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Pournara, Craig

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Mathematics-for-teaching in pre-service mathematics teacher education: The case of financial mathematics

Craig Pournara

A thesis submitted to the Wits School of Education,
Faculty of Humanities, University of the Witwatersrand
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ABSTRACT

Mathematics-for-teaching in pre-service mathematics teacher education: The case of financial mathematics

Mathematics-for-teaching (MfT) is complex, multi-faceted and topic-specific. In this study, a Financial Mathematics course for pre-service secondary mathematics teachers provides a revelatory case for investigating MfT. The course was designed and taught by the author to a class of forty-two students at a university in South Africa. Eight students, forming a purposive sample, participated as members of two focus tutorial groups and took part in individual and group interviews.

As an instance of insider research, the study makes use of a qualitative methodology that draws on a variety of data sources including lecture sessions and group tutorials, group and individual interviews, students' journals, a test and a questionnaire.

The thesis is structured in two parts. The first part explores revisiting of school mathematics with particular focus on compound interest and the related aspects of percentage change and exponential growth. Four cases are presented, in the form of analytic narrative vignettes which structure the analysis and provide insight into opportunities for learning MfT of compound interest. The evidence shows that opportunities may be provided to learn a range of aspects of MfT through revisiting school mathematics.

The second part focuses on obstacles experienced by students in learning annuities, their time-related talk, as well as their use of mathematical resources such as timelines and spreadsheets. A range of obstacles are identified. Evidence shows that students use timelines in a range of non-standard ways but that this does not necessarily determine or reflect their success in solving annuities problems. Students' use of spreadsheets reveals that spreadsheets are a powerful tool for working with annuities.

A key finding with regard to teachers' mathematical knowledge, and which cuts across both parts of the thesis, is the importance of being able to move between compressed and decompressed forms of mathematics.

The study makes three key contributions. Firstly, a framework for MfT is proposed, building on existing frameworks in the literature. This framework is used as a conceptual tool to frame the study, and as an analytic tool to explore opportunities to learn MfT as well as the obstacles experienced by. A second contribution is the theoretical and empirical elaboration of the notion of *revisiting.* Thirdly, a range of theoretical constructs related to teaching and learning introductory financial mathematics are introduced. These include separate reference landscapes for the concepts of compound interest and annuities.

Keywords

mathematics for teaching, mathematics teacher education, teachers' knowledge, pre-service, secondary, financial mathematics, revisiting

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

_____

Craig Gavin Pournara
20th February 2013

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LIST OF ABBREVIATIONS

Abbreviation	Description
AB approach	Account balance approach
A-group	Attiyah's group
B.Ed	Bachelor of Education
CBMS	Conferenced Board of the Mathematical Sciences (US)
DBE	Department of Basic Education
DoE	Department of Education
FV	Future value
GT	Group tutorial
H-group	Hailey's group
IP approach	Individual payment approach
MfT	Mathematics-for-teaching
NCA	National Credit Act
OECD	Organisation for Economic Cooperation and Development
PCK	Pedagogical content knowledge
PV	Present value
SMK	Subject matter knowledge

CHAPTER 1
Introducing the study

1.1 Introduction

Craig: … so I am going to ask them which scenario will produce more interest, R500 at 8% compounded quarterly, or compounded every four months?

Helen: But why would you want to ask about every *four* months? It never happens in the real world!

Craig: Because some of the students aren't sure whether 'quarterly' means every *three* months or every *four* months.

Helen: But 'quarterly' *always* means every three months – end of March, June, September and December.

Thus went my conversation with Helen, a highly-paid and well-respected accountant, working in treasury finance of one of the "big four" South African banks. I was juxtaposing different compounding periods, and addressing directly an issue my students were battling with. To Helen this was bizarre - in banking terms "quarterly" means every three months, there is no such thing as compounding every four months. My intention was pedagogical; her response was a rather blunt reality-check. This is just one of many vivid memories of my early encounters with teaching financial maths. I was a novice and had endless questions for the now ex-highly-paid treasury finance accountant (who is also now my wife).

As I submit this thesis, it is fifteen years later and I have taught introductory financial maths to prospective secondary maths teachers six times. It has been a journey that continues to intrigue me – learning about finance, economics and mathematics, learning about students' struggles, and thinking about teaching financial maths in high schools in South Africa. To me there is something much more compelling and urgent about teaching and learning financial maths than calculus, geometry or trigonometry. No doubt it has much to do with the central role that money plays in our lives. It also has to do with the applied nature of financial maths. Equally compelling, and arguably more important, is the production of competent mathematics teachers – both in South Africa and across the world. And so I chose to design my doctoral study around my teaching of a financial maths course offered in 2008 to a group of pre-service secondary maths teachers at the University of the Witwatersrand.

The story of the research unfolds in this chapter. I begin by tracing my journey into financial maths and my work in developing mathematics content courses in a pre-service secondary maths teacher education programme. I describe how, at first, my attention was taken up by learning the mathematics and learning about finance. Only later was I able to consider the students, and their responses to what I was offering them. I then present the focus of the study, the research questions guiding the various parts of the study, and the structure of the thesis.

1.2 First encounter – first maths, then finance, then students

My first encounter with financial maths was in 1997, the year in which I began the transition from teacher of mathematics and computer science, to mathematics teacher educator. I was required to teach

a module on introductory financial maths to a small group of students who were doing their one-year postgraduate teaching certificate (now called PGCE[1]). This was completely new terrain for me. I knew far too little about the details of the world of finance to teach the course as well as I would have liked. I depended heavily on two text books – the Additional Maths[2] text book used in schools (e.g. Kitto et al., 1990) and one on introductory financial maths written for first year Commerce students (Young, 1993). Both books adopted a strongly mathematical approach, with little reference to the world of finance, and given my lack of expertise, I had little option but to stick closely to this approach. As I recall, I was so focused on getting to grips with the maths and making sense of it in relation to the world of finance that I was not able to attend to student thinking. So, apart from the confusion about quarterly compounding mentioned earlier, I can say little about what students struggled with or how they thought about the mathematical content of the module. For me the object of attention was the mathematics and its meaning in the world, and I wanted to "get that across" to students. So my priorities were maths, then the real world, then the students. I did not have the proverbial "headspace" to think about how my students might respond to the maths I presented to them.

Helen and I had many conversations about "how it works" in the banking world. I had many questions, such as "does simple interest really exist?", "when exactly is interest capitalised?", "where does this interest sit during the month because you don't see it in your bank account?". It was not easy to get this kind of information in books so I was fortunate to be romancing an "insider" of the banking world.

1.3 Second encounter – first students, then maths and finance

My next encounter with teaching financial maths came in 2002, with a very small group of final year undergraduate (prospective) secondary maths teachers[3]. The expectations of the course were similar to 1997 but the students were different. Whereas the 1997 group was predominantly white, and from middle to upper-middle class socio-economic backgrounds, the 2002 group was predominantly black and from much poorer backgrounds[4]. As a group, they were also mathematically weaker. In 1997 I had not recognised that I was taking students' knowledge of the broader financial context for granted – aspects such as everyday dealings with banks, saving for the future, repaying home loans. But the 2002 group did not have this knowledge. One of my most vivid memories of that class was my assumption that I could revise simple and compound interest in the first hour of the course, give the students a tutorial to complete in their own time, and move on to new content. In that hour I learned that the students had little knowledge of how interest works in banks and so the notions of simple and compound interest were merely mathematical calculations to them. As I worked with the students, I became increasingly aware of the importance of a basic knowledge of finance if one wants to go beyond substituting into formulae and calculating answers in the teaching of financial maths. I thus needed to help the students learn financial maths and to learn about finance more broadly.

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1 Postgraduate Certificate in Education

2 Additional Mathematics was offered in some (historically white) schools to learners who were mathematically talented. It was generally offered only in Grades 11 and 12, and was taken in addition to the six subjects required for the final school-leaving assessment.

3 These students were registered for a four-year undergraduate teaching diploma, which was soon to be replaced by the four-year undergraduate Bachelor of Education degree.

4 Despite the abolition of apartheid in South Africa in the early nineties, racial categorisations are still widely found in many aspects of South African society. They are reinforced by government policies of redress such as Black Economic Empowerment.

My planning and decision-making took far greater account of the students, and their learning of the subject matter than it had previously. In preparing for the classes in the first few sessions of the course I paid much more attention to opening up ideas and helping students see what was happening "inside" the simple and compound growth formulae from a mathematical point of view and relating this to what happens in the bank. It seemed that spreadsheets would be helpful to do this and so I developed basic spreadsheets for simple and compound growth, and also for annuities. Since the class was so small we had regular access to a computer laboratory and so I encouraged students to construct or re-construct most of the spreadsheets for themselves. The spreadsheets were effective and I often wondered to myself why I had not made use of them five years earlier.

The most hard-hitting realisation of the 2002 course was that these students had very little opportunity to learn about financial matters in their daily context, and so teaching them financial maths without dealing with the real world was irresponsible and immoral. Furthermore, given the emphasis in the new curriculum on integration, application and relevance of knowledge (Department of Education (DoE), 2002, 2003), the students would be expected to help learners[5] connect their learning of financial maths to the working world of finance. Pre-service mathematics teacher education programmes needed to take this matter seriously in their preparation of future maths teachers.

1.4 Third encounter – conceptualising mathematics courses for pre-service maths teachers

In my role as coordinator of the undergraduate maths teaching programme, I had the privilege of leading a team of colleagues[6] to conceptualise, design, implement and revise the mathematics and mathematics education components of a four-year undergraduate Bachelor of Education (B.Ed) programme. In this study my particular interest is in the mathematics content courses for secondary teachers. As a team, our fundamental concerns about the content courses could be summed up in the two-part question: "What mathematics should future secondary school teachers learn and how should they learn it?" We were convinced, through experience and through the growing literature on teachers' mathematical knowledge (e.g. Adler, 2004; Ball & Bass, 2000; Cooney & Wiegel, 2003; Davis & Simmt, 2006) that mathematics courses which focus only on covering large amounts of content at a rapid pace, with emphasis on definitions and procedures, do not provide adequate mathematical preparation for future maths teachers. Similarly courses that do not go beyond the content of the school curriculum are also inadequate.

It is obvious to say that maths teachers need to know the maths they will teach, and there is general agreement in the literature that they also need to know more than school maths (e.g. Cooney & Wiegel, 2003; Stacey, 2008). But there is little insight into "how" teachers need to know the maths they will teach and similarly little guidance on what constitutes the "more maths". In many parts of the world, the mathematical content preparation of secondary teachers is done by mathematicians in

[5] In post-apartheid South Africa, the term "learners" refers to scholars in schools. I use this term throughout the thesis to refer to those in schools. I use the terms "students", "student teachers" and "pre-service teachers" to refer to those in the course or in pre-service teacher education more generally. In the international literature "students" refers to those in schools and in post-school institutions. I capitalise on this ambiguity in cases where the issue under discussion applies both to "learners" and "students".

[6] The colleagues with whom I worked most closely on this programme were Jacques du Plessis, Erna Lampen, Bharati Parshotam and Louise Sheinuk. Others who were involved in various ways and to various extents were Jill Adler, Lynn Bowie, Julie Hannah, Caroline Long, Dale Taylor, Retha van Niekerk and Hamsa Venkat. These are the colleagues who most impacted my thinking at the time.

mathematics departments, and often there is limited interaction between mathematicians and the mathematics educators who take responsibility for the methodology/didactic components. Furthermore, the content and pedagogy of the mathematics courses themselves may not be open to negotiation because the courses may be offered to a wide range of other students too – not just prospective teachers (Conference Board of the Mathematical Sciences (CBMS), 2001). This explains, in part, the gap in the literature on what content should be included in maths courses for secondary teachers, although there are some attempts to address the gap (e.g. Conference Board of the Mathematical Sciences (CBMS), 2001) and the proposal of capstone courses (e.g. Artzt, Sultan, Curcio, & Gurl, 2011).

In designing the programme, an additional factor that strongly influenced our decisions was that many students entering our programme are underprepared for university mathematics. This is discussed in more detail in chapter 4. In much of the mathematics teacher education research reported from the developed world (e.g. Stacey, 2008), assumptions are made about minimum levels of mathematical knowledge of pre-service and practising secondary mathematics teachers. These assumptions do not necessarily hold in the South African context. In instances where pre-service teachers' mathematical knowledge is problematised, it is largely from the perspective of a traditional/reform distinction. For example, Cooney and Weigel (2003) portray pre-service secondary teachers' mathematical knowledge as being in deficit because their experiences of traditional mathematics teaching at school and in their undergraduate mathematics studies, is perceived to be inadequate for them to teach reform curricula.

Our programme thus needed to address multiple goals which included providing access to advanced mathematics, addressing students' poor knowledge of school mathematics, providing experiences of learning mathematics to develop students' confidence, to strengthen their mathematical identities, and to prepare them to deliver an ambitious reform-oriented school mathematics curriculum. And all this had to be done with little guidance from the literature.

We took the decision to offer a different suite of mathematics courses to the typical undergraduate mathematics programme, to focus on less content and study it more deeply, and to adopt a pedagogy that promoted investigative work, that valued students' contributions, and that encouraged and demanded class interaction. Throughout analysis of the data and the write-up of this thesis, I have been well aware of the benefits, limitations and unintended consequences of these decisions[7]. I reflect on some of these issues in chapters 11, 16 and 17, with particular reference to the Financial Maths course.

1.5 Fourth encounter – redesigning a Financial Maths course for pre-service maths teachers

As part of the suite of content courses, we chose to offer an entire course on introductory financial maths to third/fourth year students. As far as I know, we were the only university in the country to do this. The fact that we could make this decision was a consequence of the space for innovation that had been opened up in the restructuring of teacher education in South Africa (Parker & Adler, 2005). There were at least six reasons for our decision.

[7] See Pournara (2009a) for a discussion of the challenges and dilemmas that arose in the implementation of the programme.

1. *School curriculum*[8] – The focus on financial maths had increased in the new secondary school curriculum and all learners would be studying financial maths. We therefore needed to give our students the opportunity to learn the new content they would be teaching, particularly since many of them had not learnt financial maths during their schooling.
2. *Modelling and applications* – By taking a strongly contextualised approach, the course could provide opportunity to introduce students to mathematical modelling and the application of mathematics in daily life, thus broadening their experience of mathematics in line with current trends in mathematics more generally (Lovasz, 2008). In addition, given the emphasis on mathematical modelling in the school curriculum that was to be implemented at the time, students' own mathematical learning experiences would potentially contribute to their future teaching competence.
3. *Financial literacy* – The contextualised approach would also help students to become more financially literate, which would benefit them personally, and potentially be of benefit in their teaching. As noted earlier, this was of particular importance for me in the light of my experiences with the students in the 2002 course.
4. *Mathematical depth* – By designing a course focused on a small amount of mathematical content, there would be opportunity to work on the mathematical content in depth – going beyond derivations of formulae and simple substitutions, to exploring relationships between concepts, working with multiple representations as well as working with technology.
5. *Revisiting school maths* – There would be opportunity to revisit school mathematics such as percentage change, and linear and exponential growth, and for future teachers to study this work as adults, with insights beyond those they would have had when they studied the content at school (Cooney and Weigel, 2003).
6. *Focus on teaching and learning issues* – There would be opportunity within a *mathematics* course to pay more attention to teaching and learning issues, and thus to prepare teachers for the work of teaching financial maths concepts not only in schools but in community contexts more broadly. This would include dealing with learners' difficulties, understanding the progression in the curriculum, selecting and developing tasks, and linking financial maths to other work on functions and algebra.

I was responsible for conceptualising and designing the Financial Maths course. As already noted, there was little guidance in the literature for developing mathematics content courses for secondary teachers, although in some content areas, such as calculus and functions, one could draw on the literature about student thinking (e.g. Bezuidenhout, 2001; Zandieh, 2000) or even pre-service teacher thinking (e.g. Doerr, 2006; Even, 1990, 1993). However, given that introductory financial maths is an un(der)-researched area, there was no substantial research evidence on teaching or on student thinking to draw on. Inevitably I drew on my previous experiences. I needed to ensure that the course articulated carefully with the rest of the programme and that it made a contribution to pre-service teacher learning that went beyond the content of financial maths. The details of the 2008 course are described in chapter 4.

[8] As I write this thesis, the school curriculum in South Africa has undergone further revisions. The new curriculum and the available text books (at the time of writing) reflect far less emphasis on "reform" aspects such as modelling and reasoning.

Taken together these four encounters signify key points in a ten-year journey, which I began as a novice maths teacher educator who knew very little about financial maths. By 2008 when the doctoral study commenced, I had learned a great deal more about introductory financial maths and even more about mathematics teacher education. I had developed a growing awareness of the students as they engaged with financial maths, and I was also in a position to influence, in substantial ways, the direction of the undergraduate secondary mathematics teacher education programme in the University.

1.6 Focus and purpose of the study

The study brings together three areas of personal interest: my concerns about the preparation of future secondary maths teachers; my attempts to grapple with the issue of teachers' mathematical knowledge for teaching; and my fascination with learning financial maths.

There was much I wanted to investigate in the study and much I wanted to explore in teaching the course – too much for one doctoral study and too much for one semester-long course. I did not begin the study with a neat set of tightly-focused research questions. I hoped the research questions would emerge as the course progressed. They didn't. In fact the experience of teaching the course, and my close proximity to the data (or more precisely the *potential data resources*) gathered from the course served more to scramble my ideas than to focus them.

In my initial design, I intended to focus on three "mini-phases" of the course: working with "known maths" (i.e. school maths, specifically exponential growth and compound interest), dealing with "new maths" (annuities) and dealing with "advanced maths" (e.g. Newton's method). As the study progressed it became clear that dealing with all three phases was beyond the scope of a doctoral study. Later I began to question the value of including the school maths phase, and my doubt was deepened by the contrasting richness of the data on students' conceptions of annuities. However, letting go of the school maths component would essentially reduce the study to students' conceptions of annuities which would substantially diminish the attention to teacher education and to mathematics-for-teaching. As the notion of *revisiting* grew, it re-ignited my commitment to the school maths data, and to a story I wanted to tell – a story that I believed had not yet been told in the mathematics teacher education research arena. The consequence for the study, as reported in this thesis, is a dual focus: on revisiting school mathematics and on student's conceptions of annuities. In both foci I am concerned with students' mathematical knowledge for teaching. The two parts mean that the study is inevitably broad, and the thesis is lengthy. Taken together, the two parts provide an opportunity to identify similarities and differences when working with known maths and new maths, and thus provide richer insight into mathematics for teaching in general.

1.7 Structure of the thesis and research questions

As I complete the write-up of the thesis, I confidently make the statement: "learning mathematics-for-teaching includes revisiting school mathematics and learning new mathematics". I say it as if it structured my thinking, and this thesis, from the outset. This is not the case. In fact, the notion of revisiting as a theoretical construct is an *outcome* of the study.

As a consequence of the dual focus, the different parts of the study are guided by different research questions. Below I provide an overview of the study, the structure of the thesis and I indicate the broad research questions relevant to the different parts.

Part 1 deals with revisiting school mathematics, with particular focus on the mathematics of compound interest, and the associated concepts of percentage change and exponential growth. I focus on a range of critical incidents drawn from weeks 2, 3 and 4 of the course. These critical incidents are written up as vignettes in the chapters. Through the vignettes I explore two issues. Firstly, what might constitute mathematics for teaching compound interest, and secondly, what opportunities to learn mathematics-for-teaching might be possible in revisiting school mathematics. I do not claim that students in the course were provided with these opportunities, nor that they learned from the opportunities they were given. Essentially I am asking the question "what can we learn about revisiting school mathematics by studying an instance of revisiting?" In so doing I focus on what opportunities "present themselves" as a result of the co-construction by myself as the lecturer and researcher, and by the students. I also consider which aspects of mathematics-for-teaching are emphasised, and which are silent or absent.

Part 2 deals with students' conceptions of annuities, with particular focus on the obstacles encountered by students. I argue that this may be considered students' first substantial engagement with annuities. I focus on three different aspects of learning and working annuities. The first aspect concerns initial engagement with what I have called an *individual payment approach* to annuities, focusing on the growth of each payment over time. The second aspect concerns students' talk about time and their use of timelines. The third aspect relates to students' use of spreadsheets and what I call *spreadsheet thinking*. In each chapter I consider how students' concerns, difficulties and errors provide insight into mathematics-for-teaching annuities. This provides the common thread for the three chapters.

OVERVIEW OF THE THESIS

Chapter 1: Introduction to the study and research questions

Chapter 2: Review of some frameworks for teachers' mathematical knowledge and development of my framework for mathematics-for-teaching (MfT)

Chapter 3: Overview of perspectives on learning used in the thesis

Chapter 4: Overview of the Financial Mathematics course

Chapter 5: Research design and methodology

PART 1 – REVISITING SCHOOL MATHEMATICS

Research questions that guide chapters 6 – 11:

- What is mathematics-for-teaching compound interest?
- What opportunities and potential opportunities are made available for learning mathematics-for teaching compound interest by revisiting school mathematics?

Chapter 6: A reference landscape for compound interest for teaching – where I identify conceptions of compound interest, and key elements of teachers' knowledge of compound interest for teaching, as seen through the lens of the MfT framework which is elaborated in chapter 2.

Chapters 7 – 10: Four cases of revisiting school mathematics, each offering particular opportunities and potential opportunities for learning MfT of compound interest.

Chapter 11: Conclusion to part 1, drawing together findings from chapters 6 to 10, reflecting on MfT framework in use, and reflecting on revisiting school maths in pre-service maths teacher education.

PART 2 – LEARNING NEW MATHEMATICS

The general research questions that guide chapters 12 – 16 are:

- What is mathematics-for-teaching annuities?
- In what ways do students use an IP approach, timelines and spreadsheets to learn annuities?
- What errors and difficulties do students encounter as they work with an IP approach, timelines and spreadsheets to learn annuities?
- What insights may be gained into mathematics-for-teaching by investigating students' use of IP approaches, timelines and spreadsheets?

The latter 3 questions are elaborated in relation to the specific topic in the relevant chapter.

Chapter 12: A reference landscape for annuities for teaching – where I identify conceptions of annuities, propose two different approaches to annuities (an account balance approach and an individual payment approach), and identify key elements of teachers' knowledge of compound interest for teaching, as seen through the lens of the MfT framework

Chapter 13: Students' first encounter with an individual payment (IP) approach for working with annuities. I identify the obstacles students encountered in making sense of an IP approach and discuss what opportunities emerged for learning MfT of annuities.

Chapter 14: Students' talk about time and their use of timelines in working on annuities tasks. I also discuss the opportunities that emerged for learning MfT of annuities.

Chapter 15: Students' use of spreadsheets in group tutorial settings. I propose the notion of spreadsheet thinking and argue that spreadsheets and spreadsheet thinking provide useful resources for "getting inside" annuities and thus for teaching annuities in ways that go beyond substituting into formulae. Once again, I discuss opportunities that emerge for learning MfT of annuities.

Chapter 16: Conclusion to part 2, drawing together findings from chapters 12 to 15, reflecting further on the MfT framework in use, and reflecting on learning new maths in pre-service maths teacher education.

Chapter 17: Conclusions, reflections and recommendations

CHAPTER 2

Developing a framework for *mathematics-for-teaching*

2.1 Introduction

In the previous chapter I posed the questions: "what is mathematics-for-teaching compound interest?" and "what is mathematics-for-teaching annuities?" In this chapter I take a step back to pose the more fundamental question: "what is mathematics-for-teaching?" Given the lack of consensus on what constitutes teachers' mathematical knowledge, I begin by generating the construct theoretically, and then exploring and refining it through the study. In this chapter I propose a framework for mathematics-for-teaching, thus providing a starting point to answer the third question and thus to propose *what* is to be learned.

The issue of teachers' knowledge for teaching is both a theoretical and a practical problem within teacher education research and practice in general, and in mathematics teacher education research and practice in particular. At a theoretical level, there is little agreement on exactly what constitutes subject matter knowledge, and the distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK) is not as clear-cut as the terminology would suggest. At a practical level there is debate about how SMK is learned and the kinds of teacher education offerings that might best promote its learning.

I begin with a brief overview of the practical concerns regarding SMK, focusing specifically on pre-service secondary mathematics programmes. Thereafter I move to the theoretical problem, and discuss several frameworks for mathematics teachers' knowledge proposed in the literature. I then outline the framework I will use for the study, and why I choose to use the term *maths-for-teaching* rather than subject matter knowledge.

2.2 The practical problem of subject matter knowledge

Across the world there are a variety of models for the mathematical preparation of secondary mathematics teachers (Tatto, Lerman, & Novotná, 2009). The students in this study were enrolled in a four-year professional Bachelor of Education programme. This is described in more detail in chapter 4. However, internationally the most typical model for secondary teachers is an undergraduate degree in mathematics followed by certification in mathematics teaching, such as a post-graduate certificate in education (PGCE) although some suggest that this model does not adequately prepare secondary mathematics teachers for the mathematical demands of teaching (e.g. Gellert, 2009). This argument tends to be located within a discourse of dichotomies, characterised by assumptions that during their schooling pre-service teachers experienced only procedural/ traditional forms of mathematics which are "bad", and by contrast, they should (or will be expected to) teach mathematics in conceptual/reform ways which are "good" (Cooney & Wiegel, 2003). Similarly, it is assumed that typical university mathematics courses perpetuate the "bad" procedural/ traditional experiences of mathematics which are perceived not to be productive for teaching, and that teacher education must therefore overcome these experiences and prepare students for school teaching by promoting and modelling conceptual, reform-oriented pedagogies. It appears that implicit in this argument is the

assumption that pre-service secondary teachers have the necessary procedural knowledge, and that teacher education courses must complement this knowledge with greater emphasis on conceptual understanding. In the South African context, the assumptions about procedural competence do not necessarily hold.

Despite a lack of wide-spread agreement regarding structure and content of programmes, there is some agreement at the level of broad principles concerning the goals of programmes that prepare secondary mathematics teachers. The following list has been compiled from a variety of sources (Conference Board of the Mathematical Sciences (CBMS), 2001; Cooney & Wiegel, 2003; Stacey, 2008; Watson, 2008):

- university mathematics should illuminate high school mathematics by providing an advanced perspective on it;
- students should gain knowledge *of* mathematics beyond the secondary school curriculum;
- students should gain knowledge *about* mathematics, including its history, philosophy, recent developments and popularist ideas that stimulate public engagement in mathematics;
- students should develop a deep understanding of the core of school mathematics including the important procedures, algorithms and related skills;
- students should have experiences of doing mathematics that go beyond the formalism of typical undergraduate pure mathematics courses, to include investigations, problem solving and mathematical modelling; and
- students should develop the skills to continue their own learning of mathematics and thus engage in mathematical inquiry as lifelong learners.

As with all "wish lists", this is an ambitious set of expectations, which I shall take as a guide in developing the framework for this study.

2.3 The theoretical problem of subject matter knowledge

2.3.1 A problem of definition

The rise in interest in teachers' subject matter knowledge can be linked to Shulman's work (1986, 1987) on teacher knowledge. Shulman coined the term *pedagogical content knowledge* in an attempt to re-insert knowledge of subject matter as a key component of teacher's professional knowledge and to distinguish knowledge of the discipline from the knowledge required to transform that disciplinary knowledge for teaching. In his 1987 paper Shulman distinguished seven categories of teacher knowledge: content knowledge, pedagogical content knowledge, curriculum knowledge, general pedagogical knowledge, knowledge of learners' characteristics, knowledge of educational contexts, and knowledge of educational purposes and values. He and his colleagues paid specific attention to three categories of content knowledge: subject matter knowledge, pedagogical content knowledge and curriculum knowledge.

Shulman suggested that a teacher's SMK should "be at least equal to that of his or her lay colleague, the mere subject matter major" (p9), meaning that knowledge of the basic content should be the same as someone who had studied it although not for the purposes of teaching. He claimed that in addition

to this common baseline which might be described as "knowing that", teachers need to "know why". In mathematics, this knowledge extends beyond definitions, theorems and algorithms to understand the structure of mathematics as a discipline, the principles of conceptual organisation, the relative importance of particular mathematical ideas within the discipline, and the principles for mathematical inquiry - how new ideas are added to the body of knowledge and erroneous ideas are rejected.

For Shulman, PCK is primarily concerned with the transformation of knowledge into forms that make it comprehensible for others. He attempted to define it as "the most useful forms of representations ..., the most powerful analogies, illustrations, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make is comprehensible to others." (1986, p9).

Ball, Thames and Phelps (2008) note that there has been unprecedented take-up of the notion of PCK since the mid-eighties. However, they claim that, as a notion, it has remained underdeveloped and its usefulness has been hampered because it lacks clear definition and has little empirical foundation. In addition, some researchers (Chick, Baker, Pham, & Cheng, 2006; Huillet, 2007; Petrou & Goulding, 2011) argue that that the distinction between SMK and PCK is not easy to make, and McEwan & Bull (1991) reject the SMK-PCK dichotomy claiming that all SMK is pedagogic.

2.3.2 Multiple frameworks

In addition to the debate about its definition, there is a proliferation of perspectives and approaches to researching SMK, its acquisition and its deployment in teaching. While some begin in the practice of mathematics teaching (e.g. Ball, et al., 2008; Turner & Rowland, 2011), others begin with the nature of mathematical enquiry (e.g. Watson & Barton, 2011) and yet others emerge from theoretical considerations (e.g. Huillet, 2007). Several different frameworks have been proposed (e.g. Ball, et al., 2008; Chick, et al., 2006; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burril, & Sandow, 2006; Huillet, 2007; Turner & Rowland, 2011) all of which show that *teachers' mathematical knowledge is complex*. Another common feature of these frameworks is the *difficulty in separating mathematical and teaching aspects*. Based on factor analysis of their measures of mathematical knowledge for teaching, Hill, Ball and Schilling (2008) suggest that content knowledge for teaching is "multidimensional". Their findings lend further support to the problem of separation, showing that it is very difficult to separate mathematics from teaching when attempting to distinguish categories of knowledge or tasks of teaching.

I seek to build on the existing frameworks. The extent to which I may be successful in this quest, will be an outcome of the study. I take as starting point Even's (1990) framework and Huillet's (2007) reformulation and extension of the framework. I also draw on the work of Ferrini-Mundy et al (2006) and Ball and her colleagues (e.g. Ball, Bass, & Hill, 2004; Ball, et al., 2008). I begin therefore with a discussion of the main features of each of these frameworks and identify the aspects that pertain to my framework.

2.3.3 Frameworks of Even and Huillet

Even (1990, 1993) argues that SMK is topic-specific and provides what in her terms is an analytic framework of seven aspects of SMK for the concept of function. The seven aspects are listed in fig. 2.1.

1. Essential features of a concept
2. Different representations
3. Alternate ways of approaching a concept
4. Strength of the concept
5. Basic repertoire of examples and routines
6. Knowledge and understanding of the concept
7. Knowledge about mathematics

Fig. 2.1 Even's seven aspects of SMK

Huillet's (2007) critique of Even's framework is done from the perspective of the Anthropological Theory of Didactics (ATD). She thus works from the position that there is no mathematics outside the activity of an institution, and hence critiques the lack of systematicity in Even's framework. She argues that although most aspects of the framework are relevant, it is under-theorised and does not distinguish adequately between SMK and PCK. For example, Huillet argues that the category *essential features* contains both epistemological and cognitive dimensions and hence relates to both SMK and PCK. She argues that *different representations*, *alternate ways of approaching a concept* and *basic repertoire* also relate to both SMK and PCK since they all refer to teaching practice. She argues that the aspects *strength of concept* and *knowledge of mathematics* belong to scholarly knowledge and are thus "strongly SMK". Finally, she claims that *knowledge and understanding of a concept* is a different kind of category since it relates to the quality of teachers' knowledge whereas the others focus on mathematical activity.

I agree with Huillet's critique of Even's framework, with one exception. Huillet links *alternate ways of approaching a concept* with Chevallard's "first encounter" (Barbe, Bosch, Espinoza, & Gascon, 2005) and thus appears to consider it as an alternate way of *introducing* a concept. From my reading of Even (1990, 1998), she is not referring to introducing a concept. In her exemplification of ways of approaching functions she distinguishes *point-wise* and *global* approaches, and argues that one must choose the most appropriate approach based on the nature of the task. She also links the choice of an approach with the ability to move between representations.

Huillet concludes her critique of Even's framework by rejecting the SMK-PCK distinction on theoretical grounds and also on practical grounds since, for her, they are not easily distinguished in practice. She adopts Adler's term "maths for teaching" (MfT) (Adler, 2005; Adler & Davis, 2006) to refer to a combination of subject matter and pedagogical content knowledge, and draws on Ball's work (e.g. Ball, et al., 2004) to include a category relating to learners' conceptions and difficulties. She thus identifies six categories of a mathematics teacher's professional knowledge. Her naming of the categories draws heavily on the language of ATD. I have rephrased the names of the categories in fig. 2.2, but attempted to retain the essence of her meaning as far as possible by using her descriptions of notions such as *practical block* and *techniques*.

