

Symmetry Structures and Conserved Forms of Differential Equations on Curved Manifolds

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Declaration

I, the undersigned, hereby declare that the work contained in this PhD thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. This thesis has not been submitted before for any degree or examination at the University of the Witwatersrand or any other university.

September 27, 2019 Johannesburg.

A handwritten signature in blue ink, appearing to be 'GMB' with a stylized flourish underneath.

Bourgeois Biova Irénée Gadjagboui

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Abstract

We explore two methods in finding Noether symmetries and conservation laws of differential equations on Riemannian manifolds. The first one is based on the Noether's theorem while the second one is about the 'multiplier approach'. Using the first method, we try to find the variational symmetries, here, denoted X . With geodesic equations, the second method consists of finding the Lagrangian multipliers. This yields the conserved quantities when one acts the multipliers on the geodesic equations. It turns out that the total number of conserved quantities is equal to the number of variational symmetries found. The Lie algebra of infinitesimal isometries of the Riemannian manifolds studied has the dimension not exceeding $\frac{1}{2}n(n+1)$, where $n = 4$.

In the first case studied in Chapter 2, variational symmetries and conservation laws of a modified de Sitter metric are classified. We came up with the suggestion that where a finite dimensional group generated by conservation laws exists, the Noether symmetry group has at least one additional symmetry that is not given by the Killing Vectors. This is later confirmed in the rest of all the other cases studied.

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Chapter 1

Introduction

This thesis aimed at extending or completing the classification of symmetries and conservation law in spacetime.

In this introduction, we present all the necessary terminologies to understand the work done. The first section gives a brief history of *groups* and *symmetries* as well as the motivation of the thesis. The second section provides the required background of Lie groups and their algebras on manifolds. The third section presents Noether symmetries and conservation laws. Finally, in the fourth section, the outline of the chapters is given.

1.1 Background and motivation

The study of physical and mathematical properties of objects gave birth to a new mathematical tool, group theory, in the second half of the eighteenth century. The revelation and the evolution of the theory began in 1770 and extended to the twentieth century. Its considerable developments happened in the nineteenth century and are due to the Norwegian mathematician Sophus Lie (1842 - 1899), [1]. Other great mathematicians have also contributed to the establishment of the theory, Wilhelm Killing (1847 - 1926), Élie Cartan

(1869 - 1951), Essai Schur (1875 - 1941), Hermann Weyl (1885 - 1955), to name a few. Group theory and its algebras then became explicit with the study of geometric objects such as curves and surfaces, and also with the classification of spacetime and geometry itself. Later on, Lie introduced the algebras of the theory in the resolution of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs).

Symmetries of geometric figures, crystals and other physical objects in the macroscopic scale have attracted the scientific communities and many studies have been done in that regard and are well known in the literature. In the modern era of the theory developed by Lie, it is realized that symmetries of objects (physical or mathematical) form a group.

Calculus of variations has begun when one has found difficulties in solving classical problems such as Dido's problem, Plateau's problem, Johann Bernoulli's brachistochrone and Catenary are some of these problems, [2]. They all involve how one can find extrema either for a surface area or for a curve.

Symmetry, Invariance and Conservation Laws in Variational Calculus have become the basis of theoretical physics as well as mathematical physics since Noether's famous theorem proven by the mathematician Emmy Noether in 1915 and published in 1918, [3]. This theorem states that every variational symmetry of the action of a physical system has a conservation law. There exist many types of symmetry which lead to different conservation laws. Recently, a number of papers have been published in a broad spectrum of areas involving symmetries and conservation laws. One of the consequences of the symmetry-conservation laws interplay is the double reduction of Differential Equations (DEs). Symmetries help to classify equations, to determine first integrals (constant of the motion), and to find group invariant solutions (particularly for PDEs).

Systems of DEs have been successfully studied using the methods of invariance, conservation laws and a combination of these. A large number of classes of DEs arise all the time in the literature of Engineering, Mathematical Finance, Mathematical Physics, General Relativity (GR), to name a few. We study in this thesis, inter alia, the DEs in Relativity

essentially from the metrics representing the geometry of spacetime for certain models following the Einstein Field Equations (EFE). We study as well the symmetry, invariance properties and conservation laws of DEs that arise in the study of manifolds via their respective metrics. These entail the Lagrangians (obtained from the metrics) and the corresponding Euler equations. For the purpose, we explore invariance and conservation laws in variational calculus, differential geometry and manifolds. We investigate how consistent symmetry methods are in the resolution of DEs in differential geometry.

1.2 Lie groups and Riemannian manifolds

We introduce in this section the prerequisite to understand the symmetry groups of DEs and PDEs.

1.2.1 Basic definitions

Definition 1.2.1 (Manifold) . [4] *An m -dimensional manifold is a set M , together with a countable collection of subsets $U_\alpha \subset M$, called coordinate charts, and one-to-one functions $\chi_\alpha : U_\alpha \longrightarrow V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$, called local coordinate maps, which satisfy the following properties:*

(i) *The coordinate charts cover M :*

$$\bigcup_{\alpha} U_{\alpha} = M.$$

(ii) *On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map*

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \rightarrow \chi_\beta(U_\alpha \cap U_\beta)$$

is a smooth (infinitely differentiable) function.

(iii) M is Hausdorff. That is, if $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M , then there exists open subsets $W \subset V_\alpha$, $\widetilde{W} \subset V_\beta$, with $\chi_\alpha(x) \in W$, $\chi_\beta(\tilde{x}) \in \widetilde{W}$, satisfying

$$\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\widetilde{W}) = \emptyset.$$

Definition 1.2.2 [4] *A manifold is said to be connected if it cannot be written as a disjoint union of open sets.*

Definition 1.2.3 (Group) . [4] *A group is a set G together with a group operation $\cdot$, usually called multiplication, such that for any two elements, g and h of G , the product $g \cdot h$ is again an element of G . The group operation is required to satisfy the following axioms:*

(i) *Associativity:*

$$(g \cdot h) \cdot k = g \cdot (h \cdot k) \quad \forall \quad g, h, k \in G.$$

(ii) *Identity element. There is a distinguished element e of G , called the identity element, such that:*

$$e \cdot g = g \cdot e = g.$$

(iii) *Inverses. For each $g \in G$ there is an inverse, denoted g^{-1} , with the property*

$$g \cdot g^{-1} = e = g^{-1} \cdot g.$$

A set that has manifold and group properties is called a Lie group. In addition to this, if the dimension of the manifold is m (with group properties), we say it is a m -parameter Lie group.

Definition 1.2.4 (Lie group homomorphism) . [4] *A Lie group homomorphism is a smooth map $\phi : G \rightarrow H$ between two Lie groups which respects the group operations:*

$$\phi(g \cdot \tilde{g}) = \phi(g) \cdot \phi(\tilde{g}), \quad g, \tilde{g} \in G.$$

The action of Lie groups on Manifolds is known as Lie group transformation.

Definition 1.2.5 (Lie group transformation) . [4] *Let G be a Lie group, and M , a manifold. A Lie group transformation is a map $*$: $G \times M \rightarrow M$ satisfying:*

$$(i) \ (g \cdot h) * x = g * (h * x) \quad \forall \ g, h, \in G, x \in M ,$$

$$(ii) \ e * x = x.$$

If G is an n -parameter Lie group, then $*$ is an “ n -parameter Lie group of transformations”. These group actions often are local, that is, the group of transformations is defined on some points of the manifold.

Definition 1.2.6 (G -invariant function) . [4] *Let G be a local group of transformations acting on a manifold M . A function $F : M \rightarrow N$, where N is another manifold, is called a G -invariant function if for all $x \in M$ and all $g \in G$ such that $g \cdot x$ is defined,*

$$F(g \cdot x) = F(x).$$

Note that the set within what F is defined is called symmetry group.

1.2.2 Coordinates

Consider an n -dimensional manifold M . Let x^μ denote local coordinates of a point p , an element of a subset of M , where it is understood that $\mu = 1, 2, \dots, n$. An n -dimensional manifold M of points is one for which n independent real coordinates $(x^1, x^2, \dots, x^n)$ are required to specify any point completely. In other words, $(x^1, x^2, \dots, x^n)$ can be seen as a map from the subset of M to $\mathbb{R}^n$. Let f be a map defined from M to M . Then by using local coordinates x^μ , we can write $f(p)$ as $f^\mu(p) = f^\mu(x^1, x^2, \dots, x^n)$. Here, $f^\mu(p)$ denotes the components of $f(p)$ and it is clear that the map f defined from M to M is translated to some map g defined from $\mathbb{R}^n$ to $\mathbb{R}^n$.

1.2.3 Vector fields

A vector field X is defined as any smooth linear map from $C^\infty(M)$ to $C^\infty(M)$. Consider $f : M \rightarrow \mathbb{R}$ a smooth real-valued function on M . Then the action of X on f is written as $X[f]$. The collection of all tangent vectors to all possible curves passing through a given point $p \in M$ is called the tangent space to M at p and is denoted $TM|_p$. If M is an n -dimensional manifold, then $TM|_p$ is an n -dimensional vector space, with $\{\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n\}$ providing a basis for $TM|_p$ in the given local coordinates. The vector field X in the local coordinates system x^μ is given by

$$X = X^\mu \frac{\partial}{\partial x^\mu},$$

where X^μ are its components. For the action of a vector field X on the point $p \in M$, we write

$$X[f](p) = X^\mu(p) \frac{\partial f}{\partial x^\mu}(p).$$

Consider two vector fields X and Y . One can find a new vector field denoted $[X, Y]$, called the commutator of X and Y , given by

$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

This commutator is also known in the literature as the Lie bracket and has the following properties, [4]

(i) Bilinearity:

$$[cX + c'Y, Z] = c[X, Z] + c'[Y, Z],$$

$$[X, cY + c'Z] = c[X, Y] + c'[X, Z],$$

where c, c' are constants.

(ii) Skew-Symmetry

$$[X, Y] = -[Y, X].$$

(iii) Jacobi Identity

$$[X, [Y, Z]] = [Z, [X, Y]] = [Y, [Z, X]] = 0.$$

1.2.4 Basis vectors

Consider a four dimensional vector X which can be referred to as X^a . At each point P on a manifold M containing the vector field X , this vector field lies in the tangent space $TM|_P$ at P . Given that the vector field X is coordinate system independent at each point p , one can define a set of basis vectors e_a for the tangent space $TM|_P$. The number of the basis vectors is equal to the dimension of $TM|_P$ and also equal to the dimension of the manifold. Using the Einstein summation convention, the vector X can be written as a linear combination of the basis vectors e_a

$$X = X^a e_a,$$

where X^a are the contravariant components of X . The basis vectors are tangent to the coordinate line. This allows us to write them without any hesitation as partial derivatives in a particular coordinate direction as

$$e_a = \frac{\partial}{\partial x^a} = \partial_a.$$

The basis $\{\partial_a\}$ is called coordinate basis and the vector field X takes the form

$$X = X^a \partial_a,$$

previously defined. This can be viewed as an operator.

1.2.5 Differential forms

Definition 1.2.7 [4] *Let M be a smooth manifold and $TM|_x$ its tangent space at x . The space $\wedge_k T^*M|_x$ of differential k -forms at x is the set of all k -linear alternating functions*

$$w : TM|_x \times \cdots \times TM|_x \longrightarrow \mathbb{R}.$$

Conventionally, any real number (also a smooth real-valued function) is a 0-form. One can represent a vector X with its covariant components X_a and this type of vector is

called one-form. It is given by

$$\tilde{X} = X_a w^a.$$

This is a different representation of the same object X . In coordinate representation, the basis one-forms are given by

$$w^a = dx^a.$$

1.2.6 Flows

For any vector field X on M there exists a smooth map $\Phi : \mathbb{R} \times M \rightarrow M$ called the flow of X , here written $\Phi(\epsilon, x)$ with $\epsilon \in \mathbb{R}$ and $x \in M$, such that $\Phi(0, x) = x$, $\Phi(\delta, \Phi(\epsilon, x)) = \Phi(\delta + \epsilon, x)$ ($\forall \epsilon, \delta \in \mathbb{R}$), and

$$\frac{d}{d\epsilon} \Phi(\epsilon, x) = X(\Phi(\epsilon, x)).$$

The vector field turns to be an infinitesimal transformation of points. Given a vector field, from each point x one can move to a nearby point x' in the direction determined by the vector field itself:

$$x^\mu \mapsto x'^\mu = \Phi^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x) + \mathcal{O}(\epsilon^2).$$

1.2.7 Riemannian manifolds

Definition 1.2.8 [5] *A space-time (or simply spacetime in one word) is a pair (M, g) , where M is a four-dimensional differentiable manifold and g is a metric of the Lorentzian signature $(+ - - -)$ given on it. Sometimes, one uses the signature $(- + + +)$.*

Definition 1.2.9 [5] *A spacetime (M, g) is called homogeneous and isotropic if*

- (i) *the manifold is the product $M = \mathbb{R} \times S$, where $t \in \mathbb{R}$ is the time coordinate and $x \in S$ is a three-dimensional space of constant curvature endowed with a negative-definite metric;*

(ii) the metric g is invariant under point transformations generated by the isometry group of S .

Theorem 1.2.1 [5] *Let a four-dimensional manifold be the product $M = \mathbb{R} \times S$, where $t \in \mathbb{R}$ is the time coordinate and $x \in S$ is a three-dimensional space of constant curvature. We suppose that M is endowed with a sufficiently smooth metric of the Lorentzian signature. It follows that if spacetime is homogeneous and isotropic, then a coordinate system $t, x^i, i = 1, 2, 3$ exists in the neighborhood of each point such that the metric takes the form*

$$ds^2 = dt^2 + a^2(t)\bar{g}_{ij}dx^i dx^j, \quad i, j = 1, 2, 3, \quad (1.1)$$

where $a(t) > 0$ is an arbitrary function of t called scale factor, and $\bar{g}_{ij}(x)$ is a negative definite constant-curvature metric on S depending on the spatial coordinates $x \in S$ only.

A proof of this theorem can be found in [5].

Definition 1.2.10 (Riemannian manifold) . [6] *A Riemannian metric on a smooth manifold M is a 2-tensor field, g , an element of the 2-tensor field space of M that is symmetric (i.e., $g(X, Y) = g(Y, X)$) and positive definite (i.e., $g(X, X) > 0$ if $X \neq 0$). A manifold together with a given Riemannian metric is called a Riemannian manifold. We often use the word “metric” to refer to a Riemannian metric when there is no chance of confusion.*

Definition 1.2.11 [6, 5] *If (M, g) and $(\widetilde{M}, \tilde{g})$ are Riemannian manifolds, a diffeomorphism i from M to $\widetilde{M}$ is called an isometry if $i^*g = \tilde{g}$, where i^* is the induced pullback of differential forms.*

We say that two Riemannian manifolds (M, g) and $(\widetilde{M}, \tilde{g})$ are isometric if there exists an isometry between them. One can easily verify that being isometric is an equivalence relation on the class of Riemannian manifolds. An isometry $i : (M, g) \longrightarrow (\widetilde{M}, \tilde{g})$ is called an isometry of M . A composition of isometries and the inverse of an isometry are again isometries.

Proposition 1.2.1 [6, 5] *The set of all isometries of a given Riemannian manifold (M, g) is a group, called the isometry group of M denoted by $\mathbb{I}(M)$.*

A proof of the proposition can be found in [5]. The isometry group $\mathbb{I}(M)$ can be either a discrete group or a Lie group.

Definition 1.2.12 [5] *If the isometry group $\mathbb{I}(M)$ is a Lie group, we can consider infinitesimal transformations. In this situation, we are dealing with infinitesimal isometries:*

$$x^a \mapsto x'^a = x^a + \epsilon X^a, \quad \epsilon \ll 1.$$

Any infinitesimal isometry is generated by a sufficient smooth vector field $X(x) = X^a(x)\partial_a$, which is called a Killing vector.

Theorem 1.2.2 [5] *The Lie algebra of infinitesimal isometries $\mathbb{I}(M)$ of a connected Riemannian manifold M has the dimension not exceeding $\frac{1}{2}n(n+1)$, where $n := \dim M$. If the dimension is maximal, $\dim \mathbb{I}(M) = \frac{1}{2}n(n+1)$, then the manifold M is homogeneous and isotropic, being a constant-curvature space.*

Note that the definitions under this subsection also apply to pseudo-Riemannian manifolds.

1.3 Symmetry group transformations and conservation laws

This section provides details on Lie group of transformations and generalized symmetries. The symmetry group of a system of DEs (respectively PDEs) is the largest (connected) local Lie group of transformations that acts on the space of the independent and dependent variables, with the property that it preserves the set of solutions. By this mean, it

maps solutions onto other solutions. It follows that a vector field can be associated with a local one-parameter Lie group of transformations and this matched-pair is known as infinitesimal generator.

Definition 1.3.1 (Differential function) . [4] *Consider a function $f(x, u, u_{(1)}, \dots, u_{(r)})$ of a finite number of variables. It is called a differential function of order r , where $x = (x^1, x^2, \dots, x^n)$ and $u = (u^1, u^2, \dots, u^m)$ are respectively independent and dependent variables and $u_{(1)}, \dots, u_{(r)}$ denote the collections of all first, second, $\dots$, r^{th} order partial derivatives. In other terms, the superscript of u stands for the number of dependent variables while the subscript designates the partial derivative's order. The set $\{x, u, u_{(1)}, \dots, u_{(r)}\}$ forms the jet space (also called bundle) which is the total space of independent and dependent variables and the derivatives of the dependent variables.*

1.3.1 Noether symmetries

Consider an r th-order system of partial differential equations of n -independent and m -dependent variables, viz.,

$$E^\beta(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \beta = 1, \dots, \tilde{m}. \quad (1.1)$$

A conservation law of (1.1) is defined by the equation [7, 8]

$$D_i T^i = 0 \quad (1.2)$$

on the solutions of (1.1). Here D_i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n,$$

is the total differential operator and the vector $T = (T^1, \dots, T^n)$ called a *conserved vector* of (1.1).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots, \quad (1.3)$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned}\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \\ &\vdots \\ \zeta_{i_1 i_2 \dots i_s}^\alpha &= D_{i_1} D_{i_2} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 i_2 \dots i_s}^\alpha, \quad s > 1\end{aligned}\tag{1.4}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.5}$$

The formula in (1.3) is the most general form of the Lie-Bäcklund operator and is known as the general prolongation formula. In this thesis, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u .

Theorem 1.3.1 (Infinitesimal criterion) . [4] *A connected group of transformation G is a symmetry group of a system of DEs of maximal rank (1.1) if and only if the classical infinitesimal symmetry conditions*

$$X E^\beta(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{1.6}$$

wherever

$$E^\beta(x, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{1.7}$$

holds for every infinitesimal generator X of G .

The equation (1.6) are the determining equations of the symmetry group to (1.7).

