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EVALUATION OF APPLYING CRUM-BASED TRANSFORMATION IN SOLVING TWO POINT BOUNDARY VALUE PROBLEMS

Aasif Jogiat

A dissertation submitted to the Faculty of Engineering and the Built Environment, University of the Witwatersrand, in fulfillment of the requirements for the degree of Master of Science in Engineering.

Johannesburg, 2016

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DECLARATION OF AUTHENTICITY

I, the undersigned Aasif Jogiat, hereby declare that this dissertation is my own individual work. I am submitting this work to the Faculty of Engineering in the Built Environment, University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university. I further declare that where I have used source material, it is acknowledged.

.....

(Signature of Candidate)

.....day of.....year.....

ABSTRACT

The aim of this research project is evaluating the application of the Crum-based transformation in solving engineering systems modelled as two-point boundary value problems. The boundary value problems were subjected to the various combinations of Dirichlet, Non-Dirichlet and Affine boundary conditions. The engineering systems that were modelled were in the fields of electrostatics, heat conduction and longitudinal vibrations. Other methods such as the Z-transforms and iterative methods have been discussed. An attractive property of the Crum-based transformation is that it can be applied to cases where the eigenparameters (function of eigenvalues) generated in the discrete case are negative and was therefore chosen to be explored further in this dissertation. An alternative matrix method was proposed and used instead of the algebraic method in the Crum-based transformation. The matrix method was tested against the algebraic method using three unit intervals. The analysis revealed, that as the number of unit intervals increase, there is a general increase in the accuracy of the approximated continuous-case eigenvalues generated for the discrete case. The other observed general trend was that the accuracy of the approximated continuous-case eigenvalues decrease as one ascends the continuous-case eigenvalue spectrum. Three cases: (Affine, Dirichlet), (Affine, Non-Dirichlet) and (Affine, Affine) generated negative eigenparameters. The approximated continuous-case eigenvalues, derived from the negative eigenparameters, were shown not to represent true physical natural frequencies since the discrete eigenvalues, derived from negative eigenparameters, do not satisfy the condition for purely oscillatory behaviour. The research has also shown that the Crum-based transformation method was useful in approximating the shifted eigenvalues of the continuous case, in cases where the generated eigenparameters were negative: since, as the number of unit intervals increase, the post-transformed approximated eigenvalues improved in accuracy. The accuracy was also found to be better in the post-transformed case than in the pre-transformed case. Furthermore, the approximated non-shifted and shifted continuous-case eigenvalues (except the approximated continuous-case eigenvalues generated from negative eigenparameters) satisfied the condition for purely oscillatory behaviour.

DEDICATION

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1 INTRODUCTION

Mathematical modelling of physical systems is an important part of understanding how physical systems behave and predicting their behaviour when subjected to certain conditions or inputs. Physical systems can be modelled in one of three ways, namely purely continuous, purely discrete or as a hybrid system (which is a combination of both continuous and discrete modelling). Numerous sources of literature (DiPrima & Boyce, 1977; Polycarpu, 2006; Wachman, 1964; Guibout & Scheers, 2004; Lin, Enzer & Stadtherr, 2008; Cheng, 1992; Budak, Samarskii & Tikkonov, 2013) indicate that most physical systems can be modelled as second-order partial differential equations. This shows growing interest and an ever-expanding field of learning. The focus in this dissertation is therefore directed with the hope of expanding knowledge in this field in order to further broaden its scope and expand its potential. However, the dissertation looks at a second-order difference equation given by

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1, \tag{1.1}$$

where $c(n) > 0$ represents a weight function, $b(n)$ a potential function, λ represents the eigenparameters of the system model (which are a function of the continuous-case eigenvalues of the system model) and m represents the number of nodes in the difference equation (Currie & Love, 2010a). The above equation is a discretised form of a second-order partial differential equation as derived in Chapter 3.

Physical systems are sometimes subjected to conditions at their boundaries; the resultant system model is known as a boundary value problem. When physical systems are subjected to boundary conditions at two boundaries, the system model is known as a two-point boundary value problem. Many physical systems can be modelled as two-point boundary value problems (Flaherty, 2005) (where the two-point boundary value problem is formally defined in Chapter 2), either in the continuous domain or in the discrete domain. There are various boundary conditions (Currie & Love, 2010a; Atkinson, 1964) but to limit the scope of the dissertation, the following boundary conditions are considered: Dirichlet, Non-Dirichlet and Affine as listed in Table 1.1.

Table 1.1: Initial and final boundary conditions and their applications.

Case	Initial Boundary	Final Boundary	Engineering Application
1	Dirichlet	Dirichlet	Electrostatics
2	Non-Dirichlet	Dirichlet	Heat conduction
3	Dirichlet	Non-Dirichlet	Heat conduction
4	Non-Dirichlet	Non-Dirichlet	Heat conduction
5	Dirichlet	Affine	Longitudinal vibrations of an elastic bar
6	Affine	Dirichlet	Longitudinal vibrations of an elastic bar
7	Non-Dirichlet	Affine	Longitudinal vibrations of an elastic bar
8	Affine	Non-Dirichlet	Longitudinal vibrations of an elastic bar
9	Affine	Affine	Longitudinal vibrations of an elastic bar

A physical example of a Dirichlet boundary condition would be the nodal voltage of an electric circuit set at zero volts (Phillips, Parr & Riskin, 2003). Applying the radiation condition at the insulated end of a bar, in the classical heat conduction problem is a classical example of a Non-Dirichlet boundary condition (DiPrima & Boyce, 1977). An example of a physical Affine boundary

condition would be a rigid mass connected to the end of a bar undergoing longitudinal vibrations (DiPrima & Boyce, 1977).

In undamped physical systems, a condition known as resonance may occur where the system, when excited at the natural frequency of the system, produces high-amplitude oscillatory behaviour which can be dangerous (Adams, 2010). These natural frequencies are determined by computing the continuous-case eigenvalues of the physical system model (Adams, 2010). Thus, techniques needed to be investigated to determine the natural frequencies of systems that are modelled as two-point boundary value problems with combinations of Dirichlet, Non-Dirichlet and Affine conditions as listed in Table 1.1.

There are several methods for computing the eigenvalues of the two-point boundary value problems: Z-transforms (Phillips et al., 2003), iterative methods (Panju, 2011) and Crum-based transformations (Currie & Love (2010a, 2010b)). The Z-transform and Crum-based transformation are both analytical methods.

The Z-transform method seeks to transform the difference equation to the z-domain in order to generate the characteristic equation. The eigenvalues are then computed by determining the roots of the characteristic equation. The Z-transform is however limited to difference equations with constant coefficients (Phillips et al., 2003). The Iterative method seeks to solve for the eigenvalues of the system model by initially estimating the eigenvector and then iteratively refining the eigenvector estimate until the true eigenvector is reached to within reasonable accuracy. Literature shows it appears that the iterative methods are not able to handle all the boundary conditions in Table 1.1 (Panju, 2011).

The aforementioned methods, except the Crum-based transformation, are discussed further in Chapter 2 and do not have the unique property of being applied to cases where the eigenparameter generated in the discrete case is negative. One technique, found in the literature, that is able to deal with cases where the eigenparameter generated in the discrete case could be negative as well as handle the boundary conditions listed in Table 1.1 imposed on the difference equation (1.1) is the Crum-based transformation method (Currie & Love (2010a, 2010b)). The Crum-based transformation method appears to not have been applied to engineering problems and was chosen as the method to be explored in this dissertation.

1.1 Research Question

Physical systems are quite commonly modelled in the discrete domain and can be discretised from continuous system models (Phillips et al., 2003). The focus of this research is on the applicability of the Crum-based transformation in solving for the continuous-case eigenvalues of discrete two-point boundary value problems with the eigenparameter λ in the three-term recurrence relation of the form given by (1.1)(Currie & Love, 2010a)

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0. \qquad n = 1, ..., m-1,$$

The research question is not focusing on exploring the engineering systems concerned, but rather on the application of the Crum-based transformation in solving engineering systems modelled as two-point boundary value problems; subjected to various combinations of boundary conditions given in Table 1.1. The types of boundary conditions considered are Dirichlet, Non-Dirichlet and Affine.

The engineering systems considered are also listed in Table 1.1. The boundary value problems are explored for five different unit interval cases: 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. This is an arbitary selection of intervals to illustrate the range from low number of intervals (coarse discrete) to high number of intervals (fine discrete).

1.2 Scope of Work

The structure of the dissertation is as follows:

1. Chapter 2 presents the literature review of the current methods that exist for solving for the eigenvalues of second-order difference equations.
2. Chapter 3 gives a background to the mathematics used.
3. Chapter 4 illustrates the Crum-based transformation method.
4. Chapter 5 presents an alternative matrix method as opposed to the algebraic method employed in the Crum-based transformation.
5. The following analysis is applied to various engineering problems in Chapter 6 for all nine cases in Table 1.1: validating the matrix method, analysing the relationship between the number of unit intervals and the accuracy of the approximated continuous-case eigenvalues generated and in certain cases, where the eigenparameters generated are negative; applying the Crum-based transformation and performing the analysis on the pre-transformed approximated continuous-case eigenvalues and post-transformed approximated continuous-case eigenvalues. Matlab (MATLAB, 2004) and Mathematica (Wolfram, 2011) were the software tools used in the aforementioned analysis.
6. The research is then concluded and recommendations for future study are suggested in Chapter 7.

2 LITERATURE REVIEW

2.1 Literature Study

The definition of a boundary value problem, is a differential equation with solution and derivatives specified at more than one point (Gladwell, 2008). The most common type of boundary value problems are two point boundary value problems (Gladwell, 2008). The problem of solving an ordinary differential equation within an interval subject to boundary conditions at the initial and final end of the interval, is known as a two-point boundary value problem (James, 2001; Flaherty, 2005).

Two-point boundary value problems have a wide range of applications in science and engineering (DiPrima & Boyce, 1977; Polycarpu, 2006; Watchman, 1964; Guibout & Scheers, 2004; Lin, Enzer & Stadtherr, 2008; Cheng, 1992; Budak, Samarskii & Tikkonov, 2013). One classical application is Schrodinger's Equation in the field of quantum physics. Schrodinger's Equation is the fundamental equation of quantum mechanics and is given by

$$\frac{-\hbar}{2m}y''(x) + (V(x) - E)y(x) = 0,$$

where $''$ is the second differential operator, $\hbar$ is Plank's constant over 2π, m is the mass of the particle, $V(x)$ is the potential function describing a potential field and E is the energy level (Ledoux, 2007; Fatah, 2012). The following Neumann boundary conditions, which take on the Non-Dirichlet form in the discrete case,

$$a_0y(a) + b_0y'(a) = 0,$$

and

$$a_1y(b) + b_1y'(b) = 0,$$

where a_0, a_1, b_0 and b_1 are non-zero constants (Ledoux 2001).

Another application of two-point boundary value problems is in cascaded electrical LC circuits: where using nodal analysis in the frequency domain, the voltage v at the nodes $n-1, n$ and $n+1$ for a cascaded electrical LC circuit, with inductance of L and capacitance C, is given by

$$v(n+1) - [2 - (2\pi f)^2LC]v(n) + v(n-1) = 0,$$

where f = frequency of the source (Phillips et al., 2003). The boundary conditions, in the frequency domain, when excited by a sinusoidal source $v(0) = V_0\sin(2\pi ft)$ at node 0 and short-circuited at the last node m, are given by:

$$v(0) = V_0$$

and

$$v(m) = 0.$$

There are three other engineering applications that are considered, that is further discussed in Chapter 2:

1. Electrostatics (Cheng, 1992)
2. Heat conduction (DiPrima & Boyce, 1977)

3. Longitudinal vibrations of an elastic bar (DiPrima & Boyce, 1977; Budak et al., 2013).

Atkinson indicated that it is convenient that separate studies of two-point boundary value problems are done for the purely continuous case and for the purely discrete case (Atkinson, 1964). Difference equations often assist in the study of a differential equation. Atkinson has conducted studies in the mixed continuous-discrete case, which have the features of both difference and differential equations (Atkinson, 1964). The focus of this study is on a discretised model of a system and therefore the methods for the mixed continuous-discrete case cannot be applied. Furthermore, another limitation of the study conducted by Atkinson is that his study did not look at cases where the generated eigenparameters may have been negative.

The primary focus of this dissertation is on solving for the continuous-case eigenvalues of two-point boundary value problems in the discrete case. Discretisation of continuous ordinary differential equations results in a difference equation. The continuous-case eigenvalues of the system being modelled physically represent the natural modes or natural frequencies of the system and can be inferred from the eigenparameter λ, since the eigenparameter λ is a function of the continuous-case eigenvalues of the system.

Several methods exist for solving the eigenparameter λ of the three-term recurrence relations given by (1.1), subjected to boundary conditions. These are listed as follows:

1. Z-transforms (Phillips et al., 2003)
2. Iterative methods (Panju, 2011)
3. Crum-based transformation (Currie & Love (2010a, 2010b)).

These methods are expounded and their drawbacks discussed in the ensuing paragraphs.

The first method or the method using Z-transforms, transforms the difference equation to the z-domain to generate the characteristic equation, where the eigenvalues are computed by determining the roots of the characteristic equation (Phillips et al., 2003). This method cannot be used to analyse the three-term recurrence relation concerned, given by (1.1), because these methods are limited to difference equations with constant coefficients (Phillips et al., 2003). The three-term recurrence relation, given by (1.1), has the eigenvalue parameter λ appearing in the coefficient of $X(n)$ and variable constants $c(n)$. Although this method can deal with all the boundary conditions mentioned in Table 1.1, the problem still remains with cases where the boundary conditions lead to negative eigenparameters. The Crum-based transformation method provides a sound theoretical basis for dealing with this case as is later discussed in this chapter.

There are several iterative methods for determining the eigenvalues from the characteristic matrix, such as (Panju, 2011):

1. Power Iteration method.
2. Shifted Inverse Iteration method.
3. Rayleigh Quotient method.

4. Simultaneous Iteration method.

5. QR method.

The above-mentioned iterative methods seek to solve the eigenvalues of the system from the characteristic matrix and not from solving the roots of the characteristic equation. From the literature (Panju, 2011), it appears that these methods are not able to handle all the boundary conditions in Table 1.1. The Power Iteration method estimates a single unit eigenvector of the system matrix and then recursively updates the eigenvector to closely approximate the eigenvector of the characteristic matrix. The major disadvantage of the Power Iteration method is that it only computes the eigenvalues of largest magnitude and the algorithm will only be guaranteed to converge if the two largest eigenvalues are of distinct magnitude (Panju, 2011).

This problem is circumvented by the Shifted Inverse Iteration method, which is an indirect application of the Power Iteration method on the inverse matrix of the system matrix. The shift is realised when the characteristic matrix is shifted by a factor (or shift value) of the identity matrix; where the factor is an estimate of some eigenvalue of the system matrix. Thus, estimating different factors allows for the computation for any eigenvalue of the system matrix. The drawback of this method is that the algorithm has to be repetitively applied to compute every eigenvalue of the system matrix (Panju, 2011).

An improvement of the Shifted Inverse Iteration method is the Rayleigh Quotient method. In the Rayleigh Quotient method, the initial eigenvector estimate is improved by updating the shift value using the Rayleigh Quotient (Panju, 2011). The Rayleigh Quotient $r(\mathbf{x})$ is defined, for a matrix $\mathbf{A}$ with eigenvector $\mathbf{x}$, as (Panju, 2011)

$$r(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \beta,$$

where β is the eigenvalue of $\mathbf{A}$.

One very important disadvantage of the Rayleigh Quotient method is that it does not work in all general cases. Convergence of the Rayleigh Quotient method is only guaranteed if the system matrix is real and symmetric, and is known to fail in cases where the system matrix is non-symmetric (Panju, 2011). Since the systems that are being dealt with are two-point boundary value problems, the system matrix is generally non-symmetric.

All the iteration methods discussed thus far are only capable of computing the eigenvalues one at a time; and thus to compute different eigenvalues of the matrix, these methods have to be applied several times, with different initial eigenvector estimates, to compute all eigenvalues of the system (Panju, 2011).

The Simultaneous Iteration method circumvents this problem by producing all the eigenvalues of the matrix at once, by having a basis of linearly independent unit eigenvector estimates, arranged in a matrix, and thus applying the Power Iteration method to all the eigenvectors at once. A variation of the Simultaneous Iteration method is the QR method, which employs QR decomposition of the system matrix in its algorithm. However; these methods are restricted to symmetric, real, full rank matrices (Panju, 2011).

The computation of the eigenparameter λ from (1.1)

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

can be accomplished through repetitive substitution and transformation of the difference equation if the eigenparameter spectrum has a negative eigenparameter (Currie & Love (2010a, 2010b)). This method is known as the Crum-based Transformation method and is further explored in the remainder of the dissertation. The repetitive substitution method employed by Currie and Love (2012) proves tedious for large computations as the number of unit intervals extends beyond three.

The Crum-based transformation, explored by Currie and Love in the discrete case, extends to cases where eigenparameters generated by repetitive substitution are negative (Currie & Love, 2010b) and can be subjected to different boundary conditions. Negative eigenparameters may arise from the inherent error of discretising a continuous system to a discrete system with a finite number of unit intervals.

The Crum-based transformation performs a mapping on the variable $X(n)$ to a transformed variable $w(n)$ (Currie & Love, 2010a), where

$$w(n) = X(n) - \frac{X(n)z(n)}{z(n-1)},$$

where $z(n)$ is the solution to the shifted form of the difference equation (1.1), given by

$$c(n)X(n+1) - [b(n) - c(n)(\mu - \lambda_0)]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

when $\mu = 0$. The μ spectrum generated in the transformed variable domain, is essentially a shift in the eigenparameter λ spectrum, computed in the $X(n)$ domain, by the lowest negative eigenparameter λ_0; thus yielding a spectrum of non-negative eigenparameters (Currie & Love, 2010b).

Although the difference equation (1.1) caters for coefficients dependant on n, the dissertation explores, by virtue of the physical applications considered, difference equations with coefficients which are not a function of n. These difference equations yield a spectrum of a finite number of equations.

In the applications considered, the eigenparameter λ is a weighted form of the eigenvalues or natural frequencies of the system. The solution of the eigenvalues of a continuous system model will yield an infinite sequence of natural frequencies. The discretised model will yield approximated continuous-case eigenvalues that approximate the infinite sequence of natural frequencies as the number of unit intervals approaches infinity (Adams, 2010). This property is further illustrated in Chapter 6 of this dissertation.

2.2 Concluding Remarks

This chapter began by defining a two point boundary value problem and then discussed some of its applications in the field of science and engineering. The different methods for solving the eigenparamter λ in the three term recurrence relation were discussed, together with their limitations. The literature research has shown that one common limitation found in all the methods described, besides the Crum-based Transformation method, was that these methods cannot be applied to cases where the least eigenparameter in the eigenparameter spectrum is negative. The Crum-based Transformation method was introduced which is further explored in Chapter 6.

3 MATHEMATICAL PRELIMINARIES

3.1 Mathematical Background

Several methods for discretising continuous derivatives in ordinary differential equations exist, of which the common formulas are, where h is the discretisation increment and $'$ is the first differential operator(Li, 2011):

1. Forward Finite Difference:
$$X^{'}(x) \approx \frac{(X(n+1) - X(n))}{h}. \tag{3.1}$$

2. Backward Finite Difference:
$$X^{'}(x) \approx \frac{(X(n) - X(n-1))}{h}. \tag{3.2}$$

3. Central Finite Difference:
$$X^{'}(x) \approx \frac{(X(n+1) - X(n-1))}{2h}, \tag{3.3}$$
$$X^{''}(x) \approx \frac{(X(n+1) - 2X(n) + X(n-1))}{h^2}. \tag{3.4}$$

Consider a general homogeneous second-order partial differential equation

$$k\frac{\partial^2 y}{\partial x^2} + p\frac{\partial^2 y}{\partial x \partial t} + q\frac{\partial^2 y}{\partial t^2} + a\frac{\partial y}{\partial t} + b\frac{\partial y}{\partial x} + cy = 0. \tag{3.5}$$

To allow for resonance in an undamped system, we need to impose the following conditions

$$b = p = 0;\ k \neq 0. \tag{3.6}$$

Substitute conditions of (3.6) in (3.5) results in

$$k\frac{\partial^2 y}{\partial x^2} + q\frac{\partial^2 y}{\partial t^2} + a\frac{\partial y}{\partial t} + cy = 0. \tag{3.7}$$

Assume $y(x,t) = y(x)e^{i\omega t}$, where $i = \sqrt{-1}$ and ω represent the eigenvalues or angular frequency of the system, then

$$\frac{\partial y}{\partial t} = i\omega y(x)e^{i\omega t}, \tag{3.8}$$
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y(x)e^{i\omega t}, \tag{3.9}$$

and

$$\frac{\partial^2 y}{\partial x^2} = y^{''}(x)e^{i\omega t}. \tag{3.10}$$

Substitute $y(x,t) = y(x)e^{i\omega t}$, (3.8), (3.9) and (3.10) into (3.7) gives

$$ky''(x)e^{i\omega t} - \omega^2 qy(x)e^{i\omega t} + ia\omega y(x)e^{i\omega t} + cy(x)e^{i\omega t} = 0.$$

Since $e^{i\omega t} \neq 0$, it follows that

$$ky''(x) - \omega^2 qy(x) + ia\omega y(x) + cy(x) = 0. \tag{3.11}$$

To discretise (3.11), we use the second derivative central difference formulae (where h is the discretisation increment) namely

$$y''(x) \approx \frac{y(n+1) - 2y(n) + y(n-1)}{h^2}.$$

Then (3.11) can be rewritten as the discretised equation

$$\frac{k}{h^2}[y(n+1) - 2y(n) + y(n-1)] - \omega^2 qy(n) + ia\omega y(n) + cy(n) = 0.$$

Collecting the terms and multiplying the equation by $h^2 k^{-1}$ gives

$$y(n+1) - y(n)\left[2 - \frac{ch^2}{k} - h^2\omega\left(\frac{ia - q\omega}{k}\right)\right] + y(n-1) = 0. \tag{3.12}$$

The general three-term difference equation that is under consideration is given by

$$c(n)y(n+1) - [b(n) - c(n)\lambda]y(n) + c(n-1)y(n-1) = 0. \tag{3.13}$$

From (3.12), the coefficients of $y(n+1)$ and $y(n)$ do not involve terms dependant on n. For (3.12) to take on the form of (3.13) and noting that λ is a function of the eigenvalues of the system, we set the following parameters as

$$c(n) = c(n-1) = 1,$$

$$b(n) = 2 - \frac{ch^2}{k},$$

and

$$\lambda = h^2\omega\left[\frac{ia - q\omega}{k}\right].$$

The dissertation explores cases where the second-order difference equation, given by (3.13), will be subjected to the following boundary conditions with m being the number of unit intervals:

1. *Dirichlet:* Boundary conditions of the form (Currie & Love, 2010a)

$$X(k) = 0; \qquad k = 0 \text{ or } m. \tag{3.14}$$

   A physical example of a Dirichlet boundary condition would be the voltages of an electric circuit, at nodes 0 and m, set at zero volts (Phillips et al., 2003).

2. *Non-Dirichlet:* Boundary conditions of the form (Currie & Love, 2010a)

$$X(k+1) - X(k) = 0. \qquad k = 0 \text{ or } m-1. \tag{3.15}$$

A physical example of a Non-Dirichlet boundary condition would be applying the radiation condition at the insulated end of a bar, in the classical heat conduction problem (DiPrima & Boyce, 1977).

3. *Affine:* Boundary conditions of the form (Currie & Love, 2013)

$$X(0) = [a\lambda + b]X(1), \qquad a \geq 0. \tag{3.16}$$

and (Currie & Love, 2013)

$$X(m-1) = [\alpha\lambda + \beta]X(m). \qquad \alpha \leq 0. \tag{3.17}$$

A physical example of an Affine boundary condition would be a rigid mass connected to the end of a bar undergoing longitudinal vibrations (DiPrima & Boyce, 1977).

To compute the expected number of eigenparameters, Theorem 4.1 from (Currie & Love, 2013) is modified to start from $n = 1$ instead of $n = 0$: since the first node is at zero, the mathematics is simpler when determining the continuous form of the discrete boundary conditions and node 0 is a natural physical representation of the beginning of unit intervals. The theorem from (Currie & Love, 2013) is given by Theorem 3.1.

Theorem 3.1 *Consider the boundary value problem given by equation* (1.1) *for* $n = 0, \ldots, r-1$, *where* $r+1$ *is the number of unit intervals considered, together with boundary conditions*

$$X(-1) = \left[a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\right] X(0), \qquad a \geq 0,\ c_k \geq 0.$$

and

$$X(r-1) = \left[\alpha\lambda + \beta - \sum_{j=1}^{p} \frac{\gamma_j}{\lambda - \sigma_j}\right] X(r). \qquad \alpha \leq 0,\ \gamma_j < 0.$$

Then the boundary value problem has

(i) $s + p + r + 1$ *eigenparameters if* $\alpha < 0$,

(ii) $s + p + r$ *eigenparameters if* $\alpha = 0$ *and* $\beta \neq 0$,

(iii) $s + p + r - 1$ *eigenparameters if* $\alpha = \beta = 0$.

Proof: The proof of the theorem can be found in (Currie & Love, 2013) for $n = 0, \ldots, r-1$. ■

The modified theorem is given by Theorem 3.2, where the number of unit intervals = m; thus $m = r + 1$. Theorem 3.2 considers Dirichlet, Non-Dirichlet and Affine boundary conditions used in this dissertation.

Theorem 3.2 *Consider the boundary value problem given by* (1.1)*, for* $n = 1, \ldots, m-1$ *with boundary conditions*

$$X(0) = [a\lambda + b]\, X(1), \qquad a \geq 0. \tag{3.18}$$

and

$$X(m-1) = [\alpha\lambda + \beta]\, X(m), \qquad \alpha \leq 0. \tag{3.19}$$

where m *is the number of unit intervals.*

Then the boundary value problem has

(i) m *eigenparameters if* $\alpha < 0$,

(ii) $m-1$ *eigenparameters if* $\alpha = 0$ *and* $\beta \neq 0$,

(iii) $m-2$ *eigenparameters if* $\alpha = \beta = 0$

This results in one less eigenparameter compared to the cases in Theorem 3.1, as there is one less unit interval as one moves from $n = 0$ *to* $n = 1$.

Proof: The proof of the theorem can be found in (Currie & Love, 2013). ■

There needs to be made a clear distinction between the eigenvalues in the continuous case as opposed to the eigenvalues in the discrete case. This will be explained via an illustration. Consider the following differential equation

$$X''(x) + k^2 X(x) = 0, \tag{3.20}$$

where k represents the eigenvalues of the continuous system or the natural frequencies of the system.