1. Scholarly mathematical knowledge of the concept - includes definitions, notation and symbols, properties and essential features of the concept; proof, and general knowledge about mathematics
2. Knowledge of the social justification to teach the concept (strength of the concept in Even's terms)
3. Knowledge of how to organise the students' first encounter with the concept
4. Knowledge of the core tasks and techniques, including a basic repertoire of key examples, and knowledge of different representations
5. Knowledge of how to explain and justify the key techniques and procedures in a way that is age-appropriate and takes cognisance of learners' previous knowledge
6. Knowledge of learners' conceptions and difficulties in learning a concept

Fig. 2.2. Huillet's six categories of a mathematics teacher's professional knowledge

2.3.4 Deborah Ball and her colleagues

Ball and her colleagues (e.g. Ball, et al., 2004; Ball, et al., 2008) have developed a practice-based theory of mathematical knowledge for teaching based on the mathematical work that teachers actually do on a regular basis. They view mathematics teaching as a form of mathematical problem solving, and are particularly interested in *how* teachers need to know mathematics for it to be usable in the day-to-day work of teaching. In order to explore this, they developed a "job analysis" for mathematics teachers, resulting in descriptive, rather than definitive, lists of the mathematical work entailed in teaching. Following from this job analysis, they propose a typology of teachers' knowledge which elaborates Shulman's SMK-PCK distinction. (See fig. 2.3)

Within the category of SMK, they distinguish common content knowledge (CCK) from specialised content knowledge (SCK). CCK is mathematical knowledge used "in a wide variety of settings" (p. 400) and not unique to teaching. By contrast, SCK "is mathematical knowledge not typically needed for purposes other than teaching" (p. 400).

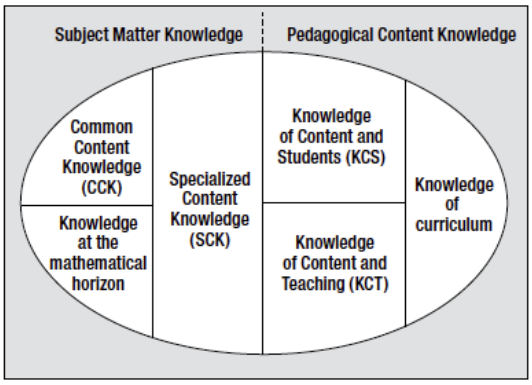


Fig. 2.3 Domains of mathematical knowledge for teaching (Hill, Ball & Schilling, 2008, p. 377)

In the context of financial maths, an example of CCK is knowledge of compound interest (and its effects) which might be considered to be knowledge that all adults should possess. By contrast an example of SCK is knowing that a nominal annual interest rate of 9.5% compounded monthly will produce a recurring decimal, and rounding of this decimal will impact the future value calculation and thus illustrate the importance of working with maximum precision in calculations.

Horizon content knowledge (HCK) concerns the trajectory of a particular topic through the curriculum and ultimately its relationship to advanced mathematics. In financial mathematics the notion of continuous compounding begins with multiple compounding periods in the school curriculum and extends to a knowledge of limits to prove the formula for continuous compounding at an advanced level.

Ball et al (2008) also propose three sub-categories of PCK: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum. These three categories reflect the intertwined nature of mathematics and pedagogy in teachers' knowledge. In the context of financial maths, KCS is exemplified in the opening dialogue in chapter 1 which points to students' confusion about the meaning of "quarterly" – does it mean every three months or every four months? An example of KCT involves deciding about sequencing of content, for example does one introduce present value or future value of annuities first? The answer to this question may depend on the order that students find more meaningful, and thus reinforces the inter-related nature of the typology. Knowledge of curriculum includes knowing what content is taught in each grade, for example knowing that compounding periods shorter than a year are only introduced in Grade 11 in the Mathematics curriculum in South Africa (and yet are introduced in Grade 10 in the Mathematical Literacy curriculum).

Ball et al (2004) identify four components of mathematics teachers' knowledge which they suggest cut across the different categories of their typography. This knowledge gives teachers the ability to:

- *unpack* or *decompress* mathematics, making it accessible to learners. In fact they suggest that the ability to unpack mathematics may be one of the distinctive features of teachers' mathematical knowledge;
- *connect* mathematical content to other topics within a grade, and to related content in previous or higher grades;
- *retain the integrity of mathematics* while still providing grade-appropriate explanations and examples that minimise the chance of learners developing misconceptions such as "multiplication makes bigger", "you can't take away a larger number from a smaller one", or "tangents only touch a curve, they never intersect it"; and
- pay attention to the features of *mathematical practice*, such as representing, defining, reasoning and proving.

While the typology appears to take the SMK-PCK divide as given, Ball et al (2008) acknowledge that the boundaries are frequently blurred.

2.3.5 Ferrini-Mundy and colleagues, and the KAT project

In the Knowledge of Algebra Teaching (KAT) project, Ferrini-Mundy et al (2006) propose a framework for conceptualising mathematical knowledge for teaching that may be seen as an attempt to coordinate and adapt the work of Even and Ball into a single framework for secondary algebra. They propose a 6×6 matrix with categories of mathematical knowledge along one axis and tasks of teaching along the other, as shown in fig. 2.4.

These categories reflect some of Even's categories, such as *core content knowledge* and *represent-tations*. The category *applications and contexts* is not explicit in either Even or Huillet's frameworks, yet is important for my study given the applied nature of financial mathematics. The tasks of teaching reflect similar emphases to the tasks identified by Ball et al (2004), and as with Ball et al, they identify three overarching features that cut across categories and tasks: *decompressing*, *bridging* and *trimming*. In essence, these features correspond to the terms used by Ball, i.e. unpacking, connecting and retaining the integrity of mathematics while simplifying it in appropriate ways for the audience.

However, as with Even, their framework is under-theorised and some of the categories of mathematics include aspects of teaching and learning, thus reflecting once again the difficulty of separating SMK and PCK. For example, in the category *content trajectories*, they list mathematical aspects such as: an understanding of the origins of core concepts and procedures, how these develop further in higher grades and in advanced maths; being able to distinguish big ideas from less important ideas, and how the main ideas are connected in a bigger mathematical picture. But they also refer to two aspects which relate to mathematics and teaching: knowing what order is most suitable to *present different aspects so that learning might be more efficient*; and knowledge of a variety of alternate approaches *to teach a sequence of ideas*.

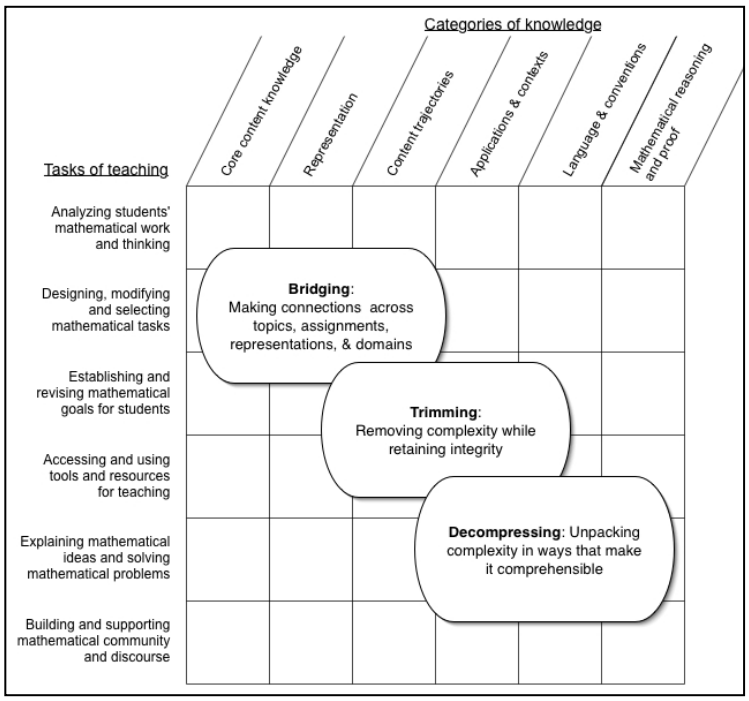


Fig. 2.4 Framework for knowledge for algebra teaching (Ferrini-Mundy et al, 2006, p. 4)

A second example comes from the category *language and notation*, where they refer to knowing possible difficulties that learners may have with language and notation. This links to Ball et al's knowledge of content and students which is classified as PCK.

It is worth noting that two of the tasks of teaching in the KAT framework are included in Huillet's framework: *explaining mathematics*, and *knowledge of learner thinking*.

2.4 Central features of teachers' mathematical knowledge

Despite the diversity of perspectives on teachers' mathematical knowledge, and the range of approaches for researching it, there is agreement on a number of issues concerning SMK. It is connected, topic-specific knowledge that cannot be pre-packaged and transmitted to teachers. It is acquired both through formal study (both pre-service and in-service), as well as in practice. But it is generally not learned within typical university mathematics content courses (since it is not a goal of these courses).

Given the difficulty in making a clear distinction between SMK and PCK, and following Huillet (2007), I shall adopt the term *mathematics-for-teaching* (MfT). MfT is thus an amalgam of SMK and PCK, but I am not concerned with categorising elements of teachers' knowledge into either or both of these categories. The following central features underpin the framework for MfT I shall propose. I organise the features in three clusters: What is MfT? How is it shaped? How is it learned?

- What is MfT?
  - It is a form of applied mathematics that is implemented to solve the problems of mathematics teaching; (Adler, 2005; Ball, et al., 2004; Bass, 2005; Stacey, 2008; Stylianides & Stylianides, 2010);
  - It consists of knowledge of school mathematics and of advanced mathematics; (Bromme, 1994; Stacey, 2008);
  - It is topic–specific (Adler & Ball, 2009; Even, 1990, 1993) ;
  - It is characterised by depth, breadth and connectedness (Ma, 1999);

- How is MfT shaped?
  - It is culturally determined with reference to the practice of teaching in schools and the kinds of knowledge needed for that particular practice (Stigler & Hiebert, 1999);
  - It is institutionally determined, as constituted by the institution in which it is acquired and used (Bingolbali & Monaghan, 2008; Chevallard, 1992; Huillet, 2007);

- How is MfT learned?
  - It is acquired through direct learning outside of the practice of teaching as in pre-service/in-service programmes and also in practice;
  - Its acquisition involves sense-making (Venkat & Graven, 2008), working with multiple representations, engaging with the mathematical ideas of others, and communicating one's own ideas.

In the remainder of this chapter I elaborate the "what" of MfT and propose a conceptual framework for studying MfT. In chapter 3, I discuss learning of mathematics in general and of MfT in particular.

2.4.1 A note on terminology

Before proceeding, two issues regarding terminology require attention. I shall refer to "mathematical entities" that are to be taught and learned. These entities include concepts, formulae, theorems, proofs, algorithms, and representations. Based on this definition, a concept is a mathematical entity but a mathematical entity is not necessarily a concept[9].

The second issue concerns the notion of topic-specific knowledge. There is increasing recognition that MfT is topic-specific (Adler & Ball, 2009). However, when one looks closely at the language used, there is slippage between the terms *topic* and *concept*. For example Even (1990, p. 253) refers, seemingly interchangeably, to both *topic* and *concept* in consecutive sentences.

> "As a result of this analysis, the following seven aspects seemed to form the main facets of teachers' subject matter knowledge about a specific mathematical *topic*.
>
> One aspect of the framework deals with the concept image, paying attention to the essence of the *concept*."
>
> (Even, 1990, p. 253, italics mine).

Huillet also tends to slip between the two terms although she appears to consider the "topic of limits" as a section of the curriculum and the "concept of limit" as a mathematical entity. We can consider compound interest as a *topic* to be taught. But in order to teach it, teachers need knowledge of the

[9] The term "entity" is vague and inadequate but I use it in an attempt to avoid the linguistic mire around mathematical objects and the objectification of mathematical knowledge (e.g. Sfard, 2008). In this study I do not enter this philosophical debate.

concept of compound interest. In contrast, we might also talk about the topic of *factorisation* but would likely not consider it as a concept. In this case the concept is *factor*. I do not see an easy resolution to this issue. For the purposes of this study, I shall refer to the *concept of compound interest* and the *concept of annuities.*

2.5 Proposing a conceptual framework for studying MfT

In constructing my framework, I began with Huillet's framework of six categories, paying particular attention to the ways in which she had drawn on Even's seven aspects. For the most part, I have not used Huillet's category-names since they are couched in the language of ATD and assume some knowledge of ATD. However, for the most part I have retained the distinctions she makes between categories. In some instances I have used Even's terms. Having done this, I extended the framework to include *modelling and applications* (from the KAT framework). This produced the following eight aspects:

1. Essential features of the concept
2. Relationship of the concept to other mathematics
3. Modelling and applications
4. Mathematical practices
5. Basic repertoire
6. Different teaching sequences and approaches
7. Explanations
8. Learners' conceptions

The descriptions of each aspect given below are an elaborated and refined version of the initial descriptions. This has come about as a result of further data analysis. One example of an extension is that *mathematical communication* was not explicitly listed in the original description of *mathematical practices*. In addition to extending and refining the eight aspects, based on the analysis it became clear that the framework did not deal adequately with personal, social, financial and economic aspects of finance. Initially these were spread across several of the original aspects but this proved inadequate. I therefore added a ninth aspect, *knowledge of contextual issues of finance*, to capture these issues in a more coherent way, and subsequently sub-divided it into two components, *financial concepts and conventions* and *socio-economic issues and financial literacy*.

2.5.1 Descriptions of the nine aspects of the framework

In this section I provide a general description of each aspect of the framework. In chapters 6 and 12 (respectively) I relate each aspect to the concepts of compound interest and annuities.

1. Knowledge of the **essential features** of the concept
   I distinguish between identifying the entity (*what* it is) and the associated techniques (*how* to work with it).

   a) *Identifying the entity* includes definitions, properties, conventions, language and terminology, axioms and theorems. These indicate instances of the entity as well as non-instances. For example, one of the ways of characterising compound interest is to show how it differs from simple interest. Duval (1999) argues that the only way to access a mathematical entity is through representations of it. Thus I include representations as part of this aspect whereas Even

[10] Duval (1999) uses the term "mathematical object".

(1990) and Ferrini-Mundy et al (2006) consider representations as a separate aspect. I include the standard representations of algebraic, numeric, and graphical, as well verbal, geometric and pictorial (such as "function machine" diagrams). It is possible to have different forms within a particular representation (or register in Duval's terms) for example $y = ax^2 + bx + c$ and $y = a(x - p)^2 + q$ are both algebraic representations of the quadratic function.

b) *Ways of working with the entity* includes deriving formulae, proving theorems, and executing standard algorithms to solve typical problems related to the concept. I also include manipulating representations such as producing equivalent forms within a particular representation and shifting between representations. Davis and Simmt (2006) argue that such shifts between representations are a key competence for teachers. I also include Even's category of alternate ways of approaching a concept here. I do so because I have absorbed her aspect of *representations* under *essential features* and she argues that moving between representations is closely linked to alternate ways of approaching a concept.

2. Knowledge of the **relationship of the concept to other mathematics**

In this aspect I distinguish between knowledge of related concepts at different levels, as well as sub-topics and sub-concepts. In the context of compound interest, percentage change is a lower level concept since it underpins the notion of interest. In the context of annuities, perpetuities is a higher level concept. As I attempted to operationalize the framework, it became necessary to distinguish between general topics in mathematics and those specific to financial maths. I elaborate these distinctions in the discussions of compound interest and annuities in chapters 6 and 12.

3. Knowledge of **modelling and applications** of a concept

This aspect focuses on the use of concept in real world applications and how real world scenarios might be modelled using the concept. It is not included in Even (1990) or Huillet's (2007) frameworks. Ferrini-Mundy et al (2006) and Stacey (2008) argue for knowledge of modelling and applications as an important element of teachers' mathematical knowledge for teaching. Watson (2008) includes modelling within *mathematical practices* (see below). I have chosen to separate it from other mathematical practices given the applied nature of financial mathematics and thus the prevalence of modelling in the study. Of course, if a curriculum does not focus on modelling and applications, then the need for this knowledge in teaching is limited which is once again a reflection of the cultural nature of teachers' knowledge.

Financial maths constitutes a well-developed area of the application of mathematics and has become an area of scholarly mathematical activity. At this level it is far more than the application of exponential growth and geometric progressions, which is how financial maths has been treated in the South African school curriculum in the past. In the context of this study, this component of the framework focuses on how the appropriate mathematics is used to model compound interest and annuity-based situations, with particular attention to how daily and monthly events in the bank are simplified and mathematized to predict yields and calculate

[11] I use "representations" as one might find the term used in mathematics text books in the sense of "multiple representations". I do not enter the domain of semiotics.

[12] Duval (2006) emphasises the distinction between producing equivalent forms within a particular representation and shifting between representations. He refers to these as "treatments" and "conversions" respectively. The English terms are not helpful and so I do not make use of them.

loan amounts.

4. Knowledge of **mathematical practices**

This aspect relates to Shulman's initial conception of teachers' content knowledge regarding how knowledge is established and grows in a particular discipline. Watson (2008) provides the following "exhausting" list of mathematical practices typical of academic mathematics:

> 'Doing mathematics' is predominantly about empirical exploration, logical deduction, seeking variance and invariance, selecting or devising representations, exemplification, observing extreme cases, conjecturing, seeking relationships, verification, reification, formalisation, locating isomorphisms, reflecting on answers as raw material for further conjecture, comparing argumentations for accuracy, validity, insight, efficiency and power. It is also about reworking to find errors in technical accuracy, and errors in argument, and looking actively for counterexamples and refutations. Mathematics is about creating methods of problem-presentation and solution for particular purposes, tinkering between physical situations and their models, and it also involves proving theorems. (p. 4)

However, she argues, as do many others (e.g. Burton, 1999, 2004; Moreira & David, 2008), that the practices and foci of school maths differ considerably from the practices of academic mathematics and mathematicians. For example, she claims that "[t]he core activity in school mathematics is to learn to use mathematical tools and ways of working so that these can be used to learn more tools and ways of working later on" (p. 6).

I work from the position that mathematics takes different institutional forms (Chevallard, 1992; Huillet, 2007) and that these differences may be found within and across the school-university divide (Bingolbali & Monaghan, 2008; Boaler, 1997).

It is therefore important to recognise the different mathematical practices that intersect in this study: the practices of scholarly mathematics, the practices of mathematics teacher education and the practices of school mathematics. The latter two are both sites where scholarly mathematics is recontextualised (Bernstein, 1996) and developed, and both have different purposes, activity and forms of participation[13].

With respect to this framework, it may therefore be appropriate to distinguish knowledge of scholarly mathematical practices from knowledge of school mathematics practices. I have chosen not to focus on this distinction. I work with the assumption that the aspect of *mathematical practices* extends beyond working with a particular concept (or topic), to mathematical practices more broadly. It might then be argued that this aspect would remain constant across different concepts although, for example, it may have slightly different emphases for algebraic concepts compared with geometric concepts.

5. Knowledge of a **basic repertoire** of key tasks and examples

Even (1990) argues that a basic repertoire should consist of "powerful examples that illustrate important principles, properties, theorems, etc." (p. 525). She suggests that such a repertoire should be readily available to a teacher and that pre-service teachers should learn such a repertoire since they will be required to teach it. From my perspective it involves knowing a range of key tasks and examples that will foreground the essential features of a concept – to enable learners to identify the concept and to provide access to typical ways of working with it.

[13] See Watson (2008) and the entire issue (vol 3 (2)) of the journal *For the Learning of Mathematics* which is devoted to the issue of disciplinary mathematics and school mathematics. See also Moreira and David (2008) for a fascinating discussion of conflicts between the knowledge of academic and school mathematics that teachers require.

This aspect is linked to curriculum knowledge (Shulman, 1986) since the selection of examples and tasks is determined by the requirements of the curriculum.

6. Knowledge of **different teaching sequences and approaches**
   I adapt this aspect from Ferrini-Mundy et al (2006) (and not Even (1990)) to focus on knowledge of different teaching sequences for a concept and knowledge of different approaches for teaching a sequence. Earlier I argued that their category *content trajectories* contained both mathematical and teaching components. The mathematical components I identified in their category are subsumed under my category *relationship of concept to other mathematics*. This aspect includes the first encounter from Huillet's framework.

7. Knowledge of how to **explain** a concept and related routines
   I shall refer to this aspect as *explanations*, and I distinguish it from *basic repertoire* to preserve Huillet's distinction between the practical block and the technical block in ATD. Also, it is possible to have knowledge of the basic repertoire and yet be unable to produce an adequate explanation of the essential features of a concept. Knowledge of how to explain a concept assumes knowledge of the relevant mathematics, the learner, and the inter-relationship between the two, such as what aspects learners will find easy and difficult, and how previous knowledge might impact on their grasp of a new concept. I discuss explanations in more detail in chapter 4 where I will also discuss the notion of unpacking.

8. Knowledge of **learners' conceptions** when studying a concept
   This aspect is drawn from the work of Ball and her colleagues (Ball, et al., 2004; Ball, et al., 2008), and Kazima and Adler (2006). It is also included in Huillet's framework. It includes knowledge of learners' errors and difficulties with regard to the concept, knowledge of how learners' new knowledge may interfere with their existing knowledge, and how learners' prior knowledge may impact the learning of the new concept. The topic-specific elements of this aspect are discussed in later chapters.

9. Knowledge of **contextual issues of finance**
   This aspect is specific to my study and its focus on financial mathematics. It includes social, economic and financial issues that constitute financial literacy. It seems likely, however, that any focus on an applied area of mathematics may require a similar knowledge component that deals with contextual issues. For example a focus on probability in relation to weather patterns will require knowledge of climate. As I operationalized this component it became necessary to distinguish financial concepts from socio-economic aspects and financial literacy

   a) *Knowledge of financial concepts and conventions* - Financial aspects include inflation, exchange rates, financial indices (all of which are included in the school mathematics curriculum). Financial conventions relate to banking practices such as how interest is calculated. In chapters 6 and 12, I discuss other examples and raise a concern about how much knowledge of finance can realistically be expected of maths teachers.

   b) *Knowledge of socio-economic issues and financial literacy* - Socio-economic aspects might best be described as knowledge of general financial literacy coupled with a critical engagement concerning how credit, debt and poverty impact people's lives. In the South African context,

with high levels of unemployment, it is important to include knowledge of credit and debt, of who can get credit and why, of the resulting impact on those who can't get credit, and those who fail to repay their debt. It also includes the impact of compound growth, both positive and negative, at various levels, ranging from personal finances to national debt

2.6 Conclusion

In this chapter I have reviewed the key frameworks of teachers' mathematical knowledge from the literature, and explained my decision to adopt the term *mathematics-for-teaching* (MfT) to avoid the difficulty of distinguishing SMK from PCK. Working from the existing frameworks, I have proposed a conceptual framework for MfT consisting of nine aspects. These are represented in fig. 2.5. At this point it seems reasonable to consider three clusters[14]:

Mainly mathematical:	Essential features, relationship to other mathematics, modelling and applications, and mathematical practices
Mainly pedagogical:	Basic repertoire, different teaching sequences and approaches, explanations, and learners' conceptions
Contextual:	Contextual aspects of finance

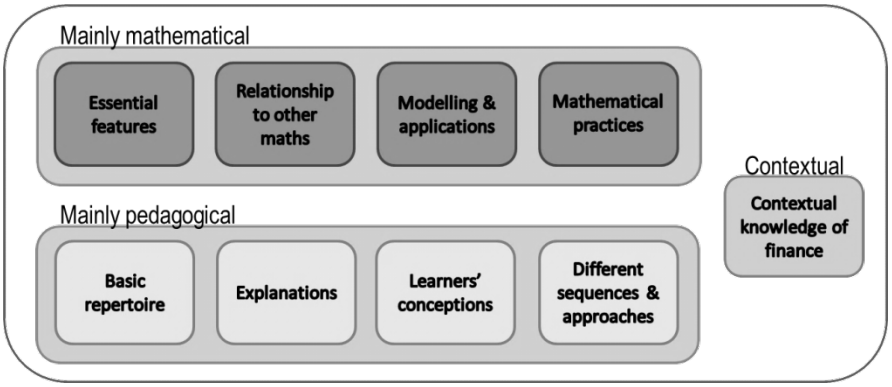


Fig. 2.5 Initial diagrammatic representation of MfT framework

Possible relationships between the aspects, as well as gaps and overlaps, will emerge through the study. These insights will lead to further structuring and refinement of the framework which will be discussed at the end of parts 1 and 2, and in the final chapter.

[14] There is resonance here with Chick et al's (2006) identification of three clusters: clearly PCK; content knowledge in a pedagogical context and pedagogical knowledge in a content context.

CHAPTER 3
Perspectives on learning mathematics

3.1 Introduction

Learning is a function of *what* is to be learned and *how* it is learned. In this study, I work essentially from a social constructivist position (Vygotsky, 1978), although I draw on work from a Piagetian tradition (Brousseau, 1997; Fischbein, 1999; Sfard, 1991) and from social practice theory (Lave & Wenger, 1991). In this chapter I shall argue that learning is thus concerned both with *acquisition* and *participation*

In chapter 2, I proposed a framework for MfT consisting of nine aspects. This constitutes the "what" that is to be learned. In this chapter I deal with perspectives on "how" these aspects might be learned. I begin by motivating for the need to consider learning as both *acquisition* and *participation*[15] (Sfard, 1998). Thereafter I focus on elements that assume an acquisitionist perspective and show how some of these may be considered to move towards participationist perspectives. In the latter part of the chapter, I consider "participationist issues" drawn from social practice theory.

3.2 The dual nature of learning

Sfard (1998) distinguishes between acquisition and participation metaphors (or perspectives) for learning. An acquisitionist perspective is associated with cognitive theories of learning such as Piaget (1970) and Vygotsky (1978, 1986), while a participationist perspective is typified by social practice theory, for example Lave and Wenger (1991) and Wenger (1998).

Acquisitionist perspectives are carried in terms and phrases such as *concept development, conception, misconception, meaning, sense, mediation, knowledge construction,* and *internalisation*. They are concerned with acquiring knowledge as a commodity. While this perspective reflects what has long been the dominant discourse on learning, it does not address what Sfard terms "foundational quandaries" (p. 7) such as the learning paradox ("how can we want to learn something that we don't already know?") and another quandary that emerges from constructivism: if we construct our *own* knowledge, how do we account for the large similarities in the knowledge that we all develop?

By contrast, participationist perspectives focus on doing and being, and are typically reflected in terms such as *practice, discourse, communication, participation, acting* and *becoming*. Such perspectives perceive learning as becoming a particular kind of participant in a particular practice. They thus promote democratic, contextualised and collective learning. However, participationist perspectives do not address the shortcomings of acquisitionist perspectives, rather they side-step the issues by rejecting, for example, the internal-external dichotomy of cognitive and socio-cultural theories.

[15] Sfard has made considerable shifts in her theoretical position, from Piagetian, to socio-cultural, to what now might be called "radically participationist" (e.g. Sfard, 2008). In this study, I draw on what I shall refer to as her "early" and "middle" work. The term "early" refers to work from the early and mid-nineties (e.g. Sfard, 1991, 1992) in which she took a strongly Piagetian position. The "middle" period refers to work from a Vygotskian and increasingly discursive position (e.g. Sfard, 2000). Her 1998 paper reflects her move towards participationist perspectives.

A major weakness of participationist theories for the learning of mathematics is that they do not offer tools to deal with the mathematical content to be learned. Consequently the mathematics tends to be backgrounded in studies that adopt a participationist perspective. For this reason, participationist theories of learning alone are inadequate for my study. Following Sfard (1998), I consider acquisition and participation perspectives on learning as complementary. Indeed the MfT framework proposed in chapter 2 assumes both perspectives. For example, *essential features* assumes an acquisitionist perspective while *mathematical practices* includes a participationist perspective in that students will gain knowledge of mathematical practices through their own participation in and reflection on these practices.

3.3 Learning MfT as moving between operational and structural conceptions

Ball and Bass (2000) argue that the compressed and abstracted forms of mathematics valued by mathematicians are inadequate for teaching, and that teachers need to be able to decompress mathematics to make it accessible and sensible for learners. Part of this decompression involves "working backwards" from reified mathematical objects to the processes that gave rise to them. I draw on Sfard's early work to elaborate this idea. I then move to the Vygotskian notion of pseudoconcepts and consider the importance of mediation in the shift from pseudoconcepts to concepts.

3.3.1 Sfard's theory of reification

Sfard (e.g. Sfard, 1991, 1992; Sfard & Linchevski, 1994) initially proposed a *cognitive* theory of learning mathematical concepts. Despite the considerable shifts in her theoretical position, as noted earlier, her primary focus has remained on learning and the objectification of knowledge. For the purposes of this study, and particularly for part 2, I draw on her earlier work as I investigate the obstacles students encounter in learning annuities. I now describe the key ideas in her cognitive theory of reification, and show that it is well-suited for working with the reification of mathematical entities[16] in the context of introductory financial maths.

Sfard (1991) argues that mathematical entities can be viewed as both processes and objects. She calls these conceptions *operational* and *structural* respectively, and argues that learners must be able to work with both conceptions, and to shift between them, in order to succeed at mathematics. An operational conception is "dynamic, sequential and detailed" (p. 4) focusing on processes, actions and algorithms, whereas a structural conception is "static ..., instantaneous and integrative" (p. 4). The two conceptions are complementary hence the need to be able to shift between them. Sfard argues that operational conceptions precede structural conceptions, and that a structural conception is not always easy to achieve.

According to Sfard (1991), learners proceed through a three-stage hierarchy in developing a structural conception: interiorisation, condensation and reification[17]. Interiorisation is a gradual process during which the learner becomes skilled at performing operations. A process has been interiorised if the learner can carry out the operations through mental representations without actually performing them. Condensation is also a gradual process where the learner develops the ability to refer to the process in terms of its inputs and outputs without having to go into the details of the manipulation. The learner

[16] As indicated in chapter 2, I use the generic term "entities" to avoid the multiple uses and meanings of "object" in instances such as these.

[17] I provide only a brief description of the three stages since I do not make substantial use of them in the study.

develops an increasing ability to think about the process as a whole, to combine it with other processes, and to shift between different representations. During this stage the concept is still tightly connected to a particular process. In contrast to the previous two stages, the final stage of reification is a sudden ability to see a familiar process as an abstract object, separate from the process that created it. This is the indicator of a structural conception. Thereafter the new object can be used as an input to a higher level process.

It is possible to show three levels of processes and objects within introductory financial maths. I illustrate this with reference to a scenario where a principal amount of R400 gains interest at 10% p.a., compounded annually for 5 years. See fig. 3.1.

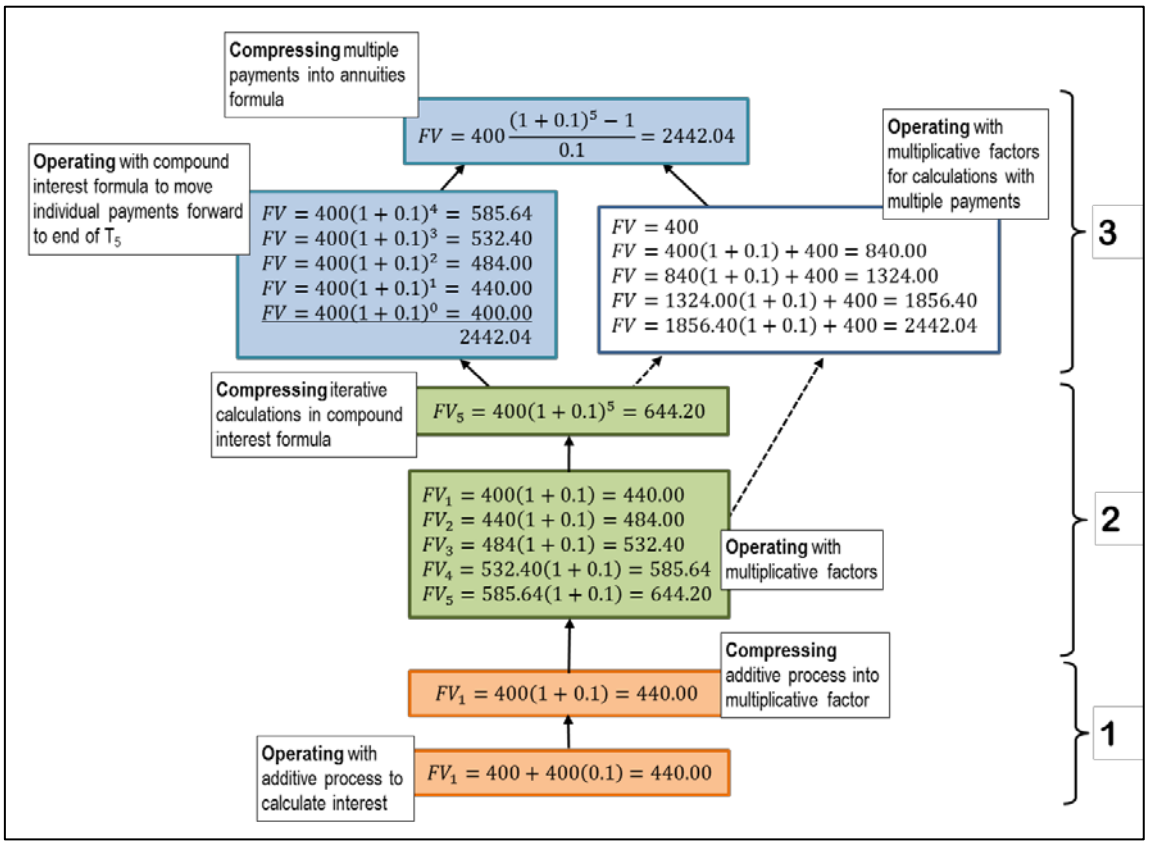


Fig. 3.1 Three levels of processes and objects

The first (bottom) level involves the process of adding interest to the principal amount (R400) using an additive approach. The general expression $P + Pi$ can be factorised, giving $P(1 + i)$. Here $P(1 + i)$ can be seen as the outcome of the process of factorising or it can be seen as the object to multiply by P to determine the new future value. Seen as an object, it becomes a factor in a multiplicative process to produce a new future value.