The Euler-Lagrange equations, if they exist, associated with (1.1) are the system

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m,$$

where

$$\frac{\delta}{\delta u^\alpha}$$

is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m. \quad (1.8)$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_{\Omega} L(x, u, u_{(1)}, \dots, u_{(r)}) dx \quad (1.9)$$

defined over Ω . If we include point dependent gauge terms $f_1, \dots, f_n$, the Noether symmetries X are given by the expression,

$$XL + LD_i \xi^i = D_i f^i. \quad (1.10)$$

Corresponding to each X , a conserved vector $T = (T^1, \dots, T^n)$ is obtained via Noether's Theorem. For our purposes, the Lagrangian is obtained directly from the metric which gives rise to the geodesic equations.

Theorem 1.3.2 (Noether's theorem) [4] *For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a corresponding vector $T = (T^1, \dots, T^n)$, $T^i \in \mathcal{A}$, called conserved vector or conserved quantity and defined by*

$$T^i = f^i - N^i L, \quad i = 1, \dots, n, \quad (1.11)$$

where N^i is called Noether operator and is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} D_{i_2} \cdots D_{i_s} (W^\alpha) + \xi^j u_{j i_1 i_2 \dots i_s}^\alpha \frac{\delta}{\delta u_{i_1 \dots i_s}^\alpha}. \quad (1.12)$$

T^i is the conserved current associated with the Euler-Lagrange equations.

Theorem 1.3.3 [9] *A Lie-Bäcklund symmetry generator X of the form (1.3) is associated with a conserved vector T of the system (1.1) if X and T satisfy the relation*

$$[T^i, X] = X(T^i) + T^i D_j (\xi^j) - T^j D_j (\xi^i) = 0, \quad i = 1, \dots, n. \quad (1.13)$$

Theorem 1.3.4 [10, 11] *Suppose that X is a Lie-Bäcklund symmetry generator of the form (1.3) and T^i are the components of the conserved vector of the system (1.1), then*

$$T^{*i} = [T^i, X] = X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, \dots, n, \quad (1.14)$$

constitute as well the components of the conserved vector of (1.1), that is

$$D_i T^{*i} = 0|_{(1.1)}. \quad (1.15)$$

There are several ways of constructing conserved quantities. In this thesis, we will use two approaches: the partial Noether theorem and the multipliers approach which mostly applies for non variational problems. This latter approach fills the gap of the lack of Noether theorem which only apply for variational problems. Göktaş and Hereman in [12] have proven that every admitted conservation law is obtained from a multiplier $Q_\mu(x, u, u_{(1)}, \dots, u_{(r)}) \in \mathcal{A}$ such that

$$Q_\mu E^\mu = D_i T^i. \quad (1.16)$$

Since the right hand side in (1.16) is a total divergence, the action of the Euler-Lagrange operator on (1.16) gives

$$\frac{\delta}{\delta u^\alpha} (Q_\mu E^\mu) = \frac{\delta}{\delta u^\alpha} (D_i T^i) = 0. \quad (1.17)$$

It is important to remark that if the system of differential equations is variational, so does multipliers. Many authors have used the homotopy operator to obtain the conserved vectors [4, 13, 14, 15].

1.3.2 Point transformations

A point transformation is a one-to-one transformation which acts on the $n+m$ -dimensional space (x, u) . It is of the form

$$\begin{aligned} \tilde{x} &= f(x, u), \\ \tilde{u} &= g(x, u). \end{aligned} \quad (1.18)$$

The infinitesimal transformations are given up to the first order Taylor expansion by

$$\begin{aligned}\tilde{x} &= x + \epsilon \xi(x, u), \\ \tilde{u} &= u + \epsilon \eta(x, u),\end{aligned}\tag{1.19}$$

where ξ and η are previously defined in (1.3). In this situation, the point transformation is a one parameter (ϵ) Lie group of transformations.

1.3.3 Contact transformations

A contact transformation is a transformation of the form

$$\begin{aligned}\tilde{x} &= f\left(x, u, \frac{du}{dx}\right), \\ \tilde{u} &= g\left(x, u, \frac{du}{dx}\right), \\ \tilde{u}_i &= h\left(x, u, \frac{du}{dx}\right),\end{aligned}\tag{1.20}$$

where $i = 1, \dots, n$. It is a one-to-one transformation on some domain D in $(n + m + nm)$ -space and leaves invariant the contact condition

$$d\tilde{u} = \tilde{u}_i d\tilde{x},\tag{1.21}$$

where it is assumed that f and g depend only on x, u and the first derivative of u given by $u_i = \frac{du}{dx}$. The transformation, if not of that assumption would be a point transformation.

The infinitesimal transformations are given up to the first order Taylor expansion by

$$\begin{aligned}\tilde{x} &= x + \epsilon \xi\left(x, u, \frac{du}{dx}\right), \\ \tilde{u} &= u + \epsilon \eta\left(x, u, \frac{du}{dx}\right), \\ \tilde{u}_i &= u_i + \epsilon \zeta\left(x, u, \frac{du}{dx}\right),\end{aligned}\tag{1.22}$$

where ζ is previously defined in (1.3).

1.3.4 Double reduction theory

The application of a Lie point or Lie-Bäcklund symmetry generator to a conserved vector provides either conservation law associated with that symmetry or conservation law that may be trivial, known already or new, see [16]. The theory of the double reduction then results from the association of symmetries to conserved vector. This means that having a system of PDEs of order r with n independent variables and m dependent variables, we can reduce the number of independent variables followed by the reduction of the order of the system of PDEs, which becomes a system of ODEs of order $(r - 1)$.

Theorem 1.3.5 [11, 16] *Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (1.1). Then Under a contact transformation, there exist functions $\widetilde{T}^i$ such that $J D_i T^i = \widetilde{D}_i \widetilde{T}^i$ where $\widetilde{T}^i$ is given by*

$$\begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \vdots \\ \widetilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \vdots \\ \widetilde{T}^n \end{pmatrix}, \quad (1.23)$$

in which

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}, \quad (1.24)$$

and $J = \det(A)$. A^T is the transpose of the matrix A .

The proof of theorem (1.3.5) is provided in full details in [16].

Theorem 1.3.6 (Fundamental theorem of double reduction) [16] *Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (1.1). Then Under a similarity transformation of a symmetry X of the form (1.3) for the PDE, there exist functions $\widetilde{T}^i$ such that X is*

still a symmetry for the PDE $\widetilde{D}_i \widetilde{T}^i = 0$ and

$$\begin{pmatrix} X\widetilde{T}^1 \\ X\widetilde{T}^2 \\ \vdots \\ X\widetilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \quad (1.25)$$

where

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}, \quad (1.26)$$

and $J = \det(A)$.

The proof of theorem (1.3.6) is provided as well in full details in [16].

1.4 Outline of chapters

In Chapter 2, variational symmetries and conservation laws of a modified de Sitter metric are classified. The resulting Noether symmetries in each case are compared with conservation laws given by Killing vectors.

Chapter 3 presents the algebra of symmetries associated with the de Sitter-Schwarzschild metric and the corresponding wave & Klein-Gordon equations. These symmetries are studied and classified. They are the Lie symmetries of the geodesic and wave equations. The conservation laws are obtained via the natural Lagrangians and variational symmetries in each of the cases.

In Chapter 4, we focus on the symmetry structure (and the corresponding conservation laws) of the Schwartzchild and other asymptotically flat metrics (in particular, the variational/divergence symmetries of the respective geodesic equations following from the ‘natural’ Lagrangian).

In Chapter 5, we undertake a detailed analysis of the symmetry structures of the plane symmetric and the locally rotationally symmetric Bianchi type I space-times in the $f(R, T)$ gravity. In particular, we construct all the variational (Noether) symmetries and, in some cases, construct the associated conservation law using Noethers theorem. Some comparisons are made with the symmetry structures of some well known space-times.

Chapter 2

Classification of a Modified de Sitter Metric by Variational Symmetries and Conservation Laws

2.1 Introduction

The recent observations of Type-I supernovae have led us to believe that the current universe is undergoing accelerated expansion [17, 18]. Looking back at the evolution of the universe, earlier on, it was in a decelerating stage, and this means that there was a transition from deceleration to acceleration. Hence the deceleration parameter q which is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}, \quad (2.1)$$

where $a \equiv a(t)$ is the scale factor of the universe, must show a signature change from positive to negative [19]. One way in which this can be achieved in cosmology is to take the deceleration parameter q to have the form [20]

$$q = \alpha + \beta H, \quad (2.2)$$

where α, β are constants, and H is the Hubble parameter given by

$$H = \frac{\dot{a}}{a}. \quad (2.3)$$

Equation (2.2) is the next possible generalisation (see also [21]) of a constant deceleration parameter (Equation (2.2) with $\beta = 0$), [22] which was popular before the discovery of the accelerated expansion of the universe. Combining (2.1), (2.2) and (2.3), we have the following differential equation

$$\frac{\ddot{a}a}{\dot{a}^2} + \beta \frac{\dot{a}}{a} + \alpha = 0, \quad (2.4)$$

whose solution is

$$a = \exp \left[-\frac{(1+\alpha)}{\beta}t - \frac{1}{1+\alpha} + \frac{c}{\beta} \right],$$

for $\alpha \neq 1$, where c is an integration constant.

Without loss of generality, we make take $\alpha = -1$ in equation (2.2). This then leads to the following solution for the scale factor a :

$$a = e^{\frac{1}{\beta}\sqrt{2\beta t+k}}. \quad (2.5)$$

Such a scale factor has been studied by several authors in a variety of different cosmological scenarios [23, 24, 25, 26, 27, 28, 29, 30]. Pradhan *et al* [20] have shown that this solution is in agreement with current observations for certain values of the parameters.

The general theory of relativity, which is a field theory of gravitation, is governed by the Einstein field equations. These equations explain how spacetime geometry and distribution of matter are related through them via the Einstein field equations. In order to understand the physics of the solutions of the Einstein's equations, a lot of work has been published. Most of this work includes investigations of the Lie symmetries or conservation laws admitted by these solutions. A tensor $\mathbf{A}$ is said to admit a Lie symmetry, X , if its Lie derivative along X is zero, i.e, $\mathcal{L}_X \mathbf{A} = 0$, where $\mathbf{A}$ represents a metric tensor g_{ab} in $ds^2 = g_{ab}dx^a dx^b$. In this chapter, we focus on finding variational or Noether symmetries and conservation laws of the modified de Sitter metric

$$ds^2 = -dt^2 + e^{2\frac{\sqrt{2\beta t+k}}{\beta}} \left(\frac{1}{1-kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (2.6)$$

Recently, a study of the role of the Noether symmetries in the classification of a large class of Lorentzian metrics have been done; this was shown to play a significant role in the classic studies of the classification of Killing vectors (isometries), homotheties, etc. that arise in Relativity via certain solutions of the Einstein equations. Some references in this regard are [31, 32, 33]. This work has also been applied to some manifolds of metrics with ‘neutral signatures’, like ASD manifolds, [34]. It has been especially useful in the construction of conservation laws. It is well-known that the classical conservation laws are fundamental tools for establishing the existence and regularity of solutions for differential equations, [35]. Thus, the classification that we resort to in the sequel is not in the sense of the ‘Lie point symmetry classification’ carried out by a number of authors in the recent past following the pioneering work of Ovsiannikov, Ibragimov, inter alia (see [36, 37, 38, 39]).

The content of this chapter is appeared in the paper [40].

2.2 Friedmann-Robertson-Walker (FRW) metric

This section provides the derivation of the FRW metric of which for the particular value of the scalar factor $a(t)$ defined in (2.5) gives rise to the modified de Sitter metric studied herein.

2.2.1 Spatial homogeneity and isotropy

We start with the definition of cosmology. Cosmology is the study of the dynamics of the whole universe. This study seems quiet simpler for two reasons. On a very large scale, only the gravity dominates and in the meantime, the universe is assumed to be homogeneous and isotropic. This latter assumption applies only to the spatial term in the metric, [41, 42].

Definition 2.2.1 (Cosmological principle) *At any particular time the universe looks the same from all positions in space at a particular time and all directions in space at any point are equivalent.*

2.2.2 FRW metric

The equations under this subsection can be found in [41, 42].

The cosmological principle allows us to consider spaces with constant curvature denoted by K . the constant curvature together with the metric tensor $g_{\mu\nu}$ defined the Riemann tensor

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}). \quad (2.7)$$

According to observation, the universe evolves in time. Thus, one has imposed to the line element the form

$$ds^2 = g_{ij}dx^i dx^j, \quad i, j = 0, 1, 2, 3, \quad (2.8)$$

since the universe by assumption is homogeneous and isotropic in spatial coordinates only, where g_{ij} are functions of the coordinates (t, x^1, x^2, x^3) and the signature $(+, -, -, -)$ is used. One introduces the scale factor $a(t)$ into the line element (2.8) and has

$$ds^2 = dt^2 - a^2(t)d\sigma^2, \quad (2.9)$$

where $d\sigma^2$ is the spatial part of the metric. We recall that the Schwarzschild line element has the general form [41, 42]

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.10)$$

Using this as a guide, we write the spatial part in (2.10) as

$$d\sigma^2 = B(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (2.11)$$

In order to find $B(r)$, we compute in the following paragraphs the Christoffel symbols also known as connection coefficients denoted Γ_{jk}^i and the Ricci tensor denoted R_{ij} .

We recall that in n -dimension space, there are n^3 connection coefficients defined by

$$\Gamma^i_{jk} \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}), \quad (2.12)$$

where g^{jl} is the inverse of the metric tensor. The Ricci tensor in terms of the connection coefficients is given by

$$R_{ij} = \partial_j \Gamma^k_{ik} - \partial_k \Gamma^k_{ij} + \Gamma^l_{ik} \Gamma^k_{lj} - \Gamma^l_{ij} \Gamma^k_{lk}. \quad (2.13)$$

We can obtain as the Ricci tensor in terms of the constant curvature K by using the Riemann tensor in the following way

$$\begin{aligned} R_{jk} &= R^l_{jlk} \\ &= g^{li} R_{ijkl} \\ &= K g^{il} (g_{ik} g_{jl} - g_{il} g_{jk}) \\ &= K (g^{il} g_{ik} g_{jl} - g^{il} g_{il} g_{jk}) \\ &= K (\delta^l_k g_{jl} - \delta^l_l g_{jk}) \\ &= K (g_{jk} - \delta^l_l g_{jk}) \\ &= K (g_{jk} - 3g_{jk}) \\ &= -2K g_{jk}, \end{aligned} \quad (2.14)$$

where we have used the Einstein summation convention and with three spatial dimensions,

$$\delta^l_l = 1 + 1 + 1 = 3. \quad (2.15)$$

The metric tensor g_{ij} associated with the line element (2.11) writes

$$g_{ij} = \begin{pmatrix} B(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (2.16)$$

The computation of the non-zero connections gives

$$\begin{aligned} \Gamma^r_{rr} &= \frac{1}{2B(r)} \frac{dB(r)}{dr}, \\ \Gamma^r_{\theta\theta} &= -\frac{r}{B(r)}, \end{aligned}$$

$$\begin{aligned}
\Gamma_{\phi\phi}^r &= -\frac{r \sin^2 \theta}{B(r)}, \\
\Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta, \\
\Gamma_{r\theta}^\theta &= \frac{1}{r}, \\
\Gamma_{r\phi}^\phi &= \frac{1}{r}, \\
\Gamma_{\phi\theta}^\phi &= \cot \theta.
\end{aligned} \tag{2.17}$$

These connection coefficients help us to find the non-zero components of the Ricci tensor

$$\begin{aligned}
R_{rr} &= -\frac{1}{rB(r)} \frac{dB(r)}{dr}, \\
R_{\theta\theta} &= -1 + \frac{1}{B(r)} - \frac{r}{2B^2(r)}, \\
R_{\phi\phi} &= R_{\theta\theta} \sin^2 \theta.
\end{aligned} \tag{2.18}$$

Substituting R_{rr} and g_{rr} in (2.14) we obtain the following differential equation

$$\frac{1}{rB(r)} \frac{dB(r)}{dr} = 2KB(r). \tag{2.19}$$

In a similar way by substituting $R_{\theta\theta}$ and $g_{\theta\theta}$ in (2.14) we obtain the following differential equation

$$1 - \frac{1}{B(r)} + \frac{r}{2B^2(r)} = 2Kr^2. \tag{2.20}$$

The integration of (2.19) yields

$$B(r) = \frac{1}{C - Kr^2}, \tag{2.21}$$

where C is a constant of integration. Replacing $B(r)$ into (2.20) gives

$$1 - C + Kr^2 = Kr^2, \tag{2.22}$$

that is $C = 1$ and

$$B(r) = \frac{1}{1 - Kr^2}. \tag{2.23}$$

Thus, the line element of the spatial part writes

$$d\sigma^2 = \frac{1}{1 - Kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{2.24}$$

and the FRW line element becomes

$$ds^2 = dt^2 - a^2(t) \left(\frac{1}{1 - Kr^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right). \tag{2.25}$$

We can normalize the curvature K by defining the parameter $k = \frac{K}{|K|}$ in such a way that $k = \pm 1$. Substituting k into (2.25) gives rise to the standard form for the FRW line element

$$ds^2 = dt^2 - a^2(t) \left(\frac{1}{1 - kr^2} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right). \quad (2.26)$$

The normalized constant curvature can take the values $-1, 0, 1$ which are respectively associated with the negative, zero or positive curvature of space (the spatial part).

In order to compute the symmetries we consider the scale factor $a = e^{\frac{1}{\beta}\sqrt{2\beta t + k}}$.

