ BesselK}[1, b1 D1] + cond2 D3 \text{ Hs3 Bessel}[0, b2 D3] \text{ BesselK}[1, b0 D1] \text{ BesselK}[1, b1 D2] \} - \\
& \text{Hs3 Bessel}[1, b1 D2] \text{ BesselK}[1, b0 D1] \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[0, b2 D2] + cond2 D3 \text{ Bessel}[0, b2 D2] \text{ BesselK}[1, b2 D2]\} \} / \\
& cond3 D2 \text{ Bessel}[0, b2 D2] \{-\text{Hs3 Bessel}[1, b2 D3] \text{ BesselK}[1, b0 D1] \text{ BesselK}[1, b1 D2] + \text{Hs0 Bessel}[1, b3 D3] \text{ BesselK}[1, b1 D1] \text{ BesselK}[1, b2 D2]\} \} \\
& \{cond1 D2 \text{ Bessel}[1, b1 D1] \text{ BesselK}[1, b0 D1] \{cond2 D3 \text{ Bessel}[1, b2 D2] \text{ BesselK}[0, b1 D2] \{ \text{Bessel}[1, b3 D3] \text{ BesselK}[0, b2 D3] + \text{Bessel}[0, b2 D3] \text{ BesselK}[1, b3 D3] \} + \\
& cond3 D2 \text{ Bessel}[0, b2 D2] \text{ BesselK}[0, b1 D2] \{ \text{Bessel}[1, b3 D3] \text{ BesselK}[1, b2 D3] - \text{Bessel}[1, b2 D3] \text{ BesselK}[1, b3 D3] \} \} \\
& \text{Bessel}[0, b1 D2] \{ \text{Bessel}[1, b3 D3] \{cond2 D3 \text{ BesselK}[0, b2 D3] \text{ BesselK}[1, b2 D2] - cond3 D2 \text{ BesselK}[0, b2 D2] \text{ BesselK}[1, b2 D3] \} + \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[1, b2 D2] \text{ BesselK}[1, b3 D3] \} \\
& - cond2 D1 \text{ Bessel}[0, b1 D1] \text{ BesselK}[1, b0 D1] \{-cond2 D3 \text{ Bessel}[1, b2 D2] \text{ BesselK}[1, b1 D2] \text{ BesselK}[1, b3 D3] \text{ BesselK}[0, b2 D3] + \text{Bessel}[0, b2 D3] \text{ BesselK}[1, b3 D3] \} + \\
& cond3 D2 \text{ Bessel}[0, b2 D2] \text{ BesselK}[1, b1 D2] \{-\text{Bessel}[1, b3 D3] \text{ BesselK}[1, b2 D3] + \text{Bessel}[1, b2 D3] \text{ BesselK}[1, b3 D3] \} + \\
& \text{Bessel}[1, b1 D2] \{ \text{Bessel}[1, b3 D3] \{cond2 D3 \text{ BesselK}[0, b2 D3] \text{ BesselK}[1, b2 D2] - cond3 D2 \text{ BesselK}[0, b2 D2] \text{ BesselK}[1, b2 D3] \} + \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[1, b2 D2] \text{ BesselK}[1, b3 D3] \} \\
& - \text{Bessel}[1, b0 D1] \{cond2 D3 \text{ Bessel}[1, b2 D2] \text{ BesselK}[0, b1 D2] \text{ BesselK}[1, b1 D1] - cond2 D1 \text{ BesselK}[0, b1 D1] \text{ BesselK}[1, b1 D2] \} \{ \text{Bessel}[1, b3 D3] \text{ BesselK}[0, b2 D3] + \text{Bessel}[0, b2 D3] \text{ BesselK}[1, b3 D3] \} + \\
& cond2 D1 \text{ Bessel}[1, b1 D2] \text{ BesselK}[0, b1 D1] \{ \text{Bessel}[1, b3 D3] \{cond2 D3 \text{ BesselK}[0, b2 D3] \text{ BesselK}[1, b2 D2] - cond3 D2 \text{ BesselK}[0, b2 D2] \text{ BesselK}[1, b2 D3] \} + \\
& \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[0, b2 D2] + cond2 D3 \text{ Bessel}[0, b2 D3] \text{ BesselK}[1, b2 D2] \text{ BesselK}[1, b3 D3] \} \} \\
& D2 \{cond3 \text{ Bessel}[0, b2 D2] \{cond1 D2 \text{ BesselK}[0, b1 D2] \text{ BesselK}[1, b1 D1] - cond2 D1 \text{ BesselK}[0, b1 D1] \text{ BesselK}[1, b1 D2] \} \{ \text{Bessel}[1, b3 D3] \text{ BesselK}[1, b2 D3] - \text{Bessel}[1, b2 D3] \text{ BesselK}[1, b3 D3] \} + \\
& cond1 \text{ Bessel}[0, b1 D2] \text{ BesselK}[1, b1 D1] \{ \text{Bessel}[1, b3 D3] \{cond2 D3 \text{ BesselK}[0, b2 D3] \text{ BesselK}[1, b2 D2] - cond3 D2 \text{ BesselK}[0, b2 D2] \text{ BesselK}[1, b2 D3] \} + \\
& \{cond3 D2 \text{ Bessel}[1, b2 D3] \text{ BesselK}[0, b2 D2] + cond2 D3 \text{ Bessel}[0, b2 D3] \text{ BesselK}[1, b2 D2] \text{ BesselK}[1, b3 D3] \} \} \}
\end{aligned}$$

Appendix H

External Inductance Determination from a Fundamental Perspective

H.1. Introduction

In chapter 5 the external inductance for the flat and cylindrical configurations were determined from an energy perspective. In order to show compliance to the fundamental definition, $L=N\phi/I$, the same results will be obtained here by deriving them from a flux density perspective.