At the second level, a sequence of five iterative calculations (process) generates the future value of R400 at the end of five years. This process can be compressed in the compound interest formula, thus producing a new mathematical object at the second level.

In shifting from single payments to summing multiple annual payments, we make a shift to annuities (at the third level). For the purposes of this explanation I assume payments are made at the end of the

year. In order to determine the future value, we may take two different approaches. The first approach models the growth of the five individual payments, moving each forward to T_5 and then summing their future values. The second approach tracks the account balance at the end of each year, where interest accumulates before the payment is made[18].

In the first approach, the five compound interest calculations can be summed to produce a future value of R 2686.24 at the end of year 5. These individual payments and their accumulated interest can be compressed into the formula for the future value of an ordinary annuity, thus generating another new mathematical object at the third level. The second approach provides an alternate route through a series of iterative calculations where the new payment is added to the account at the end of each year. This sequence of calculations also leads to the annuities formula although it requires more careful algebraic manipulation than the first approach.

The derivations of the formulae are discussed in more detail in chapters 6 and 12 where I elaborate reference landscapes for compound interest and annuities respectively. I draw substantially on the operational-structural distinction in the analysis in chapters 10, 13 and 15.

3.3.2 Pseudostructural conceptions

In her early work Sfard (1991) argued that one of the difficulties of reification lies in the fact that a learner cannot "convert" a lower level process into an object until s/he has to perform a higher level process on it. This leads to the "vicious circle of reification" (p. 31) because higher level interiorisation is not possible without an object on which to perform the process, but the lower level object will not emerge until required by a higher level process. The relationship is thus dialectic and suggests that lower level reification must occur simultaneously with higher level interiorisation. If learners are unable to reify a process, they may develop a *pseudostructural conception* (Sfard and Linchevski, 1994) which occurs because they have to devise a means of treating the should-be object as an object, and they tend to do so by associating the object with its representation, rather than with its referent. When learners create such a *pseudo-object*, they sever the object from the processes that gave rise to it, thus it may lose meaning in relation to its origins but is not meaningless for the learner.

In her later work from a socio-cultural perspective, Sfard (2000) argues that symbols and objects create each other and that, in working with a new symbol, the mathematical object signified by the symbol, will emerge and come to have meaning for the learner. In a similar vein, Berger (2004) elaborates Vygotsky's (1986) work on preconceptual thinking and shows that pseudoconcepts are a necessary part of learning. Vygotsky (1986) argues that the use of pseudoconcepts enables learners to make the transition from complexes to concepts. The *use* of pseudoconcepts resembles that of true concepts. However, the *thinking behind* pseudoconcepts is complex-type thinking where the links between elements of a pseudoconcept are associative and factual, not logical and abstract. Nevertheless, the learner uses the pseudoconcept *as if* it were a true concept. Berger (2002) argues that:

> "The use of pseudoconcepts is ubiquitous in mathematics and is analogous to the way in which a child uses a word in conversation with adults before he fully understands the meaning of that word. Pseudoconcepts occur whenever a student uses a particular mathematical object in a way that coincides with the use of a genuine concept, even though the student has not fully constructed that concept for himself." (p. 43)

[18] I refer to the two different approaches as individual payment (IP) and account balance (AB) respectively. These are described in detail in chapter 12.

It is through the use of pseudoconcepts, and through the mediation by more knowledgeable others, that the learner (potentially) comes to transform the pseudoconcept into a true concept. I shall not elaborate further on this aspect of Berger's work since I do not focus in depth on pseudoconcepts in the analysis.

However, the Vygotskian perspective, and Berger's elaboration of it for learning mathematics, exemplifies the dual nature of learning – the acquisition of a concept comes through its use (initially as a pseudoconcept) within the mathematical community, and through mediation by members of the community, as well as through the resources used in the community such as text books.

If we consider pseudoconcepts as a necessary part of mathematical learning, then there are important implications for MfT, in particular knowledge of *learners' conceptions*: teachers need to know which pseudoconcepts learners are likely to develop, and then how to move them beyond the pseudoconcept towards the target concept. If such development is dependent on learners' participation in mathematical practices, then it has implications for teachers' choice of pedagogy and for task selection and design.

3.4 Learning MfT as deepening, broadening and connecting mathematical knowledge

Many discussions of teachers' knowledge include some reference to *depth*, *breadth* and *connectedness* (e.g. Ball, et al., 2004; Conference Board of the Mathematical Sciences (CBMS), 2001; Shulman, 1986, 1987). However, seldom are these terms defined. A central contribution of Ma's (1999) study is her operationalising of the notions of *depth* and *breadth* with regard to teacher's mathematical knowledge. Her findings foreground the importance of *mathematical structure* as a component of teacher's knowledge, and an ability to make connections between mathematical concepts. She coined the term *profound understanding of fundamental mathematics* (PUFM), and proposed that profound understanding consists of three interconnected components: depth, breadth and thoroughness, all of which are related to the structure of mathematics. *Depth of understanding* concerns the ability to connect a concept or a topic "with more conceptually powerful ideas of the subject" (p. 121) where the power of a mathematical idea is related to its proximity to the structure of the discipline. A mathematical idea that is closer to the structure of the discipline underpins more topics and hence has more "mathematical influence" (my term) and thus mathematical power. For example a teacher with a deep understanding of number and number operations will link the notion of subtraction-with-regrouping with the core mathematical idea that addition and subtraction are inverse operations.

By contrast, *breadth of understanding* is related to the ability to connect a concept or topic with topics of similar or less conceptual power, for example connecting subtraction-with-regrouping to addition-with-carrying. Ma argues that these two ideas have similar conceptual power since they are analogues of each other with regard to the respective operations. The notion of *thoroughness* concerns the ability to make connections. She argues that "it is this thoroughness which 'glues' knowledge of mathematics into a coherent whole" (p. 121).

In terms of the MfT framework, depth and breadth of knowledge relate directly to the aspect *relationship to other mathematics*, and I explore these relationships in chapters 6 and 12, where I propose hierarchies of concepts for compound interest and annuities respectively. In both cases I

distinguish concepts at different levels and identify relationships between the concepts. In chapter 6, I also elaborate a network of related concepts, and hence connections, for the notion of *growth factor* which I argue is a key component of (compound) interest. Both the hierarchies and network are inspired by Ma's (1999) diagrammatic representations of *knowledge packages* although I do not propose them as knowledge packages in this study.

One of the ways in which teachers' mathematical knowledge may be deepened and broadened is by revisiting school mathematics. This is the focus of the next section.

3.5 Learning MfT as revisiting school mathematics

There is increasing acknowledgement that teacher education programmes cannot assume prospective secondary mathematics teachers have adequate knowledge of secondary school mathematics (e.g. Bromme, 1994; Bryan, 1999; Conference Board of the Mathematical Sciences (CBMS), 2001; Cooney & Wiegel, 2003; Stacey, 2008). The literature tends to assume that students have had adequate opportunity to learn the mathematical content at school, and now need to deepen and to "top up" their knowledge by gaining new insights and making new connections between different aspects of school maths. In addition, the literature suggests that learning of mathematics at school was traditional and procedural, and therefore "bad", and that revisiting should be done from a reform-based, constructivist pedagogy which is "good". Apart from subscribing to the traditional-reform, procedural-conceptual dichotomies, few suggestions are given as to how this revisiting might be done.

In South Africa many pre-service mathematics teachers have not had the kinds of opportunities to learn mathematics at school that might be assumed in other parts of the world. For example, they may have attended schools characterised by some or all of the following features: teachers were un(der)-qualified to teach mathematics; learners had limited access to text books and other learning resources; there was limited coverage of the syllabus due to poor pacing; teachers were unable to teach difficult sections of the syllabus (e.g. calculus and Euclidean geometry); there was an over-dependence on teaching mathematics in the vernacular thus limiting access to mathematical discourse in English. Under such conditions, it cannot be assumed that students even had access to "good" procedural teaching. Despite these odds, there are students who gain entrance to universities across the country to study mathematics teaching, many of whom are determined that their future learners will have better experiences of school mathematics than they had themselves.

For these students, revisiting school mathematics is much more than "topping up". In the most extreme cases, it may involve learning the mathematics they will teach for the first time. In most cases revisiting will involve some or all of the following aspects: identifying the key concepts in an area of mathematics and how sub-concepts and a range of procedures are linked to the key concept; developing fluency and confidence with these procedures through their repeated use; making sense of the procedures and making connections between isolated pieces of content and different representations.

In this study, I define *revisiting* as a technical term, and provide an analytic elaboration from an empirical base. In this section I describe revisiting as the re-learning of known mathematical content, drawing on Zazkis's (2011) work, but first I illustrate revisiting from a metaphorical position.

The term *revisiting* assumes a travel metaphor which is useful in elaborating the notion. When we travel to a new destination for the first time, we are generally focused on reaching the destination, following the directions and not getting lost. As a result, we may not notice the view nor see shorter, alternate routes different from the directions we were given. Also, it is unlikely that we will notice distances between particular landmarks or be aware that particular "special" landmarks, such as a brightly flowering shrub, may be useful indicators of where to turn. We also cannot know the relationships between some landmark A and further landmark B that is closer to the destination. However, on subsequent trips over the same route, we start to notice all these things. Also, it is likely that the second time we travel the route, we do so in the opposite direction, and this may open up new insights to the route.

There are analogous situations in revisiting mathematics. Revisiting assumes we do not (or cannot?) learn everything about an idea at once. But revisiting is not simply repetition for mastery. In general, the goal of mastery through repetition is to become increasingly able to complete a specified range of tasks with greater accuracy and efficiency. While revisiting involves some forms of repetition, it must provide opportunity to see new things and to journey to deeper layers of the concept[19].

3.5.1 Zazkis's notion of re-learning

Zazkis (2011) explored the similarities and differences between learning and re-learning mathematical content in the context of a mathematics course for pre-service elementary teachers[20]. She describes this course as one striving to develop profound understanding of mathematics (Ma, 1999) and as one that is concerned with unpacking topics in elementary mathematics. She argues that for the most part her students' were *re-learning* mathematics rather than *learning* it (for the first time) since the mathematics they engaged with was first encountered in elementary and middle school. Working from a broadly constructivist position, she acknowledges that one cannot erase students' initial mathematical learning and must therefore work with the mathematical ideas they bring. Thus she treats old (or existing) mathematical knowledge as both a support and a barrier to new learning.

Zazkis argues that re-learning mathematics involves reconstructing and restructuring which includes reorganising, expanding, discarding and replacing some ideas with other more robust and general conceptions. She also argues that relearning requires a shift from *operational* to *structural* conceptions (Sfard, 1991). Furthermore, she proposes that unpacking and repacking are central components of relearning mathematics, and suggests that, in the context of teacher education, unpacking is concerned with "examining components of knowledge that may be compressed and initially inaccessible" (pp. 13-14). She suggests that the activity of unpacking may be the trigger for new (or re-) learning, as one attempts to reconstruct knowledge that was previously acquired and compressed. Having unpacked components of mathematical knowledge, she argues that students then need to *repack* it, which involves "organising the contents in a logical and accessible manner, possibly replacing some elements with more useful ones" (p. 14). This suggests that some elements are not repacked but are discarded and new elements are included.

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19 There are some similarities here with the notion of *layered learning* (e.g. Kahn, Noss, Hoyles, & Jones, 2006).

20 Zazkis's book was only published in 2011 and so her ideas did not inform or frame my initial thinking. However, many of her ideas resonate with my thinking although I would argue that there are substantial differences in the relearning of mathematics by prospective primary teachers compared with prospective secondary teachers.

Skemp (1971) proposed that "to understand something means to assimilate it into an appropriate schema" (p. 46). Zazkis extends this proposition by suggesting that to relearn something "means to assimilate it in a richer or more abstract schema" (p. 50). For her a "richer schema" is one where a known mathematical concept is connected to a larger class of mathematical entities thus becoming a "particular example" (p. 50) of that larger class. For example, *evenness* is considered as a particular case of *divisibility by a prime number*, rather than the opposite of *odd* or *a number ending in 0, 2, 4, 6, 8*. There are similarities here with Ma's (1999) notion of depth. For Ma, depth is concerned with connections to more powerful mathematical ideas. The same is true for Zazkis - she is including a particular case within a larger class, and thus abstracting its properties in relation to the larger class. This may involve a redefinition of the mathematical entity in relation to the properties of the elements in the larger class.

Drawing on Vinner and Tall's (1981) notions of *concept image* and *concept definition*, Zazkis argues that relearning involves expanding one's concept image. Since concept image is largely a function of the examples that one encounters, if one only encounters a sub-set of all possible examples of a concept, one's concept image is likely to be distorted by these examples. In the context of financial maths, a typical example of a limited range of examples may occur when dealing with different compounding periods. By increasing the number of compounding periods per year, one sees that the yield increases. However, the yield tends to a limiting value. Unless learners are exposed to examples with very high frequencies of compounding (despite the fact that such frequencies are not realistic) which ultimately leads to the notion of continuous compounding, they may not become aware of the limiting value. The sub-topic of *changes in compounding periods* is dealt with at Grade 11 level when learners clearly cannot work with formal mathematical explanations of continuous compounding, although they can explore the notion numerically, and spreadsheets provide a useful resource in this regard. But in teacher education programmes, students can draw on their knowledge of limits and e to produce a rigorous justification of the limiting value.

3.5.2 Operationalising revisiting for learning MfT

For the purposes of this study, I explore the notion of revisiting in relation to pre-service secondary mathematics teachers developing MfT of compound interest and annuities. So while the focus is on (re-)learning mathematical content, it is done with the ultimate goal of teaching. While Zazkis's notion of relearning is largely acquisitionist, I consider revisiting to include elements of acquisition and participation. The purpose of revisiting is to increase one's mathematical knowledge through connections that deepen and broaden knowledge of the concept, paying attention to mathematical aspects that are important for teaching. While such knowledge may be useful in many professions, it is essential for teaching. The construct of revisiting emerged through the analysis of the data. It thus reflects the interplay between the theoretical field and the empirical setting (Brown & Dowling, 1998). I do not claim that through the course all, or even most students, necessarily achieved a deeper understanding of school mathematics.

In elaborating the notion of revisiting, I have identified four inter-related aspects: content, goals/purposes, task and activity, and resources[21], each of which is discussed in more detail below. I shall refer to revisiting *a piece of mathematics*. This could include mathematical entities such as concepts, techniques, formulae, theorems or representations.

[21] There are similarities here with activity theory (e.g. Engeström, 2001) but I do not draw on activity theory in this study.

Content – Revisiting a piece of mathematics *assumes the mathematical content has been encountered* previously and the student is at least familiar with the terminology, notation and techniques. Also, the student is assumed to have an *overview* of the content, not just the first parts. In relation to the travel metaphor, it assumes the student has travelled to the destination at least once. However, it does not assume the initial journeys were smooth and accident-free. A student may have completed the journey one or several times having acquired several bumps and bruises along the way, and may arrive at the opportunity to revisit the mathematics with typical misconceptions and incomplete knowledge, of which s/he may or may not be aware.

Goals/Purposes – When revisiting a piece of mathematics, the *goal of the activity is different from the initial encounter*. If pre-service teachers are revisiting school mathematics then the goals are necessarily different since the first time they studied the mathematics, they were learning it as school learners, now they are learning it (as a requirement) to become teachers, and then to teach it to others. The ways in which they need to know it are different. One of the goals must be to deepen and broaden their knowledge, making more links, for example links between representations or links to other aspects of mathematics. This could include paying attention to issues that tend to be left on the periphery in school mathematics for example: identifying restrictions on denominators of algebraic fractions; dealing with restrictions on domain and range when working with rational exponents; interrogating all aspects of a definition to understand why each part of a definition is necessary; finding more efficient proofs of geometry riders; and identifying the necessary and sufficient conditions when there is redundancy in a peer's definition or list of properties. So the goal is a more coherent and connected grasp of the mathematics. *Students are expected to move to uncover the more conceptually powerful ideas that underpin of the mathematics they are learning* (Ma, 1999).

Task and activity – I distinguish *task* from *activity* drawing on Christiansen and Walther (1986): A *task* is set by the teacher (or teacher educator), while *activity* refers to what the students do in response to the task. Thus the task is one of the key elements in framing the mathematical activity of the students. When revisiting mathematics, the task must require more of students than it would of school learners. While the task may be drawn from a school text book, if the requirements are simply to complete the text book task as school learners would do, then this is not revisiting. A task involves revisiting if it promotes mathematical activity such as: drawing on knowledge of advanced mathematics to generalise or to extract structure, dealing with extreme cases in a definition, and working with non-examples or counter-examples. A revisiting task may include learners' responses to a school task. This provides opportunity to work on the *learners' conceptions* aspect of MfT. In addition to the nature of the task, the activity should be such that students are expected to extend themselves beyond merely getting an answer. For example, students may be challenged (or may challenge themselves) in one or more of the following ways: to ensure they understand "all" aspects of the concept, to answer the question using more than one approach and/or different representations, to identify errors in learners' responses (or those of their peers), and to ask "what if" questions – considering what would happen if something in the task changes. These may all be considered deeper layers of understanding the concept that go beyond getting answers and correct execution of techniques.

Resources – Revisiting involves the use of new or additional resources. I use the term "resources" to include tools, artefacts and knowledge. At the most simple level, pre-service teachers have more mathematical knowledge resources to bring to bear on the revisiting task than when they previously

encountered the school mathematics. This may involve seeing school mathematics from the perspective of advanced mathematics, for example investigating integer arithmetic from the perspective of group theory. It may also involve drawing on the school mathematics of a higher grade where the simpler content is a special case of a more general process or structure, for example seeing the for area of a triangle ($Area = ½\ base \times height$) as a special case of the area-formula learned in trigonometry ($Area = ½\ a.b.sin\ C$, where $\hat{C}$ lies between sides a and b). Revisiting could also involve the use of additional resources such as computer technology (which is not available for learning maths in most schools in South Africa). For example, the use of graphing software for exploring transformations of functions, or dynamic geometry software which may provide new insights into geometric relationships that are already known. These kinds of computer resources enable the student to ask new questions which may lead to further enquiry and deeper insights. For example, the following question on financial maths can be solved graphically but cannot be answered using secondary school algebra: Compare investing R250 at 10% p.a. compound interest and investing R500 at 10% p.a. simple interest. When, if ever, will the compound scenario provide a better return?

The first part of the study focuses on revisiting school maths with particular emphasis on compound interest, percentage change and exponential growth. In analysing the incidents in chapters 7 to 10, I draw on the four aspects outlined above to emphasise the ways in which revisiting the particular content of the school curriculum differs from learning the content as a learner at school. In the analysis, I explore opportunities that arose and potential opportunities that may arise for learning MfT through revisiting school maths. In the final chapter, I provide an example of a revisiting task for annuities, thus drawing together aspects from both parts of the study.

3.6 Learning MfT as overcoming obstacles

For the purposes of this study, I use the term "obstacle" to refer to the breakdowns that students experience in their attempts to make sense of a concept. I draw on Brousseau's notion of epistemological obstacle[22] as "knowledge which functions well in a certain domain of activity and therefore becomes well-established, but then fails to work satisfactorily in another context where it malfunctions" (Cornu, 1991, p. 159). Thus the notion of obstacle is concerned with a presence, rather than an absence of knowledge. Brousseau (1997) argues that obstacles are "made apparent by errors" (p. 84) and are a necessary part of learning.

While Brosseau's work is located in a Piagetian framework, I shall consider obstacles from a broader social perspective and I shall distinguish *concern*, *difficulty* and *error* as manifestations of obstacles in the learning of mathematics.

Concern – A concern is something that provokes one's interest, that troubles one, that may even cause anxiety (Oxford English Dictionary, 1989). In the context of the study, a students' mathematical concern relates to something that is not immediately obvious, that does make not make initial sense in relation to a particular frame of reference, that is not readily accepted. However, the student perceives this "something" to be worthy of further pursuit, and not to be ignored. Concerns might be phrased in terms such as: "I am not sure about", or "I am wondering". They may also be phrased as questions to peers: "Are you sure about?" or "How did you come up with that?"

[22] Brousseau (1997) draws on Bachelard's work (1938) on obstacles. Bachelard's work is in French and I do not include it in my references since I have not read it. Although Brousseau distinguishes between ontological, didactic and epistemological obstacles, I deal only with epistemological obstacles in this study.

Difficulty – The notion of difficulty is associated with struggle (Oxford English Dictionary, 1989). In the context of the study, a difficulty relates to some mathematical entity (concept, procedure, representation, etc.) that a student is not able to make sense of with the "usual" amount of effort, using known tools. The student gets stuck with one or more aspects of the object and cannot proceed further. This may or may not lead to errors. A difficulty has similarities with a concern but is likely to require more time and effort to overcome.

Error – I consider errors to be *persistent, systematic mistakes* which learners make, yet which are merely the symptoms of some underlying conceptual problem or obstacle (Brousseau, 1997; Olivier, 1989). Errors are largely the result of previous correct learning that has been overgeneralised to tasks and contexts where it is not applicable. Furthermore, errors are the natural result of learners' attempts to make sense of their world. They can be distinguished from slips which are mistakes that learners can easily correct when the mistake is brought to their attention.

At this point a comment on misconceptions is important. I have deliberately chosen not to use the term "misconception". Nesher (1987) considers a misconception as a "line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors" (p. 35). Given this description, there is insufficient evidence from the study, and no evidence in the literature to identify particular lines of thinking that systematically give rise to errors in the area of annuities in particular, or to financial maths more broadly. I would, however, suggest that there are three issues that may be considered as "potential" misconceptions, and which require further research. The first is the counter-intuitive notion that larger amounts of money do not take a longer time to double than smaller amounts at the same interest rate. Simple algebra confirms that the growth is independent of the starting amount. A second example is the assumption that as the frequency of compounding increases, the resulting amounts get larger. Mathematically it can be shown that the resulting amount tends to a limit. Thirdly, there is evidence that students treat n in the annuities formula as the number of periods rather than the number of payments. This may be considered a misconception because it is an "erroneous guiding rule" (Nesher, 1987, p. 35) yet, as Nesher argues concerning misconceptions in general, it does not necessarily lead to an error. I will treat it as a "rational error" (Ben-Zeev, 1996) because it follows logically from the simple and compound interest formulae where n represents number of periods, and it appears to follow the same rules as these earlier formulae. Furthermore, there are many cases where treating n as the number of compounding periods (for the first annuity payment) will yield the correct answer.

Given the ubiquity and persistence of obstacles with respect to learning particular concepts, I work from the assumption that the obstacles encountered by the pre-service teachers are similar to those that learners in schools will encounter when learning financial maths.

In the analysis in part 2, I identify and discuss the obstacles students encountered when approaching annuities from the perspective of individual payments (what I refer to as an *individual payment approach*). I attempt to account for the sources of particular obstacles but I make no claims as to whether students overcame the obstacles. My concern is with gaining insight into the obstacles students encountered, and how these inform our knowledge of MfT of annuities with particular reference to the aspect of *learners' conceptions*.

3.7 Learning MfT as participating in mathematical practices

In chapter 2, I quoted an extensive list of mathematical practices identified by Watson (2008). I also noted (as does Watson) that the practices of school mathematics are different from those of research mathematicians. Similarly, the practices of mathematics teacher education are different from both research mathematics and school mathematics. For the purposes of this study, I shall consider mathematical practices to include the ways of working mathematically as described by Watson, as well as those practices of mathematics teacher education concerned with developing teachers' MfT. As noted in chapter 2, these include specific attention to working with multiple representations, engaging with the mathematical ideas of others, and communicating one's own ideas through verbal and written explanations. These elements are typical indicators of participationist views on learning.

In this section I focus on four aspects of participation that are important for the study: intuition, explanation, unpacking, and use of mathematical resources.

3.7.1 Intuition and mathematical learning

Tall (1991) argues that while intuition and rigour are sometimes considered to be opposites in mathematics, this is a false dichotomy. Indeed, based on interviews with 70 research mathematicians, Burton (1999, 2004) developed a model for knowing mathematics consisting of five categories, one of which was *intuition and insight*. In part 2, I challenge the typical practice of introducing annuities by means of geometric progressions, and argue that this is not intuitive for many students. In so doing, I draw on Fischbein's (1987, 1999) seminal work on intuition.

Fischbein (1987, 1999) describes intuitive cognitions as "apparently self-evident and immediate", certain, implicit, coercive and global. The qualifier "apparent" makes explicit that the products of intuitive thinking may be neither self-evident nor immediate, and furthermore may be incorrect. The apparent self-evident nature of the intuition means that the individual readily accepts it and feels no need to check further, nor to provide any extrinsic form of justification. Consequently, intuition is generally characterised by premature closure where particular mechanisms[23] tend to mask the incompleteness and vagueness of one's thinking, giving the impression of coherence. This masking of gaps is key in enabling one to proceed with mental activity. Intuitions tend to exert a "coercive effect" (Fischbein, 1999) on one's reasoning which may lead to the rejection of alternate interpretations that contradict the intuition. In addition, intuitions tend to "extrapolate beyond any empirical support" (Fischbein, 1999, p. 30), for example extrapolating to the infinite a finding that holds for the finite. Thus intuitions serve to generalise, to eliminate conflicts and to organise thinking into an apparently coherent and compact whole.

Fischbein (1987) argues that intuitions stem from experience, being learned through participation in the world, and are thus culturally determined. Correct intuitions are productive and enabling of learning but incorrect intuitions may become obstacles because they are resistant to change and are not easily replaced by clear instruction or by systematic empirical evidence. In fact, incorrect intuitions may emerge from or defy instruction precisely because there is nothing in the learners' direct experience that provides sufficient evidence to him/her of the flaw in the intuition.

[23] Fischbein (1987) identifies the following mechanisms: over-confidence, dramatisation, premature closure, the primary effect, immediacy and globality. They are not central to my argument and so I do not discuss them here.

Fischbein (1987) identifies two categorisations of intuitions: primary and secondary, and distinguishes between them on the basis of their origins. A *primary* intuition has its origin in experience and the activity of life. It is holistic and develops independently of instruction. By contrast a *secondary* intuition is produced through instruction. He suggests that the primary-secondary distinction is a continuum rather than a dichotomy. However, a secondary intuition may not be consistent with a related primary intuition. Consider the example of tossing a fair coin: we can show analytically that there is equal probability of getting "heads" or "tails" in the toss of a fair coin. This becomes a secondary intuition. Yet, despite this secondary intuition, if a coin has landed on "heads" several times in succession, our primary intuition may lead us to expect that is *more likely* to continue to the pattern, or conversely, to change that pattern on the next throw. Regardless of our expectation, the primary intuition is *not one of equal likelihood*. This is a powerful example of the ways in which intuitions may become obstacles for learning. Fischbein calls for an awareness of one's intuitions – to recognise them and their origins, to distinguish them from formal, analytical thinking, and then either to reconcile or subordinate them to the formal, accepted concepts of the discipline.

In chapters 12 and 13, I discuss students' initial approaches to modelling annuities scenarios, and show that many students drew on their primary intuitions by calculating monthly account balances. By contrast, I argue that the use of geometric progressions to model annuities scenario is not intuitive for many students. However, I shall suggest that once students made sense of this approach, it became a secondary intuition for them.

3.7.2 Learning MfT and giving explanations

Research shows that teacher explanations have various purposes and take various forms such as summarising main ideas, using analogies, using examples to illustrate a concept or procedure, and demonstrating how to execute an algorithm (Inoue, 2009; Roscoe & Chi, 2008). Inoue (2009) proposes that pre-service maths teacher education programmes should pay explicit attention to helping students develop the necessary knowledge and skills to produce good mathematical explanations. In Inoue's study, weaknesses in student teachers' instructional explanations appeared to be linked to their poor knowledge of how learners construct mathematical knowledge rather than to the student teachers' own mathematical knowledge. This supports the inclusion of knowledge of *learners' conceptions* as an important aspect of MfT.

In chapter 2, I identified *explanation* as an aspect of the MfT framework. While explanations occur throughout the thesis, the data selected for the study does not focus on knowledge of explanations, nor on learning to produce an explanation. However, there are several instances where the content of the explanation (produced by students or by me) provides insight into MfT, and therefore forms part of the analysis.

Leinhardt (1997) distinguishes four types of explanation: common, disciplinary, self and instructional. The latter two are particularly relevant for my study, and are discussed briefly below. In addition, I propose a distinction between *self* and *peer* explanation.

Self-explanations are focused on establishing personal meaning, and generally involve incorporating new knowledge with old. The explanation is local, idiosyncratic and fragmented, only drawing on disciplinary conventions to support meaning-making. Self-explanations are frequently incomplete,

stopping when a known issue is reached or skipping over a known piece to deal with the "problematic" piece (Leinhardt, 1997).

Instructional explanations are designed to teach, i.e. to communicate about a particular aspect of subject matter to specific audience. They are systematic, containing a complete "verbal trace" and are used to convey structure, to convince and to demonstrate. While they reflect the rules of communication and conventions of the discipline, they are local in time and place and occur in response to (or in anticipation of) a query from a learner. They draw on the shared experience of the local community as well as the discipline (Leinhardt, 1997).

Leinhardt argues that self-explanations may involve an individual or a group. I wish to consider the latter as *peer explanations* which inevitably include some pedagogic purpose. Therefore I would argue that peer explanations share features of self-explanations and instructional explanations. Consequently they may be less idiosyncratic and more complete than explanations intended only for oneself. They are also likely to be more systematic than self-explanations in order to demonstrate the thinking of the person explaining. However, as with self-explanations, they may skip over aspects of mathematics that are familiar to the group, and only adhere to disciplinary conventions to the extent to which these support meaning.

As noted above, much of the analysis concerns the content of students' explanations, most of which are peer explanations. The content of the explanations vary, and include describing an approach to a problem, justifying an answer, and explaining the use of a particular spreadsheet.

3.7.3 The notion of *unpacking*

The term *unpacking* appears to have been introduced into the mathematics education research literature by Ball and her colleagues, drawing on the work of David Cohen (Ball & Bass, 2000). Ball and Bass argue that the compressed and abstracted forms of mathematics valued by mathematicians are inadequate for teaching, and that teachers need to be able to *unpack* mathematics in order to make it accessible and sensible for learners. They suggest that unpacking "may be a distinctive feature of knowledge for teaching" (Ball, et al., 2004, p. 59). However, they do not provide a definitive description of unpacking. In earlier work they refer to both *unpacking* and *decompression*, defining the latter as the need to "deconstruct one's own mathematical knowledge into [a] less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98). Ferrini-Mundy et al (2006) also refer to *decompressing* and suggest it "involves attaching fundamental meaning to symbols and algorithms that are typically employed by sophisticated users in automatic, unconscious ways" (p. 8). Zazkis (2011) describes unpacking as "examining components of knowledge that may be compressed and initially inaccessible" (pp. 13-14).

I would argue that unpacking is mathematical work that is distinguished by the mathematics being explained. It involves teasing out the relationships in compressed forms of mathematics such as formulae and algorithms. For example, investigating why a particular element comes to be part of a formula requires unpacking of the formula. In the context of financial maths, explaining "where the 1 comes from" in the compound interest formula, or explaining why there is an *i* in the denominator of the annuities formulae both require unpacking of the compressed forms. By contrast, explaining how to substitute values into the compound interest formula, or how to determine the monthly repayment in

an annuities formula do not constitute examples of unpacking because the concern is not primarily with teasing out and decompressing the fundamental relationships in the formulae.