2.3 Symmetries and conservation laws

2.3.1 Symmetry via Noether's theorem

The metric (2.6) has a corresponding Lagrangian

$$L = -\dot{t}^2 + e^{2\frac{\sqrt{2\beta t + k}}{\beta}} \left(\frac{1}{1 - kr^2} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \quad (2.27)$$

with the Euler (geodesic) equations

$$\begin{aligned} G^1 &= \ddot{t} + \frac{1}{\sqrt{2\beta t + k}} \left(e^{\frac{\sqrt{2\beta t + k}}{\beta}} \right)^2 \left(\frac{\dot{r}^2}{1 - kr^2} + r^2 \dot{\theta}^2 + r^2 (\sin \theta)^2 \dot{\phi}^2 \right) = 0, \\ G^2 &= \left(r \dot{\phi}^2 (-1 + kr^2)^2 (\sin \theta)^2 + r^5 \dot{\theta}^2 k^2 - 2 r^3 \dot{\theta}^2 k + \ddot{r} k r^2 \right. \\ &\quad \left. + \left(-\dot{r}^2 k + \dot{\theta}^2 \right) r - \ddot{r} \sqrt{2\beta t + k} + 2 \dot{r} \dot{t} (-1 + kr^2) \right) = 0, \\ G^3 &= r \sin \theta \dot{\phi}^2 \cos(\theta) \sqrt{2\beta t + k} - 2 r \dot{\theta} \dot{t} - 2 \dot{\theta} \dot{r} \sqrt{2\beta t + k} - r \ddot{\theta} \sqrt{2\beta t + k} = 0, \\ G^4 &= 4 \left(\left(\dot{\phi} \dot{r} + 1/2 r \ddot{\phi} \right) \sin \theta + r \dot{\phi} \cos \theta \dot{\theta} \right) \sqrt{2\beta t + k} + 4 r \sin(\theta) \dot{t} \dot{\phi} = 0, \end{aligned} \quad (2.28)$$

where s is the arclength variable and the derivative $\dot{x}$ is the derivative of ' x ' taken with respect to s . The variational symmetries, [4, 43, 8, 44],

$$X = \xi \partial_s + \eta_t \partial_t + \eta_r \partial_r + \eta_\theta \partial_\theta + \eta_\phi \partial_\phi$$

in which $\xi, \eta_t, \eta_r, \eta_\theta$ and η_ϕ all depend on (t, r, θ, ϕ) , satisfy for some gauge function f ,

$$X^{[1]} L + L \frac{d\xi}{ds} = \frac{df}{ds}. \quad (2.29)$$

Since the Lagrangian is in terms of the first order derivative of the variables, we must rewrite the variational symmetries by making use of the first prolongation defined in (1.3) as follows:

$$X^{[1]} = \xi \partial_s + \eta_t \partial_t + \eta_r \partial_r + \eta_\theta \partial_\theta + \eta_\phi \partial_\phi + \eta_t^{[1]} \partial_{\dot{t}} + \eta_r^{[1]} \partial_{\dot{r}} + \eta_\theta^{[1]} \partial_{\dot{\theta}} + \eta_\phi^{[1]} \partial_{\dot{\phi}}, \quad (2.30)$$

where

$$\eta_c^{[1]} = D_s \eta_c - \dot{c} D_s \xi$$

and c stands for $\{t, r, \theta, \phi\}$. The total derivative with respect to s is D_s and is given by

$$D_s = \frac{d}{ds} = \partial_s + \dot{t} \partial_t + \dot{r} \partial_r + \dot{\theta} \partial_\theta + \dot{\phi} \partial_\phi.$$

Substituting (2.30) in (2.29) leads to

$$\begin{aligned} & \frac{4}{\sqrt{2\beta t + k}} \eta_t \left[\frac{\dot{r}^2}{1 - k r^2} + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right] e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \\ & + \eta_r \left[\frac{2 k r \dot{r}^2}{(1 - k r^2)^2} + 2 r \dot{\theta}^2 + 2 r \sin^2 \theta \dot{\phi}^2 \right] e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \\ & + 2 r^2 \sin \theta \cos \theta \dot{\phi}^2 e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \\ & - 2 [\partial_s \eta_t + \dot{t} \partial_t \eta_t + \dot{r} \partial_r \eta_t + \dot{\theta} \partial_\theta \eta_t + \dot{\phi} \partial_\phi \eta_t \\ & - \dot{t} \partial_s \xi - \dot{t}^2 \partial_t \xi - \dot{t} \dot{r} \partial_r \xi - \dot{t} \dot{\theta} \partial_\theta \xi - \dot{t} \dot{\phi} \partial_\phi \xi] \dot{t} \\ & + 2 [\partial_s \eta_r + \dot{t} \partial_t \eta_r + \dot{r} \partial_r \eta_r + \dot{\theta} \partial_\theta \eta_r + \dot{\phi} \partial_\phi \eta_r \\ & - \dot{r} \partial_s \xi - \dot{t} \dot{r} \partial_t \xi - \dot{r}^2 \partial_r \xi - \dot{r} \dot{\theta} \partial_\theta \xi - \dot{r} \dot{\phi} \partial_\phi \xi] e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \dot{r} (1 - k r^2)^{-1} \\ & + 2 [\partial_s \eta_\theta + \dot{t} \partial_t \eta_\theta + \dot{r} \partial_r \eta_\theta + \dot{\theta} \partial_\theta \eta_\theta + \dot{\phi} \partial_\phi \eta_\theta \\ & - \dot{\theta} \partial_s \xi - \dot{t} \dot{\theta} \partial_t \xi - \dot{r} \dot{\theta} \partial_r \xi - \dot{\theta}^2 \partial_\theta \xi - \dot{\theta} \dot{\phi} \partial_\phi \xi] e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} r^2 \dot{\theta} \\ & + 2 [\partial_s \eta_\phi + \dot{t} \partial_t \eta_\phi + \dot{r} \partial_r \eta_\phi + \dot{\theta} \partial_\theta \eta_\phi + \dot{\phi} \partial_\phi \eta_\phi \\ & - \dot{\phi} \partial_s \xi - \dot{t} \dot{\phi} \partial_t \xi - \dot{r} \dot{\phi} \partial_r \xi - \dot{\theta} \dot{\phi} \partial_\theta \xi - \dot{\phi}^2 \partial_\phi \xi] e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} r^2 \sin^2 \theta \dot{\phi} \\ & - (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi) e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \dot{t}^2 \\ & + (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi) e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} \dot{r}^2 (1 - k r^2)^{-1} \\ & + (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi) e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} r^2 \\ & + (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi) e^{2 \frac{\sqrt{2\beta t + k}}{\beta}} r^2 \sin^2 \theta \dot{\phi}^2 \\ & = \partial_s f + \dot{t} \partial_t f + \dot{r} \partial_r f + \dot{\theta} \partial_\theta f + \dot{\phi} \partial_\phi f. \end{aligned} \quad (2.31)$$

The separation by monomials leads to the overdetermined system

$$\begin{aligned}
\dot{r}^2 &: 4\eta_t(1 - kr^2)^{-1}(2\beta t + k)^{-\frac{1}{2}} + 2\eta_r kr(1 - kr^2)^{-2} + 2\partial_r\eta_r(1 - kr^2)^{-1} \\
&\quad - \partial_s\xi(1 - kr^2)^{-1} = 0, \\
\dot{\theta}^2 &: 4\eta_t(2\beta t + k)^{-\frac{1}{2}} + 2\eta_r r + 2r^2\partial_\theta\eta_\theta - r^2\partial_s\xi = 0, \\
\dot{\phi}^2 &: 4\eta_t(2\beta t + k)^{-\frac{1}{2}}r^2\sin^2\theta + 2\eta_\theta r^2\sin\theta\cos\theta + 2\partial_\phi\eta_\phi r^2\sin^2\theta - \partial_s\xi r^2\sin^2\theta = 0, \\
\dot{t}\dot{r} &: -2\partial_r\eta_t + 2\partial_t\eta_r(1 - kr^2)^{-1}e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = 0, \\
\dot{t}\dot{\theta} &: -2\partial_\theta\eta_t + 2\partial_t\eta_\theta r^2e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = 0, \\
\dot{t}\dot{\phi} &: -2\partial_\phi\eta_t + 2\partial_t\eta_\phi r^2\sin^2\theta e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = 0, \\
\dot{r}\dot{\theta} &: 2\partial_\theta\eta_r(1 - kr^2)^{-1} + 2\partial_r\eta_\theta r^2 = 0, \\
\dot{r}\dot{\phi} &: 2\partial_\phi\eta_\theta r^2 + 2\partial_\theta\eta_\phi r^2\sin^2\theta = 0, \\
\dot{\theta}\dot{\phi} &: 2\partial_\phi\eta_r(1 - kr^2)^{-1} + 2\partial_r\eta_\phi r^2\sin^2\theta = 0, \\
1 &: \partial_s f = 0, \\
\dot{t} &: -2\partial_s\eta_t = \partial_t f, \\
\dot{r} &: 2\partial_s\eta_r(1 - kr^2)^{-1}e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = \partial_r f, \\
\dot{\theta} &: 2\partial_s\eta_\theta r^2e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = \partial_\theta f, \\
\dot{\phi} &: 2\partial_s\eta_\phi r^2\sin^2\theta e^{2\frac{\sqrt{2\beta t+k}}{\beta}} = \partial_\phi f,
\end{aligned} \tag{2.32}$$

whose solution, with zero gauge f , leads to the vector fields

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= -\sqrt{-1 + kr^2}\cos\phi\sin\theta\partial_r - \frac{\sqrt{-1 + kr^2}\cos\theta\cos\phi}{r}\partial_\theta + \frac{\sqrt{-1 + kr^2}\sin\phi}{r\sin\theta}\partial_\phi, \\
X_3 &= \sqrt{-1 + kr^2}\sin\phi\sin\theta\partial_r - \frac{\sqrt{-1 + kr^2}\cos\theta\cos\phi}{r}\partial_\theta + \frac{\sqrt{-1 + kr^2}\cos\phi}{r\sin\theta}\partial_\phi, \\
X_4 &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi, \\
X_5 &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \\
X_6 &= \partial_\phi, \\
X_7 &= -\sqrt{-1 + kr^2}\cos\theta\partial_r + \frac{1}{r}\sqrt{-1 + kr^2}\sin\theta\partial_\theta.
\end{aligned} \tag{2.33}$$

The algebra of Noether symmetries of (2.28) contains the Killing vectors/isometries of (2.6). It can be shown that all except X_1 are Killing vectors (isometries) of the manifold.

2.3.2 Killing equation

Consider a scalar function $f : M \rightarrow \mathbb{R}$ and a vector field $X = X^\mu \partial_\mu$ defined on M . The Lie derivative of the function f is defined with respect to X and its action on f tells us how much f changes along the flow of X , that is the change of f from a point x to another point x' , $(f(\Phi(\epsilon, x)) - f(x))$. Thus the Lie derivative of a scalar function f along the vector field X denoted $\mathcal{L}_X f$ is given by

$$\mathcal{L}_X f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(\Phi(\epsilon, x)) - f(x)}{\epsilon}, \quad (2.34)$$

where $\Phi(\epsilon, x)$ is the flow of X . By using the following relation

$$x^\mu \mapsto x'^\mu = \Phi^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x) + \mathcal{O}(\epsilon^2), \quad (2.35)$$

one can write

$$f(\Phi(\epsilon, x)) = f(x) + \epsilon X^\mu \frac{\partial f}{\partial x^\mu}(x) + \mathcal{O}(\epsilon^2). \quad (2.36)$$

This in the local coordinates system x^μ , The Lie derivative of a scalar function is just

$$\mathcal{L}_X f(x) = X^\mu \frac{\partial f}{\partial x^\mu}. \quad (2.37)$$

The Lie derivative can generalize the directional derivative of a scalar function to higher rank tensors.

Definition 2.3.1 *Let X and Y be vector fields on M . The Lie derivative of Y with respect to X is the object whose value at a point $p \in M$ is defined by*

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{Y(p + \epsilon X) - Y(p)}{\epsilon}, \quad (2.38)$$

where $Y(p + \epsilon X)$ is called the Lie transport of Y along a curve.

$\mathcal{L}_X Y$ has been proven to be equal to the commutator $[X, Y]$. Furthermore, using components notation, the Lie derivative of a vector field Y with respect to a vector field X is given the relation

$$\mathcal{L}_X Y_\mu = X^\nu \partial_\nu Y_\mu + Y_\nu \partial_\mu X^\nu. \quad (2.39)$$

Likewise, the Lie derivative of the covariant tensor metric $g_{\mu\nu}$ along the vector field X is

$$\mathcal{L}_X g_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu X^\rho + g_{\mu\rho} \partial_\nu X^\rho. \quad (2.40)$$

If the metric tensor $g_{\mu\nu}$ is invariant under the flow of X , that is $\mathcal{L}_X g_{\mu\nu} = 0$, then X is said to be a Killing vector and the Killing vector Equations are

$$X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu X^\rho + g_{\mu\rho} \partial_\nu X^\rho = 0. \quad (2.41)$$

We use (2.41) to verify that X_6 and X_2 are Killing vectors.

(i) Case of X_6

When we substitute $X_6 = \partial_\phi$ previously obtained into (2.40), for example, we get

$$\mathcal{L}_X g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \phi}, \quad (2.42)$$

which vanishes since the metric given in (2.6) does not depend on ϕ . Hence, we proved that X_6 is a Killing vector.

(ii) Case of X_2

The metric (2.6) takes the form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2 \theta \end{pmatrix}, \quad (2.43)$$

and

$$X_2 = \left(0, -\sqrt{-1+kr^2} \cos \phi \sin \theta, -\frac{\sqrt{-1+kr^2} \cos \theta \cos \phi}{r}, \frac{\sqrt{-1+kr^2} \sin \phi}{r \sin \theta} \right). \quad (2.44)$$

The Killing vector equation of X_2 writes

$$\mathcal{L}_{X_2} g_{\mu\nu} = X_2^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu X_2^\rho + g_{\mu\rho} \partial_\nu X_2^\rho. \quad (2.45)$$

Since $g_{\mu\nu}$ is diagonal, we can simply compute $\mathcal{L}_{X_2} g_{tt}$, $\mathcal{L}_{X_2} g_{rr}$, $\mathcal{L}_{X_2} g_{\theta\theta}$ and $\mathcal{L}_{X_2} g_{\phi\phi}$.

$$\begin{aligned}
\mathcal{L}_{X_2} g_{tt} &= X_2^\rho \partial_\rho g_{tt} + g_{\rho t} \partial_t X_2^\rho + g_{t\rho} \partial_t X_2^\rho \\
&= X_2^\rho \partial_\rho g_{tt} + 2g_{\rho t} \partial_t X_2^\rho \\
&= X_2^\rho \partial_\rho g_{tt} + 2g_{tt} \partial_t X_2^t \\
&= 0,
\end{aligned} \tag{2.46}$$

since $g_{tt} = -1$ and $X_2^t = 0$. In a similar way

$$\begin{aligned}
\mathcal{L}_{X_2} g_{rr} &= X_2^\rho \partial_\rho g_{rr} + g_{\rho r} \partial_r X_2^\rho + g_{r\rho} \partial_r X_2^\rho \\
&= X_2^\rho \partial_\rho g_{rr} + 2g_{\rho r} \partial_r X_2^\rho \\
&= X_2^r \partial_r g_{rr} + 2g_{rr} \partial_r X_2^r \\
&= 0,
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
\mathcal{L}_{X_2} g_{\theta\theta} &= X_2^\rho \partial_\rho g_{\theta\theta} + g_{\rho\theta} \partial_\theta X_2^\rho + g_{\theta\rho} \partial_\theta X_2^\rho \\
&= X_2^\rho \partial_\rho g_{\theta\theta} + 2g_{\rho\theta} \partial_\theta X_2^\rho \\
&= X_2^r \partial_r g_{\theta\theta} + 2g_{\theta\theta} \partial_\theta X_2^\theta \\
&= 0,
\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
\mathcal{L}_{X_2} g_{\phi\phi} &= X_2^\rho \partial_\rho g_{\phi\phi} + g_{\rho\phi} \partial_\phi X_2^\rho + g_{\phi\rho} \partial_\phi X_2^\rho \\
&= X_2^\rho \partial_\rho g_{\phi\phi} + 2g_{\rho\phi} \partial_\phi X_2^\rho \\
&= X_2^r \partial_r g_{\phi\phi} + X_2^\theta \partial_\theta g_{\phi\phi} + 2g_{\phi\phi} \partial_\phi X_2^\phi \\
&= 0.
\end{aligned} \tag{2.49}$$

We see that $\mathcal{L}_{X_2} g_{tt} = \mathcal{L}_{X_2} g_{rr} = \mathcal{L}_{X_2} g_{\theta\theta} = \mathcal{L}_{X_2} g_{\phi\phi} = 0$, which implies that $\mathcal{L}_{X_2} g_{\mu\nu} = 0$. Thus, X_2 is a Killing vector.

2.3.3 Conserved vectors using multipliers approach

An alternative approach that guarantees a knowledge of all the symmetries requires the construction of the multipliers of the Euler equations (2.28) that are in one-to-one

correspondence with the conservation laws of the Euler equations, [13, 14]. The approach is based on the well known result that the Euler-Lagrange operator annihilates a total divergence. We analyse the Euler-Lagrange equations (2.28). The multipliers $Q_j = (q_j^1, q_j^2, q_j^3, q_j^4)$ are chosen to satisfy the “joint” Euler-Lagrange operator as follows:

$$\frac{\delta}{\delta(t, r, \theta, \phi)}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4) = 0. \quad (2.50)$$

This is a consequence of four dependent variables and can be interpreted as the action of the Euler-Lagrange operator of each one of the variables t, r, θ, ϕ ; that is

$$\begin{aligned} \frac{\delta}{\delta t}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4) &= 0, \\ \frac{\delta}{\delta r}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4) &= 0, \\ \frac{\delta}{\delta \theta}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4) &= 0, \\ \frac{\delta}{\delta \phi}(q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4) &= 0. \end{aligned} \quad (2.51)$$

Then the relation between multipliers and conserved quantities takes the form

$$q_j^1 G^1 + q_j^2 G^2 + q_j^3 G^3 + q_j^4 G^4 = D_s T^s. \quad (2.52)$$

For simplicity, we restrict the multipliers Q_j to be functions of the independent variable s , the four dependent variables t, r, θ, ϕ and their first order derivative only, i.e.

$$\begin{aligned} q_j^1 &= h^1(s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \\ q_j^2 &= h^2(s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \\ q_j^3 &= h^3(s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \\ q_j^4 &= h^4(s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}). \end{aligned} \quad (2.53)$$

The detailed calculations of (2.51) show that the multipliers are

$$\begin{aligned}
Q_1 &= (-\dot{t}, -\dot{r}, -\dot{\theta}, -\dot{\phi}), \\
Q_2 &= (0, -\sqrt{-1+kr^2} \cos \phi \sin \theta, -\frac{\sqrt{-1+kr^2} \cos \theta \cos \phi}{r}, \frac{\sqrt{-1+kr^2} \sin \phi}{r \sin \theta}), \\
Q_3 &= (0, \sqrt{-1+kr^2} \sin \phi \sin \theta, -\frac{\sqrt{-1+kr^2} \cos \theta \sin \phi}{r}, \frac{\sqrt{-1+kr^2} \cos \phi}{r \sin \theta}), \\
Q_4 &= (0, 0, -\cos \phi, \cot \theta \sin \phi), \\
Q_5 &= (0, 0, \sin \phi, \cot \theta \cos \phi), \\
Q_6 &= (0, 0, 0, 1), \\
Q_7 &= (0, -\sqrt{-1+kr^2} \cos \theta, \frac{1}{r} \sqrt{-1+kr^2} \sin \theta, 0).
\end{aligned} \tag{2.54}$$

Since the system (2.28) is variational, each of the multipliers Q_j corresponds to a symmetry, viz., X_j . The conserved quantities (first integrals) are, respectively,

$$\begin{aligned}
T_1 &= -\frac{1}{kr^2-1} \left[-\dot{t}^2 + \dot{t}^2 kr^2 + e^{\frac{\sqrt{2\beta t+k}}{\beta}} \dot{r}^2 + e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^2 \dot{\theta}^2 - e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^4 \dot{\theta}^2 k + e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^2 \dot{\phi}^2 \right. \\
&\quad \left. - e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^2 \dot{\phi}^2 \cos^2 \theta - e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^4 \dot{\phi}^2 k + e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^4 \dot{\phi}^2 k \cos^2 \theta \right], \\
T_2 &= -\frac{1}{\sqrt{kr^2-1}} 2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(-\sin \phi r \sin \theta \dot{\phi} + \sin \phi r^3 \sin \theta \dot{\phi} k + \cos \theta \cos \phi r \dot{\theta} \right. \\
&\quad \left. - \cos \theta \cos \phi r^3 \dot{\theta} k + \cos \phi \sin \theta \dot{r} \right), \\
T_3 &= \frac{1}{\sqrt{kr^2-1}} 2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(\cos \phi r \sin \theta \dot{\phi} - \cos \phi r^3 \sin \theta \dot{\phi} k + \cos \theta \sin \phi r \dot{\theta} \right. \\
&\quad \left. - \cos \theta \sin \phi r^3 \dot{\theta} k + \sin \phi \sin \theta \dot{r} \right), \\
T_4 &= -2 r^2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(\cos \theta \sin \phi \sin \theta \dot{\phi} - \cos \phi \dot{\theta} \right), \\
T_5 &= -2 r^2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(\cos \theta \cos \phi \dot{\phi} \sin \theta + \sin \phi \dot{\theta} \right), \\
T_6 &= -2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} r^2 \dot{\phi} \sin^2 \theta, \\
T_7 &= -\frac{1}{\sqrt{kr^2-1}} 2 e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(-\sin \theta r \dot{\theta} + \sin \theta r^3 \dot{\theta} k + \cos \theta \dot{r} \right).
\end{aligned} \tag{2.55}$$

2.4 Discussion

In this chapter, we have found the variational or Noether symmetries and conservation laws of the modified de Sitter metric

$$ds^2 = -dt^2 + e^{\frac{\sqrt{2\beta t+k}}{\beta}} \left(\frac{1}{1-kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \tag{2.56}$$

corresponding to the solution for the scale factor

$$a = e^{\frac{1}{\beta}\sqrt{2\beta t+k}}. \quad (2.57)$$

This solution has been found to be compatible with observations for suitable values of the parameters, [20].