H.2. External inductance

I.2.1. Semi-infinite flat conductor configuration

Starting with the semi-finite flat differential conductor configuration used in chapter 5, and given in Figure H.1, the magnetic field intensity between the two layers is,

$$H = \frac{I}{w} \quad (1)$$

where a current, I , flows through a section of width, w . For a linear material $\vec{B} = \mu\vec{H}$ and the magnetic flux density is,

$$B = \frac{\mu I}{w} \quad (2)$$

The cross sectional area through which the magnetic flux flows is,

$$A = hl \quad (3)$$

with l being the length of the section under consideration. This allows the flux to be expressed as,

$$\phi = BA = \frac{\mu I}{w} hl \quad (4)$$

Diving by the current allows the inductance to be expressed as,

$$L = \frac{N\phi}{I} = \frac{\mu hl}{w} \quad (5)$$

where N is obviously equal to one. This result is identical to that obtained in chapter 5 from an energy perspective.

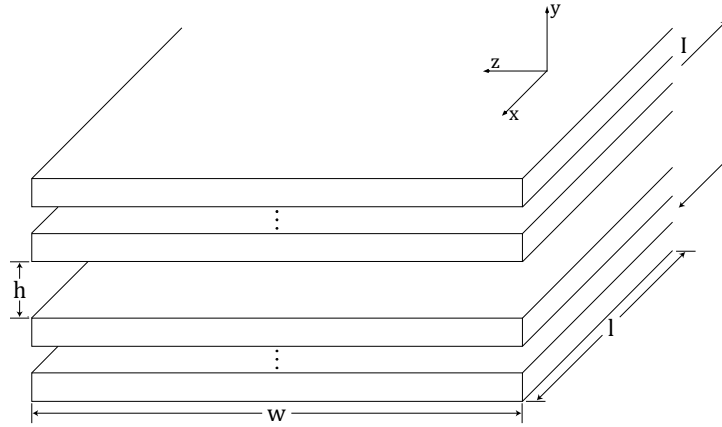


Figure H.1. Multi-layer flat differential conductor used in the derivation of the external inductance

I.2.2. Cylindrical conductor configuration

The magnetic field between the two layers ($b_{1_n1} \leq \rho \leq b_{2_0}$) is given by,

$$H = \frac{I}{2\pi\rho} \quad (6)$$

for

$$b_{1_n1} \leq \rho \leq b_{2_0}$$

The magnetic flux density is therefore given by,

$$B = \frac{\mu I}{2\pi\rho} \quad (7)$$

for a linear material. The flux contained in the area between $b_{1_n1} \leq \rho \leq b_{2_0}$ for a length l is,

$$\begin{aligned} \phi &= \iint_S B da \\ &= \int_{b_{1_n1}}^{b_{2_0}} \int_0^l \frac{\mu I}{2\pi\rho} dl d\rho \\ &= \frac{l\mu I}{2\pi} \ln \left| \frac{b_{2_0}}{b_{1_n1}} \right| \end{aligned} \quad (8)$$

The inductance can then be determined as,

$$L = \frac{N\phi}{I} = \frac{l\mu}{2\pi} \ln \left| \frac{b_{0_out}}{b_{2_in}} \right| \quad (9)$$

where N is again equal to one. Equation (9) is identical to that obtained in chapter 5 from an energy perspective.

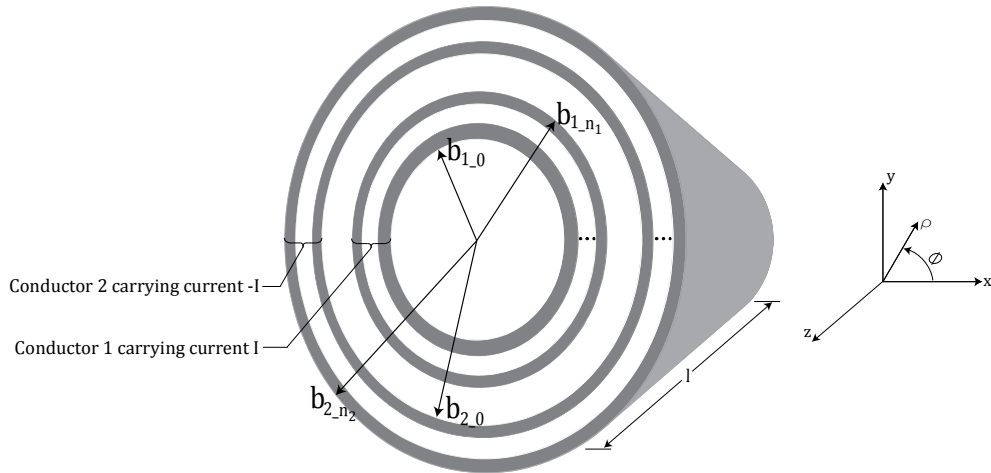


Figure H.2. Multi-layer cylindrical differential conductor used in the derivation of the external inductance