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Topologies on Groups and Semigroups [1]

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Abstract

Topological groups and semigroups form the basic building blocks of many different areas of mathematics. The aim of this work is to determine if a general cancellative semigroup can be given a left shift invariant topology. The theory behind a class of topologies that can be created on a given group or semigroup is discussed. The t-sequence proof of the Markov theorem is presented and this serves as a catalyst for further inquiry. The algebra of the Stone-Cech compactification of a discrete semigroup is utilized to prove the existence of certain ultrafilters, with which topologies can be constructed.

Declaration

I declare that this is my own, unaided work. It is being submitted for the Degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

(Signature)

(Date)

Acknowledgement

I would like to dedicate this dissertation to my family for motivating me through the times when the end felt very far away, to Shel for being supportive and understanding of my late night working habits and to Professor Zelenyuk for displaying the patience of a saint when many others would have given up.

"I have had my results for a long time: but I do not yet know how I am to arrive at them."

- Karl Friedrich Gauss (1777-1855)

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Chapter 1

Introduction

This dissertation is part a literature review and part an extension of the current theory of topologies on Groups and Semigroups. The marrying of the algebraic and naturally associated topological structures have proven to be a particularly fruitful endeavour. In this dissertation, we attempt to map out one small area of that growing body of knowledge. This area is singled out for two reasons. The first is that the literature abounds with examples of work focusing on the one side of the prism so to speak. This side takes a group and attaches to it a relevant natural topology and uses this to reveal the inner structure of this new creation. There are many books on locally compact group topologies for example. However the reverse side of the prism, the one which tries to see if a given group admits a topology has only had a relatively more ad-hoc approach to it. This dissertation seeks to collate the different pieces of work and step the reader through the problem at hand, from the framing of the Markov problem through to the construction of translation invariant topologies on arbitrary groups. Since the methodologies used for the constructions come from many different mathematical areas, the background theory is also built so that the results are relatively self contained. Since a self contained treatment of this area has not been available to the author, it seemed reasonable to create such a piece of work. This, as it happens, is the second reason behind this dissertation.

The usefulness of this framework can be seen in the next example. Furstenberg used topological ideas in [Fur55] to show that there are an infinite number of primes in the integers.

Theorem 1.1 (Euclid). *There are an infinite number of prime numbers*

Proof. Consider the family of all sets of the form $U(b) = \{nb : n \in \mathbb{Z}\}$ where $b \in \mathbb{Z}$. It can be shown, using the results of this dissertation, that these sets form a basis for the neighbourhood system of identity of a zero dimensional left shift invariant topology on $\mathbb{Z}$. Now, by the fundamental theorem of arithmetic, we know that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} U(p)$$

Now suppose that there are a finite number of primes. This implies that the set $\{-1, 1\}$ is an open set, being the finite union of a collection of clopen subsets of $\mathbb{Z}$. However, by construction, every open set in this topology is infinite since any open set must contain the translation of some $U(b)$ which itself is an infinite set. □

We will show that any cancellative semigroup admits a non-discrete left shift invariant topology so that every open set has the same cardinality as the semigroup in question. Using this result we can give a similar proof of this fact for the integers (and can extend it to a statement about irreducible elements of a certain class of semigroups).

In Chapter one, the basic definitions and terminology of topological group theory are introduced. We then introduce the theory of T-Filters and T-sequences as developed by Zelenyuk and Protosov [PZ99] in sufficient detail so that the Markov criteria for the topologizability of a countable group can be determined. The theory is first presented for Abelian groups as the constructions are easier to understand in this setting, and then are extended to the case of general countable groups. We show that Abelian groups are strongly topologizable and thus the question is moot for this family of groups. Once Markov's theorem has been proven, we give two examples of groups who only admit discrete topologies. Since every linear group is topologizable (the general linear group, $\mathrm{GL}_n(\mathbb{R})$ inherits the

natural topology from $\mathbb{R}^{n^2}$), any group that admits only a discrete topology is rather exotic. The first construction is by Shelah and depends on the continuum hypothesis. The second proof by Ol'shanskii is in ZFC and settled the question about the existence of non-topologizable groups.

In Chapter two, we give the z-ultrafilter construction for the Stone-Cech compactification of a completely regular space. Since every topological group is completely regular, this construction is the one needed for the Stone-Cech compactification of that group. The left shifts of the original topological group can be extended via the construction to the entire compactification but it is difficult to gain much more traction in this regard. However, if we treat the space as an algebraic semigroup with the discrete topology then we can give a much more explicit construction of this compact space and we use this fact, along with the theorem of Ellis, to show the famous Hindmann proof of the Finite Products theorem.

In Chapter three we give three constructions of weaker topologies on groups and semigroups. The first construction is that of a left shift invariant topology on an arbitrary group. We then extend this theorem in two ways. We first extend it so that we have a left shift invariant, non discrete topology on an embeddable semigroup. We finally drop the requirement of embeddability into a larger group and prove that every cancellative semigroup admits a left shift invariant topology. Finally, we give a slightly different version of Zelenyuk's construction (one based on the construction of this invariant topology on a semigroup) of an invariant topology on an arbitrary group.

The two appendices at the end of this dissertation serve as a repository of general topological results that will be required for this dissertation but whose proofs would move us too far afield. The first appendix deals with general topological definitions and results, which would be covered in any good topology textbook. The second appendix gives some results on filters, ultrafilters and p-limits that are required and if not explicitly required then illuminating in their own right.

Chapter 2

Topological Groups

We start this section by discussing the basic properties of a topological group. The study of topological groups is quite advanced. They form the basis of the entire field of Abstract Harmonic Analysis with Comfort [Com84] being a good general overview of the area. Hewitt [HR79], Morris [Mor77] and Pontryagin [Pon66] are the classical texts with all three books having a strong leaning towards the structure of locally compact topological groups. The focus of this section is to show how special structures centred around the identity of a group allow us to prove many interesting theorems regarding the interaction of the group operations and the topology generated by that structure. Metrizability of a topological group serves as the test case for that section. We then continue to investigate how topologies can be constructed on a given group and what algebraic properties a group must have before a compatible group topology can be constructed. The main results in this regard follow [PZ99] rather closely, though the presentation is different. A different view of this construction can be found in [CS90] as explained in [Luk06]. We use this new machinery to give a proof of the famed Markov theorem. The fact that not every group admits a group topology serves as the springboard into the latter questions of this paper.

2.1 Definition and Basic properties

The mixing of continuity with the basic algebraic operations of a group is a natural step in the study of infinite groups and their associated structures. In fact, many of the trivial examples of infinite groups already come equipped with a topology such that the operations of inversion and multiplication are already continuous, for example the Circle Group $\mathbb{T} = \{z \in \mathbb{C} \mid \|z\| = 1\}$ which inherits the subspace topology from $\mathbb{C}$ which itself is a group under multiplication. Encasing a group with a topology that respects its algebraic operations also allows the use of rather powerful topological machinery in the pursuit of answers to questions from other mathematical fields that may at first glance appear to be far removed from set theoretic topology. However, before anything of that nature can begin, a definition is in order.

Definition 2.1. *A* topological group *is a pair* $(G, \mathcal{T})$ *for which*

1. $(G, \mathcal{T})$ *is a topological space*
2. G *is a group*
3. *the function from* $G \times G$ *to* G *defined by* $\langle a, b\rangle \to ab^{-1}$ *is continuous.*

A *base of a topology* is a collection of open sets such that any open set can be written as the union of elements of that base. A base for a neighbourhood system is called *countably decreasing* if it is of the form $\{V_i : i \in \mathbb{N}\}$ such that $V_1 \supseteq V_2 \supseteq ...$ and *symmetric* if for any element U of that base it is true that $U = U^{-1}$. Any base for the neighbourhood filter at identity fully determines a group topology, as the following basic theorem shows:

Theorem 2.2. *Let* G *be a group with identity* $e \in G$ *then* $\phi \subset 2^G \setminus \{\emptyset\}$ *is the basis of the neighbourhood filter at identity of a group topology* $\mathcal{T}$ *if and only if*

1. $\cap\phi = \{e\}$
2. $\forall U, V \in \phi$ *there exists* $W \in \phi$ *such that* $W \subset U \cap V$

3. *$\forall U \in \phi$ there exists $V \in \phi$ such that $V^2 \subset U$*

4. *$\forall U \in \phi$ there exists $V \in \phi$ such that $V^{-1} \subset U$*

5. *$\forall U \in \phi$ and $g \in G$ there exists $V \in \phi$ such that $g^{-1}Vg \subset U$*

Furthermore, we may always assume that the base generated by ϕ is symmetric and that if this topology is first countable then we can choose this base so that it is countably decreasing.

Proof. Suppose $\phi \subset 2^G$ satisfies the conditions of the theorem, then by condition (2) ϕ is a basis for a filter on G (as for all $U, V \in \phi$ there exists a $W \in \phi$ such that $W \subset U \cap V \neq \emptyset$). Consider the family of subsets $\mathcal{N}_x = \{U \subset G : xV \subset U$, for some $V \in \phi\}$ where $x \in G$. Let $U \subset N_x$. There exists a $V \in \phi$ such that $xV \subset U$. Let $W \in \phi$ such that $W^2 \subset V$ and consider $xW \in \mathcal{N}_x$. Let $y \in xW$ i.e. $y = xw$ for some $w \in W$ then $yW = xwW \subset xW^2 \subset xV$ and thus $xV \in \mathcal{N}_y$ for all $y \in xV$ and thus $\mathcal{N}_x$ is the neighbourhood filter at $x \in G$ for some topology on G.

Now let $x, y \in G$ and $U \in \phi$ so that $xyU \in \mathcal{N}_{xy}$. Now there exist $V, W \in \phi$ such that $V^2 \subset U$ and $y^{-1}Wy \subset V$ but then $xW \in \mathcal{N}_x$ and $yV \in \mathcal{N}_y$ and thus $xW \circ yV = xy\,(y^{-1}Wy)\,V \subset xyV^2 \subset xyU$. Consequently, the function $\circ : G^2 \to G$ is continuous. It should be noted that for all $g \in G$ the function $\lambda_g : G \to G$ given by $\lambda_g(x) = gx$ is a homeomorphism. Now $x^{-1}U \in \mathcal{N}_{x^{-1}}$ and there exists $V, W \in \phi$ such that $xWx^{-1} \subset U$ and $V^{-1} \subset W$ but then $x\,(xV)^{-1} = xV^{-1}x^{-1} \subset xWx^{-1} \subset U$ and thus $(xV)^{-1} \subset x^{-1}U$. This implies that $x \to x^{-1}$ is continuous as well and thus ϕ is a base of the neighbourhood filter at identity of a group topology on G.

Now this topology is T_1 as if $x \neq y \in G$ then there exists a $U \in \phi$ such that $e \in U$ but $x^{-1}y \notin U$ (as if this was not true then $\cap\phi \neq \{e\}$ but then $x \in xU \in \mathcal{N}_x$ and $y \notin xU$. However, if $U \in \phi \subset \mathcal{N}_e$ then there exists $V, W \in \phi$ such that $W^2 \subset U$, $V^{-1} \subset W$. But if $x \in \overline{W}$ then $xV \cap W \neq \emptyset$ which implies that $x = w_1 v_1^{-1} \subset WV^{-1} \subset W^2 \subset U$ i.e. $\overline{W} \subset U$ but this implies that the topology is

in fact T_3 and thus Hausdorff.

Now if ϕ is a base of the neighbourhood filter at identity of a Hausdorff group topology on G then both $\circ : G^2 \to G$ and $x \to x^{-1}$ are continuous at identity and thus $e^2 = e$ implies (3), $e^{-1} = e$ implies (4) and $geg^{-1} = e$ implies (5). (1) is implied by the fact that the topology is Hausdorff and (2) is implied by the fact that ϕ is a base for a filter.

Now by considering the subsets $\tilde{U} = U \cap U^{-1}$ if necessary, we can assume that this neighbourhood basis is in fact symmetric. Now suppose the resulting topology generated by ϕ is in fact first countable. We can then find a countable base $\tilde{\phi} = \{U_i : i \in \mathbb{N}\}$ of the neighbourhood system of identity which satisfies conditions (1) to (5). Define V_i as follows : $V_1 = U_1$ and assuming V_i is defined for $i \leq k$, define V_{k+1} to be the first U_i such that $U_i^3 \subset V_k \cap U_1 \cap ... \cap U_k$ which is well-defined since $e^3 = e$. This implies that $V_{k+1} = e^2 V_{k+1} \subset V_{k+1}^3 \subset V_k$ and furthermore that the V_i's form a base for the filter $\tilde{\phi}$. □

The above proof shows that any T_0 topological group is in fact T_3. Now if the group is T_3 at $e \in G$ then by the continuity of left shifts it is T_3 everywhere. So consider any open neighbourhood U of $e \in G$ and let V be some symmetric open neighbourhood such that $e \in V \subset V^2 \subset U$. If $x \in \bar{V}$ then $xV \cap V \neq \emptyset$. This implies that $xv_1 = v_2$ and so $x \in VV^{-1} \subset V^2 \subset U$ and thus $\bar{V} \subset U$ i.e. the group is T_3 at $e \in G$ and thus T_3 everywhere. This result can be extended to the following important theorem (whose proof is beyond the scope of this dissertation)

Theorem 2.3. *Any T_0 topological group is in fact a completely regular topological group*

Proof. See [HR79] Theorem 8.2 and Theorem 8.4 □

This result is quite surprising for a number of reasons, least of which the introduction of continuous real valued functions to a discourse which seems rather foreign to them. In fact this is a rather strong condition to require any topological space to

have and this raises the question about which groups (and more generally algebraic structures) admit non-trivial T_0 topologies. Topological groups have many other interesting and surprising characteristics. Any metrizable space is first countable (i.e. every neighbourhood has a countable neighbourhood basis). To see this consider the open balls $B(x, \frac{1}{n})$ for $n \in \mathbb{N}$, these form a basis for the neighbourhood system at $x \in X$ and thus the space is first countable. Surprisingly, the converse holds for topological groups as the following theorem of Birkhoff ([Bir36], our proof using part of [Hjo00]) shows.

Theorem 2.4. *Suppose G is a first countable T_0 topological group, then G is metrizable*

Proof. Now as G is first countable there exists an countably decreasing, symmetric basis for the neighbourhood system at identity $\mathcal{N}_e$, say $\mathcal{N}_e = \{\hat{V}_1, \hat{V}_2, ...\}$. For each $\hat{V}_i$ there exists a $V_i \in \mathcal{N}_e$ such that $V_{i+1}^4 \subset V_{i+1}^3 \subset V_i \subset \hat{V}_i$. Extend this system by defining $V_0 = G$. Define the function $\varrho : G \times G \to \mathbb{R}$ by $\varrho(x, y) = \inf\{2^{-k} : xy^{-1} \in V_k\}$ which is well defined as the set $xy^{-1} \in V_0 = G$ for all $x, y \in G$. Now as $xx^{-1} = e$ we have that $\varrho(x, x) = 0$ since $e \in V_k$ for all $k \in \mathbb{N}$. Since G is T_0, $x \neq y$ implies that $xy^{-1} \neq e$ and so $\varrho(x, y) > 0$. Also as the base was chosen to be symmetric we have that $\varrho(x, y) = \varrho(y, x)$. Since $V_{k+1}^3 \subset V_k$ we have the result that $V_h V_i V_j \subset V_k$ if $k > h, i, j$ and thus if $\varrho(x, y) < \epsilon$, $\varrho(y, w) < \epsilon$ and $\varrho(w, z) < \epsilon$ then $\varrho(x, z) < 2\epsilon$. Now define a function $\rho(x, y) = \inf\{\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) : u_i \in G,$ for $i = 0...n$ and $u_0 = x$ and $u_n = y\}$. This function ρ is both positive definite and symmetric since ϱ is both positive definite and symmetric. We wish to verify the triangle inequality, so let x, y and $z \in G$ then

$$\begin{aligned}
\rho(x, y) &\leq \inf\{\sum_{k=1}^{2n} \varrho(u_{k-1}, u_k) : (u_i)_{i=0...2n} \in G, , u_0 = x, u_n = z, u_{2n} = y\} \\
&\leq \inf\{\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) : (u_i)_{i=0...n} \in G, , u_0 = x, u_n = z\} \\
&\quad + \inf\{\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) : (u_i)_{i=0...n} \in G, , u_0 = z, u_n = y\} \\
&= \rho(x, z) + \rho(z, y)
\end{aligned}$$

This verifies the triangle inequality and thus $\rho : G \times G \to \mathbb{R}$ defines a metric on G. All that is left is showing that this metric induces the original topology on G. Since $\rho(x, y)$ is defined to be the greatest lower bound of the set $\{\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) : (u_i)_{i=0\ldots n} \in G, , u_0 = x, u_n = y\}$ we have that $\rho(x, y) \leq \varrho(x, y)$. The final part of the proof is to show that $\rho(x, y) \geq \frac{\varrho(x,y)}{2}$. So suppose that there exists $x, y \in G$ and a sequence $(u_i)_{i=0\ldots n}$ with $u_0 = x$ and $u_n = y$ such that $\sum_{k=1}^{n} \varrho(u_{k-1}, u_k) < \frac{\varrho(x,y)}{2}$ is a counterexample. Since $\mathbb{N}$ is well ordered, we can choose this sequence such that k is the shortest sequence length amongst all counterexamples. Now by the construction we know that if $\varrho(x, u_j) < \frac{\varrho(x,y)}{2}$ and $\varrho(u_j, y) < \frac{\varrho(x,y)}{2}$ for any $j \leq n$ then $\varrho(x, y) < \varrho(x, y)$ which is a contradiction. So either $\varrho(x, u_j) \geq \frac{\varrho(x,y)}{2}$ or $\varrho(u_j, y) \geq \frac{\varrho(x,y)}{2}$ for all $j \leq n$. In the first case take $i_{10} = 0$ and $i_{20} = j$ else take $i_{10} = j$ and $i_{20} = n$. Then $\varrho(u_{i_{10}}, u_{i_{20}}) = \frac{\varrho(x,y)}{2}$ and thus for any $i_{10} < l < i_{20}$ exactly one of $\varrho(x, u_{i_{10}})$ and $\varrho(u_{i_{20}}, y)$ is equal to $\frac{\varrho(x,y)}{2}$. We iterate this construction to build a sequence of i_{1m} and i_{2m} such that

- $\varrho(x, u_{i_{1m}}) < \frac{\varrho(x,y)}{2}$
- $\varrho(u_{i_{1m}}, u_{i_{2m}}) = \frac{\varrho(x,y)}{2}$
- $\varrho(u_{i_{2m}}, y) < \frac{\varrho(x,y)}{2}$

This contradicts the minimality of k and so for all $x, y \in G$ we know that $\rho(x, y) \geq \frac{\varrho(x,y)}{2}$. But then $\rho(x, y)$ generates the original topology and thus the topology is metrizable. □

So we have a strong version of metrizability on a topological group. However, not all topological groups are metrizable. To show this we require strong version of Kronecker's Approximation Theorem. A proof of this theorem is beyond the scope of this dissertation

Definition 2.5. *Let $x \in \mathbb{R}$ and $\epsilon > 0$ then $|x| < \epsilon \mod 1$ if and only if there exists a $p \in \mathbb{Z}$ such $|x - p| < \epsilon$*

So a number is small in this norm if it is close to an integer. Next we recall the definition of an linearly independent set of numbers over the set of rational

numbers.

Definition 2.6. *A set of numbers Λ is linearly independent over the set of rational numbers if and only if given any finite subset $\{v_i | i \in I\} \subset \Lambda$ and collection $\{q_i \in \mathbb{Q} : i \in I\}$ of rational numbers the condition $\sum_{i \in I} q_i v_i = 0$ implies that $q_i = 0$ for all $i \in I$*

For example, the set $\{\log(p) : p \text{ is prime}\}$ is linearly independent over the set of rational numbers. To see this consider any finite subset of primes $\{p_i\}_{i \leq n}$ and collection of rational numbers $\{\frac{q_i}{r_i}\}_{i \leq n}$. Let $d = \text{LCM}(r_1, r_2, ..., r_n) \neq 0$ and note that $d_i = d\frac{q_i}{r_i} \in \mathbb{Z}$. Now if $\sum_{i \in I} \frac{q_i}{r_i} \log(p_i) = 0$ then $\sum_{i \in I} d_i \log(p_i) = 0$ but this implies that $p_1^{d_1} p_2^{d_2} ... p_n^{d_n} = 1$ which by the Fundamental Theorem of Arithmetic implies that $d_i = 0$ for all $1 \leq i \leq n$. But then $\frac{q_i}{r_i} = 0$ for all $1 \leq i \leq n$ and so $\{\log(p) : p \text{ is prime}\}$ is linearly independent.

Theorem 2.7 (Kronecker). *Let θ_i be a finite family of real numbers such that the set $\{\theta_i\} \cup 1$ is linearly independent over the rational numbers. For any finite collection $x_i \in \mathbb{R}$ and $\epsilon > 0$ there exists a $k \in \mathbb{Z}$ such that $|k\theta_i - x_i| < \epsilon \mod 1$ for all $i = 1...n$*

Proof. See [Nik98] Section 9.3 page 273 □

We now construct a non-metrizable topology on the integers. Extend the one point set $\{1\}$ to be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ i.e. a linear basis for $\mathbb{R}$ as seen as a vector space over the field $\mathbb{Q}$. Denote this basis by H. Let $\alpha_i \in H \setminus \{1\}$ for $i = 1...m$ and let

$$U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon) = \{n \in Z : |\alpha_k n - p_k| < \epsilon\}$$

Lemma 2.8. *Let $(\alpha_i) \subset H \setminus \{1\}$ be a finite subset and let $\epsilon > 0$ then*

1. *if $(\alpha_{i_j}) \subset \{\alpha_i\}$ then $U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon) \subset U(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n}, \epsilon)$*
2. *if $0 < \epsilon_1 < \epsilon$ then $U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon_1) \subset U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$*
3. *Suppose that there exists $p_{1i}, p_{2i} \in \mathbb{Z}$ such that $|n\alpha_i - p_{1i}| < \epsilon < \frac{1}{2}$ and $|n\alpha_i - p_{2i}| < \epsilon < \frac{1}{2}$ then $p_{1i} = p_{2i}$*

Proof. 1. Suppose $n \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ then trivially $n \in U(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n}, \epsilon)$ since $(\alpha_i) \subset H \setminus \{1\}$

2. Let $0 < \epsilon_1 < \epsilon$ and $n \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon_1)$ then $|\alpha_i n - p_i| < \epsilon_1 < \epsilon$ and thus $n \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$

3. By the triangle inequality we know that $|p_{1i} - p_{2i}| \leq |n\alpha_i - p_{1i}| + |n\alpha_i - p_{2i}| < \frac{1}{2} + \frac{1}{2} \leq 1$. This implies that $p_{1i} = p_{2i}$

□

We show next that none of these sets are degenerate i.e. that there does not exists a finite subset $\{\alpha_{ij}\} \subset H \setminus \{e\}$ and a collection $\epsilon_j > 0$ such

$$\bigcap_{j=1..n} U(\alpha_{1j}, \alpha_{2j}, ..., \alpha_{m_j j}, \epsilon_j) = \{e\}$$

Lemma 2.9. *$U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ is an infinite set for all $\alpha_i \in H \setminus \{1\}$ for $i = 1...m$ and $\epsilon > 0$.*

Proof. Let $\epsilon_1 = \frac{1}{2}\min(\epsilon, \frac{1}{2})$. By Kronecker's theorem, we know that there exists a $k_1 \in \mathbb{Z}$ such that $|k_1\alpha_i - \frac{\epsilon_1}{2}| < \frac{\epsilon_1}{2} \mod 1$. So there exists a sequence $(p_{1i}) \in \mathbb{Z}$ such that

$$-\epsilon_1 < 0 < k_1\alpha_i - p_i < \epsilon_1 < \epsilon$$

This implies that $k_1 \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$. We proceed by induction, so suppose we have constructed a sequence of $(\epsilon_j)_{j \leq n}$ and $(k_j)_{j \leq n}$ such that $\epsilon_{j+1} < \epsilon_j < \frac{1}{4}$ and $k_i = k_j$ implies that $i = j$. Define $\epsilon_{n+1} = \frac{1}{2}\min(\epsilon_n, |k_n\alpha_i - p_{n,i}|)$. By Kronecker's theorem there exists a $k_{n+1} \in \mathbb{Z}$ such that $|k_{n+1}\alpha_i - \frac{\epsilon_{n+1}}{2}| < \frac{\epsilon_{n+1}}{2} \mod 1$ which as previously shown implies the existence of a sequence $(p_{n+1,i}) \in \mathbb{Z}$ such that $k_{n+1} \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$. Now suppose that $k_{n+1} = k_j$ for some $j \leq n$. By the triangle inequality and the fact that $\epsilon_j < \frac{1}{4}$ we find that $p_{j,i} = p_{n+1,i}$ for all $i = 1...m$ However, this then implies that $|k_j\alpha_i - p_{n+1,i}| < |k_j\alpha_i - p_{n+1,i}|$ which is a contradiction. Thus the infinite set $\{k_i | i \in \mathbb{N}\} \subset U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ which implies that $U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ must be an infinite set. □

This family of sets generate a rather interesting topology on the integers. To show this we follow the general strategy embedded in Theorem 2.2.

Lemma 2.10. *The family of sets* $\mathcal{F} = \{U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon) : \epsilon > 0, \alpha_i \in H \setminus \{1\}\}$ *form a basis for a the neighbourhood system of identity of a non-discrete Hausdorff group topology* $\mathcal{T}_Z$ *on* $\mathbb{Z}$

Proof. Let $\alpha_k \in H \setminus \{1\}$ for $k = 1...n$ and let $x_k = 0$ for all $k = 1...n$. Let $\epsilon > 0$ be an arbitrary real number. By Kronecker's theorem, there exists a $n \in \mathbb{Z}$ such that $|\alpha_k n| < \epsilon \mod 1$ for all $k = 1...n$. This implies that $|\alpha_k n - p_k| < \epsilon$ for some $p_k \in \mathbb{Z}$. In fact, $n = 0$ satisfies this claim and thus $0 \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$. i.e. $0 \in U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ for all finite subsets $\{\alpha_k\} \subset H \setminus \{1\}$ and *epsilon* > 0. Now by Theorem 2.2 we have that the family of subsets $\mathcal{F} = \{U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon) : \alpha_i \in H \setminus \{1\}, \epsilon > 0\}$ forms a basis for the neighbourhood system of identity for a group topology $\mathcal{T}_Z$ on $\mathbb{Z}$. Most of the required properties follow trivially from the properties of $\mathbb{Z}$. The fact that $\forall U \in \mathcal{F}$ and $g \in \mathbb{Z}$ there exists $V \in \mathcal{F}$ such that $-g + V + g \subset U$ follows from the commutativity of addition in $\mathbb{Z}$. $-U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon) \subset U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$ follows from the properties of the absolute value function while

$$U(\alpha_1, \alpha_2, ..., \alpha_m, \frac{\epsilon}{2}) + U(\alpha_1, \alpha_2, ..., \alpha_m, \frac{\epsilon}{2}) \subset U(\alpha_1, \alpha_2, ..., \alpha_m, \epsilon)$$

follows from the triangle identity. All that we really need to show is that $\cap\phi = \{0\}$. Let $\alpha \in H \setminus \{1\}$ and consider the sequence of elements of $(U_j) \subset \phi$ where $U_j = U(\alpha, 2^{-j})$. We claim that $\cap U_j = \{0\}$ and thus $\cap\phi = \{0\}$. So suppose that $0 \neq x \in U_j$ for all $j \in \mathbb{N}$, which implies that there exist a sequence of integers $p_j \in \mathbb{Z}$ such that $|\alpha x - p_j| < 2^{-j}$. But $2^{-j} \to 0$ as $j \to \infty$,where the limit is taken in the normal sense and not the new topology on $\mathbb{Z}$. This implies that $\alpha x \in \mathbb{Z}$ which contradicts the linear independence of the original Hamel basis H. So $\cap U_j = \{0\}$ and thus $\cap\phi = \{0\}$. By lemma 2.9 we know that $\{e\}$ is not open and thus the generated topology is not discrete. □

Next we show that there are many elements in $\mathcal{F}$, in fact $\mathcal{F}$ is not countable.

Lemma 2.11. *Let $U = \{U_i : i \in \mathbb{N}\} \subset \mathcal{F}$ be a countable collection of elements of $\mathcal{F}$. Then there exists a $V \in \mathcal{F}$ such that $U_i \not\subset V$ for all $i \in \mathbb{N}$*

Proof. Fix an $0 < \epsilon < \frac{1}{4}$. Every $U_i = U(\alpha_{1i}, \alpha_{2i}, ..., \alpha_{m_i i}, \epsilon_i)$ for some $\alpha_{ki} \in H \setminus \{1\}$. Without loss of generality we may assume that $\epsilon_i < \epsilon < \frac{1}{4}$. Now since H is uncountable and $W = \bigcup \alpha_{ki}$ is countable we know that $H \setminus W$ is uncountable. So let $1 \neq \beta \in H \setminus W$. Now by Kronecker's theorem there exists a $n \in \mathbb{N}$ such that $|n\alpha_{ki} - \frac{\epsilon_j}{2}| < \frac{\epsilon_j}{2} \mod 1$ and $|n\beta - (\frac{\epsilon_j}{2} + \epsilon)| < \frac{\epsilon_j}{2} \mod 1$. But then there exists $p_{ki} \in \mathbb{Z}$ and a $q \in Z$ such that $|n\alpha_{ki} - p_{ki}| < \epsilon_j$ and $\epsilon < n\beta - q < \epsilon_j + \epsilon < \frac{1}{2}$. The first part implies that $n \in U_i$, so suppose there exists a $\hat{q} \in \mathbb{Z}$ such that $|n\beta - \hat{q}| < \epsilon < \frac{1}{4}$. But then by lemma 2.8 we know that $\hat{q} = q$ and thus $|n\beta - \hat{q}| > \epsilon$ which is a contradiction. Thus $n \notin U(\beta, \epsilon)$ and thus $U_i \not\subset U(\beta, \epsilon)$ for any U_i in the countable collection. □

Since no countable subset of $\mathcal{F}$ exhausts $\mathcal{F}$ we immediately see that $\mathcal{F}$ is uncountable. Recall that every metric space is first countable.

Theorem 2.12. *The cardinality of any open basis of identity of $\mathcal{T}_Z$ is $2^{\aleph_0}$ and thus $\mathcal{T}_Z$ is not metrizable.*

Proof. Let $\mathcal{B}$ be a basis for the neighbourhood system of identity of $\mathcal{T}_Z$. Trivially $|\mathcal{B}| \leq 2_0^{\aleph}$. Suppose that $|\mathcal{B}| = \aleph_0$. Enumerate $\mathcal{B} = \{B_i | i \in \mathbb{N}\}$, then for all B_i there exists a finite sequence $(\alpha_{ij})_{j=1...m_i}$ and an $\epsilon_i > 0$ such that $U_i = U(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{im_i}, \epsilon_i) \subset B_i$. But then the set $\{U(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{im_i}, \epsilon_i) | i \in \mathbb{N}\}$ is a countable subset of $\mathcal{F}$ and thus there exists a $V \in \mathcal{F}$ such that $U_i \not\subset V$ for all $i \in \mathbb{N}$. But then $B_i \not\subset V$ for all $i \in \mathbb{N}$ and so $\mathcal{B}$ cannot be a basis for the neighbourhood system of identity of $\mathcal{T}_Z$. Now any metric space is first countable as the open balls $B(e, \frac{1}{n})$ form a basis for the neighbourhood system of identity and thus $\mathcal{T}_Z$ is not metrizable □

2.2 T-Filters on arbitrary groups

We wish to be able to construct certain classes of general topologies on a given group. At first this may seem like a hopeless task, as marrying both the algebraic and topological requirements of a topological group is a very large undertaking. Given a set and a topology on that set it is not immediately clear how one could define a group structure on that set so that the resulting entity is in fact a topological group. Similarly it is not clear how one can find a topology for a given infinite group so that it is not a trivial topological group. Any given topology would have to be at least completely regular by Theorem 2.3 and that is if we only want one of the weakest separation axioms to hold. We could hope that a theorem like this would hold for some nice class of topological groups.