Applying the central finite difference equation (3.4) to (3.20) results in

$$\frac{X(n+1) - 2X(n) + X(n-1)}{h^2} + k^2 X(x) = 0,$$

$$X(n+1) - [2 - k^2h^2]X(n) + X(n-1) = 0. \tag{3.21}$$

In order for (3.21) to take on the form of (1.1)

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

and noting that the eigenparameter λ needs to be a function of the eigenvalues k of the continuous system, the parameters in (1.1) have to take on the following values

$$c(n) = c(n-1) = 1,$$

$$b(n) = 2$$

and

$$\lambda = k^2h^2.$$

Therefore, (3.21) becomes

$$X(n+1) - (2 - \lambda)X(n) + X(n-1) = 0. \tag{3.22}$$

To compute the eigenvalues of the discrete system, substitute an assumed solution of $X(n) = c_1 r^n$ in (3.22); resulting in

$$c_1 r^{n+1} - (2-\lambda)c_1 r^n + c_1 r^{n-1} = 0,$$

where r is the eigenvalue of the discrete system and c_1 is an arbitrary constant. Note that the eigenvalues r of the discrete system are different to the eigenvalues k of the continuous system. Factoring out $c_1 r^{n-1}$ results in

$$c_1 r^{n-1}\left[r^2 - (2-\lambda)r + 1\right] = 0. \tag{3.23}$$

Making the following substitution

$$(2-\lambda) = 2d, \tag{3.24}$$

in (3.23) for ease of computation yields

$$c_1 r^{n-1}\left[r^2 - 2dr + 1\right] = 0.$$

Since, for a non-trivial solution, $c_1 r^{n-1} \neq 0$. Thus

$$r^2 - 2dr + 1 = 0,$$

$$r = \frac{2d \pm \sqrt{4d^2 - 4}}{2},$$

$$r = d \pm \sqrt{d^2 - 1}. \tag{3.25}$$

Physical systems that exhibit purely oscillatory behaviour have sinusoidal solutions. Thus, natural frequencies arise from systems that exhibit purely oscillatory behaviour. For physical systems to exhibit purely oscillatory behaviour, the discrete eigenvalues r need to lie on the unit circle (Phillips et al., 2003). This imposes the condition

$$|r| = 1,$$

Using (3.25) produces

$$\left|d \pm \sqrt{d^2 - 1}\right| = 1. \tag{3.26}$$

Substituting (3.24) into (3.26) yields

$$\left|\frac{2-\lambda}{2} \pm \sqrt{\left(\frac{2-\lambda}{2}\right)^2 - 1}\right| = 1, \tag{3.27}$$

Using the assignment $\lambda = k^2 h^2$ in (3.27)

$$\left|\frac{2-k^2h^2}{2} \pm \sqrt{\left(\frac{2-k^2h^2}{2}\right)^2 - 1}\right| = 1. \tag{3.28}$$

The condition (3.27) thus restricts the values of the eigenparameter λ in order for the system to exhibit purely oscillatory behaviour.

Another phenomenon to take into consideration, when performing the analysis between the approximated continuous-case eigenvalues in the discrete case with the actual continuous-case eigenvalues,

is aliasing. Aliasing is a phenomenon whereby erroneous frequency components appear in the discretised data when the sampling frequency ω_s is too low. Shannon's sampling theorem states that the sampling frequency must be at least twice the maximum frequency component of the continuous system. The Nyquist frequency ω_N is defined as the maximum frequency component in the frequency spectrum that can be computed before the discrete model, of a particular sampling frequency ω_s, suffers the effects of aliasing (Phillips et al., 2003). The Nyquist frequency ω_N is given by

$$\omega_N = \frac{\omega_s}{2}. \tag{3.29}$$

Since

$$\omega_s = \frac{2\pi}{T_s},$$

where $T_s = \frac{1}{N}$ is the sampling period and N = number of unit intervals, this results in

$$\omega_s = 2\pi N, \tag{3.30}$$

Substituting (3.30) in (3.29) yields

$$\omega_N = \pi N. \tag{3.31}$$

Hence, in relation to the analysis between the approximated continuous-case eigenvalues in the discrete case with the continuous-case eigenvalues performed in Chapter 6, the estimated natural frequencies, derived from the approximated continuous-case eigenvalues, should be lower than the Nyquist frequency so as not to incur the effects of aliasing. Table 3.1 indicates the Nyquist frequencies for the unit-intervals cases considered in this dissertation.

Table 3.1: Nyquist frequencies for the unit-intervals cases considered in the analysis in Chapter 6.

Number of unit intervals	Nyquist frequency
6	18.850
11	34.558
19	59.690
30	94.248
100	314.159

3.2 Concluding Remarks

This chapter illustrated the background into the the mathematics used throughout the dissertation. The different Finite Difference methods were outlined. The derivation of the three-term recurrence relation, given by (1.1), from a general homogeneous second-order partial differential equation was presented. The types of boundary conditions considered i.e. Dirichlet, Non-Dirichlet and Affine were defined. The theorem for determining the expected number of eigenparameters was outlined. Furthermore, the condition for physical systems to exhibit oscilatory behaivour was stated and a condition was derived therefrom: which restricts the values the eigenparameteer λ can hold. The phenomenon of aliasing was discussed and the Nyquist frequency was calculated for the different number of unit intervals cases.

4 CRUM-BASED TRANSFORMATION OF DIFFERENCE EQUATIONS

The Crum-based Transformation method, as developed by Currie and Love (Currie & Love, 2010a), to solve the eigenparameters of the weighted second-order difference equation of the form given by (1.1)

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

subject to initial and final boundary conditions, can be separated into three procedures (Currie & Love, 2012):

1. Repetitive substitution;
2. Shifting;
3. Transformation.

The second and third procedure is only required if the least eigenvalue is negative. The subsequent sub-sections outline an illustration of the Crum-based transformation technique.

The three different procedures will be illustrated using an (Affine, Dirichlet) case. The bracket notation represents the initial and final boundary conditions respectively. The (Affine, Dirichlet) case is where the Affine is the initial boundary condition and Dirichlet is the final boundary condition. Consider the following difference equation (which is a form of (1.1) where the coefficients are not a function of n)

$$X(n+1) - (2-\lambda)X(n) + X(n-1) = 0, \tag{4.1}$$

subjected to the following initial Affine boundary condition

$$X(0) = X(1)(\lambda + 3), \tag{4.2}$$

and the final Dirichlet boundary condition

$$X(3) = 0. \tag{4.3}$$

The difference equation will be considered for the range $n \in [0, 2]$, thus making the number of intervals $m = 3$. In order for (4.1) to take on the form of (1.1), a clear assignment would be

$$c(n) = c(n-1) = 1, \tag{4.4}$$

and

$$b(n) = 2. \tag{4.5}$$

4.1 Repetitive Substitution

Considering equation (4.1) for $n = 1$ and 2 results in

$$X(2) - (2 - \lambda)X(1) + X(0) = 0, \tag{4.6}$$

and

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{4.7}$$

The system of linear equations will undergo a series of repetitive substitutions to obtain an expression for the eigenparameter λ. Substituting (4.2) into (4.6) yields

$$X(2) - (2 - \lambda)X(1) + (\lambda + 3)X(1) = 0,$$

$$X(2) = -(2\lambda + 1)X(1). \tag{4.8}$$

Substituting (4.3) into (4.7) yields

$$-(2 - \lambda)X(2) + X(1) = 0,$$

$$X(1) = (2 - \lambda)X(2). \tag{4.9}$$

Substituting (4.9) into (4.8) yields

$$X(2) = -(2\lambda + 1)(2 - \lambda)X(2),$$

$$X(2)[1 + (2\lambda + 1)(2 - \lambda)] = 0.$$

Since $X(2) \neq 0$; then

$$1 + (2\lambda + 1)(2 - \lambda) = 0. \tag{4.10}$$

Equation (4.10) can be solved in Mathematica (Wolfram, 2011); which yields the following two eigenparameter values $\lambda = -0.68614\ldots\ldots$ and $\lambda = 2.18614\ldots\ldots$; where the dots represent the remaining decimal places.

The expected number of eigenparameters can be determined from Theorem 3.2. In order for the boundary conditions (3.18) and (3.19) from Theorem 3.2 to take on the form of the boundary conditions (4.2) and (4.3) of the boundary value problem illustration; the following parameters need to take on the following values: $a = 1$, $b = 3$, $\alpha = 0$ and $\beta \neq 0$. Since $X(m = 3)$ has to be zero, we chose $\alpha = 0$ and $\beta \neq 0$ so that the factor $\alpha\lambda + \beta$ is not zero nor is there any non-zero λ factor. This satisfies the condition of case (ii) in Theorem 3.2. Therefore, the expected number of eigenparameters as determined using Theorem 3.2 $= m - 1 = 3 - 1 = 2$. This result confirms the number of actual eigenparameters being generated, which is two.

Since the least eigenparameter value λ_0 is negative; the shifting and transformation method needs to be applied, so the eigenparameter spectrum can have non-negative elements. The eigenparameters are shifted by the least eigenparameter value up to 500 decimals i.e

$$\lambda_0 = \ -0.68614\ldots\ldots$$

The notation of 500 decimals was used since a symbolic program was used in Mathematica (Wolfram, 2011) but such accuracy is impractical from an engineering perspective. The shift in the eigenparameter spectrum results in the following expression for the shifted eigenparameter μ

$$\lambda = \mu + \lambda_0. \tag{4.11}$$

The least eigenparameter value will be represented symbolically throughout the illustration by λ_0 to avoid writing long strings of digits.

4.2 Shifting

The boundary value problem is shifted by substituting(4.11) in (4.1), (4.2) and (4.3); yielding the following set of equations

$$X(n+1) - (2 - \mu - \lambda_0)X(n) + X(n-1) = 0, \tag{4.12}$$

$$X(0) = X(1)[3 + \mu + \lambda_0], \tag{4.13}$$

and

$$X(3) = 0. \tag{4.14}$$

4.3 Transformation

After the boundary value problem has been shifted, the solution $z(n)$ of the shifted boundary value problem, when $\mu = 0$, needs to be computed. The solution $z(n)$ is required in the mapping from $X(n)$ to $w(n)$. Equations (4.12), (4.13) and (4.14), for $\mu = 0$ and $z(n) > 0$ for all $n = 0, \ldots, 3$; then becomes

$$z(n+1) - (2 - \lambda_0)z(n) + z(n-1) = 0, \tag{4.15}$$

$$z(0) = z(1)[3 + \lambda_0], \tag{4.16}$$

and

$$z(3) = 0. \tag{4.17}$$

The condition imposed is that $z(n)$ obeys both boundary conditions. This is expected in physical applications as a physical system naturally obeys the boundary conditions imposed. Now consider putting $n = 1$ in (4.15)

$$z(2) - (2 - \lambda_0)z(1) + z(0) = 0. \tag{4.18}$$

Substituting (4.16) in (4.18), yields

$$z(2) - (2 - \lambda_0)z(1) + (\lambda_0 + 3)z(1) = 0,$$

$$z(2) = -(1 + 2\lambda_0)z(1). \tag{4.19}$$

The next step is to transform the shifted boundary value problem (Currie & Love, 2012), defined by (4.12), (4.13) and (4.14), to shift the eigenparameter spectrum using the following mapping (Currie & Love, 2010a)(Crum-based Transformation)

$$w(n) = X(n) - \frac{X(n-1)z(n)}{z(n-1)}. \tag{4.20}$$

This results in the transformed equation (Currie & Love, 2010a)

$$\hat{c}(n)w(n+1) - [\hat{b}(n) - \mu\hat{c}(n)]w(n) + \hat{c}(n-1)w(n-1) = 0, \tag{4.21}$$

where

$$\hat{c}(n) = \frac{c(n-1)z(n-1)}{z(n)}, \tag{4.22}$$

and

$$\hat{b}(n) = \left[\frac{c(n-1)z(n-1)}{c(n)z(n)} + \frac{z(n)}{z(n-1)}\right]\frac{c(n-1)z(n-1)}{z(n)}. \tag{4.23}$$

However, using (4.4), (4.22) and (4.23) becomes

$$\hat{c}(n) = \frac{z(n-1)}{z(n)}, \tag{4.24}$$

and

$$\hat{b}(n) = \left(\frac{z(n-1)}{z(n)}\right)^2 + 1. \tag{4.25}$$

Consider (4.21) for $n = 2$

$$\hat{c}(2)w(3) - [\hat{b}(2) - \mu\hat{c}(2)]w(2) + \hat{c}(1)w(1) = 0. \tag{4.26}$$

Dividing (4.26) by $w(2)$ results in

$$\hat{c}(2)\frac{w(3)}{w(2)} - [\hat{b}(2) - \mu\hat{c}(2)] + \hat{c}(1)\frac{w(1)}{w(2)} = 0,$$

which we denote by

$$\hat{c}(2)W_{32}(\mu) - [\hat{b}(2) - \mu\hat{c}(2)] + \hat{c}(1)W_{12}(\mu) = 0, \tag{4.27}$$

where

$$W_{32}(\mu) = \frac{w(3)}{w(2)}, \tag{4.28}$$

and

$$W_{12}(\mu) = \frac{w(1)}{w(2)}, \tag{4.29}$$

Clearly, $w(1),\ w(2),\ w(3),\ \hat{c}(1),\ \hat{c}(2)$ and $\hat{b}(2)$ have to be computed. Putting $n = 1$ in (4.24) yields

$$\hat{c}(1) = \frac{z(0)}{z(1)},$$

and using (4.16) this becomes

$$\hat{c}(1) = \lambda_0 + 3. \tag{4.30}$$

Equation (4.24), for $n = 2$, becomes

$$\hat{c}(2) = \frac{z(1)}{z(2)},$$

and from (4.19) we obtain

$$\hat{c}(2) = \frac{-1}{(1 + 2\lambda_0)}. \tag{4.31}$$

Consider (4.25) for $n = 2$, which becomes

$$\hat{b}(2) = \left(\frac{z(1)}{z(2)}\right)^2 + 1,$$

which simplifies, using (4.19) to

$$\hat{b}(2) = \frac{1}{(1 + 2\lambda_0)^2} + 1. \tag{4.32}$$

Now consider (4.20) for $n = 1$, which becomes

$$w(1) = X(1) - \frac{X(0)z(1)}{z(0)}. \tag{4.33}$$

Substituting (4.16) and (4.13) into (4.33) yields

$$w(1) = X(1) - \frac{X(1)(\mu + \lambda_0 + 3)}{\lambda_0 + 3},$$

Rewriting the first term $X(1)$ on the right hand side of the equation

$$w(1) = \frac{X(1)(\lambda_0 + 3)}{\lambda_0 + 3} - \frac{X(1)(\mu + \lambda_0 + 3)}{\lambda_0 + 3},$$

$$w(1) = \frac{X(1)(-\mu)}{\lambda_0 + 3}. \tag{4.34}$$

Putting $n = 2$ in (4.12) yields

$$X(3) - (2 - \mu - \lambda_0)X(2) + X(1) = 0. \tag{4.35}$$

Substituting (4.14) into (4.35) results in

$$X(1) = (2 - \mu - \lambda_0)X(2),$$

$$X(2) = \frac{X(1)}{(2 - \mu - \lambda_0)}. \tag{4.36}$$

Considering $n = 2$ in (4.20) becomes

$$w(2) = X(2) - \frac{X(1)z(2)}{z(1)}. \tag{4.37}$$

Substituting (4.36) and (4.19) into (4.37), results in

$$w(2) = \frac{X(1)}{(2-\mu-\lambda_0)} + X(1)(1+2\lambda_0),$$

$$w(2) = X(1)[\frac{1}{(2-\mu-\lambda_0)} + (1+2\lambda_0)]. \tag{4.38}$$

Looking at (4.20) when $n = 3$ yields

$$w(3) = X(3) - \frac{X(2)z(3)}{z(2)}. \tag{4.39}$$

Substituting (4.14), (4.19) and (4.17) into (4.39), results in

$$w(3) = 0.$$

The ratios $W_{32}(\mu)$ and $W_{12}(\mu)$ can be computed in terms of μ, where from (4.28)

$$W_{32}(\mu) = 0, \tag{4.40}$$

and from (4.29)

$$W_{12}(\mu) = \frac{-\mu X(1)}{(\lambda_0+3)[\frac{1}{(2-\mu-\lambda_0)} + (1+2\lambda_0)]X(1)}.$$

Cancelling $X(1)$, since $X(1) \neq 0$, this becomes

$$W_{12}(\mu) = \frac{-\mu}{(\lambda_0+3)[\frac{1}{(2-\mu-\lambda_0)} + (1+2\lambda_0)]}. \tag{4.41}$$

From (4.27), all the required expressions of $W_{12}(\mu)$, $W_{32}(\mu)$, $\hat{c}(1)$, $\hat{c}(2)$ and $\hat{b}(2)$ have been computed in (4.30), (4.31), (4.32), (4.40) and (4.41) respectively to obtain an expression to solve for μ. Equation (4.27) is then solved using Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.1, for Mathematica (Wolfram, 2011) code). The μ-value generated is 2.87228..... Substituting the positive λ-value computed in Section 4.1 into (4.11) yields the aforementioned μ-value. Note that the number of eigenparameters reduces by one. As per the paper (Currie & Love, 2011), the shift can be generalised to the boundary conditions in Table 1.1.

Through the illustrated boundary value problem given by (4.1), (4.2) and (4.3), it has been proven that the transformation performs the same function as shifting the eigenparameters by the least eigenparameter to ensure that the eigenparameters are non-negative. Through the analysis of the different boundary value problems in Chapter 6, the eigenparameters will be shifted by the least eigenparameter by virtue of the Crum-based transformation.

The physical importance of the shift is noted when considering the condition of the system to exhibit purely oscillatory behaviour indicated in Chapter 3 by (3.26)

$$|r| = \left|d \pm \sqrt{d^2-1}\right| = 1, \tag{4.42}$$

where from (3.24)

$$d = \frac{2-\lambda}{2}. \tag{4.43}$$

Equation (4.42) can be used since the difference equation (4.1) in the illustration is the same as equation (3.22) in Chapter 3 from which (4.42) was derived from.

Now consider when the least eigenparameter λ_0 is a real-value less than 0

$$\lambda_0 < 0,$$

$$-\lambda_0 > 0.$$

Adding 2 to both sides of the inequality yields

$$2 - \lambda_0 > 2.$$

Now dividing both sides of the inequality by 2 yields

$$\frac{2 - \lambda_0}{2} > 1. \tag{4.44}$$

Now, letting d_0 be a function of λ_0 from (4.43) produces

$$d_0 = \frac{2 - \lambda_0}{2}. \tag{4.45}$$

Rewriting (4.44) using (4.45) results in

$$d_0 > 1. \tag{4.46}$$

Squaring both sides of the inequality results in

$$(d_0)^2 > 1.$$

Subtracting 1 on both sides of the inequality yields

$$(d_0)^2 - 1 > 0.$$

Taking the positive square root on both sides of the inequality yields

$$\sqrt{(d_0)^2 - 1} > 0. \tag{4.47}$$

The left side of the inequality (4.47) does not generate complex values, since $d_0 > 1$ from (4.46). Thus, the inequality (4.47) holds true.

Adding (4.46) and (4.47), and subtracting (4.47) from (4.46) results in

$$d_0 \pm \sqrt{d_0^2 - 1} > 1.$$

Taking the absolute values on both sides of the inequality produces

$$\left| d_0 \pm \sqrt{d_0^2 - 1} \right| > 1. \tag{4.48}$$

Therefore, (4.48) does not satisfy the condition (4.42) for purely oscillatory behaviour. Thus, the negative eigenparameter λ_0 cannot yield a meaningful physical natural frequency. By shifting the

eigenparameter spectrum, as performed by the Crum-based transformation; this ensures that all the eigenparameters in the eigenparameter spectrum are non-negative so as to satisfy condition (4.42). The Crum-based transformation is thus a useful application in dealing with cases where the least eigenparamter in the eigenparameter spectrum is negative.

By computing d from the shifted difference equation given by (4.12), (4.43) can be rewritten as

$$d = \frac{2 - \mu - \lambda_0}{2},$$

$$d = \frac{2 - \lambda}{2},$$

which is the same form as (4.43). Thus, the discrete eigenvalues for the shifted eigenparameter μ can be computed from their corresponding non-shifted eigenparameter λ.

Table 4.1: Absolute value of discrete eigenvalues $|r|$ from the non-shifted case and shifted case of the illustration.

Non-Shifted Eigenparameter λ	$\|r\|$ in Non-Shifted Case	Shifted Eigenparameter μ	Eigenvalue $\|r\|$ in Shifted Case
-0.68614...	1.31685...; 0.75939...	2.87228...	1
2.18614...	1		

The test for purely oscillatory behaviour is illustrated by considering the illustration of the Crum-based transformation in this chapter. The λ and μ values are given in Table 4.1, together with their corresponding values for $|r|$. From Table 4.1, it is apparent that the positive λ-value and μ-value satisfies condition (4.42) for purely oscillatory behaviour. However, from Table 4.1, $|r| \neq 1$ for the negative eigenparameter and thus cannot be a function of a meaningful physical natural frequency.

4.4 Concluding Remarks

This chapter outlined the Crum-based transformation method through a two-point boundary value problem illustration, where the inital boundary condition is Affine and the final boundary condition is Dirichlet. The three sub-methods of the Crum-based transformation: repetitive substitution, shifting and transformation are discussed in the illustration. The expected number of eigenparameters was determined using Theorem 3.2 and found to be equal to the number of generated eigenparameters. The illustrated boundary value problem has shown that the transformation performs the same function as shifting the eigenaprameter spectrum by the least negative eigenparameter. Also, it was shown that the transformation generated shifted eigenparameters that meet the condition for the system to exhibit pure oscillatory behaviour.

5 ROOTS OF CHARACTERISTIC EQUATION USING THE MATRIX METHOD

5.1 Matrix Method

This dissertation proposes an alternative method to the repetitive substitution using algebra by solving for the eigenparameter λ from the characteristic equation of the system matrix. The characteristic equation of the system matrix is a polynomial. Finding the roots of a polynomial is generally a difficult challenge as there is no closed formula for polynomials of degree five or higher (Panju, 2011). Thus, it would make sense to use a polynomial root-finding algorithm to determine the roots of the characteristic equation.

Software tools such as Mathematica (Wolfram, 2011) and Matlab (MATLAB, 2004) are powerful mathematical tools which have the functionality that can efficiently compute the characteristic equation of a matrix as well as find the roots of the characteristic equation (Colins II, 2003; Wolfram, 2015). The choice of software used for this functionality was chosen to be Mathematica (Wolfram, 2011). The function used to solve the roots of the characteristic equation is NSolve (Wolfram, 2015). NSolve determines a numerical Grobner basis using an efficient monomial ordering. The exact numerical roots are then extracted using eigenvalue methods (Wolfram, 1999).

An illustration of the matrix method is now discussed. Consider the following difference equation

$$X(n+1) - (2-\lambda)X(n) + X(n-1) = 0, \tag{5.1}$$

which is a form of the difference equation given by (1.1),

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

where the coefficients are independent of n. The difference equation (5.1) is considered over 3 unit intervals and subjected to the following Dirichlet boundary conditions

$$X(0) = 0, \tag{5.2}$$

and

$$X(3) = 0. \tag{5.3}$$

Now consider the difference equation (5.1) for $n = 1$

$$X(2) - (2-\lambda)X(1) + X(0) = 0, \tag{5.4}$$

and for $n = 2$

$$X(3) - (2-\lambda)X(2) + X(1) = 0. \tag{5.5}$$

Equations (5.2), (5.3), (5.4) and (5.5) can be compiled in matrix form, given by

$$\mathbf{AX}=\mathbf{B}, \tag{5.6}$$

where $\mathbf{A}$ is a 4×4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.7}$$

$\mathbf{X}$ is a 4×1 matrix given by

$$\mathbf{X} = \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}, \tag{5.8}$$

and $\mathbf{B}$ is a 4×1 zero vector

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.9}$$

The determinant of the coefficient matrix $\mathbf{A}$, determined using Mathematica (Wolfram, 2011), is given by

$$\det(\mathbf{A}) = \lambda^2 - 4\lambda + 3,$$

The roots of $\det(\mathbf{A})$ yield the eigenparameters λ of the system

$$\lambda = 1 \text{ or } 3.$$

This method outlined above can be generalised to the difference equation (5.1) for m unit intervals and for the different boundary conditions given in Table 1.1.

5.2 Concluding Remarks

This chapter outlined the matrix method through an illustration. The matrix method is a more efficient method than the repetitive substituiton method for solving for the eigenparameter λ, especially where the number of unit intervals exceed three. The matrix method can be applied to a two-point boundary value problem, given by (5.1) and combination of boundary conditions listed in Table 1.1, with any number of unit intervals.

6 ENGINEERING APPLICATIONS

In this chapter, the form of the two-point boundary value problem, given by (1.1)

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

where the coefficients are independent of n will be subjected to the boundary condition cases, given in Table 1.1, and in the context of relevant engineering applications. These engineering applications will be discussed in the ensuing sub-sections.

For each boundary condition combination, the matrix method developed with Mathematica (Wolfram, 2011) code (Appendix A) is confirmed using the algebraic method for a three unit interval scenario. The continuous-case eigenvalues are then approximated using five different unit intervals in the discrete case and a comparison is then done between the approximated continuous-case eigenvalues and the actual continuous-case eigenvalues to establish if the discrete case approaches the continuous case as the number of unit intervals is increased.

6.1 Electrostatics

The effect of electric charges at rest falls under the study of electrostatics. The problem of determining the potential in the surrounding space around conducting bodies of known potential leads to boundary value problems (Cheng, 1992:178).

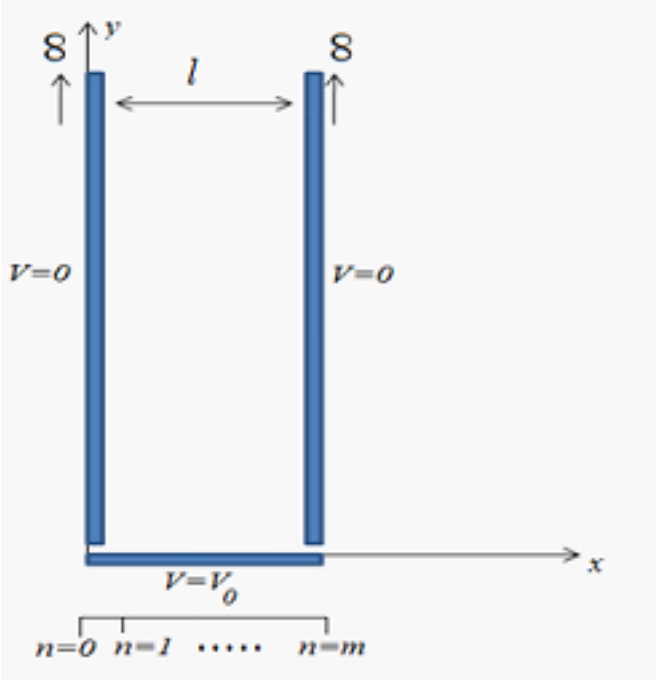

Figure 6.1: Cross-sectional figure for two grounded semi-infinite, parallel-plane electrodes with the third electrode at potential V_0.

Consider two grounded, semi-infinite, parallel-plane electrodes separated by a distance l as shown in Figure 6.1. A third plate, insulated from the grounded plates at a constant potential V_0, is situated

normal to the grounded plates. All planes are assumed to be infinite in the z-direction, so there is no dependence of the scalar electric potential on the Cartesian co-ordinate z.