The study suggests that where a finite dimensional group generated by conservation laws exists, the Noether symmetry group has at least one additional symmetry that is not given by the Killing Vectors. It will be worthwhile to develop methods for general classification of Noether symmetries with some minimal symmetry group in terms of Killing Vectors and to see how this general classification compares with Killing Vectors, Noether symmetries, Ricci and curvature collineations. It is unlikely that higher order conservation laws for the Euler equations exist. In any case, computing these would be extremely time consuming and futile.

Chapter 3

Symmetry and Conservation Law

Classification of the de

Sitter-Schwarzschild Metric and the

Corresponding Wave and

Klein-Gordon Equations

The Einstein's field equations (EFEs) comprise a system of ten non-linear coupled partial differential equations [45, 46, 47]. These equations provide an interpretation on how spacetime geometry is related to the distribution of matter by an expression given by

$$G^a_b = \kappa T^a_b,$$

where

$$T^a_b$$

and

$$G^a_b = R^a_b - \frac{1}{2}\delta^a_b R$$

respectively represent energy-momentum and Einstein tensors [46]. The value of the constant κ is given by

$$\kappa = \frac{8\pi G}{c^4},$$

where G is the Newton's gravitational constant, and c , the speed of light in vacuum. In case one uses Einstein tensor to define the corresponding energy-momentum tensor, then every metric tensor can be regarded as a solution of the EFEs. However, only those solutions are of interest which are physically plausible and satisfy certain physical assumptions such as conservation of energy, and angular momentum etc. On the other hand, the EFEs are extremely non-linear and finding their exact solutions is an uphill task. To overcome this difficulty of finding exact solutions, it is required that the spacetime metric admits certain symmetries [45, 46]. These symmetry considerations have lead relativists to find interesting exact solutions. A detailed account of these exact solutions is documented in literature [45, 46]. Most of this known work is based on the existence of symmetries (metric, variational, inter alia) and conservation laws admitted by these solutions.

A two rank tensor T_{ab} is said to admit a symmetry ξ , if it satisfies a requirement that the Lie derivative of this tensor T_{ab} along ξ is zero, i.e,

$$\mathcal{L}_\xi T_{ab} = 0.$$

The vector ξ is called a Killing vector if T_{ab} in the above equation represents a metric tensor g_{ab} . If we replace T_{ab} in this equation with Ricci tensor (R_{ab}), then the ξ defines a Ricci collineation. If T_{ab} represents the one form Lds , where L is the Lagrangian of the geodesic equations and s the arclength, then ξ are the variational symmetries and

$$\mathcal{L}_\xi A = \text{div}(B),$$

for some differential function B , ξ is a divergence symmetry [4, 8]. Thus, divergence symmetries with zero gauge B are the variational symmetries. In either case, one can construct conservation laws via Noether's theorem. These conservation laws and others, are widely discussed in the literature and interesting physical interpretations are found of

various spacetime geometries [48, 49, 50, 51, 52].

More recently, attempts have been made to explore the physics of spacetime geometries that could be given by Noether symmetries [7]. Noether symmetries have been discussed from a point of view of their comparison with Killing vectors, Ricci and Riemann tensor symmetries [45, 48, 52]. The de Sitter-Schwarzschild solution describes a black hole in a causal patch of de Sitter space. Unlike a flat-space black hole embedding, there is a largest possible de Sitter black hole, which is the Nariai spacetime. The Nariai limit has no singularities. the cosmological and black hole horizons have the same area and they can be mapped to each other by a discrete reflection symmetry in any causal patch [53, 54, 55]. The purpose of this chapter is to focus on understanding the conservation laws admitted by the e Schwarzschild-de Sitter metric. The Schwarzschild-de Sitter metric is given by

$$ds^2 = (1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2)dt^2 - (1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (3.1)$$

where the two parameters m and λ give the black hole mass and the cosmological constants respectively. The de Sitter space is the solution of Einstein's equation with a positive cosmological constant. It is spherically symmetric and it has a cosmological horizon surrounding any observer, and describes an inflating universe. Similarly, the Schwarzschild solution is a spherically symmetric static solution of the Einstein equations with zero cosmological constant and it describes a black hole event horizon in otherwise empty space. The Schwarzschild-de Sitter spacetime is a combination of the two (the Schwarzschild space and the de Sitter space), and describes a black hole horizon spherically centered in an otherwise de Sitter universe. An observer, who has not fallen into the black hole and who can still see the black hole despite the inflation, is sandwiched between the two horizons. The Nariai solution is the limit of the largest black hole in a space which is de Sitter at large distances, it has two horizons, the cosmological de Sitter horizon and a Schwarzschild black hole horizon. For small mass black holes, the two are very different. There is a singularity at the center of the black hole and there is no singularity past the cosmological horizon. But the Nariai limit considers making the black hole bigger and bigger, until its event horizon has the same area as the cosmological de Sitter horizon. At this point, the space-time becomes regular. The black hole singularity runs off to infinity

and the two horizons are related by a spacetime symmetry. In the Nariai limit, the black hole and de Sitter horizon can be interchanged. When there is additional matter density, the solution can be thought of as an Einstein spherical universe with two antipodal black holes. Whichever black hole becomes larger becomes the cosmological horizon.

The content of this chapter is under review [56].

3.1 Symmetry classification of the Schwarzschild-de Sitter metric

The geodesic equations followed by the test particles in the Schwarzschild-de Sitter spacetime are obtained by varying the action

$$S = \int L ds$$

constructed from the metric given in (3.1). The geodesic equations corresponding to the variation of this action take the form:

$$\begin{aligned} G^1 &= \left(2 \frac{m}{r^2} - \frac{2}{3} \lambda r\right) \dot{t}^2 + \frac{6 r^5 \lambda - 18 r^2 m}{(-3 r + 6 m + \lambda r^3)^2 r^2} \dot{r}^2 - 2 r \dot{\theta}^2 - 2 r \sin^2 \theta \dot{\phi}^2 - 6 \frac{r}{-3 r + 6 m + \lambda r^3} \ddot{r} = 0, \\ G^2 &= -2 r^2 \sin \theta \dot{\phi}^2 \cos \theta + 4 r \dot{\theta} \dot{r} + 2 r^2 \ddot{\theta} = 0, \\ G^3 &= 4 r \sin^2 \theta \dot{\phi} \dot{r} + 4 r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + 2 r^2 \sin^2 \theta \ddot{\phi} = 0, \\ G^4 &= \frac{2}{3} \frac{-3 r + 6 m + \lambda r^3}{r} \ddot{t} - \frac{4}{3} \frac{3 m - \lambda r^3}{r^2} \dot{t} \dot{r} = 0, \end{aligned} \tag{3.2}$$

with Lagrangian defined by,

$$L = \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2\right) \dot{t}^2 - \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2\right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2.$$

Since we are dealing with 4-spacetime geometry, we write the Noether vector field as,

$$X = \sigma(s, t, r, \theta, \phi) \partial_s + \tau(s, t, r, \theta, \phi) \partial_t + A(s, t, r, \theta, \phi) \partial_r + B(s, t, r, \theta, \phi) \partial_\theta + C(s, t, r, \theta, \phi) \partial_\phi, \tag{3.3}$$

or for simplicity

$$X = \sigma \partial_s + \tau \partial_t + A \partial_r + B \partial_\theta + C \partial_\phi, \tag{3.4}$$

where we constraint the coefficients σ, τ, A, B , and C to depend only on the curvature s and the spacetime coordinates t, r, θ , and ϕ . This vector field satisfies the equation

$$X^{[1]}L + L \frac{d\sigma}{ds} = \frac{df}{ds}. \quad (3.5)$$

with gauge $f = f(s, t, r, \theta, \phi)$. $X^{[1]}$ is the first prolongation of X , given that L depends on first order derivative, and is given as

$$X^{[1]} = \sigma \partial_s + \tau \partial_t + A \partial_r + B \partial_\theta + C \partial_\phi + \tau^{[1]} \partial_{\dot{t}} + A^{[1]} \partial_{\dot{r}} + B^{[1]} \partial_{\dot{\theta}} + C^{[1]} \partial_{\dot{\phi}}. \quad (3.6)$$

After replacing L into (3.5), we have

$$\begin{aligned} & A \left[\left(\frac{2m}{r^2} - \frac{2}{3} \lambda r \right) \dot{t}^2 + \left(\frac{2m}{r^2} - \frac{2}{3} \lambda r \right) \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right)^{-2} \dot{r}^2 - 2r \dot{\theta}^2 - 2r \sin^2 \theta \dot{\phi}^2 \right] \\ & + 2 \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) \dot{t} \left[\tau_s + \dot{t} \tau_t + \dot{r} \tau_r + \dot{\theta} \tau_\theta + \dot{\phi} \tau_\phi - \dot{t} (\sigma_s + \dot{t} \sigma_t + \dot{r} \sigma_r + \dot{\theta} \sigma_\theta + \dot{\phi} \sigma_\phi) \right] \\ & - 2 \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right)^{-1} \dot{r} \left[A_s + \dot{t} A_t + \dot{r} A_r + \dot{\theta} A_\theta + \dot{\phi} A_\phi - \dot{r} (\sigma_s + \dot{t} \sigma_t + \dot{r} \sigma_r + \dot{\theta} \sigma_\theta + \dot{\phi} \sigma_\phi) \right] \\ & - 2r^2 \dot{\theta} \left[B_s + \dot{t} B_t + \dot{r} B_r + \dot{\theta} B_\theta + \dot{\phi} B_\phi - \dot{\theta} (\sigma_s + \dot{t} \sigma_t + \dot{r} \sigma_r + \dot{\theta} \sigma_\theta + \dot{\phi} \sigma_\phi) \right] \\ & - 2r^2 \sin^2 \theta \dot{\phi} \left[C_s + \dot{t} C_t + \dot{r} C_r + \dot{\theta} C_\theta + \dot{\phi} C_\phi - \dot{\phi} (\sigma_s + \dot{t} \sigma_t + \dot{r} \sigma_r + \dot{\theta} \sigma_\theta + \dot{\phi} \sigma_\phi) \right] \\ & + \left[\left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) \dot{t}^2 - \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2 \right] \times \\ & (\sigma_s + \dot{t} \sigma_t + \dot{r} \sigma_r + \dot{\theta} \sigma_\theta + \dot{\phi} \sigma_\phi) - 2Br^2 \sin \theta \cos \theta \dot{\phi}^2 = f_s + \dot{t} f_t + \dot{r} f_r + \dot{\theta} f_\theta + \dot{\phi} f_\phi, \quad (3.7) \end{aligned}$$

followed by the separation by monomials which leads to the overdetermined system

$$\begin{aligned} \dot{t}^2 & : \left(\frac{2m}{r^2} - \frac{2}{3} \lambda r \right) A + 2 \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) (2\tau_t - \sigma_s) = 0 \\ \dot{r}^2 & : \left(\frac{2m}{r^2} - \frac{2}{3} \lambda r \right) A + (\sigma_s - 2A_r) = 0 \\ \dot{\theta}^2 & : -2rA + r^2(\sigma_s - 2B_\theta) = 0 \\ \dot{\phi}^2 & : -2r \sin^2 \theta A - 2r^2 \sin \theta \cos \theta B + r^2 \sin^2 \theta (\sigma_s - 2C_s) \\ \dot{t} & : 2 \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) \tau_s = f_t \\ \dot{r} & : -2 \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) A_s = f_r \\ \dot{\theta} & : -2r^2 B_s = f_\theta \\ \dot{\phi} & : -2r^2 \sin^2 \theta C_s = f_\phi \\ \dot{t} \dot{r} & : \tau_r - \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right)^{-2} A_t = 0 \end{aligned}$$

$$\begin{aligned}
t\dot{\theta} &: \left(1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2\right) \tau_\theta - r^2 B_t = 0 \\
t\dot{\phi} &: \left(1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2\right) \tau_\phi - r^2 \sin^2 \theta C_t = 0 \\
r\dot{\theta} &: \left(1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2\right)^{-1} A_\theta + r^2 B_\theta = 0 \\
r\dot{\phi} &: \left(1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2\right)^{-1} A_\phi + r^2 \sin^2 \theta C_r = 0 \\
\dot{\theta}\dot{\phi} &: B_\phi + \sin^2 \theta C_\theta = 0 \\
t^3 &: \sigma_t = 0 \\
r^3 &: \sigma_r = 0 \\
\dot{\theta}^3 &: \sigma_\theta = 0 \\
\dot{\phi}^3 &: \sigma_\phi = 0 \\
1 &: f_s = 0
\end{aligned} \tag{3.8}$$

From the lengthy and tedious calculations in (3.8), one can show that the solutions of the corresponding Noether symmetries is represented by five Lie symmetry generators whose Lie algebra is spanned by the following Noether symmetries with zero gauge ($f = 0$)

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_t, \\
X_3 &= \partial_\phi, \\
X_4 &= \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \\
X_5 &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi
\end{aligned}$$

Each X_i leads to an independent conservation law and X_2 - X_5 form the algebra as they are Killing vectors of the metric. For example, the first integral associate with X_1 is

$$\begin{aligned}
T_1 = & \frac{1}{3} \frac{1}{r(-3r+6m+\lambda r^3)} [-9t^2 r^2 + 36t^2 r m + 6t^2 \lambda r^4 - 36t^2 m^2 - 12t^2 m \lambda r^3 \\
& - t^2 \lambda^2 r^6 + 9r^2 \dot{r}^2 - 18r^3 \dot{\theta}^2 m - 3r^6 \dot{\theta}^2 \lambda + 9r^4 \dot{\theta}^2 - 3r^6 \dot{\phi}^2 \lambda + 9r^4 \dot{\phi}^2 \\
& - 18r^3 \dot{\phi}^2 m + 3 \cos^2 \theta r^6 \dot{\phi}^2 \lambda - 9 \cos^2 \theta r^4 \dot{\phi}^2 + 18 \cos^2 \theta r^3 \dot{\phi}^2 m],
\end{aligned}$$

with X_3 is

$$T_3 = -2r^2 \sin^2 \theta \dot{\phi},$$

and with X_4 is

$$T_4 = -2r^2 (-\cos \theta \sin \theta \sin \phi \dot{\phi} + \cos \phi \dot{\theta}),$$

and so to with T_2 and T_5 . The integral T_1 cannot be a conserved quantity determined by a Killing vector due to the vector field, X_1 's, explicit dependence on the arclength s .