Theorem 2.13. *Let X be a set and $a \in X$. Let (x_n) be a non-trivial sequence such that $x_n \neq a$ for all $n \in \mathbb{N}$. Then there exists a Hausdorff topology on X such that $x_n \to a$*

Proof. Define $A_m = \{x_n : n > m\}$ Define a topology on X as follows: For all $x \in X \setminus \{a\}$ define $\mathcal{N}_x = \{U \subset X : x \in U\}$ and define $\mathcal{N}_a = \{U \subset X : A_m \subset U$, for some $m \in \mathbb{N}\}$. Now let $U \in \mathcal{N}_x$ for some $x \in X \setminus \{a\}$ then $U \cap (X \setminus \{a\}) \in \mathcal{N}_x$ and $U \cap (X \setminus \{a\}) \subset U$ and for all $y \in U \cap (X \setminus \{a\})$ we have that $U \cap (X \setminus \{a\}) \in \mathcal{N}_y$. Now if $U \in \mathcal{N}_a$ then $A_m \subset U$ for some $m \in \mathbb{N}$ but then $A_m \in \mathcal{N}_y$ for all $y \in A_m$ and thus we have defined a topology on X such that $x_n \to a$. Obviously, this topology is Hausdorff. □

The situation is not as trivial for the case when X is in fact a group and we wish for the resulting topology to be a group topology. In fact there exists a group G and a non-trivial sequence (x_n) which does not converge to the identity for any non-discrete Hausdorff topology. Consider the p-quasicyclic group

$$\mathbb{Z}(p^\infty) = \{e^{\frac{2ni\pi}{p^m}} : n \in Z, m \in \mathbb{Z}^+\}$$

which is a subgroup of $\mathbb{T}$ and consider the sequence $x_n = e^{2\pi i\left(\frac{1}{p^n} - \frac{1}{p}\right)}$. Suppose that $x_n \to 1$ i.e. $e^{\frac{2\pi i}{p^n}} \to e^{2\pi i}$. Then $x_{n+1}^p = e^{2\pi i\left(\frac{1}{p^n} - 1\right)} = e^{-2\pi i} e^{\frac{2\pi i}{p^n}} \to 1$ and thus

$\frac{x_{n+1}^p}{x_n} \to 1$. But $\frac{x_{n+1}^p}{x_n} = e^{2\pi i\left(\frac{1}{p}-1\right)} = x_1$ and so x_1 is in every neighbourhood of the identity which contradicts the fact that the topology is Hausdorff. The discrete topology is always available but it is only useful or interesting in certain cases. We need to start from a more general position in an attempt to answer this question. That starting point is that topological groups are homogeneous spaces and this suggests another method of attack. Instead of trying to define an entire topology in one go we could instead focus on one point and build something there, allowing the homogeneity of the space to fill in the gaps. For starters that something has to be more general then a sequence.

Definition 2.14. *A filter $\phi \subset 2^G$ is said to converge to a point $x \in G$ i.e. $\phi \to x$ if and only if the neighbourhood filter at that point $x \in \mathcal{N}_x \subset \phi$*

The notion of a filter converging to a point is an extension of the standard notion of a sequence converging to a point but is better behaved topologically. Many results for convergent sequences extend to convergent filters, for example filter limits in a Hausdorff space are unique and a space is compact if and only if every ultrafilter converges on that space. Now it is very easy to define a Hausdorff topology on that set such that a given sequence converges to a given point in that topology.

Definition 2.15. *A filter $\phi \subset 2^G$ is called a* T-filter *if there exists a Hausdorff group topology on G for which $\phi \to e$. The strongest group topology for which $\phi \to e$ is denoted by $\mathcal{T}(G, \phi)$*

Let $\mathcal{N}_e$ denote the neighbourhood filter of identity of a group topology on a topological group G then it is the case that $\mathcal{T}(G, \mathcal{N}_e)$ is the original topology on G.

Definition 2.16. *Let $\mathcal{F}$ be a filter on an arbitrary group G. $\mathcal{F}$ is called* multiplicative *if and only if for all $U \in \mathcal{F}$ there exists a $V \in \mathcal{F}$ such that $V \subset U$ and $V^2 \subset U$.*

For example, let $\mathcal{F} = \langle \{(n^2)_{n \geq k} : k \in \mathbb{N}\} \rangle$ be the filter generated by the sequence of square natural numbers. Now if $U \in \mathcal{F}$ then $V = \{k^2, (k+1)^2, ...\} \subset U$ for some $k \in \mathbb{N}$. However the set $V^2 = \{(mn)^2 : m, n \geq k\} \subset U$ and thus $\mathcal{F}$ is a trivial example of a multiplicative filter.

Definition 2.17. *Let G be a group and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of G. Let S_n denote the n-element permutation group. Define*

$$SP_{i\leq n}F_i = \bigcup_{\sigma\in S_{n+1}} F_{\sigma(0)}F_{\sigma(1)}...F_{\sigma(n)}$$

and

$$SP_{i\in\mathbb{N}}F_i = \bigcup_{n\in\mathbb{N}} (SP_{i\leq n}F_i)$$

The introduction of the permutation group is necessary as we are not guaranteed that our group is Abelian. We wish to construct the maximal multiplicative filter contained in a given filter $\mathcal{F} \subset 2^G$.

Lemma 2.18. *Let $\mathcal{F}$ be a filter on a group G, then there exists a maximal multiplicative filter $\mathcal{G}$ such that $\mathcal{G} \subset \mathcal{F}$. The family of subsets*

$$SP(\mathcal{F}) = \{SP_{n\in\mathbb{N}}F_n : F_n \in \mathcal{F}, n \in \mathbb{N}\}$$

forms a basis for this multiplicative filter.

Proof. We need to show that $SP(\mathcal{F})$ generates a multiplicative filter contained in $\mathcal{F}$. So let F_n and G_n be two sequences of elements from $\mathcal{F}$. Then as $\mathcal{F}$ is a filter we know that $F_n \cap G_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. Also it is easy to see that

$$SP_{n\in\mathbb{N}}(F_n \cap G_n) \subset SP_{n\in\mathbb{N}}F_n \cap SP_{n\in\mathbb{N}}G_n$$

and thus $SP(\mathcal{F})$ is a basis for some filter on G, say $\mathcal{G}$. So let $U \in \phi$ and let $(F_n) \subset \mathcal{F}$ be a sequence of subsets such that $SP_{n\in\mathbb{N}}F_n \subset U$ and $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. We can construct the required sequence through induction. Let G_n be a sequence in $\mathcal{F}$ such that $SP_{n\in\mathbb{N}}G_n \subset U$. Define $F_0 = G_0$ and $F_{n+1} = G_n \cap F_n$ then F_n is a sequence with the required properties. Now define $V_0 = SP_{n\in\mathbb{N}}F_{2n}$ and $V_1 = SP_{n\in\mathbb{N}}F_{2n+1}$. Now $V_i \in \phi$ and since $SP_{n\in\mathbb{N}}F_n$ is defined using all the permutations of the permutation group S_{n+1} we have that $V_0V_1 \subset U$. Let $V = V_0 \cap V_1$ then

$$V^2 \subset V_0V_1 \subset U$$

i.e. $\mathcal{G}$ is a multiplicative filter. Finally, since

$$\mathcal{F} \ni F_0 \in SP_{i\leq 0}F_i \subset SP_{n\in\mathbb{N}}F_i \subset U$$

we have that $U \in \mathcal{F}$ for all $U \in \mathcal{G}$. □

We can now prove the existence of a maximal multiplicative filter contained in any filter $\mathcal{F} \subset 2^G$

Lemma 2.19. *Let $\mathcal{F}$ be a filter on a group G, then there exists a maximal multiplicative filter ϕ such that $\phi \subset \mathcal{F}$*

Proof. Let $\mathcal{Z} = \{\mathcal{G} \subset \mathcal{F} : \mathcal{G}$ is a multiplicative filter $\}$. By Lemma 2.18, $\mathcal{Z} \neq \emptyset$. Let $(\mathcal{G}_\alpha)_{\alpha\in\Lambda}$ be a chain in $\mathcal{Z}$. We claim that $\cup\mathcal{G}_\alpha$ is a multiplicative filter. Now it is easy to see that $\cup\mathcal{G}_\alpha$ must be a filter on G. For suppose $U, V \in \cup\mathcal{G}_\alpha$ then there exists $\alpha_1, \alpha_2 \in \Lambda$ such that $U \in \mathcal{G}_{\alpha_1}$ and $V \in \mathcal{G}_{\alpha_2}$. Let $\alpha_3 = \max(\alpha_1, \alpha_2)$ then $U, V \in \mathcal{G}_{\alpha_3}$ and so

$$U \cap V \in \mathcal{G}_{\alpha_3} \subset \cup\mathcal{G}_\alpha$$

Also if $W \subset G$ and $U \subset W$ with $U \in \cup\mathcal{G}_\alpha$ then there must exist an $\alpha_1 \in \Lambda$ such that $U \in \mathcal{G}_{\alpha_1}$ which implies that

$$W \in \mathcal{G}_{\alpha_1} \subset \cup\mathcal{G}_\alpha$$

Thus we have shown that $\cup\mathcal{G}_\alpha$ is a filter. Also if $U \in \cup\mathcal{G}_\alpha$ then $U \in \mathcal{G}_{\alpha 1}$ for some $\alpha_1 \in \Lambda$. But as $\mathcal{G}_{\alpha_1}$ is a multiplicative filter, there exists a $V \in \mathcal{G}_{\alpha_1}$ such that $V^2 \subset U$. But

$$V \in \mathcal{G}_{\alpha_1} \subset \cup\mathcal{G}_\alpha$$

which implies that $\cup\mathcal{G}_\alpha$ is a multiplicative filter on G which is trivially contained in $\mathcal{F}$. Thus the chain $(\mathcal{G}_\alpha)_{\alpha\in\Lambda}$ has an upper bound and thus by Zorn's lemma, $\mathcal{Z}$ contains a maximal element. Thus there exists a maximal multiplicative filter $\phi \subset \mathcal{F}$ □

We actually have constructed something stronger here, the filter constructed in Lemma 2.18 is in fact the maximal multiplicative filter contained in a given filter $\mathcal{F}$. This is the final step of our construction.

Theorem 2.20. *Let $\mathcal{F}$ be a filter on a group G, then the family of subsets*

$$SP(\mathcal{F}) = \{SP_{n\in\mathbb{N}}F_n : F_n \in \mathcal{F}, n \in \mathbb{N}\}$$

forms a basis for the maximal multiplicative filter contained in $\mathcal{F}$.

Proof. Let $\mathcal{G}$ denote the filter generated by $SP(\mathcal{F})$. By Lemma 2.18 we know that this filter is in fact a multiplicative filter contained in $\mathcal{F}$. Also, by Lemma 2.19, the maximal multiplicative filter ϕ contained in $\mathcal{F}$ exists. Let $U \in \phi$. We now construct a sequence of elements F_n of ϕ such that $SP_{n\in\mathbb{N}}F_n \subset U$. This is sufficient to prove the theorem as then $U \in \mathcal{G}$ for all $U \in \phi$ and thus $\mathcal{G} = \phi$ since ϕ is the maximal multiplicative ultrafilter contained in $\mathcal{F}$. We proceed by induction. Let $U_{-1} = U = F_{-1}$ and let $U_0 \in \phi$ such that $U_0^2 \subset U_{-1}$. Let $V_0 \in \phi$ such that $V_0^2 \subset U_0$. Define $F_0 = V_0 \cap U_0$ then

$$F_0^2 \subset U_0^2 \subset U_{-1} = F_{-1}$$

and

$$F_0^3 \subset V_0^2 \cdot U_0 \subset U_0^2 \subset U_{-1} = F_{-1}$$

Proceed by induction to define a sequence F_n such that $F_n^2 \subset F_{n-1}$ and $F_n^3 \subset F_{n-1}$. Now let $(i_m)_{m=0...n}$ be a sequence in $\mathbb{N}$ and let $i_j = \min(i_0, i_1, ..., i_n)$. We now show that

$$F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_n} \subset F_{i^*-1}$$

We proceed by induction. If $n = 0$ then by the construction of F_{i_0} we have that $F_{i_0} \subset F_{i_0-1}$. So suppose that for all $k = 0...n-1$ we have that

$$F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_k} \subset F_{i^*-1}$$

where $i_j = \min(i_0, i_1, ..., i_k)$. Now consider $i_k = \min(i_0, i_1, ..., i_n)$. If $k \neq 0$ and $k \neq n$ then we can compute $i_p = \min(i_0, i_1, ..., i_{k-1})$ and $i_q = \min(i_{k+1}, i_{k+2}, ..., i_n)$ and we know that $i_k < \min(i_p, i_q)$. So we have that

$$\left(F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_{k-1}}\right) \cdot F_{i_k} \cdot \left(F_{i_{k+1}} \cdot F_{i_{k+2}} \cdot ... \cdot F_{i_n}\right) \subset F_{i_p} \cdot F_{i_k} \cdot F_{i_q} \subset F_{i_k}^3 \subset F_{i_k-1}$$

If $k = 0$ then we have that $i_0 = i_k < i_p = \min(i_1, i_2, ..., i_n)$ and thus

$$F_{i_0} \cdot (F_{i_1} \cdot ... \cdot F_{i_n}) \subset F_{i_k} \cdot F_{i_p} \subset F_{i_k}^2 \subset F_{i_k - 1}$$

similarly for $k = n$ and $i_n = i_k < i_p = \min(i_0, i_1, ..., i_{n-1})$ we have that

$$\left(F_{i_0} \cdot F_{i_1} \cdot ... F_{i_{n-1}}\right) \cdot F_{i_n} \subset F_{i_p} \cdot F_{i_k} \subset F_{i_k}^2 \subset F_{i_k - 1}$$

We have shown that for any $\sigma \in S_{n+1}$ we have that

$$F_{\sigma(0)} \cdot F_{\sigma(1)} \cdot ... \cdot F_{\sigma(n)} \subset F_{-1}$$

and thus that $SP_{i \leq n} F_i \subset F_{-1}$ and finally that $SP_{n \in \mathbb{N}} F_n \subset F_{-1} = U$. □

Thus we have shown that every filter on a group admits a maximal multiplicative filter contained in that filter. This maximal multiplicative filters are important for the characterization of T-filters on countable groups and thus have a special name.

Definition 2.21. *Let $\mathcal{F}$ be a filter on a group G, we call the maximal multiplicative filter contained in $\mathcal{F}$ the multiplicative hull of $\mathcal{F}$ and denote it by ψ*

Once again we wish to classify the T-filters on a given countable group. The lack of computability in a general countable group complicates the proceedings considerably. The classification of a general T-filter requires three separate parts.

Definition 2.22. *Let ϕ be a filter on a group G. Define $\hat{\phi} = \{F^* | F \in \phi\}$ where $F^* = F \cup F^{-1} \cup \{e\}$*

This first part of the construction is to extend the original filter to include the inverse information of the group in question.

Lemma 2.23. *Let ϕ be a filter on a group G then $\hat{\phi}$ is a basis for a filter on G*

Proof. Let $F, G \in \phi$ so that $F^*, G^* \in \hat{\phi}$. Also, since ϕ is a filter there exists a $H \in \phi$ such that $H \in F \cap G$. We claim that $H^* \in F^* \cap G^*$. But this is easy to see as

$$F^* \cap G^* = (F \cap G) \cup (F \cap G^{-1}) \cup (F^{-1} \cap G) \cup (F^{-1} \cap G^{-1}) \cup \{e\}$$

and $H \in F \cap G$ implies that $H^{-1} \in F^{-1} \cap G^{-1}$ □

To simplify notation, from now on ϕ^* will denote the filter generated by $\hat{\phi}$. We now introduce a trick (and the second part of the definition) which will be useful for another section of the discussion.

Definition 2.24. *Let G be a group and let ϕ be a filter on this group. Define*

$$\tilde{\phi} = \{\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x | f : G \to \phi\}$$

This second part encodes the multiplicative information of the original group as it pertains to the original filter, in particular it encodes the non commutative nature of the group multiplication.

Lemma 2.25. *Let ϕ be a filter on a group G then $\tilde{\phi}$ is a basis for a filter on G*

Proof. Let $f : G \to \phi$ and $g : G \to \phi$ so that

$$\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x \in \tilde{\phi}$$

and

$$\bigcup_{x \in G} x^{-1} \cdot g(x) \cdot x \in \tilde{\phi}$$

Define $h : G \to \phi$ by $h(x) = f(x) \cap g(x) \in \phi$ since ϕ is a filter. Then

$$\bigcup_{x \in G} x^{-1} \cdot h(x) \cdot x \subset (\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x) \cap (\bigcup_{x \in G} x^{-1} \cdot g(x) \cdot x)$$

□

Once again, to simplify notation we will denote the filter generated by the basis $\tilde{\phi}$ as ϕ^G. We now have enough basic definitions to state the final part of the construction for the classification theorem.

Lemma 2.26. *Let ϕ be a filter on a group G. Denote by ψ the filter generated by the filter basis $SP\left((\phi^*)^G\right)$. Then the following statements are true*

1. *ψ is a multiplicative filter*
2. *if $U \in \psi$ then $U^{-1} \in \psi$*

3. *if* $U \in \psi$ *and* $y \in G$ *then* $y^{-1} \cdot U \cdot y \in \psi$

Proof. 1. By Lemma 2.23 and Lemma 2.25 we know that $(\phi^*)^G$ is a filter on G and thus, by Lemma 2.20 we know that $SP\left((\phi^*)^G\right)$ generates a multiplicative filter

2. Let $U \in SP\left((\phi^*)^G\right)$, so there exists a sequence of elements of $(\phi^*)^G$, say $(F_i) \subset (\phi^*)^G$ such that

$$SP_{i\in\mathbb{N}}F_i = \bigcup_{n\in\mathbb{N}} (SP_{i\leq n}F_i) \subset U$$

recalling that

$$SP_{i\leq n}F_i = \bigcup_{\sigma\in S_{n+1}} F_{\sigma(0)}F_{\sigma(1)}...F_{\sigma(n)}$$

However, for all F_i there exists a sequence of functions $g_i : G \to \phi^*$ such that

$$G_i = \bigcup_{x\in G} x^{-1} \cdot g_i(x) \cdot x \subset F_i$$

However, for all $V \in \phi^*$ we know that $V^{-1} = V$ and thus $G_i \subset F_i^{-1}$ where $G_i \in (\phi^*)^G$. This implies that $SP_{i\leq n}G_i \subset SP_{i\leq n}F_i^{-1}$ and finally that $SP_{i\in\mathbb{N}}G_i \subset SP_{i\in\mathbb{N}}F_i^{-1} \subset U$ which was the claim we needed to show.

3. Similarly as before we can find a sequence G_i of elements of $(\phi^*)^G$ such that $SP_{i\in\mathbb{N}}G_i \subset U$. Define a sequence of functions $h_i : G \to \phi^*$ so that $h_i(x) = g_i(x \cdot y)$. Then

$$y^{-1}H_iy = \bigcup_{x\in G} y^{-1} \cdot x^{-1} \cdot h_i(x) \cdot x \cdot y = \bigcup_{x\in G} (xy)^{-1} \cdot g_i(xy) \cdot (xy) = G_i$$

and thus

$$y^{-1} \cdot (SP_{i\in\mathbb{N}}H_i) \cdot y \subset y^{-1} \cdot U \cdot y$$

□

We now have sufficiently many pieces to state and prove the promised classification theorem. Recall that a filter ϕ is generated by a set A if and only if $\cap\phi = A$

Theorem 2.27. *Let ϕ be a filter on a group G, then ϕ is a T-filter if and only if the multiplicative hull of ϕ is generated by $\{e\}$. Also, if ϕ is a T-filter from some Hausdorff group topology $\mathcal{T}$, then the multiplicative hull of ϕ is a neighbourhood filter of the identity in that topology.*

Proof. Firstly, let ϕ be a T-filter on a group G with Hausdorff topology $\mathcal{T}$ and let τ_e denote a neighbourhood filter of the identity in topology $\mathcal{T}$. Now by Theorem 2.2 we know that τ_e is a multiplicative filter and since $\phi \to e$ we know that $\tau_e \subset \phi$. But the topology $\mathcal{T}$ is Hausdorff and so $\cap\tau_e = \{e\}$. But since τ_e is multiplicative it must be contained in the multiplicative hull ψ of ϕ and thus $\cap\psi \subset \cap\tau_e = \{e\}$. Furthermore, $e \in U$ for all $U \in \psi$ and thus ψ is generated by the identity $e \in G$. Also, by Lemma 2.26 and Theorem 2.2 we know that ψ must also be a neighbourhood filter of the identity in the topology $\mathcal{T}$. Now let ϕ be a filter on a group G such that its multiplicative hull is generated by $\{e\}$. But once again by Lemma 2.26 and Theorem 2.2 we know that ψ must also be a neighbourhood filter of the identity for some Hausdorff topology $\mathcal{T}$. But $\psi \subset \phi$ and so $\phi \to e$ in that topology and thus ϕ is a T-filter □

So we have now classified all the T-filters on an arbitrary group G. It is difficult to gain much more traction in this setting though, especially if instead of the more general filters we would want to define our topologies using sequences. The problem lies with the amount of freedom available when choosing the functions which form the basis of ϕ^G. For suppose ϕ is a filter on a group of cardinality β then the cardinality of the set of all functions from G into ϕ is bounded by $2^{|\phi|}$ which can be a rather large cardinal number, larger then most spaces where sequences capture sufficient structure to be useful. However, the continuum is a realm where we still have lots of leverage and this suggest we look instead at countable groups. Another line of attack would be to get rid of the multiplicative hull part of the construction entirely. This suggests that we should also look at general Abelian groups, which we look at first.

2.3 T-filters and T-sequences on Abelian Groups

In this section all groups are assumed to be Abelian i.e. commutative and because of this fact all group operations will use the customary additive notation. Our first theorem is a complete characterisation of T-filters on an Abelian group. To do that we will extract all the extra construction pieces for the arbitrary group case which are not needed when the group is in fact Abelian.

Definition 2.28. *Let $\phi \subset 2^G$ and let (F_n) be a sequence in ϕ then $\sum_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} \sum_{i=1}^{n} F_i$ and denote the set of all such combinations by $\sum \phi$ i.e. $\sum \phi = \{\sum_{n\in\mathbb{N}} F_n : (F_n)$ is a sequence in $\phi\}$.*

The sum sets are useful for simplifying the previous framework we developed for T-filters on a general group. We start with a basic observation.

Lemma 2.29. *Let ϕ be a filter on a group G. Then $\sum \phi^* \subset SP\left((\phi^*)^G\right)$*

Proof. Let $U \in \sum \phi^*$ so there exists a sequence $(F_n) \in \phi*$ such that $U = \sum_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} \sum_{i=1}^{n} F_i$. Define a sequence of functions $f_i : G \to \phi^*$ by $f_i(x) = F_n$ then $SP_{i\in\mathbb{N}}(\cup_{x\in G}(-x + f_i(x) + x)) \subset U$ since the group is Abelian and thus the introduction of the permutations does not change the final product. This implies that $\sum \phi^* \subset SP\left((\phi^*)^G\right)$ □

So we could apply all the theorems of the previous section and obtain the obvious characterization of a T-filter of an Abelian group. In particular, we know that if $\cap \sum \phi^* = \{e\}$ then ϕ is a T-filter by Theorem 2.27. However, simpler proofs can be given for this case and the converse can also be proved. Some of the techniques will be useful later so we reprove that which is useful in this regard.

Lemma 2.30. *Let ϕ be a filter on a group G then*

1. *$e \in A$ for all $A \in \sum \phi^*$*
2. *if $A \in \sum \phi^*$ then $-A \in \sum \phi^*$*
3. *if $A, B \in \sum \phi^*$ then there exists a $C \in \sum \phi^*$ such that $C \subset A \cap B$*

4. *if* $C \in \sum \phi^*$ *then there exists* $A, B \in \sum \phi^*$ *such that* $A + B \subset C$

Proof. 1. Since $e \in U \cup -U \cup \{e\}$ for all $U \in \phi$ implies that $e \in \sum_{i=1}^{n} F_i$ for any sequence $(F_i) \subset \phi^*$ and thus $e \in \sum_{i \in \mathbb{N}} F_i$

2. Since $U \cup -U \cup \{e\} = -(U \cup -U \cup \{e\})$ for all $U \in \phi$ we know that $-\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} F_i$ for any sequence $(F_i) \subset \phi$ and thus $-\sum_{i \in \mathbb{N}} F_i \in \sum \phi^*$

3. Let $C \in \sum \phi^*$ so $C = \sum C_i$. Define $A = \sum C_{2i} \in \sum \phi^*$ and $B - \sum C_{2i+1} \in \sum \phi^*$ then obviously $A + B \subset C$

□

Lemma 2.31. *Let* G *be a topological group and* ϕ *a filter on* G *such that* $\phi \to G$ *then for all* $U \in \mathcal{N}_e$ *there exists* $V \in \sum \phi^*$ *such that* $V \subset U$

Proof. Now by Theorem 2.2 if $U \in \mathcal{N}_e$ then there exists $W_1 \in \mathcal{N}_e$ such that $W_1^* = W_1 \subset U$ and $W_2 \in \mathcal{N}_e$ such that $W_2 + W_2 \subset U$ and thus $W = W_1 \cap W_2 \in \mathcal{N}_e$ satisfies $W + W \subset U$ and $W^* = W$. Define $V_1 = W$. Now suppose that V_n has been defined for $n \leq k$ then there exists a $V_{k+1} \in \mathcal{N}_e$ such that $V_{k+1} + V_{k+1} \subset V_k$ and $V_{k+1}^* = V_{k+1}$. So by induction the sequence (V_n) is well defined. Now since $\phi \to e$ we have that $(V_n) \subset \mathcal{N}_e \subset \phi$ and so $\sum V_n^* = \sum V_n \in \sum \phi^*$ and $V = \sum V_n \subset U$ □

We come to the characterisation of a T-filter for an arbitrary Abelian group

Theorem 2.32. *A filter* ϕ *on a group* G *is a T-filter if and only if* $\bigcap \sum \phi^* = \{e\}$. *If* ϕ *is a T-filter then* $\sum \phi^*$ *forms a basis for the neighbourhood filter at identity for the topology* $\mathcal{T}(G, \phi)$

Proof. If ϕ is a T-filter then $\bigcap \sum \phi^* = \{e\}$ since the topology $\mathcal{T}(G, \phi)$ is T_1 and every neighbourhood of the identity is in $\sum \phi^*$ by Lemma 2.31. Now suppose ϕ is a filter on G such that $\bigcap \sum \phi^* = \{e\}$ then by Lemma 2.30 and Theorem 2.2 the family of subsets $\sum \phi^*$ forms a basis for the neighbourhood filter at identity which generates the largest Hausdorff topology on G for which $\phi \to e$ □

We return now to the question of sequences generating Hausdorff group topologies on a given group G. Let $(x_n) \subset G$ be a sequence and define $A_m = \{x_n : n > m\}$ and $A_m^* = \{x_n : n > m\} \cup \{-x_n : n > m\} \cup \{e\}$. Any sequence generates a filter $\phi = \{U \subset G : A_n \subset U \text{ for some } n \in \mathbb{N}\}$ and we use this fact to define T-sequences on a group.

Definition 2.33. *Let $(x_n) \subset G$ be a sequence on a group G then $(x_n) \subset G$ is a T-sequence if the corresponding filter $\phi = \{U \subset G : A_n \subset U$ is a T-filter on G. We denote the strongest group topology on G for which the filter $\phi \to e$ by $\mathcal{T}(G, (x_n))$*

Now Theorem 2.32 fully characterises T-sequences by there corresponding filters on G. However, there exists a more natural characterising for the T-sequence case that utilises the special structure of the filter ϕ.

Definition 2.34. *Let $(x_n) \subset G$ be a sequence on a group G then*

$$A(k, m) = \{\sum_{i=1}^{k} g_i : g_i \in A_m^*\}$$

Theorem 2.35. *A sequence $(x_n) \subset G$ is a T-sequence on G if and only if for every $g \in G \setminus \{e\}$ and $k \in \mathbb{N}$ there exists a $m \in \mathbb{N}$ such that $g \notin A^*(k, m)$*

Proof. First suppose that $(x_n) \subset G$ is a T-sequence, then by Theorem 2.32 we know that $\bigcap \sum \phi^* = \{e\}$. Now if $g \in G$ and $g \neq e$ then there exists a sequence $\left(A_{n_i}^*\right) \in \phi^*$ such that $g \notin \sum A_{n_i}^*$. Let $m = \max\{n_1, n_2, ..., n_k\}$ then $g \notin A(k, m) \subset \sum A_n^*$. Now suppose that $(x_n) \subset G$ is a sequence on G such that for all $g \in G \setminus \{e\}$ and $k \in \mathbb{N}$ there exists a $m \in \mathbb{N}$ such that $g \notin A^*(k, m)$. Now there exists $n_1 \in \mathbb{N}$ such that $g \notin A_{n_1}^*$. We proceed by induction, suppose we have chosen $n_1, n_2, ..., n_k$ such that $g \notin \sum_{j=1}^{k} A_{n_j}^*$ but for any $n_{k+1} \in \mathbb{N} \setminus \{n_1, n_2, ..., n_k\}$ we have that $g \in \sum_{j=1}^{k+1} A_{n_j}^*$. Now by passing to sub-sequences if necessary we can assume that there exists sequences $(g_{i,j}) \in A_{n_i}^*$, $i = 1...k+1$ and $j \in \mathbb{N}$ such that $g = \sum_{i=1}^{k+1} g_{i,j}$ and one of the two following cases hold

Case 1 If $(g_{i,j}) \subset (x_n)$ or $(g_{i,j}) \subset (-x_n)$ for all $i \leq k$ then there exists a $m \in \mathbb{N}$

such that $g \notin A(k+1,m)$. However we have that $g = \sum_{i=1}^{k+1} g_{i,j} \in A(k+1,m)$ for sufficiently large $j \in \mathbb{N}$ which is a contradiction.

Case 2 Otherwise there exists $i \leq k$ for which $g_{i,j} = b_i \neq \pm a_n$ for all $n \in \mathbb{N}$. Let $J = \{j_1, ..., j_t\}$ be the set of all indexes with this property. But then $e \neq g - \sum_{l=1}^{t} b_{j_l} = \sum_{s \notin J} g_{i,j}$ furthermore there exists $m \in \mathbb{N}$ such that $g - \sum_{l=1}^{t} b_{j_l} \notin A^*(k+1-t,m)$ which contradicts the definition of the b_j's.