The Laplace equation for a scalar electric potential $V(x, y)$ in Cartesian co-ordinates is given by (Cheng, 1992:178)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{6.1}$$

Applying the method of variable separable, the scalar electric potential $V(x, y)$ can be specified as a product (Cheng, 1992:178)

$$V(x, y) = X(x)Y(y). \tag{6.2}$$

Substituting (6.2) into (6.1) results in

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k^2, \tag{6.3}$$

where k represents the eigenvalues or the natural frequencies of the system.

To determine the scalar electric potential in the x-direction, keeping y constant; results in

$$X''(x) + k^2 X(x) = 0. \tag{6.4}$$

The spatial differential equation, given by (6.4), is then discretised using the central difference formula and one obtains

$$\frac{X(n+1) - 2X(n) + X(n-1)}{h^2} + k^2 X(n) = 0.$$

Multiplying by h^2 yields

$$X(n+1) - [2 - k^2 h^2]X(n) + X(n-1) = 0. \tag{6.5}$$

In order for (6.5) to take on the form of (1.1), given by

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

and since the eigenparameter λ is a function of the eigenvalue k of the system , a clear assignment would be

$$c(n) = c(n-1) = 1, \tag{6.6}$$

$$b(n) = 2, \tag{6.7}$$

and

$$k^2 h^2 = \lambda. \tag{6.8}$$

Therefore, the final difference equation under study is given by

$$X(n+1) - [2 - \lambda]X(n) + X(n-1) = 0. \tag{6.9}$$

The boundary condition cases for this engineering problem are given in Table 1.1.

6.1.1 (Dirichlet, Dirichlet) Boundary Conditions

This case considers where both initial and final boundary conditions are Dirichlet. In the electrostatic problem shown in Figure 6.1, both semi-infinite, parallel-plane electrodes are grounded and thus at zero potential. Consequently, the initial Dirichlet boundary condition, in both the continuous case and the discrete case, is given by

$$X(0) = 0.$$

The final Dirichlet boundary condition, in the continuous case is given by

$$X(l) = 0,$$

which corresponds to a final Dirichlet boundary condition in the discrete case as

$$X(m) = 0,$$

where m is the number of unit intervals.

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.9), for 3-unit intervals, using the algebraic method. The matrix method, outlined in Chapter 5, is then computed and checked against the algebraic method for validation.

The initial Dirichlet boundary condition is given by

$$X(0) = 0, \tag{6.10}$$

while the final Dirichlet boundary condition is given by

$$X(3) = 0. \tag{6.11}$$

For $n = 1$, (6.9) becomes

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.12}$$

Substitute (6.10) into (6.12) to obtain

$$X(2) = (2 - \lambda)X(1). \tag{6.13}$$

Put $n = 2$ in (6.9) to get

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.14}$$

Substitute (6.11) and (6.13) into (6.14)

$$-(2 - \lambda)^2 X(1) + X(1) = 0,$$

$$X(1)[1 - (2 - \lambda)^2] = 0.$$

Since $X(1) \neq 0$, it is clear that

$$(2 - \lambda)^2 - 1 = 0.$$

Therefore $\lambda = 1$ and 3.

The eigenparameters λ of the boundary value problem defined by (6.9), (6.10) and (6.11) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.2 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = \lambda^2 - 4\lambda + 3,$$

Using NSolve in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.2 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 1$ and 3. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2. Consider the initial boundary condition in (3.18). For the initial boundary condition in (3.18) to take on the Dirichlet form of the boundary condition in (6.10): $a = 0,\ b = 0$.

Now consider the final boundary condition form in (3.19) from Theorem 3.2. The parameters in (3.19) have to take on the following values for the initial boundary condition to take on the Dirichlet form of (6.11): $\alpha = 0$ and $\beta = 1$. Since $X(m = 3)$ has to be zero, we chose $\alpha = 0$ and $\beta \neq 0$ so that the factor $\alpha\lambda + \beta$ is not zero nor is there any non-zero λ factor. Thus, this satisfies case (ii) in Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Dirichlet, Dirichlet) condition.

Continuous Case

The continuous case equation (6.4) resulted from using the variable separable method. The initial Dirichlet boundary condition is given by

$$X(0) = 0, \tag{6.15}$$

and the final Dirichlet boundary condition is given by

$$X(l) = 0. \tag{6.16}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx),$$

so that using (6.15) results in

$$b\cos(0) = 0.$$

Therefore,

$$b = 0.$$

In a similar manner, using (6.11) and with $b = 0$

$$a\sin(kl) = 0.$$

Since $a \neq 0$ for a non-trivial solution, it follows that

$$k = \frac{j\pi}{l}. \qquad j = 1, 2, 3 \ldots$$

Assume the length $l = 1$, then

$$k = j\pi. \qquad j = 1, 2, 3 \ldots$$

The eigenvalues k thus form an infinite spectrum of positive integer multiples of π. The solution $X_j(x)$ for a particular j is given by

$$X_j(x) = a\sin(j\pi x).$$

Since (6.4) is a linear differential equation, the superposition of the solution $X_j(x)$ for all j is a solution in itself, where

$$X(x) = a\sum_{j=1}^{\infty}\sin(j\pi x).$$

The eigenvalues thus represent the natural frequencies of the scalar electric potential $V(x, y)$ in the spatial domain x and form a Fourier series of sinusoidal solutions with the natural frequencies occurring at integer multiples of π.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

A comparison is done between the k values, derived from the continuous case, to the k values, derived from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The generated eigenparameters, in the discrete case, are computed from the same Mathematica (Wolfram, 2011) code in Appendix A, Section A.2 except that the matrices are of size 7x7, 12x12, 20x20, 31x31 and 101x101 for the 6-unit-intervals, 11-unit-intervals, 19-unit-intervals, 30-unit-intervals and 100-unit-intervals cases respectively. The Mathematica (Wolfram, 2011) code is also extended in the same manner when computing the eigenparameters in the remaining combination of boundary conditions. Noting that $N = \frac{1}{h}$, the eigenvalues or k values are derived from the eigenparameters from (6.8)

$$k = N\sqrt{\lambda}.$$

The Mathematica (Wolfram, 2011) code algorithm is based on the matrix method outlined in Chapter 5. The matrix equation is set up from the respective difference equation for $n = 1$ to $m - 1$, where m is the number of unit intervals as well as from the respective boundary condition

equations. The determinant of the respective characteristic matrix is then computed to form the characteristic equation. Consequently, the roots of the characteristic equation are then solved using the NSOLVE function in Mathematica (Wolfram, 2011), which yield the eigenparameter solution spectrum.

Table 6.1: Expected and generated number of eigenparameters for the (Dirichlet, Dirichlet) condition for the unit intervals indicated in the first column.

Number of unit intervals $=m$	α	β	Case in Theorem 3.2	Expected number of eigenparameters $=m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

The expected number of eigenparameters is determined from Theorem 3.2. The expected and generated number of eigenparameters is equal as shown in Table 6.1.

An eigenvalue index is defined as the index that a particular eigenvalue holds in the array of eigenvalues sorted in ascending order i.e. the smallest eigenvalue will be at eigenvalue index 1 and the largest eigenvalue will be at position Z, where Z is the number of eigenvalues.

Table 6.2 shows the approximated continuous-case eigenvalues derived from the discrete case, in order of their eigenvalue index, for the different unit-interval cases as well as the continuous-case eigenvalues. The comparison focuses on the first five eigenvalues as these are the minimum generated number of approximated continuous-case eigenvalues in all cases. The comparison is done against the percentage relative error of the approximated continuous-case eigenvalues derived from the discrete case measured against the continuous case eigenvalues. The percentage relative error of the ith eigenvalue e_i is given by

$$e_i = \frac{100|E_{c(i)} - E_{d(i)}|}{E_{c(i)}},$$

where $E_{c(i)} = i^{th}$ eigenvalue in the continuous case and $E_{d(i)} = i^{th}$ approximated continuous-case eigenvalue in the discrete case.

The results in Table 6.2 are shown graphically in Figure 6.2.

Table 6.2: First five eigenvalues k derived from the different unit-intervals cases and the continuous case for the (Dirichlet, Dirichlet) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	3.106	3.131	3.139	3.140	3.141	3.142
2	6.000	6.198	6.256	6.272	6.282	6.283
3	8.485	9.139	9.329	9.386	9.421	9.425
4	10.392	11.894	12.338	12.475	12.558	12.566
5	11.591	14.407	15.264	15.529	15.692	15.708

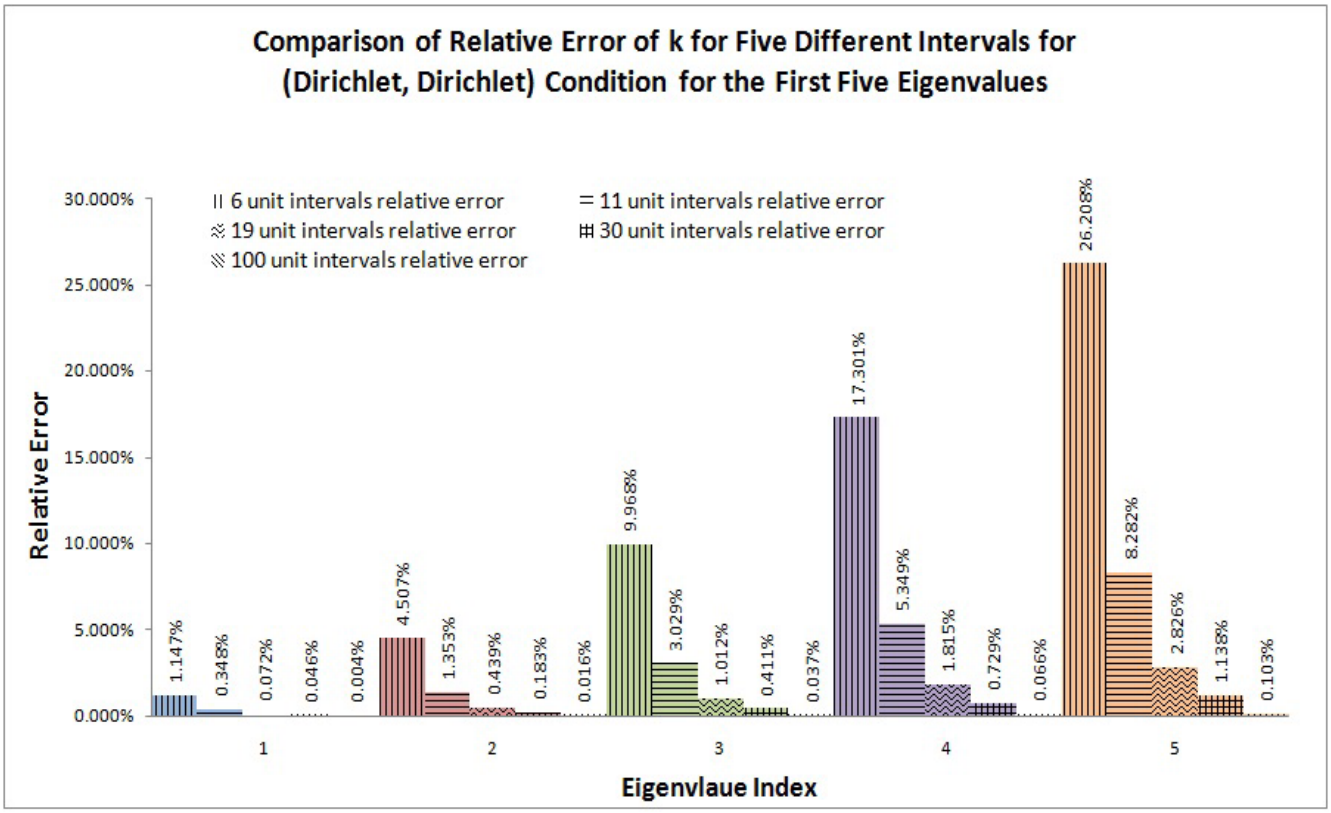

Figure 6.2: Comparison of Relative Error of k for Five Different Intervals for (Dirichlet, Dirichlet) Condition for the First Five Eigenvalues.

As can be seen from Figure 6.2; a general trend is that as the number of unit intervals increase, the relative error of the approximation to k reduces. For example, if we consider the 5th eigenvalue in Figure 6.2, the relative error reduces from 26.208 % for the 6-unit intervals case to 0.103 % for the 100-unit intervals case. Therefore, as the number of unit intervals increase, the discrete-case derived k values tends towards the continuous-case k values.

Another general observation from Figure 6.2 is that as one ascends through the eigenvalues spectrum, the discrete-case approximation tends to reduce in accuracy for the higher eigenvalues i.e. the relative error increases for higher eigenvalues. As an example, consider the relative error of 1.14 % for the 6-unit intervals case for the 1st eigenvalue as opposed to the relative error of 26.208 % for the 6-unit intervals case in the 5th eigenvalue.

To determine if the condition for purely oscillatory behaviour has been satisfied, the left-hand side of (3.28)

$$\left| \frac{2-k^2h^2}{2} \pm \sqrt{\left(\frac{2-k^2h^2}{2}\right)^2 - 1} \right|,$$

is applied to the approximated k values in Table 6.2. Equation (3.28) can be used because (6.9) is the same as (3.22). It is found that the left-hand side of (3.28) when applied to the approximated k values from Table 6.2 equals to 1, thus satisfying the condition of purely oscillatory behaviour. Therefore, the approximated k values represent natural frequencies of the system. Also, all the approximated natural frequencies as given in Table 6.2 are below their respective Nyquist frequencies listed in Table 3.1.

6.2 Heat Conduction

Heat conduction is a classical physical application of a two-point boundary value problem. It has significant importance in the analysis of heat transfer away from sources in high speed machinery (DiPrima & Boyce, 1977:453). In this study, heat conduction is considered along a straight bar of uniform cross-section and made of homogeneous material.

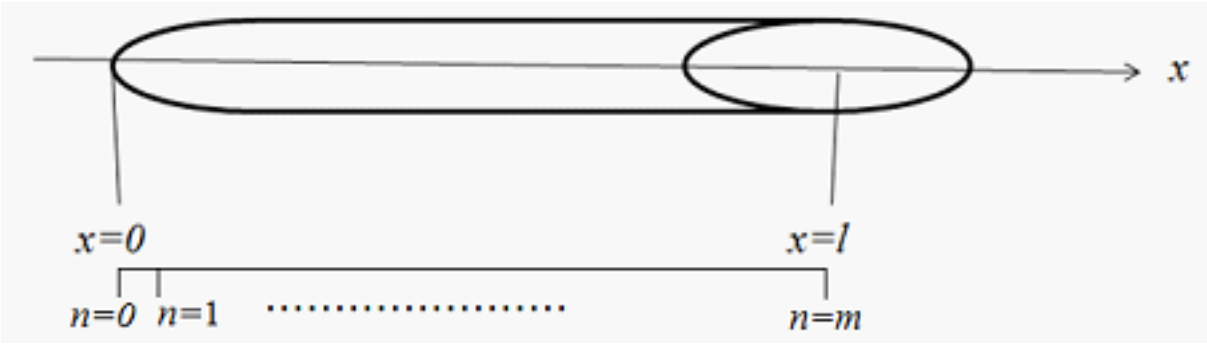

Figure 6.3: Straight bar of uniform cross-section and homogeneous materials and length l units (DiPrima & Boyce, 1977:453).

The heat conduction of a straight bar of uniform cross-section, of length l units, made of homogeneous material is shown in Figure 6.3. The x-axis lies along the axis of the bar and the ends are denoted by $x = 0$ and $x = l$. Assume that the sides of the bar are perfectly insulated so no heat passes through them and that the cross-sectional areas are significantly small so that the temperature u can be considered to be the same at a given cross-section (DiPrima & Boyce, 1977:453). The temperature u is then governed by the following equation (DiPrima & Boyce, 1977:453)

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \qquad 0 < x < l,\ t > 0. \tag{6.17}$$

where α is the thermal diffusivity of the material (cm^3/s) and $u(x,t)$ is the temperature at position x and time t.

Assume that the temperature function $u(x,t)$ is given by (DiPrima & Boyce, 1977:514)

$$u(x,t) = X(x)T(t), \tag{6.18}$$

that is, the variables can be separated. Substituting (6.18) into (6.17) results in

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -k^2, \tag{6.19}$$

where k represents the eigenvalues or the natural frequencies of the system.

Using the variable separable method, this results in the following spatial differential equation

$$X''(x) + k^2 X(x) = 0. \tag{6.20}$$

The spatial differential equation, given by (6.20), is discretised using the central difference formula and one obtains

$$\frac{[X(n+1) - 2X(n) + X(n-1)]}{h^2} + k^2 X(n) = 0.$$

Multiplying by h^2 results

$$X(n+1) - [2 - k^2h^2]X(n) + X(n-1) = 0. \tag{6.21}$$

In order for (6.21) to take on the form of (1.1), given by

$$c(n)X(n+1) - [b(n) - c(n)\lambda]X(n) + c(n-1)X(n-1) = 0, \qquad n = 1, ..., m-1,$$

and since the eigenparameter λ is a function of the eigenvalue k of the system , a clear assignment would be

$$c(n) = c(n-1) = 1, \tag{6.22}$$

$$b(n) = 2, \tag{6.23}$$

and

$$k^2h^2 = \lambda. \tag{6.24}$$

Therefore, the final difference equation under study is given by

$$X(n+1) - [2 - \lambda]X(n) + X(n-1) = 0. \tag{6.25}$$

The boundary condition cases for this engineering problem are given in Table 1.1.

6.2.1 (Non-Dirichlet, Dirichlet) Boundary Conditions

This case considers where the initial boundary condition is a Non-Dirichlet condition and the final boundary condition is a Dirichlet condition. Consider the heat flow occurring through the end of the rod at $x = 0$ and is proportional to the temperature at point $x = 0$. This yields the radiation condition (DiPrima & Boyce, 1977:514)

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad t > 0.$$

where $u_x(0,t)$ is the first derivative of the temperature function $u(x,t)$ with respect to the spatial variable x and at spatial position $x = 0$. Since the bar is insulated at the bar end: $x = 0$, this implies that $h_1 = 0$ (DiPrima & Boyce, 1977:514) yielding

$$u_x(0,t) = X^{'}(0)T(t) = 0.$$

Since $T(t) \neq 0$ for all t>0; we have an initial boundary condition in the continuous case as

$$X^{'}(0) = 0.$$

Using the forward finite difference formula

$$X^{'}(0) \approx \frac{X(1) - X(0)}{h},$$

Then, the resulting discrete-case initial Non-Dirichlet boundary condition is given by

$$X(1) - X(0) = 0.$$

The final boundary condition in the continuous case is given by

$$X(l) = 0,$$

with the corresponding final Dirichlet boundary condition in the discrete case as

$$X(m) = 0,$$

where m is the number of unit intervals.

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.25), for 3 unit intervals, using the algebraic method. The matrix method, discussed in Chapter 5, is then computed and checked against the algebraic method for validation.

The initial Non-Dirichlet boundary condition is given by

$$X(1) - X(0) = 0,$$

$$X(1) = X(0), \tag{6.26}$$

and the final Dirichlet boundary condition is given by

$$X(3) = 0. \tag{6.27}$$

For $n = 1$, (6.25) becomes

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.28}$$

Substitute (6.26) into (6.28) to obtain

$$X(2) - (2 - \lambda)X(1) + X(1) = 0,$$

$$X(2) = [(2 - \lambda) - 1]X(1). \tag{6.29}$$

Put $n = 2$ in (6.25) to get

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.30}$$

Substitute (6.27) and (6.29) into (6.30) results in

$$-(2 - \lambda)[(2 - \lambda) - 1]X(1) + X(1) = 0,$$

$$X(1)\{(2 - \lambda)(1 - \lambda) - 1\} = 0.$$

Since $X(1) \neq 0$, this becomes

$$(2 - \lambda)(1 - \lambda) - 1 = 0,$$

$$\lambda^2 - 3\lambda + 1 = 0,$$

Therefore

$$\lambda = 0.382 \text{ and } 2.618.$$

The eigenparameters λ of the boundary value problem defined by (6.25), (6.26) and (6.27) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.3 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = -\lambda^2 + 3\lambda - 1,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.3 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 0.382$ and 2.618. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2. In order for the initial boundary condition in (3.18) to take on the Non-Dirichlet form in (6.26), the following parameters must be set as follows: $a = 0$ and $b = 1$.

The parameters from the Dirichlet form of the final boundary condition was determined in Section 6.1.1 as $\alpha = 0$ and $\beta = 1$.

Thus, this satisfies case (ii) in Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with the number of eigenparameters that have been computed in the three unit interval boundary problem for the (Non-Dirichlet, Dirichlet) condition.

Continuous Case

The continuous case equation is given by (6.20) which has been obtained by using the variable separable method.The initial boundary condition is given by

$$X'(0) = 0, \tag{6.31}$$

while the final boundary condition is given by

$$X(l) = 0. \tag{6.32}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx).$$

From (6.32), it follows that

$$a\sin(kl) + b\cos(kl) = 0. \tag{6.33}$$

From (6.31), we have

$$ak\cos(0) - bk\sin(0) = 0,$$

from which we obtain

$$ak = 0.$$

Since $k \neq 0$, it is clear that

$$a = 0.$$

From (6.33),

$$b\cos(kl) = 0.$$

Since $b \neq 0$ for a non-trivial solution, we obtain, for odd values of j

$$k = \frac{j\pi}{2l}. \qquad j = 1, 3, 5 \ldots\ldots$$

Assume the length $l = 1$, then

$$k = \frac{j\pi}{2}. \qquad j = 1, 3, 5 \ldots\ldots$$

The solution $X_j(x)$ at a particular j is given by

$$X_j(x) = b\cos\left(\frac{j\pi}{2}x\right). \qquad j = 1, 3, 5 \ldots\ldots$$

Also, the superposition of the solution $X(x)$ is a solution in itself since (6.20) is a linear differential equation. Consequently $X(x)$ takes on the form

$$X(x) = b\sum_{j=odd}^{\infty}\cos\left(\frac{j\pi}{2}x\right).$$

The eigenvalues thus do represent the natural frequencies of the temperature $u(x,t)$ along the bar in the spatial domain x and these natural frequencies occur at odd-integer multiples of $\frac{\pi}{2}$.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

A comparison is done between the k values, derived from the continuous case, to the k values, derived from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) Code in Appendix A, Section A.3, for the respective unit interval as was described in Section 6.1.1. Noting that $N = \frac{1}{h}$, the approximated continuous-case eigenvalues k values are derived from the eigenparameters from (6.24)

$$k = N\sqrt{\lambda}.$$

The formula for k is used in the remaining Sections 6.2.2 and 6.2.3.

Table 6.3 shows the expected and generated number of eigenparameters to be equal. The expected number of eigenparameters is determined from Theorem 3.2.

Table 6.3: Expected and generated number of eigenparameters for the (Non-Dirichlet, Dirichlet) condition for the unit intervals indicated in the first column.

Number of unit intervals $=m$	α	β	Case in Theorem 3.2	Expected number of eigenparameters $=m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

Table 6.4 indicates the approximated continuous-case eigenvalues derived from the discrete case as well as the eigenvalues from the continuous case. Not all the eigenvalues are shown in Table 6.4 as the comparison focuses on the first five eigenvalues as these are the minimum generated number of approximated continuous-case eigenvalues of all the unit-intervals cases. The comparison is done against the percentage relative error of the approximated continuous-case eigenvalues derived from the discrete case measured against the continuous-case eigenvalues.

Table 6.4: First five eigenvalues k derived from the different unit-intervals cases and the continuous case for the (Non-Dirichlet, Dirichlet) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	1.708	1.643	1.612	1.597	1.579	1.571
2	4.985	4.896	4.825	4.787	4.736	4.712
3	7.858	8.038	8.005	7.964	7.891	7.854
4	10.095	11.000	11.128	11.117	11.045	10.996
5	11.514	13.717	14.169	14.240	14.196	14.137

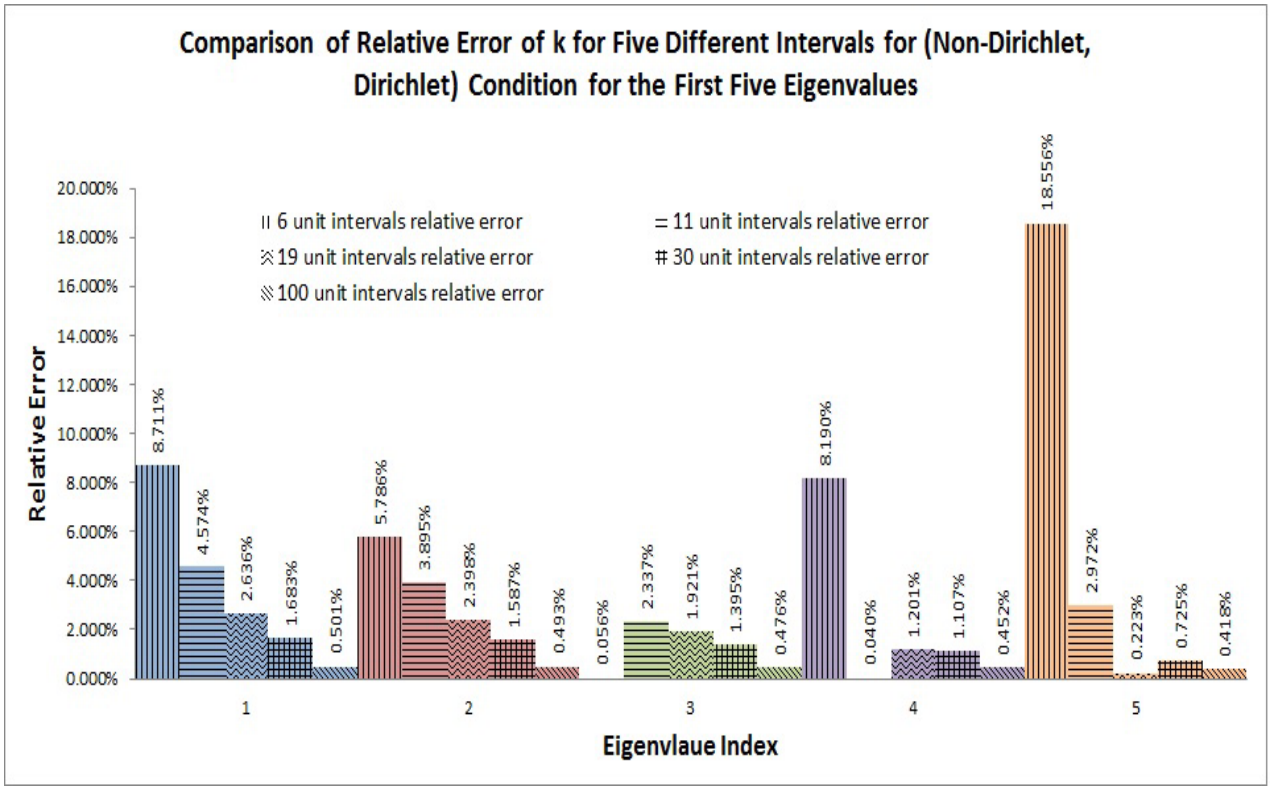

Figure 6.4: Comparison of Relative Error of k for Five Different Intervals for (Non-Dirichlet, Dirichlet) Condition for the First Five Eigenvalues.