3.2 Symmetry classification of the wave and Klein-Gordon equation

The general wave equation is given by

$$\frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|-g|} g^{ik} \frac{\partial}{\partial x^k} u \right) = 0, \quad (3.9)$$

where u is a field, g is the determinant of the metric g_{ik} and g^{ik} its inverse. The metric writes

$$g_{ik} = \begin{pmatrix} (1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2) & 0 & 0 & 0 \\ 0 & -(1 - \frac{2m}{r} - \frac{1}{3}\lambda r^2)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \quad (3.10)$$

We find that $\sqrt{|-g|} = r^2 \sin \theta$ for this metric and (3.9) can be written in detail as

$$\begin{aligned} \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial t} \left(\sqrt{|-g|} g^{tt} \frac{\partial}{\partial t} u \right) + \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial r} \left(\sqrt{|-g|} g^{rr} \frac{\partial}{\partial r} u \right) + \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial \theta} \left(\sqrt{|-g|} g^{\theta\theta} \frac{\partial}{\partial \theta} u \right) \\ + \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial \phi} \left(\sqrt{|-g|} g^{\phi\phi} \frac{\partial}{\partial \phi} u \right) = 0, \end{aligned}$$

or

$$\begin{aligned} g^{tt} u_{tt} + g^{rr} u_{rr} + \frac{1}{\sqrt{|-g|}} \left(2r \sin \theta g^{rr} - \sqrt{|-g|} \left(\frac{2m}{r^2} - \frac{2}{3} \lambda r \right) \right) u_r + g^{\theta\theta} u_{\theta\theta} \\ + \frac{r^2 g^{\theta\theta} \cos \theta}{\sqrt{|-g|}} u_\theta + g^{\phi\phi} u_{\phi\phi} = 0. \end{aligned} \quad (3.11)$$

Substituting g_{ik} given in (3.10) into (3.11) yields

$$\begin{aligned} \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right)^{-1} u_{tt} - \left(1 - \frac{2m}{r} - \frac{1}{3} \lambda r^2 \right) u_{rr} + \left(-\frac{2}{r} + \frac{2m}{r^2} + \frac{4}{3} \lambda r \right) u_r \\ - r^{-2} \cot \theta u_\theta - r^{-2} u_{\theta\theta} - (r^2 \sin^2 \theta)^{-1} u_{\phi\phi} = 0. \end{aligned}$$

We multiply the above equation by $r^2 \sin \theta$ and we obtain, after simplification and rearrangement, the equation below:

$$\begin{aligned} & \frac{3r^3 \sin \theta}{3r-6m-\lambda r^3} u_{tt} + \frac{1}{3} \sin \theta [(-3r^2 + 6mr + \lambda r^4) u_{rr} + (-6r + 6m + 4\lambda r^3) u_r] \\ & - \sin \theta u_{\theta\theta} - \cos \theta u_\theta - \frac{1}{\sin \theta} u_{\phi\phi} = 0 \end{aligned} \quad (3.12)$$

which admits a Lagrangian depending on u given by

$$L = \frac{3r^3 \sin \theta}{3r-6m-\lambda r^3} u_t^2 + (r^3 - 2mr \frac{1}{3} \lambda r^4) \sin \theta u_r^2 - \sin \theta u_\theta^2 - \frac{1}{\cos \theta} u_\phi^2.$$

Via L , it can be shown that the variational symmetries of the wave equation are

$$\begin{aligned} Y_1 &= \partial_u, \\ Y_2 &= \partial_t, \\ Y_3 &= \partial_\phi, \\ Y_4 &= \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \\ Y_5 &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \end{aligned} \quad (3.13)$$

and the Lie symmetries of the wave equation are Y_2 to Y_5 with Y_1 replaced by the ‘infinite’ symmetry $Y_\infty = f(t, r, \theta, \phi) \partial_u$, where f satisfies the wave equation (3.9). Suppose the left hand side is some function $G(t, r, \theta, \phi, u_\theta, u_{tt}, u_{rr}, u_{\theta\theta}, u_{\phi\phi})$ or G for simplicity. That is

$$G(t, r, \theta, \phi, u_\theta, u_{tt}, u_{rr}, u_{\theta\theta}, u_{\phi\phi}) = 0. \quad (3.14)$$

Then to find the n^{th} order invariant G of a vector field symmetry X , we need to solve

$$X^{[n]} G(t, r, \theta, \phi, u_\theta, u_{tt}, u_{rr}, u_{\theta\theta}, u_{\phi\phi}) = 0.$$

This leads to the reduction of the independent variables. For example if we consider Y_4 , then a 0-order invariant G satisfies

$$Y_4 G = 0,$$

i.e.

$$(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi) G = 0, \quad (3.15)$$

or

$$\frac{d\theta}{\cos \phi} = -\frac{d\phi}{\cot \theta \sin \phi} \left(= \frac{dG}{0} \right). \quad (3.16)$$

Combining the first two elements in the above equality and after integrating, we get a new independent variable $\alpha = \sin \theta \sin \phi$ which is an integration constant depending on θ and ϕ . Thus, by a symmetry reduction using Y_4 , the wave equation reduces to

$$\frac{3r^3}{3r-6m-\lambda r^3}u_{tt} + \frac{1}{3}[(-3r^2 + 6mr + \lambda r^4)u_{rr} + (-6r + 6m + 4\lambda r^3)u_r] + 2\alpha u_\alpha - (1 - \alpha^2)u_{\alpha\alpha} = 0,$$

where $\alpha = \sin \theta \sin \phi$. This retains Y_1 and Y_2 for the next reduction. Taking a linear combination $Y_2 + \beta Y_1$ for some constant β , we get the scaling symmetry generator $Y_c = \partial_t + \beta \partial_u$ leading to the invariant and transformed variable $u = v(r, \alpha) + \beta t$, where $v(r, \alpha)$ is an integration constant obtained from

$$\frac{dt}{1} = \frac{du}{\beta}. \quad (3.17)$$

The symmetry reduction therefore leads to

$$\frac{1}{3}[(-3r^2 + 6mr + \lambda r^4)v_{rr} + (-6r + 6m + 4\lambda r^3)v_r] + 2\alpha v_\alpha - (1 - \alpha^2)v_{\alpha\alpha} = 0.$$

We remark that in the case of the wave equation with a source term, the right hand side of (3.12) is some function of u , say $F(u)$; if $F = u$, we have the Klein-Gordon equation. For any $F(u)$, the equation loses the symmetry ∂_u from the list in (3.13).

3.3 Summary and discussions

In this chapter, we have studied and classified the algebra of symmetries associated with the Schwarzschild-de Sitter metric and the corresponding wave and Klein-Gordon equations. Separately, these were the Lie symmetries of the geodesic and wave equations. The conservation laws were obtained via the natural Lagrangians and variational symmetries in each of the cases.

Chapter 4

Gravitational Waves and the Classification of Spacetimes that are Asymptotically Flat

4.1 Introduction

Relativity, in its general form, is well known to be dependent on the equations of Einstein and centres around the spread of matter and the geometry of a spacetime manifold. These equations constitute ten partial differential equations (pdes) [45, 46, 47] and since its inception, much work has been done in attempting to obtain exact solutions leading to various models of the universe; the symmetries and corresponding conservation laws that underpin these models lies at the core of the various models.

A symmetry, X , admitted by a metric tensor satisfies the annihilation equation that the Lie derivative of the tensor along X is zero, i.e, $\mathcal{L}_X g_{ab} = 0$, where g_{ab} is a metric tensor and X is called a Killing vector. If instead, we consider the ‘annihilation’ equation $\mathcal{L}_X L ds = 0$, where L is the Lagrangian of the geodesic equations and s the arclength variable, then the X s are the variational symmetries and when $\mathcal{L}_X L ds = \text{div}(B)$, for some

differential function B , X is a divergence symmetry [4, 8]. Thus, divergence symmetries with zero gauge B are the variational symmetries. In both cases, conservation laws are obtainable using the celebrated Noether's theorem. The relation between the variational or divergence symmetries and Killing vectors have been expounded in [48]–[52].

In [53, 54], a detailed account on how the theoretical existence of gravitational waves occur and the mathematics behind gravitational waves is presented. The articles describe the history and theoretical development of gravitational waves and the recent results of the LIGO detector. Einstein, in 1916, linearized his field equations by assuming that the metric $g_{\mu\nu}$ representing the gravitational field has the form of a slightly perturbed Minkowski metric $\eta_{\mu\nu}$, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad (4.1)$$

where $0 < \epsilon < 1$.

The example

$$\begin{aligned} ds^2 &= dt^2 - dx^2 - dy^2 - dz^2 + \cos(t-x)(2 + \cos(t-x))dt^2 \\ &\quad - 2\cos(t-x)(1 + \cos(t-x))dtdx + \cos^2(t-x)dx^2 \end{aligned} \quad (4.2)$$

is cited in [53] and has been shown not be a plane gravitational wave [57] although it satisfies properties of a ‘gravitational wave’ in the linearized Einstein theory. What one obtains is that gravitational waves which transport no energy can be generated from a ‘field-free system’ by a simple coordinate transformation. In this example, the straight transformation of a Minkowski metric is a change of the time coordinate $\bar{t} = t + \sin(t-x)$, that is

$$\begin{aligned} ds^2 &= d\bar{t}^2 - dx^2 - dy^2 - dz^2 \\ &= [d(t + \sin(t-x))]^2 - dx^2 - dy^2 - dz^2 \\ &= [dt + \cos(t-x)dt - \cos(t-x)dx]^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2 - dx^2 - dy^2 - dz^2 + \cos(t-x)(2 + \cos(t-x))dt^2 \\ &\quad - 2\cos(t-x)(1 + \cos(t-x))dtdx + \cos^2(t-x)dx^2. \end{aligned} \quad (4.3)$$

The algebra of isometries of the metric (4.2) has basis $(SO(3) \otimes \mathbb{R}^4)$

$$\begin{aligned}
X_1 &= -\frac{z(\cos(t-x)-1)\partial_t}{1+\cos(t-x)} - \frac{z\cos(t-x)\partial_x}{1+\cos(t-x)} + t\partial_z, \\
X_2 &= -\frac{z\cos(t-x)\partial_t}{1+\cos(t-x)} - z\partial_x + x\partial_z, \\
X_3 &= -\frac{y\cos(t-x)\partial_t}{1+\cos(t-x)} - y\partial_x + x\partial_y, \\
X_4 &= \partial_t, \\
X_5 &= -\frac{(\cos(t-x)t - \cos(t-x)x + x)\partial_t}{\cos(t-x)} - \frac{(\cos(t-x)t - \cos(t-x)x + t)\partial_x}{\cos(t-x)}, \\
X_6 &= -z\partial_y + y\partial_z, \\
X_7 &= -\frac{y(\cos(t-x)-1)\partial_t}{1+\cos(t-x)} - \frac{y\cos(t-x)\partial_x}{1+\cos(t-x)} + t\partial_y, \\
X_8 &= \partial_x, \\
X_9 &= \partial_y, \\
X_{10} &= \partial_z
\end{aligned}$$

with an additional two variational (zero divergence) symmetries

$$\begin{aligned}
X_{11} &= 2s\partial_s + \frac{(2\cos(t-x)t - \cos(t-x)x + x)\partial_t}{\cos(t-x)} + \frac{(1+\cos(t-x))t\partial_x}{\cos(t-x)} + y\partial_y + z\partial_z, \\
X_{12} &= \partial_s.
\end{aligned}$$

In fact, there exists a further five divergence symmetries. The respective algebras are all isomorphic to the algebras in regard to the Minkowski metric.

Gravitational waves are waves in a space time that propagate at the speed of light away from their source. They may be produced, for example, when black holes merge and are, inter alia, metrics that are asymptotically flat. Examples of asymptotically flat metrics are the Kerr metric, the Reissner-Nördstrom metric, the Kerr-Newman metric, the Friedman-Robertson-Walker (FRW) metric, the Vadyia metric.

An example is the group of Schwarzschild metrics which are solutions of the Einstein vacuum equations that describe space times containing a black hole,

$$g_S = -c^2 \frac{(1 - \frac{r_s}{4\rho})^2}{(1 + \frac{r_s}{4\rho})^2} dt^2 + (1 + \frac{r_s}{4\rho})^4 [dx^2 + dy^2 + dz^2], \quad (4.4)$$

where $r_s = \frac{2Gm}{c^2}$, $m > 0$ and $\rho = \sqrt{x^2 + y^2 + z^2}$. As $\rho \rightarrow \infty$, the space is asymptotically flat (c.f., the Minkowski metric $\eta = -dt^2 + dx^2 + dy^2 + dz^2$).

One aspect of gravitational waves is to study what happens to its source when it radiates energy. One way to proceed this study is to calculate how much energy the gravitational waves are carrying away and then make use of the principle of conservation of energy to work out the effect a loss of such a quantity of energy would have on the motion of the source [58].

In what follows, we focus on the symmetry structure (and the corresponding conservation laws) of the Schwartzchild and other asymptotically flat metrics, in particular, the variational/divergence symmetries of the respective geodesic equations following from the ‘natural’ Lagrangian. We sometimes refer to these as Noether symmetries which contain the algebra of isometries/Killing vectors of the metric. Each of the symmetries give rise to an independent conservation law.

The content of this chapter is submitted to journal [59].

4.2 Symmetries of the static, spherically symmetric space time metrics

We recall an established result involving the general static spherically symmetric space time metric given by $ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$ for which the corresponding Lagrangian is given by $L = e^{\nu(r)}\dot{t}^2 - e^{\lambda(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2$, where dot represents derivative with respect to s . The corresponding five dimensional algebra of Noether symmetries of the geodesic equations is spanned by

$$X_1 = \partial_s,$$

$$X_2 = \partial_t,$$

$$X_3 = \partial_r,$$

$$X_4 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi,$$

$$X_5 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi.$$

It turns out that the Schwarzschild metric representing vacuum solution of the Einstein equations generates, in spherical coordinates, the same algebra.

Reissner-Nordström metric.

The Reissner-Nordström metric is a static solution to the Einstein-Maxwell field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body. In spherical coordinates, we have

$$ds_R^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q}{r^2}\right) dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (4.5)$$

where r_s is the Schwarzschild radius and r_Q is a characteristic length scale (or equivalently, the length-scale). In the limit that the electric charge Q goes to zero, one recovers the Schwarzschild metric. The complete algebra of Noether symmetries has the same basis as that of the Schwarzschild metric above.

4.3 Classification of some Kerr metrics

Case 1. Kerr-Newman.

The Kerr-Newman spacetime is a stationary, axisymmetrical, and asymptotically flat solution of the Einstein-Maxwell equation which in the Boyer-Lindquist coordinates is

$$ds_{KN}^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\Delta}{\rho^2} [a \sin^2 \theta d\phi - dt]^2, \quad (4.6)$$

where $\Delta = r^2 - 2mr + a^2 + Q^2 + P^2$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $m > 0$ the mass, a the specific angular momentum, Q the electric charge and P the magnetic charge of the gravitating source. The corresponding Lagrangian writes

$$L = - \left(1 - \frac{2mr}{\rho^2}\right) \dot{t}^2 + \rho^2 \left(\frac{\dot{r}^2}{\Delta} + \dot{\theta}^2\right) - \frac{4mar \sin^2 \theta}{\rho^2} \dot{t} \dot{\phi} + \left((r^2 + a^2) \sin^2 \theta + \frac{2ma^2 r \sin^4 \theta}{\rho^2}\right) \dot{\phi}^2. \quad (4.7)$$

Using the Lagrangian in (4.7) with the divergence symmetry

$$X = \xi \partial_s + \eta_t \partial_t + \eta_r \partial_r + \eta_\theta \partial_\theta + \eta_\phi \partial_\phi,$$

the equation

$$XL + LD_i \xi^i = D_i f^i. \quad (4.8)$$

becomes

$$\begin{aligned} & \eta_r \left[\left(\frac{2m\rho^2 - 4mr^2}{\rho^4} \right) \dot{t}^2 + \left(\frac{2r\Delta - \rho^2(2r-m)}{\Delta^2} \right) \dot{r}^2 + 2r\dot{\theta}^2 - \right. \\ & \left. \left(\frac{4ma\rho^2 \sin^2 \theta 8mr^2 a \sin^2 \theta}{\rho^4} \right) \dot{t}\dot{\phi} + \left(2r \sin^2 \theta + \frac{2ma^2 \rho^2 \sin^4 \theta - 4ma^2 r^2 \sin^4 \theta}{\rho^4} \right) \dot{\phi} + \right] \\ & \eta_\theta \left[\left(\frac{4mra^2 \sin \theta \cos \theta}{\rho^4} \right) \dot{t}^2 - \frac{2a^2 \sin \theta \cos \theta}{\Delta} \dot{r}^2 \left(\frac{8mar \sin \theta \cos \theta (\rho^2 + a^2 \sin^2 \theta)}{\rho^4} \right) \dot{t}\dot{\phi} \right. \\ & \left. - 2a^2 \sin \theta \cos \theta \dot{\theta}^2 + \left(2(r^2 + a^2) \sin \theta \cos \theta + \frac{4ma^2 r \sin^3 \theta \cos \theta (2\rho^2 + a^2 \sin^2 \theta)}{\rho^4} \right) \dot{\phi}^2 \right] \\ & - \left[2\dot{t} \left(1 - \frac{2mr}{\rho^2} \right) + \frac{4mar\dot{\phi} \sin^2 \theta}{\rho^2} \right] (\partial_s \eta_t + \dot{t} \partial_t \eta_t + \dot{r} \partial_r \eta_t + \dot{\theta} \partial_\theta \eta_t + \dot{\phi} \partial_\phi \eta_t \\ & - \dot{t} (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi)) + \frac{2\rho^2 \dot{r}}{\Delta} (\partial_s \eta_r + \dot{t} \partial_t \eta_r + \dot{r} \partial_r \eta_r + \dot{\theta} \partial_\theta \eta_r + \dot{\phi} \partial_\phi \eta_r \\ & - \dot{r} (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi)) + 2\rho^2 (\partial_s \eta_\theta + \dot{t} \partial_t \eta_\theta + \dot{r} \partial_r \eta_\theta + \dot{\theta} \partial_\theta \eta_\theta + \dot{\phi} \partial_\phi \eta_\theta \\ & - \dot{\theta} (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi)) + \left[2\dot{\phi} \sin^2 \theta \left(r^2 + a^2 + \frac{2ma^2 r \sin^2 \theta}{\rho^2} \right) \right. \\ & \left. - \frac{4mar \sin^2 \theta}{\rho^2} \right] (\partial_s \eta_\phi + \dot{t} \partial_t \eta_\phi + \dot{r} \partial_r \eta_\phi + \dot{\theta} \partial_\theta \eta_\phi + \dot{\phi} \partial_\phi \eta_\phi - \dot{\phi} (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi \\ & + \dot{\phi} \partial_\phi \xi)) + (\partial_s \xi + \dot{t} \partial_t \xi + \dot{r} \partial_r \xi + \dot{\theta} \partial_\theta \xi + \dot{\phi} \partial_\phi \xi) L = f_s + \dot{t} f_t + \dot{r} f_r + \dot{\theta} f_\theta + \dot{\phi} f_\phi. \end{aligned} \quad (4.9)$$