□

Definition 2.36. *A sequence (x_n) is called* trivial *if there exists a $m \in \mathbb{N}$ and $b \in G$ such that $x_n = e$ for all $n \geq m$*

Now as noted for the p-quasicyclic group, not every sequence on a group is a T-sequence. Any trivial sequence is a T-sequence but the resulting topology is the discrete topology which is not very interesting in this regard. In fact it is easy to see that the discrete topology on a group G is generated by the trivial T-sequence $x_n = e$.

Lemma 2.37. *Let A be an infinite subset of a countable Abelian group G such that for every element $g \notin G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for every integer $k \in \mathbb{Z}$ then there exists a T-sequence $(a_n) \subset A$*

Proof. Let $G \setminus \{e\} = \{g_n : n \in \mathbb{N}\}$. We wish to define a T-sequence on G through the enumeration of $G \setminus \{e\}$ and induction. Define

$$X(a_0, a_1, ..., a_n) = \{x_0 a_0 + x_1 a_1 + ... + x_n a_n : x_i \in \{0, \pm 1, ..., \pm(i+1)\}, i \leq n\}$$

where $a_i \in G$. Note that

$$\sum_{n \in \mathbb{N}} A_n^* = \bigcup_{n \in \mathbb{N}} X(a_0, a_1, ..., a_n)$$

for any sequence $(a_i)_{i \in \mathbb{N}} \subset G$. Now since the set $\{x \in A : \pm x = g_0\}$ is finite there exists an a_0 such that $g_0 \notin X(a_0)$. Now suppose the sequence $a_0, a_1, ..., a_n$ has been defined so that $g_m \notin X(a_m, a_{m+1}, ..., a_n)$ for any $m \leq n$. Now the set

$$X(a_m, a_{m+1}, ..., a_n, x) = X(a_m, a_{m+1}, ..., a_n) + \{kx : k \in \{0, \pm 1, ..., \pm(n-m+1)\}$$

and is thus finite by the definition of $X(a_m, a_{m+1}, ..., a_n)$ and the finiteness of the sets $\{x \in A : kx = g_m\}$ and thus there exists an a_{n+1} such that $g_m \notin X(a_m, a_{m+1}, ..., a_{n+1})$ for $m \leq n+1$. This is sufficient to show that $g_m \notin \sum_{n \geq m} A_n^* = \bigcup_{n \in \mathbb{N}} X(a_m, a_{m+1}, ..., a_n$. But by Theorem 2.32 this implies that $(a_n)_{n \in \mathbb{N}}$ is a T-sequence in G □

The next technical lemma allows us to extend the previous lemma to general Abelian groups

Lemma 2.38. *Let G be an arbitrary Abelian group and $A \subset G$ an infinite set, then their exists a countable subgroup H of G such that $|H \cap A| = \aleph_0$*

Proof. If G contains an element $x \in G$ such that $x^n \neq e$ for all $n \in N$ (i.e. of infinite order) then we are done with $H = \{x^n : n \in \mathbb{Z}\}$. So suppose that for all $x \in G$ there exists an $n \in N$ such that $x_n = e$. Let $(a_n)_{n \in \mathbb{N}}$ be any non-repeating sequence of elements of $G \cap A$. Define $H = \{a_0^{p_0} a_1^{p_2} ... a_n^{p_n} : n \in N, p_i \in N\}$ then it is easy to see that H is a subgroup of G and that H, being the union of countably many finite sets, is countable. Also since $(a_n)_{n \in \mathbb{N}} \subset G \cap A$ we have that $|H \cap A| = \aleph_0$. □

We can now prove the result for general Abelian groups.

Theorem 2.39. *Let A be an infinite subset of an Abelian group G such that for every element $g \notin G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for every integer $k \in \mathbb{Z}$ then there exists a T-sequence $(x_n) \subset A$*

Proof. If G is countable then we are can use Lemma 2.37. So suppose G is uncountable, then by Lemma 2.38 there exists a countable subgroup $H \subset G$ such that $|A \cap H| = \aleph_0$. But then by Lemma 2.38 there exists a T-sequence $(a_n)_{n \in \mathbb{N}} \subset |A \cap H| \subset A$ for the group H. Now define a topology $\mathcal{T}$ with the base $\mathcal{T}(a_n) \cup H$ then $(a_n)_{n \in \mathbb{N}}$ is a T-sequence for the group G with topology $\mathcal{T}$. □

For example the set $A = (\mathbb{R} \backslash \mathbb{Q}) \subset \mathbb{R}$ obviously contains at least one T-sequence. In fact, A contains an uncountable number of $T-sequences$ since we can continue to

remove countable collections of elements from A and still have an uncountable set. The previous theorem admits a converse in the form that if A is an infinite subset of G and there exists a T-sequence taking values in A then the $\{x \in A : kx = g\}$ must be finite for and $k \in \mathbb{N}$ and $g \in G \setminus \{e\}$.

Theorem 2.40. *For every infinite group G the following statements are equivalent*

1. *For every infinite subset $A \subset G$ there exists a T-sequence $(x_n) \subset A$*
2. *For every $g \in G \setminus \{e\}$ and $k \in Z$ the set $\{x \in A : kx = g\}$ is finite*

Proof. (1) $\Rightarrow$ (2) Follows from Lemma 2.39

(2) $\Rightarrow$ (1) Suppose $\{x \in A : kx = g\}$ is infinite for some $k \in \mathbb{Z}$ and $g \in G \setminus \{e\}$ but A contains a T-sequence (x_n) then $g \in A(k, m)$ for every $m \in \mathbb{N}$ which implies $e \neq g \in \cap \sum \phi^*$ which contradicts the characterisation of T-sequences.

□

The previous theorem actually shows that there are $2^{\aleph_0}$ T-sequences on an infinite group and thus the example on $\mathbb{R}$ mentioned previously is not a special case and is in fact indicative of the general state of affairs, as the following corollary shows

Corollary 2.41. *Every infinite group admits at least one non-trivial T-sequence and thus admits at least $2^{\aleph_0}$ non-trivial T-sequences*

Proof. The second result follows from the fact that any subsequence of a T-sequence is obviously also a T-sequence. Now if $\{x \in A : kx = g\}$ is finite then $A \subset G$ contains a T-sequence. Let S_p denote all elements of G with order p for some prime $p \in \mathbb{N}$. Now if $|S_p| < \infty$ for all $p \in \mathbb{N}$ then if $g \in G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for $A = G$ and thus G contains a T-sequence. Now if $|S_p| = \infty$ for some prime $p \in \mathbb{N}$ then $\{x \in A : kx = g\}$ is finite for $A = S_p$ and thus G contains a T-sequence. □

The previous corollary also shows that every Abelian group is topologizable, as every Abelian group admits at least one non-trivial T-sequence (x_n) and the group

topology $\mathcal{T}(G,(x_n))$ is Hausdorff by the definition of a T-sequence and is non-discrete as (x_n) is not a trivial sequence. We have $2^{\aleph_0}$ non-trivial T-sequences to utilize though, but we are not guaranteed that each one generates a unique group topology. The next result, which we merely sketch, states that this is indeed the case.

Definition 2.42. *Two T-sequences (x_n) and (y_n) are called* strongly incomparable *if and only if there exist $U \in \mathcal{T}(G,(x_n))$ and $V \in \mathcal{T}(G,(y_n))$ such that $U \cap V = \{e\}$*

Two strongly incomparable T-sequences cannot generate the same topology, for suppose that they did generate the same topology then U and V would be two open subsets of that topology and thus $\{e\} = U \cap V$ would be an open subset of that topology and thus the topology is discrete. This contradicts the claim that these two sequences are in fact T-sequences.

Theorem 2.43. *For every infinite group there exists $2^{\aleph_0}$ strongly incomparable non-trivial T-sequences*

Proof. We call a non-trivial T-sequence $(a_n)_{n\in\mathbb{N}}$ alternative if for every partition of $\mathbb{N}$ into two infinite subsets N_1 and N_2 we have that $(a_n)_{n\in N_1}$ and $(a_n)_{n\in N_2}$ are strongly incomparable. We first prove that every T-sequence contains an alternative T-sequence and thus by corollary 2.41 that there exists an alternative T-sequence $(b_n)_{n\in\mathbb{N}}$ on any infinite group G. Recall that any family of subsets $\{W_\alpha \subset \mathbb{N} : \alpha < 2^{\aleph_0}\}$ is called almost disjoint if $|W_\alpha \cap W_\beta| < \infty$ for all $\alpha \neq \beta$. It can be shown that this non-disjoint family exists and thus the collection of sequences $\{(b_n)_{n\in W_\alpha} : \alpha < 2^{\aleph_0}\}$ is the required family of strongly incomparable T-sequences. □

Thus every Abelian group admits $2^{\aleph_0}$ strongly incomparable, non-discrete group topologies. This result can be considerably strengthened, for not only does an Abelian group admit a continuum of strongly incomparable T-sequences but these resulting group topologies can be linearly ordered. Once again we merely sketch the proof.

Theorem 2.44. *Let G be an infinite Abelian topological group, then there exists a family of strongly incomparable T-sequences with cardinality that of the continuum such that the resulting group topologies can be linearly ordered.*

Proof. Once again, we construct an alternative T-sequence $(b_n)_{b\in\mathbb{N}}$ in G. Since $\mathbb{Q}$ is countable, there exists a bijection $f : \mathbb{N} \to \mathbb{Q}$. Denote by

$$W_r = \{n \in \mathbb{N} : f(n) \geq r\}$$

Where $r \in \mathbb{R}$. These subsets of $\mathbb{N}$ are obviously non-empty. Now if $r, t \in \mathbb{R}$ and $r < t$ then the group topology generated by $(b_n)_{n\in W_t}$ is stronger then the one generated by $(b_n)_{n\in W_r}$ since $W_t \subset W_r$. □

It should now be evident that Abelian groups are strongly topologizable. This is due to the nice nature of Abelian groups, in general the class of Abelian groups is free of many of the pathologies a general class of groups could have which limit the ability to generate group topologies for the members of that class. This can be seen in how a whole level of complexity is lifted when trying to characterise a T-filter on an Abelian group as compared to a general group. The whole framework developed in this section can be focused onto one specific Abelian group as well, $\mathbb{Z}$ being an obvious first port of call. There are many interesting results in this field, mainly concerning what types of sequences in $\mathbb{Z}$ are in fact T-sequences. For example

Theorem 2.45. *Let $(a_n) \subset \mathbb{Z}$ be a sequence such that $\frac{a_{n+1}}{a_n} \to r$ where r is a transcendental number. Then (a_n) is a T-sequence*

Proof. See [PZ99] theorem 2.2.3 □

Interested readers are urged to consult [PZ99] and [ZP90]. The question of characterising T-filters and T-sequences in a general countable group will be considered next.

2.4 T-filters and T-sequences on Countable Groups

As mentioned previously, we restrict ourselves to countable groups so that the rather useful T-sequence construction still has a lot of power. It should be noted that since all group topologies are completely regular we know that each one of these topologies is zero-dimensional.

Definition 2.46. *Let G be a countable group with some enumeration $G = \{g_n | n \in \mathbb{N}\}$. Let ϕ be an arbitrary filter on G and and let $(F_i) \subset \phi^*$ be an arbitrary sequence. Define a sequence by*

$$U_k = (g_1^{-1} \cdot F_k \cdot g_1) \cup (g_2^{-1} \cdot F_{k+1} \cdot g_2) \cup ...$$

Then the subset $SP_{k\in\mathbb{N}}(U_k)$ is called a standard element of the family $SP((\phi^*)^G)$

We can use these standard elements to give a different version of the T-filter classification theorem which is better suited to the class of countable groups.

Definition 2.47. *Let G be a countable group and let ϕ be a filter on that group. Denote by $\prod_{i\in\mathbb{N}} \phi^*$ the set of all standard elements of $SP((\phi^*)^G)$*

Define a function $f_i : G \to \phi^*$ where $G = \{g_j\}$ is a countable group by $f_i(g_j) = F_{i+j-1}$. Then $\cup_{j\in\mathbb{N}} g_j^{-1} \cdot f_i(g_j) \cdot g_j \subset U_i$ and thus $SP_{k\in\mathbb{N}}(U_k) \in SP((\phi^*)^G)$. So once again a sufficient condition that ϕ is a T-filter is that $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$. We can show that this condition is necessary as well.

Theorem 2.48. *Let ϕ be a filter on a countable group G, then ϕ is a T-filter if and only if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$*

Proof. We have already shown the sufficiency of the claim, so all that is left is to show that the condition is necessary for ϕ to be a T-filter. So let $V \in SP((\phi^*)^G)$, so there exists a sequence $V_k \in (\phi^*)^G$ such that $SP_{k\in\mathbb{N}}(V_k) \subset V$. But there exists a function $f_k : G \to \phi^*$ such that $\cup_{i\in\mathbb{N}} g_i^{-1} \cdot f_k(g_i) \cdot g_i \subset V_k$. Define a sequence of subsets $U_k \in \phi^*$ by

$$U_k = \bigcap_{m+n=k+1} f_m(g_n)$$

which is well-defined as ϕ^* is a filter. Now obviously, $SP_{k\in\mathbb{N}}(U_k) \in \prod_{i\in\mathbb{N}} \phi^*$ so all we need to show is that $U_k \subset V_k$ as then $SP_{k\in\mathbb{N}}(U_k) \subset V$. Now

$$U_{k+i-1} = \bigcap_{m+n=k+i} f_m(g_n) \subset f_m(g_n)$$

For all $m, n \in \mathbb{N}$ such that $m + n = k + i$ but then $U_{k+i-1} \subset f_k(g_i)$ which is the start of the union cascade we needed to prove. Thus $SP((\phi^*)^G) \subset \prod_{i\in\mathbb{N}} \phi^*$ and so if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$ then $\cap SP((\phi^*)^G) = \{e\}$ and so by Theorem 2.27 we know that ϕ is a T-filter. □

There is a more algebraic version of the previous theorem, which will be important for the next section.

Definition 2.49. *Let X be a set and Y another set. Denote by $X[Y]$ the group of words generated by $A \cup B$.*

So for example if $X = \{a, b, c\}$ and $Y = \{t\}$ then $a^2 \cdot t \cdot b^{-1} \in X[Y]$. The set Y is called the set of variables while X is called the set of constants. We use these group words to give an algebraic characterisation of T-filters.

Definition 2.50. *Let $X = \{x_{ij} | i, j \in \mathbb{N}\}$ be a set of indeterminants and let G be a countable group with some enumeration $G = \{g_i : i \in \mathbb{N}\}$. Define $f : X^n \to G[X]$ by*

$$f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) = g_{j_1}^{-1} \cdot x_{i_1,j_1} \cdot g_{j_1} \cdot g_{j_2}^{-1} \cdot x_{i_2,j_2} \cdot g_{j_2} \cdot ...$$

and $\mathcal{F}_n = \{f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) | i_k + j_k \leq n \text{ and } i_j = i_k \Leftrightarrow j = k\}$

Lemma 2.51. *Let $G = \{g_n | n \in \mathbb{N}\}$ be a countable group and let ϕ be a filter on G. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in ϕ^* and let $SP_{k\in\mathbb{N}}U_k$ be the standard element of $SP((\phi^*)^G)$ generated by $(F_k)_{k\in\mathbb{N}}$. Then $g \in SP_{k\in\mathbb{N}}U_k$ if and only if there exists a $n \in \mathbb{N}$ and an element $f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) \in \mathcal{F}_n$ such that*

$$g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

Proof. If $g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$ then $g \in U_{\sigma(1)} \cdot U_{\sigma(2)} \cdot ... \cdot U_{\sigma(n)}$ for some $\sigma \in S_n$ and thus $g \in SP_{k\in\mathbb{N}}U_k$. So suppose that $g \in SP_{k\in\mathbb{N}}U_k$ and recall that

$U_k = (g_1^{-1} \cdot F_k \cdot g_1) \cup (g_2^{-1} \cdot F_{k+1} \cdot g_2) \cup ....$ Now since $g \in SP_{k\in\mathbb{N}} U_k$ there exists a $m \in \mathbb{N}$ and a permutation $\sigma \in S_m$ such that $g \in U_{\sigma(1)} \cdot U_{\sigma(2)} \cdot ... \cdot U_{\sigma(m)}$ but then there exists a sequence $(j_l)_{l=1...m} \subset \mathbb{N}$ such that

$$g \in g_1^{-1} F_{\sigma(1)+j_1} g_1 \cdot g_2^{-1} F_{\sigma(2)+j_2} g_2] \cdot ... \cdot g_m^{-1} F_{\sigma(m)+j_m} g_m$$

Define $i_l = \sigma(l)$ for $l = 1...m$ and $i_l = l$ for $l = m+1... \max(i_1 + j_1, i_2 + j_2, ..., i_m + j_m)$. Define $j_l = 0$ for $l = m+1... \max(i_1 + j_1, i_2 + j_2, ..., i_m + j_m)$ and let $f(x_{i_1 j_1}, x_{i_2 j_2}, ..., x_{i_n j_n}) \in \mathcal{F}$. Then

$$g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

□

We use this lemma to give the obvious algebraic version of Theorem 2.48. Recall that for any family of subsets $\{W_\alpha\}$ of a set X such that $z \in W_\alpha$ for all α. We know that $\cap W_\alpha = \{z\}$ if and only if for all $x \in X \setminus \{z\}$ there exists a W_α such that $x \notin W_\alpha$. This trivial observation forms the basis of the next theorem.

Theorem 2.52. *Let $G = \{g_n | n \in \mathbb{N}\}$ be a countable group and let ϕ be a filter on G. ϕ is a T-filter on G if and only if for all $x \in G \setminus \{e\}$ there exists a sequence of elements of ϕ^* such that*

$$g \notin f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

for any $n \in \mathbb{N}$ and $f(x_{i_1 j_1}, x_{i_2 j_2}, ..., x_{i_n j_n}) \in \mathcal{F}_n$

Proof. By Theorem 2.48 we know that ϕ is an ultrafilter if and only if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$ but this is true if and only if for all $x \in G \setminus \{e\}$ there exists a $U \in \prod_{i\in\mathbb{N}} \phi^*$ such that $x \notin U$ but then by Lemma 2.51 this is true if and only if for all $f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) \in \mathcal{F}_n$ and $n \in \mathbb{N}$,

$$g \notin f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

□

We now return to the question of T-sequences on an arbitrary countable group. Recall that a sequence is a T-sequence if the filter corresponding to that sequence is a T-filter.

Definition 2.53. *Let (x_n) be a sequence in a group G. Define $A_m^* = \{e, x_m^{\pm 1}, x_{m+1}^{\pm 1}, ...\}$*

These are the same star sets we encountered when we where studying Abelian groups, just written in multiplicative notation.

Theorem 2.54. *Let (x_n) be a sequence in $G = \{g_n | n \in \mathbb{N}\}$, a countable group. (x_n) is a T-sequence if and only if for any group word $f(x_1, x_2, ..., x_n)$ in $G[x_1, x_2, ..., x_n]$ such that $f(e, e, ..., e) = e$ and any $g \in G$ there exists a $m \in \mathbb{N}$ such that $g \notin f(A_m^*, A_m^*, ..., A_m^*)$*

Proof. The necessity of the statement follows directly from the fact that any topological group is a Hausdorff space. For let $f : G^n \to G$ be a group word in $G[x_1, x_2, ..., x_n]$ then f must be a continuous function. Let $e \neq g \in G$ then there exists an open neighbourhood of the identity V such that $g \notin V$ and $g^{-1} \notin V$. However, since $f(e, e, ..., e) = e$ we know that there must exist a symmetric open neighbourhood of the identity $U \subset G$ such that $f(U, U, ..., U) \subset V$ i.e. $g \notin f(U, U, ..., U)$ and $g^{-1} \notin f(U, U, ..., U)$. But $x_n \to e$ and thus there exists an integer $m \in \mathbb{Z}$ such that $x_n \in U$ for all $n \geq m$. But this implies that $g \notin A_m^* \subset U$ which gives the required result. The sufficiency follows from Theorem 2.52. □

For example, consider the group G generated by the two element set $\{a, b\}$ with relation $a \cdot b \cdot a = b$. Consider the sequence, $x_n = a^n$ for $n = 1, 2, ...$ We can show that this sequence is a T-sequence and thus this group is topologizable. To see this consider any group word f on $G[x_1, x_2, ..., x_n]$ such that $f(e, e, ..., e) = e$ then by iterating the group relation we can show that $f(x_1, x_2, ..., x_n) = \prod x_i^{\alpha_i}$ for some sequence of $\alpha_i \in \{-1, +1\}$. But then $f(A_m^*, A_m^*, ..., A_m^*) = (A_m^*)^n$ and thus if $x \in f(A_m^*, A_m^*, ..., A_m^*)$ then $x = a^k$ for some $k \in \mathbb{N}$. This is sufficient to show that Theorem 2.54 is applicable and thus the given sequence is in fact a T-sequence.

2.5 Markov's criteria

Definition 2.55. *A group G is called* topologizable *if there exists a topology $\mathcal{T}$ such that*

1. *$(G, \mathcal{T})$ is a topological group*
2. *$\mathcal{T}$ is a Hausdorff topology on G*
3. *$\mathcal{T}$ is non-discrete*

Markov [Mar50] posed the following question in this regard : Is every infinite group topologizable? It is a trivial fact of set theoretic topology that any group endowed with the discrete or trivial topology is a topological group and the Hausdorff and non-discrete conditions are meant to exclude these two possibilities. A topological space being Hausdorff is also a natural requirement as all but the weakest separation axioms result in a topological group being Hausdorff anyway. Furthermore, topological group theory has its genesis in analysis where Hausdorff separation conditions allow for the uniqueness of limits etc. We start off by linking the notion of a T-sequence with that of the topologiziablilty of a countable group.

Theorem 2.56. *A countable topological group $G = \{g_n\}$ is topologizable if and only if there exists a non-trivial T-sequence on G*

Proof. If there exists a non-trivial T-sequence on G then trivially G must be topologizable. So suppose that G admits a non-discrete Hausdorff group topology and define the following sequence of open subsets. Enumerate G such that $g_0 = e$. Let $U_0 = G$ and choose V_1 such that $V_1 = V_1^{-1}$, $V_1^2 \subset U_0$ and $g_1 \notin V_1$. Now since the function $x \to g_0^{-1} \cdot x \cdot g_0$ is continuous we can find an open neighbourhood $U_1 \subset V_1$ such that $g_0^{-1} \cdot U_1 \cdot g_0 \subset U_0$. We can iterate this construction and thus obtain a sequence U_n of open subsets of G such that $U_n = U_n^{-1}$, $U_{n+1}^2 \subset U_n$, $g_n \notin U_n$ and $g_i^{-1} \cdot U_{n+1} \cdot g_i \subset U_n$ for $i \leq n$. By Theorem 2.2, the subsets V_n generate the neighbourhood filter of identity for some group topology on G. Since this topology is first countable it is metrizable by Theorem 2.4, denote this metric

by μ. Define a sequence a_n by choosing $a_n \in \{g \in G | \mu(g, e) < \frac{1}{n}\}$. Then a_n converges to the identity in this topology and thus is a T-sequence. $\square$

The next step is to link the existence of T-sequences on a countable group to some algebraic property of that group. Recall that any function $f : G \to G$ that is continuous must inverse map the identity to a closed subset of G, provided that the topology on G is T_1. This observation serves as the spring board into the classification. Before we begin, some clarification is required. There is confusion in the literature around the concept of an algebraic set. Markov termed what we will define as an algebraic set an additive algebraic set (see [Mar50]). This is contradicted in the modern literature, with Zelenyuk et al using the next definition (see [PZ99]) but Sipacheva using the original Markov terminology (see [Sip08]). We choose to use the Zelenyuk terminology and will refer to Markov's algebraic sets as weakly algebraic.

Definition 2.57. *A subset U of a group G is called* algebraic *if there exists a finite collection of group words $f_i \in G[x]$ such that*

$$U = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

A subset U of a group G is called weakly algebraic *if it is the intersection of some family of algebraic subsets of G*

It is easy to see that an algebraic set and weakly algebraic set must be closed in any Hausdorff group topology on G. For any group word on $G[x]$ defines a continuous function via the evaluation map on G, and thus the pre-image of $\{e\}$, a closed subset of any Hausdorff group topology, must be closed. Since finite unions and arbitrary intersections of closed sets are closed, we have the result. Secondly, both the empty set and the entire group are trivially algebraic sets[1].

Definition 2.58. *A point $g \in G$ is called* algebraically isolated *if the set $G \setminus \{g\}$ is algebraic*

[1]By considering the empty collection of group words we can see that the entire group is algebraic

Lemma 2.59. *A point $g \in G$ is algebraically isolated if and only if $G \setminus \{g\}$ is weakly algebraic*

Proof. The necessity of the claim is trivial, since any algebraic set must be weakly algebraic. So suppose that $G \setminus \{g\}$ is weakly algebraic, so there exist a family of algebraic subsets of G, say U_α such that

$$G \setminus \{g\} = \bigcap U_\alpha$$

Consider the subfamily of the U_α's which do not contain the point $g \in G$. Then it is easy to see that every element of this subfamily must be the same set as $G \setminus \{g\}$. This implies the existence of a finite set of group words $f_i \in G[x]$ such that

$$G \setminus \{g\} = \bigcup_{i \leq n} \{y \in G | f_i(y) = e\}$$

i.e. $G \setminus \{g\}$ is algebraic. □

The concept of algebraic isolation is what is required to classify those countable groups which admit group topologies.

Lemma 2.60. *Suppose $e \in G$ is algebraically isolated, then any point of G is algebraically isolated. Furthermore, any group topology on G must be discrete*

Proof. Firstly, if $e \in G$ is algebraically isolated then every point of G is algebraically isolated. To see this consider any point $x \in G$, now since e is algebraically isolated we know that there exist group words f_i such that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Define $\hat{f}_i(g) = f_i(x^{-1} \cdot g)$ then if $g \neq x$ then $x^{-1} \cdot g \neq e$. But this implies the existence of a $i \leq n$ such that $\hat{f}_i(g) = f_i(x^{-1} \cdot g) = e$ which implies that $G \setminus \{x\}$ is algebraically isolated. But since any additively algebraic set is closed in a group topology this implies that $G \setminus \{x\}$ is closed for all $x \in G$ i.e. $\{x\}$ is an open subset of G for all $x \in G$. This implies that the topology is discrete. □

The theme of the next lemma is more important then the lemma itself, but it is interesting in its own right.

Lemma 2.61. *Let G be a group with a non-discrete T_1 group topology and let $\{f_i : G \to G\}$ be a finite collection of continuous functions of this group such that $f_i(e) \neq e$ for all $i \leq n$. Then there exists a $g \in G \setminus \{e\}$ such that $f_i(g) \neq e$ and $f_i(g_{-1}) \neq e$.*

Proof. Firstly, extend the finite collection of functions $\{f_i : G \to G\}$ by including all functions of the form $\hat{f}_i(x) = f_i(x^{-1})$. We then have that $\hat{f}_i(e) = f_i(e^{-1}) = f_i(e) \neq e$. Also, since G is a topological group we know that the function $x \to x^{-1}$ is continuous and thus each $\hat{f}_i$ is continuous, being the composition of two continuous functions. So without loss of generality we may assume that the collection $\{f_i : G \to G\}$ contains all the required hat functions. Now consider the collection of subsets $A_i = f_i^{-1}(G \setminus \{e\})$. Since G is a T_1 space and each f_i is continuous we know that A_i is an open neighbourhood of the identity. Thus $\cap A_i$ is an open neighbourhood of the identity and since the topology is not discrete there must exist a $g \in \cap A_i$ such that $g \neq e$. □

Lemma 2.62. *Let G be a group. If the identity is not algebraically isolated then for any finite collection of group works $f_i(x)$ with $f_i(e) \neq e$ there exists a $g \in G \setminus \{e\}$ with $f_i(g) \neq e$ and $f_i(g^{-1}) \neq e$*

Proof. Once again extend the collection of group words by defining $\hat{f}_i(x) = f_i(x^{-1})$. Without loss of generality we may assume that the collection $\{f_i : G \to G\}$ contains all the required hat functions. Now suppose that for all $g \in G \setminus \{e\}$ there exists a i such that $f_i(g) = e$. This would imply that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Which contradicts the fact that $\{e\}$ is not algebraically isolated. □

Theorem 2.63 (Markov Criteria). *Let G be a countable group. G is topologizable if and only if the identity element $e \in G$ is not algebraically isolated*

Proof. The necessity of the claim has been shown in Lemma 2.60 so we only require the sufficiency of the claim. So let G be a countable group such that its identity is not algebraically isolated. Enumerate G as $\{g_n\}$ with $g_0 = e$. Since G is countable, and the countable union of a countable family of sets is also countable, we can enumerate the following subset of the set of all words on G

$$\{f_k\} = \bigcup_{i\in\mathbb{N}} \{f \in G[x_1, x_2, ..., x_i] | f(e, e, ..., e) = e\}$$

Define a sequence in G as follows. Since $g_0 \in G$ satisfies $f_0(g_0, g_0, ..., g_0) = g_0 \neq g_1$ we know by Lemma 2.62 that there exists a $x_1 \in G \setminus \{g_0\}$ such that

$$f_0(\{e, x_1^{\pm 1}\}, \{e, x_1^{\pm 1}\}, ..., \{e, x_1^{\pm 1}\}) \neq g_1$$

We can continue iterating Lemma 2.62 to form a sequence x_i such that $x_{i+1} \in (G \setminus \{e\}) \setminus \{x_n\}_{n\leq i}$ and

$$f_m(\{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}, \{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}, ..., \{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}) \neq g_{m+1}$$

But this implies that the sequence $\{x_m\}$ is a non-trivial T-sequence, by Theorem 2.54 and thus the group G admits a non-discrete Hausdorff topology. □

We can use this theorem to show once again that every countable Abelian group is topologizable. For suppose that G is a countable Abelian group such that the identity e of G is algebraically isolated. Then there must exist a finite family of group words f_i such that

$$G \setminus \{e\} = \bigcup_{i\leq n} \{g \in G | f_i(g) = e\}$$

However, any group word on an Abelian group is of the form $f_i(x) = a_i \cdot x^{k_i}$ where $a_i \in G \setminus \{e\}$ and $k_i \in \mathbb{Z} \setminus \{0\}$. Without loss of generality we may assume that k_i is positive for if $k_i < 0$ we can consider the new group word $a_i^{-1} \cdot x^{-k_i} = e$ where $-k_i > 0$. So $y \in G$ is a solution to this equation if and only if y^{-1} is a solution to the original equation. So choose your favourite $f_i(x) = a_i \cdot x^{n_i}$ and consider two cases. If G has an element of order n_i for some $i \leq n$, say y then $f_i(y) \neq e$ for

$f_i(y) = a_i \cdot y^{n_i} = a_i \neq e$. If for all $y \in G \setminus \{e\}$ and $i \leq n$, $y^{n_i} \neq e$ we have to argue slightly more carefully. So suppose that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

It is obvious that $G \setminus \{e\}$ is a countable set, thus one of the $\{g \in G | f_i(g) = e\}$ must be countable. Choose n_i distinct elements of the countable set $\{g \in G | f_i(g) = e\}$, label them $\{y_k\}$ for $k = 1...n_i$. Then $a_i \cdot y_k^{n_i} = e$ for $k = 1...n_i$, but this implies that

$$a_i^{n_i} \cdot \left(\prod_{j=1}^{n_i} y_j \right)^{n_i} = \left(a_i \cdot \prod_{j=1}^{n_i} y_j \right)^{n_i} = e$$

But then $a_i \cdot \prod_{j=1}^{n_i} y_j$ is an element of order n_i which is a contradiction. So the identity e of the countable Abelian group G cannot be algebraically isolated, which implies that G is topologizable.