The same analysis holds for Figure 6.4 as was done for the comparison in Figure 6.2. Also, the same analysis is applied to Table 6.4 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.25) is the same as (3.22). The approximated k values from Table 6.4 were also found to satisfy condition (3.28) and thus represent natural frequencies of the system. All the approximated natural frequencies, as given in Table 6.4, are below their respective Nyquist frequencies listed in Table 3.1.

6.2.2 (Dirichlet, Non-Dirichlet) Boundary Conditions

Here we consider what happens when the initial boundary condition is a Dirichlet condition and the final boundary condition is a Non-Dirichlet condition. Suppose that the heat flow occurs through the end of the rod at $x = l$ and is proportional to the temperature at point $x = l$. This yields the radiation condition (DiPrima & Boyce, 1977:514)

$$u_x(l,t) + h_1 u(l,t) = 0. \qquad t > 0.$$

Assume the bar is insulated at the bar end $x = l$ which implies that $h_1 = 0$ (DiPrima & Boyce, 1977:514), yielding

$$u_x(l,t) = X^{'}(l)T(t) = 0.$$

Since $T(t) \neq 0$ for all $t > 0$, this implies that the final continuous boundary condition is

$$X^{'}(l) = 0,$$

and hence using the backward finite difference formula

$$X^{'}(l) \approx \frac{X(m) - X(m-1)}{h},$$

where m is the number of unit intervals; results in the final Non-Dirichlet boundary condition, in the discrete case, of

$$X(m) - X(m-1) = 0.$$

The initial continuous and discrete Dirichlet boundary condition is given by

$$X(0) = 0.$$

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.25), for 3 unit intervals, using the algebraic method to confirm the results obtained using the matrix method. The initial Dirichlet boundary condition is given by

$$X(0) = 0, \tag{6.34}$$

and the final Non-Dirichlet boundary condition is given by

$$X(3) - X(2) = 0,$$

$$X(3) = X(2). \tag{6.35}$$

For $n = 1$ in (6.25) so that

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.36}$$

Substitute (6.34) into (6.36)

$$X(2) = (2 - \lambda)X(1). \tag{6.37}$$

Suppose $n = 2$ in (6.25), then we obtain

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.38}$$

Substitute (6.35) into (6.38) to get

$$X(2) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.39}$$

Substituting (6.37) into (6.39) gives

$$(2 - \lambda)\ X(1) - (2 - \lambda)^2 X(1) + X(1) = 0,$$

that is,

$$X(1)[\lambda^2 - 3\lambda + 1] = 0.$$

Since $X(1) \neq 0$, it follows that

$$\lambda^2 - 3\lambda + 1 = 0.$$

and hence,

$$\lambda = 0.3820 \text{ and } 2.618.$$

The eigenparameters λ of the boundary value problem defined by (6.25), (6.34) and (6.35) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.4 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = \lambda^2 - 3\lambda + 1,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.4 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 0.3820$ and 2.618. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2. As determined in the Section 6.1.1 for Dirichlet initial boundary condition, $a = 0$ and $b = 0$.

For the final boundary condition to take on the form of the Non-Dirichlet boundary condition in (6.35), the parameters must take on the following values: $\alpha = 0$ and $\beta = 1$.

Therefore, this set of parameters satisfies the requirements in case (ii) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Dirichlet, Non-Dirichlet) condition.

Continuous Case

The continuous case equation is given by (6.20) which has been obtained using the variable separable method.The final boundary condition is given by

$$X^{'}(l) = 0, \tag{6.40}$$

while the initial boundary condition is given by

$$X(0) = 0. \tag{6.41}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx).$$

From (6.41), it follows that

$$X(0) = b = 0,$$

and thus we obtain

$$X(x) = a\sin(kx).$$

From (6.40), we have

$$ak\cos(kl) = 0.$$

Since $a \neq 0$ and $k \neq 0$, it is clear that

$$\cos(kl) = 0.$$

Therefore, for odd values of j,

$$k = \frac{j\pi}{l(2)}. \qquad j = 1, 3, 5 \ldots\ldots$$

Assume the length $l = 1$, then

$$k = \frac{j\pi}{2}. \qquad j = 1, 3, 5 \ldots\ldots$$

The solution $X_j(x)$ for a particular j is given by

$$X_j(x) = a \sin\left(\frac{j\pi}{2}x\right). \qquad j = 1, 3, 5 \ldots\ldots$$

By virtue of (6.20) being a linear differential equation, the superposition of the solution $X(x)$ is a solution in itself, given by

$$X(x) = a \sum_{j=odd}^{\infty} \sin\left(\frac{j\pi}{2}x\right).$$

The eigenvalues k occur as odd integer multiples of $\frac{\pi}{2}$ and represent the natural frequencies of the temperature $u(x,t)$ along the spatial domain x.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

A comparison is done between the k values, derived from the the continuous case, to the k values, derived from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.4, for the respective unit interval as was described in Section 6.1.1. The generated and expected number of eigenparameters is found to be equal. This is illustrated in Table 6.5 where the expected number of eigenparameters is determined from Theorem 3.2.

Table 6.5: Expected and generated number of eigenparameters for the (Dirichlet, Non-Dirichlet) condition for the unit intervals indicated in the first column.

Number of unit intervals $= m$	α	β	Case in Theorem (3.2)	Expected number of eigenparameters $= m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

Table 6.6 does not show all the eigenvalues as the comparison focuses on the first five eigenvalues as these are the minimum generated number of approximated continuous-case eigenvalues of all cases. The comparison is done against the percentage relative error of the approximated continuous-case eigenvalues derived from the discrete case measured against the continuous case eigenvalues.

Table 6.6: First five k-values derived from the different unit-intervals cases and the continuous case for the (Dirichlet, Non-Dirichlet) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	1.708	1.643	1.612	1.597	1.579	1.571
2	4.985	4.896	4.825	4.787	4.736	4.712
3	7.858	8.038	8.005	7.964	7.891	7.854
4	10.095	11.000	11.128	11.117	11.045	10.996
5	11.514	13.717	14.169	14.240	14.196	14.137

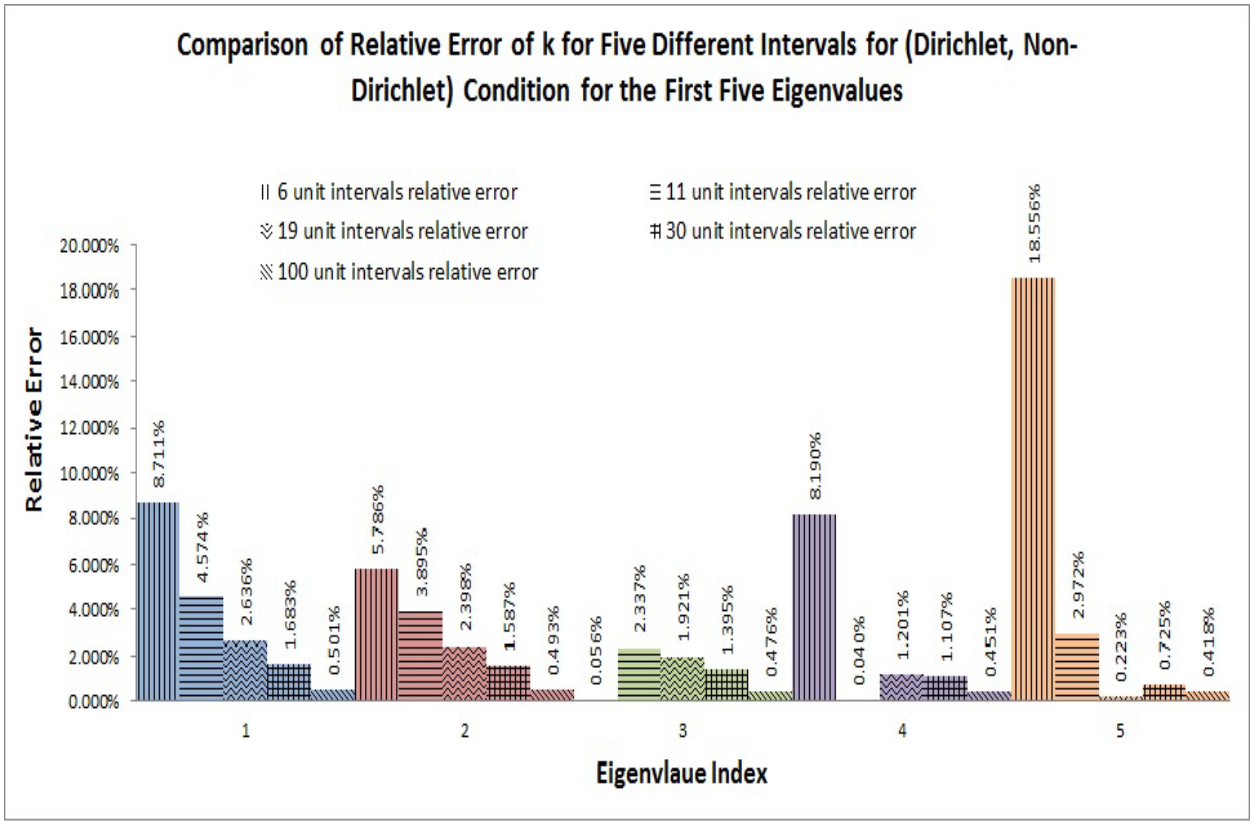

Figure 6.5: Comparison of Relative Error of k for Five Different Intervals for (Dirichlet, Non-Dirichlet) Condition for the First Five Eigenvalues.

The same analysis holds for Figure 6.5 as was done for the comparison in Figure 6.2. Also, the same analysis is applied to Table 6.6 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.25) is the same as (3.22). The approximated continuous-case eigenvalues from Table 6.6 were also found to satisfy condition (3.28) and thus represent natural

frequencies of the system. All the approximated k values, as indicated in Table 6.6, are below their respective Nyquist frequencies listed in Table 3.1.

6.2.3 (Non-Dirichlet, Non-Dirichlet) Boundary Conditions

Now consider what happens when the initial boundary condition is a Non-Dirichlet condition and the final boundary condition is a Non-Dirichlet condition. Consider the heat flow occurring through both ends of the rod at $x = 0$ and $x = l$ and suppose this heat flow is proportional to the temperature at point $x = 0$ and $x = l$ respectively. This yields the radiation conditions (DiPrima & Boyce, 1977:514)

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad t > 0.$$

and

$$u_x(l,t) + h_1 u(l,t) = 0. \qquad t > 0.$$

Since the bar is insulated at the bar end $x = 0$, this implies that $h_1 = 0$ (DiPrima & Boyce, 1977:514) yielding

$$u_x(0,t) = X^{'}(0)T(t) = 0.$$

Since $T(t) \neq 0$ for all $t > 0$, this implies that the continuous initial boundary condition is given as

$$X^{'}(0) = 0.$$

Using the forward finite difference formula, the initial Non-Dirichlet boundary condition, in the discrete case, is computed, as

$$X^{'}(0) \approx \frac{X(1) - X(0)}{h} = 0,$$

$$X(1) - X(0) = 0.$$

Since the bar is also insulated at the bar end $x = l$, this implies that $h_1 = 0$ (DiPrima & Boyce, 1977:514) yielding

$$u_x(l,t) = X^{'}(l)T(t) = 0.$$

Since $T(t) \neq 0$ for all t>0, this implies that the continuous final boundary condition is

$$X^{'}(l) = 0.$$

Using the backward finite difference formula, the final Non-Dirichlet boundary condition, in the discrete case, is computed as

$$X^{'}(l) \approx \frac{X(m) - X(m-1)}{h} = 0,$$

$$X(m) - X(m-1) = 0.$$

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.25), for 3 unit intervals, using the algebraic method to validate the matrix method computed in Mathematica (Wolfram, 2011). The initial Non-Drichlet boundary condition is given by

$$X(1) - X(0) = 0,$$

$$X(1) = X(0), \tag{6.42}$$

while the final Non-Dirichlet boundary condition is given by

$$X(3) - X(2) = 0,$$

$$X(3) = X(2). \tag{6.43}$$

For $n = 1$ in (6.25) so that

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.44}$$

Substitute (6.42) into (6.44)

$$X(2) - (2 - \lambda)X(1) + X(1) = 0,$$

$$X(2) = [1 - \lambda]X(1). \tag{6.45}$$

Suppose $n = 2$ in (6.25), then we obtain

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.46}$$

Substitute (6.43) into (6.46) to get

$$X(2) - (2 - \lambda)X(2) + X(1) = 0,$$

$$[-1 + \lambda]X(2) + X(1) = 0. \tag{6.47}$$

Substitute (6.45) into (6.47) to obtain

$$(\lambda - 1)(1 - \lambda)X(1) + X(1) = 0,$$

$$X(1)\{1 - (1 - \lambda)^2\} = 0.$$

Since $X(1) \neq 0$, then

$$1 - (1 - \lambda)^2 = 0.$$

Therefore

$$\lambda = 0 \text{ and } 2.$$

The eigenparameters λ of the boundary value problem defined by (6.25), (6.42) and (6.43) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.5 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 2\lambda - \lambda^2,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.5 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 0$ and 2. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2. The parameters required for the (Non-Dirichlet, Non-Dirichlet) condition was determined in Sections 6.2.1 and 6.2.2 respectively to match (6.42) and (6.43). The parameter values are given as follows: $a = 0,\ b = 1,\ \alpha = 0$ and $\beta = 1$.

The parameters satisfies the criteria in case (ii) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Non-Dirichlet, Non-Dirichlet) condition.

Continuous Case

The continuous case equation is given by (6.20) which has been obtained using the variable separable method.The initial boundary condition is given by

$$X^{'}(0) = 0, \tag{6.48}$$

and the final boundary condition is given by

$$X^{'}(l) = 0. \tag{6.49}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx).$$

Therefore,

$$X^{'}(x) = ak\cos(kx) - bk\sin(kx).$$

From (6.48), we obtain

$$X^{'}(0) = 0,$$
$$ak = 0.$$

Since $k \neq 0$, it is clear that

$$a = 0.$$

From (6.49), we have

$$X^{'}(l) = 0,$$
$$-bk\sin(kl) = 0.$$

Since $b \neq 0$ and $k \neq 0$, it is clear that

$$k = \frac{j\pi}{l}. \qquad j = 1, 2, 3 \ldots\ldots$$

Assume the length $l = 1$, then

$$k = j\pi. \qquad j = 1, 2, 3 \ldots\ldots$$

The solution $X_j(x)$ for a particular j is given by

$$X_j(x) = b\cos(j\pi x).$$

Since (6.20) is a linear differential equation, the superposition of the solution $X_j(x)$ for all j is a solution in itself, where

$$X(x) = b\sum_{j=1}^{\infty}\cos(j\pi x)\ .$$

The eigenvalues thus occur at integer values of π, representing the natural frequencies of the temperature $u(x,t)$ in the spatial domain x.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

In this section, a comparison is done between the k values, obtained from the continuous case, to the k values, obtained from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.5, for the respective unit interval as was described in Section 6.1.1. Table 6.7 indicates that the expected and generated number of eigenparameters is equal. The expected number of eigenparameters was determined using Theorem 3.2.

Table 6.7: Expected and generated eigenparameters for the different unit-intervals cases for the (Non-Dirichlet, Non-Dirichlet) condition.

Number of unit intervals=m	α	β	Case in Theorem 3.2	Expected number of eigenparameters= $m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

Since the comparison focuses on the first five eigenvalues, as these are the minimum generated number of approximated k values of all unit-intervals cases, not all eigenvalues are shown in Table 6.8. The comparison is done against the percentage relative error of the approximated k values measured against the continuous case eigenvalues.

Table 6.8: First five eigenvalues k derived from the different unit-intervals cases and the continuous case for the (Non-Dirichlet, Non-Dirichlet) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	0.000	0.000	0.000	0.000	0.000	0.000
2	3.708	3.442	3.313	3.248	3.173	3.142
3	7.054	6.799	6.598	6.487	6.346	6.283
4	9.708	9.988	9.834	9.707	9.516	9.425
5	11.413	12.931	12.997	12.898	12.685	12.566

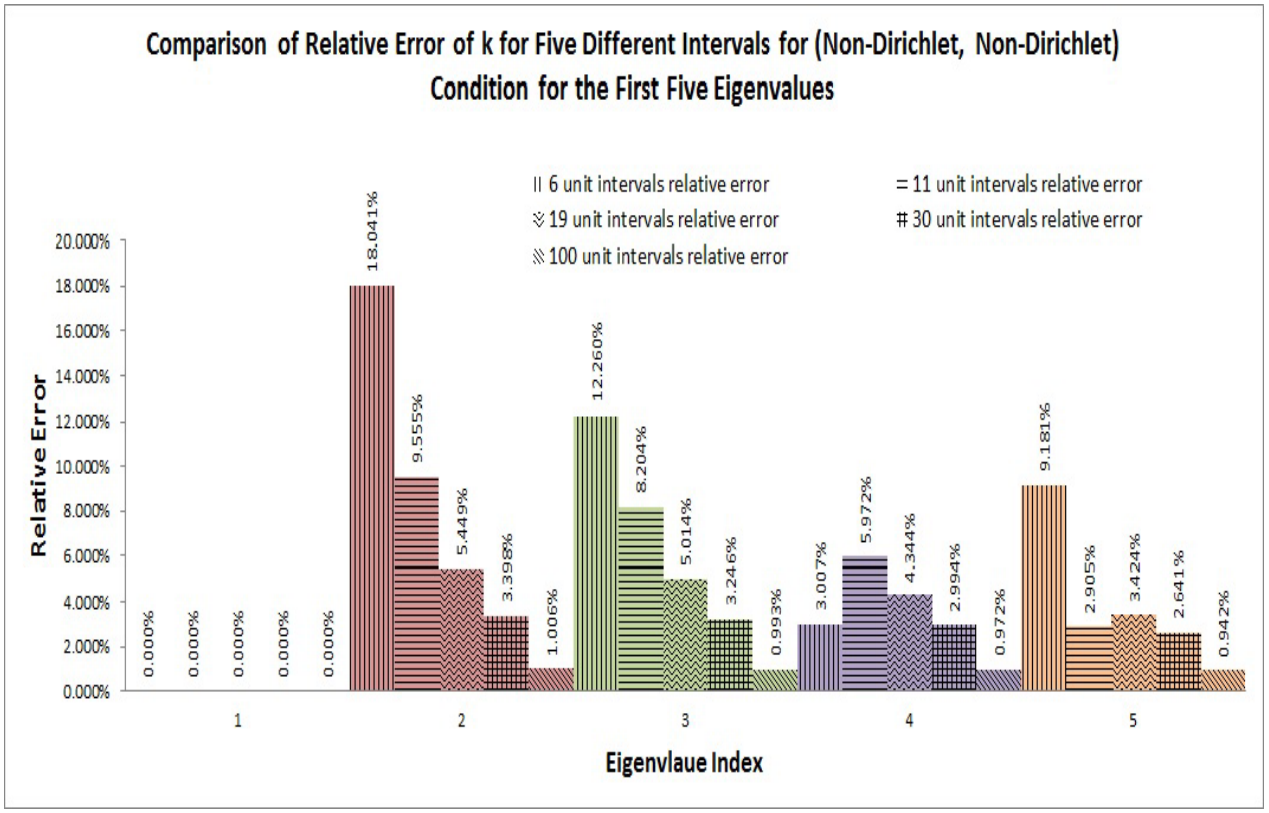

Figure 6.6: Comparison of Relative Error of k for Five Different Intervals for (Non-Dirichlet, Non-Dirichlet) Condition for the First Five Eigenvalues.

The same analysis holds for Figure 6.6 as was done for the comparison in Figure 6.2. Also, the same analysis is applied to Table 6.8 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.25) is the same as (3.22). The approximated continuous-case eigenvalues from Table 6.8 were also found to satisfy condition (3.28) and thus represent natural frequencies of the system. Further to that, all the approximated k values, as indicated in Table 6.8, are below their respective Nyquist frequencies listed in Table 3.1.

6.3 Longitudinal Vibrations of an Elastic Bar

Another physical application that is considered is the longitudinal vibrations of a straight uniform elastic bar. Vibrations are of particular importance in the study of mechanical systems, as vibrations could result in poor performance or even safety implications of mechanical systems especially when mechanical systems vibrate at the resonant frequencies (Alauf, 2011; Adams, 2010).

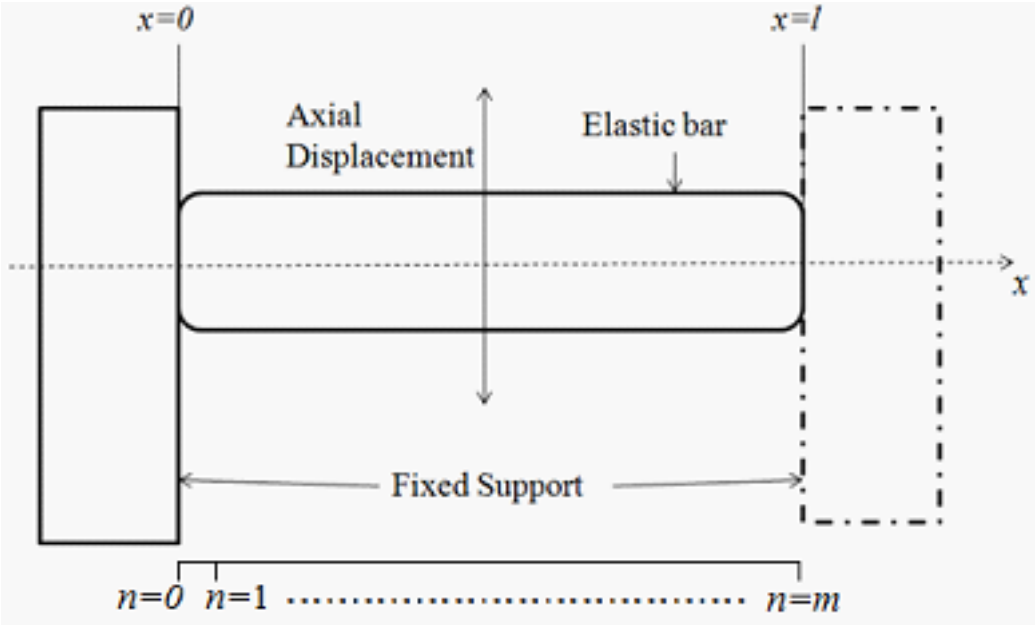

Figure 6.7: Uniform straight elastic bar with fixed supports at either $x=0$ or $x=l$.

Consider the longitudinal vibrations of a uniform straight elastic bar of length l as shown in Figure 6.7, either with a fixed support at $x = 0$ or $x = l$. The axial displacement $u(x,t)$ is governed by (DiPrima & Boyce, 1977:531)

$$\frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \qquad 0 < x < l,\ t > 0. \tag{6.50}$$

where E is Young's modulus of the bar material and ρ is the density of the bar material.

Assume that the axial displacement $u(x,t)$ is given by

$$u(x,t) = X(x)T(t). \tag{6.51}$$

Substituting (6.51) in (6.50) yields

$$\frac{X''(x)}{X(x)} = \frac{\rho T''(t)}{ET(t)} = -k^2, \tag{6.52}$$

where k represents the eigenvalues or natural frequencies of the system.

Using the variable separable method, this results in the following spatial differential equation

$$X''(x) + k^2 X(x) = 0. \tag{6.53}$$

The spatial differential equation, given by (6.53), is then discretised using the central difference formula and one obtains

$$\frac{X(n+1)-2X(n)+X(n-1)}{h^2}+k^2X(n)=0,$$

$$X(n+1)-[2-k^2h^2]X(n)+X(n-1)=0. \tag{6.54}$$

In order for (6.54) to take on the form of (1.1), given by

$$c(n)X(n+1)-[b(n)-c(n)\lambda]X(n)+c(n-1)X(n-1)=0, \qquad n=1,...,m-1,$$

and since the eigenparameter λ is a function of the eigenvalue k of the system, a clear assignment would be

$$c(n)=c(n-1)=1, \tag{6.55}$$

$$b(n)=2, \tag{6.56}$$

and

$$k^2h^2=\lambda. \tag{6.57}$$

The resultant difference equation is given by

$$X(n+1)-[2-\lambda]X(n)+X(n-1)=0. \tag{6.58}$$

The boundary condition cases for this engineering problem are given in Table 1.1.

6.3.1 (Dirichlet, Affine) Boundary Conditions

Let us consider where the initial boundary condition is a Dirichlet condition and the final boundary condition is an Affine condition. If the end $x=0$ is rigidly fixed, then the initial continuous-case and discrete-case boundary condition is given by (DiPrima & Boyce, 1977:531)

$$X(0)=0.$$

Now consider the end $x=l$ being connected to an unrestrained rigid mass M. The final continuous-case boundary condition is as follows (DiPrima & Boyce, 1977:531)

$$EAu_x(l,t)+Mu_{tt}(l,t)=0,$$

where A = cross-sectional area of the bar, $u_x(l,t)$ is the first derivative of the axial displacement function $u(x,t)$ with respect to the spatial variable x and at spatial position $x=l$, and $u_{tt}(l,t)$ is the second derivative of the axial displacement function $u(x,t)$ with respect to the time variable t and at spatial position $x=l$.

The specifications of the elastic bar and unrestrained mass are $M=0.5$ kg, $A=0.5m^2$ and $\rho=0.5kg.m^{-3}$. From Appendix B, the final continuous-case boundary condition is given by

$$X^{'}(l)-2\gamma k^2X(l)=0.$$

The discretised form of the final continuous-case boundary condition yields the final discrete-case Affine boundary condition

$$X(m)[1-2\lambda N]-X(m-1)=0.$$

where $N=m=$ number of unit intervals. The derivation of the final boundary condition is illustrated in Appendix B.

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.58), for 3 unit intervals, using the algebraic method to validate the matrix method computed in Mathematica (Wolfram, 2011). The initial Dirichlet boundary condition is given by

$$X(0) = 0, \tag{6.59}$$

and the final Affine boundary condition is given by

$$X(3)[1 - 2\lambda N] - X(2) = 0.$$

For $N = 3$, this becomes

$$X(3)[1 - 6\lambda] - X(2) = 0,$$

$$X(3) = \frac{X(2)}{1 - 6\lambda}. \tag{6.60}$$

Put $n = 1$ in (6.58), which results in

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.61}$$

Substitute (6.59) into (6.61) yields

$$X(2) - (2 - \lambda)X(1) = 0,$$

$$X(2) = (2 - \lambda)X(1). \tag{6.62}$$

Suppose $n = 2$ in (6.58), then we obtain

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.63}$$

Substituting (6.60) into (6.63) yields

$$\frac{X(2)}{1 - 6\lambda} - (2 - \lambda)X(2) + X(1) = 0,$$

$$X(2)\left[\frac{1 - (2 - \lambda)(1 - 6\lambda)}{1 - 6\lambda}\right] + X(1) = 0. \tag{6.64}$$

Substituting (6.62) into (6.64) then results in

$$\left[\frac{(2 - \lambda)[1 - (2 - \lambda)(1 - 6\lambda)] + (1 - 6\lambda)}{1 - 6\lambda}\right] X(1) = 0.$$

Since $X(1) \neq 0$, it is clear that

$$\frac{(2 - \lambda)[1 - (2 - \lambda)(1 - 6\lambda)] + (1 - 6\lambda)}{1 - 6\lambda} = 0.$$

Consider the numerator to be equal to zero, thus

$$(2 - \lambda)[1 - (2 - \lambda)(1 - 6\lambda)] + (1 - 6\lambda) = 0,$$

$$6\lambda^3 - 25\lambda^2 + 21\lambda - 2 = 0.$$

Therefore, using Matlab (MATLAB, 2004),

$$\lambda = 0.0506,\ 1.0865 \text{ and } 3.0295.$$

The eigenparameters λ of the boundary value problem defined by (6.58), (6.59) and (6.60) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & (1-6\lambda) \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.6 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 1 - 21\lambda + 25\lambda^2 - 6\lambda^3,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.6 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 0.0506,\ 1.0865$ and 3.0295. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2 As determined in the Section 6.1.1 for Dirichlet initial boundary condition, $a = 0$ and $b = 0$.