We get, after separation of the above equation by monomials, the following system of overdetermined pdes

$$\begin{aligned}
t^2 &: -2 \partial_t \eta_t + \partial_s \eta_s = 0, \\
\dot{t}^2 &: \frac{2m\rho^2 - 4mr^2}{\rho^4} \eta_r + \frac{4mra^2 \sin \theta \cos \theta}{\rho^4} \eta_\theta - 2 \left(1 - \frac{2mr}{\rho^2}\right) (\partial_t \eta_t - \partial_s \xi) \\
&\quad - \frac{4mra \sin^2 \theta}{\rho^2} \partial_t \eta_\phi - \left(1 - \frac{2mr}{\rho^2}\right) \partial_s \xi = 0, \\
\dot{r}^2 &: \left(\frac{2r\Delta - \rho^2(2r-m)}{\Delta^2}\right) \eta_r - \frac{2a^2 \sin \theta \cos \theta}{\Delta} \eta_\theta + \frac{2\rho^2}{\Delta} (\partial_r \eta_r - \partial_s \xi) + \frac{\rho^2}{\Delta} \partial_s \xi = 0, \\
\dot{\theta}^2 &: 2r \eta_r - (2a^2 \sin^2 \theta \cos \theta) \eta_\theta + 2\rho^2 (\partial_\theta \eta_\theta - \partial_s \xi) + \rho^2 \partial_s \xi = 0, \\
\dot{\phi}^2 &: \left(2r \sin^2 \theta + \frac{2ma^2 \sin^4 \theta}{\rho^2} - \frac{4mr^2 a^2 \sin^4 \theta}{\rho^4}\right) \eta_r + \left(2(r^2 + a^2) \sin \theta \cos \theta + \frac{8mra^2 \sin^3 \theta \cos \theta}{\rho^2}\right. \\
&\quad \left.+ \frac{4mra^4 \sin^5 \theta \cos \theta}{\rho^4}\right) \eta_\theta + 2 \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) (\partial_\phi \eta_\phi - \partial_s \xi) \\
&\quad - \frac{4mra \sin^2 \theta}{\rho^2} \partial_\phi \eta_t + \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) \partial_s \xi = 0, \\
\dot{t}\dot{r} &: -2 \left(1 - \frac{2mr}{\rho^2}\right) \partial_r \eta_t + \frac{2\rho^2}{\Delta} \partial_t \eta_r - \frac{4mra \sin^2 \theta}{\rho^2} \partial_r \eta_\phi = 0, \\
\dot{t}\dot{\theta} &: -2 \left(1 - \frac{2mr}{\rho^2}\right) \partial_\theta \eta_t + 2\rho^2 \partial_t \eta_\theta - \frac{4mra \sin^2 \theta}{\rho^2} \partial_\theta \eta_\phi = 0, \\
\dot{t}\dot{\phi} &: -\left(\frac{4ma \sin^2 \theta}{\rho^2} - \frac{8mr^2 a \sin^2 \theta}{\rho^4}\right) \eta_r - \left(\frac{8mra \sin \theta \cos \theta}{\rho^2} + \frac{8mra^3 \sin^3 \theta \cos \theta}{\rho^4}\right) \eta_\theta - 2 \left(1 - \frac{2mr}{\rho^2}\right) \\
&\quad \times \partial_\phi \eta_t + 2 \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) \partial_t \eta_\phi - \frac{4mra \sin^2 \theta}{\rho^2} (\partial_t \eta_t + \partial_\phi \eta_\phi - 3 \partial_s \xi) = 0, \\
\dot{r}\dot{\theta} &: \frac{2\rho^2}{\Delta} \partial_\theta \eta_r + 2\rho^2 \partial_r \eta_\theta = 0, \\
\dot{r}\dot{\phi} &: \frac{2\rho^2}{\Delta} \partial_\phi \eta_r + 2 \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) \partial_r \eta_\phi - \frac{4mra \sin^2 \theta}{\rho^2} \partial_r \eta_t = 0, \\
\dot{\theta}\dot{\phi} &: -2\rho^2 \partial_\phi \eta_\theta + 2 \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) \partial_\theta \eta_\phi - \frac{4mra \sin^2 \theta}{\rho^2} \partial_\theta \eta_t = 0, \\
1 &: \partial_s f = 0, \\
\dot{t} &: -2 \left(1 - \frac{2mr}{\rho^2}\right) \partial_s \eta_t - \frac{4mra \sin^2 \theta}{\rho^2} \partial_s \eta_\phi = \partial_t f, \\
\dot{r} &: \frac{2\rho^2}{\Delta} \partial_s \eta_r = \partial_r f, \\
\dot{\theta} &: 2\rho^2 \partial_s \eta_\theta = \partial_\theta f, \\
\dot{\phi} &: -\frac{4ma \sin^2 \theta}{\rho^2} \partial_s \eta_t + 2 \left((r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^4 \theta}{\rho^2}\right) \partial_s \eta_\phi = \partial_\phi f.
\end{aligned} \tag{4.10}$$

It turns out that the algebra of Noether symmetries of the geodesic equations is spanned by

$$\partial_s, \quad \partial_t, \quad \partial_\phi.$$

Case 2. Kerr-Sen.

The Kerr-Sen solution [60] is an exact classical four-dimensional black hole solution in

the low energy limit of heterotic string field theory. The metric being

$$ds_{KS}^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\phi - a dt]^2 - \frac{\Delta}{\rho^2} [a \sin^2 \theta d\phi - dt]^2, \quad (4.11)$$

where $\Delta = r^2 - 2(M - b)r + a^2$ and $\rho^2 = r^2 + 2br + a^2 \cos^2 \theta$ and $M > 0$ the mass, a the specific angular momentum and $b = Q^2/2M$ (Q is the electric charge). The algebra of Noether symmetries of the geodesic equations is spanned by

$$\partial_s, \quad \partial_t, \quad \partial_\phi.$$

4.4 Classification of the Friedman-Lemaitre-Robertson-Walker (FLRW) spacetime

The FLRW space times describe homogeneous and isotropic universes, viz.,

$$ds_F^2 = -dt^2 + a^2(t) \left[\frac{1}{1-r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta dp^2 \right]. \quad (4.12)$$

For $a(t) = 1$, the dimension of the gauge symmetries is nine (seven Killing vectors, one additional zero gauge Noether symmetry and one non zero gauge Noether symmetry).

These are

$$\begin{aligned} X_1 &= -\sqrt{-1+kr^2} \cos(\phi) \sin(\theta) \partial_r - \frac{\sqrt{-1+kr^2} \cos(\theta) \cos(\phi) \partial_\theta}{r} + \frac{\sqrt{-1+kr^2} \sin(\phi) \partial_\phi}{r \sin(\theta)}, \\ X_2 &= \sqrt{-1+kr^2} \sin(\phi) \sin(\theta) \partial_r + \frac{\sqrt{-1+kr^2} \cos(\theta) \sin(\phi) \partial_\theta}{r} + \frac{\sqrt{-1+kr^2} \cos(\phi) \partial_\phi}{r \sin(\theta)}, \\ X_3 &= -\cos(\phi) \partial_\theta + \frac{\cos(\theta) \sin(\phi) \partial_\phi}{\sin(\theta)}, \\ X_4 &= \sin(\phi) \partial_\theta + \frac{\cos(\theta) \cos(\phi) \partial_\phi}{\sin(\theta)}, \\ X_5 &= -\sqrt{-1+kr^2} \cos(\theta) \partial_r + \frac{\sqrt{-1+kr^2} \sin(\theta) \partial_\theta}{r}, \\ X_6 &= \partial_t, \\ X_7 &= \partial_\phi, \\ X_8 &= \partial_s, \\ X_9 &= s \partial_t. \end{aligned}$$

If $a(t)$ is arbitrary, one obtains a seven-dimensional algebra (six Killing vectors and one additional zero gauge Noether symmetry). In the latter case, energy conservation ∂_t and $s \partial_t$ are lost.

4.5 Summary and discussions

In this chapter, we have studied some metrics that are asymptotically flat (i.e., Minkowski ‘in the limit’) and, for illustrative purposes, Minkowski by a coordinate transformation. It turns out that, their respective symmetry algebras are radically different and in the former case it is not the maximal Minkowski metric. Inter alia, this is a distinguishing factor between a flat Minkowski type metric and one that generates gravitational waves being Minkowski in the limit.

Chapter 5

Symmetries and Conservation Laws of some Spacetimes in $f(R, T)$ Gravity

5.1 Introduction

The universe, on large enough scales, is homogeneous and isotropic. Recent cosmological observations from different sources involving geometrical and dynamical methods have confirmed that the universe is at the phase of accelerated expansion. The generalization of the Einstein's general relativity known as $f(R)$ gravity theory was first proposed in 1970 by Buchdahl [61]. Recently, Harko et al. [62] proposed a generalization of the $f(R)$ theory of gravity to the $f(R, T)$ theory of gravity where R and T respectively represent the scalar curvature and the trace of the energy-momentum tensor. For having potential applications in cosmology, this work turned out to be quite interesting and has opened up a number of investigations, for example, Jamil et al. [63] reconstructed some cosmological models in $f(R, T)$ gravity where it was proved that the dust fluid reproduced Lambda cold dark matter (Λ CDM) model. Keeping in view the importance the role Ricci collineations

play in general Relativity, Sharif and Zubair [64] provided a reconstruction and stability criterion of $f(R, T)$ gravity with a Ricci and modified Ricci dark energy and discussed the laws of thermodynamics in this modified theory of gravity [65, 66]. Santos [67] investigated a Gödel type universe model, while Harko [62] reconstructed $f(R, T)$ gravity for a special form given by $f(R, T) = f_1(R) + f_2(T)$. More recently, Shamir obtained some interesting exact solutions of the $f(R, T)$ gravity in a plane symmetric spacetime [68]; the results correspond to an exterior metric of cosmic string and a non-null electromagnetic field. They also considered locally rotationally symmetric Bianchi type I spacetimes in the $f(R, T)$ gravity for some specific cases of $f(R, T)$.

We briefly recall some aspects of invariance properties and vector fields which will impact our discussions on the symmetry properties of spacetimes. Recall that $\mathcal{L}_X T$ is the Lie derivative of the tensor T along a vector field X . Then X defines a Lie symmetry of the Tensor T if $\mathcal{L}_X T = 0$. In case the tensor $T = g_{ab}$ and its Lie derivative along X is zero, then such a vector is called a Killing's vector defined mathematically by $\mathcal{L}_X g_{ab} = 0$. A detailed overview of Killing's vectors is given in [45], [69] and [46].

In this chapter, we follow the detailed analysis and results given in [70], [71] and [72]. Therein, the exact solutions of plane symmetric spacetime in the framework of $f(R, T)$ gravity, the exact solutions of locally rotationally symmetric (LRS) Bianchi type I spacetime and Bianchi type V in $f(R, T)$ gravity are investigated. In what follows, we undertake a detailed analysis of the symmetry structures of the plane symmetric and the locally rotationally symmetric Bianchi type I spacetimes in the $f(R, T)$ gravity expounded in [70], [71] and [72]. In particular, we construct all the variational (Noether) symmetries and, in some cases, construct the associated conservation law using Noether's theorem. We summarize results in the concluding section with particular reference to the symmetry structures of some well known spacetimes.

Some results of this chapter have been accepted for publication and will appear in [73].

5.2 Model and basic equations of the $f(R, T)$ gravity theory

5.2.1 $f(R, T)$ field equations

The $f(R, T)$ gravity formalism is the modification of the general theory of relativity. The action of the theory is given in [62] by

$$S = \int \sqrt{-g} \left(\frac{1}{2\kappa} f(R, T) + L_m \right) d^4x. \quad (5.1)$$

Here g is the determinant of $g^{\mu\nu}$, $\kappa = \frac{8\pi G}{c^2}$, $f(R, T)$ is an arbitrary function of the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ and $T = g^{\mu\nu} T_{\mu\nu}$ the trace of the matter energy-momentum tensor $T_{\mu\nu}$. L_m represents the matter Lagrangian density. Let us now consider varying the action (5.1) with respect to the inverse metric $g^{\mu\nu}$. We have

$$\frac{\delta S}{\delta g^{\mu\nu}} = \int \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{-g} \left(\frac{1}{2\kappa} f(R, T) + L_m \right) d^4x \right) \quad (5.2)$$

which vanishes accordingly with the variational principle. Thus the $f(R, T)$ gravity field equations as given by [62] are

$$f_R(R, T) R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f(R, T) = \kappa T_{\mu\nu} - f_T(R, T) (T_{\mu\nu} + \Theta_{\mu\nu}), \quad (5.3)$$

where ∇_μ denotes the covariant derivative, and

$$\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = g_{\mu\nu} L_m - 2T_{\mu\nu} - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}, \quad (5.4)$$

$$\square \equiv \nabla^\mu \nabla_\mu, \quad f_R(R, T) = \frac{\partial f(R, T)}{\partial R}, \quad f_T(R, T) = \frac{\partial f(R, T)}{\partial T}. \quad (5.5)$$

We can now get a simple relationship between the Ricci scalar and the trace of the matter energy-momentum tensor by contracting (5.3); that is we multiply (5.3) by the inverse metric tensor $g^{\mu\nu}$. We get

$$f_R(R, T) - 2f(R, T) + 3\square f(R, T) = \kappa T - f_T(R, T)(T + \Theta),$$

where $\Theta = \Theta_\mu{}^\mu$. We recall that the covariant derivative of the energy-momentum tensor vanishes; that is

$$\nabla^\mu T_{\mu\nu} = 0.$$

With this divergence property, we have the following relations, see [62, 74, 75]

$$\frac{f_T(R, T)}{\kappa - f_T(R, T)} ((T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu \ln f_T(R, T) + \nabla^\mu \Theta) = 0$$

and

$$\nabla^\mu \left(f_R(R, T) R_{\mu\nu} \frac{1}{2} f(R, T) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f_R(R, T) \right) = 0.$$

5.2.2 Models of the $f(R, T)$ theory

Due to the non-uniqueness definition of the matter Lagrangian, it is known that the perfect fluid may be described by the energy density ρ , the pressure p and the four velocity u_μ . We therefore assume the matter Lagrangian in the framework of this study as $L_m = -p$. With this assumption, the standard form of the energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \quad (5.6)$$

where $u_\mu = \sqrt{g_{00}}(1, 0, 0, 0)$ is the four velocity in co-moving coordinates. The substitution of $L_m = -p$ in (5.4) gives

$$\Theta_{\mu\nu} = -p g_{\mu\nu} - 2T_{\mu\nu}.$$

Hence, the substitution of $\Theta_{\mu\nu}$ in (5.3) returns

$$f_R(R, T) R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f_R(R, T) = \kappa T_{\mu\nu} + f_T(R, T) (T_{\mu\nu} + p g_{\mu\nu}). \quad (5.7)$$

The detailed computations given above showed that the field equations above depend on the physical nature of the matter source $\Theta_{\mu\nu}$. A number of model appear in this framework and it was Harko *et al.*, see [62], who came up with the classification of these into three classes

$$f(R, T) = \begin{cases} R + 2f(T), \\ f(R) + g(T), \\ f(R) + g(R)h(R). \end{cases}$$

Remark: In what follow we use [70], [71] and [72]. We verify the results in those papers but some do not match with our computations.

5.3 Exact plane symmetric solutions

In the first case, we consider plane symmetric static solutions of the field equations in $f(R, T)$ gravity. The general static plane symmetric spacetime is given by

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2), \quad (5.8)$$

where A and B are the cosmic scale factors. A natural Lagrangian for the corresponding geodesic equations is then

$$L = A(x)\dot{t}^2 - \dot{x}^2 - B(x)(\dot{y}^2 + \dot{z}^2). \quad (5.9)$$

From this Lagrangian of the geodesic equations, we can construct the variational and gauge symmetries using the Killing type equation

$$X^{[1]}L + L\frac{d\xi}{ds} = \frac{df}{ds}, \quad (5.10)$$

where f is a functions of (s, t, x, y, z) , $X^{[1]}$ is the first prolongation (in the jet space) of the vector field

$$X = \xi\partial_s + \tau\partial_t + \eta^x\partial_x + \eta^y\partial_y + \eta^z\partial_z \quad (5.11)$$

while ξ , τ and η^i are functions of (s, t, x, y, z) . For $f = 0$, we obtain the zero gauge (variational) symmetries; the variational symmetries include, as a subalgebra, the Killing vectors or isometries of the corresponding metric. We note that the variational symmetries are those that leave the functional $\int Lds$ invariant whilst the isometries are the vector fields that correspond to the one-to-one onto functions that leave the distance function, defined by the metric, invariant. In the former case, the conserved quantities (conservation laws) are obtained by Noether's theorem; for each variational symmetry, there exists a conservation law. For example, Noether showed that the symmetry ∂_t leads to energy conservation, ∂_x , p_y and p_z are conservation of linear momentum, $x\partial_y - y\partial_x$ leads to

an angular momentum and $t\partial_x + x\partial_t$ is related to the spin. These may be seen in the cases expounded below. Some of the symmetries may not lead to physical conservation laws like ‘scaling’ but have their importance in the mathematical analysis of the geodesic equations.

Following the analysis given in [70], we look at the cases $f(R, T) = R + 2f(T)$ and $f(R, T) = f(R) + g(T)$ dealing with the vector fields of the variational symmetries. These would necessarily include the Killing vectors/isometries. For illustration, in some of the cases, we construct the conservation laws.

For convenience, the Ricci scalar is obtained by using the formula

$$R = g^{\mu\nu} R_{\mu\nu}.$$

The detailed calculation gives

$$R = \frac{1}{2} \left[\frac{2A''}{A} - \left(\frac{A'}{A} \right)^2 + \frac{2A'B'}{AB} + \frac{4B''}{B} - \left(\frac{B'}{B} \right)^2 \right], \quad (5.12)$$

where prime represents the derivative with respect to x .

5.3.1 Models $f(R, T) = R + 2f(T)$

When $f(R, T) = R + 2f(T)$, we obtain a number of possibilities. From now on, we use the natural system of unit where $\kappa = 1$. By taking $f(T) = \lambda T$ for some dynamical parameter λ , the field equations become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \lambda(T + 2p)g_{\mu\nu} = (1 + 2\lambda)T_{\mu\nu}. \quad (5.13)$$

If we introduce R into (5.13), we obtain the system of differential equations below

$$\left(\frac{B'}{2B} \right)^2 - \frac{B''}{B} = (1 + 3\lambda)\rho - \lambda p, \quad (5.14)$$

$$\left(\frac{A'B'}{2AB} \right) + \left(\frac{B'}{2B} \right)^2 = (1 + 3\lambda)p - \lambda\rho, \quad (5.15)$$

$$\left(\frac{A'B'}{4AB}\right) + \left(\frac{B''}{2B}\right) - \left(\frac{A'}{2A}\right)^2 + \left(\frac{A''}{2A}\right) - \left(\frac{B'}{2B}\right)^2 = (1 + 3\lambda)p - \lambda\rho, \quad (5.16)$$

which is a system of three independent equations with four variables namely A , B , ρ and p . The system above is solved with different assumptions, see [70]. Since we are interested in the form of the metric, we shall just consider solutions for A and B . However, we proceed at the elimination of ρ and p by subtracting (5.15) from (5.16) and we get

$$\left[\left(\frac{A'}{A}\right)^2 + 2\left(\frac{B'}{B}\right)^2 - 2\left(\frac{A''}{A} + \frac{B''}{B}\right) + \left(\frac{A'B'}{AB}\right)^2\right] = 0. \quad (5.17)$$

(i) Power law model

We assume $A = k_1 x^r$ and $B = k_2 x^l$ where k_1 and k_2 are constants of proportionality. It results that $A = k_1 x^{-\frac{2}{3}}$ and $B = k_2 x^{\frac{4}{3}}$. Hence, the power law assumption on A and B leads to a metric with corresponding Lagrangian given by,

$$L_1 = k_1 x^{-2/3} \dot{t}^2 - \dot{x}^2 - k_2 x^{4/3} (\dot{y}^2 + \dot{z}^2).$$

The associated symmetries are:

$$X_1 = \partial_s,$$

$$X_2 = \partial_t,$$

$$X_3 = \partial_y,$$

$$X_4 = \partial_z,$$

$$X_5 = -z\partial_y + y\partial_z,$$

$$X_6 = 6s\partial_s + 4t\partial_t + 3x\partial_x + y\partial_y + z\partial_z.$$

These variational symmetries are all killing vectors except X_1 and X_6 . The conserved quantity associated with X_6 , for e.g., not attained through the isometries is

$$T_6 = \frac{2}{x^{\frac{2}{3}}} [k_2 x^2 \dot{z}z - 3k_2 x^2 \dot{z}^2 s + k_2 x^2 \dot{y}y - 3k_2 x^2 y_1^2 s + 3\dot{x}x^{5/3} - 3\dot{x}^2 x^{2/3} s - 4k_1 \dot{t}t + 3k_1 \dot{t}^2 s]$$

while with X_5 it is

$$T_5 = -2k_2 x^{\frac{4}{3}} (z\dot{y} - y\dot{z})$$

which is an angular momentum in $y - z$.