Definition 2.64. *A subset A of a group G is called* unconditionally closed *if it is closed in any Hausdorff group topology on G*

The family of all unconditionally closed subsets of a group, as well as the set of all weakly algebraic sets of a group define two topologies on that group. Since the set of all weakly algebraic sets on a group G is closed under finite unions and arbitrary intersections, its forms a base for a topology on G. We call this topology the Zariski topology. Similarly, since the family of all unconditionally closed subsets is also closed under arbitrary intersections and finite unions, it to generates a topology on G. This topology is called the Markov topology.

Definition 2.65.

1. *The* Markov topology $\mathcal{T}_M$ *on a group G is the topology generated by all unconditionally closed subsets of G*

2. *The* Zariski topology $\mathcal{T}_Z$ *on a group G is the topology generated by all weakly algebraic subsets of G*

The Markov theorem can be rephrased as asking when these two topologies coincide. Markov showed that this was true for an arbitrary countable group.

Theorem 2.66 (General Markov's Theorem). *Let G be a countable group. A set A of G is weakly algebraic if and only if it is unconditionally closed*

Proof. We sketch a proof (for the full proof see [Mar50]). Since the arbitrary intersection of weakly algebraic sets is once again weakly algebraic, and any subset of a group G is contained in at least one weakly algebraic subset of G (namely G itself) we can define a weak algebraic closure operator on the group. We define

$$\mathrm{cl}_{\mathcal{T}_Z}(A) = \bigcap \{U \subset G : A \subset U \text{ and } U \text{ is weakly algebraic}\}$$

Now since the weak algebraic closure of a subset of G is a closed subset of G in any group topology $\mathcal{T}$ on G, the weak algebraic closure must contain the topological closure of that subset. Now using the theory of multi-norms on a group, as explored in [Mar50], it is possible to define a group topology on G on which the weak algebraic closure of a subset equals the topological closure of that same subset. This step depends strongly on the countability of the group in question. Now suppose the subset in question is in fact unconditionally closed, then it closed in the group topology where the topological closure and weak algebraic closure coincide, which implies that the subset must be weakly algebraic. □

Taking $A = G \setminus \{e\}$ shows that Theorem 2.63 is merely a special case of this theorem. A reasonable question to ask is if this theorem extends to general infinite groups, as this would serve as a characterization of topologizable groups. Sadly, this is not the case. It was shown in [Sip06] that there exists a group, assuming CH, which contains an unconditionally closed subset which is not algebraic.

Theorem 2.67. *Let G be a non-topologizable group such that any finite subset of G is contained in a topologizable subgroup of G. Then $G\setminus\{e\}$ is an unconditionally closed subset of G which is not weakly closed*

Proof. Since G is non-topologizable, $\{e\}$ is open in every Hausdorff group topology on G and thus $G \setminus \{e\}$ is unconditionally closed. Suppose that $G \setminus \{e\}$ is also algebraic (and thus by Lemma 2.59 also weakly algebraic). This implies the

existence of a finite collection of group words $\{f_i\}$ on G such that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Consider the word $f_k(x) = \prod_{j=1}^{n} \alpha_j x^{p_j}$ where $\alpha_j \in G$ and $p_j \in \mathbb{Z}$. Since the set $\{\alpha_j\}$ is finite for any word f_k, the collection of all the α_j's over all the words f_i is contained in some topologizable subgroup H of G. But then the restriction of the words to this subgroup satisfies

$$H \setminus \{e\} = \bigcup_{i \leq n} \{g \in H | f_i(g) = e\}$$

which contradicts the fact that H is topologizable. □

The existence of such a group will be expanded upon in the next section. Seeing that the general characterization is beyond our grasp, we can ask for which class of groups we can extend the Markov-Zariski equivalence and thus have a limited characterization of topologizable groups. Markov mentioned in [Mar50] the fact that his theorem can be extended to arbitrary infinite Abelian groups, this fact was credited to Perel'man but no proof was provided. Recently, Dikranjan and Shakhmatov ([DS08]) have given a complete characterization of the groups for which the Markov-Zariski equivalence does hold. Let $G^{<\omega}$ denote the family of all finite subsets of a group G and $G^{\leq\omega}$ denote the family of all countable subsets of a group G.

Definition 2.68. *Let $\mathcal{C}$ be a family of elements of $G^{\leq\omega}$ for some group G.*

1. *$\mathcal{C}$ is* closed *if for any increasing sequence of elements $\{C_i\}$ of $\mathcal{C}$, $\cup C_i \in \mathcal{C}$*
2. *$\mathcal{C}$ is* unbounded *if for any element $U \in G^{\leq\omega}$ there exists a $V \in \mathcal{C}$ such that $U \subset V$*
3. *$\mathcal{C}$ is a* club *if it is both closed and unbounded*

Definition 2.69. *Let G be a group, then $\mathcal{P}(G) = \{H \in G^{\leq\omega} : H$ is a subgroup of $G\}$*

Since the union of an increasing family of countable subsets of a group G is still a countable subgroup of G and any countable subset of G is contained in the countable subgroup it generates, we have shown the following lemma.

Lemma 2.70. *$\mathcal{P}(G)$ is a club, for any group G*

Recall that $\mathrm{cl}_{\mathcal{T}_Z}(A)$ denotes the weak algebraic closure of a subset A of G. This is the same as the topological closure operator of the Zariski topology (hence the notation). Let H be a subgroup of G, then $\mathrm{cl}_{\mathcal{T}_Z}\{H\}(A)$ denotes the topological closure operator of the subspace topology inherited from the Zariski topology on H. So $\mathrm{cl}_{\mathcal{T}_Z}\{G\}(A) = \mathrm{cl}_{\mathcal{T}_Z}(A)$ Similarly, let $\mathrm{cl}_{\mathcal{T}_M}\{H\}(A)$ denote the topological closure operator of the subspace topology inherited from the Markov topology on H.

Theorem 2.71. *Let G be an arbitrary group, then $\mathcal{T}_M = \mathcal{T}_Z$ if and only if for all $A \subset G$, the family of subgroups $\{H \in \mathcal{P}(G) : \mathrm{cl}_{\mathcal{T}_M}\{H\}(A \cap H) = \mathrm{cl}_{\mathcal{T}_M}\{G\}(A) \cap H\}$ contains a club of elements from $G^{\leq \omega}$*

Proof. See [DS08] □

This theorem is sufficient to generalize the Markov criteria to arbitrary Abelian groups and the direct product of a family of countable groups. Interested readers are referred to [DS08] for the more set theoretic exposition of these ideas. Furthermore, with this theorem now in our arsenal we can extend the previous Markov-type proof of the existence of group topologies on countable Abelian groups to general Abelian groups. This is because that proof only required the countability of the Abelian group in question to apply Markov's criteria, and thus an extended criteria is immediately beneficial. We thus have this general Markov criteria

Theorem 2.72. *Let G be a group which is either the direct product of some family of countable groups and an Abelian group. There exists a non-discrete Hausdorff group topology on G if and only if the identity of that group is not algebraically isolated*

Proof. If the subset $G \setminus \{e\}$ is unconditionally closed, then it is in $\mathcal{T}_M$. But by Theorem 2.71 we know that $\mathcal{T}_M = \mathcal{T}_Z$ and thus $G \setminus \{e\}$ is algebraic i.e. the identity of G is algebraically isolated. Similarly if the identity of G is algebraically isolated, then $G \setminus \{e\} \in \mathcal{T}_Z = \mathcal{T}_M$ and thus $G \setminus \{e\}$ is unconditionally closed. □

2.6 Examples of non-topologizable groups

Given the strict conditions a natural topology places on the algebraic structure of a group it is surprising that many common classes of groups are topologizable. In fact every infinite Abelian group is extremely topologizable, as was previously mentioned. The general question of Markov regarding the existence of group topologies of infinite groups was answered in the negative by Shelah [She80] when he constructed an infinite group (assuming the continuum hypothesis) which only admits a discrete group topology using the techniques of Small Cancellation theory. The fact that all linear groups are topologizable vastly complicates the situation here as one must introduce amalgams of groups and other more exotic construction methods to find a counter example. His proof does not fall into the ZFC framework as well since the Continuum Hypothesis must be assumed.

Theorem 2.73. *Assume CH. There exists a group G with $|G| = \omega^+$ such that the only group topology on G is discrete. Furthermore, any finite subset of G is contained in a topologizable subgroup of G.*

Proof. See [She80] □

This construction and Theorem 2.67 show the existence of groups whose topological nature cannot be determined solely on the finite algebraic structure of its subsets. It is possible to extend the notion of an algebraic set so that we have a partial strengthening of the equivalence between unconditionally closed and algebraic sets. The idea is to extend the number of words allowed to define the algebraic set from a finite limit to some cardinal number. The problem is that this generalisation breaks the correspondence between normal topological closure of a set and the algebraic nature of that set. Interested readers are directed to [Sip08].

Shelah also gave a sufficient condition for an uncountable group G to be non-topologizable. Since these conditions are much more tractable then the actual construction he presented in connection with Kurosh groups and since this conditions are easy to state it is worthwhile going through the short proof. The group

constructed in [She80] essentially satisfies these two criteria, and thus cannot be topologizable.

Theorem 2.74. *An uncountable group G is non-topologizable if the following two conditions hold*

1. *There exists $m \in \mathbb{N}$ such that $A^m = G$ for every $A \subset G$ such that $|A| = |G|$*
2. *If H is a subgroup of G such that $|H| < |G|$ then there exists $n \in \mathbb{N}$ and elements $x_1, x_2, ..., x_n \in G$ such that $\cap_{i=1..n} x_i^{-1} H x_i$ is a finite set.*

Proof. Let $\mathcal{T}$ be a group topology on G and let $V \subset G$ be an open neighbourhood of $e \in G$ such that $V \neq G$. Now since there exists an $m \in \mathbb{N}$ such that $A^m = G$ for every subset $A \subset G$ such that $|A| = |G|$ and $\mathcal{T}$ is a group topology there exists an open neighbourhood W of e such that $W^m \subset V$ and thus $|W| < |G|$. Let H be the subgroup generated by W. Since H consists of all finite products of elements of W we have that $|H| < |G|$. But then there exists an $n \in \mathbb{N}$ such that $\cap_{i=1..n} x_i^{-1} H x_i$ is a finite set, for some $x_1, x_2, ..., x_n \in G$. But $e \in x_i^{-1} H x_i$ for all $i = 1...n$ and thus

$$e \in \cap_{i=1..n} x_i^{-1} H x_i = \{e, y_1, y_2, ..., y_k\}$$

Where the y_i are distinct and k is some natural number. But $\mathcal{T}$ is a group topology and is thus Hausdorff so is T_1 and thus there exist open neighbourhoods Y_i of e such that $y_i \notin T_i$. But then

$$\{e\} = \bigcap_{i=1..k} Y_i \cap \bigcap_{i=1..n} x_i^{-1} H x_i$$

and thus $\mathcal{T}$ is discrete. □

The second condition is not required if we wish to extend this result to countable groups. This new sufficient condition should be compared with the characterisation theorem previously stated.

Theorem 2.75. *Let G be a countable group. If there exists $m \in \mathbb{N}$ such that $A^m = G$ for every countable $A \subset G$ then G is non-topologizable.*

Proof. Let $\mathcal{T}$ be a group topology on G and let $V \subset G$ be an open neighbourhood of $e \in G$ such that $V \neq G$. Now since there exists an $m \in \mathbb{N}$ such that $A^m = G$ for every subset $A \subset G$ such that $|A| = |G|$ and $\mathcal{T}$ is a group topology there exists an open neighbourhood W of e such that $W^m \subset V$ and thus $|W| < |G|$. But this implies that W is a finite open neighbourhood of e and thus by the proof of Theorem 2.74 we have that G is non-topologizable. □

The final demonstration of a countable non-topologizable group in *ZFC* was obtained by Hesse in [Hes79] and Ol'shanskii in [Ols80]. Hesse essentially showed that the continuum hypothesis was not a necessary requirement for the Shelah construction, at least in a special case sufficient to show the theorem. The Ol'shanskii construction is something completely different. Once again this constructed group is of rather exotic structure, as was to be expected. We give a version of his construction below, however we first require a "technical" lemma.

Lemma 2.76. *Let $m, n \in \mathbb{N}$ such that $n \geq 2$ and $m \geq 665$, with m an odd number. There exists a group $A(n, m)$ generated by n elements with the following three properties*

1. *$A(n, m)$ is a torsion free group*
2. *The centre C of $A(n, m)$ is an infinite cyclic group*
3. *$A(n, m)/C$ is an infinite group of period m*

Proof. See Adian and Novikov [NA68a], [NA68b] and [NA68c] □

This lemma is only technical in the sense that its construction does not illuminate the question of topologizing arbitrary countable groups. The groups constructed in the above lemma are called Adian groups. These groups are in fact the counterexamples to the famous Bounded Burnside problem (Is every finitely generated groups of finite exponent finite?) and takes up over 330 journal pages to construct. We can now prove the existence of a countable non-topologizable group in ZFC.

Theorem 2.77. *Let $A(n, m)$ denote the Adian group and let $C_m = \{x^m : x \in C\}$ then the infinite group $A(n, m)/C_m$ is not topologizable.*

Proof. Firstly, C_m is a subgroup of $A(n,m)$ for suppose $x, y \in C$ then $y^{-1} \in C$ and $xy^{-1} \in C$. Thus $x^m \cdot (y^{-1})^m = (x \cdot y^{-1})^m \in C_m$ since C is the centre of $A(n,m)$. Also since C is the centre of $A(n,m)$ we have that $x^{-1}C_m x \subset C_m$ for all $x \in G$ and thus we have that C_m is a normal subgroup of $A(n,m)$ and so $G = A(n,m)/C_m$ is well-defined. Since the centre C of $A(n,m)$ is an infinite cyclic group, we know that $C = \{c^n : n \in \mathbb{Z}\}$ for some $c \in C$. Now if $g \in A(n,m) \setminus C$ then $g \neq c^n$ for all $n \in \mathbb{Z}$. However, since $A(n,m)/C$ is an infinite group of period m we know that $g^m \in C$. But if $g^m \in C_m$ then $g^m = x^m$ for some $x \in C$. This implies that $(gx^{-1})^m = e$ with $gx^{-1} \neq e$ which contradicts the fact that $A(n,m)$ is a torsion free group. Thus if $g \in A(n,m) \setminus C$ then $g^m \in C \setminus C_m$. Now let $f : A(n,m) \to G$ be the natural homeomorphism. Let $a_1 = f(c)$, $a_2 = f(c^2), ..., a_{m-1} = f(c^{m-1})$ then for every $a \in G \setminus \{e\}$ we have that either $a \in \{a_1, a_2, ..., a_{m-1}\}$ or $a^m \in \{a_1, a_2, ..., a_{m-1}\}$. For suppose that $a \notin \{a_1, a_2, ..., a_{m-1}\}$ then for all $g \in f^{-1}(a)$ we have that $g \notin A(n,m) \setminus C$ which implies that $g^m \in C \setminus C_m$ which implies that $a^m = f(g)^m \in \{a_1, a_2, ..., a_{m-1}\}$. Now let $\mathcal{T}$ be any Hausdorff topology on G such that the mapping $h : G \to G$ where $h(x) = x^m$ is continuous. But then $h^{-1}(\{a_1, a_2, ..., a_{m-1}\}) = G \setminus \{e, a_1, a_2, ..., a_m\}$. This implies that $\{e, a_1, a_2, ..., a_m\}$ is an open subset and thus that $\{e\}$ is an open subset i.e. $\mathcal{T}$ is discrete and thus G is a non-topologizable group. □

So not every group admits a non-discrete Hausdorff topology, even if we limit ourselves to ZFC. Since we have characterised this problem for countable groups and partially characterised it for general groups we may ask if there is a general characterisation for the topologizability of a general group. This seems like a very difficult problem and to the author's knowledge there has not been much progress in this regard. The second question one should ask is if there are weaker topology and algebra hybrids that exist for all groups, or at least an easy to define subclass of them. We consider these questions next.

Chapter 3

The Stone-Cech compactification of a discrete semigroup

As shown in the previous section, not every group admits a non-discrete group topology. The next question to ask would be what class of groups do admit non-discrete group topologies then and can they be characterised. This question requires some deep machinery to answer and in this section we develop that machinery. We start by giving a general construction of the Stone-Cech compactification of a general completely regular space. We use the Stone-Cech compactification to construct a general compact one-sided extension of a general semigroup. The theorem of Ellis then comes into play and we can prove the existence of an idempotent in the Stone-Cech compactification of a semigroup. This has interesting combinatorial implications, we limit ourselves to the Finite Product Theorem and use it to count the number of solutions to a general inequality in a group. For the Stone-Cech construction we follow [CN74] while the construction of the semigroup operations follows [HPS98], [HS02] and [Hin96].

3.1 The Stone-Cech compactification of a completely regular space

The Stone-Cech compactification is the largest Hausdorff, compact topological space into which a given topological space can be continuously embedded. Any completely regular topological space has a unique Stone-Cech compactification(up to homeomorphism that is) which is characterised by the universal property that any continuous mapping from the original topological space into a Hausdorff compact space can be extended to a unique map on the Stone-Cech compactification of the original topological space. Any compactification with this property must be homeomorphic to the Stone-Cech compactification of that space. It is not the purpose of this dissertation to give an overview of category theory however and in that regard the reader is referred to [ML98]. As an aside, it is possible to do these constructions merely using the normal definition of filters and ultrafilters if we restrict ourselves to discrete spaces instead of all completely regular spaces. In fact, it will be shown in the next section that the Stone-Cech compactification of a discrete space is homeomorphic to the set of all ultrafilters on that space. This should be contrasted with the Lemma 3.10 where we show that the Stone-Cech compactification of any completely regular space is homeomorphic to the set of Z-ultrafilters of that space. We start with a few definitions:

Definition 3.1. *A family of non-empty, zero-sets $\mathcal{F} \subset 2^X$ of a topological space $(X, \mathcal{T})$ is a* z-filter *if*

1. *if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$*
2. *if $A \in \mathcal{F}$ and $A \subset B$ with $B \in \mathcal{Z}(X)$ then $B \in \mathcal{F}$*
3. *$\mathcal{F} \neq \mathcal{Z}(X)$*

Definition 3.2. *A maximal z-filter is a* z-ultrafilter.

Z-ultrafilters and z-filters share many of the same characteristics as the sibling vanilla filters and ultrafilters as the following few theorems show. For example

any family of subsets of a space which satisfy the finite intersection property can be extended to a z-ultrafilter (as compared to a normal ultrafilter in the vanilla case).

Theorem 3.3. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F} \subset \mathcal{Z}(X)$ have the finite intersection property. There exists a z-ultrafilter p on X such that $\mathcal{F} \subset p$*

Proof. Let $\mathcal{C} = \{\mathcal{G} \in 2^{2^X} : \mathcal{G}$ be a z-filter and $\mathcal{F} \subset \mathcal{G}\}$ and order this by set inclusion. Now $\mathcal{F} \in \mathcal{C}$ as finite intersections of open subsets are open and so $\mathcal{C} \neq \emptyset$. Let $\{\mathcal{G}_\alpha\}_{\alpha \in I} \subset \mathcal{C}$ be a chain, then $\mathcal{G}_\beta \subset \cup_{\alpha \in I} \mathcal{G}_\alpha \forall \beta \in I$ and it is easy to see that $\cup_{\alpha \in I} \mathcal{G}_\alpha \in \mathcal{C}$ so $\{\mathcal{G}_\alpha\}_{\alpha \in I}$ has an upper bound and thus by Zorn's Lemma $\mathcal{C}$ has a maximal element. Let p be a maximal element of $\mathcal{C}$. Now if p is not a z-ultrafilter then there exists a filter $\mathcal{H}$ such that $p \subset \mathcal{H}$. But then $\mathcal{H} \in \mathcal{C}$ and so $p = \mathcal{H}$ i.e. p is an z-ultrafilter. □

The following lemma partially characterises the z-ultrafilters on a topological space:

Lemma 3.4. *Let p and q be z-ultrafilters on a topological space $(X, \mathcal{T})$.*

1. *if $B \in \mathcal{T}$ and $A \cap B \neq \emptyset$ for $A \in p$ then $B \in p$.*
2. *if $A, B \in \mathcal{T}$ and $A \cup B \in \mathcal{T}$ then $A \in p$ or $B \in P$*
3. *if $p \neq q$ then there are $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$*

Proof.

1. The family $\mathcal{F} = \{A \cap B : A \in p\}$ has the finite intersection property and thus by theorem 3.3 there must exist a z-ultrafilter r on X such that $\mathcal{F} \subset r$. Let $A \in p$ then $A \cap B \subset A$ so $A \in r$ i.e. $p \subset r$ but p is a z-ultrafilter and thus $p = r$ which implies that $B \in p$.

2. Let $A, B \in \mathcal{F}(X)$ and assume that $A \notin p$ and $B \notin p$ then there exists $C, D \in p$ such that $A \cap C = B \cap D = \emptyset$. So $(A \cup B) \cap (C \cap D) = \emptyset$ and thus $A \cup B \notin p$

3. Now there exists $B \in q$ such that $B \notin p$ so there exists $A \in p$ such that $A \cap B = \emptyset$.

□

Z-ultrafilters also have a corresponding notion of a fixed z-ultrafilter (and thus a corresponding idea of the much less trivial free z-ultrafilter) which is similar to the division between fixed and free ultrafilters on a space. The following theorem shows firstly that z-ultrafilters exist and secondly that any space of cardinality κ has κ many free z-ultrafilters defined on it.

Theorem 3.5. *Let $x \in X$ then $\{A \in \mathcal{Z}(X) : x \in A\}$ is a z-ultrafilter*

Proof. Let $A, B \in \{A \in \mathcal{Z}(X) : x \in A\}$. So there exists continuous functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = A$ and $g^{-1}(\{0\}) = B$. Define $h : X \to \mathbb{R}$ by $h(t) = f^2(t) + g^2(t)$ then $h^{-1}(\{0\}) = A \cap B$ so $\emptyset \neq A \cap B \in \mathcal{Z}(X)$. Now let $A \in \{A \in \mathcal{Z}(X) : x \in A\}$ and let $A \subset B \in \mathcal{Z}(X)$ then $x \in A \subset B$ and so $B \in \{A \in \mathcal{Z}(X) : x \in A\}$ i.e. $\{A \in \mathcal{Z}(X) : x \in A\}$ is a z-filter. Now by theorem 3.3 there exists a z-ultrafilter p with $\{A \in \mathcal{Z}(X) : x \in A\} \subset p$. Now let $C \in p \setminus \{A \in \mathcal{Z}(X) : x \in A\}$ and thus $x \notin C \in \mathcal{Z}(X)$. However, X is $T_{3\frac{1}{3}}$ and so there exists a $D \in \mathcal{Z}(X)$ such that $x \in D$ and $C \cap D = \emptyset$. □

The Stone-Cech compactification is the maximal compactification of a completely regular space. However, this notion does not make an easy to work with definition and we choose instead to define the Stone-Cech compactification via the following universal property.

Definition 3.6. *Let $(Y, \mathcal{T})$ be a compact, Hausdorff topological space with a dense subspace X such that any continuous function $f : X \to Z$ where Z is a compact Hausdorff space uniquely extends to a continuous function $\bar{f} : Y \to Z$ then Y is the Stone-Cech compactification of X*

We now proceed to construct the Stone-Cech compactification of a completely regular space using the family of z-ultrafilters on that space. We first prove some basic properties of Z-ultrafilters in the first few lemmas and then prove that the

family of Z-ultrafilters satisfy the conditions of the universal property that defines the Stone-Cech compactification.

Lemma 3.7. *Let $(X, \mathcal{T})$ be a topological space and define $\beta X = \{\ p\ :\ p$ is an z-ultrafilter on $X\ \}$. Let $\mathcal{B} = \{\{p \in \beta X : A \notin p\} : A \in \mathcal{Z}(X)\}$. Define $e : X \to \beta X$ by $e(x) = \{A \in \mathcal{Z}(X) : x \in A\}$ then*

1. *$\mathcal{B}$ is a base for a compact, Hausdorff topology on βX*
2. *$e[X \setminus A] = e[X] \cap \{p \in \beta X : A \notin p\}$ for $A \in \mathcal{Z}(X)$*
3. *$e : X \to \beta X$ is a topological embedding of X into βX*
4. *$\overline{e[A]} = \{p \in \beta X : A \in p\}$ for $A \in \mathcal{Z}(X)$*
5. *$\overline{e[X]} = \beta X$*

Proof. 1. Let $A_i \in \mathcal{Z}(X)$ and let $B_i = \{p \in \beta X : A_i \notin p\}$ for $i \in \{1, 2\}$. Now $A_1 \cup A_2 \notin p \Leftrightarrow A_1 \notin p$ and $A_2 \notin p$ and thus $B_1 \cap B_2 = \{p \in \beta X : A_1 \cup A_2 \notin p\} \in \mathcal{B}$ and thus $\mathcal{B}$ is a base for a topology on X. Now let $p, q \in \beta X$ such that $p \neq q$ so there exists $A, B \in \mathcal{Z}(X)$ such that $A \in p$, $B \in q$ and $A \cap B \neq \emptyset$. Now since disjoint zero sets are actually completely separated there are co-zero sets $C, D \subset X$ and zero sets $E, F \subset X$ such that $A \subset C \subset E$ and $B \subset D \subset E$ with $E \cap F = \emptyset$. But then $p \in \{t \in \beta X : X \setminus C \notin t\}$ and $q \in \{t \in \beta X : X \setminus D \notin t\}$ $\{t \in \beta X : X \setminus C \notin t\} \cap \{t \in \beta X : X \setminus D \notin t\} \subset \{t \in \beta X : E \in t\} \cap \{t \in \beta X : F \in t\} = \emptyset$ and thus $\mathcal{B}$ generates a Hausdorff topological space. Now let $\{A_i : i \in I\} \subset \mathcal{Z}(X)$ and suppose the family $\{\{p \in \beta X : A_i \in p\} : i \in I\}$ has the finite intersection property then there exists $q \in \beta X$ such that $\{A_i : i \in I\} \subset q$ and thus

$$q \in \bigcap_{A_i \in q} \{p \in \beta X : A_i \in p\} \subset \bigcap_{i \in I} \{p \in \beta X : A_i \in p\} \neq \emptyset$$

and thus $\mathcal{B}$ generates a compact, Hausdorff topology

2. $p \in [X \setminus A] \Leftrightarrow \exists t \in X \setminus A \subset X$ such that $p = \{B \in \mathcal{B}(Z) : t \in B\} \Leftrightarrow p \in e[X]$ and $A \notin p$ as $A \in \mathcal{Z}(X) \Leftrightarrow p \in e[X] \cap \{q \in \beta X : A \notin q\}$

3. $e : X \to \beta X$ is obviously well defined. Now if $x \neq y$ then as $\{x\}$ and $\{y\}$ are closed subsets of X there exist subsets $A, B \in \mathcal{Z}(X)$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$ but then $\{C \in \mathcal{Z}(X) : x \in C\} \neq \{C \in \mathcal{Z}(X) : x \in C\}$ and thus $e : X \to \beta X$ is an injection. Also as $e[X \setminus A] = e[X] \cap \{p \in \beta X : A \notin p\}$ and base for $(X, \mathcal{T})$ is carried onto a base for the subspace topology of $e[X] \subset \beta X$

4. $\{p \in \beta X : A \in p\}$ is closed in βX and obviously $e[A] \subset \{p \in \beta X : A \in p\}$ and thus $\overline{e[A]} \subset \{p \in \beta X : A \in p\}$. Now suppose $p \in \{p \in \beta X : A \in p\} \setminus \overline{e[A]}$ then there exists $B \in \mathcal{Z}(X)$ such that $B \notin p$ and $e[A] \cap \{t \in \beta X : B \notin t\} = \emptyset$. However $e[A] \subset e[X] \cap \{t \in \beta X : B \notin t\} = e[B]$ and thus $B \in p$ which is a contradiction.

5. $\overline{e[X]} = \{p \in \beta X : X \in p\} = \beta X$

□

So we have shown the basic construction blocks of the topological space level construction of the Stone-Cech compactification. However, our job is only half completed as we wish to be able to talk about functions as well. We wish to determine how the functions on the original space X behave on our new extended space. The next sequence of lemmas achieve that very goal, and allow us to determine the Stone extension of a continuous function on X.

Lemma 3.8. *Let X be a dense subset of a space Y and let $f : X \to K$ be a continuous function where K is a compact space.*

1. *If for $p \in Y$ there is a continuous function $f_p : X \cup \{p\} \to K$ such that $f \subset f_p$ then there exists a continuous function $g : Y \to K$ such that $f \subset f$*

2. *If $p \in Y$ and there is no continuous function $g : X \cup \{x\} \to K$ such that $f \subset g$ then there exists $A, B \in \mathcal{Z}(K)$ such that $A \cap B = \emptyset$ and $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$*

Proof. 1. Define $g = \bigcup\{f_p : p \in Y\}$. Now let $p \in Y$ and let V and W be open neighbourhoods of $g(p)$ such that $\overline{W} \subset V$ (which is possible as X is

a compact, completely regular space and is thus T_4) and let U be an open neighbourhood of $p \in Y$ such that $f(q) \in W$ for $q \in U \cap X$. If $q \in U$ then $q \in \overline{U \cap X}$ and thus $g(q) = f_q(q) \in \overline{f_q[U \cap X]} = \overline{f[U \cap X]} \subset \overline{W} \subset V$

2. The family $\mathcal{F} = \{\overline{f[U \cap X]} : U \text{ is a neighbourhood of } p \in Y\}$ is a family of closed sets with the finite intersection property and thus $\bigcap \mathcal{F} \neq \emptyset$. Now $|\bigcap \mathcal{F}| \geq 2$ as else there exists a continuous extension of f. So there exists $s, t \in \bigcap \mathcal{F}$ with $s \neq t$ and thus there exists $A, B \in \mathcal{Z}(Y)$ with $s \in A$, $t \in B$ and $A \cap B = \emptyset$ and so $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$

□

So we know what type of functions on a space X are determined by their values on a dense subset Y of X. Interestingly enough, the proof gives a finite step condition for this extension. We need only be able to extend our function one point at a time to be able to find a proper extension to the whole space. For example, we can use limit operations to extend any function continuously defined on the rationals to a continuous function on the reals.

Definition 3.9. *Let X be a space*

1. $C(X) = \{f : X \to \mathbb{R} : f \text{ is continuous}\}$
2. $C^*(X) = \{f : X \to \mathbb{R} : f(X) \subset (-a, a) \text{ for some } a \in \mathbb{R}\}$
3. *Let $A \subset X$ then A is C^*-embedded in X if for $f \in C^*(A)$ there exists a $g \in C^*(X)$ such that $f \subset g$*

So C^*-embeddings are well behaved in the sense that they preserve bounded functions. Since any continuous functional on a compact space must be bounded, this restriction seems natural in this setting. We certainly do not want to be losing bounded functionals during the transition from our original locally compact space to its compactification.