For the final boundary condition to take on the form of the Affine boundary condition in (6.60), the parameters must take on the following values: $\alpha < 0$ and $\beta = 1$.

Therefore, this set of parameters satisfies the requirements in case (i) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m = 3$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Dirichlet, Affine) condition.

Continuous Case

The continuous case equation is given by (6.53) which has been obtained using the variable separable method. The initial boundary condition is given by

$$X(0) = 0, \tag{6.65}$$

while the final boundary condition is given by

$$X^{'}(l) - 2k^2 X(l) = 0. \tag{6.66}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx),$$

of which (6.65) will result in

$$X(0) = b = 0.$$

Therefore

$$X(x) = a\sin(kx), \tag{6.67}$$

and

$$X^{'}(x) = ak\cos(kx). \tag{6.68}$$

Rewriting (6.66) using the forms (6.67) and (6.68) results in

$$ak\cos(kl) - 2k^2 a\sin(kl) = 0.$$

Since $a \neq 0$ for a non-trivial solution,

$$k[\cos(kl) - 2k\sin(kl)] = 0.$$

Since $k \neq 0$ and assume length $l = 1$, then

$$\cos(k) - 2k\sin(k) = 0,$$

$$2k\tan(k) - 1 = 0. \tag{6.69}$$

The transcendental equation is solved using Matlab (MATLAB, 2004).

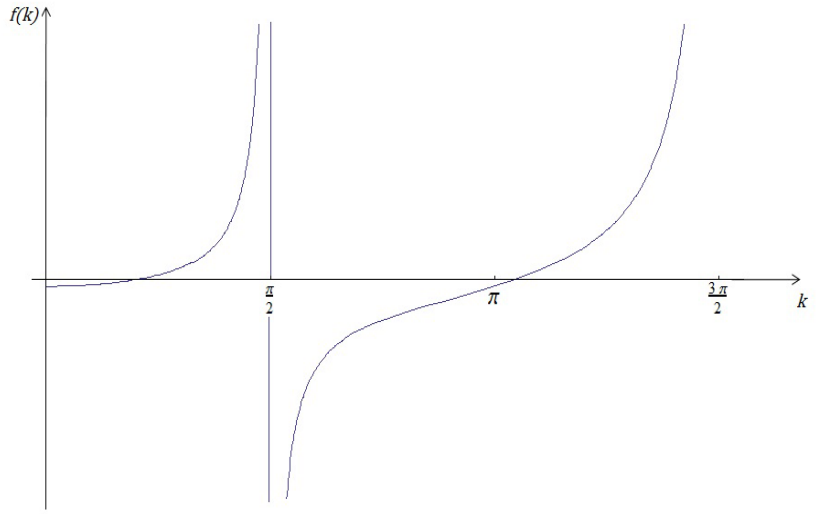

Figure 6.8: Graph of $f(k) = 2k\tan(k) - 1$ for the period $[0, 3\pi/2]$.

The graph of $f(k) = 2k\tan(k) - 1$ for the period $[0, 3\pi/2]$ is shown in Figure 6.8. From Figure 6.8, it can be seen, for $k > 0$, that the zero-crossing lies in the interval $[a\pi, (2a+1)\frac{\pi}{2}]$ for $a \in \mathbb{Z};\ a \geq 0$. The first six eigenvalues are shown in Table 6.59; computed using Matlab (MATLAB, 2004) coding (cf. to Appendix C, Section C.1). The software coding was done in Matlab (MATLAB, 2004). The software code is a simple numerical iterative algorithm which searches for the minimum value of $f(k) = 2k\tan k - 1$ for the period $[a\pi, (2a+1)\frac{\pi}{2}]$ using an iterative step $\delta k = 0.0001\pi$. The k value for which $f(k)$ is closest to zero is regarded as the k value of the zero-crossing. The process is repeated six times to compute the first six eigenvalues.

Table 6.9: First six eigenvalues k derived from the different unit-intervals cases and the continuous case for the (Dirichlet, Affine) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	0.665	0.660	0.658	0.656	0.654	0.653
2	3.253	3.282	3.291	3.292	3.293	3.292
3	6.064	6.273	6.333	6.350	6.361	6.326
4	8.516	9.185	9.380	9.438	9.474	9.478
5	10.405	11.924	12.375	12.513	12.598	12.606
6	11.594	14.427	15.292	15.559	15.723	15.740

Equation (6.53) is a linear differential equation, the solution $X(x)$ takes on the form as a Fourier series

$$X_j(x) = a\sum_{j=1}^{\infty} \sin(k_j x),$$

where the k-values form an infinite spectrum and are determined from (6.69). The eigenvalues k represent the natural frequencies of the axial displacement $u(x,t)$ in the spatial domain x. These natural frequencies are determined by solving for the roots of $f(k) = 2k\tan(k) - 1$ for $k > 0$ as discussed previously.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

In this section, a comparison is done between the k values, generated from the continuous case, to the k values, generated from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals as shown in Table 6.9. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.6, for the respective unit interval as was described in Section 6.1.1. Noting that $N = \frac{1}{h}$, the eigenvalues or k values, approximated in the discrete case, are derived from the eigenparameters from (6.57)

$$k = N\sqrt{\lambda}$$

The formula for k is then also used to compute the eigenvalues for Sections 6.3.2, 6.3.3, 6.3.4 and 6.3.5. As illustrated in Table 6.10, the number of expected eigenparameters and generated eigenparameters is equal. The expected number of eigenparameters was determined using Theorem 3.2.

Table 6.10: Expected and generated eigenparameters for the different unit-intervals cases for the (Dirichlet, Affine) condition.

Number of unit intervals = m	α	β	Case in Theorem 3.2	Expected number of eigenparameters = m	Number of eigenparameters generated
6	<0	1	(i)	6	6
11	<0	1	(i)	11	11
19	<0	1	(i)	19	19
30	<0	1	(i)	30	30
100	<0	1	(i)	100	100

Table 6.9 only illustrates the first six eigenvalues as these are the minimum generated number of approximated continuous-case k values of all unit-intervals cases. The comparison is thus done only against the first six eigenvalues. The comparison is done against the percentage relative error of the approximated k values measured against the continuous-case eigenvalues.

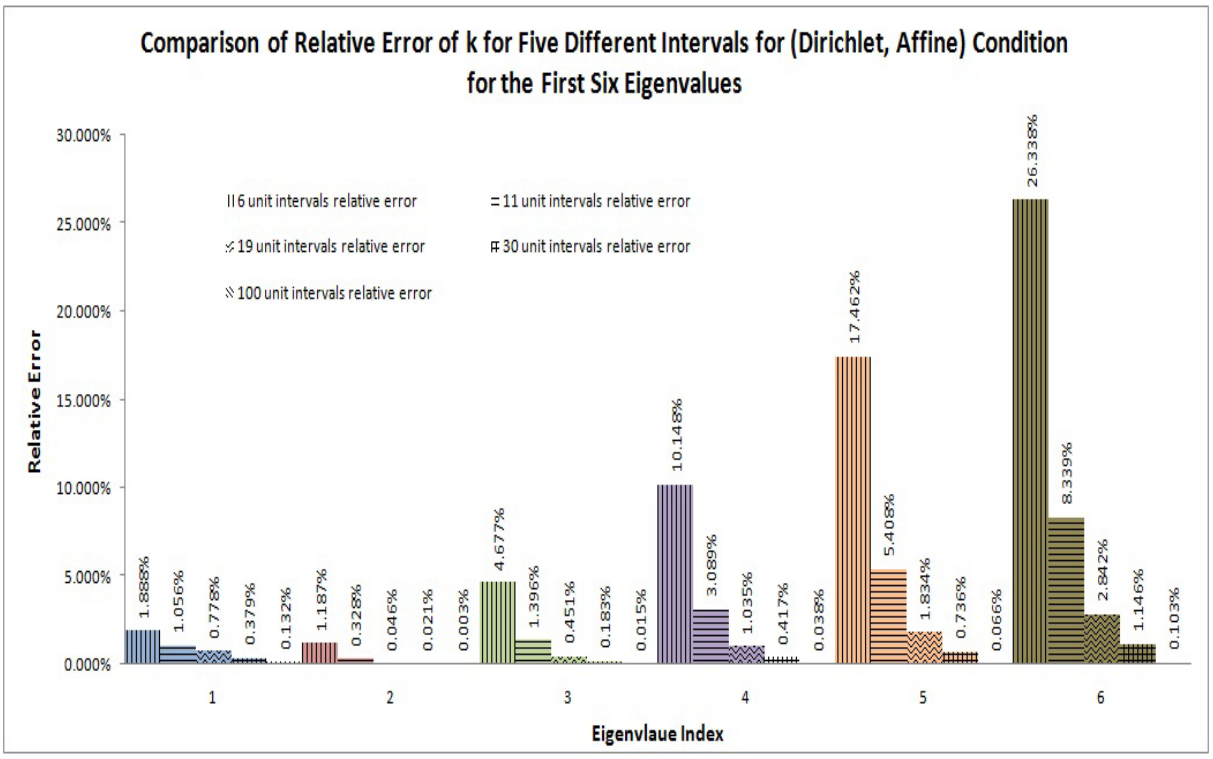

Figure 6.9: Comparison of Relative Error of k for Five Different Intervals for (Dirichlet, Affine) Condition for the First Six Eigenvalues.

The same analysis holds for Figure 6.9 as was done for the comparison in Figure 6.2. Further to that, the same analysis is applied to Table 6.9 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.58) is the same as (3.22). The approximated continuous-case eigenvalues from Table 6.9 were also found to satisfy condition (3.28) and thus

represent natural frequencies of the system. All the approximated k values, as indicated in Table 6.9, are below their corresponding Nyquist frequencies listed in Table 3.1.

6.3.2 (Affine, Dirichlet) Boundary Conditions

This case considers where the initial boundary condition is an Affine condition and the final boundary condition is a Dirichlet condition. If the end $n = m$ is rigidly fixed, the final continuous-case and discrete-case Dirichlet boundary condition is as follows (DiPrima & Boyce, 1977:531)

$$X(m) = 0.$$

Since no real physical initial Affine boundary condition could be found that meets the initial Affine boundary condition of the form given by equation (3.16), a hypothetical initial Affine discrete boundary condition was contrived as

$$X(0) - X(1)[1 + \frac{300}{N} + \lambda N] = 0,$$

where N = number of unit intervals. The corresponding final continuous-case boundary condition is derived in the ensuing *Continuous Case* sub-section as

$$X^{'}(0) + (k^2 + 300)X(0) = 0.$$

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.58), for 3 unit intervals, using the algebraic method as validation of the matrix method.

The initial Affine boundary condition is given by

$$X(0) = X(1)[1 + \frac{300}{N} + \lambda N],$$

For $N = 3$, this becomes

$$X(0) = X(1)[3\lambda + 101]. \tag{6.70}$$

The final Dirichlet boundary condition is given by

$$X(3) = 0. \tag{6.71}$$

Put $n = 1$ in (6.58) to obtain

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.72}$$

Substituting (6.70) into (6.72) yields

$$X(2) - (2 - \lambda)X(1) + X(1)[3\lambda + 101],$$

$$X(2) = -[4\lambda + 99]X(1). \tag{6.73}$$

Put $n = 2$ in (6.58) yields

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.74}$$

Substituting (6.71) into (6.74) results in

$$X(1) = (2 - \lambda)X(2). \tag{6.75}$$

Substituting (6.75) into (6.73) yields

$$X(2) = -(2 - \lambda)(4\lambda + 99)X(2),$$
$$X(2)[1 + (2 - \lambda)(4\lambda + 99)] = 0.$$

Since $X(2) \neq 0$, then

$$1 + (2 - \lambda)(4\lambda + 99) = 0.$$

Using Mathematica (Wolfram, 2011), this gives the solution

$$\lambda = -24.7593 \ldots\ldots \text{and } 2.0093 \ldots\ldots$$

The eigenparameters λ of the boundary value problem defined by (6.58), (6.70) and (6.71) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & (3\lambda + 101) & 0 & 0 \\ 1 & -(2 - \lambda) & 1 & 0 \\ 0 & 1 & -(2 - \lambda) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.7 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 199 - 91\lambda - 4\lambda^2,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.7 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = -24.7593$ and 2.0093. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

One observation of the solution is that the least eigenparameter is negative. The possible reasons for a negative eigenparameter being generated are two-fold: the discrete case is an approximation to the continuous case model of the system and thus the approximated continuous-case eigenvalues have inherent errors, and the initial Affine boundary condition was hypothesised and does not represent an actual physical boundary condition (as no physical boundary condition could be found to match the form of the initial Affine boundary condition in (3.16)). It was proven in Chapter 4 that negative eigenparameters cannot be accepted for the system to exhibit purely oscillatory behaviour and the Crum-based transformation must be applied.

The expected number of eigenparameters can be determined from Theorem 3.2. The following parameters must be set as $a > 0$ and $b > 0$ in order for the initial boundary condition in (3.18) to take on the Affine form of the boundary condition in (6.70). The parameters $\alpha = 0$ and $\beta = 1$ have been determined from Section 6.1.1 for the final Dirichlet boundary condition given in (6.71).

Therefore, this set of parameters satisfies the requirements in case (ii) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with the number of eigenparameters that have been computed in the three unit interval boundary problem for the (Affine, Dirichlet) condition.

Continuous Case

The continuous case equation is given by (6.53) which have been obtained using the variable separable method. The initial boundary condition, in the discrete case, is given by

$$X(0) = X(1)[1 + \frac{300}{N} + \lambda N],$$

where N = number of unit intervals. Since $\lambda = k^2h^2$, then

$$X(0) = X(1)[1 + \frac{300}{N} + k^2h^2N],$$

Since $N = \frac{1}{h}$, then

$$X(0) - X(1) = X(1)[300h + k^2h],$$

Dividing by h yields

$$-\frac{X(1) - X(0)}{h} = X(1)[300 + k^2],$$

$$-\frac{X(1) - X(0)}{h} = X(1)[300 + k^2].$$

Using forward-finite difference method and noting that $X(n = 1)$ in the discrete domain corresponds to $X(x = h)$ in the continuous domain, taking the limit as $h \to 0$ yields

$$X^{'}(0) + (k^2 + 300)X(0) = 0. \tag{6.76}$$

The final boundary condition is given by

$$X(l) = 0. \tag{6.77}$$

We begin by assuming a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx). \tag{6.78}$$

Consequently,

$$X^{'}(x) = ak\cos(kx) - bk\sin(kx). \tag{6.79}$$

Using (6.78) in (6.77), yields

$$a\sin(kl) + b\cos(kl) = 0,$$

Dividing by $b\cos(kl)$ and re-arranging terms yields

$$\tan(kl) = \frac{-b}{a}. \tag{6.80}$$

Rewriting (6.76) using (6.78) and (6.79)

$$ak + b[k^2 + 300] = 0,$$

$$bk^2 + ak + 300b = 0.$$

Dividing by $-a$ yields

$$-\frac{bk^2}{a} - k - \frac{300b}{a} = 0. \tag{6.81}$$

Substituting (6.80) into (6.81) yields

$$k^2 \tan(kl) - k + 300 \tan(kl) = 0,$$

Assuming the length $l = 1$, then

$$k^2 \tan(k) - k + 300 \tan(k) = 0. \tag{6.82}$$

The transcendental equation is solved using Matlab (MATLAB, 2004).

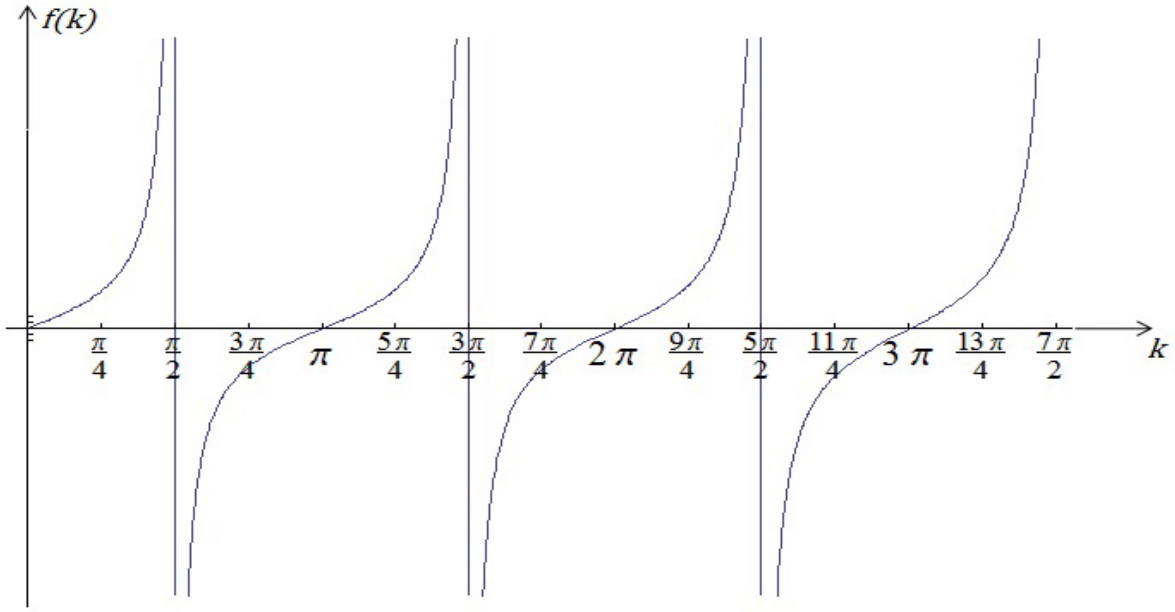

Figure 6.10: Graph of $f(k) = k^2 \tan(k) - k + 300 \tan(k)$ for the period $[0, 7\pi/2]$.

The graph of $f(k) = k^2 \tan(k) - k + 300 \tan(k)$ for the period $[0, 7\pi/2]$ is shown in Figure 6.10. From Figure 6.10, there is a obvious root at $k = 0$ and it can be seen for $k > 0$ that the zero-crossing lies in the interval $[a\pi, (4a + 1)\frac{\pi}{4}]$ for $a \in \mathbb{Z};\ a > 0$.

The first five eigenvalues are shown in Table 6.11; computed using Matlab (MATLAB, 2004) coding (cf. to Appendix C, Section C.2). The software coding was done in Matlab (MATLAB, 2004). The software code is a simple numerical iterative algorithm which searches for the minimum value of $f(k) = k^2 \tan(k) - k + 300 \tan(k)$ for the period $[a\pi, (4a+1)\frac{\pi}{4}]$ using an iterative step $\delta k = 0.000025\pi$. The k value for which $f(k)$ is closest to zero is regarded as the k value of the zero-crossing. The process is repeated four times to compute the second to the fifth eigenvalues.

Table 6.11: k-values from the first five eigenparameters for the different unit-intervals cases and the first four eigenvalues of the continuous case for the (Affine, Dirichlet) condition in the non-shifted case.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	k^2=-252.577	k^2=-267.464	k^2=-274.690	k^2=-278.059	k^2=-281.747	0
2	3.721	3.453	3.323	3.259	3.184	3.152
3	7.069	6.818	6.618	6.507	6.364	6.302
4	9.718	10.009	9.859	9.732	9.541	9.449
5	11.416	12.951	13.022	12.925	12.712	12.594

Since (6.53) is a linear ordinary differential equation, the solution $X(x)$ takes on the form as a Fourier series, given by

$$X(x) = \sum_{j=1}^{\infty} a\sin(k_j x) + b\cos(k_j x) \ ,$$

where the k_j-values form an infinite spectrum. Thus, these eigenvalues k have a physical interpretation in that they are the natural frequencies of the axial displacement $u(x,t)$ in the spatial domain and are determined from (6.82).

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

The same analysis is applied to Table 6.11 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.58) is the same as (3.22). The approximated continuous-case eigenvalues from Table 6.11 were also found to satisfy condition (3.28), with the exception of the negative eigenparameters which yielded absolute value of discrete eigenvalues $|r|$ not equal to 1, as shown in Table 6.12.

Table 6.12: Absolute Value of Discrete Eigenvalues $|r|$ for the Negative k^2 values from Table 6.11.

Number of unit intervals $=m$	Negative k^2 values	$\|r\|$
6	-252.577	8.904 ; 0.112
11	-267.464	3.958 ; 0.253
19	-274.690	2.332 ; 0.429
30	-278.059	1.731 ; 0.578
100	-281.747	1.183 ; 0.846

Therefore, it can be seen from Table 6.12, that the negative eigenparameters do not satisfy the condition for purely oscillatory behaviour and need to be dealt with using the Crum-based transformation method.

A comparison is done between the k values, generated from the the continuous case, to the k values, derived from the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals as shown in Table 6.11. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.7, for the respective unit interval as was described in Section 6.1.1. Table 6.13 shows that the expected and generated number of eigenparameters is equal; where the expected number of eigenparameters is determined from Theorem 3.2.

Table 6.13: Expected and generated number of eigenparameters for the (Affine, Dirichlet) condition for the unit intervals indicated in the first column.

Number of unit intervals $=m$	α	β	Case in Theorem 3.2	Expected number of eigenparameters $=m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

Table 6.11 indicates the comparison of the first five eigenvalues as these are the minimum generated number of approximated k values in all the unit-intervals cases. The comparison is done against the percentage relative error of the approximated continuous-case eigenvalues, derived from the discrete case, measured against the continuous case eigenvalues. As can be seen from Table 6.11, the first eigenparameter for each of the unit interval cases is negative. Figure 6.11 is an extraction from Table 6.11.

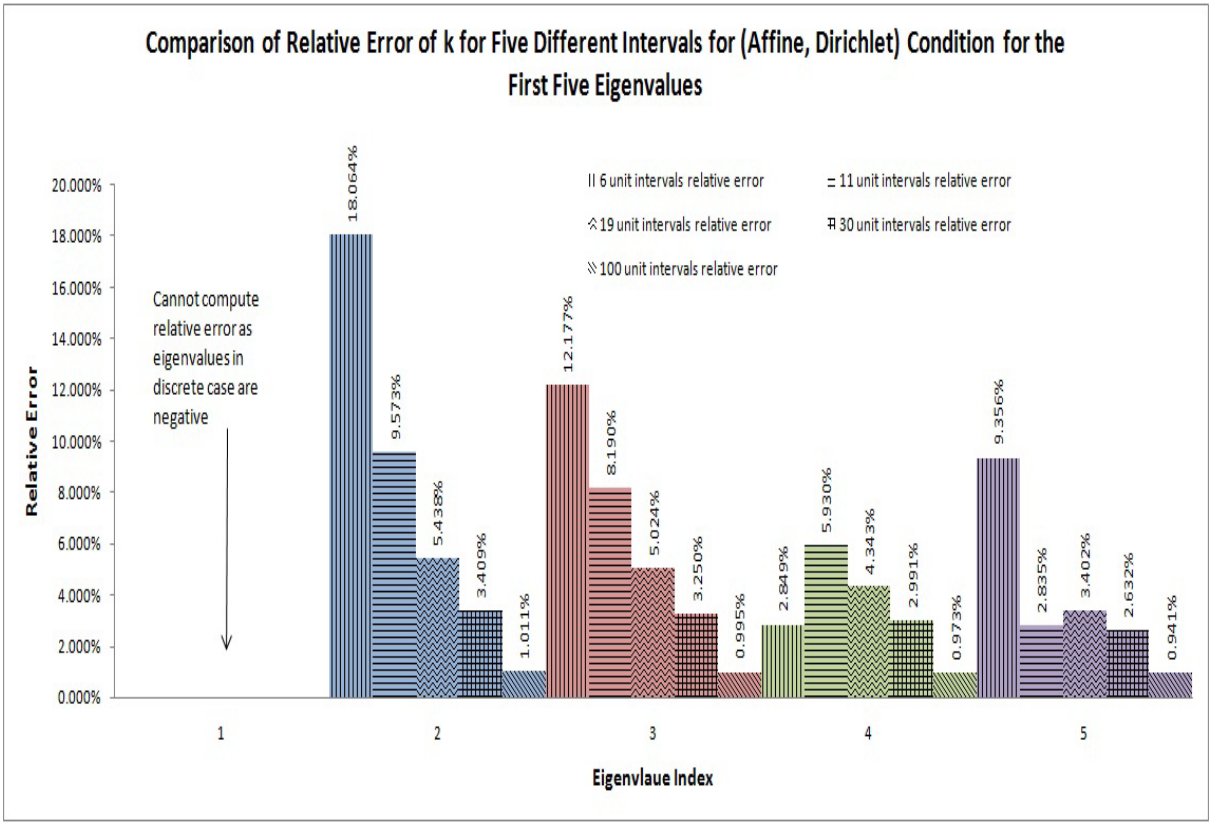

Figure 6.11: Comparison of Relative Error of k for Five Different Intervals for (Affine, Dirichlet) Condition for the First Five Eigenvalues.

From Figure 6.11, the relative error for the first approximated continuous-case eigenvalue could not be computed because the values are negative. The same analysis holds for Figure 6.11 as was done for the comparison in Figure 6.2. Also, all the approximated k values, as indicated in Table 6.11, are below their corresponding Nyquist frequencies listed in Table 3.1.

Since the least eigenparameter from the discrete cases are negative, the eigenparameter spectrum is shifted by the least eigenparameter according to the Crum-based transformation. This results in the number of eigenparameters reducing by one as the resultant zero-valued eigenparameter does not form part of the eigenparameter spectrum. Shifting the eigenparameter spectrum by the least eigenparameter λ_0 corresponds to a shift in the continuous-case eigenvalue spectrum. Consider the relationship between the eigenparameter λ and the continuous-case eigenvalue k from (6.57)

$$\lambda = k^2 h^2. \tag{6.83}$$

Thus, for λ_0

$$\lambda_0 = k_0^2 h^2. \tag{6.84}$$

where k_0 is the corresponding continuous-case eigenvalue for the least eigenparameter λ_0. Subtracting (6.84) from (6.83) results in

$$\lambda - \lambda_0 = (k^2 - k_0^2)h^2,$$
$$\mu = (k^2 - k_0^2)h^2,$$
$$\mu = k_\mu^2 h^2,$$

where k_μ is the shifted continuous-case eigenvalue, thus given by

$$k_\mu = \sqrt{k^2 - k_0^2}.$$

A comparison is thus done between the shifted continuous-case eigenvalues k_μ and the approximated k_μ values, derived from the discrete case. The results are tabulated in Table 6.14. Not all the eigenvalues are shown in Table 6.14 as the comparison focuses on the first four eigenvalues as these are the minimum generated number of approximated k_μ values of all unit-intervals cases. The comparison is done against the percentage relative error of the approximated k_μ values measured against the shifted continuous-case eigenvalues k_μ.