(ii) $B = A^n$ assumption for n an arbitrary real number

We substitute B into (5.17) and we obtain

$$(3n + 1)A'^2 - 2(n + 1)AA'' = 0,$$

whose solution is

$$A = k_3 [(n - 1)x + 2k_4(n + 1)]^{2\frac{n+1}{1-n}}, \quad n \neq 1,$$

where k_3 and k_4 are integration constants. Taking $k_3 = 1$ and $k_4 = 0$, the cosmic scale factors give

$$A = [(n - 1)x]^{2\frac{n+1}{1-n}},$$

and

$$B = [(n - 1)x]^{2n\frac{n+1}{1-n}},$$

see [70]. Thus the assumption $B = A^n$ lead to the Lagrangian

$$L = [(n - 1)x]^{2(n+1)/(1-n)} \dot{t}^2 - \dot{x}^2 - [(n - 1)x]^{2n(n+1)/(1-n)} (\dot{y}^2 + \dot{z}^2),$$

for which associated symmetries are

$$X_1 = \partial_s,$$

$$X_2 = \partial_t,$$

$$X_3 = \partial_y,$$

$$X_4 = \partial_z,$$

$$X_5 = -z\partial_y + y\partial_z,$$

$$X_6 = \frac{2(n-1)s}{n^2+2n-1}\partial_s + \frac{2nt}{n^2+2n-1}\partial_t + \frac{(n-1)x}{n^2+2n-1}\partial_x + y\partial_y + z\partial_z.$$

These Noether symmetries are all Killing vectors except X_1 and X_6 . Similarly, the conserved quantity associated with X_4 is

$$T_4 = 2 [(n - 1)x]^{\frac{-2n(n+1)}{n-1}} \dot{z}$$

which is linear momentum in z .

5.3.2 $f(R, T) = f(R) + g(T)$

The field equations (5.7) become

$$f_R(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)f_R(R) = (1 + g_T(T))T_{\mu\nu} + \left(pg_T(T) + \frac{1}{2}g(T)\right)g_{\mu\nu}. \quad (5.18)$$

If we contract the above equation with the inverse metric $g^{\mu\nu}$, we have

$$f_R(R)R - 2f(R) + 3\square f_R(R) = T + 2g(T) + (T + 4p)g_T(T) \quad (5.19)$$

from which we obtain $f(R)$ whose substitution into (5.18) with rearrangement gives

$$\frac{f_R(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu f_R(R) - (1 + g_T(T))T_{\mu\nu}}{g_{\mu\nu}} = \frac{(R - \square)f_R(R) - (1 + g_T(T))}{4}.$$

However, since we are dealing with cosmic scale factors depending on the spatial coordinate x and due to the fact that the metric (5.8) is diagonal, we can write

$$\frac{f_R(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu f_R(R) - (1 + g_T(T))T_{\mu\mu}}{g_{\mu\mu}} = \frac{(R - \square)f_R(R) - (1 + g_T(T))}{4}. \quad (5.20)$$

One can remark that the right hand side of (5.20) does not depend on index μ . Hence, one can define a quantity Q_μ by

$$Q_\mu = \frac{f_R(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu f_R(R) - (1 + g_T(T))T_{\mu\mu}}{g_{\mu\mu}} \quad (5.21)$$

such that $Q_\mu - Q_\nu = 0 \forall \mu, \nu$. The detailed calculation of the nonzero Christoffel symbols gives

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}A', & \Gamma^0_{10} &= \Gamma^0_{01} = \frac{1}{2}\frac{A'}{A}, & \Gamma^1_{22} &= \Gamma^1_{33} = -\frac{1}{2}B', \\ \Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}\frac{B'}{B}. \end{aligned} \quad (5.22)$$

We use these coefficients to compute the nonzero components of the Ricci tensor given below

$$R_{00} = \frac{1}{4} \left[2\frac{A'B'}{B} - \frac{A'^2}{A} + 2A'' \right], \quad (5.23)$$

$$R_{11} = \frac{1}{4} \left[2\left(\frac{B'}{B}\right)^2 - 4\frac{B''}{B} + \left(\frac{A'}{A}\right)^2 - 2\frac{A''}{A} \right], \quad (5.24)$$

$$R_{22} = R_{33} = \frac{A'B'}{4A} - \frac{B''}{2}. \quad (5.25)$$

We also use the formula

$$\nabla_\mu \nabla_\mu f_R(R) = \nabla_\mu (\partial_\mu f_R(R)) = \partial_\mu p_\mu f_R(R) + \Gamma_{\mu\mu}^\alpha \partial_\mu f_R(R)$$

to find the second covariant derivative of the function $f_R(R)$. Finally, we obtain

$$\begin{aligned} Q_0 &= f_R(R) \left[\frac{A'B'}{2AB} - \left(\frac{A'}{2A} \right)^2 + \frac{A''}{2A} \right] - \frac{A'}{2A} f'_R(R) - (1 + g_T(T))\rho, \\ Q_1 &= f_R(R) \left[- \left(\frac{A'}{2A} \right)^2 - \frac{1}{2} \left(\frac{B'}{B} \right)^2 + \frac{A''}{2A} + \frac{B''}{B} \right] + f''_R(R) - (1 + g_T(T))p, \\ Q_2 &= Q_3 = f_R(R) \left[\frac{A'B'}{4AB} + \frac{B''}{2B} \right] - \frac{B'}{2B} f'_R(R) - (1 + g_T(T))p. \end{aligned}$$

By subtracting Q_1 from Q_2 , we eliminate the variable ρ and p and obtain the following non-linear equation

$$\left[\frac{A'B'}{AB} + \left(\frac{A'}{A} \right)^2 + 2 \left(\frac{B'}{B} \right)^2 - 2 \left(\frac{A''}{A} + \frac{B''}{B} \right) \right] - 2 \frac{B'}{B} \frac{f'_R(R)}{f_R(R)} - 4 \frac{f''_R(R)}{f_R(R)} = 0. \quad (5.26)$$

Now, we assume that $B = A^n$ and $f_R(R) \propto R^m$, where

$$R = \frac{1}{2} \left[(3n^2 - 2n - 1) \left(\frac{A'}{A} \right)^2 + (4n + 2) \frac{A''}{A} \right].$$

Then, (5.26) becomes

$$(3n + 1) \left(\frac{A'}{A} \right)^2 - 2(n + 1) \frac{A''}{A} - 2n \frac{A'}{A} \frac{f'_R(R)}{f_R(R)} - 4 \frac{f''_R(R)}{f_R(R)} = 0. \quad (5.27)$$

(iii) $A = x^{-1}$ model

We can show that $B = x^{-\frac{16m^2+8m+3}{1+4m}}$ instead of $B = x^{-(16m^2+8m+3)/3}$ as given in [70] so that

$$L = x^{-1} \dot{t}^2 - \dot{x}^2 - x^{-\frac{16m^2+8m+3}{1+4m}} (\dot{y}^2 + \dot{z}^2),$$

where $n = \frac{16m^2+8m+3}{1+4m}$ is obtained from the constraint equation

$$16m^2 + 8m + 3 - n(1 + 4m) = 0.$$

We notice that this constraint equation results from the substitution of $A = x^{-1}$, $f_R(R) = R^m$ into (5.27). The associated symmetries are

$$X_1 = \partial_s,$$

$$X_2 = \partial_t,$$

$$X_3 = \partial_y,$$

$$X_4 = \partial_z,$$

$$X_5 = -z\partial_y + y\partial_z,$$

$$X_6 = \frac{4s(1+4m)}{16m^2+8m+5}\partial_s + \frac{3t(1+4m)}{16m^2+8m+5}\partial_t + \frac{2x(1+4m)}{16m^2+8m+5}\partial_x + y\partial_y + z\partial_z,$$

where X_1 and X_6 are Noether symmetries but are not Killing vectors. We can find all the associated conserved quantities so that choosing X_1 for example, leads to the conserved quantity

$$T_1 = -\frac{2}{x}\dot{t}.$$

(iv) $A = e^{\alpha x}$ model

We get

$$R = \frac{1}{2}(3n^2 + 2n + 1)\alpha^2.$$

This leads to the constraint equation $n - 1 = 0$ instead of $(n - 1)(3n^2 + n2 + 1)$ obtained in [70]. The corresponding Lagrangian is given by

$$L = e^{\alpha x}(\dot{t}^2 - \dot{y}^2 - \dot{z}^2) - \dot{x}^2$$

for which the associated symmetries are

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_t, \\
X_3 &= \partial_y, \\
X_4 &= \partial_z, \\
X_5 &= -z\partial_y + y\partial_z, \\
X_6 &= z\partial_t + t\partial_z, \\
X_7 &= y\partial_t + t\partial_y, \\
X_8 &= t\partial_t - \frac{2}{\alpha}\partial_x + y\partial_y + z\partial_z, \\
X_9 &= ty\partial_t - \frac{2y}{\alpha}\partial_x + \left[\frac{1}{2}(t^2 + y^2 - z^2) - \frac{2}{\alpha^2}e^{-\alpha x}\right]\partial_y + yz\partial_z, \\
X_{10} &= \left[\frac{1}{2}(t^2 + y^2 + z^2) - \frac{2}{\alpha^2}e^{-\alpha x}\right]\partial_t - \frac{2t}{\alpha}\partial_x + ty\partial_y + tz\partial_z, \\
X_{11} &= -\frac{\alpha^2 tz}{2}\partial_t + \alpha z\partial_x - \frac{\alpha^2 yz}{2}\partial_y + \left[\frac{\alpha^2}{4}(t^2 - y^2 + z^2) - e^{-\alpha x}\right]\partial_z.
\end{aligned}$$

Except X_1 , these variational symmetries are Killing vectors.

5.4 Locally rotationally symmetric (LRS) Bianchi type I cosmology

The line element of LRS Bianchi type I spacetime used in this study is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)[dy^2 + dz^2], \quad (5.28)$$

where A and B are cosmic scale factors with corresponding Ricci scalar

$$R = -2 \left[\frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right]. \quad (5.29)$$

The ‘dot’ denotes the derivative with respect to t . Here too, we consider the cases $f(R, T) = R + 2f(T)$ and $f(R, T) = f_1(R) + f_2(T)$.

5.4.1 $f(R, T) = R + 2f(T)$ model

The field equations of the model are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} + 2f_T(T)T_{\mu\nu} + (f(T) + 2pf_T(T))g_{\mu\nu}. \quad (5.30)$$

When we take $f(T) = \lambda T$ for some dynamical parameter λ , the field equations then become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \lambda(T + 2p)g_{\mu\nu} = (1 + 2\lambda)T_{\mu\nu}. \quad (5.31)$$

Introducing R into (5.31) yields a system of three differential equations with four unknowns below

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = (1 + 3\lambda)\rho - \lambda p, \quad (5.32)$$

$$-2\frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 = (1 + 3\lambda)p - \lambda\rho, \quad (5.33)$$

$$-\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = (1 + 3\lambda)p - \lambda\rho, \quad (5.34)$$

We eliminate ρ and p from the system above by subtracting (5.34) from (5.33). It results the following non-linear differential equation

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 = 0. \quad (5.35)$$

We study some cases for a precised physical conditions as given in [71].

(i) $A = B^n$ assumption

The substitution of $A = B^n$ into (5.35) yields

$$\frac{\ddot{B}}{B} + (n + 1)\frac{\dot{B}^2}{B^2} = 0, \quad (5.36)$$

whose solution is

$$B = c_1[(n + 2)t + c_2]^{\frac{1}{n+2}},$$

where c_1 and c_2 are constants of integration. The corresponding Lagrangian is therefore

$$L = \dot{t}^2 - c_1^{2n}[(n+2)t + c_2]^{\frac{2n}{n+2}} \dot{x}^2 - c_1^2[(n+2)t + c_2]^{\frac{2}{n+2}} [\dot{y}^2 + \dot{z}^2].$$

The associated symmetries are

$$\begin{aligned} X_1 &= \partial_s, \\ X_2 &= \partial_x, \\ X_3 &= \partial_y, \\ X_4 &= \partial_z, \\ X_5 &= -z\partial_y + y\partial_z, \\ X_6 &= \frac{2(n+2)s}{n+1}\partial_s + \frac{(n+2)t+c_2}{n+1}\partial_t - \frac{2x}{n+1}\partial_x + y\partial_y + z\partial_z. \end{aligned}$$

Here, again X_1 and X_6 are not isometries.

(ii) For $n = -\frac{1}{2}$

The Ricci scalar vanishes and the Lagrangian of the model becomes

$$L = \dot{t}^2 - \left[c_1 \left(\frac{3}{2}t + c_2 \right) \right]^{-\frac{2}{3}} \dot{x}^2 - \left[c_1 \left(\frac{3}{2}t + c_2 \right) \right]^{\frac{4}{3}} [\dot{y}^2 + \dot{z}^2].$$

The associated symmetries are

$$\begin{aligned} X_1 &= \partial_s, \\ X_2 &= \partial_x, \\ X_3 &= \partial_y, \\ X_4 &= \partial_z, \\ X_5 &= -z\partial_y + y\partial_z, \\ X_6 &= 6s\partial_s + (3t + 2c_2)\partial_t + 4x\partial_x + y\partial_y + z\partial_z. \end{aligned}$$

5.4.2 $f(R, T) = f(R) + g(T)$

We obtain the field equations of the model in a similar way as given

$$f_R(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu)\nabla_\nu - g_{\mu\nu}\square)f_R(R) = (1 + g_T(T))T_{\mu\nu} + \left(pg_T(T) + \frac{1}{2}g(T) \right) g_{\mu\nu}. \quad (5.37)$$

We multiply the equation above by the inverse metric $g^{\mu\nu}$ and it results

$$f_R(R)R - 2f(R) + 3\Box f_R(R) = T + 2g(T) + (T + 4p)g_T(T), \quad (5.38)$$

from which we obtain $f(R)$. Substituting $f(R)$ into (5.37) yields

$$\frac{f_R(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu f_R(R) - (1 + g_T(T))T_{\mu\nu}}{g_{\mu\nu}} = \frac{(R - \Box)f_R(R) - (1 + g_T(T))}{4}.$$

Since the cosmic scale factors depend on the temporal coordinate t and the metric (5.28) is diagonal, we can write

$$\frac{f_R(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu f_R(R) - (1 + g_T(T))T_{\mu\mu}}{g_{\mu\mu}} = \frac{(R - \Box)f_R(R) - (1 + g_T(T))}{4}. \quad (5.39)$$

Similarly, the right hand side of (5.39) is independent of the index μ . Therefore, we can define a quantity Q'_μ by

$$Q'_\mu = \frac{f_R(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu f_R(R) - (1 + g_T(T))T_{\mu\mu}}{g_{\mu\mu}} \quad (5.40)$$

such that $Q'_\mu - Q'_\nu = 0$ for all μ and ν . We calculate the nonzero Christoffel symbols as given below

$$\begin{aligned} \Gamma^0_{11} &= A\dot{A}, \quad \Gamma^0_{22} = \Gamma^0_{33} = B\dot{B}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{A}}{A}, \\ \Gamma^2_{02} &= \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{B}}{B}. \end{aligned} \quad (5.41)$$

Similarly, with these coefficients, we find the nonzero components of the Ricci tensor

$$R_{00} = -\frac{\ddot{A}}{A} - 2\frac{\ddot{B}}{B}, \quad (5.42)$$

$$R_{11} = A\ddot{A} + 2\frac{A\dot{A}\dot{B}}{B}, \quad (5.43)$$

$$R_{22} = R_{33} = B\ddot{B} + \dot{B}^2 + \frac{\dot{A}\dot{B}B}{A}. \quad (5.44)$$

Using these components, we obtain

$$Q'_0 = -f_R(R) \left(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} \right) - \ddot{f}_R(R) - (1 + g_T(T))\rho,$$

$$Q'_1 = -f_R(R) \left(\frac{\ddot{A}}{A} + 2 \frac{\dot{A}\dot{B}}{AB} \right) + \frac{\dot{A}}{A} \dot{f}_R(R) + (1 + g_T(T))p,$$

$$Q'_2 = Q'_3 = -f_R(R) \left(\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} \right) + \frac{\dot{B}}{B} \dot{f}_R(R) + (1 + g_T(T))p.$$

We subtract Q'_1 from Q'_2 to eliminate the variable ρ and p and we obtain the following non-linear equation

$$\left[-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right] f_R(R) + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \dot{f}_R(R) = 0. \quad (5.45)$$

(iii) Exponential law solutions

We suppose that $A = B^n$ and $f_R(R) \propto R^m$, where

$$R = -2 \left[(n^2 + n + 1) \frac{\dot{B}^2}{B^2} + (n + 2) \frac{\ddot{B}}{B} \right].$$

Hence, (5.45) becomes

$$(n^3 + 2n^2 + 2n + 2)\dot{B}^4 + (2n^2 + 4n + 3)B\dot{B}^2\ddot{B} + (n + 2)B^2\ddot{B}^2 + 2m(n^2 + n + 1)\dot{B}^4 - m(2n^2 + n)B\dot{B}^2\ddot{B} - m(n + 2)B^2\ddot{B}\ddot{B} = 0. \quad (5.46)$$

A solution of the above equation can be of the form

$$B(t) = e^{c_3 t + c_4}$$

whose substitution into (5.46) gives the following constraint equation

$$n^3 + 4n^2 + 7n + 7 = 0.$$

This equation admits two complex conjugate solutions and one real solution. The real solution is given by

$$n = -\frac{1}{6}(260 + 60\sqrt{21})^{\frac{1}{3}} + \frac{10}{3(260 + 60\sqrt{21})^{\frac{1}{2}}} - \frac{4}{3}. \quad (5.47)$$

Hence, the resulting Lagrangian is

$$L = \dot{t}^2 - e^{2n(c_3 t + c_4)} \dot{x}^2 - e^{2(c_3 t + c_4)} [\dot{y}^2 + \dot{z}^2],$$

where n is given in (5.47). The associated symmetries are

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_x, \\
X_3 &= \partial_y, \\
X_4 &= \partial_z, \\
X_5 &= -z\partial_y + y\partial_z, \\
X_6 &= -\frac{1}{c_3}\partial_t + \frac{(260+60\sqrt{21})^{\frac{1}{3}}(65+15\sqrt{21})^{\frac{1}{2}}+8(65+15\sqrt{21})^{\frac{1}{2}}-30}{12(65+15\sqrt{21})^{\frac{1}{2}}}x\partial_x + y\partial_y + z\partial_z.
\end{aligned}$$

Only X_1 is not an isometry.