The notion of unpacking presupposes a pedagogic subject. In some of the literature (e.g. Ball, et al., 2004; Ferrini-Mundy, et al., 2006) it is assumed that teachers unpack mathematics *for learners*. By contrast, Zazkis's (2011) use of unpacking is in relation to pre-service teachers unpacking (and then repacking) mathematics *for themselves*. I would argue that the difference in *audience* has important consequences for the *purpose* of unpacking. Within this thesis I shall distinguish between unpacking for oneself, for peers (or those with similar mathematical knowledge to oneself), and for learners (or any other who knows less mathematics). In all cases I shall take unpacking to refer to mathematics that is already known by the unpacker and where the purpose is ultimately pedagogic, even if the initial unpacking is to clarify an issue for oneself.

Different audiences of unpacking suggest different *purposes* for unpacking. When unpacking for oneself, the purpose is mainly mathematical. Typical examples include making sense of an unfamiliar representation, algorithm or formula, or making sense of an unexpected response from a learner. This may involve reconsidering the meaning of particular symbols or the logic of a particular procedure. It may take only a few seconds but it may take much longer, and will likely lead to deepening and/or broadening of one's knowledge depending on connections that are made. Unpacking for oneself has many similarities with Leinhardt's (1997) notion of self-explanation. It may be unsystematic and messy, and involve repacking to connect old and new knowledge, to add new insights, discard unwanted elements, and thus to reorganise and restructure one's mental schema.

Unpacking for peers and learners has both a mathematical and a pedagogic purpose, and requires instructional explanation. Unpacking for learners requires careful attention to aspects such as sequencing, coherence and choice of representations. By contrast, unpacking for peers may be less detailed, less precise, and less carefully sequenced because of assumptions that are made about shared mathematical knowledge. Consequently there are similarities with self- and peer explanation in that known aspects may be skipped over and use of notation may not be precise. I do not consider students' explanations of their thinking to the teacher/lecturer as unpacking.

I therefore propose that unpacking is a *form of explanation* with the following features:

- The mathematical object occurs in a compressed form (e.g. formula or algorithm), is familiar to the "unpacker", and has previously been compressed and/or used regularly in a compressed form;
- Unpacking involves identifying and foregrounding the essential features of the mathematical object, distinguishing essential features and their interrelationship, and using appropriate sequencing, representations and language to communicate the essential features and their interrelationship;
- The audience of unpacking may be oneself, those with similar mathematical knowledge (e.g. peers), or those with less mathematical knowledge (e.g. learners/students);
- The purpose of unpacking depends on the audience, e.g. it will be mainly mathematical if the audience is oneself, but will be mainly pedagogic of the audience is learners.

In the study I draw on the notion of unpacking mainly with reference to mathematical work that I did in delving deeper into formulae, in response to students' productions and concerns. While there are instances of student's decompressing mathematics and sharing their ideas, I have not treated these as unpacking since they do not sufficiently fit the criteria listed above.

3.7.4 Working with resources of mathematical practice

A fundamental assumption about learning as participation in a practice is that participants come to use the resources of the practice in increasingly appropriate ways. In the context of mathematical practices such resources include mathematical notation, representations, definitions, techniques and language. Other resources include text books, and digital technologies such as spreadsheets.

In this study I focus on students' use of various resources including notation (chapter 7), timelines and time-related language (chapter 14), and spreadsheets (chapter 15). In addition, I argue in chapter 9 that learners' responses to tasks may be considered a resource for learning MfT.

In studying student's use of mathematical resources, I draw on Lave and Wenger's (1991) notion of *transparency*. In the context of mathematics, transparency is concerned with the ways in which students' use of resources enables them to gain access to the knowledge, processes and practices of mathematics. Transparency is not an inherent characteristic of a resource. Rather a resource becomes transparent for a user through its use for a particular purpose in a particular setting. The notion of transparency has been taken up in mathematics education in the context of language and multilingualism (Adler, 1999; Setati, Molefe, & Langa, 2008), and in the context of resources in mathematics teaching and learning more generally (Adler, 2000; Ainley, 2000; Meira, 1998). The transparency of a resource emerges through its dual characteristics of visibility and invisibility: it must be visible so that it can be seen and used, but must also be invisible so that through its use, it supports access to mathematics. When students first work with new resources such as spreadsheets, they often struggle to see the mathematics they are doing because the new technology is the object of their attention, too much in the foreground and hence too visible to provide access to the mathematics. Once they become more familiar with the technology, they are able to attend more to the mathematical ideas, the spreadsheet recedes into the background and thus becomes invisible. Familiarity with a resource is powerful because the use of the resource becomes tacit. However, its power is potentially also its poison: an invisible resource may be disabling because its properties cannot be distinguished in order to be used. In chapter 8, I discuss my experience in this regard while working with the compound interest formula.

3.7.5 The role of power in pedagogic settings

Pedagogic settings and the relations between actors in those settings are characterised by an imbalance in power (Bernstein, 1996). Such imbalances exist between teacher and students, and amongst the students themselves. Consequently *what* students might learn and *how* they might learn is a function of a range of factors such as: the content that is selected, the choice of pedagogy, the roles that students adopt, and the extent to which the teacher and fellow students enable and constrain the participation of individual students and groups of students. While I acknowledge the power relations that were continually at work, I do not focus on their impact on the research. Suffice it to say that what came to be produced in the course, and through the analysis of the data, are products of the exercise of power.

3.8 Conclusion

In chapters 2 and 3, I have positioned the study in relation to teacher's knowledge and in relation to learning. In chapter 4, I describe the empirical field, providing an overview of the Financial Maths course. While I refer to student's learning and their opportunities to learn[24], I do not make any claims about *what* students' learned in the course. For the most part I avoid claims about individual students' knowledge based on their utterances or written responses. I do, however, refer to students' "take-up", particularly with respect to their use of spreadsheets. I use the term "take-up" in a similar way to Adler (2002), to refer to aspects from the course that students' demonstrated/used, for example the extent to which their talk about time was precise, and their use of spreadsheets.

[24] I use the terms "opportunity/ies to learn" in a general sense, and not in the various technical senses as used by others such as Reeves (e.g. Reeves & Muller, 2005), McDonnell (1995) and Porter (1995).

CHAPTER 4
Overview of the Financial Maths course

4.1 Introduction

There are many descriptions that could be written of the Financial Maths course. No doubt each of the students, if asked, would have produced their own distinct version, and I could produce at least two versions – as lecturer and as researcher. The description I give in this chapter comes from the lecturer's perspective although the researcher lurks in the shadows. The description places the spot-light on the students, course content, pedagogical choices, use of technology and assessment. In so doing I provide the reader with an overview of key elements of the course using broad brush-strokes, and I show how the research focus emerged. The description is framed by my MfT framework and my perspectives on learning, both of which are discussed in the preceding two chapters.

4.2 The students

I begin my description with the students because they influenced many decisions regarding content selection, and my pedagogical choices.

4.2.1 The general experiences of the students prior to the course

The 42 students who took the course were registered for the undergraduate Bachelor of Education (B.Ed) degree with specialisation in secondary mathematics. Thirty students were in their third year of study and 12 in their fourth year of study. The group was racially mixed[25] with black male being the largest sub-group (23 students). This is significant because race and socio-economic status are closely linked in South Africa (see for example, National Income Dynamics Study (NIDS), 2012). Students came from a wide range of socio-economic backgrounds. Some lived in wealthy areas of Johannesburg, others came from rural areas far away from the city. The majority of students held study bursaries provided by the state, and some used a portion of this money to support their families. Students therefore had vastly different experiences of managing finances – their own and their family's. The majority had limited exposure to and knowledge of the formal financial sector. As I noted in chapter 1, one of my goals for the course was to address students' levels of financial literacy, and so this had significant bearing on course design and content selection.

In their earlier years in the B.Ed programme, the students had done courses on functions and college algebra, geometry and trigonometry, introductory calculus and linear algebra. Fourth year students had also completed courses in mathematical modelling and statistics. Thus they had more experience of applied mathematics than the third year students. All students had taken one maths methodology course, and the fourth year students were doing a second maths methodology course concurrently with the Financial Maths course. In chapter 3, I noted that many students were underprepared for university mathematics when they started in first year. One might assume that deficits from school mathematics should have been overcome by third/fourth year. However, there is evidence in the data to suggest that this was not the case.

[25] As noted in chapter 1, racial categorisations are still common in post-apartheid South Africa.

I had taught all the students at least one maths course and one maths methodology course. I therefore had some knowledge of their commitment to their studies and their mathematical competence, based on marks in previous courses. It was the first time that the two year-groups were in the same class. I therefore needed to work hard at establishing a classroom community where the students from the different year-groups were comfortable to work together, and where one group did not feel dominated by or superior to the other.

4.2.2 Students' previous experiences of financial maths and expectations of the course

I was aware that students had done some financial maths in the past. Half the class indicated that they had learned about simple and compound interest at school (annuities was not part of the school curriculum at that stage). All students had also done some financial maths in a compulsory maths literacy course in first year. Most recalled learning simple and compound growth, interest rates, and different compounding periods but only thirteen students referred specifically to learning annuities[26].

Less than half the class had experience of teaching or tutoring financial maths. For those who indicated some experience, this had taken place on Teaching Experience (practicum) and had generally focused on simple and compound interest. No-one referred to experience of teaching annuities. Some had also helped other B.Ed students with their coursework in the first year course.

At the end of week 2, students wrote in a journal entry the three things they wanted to learn in the course and why these were important for them[27]. I provide only a brief summary here. Students wanted to increase their general levels of financial and economic literacy, and learn how to manage personal finances. The most noticeable trend across all responses was that they wanted more than descriptions and definitions of financial and economic phenomena. They wanted to understand relationships between phenomena – the causes, impacting factors, and consequences of changes in these factors. Students also gave high priority to social issues relating to finance such as how the economy affects the general population, and how to help people get out of debt. It is possible that the extent of their concern here may have been a consequence of work done in the first week on the informal microlending sector.

Students' references to mathematics also showed very specific purposes for their learning. Some wanted to learn particular formulae and procedures because they could then work more efficiently than using iterative calculations. Others wanted to be able to do the necessary mathematics to check printouts from financial institutions and to project the value of investments and loans. In both cases, learning of the mathematics was not the key issue – the students sought the mathematical power which they believed the new mathematics would provide. In their references to teaching, some students focused on expanding their own knowledge, particularly of financial and economic issues because their learners might ask them about these. Others referred to learning appropriate strategies for teaching financial maths.

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[26] When I looked at the main assessment tasks for the years in which these students were in first year, it seemed that the fourth year students had dealt mainly with the annuities formulae and substituting values into the formulae, but the third years had done very little work on annuities, likely not learning the formulae at all.

[27] Thirty-one journal entries were analysed. A coding scheme was developed to categorise the data. I was particularly interested in distinguishing reference to maths, finance/economics and teaching in their responses. In the analysis I also distinguished *content* that students wanted to learn from the *purpose* they had for learning it

Students thus saw the course as an opportunity to increase their own levels of financial literacy, to learn the relevant mathematics, and to learn to teach it. From the perspective of the MfT framework, their expectations related most closely to *essential features* and *contextual knowledge of finance*.

4.3 Overview of the course

The course ran for 12 weeks from February to May 2008 with final assessments in June 2008. There was a three-week gap between weeks 9 and 10 when students were in local secondary schools for their Teaching Experience. Most weeks consisted of two class sessions of approximately two hours each (on Tuesday and Friday) with a two-hour group tutorial (on Thursday). Students were assigned to a tutorial group, and worked on seven group tutorials over the course.

4.3.1 Course content

The course was designed to deal with aspects of mathematics, finance and economics, and teaching. Within the mathematical work, I planned to revisit aspects of school mathematics as well as to deal with the mathematical content of annuities, Newton's method and continuous compounding. The first four class sessions were designed to give students a "taster" of the different course components. These are described briefly below.

The tasks in the first session were structured around hire purchase adverts and a flyer from a microlender. By the end of the session students recognised that they were not able to explain and account for the figures provided in the hire purchase adverts. Many left the session shocked by the punitive consequences of exorbitant interest rates (such as 19.5% per month charged by the microlender), the consequences of not being able to repay the microloan in the stipulated time, and the impact of various repayment plans such as doubling the first instalment. In the second session, they set up spreadsheets to model the microloan scenario. The spreadsheet work enabled them to explore the financial situation more fully, and to develop their spreadsheet skills. By the end of the session, many had seen evidence that "pay as much as you can as soon as you can" is sound financial advice. In the third session students were required to discuss and reflect on various newspaper articles about microlending, and an article from a financial magazine on the National Credit Act (NCA)[28]. During the whole-class discussion many students shared personal stories about their experiences of microlending and *mashonisa* (a Zulu word for microlender)[29]. The fourth session was the first group tutorial which included two tasks on compound growth. One was a mathematical modelling task, and the other was a teaching task requiring students to engage with learners' responses to a compound growth problem. The latter is discussed in more detail in chapter 9.

Revisiting of school maths was done in weeks 2 to 4. The mathematical content covered in this period included exponential growth, percentage change, simple and compound interest, and nominal and effective interest rates. This provided opportunity to deal with various aspects of mathematics-for-teaching such as *essential features* (e.g. notation, definitions and representations), links to *other aspects of mathematics* (e.g. percentage change and exponential growth) and *learners' conceptions*. This work forms the focus of part 1 of the study.

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[28] The National Credit Act (Government Gazette, 2006) came into effect in June 2007 to protect consumers against reckless lending and borrowing.

[29] This session provided a typical example of the moral and political projects that formed part of the course – to increase students' levels of personal financial literacy, to develop a critical awareness of the financial sector, and of the impact of compound interest on society, particularly on the poor. It may be considered an instance of *reading and writing the world with mathematics* (Gutstein, 2006). However, these aspects of the course are backgrounded in the thesis.

The new mathematics (of annuities) was dealt with in weeks 5 to 9, beginning with investigating a formula for future value of an ordinary annuity, and ending with a group tutorial on complex annuities. We also dealt with aspects of MfT and curriculum knowledge such as the school curriculum for financial maths in preparation for an assignment students would do during their Teaching Experience. Part 2 of the study focuses mainly on the content of weeks 5 and 6.

4.3.2 Course pedagogy

The pedagogy of the course, at least for the first nine weeks, could be described as "participatory-inquiry" (Adler, 1996). Similar terms used in the literature include "reform" (Boaler, 1997) and "process-oriented" (Cooney & Wiegel, 2003). Adler (1996) characterises a participatory-inquiry approach as follows:

> The underlying assumptions … are that knowledge and learning are social and that a participatory approach to learning and teaching mathematics is more appropriate, meaningful and effective than a traditional teacher-centred approach. In a participatory-inquiry approach, pupils are expected to take responsibility for their learning. Typically, they are provided opportunity to engage with challenging mathematical tasks, either alone, but more likely in pairs or groups. The knowledge pupils bring to class in recognised and valued. Diverse and creative responses are encouraged, and justifications for mathematical ideas [are] sought, often through having pupils explain their ideas to the rest of the class" (pp. 217-8)

During class sessions there was a great deal of student-student and student-lecturer interaction with frequent and extended periods where students could be found at the front of the class explaining their ideas, and challenging the contributions of others. The tasks I used were carefully designed for specific purposes. For example, in revisiting school maths I designed a task with four carefully-selected learners' responses to a typical compound growth problem. My intention was that the learners' responses would provoke students to reconsider their own knowledge of compound growth and their use of the compound interest formula. This is discussed in detail in chapter 9. Several of the annuities tasks required students to produce models for a given scenario, and then to extract the relevant mathematics from the problem in a way that Julie (2002) might describe as using modelling as a *vehicle* for learning new mathematics.

During class sessions, I usually solicited contributions from students and orchestrated the whole-class discussions, opening up opportunities for discussion of some issues and closing down discussion of others. In most cases my goals were mathematical and my decisions were informed by the extent to which a discussion was moving in the direction of the mathematical issue I wished to foreground. However, there were also instances where I allowed discussions to flow freely. Typically this occurred when a student's contribution took me by surprise and I wanted to explore the extent to which other students shared the concern or difficulty. Such decisions were informed by the researcher's concerns and provide an instance of the opportunities provided by insider research where the researcher can direct the flow of events because s/he is also the teacher[30]. A typical example of this is Palesa's concern about the use of subscript notation, which is discussed in detail in chapter 7. If I had not been conducting research, I would likely have moved on much sooner than I did.

One of the consequences of this participative pedagogy, and of the research, was that pacing was slower than anticipated and content coverage was consequently constrained. In weeks 2 to 7, I

[30] Insider research is discussed in chapter 5.

frequently found myself in the situation where content that I expected to cover in a particular session overflowed into the next (or a subsequent) session. In weeks 10 to 12, I adopted a much more typical university-lecture pedagogy in order to increase the pace and coverage.

In the group tutorial sessions students were given the task at the beginning of the session and were expected to submit a report on their workings at the end of the session. During the session I moved around to observe the progress of the groups. I offered input if students requested, and I made suggestions for redirecting or focusing their work where I thought they were drifting from the task. On several occasions I encouraged the groups to pay attention to their reports and not to leave the write-up of the report too late. During the first few group tutorials I did not engage with the focus groups because I was concerned with "contaminating" the data. However, I soon realised that I may be disadvantaging them by ignoring them (since they were in a different venue) and so from the fourth group tutorial I attempted to interact with both focus groups in the same ways that I was interacting with the other eight groups.

4.3.3 Revising content selections

In weeks to 2 to 9, the mathematical work frequently took longer than anticipated, with the result that aspects of finance, economics and teaching got less attention than intended. At the end of week 9, I revised my plans for the final three weeks, taking into account what content had been privileged, what had been marginalised, and what still remained to be dealt with. My overall decision was that more time needed to be given to finance, economics and teaching issues. This forced a choice regarding the advanced maths content: I had to choose between Newton's method and continuous compounding. I chose to include Newton's method for three reasons. Firstly, students had no prior experience of working with numerical methods and Newton's method would thus expose them to new ways of working mathematically – and thus broaden their exposure to mathematical tools and practices. Secondly, I felt their knowledge of annuities would be incomplete without knowledge of how to solve for any of the unknowns in the annuities formulae. Thirdly, there are parallels between the demands of determining the interest rate in an annuities problem and determining the number of compounding periods in the compound interest formula. In both cases students require new mathematics – the former requires Newton's method and the latter requires logs which are only taught in Grade 12 in South African schools. I thus used this as an opportunity to point out parallels between the students' learning and that of their future learners in school.

The decision to exclude continuous compounding was a difficult one. In chapter 3, I used continuous compounding as an instance of expanding the example space, as proposed by Zazkis (2011). A consequence of not expanding the example space was that I did not explicitly address the limiting value of continuous compounding.

Based on these revised plan, we dealt with Newton's method in weeks 10 to 12, and paid more attention to financial, economic and teaching issues, including inflation, index numbers, hire purchase, and teaching-focused work involving task design. These content selections address two components of the MfT framework: knowledge of *mathematical practices* and *contextual knowledge of finance*.

Table 4.1 provides a list of the content dealt with in the course.

Content dealt with in the course
Mathematics Simple and compound interest, different compounding periods, nominal and effective interest, daily interest calculations, depreciation, future value of ordinary annuities and annuities due, present value of ordinary annuities, missed payments, outstanding balance, and future value of complex annuities. In addition the following more general mathematical aspects were also covered: percentage change, exponential growth, working with index numbers, mathematical modelling and Newton's method, specifically for application to annuities.
Finance and economics Repo and prime rate, percentage and basis points, gross domestic product (GDP), gross national income (GNI), interest rates in banks, history of interest and usury from ancient civilisations to current times, international day count conventions, different kinds of saving options, inflation - consumer price index (CPI and CPIX), producer price index (PPI), buying power, time value of money, dealing with debt in the developing world, microfinance, National Credit Act (NCA), Usury Act and hire purchase.
Teaching Requirements for financial mathematics in the school curriculum both Mathematics and Mathematical Literacy[31], analysing text book tasks, dealing with learners' thinking, practicing teachers' and learners' experiences of teaching and learning financial maths, teachers' knowledge for teaching algebra[32], task design and use of technology in teaching financial maths.

Table 4.1 Overview of course content

The course was assessed by means of a wide range of assessment tasks including small coursework tasks, individual and group tutorials, assignments on various topics, two tests and an examination equivalent assignment. Most assessments focused on mathematical aspects but finance/economics and teaching were also assessed on several occasions. Since assessment is not in focus in the study, I do not provide further details on assessment.

4.3.4 Use of technology

I made extensive use of computer technology in my teaching. Spreadsheets were used regularly in the sections on simple and compound interest, nominal and effective interest rates, and annuities. I made use of Autograph software for teaching the section on Newton's method because of its built-in Newton-Raphson functionality. I also used the graphing functionality of Geometer's Sketchpad at various stages.

I expected students to make use of the available software, particularly spreadsheets, for their own learning of mathematics and the related financial content. As noted above, in the first week I allocated one session in the computer room to technical aspects of spreadsheets such as setting up formulae, absolute cell referencing and graphing. Thereafter students were expected to develop their own spreadsheets, and to adapt existing spreadsheets.

Students used technology to varying extents for their own learning. From the outset, some students were already very proficient with spreadsheets and other packages such as Geometer's Sketchpad. Some made a concerted effort to develop their spreadsheet skills during the course but there were others who made little effort. Although students were not familiar with Autograph, approximately half the class made some use of it for a task on Newton's method. Several students made use of Geometer's Sketchpad for graphical work, and a few made use of their graphing calculators although this was not prioritised in the course.

[31] In Grades 10 – 12 learners must select either Mathematics or Mathematical Literacy, the latter option being intended for learners who struggle with Mathematics and/or learners who don't require abstract and formal mathematical content in their intended careers.

[32] This work was based on the Knowledge for Algebra Teaching Project (KAT) (Ferrini-Mundy et al (2006)).

I also expected students to make use of spreadsheets for future teaching purposes. For example, students were expected to produce spreadsheets to illustrate simple and compound growth numerically and graphically for their future learners. This included attention to the user-interface, such as learning to use sliders. It also included attention to potential obstacles in the use of spreadsheets, such as rounding, and auto-scaling of axes when parameters are changed.

Students' differential use of technology can partly attributed to unequal access to computers. For example, some students owned laptops while others were dependent on university computer laboratories for access. At the time, the Education campus did not have adequate facilities for the rapidly increasing numbers of students and so students frequently struggled to get computer access particularly during the day. In order to address this issue, some class sessions were held in a computer laboratory so that students could work hands-on during the session. In the middle of the course I organised for some stand-alone PC's to be set up in the teaching venue so that students had computer access in class-time.

4.4 Reflecting on the course

Reflecting on the course in mid-2012, as I complete the write-up of this thesis is a very different task, and comes from a very different perspective to reflecting on it in mid-2008, soon after I finished teaching the course. In fact, reflecting on the course at any stage within this time period would also have come from a different perspective, informed by my thinking at the time and the data that had been analysed up to that point. I reflect as the teacher educator, although the researcher's perspective cannot be excluded. There are many issues that could be included in the reflection any many stories that can be told, but I have chosen to focus on five key issues.

4.4.1 Pedagogical choices and the marginalising of teaching issues

In much of the mathematics teacher education research reported from the developed world, assumptions are made about minimum levels of mathematical knowledge of pre-service secondary mathematics teachers. As I have previously noted, these assumptions do not necessarily hold in the South African context. We do not yet know what it takes to support these students to learn the knowledge and skills they will need to become effective teachers of secondary mathematics.

The literature (e.g. Cooney & Wiegel, 2003; Stacey, 2008) suggests that reform approaches to learning mathematics are key for pre-service teachers. This tends to be motivated by the desire to overcome traditional experiences of mathematics at school and in undergraduate mathematics programmes (Ruthven, 2011).

Earlier I described the pedagogy of the course as participatory-inquiry. While this pedagogy gave students many opportunities to engage deeply with the mathematics in question and to engage with each other's ideas, one of the consequences was a slow pace and associated struggle to cover the intended content. When I prepared to teach the section on Newton's method, I was very conscious of time pressures. The decision to adopt a lecture style was thus a pragmatic one. However, after finishing the three sessions on the topic, and covering the content quickly, I got a strong sense that students had "got it". The experience seemed far cleaner and less stressful than the messiness and complexity of the participatory and student-centred approach to compound interest and annuities. This caused me to reconsider my pedagogical choices in the light of their affordances and constraints for

learning mathematics and for learning about teaching. This is not in focus in the body of this study, but I shall return to the issue in the final chapter.

Table 4.2 indicates the number of sessions spent on various topics. This time allocation is different from what was planned. I have separated lecture time from group tutorial time in order to show the emphasis on mathematics, specifically annuities, in the group tutorials. The contextual work in the table such as hire purchase and inflation included substantial quantitative work. Broader economic issues such as gross domestic product were dealt with via readings, and were not given much attention in class. Consequently they are not included in the table.

Topic	Lecture time	Time in group tutorials	Total [33]
Annuities - future value	5	6	11
Annuities - present value	8	4	12
Hire purchase	3	0	3
Inflation, buying power, time value of money	3	0	3
Microfinance	3	0	3
Modelling	2	0	2
Newton's method	3	0	3
Nominal and effective interest	6	0	6
Percentage increase, exponential growth	4	2	6
Simple & compound interest, compounding periods	8	2	10
Teaching, curriculum	5	0	5
TOTAL	50	14	64

Table 4.2 No. of sessions spent on each topic

The table shows that the topic of annuities was given the most attention (26 sessions including Newton's method). The next biggest cluster focuses on school mathematics (16 sessions) covering percentage increase, exponential growth, and simple and compound interest. In both cases these areas were given more time than was intended. The most noticeable absence in the table is content reflecting general financial and economic issues. Where these issues were covered in contact sessions, they were generally handled within sessions that included mathematical content, such as microlending, hire purchase and the sessions dealing with inflation. Despite this, far less time was spent in class on general financial literacy than on mathematics.

My plan was that the selected teaching issues should lever up general principles of maths teaching such as the role of formulae in learning mathematics, selecting and designing tasks, and dealing with learners' misconceptions. However, since teaching issues were continually sidelined, this did not happen. For example, students were not given new tools to engage with task selection and design. They were frequently left to draw on their knowledge of mathematics teaching from methodology courses. While one necessarily assumes some prior learning about the tasks of teaching, this aspect of the course needed to be more carefully considered. It also raises the question as to what can be realistically achieved regarding teaching aspects in a *mathematics* course for teachers. The fact that teaching issues were continually marginalised was a concern for me because there is slim chance of dealing with the teaching of a small section of the curriculum such as financial maths in a methodology course, when the teaching of algebra, function, geometry and trigonometry require a

[33] All times are measured in one hour units.

great deal of attention. However, it was important for me that students got to grips with the maths first. It also seemed to me that students were more interested in doing maths tasks than teaching-related tasks and so it could be argued that we colluded to sideline teaching issues.

4.4.2 Content selections

Although I have noted that reflecting on the course at different times would be done from different perspectives and would thus produce different stories, I have no doubt that the on-going decisions I faced over content selection and sequencing would feature strongly in all of these potential stories. Those decisions, and the associated tensions, were all broadly informed by various aspects of the MfT framework although I was not able to articulate the framework at that stage. I have already discussed my decision to include Newton's method and to exclude continuous compounding. If I had made different choices earlier in the course regarding pacing and the amount of time allocated to financial and economic issues, I would have had sufficient time to deal with continuous compounding in detail.

I also chose not to deal in depth with depreciation. Students were required to work through a text book extract involving depreciation, as part of a teaching task, and one group chose depreciation as their exam equivalent topic. I chose not to focus on it in the course because I felt that students could learn it for themselves, drawing on their knowledge of simple and compound increase. Furthermore, within the contexts of financial mathematics and financial management, depreciation is concerned with valuation of assets and not primarily with the time value of money. In texts on financial management (e.g. Brigham & Ehrhardt, 2008; Correira, Flynn, Uliana, & Wormald, 2003), there are typically separate chapters on the time value of money (covering simple and compound interest, interest rates, and annuities) and on depreciation.

However, on reflection I may have made a poor choice in excluding depreciation since it provides opportunity to engage with at least two aspects of the MfT framework. Firstly, by dealing with simple and compound decrease there is opportunity to deal with *links to other mathematics*. For example, straight line depreciation provides a powerful illustration of "equal divisions of the whole" regardless of the interest rate, although this may not be immediately obvious. Similarly, reducing balance depreciation provides a meaningful context for the notion of "tending to zero" and thus an interpretation of the horizontal asymptote. Secondly, from the perspective of *financial issues*, depreciation raises useful and important contextual issues for financial literacy like writing off assets.

A largely unintended omission from the course was the lack of attention to graphical representations for present and future value of annuities. By contrast, I worked extensively with graphical representations for simple and compound growth, and Newton's method. Although I became aware during the course that I was not attending to graphical representations, I chose to ignore the "prompt in my head" because I did not consider the graphs of annuities to be particularly helpful and by contrast the spreadsheet representations were very powerful. I still cannot fully reconcile this decision with my commitment to working with multiple representations of functions.

4.4.3 Use of technology for learning maths in maths teacher education

The extensive use of technology in teachers' learning of mathematics, and the related assumption that they have access to such technology in their teaching, poses dilemmas for teacher education when technology is not readily available in the majority of South African schools. As noted above, students indicated that the use of technology was valuable for their own learning. Some students commented at

various stages of the course that learners would not have access to spreadsheets and so they needed to consider how they would teach without access to technology. The provision of technology in schools is beyond the scope of this study, as is the preparation of teachers to teach with technology. However, since introducing spreadsheets into the Financial Maths course in 2002, I have become increasingly interested in the extent to which teachers may be able to employ what I shall call "spreadsheet thinking" in their teaching of financial maths, irrespective of their access to technology for teaching. In Part 2, I analyse the unanticipated ways in which the two groups made use of spreadsheets for working with annuities, and argue that this holds potential for developing spreadsheet thinking that can be deployed in settings where technology is not available.

4.4.4 Mathematical modelling – present but not in focus

As an applied area of mathematics, financial maths is fundamentally concerned with mathematical modelling. However, at school level the formulae tend to be treated only as objects for substituting values, rather than mathematical models of a real world process. I take mathematical modelling at school level to be a complex iterative process that involves engaging with a real world problem, simplifying it to produce a mathematical model, doing the necessary mathematics to produce a mathematical model which is then interpreted in relation to the real world problem, and revised and refined as necessary. My thinking regarding the modelling process has been informed by the work of Galbraith and Clatworthy (1990), Stillman, Galbraith, Brown and Edwards (2007), Blum and Leiss (e.g. Leiss, Schukajlow, Blum, Messner, & Pekrun, 2010), Maaβ (e.g. Maaß, 2006; Maaß, 2010) and Borromeo-Ferri (2010). Elsewhere (Pournara, 2011a) I have argued that learners cannot see the compound interest formula as a model of something because they do not have access to the processes of calculating and compounding interest in banks. In chapter 12, I shall argue that in simplifying the real world scenario of annuity payments, by considering only payments made at the beginning and end of a period, we create potential for students' confusion. Beyond this, I do not focus specifically on the mathematical modelling process in the thesis.

4.4.5 Engaging the world of finance and economics

As I noted in the first chapter, I have come to financial maths out of personal interest, with no academic background or experience in finance or economics. Consequently the confidence I have in selecting mathematical content and designing mathematical tasks does not extend to finance or economics. I identified four distinct categories when working with financial and economic content in the course:

- *Category 1* involved those aspects which have simple quantitative features and where background information is reasonably accessible in books or on the web. Examples include repo rate, prime rates, daily and monthly interest calculations in banks, exchange rates, and index numbers such as Consumer Price Index (CPI). It was easy to develop spreadsheets to engage with the mathematics embedded in these aspects.
- *Category 2* involved aspects with strongly quantitative features but where background information is far more difficult to obtain, for example hire purchase agreements and vehicle finance. I was privileged to have access to expertise on hire purchase that is not available to the general consumer and is also not documented for public access. In these cases it was possible to do quantitative work with spreadsheets to develop various models of each process in similar ways to those described by Hoyles et al (2010) in their work on techno-mathematical literacies.

- *Category 3* involved what I shall call "terms and concepts" that one encounters regularly in the media such as inflation, gross domestic product, current account deficit, markets, bonds and equities. In a financial maths course such as the one in focus, these elements might be considered as theory or bookwork that can be accessed in readings. As the lecturer I could do little more than act as a conduit, passing information from a textbook or website to my students with very minor transformations along the way. Students were also provided with readings and accessible references on these issues.
- *Category 4* involved knowledge of relationships between financial and economic phenomena, and sought answers to questions such as "what causes exchanges rates to change?" and "should the policy of inflation-targeting be retained?" Answers to such questions required a great deal of reading but do not necessarily have simple answers. I had (and still have) no expertise in these areas and I did not feel confident to deal with them in class nor to suggest suitable readings. Needless to say, I was relieved that the stock market crash of October 2008 took place after the course ended!