Lemma 3.10. *Let X be a dense subset of a compact space Y. The following statements are equivalent*

1. *For $f : X \to K$ with f continuous and K compact there is a continuous function $g : Y \to K$ such that $f \subset g$*

2. *X is C^*-embedded in Y*

3. *If $A, B \in \mathcal{Z}(X)$ and $A \cap B = \emptyset$ then $\overline{A} \cap \overline{B} = \emptyset$*

4. *If $A, B \in \mathcal{Z}(X)$ then $\overline{A} \cap \overline{B} = \overline{A \cap B}$*

5. *The family $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ is a z-filter for $p \in Y$*

Proof. **(1) $\Rightarrow$ (2)** This is essentially the definition of X being C^*-embedded in Y, as on a compact space $C(Y) \subset C^*(Y)$

(2) $\Rightarrow$ (3) Let $A, B \in \mathcal{Z}(X)$ with $A \cap B = \emptyset$. Now there exists a continuous function $f : X \to \mathbb{R}$ such that $A \subset f^{-1}(\{0\})$ and $B \subset f^{-1}(\{1\})$. However as X is C^*-embedded in Y there exists a continuous $g : Y \to \mathbb{R}$ with $f \subset g$. However then $g[\overline{A}] = \{0\}$ and $g[\overline{B}] = \{1\}$ and thus $\overline{A} \cap \overline{B} = \emptyset$

(3) $\Rightarrow$ (4) Now $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and suppose $p \notin \overline{A \cap B}$ then there exists a $f \in C(Y)$ such that $f(p) = 0$ and $f(\overline{A \cap B}) = \{1\}$ and thus $f(A \cap B) = \{1\}$. Now let $C = \{q \in Y : f(q) \leq \frac{1}{2}\}$ which is an open subset of Y as $f \in C(Y)$ and also $A \cap B \cap C = \emptyset$. Now $A \cap C \in \mathcal{Z}(X)$ as $A \in \mathcal{Z}(X)$ and A is a dense subset of Y. Similarly, $B \cap C \in \mathcal{Z}(X)$ and thus $\overline{A \cap C} \cap \overline{B \cap C} = \emptyset$ and thus $p \notin \overline{A} \cap \overline{B}$ and thus $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$ so $\overline{A} \cap \overline{B} = \overline{A \cap B}$

(4) $\Rightarrow$ (5) Let $A, B \in \mathcal{Z}(X)$ such that $p \in \overline{A}$ and $A \subset B$ then $p \in \overline{A} \subset \overline{B}$ and thus $B \in \{A \in \mathcal{Z}(X) : p \in \overline{A}\}$. Now let $C \in \mathcal{Z}(X)$ with $p \in \overline{A}$ then $p \in \overline{A} \cap \overline{C} = \overline{A \cap C}$ and $A \cap C \in \mathcal{Z}(X)$ and thus $A \cap C \in \{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ and so $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ is a z-filter on X.

(5) $\Rightarrow$ (1) Now if there exists a continuous function $f : X \to K$ such that for all $g : Y \to K$, $f \not\subset g$ then there must exist a $p \in Y$ and $A, B \in \mathcal{Z}(K)$ such that $A \cap B = \emptyset$ and $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$ but since $f^{-1}(A), f^{-1}(B) \in \mathcal{Z}(X)$ (since if $g \in C(K)$ such that $g(A) = \{0\}$ then $g \circ f \in C(X)$ and $g \circ f(f^{-1}(A)) \subset$

$g(A) = \{0\}$) and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ the family $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ cannot be a z-filter on X.

□

The Stone-Cech compactification of a space is surprisingly difficult to describe explicitly. For example $\beta(0,1] \neq [0,1]$ as the function $f : (0,1] \to [0,1]$ given by $f(t) = \sin(\frac{\pi}{2t})$ cannot be continuously extended to $[0,1]$ as the $\lim_{t\to 0} f(t)$ does not exist. Also even though $\beta\mathbb{N}$ is the "closure" of a countable set it can be shown that $|\beta\mathbb{N}| = 2^{2^\omega}$ and thus is dramatically larger then $\mathbb{N}$.

Theorem 3.11. *βX is the Stone-Cech compactification of X*

Proof. By Lemma 3.7 the space βX is compact and $e[X]$ is a dense subset of βX. Let K be a compact, Hausdorff space and let $f : e[X] \to K$. Now let $p \in \beta X$ i.e. p is a z-ultrafilter on X and consider the family of subsets $\{A \in \mathcal{Z}(X) : p \in \overline{e[A]}\} = \{A \in \mathcal{Z}(X) : p \in \{q \in \beta X : A \in q\}\} = p$ and so $\{A \in \mathcal{Z}(X) : p \in \overline{e[A]}\}$ is a z-filter on $e[X]$ and thus by Lemma 3.10 we can find $g \in C(\beta X)$ such that $f \subset g$. This implies that βX is the Stone-Cech compactification of $e[X]$ which is homeomorphic to X and thus βX is the Stone-Cech compactification of X. □

This theorem also provides another characterization of compactness for completely regular spaces. A completely regular space is compact if and only if every z-ultrafilter on that space is fixed. This follows immediately from the previous theorem and the embedding $e : X \to \beta X$. Now suppose X is not only completely regular but also locally compact. This has implications for the embedding of X into the Stone-Cech compactification of X. First we require a lemma

Lemma 3.12. *Let A be a locally compact, dense subset of a Hausdorff space X then A is an open subset of X.*

Proof. We proceed by a number of quick, light steps. For all $x \in A$ we can choose an open U_x such that $\bar{U_x} \cap A$ is compact. This implies that the set $A \cap U_x$ must be closed in the subspace topology of U_x which in turn means that $U_x \setminus (A \cap U_x)$

is open in the same subspace topology. This implies that

$$A = \bigcup_{x \in A} U_x \setminus \bigcup_{x \in A} U_x \setminus (A \cap U_x)$$

Must be a closed subset relative to the subspace topology on $\bigcup_{x \in A} U_x$. But then $A = \bar{A} \cap \bigcup_{x \in A} U_x = X \cap \bigcup_{x \in A} U_x = \bigcup_{x \in A} U_x$ is open. □

With this lemma in hand we can reveal more structure about the embedding of a locally compact space into its Stone-Cech compactification.

Theorem 3.13. *If X is a locally compact, completely regular space then $\beta X \setminus e(X)$ is a compact subset of the Stone-Cech compactification of X*

Proof. Since X is locally compact, we know that e(X) must also be locally compact. However, e(X) is also a dense subset of the Hausdorff space βX and thus $e(X)$ is open. This implies that $\beta X \setminus e(X)$ is closed and thus compact. □

The extension of functions from an arbitrary topological space to a compact, Hausdorff space to the Stone-Cech compactification is a powerful and useful technique. We proceed to define these Stone extensions and consider a few of their properties that will be useful in further sections.

Definition 3.14. *Let $f : X \to K$ be a continuous function and K a compact Hausdorff space then the unique continuous function $\overline{f} : \beta X \to K$ such that $f \subset \overline{f}$ is the* Stone extension *of f*

For example, let X be any T_1 topological group and thus is a completely regular space. It thus has a Stone-Cech compactification βX. Consider the family of continuous left shifts $l_a(x) = a \cdot x$ for $a \in X$. By considering the embedding of X in βX we can treat this as a family of continuous function from X to the compact space βX and thus the Stone extensions of this family exist. This allows us to carry some of the algebraic structure of the original group onto its Stone-Cech compactification. The resulting algebraic structure need not be a topological group though.

Corollary 3.15. *If X is a discrete topological space then $\beta X = \{p \subset 2^X : p$ is an ultrafilter on $X\}$ furthermore βX is a zero dimensional topological space*

Proof. As X is a discrete topological space it is locally compact and completely regular and thus the Stone-Cech compactification exists. Let $A \subset X$ and define $f : X \to \mathbb{R}$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ otherwise. Now as X is discrete this function is continuous and $f^{-1}(A) = \{0\}$ and thus A is a zero set. So any z-ultrafilter is in fact a standard ultrafilter on X and so $\beta X = \{p \subset 2^X : p$ is an ultrafilter on $X\}$. Now the family of sets $\mathcal{B} = \{\{p \in \beta X : A \notin p\} : A \subset X\}$ forms a base for the compact Hausdorff topology on X. Let $A \subset X$ and $U = \{p \in \beta X : A \notin p\}$ then $\beta X \setminus U = \{p \in \beta X : A \in p\} = \{p \in \beta X : X \setminus A \notin p\} \in \mathcal{B}$ as p is an ultrafilter on X. This implies that $\mathcal{B}$ is a clopen base for the compact topology on βX and thus is zero dimensional. □

Definition 3.16. *A topological space $(X, \mathcal{T})$ is strongly zero dimensional if βX is zero dimensional*

So every discrete space is strongly zero dimensional by the previous corollary. The following combinatorial fact will be useful for the next theorem and is stated without proof

Lemma 3.17. *Let X be a set and let $f : X \to X$ be a function such that $f(x) \neq x$ for all $x \in X$ then there exist subsets A_0, A_1 and A_2 of X such that*

1. *$X = A_0 \cup A_1 \cup A_2$*
2. *if $i \neq j$ then $A_i \cap A_j = \emptyset$*
3. *$A_i \cap f[A_i] = \emptyset$*

Proof. See [Kat67] □

If X is a locally compact, Hausdorff topological space and $f : X \to X$ is a continuous function then we can extend the codomain of f to a compact, Hausdorff space by using the embedding $e : X \to \beta X$. The function $e \circ h : X \to \beta X$ is then

a continuous function from X to a compact, Hausdorff space and thus has a Stone extension. We write $\overline{f}$ for the Stone extension of $e \circ h$.

Theorem 3.18. *Let X be a discrete topological space, $f : X \to X$ a function and let $p \in \beta X$ then $\overline{f}(p) = p$ if and only if $\{x \in X : f(x) = x\} \in p$*

Proof. Let $A = \{x \in X : f(x) = x\}$ and consider the set $e[A] \subset \beta X$. Now if $x \in A$ then $e(x) = \{C \subset X : x \in C\}$ and $\overline{f}(e(x)) = e(f(x)) = e(x)$. Now let $p \in \beta X$ and suppose $A \in p$ then $p \in \overline{e[A]}$ and since $\overline{f}$ is continuous we have that $\overline{f}(p) = p$.

Now if $A \notin p$ then $X \setminus A \in p$ as p is an ultrafilter on X and so $X \setminus A \neq \emptyset$ and if $x \in X \setminus A$ then $f(x) \neq x$. Now by Lemma 3.17 there exist subsets A_0, A_1 and A_2 of $X \setminus A$ such that

1. $X \setminus A = A_0 \cup A_1 \cup A_2$
2. if $i \neq j$ then $A_i \cap A_j = \emptyset$
3. $A_i \cap f[A_i] = \emptyset$

Now if $A_i \notin p$ for $i = 0, 1, 2$ then $X \setminus A_i \in p$ but then $\bigcap X \setminus A_i = X \setminus \bigcup A_i = X \setminus (X \setminus A) = A \in p$ which is a contradiction and so with out loss of generality we may assume that $A_1 \in p$. If $A_2 \in p$ then $\emptyset = A_1 \cap A_2 \in p$ which is a contradiction, similarly $A_3 \notin p$. However, $p \in \overline{e[A_1]}$ and since $\overline{f}$ is continuous $\overline{f}(p) \in \overline{e[f[A_1]]} \subset \overline{e[A_2 \cup A_3]}$ as $f[A_1] \cap A_1 = \emptyset$ and thus $\overline{f}(p) \neq p$ □

So it is possible to fully characterise the presence of fixed points for the Stone extension of a function defined on a discrete topological space X. In a sense, the extended function will have a point $p \in \beta X$ as a fixed point if the original function f on X had in some sense a large collection of fixed points in X. For example, any function with only a finite number of fixed points will not suddenly find more when extended to the Stone-Cech compactification. This will we important in later sections.

3.2 The semigroup βS and its properties

We have defined the Stone-Cech compactification for general completely regular spaces, via the z-ultrafilters on that space. Now any T_1 topological group is in fact completely regular and thus the Stone-Cech compactification of that group exists. Secondly, the group multiplication operation can be continuously extended in each of its arguments to the Stone-Cech compactification of that group. However, the resulting Stone extensions need not be associative, and thus the Stone-Cech compactification of an arbitrary completely regular group topology need not be a semitopological semigroup (i.e. a semigroup where the multiplication is continuous in each argument separately). Classifying those topologies for which this result would be true seems to be a difficult problem, however it is true if the original group topology is discrete.

An important construction required for this discussion is the $p-\lim$ which was first introduced in [Fro67].

Definition 3.19. *Let X be a set and p an ultrafilter in D. Let $<x_s>_{s\in D}$ be an indexed family in a topological space X and let $y \in X$. Then the $p-\lim_{s\in D} x_s = p\text{–}\lim <x_s>_{s\in D} = y$ if and only if for every neighbourhood U of y we have that $\{s \in D \mid x_s \in U \} \in p$*

So a net converges to a point in the p–lim if it is in every neighbourhood of that point almost always. The idea of a p–lim is important in our view as in a compact space p–lim's always exist and the required functions defined by the use of a p–lim on the Stone-Cech compactification will be continuous. The following theorem will be useful when defining semigroup operations on the Stone-Cech compactification of a semigroup. The proofs for part 2 and 3 where adapted from [Hin96]

Theorem 3.20. *Let D be a set and let p be an ultrafilter on D. Let $<x_s>_{s\in D}$ be an indexed family in a topological space X.*

1. *If X is Hausdorff and $p\text{–}\lim <x_s>_{s\in D}$ exists then it is unique.*

2. *If X is compact then $p\text{–}\lim <x_s>_{s\in D}$ exists.*

3. *Let X and Y be Hausdorff topological spaces and let $f : X \to Y$ be a continuous function. If p–lim$<x_s>_{s\in D}$ exists then $f(p\text{–lim}<x_s>_{s\in D}) = p\text{–lim}<f(x_s)>_{s\in D}$*

Proof. 1. Suppose p–lim $<x_s>_{s\in D}= y_1$ and p–lim $<x_s>_{s\in D}= y_2$ with $y_1 \neq y_2$. Now as X is Hausdorff, there exists open disjoint neighbourhoods of y_1 and y_2, say V and W respectively. But then $\{s \in D \mid x_s \in V \} \in p$ and $\{s \in D \mid x_s \in W \} \in p$ and p is a filter so $\{s \in D \mid x_s \in V \} \cap \{s \in D \mid x_s \in W \} \neq \emptyset$. So $x_s \in V \cap W$ for some $s \in D$, which is a contradiction.

2. Now suppose that p–lim $<x_s>_{s\in D}$ does not exist. For each $y \in X$ choose a neighbourhood U_y of y such that $\{s \in D \mid x_s \in U_y \} \notin p$. The family $\{U_y\}_{y\in X}$ then forms an open cover of X, so let $\{U_{y_n}\}_{n=1...m}$ be a finite subcover. However $D = \bigcup_{n=1...m}\{s \in D \mid x_s \in U_{y_n} \}$ and so we can find a $n \in \{1...m\}$ such that $\{s \in D \mid x_s \in U_{y_n} \} \in p$ as p is an ultrafilter. This contradicts the choice of U_{y_n}.

3. Let V be a neighbourhood of $f(p\text{–lim}<x_s>_{s\in D})$ then there exists a neighbourhood $W \ni p-\lim <x_s>_{s\in D}$ such that $f(W) \subset V$ as f is continuous. Now $\{s \in D \mid x_s \in W \} \in p$ and $\{s \in D \mid x_s \in W \} \subset \{s \in D \mid f(x_s) \in V \}$ so $\{s \in D \mid f(x_s) \in V \} \in p$

□

Now if S happens to be a semigroup then we can extend the semigroup operations of S to βS to obtain a compact semigroup with continuous left shifts. This can be done as follows (for another view of this construction see [Ell69]):

Definition 3.21. *Let S be a semigroup with binary operation $\cdot : S \times S \to S$. Extend $\cdot$ to βS in three steps*

1. *Given $q \in \beta S$ and $s \in S$ define $s \odot q = q- \lim_{t\in S} s \cdot t$*

2. *Given $p \in \beta S$ and $q \in \beta S$ define $p \odot q = p- \lim_{s\in S} s \cdot q$*

3. *Given $p \in \beta S$ and $q \in \beta S$ define $p \cdot q = q \odot p$*

Where the identification between $e(S) \subset \beta S$ and S is required for this definition to work. With the extended definition of $\cdot$ we find that βS is a compact semigroup with continuous left shifts. This follows from the properties of p-limits given in the appendix and the following theorem. However, first we require a technical lemma.

Lemma 3.22. *Let D be a set, X a compact Hausdorff space and $(x_s)_{s\in D}$ be a D indexed subset of X. Let p be an ultrafilter on D and let $U \subset X$ be a subset of X such that $\{s \in D : x_s \in U\} \in p$. Then $p-\lim_{s\in D} x_s \in cl(U)$*

Proof. Suppose that $p-\lim_{s\in D} x_s \notin U$ then for all neighbourhoods $V \subset X$ of $p-\lim_{s\in D} x_s$ we have that $\{s \in D : x_s \in V\} \in p$. But $\{s \in D : x_s \in U\} \in p$ and so $\{s \in D : x_s \in U \cap V\} \in p$ which implies that $U \cap V \neq \emptyset$ for all neighbourhoods $V \ni p-\lim_{s\in D} x_s$ i.e. $p-\lim_{s\in D} x_s \in cl(U)$ □

With this lemma in hand, the continuity of the extended operations on βS is in fact continuous in its right argument. In fact we can prove something stronger, with the problem of interest being a special case of the following theorem.

Theorem 3.23. *Let D be a discrete space, X a compact Hausdorff space and $(x_s)_{s\in D}$ be a D indexed subset of X. Then the function $f : \beta D \to X$ defined by $f(p) = p-\lim_{s\in D} x_s$ is continuous.*

Proof. Since the space X is compact and Hausdorff, we have that f is well defined by theorem 3.20 and that X is in fact T_3. All that is required is to show that $f : \beta D \to X$ is in fact continuous. So let $V \subset X$ be an open neighbourhood of $f(p)$ for some fixed $p \in \beta D$. Since X is T_3 there exists an open $U \subset X$ such that $f(p) \in U \subset \bar{U} \subset V$. Now the set $A = \{s \in D : x_s \in U\} \in p$ by the definition of a p-limit and thus $p \in \bar{A}$, which is an open subset of βD. But then $f(\bar{A}) \subset V$ as if $q \in \bar{A}$ then $A \in q$ i.e. $\{s \in D : x_s \in U\} \in q$ which by Lemma 3.22 implies that $f(q) \in cl(U) \subset V$. □

This theorem shows one of the benefits of using $p-\lim'$s as the basis for this construction. We get a whole family of continuous functions for free. We know

that since βS is completely regular these functions exist, but a $p-$ lim seems to make them more tangible. However, that is merely something of interest. More importantly, we can now show the continuity requirement of the result we are looking for.

Corollary 3.24. *Let S be a semigroup endowed with the discrete topology, then the extension of the operator to βS is right continuous on βS.*

Proof. Take X to be βS and let $q \in \beta S$. Define

$$f(p) = q \cdot p = p\text{-}\lim_{s \in D} x_s$$

where $x_s = q\text{-}\lim_{t \in D} s \cdot t$. Then by theorem 3.23, $f : \beta S \to \beta S$ is continuous. □

So we have almost shown that βS is in fact a left invariant, zero dimensional semigroup if S is a discrete semigroup. All that is left to show is that the operation defined on βS is in fact associative. For this we require the following lemma.

Lemma 3.25. *Let S be a semigroup and let X be a Hausdroff space with $< x_s >_{s \in S}$ an indexed family of elements of X. Let $p, q \in \beta S$. Then if all the limits involved exist then $(p \cdot q)\text{-}\lim_{v \in S} x_v = q\text{-}\lim_{s \in S}(p\text{-}\lim_{t \in S} x_{s \cdot t})$*

Proof. Let $z = (p \cdot q)\text{-}\lim_{v \ in S} x_v$ and for all $s \in S$ let $y_s = (p\text{-}\lim_{t \in S} x_{s \cdot t})$. Now if $q\text{–}\lim_{s \in S} y_s \neq z$ we can find two disjoint open neighbourhoods of z and $q\text{–}\lim_{s \in S} y_s$, say U and V respectively. Let $A = \{v \in S : x_v \in V\}$ and $B = \{s \in S : y_s \in U\}$. Then it can be easily shown that $A \in p \cdot q$ and $B \in p$. Let $C = \{s \in S : s \cdot p \in \bar{A}\}$ but $\bar{A}$ is a neighbourhood of $q\text{-}\lim_{v \in S} s \cdot p$ and so $C \in p$. But then there exists a $s \in B \cap C$. Since $s \in B$ we know that $p\text{-}\lim_{t \in S} x_{s \cdot t} \in U$ so let $D = \{t \in S : x_{s \cdot t} \in U\} \in q$. Also since $s \in C$ we know that $s \cdot p \in \bar{A}$ and so $\bar{A}$ is a neighbourhood of $p\text{-}\lim_{t \in S} s \cdot t$. Define $E = \{t \in S : s \cdot t \in \bar{A}\}$. We know that $\emptyset \neq D \cap E \in p$ so choose $t \in D \cap E$. Since $t \in D$ we know that $x_{s \cdot t} \in U$. But then $s \cdot t \in A$ since $t \in E$ and so $x_{s \cdot t} \in V$ which implies that $U \cap V \neq \emptyset$. □

We can now show the final theorem, that if S is a discrete topological semigroup then it is possible to extend the operation on S to the Stone-Cech compactifica-

tion of S, βS in such a way that βS is a left shift continuous, zero dimensional semigroup.

Theorem 3.26. *Let S be a discrete topological semigroup. Then it is possible to extend the operation on S in such a way that βS is a left shift continuous, compact and Hausdorff, zero dimensional topological semigroup.*

Proof. We have already shown that βS is a left shift continuous, compact and Hausdorff, zero dimensional space. All that is required is to show that the extended shift operation is in fact associative. For this we use Lemma 3.25. For starters, since βS is compact we know that all $p-\lim$'s exist by theorem 3.20. Now let p, q and $r \in \beta S$ then

$$\begin{aligned} p \cdot (q \cdot r) &= (q \cdot r) - \lim_{s \in S} (p - \lim_{w \in S} s \cdot w) \\ &= r - \lim_{v \in S} (q - \lim_{t \in S} (p - \lim s \in Ss \cdot (t \cdot v))) \\ &= r - \lim_{v \in S} (q - \lim_{t \in S} (p - \lim s \in S(s \cdot t) \cdot v)) \\ &= (p \cdot q) \cdot r \end{aligned}$$

□

We have shown that any semigroup can be embedded as a dense subset of a compact left shift continuous semigroup. We call the subset $\beta S \setminus S$ the remainder of Stone-Cech compactification. If the semigroup in question has a little more structure then we can say something about the algebraic nature of the remainder. For example, it is possible to show that if the semigroup starts off as cancellative then its remainder in βS is an ideal. This is the starting point of Zelenyuk's theorem regarding nontrivial subgroups of $\beta\mathbb{N}$.

Theorem 3.27 (Zelenyuk). *Let G be a countable group endowed with the discrete topology and let A be a subgroup of the remainder of G, $\beta G \setminus G$. If A is finite then A is a trivial subgroup. Furthermore, if βG contains a nontrivial, finite subgroup then so must G.*

This proof is long and involved though and so a proof will not be given. Interested readers are urged to consult the original paper (see [Zel97]) or Hindman's write

up on the result in (see [Hin96]). It is interesting to note that despite the fact that the one sided multiplication operator of a group, endowed with any T_1 group topology, can be extended to an operator on the Stone-Cech compactification of that group we still require a rather special topology on G to allow us to show something about the algebraic nature of this extension i.e. That the Stone-Cech compactification endowed with the Stone extension of this operator is merely a semigroup, ignoring any topological considerations of that extended operator, required a strong statement about the strength of the original group topology. Also, the existence of certain types of subgroups was also dependant on the original group topology in question being discrete. Whether the discrete topology is the only such topology on a group G with this property is unknown, as far as the author can tell.

3.3 Idempotents of compact right topological semigroups

In this section we prove the two theorems that are required to determine the existence of certain ultrafilters in the Stone-Cech compactification of a discrete semigroup.

Theorem 3.28 (Ellis). *Let S be a compact, Hausdorff semigroup with continuous left shifts. Then S contains an idempotent*

Proof. Let $\mathcal{A} = \{A \subset S : A^2 \subset A, A \neq \emptyset$, and A is compact$\}$ with $\subset$ as a partial ordering on $\mathcal{A}$. Now as S is compact, $\mathcal{A} \neq \emptyset$. Let $\mathcal{C} = \{A_\alpha\}$ be a chain in $\mathcal{A}$. Now $\cap\mathcal{C} \in \mathcal{A}$ and so $\mathcal{C}$ has a lower bound and thus by Zorn's Lemma, $\mathcal{A}$ has a minimal element. Let $B \in \mathcal{A}$ be a minimal element and let $x \in B$. Now consider the subset $C = xB \neq \emptyset$ as $x^2 \in C$. Also $C \subset B^2 \subset B$. Now $C^2 = (xB)(xB) \subset xB^3 \subset xB = C$ and since left shifts are continuous C is a compact subset of S and thus $C \in \mathcal{A}$. However, B is a minimal element of $\mathcal{A}$ and thus $B = C$. Finally, define $D = \{y \in B : xy = x\} \neq \emptyset$ as $C = xB = B$. Also as left shifts are continuous we have that D is a compact subset of S. Now let $y, z \in C \subset B$ then $yz \in B$ and $xyz = xz = x$ and thus $D^2 \subset D$ i.e. $D \in \mathcal{A}$. As B is a minimal element of $\mathcal{A}$ we have that $x \in B = D$. But this implies that $x \in D$ and so $x^2 = x$. □

Similarly, it can be shown that any compact, Hausdorff semigroup with continuous right shifts must contain an idempotent. This is achieved by replacing all left shifts in the above proof by the appropriate right shift.

A semigroup with identity can have more then one idempotent, as the following example shows. Let X be an infinite Hausdorff, compact topological space and consider the semigroup $X^X = \{f : X \to X\}$ with the binary operation being composition of functions. We can endow X^X with the product topology and thus by Tychonoff's theorem we know that X^X is a compact topological space. A basic

open neighbourhood of a function f is given as follows: Choose a finite number of points $\{x_i\}_{i=1..n} \subset X$ and choose open neighbourhoods $U_i \subset X$ of $f(x_i)$. A basic open neighbourhood of f is then of the form $V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f) = \{h : X \to X : h(x_i) \in U_i\}$. Now the right shift $f \to f \circ g$ is continuous (as $V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f) \circ g \subset V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f)$) and thus X^X satisfies all the criteria of Theorem 3.28 and thus contains idempotents. In fact, every constant function on X is an idempotent and thus X^X contains at least $|X|$ many idempotents, one for each constant function.

Now there exists a natural ordering on the set of all idempotents of a semigroup. This ordering relates to the ideal structure of the semigroup. Recall that a nonempty subset H of a semigroup S is called a left ideal if $S \cdot H \subset H$, a right ideal if $H \cdot S \subset H$ and an ideal of S if it is both a left ideal and a right ideal.

Definition 3.29. *An idempotent $p \in S$ is minimal if it belongs to some minimal ideal of S*

We first prove that the Stone-cech compactification of a semigroup contains minimal right ideals.

Lemma 3.30. *Let S be a compact, left shift continuous topological semigroup and let T be a right ideal of S. Then T contains a minimal right ideal. Furthermore, this right ideal is closed.*

Proof. Let T be a right ideal and let $x \in T$. Then xS is a compact right ideal contained in T. The compactness follows from the continuity of left shifts in S while $x \cdot S \subset T \cdot S \subset T$. Let $\mathcal{H} = \{H : H \text{ is a compact right ideal }, H \subset T\} \neq \emptyset$. Let H_α be a chain in $\mathcal{H}$ then by the finite intersection property of S we know that $\cap H_\alpha$ is a lower bound for the chain. So by Zorn's Lemma we know that $\mathcal{H}$ contains a minimal element i.e. Any right ideal contains a minimal right ideal. □

We know show the existence of minimal ideals in a left shift invariant compact topological semigroup.

Lemma 3.31. *Let S be a compact, left shift continuous topological semigroup. Then S contains a minimal ideal*

Proof. We proceed by constructing a minimal ideal as the largest minimal right ideal in the semigroup. So define

$$\mathcal{K} = \bigcup \{T : T \text{ is a minimal right ideal of } S\}$$

We can use this lemma to show the existence of minimal idempotents. By Lemma 3.30 we know that $\mathcal{K} \neq \emptyset$. Let $x \in S$ then $\mathcal{K} \cdot x = T_\alpha \cdot x \subset T_\alpha \subset \mathcal{K}$ for some α and thus $\mathcal{K}$ is a right ideal. To show that $\mathcal{K}$ is a left ideal, let $x \in S$ and $y \in \mathcal{T}$ which implies the existence of an α such that $y \in T_\alpha$. We know that $x \cdot y \in x \cdot T_\alpha$ and that $x \cdot T_\alpha$ is trivially a right ideal. However, it can be shown that $x \cdot T_\alpha$ is in fact a minimal right ideal of S and so $x \cdot y \in x \cdot T_\alpha \in \mathcal{K}$ and thus $\mathcal{T}$ is in fact an ideal of the semigroup S. To show this claim, consider the set $A = \{z \in T_\alpha : y \cdot z \in H\}$ where H is a minimal right ideal contained in $x \cdot T_\alpha$. But then A is a trivially a right ideal which implies that $H = x \cdot T_\alpha$. Finally, let I denote any other ideal in S. We know that for any minimal left ideal T that $T \cap I \neq \emptyset$ and that $T \cap I$ is a right ideal contained in T and so $T = T \cap I$. But this implies that $T \subset I$ and thus $\mathcal{T} \subset I$ □

Theorem 3.32. *Let S be a compact, left shift continuous topological semigroup. Then S contains a minimal idempotent.*

Proof. We know by Lemma 3.31 that S contains a minimal ideal, say I. By the proof of Lemma 3.31 we know that we may assume that there exists a minimal closed right ideal, say L, contained in this minimal ideal. The restriction of the extended semigroup operations to this minimal closed right ideal are well defined (since it is a right ideal) and are continuous (since it is closed). Also since the minimal right ideal is a closed subset of a compact space we know that it is compact too. So we can apply the Ellis theorem (theorem 3.28) to show the existence of an idempotent in $L \subset I$ □

It is possible to define a natural ordering on the set of idempotents of a semigroup. The notion of a minimal idempotent in that setting is equivalent to this one.

3.4 Finite Product Theorems

The Finite Product Theorem has a long and interesting history. After originally being proven combinatorially by Neil Hindman in 1975 (in a proof that was described by Hindman as perfect for anyone with a masochistic bent) it was proven using the methods given previously by Galvin and Glazer. As it forms the very basis of the construction required in Chapter 4, the full proof of this important result will be given. Since it uses almost all the Stone-Cech machinery developed so far it also serves as an illustrative example for the power of the method. Finally we finish with the celebrated Van Der Warden's theorem.