Table 6.14: k_μ-values from the first four eigenparameters for the different unit-intervals cases and the second to the fifth shifted continuous case eigenvalues for the (Affine, Dirichlet) condition shifted by the least eigenvalue.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1 (2 for continuous)	16.322	16.715	16.904	16.991	17.085	17.079
2 (3 for continuous)	17.394	17.719	17.846	17.900	17.951	17.929
3 (4 for continuous)	18.628	19.174	19.285	19.307	19.307	19.262
4 (5 for continuous)	19.568	20.861	21.078	21.098	21.056	20.985

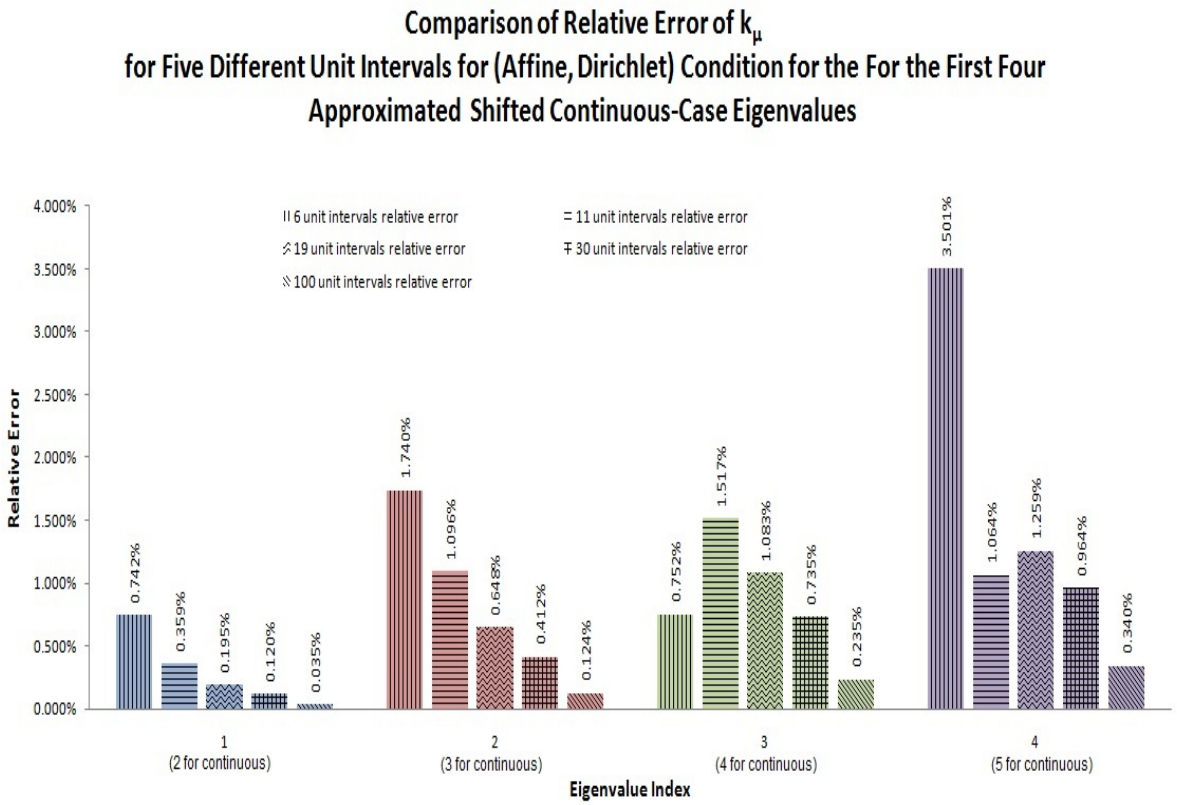

Figure 6.12: Comparison of Relative Error of k_μ for Five Different Unit Intervals for (Affine, Dirichlet) Condition for the First Four Approximated Shifted Continuous-Case Eigenvalues.

Applying the test for purely oscillatory behaviour on Table 6.14 using the left-hand side of (3.28) results in all the discrete eigenvalues r, derived from the shifted continuous-case eigenvalues, satisfying the purely oscillatory behaviour condition $|r| = 1$. Therefore, the Crum-based transformation ensures that the discrete eigenvalues lie on the unit circle and that the eigenparameters are a function of real physical natural frequencies.

It can be seen from Figure 6.12 that as the number of unit intervals increase for a particular eigenvalue index, there is a general trend that the accuracy of the approximated shifted continuous-case eigenvalues improves. For example, for the first shifted eigenvalue, the accuracy improves from 0.742 % for 6 unit intervals to 0.035 % for 100 unit intervals.

Also from Figure 6.12, as one ascends through the shifted eigenvalue spectrum, the accuracy of the approximated shifted continuous-case eigenvalues worsens. Consider the six unit-intervals case: where the accuracy of the approximated shifted continuous-case eigenvalues worsens from 0.742 % for the 1st shifted continuous-case eigenvalue to 3.501 % for the 4th shifted continuous-case eigenvalue.

On comparing Figures 6.11 and 6.12, the approximated shifted continuous-case eigenvalues show better accuracy than the approximated non-shifted continuous-case eigenvalues when compared to their respective continuous-case eigenvalues. This is explained via an example. Consider the 1st non-shifted eigenvalue and shifted eigenvalue for the 6 unit-intervals cases in Figures 6.11 and 6.12 respectively. The accuracy of the approximated non-shifted continuous-case eigenvalue for the 1st non-shifted eigenvalue for 6 unit intervals is 18.064 % from Figure 6.11, whereas the accuracy of the approximated shifted continuous-case eigenvalue for the 1st shifted eigenvalue for 6 unit intervals is 0.742 % from Figure 6.12.

All the approximated shifted continuous-case values, as indicated in Table 6.14, are below their corresponding Nyquist frequencies listed in Table 3.1 besides the 4th approximated shifted continuous-case eigenvalue for 6 unit intervals. The effects of aliasing are not very strong for the 4th approximated shifted continuous-case eigenvalue for 6 unit intervals, given the respective relative error is only 3.501%. This can be understood as the approximated k_μ value of 19.568 rad/m which is not much more than the Nyquist frequency of 18.850 rad/m.

6.3.3 (Non-Dirichlet, Affine) Boundary Conditions

This case considers where the initial boundary condition is a Non-Dirichlet condition and the final boundary condition is an Affine condition. When the end of the rod at $x = 0$ is free; this yields the initial boundary condition (Budak et al., 2013)

$$u_x(0,t) = 0. \qquad 0 < t < \infty.$$

Using the variable separable method yields

$$u_x(0,t) = X^{'}(0)T(t) = 0.$$

Since $T(t) \neq 0$ for all $t > 0$, this implies that the continuous-case initial boundary condition is

$$X^{'}(0) = 0.$$

Using the forward finite difference formula, the initial boundary condition, in the discrete case, is computed as

$$X'(0) \approx \frac{X(1) - X(0)}{h} = 0,$$
$$X(1) - X(0) = 0.$$

The initial Non-Dirichlet boundary condition, in the discrete case, is given by

$$X(1) - X(0) = 0.$$

When the end of the rod at $x = l$ is connected to an unrestrained rigid mass M, the final continuous-case boundary condition is given by

$$X'(l) - 2\gamma k^2 X(l) = 0.$$

and the corresponding discrete-case Affine boundary condition is given by

$$X(m)[1 - 2\lambda N] - X(m-1) = 0,$$

where N = number of unit intervals. The derivation of the discrete final Affine boundary condition is given in Appendix B.

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.58), for 3 unit intervals, using the algebraic method to validate the matrix method computed in Mathematica (Wolfram, 2011). The initial Non-Dirichlet boundary condition is given by

$$X(1) - X(0) = 0,$$
$$X(1) = X(0), \tag{6.85}$$

while the final Affine boundary condition, for $N = 3$, is given by

$$X(3)[1 - 6\lambda] - X(2) = 0,$$
$$X(3) = \frac{X(2)}{1 - 6\lambda}. \tag{6.86}$$

Consider (6.58) for $n = 1$ results in

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.87}$$

Substitute (6.85) into (6.87) yields

$$X(2) - (2 - \lambda)X(1) + X(1) = 0,$$
$$X(2) = (1 - \lambda)X(1). \tag{6.88}$$

Now considering (6.58) for $n = 2$ results in

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.89}$$

Substituting (6.86) into (6.89) produces

$$\frac{X(2)}{1-6\lambda} - (2-\lambda)X(2) + X(1) = 0. \tag{6.90}$$

Substitute (6.88) into (6.90)

$$\frac{(1-\lambda)X(1)}{1-6\lambda} - \frac{(2-\lambda)(1-\lambda)X(1)}{1} + X(1) = 0,$$

$$\left[\frac{(1-\lambda) - (2-\lambda)(1-\lambda)(1-6\lambda) + (1-6\lambda)}{1-6\lambda}\right] X(1) = 0$$

Since $X(1) \neq 0$, then

$$\frac{(1-\lambda) - (2-\lambda)(1-\lambda)(1-6\lambda) + (1-6\lambda)}{1-6\lambda} = 0$$

Using Mathematica (Wolfram, 2011), this gives the solution $\lambda = 0,\ 0.5$ and 2.667.

The eigenparameters λ of the boundary value problem defined by (6.58), (6.85) and (6.86) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & (1-6\lambda) \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.8 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 8\lambda - 19\lambda^2 + 6\lambda^3,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.8 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = 0,\ 0.5$ and 2.667. The eigenparameter spectrum using the matrix method is thus the same as the eigenparameter spectrum using the algebraic method.

The expected number of eigenparameters can be determined from Theorem 3.2. The following parameters a and b is set as $a = 0$ and $b = 1$ in order for the initial boundary condition in (3.18) to take on the Non-Dirichlet form of the boundary condition in (6.85).

As determined in Section 6.3.1, for the final boundary condition in (3.19) to take on the form of the Affine boundary condition in (6.86), the parameters must take on the following values: $\alpha < 0$ and $\beta = 1$.

Therefore, this set of parameters satisfies the requirements in case (i) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m = 3$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Non-Dirichlet, Affine) condition.

Continuous Case

The continuous case equation is given by (6.53) which have been obtained using the variable separable method. The initial boundary condition is given by

$$X^{'}(0) = 0. \tag{6.91}$$

and the final boundary condition is given by

$$X^{'}(l) - 2k^2 X(l) = 0. \tag{6.92}$$

The following form of solution is assumed

$$X(x) = a\sin(kx) + b\cos(kx), \tag{6.93}$$

with the first derivative being

$$X^{'}(x) = ak\cos(kx) - bk\sin(kx). \tag{6.94}$$

Substituting (6.94) in (6.91) yields

$$ak = 0.$$

Since $k \neq 0$, this indicates that

$$a = 0.$$

Using $a = 0$, (6.93) and (6.94) can be rewritten respectively as

$$X(x) = b\cos(kx), \tag{6.95}$$

and

$$X^{'}(x) = -bk\sin(kx). \tag{6.96}$$

Rewriting (6.92) using (6.95) and (6.96) produces

$$-bk\sin(kl) - 2k^2 b\cos(kl) = 0,$$

$$bk[\sin(kl) + 2k\cos(kl)] = 0,$$

Since $k \neq 0$ and $b \neq 0$ for a non-trivial solution, it indicates that

$$\sin(kl) + 2k\cos(kl) = 0.$$

Dividing by $\cos(kl)$ results in

$$\tan(kl) + 2k = 0.$$

Assume the length $l = 1$,

$$\tan(k) + 2k = 0. \tag{6.97}$$

Matlab (MATLAB, 2004) is then used to solve the transcendental equation.

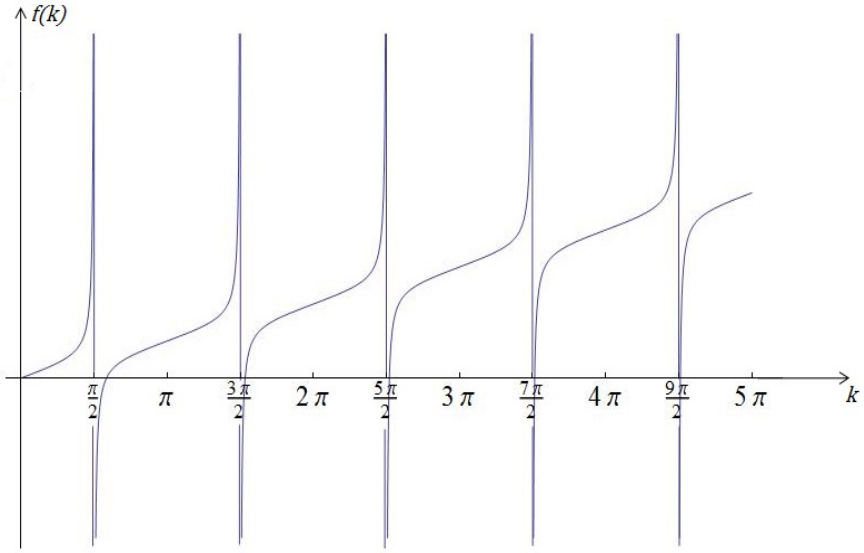

Figure 6.13: Graph of $f(k) = \tan(k) + 2k$ for the period $[0, 5\pi]$.

The graph of $f(k) = \tan(k) + 2k$ for the period [0,5π] is shown in Figure 6.13. From Figure 6.13, it can be seen, for $k \geq 0$, that the zero-crossing lies in the interval $[(2a-1)\frac{\pi}{2}, a\pi]$ for $a \in \mathbb{Z};\ a > 0$ and the obvious root at $k = 0$. The first six eigenvalues are shown in Table 6.15; computed using Matlab (MATLAB, 2004) coding (cf. to Appendix C, Section C.3). The software coding was done in Matlab (MATLAB, 2004). The software code is a simple numerical iterative algorithm which searches for the minimum value of $f(k) = \tan k + 2k$ for the period $[(2a-1)\frac{\pi}{2}, a\pi]$ using an iterative step $\delta k = 0.0001\pi$. The k value for which $f(k)$ is closest to zero is regarded as the k value of the zero-crossing. The process is repeated five times to compute the second to the sixth eigenvalues.

Table 6.15: k-values derived from the first six eigenparameters for the different unit-intervals cases and the first six continuous case eigenvalues for the (Non-Dirichlet, Affine) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	0.000	0.000	0.000	0.000	0.000	0.000
2	1.982	1.915	1.881	1.865	1.845	1.837
3	5.077	4.997	4.929	4.891	4.839	4.816
4	7.899	8.095	8.068	8.026	7.955	7.917
5	10.111	11.036	11.170	11.162	11.090	11.041
6	11.518	13.741	14.201	14.273	14.231	14.172

Since (6.53) is a linear ordinary differential equation, the solution $X(x)$ takes on the form as a Fourier series, given by

$$X(x) = \sum_{j=1}^{\infty} b \cos(k_j x),$$

where the k-values form an infinite spectrum. The eigenvalues k represent the natural frequencies of the axial displacement $u(x,t)$ and are determined from (6.97).

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

A comparison is done between the k values obtained from the continuous case, to the k values, derived from the eigenparameters for the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.8, for the respective unit interval as was described in Section 6.1.1. Theorem 3.2 is used to calculate the expected number of eigenparameters as shown in Table 6.16. The generated number of eigenparameters equals the number of expected number of eigenparameters for the different unit-interval cases, as shown in Table 6.16.

Table 6.16: Expected and generated number of eigenparameters for the (Non-Dirichlet, Affine) condition for the unit intervals indicated in the first column.

Number of unit intervals $=m$	α	β	Case in Theorem (4.1)	Expected number of eigenparameters $=m$	Number of eigenparameters generated
6	<0	1	(i)	6	6
11	<0	1	(i)	11	11
19	<0	1	(i)	19	19
30	<0	1	(i)	30	30
100	<0	1	(i)	100	100

The comparison focuses on the first six eigenvalues as these are the minimum generated number of eigenparameters of all unit-interval cases, as shown in Table 6.15. The comparison is done against the percentage relative error of the approximated k values, derived from the discrete case, measured against the continuous-case eigenvalues.

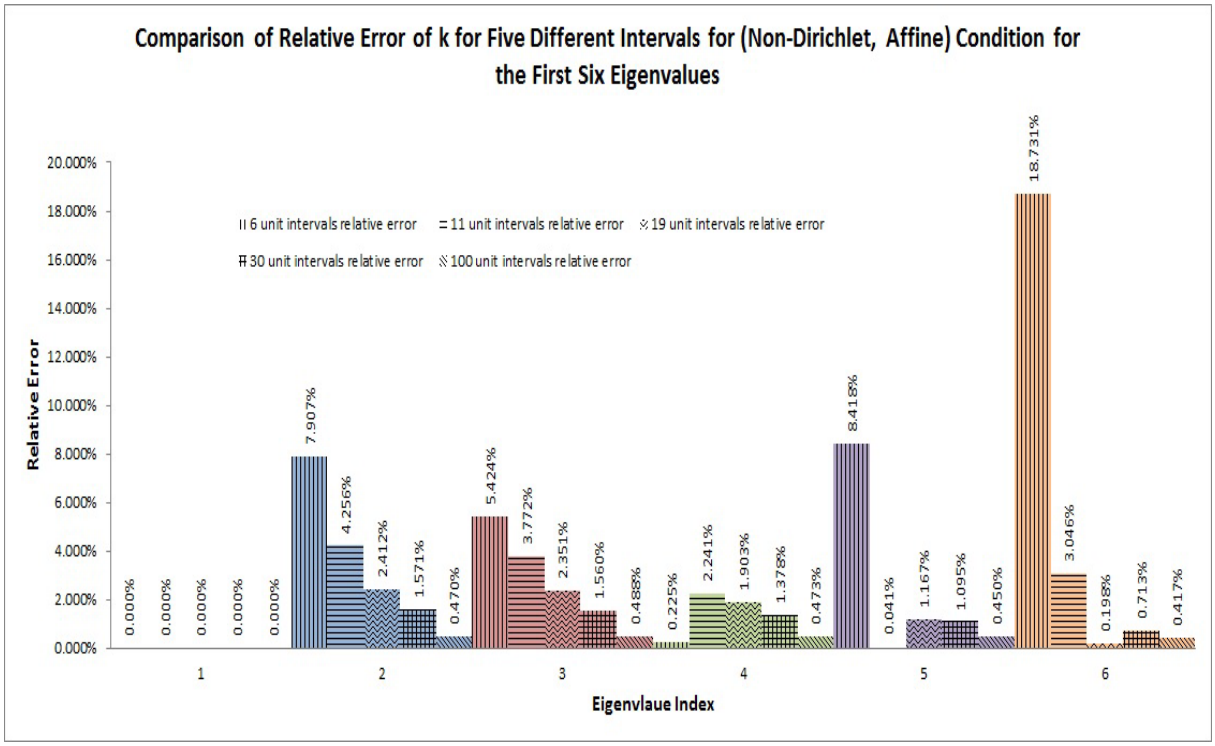

Figure 6.14: Comparison of Relative Error of k for Five Different Intervals for (Non-Dirichlet, Affine) Condition for the First Six Eigenvalues.

The same analysis holds for Figure 6.14 as was done for the comparison in Figure 6.2 in Section 6.1.1. Also, the same analysis is applied to Table 6.15 to test for purely oscillatory behaviour as was done in Section 6.1.1; as the difference equation (6.58) is the same as (3.22). The approximated continuous-case eigenvalues from Table 6.15 were also found to satisfy condition (3.28) and thus represent natural frequencies of the system. The approximated continuous-case eigenvalues are given in Table 6.15, of which all are below their corresponding Nyquist frequencies indicated in Table 3.1.

6.3.4 (Affine, Non-Dirichlet) Boundary Conditions

This case considers where the initial boundary condition is an Affine condition and the final boundary condition is a Non-Dirichlet condition. Since no real physical initial Affine boundary condition could be found that meets the initial Affine boundary condition of the form given by (3.16), a hypothetical initial Affine boundary condition, in the discrete case, was contrived as

$$X(0) - X(1)[1 + \frac{300}{N} + \lambda N] = 0,$$

where N = number of unit intervals. The corresponding final continuous-case boundary condition has been derived in Section 6.3.2 as

$$X^{'}(0) + (k^2 + 300)X(0) = 0.$$

When the end of the rod at $x = l$ is free; this yields the final boundary condition (Budak et al., 2013)

$$u_x(l,t) = 0. \qquad 0 < t < \infty.$$

Using the variable separable method yields

$$u_x(l,t) = X^{'}(l)T(t) = 0.$$

Since $T(t) \neq 0$ for all $t > 0$, this implies that the final boundary condition in the continuous case is

$$X^{'}(l) = 0.$$

Using the backward finite difference formula, the final boundary condition is computed, as

$$X^{'}(l) \approx \frac{X(m) - X(m-1)}{h} = 0,$$

where m is the number of unit intervals. The final Non-Dirichlet boundary condition, in the discrete case, is given by

$$X(m) - X(m-1) = 0.$$

Confirmation of the Matrix Method

In this section, the eigenparameter λ is computed from (6.58), for 3 unit intervals, using the algebraic method. The results are then used to validate the matrix method computed in Mathematica (Wolfram, 2011).

The initial Affine boundary condition is given by

$$X(0) = X(1)[1 + \frac{300}{N} + \lambda N].$$

For $N = 3$, this becomes

$$X(0) = X(1)[3\lambda + 101]. \tag{6.98}$$

The final Non-Dirichlet boundary condition is given by

$$X(3) - X(2) = 0,$$

$$X(3) = X(2). \tag{6.99}$$

Put $n = 1$ in (6.58) to obtain

$$X(2) - (2 - \lambda)X(1) + X(0) = 0. \tag{6.100}$$

Substituting (6.98) into (6.100) yields

$$X(2) - (2 - \lambda)X(1) + X(1)(3\lambda + 101) = 0,$$

$$X(2) = -[4\lambda + 99]X(1). \tag{6.101}$$

Consider (6.58) for $n = 2$ results in

$$X(3) - (2 - \lambda)X(2) + X(1) = 0. \tag{6.102}$$

Substitute (6.99) into (6.102)

$$X(2) - X(2)(2-\lambda) + X(1) = 0,$$

$$X(1) = X(2)(1-\lambda). \tag{6.103}$$

Substituting (6.103) into (6.101) yields

$$X(2) = -[4\lambda + 99][1-\lambda]X(2),$$

$$X(2)[1 + (4\lambda + 99)(1-\lambda)] = 0.$$

Since $X(2) \neq 0$, this becomes

$$1 + (4\lambda + 99)(1-\lambda) = 0.$$

Using Matlab (MATLAB, 2004), this gives the solution $\lambda = -24.75971\ldots\ldots$ and $1.00971\ldots\ldots$

The eigenparameters λ of the boundary value problem defined by (6.58), (6.98) and (6.99) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & (3\lambda + 101) & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.9 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 100 - 95\lambda - 4\lambda^2,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.9 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = -24.75971$ and 1.00971. The eigenparameter spectrum generated using the matrix method is thus proven to be the same as the eigenparameter spectrum generated using the algebraic method.

Theorem 3.2 is used to determine the expected number of eigenparameters. The following parameters a and b are set as $a > 0$ and $b > 1$ in order for the initial boundary condition in (3.18) to take on the Affine form of the boundary condition in (6.98).

As determined in Section 6.3.1, for the final boundary condition in (3.19) to take on the form of the Non-Dirichlet boundary condition in (6.99), the parameters must take on the following values: $\alpha = 0$ and $\beta = 1$.

Therefore, this set of parameters satisfies the requirements in case (ii) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m - 1 = 3 - 1 = 2$. This result ties up with number of eigenparameters that have been computed in the three unit interval boundary problem for the (Affine, Non-Dirichlet) condition.

Continuous Case

The continuous case equation is given by (6.53) which have been obtained using the variable separable method. From the (Affine, Dirichlet) case in Section 6.3.2, the initial boundary condition, in the continuous case, is given by

$$X^{'}(0) + (k^2 + 300)X(0) = 0. \tag{6.104}$$

The final boundary condition is given by

$$X^{'}(l) = 0. \tag{6.105}$$

Assume a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx), \tag{6.106}$$

and the first derivative being

$$X^{'}(x) = ak\cos(kx) - bk\sin(kx). \tag{6.107}$$

Rewriting (6.104) using (6.106) and (6.107)

$$ak + b(k^2 + 300) = 0,$$

$$ak + bk^2 + 300b = 0.$$

Dividing by b yields

$$\frac{ak}{b} + k^2 + 300 = 0. \tag{6.108}$$

Considering (6.105) using (6.107) results in

$$ak\cos(kl) - bk\sin(kl) = 0,$$

$$k[a\cos(kl) - b\sin(kl)] = 0.$$

Since $k \neq 0$, this becomes

$$a\cos(kl) - b\sin(kl) = 0.$$

Dividing by $a\cos(kl)$, yields

$$\tan(kl) = \frac{a}{b}. \tag{6.109}$$

Substituting (6.109) into (6.108) and assuming the length $l = 1$, then

$$k\tan(k) + k^2 + 300 = 0. \tag{6.110}$$

Matlab (MATLAB, 2004) is then used to solve the transcendental equation.

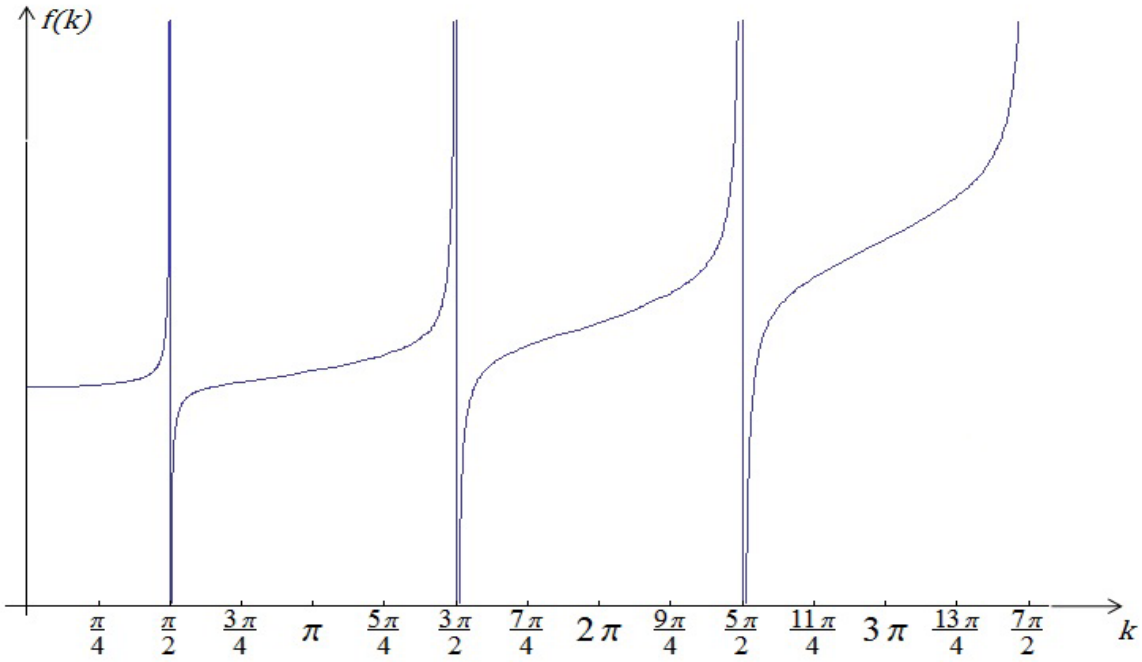

Figure 6.15: Graph of $f(k) = k\tan(k) + k^2 + 300$ for the period $[0, 7\pi/2]$.