All this raises the question about appropriate criteria for selecting financial and economic content into a financial maths course for teachers. Students taking a financial maths course as part of a commerce or business programme are likely to be taking other courses that deal with the key financial and economic issues. Therefore the problem of content selection is restricted to situations where students do not have other opportunities built into their academic programme to gain the necessary contextual knowledge. A related issue concerns the challenges that maths teachers face if they want to take contexts seriously when teaching maths. Both these issues are beyond the scope of this study.

4.5 Conclusion

In this chapter I have provided a description of the course from my perspective, and informed by the theoretical perspectives on MfT and learning provided in the previous chapters. Together these constitute the theoretical and empirical fields of the study (Brown & Dowling, 1998). The course was characterised by a participatory-inquiry pedagogy, which will be evident throughout the thesis, but particularly in part 1. The mathematical aspects of the course are foregrounded throughout the study, and the use of spreadsheet technology comes into focus in chapter 15. In the next chapter I discuss the methodology and design of the study. Thereafter I move to the substance of the study which is separated into two parts: revisiting school mathematics and learning new mathematics.

CHAPTER 5
Research design and methodology

> "If one … assumes that the goal of the researcher's work is to understand and learn about the phenomena being studied, then research is simply a form of learning. If one assumes that research, like any other learning process, can be described by the phenomenology of human learning, it then becomes clear that the most advanced form of understanding is achieved when the researchers place themselves within the context being studied. Only in this way can researchers understand the viewpoints and the behavio[u]r which characterises social actors"
>
> (Flyvbjerg, 2006, p. 236)

5.1 Introduction

The introductory quote from Flyvberg (2006) reflects my position of research as learning. In seeking to understand more fully the notion of mathematics-for-teaching and how pre-service teachers might learn it, I have placed myself in the centre of the study – as teacher-researcher, and thus designer, enactor, re-designer, co-constructor and researcher. If I had sought only (sic!) to learn about students' conceptions of compound interest and annuities, I could have done so from "outside" the course. I place "outside" in quotes because I could never be outside a course I was teaching, but studying students' conceptions through task-based interviews that were focused on their thinking, and through analysis of their test scripts would have moved me away from the centre of the enterprise. So while I am always "inside" the study, at times I am on the periphery of the action.

In this chapter I describe the research design and methodology of the study. I begin by aligning my research with critical realist perspectives, followed by a brief discussion of case-study and first-person research. I then move on to describe the sample, the empirical settings, and the data resources that were selected from the course for analysis in the study. I describe how, in identifying analytic foci and mounting the necessary analytic resources, I shifted from critical incidents to analytic narrative vignettes. I close with a discussion of reliability, validity and generalisation.

As a social theory, critical realism steers a productive path between the extremes of positivism and interpretivism. Shipway (2011) refers to the "ontological realism" and "epistemological relativism" (p. 211) of critical relativism. Scott and Morrison (2006) capture Bhaskar's early work on critical realism (e.g. Bhaskar 1978, 1989) as follows:

> "there are objects in the world whether they are known or not; knowledge is fallible because any claims to knowledge that are made are open to refutation; there are trans-social truths where the observer can only access appearances, but these appearances refer to underlying mechanisms which are not easily apprehended; [and] those … mechanisms may actually contradict … their appearances" (p. 47).

When we speak of mathematical concepts, formulae, algorithms, representations etc., we imply that they exist independently of ourselves. For example, we speak of "teaching the compound interest formula", "calculating the present value of an annuity" and "drawing the graph of an exponential

function". In each case this suggests that we are operating on an object that exists "out there" and which is available for us to operate on[34]. This reflects an ontological realism.

However, I consider knowledge as personal and subjective, not directly perceptible from the empirical – hence the relativism of the epistemological. Nevertheless, a critical realist perspective supports the misconceptions literature in mathematics (e.g. Ben-Zeev, 1996, 1998; Nesher, 1987; Olivier, 1989) and the claims that learners' misconceptions are frequently widespread across time and place, thus . Fundamental to a critical realist perspective is an emancipatory agenda which considers human beings as being able to act on the world, and to change it, in line with their needs and goals. This agenda is reflected in the moral project of the Financial Maths course to provide students access to knowledge of finance and economics that would enable them to act in the world in ways that are not pre-determined by their past or current circumstances.

5.2 Case study research

Yin (1994) describes case study research as a "comprehensive research strategy" (p. 13) that includes research design and data collection procedures. Case studies are useful when one seeks to gain insight into the significant factors that characterise a phenomenon, such as teachers' mathematical knowledge for teaching. While Merriam (1998) suggests that case study research does not have particular methods of data collection or analysis, Yin argues that the data from multiple sources of evidence must converge. The data for this study are drawn from a variety of sources including video, interview and documentary evidence, and I shall show in later chapters the extent to which the data converge.

The case within this case study may be considered a "revelatory" (Yin, 1994) or "extreme" (Flyvberg, 2006) case for studying mathematics-for-teaching, for two reasons. Firstly, the students are prospective *secondary* mathematics teachers and the course is specifically designed for pre-service teachers. This differs from most parts of the world where the preparation of secondary mathematics teachers takes place in Science faculties and courses are not specifically designed for teachers. Secondly, the course deals with *financial mathematics*, which is not a major component of the school mathematics curriculum and is not included in school curricula in many parts of the world. The focus on an area of applied mathematics foregrounds contextual factors such as financial and socio-economic issues, as well as mathematical modelling. A third factor which may be more important than I had initially appreciated is the experience of school mathematics which the students brought with them to the course. As noted in chapter 4, many students came to university with poor mathematical backgrounds due largely to inadequate opportunities to learn mathematics at school.

One of the many criticisms levelled at case study research is that the holistic design is susceptible to "slippage" where the nature of the case may shift during the study (Yin, 1994). While such flexibility may be considered a strength of case research, Yin argues that it needs to be avoided, and if slippage does occur, a new research design needs to be developed. In chapter 4, I referred to such slippage as I discussed how my intended focus on three mini-phases of the course shifted to a dual focus on revisiting school mathematics and on student's conceptions of annuities.

[34] In chapter 2, I noted that I do not enter the philosophical debate on the objectification of mathematical knowledge. This paragraph provides one minor exception, and apart from this paragraph, I refer to "mathematical entities" and not "objects" as a generic term. I use the term "objects" with reference to Sfard's (1991) process-object duality.

5.3 Insider research

Although I have used my teaching as a site for studying mathematics-for-teaching, my intention was not to study my own teaching. This study is thus not concerned with action research, nor is it a self-study of teacher education practice (Loughran, Hamilton, LaBoskey, & Russell, 2004).

There is a diverse range of work in mathematics education that takes an insider perspective. For example, Cotton (2005, 2007) works from a critical perspective and argues that even as a researcher in other teachers' classrooms, he is still an insider in constructing the narrative. Mason (e.g. Mason, 2002) considers research as the increasing development of *noticing* in the observer/teacher. I have drawn on Ball's (2000) on insider research in mathematics teacher education.

Ball (2000) suggests that first-person (or insider) research shares many of the features of other case study research as it "interweaves the empirical with the conceptual …, strives to illuminate a broader point, probe a theoretical issue, develop an argument or framework" (p. 374). It also shares the same "challenges of convincing others that something worthwhile can be learned from the close probing of a single instance" (p. 374).

In addition, she argues that insider research differs from other case study research in two important ways. Firstly, the researcher must *design* the phenomenon s/he wishes to study, and locate it within some local context. Secondly, the researcher must *build* the instance to be studied, working iteratively and making appropriate adjustments as the implementation unfolds. In this way the design and construction work shape each other in dialectic relationship.

With regard to first-person research *on teaching*, Ball (2000) argues that it "offers the researcher a role in creating the phenomenon to be investigated coupled with the capacity to examine it from the inside, to learn that which is less visible. As architect, builder and critic, the researcher-teacher moves fluidly and without interference across roles and functions of the work. These features make possible many inquiries that would be difficult to pursue from the outside" (p. 388).

In this study the roles of designer and builder operated at two levels. Firstly I had to design and build the Financial Maths course within the undergraduate mathematics teacher education programme. As I implemented the course, I faced constant challenges regarding pacing and coverage, both of which were derived from a particular pedagogy I had chosen, as well as students' responses to the tasks they were given. This required continual adjustment of plans which eventually led to a change of pedagogy in the last three weeks of the course. While these elements are not in focus in the study, they reflect the interplay of designing and building as the course took shape.

The roles of designer and builder operated at a theoretical level too. In the absence of adequately-developed theoretical tools, I built a framework for mathematics-for-teaching in order to study it (i.e. MfT) in the context of pre-service maths teacher education. Similarly, in the absence of an adequate literature base, I have built conceptions of compound interest for teaching and annuities for teaching, against which to study these elements of the course. I refer to these as *reference landscapes* for compound interest and annuities. (See chapters 6 and 12)

Ball (2000) warns of the importance of balancing distance and insight in first-person research. While the researcher needs to establish distance to develop an inquiry stance, s/he also needs to capitalise on the insights gained from the intimacy of the teaching setting. One of my challenges in creating distance concerned my emotional response to incidents – both to incidents as they had occurred in class sessions, and to some of the video footage. For example, in class session 4, when Zwaii was explaining his derivation of the compound interest formula, I was extremely frustrated by the class's lack of concern about the level of his explanation and their seeming inability/unwillingness to consider that Grade 10/11 learners would likely not be able to make sense of his derivation. In the days following that session, my recall of the incident was one of frustration with the students and with my inability to get them to appreciate the importance of considering the audience when producing an explanation. Yet eight months later when I began the analysis of the incident, I was astonished that I could not see evidence in the video footage of the emotional intensity I had felt during the session. My initial response was to exclude the incident from the study because the video footage was not an accurate reflection of my experience of it. I subsequently reversed that decision and chapter 10 is the result of the analysis.

In reflecting on my initial, irrational response to the video footage, I was reminded that "if you can't see it or hear it, the video camera won't capture it". So the camera could not capture the emotions I felt during that session. Interestingly, having watched the video footage many more times since, I am still unable to point to evidence in the tone of my voice or my body language that reflects the frustration I felt. Space does not permit a psychological analysis of my perceptions of the incident at the time nor of my various subsequent responses. Suffice it to say that the experience was a very personal reminder of the subjective and complex nature of insider research.

There were several interrelated factors that contributed to establishing distance from the data, some of which I am not easily able to separate. One "cluster of factors" involves the *passing of time, repeated viewings of the footage* and *greater familiarity with the data* through which I came to respond less emotionally to the footage, to the extent that I am now surprised by the "inert-ness" of my reaction to it. Another factor is the *use of theoretical tools* that promote distance from the data. The MfT framework provides a lens that enables me to look for both presences and absences in the data. Initially all I could see in the video footage was an absence of what I wanted students to attend to. By looking systematically for evidence of each of the nine aspects of MfT, I began to see presences that had previously been obscured. This was complemented by Sfard's (1991) process-object distinction through which I was able to appreciate the strengths of Zwaii's explanation but also the demands it places on learners. In short then, the distancing process facilitates insider research.

5.4 The sample

There are two aspects to the sample: the students in the course, and the two focus groups. I discuss these separately.

5.4.1 Participating in the course vs participating in the research project

In chapter 4, I described the students who registered for the course. Together with me, they constituted the participants in the course, and the *potential* participants in the research project. It was therefore important to distinguish participation in the course from participation in the research although I cannot know the extent to which students made this distinction as the course proceeded.

Since the course commenced on the first day of the University academic year, it was not possible to give students advance warning of the research project. In the first session I informed them of the research project, and explained the need to distinguish *taking* the Financial Maths course from *participating in the research*, and hence the need for me to obtain their consent to participate in the research. I assured students that choosing not to participate in the research would not impact their marks, that they were free to withdraw at any point, and that I would not consider this to impact their participation in the course. I was also concerned that students should have a mechanism to raise concerns about the course or the research. I therefore requested a colleague who had taught the students previously to play this role and to attend the first session. I informed students that they could approach her or my supervisor with any concerns regarding the course or the research. (See Appendix A1 – A5 for information and consent forms)

Thirty-six of the 42 students consented to participate in all aspects of the research relating to the lecture sessions. Initially three students did not give consent for their class participation to be videoed. In consultation with my colleague and my supervisor, we agreed that I should approach each student individually to discuss how we might maximise their participation in the course and yet not include them in the video records. To my surprise, as I explained the situation to each student and suggested possible ways of accommodating their participation, each of them agreed to have his/her class participation videoed.

5.5 Selecting participants for the focus groups

I begin this section by describing the initial selection of participants for the focus groups. Thereafter I discuss how the role of the focus groups shifted as the study developed.

As part of the initial design of the study I wanted to access students' thinking and to explore the opportunities for learning mathematics-for-teaching within peer group settings. Tutorial groups would provide a means to do this, and so I designed the study to include a focus on two tutorial groups. I refer to these as the *focus groups*. I do not use the term "focus group" in exactly the same way as the literature. For example, Barbour (2007) takes a broad definition of "focus group" as any group discussion where the researcher deliberately encourages and attends to interactions between the group members. From the outset my concern was with the mathematical content of students' interactions, and not their social relations.

The focus groups constituted a purposive sample, with several selection criteria. There were three primary selection criteria for group members. The first two relate to individual students: (1) evidence of commitment to studies and a track record of regular attendance at lectures; and (2) evidence of commitment to becoming a secondary maths teacher. The third criterion concerned the composition of the group where I sought a diverse range of mathematical achievement. I used performance in previous mathematics courses as a proxy for mathematical achievement. The first two criteria were important because I sought students who would participate actively in class, prepare for lectures, and complete all the necessary task submissions. The third criterion was equally important because I was planning for mixed ability tutorial groups across the class. Furthermore, the range of mathematical abilities would also increase the potential for generalising findings.

The secondary selection criteria were race, gender and year of study. As the lecturer, I wanted to set up heterogeneous tutorial groups, and so the focus groups also needed to be heterogeneous. Ideally I wanted all groups to consist of two males and two females, two fourth year and two third year students, and to be racially diverse. While this was not entirely possible because of the composition of the class, I wanted to prioritise this level of heterogeneity for the focus groups. However, this was dependent on which students volunteered to participate in the focus groups.

While it seemed bizarre to *invite* students to participate in research on a compulsory course in their programme, I could reasonably invite them to participate in the focus groups. When I introduced the study I described the additional commitments for students in the focus groups and asked for volunteers, but I did not indicate my selection criteria for focus group members. Thirty-five students volunteered. I was therefore able to meet my primary or secondary criteria in selecting all eight students. I allocated these students to two tutorial groups and then allocated the rest of the class to the remaining eight groups, as discussed in chapter 4.

The two focus groups were referred to as groups 6 and 10 respectively during the course. During the analysis I have come to refer to them as *Attiyah's group* and *Hailey's group* since it is clear from the video footage of the group tutorials that these two students were considered as the group leaders by their respective group members. I shall also refer to the groups as *A-group* and *H-group*, using the first letter of their names. The letters *A* and *H* should not be taken to imply anything about the students' or the groups' mathematical knowledge or participation in the course. The members of the focus groups are listed in table 5.1.

Attiyah's group		Hailey's group	
Attiyah[35]	Indian, female, 4th year	Hailey	White, female, 4th year
Vusi	Black, male, 4th year	Sakhile	Black, male, 4th year
Shaun	White, male, 3rd year	Jefferson	Black, male, 3rd year
Palesa	Black, female, 3rd year	Lina	Chinese, female, 3rd year

Table 5.1 Focus group students

As the study developed, the focus on learning in peer group settings shifted into the background. I chose to focus on the mathematical content of students' interactions, the obstacles they faced, and the resources used such as timelines and spreadsheets. While there is much evidence that Hailey and Attiyah influenced what was done in the group tutorials and how it was done, these aspects are not dealt with in the thesis. Similarly, I did not explore issues of race and gender in students' participation in the course as a whole nor in the focus groups. Given the socio-economic consequences of apartheid which are typically manifested along racial lines, there is inevitably much that could be explored in relation to students' personal financial experiences, and how these contributed to their learning in the course. However, these issues were not pursued in this study.

[35] As a devout Muslim, Attiyah took a pragmatic approach to dealing with compound interest in the course. Her perspective was that she needed to know about it in order to teach about it but that she would not benefit from compound interest in dealing with her personal finances. In an assignment involving a hypothetical investment, she chose to investigate options in Islamic banking. We talked at length about her perspective in the mid-course interview but this does not form part of the study.

5.5.1 Confidentiality and anonymity

As noted above, when students were informed about the research, they were guaranteed anonymity and confidentiality. At the end of the course I invited them to choose pseudonyms. I also indicated that they could choose to use their own names but that this would have implications for confidentiality and anonymity. Many students chose to use their own names. Consequently the names used in the thesis are a combination of students' actual names and pseudonyms. Where students chose pseudonyms, I have tried as far as possible to conceal their identity. In cases where students requested their own names be used, I clearly cannot hide their identity. However, I have ensured that in such instances I have not reported on personal aspects that were shared in questionnaires or in the interview context. For similar reasons I have not indicated the mathematics marks used in the selection of the focus group students in table 5.1.

5.6 Empirical settings

In chapter 4, I provided a description of the course which may be considered the *empirical field*. In this chapter I delineate the *empirical settings* (Brown & Dowling, 1998) of the two parts of the thesis. I provide an overview of the two periods of the course that are in focus, and then describe the range of data that was selected for analysis. I distinguish between primary and secondary data resources.

A primary data resource is one that has been *systematically analysed over time* to identify critical incidents and obstacles. In the case of a critical incident, I worked systematically through the incident describing in the detail the sequence of events, and studying the transcript to make sense of the mathematical content of students' interactions and their boardwork. In the case of obstacles, I trawled through the data resource for further evidence of the obstacle/s I had identified. This was done repeatedly to ensure that all obstacles had been identified.

A secondary data resource is one where the information was *summarised* in the sequence in which it occurred, but was not *systematically analysed* to identify all occurrences of a particular issue, or to identify themes, students' obstacles, etc. The summaries were searched for evidence to support incidents, themes, etc, that had been identified in the primary data sources. Where the data source was electronically searchable, I searched for instances of key words such as "spreadsheet" or "timeline". For hard copy data sources such as students' journal entries, I searched manually through the annotations I had previously made on the hard copies.

5.6.1 Empirical setting for Part 1 – revisiting school mathematics

The first part of the study deals with revisiting school mathematics, with particular focus on compound interest and the related topics of percentage change and exponential growth. I seek to identify and explore opportunities and potential opportunities that emerge for learning mathematics-for-teaching compound interest, as interpreted through the MfT framework developed in chapter 2. I focus on four consecutive two-hour class sessions spanning twelve days. I refer to *twelve* days, rather than *four* days to acknowledge that students were expected to do independent work outside of contact sessions and thus may potentially have been working on aspects of the course for twelve days. Part 1 includes four analysis chapters which focus on opportunities and potential opportunities for learning the following aspects of mathematics-for-teaching: mathematical practices and conventions, connections with other areas of mathematics and further mathematical inquiry, learners' conceptions, and explaining essential features. Chapters 7, 8 and 9 are constructed around analytic narrative vignettes which are discussed later this chapter. Chapter 10 is structured around two student explanations.

Overview of the twelve-day period of revisiting school mathematics

Revisiting of school maths was done in weeks 2, 3, and 4. The value of selecting data that spans a twelve-day period is that it gives a sense of how the course ran at that time, the pedagogy that was employed and the decisions taken to adjust tasks, emphasising some aspects and ignoring others. Table 5.2 provides brief details of the four class sessions, in the sequence in which they occurred. For the purposes of the study, I refer to *class sessions 1 to 4*, although these were not the first four sessions of the course.

Date	Content of session (each session lasted approximately 2 hours)
Fri 15 Feb	*Class session 1* Wage Doubling Task[36], general form of exponential function, converting wage doubling scenario to percentage growth problem
Tues 19 Feb	*Class session 2* Continue with percentage growth issues from previous session, Computer Operator's Salary Task (involved compound growth scenario and included responses of four learners)
Fri 22 Feb	*Class session 3* Continue with issues emerging from Computer Operator's Salary Task Input from student on microlending in casinos in Macau (not included in the analysis)
Tues 26 Feb	*Class session 4* Task involving unpacking of compound interest formula, and two other tasks dealing with different compounding periods (latter two tasks not included in the analysis)

Table 5.2 Summary of class sessions 1 – 4

Selection of data for Part 1

The majority of the primary data were drawn from class sessions 1 to 4. Secondary data sources for part 1 include a journal entry relating to class session 3, and the mid-course and final interviews with focus group students. Table 5.3 provides a summary of the data resources that were analysed for this part of the study.

	Event/data source	Data	Case 1	Case 2	Case 3	Case 4	Comments
PRIMARY	Class session 1	Video records	x	x			Detailed summaries of each session, transcriptions of selected sections of each session
	Class session 2	Video records		x	x		
	Class session 3	Video records			x		
	Class session 4	Video records				x	
SECONDARY	Students' journals (35)	Written response to prompt			x		
	Mid-course interviews with focus group students (week 8)	Audio records			x		To illustrate and support claims made in the analysis of primary data; involves summaries of interviews and transcriptions of selected student comments
	Final interviews with focus group students (Nov 2008)	Audio records			x		

Table 5.3 Data resources for part 1

5.6.2 Empirical setting for Part 2 – learning new mathematics

The second part of the study focuses on students' thinking in relation to annuities, with particular attention on the obstacles they encountered. I focus on the eight-day period where annuities were

[36] See Appendices C3, C4 and C5 for the handouts of the tasks.

introduced. Part 2 includes three analysis chapters which deal with data from this period. Chapter 13 deals with the introduction of the individual payment approach, and students' take-up of the approach. I distinguish two approaches to annuities: an account balance (AB) approach is concerned with tracking changes in the account balance, which result from transactions and the accumulation of interest; an individual payment (IP) approach tracks the behaviour of each payment over time, as modelled by a geometric progression. (See chapter 12 for a detailed discussion of each approach). Chapter 14 focuses on students' talk about time and use of timelines when working on annuities tasks. In chapter 15, I explore students' use of spreadsheets to solve annuities problems.

Overview of the eight-day period

The eight-day period took place over weeks 5 and 6. It began and ended with a group tutorial, with two two-hour class sessions in between. The value of working over this eight-day period is that it gives insight into students' initial responses to an IP approach, the obstacles they encountered, as well as their take-up of IP thinking by the end of the eight-day period. Table 5.4 provides brief details of the four two-hour sessions in the sequence in which they occurred. I refer to *class session A* and *B* to distinguish these from the class sessions in the twelve-day period in weeks 2 to 4.

Selection of data for Part 2

As with part 1, much of the primary data were drawn from four sessions. In this case, two of these sessions were group tutorials. In addition, I conducted three interviews with the focus groups. The class sessions, group tutorials and focus group interviews were video recorded. Documentary evidence is drawn from two sets of group tutorial reports, and two journal entries for each of the focus group students. I have also included two additional sources that were gathered after the eight-day period: students' test scripts from the test written in week 7, and Hailey's group working on GT5. The analysis of the primary data is supported by extracts from the individual interviews with focus group students. Other documentary evidence is drawn from the questionnaire completed at the start of the course. Tables 5.5a, b provide a summary of the data that was analysed for part 2 of the study. It also indicates the chapters where the analysis of the data is reported.

Date	Content of session (each session lasted approximately 2 hours)
Thurs 6 March	*Group tutorial 3* Students worked on a contextualised problem intended to lead to formula for future value of an ordinary annuity.
Fri 7 March	*Class session A* Further attempts in tutorial groups to produce a formula, followed by the introduction of IP approach. A student explained her group's IP approach, illustrated with calculations of several monthly payments. Another student explained her group's IP approach, making use of a diagram showing individual payments gaining interest each month. Described in detail in chapter 14. I introduced the triangular spreadsheet described in chapters 12 and 15. Journal entry on to IP approach including the questions "What makes sense?", What does not make sense?"
Tues 11 March	*Class session B* Students discussed their responses to the journal entry prompts. I dealt with some questions of clarification about the spreadsheet introduced in class session A. I derived formula for future value of ordinary annuity focusing on a specific case of 12 payments, beginning with representing the scenario on a timeline, and making links to the triangular spreadsheet. I then derived the general case for *n* payments.
Thurs 13 March	*Group tutorial 4* Students worked on another contextualised problem involving future value of an annuity due with missed and double payments.

Table 5.4 Summary of four sessions for part 2

	Event/ Data source	Data	Introduction to IP approach	Time and timelines	Spreadsheets	Comments
PRIMARY	Group tutorial 3	Video records of both focus groups Journal entries of focus group students (8) Group tutorial reports (10)	x x x	x x		Detailed descriptions of each group's mathematical work, full transcription of Hailey's group, partial transcription of Attiyah's group
	Class session A	Video records Journal entries of focus group students (8)	x x	x		Detailed description of mathematical content of session, transcription of sections relating to IP approach
	Focus group interviews	Video records	x	x		Detailed description of mathematical content of each interview, full transcriptions
	Class session B	Video records	x	x		Detailed description of mathematical content of session, transcription of setting up timeline in preparation for derivation of formula
	Group tutorial 4	Video records of both focus groups Group tutorial reports (10)	x x	x x	x x	Detailed descriptions of each group's mathematical work, full transcription of H-group, partial transcription of A-group
	Test scripts (week 7)	Responses to sub-question relating to timelines (35)		x		Quantitative analysis of representations used on timeline, qualitative analysis of use of conventions of timeline
	Group tutorial 5 (week 8)	Video records for H-group			x	Detailed descriptions of group's mathematical work, full transcription
	Interview with H- group on GT5 (week 9)	Video records			x	Detailed description of mathematical content of discussions

Table 5.5a Primary data resources for part 2

	Event/ Data source	Data	Introduction to IP approach	Time and timelines	Spreadsheets	Comments
SECONDARY	Questionnaire (week 1)	Responses to question involving present value of annuity calculation (35)	x			Qualitative analysis and categorisation according to strategy used
	Mid-course interviews with focus group students (week 8)	Audio records	x	x	x	To illustrate and support claims made in the analysis of primary data; involves summaries of interviews and transcriptions of selected student comments
	Final interviews with focus group students (Nov 2008)	Audio records	x	x	x	

Table 5.5b Secondary data resources for part 2

5.7 Collecting and analysing data resources

In this section I describe how the primary and secondary data resources were collected and analysed. I begin with the primary records.

5.7.1 Video footage of class sessions

All class sessions were video recorded by a single, stationary camera which was focused on me, as the lecturer, and tracked my movements. A wire-less microphone linked to the video camera, provided an audio record of my interactions with students as I moved around the room.

I produced detailed summaries of each class session from the video records, focusing on the mathematical content of interactions and the boardwork. In class sessions 1 to 4, transcriptions were produced of the interactions relating to the critical incidents which form the basis of the vignettes. In class sessions A and B, transcriptions were produced of all public interactions relating to the IP approach and timelines. I focused particularly on evidence of students' difficulties and errors, as well as their explicit references to time.

5.7.2 Video footage of group tutorials

The group tutorials form a central part of the empirical setting for part 2, and thus constitute a significant portion of the primary data. GT3 and GT4 mark the beginning and end, respectively, of the eight-day period. I included GT5 for Hailey's group because of their use of spreadsheets in that tutorial. Each of the three group tutorial tasks is described in more detail at the beginning of part 2. (See Appendix C4, C5 and C6 for the details of each group tutorial task)

During the tutorial sessions, the two focus groups worked in a separate venue from the rest of the class, and were video-recorded as they worked. The video camera was "unmanned", and was positioned near the group, with the zoom set so that all four students were in the frame. Given this zoom setting, it was seldom possible to see from the video-records what students were writing. In some instances there was an accompanying audio-record. With hindsight, I should have collected audio-records as backups for all the video-records. The group tutorial reports were the only written records collected from the focus groups. This was an oversight. I should have requested the groups to submit their rough written work in addition to the reports.

I produced detailed descriptions of the focus groups working on the group tutorials from the video records and from their written reports. The descriptions focused on students' calculations, use of AB and/or IP approaches, timelines, formulae, and spreadsheets (where relevant) as well as the mathematical content of their explanations and their references to time. Hailey's group tended to begin each tutorial by working individually, and then discuss their ideas and strategies after an agreed period of time. In that group pairs of students frequently whispered to each other so as not to disrupt other group members, but these whispers were seldom audible from either the video- or audio-records. Attiyah's group worked individually and in silence for much of GT3 but in GT4 they began by discussing all three questions, choosing a common interpretation, and then agreeing upon a strategy for approaching the problem. This provided access to their thinking from the early stages of the tutorial.

Full transcriptions were produced of all three tutorials for Hailey's group. Transcriptions of only key sections in the two tutorials were produced for Attiyah's group. There were several reasons for this. Much of the discussion in GT3 involved a model of consecutive differences which was not relevant to my focus on AB and IP approaches. In GT4 there were technical difficulties with the recording equipment which made it difficult to hear the students' talk. This was exacerbated by the students' tendency to work in pairs, and then to change pairings frequently. As a result, students tended to talk over each other which made transcription very difficult. It was therefore generally easier to work directly from the detailed descriptions and video-records in order to keep track of which pairs were working together. Selected selections were transcribed and analysed in chapters 14 and 15.

In analysing the video footage I sought evidence of the obstacles students faced relating to AB and IP approaches. Having identified these obstacles, I then sought incidents to exemplify the obstacles. When looking for typical and critical examples of students' talk about time, I restricted myself to the incidents that had already been selected.

Once the incidents had been identified, I re-analysed the relevant sections of the video. This involved identifying the central footage for the incident as well as shorter interactions that took place earlier or later in the tutorial relating to the content of the incident. Drawing on Tripp's (1993) metaphor of zooming in and out, it was important to zoom in to study the details of the incident, to reconstruct as far as possible the details of diagrams, and to clarify references to particular parts of documents such as spreadsheets. Zooming out provided a means of linking separate sections of the footage that dealt with the same issue at different points in the tutorial. For example, in GT5 Hailey had to repeat the explanation of her thinking on more than one occasion so it was necessary to identify each instance of the explanation and then link these together to gain a comprehensive picture of her thinking. In analysing the data on students' talk about time, I had to use different strategies for the two groups. Since I had transcriptions of both tutorials for H-group, I was able to trace manually through the transcripts for evidence of explicit talk about time such as references to beginning/end of month, and number of compounding periods for a particular payment. In the case of A-group I worked from the detailed descriptions and video-records, and transcribed the relevant sections. This clearly was not as reliable as working from a transcript.

5.7.3 Group tutorial reports

Initially I did not intend to use the group tutorial reports as a primary data source because some students had not given consent. I later decided to include all the reports because they complemented the data from the two focus groups. However, this was only possible after obtaining retrospective consent from six students after the course had finished. The video-records of the focus groups show that much of the mathematical reasoning and discussion that takes place during the tutorial is not captured in the report. This means that absences in the reports of the focus groups cannot be construed as absences in their discussions and reasoning, and it seems reasonable to extend this assumption to the other groups. Therefore, I do not claim that a report reflects the thinking of the whole group or that it reflects all the work done by the group. Neither do I make claims about the thinking of any particular student from evidence in the report. I can merely consider the reports as evidence of some of the thinking of the group as a collective.

5.7.4 Group interviews

I conducted separate interviews with the focus groups between class sessions A and B. The interviews were held in small tutorial room. They were video- and audio-recorded, and fully transcribed. In the interviews I probed the students about the mathematical work they had produced in GT3, and on their responses to the introduction of the IP approach in class session A. Before each interview, I watched the video footage of the group tutorial, and made note of issues I wished to pursue with the students. These notes formed the basis of an informal interview schedule. For example, when I interviewed Hailey's group, I wanted to track the key points in their progress to deriving a formula for GT3. This required them to report on what they had done together during the GT3 session, then individual reporting by each student on what s/he had done between GT3 and class session A, and then reporting on the sequence of events that led them to obtain the formula in the first hour of class session A. We did not watch footage of the group tutorials during the interviews.

In analysing the interviews I focused on those parts that clarified and/or elaborated the obstacles students faced relating to the IP approach. This complemented and triangulated the analysis of the video footage of the group tutorial, extracts of the video footage of class session A, the tutorial report and students' journal entries. The findings are presented in chapter 13.