Definition 3.33. *Let S be an infinite semigroup with $(x_s)_{s\in\mathbb{N}}$ a sequence in S. Let $\mathcal{P}_f(\mathbb{N})$ denote the set of all finite subsets of $\mathbb{N}$ and let $\prod_{n\in F} x_n$ denote the product in increasing order of indices $x_{n_1}\cdot x_{n_2}\cdot ...\cdot x_{n_m}$ for all $F\in\mathcal{P}_f(\mathbb{N})$. Then the set of all finite products is $FP(x_n)_{n\in\mathbb{N}} = \{\prod_{n\in F} x_n : F\in\mathcal{P}_f(\mathbb{N})$*

The Finite Product Theorem is a combinatorial theorem that states that given any finite partition $\{U_i : i = 1...r\}$ of a semigroup S there exists both a sequence $(x_n)_{n\in\mathbb{N}}$ in S and an element of the partition U_j such that $FP(x_n)_{n\in\mathbb{N}}\subset U_j$. For example consider the natural numbers under addition as the semigroup in question and consider the partition $\mathbb{N} = \{2n : n\in\mathbb{N}\}\cup\{2n-1 : n\in N\}$. Then the subset $\{2n : n\in\mathbb{N}\}\subset\mathbb{N}$ and the sequence $x_n = 4n$ satisfy the claims of the Finite Product Theorem. Before we can prove the theorem though, we require a lemma.

Definition 3.34. *For all $A\subset S$ and $p\in\beta S$, let $\Omega_p(A) = \{x\in S : x^{-1}\cdot A\in p\}$*

Recall that we constructed the extension of the semigroup operation $\cdot : S\times S\to S$ using p-limits as these guaranteed that our operators were left continuous and well defined on βS, the Stone-Cech compactification of our semigroup S. We can use the sets Ω to construct these limits explicitly, as the following lemma demonstrates. Recall that the subsets $\bar{A} = \{p\in\beta S : A\in p\}$ form a clopen basis for the compact topology on βS.

Lemma 3.35. *Let $p, q \in \beta S$ then*

$$q\text{–}\lim_{s\in S} p\text{–}\lim_{t\in S} s\cdot t = p\cdot q = \{A \subset S : \Omega_q(A) \in p\} \tag{3.1}$$

Proof. Let $U \subset S$ such that $\{A \subset S : \Omega_q(A) \in p\} \subset \bar{U}$. This implies that:

$$\begin{aligned} & \Omega_q(U) \in p \\ \Leftrightarrow \quad & \{s : s^{-1}\cdot U \in q\} \in p \\ \Leftrightarrow \quad & \{s : \{t : s\cdot t \in U\} \in q\} \in p \end{aligned}$$

Now let $\alpha(s)$ denote the ultrafilter generated by the set $\{s\cdot B : B \in p\}$. We claim that $p\text{–}\lim_{t\in S} s\cdot t = \alpha(s)$. For suppose $\alpha(s) \in \bar{V}$ i.e. $V \in \alpha(s)$ then $V \cap (s\cdot B) \neq \emptyset$ for all $B \in p$ which implies that $\{t \in S : s\cdot t \in V\} \cap B \neq \emptyset$ for all $B \in p$ and thus $\{t \in S : s\cdot t \in V\} \in p$. This is sufficient to prove the first claim as well as implying that

$$\{s : p\text{–}\lim_{t\in S} s\cdot t \in \bar{V}\} = \{s : V \in \alpha(s)\} \tag{3.2}$$

Now let $s \in \{s : \{t : s\cdot t \in V\} \in q\}$ i.e. $\{t : s\cdot t \in V\} \in q$ and suppose that $V \notin \alpha(s)$ then as $\alpha(s)$ is an ultrafilter we have that $X \setminus V \in \alpha(s)$. This implies that $(X \setminus V) \cap s\cdot B \neq \emptyset$ for all $B \in q$. However, $\{t : s\cdot t \in V\} \in q$ and so $(X \setminus V) \cap s\cdot\{t : s\cdot t \in V\} \subset (X \setminus V) \cap V = \emptyset$ which is a contradiction. So $\{s : \{t : s\cdot t \in U\} \in q\} \subset \{s : U \in \alpha(s)\}$ which by equation 3.2 implies that $\{s : p\text{–}\lim_{t\in S} s\cdot t \in \bar{U}\} \in p$ □

We know have sufficiently many pieces to prove the following lemma, from which the Finite Product Theorem follows trivially.

Lemma 3.36. *Let S be a discrete semigroup with $p \in \beta S$ an idempotent and $A \in p$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $FP(x_n)_{n\in\mathbb{N}} \subset A$*

Proof. Now $p \in \beta S$ is an idempotent, so $p\cdot p = p$. But by Lemma 3.35 we know that $\{A \subset S : \Omega_p(A) \in p\} = p$ therefore $\Omega_p(A) = \{x \in S : x^{-1}\cdot A \in p\} \in p$. Let $B_1 = A$ and define a sequence of $B_n \in p$ such that $B_{n+1} = B_n \cap x_n^{-1}B_n$ where $x_n \in B_n \cap \Omega_p(B_n)$. Notice that this implies that $x_n^{-1}B_n \in p$ and so $B_{n+1} =$

$B_n \cap x_n^{-1} B_n \in p$ as claimed. We claim that (x_n) is the required sequence. We proceed by induction on the length of the finite product. If $k = 1$ then

$$\prod_{i=1}^{k} = x_r \in B_r$$

for some $r \in \mathbb{N}$. So suppose the claim is true for all products $k < m$, then consider some finite product of elements of (x_n) of length $k = m$. Let F denote the set of all subscripts of that finite product with $r = \min(F)$ and $H = F \setminus \{r\}$ and $s = \min(H)$. Then

$$\prod_{n \in H} x_n \in B_s \subset B_{r+1} \subset x_r^{-1} B_r$$

But then

$$\prod_{n \in F} x_n = x_r \prod_{n \in H} x_n \subset x_r \cdot x_r^{-1} B_r \subset B_r \subset A$$

. □

There is a different way of looking at the finite product theorem that can aid intuition.

Definition 3.37. *A collection of subsets $\mathcal{P}$ of a set S is called* Partition Regular *if for any finite partition $\{U_i\}$ of S there exists a n such that $U_n \in \mathcal{P}$*

For example, any ultrafilter is partition regular. A partition regular collection is in some sense a large collection of subsets of S. We call a set an IP set if it contains any finite product of its elements. The Finite product theorem states that the family of all IP subsets of a semigroup are partition regular. We can now prove the original Finite Product Theorem.

Theorem 3.38 (Finite Product Theorem). *Let S be a semigroup and let A_i, $i = 1...r$ be a collection of finite disjoint subsets of S such that $S = \bigcup A_i$ i.e. The A_i form a finite partition of the semigroup. Then there exists a sequence (x_n) such that $FP(x_n) \subset A_m$ for some $m \in \{1...r\}$*

Proof. Consider the Stone-Cech compactification of S when considered as a discrete topological space. Then by the Ellis theorem (theorem 3.28) we know that

there exists an idempotent $p \in \beta S$ such that $A_m \in p$ for some $m \in \{1...r\}$. But then by Lemma 3.36 we know that there must exist a sequence (x_n) such that $FP(x_n)_{n\in\mathbb{N}} \subset A_m$ □

If the semigroup in question is abelian then this theorem is also known as the Finite Sums Theorem (and was originally proved as such with $\mathbb{N}$ as the underlying semigroup). We know give one application of the Finite Product theorem that is important for a later construction. Before we can do that though we need to give an important corollary of the Finite Product Theorem which is useful for when we have binary partitions with a particular structure. It allows us to choose the element of the partition we are interested in, subject to a cardinality restriction.

Corollary 3.39. *Let S be a discrete semigroup which is algebraically a group. Let A_i, $i = 1...r$ be a finite partition of S. Then not only is the finite product theorem true, it is possible to find a sequence (x_n) such that $FP(x_n)_{n\in\mathbb{N}} \subset A_m$ with $|A_m| = |S|$*

Proof. Reconsider the proof of Lemma 3.36. Let $A \subset S$ such that $|A| = |S|$. If we can find an idempotent ultrafilter p such that $A \in p$ we are done. We claim that for any subset $A \subset S$ with $|A| = |S|$ there exists an idempotent ultrafilter p such that $A \in p$. Now the set of all cofinite ultrafilters $\mathcal{C}$ forms a closed subsemigroup of βS and thus is compact. This implies that the set of all cofinite ultrafilters must contain at least one idempotent, say p, and since any subset $A \subset S$ such that $|A| = |S|$ is a member of any cofinite ultrafilter we know that $A \in p$. So we need only show that $\mathcal{C}$ is in fact a closed subsemigroup of S. Firstly suppose that $p, q \in \mathcal{C}$ i.e. p, q are cofinite ultrafilters. Let $A \subset S$ be a cofinite set. Now for any $x \in S$ we know that $x^{-1}A$ is a cofinite set and thus $\Omega_q(A) = S$. But then any cofinite subset of S is a member of $p \cdot q$ and thus $\mathcal{C}$ is a subsemigroup of S. To show that $\mathcal{F}$ is a closed subset of βS we take a round about route. Let (A_α) be an enumeration of all cofinite subsets of S. Now $B_\alpha = cl_{\beta S}(A_\alpha) = \{p \in \beta S : A_\alpha \in p\}$ is a closed subset of βS. Also the family (B_α) has the finite intersection property

(as any co-finite ultrafilter is in their finite intersection) and thus

$$B = \bigcap_{\alpha} B_\alpha \neq \emptyset$$

But $\mathcal{C} = B$ and B is a closed subset of βS and so $\mathcal{C}$ is closed. □

As was seen in the proof of Markov's theorem, the question of the cardinality of a system of inequalities on a group can lead to answers about possible topological structures permissible on that group. We can use the Finite Product Theorem to show that a certain system of inequalities has many solutions.

Theorem 3.40. *Let G be a group with identity $e \in G$ and let $\mathcal{S}$ be a system of inequalities of the form $a \cdot y \cdot b \neq y^\epsilon$ where $a \cdot b \neq e$ and $\epsilon = \pm 1$ where y is a variable. Then $\mathcal{S}$ has $|G|$ solutions*

Proof. Suppose the theorem is false, so there exists some system of inequalites $\mathcal{S}$ and a subset $F \subset G$ such that $|F| < |G|$ and such that for any $x \in G \setminus F$ there exists an inequality of the system $\mathcal{S}$, say $a \cdot y \cdot b \neq y^\epsilon$ such that $a \cdot x \cdot b = x^\epsilon$. Now $|G \setminus F| = |G|$ and so by corollary 3.39 we know that there exists a sequence (x_n) such that $FP(x_n) \subset G \setminus F$. We know consider two cases.

Case 1. If $\epsilon = 1$ then for any $i, j \in \mathbb{N}$ and $x_i, x_j \in (x_n)$ we know that $a \cdot x_i \cdot x_j \cdot b = x_i \cdot x_j$. But then

$$a \cdot x_i \cdot x_j \cdot b = a \cdot x_i \cdot b \cdot b^{-1} \cdot a^{-1} \cdot a \cdot x_j \cdot b = x_i (a \cdot b)^{-1} x_j$$

This implies that $a \cdot b = e$ which is a contradiction.

Case 2. If $\epsilon = -1$ then for any $i, j \in \mathbb{N}$ and $x_i, x_j \in (x_n)$ we know that $a \cdot x_i \cdot x_j \cdot b = x_j^{-1} \cdot x_i^{-1}$. But then $a \cdot x_i \cdot x_j \cdot b = x_j^{-1} \cdot x_i^{-1} = x_i^{-1}(ab)^{-1} x_j^{-1}$ which implies that $x_i^{-1} \cdot x_j \cdot (a \cdot b \cdot x_i)^{-1} = x_j$ However, this is exactly the first case considered as $x_i^{-1} \cdot a \cdot b \cdot x_i \neq e$.

□

We know give the final theorem in this sequence, the famous Van Der Warden's theorem regarding the existence of arbitrary long geometric sequences in any abelian semigroup. The proof is partially based on [Ber03]

Theorem 3.41 (Van Der Warden). *Let S be a discrete abelian semigroup with identity e and let $p \in \beta S$ a minimal idempotent. If $A \in p$ then A contains arbitrary long geometric sequences.*

Proof. Consider the compact, left shift invariant semigroup given by the direct product of $k \in \mathbb{N}$ copies of βS. This is compact as a topological space since the product of compact spaces is compact and the continuity of the operator follows from the continuity of each projection onto the underlying space and the definition of the product topology. Let $p \in \beta S$ be any minimal idempotent of βS, whose existence is guaranteed by Theorem 3.32. Define $P = (p, p, ..., p) \in (\beta S)^k$ and consider the subsemigroup

$$E = cl(\{(a, a \cdot b, ..., a \cdot b^{k-1}) : a \in S \setminus \{e\}, b \in S\})$$

and the two sided ideal

$$I = cl(\{(a, a \cdot b, ..., a \cdot b^{k-1}) : a, b \in S \setminus \{e\}\})$$

Now $P \in E$ since if A_1, A_2, ..., $A_k \in p$ then $P \in \bar{A}_1 \times \bar{A}_2 \times ... \times \bar{A}_k$. But then if $a \in \bigcap A_i$ then

$$(a, a, ..., a) \in (\bar{A}_1 \times \bar{A}_2 \times ... \times \bar{A}_k) \cap \{(a, a \cdot b, ..., a \cdot b^{k-1}) : a \in S \setminus \{e\}, b \in S\}$$

which is our claim. Now let R be the minimal right ideal of βS such that $p \in R$. But $P \in E$ and so $P \cdot E$ is also a right ideal and thus their exists a minimal right ideal $\tilde{R}$ such that $\tilde{R} \subset P \cdot E$. Now since every minimal right ideal contains an idempotent, we can find a such an idempotent $q \in \tilde{R}$ such that $q = (q_1, q_2, ..., q_k)$. But then since $q \in P \cdot E$ there must exist $s \in E$ such that $q = P \cdot s$ where $s = (s_1, s_2, ..., s_k)$. But then $q_i \in R$ and since R is a minimal right ideal we know that $q_i \cdot \beta S = R$. This statement implies that $P = q_i \cdot t_i$ and thus $q_i \cdot P = q_i \cdot q_i \cdot t_i = q_i \cdot t_i = P$ so $P = q + P$. But then $P \in R$ and so P is in the smallest two sided ideal in $(\beta S)^k$ and so $P \in I$. Now suppose $A \in p$ and so $P \in (\bar{A})^k$. But then $P \in (\bar{A})^k \cap I$ which shows that

$$A^k \cap \{(a, a \cdot b, ..., a \cdot b^{k-1}) : a, b \in S \setminus \{e\}\} \neq \emptyset$$

□

Chapter 4

Topologizable groups and semigroups

We showed previously that there exist groups and semigroups which do not admit nondiscrete Hausdorff group or semigroup topologies respectively. This is not a totally devastating state of affairs as many groups and semigroups admit slightly weaker generalisations of these structures. This chapter will show this result for a number of groups and semigroups satisfying a limited number of conditions. Namely, it will be shown that every embeddable semigroup admits a left invariant topology (we follow [Pap91]) and we extend this result to show that the embeddability of the group is not required i.e. every semigroup admits a left invariant topology. We finish this section with a recent result by Zelenyuk ([Zel07]) where he proves that in fact every group admits an invariant topology. This result uses most of the apparatus we have developed so far, especially the finite product theorem. The proof that every semigroup admits a left invariant topology is based partially on [Zel07], namely theorem 4.21 is based on [Zel07].

4.1 Left shift invariant group topologies determined by idempotent ultrafilters

Every semigroup has a maximal semigroup compactification, namely the Stone-Cech compactification of the semigroup when considered as a discrete topological space. As mentioned previously, the operation of left translation can be extended in a continuous way on the Stone-Cech compactification so that this compactification is in fact a left translation invariant semigroup. Now by Ellis's theorem (Theorem 3.28) this semigroup contains idempotents and, as will be shown in this section, these idempotents generate in some sense topologies on the original semigroup. Furthermore, if this semigroup is a group these topologies are Hausdorff. So the point of this section is to prove that every group admits both left invariant, zero dimensional group topology.

Definition 4.1. *Let $\mathcal{T}$ be a topology on a Semigroup S*

1. *$\mathcal{T}$ is left shift invariant if for every $U \in \mathcal{T}$ and every $x \in S$ we have that $xU \in \mathcal{T}$ and $x^{-1}U \in \mathcal{T}$*

2. *$\mathcal{T}$ is right shift invariant if for every $U \in \mathcal{T}$ and every $x \in S$ we have that $Ux \in \mathcal{T}$ and $Ux^{-1} \in \mathcal{T}$*

3. *$\mathcal{T}$ is shift invariant if it is both left and right shift invariant*

For example, the discrete topology is a left shift invariant topology on the natural number $\mathbb{N}$ under addition. As a less trivial example consider the topology generated by sets of the form $V(a,d) = a + \{nd : n \in \mathbb{Z}\}$ form a left shift invariant topology on the integers $\mathbb{Z}$. As both these semigroups are Abelian, they give trivial examples of invariant topologies. Not every infinite semigroup admits a non-discrete left shift invariant topology. Let S be an infinite set and define a multiplication on S as follows: $x \cdot y = x$ for all $x, y \in S$. Now if $\mathcal{T}$ is a left shift invariant topology on S then for all $x \in S$ we have that $xS = \{x\} \in \mathcal{T}$ which implies that $\mathcal{T}$ is discrete. This semigroup is the infinite left-zero semigroup. However, a

large class of infinite semigroups with some form of cancellative property do admit a left shift invariant topology.

Let S be an arbitrary infinite cancellative semigroup with identity $e \in S$

The existence of an identity element results in no loss of generality. For suppose S is a semigroup without identity and $\mathcal{T}$ is a Hausdorff left shift invariant topology on the Semigroup with identity $S' = S \cup \{e\}$. The subset $S \subset S'$ is an open subset of S' and thus the subspace topology on S is a Hausdorff, left shift invariant topology. The notion of a left shift invariant topology allows a simpler classification, as the next lemma demonstrates.

Lemma 4.2. *A topology on S is left shift invariant if and only if every left shift by an arbitrary element $x \in S$ is both open and continuous*

Proof. First suppose that $\mathcal{T}$ is a left shift invariant topology on S and let $x, y \in S$ and $V \in \mathcal{T}$ such that $xy \in V$. Now $y \in x^{-1}V$ as $xy \in V$ and further $x(x^{-1}V) \subset V$ and so the left shift $y \to xy$ is continuous. Further, as $xU \in \mathcal{T}$ for all $x \in S$ we have that the left shift $y \to xy$ is also open. The reverse implication is trivial. □

The fact that all left shifts are open and continuous allows any left shift invariant topology to be generated entirely by the neighbourhood basis of identity.

Lemma 4.3. *Let $\mathcal{N}_e$ be a basis for the neighbourhood system of identity of a left shift invariant topology on S. The family of subsets $\mathcal{B} = \{xF : x \in S \text{ and } F \in \mathcal{N}_e\}$ forms a basis for this topology. Furthermore the family of subsets $x\mathcal{N}_e$ forms a basis for the neighbourhood system at $x \in X$.*

Proof. Let U be an open subset of S and let $y \in U$. The function $l_y : S \to S$ defined by $l_y(t) = yt$ is continuous and open as the topology is left shift invariant. So if $V \subset S$ is open then yV is also open i.e. $\mathcal{B}$ is a family of open subsets of S. Furthermore there must exist an open neighbourhood of identity $\tilde{U} \ni e$ such that $y\tilde{U} \subset U$. However, this implies that there exists a $V_y \in \mathcal{N}_e$ such that $yV_y \subset y\tilde{U} \subset U$. Hence $\{yV_y\}_{y \in U}$ is a subset of $\mathcal{B}$ with $U = \bigcup_{y \in U} yV_y$ and so $\mathcal{B}$ is a basis for the topology and $y\mathcal{N}_e$ forms a basis for $\mathcal{N}_y$. □

The idea that a basis for a neighbourhood system at identity fully determines the topology on a left shift invariant space suggests that there is a characterisation of the topology is the same vain as that proposed by Pontryagin for general topological groups. The following lemma shows that this is in fact the case.

Lemma 4.4. *Let $\mathcal{F}$ be a filter on S then $\mathcal{F}$ is a basis for the neighbourhood system of identity $e \in S$ of a left shift invariant topology on S if and only if:*

1. *$e \in F$, $\forall F \in \mathcal{F}$*
2. *if $F_1 \in \mathcal{F}$ and $f \in F_1$ then $\exists F_2 \in \mathcal{F}$ such that $f \cdot F_2 \subseteq F_1$*

Proof. Firstly suppose that $\mathcal{F}$ is the neighbourhood system of identity of a left shift invariant topology on S. Then

- $e \in F$, $\forall F \in \mathcal{F}$ as F is an open neighbourhood of e.
- The family of subsets $\{xF : x \in S \text{ and } F \in \mathcal{F}\}$ form a basis for the topology on S. Now $F_1 \in \mathcal{F}$ if and only if $\forall f \in F_1$, $F_1 \in \mathcal{N}_f$ but $f\mathcal{F}$ forms a basis for $\mathcal{N}_f$ so $\exists F_2 \in \mathcal{F}$ such that $fF_2 \subseteq F_1$

Now suppose that $\mathcal{F}$ is a filter on S such that:

1. $e \in F$, $\forall F \in \mathcal{F}$
2. if $F_1 \in \mathcal{F}$ and $f \in F_1$ then $\exists F_2 \in \mathcal{F}$ such that $f \cdot F_2 \subseteq F_1$

Firstly the family of subsets $\{xF : x \in S \text{ and } F \in \mathcal{F}\}$ is a basis for a topology on S. This is because:

- As $e \in F$ $\forall F \in \mathcal{F}$ which implies that $X = \bigcup_{x \in S} xF$ where $F \in \mathcal{F}$
- Suppose $t \in xF \cap yG$ where $F, G \in \mathcal{F}$ and $x, y \in S$. This implies that $t = xf_1 = yf_2$ where $f_1 \in F$ and $g_1 \in G$. So $\exists \tilde{F}, \tilde{G} \in \mathcal{F}$ such that $f_1\tilde{F} \subseteq F$ and $g_1\tilde{G} \subseteq G$. Now $\mathcal{F}$ is a filter so $H = \tilde{F} \cap \tilde{G} \in \mathcal{F}$ and $tH \subseteq xF \cap yG$.

So $\mathcal{B} = \{xF : x \in S \text{ and } F \in \mathcal{N}_e\}$ is a basis for a topology on S. Now this topology is in left shift invariant as

- l_x is continuous : let $t \in S$ and let U be an open neighbourhood of xt. Then there exists a $\tilde{U} \in \mathcal{F}$ such that $xt\tilde{U} \subseteq U$. But $t\tilde{U}$ is a basic open neighbourhood of $t \in S$ so $\forall$ open neighbourhoods $U \ni xt$ there exists an open neighbourhood $V = t\tilde{U}$ such that $l_x(V) = xV \subseteq U$

- l_x is open : Let U be an open subset of S so $U = \bigcup_{\alpha \in I} x_\alpha F_\alpha$ where $\{x_\alpha\} \subseteq S$ and $\{F_\alpha\} \subseteq \mathcal{F}$. Then $l_x(U) = x \bigcup_{\alpha \in I} x_\alpha F_\alpha = \bigcup_{\alpha \in I} x x_\alpha F_\alpha$ so $l_x(U)$ is open.

So the basis $\mathcal{B}$ is a basis for a left shift invariant topology on the S. □

So any filter satisfying the conditions of Lemma 4.4 generates a left invariant topology on the Semigroup S. Now as any group G is indeed a semigroup under the same operation we can compactify our group when considered as a discrete space and use the results of this section. The method of proof here will be to use the existence of idempotents on any compact, left continuous semigroup to show the existence of a filter satisfying the conditions of Lemma 4.4 and such that the resulting left invariant topology is Hausdorff. However, before we can show this, some work on the algebraic structure of βS must be done.

Recall the definition of the extended semigroup operation on βS and the characterization of this operation given in Lemma 3.35. With this characterisation and explicit construction of the extended operation of interest, we prove some algebraic properties of βS

Lemma 4.5. *Let $x \in S$ and $p \in \beta S \setminus S$ then*

1. *$x \cdot p = p$ if and only if $\{y \in S : x \cdot y = y\} \in p$*

2. *$p \cdot x = p$ if and only if $\{y \in S : y \cdot x = y\} \in p$*

3. *if $|\{y \in S : x \cdot y = y\}| < \infty$ then for all $p \in \beta S \setminus S$, $x \cdot p \neq p$*

4. *if $y \in S$ is another distinct element then $x \cdot p \neq y \cdot p$*

Proof. Statement 1 and statement 2 follow from Theorem 3.18 by considering the left and right shifts on the semigroup S. Now suppose $|\{y \in S : x \cdot y = y\}| < \infty$ but

we have that $x \cdot p = p$. By statement 1, this implies that $\{y \in S : x \cdot y = y\} \in p$. But then the ultrafilter p contains a finite set and is thus principle. This implies that $p \in S$, where we have once again identified the semigroup S with its embedding in βS. Now suppose that $p \in S \setminus \{x\}$ but $x \cdot p = y \cdot p$, where $x, y \in S$. This implies that $(y^{-1}x) \cdot p = p$ which implies that $\{z \in S : y^{-1}xz = z\} \in p$ i.e. there exists a $z \in S$ such that $xz = yz$. But then $x = y$, a contradiction. □

We now come to the first construction in this series, namely a Hausdorff left shift invariant topology on an arbitrary group G.

Theorem 4.6. *Let p be an idempotent in $\beta G \setminus G$ then the family of subsets $B = \{(\Omega_p(U) \cap U) \cup \{e\} : U \in p\}$ forms the neighbourhood basis of identity for a Hausdorff, left invariant group topology on G.*

Proof. We break the proof into a number of substeps

Step 1 The family $B = \{(\Omega_p(U) \cap U) \cup \{e\} : U \in p\}$ forms a filter basis. To see this, let $U, V \in p$, then $U \cap V \in p$. So suppose $x \in \Omega_p(U \cap V) \in B$ then $x^{-1} \cdot (U \cap V) \in p$. However, $x^{-1} \cdot (U \cap V) \subset x^{-1} \cdot U$ and thus $x \in \Omega_p(U)$. Similarly, $x \in \Omega_p(V)$ and thus $\Omega_p(U \cap V) \subset \Omega_p(U) \cap \Omega_p(V)$ But then $\Omega_p(U \cap V) \cap (U \cap V) \subset (\Omega_p(U) \cap U) \cap (\Omega_p(V) \cap V) \in B$.

Step 2 The family B forms a filter basis for the neighbourhood filter of identity of a left shift invariant topology on G. To see this, we first note that by construction $e \in U$ for all $U \in B$. We need only check condition two of Lemma 4.4. So let $\tilde{U} \in B$ and let $x \in \tilde{U}$. Now since $x \in \tilde{U}$ we know that $x \in (\Omega_p(U) \cap U) \cup \{e\}$ for some $U \in p$. If $x = e$ then trivially, $x\tilde{U} \subset \tilde{U}$ so suppose that $x \in (\Omega_p(U) \cap U)$ and $x \neq e$. This implies that $x^{-1} \cdot U \in p$. So consider $\tilde{V} = (\Omega_p(x^{-1} \cdot U) \cap x^{-1} \cdot U) \cup \{e\}$ and $z \in x \cdot \tilde{V}$. The one case is trivial, since if $z \in \{x\}$ then $z = x \in \tilde{U}$. So suppose there exists a $t \in (\Omega_p(x^{-1} \cdot U) \cap x^{-1} \cdot U)$ such that $z = xt$. But then $(xt)^{-1} \cdot U \in p$ which shows that $z^{-1} \cdot U \in p$. Also $t \in x^{-1} \cdot U$ which implies that $z \in U$ i.e. $z \in \tilde{U}$. So by Lemma 4.4 we know that there exists a left shift continuous topology on G

Step 3 This topology is Hausdorff and is not discrete. The second case is easy to show. For suppose this left shift continuous topology is discrete, then $\cap B = \{e\}$. But the idempotent ultrafilter p was chosen to be free, and thus there exists a $U \in p$ such that $e \notin U$. This contradicts the assertion. The Hausdorff claim is more difficult to show. Recall that a topological space G with topology $\mathcal{T}$ is Hausdorff if for all $x \in G$,

$$\bigcap\{\bar{U} : x \in U \in \mathcal{T}\} = \{x\}$$

So suppose there exists a $t \in \bigcap\{\bar{U} : U \in B\}$ with $t \neq e$. This means that for any $U \in B$ and $V \in B$, $t \cdot V \cap U \neq \emptyset$. So let $y \in t \cdot V \cap U$ i.e. There exist $z_1 \in V = (\Omega_p(V_1) \cap V_1) \cup \{e\}$ and $z_2 \in (\Omega_p(U_1) \cap U_1) \cup \{e\}$ such that $y = tz_1 = z_2$, for some $U_1, V_1 \in p$. Now if $U_1, V_1 \in p$, then we must have that $U_2 = U_1 \setminus \{t, t^{-1}\}, V_2 = V_1 \setminus \{t, t^{-1}\} \in p$ since p is a free ultrafilter. But then there must exist $z_3 \in (\Omega_p(V_2) \cap V_2) \cup \{e\}$ and $z_4 \in (\Omega_p(U_2) \cap U_2) \cup \{e\}$ such that $tz_3 = z_4$. But since $t \notin U_2$ and $t^{-1} \notin V_2$ we must have that $(t \cdot (\Omega_p(V_2) \cap V_2)) \cap (\Omega_p(U_2) \cap U_2) \neq \emptyset$ which shows that $(t \cdot (\Omega_p(V_1) \cap V_1)) \cap (\Omega_p(U_1) \cap U_1) \neq \emptyset$. So we have shown that for all $U_1, V_1 \in p$, $t \cdot V_1 \cap U_1 \neq \emptyset$. However, this statement implies that the family of subsets $\{t \cdot V_1 \cap U_1 : U_1, V_1 \in p\}$ is a filter base for the ultrafilter p. Now if we can show that $t \cdot p = p$ then by Lemma 4.5 we have the contradiction that $t = e$ and $t \neq e$. So let $A \in t \cdot p$, then $t^{-1} \cdot A \in p$. But $\{t \cdot V_1 \cap U_1 : U_1, V_1 \in p\}$ is a filter base for the ultrafilter p and so $t \cdot V_1 \in p$ for all $V_1 \in p$. This implies that $(t \cdot t^{-1}) \cdot A = A \in p$. So if $A \in t \cdot p$ then $A \in p$. Also if $A \in p$ then by the above argument we have that $t \cdot A \in p$. This shows that $t \cdot p = p$, and thus we arrive at the required contradiction. This implies that $t = e$ and so the topology must be Hausdorff.

□

So there is a strong link between the existence of idempotent ultrafilters and the existence of certain types of topologies on a group. All that is left to do is show that the required non principal idempotent ultrafilters do indeed exist.

Since the Stone-Cech compactification of G is compact and G is locally compact in the discrete topology we have that $\beta G \setminus G$ is a compact subset of βG. So we know by Ellis' theorem and Theorem 3.13 that idempotents exist in $\beta G \setminus G$. This proves the existence of left shift invariant topologies on arbitrary groups. Now suppose that we have a semigroup which is embeddable in a group, then we can embed our semigroup in a non-discrete, Hausdorff topological group by Theorem 4.6. However, the subspace topology imposed onto the semigroup from its parent group may be discrete. To cover any more ground, we first require a few lemmas.