The graph of $f(k) = k\tan(k) + k^2 + 300$ for the period $[0, 7\pi/2]$ is shown in Figure 6.15. From Figure 6.15, it can be seen, for $k > 0$, that the zero-crossing lies in the interval $[(2a-1)\frac{\pi}{2}, (4a-1)\frac{\pi}{4}]$ for $a \in \mathbb{Z}$; $a > 0$. The first four eigenvalues are shown in Table 6.17; computed using Matlab (MATLAB, 2004) coding (cf. to Appendix C, Section C.4). The software coding was done in Matlab (MATLAB, 2004). The software code is a simple numerical iterative algorithm which searches for the minimum value of $f(k) = k\tan(k) + k^2 + 300$ for the period $[(2a-1)\frac{\pi}{2}, (4a-1)\frac{\pi}{4}]$ using an iterative step $\delta k = 0.00001\pi$. The k value for which $f(k)$ is closest to zero is regarded as the k value of the zero-crossing. The process is repeated four times to compute the first four eigenvalues.

Table 6.17: First five approximated k values of the discrete case for the different unit-intervals cases and the first four eigenvalues of the continuous case for the (Affine, Non-Dirichlet) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	k^2=-252.57744	k^2=-267.464	k^2=-274.690	k^2=-278.059	k^2=-281.747	
2 (1 for continuous)	2.093	1.824	1.711	1.659	1.600	1.576
3 (2 for continuous)	6.018	5.419	5.118	4.971	4.799	4.727
4 (3 for continuous)	9.205	8.860	8.479	8.264	7.994	7.876
5 (4 for continuous)	11.280	12.054	11.769	11.529	11.184	11.022

Since (6.53) is a linear ordinary differential equation, the solution $X(x)$ takes on the form as a Fourier series, given by

$$X(x) = \sum_{j=1}^{\infty} a\sin(k_j x) \ + b\cos(k_j x) \ ,$$

where the k_j-values form an infinite spectrum and the eigenvalues k represent the natural frequencies of the axial displacement $u(x,t)$ and are determined from (6.110).

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

Since the difference equation (6.58) is the same as (3.22), the same analysis is applied to Table 6.17 to test for purely oscillatory behaviour as was done in Section 6.1.1. The approximated k values from Table 6.17 were also found to satisfy condition (3.28), with the exception of the negative eigenparameters which produces discrete eigenvalues, whose absolute value $|r|$ is not equal to 1, as shown in Table 6.18.

Table 6.18: Absolute Value of Discrete Eigenvalues $|r|$ for the Negative k^2 values from Table 6.17.

Number of unit intervals $=m$	Negative k^2 values	$\|r\|$
6	-252.577	8.904 ; 0.112
11	-267.464	3.958 ; 0.253
19	-274.690	2.332 ; 0.429
30	-278.059	1.731 ; 0.578
100	-281.747	1.183 ; 0.846

It is observed from Table 6.18 that the negative eigenparameters do not satisfy the condition for purely oscillatory behaviour and needs to be dealt with using the Crum-based transformation method.

A comparison is done between the k values from the continuous case to the k values, derived from the eigenparameters for the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.9, for the respective unit interval as was described in Section 6.1.1. For all the unit-interval cases, the expected number of eigenparameters equals the number of generated eigenparameters as seen in Table 6.19.

Table 6.19: Expected and generated number of eigenparameters for the (Affine, Non-Dirichlet) condition for the unit intervals indicated in the first column.

Number of unit intervals $=m$	α	β	Case in Theorem (4.1)	Expected number of eigenparameters $=m-1$	Number of eigenparameters generated
6	0	1	(ii)	5	5
11	0	1	(ii)	10	10
19	0	1	(ii)	18	18
30	0	1	(ii)	29	29
100	0	1	(ii)	99	99

The comparison focuses on the first five eigenvalues as these are the minimum generated number of approximated continuous-case eigenvalues of all unit-intervals cases. The generated approximated k values in each unit-interval case are shown in Table 6.17. The comparison is done against the percentage relative error of the approximated k values in the discrete case measured against the continuous-case eigenvalues.

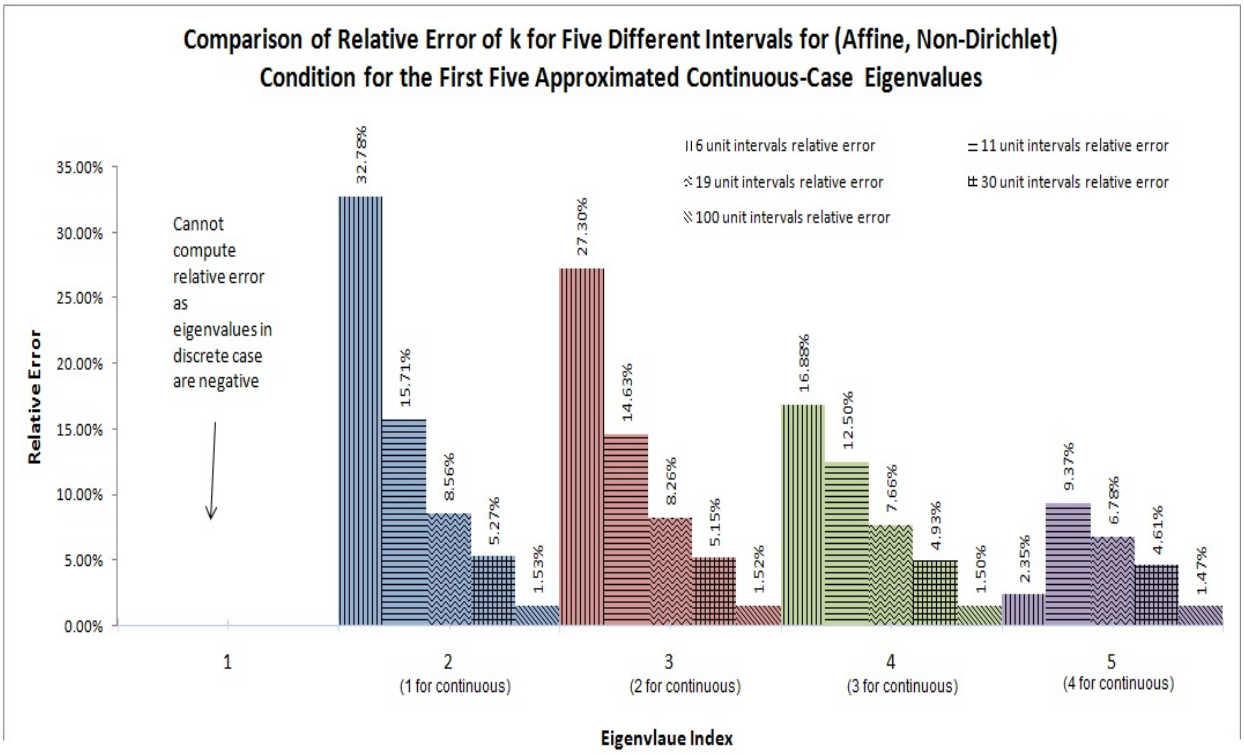

Figure 6.16: Comparison of Relative Error of k for Five Different Intervals for (Affine, Non-Dirichlet) Condition for the First Five Approximated Continuous-Case Eigenvalues.

The same analysis holds for Figure 6.16 as was done for the comparison in Figure 6.2. All the approximated k values, as indicated in Table 6.17, are below their corresponding Nyquist frequencies listed in Table 3.1. The Crum-based transformation can be applied since the least eigenparameter from the discrete cases are negative. Thus, the eigenparameter spectrum is shifted by the least eigenparameter. The consequence is that the number of eigenparameters is reduced by one as the resultant zero-valued eigenparameter, as a result of the shift, is not part of the eigenvalue spectrum.

A comparison is thus done between the shifted continuous-case eigenvalues k_μ and the shifted approximated continuous-case eigenvalues. The results are tabulated in Table 6.20. The comparison focuses on the first four eigenvalues as these are the minimum generated number of approximated k_μ values of all unit-intervals cases. The comparison is done against the percentage relative error of the approximated k_μ values in the discrete case measured against the shifted continuous-case eigenvalues k_μ.

Table 6.20: First four shifted eigenvalues k_μ derived in the discrete case for the different unit-intervals cases and the shifted eigenvalues for the continuous case for the (Affine, Non-Dirichlet) condition, where the eigenparameter spectrum is shifted by the least eigenparameter.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	16.030	16.456	16.662	16.757	16.861	16.479
2	16.994	17.229	17.346	17.400	17.458	17.332
3	18.366	18.600	18.617	18.611	18.592	18.441
4	19.489	20.317	20.327	20.273	20.170	19.988

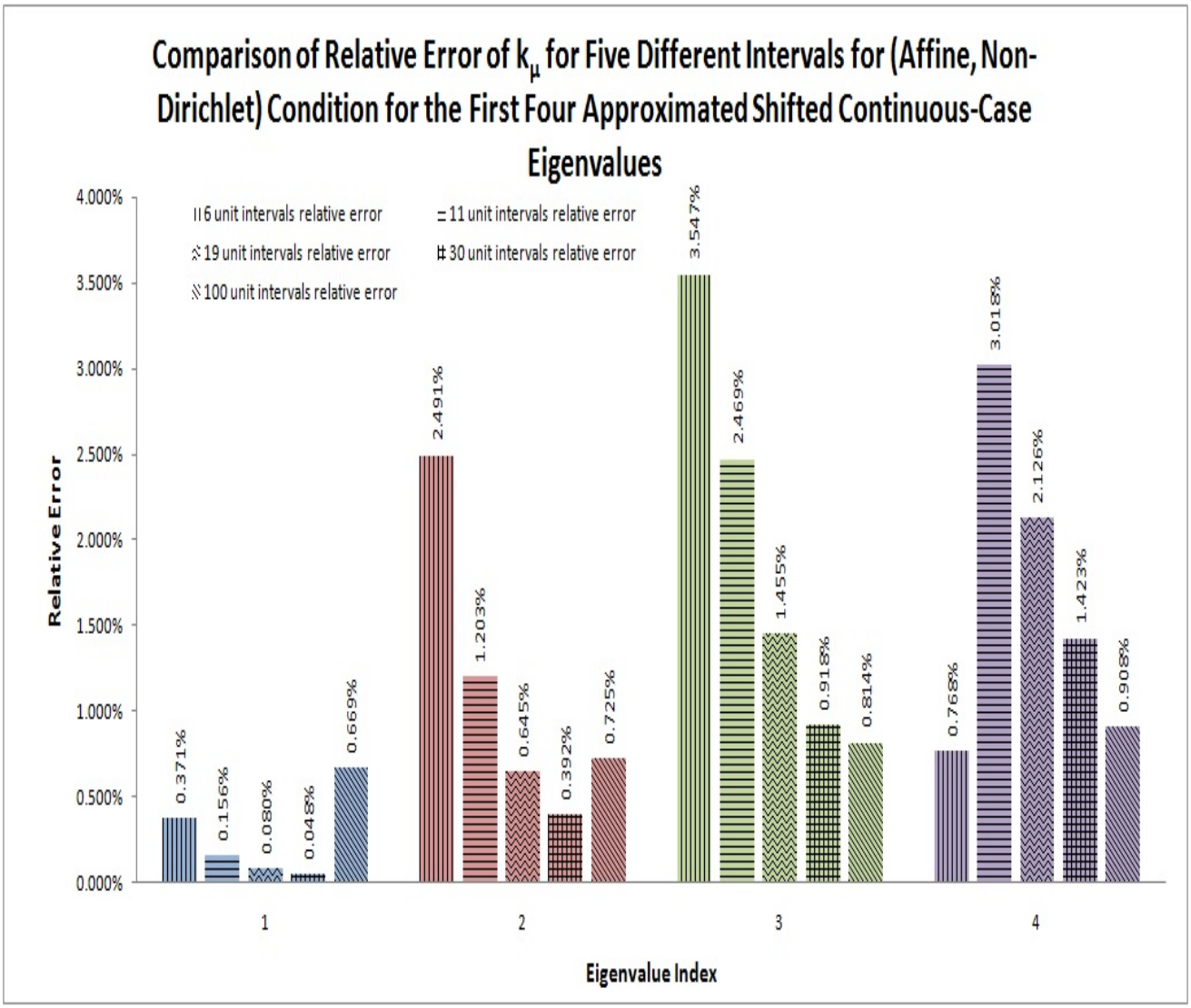

Figure 6.17: Comparison of Relative Error of k for Five Different Intervals for (Affine, Non-Dirichlet) Condition for the First Four Approximated Shifted Continuous-Case Eigenvalues.

The test for purely oscillatory behaviour is applied on Table 6.20 using the left-hand side of (3.28), which results in all the discrete eigenvalues r, derived from the shifted approximated continuous-case eigenvalues, satisfying the purely oscillatory behaviour condition $|r| = 1$. Therefore, the

Crum-based transformation ensures that the discrete eigenvalues r lie on the unit circle and that the eigenparameters are a function of real physical natural frequencies.

The same analysis holds for Figure 6.17 as was done for the comparison in Figure 6.12. All the approximated k_μ values, as indicated in Table 6.20, are below their corresponding Nyquist frequencies listed in Table 3.1 besides the 4th approximated continuous-case eigenvalue for the 6 unit-interval case. The effects of aliasing are visible since the relative error of the 4th approximated continuous-case eigenvalue for the 6 unit-interval case is 0.786%, which is lower than the relative errors of the subsequent approximated continuous-case eigenvalues in the higher number of unit-interval cases for the 4th shifted continuous-case eigenvalue and deviates from the general trend that as the number of unit intervals increase, the accuracy of the approximated shifted continuous-case eigenvalues increase. However, the relative error 4th approximated continuous-case eigenvalue for the 6 unit-interval case is still low and thus not significantly affected by aliasing.

6.3.5 (Affine, Affine) Boundary Conditions

This case considers where the initial boundary condition is an Affine condition and the final boundary condition is an Affine condition. The hypothetical initial Affine boundary condition is given by

$$X(0) = X(1)[1 + \frac{300}{N} + \lambda N].$$

The corresponding final continuous-case boundary condition has been derived in Section 6.3.2 as

$$X^{'}(0) + (k^2 + 300)X(0) = 0.$$

When the end of the rod at $x = l$ is connected to an unrestrained rigid mass M, the final continuous-case boundary condition is given by

$$X^{'}(l) - 2\gamma k^2 X(l) = 0.$$

This corresponds to a final Affine boundary condition

$$X(m)[1 - 2\lambda N] - X(m-1) = 0,$$

where N = number of unit intervals.

Confirmation of the Matrix Method

The eigenparameter λ is computed from (6.58), for three unit intervals, using the algebraic method. The results are then used to validate the matrix method. The initial Affine boundary condition is given by

$$X(0) = X(1)[1 + \frac{300}{N} + \lambda N].$$

For $N = 3$, this becomes

$$X(0) = X(1)[3\lambda + 101]. \tag{6.111}$$

The final Affine boundary condition is given by

$$X(3)[1-2\lambda N]-X(2)=0.$$

For $N=3$, this results in

$$X(3)[1-6\lambda]-X(2)=0. \tag{6.112}$$

Consider (6.58) for $n=1$ to obtain

$$X(2)-(2-\lambda)X(1)+X(0)=0. \tag{6.113}$$

Substituting (6.111) into (6.113) yields

$$X(2)-(2-\lambda)X(1)+X(1)(3\lambda+101)=0,$$

$$X(2)=-[4\lambda+99]X(1). \tag{6.114}$$

Looking at (6.58) for $n=2$ becomes

$$X(3)-(2-\lambda)X(2)+X(1)=0. \tag{6.115}$$

Rewriting (6.112)

$$X(3)=\frac{X(2)}{1-6\lambda}. \tag{6.116}$$

Substitute (6.116) into (6.115)

$$\frac{X(2)}{1-6\lambda}-(2-\lambda)X(2)+X(1)=0,$$

$$X(1)=\left[(2-\lambda)-\frac{1}{1-6\lambda}\right]X(2). \tag{6.117}$$

Substituting (6.117) into (6.114) results in

$$X(1)=-\left[(2-\lambda)-\frac{1}{1-6\lambda}\right][4\lambda+99]\,X(1),$$

$$X(1)\left[\left((2-\lambda)-\frac{1}{1-6\lambda}\right)(4\lambda+99)+1\right]=0.$$

Since$X(1)\neq 0$, this becomes

$$\left[(2-\lambda)-\frac{1}{1-6\lambda}\right][4\lambda+99]+1=0.$$

Using Matlab (MATLAB, 2004), this gives the solution $\lambda=-24.75934\cdots-0.33012\cdots\times10^{-31}i$, $0.08030\cdots+0.83301\cdots\times10^{-30}i$ and $2.09571\cdots-0.9\times10^{-30}i$, where $i=\sqrt{-1}$. Since the imaginary part of λ is so small compared to the real part, the imaginary will be regarded as negligible. Thus, the eigenparameter λ solution is $\lambda=-24.75934\ldots\ldots$, $0.08030\ldots\ldots$ and $2.09571\ldots\ldots$

The eigenparameters λ of the boundary value problem defined by (6.58), (6.111) and (6.112) will now be computed using the matrix method outlined in Chapter 5. The coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & (3\lambda + 101) & 0 & 0 \\ 1 & -(2-\lambda) & 1 & 0 \\ 0 & 1 & -(2-\lambda) & 1 \\ 0 & 0 & -1 & (1-6\lambda) \end{pmatrix},$$

with the corresponding determinant of $\mathbf{A}$ $\det(\mathbf{A})$, generated in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.10 for Mathematica (Wolfram, 2011) code), given by

$$\det(\mathbf{A}) = 100 - 1289\lambda + 542\lambda^2 + 24\lambda^3,$$

Using the NSolve function in Mathematica (Wolfram, 2011) (cf. to Appendix A, Section A.10 for Mathematica (Wolfram, 2011) code), the roots of the determinant $\det(\mathbf{A})$ is thus calculated as $\lambda = -24.75934\ldots\ldots,\quad 0.08030\ldots\ldots$ and $2.09571\ldots\ldots$, neglecting the small imaginary value of the roots. The eigenparameter spectrum generated using the matrix method is thus proven to be the same as the eigenparameter spectrum generated using the algebraic method. Section 6.3.2 discusses the same reason for the generated negative eigenparameter as found in the aforementioned solution.

Theorem 3.2 is used to determine the expected number of eigenparameters. From Sections 6.3.1 and 6.3.2, the parameters for the Affine initial and final boundary conditions were set as follows: $a > 0,\ b > 1,\ \alpha < 0$ and $\beta = 1$. Therefore, this set of parameters satisfies the requirements in case (i) of Theorem 3.2, so the expected number of eigenparameters is calculated as $m = 3$. This result ties up with the number of eigenparameters that have been computed in the three unit interval boundary problem for the (Affine, Affine) condition.

Continuous Case

The continuous case equation is given by (6.53) which have been obtained using the variable separable method. From the (Affine, Dirichlet) case, the initial Affine boundary condition, in the continuous case, was determined to be

$$X^{'}(0) + (300 + k^2)X(0) = 0. \tag{6.118}$$

The final Affine boundary condition is given by

$$X^{'}(l) - 2k^2X(l) = 0. \tag{6.119}$$

We begin by assuming a solution of the form

$$X(x) = a\sin(kx) + b\cos(kx). \tag{6.120}$$

Consequently, the first derivative is given by

$$X^{'}(x) = ak\cos(kx) - bk\sin(kx). \tag{6.121}$$

Using (6.120) and (6.121) in (6.118)

$$ak + (300 + k^2)b = 0,$$

$$bk^2 + ak + 300b = 0.$$

Dividing by b yields

$$\frac{a}{b}k + k^2 + 300 = 0,$$

$$\frac{a}{b} = \frac{-(k^2 + 300)}{k}. \tag{6.122}$$

Rewriting (6.119) using (6.120) and (6.121) produces

$$ak\cos(kl) - bk\sin(kl) - 2k^2[a\sin(kl) + b\cos(kl)] = 0,$$

$$k[(a - 2kb)\cos(kl) - (b + 2ak)\sin(kl)] = 0.$$

Dividing by $b\cos(kl)$ and noting that $k \neq 0$ for a non-trivial solution to exist, produces

$$\left[\frac{a}{b} - 2k\right] - \left[1 + \frac{2ak}{b}\right]\tan(kl) = 0. \tag{6.123}$$

Continue by substituting (6.122) into (6.123) producing

$$\left[\frac{-(k^2 + 300)}{k} - 2k\right] - \left[1 + 2k\frac{-(k^2 + 300)}{k}\right]\tan(kl) = 0.$$

Rearranging terms

$$\tan(kl)[2k^2 + 599] - 3k - \frac{300}{k} = 0.$$

Assuming the length $l = 1$, this becomes

$$\tan(k)[2k^2 + 599] - 3k - \frac{300}{k} = 0. \tag{6.124}$$

The transcendental equation is then solved using Matlab (MATLAB, 2004).

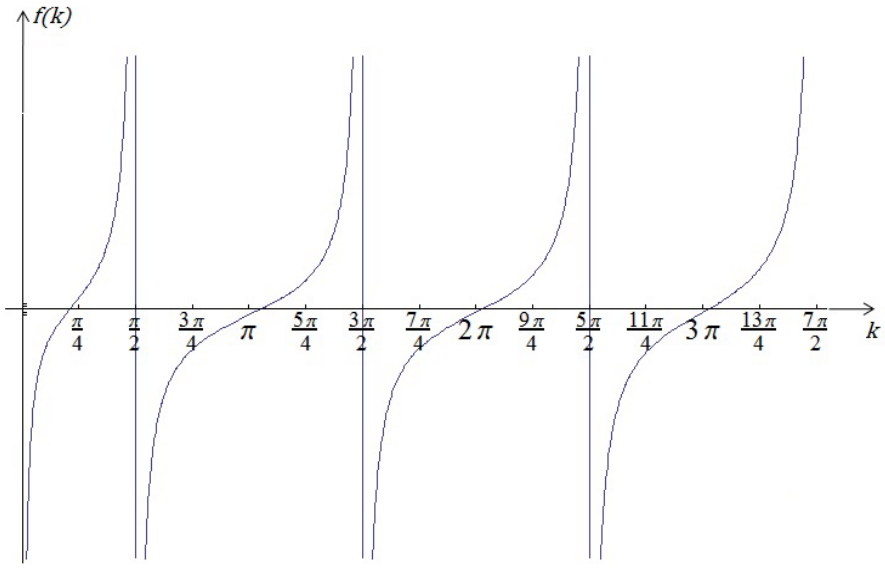

Figure 6.18: Graph of $f(k) = \tan(k)[2k^2 + 599] - 3k - \frac{300}{k}$ for the period $[0, 7\pi/2]$.

The graph of $f(k) = \tan(k)[2k^2+599]-3k-\frac{300}{k}$ for the period $[0, 7\pi/2]$ is shown in Figure 6.18. From Figure 6.18, it can be seen for $k \geq 0$, that the zero-crossing lies in the interval $[a\pi,\ (4a+1)\frac{\pi}{4}]$ for $a \in \mathbb{Z};\ a \geq 0$. The first five eigenvalues are shown in Table 6.21; computed using Matlab (MATLAB, 2004) coding (cf. to Appendix C, Section C.5). The software coding was done in Matlab (MATLAB, 2004). The software code is a simple numerical iterative algorithm which searches for the minimum value of $f(k) = \tan(k)[2k^2 + 599] - 3k - \frac{300}{k}$ for the period $[a\pi,\ (4a+1)\frac{\pi}{4}]$ using an iterative step $\delta k = 0.000001\pi$. The k value for which $f(k)$ is closest to zero is regarded as the k value of the zero-crossing. The process is repeated five times to compute the first five eigenvalues.

Table 6.21: First six approximated continuous-case eigenvalues derived from the discrete case for the different unit-intervals cases and the first five eigenvalues for the continuous case for the (Affine, Affine) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	k^2= -252.577	k^2= -267.464	k^2= -274.690	k^2= -278.059	k^2= -281.747	
2 (1 for continuous)	0.739	0.698	0.679	0.670	0.659	0.655
3 (2 for continuous)	3.866	3.605	3.475	3.411	3.335	3.302
4 (3 for continuous)	7.126	6.892	6.696	6.585	6.443	6.380
5 (4 for continuous)	9.740	10.054	9.910	9.784	9.594	9.502
6 (5 for continuous)	11.421	12.979	13.058	12.964	12.752	12.633

The linear ordinary differential equation (6.53) enables the solution $X(x)$to take on the form as a Fourier series, given by

$$X(x) = \sum_{j=1}^{\infty} a\sin(k_j x)\ + b\cos(k_j x)\ ,$$

where the k_j-values form an infinite spectrum. The k_j-values are determined from (6.124) and represent the natural frequencies of the axial displacement $u(x,t)$ in the spatial domain.

Comparison Between Continuous-Case Eigenvalues and Approximated Continuous-Case Eigenvalues Derived From the Discrete Case

Since the difference equation (6.58) is the same as (3.22), the same analysis is applied to Table 6.21 to test for purely oscillatory behaviour as was done in Section 6.1.1. The approximated k values

from Table 6.21 were also found to satisfy condition (3.28), with the exception of the negative eigenparameters which yield discrete eigenvalues r whose absolute value is not equal to 1, as shown in Table 6.22.

Table 6.22: Absolute Value of Discrete Eigenvalues $|r|$ for the Negative k^2 values from Table 6.21.

Number of unit intervals $=m$	Negative k^2 values	$\|r\|$
6	-252.577	8.904 ; 0.112
11	-267.464	3.958 ; 0.253
19	-274.690	2.332 ; 0.429
30	-278.059	1.731 ; 0.578
100	-281.747	1.183 ; 0.846

It is observed from Table 6.22 that the negative eigenparameters does not fulfill the condition for purely oscillatory behaviour and needs to be handled using the Crum-based transformation method.

A comparison is done between the k values obtained for the continuous case to the k values, derived from the eigenparameters of the discrete case for scenarios with 6 unit intervals, 11 unit intervals, 19 unit intervals, 30 unit intervals and 100 unit intervals. The discrete case eigenparameters are determined using an extended version of the Mathematica (Wolfram, 2011) code in Appendix A, Section A.10, for the respective unit interval as was described in Section 6.1.1. The generated number of eigenparameters is found to equal the expected number of eigenparameters which were computed from Theorem 3.2. The number of expected and generated number of eigenparameters is shown in Table 6.23.