I conducted an additional interview with Hailey's group after GT5 to find out more about their use of spreadsheets in making sense of a formula. Hailey had proposed the initial ideas and I wanted to probe the extent to which the other students had made sense of her explanation. Their thinking is discussed in detail in chapter 15. I produced a summary of the students' responses, focusing on evidence of correct and incorrect interpretation of Hailey's explanation. I also paid attention to the aspects that students struggled with, and to evidence of new learning during the interview. For example, Hailey and Sakhile both shared about new relationships in the spreadsheet which they had noticed. The interview was not fully transcribed because I already had a transcription of the video-record of GT5 and there was a great deal of overlap in the content.

5.7.5 Test scripts

One question in the first test required the explicit use of timelines. I did a quantitative and qualitative analysis of students' responses to this question. The test question can be found in Appendix C7. The analysis is provided in chapter 14.

5.7.6 Individual interviews

I conducted two individual semi-structured interviews with each of the eight focus group students. These were held in a small tutorial room and frequently ran longer than intended, ranging in length from approximately 80 minutes to two hours. In two instances the students involved came back to complete the interview on another day. I requested them not to discuss their interviews with each other but I cannot know whether any discussion did take place.

The first set of interviews was done during week 8. The interview schedule consisted of two parts: a set of common questions about students' experiences of the course, and a set of individualised questions concerning issues such as particular journal entries, an incident in a group tutorial, or a response to a question in the first test. The common questions provided opportunity to compare responses across students. The individualised questions enabled insight into students' thinking about compound interest and annuities. Given the specificity of the second part of the interview, I chose to conduct the interviews myself because I could not realistically expect an "external" interviewer to become familiar with the details of each student's coursework. The second set of interviews took place in November 2008, approximately six months after the course had finished. I gave students a set of prompts prior to the interview in order to stimulate their recall of specific sections of the course. I also gave two text book extracts, and asked them to do the tasks and then to consider whether they would use the tasks in their teaching and what difficulties they anticipated learners might experience.

All interviews were audio-recorded, and all written work produced in the interviews, as well as responses to the text books tasks, were collected. (See Appendix B3, B4 and B5 for examples of the interview schedules)

My initial intention was that the interviews would constitute the primary data for the study. However, when the notion of revisiting emerged, it was clear the interview data concerning the early weeks of the course was inadequate because several students did not recall the events in sufficient detail in the interviews and I had not probed adequately. I therefore chose to work primarily with the video records of the class sessions. With regard to knowledge of annuities, I chose to focus first on the eight-day period when the IP approach was introduced. This yielded a great deal of data across several sources. However, given the two-part structure of the study, and thus its breadth, as well as the richness of the data from the group tutorials, I chose to focus primarily on the group tutorial data. Thus the interviews have become a secondary data source for the thesis, and so I draw on them only to support and illustrate my arguments based on the primary data. This frequently takes the form of quotes from students which were transcribed from the audio-recording.

5.7.7 Students' journals

Students were required to keep journals, in which they responded to various prompts I gave periodically during the course. Some prompts related to the content of the session, for example after class session A, I asked students to write about "what makes sense" and "what does not make sense". Other journal entries involved further mathematical work, for example after GT3 students were required to continue their work on finding an annuity formula. The journals were submitted periodically, and copies of journals were made at the end of the course. I have drawn only on the journals of the focus groups students, and then only to complement, support and illustrate arguments emerging from the primary data.

5.7.8 Questionnaire

A questionnaire was administered in the first week of the course. It was designed to gather background information from students on their experiences of learning financial maths at school and in the maths literacy course in first year, and any experiences of teaching financial maths. Other questions addressed experience relating to personal finance. The final question required calculations for a loan repayment scenario. This was intended to gauge what sense students had of the time-value of money. The responses are reported in chapter 13. (See Appendix B1 for questionnaire)

5.7.9 Use of transcripts

I make extensive use of transcripts as part of the analytic narrative vignettes. I checked all transcriptions several times for accuracy against the video- and/or audio-records. Where appropriate, details of students' movements and other non-verbal communication have been included. Although some transcriptions are long, they have been included within the text rather than in an appendix, to help the reader follow the references to particular utterances. Conventions used in the transcripts are given in the introduction to parts 1 and 2.

5.8 Reporting the research

I analyse and report the research in different forms, and using different unit of analysis. By "unit of analysis" I refer to the main entity that is being analysed in that part of the study. In part 1 the unit of analysis is the critical incident, whereas in part 2 the unit of analysis is obstacles the students encountered. The predominant form for reporting is the analytic narrative vignette (Erickson, 1986) but this is not always appropriate for analysing and reporting student thinking, in which case I have reported on the various concerns, difficulties and errors. The units of analysis and forms of reporting are summarised in table 5.6 where I include only the chapters dealing with empirical analysis.

Chapter	Unit of analysis and form of reporting	Brief description
7	Analytic narrative vignette	Challenging definitions and mathematical conventions
8	Analytic narrative vignette	Students' attempt to make connections between formulae and the class's response
9	Analytic narrative vignette	Interpreting timeframes in the Computer Operator's Salary Task
10	Student explanations	Pair of students explaining to me, student explanation to class
13	Analytic narrative vignette, obstacles	Student's introduction of IP approach to class, H-group working on GT4 Students' concerns, difficulties and errors in relation to IP approach
14	Obstacles	Students' difficulties and errors in using timelines and talking about time
15	Analytic narrative vignettes	A-group working on GT4, H-group working on GT5

Table 5.6 Summary of units of analysis for part 1 and part 2

5.8.1 From critical incidents to analytic narrative vignettes

The analytic narrative vignettes were derived from critical incidents which in turn emerged from what might be called "moments of breakdown" as perceived from my perspective as teacher and/or researcher. In class sessions they typically came as surprises to me or as interruptions to the flow of the interaction, for example a student raising an unexpected concern or challenging the thinking of a peer. Some moments of breakdown may be recognised in relation to theoretical and/or analytic constructs such as the operational-structural distinction (Sfard, 1991) or AB and IP approaches to annuities. Other moments emerged in a grounded way, for example Palesa's concern about the use of subscript notation which is reported in chapter 7.

I trace briefly my journey from the production of critical incidents (Tripp, 1993) to the construction of analytic narrative vignettes for part 1. I identify similarities and differences between critical incidents and analytic narrative vignettes, and motivate why I chose analytic narrative vignettes rather than critical incidents to report the research.

In the early stages of the course, I identified several incidents (or moments of breakdown) which I considered to be significant. Inevitably I was a co-constructor in making these incidents significant because of the way in which I chose to respond. In several cases I altered my plans for the session in order to give attention to the incident. As the analysis progressed and the notion of *revisiting* emerged, I began to reconsider these incidents in relation to revisiting school mathematics, and I chose to work with them as critical incidents, following Tripp (1993). For Tripp "critical incidents are produced by the way we look at a situation: a critical incident is an interpretation of an event. To take something as a critical incident is a value judgement we make, and the basis of that value judgement is the significance we attach to the meaning of the incident" (p. 8).

The production of the critical incidents required description followed by analysis. Drawing on Tripp's photography metaphor, I employed the complementary actions of *focus* and *zoom* to produce the description. Focus is concerned with increasing the definition of detail and clarifying the whole picture in order to produce a comprehensive account of the event. Zoom involves both zooming in and zooming out. When zooming in, one obtains a larger image of a part of the picture. This enables further focus on the isolated element/s such as verifying a particular utterance. When zooming out, one sees the smaller part in relation to the broader context.

Although Tripp notes that the level of detail in the description of a critical incident is always relative to its purpose, he emphasises the need to represent an incident as accurately and completely as

possible before engaging in analysis and interpretation. I therefore persisted in elaborating the detail in order to meet, what I perceived to be, Tripp's requirements for rigour with regard to critical incidents. However, the consequence of increasing the level of detail was a blurring of the issues I was trying to foreground. This led to a dilemma – if I increased the level of detail in the description, another researcher may stand more chance of gaining deeper insight into the incident and thus might be better able to judge the appropriateness of my interpretation of the data. Alternatively, the "noise" of the additional detail may overshadow what I considered to be the key aspects of the incident, with the result that another researcher may not be able to see what I was seeing.

I then turned to Erickson's (1986) notion of *analytic narrative vignette* which he describes as "a vivid portrayal of the conduct of an event of everyday life, in which the sights and sounds of what was being said and done are described in the natural sequence of their occurrence in real time. The moment-to-moment style of description in narrative vignette gives the reader a sense of *being there* in the scene" (pp. 149-150, italics in original).

Based on this description, the analytic narrative vignette sounds very similar to the critical incident. However, Erickson qualifies this description to foreground the analytic nature of the narrative vignette, arguing that the analytic narrative vignette is by nature "an abstraction; an analytic caricature … in which some details are sketched in and others are left out; some features are sharpened and heightened in their portrayal … and other features are softened, or left to merge with the background" (p. 150).

Thus Erickson gives "license" to edit the description of an incident for the researcher's purpose. This reflects a shift from Tripp's photographic metaphor to the metaphor of a painter/sketcher who interprets the image (whether photographic or live) and selects some aspects to emphasise and others to background or omit. While a photograph is always an interpretation of some reality, a painting/sketch involves further selection and interpretation, permitting the painter/sketcher both to emphasise and to ignore what the photographer cannot.

So, having begun with the critical incidents, I then adapted and reworked them into narrative vignettes to construct and represent the interpretation that I wished to convey, supported by transcript and evidence of boardwork or other written work.

The description of analytic narrative vignettes provided above may lead to the concern, as Erickson notes, than an analytic narrative vignette is a "potentially dangerous tool" if it is not accompanied by "interpretive commentary, and … other instances of analogous events" (p. 150). Defending her use of analytic narrative vignettes, Adler (1996) notes that "significance is not internal to the incident, but rather in its instantiation and/or illumination of assertions in relation to the study as a whole. In short, the particular must be located in the general" (p. 110). In moving from critical incidents to analytic narrative vignettes, I run the risk of portraying the particular as the general without reference to analogous events. I provide supporting evidence for my interpretation of the vignettes in various ways. For example, in chapter 7, I provide explicit reference to similar events. In most other cases, I refer to data from other sources and other parts of the course to support the assertions made in my interpretation of the vignette.

5.8.2 Analysing and reporting cognitive error and student explanations

In part 2 my focus is on students' thinking as they learn annuities, with particular attention to the part of the course where the IP approach was introduced. Given the lack of research on students' conceptions of annuities, and my conjecture than an IP approach is not intuitive for students, I focused on their concerns, difficulties and errors as they encountered the IP approach. I identified these obstacles by mapping them against the reference landscapes for compound interest and annuities in chapters 6 and 12 respectively. They were found across the range of data sources listed above, and I have reported them in detail in chapter 13.

All chapters dealing with empirical analysis contain examples of students' explanations. My attention to explanations generally focuses on the *content* of the explanation and not on the features. An exception is in chapter 10 where I consider some features of the students' explanations in relation to other aspects of the MfT framework.

5.9 Reliability, validity and generalizability

5.9.1 Reliability

Silverman (2006) argues that credibility of qualitative research lies in the validity of the findings, and not in the reliability of the data and methods. So the question, then, is not whether another researcher would have identified the same critical incidents, or constructed the analytic narrative vignettes in the same way, or produced the same interpretation of students' errors. Rather, the concern must be with whether another researcher using my lenses (e.g. the MfT framework and Sfard's (1991) process-object distinction) *recognises* the critical incidents, analytic narrative vignettes and errors that I identified. Similarly, would the researcher identify the same categories, errors, critical incidents, etc. from multiple viewings of the data over time. I began the data analysis in late 2008 and have worked with the data over a four-year period. Since the study consists of two parts, there were long periods when I worked on data from only one part of the study. For example, from September to December 2009, I worked only on data from part 2. When I returned to the data from part 1, I brought with me "modified lenses" that had been adjusted in the analysis of part 2. This influenced my ideas about issues such as opportunities for learning school maths, aspects of mathematics-for-teaching, and the relationship of compound interest to annuities. Each time I returned to a "dormant" part of the study, I had to work myself back into the data and my earlier interpretations. This continually led to new insights into the data, and refining of interpretations but it did not cause me to reject outright the incidents I had perceived as critical, nor the obstacles experienced by students. This may be taken as evidence of the reliability of my study.

5.9.2 Validity

The validity of an account is determined by its coherence and the extent to which it measures what it is intended to measure. Brown and Dowling (1998) argue that *coherence* is the "fundamental criterion by which educational research is to be judged" (p. 137) and Maxwell (1992) suggests that the validity of an account should be evaluated "on its relationship to those things that it is intended to be an account *of*" (p. 281, italics in original) and not only on its "inner logic and coherence" (p. 285). I draw on Maxwell's distinction between *descriptive*, *interpretative* and theoretical validity to demonstrate the validity of my account.

Descriptive validity concerns the factual accuracy of the account, i.e. the accuracy of the *reporting* of events rather than with the meaning of the events. I have attempted to produce accurate descriptions from the wide range of data resources, including class sessions, group tutorials and interviews, and students' written work. This is reflected, for example, in the transcripts which were carefully checked against the original sources, and scans of students' written work. In several instances I have reproduced boardwork and this has been accurately checked against the original in relation to what was written and its relation to other images on the board.

Interpretive validity is fundamentally concerned with the "participants' perspective" (p. 10), that is with "what [the] object, events and behavio[u]rs *mean* to the people engaged in and with them" (p. 10, italics in original). In my study, interpretative validity refers to inferences about what took place in a class session, a group tutorial, students' thinking, etc. All the reporting in this study is done from my perspective. However, in drawing on multiple sources of data, I have sought to build a plausible interpretation of actions, events and student's thinking from their perspective. For example, the focus group interviews provided opportunity to explore and confirm my interpretation of what had taken place during the group tutorial, and individual interviews, provided opportunity to seek clarification on aspects of students' journal entries.

Theoretical validity "refers to an account's validity as a *theory* of some phenomenon" (p. 291, italics in original) and is concerned with the validity of the concepts that are applied to phenomena and the "postulated relationships among the concepts" (p. 291). For example, in my study I have proposed the constructs of *account balance* approach and *individual payment* approach to annuities. These are descriptive terms for the way in which a student might approach an annuities problem and reflect the strategy the student might employ to solve the problem. Theoretical validity is concerned with the extent to which these are valid constructs and then how these are related. Maxwell argues that "[w]hat counts as theoretical validity … depends on whether these is consensus within the community concerned with the research about the terms used to characterise the phenomenon" (p. 292). Since I have presented these ideas, and elaborated these constructs in doctoral seminars, local and international conferences and in previous publications (e.g. Pournara, 2009b, 2011a, 2011b; Pournara, 2012), and have received support for them, I shall consider the constructs to have been endorsed by the relevant community.

5.9.3 Generalizability

How does one generalise the findings of a case study on a financial maths course for pre-service secondary mathematics teachers? If one wishes to generalize from sample to population, as is typical of quantitative research, then one would need to find other instances of such a course. While this is currently possible within the university where the study was conducted, it could hardly be considered as useful generalisation. However, if one considers generalizability in relation to the theoretical constructs and assertions of the study, then the test of generalizability lies in the extent to which the language of description and associated constructs are usable in further research and practice. For example, to what extent is the MfT framework that I have proposed, and will refine through the thesis, both usable and useful in other studies of mathematics teachers' knowledge? To what extent is the language of "account balance approach" and "individual payment approach" useful in other studies of students' learning of annuities? And, is the notion of "revisiting" school mathematics useful for distinguishing mere revision of the content of school mathematics from opportunities to grapple with the essential features of key content areas of school mathematics?

Furthermore, Flyvberg (2006) argues that extreme cases (and this course may be considered an extreme case as I have already argued) may yield greater insights than typical cases into the phenomenon being investigated because they illuminate more of the basic components which may otherwise be taken-for-granted and thus not be visible. I shall return to this issue in the final chapter.

Ball (2000) argues that the problems of making claims based on a single case study are exacerbated in first-person research because of the "fusion of researcher and teacher" (p. 393). In the analysis chapters in part 1 (i.e. chapters 7, 8, 9 and 10) I begin with an analysis of the vignettes, making claims that are specific to the task and the local context of the vignette. In certain cases I then extend these claims more broadly through reference to the MfT framework through which I consider the opportunities that *were* provided, and the potential opportunities that *may be* provided, for learning particular aspects of MfT of compound interest and/or annuities. In so doing, I do not make claims for "what must be", but rather propose "what is possible" with regard to learning MfT, in similar ways to Lampert's (2001) study of teaching and learning multiplication in a fourth grade classroom. In part 2, the problem of "fusion" of teacher and researcher is less acute because the focus is on obstacles encountered by the students and because most of the data comes from episodes where I was not teaching, for example the group tutorials. While this does not change the fact that I was the teacher-researcher, my role as designer is more prominent than my role as teacher and so I am less centrally involved in the co-construction of what transpired.

5.10 Conclusion

I conclude the introductory part of the thesis by acknowledging once again that this study is my interpretation of data drawn from a course that I taught, to students whom I had previously taught. The analysis and findings are reported in my voice, drawing extensively on transcripts of students' talk and their written work. The incidents I report on in part 1 were critical for me. They were likely not critical for many of the students. The use of analytic narrative vignettes has enabled me to establish distance from the incidents and thus to see the general in the particular. I shall argue that they offer important insights into opportunities and potential opportunities for learning mathematics-for-teaching through revisiting school maths. The use of analytic narrative vignettes has enabled me to establish distance from the incidents and thus to analyse them in relation to the MfT framework.

In part 2, I turn my attention to obstacles students encountered in learning annuities. While I provide evidence of the obstacles from a variety of sources, I am well aware that these obstacles were not experienced by all students. I am also aware that, based on the final assessments, some students completed the course and were still struggling with some aspects of annuities.

INTERLUDE

Introducing Part 1 – Revisiting school mathematics

Part 1 deals with revisiting school mathematics, with particular focus on compound interest and related areas of percentage change and exponential growth. As noted in chapter 3, the notion of *revisiting* is an outcome of the study. While I had always intended to deal with these aspects of school mathematics in the course, I did not think or speak of them in terms of revisiting during the course.

Part 1 consists of six chapters. Chapter 6 provides a *reference landscape* for compound interest, in the context of teaching. I elaborate the concept of compound interest from three perspectives: the underlying mathematical concepts and procedures, typical student errors and difficulties, and daily banking practice. I then draw these together in relation to the MfT framework proposed in Chapter 2. Chapters 7, 8, 9 and 10 consist of four cases of revisiting compound growth, each structured around one or more analytic narrative vignettes, and each read through the lens of the MfT framework. As noted in prior chapters, my focus is on *opportunities and potential opportunities to learn mathematics-for-teaching* by revisiting school maths. I make no claims about what students *learned* in revisiting school maths. The four cases deal with the following aspects of mathematics-for-teaching:

Case 1 – Opportunity to engage with mathematical practices and conventions
Case 2 – Opportunity for making connections with other mathematics and for further mathematical inquiry
Case 3 – Opportunity to engage with learners' conceptions
Case 4 – Opportunities for explaining essential features

There are several instances in these four chapters where it is important to distinguish between the task given to learners in schools and the task designed for the pre-service teachers. I refer to the former as the *learner task* and the latter as the *revisiting task*. While the selection and design of the revisiting tasks was an important element of designing the course, it is not a focus of the study.
At the end of each case I identify the aspects of the MfT framework that are in focus in the chapter. These are drawn together in chapter 11 where I summarise the key findings from each case and relate them to the MfT framework. I also begin to reflect on the framework itself.

A note on transcripts

I make extensive use of transcripts throughout the study. Since I am concerned with a *content analysis* of the transcripts, I have attempted to capture as accurately as possible *what* the speaker said. I have only included how it was said (e.g. intonation, implied meaning) when this seemed important for the content analysis. I have attempted to capture the timing of utterances to reflect ways in which students interrupted or spoke over each other. This is done by indenting the text from the left margin. I make use of ellipsis (…) to indicate short pauses or apparent intentions of the speaker to continue. Underlined text reflects emphasis of a word or phrase by the speaker. Bold text is used to emphasise an issue in the analysis of the transcript.
When referring to the transcripts within the text I use the following conventions:

Utterance numbers are given in square brackets, e.g. [7] refers to utterance 7 in the transcript;

Reference to consecutive utterances is indicated by [20-26];

Reference to a collection of individual utterances is separated by commas, e.g. [2, 3, 12].

CHAPTER 6

A reference landscape for compound interest for teaching

> While mathematics provides the basis to show that amounts grow in mind-bogglingly fast ways due to the compounding effect, society needs to appreciate more deeply the consequence of this growth when borrowing money. There is something about the power of the compounding effect that is very difficult to grasp, even when one knows and understands how the mathematics works. It's really hard to **believe** that 1c takes as long to double as R1 000 000 even though I can prove it algebraically and show it on a spreadsheet. If teachers only teach the formula and get learners to cope with typical exam questions, the learners might get marks but that's not going to help them manage their finances.
>
> My reflections, November 2011

6.1 Introduction

In this chapter I set up what I shall call a *reference landscape* organisation for compound interest. This landscape provides a framework against which to interpret the course and to study students' learning of MfT in the chapters that follow. I do not provide an epistemological genesis of compound interest (for a fascinating history of compound interest starting in Babylonian times, see Hudson, 2000). Rather, I focus on mathematical and financial aspects of compound interest that are pertinent to teaching at school level. I also attend to the role of compound interest in banking.

The ideas I discuss here have emerged through the teaching of the course and the analysis of the data. I did not set out with all of them in mind when I started teaching the course, and some have only emerged deep into the process of analysis. The network diagrams in this chapter are largely an output of the study, and have emerged from the interplay between the literature (e.g. Ma, 1999) and the analysis of the data which is reported in later chapters. It is necessary to include them at this point of the thesis (as opposed to the concluding chapters) because they provide structure for the analysis that follows in later chapters.

In this chapter I deal with many elements of compound interest. I am not claiming that one can learn all these elements at once; they are learned over time. Taken together, they reflect a knowledge of compound interest to which teachers might aspire, but there is no doubt that one can teach compound interest well without much of this knowledge. However, I argue that teachers who deepen and broaden their knowledge of the technical aspects of compound interest, how it operates in banks, and an appreciation of the power of the compounding effect, will be better able to help learners to deal with typical curriculum questions on compound interest, to deal with annuities in higher grades and to understand the impact of compound growth on their personal finances. Likewise, I am not suggesting that learners in school need all this knowledge, although there are aspects that are useful but which are generally not taught in schools. One of my concerns is that compound interest is treated as an end in itself for too long (from Grade 9 to Grade 11). It is only in Grade 12, when annuities are introduced, that compound interest is backgrounded and becomes a tool for doing more meaningful calculations with respect to finance.

In this chapter I locate compound interest in the South African school curriculum and use examples from local text books to illustrate typical tasks. I then discuss research findings that are pertinent to my study, followed by a focus on issues relating to interest and time. I locate compound interest within a hierarchy of related concepts and discuss a network of concepts related to interest with particular emphasis on the notions of *growth factor* and *rate per period*. Thereafter I discuss the derivation of the compound interest formula, distinguishing additive and multiplicative approaches. I then argue for an expanded view of the compound interest formula, introducing the notions of an *accumulation view* and an *adjustment view* of the formula. This is followed by considering how compounding takes place in the banking sector and I show that what is taught in schools does not take into account how banks calculate interest. Thereafter I show that the compound interest formula is both an efficient and accurate model. Finally, I draw the chapter together by returning to the first research question "what is MfT of compound interest?" and answer this through the lens of my MfT framework.

6.2 Compound interest in the South African school curriculum

The notion of interest on money is introduced in the South African Mathematics[37] curriculum in Grade 7 (Department of Education (DoE), 2002). In most texts, learners' first encounter with interest deals with simple interest, followed by compound interest with annual compounding. Thereafter shorter compounding periods, such as quarterly and monthly, are introduced. In Grade 11 the notions of nominal and effective interest rates are introduced, together with problems involving changes in interest rates (and depreciation). Three typical examples of text book questions on compound interest are given below.

Question 1 Brett has R7 500, which he wants to invest for 5 years. Which savings plan will yield more interest: simple interest at 14% p.a. or compound interest at 12% p.a.?

(Cross et al., 2005, p. 60)

Question 2 A sum of money was invested at a nominal annual interest rate of 4.25% per annum compounded quarterly. After five years this investment was worth R4500.

- What sum of money was invested?
- Give the effective annual interest rate for this example.

(Bennie, Blake, & Fitton, 2006, p. 140)

Question 3 R28 000 is invested for 10 years. The interest is calculated at 9.3% p.a. compounded monthly for the first four years. After four years the interest rate is increased to 11.8% p.a. compounded quarterly. Calculate the value of the investment at the end of 10 years.

(Laridon et al., 2006, p. 38)

The purpose of Question 1 is that students should distinguish between calculating simple and compound interest. It is a typical question in both local and international texts. To answer such questions, students will either do the necessary iterative calculations for five years, or they will use the simple and compound interest formulae. It is assumed here that compounding is annual since it is not stated otherwise. By contrast, in the world of banking, compounding is assumed to be monthly unless specified otherwise.

[37] In this study I focus only on the secondary Mathematics curriculum. In South Africa learners are required to choose between the subjects Mathematics and Mathematical Literacy in Grades 10, 11 and 12. Mathematical Literacy is aimed at learners who struggle with mathematics or who do not require the abstraction and formalism of school mathematics for post-secondary education.

Question 2 deals with quarterly compounding over five years, and with nominal and effective interest rates. It requires students to identify the *rate per period* and the associated number of compounding periods. As a typical question for Grade 11 level, learners are expected to be familiar with the compound interest formula, and to convert and substitute the inputs as required. Question 3 requires students to deal with a change in interest rates and in compounding periods. A key aspect here is the different rate per period for two time segments of the question. This question is typical of text book questions that make use of timelines.

All three questions reflect very simplified scenarios from the point of view of the real world. For example, it is assumed that deposits remain in the bank for full periods and, by implication, that they are made at the beginning of the period. No attention is given to the number of days in the year or the number of days in a month. In question 3 the big increase in interest rate and the shift from monthly to quarterly compounding are not likely in the banking world.

One of the consequences of the over-simplified nature of text book questions is that students may assume that maths text books reflect the reality of banking practice, thus leading to potential misunderstandings about how the banking world operates. This may be exacerbated by the South African curriculum reforms of the past fifteen years where connections between the school curriculum and the real world have been over-emphasised. Thus students may think there are banks that really do offer simple interest on standard products, or allow the client to choose between simple and compound interest in a simplistic way. Furthermore, using compounding periods such as quarterly, bi-monthly and daily erroneously suggests that all these options are offered in retail banking. In addition, students tend to assume banks actually use the compound interest formula at the end of every period to determine the accumulated interest. I shall discuss in detail later in this chapter that this is not the case.

The pervasiveness of the tasks described above, both in local and international texts, reflects the strength of the institutional recontextualisation of this knowledge. It may be that these are appropriate ways of introducing students to the notions of simple and compound interest but this is an empirical question, deserving further investigation. Similarly, tasks that refer to unrealistic compounding periods or quote monthly interest rates may reflect choices made for pedagogical purposes, such as getting students to think about the compounding periods in relation to the formula. Such choices may be justifiable from a pedagogical perspective, but I would then argue that it should be made clear that these options are not actually available in the world of banking.

I would therefore argue that the mathematics teacher has a responsibility to give learners access to the everyday world of finance, and to make learners aware that the financial maths tasks found in most school texts do not necessarily reflect the reality of the everyday world of finance. This places additional demands on the teacher in two ways: firstly, to pay explicit attention to values in the curriculum; and secondly, to show that school mathematics is not severed from practical action, despite the nature of some text book tasks which might suggest otherwise.

6.3 Research on conceptions of compound interest

Knowledge of compound interest is not only fundamental to the financial sector but also an essential element of general financial literacy. A report by the Organisation for Economic Cooperation and Development (OECD) (2005) based on survey results indicates that only 28% of respondents in

Australia and only 18% in the United States could calculate compound interest. However, there is little research on student thinking in financial maths, and I have found no studies that focus on students' conceptions of compound interest or annuities at school or university level The studies I discuss below are either situated in the workplace (e.g. Hoyles, et al., 2010; Noss & Hoyles, 1996a, 1996b), have financial literacy or economics as a focus (e.g. Beal & Delpachitra, 2003; Chen & Volpe, 1998; Parramore, 2011), focus on teaching approaches (e.g. Dempsey, 2003; Eddy & Swanson, 1996; Gardner, 2004; Jalbert, Jalbert, & Chan, 2004) or have student thinking in financial maths on the periphery (e.g. Geiger & Goos, 1996; Goos & Geiger, 1995).

The following findings can be gleaned from the limited research on students' knowledge of compound interest and their ability to work with it. While it is difficult to determine the extent to which the findings may be generalisable beyond the original studies, the findings all resonate with my experiences of students' learning and knowledge of compound interest.

- University students as well as the broader population have difficulty in executing compound interest calculations (Dempsey, 2003; 2005);
- University students cannot easily distinguish the impact of simple interest from compound interest on the growth of a principal amount (Beal & Delpachitra, 2003);
- University students, including "business majors", lack knowledge of the impact of increasing the frequency of compounding (Chen & Volpe, 1998);
- University students have difficulty in identifying which formula/procedure to use for time value calculations involving single amounts or multiple payments (Jalbert, et al., 2004);
- High school learners may not convert the nominal annual rate to an appropriate rate per period, and may not easily distinguish whether a formula calculates the accumulated interest or the cumulative balance (Geiger & Goos, 1996).

The work by Hoyles and Noss, and their colleagues (e.g. Hoyles, et al., 2010; Noss & Hoyles, 1996a, 1996b) provides the greatest insight into the learning of compound interest. Their research on mathematical learning in the financial sector formed part of two larger projects in the past twenty years. In both studies, their goal was for employees to see and make sense of the mathematics that underpinned their daily work through studying artefacts of and scenarios within their practice. They found that percentage, compound growth and graphs were pervasive in daily financial work, and that employees' grasp of percentage and interest was deeply intertwined with their working contexts but generally lacked a mathematical basis. For example employees did not see the mathematical similarities between instruments with different payment frequencies or different compounding periods, nor did they see a connection between monthly interest rates and effective annual rates.

Based on their interventions with employees, Bakker and colleagues (Bakker, Kent, Noss, Hoyles, & Bhinder, 2006) argue that an approach of multiplying factors is preferable to adding or subtracting percentages. For example, in the case of adding interest of 6% p.a. or calculating a discount of 25% on some amount, P, they suggest that learners (students or employees) should be encouraged to work with the multiplicative forms of $P(1 + 0.06)$ and $0.75 \times P$ rather than $P + \frac{6}{100}P$, or $P - P \times \frac{25}{100}$. The multiplicative form has several advantages, including its efficiency and its similarity to the compound

interest formula. The findings from both studies are important for my study because they highlight the fundamental role of percentage and compound interest in financial literacy.

In their later work Hoyles et al (2010), emphasised the ability to communicate with the customer to explain printouts, answer queries, and provide meaningful explanations about the benefits of one product over another. There are similarities between the demands of this work and teaching. Both require an ability to distil the essence of the issue and to communicate this to another who is less knowledgeable. This requires explaining an idea in a way that is appropriate for the audience. It requires knowledge that goes beyond substituting into formulae, that appreciates the impact of different variables and parameters on the outputs, and the meanings of those outputs. It may also involve the use of a resource such as a spreadsheet to illustrate relationships between inputs and outputs.

In conclusion, the research findings generally resonate with my previous experiences of teaching financial maths. The importance of multiplicative thinking in working with interest is taken up later in this chapter, and I return to it in several other places in the thesis. The need to help others make sense of mathematical relationships embedded in compound interest and annuities scenarios is a recurring theme in the analysis.

6.4 Conceptualising interest

In this section I distinguish the key features of interest in general and compound interest in particular. In terms of the MfT framework this section relates to the "what" of *essential features* of teachers' knowledge and to *relationship to other mathematics*.

6.4.1 A hierarchy of interest concepts

Drawing on Ma's (1999) notions of depth and breadth of teachers' knowledge and her knowledge packages of mathematical concepts, I propose the hierarchy shown in fig. 6.1 which represents the network of links between key mathematical concepts in simple and compound growth. Concepts that are higher in the diagram build on those that are lower down. I use small-caps font (SMALL CAPS) when referring to the nodes (text boxes) in the diagram.