Lemma 4.7. *Let B be some base for the neighbourhood system of identity of a left shift invariant topology and let C be another base for the neighbourhood system of identity of a left shift invariant topology such that for all $U \in B$ there exists a $V \in C$ such that $V \subset U$. Then the topology generated by B is weaker then the one generated by C*

Proof. Let U be an element of the topological base B, then since the topology is left shift invariant we know that for all $x \in U$ there exists a $U_x \in B$ such that $x \cdot U_x \subset U$. Since $e \in U_x$ for all $x \in B$ we have in fact that $U = \cup_{x \in U} x \cdot U_x$. But for all U_x there exists a $V_x \in C$ such that $V_x \subset U_x$ and so $U = \cup_{x \in U} x \cdot V_x$. So every element of B is an open set in the topology generated by C and thus generates a weaker topology □

We wish to extend the domain of certain ultrafilters for this construction. The following lemma is the first such extension theorem we require.

Lemma 4.8. *Let X be a set with subset $A \subset X$. Any ultrafilter $p \in 2^{2^A}$ is of the form $\{A \cap U : U \in \hat{p}\}$ for some ultrafilter $\hat{p} \in 2^{2^X}$. If p is a non-principle ultrafilter then so is $\hat{p}$*

Proof. The family p of subsets of $A \subset X$ has the finite intersection property and can thus be extended to some ultrafilter $\hat{p}$ on X. Since $A \in p \subset \hat{p}$ we know that $A \cap U \neq \emptyset$ for all $U \in \hat{p}$ and so $\phi = \{A \cap U : U \in \hat{p}\}$ is a filter on A. This is easy to see since if $U, V \in \hat{p}$ then $A \cap U, A \cap V \in \phi$ and since $U \cap V \in \hat{p}$ we know that

$(A \cap U) \cap (A \cap V) = A \cap (U \cap V) \in \phi$. Secondly if $B \subset A$ and $A \cap U \subset B$ then since $B \subset X$ and $U \subset B$ we know that $B \in \hat{p}$. But then $B = B \cap A \in \phi$. Finally, $p \subset \phi$ since if $W \in p \subset \hat{p}$ then $W = A \cap W \in \phi$. However, p is an ultrafilter and thus

$$p = \phi = \{A \cap U : U \in \hat{p}\}$$

Finally, suppose p is a non-principal ultrafilter but its extension $\hat{p}$ is a principle ultrafilter, say generated by $x \in X$. If $x \in A$ then $x \in A \cap U$ for all $U \in \hat{p}$ which contradicts the freeness of p. So $x \in X \setminus A$ but then $A \in \hat{p}$ and $X \setminus A \in \hat{p}$ which contradicts the fact that $\hat{p}$ is a filter itself. So $\hat{p}$ is a non-principle ultrafilter. □

Recall the definition of the omega sets, $\Omega_q(U) = \{x \in S : x^{-1} \cdot U \in q\}$ where $U \subset S$ and q is a filter on some semigroup S.

Lemma 4.9. *1. Let $U \subset V \subset S$ then $\Omega_q(U) \subset \Omega_q(V)$*

2. Let p and q be filters on S with $p \subset q$, then $\Omega_p(U) \subset \Omega_q(U)$

Proof. 1. Let $x \in \Omega_q(U)$ (i.e. $x^{-1} \cdot U \in q$) but then $\{t \in S : x \cdot t \in U\} \in q$. But $\{t \in S : x \cdot t \in U\} \subset \{t \in S : x \cdot t \in V\}$ and so $\{t \in S : x \cdot t \in V\} \in q$ which implies that $x \in \Omega_q(V)$

2. Let $x \in \Omega_p(U)$, so $x^{-1} \cdot U \in p$ but then $x^{-1} \cdot U \in q$ since $p \subset q$ and thus $x \in \Omega_q(U)$

□

The final ingredient required is the ability to extend idempotent ultrafilters of a subspace to idempotent ultrafilters of the larger parent space.

Lemma 4.10. *Suppose S is a semigroup that can be algebraically embedded into a group G. Then any idempotent ultrafilter p in $\beta S \setminus S$ can be extended to an idempotent ultrafilter $\hat{p}$ in $\beta G \setminus G$*

Proof. We know by Lemma 4.8 that p can be extended to some non-principle ultrafilter $\hat{p}$ on G. We need only show that this extended ultrafilter is in fact an

idempotent in βG. Recall that an ultrafilter $q \in \beta G$ is idempotent if for all $U \in q$ we have that $\Omega_q(U) \in q$ (this follows from Lemma 3.35, since $q \cdot q = \{U \in G : \Omega_q(U) \in q\}$). So let $U \in \hat{p}$, so $S \cap U \in p$ and so $\Omega_p(S \cap U) \in p$ since p is an idempotent ultrafilter. But then $\Omega_p(S \cap U) \in \hat{p}$ and since

$$\Omega_p(S \cap U) \subset \Omega_p(U) \subset \Omega_{\hat{p}}(U)$$

we have that $\Omega_{\hat{p}}(U) \in \hat{p}$ □

We can now prove the extension of Papazyan's theorem, namely that any semigroup which can be algebraically embedded in a group can be endowed with a non discrete, Hausdorff, left shift invariant topology via the topology of the larger group.

Theorem 4.11. *Let S be a semigroup which can be algebraically embedded into some group G, then there exists a non discrete, left shift invariant, Hausdorff topology on S*

Proof. Since G is a group, we know that there exists a non discrete, left shift invariant, Hausdorff topology on G whose neighbourhood system at identity is generated by sets of the form

$$(\Omega_p(U) \cap U) \cup \{e\} : U \in p\}$$

where $p \in \beta G \setminus G$ is any idempotent, non principle ultrafilter. Now endow S with the discrete topology, in which it is locally compact, then by the theorem of Ellis and Theorem 3.13 we know that $\beta S \setminus S$ contains idempotent ultrafilters. Let q be such an ultrafilter. Then q can be extended to a non-principle idempotent ultrafilter $\hat{q}$ on G, with which the sets

$$\{(\Omega_{\hat{q}}(U) \cap U) \cup \{e\} : U \in \hat{q}\}$$

form a basis for the neighbourhood system of identity. We now claim that the family of sets

$$B = \{((\Omega_{\hat{q}}(U) \cap U) \cup \{e\}) \cap S : U \in \hat{q}\}$$

form a basis for the neighbourhood system of identity of a left shift invariant topology. Furthermore, we claim that this topology is stronger then the subspace topology on $S \subset G$. We take each claim in turn

- It is easy to see that B is a base for some filter on S, since it has the finite intersection property. So let $F_1 \in B$ i.e. $F_1 = (\Omega_{\hat{q}}(U_1) \cap (U_1 \cap S)) \cup \{e\}$ and $x \in F_1$, then by Theorem 4.6 we know there exists a $F_2 \in \{(\Omega_{\hat{q}}(U) \cap U) \cup \{e\} : U \in \hat{q}\}$, say $F_2 = (\Omega_{\hat{q}}(U_2) \cap U_2) \cup \{e\}$ such that $x \cdot F_2 \subset (\Omega_{\hat{q}}(U_1) \cap U_1) \cup \{e\}$. But S is a semigroup and thus $x \cdot (F_2 \cap S) \subset F_1$. So by Lemma 4.3, we have a left shift invariant topology on S

- The subspace topology on S is generated by the topological base $\{U \cap S : U \in \mathcal{T}\}$ where $\mathcal{T}$ is the parent topology. So let U be an open set in the parent topology, so $U = \cup_{x \in I} x \cdot U_x$ for some $I \subset G$ and U_x is a member of the neighbourhood basis of identity in the parent topology. Now $U_x \cap S$ is an element of the new neighbourhood basis and $x \cdot (U_x \cap S) \subset (x \cdot U_x) \cap S$ and so by Lemma 4.7 this topology is stronger then the subspace topology

Finally, since any subspace of a Hausdorff space is itself Hausdorff, and a topology stronger then a Hausdorff topology is still Hausdorff, we know that this topology is Hausdorff. Finally, if this topology is discrete then $((\Omega_{\hat{q}}(U) \cap U) \cup \{e\}) \cap S = \{e\}$ for some $U \in \hat{p}$. But then

$$\emptyset \neq ((\Omega_{\hat{q}}(U \cap S) \cap (U \cap S) \cup \{e\}) \subset ((\Omega_{\hat{q}}(U) \cap U) \cup \{e\}) \cap S = \{e\} = \{e\}$$

But since $S \in \hat{p}$ and $U \in \hat{p}$ we know that $U \cap S \in \hat{p}$ and thus the parent topology generated by $\hat{p}$ on the group G is discrete, a contradiction. $\square$

4.2 Left shift invariant topologies on cancellative semigroups

There are semigroups which do not admit semigroup topologies. Luckily, the construction is a lot easier in the semigroup case when compared to the construction and labour that went into showing the same result for groups. The example here is originally by Taĭmanov ([Taĭ73])

Theorem 4.12. *There exists an infinite semigroup X such that all semigroup topologies on it are discrete*

Proof. Let X be an infinite set with $a, b \in X$ such that $a \neq b$. Define a multiplication on X by

$$x \cdot y = \begin{cases} b & \text{if } x = y \text{ or } x, y \in \{a, b\} \\ a & \text{otherwise} \end{cases}$$

It is easy to see that this operation is associative and thus we have defined an infinite semigroup. We start by showing that the point $a \in X$ is an isolated point in any semigroup topology. So let $\mathcal{T}$ be a Hausdorff topology on X such that $f(x, y) = x \cdot y$ is continuous. There exists an open set U such that $b \in U$ but $a \notin U$. Since $a^2 = b$ there exists an open neighbourhood V of a such that $V^2 \subset U$. So suppose there exists two distinct $z_1, z_2 \in V$ such that neither $z_1 = a$ nor $z_2 = a$. But then $a = z_1 \cdot z_2 \in V^2 \subset U$ which is a contradiction. So $U = \{a, z\}$ for some $z \in X$. Choose an open neighbourhood W of a such that $z \notin W$ then $\{a\} = W \cap U$ is open and so a is isolated. Similarly we can show that b is also isolated. Finally let $x \in X \setminus \{a, b\}$. Let V be an open neighbourhood of b that misses a. Since $x^2 = b$ we know that there exists an open neighbourhood U of x such that $U^2 \subset V$. Let $x \neq z \in U$ then $x \cdot z = a \in U^2 \subset V$ which is a contradiction and thus $U = \{x\}$. This implies that $\mathcal{T}$ is a discrete topology □

The situation is actually much more restrictive then the above theorem implies. The previously constructed semigroup is quite perverse and it would be reasonable to hope that better behaved semigroups are better behaved topologically as well.

This is not the case, as the next construction based on Eberhart and Selden (see [ES69]) shows.

Theorem 4.13. *There exists an infinite, countable cancellative semigroup X with identity e such that all topologies with continuous right and left shifts are discrete*

Proof. Consider the bicyclic group B generated by two distinct elements p and q with the relation that $p \cdot q = e$. Choose some prime ρ and add the relation that $p^\rho = e$. Denote this new semigroup by B_ρ. Firstly, B_ρ is a cancellative semigroup since if $a \cdot x = a \cdot y$ then $a = q^m \cdot p^n$ for some $m, n \in \mathbb{N}$ and $n < \rho$. But then $p^{m+\rho-n}) \cdot (a \cdot x) = p^{m+\rho-n} \cdot (a \cdot y)$ and since $(p^{m+\rho-n}) \cdot a = p^{m+\rho-n}) \cdot q^m \cdot p^n = e$ we have our cancellative result. It is easy to see that this semigroup is countable and infinite. Suppose that $\mathcal{T}$ is a non discrete topology on B_ρ with continuous right and left shifts. Now $q \cdot p \cdot B_\rho$ and $B_\rho \cdot q \cdot p$ are both closed sets and thus $\{1\} = B_\rho \setminus (q \cdot p \cdot B_\rho) B_\rho \setminus (\cap B_\rho \cdot q \cdot p)$ is open. This is a contradiction as it implies that the topology $\mathcal{T}$ is discrete. □

So we cannot even settle for the weaker property of shift invariant topologies on cancellative semigroups. The situation changes if we limit ourselves to one sided continuity. First, we describe the basis for the neighbourhood filter of identity for a cancellative semigroup with identity.

Lemma 4.14. *Let $\mathcal{F}$ be a filter on S and let $\mathcal{T}$ be the maximal left shift invariant topology on S. Then $\phi = \{F \cup \{e\} : F \in \mathcal{F}\}$ is a filter on S and if $\mathcal{F} \to \{e\}$ in $\mathcal{T}$ then the family of subsets of S of the form $[f] = \bigcup_{n \in \mathbb{N}} U_n$ where:*

- *$f : S \to \phi$ is a function and*
- *$U_1 = f(e)$ and $U_{n+1} = \bigcup_{t \in U_n} tf(t)$*

form a neighbourhood basis of $e \in S$ for the topology $\mathcal{T}$

Proof. Define $N = \{[f] : f : S \to \Phi\}$

1. ϕ is a filter on S as :

- let $U, V \in \phi$ so $\exists \tilde{U}, \tilde{V} \in \mathcal{F}$ such that $U = \tilde{U} \cup \{e\}$ and $V = \tilde{V} \cup \{e\}$. Then $U \cap V = \left(\tilde{U} \cap \tilde{V}\right) \cup \{e\}$ which implies that $U \cap V \in \phi$
- Let $U \in \phi$ and $V \subset S$ such that $U \subset V$. Now $\exists \tilde{U} \in \mathcal{F}$ such that $\tilde{U} \cup \{e\} = U$ and so $\tilde{U} \subset U \subset V$ which implies that $V \in \mathcal{F}$ and so $V \cup \{e\} = V \in \phi$

2. By Lemma 4.4, N is a neighbourhood basis of $e \in S$ for a left shift invariant topology on S as :
   - $\forall\, [f] \in N$ we have that $e \in [f]$ as $e \in f(e) = U_1 \subset [f]$
   - Let $[f] \in N$ with $t \in [f]$. Define $g : S \to \phi$ by $g(x) = f(tx)$ then $t\,[g] \subset [f]$: Now there exists an $n \in \mathbb{N}$ such that $t \in U_{n-1}$ and so $tf(t) \subset [f]$. Now $tV_1 = tg(e) = tf(t) \subset \bigcup_{s \in U_{n-1}} sf(s) = U_n$. So suppose that $tV_m \subset U_{n+m-1}$ then $tV_{m+1} = \bigcup_{x \in V_m} txg(x) = \bigcup_{x \in V_m} txf(tx) \subset \bigcup_{s \in tV_m} sf(s) \subset \bigcup_{s \in U_{n+m-1}} sf(s) \subset U_{n+m}$ and so by induction we have that $tV_m \subset U_{n+m-1}$ for all $m \in \mathbb{N}$. So $t\,[g] = t \bigcup V_m \subset \bigcup tV_m \subset \bigcup U_{n+m-1} \subset \bigcup U_m = [f]$

Let us denote this topology with neighbourhood basis of identity N generated by $\mathcal{F}$ by $\mathcal{T}(\mathcal{F})$. Now suppose $\mathcal{F} \to \{e\}$ in $\mathcal{T}$ then the neighbourhood system at $e \in S$ (denoted by $\mathcal{N}_e$) is a subset of $\mathcal{F}$ thus $\mathcal{N}_e \subset \phi \subset \mathcal{F}$ i.e. $\mathcal{F} \to \{e\}$ in $\mathcal{T}(\mathcal{F})$.

Now let $U \in \mathcal{N}_e$. As $\mathcal{T}$ is a left shift invariant topology $\exists V_t \in \mathcal{N}_e$ such that $tV_t \subset U$. Define $f : S \to \mathcal{N}_e \subset \phi$ by

$$f(x) = \begin{cases} V_x, & \text{if } x \in U - \{e\} \\ U, & \text{otherwise} \end{cases} \tag{4.1}$$

Then $[f] \subset U$ as $U_1 = f(e) = U$ and $U_2 = \bigcup_{t \in U} tf(t) = U \cup \bigcup_{t \in U - \{e\}} tV_t = U$. Thus $\mathcal{N}_e \subset N$ and so $\mathcal{N}_e = N$ as $\mathcal{T}$ is the maximal left shift invariant topology on S. □

Lemma 4.15. *The left shift invariant topology $\mathcal{T}(\mathcal{F})$ generated by a filter $\mathcal{F}$ on S is discrete if and only if $\{e\} \in \mathcal{F}$*

Proof. $\mathcal{F}$ is discrete $\Leftrightarrow$ $\{e\}$ is open $\Leftrightarrow$ $\exists f : S \to \phi$ such that $[f] = \{e\} \Leftrightarrow f(e) = \{e\} \Leftrightarrow \{e\} \in \phi \Leftrightarrow \{e\} \in \mathcal{F}$ □

Lemma 4.16. *Let $\mathcal{F}$ be a filter on S and let $f : S \to \phi = \{F \cup \{e\} : F \in \mathcal{F}\}$ be a function. If $t \in V_n$ where $V_1 = f(e)$ and $V_{n+1} = \bigcup_{t \in V_n} tf(t)$ then we can find a sequence $(t_i)_{i=1..n}$ such that $t = t_1 t_2 ... t_n$ and $t_{i+1} \in f(et_1 t_2 ... t_i)$*

Proof. Proof by induction:

1. if $t \in V_1$ then $t \in f(e)$ so this case is trivial.

2. Now suppose that if $t \in V_n$ we can find a sequence $(t_i)_{i=1..n}$ such that $t = t_1 t_2 ... t_n$ and $t_{i+1} \in f(et_1 t_2 ... t_i)$. Let $\tilde{t} \in V_{n+1} = \bigcup_{s \in V_n} sf(s)$ so $\exists t \in V_n$ with $\tilde{t} \in tf(t)$. So there exists a sequence $(t_i)_{i=1..n}$ such that $t = t_1 t_2 ... t_n$ and $t_{i+1} \in f(et_1 t_2 ... t_i)$ i.e. $\tilde{t} \in t_1 t_2 ... t_n f(t_1 t_2 ... t_n)$ so define t_{n+1} to be that unique element of $f(t_1 t_2 ... t_n)$ such that $\tilde{t} = t_1 t_2 ... t_n t_{n+1}$ with $t_{n+1} \in f(t_1 t_2 ... t_n)$.

□

Lemma 4.17. *Let $\mathcal{F}$ be a filter on S and suppose there exists a function $f : S \to \phi = \{F \cup \{e\} : F \in \mathcal{F}\}$ such that the family of sets $\{xf(x) : x \in S\}$ is pairwise disjoint. Now if $x_0 x_1 ... x_n = y_0 y_1 ... y_m$ where $m < n$ and $x_{i+1} \in f(x_1 x_2 ... x_i)$ and $y_{i+1} \in f(y_1 y_2 ... y_i)$ then $x_0 x_1 ... x_{n-m} = y_0$ and $x_{n-m+j} = y_j$ for $0 < j \leq m$*

Proof. Proof by induction:

1. Suppose $m = 0$ then $x_0 x_1 ... x_n = y_0$ so in this case it is trivial.

2. Now suppose $x_0 x_1 ... x_n = y_0 y_1 ... y_{k+1}$ but $x_n \neq y_{k+1}$. As S is a cancellative semigroup we have that $x_0 x_1 ... x_{n-1} \neq y_0 y_1 ... y_k$ and thus

$$x_0 x_1 ... x_{n-1} f(x_0 x_1 ... x_{n-1}) \cap y_0 y_1 ... y_k f(y_0 y_1 ... y_k) = \emptyset \tag{4.2}$$

   which contradicts that $x_0 x_1 ... x_n = y_0 y_1 ... y_{k+1}$ and thus $x_n = y_{k+1}$

□

Lemma 4.18. *Let $\mathcal{F}$ be a filter on S and suppose $\exists M : S \to \mathcal{F}$ such that the family of subsets $\{xM(x) : x \in S\}$ is pairwise disjoint. If $\cap\mathcal{F} = \emptyset$ then the topology $\mathcal{T}(\mathcal{F})$ is Hausdorff and non-discrete.*

Proof. We treat each claim separately:

1. Suppose $\mathcal{T}(\mathcal{F})$ is discrete then $\{e\} \in \mathcal{T}(\mathcal{F})$ which implies $\{e\} \in \mathcal{F}$ and thus $\cap\mathcal{F} = \{e\} \neq \emptyset$ so $\mathcal{T}(\mathcal{F})$ is non-discrete.

2. Let $x, y \in S$ and suppose $x \neq y$. Now as $\cap\mathcal{F} = \emptyset$ we can assume without lose of generality that $x, y \notin tM(t)$, $\forall t \in S$ for suppose $\exists t \in S$ such that $x \in tM(t)$ i.e. $\exists! m_1 \in M(t)$ such that $x = tm_1$. Now as $\exists U \in \mathcal{F}$ such that $m_1 \notin U$ and thus $m_1 \notin M(t) \cap U \in \mathcal{F}$ and thus $x \notin t\,(M(t) \cap U)$.

   Now define $h : S \to \phi$ by $h(t) = (M(xt) \cap M(yt)) \cup \{e\} \in \phi$ so $h(t)$ is well-defined. Now suppose $\tilde{t} \in x\,[h] \cap y\,[h] = \bigcup_{n\in\mathbb{N}} xV_n \cap yV_n$. So $\exists n \in \mathbb{N}$ such that $\tilde{t} \in xV_{n+1} \cap yV_{n+1} = \bigcup_{s,t\in V_n} xt \cdot h(t) \cap ys \cdot h(s)$. So $\exists s, t \in V_n$ such that

   $$\tilde{t} \in (xt{\cdot}M(xt){\cap}ys{\cdot}M(ys)){\cup}(\{xt\}{\cap}\{ys\}){\cup}(xt{\cdot}M(xt){\cap}\{ys\}){\cup}(ys{\cdot}M(ys){\cap}\{xt\}) \tag{4.3}$$

   Now we consider two cases

   Case 1: Suppose $\tilde{t} \in (xt \cdot M(xt) \cap ys \cdot M(ys))$ or $\tilde{t} \in (\{xt\} \cap \{ys\})$ then we have that $xt = ys$ and thus by Lemma 4.17 we have that $x = y$.

   Case 2: Suppose $\tilde{t} \in (xt \cdot M(xt) \cap \{ys\})$ or $\tilde{t} \in (ys \cdot M(ys) \cap \{xt\})$. Now if $\tilde{t} \in (xt \cdot M(xt) \cap \{ys\})$ (the other case is handled similarly) then $xt_1t_2...t_{n+1} = ys_1s_2...s_n$ and so by Lemma 4.17 we have that $xt_1 = y$ and thus $y \in xM(x)$ which contradicts our construction.

So if $x \neq y$ we can find open neighbourhoods $x\,[h] \ni x$ and $y\,[h] \ni y$ such that $x\,[h] \cap y\,[h] = \emptyset$ and thus $\mathcal{T}(\mathcal{F})$ is Hausdorff. □

Lemma 4.19. *Let $\mathcal{T}$ be a left shift invariant topology on S and suppose that if $\forall U \in \mathcal{N}_e$ there exists a $V \in \mathcal{N}_e$ such that V is clopen then $\mathcal{T}$ is zero dimensional.*

Proof. Let $x \in S$ and $\tilde{U} \in \mathcal{T}$ be an open set with $x \in \tilde{U}$. Then there $\exists U \in \mathcal{N}_e$ with $xU \subset \tilde{U}$ but then there exists a $V \in \mathcal{N}_e$ with V clopen so $xV \subset xU \subset \tilde{U}$ and xV is clopen as $\mathcal{T}$ is left shift invariant. Thus $\mathcal{T}$ is zero dimensional. □

Lemma 4.20. *Let $\mathcal{F}$ be a filter on S and suppose $\exists M : S \to \mathcal{F}$ such that the family of subsets $\{xM(x) : x \in S\}$ is pairwise disjoint. If $\cap\mathcal{F} = \emptyset$ then the topology $\mathcal{T}(\mathcal{F})$ is zero dimensional.*

Proof. let $[f] \in \mathcal{T}(\mathcal{F})$ and define $h : S \to \phi$ by $h(t) = (M(t) \cap f(t)) \cup \{e\}$. Now as $\cap\mathcal{F} = \emptyset$ we can assume that $e \notin tM(t)$, $\forall t \in S$ for suppose $\exists t \in S$ such that $e \in tM(t)$ i.e. $\exists! m_1 \in M(t)$ such that $e = tm_1$. Now as $\exists U \in \mathcal{F}$ such that $m_1 \notin U$ and thus $m_1 \notin M(t) \cap U \in \mathcal{F}$ and thus $e \notin t\,(M(t) \cap U)$. Also it is the case that $[h] \subset [f]$. Let $a \in S \setminus [h]$. Define $\tilde{h} : S \to \phi$ by $\tilde{h}(t) = (h(t) \cap M(at)) \cup \{e\}$ so $[\tilde{h}] \subset [h]$. Now suppose $\tilde{t} \in [h] \cap a[\tilde{h}] = \bigcup_{n,m\in\mathbb{N}} V_n \cap a\tilde{V}_m$. Now as $[\tilde{h}] \subset [h]$, $\exists n \in \mathbb{N}$ such that $\tilde{t} \in V_{n+1} \cap a\tilde{V}_{n+1}$. However, as $\tilde{V}_n \subset V_n$ we know that there $\exists s, t \in V_n$ with $\tilde{t} \in th(t) \cap as\tilde{h}(s)$. Thus

$$\tilde{t} \in (tM(t) \cap asM(as)) \cup (\{t\} \cap \{as\}) \cup (tM(t) \cap \{as\}) \cup (asM(as) \cap \{t\}) \quad (4.4)$$

We consider three cases

Case 1: If $\tilde{t} \in (tM(t) \cap asM(as))$ or $\tilde{t} \in (\{t\} \cap \{as\})$ then $t = as$ and so by Lemma 4.18 we have that $e = a$ which is a contradiction.

Case 2: if $\tilde{t} \in (tM(t) \cap \{as\})$ then $et_1t_2...t_{n+1} = as_1s_2...s_n$ which by Lemma 4.17 implies that $et_1 = a$ and thus $a \in V_1 \subset [h]$ which is a contradiction.

Case 3: if $\tilde{t} \in (asM(as) \cap \{t\})$ then $et_1t_2...t_n = as_1s_2...s_{n+1}$ which by Lemma 4.18 implies that $e = as_1$ and thus $e \in a\tilde{V}_1 \subset aM(a)$ which is a contradiction.

Thus $X \setminus [h]$ is open and so by Lemma 4.19, $\mathcal{T}(\mathcal{F})$ is zero dimensional. □

Lemma 4.21. *Let S be a cancellative semigroup then there exists a filter F on S such that*

1. *$\cap\mathcal{F} = \emptyset$ and*

2. *$\exists M : S \to \mathcal{F}$ such that the family of sets $\{xM(x) : x \in S\}$ is pairwise disjoint. Furthermore $\Delta(F) = |S|$*

Proof. We proceed by transfinite induction. Firstly, enumerate the set $S = \{x_\alpha\}$ for $\alpha \leq |S|$ and define an α-sequence as follows: Let $y_0 = e$ and suppose that the sequence y_α has been defined for $\alpha < \beta$. We wish to define y_β such that if $\gamma \leq \delta \leq \beta$ then

$$(x_\gamma \cdot \{y_\alpha : \gamma \leq \alpha \leq \beta\}) \bigcap (x_\delta \cdot \{y_\alpha : \delta \leq \alpha \leq \beta\}) = \emptyset \tag{4.5}$$

So suppose that the sequence $(y_\alpha)_{\alpha<\beta}$ satisfies the equation

$$(x_\gamma \cdot \{y_\alpha : \gamma \leq \alpha < \beta\}) \bigcap (x_\delta \cdot \{y_\alpha : \delta \leq \alpha < \beta\}) = \emptyset \tag{4.6}$$

For all $\gamma \leq \delta < \beta$ and choose y_β such that equation 4.5 is satisfied. This choice is possible as the semigroup is cancellative and thus

$$|\bigcup_{\alpha<\beta} x_\alpha \cdot \{y_\delta : \gamma \leq \delta < \beta\}| < |S| \tag{4.7}$$

Define a filter $\mathcal{F}$ on S by taken as a filter base the subsets $\{y_\alpha : \gamma \leq \alpha \leq \beta\}$ for $\gamma < |S|$. Define the function $M : S \to \mathcal{F}$ by $M(x_\gamma) = \{y_\alpha : \gamma \leq \alpha < |S|\}$. Furthermore, $|\{y_\alpha : \gamma \leq \alpha < |S|\}| = |S|$ for all $\gamma < |S|$ and thus $\Delta(F) = |S|$ □

Theorem 4.22. *Let S be a cancellative semigroup then there exists a non-discrete, left shift invariant, zero dimensional Hausdorff topology on S such that $\Delta(X) = |S|$.*

Proof. By Lemma 4.21 there exists a filter F on S such that $\cap\mathcal{F} = \emptyset$ and $\exists M : S \to \mathcal{F}$ such that the family of sets $\{xM(x) : x \in S\}$ is pairwise disjoint. But then by lemma 4.18 the topology $\mathcal{T}(\mathcal{F})$ is Hausdorff and non-discrete and by Lemma 4.20 the topology is zero dimensional. Furthermore by lemma 4.4 this topology is left shift invariant. Now as $\Delta(F) = |S|$ and all left shifts are injections (as S is a cancellative semigroup) we have that $\Delta(S) = |S|$ □

So we have shown that every cancellative semigroup admits a non-discrete, left shift invariant, zero dimensional Hausdorff topology with dispersion character

equal to the cardinality of the semigroup. Now it is easy to see that $\mathbb{Z}$ is such a semigroup, and thus there exists a non-discrete, left shift invariant, zero dimensional topology on S. Let $P \subset \mathbb{Z}$ be the set of all primes in $\mathbb{Z}$ and suppose that $|P| < \infty$. Since the left shifts on $\mathbb{Z}$ are continuous and open we know that the set

$$P \cdot \mathbb{Z} = \bigcup_{p \in P} p \cdot \mathbb{Z}$$

Must be open, since it is the union of open subsets. However, since P is assumed to only have a finite number of distinct elements, $P \cdot \mathbb{Z}$ is also a closed set and thus $\mathbb{Z} \setminus (P \cdot \mathbb{Z})$ is a non empty open set, since $1 \in \mathbb{Z} \setminus (P \cdot \mathbb{Z})$. But then

$$|\mathbb{Z} \setminus (P \cdot \mathbb{Z})| = |\mathbb{Z}| = \omega$$

and so there must exist an integer $z \in \mathbb{Z} \setminus \{-1, 1\}$ which does not have a prime divisor. This contradicts the Fundamental Theorem of Arithmetic and so P must have an infinite number of distinct elements.

Definition 4.23. *An element x in a semigroup S with identity is called irreducible if for all $p, q \in S$ such that $x = p \cdot q$ either $p = x$ and $q = e$ or $p = e$ and $q = x$*

The primes, which are irreducible elements replace the vanilla irreducible elements utilized in the special case of this result for $\mathbb{Z}$.