Table 6.23: Expected and generated number of eigenparameters for the (Affine, Affine) condition for the unit intervals indicated in the first column

Number of unit intervals $=m$	α	β	Case in Theorem 3.2	Expected number of eigenparameters $=m$	Number of eigenparameters generated
6	<0	1	(i)	6	6
11	<0	1	(i)	11	11
19	<0	1	(i)	19	19
30	<0	1	(i)	30	30
100	<0	1	(i)	100	100

The comparison in Table 6.21 focuses on the first six eigenvalues as these are the minimum generated number of approximated continuous-case eigenvalues of all unit-intervals cases. The comparison is done against the percentage relative error of the approximated k values measured against the continuous-case eigenvalues.

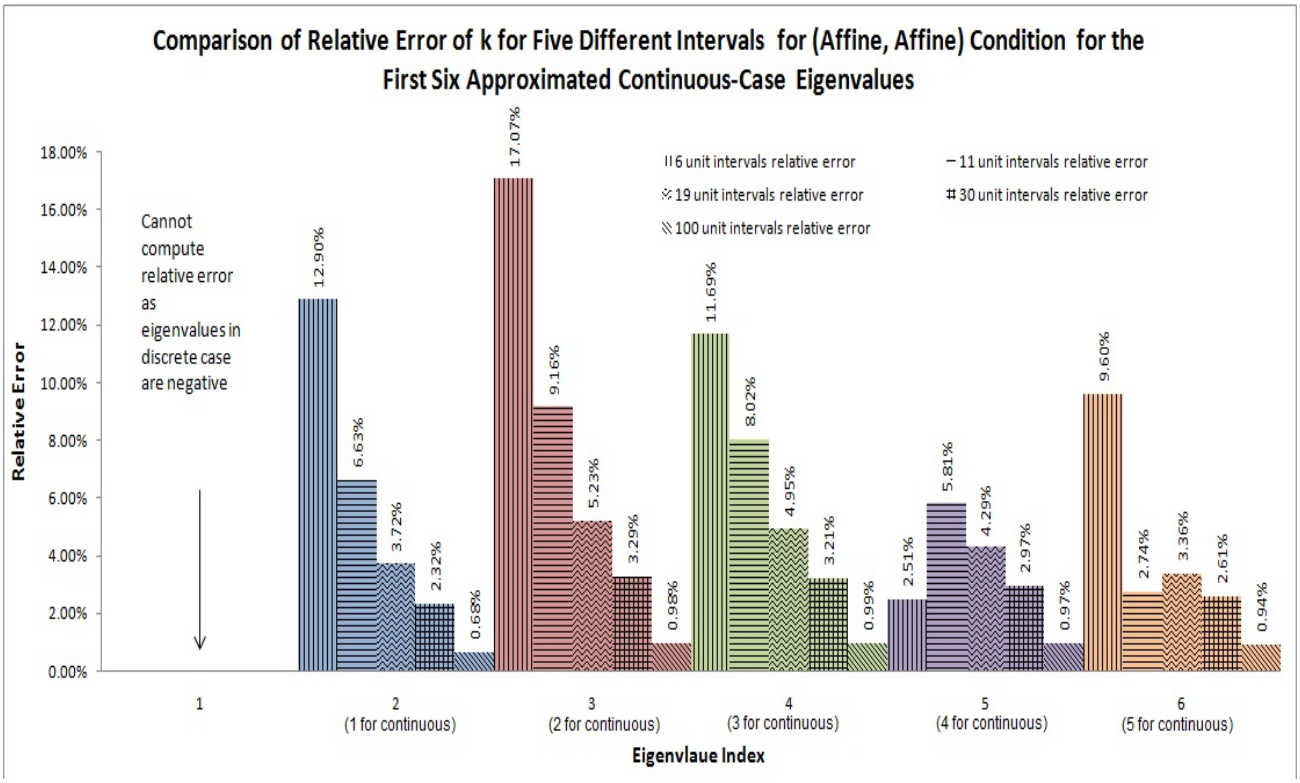

Figure 6.19: Comparison of Relative Error of k for Five Different Intervals for (Affine, Affine) Condition for the First Six Approximated Continuous-Case Eigenvalues.

The same analysis holds for Figure 6.19 as was done for the comparison in Figure 6.2. There are no effects of aliasing since all the approximated k values, as indicated in Table 6.21, are below their corresponding Nyquist frequencies listed in Table 3.1. The fact that the least eigenparameter from the discrete cases are negative allows for the Crum-based transformation to be applied by shifting the eigenvalue spectrum by the least eigenparameter. This results in the number eigenparameters reducing by one as the resultant zero-valued eigenparameter is not part of the eigenparameter spectrum.

A comparison is thus done between the first five shifted continuous-case eigenvalues k_μ and the first five approximated k_μ values derived from the discrete case. The results are tabulated in Table 6.24. Table 6.24 does not show all the eigenvalues as the comparison focuses on the first five eigenvalues as these are the minimum generated number of approximated k_μ values of all unit-intervals cases. The comparison is done against the percentage relative error of the approximated k_μ values measured against the shifted continuous-case eigenvalues.

Table 6.24: k_μ-values for the different unit-intervals cases and the continuous case for the (Affine, Affine) condition.

Eigenvalue Index	6 unit intervals	11 unit intervals	19 unit intervals	30 unit intervals	100 unit intervals	Continuous
1	15.910	16.369	16.588	16.689	16.798	16.798
2	16.356	16.747	16.934	17.020	17.113	17.107
3	17.417	17.747	17.875	17.928	17.979	17.957
4	18.640	19.197	19.310	19.333	19.334	19.288
5	19.571	20.879	21.100	21.121	21.080	21.008

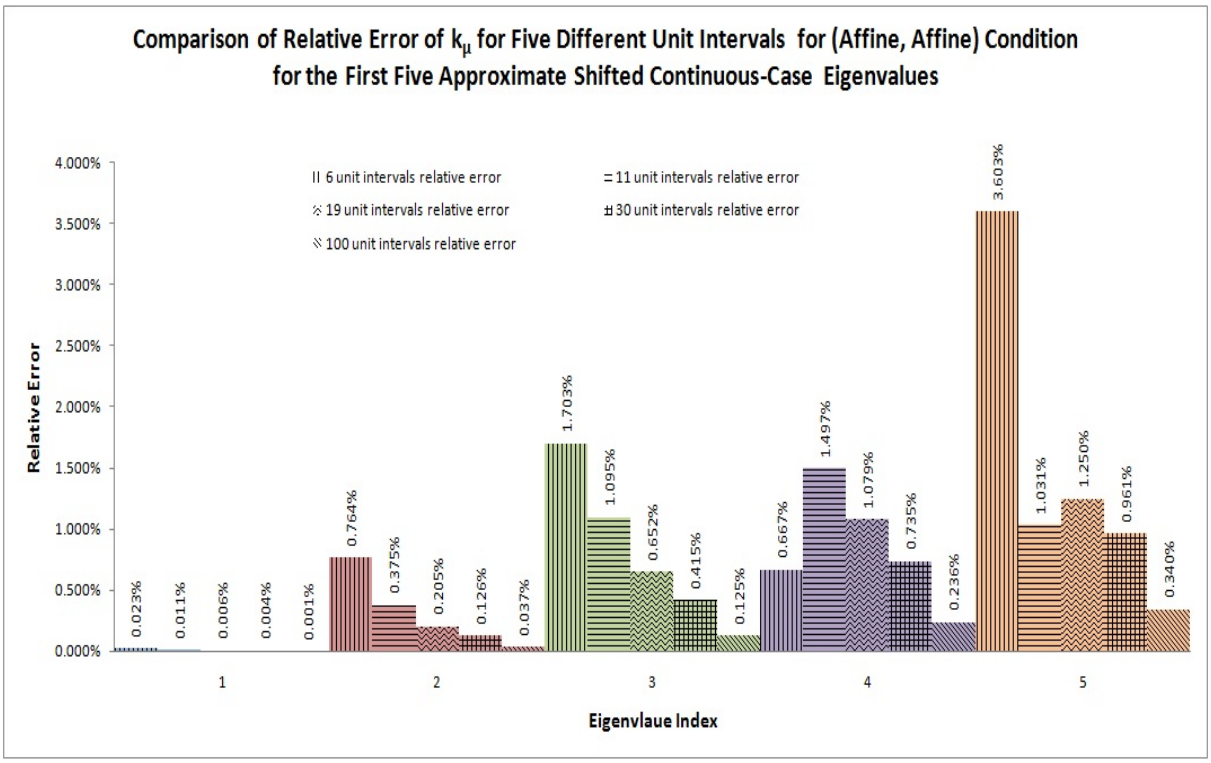

Figure 6.20: Comparison of Relative Error of k_μ for Five Different Unit Intervals for (Affine, Affine) Condition for the First Five Approximated Shifted Continuous-Case Eigenvalues.

The approximated shifted continuous-case eigenvalues in Table 6.24 are subjected to the test for purely oscillatory behaviour using the left-hand side of (3.28). This results in all the discrete eigenvalues r, derived from the shifted continuous-case eigenvalues, satisfying the purely oscillatory behaviour condition $|r| = 1$. Therefore, the Crum-based transformation ensures that the discrete eigenvalues r lie on the unit circle and that the eigenparameters are a function of real physical natural frequencies.

The same analysis holds for Figure 6.20 as was done for the comparison in Figure 6.12. all the approximated k values, as indicated in Table 6.24, are below their corresponding Nyquist frequencies listed in Table 3.1 besides the 5th approximated k_μ value for the six unit-interval case. The effects of aliasing are not so strong as the relative error of the 5th approximated k_μ value for the six unit-interval case is 3.603% since this can be attributed to the 5th approximated k_μ value for the six

unit-interval case of 19.571 rad/m not being much larger than the corresponding Nyquist frequncy of 18.850 rad/m.

6.4 Concluding Remarks

This chapter looked at subjecting the two-point boundary value problems defined by (1.1) and Table 1.1 in the context of engineering applications in the fields of electrostatics and heat conduction. In all nine cases listed in Table 1.1, the matrix method has been confirmed by the algebraic method using a three-unit interval case. Figures 6.2, 6.4, 6.5, 6.6, 6.9, 6.11, 6.14, 6.16 and 6.19 of the pre-transformed approximated continuous-case eigenvalues and Figures 6.12, 6.17 and 6.20 of the post-transformed approximated continuous-case eigenvalues reveal a general trend that the accuracy of the approximated continuous-case eigenvalues improves as one ascends the eigenvalue spectrum and as the number of unit intervals increase. Table 6.25 summarises the approximated pre-transformed and post-transformed (if required) continuous-case eigenvalues for the non-shifted and shifted (if required) continuous-case eigenvalues generated in all nine cases listed in Table 1.1 against the first continuous-case eigenvalue for the 100 unit-intervals case for a particular eigenvalue index.

Table 6.25: Summary of the approximated pre-transformed and post-transformed (if required) continuous-case eigenvalues, generated in all nine cases listed in Table 1.1, against the non-shifted and shifted (if required) continuous-case eigenvalue for the 100 unit-intervals case for a particular eigenvalue index.

Case	Eigenvalue Index	k_μ-value as approximation of 1st continuous-case eigenvalue	1st continuous-case eigenvalue
(Dirichlet, Dirichlet)	1	3.141	3.142
(Non-Dirichlet, Dirichlet)	1	1.579	1.571
(Dirichlet, Non-Dirichlet)	1	1.579	1.571
(Non-Dirichlet, Non-Dirichlet)	1	0.000	0.000
(Dirichlet, Affine)	1	0.654	0.653
(Affine, Dirichlet)	2	3.184	3.152
Shifted (Affine, Dirichlet)	1 (2 for continuous)	17.085	17.079
(Non-Dirichlet, Affine)	1	0.000	0.000
(Affine, Non-Dirichlet)	2 (1 for continuous)	1.600	1.576
Shifted (Affine, Non-Dirichlet)	1	16.861	16.749
(Affine, Affine)	2 (1 for continuous)	0.659	0.655
Shifted(Affine, Affine)	1	16.798	16.798

All the approximated non-shifted continuous-case eigenvalues yielded discrete eigenvalues that obey the condition for purely oscillatory behaviour with the exception of the negative eigenparameters generated in the following three cases: (Affine, Dirichlet), (Affine, Non-Dirichlet) and (Affine, Affine). However, the approximated shifted continuous-case eigenvalues yielded discrete eigenvalues that satisfy the condition for purely oscillatory behaviour.

7 CONCLUSION

The matrix method, outlined in Chapter 5, is the same as the algebraic method and has been confirmed, using the three-unit-interval case, for all of the boundary-condition combination cases indicated in Table 1.1 and thus would be an easier method of computing the eigenparameters for unit-interval cases extending beyond three unit intervals.

From the comparative analysis of the discrete case and continuous case of the engineering boundary value problems in Chapter 6, there was a general trend in the following Figures: 6.2, 6.4, 6.5, 6.6, 6.9, 6.11, 6.14, 6.16 and 6.19; where, as the number of unit intervals increase, the accuracy of the approximated continuous-case eigenvalues generated were found to increase. This is summarised in Table 6.25 in Section 6.4. This implies that the approximation of the continuous-case eigenvalues in the discrete case approaches the continuous case as the number of unit intervals increases towards infinity. Also, all the approximated non-shifted continuous-case eigenvalues yielded discrete eigenvalues that obey the condition for purely oscillatory behaviour with the exception of the negative eigenparameters generated in the following three cases: (Affine, Dirichlet), (Affine, Non-Dirichlet) and (Affine, Affine).

There are two possible reasons for negative eigenparameters being generated: the discrete case is an approximation to the continuous-case model of the system and thus the approximated continuous-case eigenvalues have inherent errors, and the initial Affine boundary condition was contrived and does not represent an actual physical boundary condition (as no physical boundary condition could be found to match the form of the initial boundary condition in (3.16)). The discrete eigenvalues, generated from the negative eigenparameters, did not lie on the unit circle and thus cannot be representative of physical natural frequencies. These negative eigenparameters can be handled by the Crum-based transformation, which shifts the approximated eigenvalue spectrum; such that the condition for purely oscillatory behaviour can be met. Thus, an outcome of the research is also the importance of having a discretised model having the structure and properties to represent the continuous system behaviour. The Crum-based transformation thus provides a useful method when dealing with boundary value problems, in the discrete case, which generate negative eigenparameters.

Applying the Crum-based transformation to the boundary value problem in the discrete domain corresponds to a shift in the continuous domain as well. Hence, the approximated shifted continuous-case eigenvalues was compared to the shifted continuous-case eigenvalues. After applying the Crum-based transformation, Figures 6.12, 6.17 and 6.20 reveal a general trend that as the number of unit intervals increase, for a particular boundary condition case, the accuracy of the approximated shifted continuous-case eigenvalues increase. Also, another general observation is that the accuracy of the post-transformed approximated eigenvalues decreases as one ascends through the eigenvalue spectrum.

Another observation is that when the relative error of the pre-transformed approximated eigenvalues (Figures 6.11, 6.16 and 6.19) are compared with their respective post-transformed approximated eigenvalues (Figures 6.12, 6.17 and 6.20) for the same unit interval case, the post-transformed approximated eigenvalues generally show a greater accuracy than the pre-transformed approximated eigenvalues. All the post-transformed approximated eigenvalues in the three cases (Affine, Dirichlet), (Affine, Non-Dirichlet) and (Affine, Affine) yielded discrete eigenvalues that fulfill the condition

for purely oscillatory behaviour. Also, the effects of aliasing was not present in both the non-shifted and shifted cases of all boundary condition cases except the shifted 6 unit-intervals cases for the (Affine, Dirichlet), (Affine, Non-Dirichlet) and (Affine, Affine) cases; of which the effects of aliasing was found to be minimal since the highest natural frequency was not much higher than the corresponding Nyquist frequency of 18.850 rad/m. Thus, the Crum-based transformation has been proven to be a useful method, in engineering applications modelled by second-order partial differential equations, when estimating shifted continuous-case eigenvalues in the discrete case and dealing with discrete boundary value problems where negative eigenparameters are generated.

7.1 Recommendations For Future Study

It would be useful for future research to evaluate the application of the Crum-based transformation to a wider range of engineering applications. This can be done by looking at engineering applications with boundary conditions that take on the general form given by

$$X(0) = \left[a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k} \right] X(1), \qquad a \geq 0,\ c_k \geq 0.$$

and

$$X(m-1) = \left[\alpha\lambda + \beta - \sum_{j=1}^{p} \frac{\gamma_j}{\lambda - \sigma_j} \right] X(m), \qquad \alpha \leq 0,\ \gamma_j < 0.$$

where m is the number of unit intervals (Currie & Love, 2013).

The paper (Currie & Love, 2010a) also looks at transforming boundary conditions from one form to another, that results in a transformed difference equation with transformed boundary conditions using the Crum-based transformation eg. transforming a Non-Dirichlet boundary condition to a Dirichlet boundary condition. Another recommendation would thus be to look at the effects of the relative error of continuous-case eigenvalues, generated in the discrete case, for boundary value problems in which the boundary conditions can be transformed through a hierarchy of λ-dependence.

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Appendix A

MATHEMATICA (Wolfram, 2011) CODE

A.1 Illustration

```
LamdaEquation = (2*lambda+1)*(2-lambda)+1;

Lambda = NSolve[LamdaEquation==0,500];

Lambda//TableForm

Lambda0 = -0.68614066163450716496265286705473232955506611449569809192496936764147518036435
11567567813413991970306048893692364127094674837056538008508504046330048905769431
81994504151478951648428493574014172305209825499367647783021748243540662250519441
06420270257908247593716662602826643265214511587263263678448197276179373367007371
68970637608049263852370190493609588457231443830319340838471513955790626667512093 0
46784348642538902647657210266909495207385177134842133128151161747618298331286585 2
33497657590401861477762470063 29;

C0 = Lambda0+3;

C1 = -(1+2*Lambda0)^(-1);

B1 = C1^2 + 1;

w1 = -mu/(Lambda0+3);

w2 = (2-mu-Lambda0)^(-1) + (1+2*Lambda0);

w3 = 0;

W12 = w1/w2;

W32 = w3/w2;

DTSE1 = C1*W32 - (B1-mu*C1) + C0*W12;

Mu = NSolve[DTSE1==0,500];

Mu//TableForm
```

A.2 (Dirichlet, Dirichlet) Case

```
n = 3;

A = {{1,    0,    0,    0},
```

```
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    0,    1}};
d = Det[A];
lambda = NSolve[d==0,x];
lambda//TableForm
```

A.3 (Non-Dirichlet, Dirichlet) Case

```
n = 3;
A = {{-1,    1,    0,    0},
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    0,    1}};
d = Det[A];
lambda = NSolve[d==0,x];
lambda//TableForm
```

A.4 (Dirichlet, Non-Dirichlet) Case

```
n = 3;
A = {{1,    0,    0,    0},
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    -1,    1}};
d = Det[A];
lambda = NSolve[d==0,x];
lambda //TableForm
```

A.5 (Non-Dirichlet, Non-Dirichlet) Case

```
n = 3;
A = {{-1,    1,    0,    0},
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    -1,    1}};
d = Det[A];
lambda = NSolve[d==0,x];
lambda //TableForm
```

A.6 (Dirichlet, Affine) Case

```
n = 3;
A = {{1,    0,    0,    0},
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    -1,    (1-2*x*n) }};
d = Det[A];
lambda = NSolve[d==0,x];
lambda //TableForm
```

A.7 (Affine, Dirichlet) Case

```
n = 3;
A = {{-1 ,    (1+(300/n)+x*n),    0,    0},
{1,    -(2-x),    1,    0},
{0,    1,    -(2-x),    1},
{0,    0,    0,    1}};
d = Det[A];
```

```
lambda = NSolve[d==0,x];

lambda //TableForm
```

A.8 (Non-Dirichlet, Affine) Case

```
n = 3;

A = {{-1,    1,    0,    0},

{1,    -(2-x),    1,    0},

{0,    1,    -(2-x),    1},

{0,    0,    -1,    (1-2*x*n) }};

d = Det[A];

lambda = NSolve[d==0,x];

lambda //TableForm
```

A.9 (Affine, Non-Dirichlet) Case

```
n = 3;

A = {{-1 ,    (1+(300/n)+x*n),    0,    0},

{1,    -(2-x),    1,    0},

{0,    1,    -(2-x),    1},

{0,    0,    -1,    1}};

d = Det[A];

lambda = NSolve[d==0,x];

lambda //TableForm
```

A.10 (Affine, Affine) Case

```
n = 3;

A = {{-1 ,    (1+(300/n)+x*n),    0,    0},

{1,    -(2-x),    1,    0},
```

```
{0,    1,    -(2-x),    1},

{0,    0,    -1,    (1-2*x*n) }};

d = Det[A];

lambda = NSolve[d==0,x];

lambda //TableForm
```

Appendix B

DISCRETISATION OF FINAL BOUNDARY CONDITION OF LONGITUDINAL VIBRATION OF AN ELASTIC BAR WITH MASS ATTACHED AT THE FINAL END

Consider the longitudinal vibrations of a uniform straight elastic bar of length l. The axial displacement $u(x,t)$ is governed by the equation (DiPrima & Boyce, 1977:531)

$$\frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \qquad 0 < x < l,\ t > 0. \tag{B.1}$$

where E is the Young's modulus of the bar material and ρ is the density of the bar material.

Assume that the axial displacement $u(x,t)$ is given by

$$u(x,t) = X(x)T(t). \tag{B.2}$$

Then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = X^{"}(x)T(t), \tag{B.3}$$

and

$$\frac{\partial^2 u(x,t)}{\partial t^2} = X(x)T^{"}(t). \tag{B.4}$$

Substituting equation (B.3) and equation (B.4) into equation (B.1) yields

$$\frac{E}{\rho}X^{"}(x)T(t) = X(x)T^{"}(t),$$

and by separation of variables,

$$\frac{X^{''}(x)}{X(x)} = \frac{\rho T^{''}(t)}{ET(t)} = -k^2, \tag{B.5}$$

where k^2 is a separation constant.

Suppose that the end where $x = l$ is connected to an unrestrained rigid mass M. The final continuous-case boundary condition is as follows

$$EAu_x(l,t) + Mu_{tt}(l,t) = 0, \tag{B.6}$$

where A denotes cross-sectional area of the bar. Using appropriate derivatives from equation (B.2), it is possible to express equation (B.6) as

$$EAX^{'}(l)T(t) + MX(l)T^{''}(t) = 0,$$

and separating the variables gives

$$\frac{T^{''}(t)}{T(t)} = -\frac{EAX^{'}(l)}{MX(l)}. \tag{B.7}$$

Substitute equation (B.5) into equation (B.7) leads to

$$\frac{EAX^{'}(l)}{MX(l)} = \frac{k^2E}{\rho}. \tag{B.8}$$

Rearranging equation (B.8) gives

$$AX^{'}(l) - \frac{k^2MX(l)}{\rho} = 0,$$

that is, the continuous-case final boundary condition being

$$X^{'}(l) - 2\gamma k^2X(l) = 0. \tag{B.9}$$

where $\gamma = \frac{M}{2\rho A}$. If the following specifications of the elastic bar and unrestrained mass are given as $M = 0.5$ kg, A $= 0.5$ m^2, $\rho = 0.5$ kg.m^{-3}; then $\gamma = 1$.

Discretising equation (B.9) using the backward difference formula, results in the following discrete boundary condition

$$\frac{X(m) - X(m-1)}{h} - 2k^2\gamma X(m) = 0,$$

where m is the number of unit intervals. Multiplying by h results in

$$X(m) - X(m-1) - 2k^2h\gamma X(m) = 0.$$

Therefore, with $\gamma = 1$ and $k^2 = \lambda N^2$ we obtain the discretised final Affine boundary condition

$$X(m)[1 - 2\lambda N] - X(m-1) = 0.$$

where $N = \frac{1}{h}$.

Appendix C

MATLAB (MATLAB, 2004) CODE

C.1 Continuous Case for (Dirichlet, Affine) condition

```
ZeroCrossings = zeros(6,1);
Eqn = 10000000*ones(10000,1);
MinValues = ones(6,1);
m = 1;
for a = 0:5
for k = a*pi:pi/10000:(2*a+1)*pi/2
Eqn(m) = abs(2*k*tan(k)-1);
m = m + 1;
%If current minumum is less than previous iteration minimum
if (MinValues(a+1) > min(Eqn))
MinValues(a+1) = min(Eqn);
ZeroCrossings(a+1) = k;
end
end
%Re-initialise Eqn
for n = 1:10000
Eqn(n) = 10000000;
end
m = 1;
end
ZeroCrossings
```

C.2 Continuous Case for (Affine, Dirichlet) condition

```
ZeroCrossings = zeros(5,1);

Eqn = 10000000*ones(40000,1);

MinValues = ones(5,1);

m = 1;

%Obvious root at k=0

ZeroCrossings(1)=0;

for a = 2:5

for k = (a-1)*pi:pi/40000:(4*(a-1)+1)*pi/4

Eqn(m) = abs( (k^2)*tan(k)-k+300*tan(k) );

m = m + 1;

%If current minimum is less than previous iteration minimum

if (MinValues(a) > min(Eqn))

MinValues(a) = min(Eqn);

ZeroCrossings(a) = k;

end

end

%Re-initialise Eqn

for n = 1:40000

Eqn(n) = 10000000;

end

m = 1;

end

ZeroCrossings
```

C.3 Continuous case for (Non-Dirichlet, Affine)

```
ZeroCrossings = zeros(5,1);
```

```
Eqn = 10000000*ones(10000,1);

MinValues = ones(5,1);

m = 1;

for a = 1:5

for k = (2*a-1)*pi/2:pi/10000:a*pi

Eqn(m) = abs(tan(k)+2*k);

m = m + 1;

%If current minimum is less than previous iteration minimum

if (MinValues(a) > min(Eqn))

MinValues(a) = min(Eqn);

ZeroCrossings(a) = k;

end

end

%Re-initialise Eqn

for n = 1:10000

Eqn(n) = 10000000;

end

m = 1;

end

ZeroCrossings
```

C.4 Continuous case for (Affine, Non-Dirichlet)

```
ZeroCrossings = zeros(4,1);

Eqn = 10000000*ones(100000,1);

MinValues = ones(4,1);

m = 1;

for a = 1:4
```

```
for k = (2*a-1)*pi/2:pi/100000:(4*a-1)*pi/4
Eqn(m) = abs(k*tan(k)+k^2+300);
m = m + 1;
%If current minimum is less than previous iteration minimum
if (MinValues(a) > min(Eqn))
MinValues(a) = min(Eqn);
ZeroCrossings(a) = k;
end
end
%Re-initialise Eqn
for n = 1:100000
Eqn(n) = 10000000;
end
m = 1;
end
ZeroCrossings
```

C.5 Continuous case for (Affine, Affine)

```
ZeroCrossings = zeros(5,1);
Eqn = 10000000*ones(40000,1);
MinValues = ones(5,1);
m = 1;
for a = 1:5
for k = (a-1)*pi:pi/40000:(4*(a-1)+1)*pi/4
Eqn(m) = abs( (2*k^2+599)*tan(k)-3*k-(300/k) );
m = m + 1;
%If current minimum is less than previous iteration minimum
```

```
if (MinValues(a) > min(Eqn))
MinValues(a) = min(Eqn);
ZeroCrossings(a) = k;
end
end
%Re-initialise Eqn
for n = 1:40000
Eqn(n) = 10000000;
end
m = 1;
end
ZeroCrossings
```