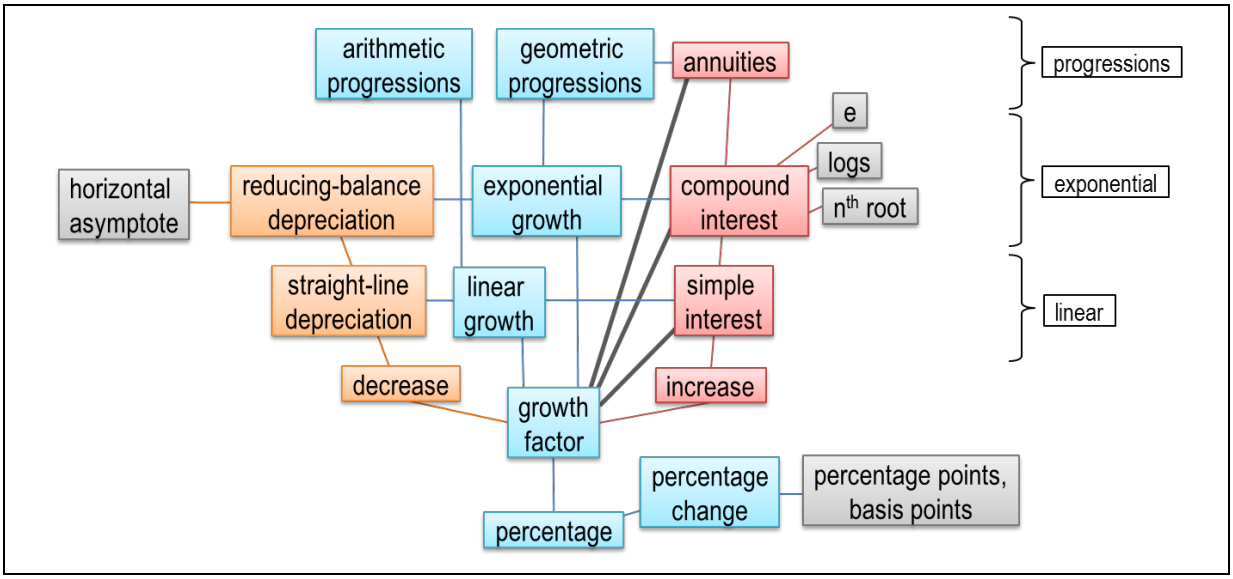


Fig. 6.1 A hierarchy of interest concepts

In network diagrams such as this it is often possible to argue that all components are linked in some way to each other, and thus to insert links between all nodes in the diagram. This is seldom productive because it obscures the key relationships in the mass of connecting lines. I have therefore chosen to represent only the key links between components, and have omitted, for example, the SIMPLE INTEREST – ARITHMETIC PROGRESSIONS link as well as the STRAIGHT-LINE DEPRECIATION – ARITHMETIC PROGRESSIONS link since these are not in focus in this study.

The central blue cluster indicates mathematical concepts, with PERCENTAGE at the lowest level, and extends to PROGRESSIONS. Following Ma (1999) this arrangement reflects that, for example, linear growth is a mathematically more powerful concept than geometric progressions because it underpins a larger number of mathematical topics (although other topics which it underpins, such as rates of change, are not indicated here). The red cluster and orange clusters indicate concepts dealing with increasing and decreasing growth respectively.

At the lowest level of the diagram links are shown between PERCENTAGE, PERCENTAGE CHANGE and their application in financial contexts in the form of PERCENTAGE POINTS and BASIS POINTS.

Building on percentage, and based on the recommendations of Bakker et al (2006), is the concept of GROWTH FACTOR. I use this term for the factor that is multiplied by the "starting value" which could be the principal amount, original price, etc. In the case of simple increase/decrease it is $(1 \pm in)$ and in the case of compound increase/decrease it is $(1 \pm i)^n$. I shall use the term UNIT GROWTH FACTOR for $(1 \pm i)$ in both simple and compound increase/decrease. This term is not indicated in the diagram.

The next two levels involve linear growth and exponential growth, both of which build on GROWTH FACTOR. I place LINEAR GROWTH below EXPONENTIAL GROWTH since constant additive change is a simpler concept than constant multiplicative change (Brown, Küchemann, & Hodgen, 2010). The top level involves PROGRESSIONS, each of which builds on the relevant growth from the levels below.

The diagram also contains three horizontal clusters (linear, exponential and progressions) that reflect the application of the mathematical concepts to financial maths. LINEAR GROWTH is applied in the contexts of SIMPLE INTEREST and STRAIGHT-LINE DEPRECIATION while EXPONENTIAL GROWTH is applied in COMPOUND INTEREST and REDUCING-BALANCE DEPRECIATION. GEOMETRIC PROGRESSIONS are applied in ANNUITIES. In chapter 12, I deal with a hierarchy of concepts related to annuities.

The bold grey lines connecting GROWTH FACTOR to SIMPLE INTEREST, COMPOUND INTEREST and ANNUITIES indicate the importance of the notion of growth factor in calculations involving each of these concepts. This will become particularly evident in chapter 15 in relation to spreadsheets.

The grey boxes indicate connections between the aspects of financial maths and other key mathematical concepts such as n^{TH} ROOTS, LOGS and HORIZONTAL ASYMPTOTES. This reflects Ma's (1999) notion of breadth. These mathematical concepts are necessary when dealing with various aspects of compound increase and decrease at school level. For example, logs are required when determining the length (or number of compounding periods) of an investment. The connection with e applies to continuous compounding at a more advanced level.

6.4.2 A network of concepts relating to *growth factor*

In fig. 6.2, I represent the key ideas regarding interest at the level of the school curriculum and the course. The diagram expands the *growth factor* component of fig. 6.1. I distinguish three main components: the calculation method, the interest rate and the compounding frequency. Each of these has several interrelated sub-components. Different colours have been used to distinguish the three main components, their sub-components and the links. I have represented only the key links between components, and have omitted, for example, the NOMINAL INTEREST RATE – COMPOUND INTEREST link as well as the COMPOUND INTEREST – CONTINUOUS COMPOUNDING link since both links are dependent on the notion of RATE PER PERIOD.

I begin with the CALCULATION METHOD: interest calculated on the principal amount is SIMPLE INTEREST, whereas interest calculated on the latest balance is COMPOUND INTEREST. DAILY INTEREST CALCULATIONS, as done in banks (and which are discussed in detail below) are simple interest calculations which produce the balance on which interest is compounded at the end of the month, hence the links to SIMPLE INTEREST and COMPOUND INTEREST. (See later in this chapter for a discussion of *accumulation view*.)

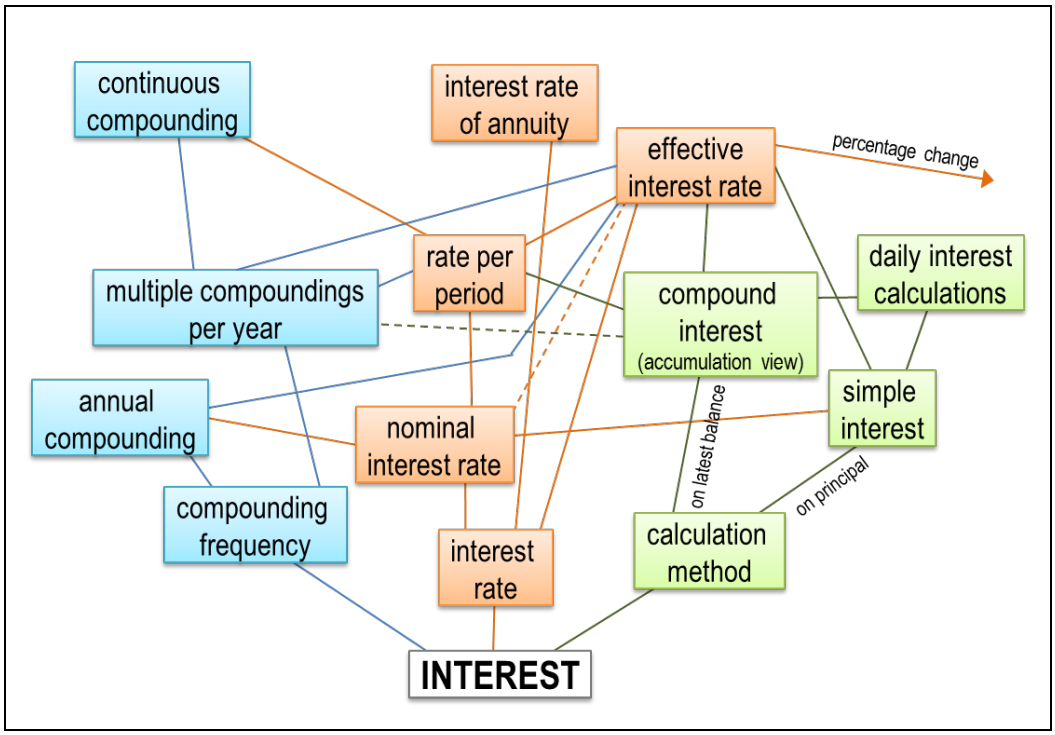


Fig. 6.2 A network of concepts relating to growth factor

With regard to INTEREST RATES, the key distinction is between nominal rates (referred to as NOMINAL INTEREST RATE) and effective rates (referred to as EFFECTIVE INTEREST RATE). In addition, the RATE PER PERIOD is an essential component and might even be considered the central idea in the entire network since it connects INTEREST RATE, COMPOUNDING FREQUENCY and CALCULATION METHOD. I have also distinguished the INTEREST RATE OF AN ANNUITY. This is included for completeness with regard to the scope of the course but is not required in the school curriculum. Consequently I have not inserted links to other sub-components such as RATE PER PERIOD and COMPOUND INTEREST since this would complicate the diagram. Calculating the interest rate of an annuity is a complex process

(requiring the generation of a polynomial function and then the application of Newton's method) and thus the process of determining this rate differs substantially from the other rates.

The COMPOUNDING FREQUENCY determines the amount of interest that will accumulate. I have distinguished ANNUAL COMPOUNDING from MULTIPLE COMPOUNDING PERIODS PER YEAR, and CONTINUOUS COMPOUNDING.

I shall not discuss all the links between sub-components of the same colour as these are generally different instances of each component. I focus mainly on the links between sub-components of different colours.

NOMINAL INTEREST RATE is connected to EFFECTIVE INTEREST RATE (by means of a broken line) since the two notions are dependent on each other. NOMINAL RATE is linked to ANNUAL COMPOUNDING, RATE PER PERIOD and SIMPLE INTEREST because it is the rate used in the respective procedures. It is also linked to RATE PER PERIOD since it is one of the inputs that determines the RATE PER PERIOD.

RATE PER PERIOD also has links from COMPOUND INTEREST and MULTIPLE COMPOUNDINGS PER YEAR. The nominal rate, together with number of compounding periods, determines the interest rate for the smaller periods. The link from COMPOUND INTEREST indicates the need to determine a rate per period when there are multiple compoundings per year. There is also has a link to EFFECTIVE INTEREST RATE since the rate per period is used in calculating the effective rate.

EFFECTIVE INTEREST RATE has links from COMPOUND INTEREST, MULTIPLE COMPOUNDING PERIODS, NOMINAL RATE and RATE PER PERIOD. These three sub-components work together to produce "interest on interest" and the effective rate is the annual interest rate, compounded once, that generates the same yield as interest-on-interest. EFFECTIVE RATE also has a link from ANNUAL COMPOUNDING and SIMPLE INTEREST. The annual compounding link exists because the most important effective rate is the annual effective rate which assumes annual compounding. The simple interest link is important because it supports an explanation of the effective rate in the following manner: if accumulated interest is removed from the account after each interest period, the balance returns to the principal amount. This becomes a simple interest situation because interest is calculated on the principal amount each time. By contrast, if the interest remains in the account, then applying the same rate per period and appropriate compounding frequency will produce a higher yield. This yield, expressed as a percentage of the principal, is the effective annual rate. Since effective rate can be seen as percentage increase, the link to PERCENTAGE CHANGE in fig. 6.1 has been included.

CONTINUOUS COMPOUNDING has links from COMPOUND INTEREST and RATE PER PERIOD. It assumes the number of compoundings per year tends to infinity, thus reaching the limiting situation which can be shown to equal Pe^{it} where P = principal, i = nominal annual rate, and t = time in years.

It is possible to make a link between RATE PER PERIOD and SIMPLE INTEREST. This would be relevant if simple interest is determined more frequently than annually. A theoretical example might be a notice deposit where interest is paid out quarterly and thus a quarterly rate would be used to determine the accumulated interest per quarter.

Based on the teaching of the course and conversations with teachers, it appears that the notion of RATE PER PERIOD may be a key element of learning compound interest. It brings together COMPOUND INTEREST, MULTIPLE COMPOUNDING PERIODS and NOMINAL RATE. Students need to grasp that the nominal rate is sub-divided into k equal portions corresponding with the number of compounding periods per annum. Thus we get one portion of interest per compounding period and this is a linear sub-division. However, this does not lead to equal amounts of interest per period because the "whole" is changing each time the interest is compounded. This is discussed in more detail later in this chapter.

6.5 Deriving a formula for compound interest

In this section I focus on the derivation of the formula which also relates to the aspect of *essential features* in the MfT framework. I show what it might take to appreciate the formula as an algebraic representation of exponential growth. I shall describe two different derivations – one that requires an operational view of the compounding process, and one that requires a structural view (Sfard, 1991).

The compound interest formula is deceptively elegant, particularly for someone learning it for the first time, and in my experience of teaching secondary school students and pre-service maths teachers, the subtleties of the formula take time to grasp. I shall illustrate this through a discussion of the key aspects of the formula and the process of its derivation. I do this from both a mathematical and a pedagogical perspective since I am interested in what it takes to make sense of the origins of the formula for oneself and in what it takes to help others make sense of it.

The first approach builds directly on the process of adding interest to the principal amount, where there is a separation between the principal and the interest which accumulates on the principal. In table 6.1 I have illustrated steps in the derivation that uses this approach, using the following variables[38]:

A_n = amount accumulated at the end of period n
P = principal amount invested
i = nominal interest rate per period

Line	End of period	Expression for process of compounding	Simplified expression
1	1	$A_1 = P + Pi$	$A_1 = P(1+i)$
2	2	$A_2 = P(1+i) + P(1+i)\,i = P(1+i)[1+i]$	$A_2 = P(1+i)^2$
3	3	$A_3 = P(1+i)^2 + P(1+i)^2\,i = P(1+i)^2[1+i]$	$A_3 = P(1+i)^3$
4	...		
5	n	$A_n = P(1+i)^{n-1} + P(1+i)^{n-1}\,i = P(1+i)^{n-1}[1+i]$	$A_n = P(1+i)^n$

Table 6.1 An additive approach to deriving the compound interest formula

Line 1 shows that the principal amount is gaining interest for one period, so the accumulated amount is the sum of P and the interest on P. This is then factorised to simplify the algebraic expression. The new expression, $P(1+i)$, needs to be seen as a single entity rather than the product of a quantity and scalar multiple – it is the closing balance at the end of period 1. The same process repeats at the end of each compounding period, but in choosing to work each time with the principal, P, rather than an iterative formula (that would substitute A_n), the algebraic representation starts to get clumsy. For example in line 2, the student must recognise that $P(1+i)$ is the new amount on which interest will be gained, and be able to separate $P(1+i)$ from the adjacent i in order to recognise the similarity with

[38] I have deliberately chosen to work with A and P, rather than PV (present value) and FV (future value) because I am focusing here on an accumulation view of the formula.

the structure in line 1. As the process continues, it leads to the general formula for the accumulated amount at the end of period n, $P(1+i)^n$. In this context, it is obvious that n represents the number of times that interest is compounded. It will be important to contrast this later with the meaning of n in the annuity formulae.

The intermediate steps shown in the "process" column enable one to distinguish between the opening balance for each period and the *amount* of interest that accumulates in that period. In my experience, when this distinction is emphasised in numeric work, it assists in developing and reinforcing an appreciation of how interest accumulates.

From the point of view of algebraic manipulation, it is fortunate that the expression simplifies elegantly each time, reducing to an exponential form. In the "process" column I have deliberately used square brackets to distinguish between the accumulated amount at the beginning of the period, for example $P(1+i)^2$ at the beginning of period 3, and the new factor of $[1+i]$ that emerges when one factorises the expression for the total amount accumulated at the end of the period. This step of factorising is key in establishing the formula. If one chooses to expand rather than factorise, one ends up with a very messy expression such as $A_3 = Pi^3 + 3Pi^2 + 3Pi + P$ at the end of period 3.

This expression does not point obviously towards the elegant formula. While those who have done more advanced maths may recognise in this expression the binomial expansion of $(1+i)^3$, those who are deriving the formula at school level are unlikely to have encountered such binomial expansion yet and so their manipulation takes them *away* from the desired formula rather than closer to it. This may not be obvious to them – they have expanded and then collected like terms, so they may initially feel they are making progress. However, in my experience, they soon realise that this formula gets increasingly complex and, although there may be a recognisable pattern and a repeatable process, the output for each period cannot be succinctly reduced to a "simple" formula.

In my experience students do not easily recognise the multiplicative structure in the process until they are able to produce the simplified expression. They then seem to accept that the growth is exponential because of the *form* of the expression, not because of the process that has led to the formula. This makes it even more important that they understand the derivation of the formula – to recognise that adding a scalar multiple of the whole, to the whole, produces an exponential relationship.

However, a disadvantage of the algebraic compression in the last column is that it may obscure the interest-gaining process until one recognises that multiplying repeatedly by a factor of $(1+i)$ models the process of growing by a proportion of the original quantity which, in financial terms, is gaining interest on interest. This is not a simple transition. It requires that students operate with the object $(1+i)$. Drawing on Sfard's (1991, 1992; Sfard & Linchevski, 1994) operational-structural distinction, initially students may see $P(1+i)$ only as the outcome of the process of algebraic simplification and not as an object to be operated with. Once a student accepts $P(1+i)^k$ and $(1+i)$ both as objects, s/he may be satisfied to work as follows:

Line	End of period	Expression for compounding	Expression for accumulated amount
1	1	$A_1 = P(1+i)$	$A_1 = P(1+i)$
2	2	$A_2 = P(1+i)(1+i)$	$A_2 = P(1+i)^2$
3	3	$A_3 = P(1+i)^2(1+i)$	$A_3 = P(1+i)^3$
4	...		
5	n	$A_n = P(1+i)^{n-1}(1+i)$	$A_n = P(1+i)^n$

Table 6.2 A multiplicative approach to deriving the compound interest formula

In table 6.2, the student must recognise that for any month, k, the expression $P(1+i)^{k-1}$ represents the accumulated amount from the previous month and that the compounding process can be modelled by multiplying by a factor of $(1+i)$. There is no explicit separation of the accumulated amount from the interest, and when you *one-plus-i* an amount, you get the new amount with interest for one period. Thus each factor of $(1+i)$ should be seen as a *unit growth factor* over a particular time period but it is not possible to determine the interest accumulated for any time period without further (albeit simple) calculations. As Bakker et al (2006) argue, becoming familiar with this multiplicative approach is an essential step in working with percentage. Once students accept the exponential form, they are likely to recognise that the increase is always the same multiple of the previous amount. They may be able to link this to problems involving "discount and a further discount" on the sale price of some item. By contrast the current scenario focuses on "interest and further interest" on the accumulated amount. If the unit growth factor should change after a certain number of periods, m (where $m < n$), in the future, it may be possible for them to appreciate that the new formula will be $A_n = P(1+i_1)^m (1+i_2)^{n-m}$ where i_1 and i_2 are the two nominal interest rates.

6.6 Students' errors and difficulties when working with compound interest

In the first chapter I described how I became increasingly aware of students' learning of financial maths once I became more familiar with the mathematical and financial content. In this section I focus on three areas of difficulty that students experience with compound interest. These are drawn from an amalgam of my previous experience as well as from the literature. I begin with students' difficulties relating to percentage, followed by issues involving interest and time.

6.6.1 Working with percentage

Percentage is one of the key concepts underpinning financial maths, yet it is well-known that percentage is a difficult concept to learn (Parker, 1997). As noted earlier in the chapter, Hoyles and Noss and their colleagues (e.g. Hoyles, et al., 2010; Noss & Hoyles, 1996a) found that employees in financial institutions lacked an adequate grasp of percentage despite it being a key part of their daily work. One of the biggest sources of difficulty with percentage stems from the non-symmetric nature of percentage increase and decrease, and this difficulty extends into adulthood (Parker, 1994, in Parker, 1997). For example, increasing some amount, A, by 10% produces a new amount, A', but decreasing A' by 10% does not undo the increase and so the answer is not A. The symmetric nature of the operations of addition-subtraction and multiplication-division is likely the source of the misconception that percentage increase and decrease are also symmetric. In financial maths this may be the source of students' difficulties in distinguishing depreciation from discounting. I focus in detail on this issue to provide an example of the knowledge of essential features of compound interest that teachers require for teaching.

Firstly, I define the following processes in ways that are appropriate for the school curriculum. I also provide the formula for each process.

Appreciation	: A sum of money gaining interest (on the latest balance) as we move forward in time	$FV = PV(1+i)^n$
Depreciation	: An asset that loses value (as a percentage of its diminishing value) as we move forward in time	$FV = PV(1-i)^n$
Compounding	: Moving an amount of money forward in time at a given interest rate	$FV = PV(1+i)^n$
Discounting	: Moving an amount of money backward in time at a given interest rate	$PV = FV(1+i)^{-n}$

It is important to note that the compound interest formula is used for both appreciation and compounding. In both cases we are dealing with time in the future. However, depreciation and discounting do not work in the same way. Depreciation refers to some point in the future but discounting refers to time past. The reverse operations of compounding and discounting can be seen in the formulae. For discounting we have simply made *PV* the subject of the formula. Consequently these two operations are symmetrical – the one operation will "undo" the other. In terms of timelines, compounding moves an amount to the right on the timeline whereas discounting moves the amount to the left.

One of the difficulties in using the term "compounding", is that it is used in various ways. For example, in the above discussion I could have replaced "appreciation" with "compounding" with little or no consequence. Later in this chapter I shall distinguish carefully between *calculating* and *compounding* interest where the latter applies to gaining "interest on interest" but calculating interest works off the original balance for the period.

Earlier in this chapter I referred to the term *unit growth factor* in relation to simple and compound increase/decrease. I now expand its use to refer to appreciation, depreciation, compounding and discounting – essentially it is the term $(1 \pm i)$.

Next, I return to Question 2 from the text book examples discussed earlier in the chapter:

> A sum of money was invested at a nominal annual interest rate of 4.25% per annum compounded quarterly. After five years this investment was worth R4500. What sum of money was invested?

Using the compound interest formula, we can determine that:

$$\text{PV} = \frac{4500}{\left(1+\frac{4.25\%}{4}\right)^{20}} = \text{R } 3642.60.$$

However, I have seen students attempt similar problems by depreciating the future value for 20 periods as follows: $PV = 4500\left(1-\frac{4.25\%}{4}\right)^{20} = R\ 3634.39.$ This error suggests students are treating compounding and depreciation (with reducing balance) as reverse operations. This is not surprising at one level because the compounding process *"adds" on* a percentage of the latest balance at the end of each compounding period, while the depreciation process *removes* a percentage of the balance at the end of each compounding period. Of course the error lies in the referent to which the percentage applies but this is less obvious when using the formulae than it is when doing a single percentage change calculation.

6.6.2 Working with multiple compounding periods and interest

Based on my experience, when students work with multiple compounding periods they make several typical errors. Consider, for example, the first part of Question 2 of the text book questions discussed earlier:

> A sum of money was invested at a nominal annual interest rate of 4.25% per annum compounded quarterly. After five years this investment was worth R4500. What sum of money was invested?

Two typical errors that students make are: failing to divide the nominal annual rate (4.25%) by 4 to get an effective rate per quarter; and using an exponent of 5 (years) instead of 20 (quarters). There was evidence of both these errors in students' work in the course.

However, a more serious issue is that students have difficulty in recognising the difference between *equal percentages* of interest but *different amounts* of interest. In my experience, some students assume that equal interest rates per period imply equal amounts of interest per period. This may be evidence of overgeneralising of linear thinking (De Bock, Van Dooren, Verschaffel, & Janssens, 2002; Esteley, Villareal, & Alagia, 2010), an issue that I shall return to in part 2 of the thesis.

The arithmetic of dividing the nominal annual rate to produce a rate per compounding period is simple, and the reason for dividing can glibly be given as "because there are 4 quarters in a year", for example. However, when pushed to explain how interest accumulates each period, I have seen students struggle to explain what the rate per period does in terms of gaining interest. I explain this issue in detail below as an example of a challenge that teachers may face in helping students make sense of a difficult piece of mathematics.

Firstly, it is important to recognise that when dealing with multiple compoundings of interest per year, the nominal annual interest rate is sub-divided into *equal* portions, depending on the number of compounding periods in the year, to obtain an effective (which is also the nominal) interest rate per period. So for example, a payment of R300 growing at a nominal annual rate of 9% p.a. compounded monthly gains 0.75% interest each month, since 9 is divided into 12 equal portions of 0.75. However, the *amount* of interest associated with this monthly rate is different each month because of compounding on the latest balance, i.e. because "the whole" changes. In fig. 6.3 it can be seen that interest for the first month is R2.25 but interest for the second month is R2.27 (rounded to two decimal places).

	1	2	3	4	5	6	7	8	9	10	11	12
Portion of interest (%)	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
Interest per month (R)	2.25	2.27	2.28	2.30	2.32	2.34	2.35	2.37	2.39	2.41	2.43	2.44

Fig. 6.3 Different interest amounts per month based on increasing balance

Adding the twelve values in the table gives total interest of R28.14. Using the compound interest formula: $\text{FV} = 300\left(1 + \frac{9\%}{12}\right)^{12} = \text{R}328.142$, and thus the interest is also R28.14. While the formula is efficient, it hides the calculation of interest each month and thus the fact that the accumulated interest differs each month.

This example illustrates the work of unpacking that teachers might do in order to make visible the process that is hidden by the formula. Later in this chapter I shall show that this method of compounding interest is not the one used daily in banks, although the above explanation is appropriate and sufficient for the purposes of unpacking the monthly compounding of interest as modelled by the compound interest formula.

Finally, it is worth noting that while the *ratios* of the interest from month to month are clearly constant (at 1.0075), the *difference* in monthly interest amounts is not constant. For example the differences in interest between months 9 and 10, months 10 and 11, and months 11 and 12 are 2c, 2c, and 1c respectively. This appears to contradict the idea that amount of interest increases each month, and the amount by which it increases also increases (because the balance grows each month). It is important to note that the gaps (2c, 2c and 1c) are not essentially a result of rounding although the actual figures are a result of rounding to two decimal places. The source of the non-constant difference lies in the fact that the interest amounts have been generated by a multiplicative relationship and not an additive one. Yet, we speak of *adding* interest to the balance at the end of each month, and so it is not surprising that our tendency is to reason additively when comparing the interest amounts in the table.

While the discussion above may not be appropriately calibrated for school level, it highlights some of the knowledge required by teachers. Firstly, teachers need to know that the "multiplicative differences" (i.e. ratios) are constant but the "additive differences" are not. Then they require knowledge that: (1) learners may notice the different gaps between amounts and question the apparent contradiction that the rate of increase of the interest is not always increasing; (2) that a comparison based on additive reasoning is intuitive and sensible; (3) that the problem does not lie in rounding of monetary amounts; but (4) that a comparison based on ratios is constant (and is based on the factor $(1 + i)$). Furthermore they require knowledge of how to explain that the ratios are equal, and why the differences are not. This is indeed a challenging task.

6.6.3 Dealing with time-related issues

While we cannot speak of the value of money independently of time, the curriculum documents do not emphasise the relationship between time and value with respect to money. Consequently, school maths text books tend not to emphasise it either. From the perspective of the school financial maths curriculum, time-related issues largely focus on whether one is referring to the beginning or to the end of a period. In many tasks involving single payments, the context provides sufficient clarity for timeframes, provided students are familiar with the meaning of words used in the context of time, and the genre of word problems. The following phrases related to time have been extracted from the text book questions discussed earlier in the chapter.

Question		Phrases referring to time
1	-	**for** 5 years
2	-	**after** 5 years
3	-	**for** 10 years; **for** the first four years; **after** four years; **at the end of** 10 years

In each case above the learner is required to focus on the end of the period. Two other phrases that frequently also imply the end of the period are: "*in* 5 years" and "*over* 5 years". In chapter 9, I shall discuss an incident where students raised a concern about the meaning of the phrase "after 3 years". Not with standing this incident, in my experience tasks involving single payments seldom provoke a

critical consideration of the timeframes involved in a question. In annuity contexts attention to time becomes much more important. For example, it is important to distinguish between paying at the beginning versus the end of a month. I shall discuss this in more detail in chapter 12. Timelines are introduced in most financial maths texts as a way to represent key events such as deposits, withdrawals and interest rate changes. This is usually done in the context of multiple payments, and I shall discuss this too in more detail in chapter 12.

However, my experience suggests that timelines should be introduced before dealing with multiple payment scenarios, and that this should be done as the need arises, rather than teaching them explicitly as a "sub-topic" of financial maths. By introducing timelines at an opportune time, students (and learners) come to see them as useful representations rather than "something to learn".

6.7 An expanded view of the compound interest formula

Students are introduced to the compound interest formula to determine the accumulated value of a certain amount over a period of time. While this is a necessary and important use of the formula, it is inadequate for developing a full grasp of annuities. In this section I shall argue for an expanded view of the compound interest formula, similar to Kieran's (1981) call for an expanded view of the equals sign to include both operational and equivalence conceptions of the sign. I shall distinguish between two different views of the compound interest formula. The first I call an *accumulation* view; the second I call an *adjustment* view.

The accumulation view is a static view, best exemplified in questions of the form: an amount, P, is invested at a certain rate, r, compounded monthly for a certain period, n, and we want to determine, A, the amount that accumulates. In this instance we focus on the original amount and then on the final amount, and compare (in an additive sense) the amount by which the original has grown because this tells us how much interest has accumulated. We are only interested in the magnitude of A, not its relative value in relation to the passage of time, or corresponding changes in its buying power. The dominant message is that the principal amount accumulates interest and becomes "more". The focus is thus on the *nominal* value of the principal amount which is separated from time. The only purpose for considering time is to determine how many times compounding takes place. The accumulation view is associated with appreciation and depreciation.

The adjustment view focuses on the time value of money. It is a dynamic view where we are interested in adjusting for the effects of time, or more correctly, the effects of inflation, exchange rates, etc. So we are concerned with the value of an amount at different points in time, and we can use the compound interest formula as a mechanism to move our amount of money to different points in time along a timeline. It is therefore associated with compounding and discounting. Here it may be better to refer to *present value* (rather than principal) and *future value* (rather an accumulated amount), thus $FV = PV(1 + i)^n$. A useful image might be to think of a slider on a computer application. Imagine a horizontal line representing time and a dot on the line representing where we are in time. As we slide the dot left or right along the line, the value of the lump sum will decrease or increase based on how far we move left or right along the timeline. If the dot is placed on "now", then our money shows the current amount. If we move left, the amount goes down, and when we move right it goes up. The amount of change is determined by the interest rate and by how far we move along the line. So the value of the money is a function of where we are on the timeline (and the interest rate) and thus

inextricably linked to time. This is similar to the first level skill proposed by Eddy and Swanson (1996) with regard to timeline operations. From an adjustment point of view, we are less interested in the actual magnitude of the number, and more concerned with its relative value at a different point in time. In real terms, the new amount may not have a higher value than the principal amount had in the past, although in nominal terms the new amount is "more" because it is a higher number. But since the time value of money is linked to its buying power, the magnitude of the number must always be seen in relation to what it can buy, such as groceries or foreign currency.

The accumulation view is emphasised throughout the school curriculum. Students are asked questions that require them to solve for an unknown in the compound interest formula and the focus is generally on the difference between the initial and final amounts, even when required to calculate the time period of the investment. However, when dealing with annuities, the compound interest formula needs to be viewed from an adjustment perspective because the individual payments are being moved forwards or backwards in time to determine their contribution to a loan, outstanding balance or projected savings. However, annuities are only introduced in Grade 12 and so the need for an expanded view of the formula comes very late. Yet, based on discussions with actuaries working in the financial sector and in academia, the adjustment view is the one which they use most often. I did not have the language of accumulation and adjustment at the time of these discussions but typical statements made by these actuaries, such as 'discounting a payment back to T_0' (i.e. the point at which a loan is taken or an annuity is purchased) reflect an adjustment view.

In school mathematics the prevalent view of the compound interest formula does not appear to reflect its dominant use in the banking sector and in actuarial science. This may be a consequence of an over-emphasis on basic interest calculations for several years of schooling, an under-emphasis of annuities, and little attention to the notion of the time value of money.

When we adopt an adjustment view of the compound interest formula, we are working with a different model to the one associated with an accumulation view. While the actual calculations are the same as the model described above, the interpretation of the numbers is different. In this case the model predicts the relative value of a sum of money at different points in time. The accuracy of the model is largely dependent on the interest rate and the change in rate between the present and predicted point in time. If we are working with time past, then the accuracy of the model can be determined based on existing data about that period. If we are predicting future values, then the accuracy of the model is based on historic trends and the extent to which predictions of future behaviour of the markets seems reasonable. However, there are far more sophisticated models that deal with predicting future values and so it would probably be more accurate to see the compound interest formula simply as a mechanism for moving a sum of money up and down the timeline.