Theorem 4.24. *Let S be a cancellative semigroup with identity e which is not a group. Then if there exists a generating set of irreducible elements for S, this set of irreducible elements cannot be finite.*

Proof. By 4.22 we know that S can be endowed with a non-discrete, left shift invariant, zero dimensional Hausdorff topology. Let P be a minimal generating set for S consisting of irreducible elements (if one exists), and if necessary remove e from P. As before we consider the set $I = P \cdot S$. If the cardinality of P is finite, then I is a closed subset of S and thus $S \setminus I$ is an open subset of S. However, $\{e\} = I$, since in a minimal set of generating irreducible elements, not one of them may have an inverse as this contradicts the minimality of the generating set or the fact that S is not a group. But then the original topology cannot be non-discrete, a contradiction. □

4.3 Invariant topologies on groups

The fact that every infinite group admits a zero dimensional topology where all left translations are continuous and open was shown in a previous section. However, this is not the best result that can be shown in this regard. In this section, it will be shown that this conclusion can be strengthened and that every infinite group in fact admits a translation invariant zero dimensional anti discrete topology. This result uses much of the machinery we have developed in the previous chapters, especially the *Finite Product Theorem* and its combinatorial implications on a group. To simplify notation, in this section G will denote an arbitrary infinite group with identity element denoted by $e \in G$

Definition 4.25. *Let $\mathcal{T}$ be a topology on a group G*

1. *$\mathcal{T}$ is left shift invariant if and only if for every $U \in \mathcal{T}$ and every $x \in G$ we have that $xU \in \mathcal{T}$*

2. *$\mathcal{T}$ is right shift invariant if and only if for every $U \in \mathcal{T}$ and every $x \in G$ we have that $Ux \in \mathcal{T}$*

3. *$\mathcal{T}$ is inverse invariant if and only if for every $U \in \mathcal{T}$ it is true that $U^{-1} \in \mathcal{T}$*

4. *$\mathcal{T}$ is shift invariant if it is left and right shift invariant as well as inverse invariant*

The following lemma gives a more map orientated classification of these definitions and as it follows the proof of Lemma 4.2 we only give a proof of the result which deals directly with inversions which are not a part of the Semigroup treatment.

Lemma 4.26. *A topology on G is*

1. *Left shift invariant if and only if every left shift by an arbitrary element $x \in G$ is both open and continuous*

2. *Right shift invariant if and only if every right shift by an arbitrary element $x \in G$ is both open and continuous*

3. *Inverse invariant if and only if the inverse map $i : x \to x^{-1}$ is both open and continuous*

4. *Shift invariant if and only if all translations and the inverse map are continuous*

Proof. We only show the third implication as it demonstrates all the others. Suppose that $\mathcal{T}$ is an inverse invariant topology on G and let $x \in G$ and $V \in \mathcal{T}$ such that $x^{-1} \in V$. Now $x \in V^{-1}$ as $xy \in V$ and further $x(x^{-1}V) \subset V$ and so the left shift $y \to xy$ is continuous. Further, as $xU \in \mathcal{T}$ for all $x \in G$ we have that the left shift $y \to xy$ is also open. The reverse implication is trivial. □

We wish to classify the filters that generate invariant topologies on a group G. Now as every group is in fact a semigroup we have by Theorem 4.22 that every group admits a zero dimensional, left shift invariant topology. The next lemma gives a neighbourhood level classification of an invariant topology with respect to the neighbourhood filter of identity.

Lemma 4.27. *Let $\mathcal{T}$ be a left shift invariant topology on G with $\mathcal{F}$ the neighbourhood filter of identity in that topology. The topology $\mathcal{T}$ is an invariant topology if and only if $\mathcal{F}^{-1} = \mathcal{F}$ and $x\mathcal{F}x^{-1} = \mathcal{F}$ for all $x \in G$. Furthermore if $\mathcal{F}$ is a filter such that $\mathcal{F}^{-1} = \mathcal{F}$ and $x\mathcal{F}x^{-1} = \mathcal{F}$ then $\mathcal{T}(\mathcal{F})$ is a left shift invariant topology on G*

Proof. $\Rightarrow$ Firstly, if $\mathcal{F}$ is the neighbourhood filter of identity of some invariant topology then since $e = e^{-1}$ we have that for all $U \in \mathcal{F}$ that there exists a $V \in \mathcal{F}$ so that $V^{-1} \subset U$. This implies that $\mathcal{F} \subseteq \mathcal{F}^{-1}$. The other containment is obtained similarly and thus $\mathcal{F} = \mathcal{F}^{-1}$. Now consider the identity $e = x \cdot e \cdot x^{-1}$ where $x \in G$. This implies that for all $U \in \mathcal{F}$ there exits a $V \in \mathcal{F}$ so that $xVx^{-1} \subset U$. This implies that $\mathcal{F} \subset x\mathcal{F}x^{-1}$. The other containment is obtained with a similar argument and so $\mathcal{F} = x\mathcal{F}x^{-1}$.

$\Leftarrow$ Now suppose that $\mathcal{F}$ is the neighbourhood filter of identity for some left shift invariant topology on G such that $\mathcal{F} = \mathcal{F}^{-1}$ and $\mathcal{F} = x\mathcal{F}x^{-1}$ for all $x \in G$.

Now let $x, y \in G$, then xU is an arbitrary base element of $\mathcal{T}$ for all $U \in \mathcal{F}$. However, since $(xU) \cdot y = (x \cdot y) \cdot (yUy^{-1})$ we have that $(xU) \cdot y \in \mathcal{T}$. Also since $(xU)^{-1} = x^{-1} \cdot (xU^{-1}x^{-1})$ we have that $(xU)^{-1} \in \mathcal{T}$ i.e. $\mathcal{T}$ is an invariant topology on G.

□

The next lemma in the series shows that if we can find a filter $\mathcal{F}$ on G such that $\mathcal{F} = \mathcal{F}^{-1}$ and $x\mathcal{F}x^{-1} = \mathcal{F}$ for all $x \in G$ then the left invariant topology generated by this filter (i.e. Lemma 4.14 and the previous lemma) is in fact an invariant topology. Indeed to show this we need only show that our filter $\mathcal{F}$ satisfies the conditions of those two lemmas. Recall that $\phi = \{U \cup \{e\} : U \in \mathcal{F}\}$. Now suppose that $f : G \to \phi$ so that $[f]$ is a basic open neighbourhood of the identity. We wish to find a function $g : G \to \phi$ so that $[g]^{-1} \subset [f]$ and $x\,[g]\,x^{-1} \subset [f]$ for all $x \in G$. Now for all $t \in G$, there exists a $U \in \mathcal{F}$ such that $tUt^{-1} \subset f(t)$ and a V such that Thus we have proven the following lemma

Lemma 4.28. *Let $\mathcal{F}$ be a filter on G such that $\mathcal{F} = \mathcal{F}^{-1}$ and for all $x \in G$ we have that $x\mathcal{F}x^{-1} = \mathcal{F}$. Then $\mathcal{T}_l(\mathcal{F}) = \mathcal{T}(\mathcal{F})$ where $\mathcal{T}_l$ denotes the left invariant topology generated by $\mathcal{F}$ and $\mathcal{T}$ denotes the invariant topology generated by that same filter*

Following a similar theme as the previous section, we wish to find out which subsets of our group G generate the largest filter $\mathcal{G} \subset \mathcal{F}$ where $\mathcal{F}$ is an arbitrary filter on G and $\mathcal{G}$ is a filter such that $\mathcal{G} = \mathcal{G}^{-1}$ and for all $x \in G$ we have that $x\mathcal{G}x^{-1} = \mathcal{G}$.

Definition 4.29. *Let $\mathcal{G}$ be a filter on a group G. We call the filter invariant if $\mathcal{G} = \mathcal{G}^{-1}$ and for all $x \in G$ we have that $x\mathcal{G}x^{-1} = \mathcal{G}$*

Invariant ultrafilters on groups play a similar role as the did on semigroups. They will form the basis of the required construction for the left invariant topology. The following lemma constructs the subsets in question.

Lemma 4.30. *Let $\mathcal{F}$ be a filter on a group G. Let $f : G \to \mathcal{F}$ be a function then*

the subsets

$$[f] = \bigcup_{x \in G} x \left(f(x) \cup f(x)^{-1} \right) x^{-1} \tag{4.8}$$

form a base for the largest invariant filter $\mathcal{G} \subset \mathcal{F}$

Proof. Once again we break up this proof into three steps

Step 1 The family of sets $B = \{[f] : f : G \to \mathcal{F}\}$ forms a filter base for some filter on G. Let $f, g \in \mathcal{F}^G$ so that $[f], [g] \in B$. Define a function $h : G \to \mathcal{F}$ by $h(x) = f(x) \cap g(x)$. Now let $x \in G$, then it is easy to see that $x \cdot h(x) \cup h(x)^{-1} \cdot x^{-1} \subset x \cdot f(x) \cup f(x)^{-1} \cdot x^{-1}$. Similarly, it is easy to see that $x \cdot h(x) \cup h(x)^{-1} \cdot x^{-1} \subset x \cdot g(x) \cup g(x)^{-1} \cdot x^{-1}$. Now consider

$$\begin{aligned}
& \left(\bigcup_{x \in G} x \cdot f(x) \cup f(x)^{-1} \cdot x^{-1} \right) \cap \left(\bigcup_{y \in G} y \cdot g(y) \cup g(y)^{-1} \cdot y^{-1} \right) \\
= & \bigcup_{x,y \in G} \left(x \cdot f(x) \cup f(x)^{-1} \cdot x^{-1} \right) \cap \left(y \cdot g(y) \cup g(y)^{-1} \cdot y^{-1} \right) \\
\supseteq & \bigcup_{x \in G} \left(x \cdot f(x) \cup f(x)^{-1} \cdot x^{-1} \right) \cap \left(x \cdot g(x) \cup g(x)^{-1} \cdot x^{-1} \right) \\
\supseteq & \bigcup_{x \in G} (x \cdot \left(\left(f(x) \cup f(x)^{-1} \right) \cap \left(\left(g(x) \cup g(x)^{-1} \right) \right) \cdot x^{-1} \right) \\
\supseteq & \bigcup_{x \in G} x \cdot \left((f(x) \cap g(x)) \cup (f(x) \cap g(x))^{-1} \right) \cdot x^{-1}
\end{aligned}$$

This shows that $[h] \subset [f] \cap [g]$

Step 2 The filter generated by B is an invariant filter. Now direct computation shows that $[f]^{-1} = [f] \in B$ and thus we only need to show that $y\,[f]\,y^{-1} \in B$ for all $y \in G$. Define $h : G \to \mathcal{F}$ by $h(x) = f(y^{-1} \cdot x)$. We claim that $y\,[f]\,y^{-1} = [h]$.

$$\begin{aligned}
& y \left(\bigcup x \left(f(x) \cup f(x)^{-1} \right) x^{-1} \right) y^{-1} \\
= & \bigcup y \cdot x \left(f(x) \cup f(x)^{-1} \right) x^{-1} \cdot y^{-1} \\
= & \bigcup t \left(f(y^{-1}t) \cup f(y^{-1}t)^{-1} \right) t^{-1} \\
= & \bigcup t \left(h(t) \cup h(t)^{-1} \right) t^{-1}
\end{aligned}$$

Step 3 The filter generated by B is the largest invariant filter contained in $\mathcal{F}$. Suppose that $\mathcal{H}$ is another invariant filter contained in $\mathcal{F}$. Let $x \in G$ and $U \in \mathcal{H}$, now since $\mathcal{H}$ is an invariant filter there exists $V_x, W \in \mathcal{H} \subset \mathcal{F}$ such that $x \cdot V_x \cdot x^{-1} \subset U$ and $W^{-1} \subset U$. Define a function $f : G \to \mathcal{F}$ by $f(x) = V_x \cap W$. Now

$$[g] = \bigcup_{x \in G} x \cdot \left(f(x) \cup f(x)^{-1}\right) \cdot x^{-1}$$

But $f(x) \cup f(x)^{-1} \subset x^{-1} \cdot U \cdot x$ and thus $x \cdot (f(x) \cup f(x)^{-1}) \cdot x^{-1} \subset U$. But this implies that $[g] \subset U$ and so $\mathcal{H}$ is contained in the filter generated by B

□

We now need to construct a filter with the same pairwise disjoint structure as was needed in the previous section

Lemma 4.31. *Let G be a group, then for every countably infinite subgroup H of G and every subset $F \subset G$ with $|F| < |G|$ there exists a filter $\mathcal{F}$ and a function $f : H \to \mathcal{F}$ such that*

1. $\bigcap \mathcal{F} = \emptyset$
2. *$\mathcal{F}$ is an invariant filter*
3. *$x, y \in G$ with $x \neq y$ implies that $x \cdot f(x) \cap y \cdot f(y) = \emptyset$*

Proof. So suppose H is countable and thus enumerate $H = \{x_n\}$. Pick $y_0 \in G$ such that $x_0 \cdot y_0 \cdot x_0^{-1} \notin F$ and $x_0 \cdot y_0^{-1} \cdot x_0^{-1} \notin F$. Proceed by induction to construct a sequence (y_n) such that for all $n < k$ the sequence of subsets

$$Z_{n,k} = \{x_i \cdot y_j^{\pm 1} \cdot x_i^{-1} : i \leq j, n \leq j < k\}$$

is pairwise disjoint and disjoint from F. We can construct such a sequence by Theorem 3.40, as we never exhaust the set of elements we can choose from.

Now define a filter $\mathcal{F}$ by using sets of the form $\bigcup_{i \in \mathbb{N}} \{x_i y_j^{\pm 1} x_i^{-1} : n(i) \leq j\}$ where $n : \mathbb{N} \to \mathbb{N}$ is an arbitrary function. The function f is defined by

$$f(x_n) = Z_n = \bigcup_{k \in \mathbb{N}} Z_{n,k}$$

By construction, this filter and function satisfy the constraints for countable subgroups of G $\square$

Lemma 4.32. *There exists a filter $\mathcal{F}$ on a group G and a function $f : G \to \mathcal{F}$ such that*

1. $\bigcap \mathcal{F} = \emptyset$
2. *$\mathcal{F}$ is an invariant filter when restricted to H*
3. *$x, y \in H$ with $x \neq y$ implies that $x \cdot f(x) \cap y \cdot f(y) = \emptyset$ and $|f(x)| = |H|$*

Proof. We proceed by transfinite induction on the cardinality of H. The countable case is handled by Lemma 4.31 and so suppose the required filters have been constructed for all subsgroups H of cardinality less then κ. Choose an increasing sequence of subgroups $(H_\alpha)_{\alpha<\kappa}$ with $|H_\alpha| < \kappa$ and define

$$H = \bigcup H_\alpha$$

For every $\alpha < \kappa$ we can construct an invariant filter $\mathcal{F}_\alpha$ and a function $f_\alpha : H_\alpha \to \mathcal{F}_\alpha$ such that the family of subsets $x \cdot f_\alpha(x)$ is pairwise disjoint. Furthermore, we demand that $|f_\alpha(x)| = |H_\alpha|$. Also for each α define a subset S_α such that $S_0 = \emptyset$, $S_{\beta+1} = S_\beta \cup \bigcup_{c \in H_\beta} x \cdot f_\beta(x)$ and if α is a limit ordinal we define $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Define the required filter by taken the family of subsets $\bigcup_{\beta \leq \alpha < \kappa} U_\alpha$ where $U_\alpha \in \mathcal{F}_\alpha$ and define the function $f : H \to \mathcal{F}$ by

$$f(x) = \bigcup_{\min\{\alpha : x \in H_\alpha\} \leq \alpha < \kappa} f_\alpha(x)$$

$\square$

We now have sufficiently many pieces to prove the theorem.

Theorem 4.33. *Let G be a group, then G admits an invariant topology $\mathcal{T}$*

Proof. By Lemma 4.32 and Lemma 4.20 there exists a left shift invariant, zero dimensional topology on G, since G is definitely a subgroup of itself. However, by Lemma 4.20 this left shift invariant topology is in fact an invariant topology. $\square$

Chapter 5

Conclusion

We have thus shown that every cancellative semigroup admits a non-discrete left shift invariant topology and used this result to show that every group admits an invariant topology by giving a slightly modified version of Zelenyuk's construction. This was achieved by constructing a suitable filter on a cancellative semigroup and then using this filter to generate the neighbourhood system of identity of a left shift invariant topology. We showed that this topology is not discrete. This result serves as a template for the theorem that every group admits an invariant topology. For this proof we required the finite products theorem to extend the induction argument, and thus the Stone-Cech compactification of an arbitrary discrete semigroup was constructed and the existence of idempotents in the remainder of the compactification shown. The question of which, if any other locally compact Tychonoff topologies extend the algebraic structure of a semigroup to the Stone-Cech compactification of that group remains open. The proper extension of the concept of a T-filter and a T-sequence to the family of infinite semigroups is also an open problem. Finally, the converse of the existence of left shift invariant topologies on cancellative semigroups is open, does every homogenous space admit a semigroup structure on which the semigroup topology is left shift invariant? The case for countable groups is known, as Zelenyuk has shown that every countable homogenous space admits a group structure on which the group topology is left shift invariant (see [Zel04]).

Appendix A

Topologies, Filters and Ultrafilters

A.1 Topology definitions

In this section, we cover the basic topological definitions that will be required in this text. (For a more general introduction see [Kel75]) Ordinal numbers are denoted by α, β, γ etc. while ω will denote the first infinite ordinal. The Axiom of Choice or any of its equivalent forms are assumed as required.

Definition A.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*

1. *A function $f : X \to Y$ is called* continuous *if $\forall\ U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$*
2. *A function $f : X \to Y$ is called an* open (closed) map *if for all $U \subset X$ with U open (closed) we have that $f(U)$ is open (closed)*
3. *a function $f : X \to Y$ is called a homeomorphism if it is continuous, open and a bijection*

Definition A.2. *Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A subset $U \subset X$ is a* neighbourhood of the point *x if there exists an open subset V of X such that $x \in V \subset U \subset X$.*

Definition A.3. *A* base *B for a topological space $(X, \mathcal{T})$ is a family of subsets $\mathcal{B} \subset 2^X$ such that $\forall U \in \mathcal{T}$ there exists a collection $\{B_i\}_{i \in I} \subset \mathcal{B}$ such that $U = \bigcup_{i \in I} B_i$*

Definition A.4. *1. A topological space $(X, \mathcal{T})$ is called T_0 if $\forall x, y \in X$ with $x \neq y$ there exists $U \in \mathcal{T}$ such that $(x \in U \wedge y \notin U)$ or $(y \in U \wedge x \notin U)$*

2. A topological space $(X, \mathcal{T})$ is called T_1 if $\forall x, y \in X$ with $x \neq y$ there exists $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$

3. A topological space $(X, \mathcal{T})$ is called T_2 or Hausdorff if $\forall x, y \in X$ with $x \neq y$ there exists U and $V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$

4. A topological space $(X, \mathcal{T})$ is called T_3 or regular *if it is T_1 and $\forall x \in X$ and $Y \subset X$, Y closed and $x \notin Y$ there exists U and $V \in \mathcal{T}$ such that $x \in U$, $Y \subset V$ and $U \cap V = \emptyset$. Equivalently, a topological space is called T_3 or* regular *if for all $x \in X$ and open neighbourhoods $U \subset X$ there exists an open set $V \subset X$ such that $x \in V \subset \bar{V} \subset U$*

5. A topological space $(X, \mathcal{T})$ is called $T_{3\frac{1}{2}}$ or completely regular *if it is T_1 and $\forall x \in X$ and $Y \subset X$, Y closed and $x \notin Y$ there exists a continuous function $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f(Y) = \{1\}$*

6. A topological space $(X, \mathcal{T})$ is called T_4 or normal *if $\forall A, B \subset X$, A, B closed and disjoint there exists open subsets $U, V \in \mathcal{T}$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$*

Theorem A.5. *A compact, $T2$ space is in fact $T4$, and is thus $T3$*

Definition A.6. *Let $(X, \mathcal{T})$ be a topological space and $A, B \subset X$. These subsets are* completely separated *if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$*

Theorem A.7 (Urysohn's Lemma). *Let $(X, \mathcal{T})$ be a T_4 space and let $U, V \subset X$ be disjoint closed subsets then U and V are completely separated*

Definition A.8. *A set $A \subset X$ of a topological space $(X, \mathcal{T})$ is a* zero set *if there exists a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(\{0\})$. The collection of all zero sets of a topological space $(X, \mathcal{T})$ is denoted by $\mathcal{Z}(X)$. Let $A \subset X$ such that $X \setminus A \in \mathcal{Z}(X)$ then A is a* cozero set

Theorem A.9. *Let $(X, \mathcal{T})$ be a completely regular topological space then $A, B \subset X$ are completely separated in $X \Leftrightarrow A$ and B are contained in disjoint zero sets*

Definition A.10. *A topological space $(X, \mathcal{T})$ is* compact *if every such open cover has a finite subcover*

Definition A.11. *A topological space $(X, \mathcal{T})$ is called* locally compact *if $\forall x \in X$ we can find an open neighbourhood $U \ni x$ such that $\overline{U}$ is compact*

Definition A.12. *A* compactification of a topological space $(X, \mathcal{T})$ *is a pair $\left(\left(\hat{X}, \hat{\mathcal{T}}\right), h\right)$ where $\left(\hat{X}, \hat{\mathcal{T}}\right)$ is a compact topological space and $h : X \to \hat{X}$ is a homeomorphism onto a dense subset of $\hat{X}$.*

Definition A.13.

1. *A topological space $(X, \mathcal{T})$ is* connected *if $X \neq U \cup V$ $\forall U, V \in \mathcal{T}$ with $U \cap V = \emptyset$.*

2. *A* connected component *of a topological space $(X, \mathcal{T})$ is an equivalence class under the equivalence relation $x \sim y$ if and only if there exists a connected subset $U \in \mathcal{T}$ such that $x, y \in U$*

3. *A topological space $(X, \mathcal{T})$ is* totally disconnected *if its components are singletons*

Definition A.14. *A topological space $(X, \mathcal{T})$ is* zero dimensional *if there exists a basis for that topology $\mathcal{T}$ consisting of clopen subsets of X*

Theorem A.15. *A countable T_3 topological space is zero dimensional*

Theorem A.16. *A totally disconnected, Hausdorff, locally compact space is zero dimensional*

Theorem A.17. *Let $(X, \mathcal{T})$ be a discrete topological space, then $(X, \mathcal{T})$ is T_4, locally compact, totally disconnected and zero dimensional.*

A.2 Filters, Ultrafilters and p-limits

The theory of filters and ultrafilters is a huge area and this appendix's purpose is to give the definitions of the basics that are required for this dissertation as well as prove some of the p-limit theorems are are utilized in the construction of the Stone-Cech compactification of a discrete semigroup. In no way does this even begin to give the depth and breadth of this subject and the reader is directed to [CN74] which the writer found indispensable in developing an understanding of filters and ultrafilters in general.

Definition A.18. *A* filter $\mathcal{F}$ *in a set* X *is a family of non-void subsets of* X *such that*

1. *if* A *and* $B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$
2. *if* $A \in \mathcal{F}$ *and* $A \subset B \subset X$ *then* $B \in \mathcal{F}$

Definition A.19. *A family of subsets* $\mathcal{C}$ *of a set* X *has the* finite intersection property *if for any finite collection of sets* $\{A_i\}_{i=1\ldots n}$ *we have that* $\cap_{i=1\ldots n} A_i \neq \emptyset$

Definition A.20. *An* ultrafilter $\mathcal{F}$ *in a set* X *is a filter on* X *which is maximal with respect to filter containment.*

The following theorem is a standard application of Zorn's lemma.

Theorem A.21. *Let* $\mathcal{C} \in 2^{2^X}$ *be a family of subsets with the finite intersection property then there exists an ultrafilter* $\mathcal{F}$ *such that* $\mathcal{C} \subset \mathcal{F}$

An ultrafilter is a filter that cannot be properly contained in any other filter on X. Ultrafilters can be characterised by the simple criteria that if $A \subset X$ then either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$. This criteria suggests that an ultrafilter supplies a measure of largeness to a family of subsets of X. In fact an ultrafilter can be seen as a $\{0, 1\}$ valued measure on 2^X, by the characterisation, so that if a property holds on the members of an ultrafilter then it holds *all most everywhere.*

Definition A.22. *An ultrafilter* $\mathcal{F}$ *is called* principle or fixed *if there exists a* $x \in X$ *such that* $\mathcal{F} = \{A \subset X : x \in A\}$. *An ultrafilter is free if it is not fixed.*

As a consequence of Theorem A.21 we have the following result:

Theorem A.23. *Let X be an infinite set then there exists a free ultrafilter on X*

The following theorem characterises the subsets of a set X which are elements of an ultrafilter p. This theorem usually serves as the only leverage available when working with free ultrafilters.

Theorem A.24. *Let X be an infinite set and $U \subset X$. Let p be an ultrafilter on X. Then $U \in p$ if and only if $U \cap V \neq \emptyset$ for all $V \in p$*

Proof. The necessity is easy, since filters are closed under intersection and the empty set is never an element of a filter. For the sufficiency let $U \subset X$ such that $X \setminus U \in p$ but $U \cap V \neq \emptyset$ for all $V \in p$. But then $U \cap (X \setminus U) \neq \emptyset$ which is a contradiction. □

Theorem A.25. *Let X be a set. The number of ultrafilters on X is $|X|$ if X is finite, and $2^{2^{|X|}}$ if X is infinite.*

Proof. If X is finite then each ultrafilter on X is principal, and so there are exactly $|X|$ ultrafilters. In the rest of the proof we will assume that X is infinite.

Let F be the set of all finite subsets of X, and let Φ be the set of all finite subsets of F.

For each $A \subseteq X$ define $B_A = \{(f, \phi) \in F \times \Phi \mid A \cap f \in \phi\}$, and $B_A^C = (F \times \Phi) \setminus B_A$. For each $\mathcal{S} \subseteq 2^X$ define $\mathcal{B}_\mathcal{S} = \{B_A \mid A \in \mathcal{S}\} \cup \{B_A^C \mid A \notin \mathcal{S}\}$.

Let $\mathcal{S} \subseteq 2^X$, and suppose $A_1, \ldots, A_m \in \mathcal{S}$ and $D_1, \ldots, D_n \in 2^X \setminus \mathcal{S}$, so that we have $B_{A_1}, \ldots, B_{A_m}, B_{D_1}^C, \ldots, B_{D_n}^C \in \mathcal{B}_\mathcal{S}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ let $a_{i,j}$ be such that either $a_{i,j} \in A_i \setminus D_j$ or $a_{i,j} \in D_j \setminus A_i$. This is always possible, since $A_i \neq D_j$. Let $f = \{a_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$, and put $\phi = \{A_i \cap f \mid 1 \leq i \leq m\}$. Then $(f, \phi) \in B_{A_i}$, for $i = 1, \ldots, m$. If for some $j \in \{1, \ldots, n\}$ we have $D_j \cap f \in \phi$, then $D_j \cap f = A_i \cap f$ for some $i \in \{1, \ldots, m\}$, which is impossible, as $a_{i,j}$ is in one of these sets but not the other. So $D_j \cap f \notin \phi$, and therefore $(f, \phi) \in B_{D_j}^C$. So $(f, \phi) \in B_{A_1} \cap \cdots \cap B_{A_m} \cap B_{D_1}^C \cap \cdots \cap B_{D_n}^C$. This shows that any finite subset of

$\mathcal{B}_\mathcal{S}$ has non-empty intersection, and therefore $\mathcal{B}_\mathcal{S}$ can be extended to an ultrafilter $\mathcal{U}_\mathcal{S}$.

Suppose $\mathcal{R}, \mathcal{S} \subseteq 2^X$ are distinct. Then, relabelling if necessary, $\mathcal{R} \setminus \mathcal{S}$ is nonempty. Let $A \in \mathcal{R} \setminus \mathcal{S}$. Then $B_A \in \mathcal{U}_\mathcal{R}$, but $B_A \notin \mathcal{U}_\mathcal{S}$ since $B_A^C \in \mathcal{U}_\mathcal{S}$. So $\mathcal{U}_\mathcal{R}$ and $\mathcal{U}_\mathcal{S}$ are distinct for distinct $\mathcal{R}$ and $\mathcal{S}$. Therefore $\{\mathcal{U}_\mathcal{S} \mid \mathcal{S} \subseteq 2^X\}$ is a set of $2^{2^{|X|}}$ ultrafilters on $F \times \Phi$. But $|F \times \Phi| = |X|$, so $F \times \Phi$ has the same number of ultrafilters as X. So there are at least $2^{2^{|X|}}$ ultrafilters on X, and there cannot be more than $2^{2^{|X|}}$ as each ultrafilter is an element of 2^{2^X}. □

The dispersion character cardinal function has a natural extension to the space of all ultrafilters on an infinite set X. This cardinal function is called the norm of an ultrafilter in [CN74]

Definition A.26. *The* dispersion character *of an ultrafilter $\mathcal{F}$ is* $\Delta(\mathcal{F}) = \min\{|A| : A \in \mathcal{F}\}$.

The following theorem is also an easy consequence of Theorem A.21

Theorem A.27. *Let X be set with $|X| = \kappa$ and let $\beta \leq \kappa$, then there exists an ultrafilter on X such that $\Delta(\mathcal{F}) = \beta$*

Proof. Let $\mathcal{C} = \{U \subset X : |X \setminus U| < \beta\}$ which obviously has the finite intersection property so by Theorem A.21 there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{C} \subset \mathcal{F}$. Suppose $\Delta(\mathcal{F}) < \beta$ then there exists a $U \in \mathcal{F}$ such that $|U| < \beta$. However then $X \setminus U \in \mathcal{C}$ as $|X \setminus (X \setminus U)| = |U| < \beta$ and thus $U \cap X \setminus U = \emptyset \in \mathcal{F}$ which is a contradiction. □

Definition A.28. *Let κ be a cardinal. An ultrafilter $\mathcal{F}$ is called* κ-uniform *if* $\Delta(\mathcal{F}) \geq \kappa$

Recall the definition of a p-limit:

Definition A.29. *Let X be a set and p an ultrafilter in D. Let $<x_s>_{s\in D}$ be an indexed family in a topological space X and let $y \in X$. Then the $p-\lim_{s\in D} x_s = p\text{–}\lim <x_s>_{s\in D} = y$ if and only if for every neighbourhood U of y we have that $\{s \in D \mid x_s \in U\} \in p$*

A space X is called *p-compact* if every p–lim exists in X. It was shown previously that every compact space is in fact p-compact. The idea of a p–lim is also useful to characterise countable compactness of a product space.

Definition A.30. *A topological space $(X, \mathcal{T})$ is countably compact if every countable open cover of X has a finite sub cover*

Not every countably compact space is a compact space, with the order topology on the first uncountable ordinal c being the standard textbook example of a countably compact space which is not compact. This is not a compact space as the open cover $\{(0, \alpha) : \alpha < c\}$ admits no finite sub cover (as the finite union of countable sets is countable). Since countably compact spaces are a generalization of compact spaces, the question of a Tychonoff type product theorem is an interesting one. The following theorem, stated without proof gives a characterisation of the product of countably compact spaces.

Theorem A.31. *Let $\{X_\alpha : \alpha \in I\}$ be a family of spaces. Then the product space $X = \prod\{X_\alpha : \alpha \in I\}$ is countably compact if and only if for every sequence $f : \mathbb{N} \to X$, $p\text{–}\lim_{n\in\mathbb{N}} \pi_\alpha \circ f \in X_\alpha$ for all $\alpha \in I$*

Proof. See [Ber70] □

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