(iv) Power law solutions

The Lagrangian of this model is given by

$$L = \dot{t}^2 - (c_5 t + c_6)^{-2(2m+1)} \dot{x}^2 - (c_5 t + c_6)^{2(2m+1)} [\dot{y}^2 + \dot{z}^2].$$

The associated symmetries are

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_x, \\
X_3 &= \partial_y, \\
X_4 &= \partial_z, \\
X_5 &= -z\partial_y + y\partial_z, \\
X_6 &= -\frac{s}{m}\partial_s - \frac{c_5 t + c_6}{2c_5 m}\partial_t - \frac{(m+1)x}{m}\partial_x + y\partial_y + z\partial_z.
\end{aligned}$$

(v) For $m = \frac{1}{2}$ or $m = \frac{1}{4}$

The Ricci scalar turns out to be constant. In particular, $m = \frac{1}{2}$ corresponds to Minkowski space-time. For $m = \frac{1}{4}$ the Lagrangian for the metric takes the form

$$L = \dot{t}^2 - (c_5 t + c_6)^{-1} \dot{x}^2 - (c_5 t + c_6) [\dot{y}^2 + \dot{z}^2]$$

and the associated symmetries are

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_x, \\
X_3 &= \partial_y, \\
X_4 &= \partial_z, \\
X_5 &= -z\partial_y + y\partial_z, \\
X_6 &= 4s\partial_s + \frac{2(c_5t+c_6)}{c_5}\partial_t + 3x\partial_x + y\partial_y + z\partial_z.
\end{aligned}$$

5.5 Bianchi Type V

The line element of the Bianchi type V spacetime, see [72], is

$$ds^2 = dt^2 - A^2(t)dx^2 - e^{2mx}[B^2(t)dy^2 + C^2(t)dz^2],$$

where A , B and C are cosmic scale factors and m is an arbitrary constant. The corresponding Lagrangian is

$$L = \dot{t}^2 - A^2(t)\dot{x}^2 - e^{2mx}[B^2(t)\dot{y}^2 + C^2(t)\dot{z}^2]. \quad (5.48)$$

To obtain the Einstein field equations of the model, we first compute the nonzero Christoffel symbols as given

$$\begin{aligned}
\Gamma_{11}^0 &= A\dot{A}, \quad \Gamma_{22}^0 = e^{2mx}B\dot{B}, \quad \Gamma_{33}^0 = e^{2mx}C\dot{C}, \quad \Gamma_{01}^1 = \Gamma_{10}^1 = \frac{\dot{A}}{A}, \\
\Gamma_{02}^2 &= \Gamma_{20}^2 = \frac{\dot{B}}{B}, \quad \Gamma_{03}^3 = \Gamma_{30}^3 = \frac{\dot{C}}{C} \\
\Gamma_{22}^1 &= -me^{2mx}\frac{B^2}{A^2}, \quad \Gamma_{33}^1 = -me^{2mx}\frac{C^2}{A^2}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = m. \quad (5.49)
\end{aligned}$$

Then, we use them to find five nonzero components of the Ricci tensor which are the 01-component and the four diagonal components

$$R_{01} = -m \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \quad (5.50)$$

$$R_{00} = -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}, \quad (5.51)$$

$$R_{11} = A\ddot{A} + \frac{A\dot{A}\dot{B}}{B} + \frac{A\dot{A}\dot{C}}{C} - 2m^2, \quad (5.52)$$

$$R_{22} = e^{2mx} \left(\frac{A\dot{A}\dot{B}}{B} + \frac{A\dot{A}\dot{C}}{C} + A\ddot{A} - 2m^2 \frac{B^2}{A^2} \right), \quad (5.53)$$

$$R_{33} = e^{2mx} \left(\frac{\dot{A}\dot{B}\dot{B}}{A} + \frac{\dot{B}\dot{C}B}{C} + B\ddot{B} - 2m^2 \frac{C^2}{A^2} \right). \quad (5.54)$$

The corresponding Ricci scalar is

$$R = -2 \left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3m^2}{A^2} \right]. \quad (5.55)$$

Hence, substituting the five nonzero components of the Ricci tensor into (5.13) leads to a system of the following five field equations

$$2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0, \quad (5.56)$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{3m^2}{A^2} = (1 + 3\lambda)\rho - \lambda p, \quad (5.57)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} = \lambda\rho - (1 + 3\lambda)p, \quad (5.58)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} = \lambda\rho - (1 + 3\lambda)p, \quad (5.59)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = \lambda\rho - (1 + 3\lambda)p. \quad (5.60)$$

The system above is solved in [72] and the values of the cosmic scale factors are provided in two cases.

5.5.1 Case $n \neq 0$: singular model of the universe

The metric functions A , B and C for this model are provided in [72] as

$$\begin{aligned} A &= (nlt + k_1)^{\frac{1}{n}}, \\ B &= P(nlt + k_1)^{\frac{1}{n}} \exp \left[\frac{Q(nlt+k_1)^{\frac{n-3}{n}}}{l(n-3)} \right], \quad n \neq 3, \\ C &= P^{-1}(nlt + k_1)^{\frac{1}{n}} \exp \left[\frac{-Q(nlt+k_1)^{\frac{n-3}{n}}}{l(n-3)} \right], \quad n \neq 3, \end{aligned}$$

where P and Q are constants. The associated Noether Symmetries X satisfy the following equation

$$\begin{aligned} &\eta(2lA^{2-n})\dot{x}^2 - \eta e^{2mx} \left[2P^2(lA^{2-n} + QA^{-1}) \exp \left[\frac{2QA^{n-3}}{l(n-3)} \right] \dot{y}^2 + 2P^{-2}(lA^{2-n} + QA^{-1}) \right. \\ &\quad \left. \exp \left[\frac{-2QA^{n-3}}{l(n-3)} \right] \dot{z}^2 \right] - 2m\alpha e^{2mx}(B^2\dot{y}^2 + C^2\dot{z}^2) + 2\dot{t}(\eta_s + \dot{t}\eta_t + \dot{x}\eta_x + \dot{y}\eta_y + \dot{z}\eta_z) \\ &\quad - 2\dot{t}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{x}A^2(\alpha_s + \dot{t}\alpha_t + \dot{x}\alpha_x + \dot{y}\alpha_y + \dot{z}\alpha_z) \\ &\quad + 2\dot{x}^2A^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{y}B^2e^{2mx}(\beta_s + \dot{t}\beta_t + \dot{x}\beta_x + \dot{y}\beta_y + \dot{z}\beta_z) \\ &\quad + 2\dot{y}^2B^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{z}C^2e^{2mx}(\gamma_s + \dot{t}\gamma_t + \dot{x}\gamma_x + \dot{y}\gamma_y + \dot{z}\gamma_z) \\ &\quad + 2\dot{z}^2C^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) + \dot{t}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) \\ &\quad - A^2\dot{x}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - \dot{y}^2B^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) \\ &\quad - \dot{z}^2C^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) = f_s + \dot{t}f_t + \dot{x}f_x + \dot{y}f_y + \dot{z}f_z. \end{aligned}$$

The separation by monomials is given below

$$\begin{aligned} \dot{t}^2 &: 2\eta_t - \xi_s = 0, \\ \dot{x}^2 &: 2\eta lA^{-n} - 2\alpha_x + \xi_s = 0, \\ \dot{y}^2 &: 2\eta(lA^{-n} + QA^{-3}) - 2m\alpha - 2\beta_z + \xi_s = 0, \\ \dot{z}^2 &: 2\eta(lA^{-n} - QA^{-3}) - 2m\alpha - 2\gamma_z + \xi_s = 0, \\ \dot{t}\dot{x} &: \eta_x - \alpha_t = 0, \\ \dot{t}\dot{y} &: \eta_y - \beta_t = 0, \\ \dot{t}\dot{z} &: \eta_z - \gamma_t = 0, \\ \dot{x}\dot{y} &: \alpha_y + P^2\beta_x \exp \left(\frac{2QA^{n-3}}{l(n-3)} \right) = 0, \\ \dot{x}\dot{z} &: \alpha_z + P^{-2}\gamma_x \exp \left(-\frac{2QA^{n-3}}{l(n-3)} \right) = 0, \\ \dot{y}\dot{z} &: P^2\beta_z \exp \left(\frac{2QA^{n-3}}{l(n-3)} \right) + P^{-2}\gamma_y \exp \left(-\frac{2QA^{n-3}}{l(n-3)} \right) = 0, \end{aligned}$$

$$\begin{aligned}
\dot{t}^3 & : \xi_s = 0, \\
\dot{x}^3 & : \xi_x = 0, \\
\dot{y}^3 & : \xi_y = 0, \\
\dot{z}^3 & : \xi_z = 0, \\
\dot{t} & : 2\eta_s = f_t, \\
\dot{x} & : 2A^2\alpha_s = f_x, \\
\dot{y} & : 2B^2\beta_s = f_y, \\
\dot{z} & : 2C^2\gamma_s = f_z, \\
1 & : f_s = 0.
\end{aligned} \tag{5.61}$$

We finally obtain twelve Noether symmetries including two non Killing vectors as given below

$$\begin{aligned}
X_1 & = \partial_s, \\
X_2 & = \partial_t, \\
X_3 & = \partial_x, \\
X_4 & = \partial_y, \\
X_5 & = \partial_z, \\
X_6 & = -P^{-2} \exp \left[\frac{-2Q(nlt + k_1)^{\frac{n-3}{n}}}{l(n-3)} \right] e^{2mx} z\partial_x + x\partial_z, \\
X_7 & = P^{-2}(nlt + k_1)^{\frac{2}{n}} \exp \left[\frac{-2Q(nlt + k_1)^{\frac{n-3}{n}}}{l(n-3)} \right] e^{2mx} z\partial_t + t\partial_z, \\
X_8 & = -P^{-2} \exp \left[\frac{-4Q(nlt + k_1)^{\frac{n-3}{n}}}{l(n-3)} \right] z\partial_y + y\partial_z, \\
X_9 & = -P^2 \exp \left[\frac{2Q(nlt + k_1)^{\frac{n-3}{n}}}{l(n-3)} \right] e^{2mx} y\partial_x + x\partial_y, \\
X_{10} & = P^2(nlt + k_1)^{\frac{2}{n}} \exp \left[\frac{2Q(nlt + k_1)^{\frac{n-3}{n}}}{l(n-3)} \right] e^{2mx} y\partial_t + t\partial_y, \\
X_{11} & = (nlt + k_1)^{\frac{2}{n}} x\partial_t + t\partial_x, \\
X_{12} & = 2s\partial_s + t\partial_t + x\partial_x + y\partial_y + z\partial_z.
\end{aligned} \tag{5.62}$$

5.5.2 Case $n = 0$: singular model of the universe

The metric functions A , B and C for this model are, see [72],

$$\begin{aligned} A &= k_2 \exp(lt), \\ B &= PA \exp \left[-Q \frac{\exp(-3lt)}{3lk_2^3} \right], \\ C &= P^{-1}A \exp \left[Q \frac{\exp(-3lt)}{3lk_2^3} \right], \end{aligned}$$

where P and Q are constants. The corresponding Noether Symmetries X satisfy the following equation

$$\begin{aligned} &\eta(2lA^2) \dot{x}^2 - \eta e^{2mx} [2B^2(l + 2QA^{-3})\dot{y}^2 + 2C^2(l - 2QA^{-3})\dot{z}^2 \\ &- 2m\alpha e^{2mx}(B^2\dot{y}^2 + C^2\dot{z}^2) + 2\dot{t}(\eta_s + \dot{t}\eta_t + \dot{x}\eta_x + \dot{y}\eta_y + \dot{z}\eta_z) \\ &- 2\dot{t}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{x}A^2(\alpha_s + \dot{t}\alpha_t + \dot{x}\alpha_x + \dot{y}\alpha_y + \dot{z}\alpha_z) \\ &+ 2\dot{x}^2A^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{y}B^2e^{2mx}(\beta_s + \dot{t}\beta_t + \dot{x}\beta_x + \dot{y}\beta_y + \dot{z}\beta_z) \\ &+ 2\dot{y}^2B^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - 2\dot{z}C^2e^{2mx}(\gamma_s + \dot{t}\gamma_t + \dot{x}\gamma_x + \dot{y}\gamma_y + \dot{z}\gamma_z) \\ &+ 2\dot{z}^2C^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) + \dot{t}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) \\ &- A^2\dot{x}^2(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) - \dot{y}^2B^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) \\ &- \dot{z}^2C^2e^{2mx}(\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) = f_s + \dot{t}f_t + \dot{x}f_x + \dot{y}f_y + \dot{z}f_z. \end{aligned}$$

In a similar way, the separation by monomials is given by

$$\begin{aligned} \dot{t}^2 &: 2\eta_t - \xi_s = 0, \\ \dot{x}^2 &: 2\eta l - 2\alpha_x + \xi_s = 0, \\ \dot{y}^2 &: 2\eta(l + 2QA^{-3}) - 2m\alpha - 2\beta_z + \xi_s = 0, \\ \dot{z}^2 &: 2\eta(l - 2QA^{-3}) - 2m\alpha - 2\gamma_z + \xi_s = 0, \\ \dot{t}\dot{x} &: \eta_x - \alpha_t = 0, \\ \dot{t}\dot{y} &: \eta_y - \beta_t = 0, \\ \dot{t}\dot{z} &: \eta_z - \gamma_t = 0, \\ \dot{x}\dot{y} &: \alpha_y + P^2\beta_x \exp\left(\frac{2QA^{n-3}}{l(n-3)}\right) = 0, \\ \dot{x}\dot{z} &: \alpha_z + P^{-2}\gamma_x \exp\left(-\frac{2QA^{n-3}}{l(n-3)}\right) = 0, \\ \dot{y}\dot{z} &: P^2\beta_z \exp\left(\frac{2QA^{n-3}}{l(n-3)}\right) + P^{-2}\gamma_y \exp\left(-\frac{2QA^{n-3}}{l(n-3)}\right) = 0, \\ \dot{t}^3 &: \xi_s = 0, \end{aligned}$$

$$\begin{aligned}
\dot{x}^3 &: \xi_x = 0, \\
\dot{y}^3 &: \xi_y = 0, \\
\dot{z}^3 &: \xi_z = 0, \\
\dot{t} &: 2\eta_s = f_t, \\
\dot{x} &: 2A^2\alpha_s = f_x, \\
\dot{y} &: 2B^2\beta_s = f_y, \\
\dot{z} &: 2C^2\gamma_s = f_z, \\
1 &: f_s = 0.
\end{aligned} \tag{5.63}$$

We obtain the following symmetries

$$\begin{aligned}
X_1 &= \partial_s, \\
X_2 &= \partial_t, \\
X_3 &= \partial_x, \\
X_4 &= \partial_y, \\
X_5 &= \partial_z, \\
X_6 &= -P^{-2} \exp \left[2Q \frac{\exp(-3lt)}{3lk_2^3} \right] e^{2mx} z \partial_x + x \partial_z, \\
X_7 &= P^{-2} (k_2 \exp(lt))^2 \exp \left[2Q \frac{\exp(-3lt)}{3lk_2^3} \right] e^{2mx} z \partial_t + t \partial_z, \\
X_8 &= -P^{-4} \exp \left[4Q \frac{\exp(-3lt)}{3lk_2^3} \right] z \partial_y + y \partial_z, \\
X_9 &= -P^2 \exp \left[-2Q \frac{\exp(-3lt)}{3lk_2^3} \right] e^{2mx} y \partial_x + x \partial_y, \\
X_{10} &= P^2 (k_2 \exp(lt))^2 \exp \left[-2Q \frac{\exp(-3lt)}{3lk_2^3} \right] e^{2mx} y \partial_t + t \partial_y, \\
X_{11} &= (k_2 \exp(lt))^2 x \partial_t + t \partial_x, \\
X_{12} &= 2s \partial_s + t \partial_t + x \partial_x + y \partial_y + z \partial_z.
\end{aligned} \tag{5.64}$$

Overall we see that the Bianchi type V gives twelve symmetries among which two are not Killing vectors.

Discussion

We have constructed the variational symmetries associated with some spacetimes following certain $f(R, T)$ models in a plane symmetric case and a locally rotationally symmetric Bianchi type I spacetime. Each symmetry may be used to construct a conservation law via Noether's theorem; we have done this for some cases. In most of the results above, the algebra of variational symmetries is six of which four are Killing vectors or isometries and the additional variational symmetries are ∂_s and a scaling symmetry involving s . This does not coincide with the well known algebra of isometries and variational symmetries of the Schwartzchild metric which generates four isometries but just one additional variational symmetry ∂_s . In the fourth case of Subsection 5.3.2, the algebra seems to mimic the de Sitter algebra producing ten Killing vectors with one additional variational symmetry.

Chapter 6

Conclusion

Symmetry analysis became a powerful tool in investigating differential equations. In this thesis we examined symmetry, invariance methods and conservation laws of some riemannian metrics of current interest.

We began our study with the modified de Sitter metric. We investigated the symmetry classification of the corresponding geodesic equations and came up with the conclusion that where a finite group generated by conservation laws exists, the Noether symmetry group has at least one additional symmetry that is not given by Killing vectors. Moreover, we saw that the dimension of the algebra formed by the Killing vectors, obtained in all the cases we have studied, is less than or equal to 10 which is in agreement with Theorem 1.2.2 since we worked in four-dimensional spactimes.

The application of invariance methods in solving equations of our interest urged us to choose dependent inavriants which lead to appropriate reduction in variables. In this dynamic, we utilized the generalized fundamental theorem of double reduction to reduce the number of variables from four to two in the wave equation studied. The wave equation, initially written in terms of a field u depending on four variables, turned to be in terms of a field v which only depends on two variables.

We saw that the isometry group is always a finite-dimensional Lie group acting smoothly on the manifold considered. Hence, the Lie derivative is one of the right tools in this study for finding symmetry of a metric.

We constructed the variational symmetries associated with some spacetimes that are asymptotically flat as well as some $f(R, T)$ -gravity theory. In these cases, we derived the Einstein field equations that gave solutions to the metrics. Most of the complicated calculations done in this thesis were made possible with the help of Maple and Mathematica.

In future, we shall develop methods to provide a general classification of Noether symmetries with some minimal symmetry group in terms of Killing vectors. If this is successful, it will help to compare this general classification to the Killing vectors algebra.

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