The distinction between an accumulation and adjustment view emerged through the analysis and therefore did not form part of the course.

6.8 Compound interest and the real world

In my earlier discussion of compound interest in the school curriculum, I noted that the selected text book examples did not reflect the complexity of the banking world. When it comes to simple interest, one of the unintended consequences of the school curriculum is that learners tend to associate simple

interest with equal amounts of interest for each period, rather than interest calculated on the principal amount. This issue was raised by Shaun in an interview midway through the course regarding a spreadsheet I had provided. He was unsure whether he was working with simple interest because the interest amounts each month were different. Although we may define simple interest as interest which is calculated on the principal amount, learners are generally only exposed to examples where the interest amounts are the same each period. In the absence of evidence to the contrary, it is therefore not surprising that they may conceive of simple interest scenarios as adding constant amounts of interest. Thus teachers' choice of examples plays an important role in supporting the definition of simple interest. One way of addressing the above-mentioned error is by dealing with daily compounding. However, as I shall show, this requires teachers to integrate a broad range of knowledge in order to shift from sterile text book tasks to tasks that deal with typical realities such as the actual number of days in each month. An explanation of daily interest follows in which I show the accuracy and efficiency of the compound interest formula as a model for calculating interest.

6.8.1 Interest in the world of banking

In the world of banking, interest is calculated daily and compound monthly. This means that interest calculations done each day use a *daily interest rate*, and interest is calculated on the balance in the account at midnight[39]. However, the interest is not added to the account until midnight on the last day of the month. So we need to distinguish between *calculating* interest and *compounding* interest. I find it useful to think about this using the analogy of a large bucket and a small bucket. The account balance is stored in the large bucket and any transactions during the month (such as making a withdrawal or a deposit) impact the balance in this bucket. The small bucket accumulates the interest each day but this is not visible to the client during the month. At month end, the small bucket is emptied into the large bucket. This increases the opening balance for the new month. Interest will then be calculated on the new balance which consists of the closing balance at month end plus the interest from the previous month – hence compounding of interest monthly. In the new month, interest accumulates again in the small bucket until the end of the month when it will be tipped into the large bucket again. The interest calculated each day is simple interest since it accumulates on the daily balance and does not include interest from previous days in the month. At the start of the following month, the new balance includes interest from the previous month. Thus by definition, interest is being compounded at this point since it is being calculated on the latest balance which includes interest. So *calculating* interest is represented by adding interest to the small bucket every night at midnight, whereas *compounding* interest is represented by tipping the small bucket into the big bucket at month end.

Although this reflects a combination of simple and compound interest, banks are not using the simple or compound interest formulae in their daily calculations of interest. They are merely doing a percentage calculation to determine the amount of interest for the day. These daily interest calculations provide the necessary flexibility for banks to deal with transactions at any stage of the month. The simple and compound interest formulae are simply models that enable us to make sense of what is happening to a particular sum of money over time. The compounding effect happens as a result of

[39] Throughout the thesis I refer to interest being calculated or compounded at "midnight". In reality banks may not carry out this process exactly at midnight – it will take place at some point after close of business, and forms part of a broader daily reconciliation process that may take several hours to complete. I therefore refer to "midnight" in a metaphorical sense. The key issue is that interest is calculated/compounded at *discrete* points in time and not continuously.

adding the accumulated interest to the account and then calculating interest on this new amount the following month.

The daily calculation of interest is a tedious exercise if done manually. In order to predict future values of money, we need an efficient method that will provide a good prediction under given constraints. The compound interest formula provides a very accurate and efficient model of daily interest calculations with monthly compounding. Consider the example: R5000 is invested at 12% p.a. for a year with daily calculation of interest and monthly compounding. This scenario can be represented by the spreadsheet below (table 6.3) which calculates daily interest for each month, assuming 28 days in February. The accumulated interest is added to the account at the end of each month. All calculations have been done with maximum accuracy although only some columns show six or more decimal places. The daily interest rate is based on a day count convention of 365 days, where the daily rate is calculated by dividing the nominal annual rate by 365 irrespective of the actual number of days in the year. Thus the cumulative interest for January is calculated as follows: $12\% \div 365 \times 31 \times 5000 = 50.96$ (to 2 decimal places). It is worth noting that the cumulative interest for February is less than that of January despite the opening balance being higher. This difference results from fewer days in February. These figures should dispel the belief that, in simple interest scenarios, the interest is constant for all periods.

Month	No of days	Opening balance	Daily interest rate	Daily interest	Cumulative interest for period	Closing balance
Jan	31	5000.00	0.00032876712	1.643836	50.958904	5050.96
Feb	28	5050.96	0.00032876712	1.660589	46.496498	5097.46
Mar	31	5097.46	0.00032876712	1.675876	51.952148	5149.41
Apr	30	5149.41	0.00032876712	1.692956	50.788677	5200.20
May	31	5200.20	0.00032876712	1.709654	52.999260	5253.20
Jun	30	5253.20	0.00032876712	1.727078	51.812339	5305.01
Jul	31	5305.01	0.00032876712	1.744112	54.067477	5359.08
Aug	31	5359.08	0.00032876712	1.761888	54.618521	5413.69
Sep	30	5413.69	0.00032876712	1.779845	53.395336	5467.09
Oct	31	5467.09	0.00032876712	1.797399	55.719374	5522.81
Nov	30	5522.81	0.00032876712	1.815718	54.471536	5577.28
Dec	31	5577.28	0.00032876712	1.833626	56.842416	5634.12
TOTAL	365				634.122488	

Table 6.3 Cumulative interest based on daily interest calculations and monthly compounding

The spreadsheet gives an accumulated amount of R5634.122488 (to 6 decimal places) while the compound interest formula gives an answer of R5634.125151 (to 6 decimal places). The difference is less than 0.0004%. For a leap year, the spreadsheet answer would be R635.957907 while the compound interest formula remains unchanged – a difference of less than 0.2882%.

I have not dealt with portions of months but the procedures discussed above can easily be adjusted to make provision for a deposit made or withdrawn in the middle of a month.

In order to compare the accuracy of the compound interest formula, one needs to consider its implicit assumptions as well as the different day count conventions. The formula assumes that all months have the same number of days and that there are an equal number of days in each year. There is a range of

different day count conventions in use across the world. As mentioned above, in general, Rand-based products use a 365-day convention. By contrast the typical standard for US dollar-based markets is 360 days, and there are other markets that use an actual-day convention where the daily interest rate is calculated by dividing the nominal annual rate by the actual number of days in the year. Fig. 6.4 provides a summary of the accuracy of the compound interest formula as model for these three different scenarios. I will consider the spreadsheet answer as the *actual* answer and the answer from the compound interest formula as the *predicted* answer. I have chosen the most extreme case for each scenario, where possible, in order to indicate the maximum possible error. The table shows that the maximum error for the chosen scenarios is less than 1.74%, and there is an outlier amongst the four cases.

Day count convention	Actual number of days in year	Actual (from spreadsheet)	Predicted (CI formula)	Difference (actual – predicted)	% error
360	366	645.290428	634.125151	11.165278	1.7303%
365	366	635.957907	634.125151	1.832756	0.2882%
actual	365	634.122488	634.125151	-0.002662	-0.0004%
actual	366	634.123666	634.125151	-0.001485	-0.0002%

Table 6.4 Comparison of actual vs predicted amounts

Up to this point I have focused on full years. If one considers only parts of years, then additional comparisons could be done dealing with the actual months involved. For example there are 181 days in the first six months of a non-leap year and 184 days in the second six months of the year. This makes a difference to the actual interest gained when daily interest calculations are considered but there would be no difference in the predicted values.

The above discussion reflects the gap between the notion of compound interest as presented in text books and the compounding of interest in the world of banking. If teachers are to bridge this gap for themselves and ultimately for their learners, they require a wide range of knowledge which is generally not found in school texts or university-level introductory financial maths texts. For example, they require contextual knowledge such as daily interest calculations and international day count conventions. They also require knowledge of basic modelling practices including the notion of error in mathematical models, and appropriate metaphors and analogies to explain the daily banking process. The bucket-analogy presented above is one such example.

6.9 What is Maths-for-Teaching compound interest?

Thus far I have focused on several aspects that are important for teachers to know in teaching compound interest. As I noted in the introduction, I am not claiming that teachers should know all these aspects before they can teach compound interest successfully. Typical tasks from text books, as shown at the start of this chapter, tend to focus on substituting into the formula, and thus make fewer demands on teachers' knowledge of compound interest. By contrast, dealing with the banking practice of daily interest calculations requires teachers to coordinate knowledge from a far wider range of sources. In the remainder of this chapter I consider the nine aspects of MfT of compound interest, discussing each aspect of the MfT framework. (See chapter 2 for initial discussions of the MfT framework)

6.9.1 Essential features

Identifying the concept of compound interest – Knowledge of what constitutes compound interest includes knowledge of the definition, the compound interest formula, and the meaning of each symbol in the formula. It also includes knowledge of appropriate representations such as numeric tables of values (which could include a spreadsheet), algebraic forms (the formula), and graphical representations showing exponential growth, as well as timelines. Teachers also need knowledge of the different symbolic forms that are used in texts. For example some texts give the compound interest formula as $A = P\left(1 + \frac{r}{100}\right)^n$ while others use $FV = PV(1 + i)^n$. Similarly, in order to represent multiple compounding periods, some texts (e.g. Young, 1993) use $\frac{i_{12}}{12}$ to represent monthly compounding where i is the nominal annual rate already expressed as a decimal value. By contrast Laridon et al (2007) use the unusual notation of $\frac{i^{(m)}}{m}$ to represent and determine nominal rate per period, where $i^{(m)}$ represents the nominal annual rate that will be compounded m times in the year. The symbol v which represents the discount factor $(1 + i)^{-n}$, is commonly used in actuarial work but is not found in school texts. For this reason it was not included in the course.

In order to identify a concept, one should also be able to identify what is not an instance of the concept, for example, distinguishing compound from simple interest in terms of definition, formulae and representations. Earlier in the chapter I discussed the similarities and differences between appreciation, compounding, depreciation and discounting. It is also important to recognise instances where the compound interest formula is not applicable, such as annuity scenarios. In addition, there are two other important, counter-intuitive (Fischbein, 1999) properties of compound interest: that simple interest will have a higher yield than compound interest, at the same rate, for periods shorter than one compounding period; and that time taken for an amount to double/treble etc. is independent of the amount.

Ways of working with compound interest – Knowledge of how to work with compound interest includes working with and without the formula. This foregrounds the importance of a *growth factor* such as $(1 + i)^n$ and the *unit growth factor*, $(1 + i)$. While one might assume that repeated percentage calculations are only necessary until the formula is introduced to learners, in a later chapter I shall show evidence of learner work where the formula is not used to answer an apparently simple problem involving compound growth. This requires teachers to move flexibly between the formula and the inductive processes that produced it.

Working with the formula involves several aspects. Earlier in the chapter I referred to different derivations of the formula where one may take either an additive or a multiplicative approach. I argued that the multiplicative approach requires students to have a structural conception of the symbols being operated on, and thus is more cognitively demanding. Teachers require knowledge of both derivations. Knowledge of working with compound interest also includes substituting into the formula, and manipulating the formula to determine any of the unknowns, such as solving for n to determine the time taken to accumulate a certain amount at a given interest rate. This also requires knowledge that solving for i and n requires mathematics that is only introduced in Grades 11 and 12 respectively. All numeric work assumes knowledge of how to convert inputs to the same unit of time (e.g. years, months or days) and central to this conversion is determining a *rate per period*. This coordinates the nominal rate with the number of compounding periods per annum as discussed earlier in the chapter. Working with the concept also includes knowledge of how to use the formula to show that the amount

of interest increases as the number of compounding periods increases (but that this tends to a limit). Finally, working with compound interest includes ways of talking about timeframes – using appropriate levels of precision, such as making explicit reference to the beginning or end of a period, and the importance of distinguishing discrete points in time from the intervals between them.

6.9.2 Relationship to other mathematics

When dealing with the relationship of compound interest to other mathematics, it is difficult to make hard distinctions between "pure" mathematics and financial mathematics. Applying Ma's (1999) notions of depth and breadth to teachers' mathematical knowledge, I identify concepts that lie at a lower level than compound interest and those that lie higher up (see fig. 6.1). The key mathematical concepts at a lower level include percentage, percentage change and linear growth. At a higher level are (geometric) progressions and annuities. At an even higher level, there are links to e through the limit: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ within the context of continuous compounding. With regard to breadth, there are connections to other concepts of school mathematics, for example use of logs when solving for n, and the use of n^{th} roots when determining the interest rate.

Connections can be made with the binomial expansion of $(1+i)^n$ although this may not be typical of the work done in financial maths at school (and therefore was not included in fig. 6.1). However, in chapters 10, 12 and 13, I shall show evidence of students working with the expansion, either implicitly or explicitly, when revisiting school maths and when deriving an annuity formula.

I wish to note one further aspect that is central to percentages and interest and which occurs regularly in the media, viz. *percentages of percentages*. Mathematically speaking, it is incorrect to refer to an increase in interest rate from 9% to 9.5% as a 0.5% increase. The notions of *percentage point* and *basis point*, which are both basic mathematical constructs within financial maths, address the problem of percentages of percentages.

6.9.3 Modelling and applications

I have already noted that the importance of this aspect of MfT depends on the emphasis placed on modelling and applications in the school curriculum. Such knowledge is predominantly concerned with how compound interest works in banks. For example banks use daily interest calculations with monthly compounding, but the compound interest formula is not actually used for daily interest calculations but only for predicting future values at a given interest rate. It also includes knowledge of day count conventions and an appreciation that the compound interest formula is both an accurate and efficient model of the compounding of money that takes place in banks. Teachers also require knowledge of other contexts that can be modelled by the compound interest formula, for example, simple population growth.

6.9.4 Mathematical practices

Compound interest tasks at school level tend to focus on numeric work, with particular emphasis on conversions to appropriate time periods and substitution into the formula. Thus a great deal of attention is paid to getting answers, as lamented by Watson (2008). However, within the topic of financial maths there are various opportunities to deal with other mathematical practices. I list several below:

- Working with accuracy – Working to different levels of accuracy with decimal values for rate per period, and the impact of rounding;
- Working inductively – Inductive reasoning is exemplified in the derivation of the simple and compound interest formulae;
- Working with inverse and reversible operations – Earlier in the chapter I discussed the issue of percentage change and reversible operations. I also discussed similarities and differences between appreciation, depreciation, compounding and discounting. All these notions provide examples and non-examples of reversible operations and inverse operations which are accessible at school level and which continue to intrigue and perplex many adults in the financial sector (Noss & Hoyles, 1996a);
- Generalising (with limits) – Increasing the number of compounding periods per year, and the resulting impact on growth, provides opportunity to generalise from examples. It also provides opportunity to show that the generalisation has limits and that continuous compounding has a limiting value, although the mathematics to prove this is beyond that of the school curriculum;
- Proving – Earlier I noted two examples of counter-intuitive phenomena (see essential features): students can use school-level mathematics to prove that the time for an amount to double is independent of the amount of money; graphical illustrations can be used to show powerfully that compound growth is slower than linear growth within the first compounding period.

A note on mathematical notation in financial maths is important at this point. Precision in the use of notation is not characteristic of introductory financial maths, and some might even argue that the use of notation is sloppy. Consider, for example, the sample text books questions provided at the beginning of this chapter. Question 1 involves annual compounding, question 2 involves quarterly compounding and question 3 involves both monthly and quarterly compounding. This requires that interest rates be adjusted to calculate rates per period, and that exponents be adjusted to deal with the number of compounding periods. Some texts will use a formula similar to $FV = PV\left(1 + \frac{i_m}{m}\right)^{n \times m}$ to show the adjustment of the nominal rate and the exponent. In the case of question 2 this would become $FV = PV\left(1 + \frac{i_4}{4}\right)^{5 \times 4}$. However, many teachers (myself included) will simply work with the "standard formula" in all three cases, i.e. $FV = PV(1 + i)^n$ and will emphasise to learners that the necessary adjustments must be made. This may leave some ambiguity as to the meaning of n – does it refer to the number of compounding periods or to the number of years? The answer will be different in each of the above formulae.

6.9.5 Different teaching sequences and approaches

Approaches to teaching compound interest have shifted in recent years as a result of curriculum change in South Africa. Previously compound interest was taught as an application of sequences and series (Department of Education (DoE), 1986a, 1986b, 1995a, 1995b). Consequently it was only introduced in the final year of schooling when sequences and series were taught. In the current curriculum (Department of Basic Education (DBE), 2011; Department of Education (DoE), 2002), compound interest is introduced at Grade 9 level where learners are generally introduced to the concept by means of iterative calculations that model the process of compounding. The formula is then typically introduced at Grade 10 level by means of an inductive approach. This therefore constitutes both a different teaching approach and a different sequence in teaching compound interest.

Another way of distinguishing the approach to compound interest relates to its purpose: is compound interest treated as an end in itself or is it considered as a component of a larger picture? The former is typical of many text book tasks at Grade 9 and 10 level, and promotes an *accumulation view* of the compound interest formula. The latter might take different forms. For example, the approach might point towards multiple-payment scenarios and ultimately annuities, thus potentially promoting an *adjustment view* of the formula by considering changing values over time. Alternately the approach might focus on the positive and negative financial impact of compounding interest over time, thus promoting a critical perspective on compound growth that emphasises its financial and socio-economic impact.

6.9.6 Basic repertoire

It seems difficult to treat this component separately from *essential features*, *learners' conceptions*, and *explanations*. Knowledge of a basic repertoire involves a collection of examples and extended tasks that foreground the key ideas of compound interest. This should include dealing with ambiguities in timeframes (e.g. *after* 3 years). It should also include strategic choices for interest rates that do and do not recur in relation to the compounding period, for example 6.5% p.a. compounded monthly gives a nominal monthly rate of $0.541\dot{6}\%$. The choice of such interest rates will be determined by the goals of the task – it may not be desirable to include the complexity of a recurring nominal rate when focusing on the impact of the number of compounding periods. But it is essential to work with a recurring rate to emphasise the impact of rounding errors in financial calculations.

6.9.7 Explanations

Knowledge of how to explain the mathematics of compound interest includes demonstrations of how to use the compound interest formula. This involves dealing with the calculation of a rate per period. It also involves illustrating by means of example that the effective rate produces the same future value as the nominal rate compounded the requisite number of times. Explanations will likely also include the importance of the order of operations when applying the formula, as well as demonstrations of the use of timelines and the associated conventions. A further element which I will discuss in detail in chapter 10 concerns the cognitive demand that different explanations place on learners. For example, deriving the formula using a multiplicative approach, as shown in table 6.2, demands a structural perspective of learners, and this may be beyond the reach of many Grade 10 and 11 learners.

6.9.8 Learners' conceptions

Knowledge of learners' conceptions and difficulties comes from experience of working with learners and from the research literature, which is limited to observations that many have a poor grasp of compound interest and experience difficulties with the procedures. However, it gives little or no indication about the underlying sources of these difficulties. Earlier in the chapter I suggested possible sources of these difficulties, based on anecdotal evidence. These include lack of explicit attention to timeframes, the adjusting of annual interest rates to determine a rate per period, and to dealing with nominal and effective rates. I also discussed learners' over-generalisation of reversible operations and the related difficulties in distinguishing depreciation from discounting. In chapter 9, I provide examples of learners' work on a compound growth task where learners did not make use of the compound interest formula. The errors in their strategies suggest the need for attention to modelling of compound growth scenarios other than compound interest.

6.9.9 Contextual issues of finance

Financial concepts and conventions – The notion of the *time value of money* is a fundamental construct underpinning all of financial maths. Knowledge of interest rates such as the repo rate and prime rate is important since these ultimately impact the banks' rates. Changes in the repo and prime rates provide opportunity to gain an appreciation of the substantial impact over time of small changes in interest rates, particularly on loans. The distinction between nominal and real interest rates is important because the latter takes inflation into account, thus some knowledge of inflation is also beneficial for teachers. Knowledge of how banks calculate interest (i.e. daily calculation and monthly compounding), as discussed above, is relevant here.

Socio-economic issues and financial literacy – It is well-known that levels of financial literacy in South Africa are low (Eighty20, 2008), that levels of national and personal debt are extremely high, and that the general public does not appreciate the negative impact of compound interest on borrowed money (South African Reserve Bank, 2012). I would therefore argue that mathematics teachers have a moral imperative to help learners appreciate the power of compound interest on both investments and loans, thus increasing learners' levels of financial literacy. In order to do this, teachers require the necessary background knowledge themselves and need to know where to access relevant information in order to remain up-to-date. Others might argue that this is not the responsibility of the mathematics teacher but rather of teachers of commercial subjects. In response, I argue that an appreciation of the power of compounding comes from knowledge of exponential growth and it is the mathematics teacher who will open up learners' access to this knowledge.

Inevitably this raises concern about the breadth of knowledge required of mathematics teachers. It could be argued that the knowledge of financial and economic issues identified above forms part of the knowledge base for general financial literacy of economically-active citizens, and so a mathematics teacher who is financially literate him/herself will possess sufficient contextual knowledge. However, the detailed knowledge of daily compounding and day count conventions discussed earlier in the chapter is not part of general financial literacy. This suggests that mathematics teachers' knowledge of financial and economic issues is specialised and extends beyond general levels of financial literacy.

6.10 Conclusion and reflection on MfT framework

In this chapter I have set up a *reference landscape* for the concept of compound interest. This landscape spans school financial mathematics, introductory financial mathematics at university level, mathematics teacher education, and (aspects of) the world of banking. In the absence of a research-based literature on conceptions of compound interest, I have proposed networks of related concepts, and discussed aspects that I consider to be important within the landscape and for my data analysis. The chapter also reflects the first attempt within the thesis to put the MfT framework "to work". Each aspect of the framework has been considered in relation to teachers' knowledge for teaching compound interest.

Aspects of MfT in focus in this chapter

Essential features

Network of concepts relating to *growth factor* and centring on *rate per period*

Key representations: formulae, graphs, timelines

Two approaches to deriving the compound interest formula: *additive* and *multiplicative (*which requires a *structural* conception)

Modelling and applications

Comparing answers from the compound interest formula with the daily interest calculations produced on a spreadsheet, and showing the accuracy and efficiency of the compound interest formula

Different teaching sequences and approaches

Promoting *accumulation* and *adjustment* views of the compound interest formula

Learners' conceptions

Precision in references to time

Importance of pre-knowledge of percentage and percentage change

Misconception that equal *percentages* of interest translate into equal *amounts* of interest

Distinguishing between *depreciation* and *discounting*

Contextual knowledge of finance

Knowledge of how interest is calculated in banks using daily interest calculations and monthly compounding

CHAPTER 7

Revisiting school mathematics - case 1

Opportunity to engage with mathematical practices and conventions

7.1 Introduction

The first case of revisiting emerges from what I have called the Wage Doubling Task. The incident on which the vignette is based led to a focus on knowledge of mathematical practices such as defining, and the conventions of mathematical notation.

I begin with a discussion of the revisiting task, its intended goals, and the resources brought to bear on the task by the students. This is followed by the vignette which draws extensively on transcripts of the students' interactions. Thereafter I focus on mathematical practices and learners' conceptions as two aspects of MfT of compound growth that are prominent in the vignette. I conclude by reflecting on challenges for revisiting that emerge in this case.

7.2 How is revisiting in this case different from learning at school?

7.2.1 The task

The learner task was an adaptation of the well-known doubling problem involving rice grains on a chessboard, and formed part of a unit to introduce Grade 9 learners to exponential growth (Gauteng Institute for Education Development (GIED), 2001). In this particular scenario a worker was negotiating with his employer for a salary increase. The worker had rejected the employer's offer of a R200 increase per month, and suggested the employer rather pay him 1c on the first working day of the month, 2c on the second day, 4c on the third day etc., thus doubling the wage of the previous day for every working day of the month. Learners are asked to explore who was better off (the boss or the worker) at the end of 2 weeks, and at the end of the month, using "whatever calculations you need". Thereafter the introduction of exponential growth is scaffolded by means of a table where learners record day number, daily wage and cumulative wage, thus revealing the exponential pattern. This is followed by questions requiring learners to extract information from the table.

The revisiting task contained the narrative about the worker and his employer but none of the scaffolding questions nor the table. The task contained three questions: (1) to produce a formula for the worker's wage; (2) to describe the impact on the formula if the starting wage were 3c, not 1c; (3) to compare the problem with a compound growth problem dealing with percentages, done the previous day (see Appendix C1). The complete learner task was not given to the pre-service teachers (i.e. the students).

The learner task thus provided a context and starting point for the revisiting task but the students were not required to complete the learner task. Most students had worked on aspects of the learner task in a methodology course earlier in their degree programmes.

7.2.2 Goals

One of my goals for the course was that students should locate compound growth within the broader context of exponential growth. The intended focus of the revisiting task was on defining variables in exponential relationships, attending to structure of exponential expressions, and making links between percentage and exponential growth. These all constituted a far higher cognitive and mathematical demand (Stein, Smith, Henningsen, & Silver, 2000) than the learner task.

I deliberately structured the task around a mathematical problem that would be simple for the students to solve, so that the focus would not be on obtaining correct answers. Previously I had used a similar version of the task in mathematics methodology classes, and so I expected that students would need to pay careful attention to the definitions of variables in their formulae for daily wage and cumulative wage. For example, the two formulae may be expressed as follows[40]:

Daily wage on day $n = 2^{n-1}$	where n represents the *day number*
Cumulative wage on day $n = 2^n - 1$	where n represents the *number of days* that have been worked

At Grade 9 level learners would most likely experience difficulty with the fact that the two formulae are similar and sound the same when spoken verbally. While I intended to draw this to students' attention, I did not expect it to be a source of difficulty for them.

The second question was intended to lead to the mathematical structure of the general formula for exponential growth, $f(x) = a.b^{x+c} + d$. In the wage doubling context $a = 1$ and so the impact of the initial value is hidden. By changing the starting salary to 3c, the impact of $a = 3$ would make explicit the impact of a on the formula. The third question referred to the group tutorial task from the previous day which involved percentage growth over several years. Through this I wanted students to make explicit links between percentage growth and doubling.

7.2.3 Resources

The revisiting task differed from the original learner task in that fewer resources were provided in the text – no scaffolding questions and no summarising table to show the patterns in the input-output relationship. Since students already knew about exponential growth, they brought this to bear on a task. The learner task made use of a "real life" context to introduce *new* mathematics (exponential growth). It also emphasised the process of repeated multiplication leading to the exponential form (e.g. $2 \times 2 \times 2 \times 2 \times 2 = 2^5$). By contrast, the student teachers were required to treat the task as an *application of known mathematics,* and to produce a general form of the input-output relationship. As will be seen below, students introduced their own resources, drawing on subscript notation to deal with concerns about how the independent variable was being defined. This provides an example of how revisiting provides opportunity to bring "new" knowledge to bear on "old" content. I use "new" in the sense that Grade 9 learners would not yet have learned subscript notation. However, the student teachers were making use of knowledge they had learned after they had first learned about exponential growth.

[40] There is some resonance here with ordinality and cardinality of natural numbers.

7.3 The vignette – Challenging definitions and mathematical conventions

When working on the first part of the revisiting task, the students took longer than expected. Most students focused first on daily wage and we agreed on the formula, as shown above. I then shifted attention to defining n since students had not offered this automatically when giving the formula. One student defined n as "number of days" while another suggested "day number". Mpho questioned whether these had the same meaning. Thabo suggested that "day number" indicated a single day while "number of days" represented several days.

Line	Speaker	Utterance
138	Mpho:	Maybe I don't, I'm not sure whether I'm gonna say it exact, the way that, that phrase: *number of days* and, and the *day number,* is it the same thing if I say 'number of days' or I say *n* represent 'day number'? I don't know whether it is the same thing or …
139	Craig:	Okay, did you get the question? Mpho is asking whether you would consider that to be the same: *number of days* or *day number*. Yes, no? *(To the class).*
140	Thabo:	Not the same.
141	Craig:	Not the same. Tell us?
142	Thabo:	Uh I think it's not the same, cos if you say number of days, uh, to give us, maybe if you say maybe five days it means you are looking for the money that's gonna be *n* for the total, I mean the salaries for ev- every, each, I mean if, maybe for five days, each week, it's a week then *P* is gonna be the sum of, the all the money that's *n* on Monday, Tuesday up to Friday, but if he says the *day number* then he means maybe day five, will mean the money he's going to earn on Friday.

After some discussion Zwaii suggested we resolve the issue by using P_n to represent the amount earned on day n, and S_n to represent the total salary after n days.

Line	Speaker	Utterance
156	Zwaii:	I think, sir, because of, if we can say, like Ronald has said that we can say *P* for, I'm not sure what to say, but P_n , so for P_n, it's a certain day, maybe if you can use S_n, we can talk about the total or the sum of, number of. It will depend on the *P* or the *S* that we use P_n or S_n, maybe P_n is equal to two raised by *n* minus one (i.e. 2^{n-1}) that will mean, uh, then the number of days at the, the amount at a certain day.
157	Craig:	So you're saying that whatever we do with this (referring to *P* or *S)* will help us to understand what that (i.e. *n*) means.
158	Zwaii:	Yes, what that means.
…		
170	Craig:	That it's one day. Okay … and so you, just following your argument, you said we could have *S*?
171	Zwaii:	Maybe if I can talk about the sum of the days, maybe, from, uh, the first day to the thirty-first of that month then I can use S_n to say this is the total amount for the month.
172	Craig:	In this case it would be total amount for? [On board: S_n].
173	Zwaii:	Total amount for *n* days.
174	Craig:	So … so if we said S-thirteen (S_{13}) we would be saying this is the total salary for thirteen days?
175	Few students:	Yes.

Although, this suggestion did not fully resolve the concern about "day number" and "number of days", the students accepted it which enabled us to move forward. We then agreed on the formula for cumulative wage, and I was about to move on to the second question when Palesa challenged our use of notation. She argued that our use of n was not consistent[41] across P_n and S_n. She said that when we talked about the wage for day n, we were working with a *single* value of n. But when we calculated the accumulated wage, we were working with *multiple* values of n. For her this was not consistent use of n and revealed her assumption that since both formulae related to the same scenario, n should be used in the same way.

[41] She actually used the word "constant" rather than "consistent" which caused difficulty for some students and so they initially dismissed her argument because *n* was the independent variable and not a constant.

Line	Speaker	Utterance
224	Palesa:	Craig, I don't know whether I'm understanding this correctly, but, to me it seems as if the *n*, it's not constant ... like it's changing, *(class murmurs)* but if, I mean, let's, let's take this one, P_n equals to two to the power of *n* minus one ($P_n=2^{n-1}$). Then the *n* ...
225	Some students:	*Inaudible mumblings*
226	Craig:	Let's listen please.
227	Palesa:	The *n* it's that day. Th-, th- that's *n*, that day, we are st.., we are talking of one day. But when you go to that one where we talk about S_n …
228	Craig:	Umhmm.
229	Palesa:	… the *n* changes now. It's days, not a day. So it seems as if the *n* it's not constant. It's changing. But we're giving it, I mean a definition that it's changing, because we said that *n* influences that *n*, I mean that *n* in the equation so, I don't know whether I'm making sense or what …. *(Voice trails off).*
230	Craig:	*(To class)* Do you get Palesa's point. You don't have to agree with her, do you get it?
231	Various students:	Yes. Yes. Ja.
232	Craig:	(*To class*) So what do you think about it?
233	Trevor:	I think the *n* has different, uh, defined differently because we want to work out a wage, if I may say, a different wage because with that P_n we want to work out, work out a wage on a specific day, let's say on the fifth day. But in S_n the *n* has a different meaning because we looking at the sum for a number of days.
234	Palesa:	That's the thing I'm talking about. Why should it be different because it's the same thing. It's the same as if we are saying *x*, *x* is number of days. Whatever you do or when you are calculating it must be the same thing.

Trevor acknowledged that n was being used in different ways in the two formulae and it did not appear to concern him, but Palesa felt that the meaning of n should not change in the two related formulae.

I allowed the class discussion to proceed for a few minutes because I wanted to gauge the responses of other students to Palesa's concern[42]. There were many students who wanted to comment so I then asked the class to discuss in small groups whether they agreed that the use of n was inconsistent. Lively discussion followed almost immediately. I moved around the class listening to the discussions of a few groups. This included eavesdropping on some of Palesa's discussion with Trevor in which he continued to acknowledge her concern but suggested that P and S determined how to interpret n.

Line	Speaker	Utterance
255	Trevor:	Does that make sense if you say *a* is number of days. (*Craig listening to discussion between Trevor and Palesa*)
256	Palesa:	Ja.
257	Trevor:	Does that make sense to you? So the problem is you saying because we saying *n* is number of days and S_n and *n* is a specific day (unclear) just doesn't, but the fact that the *n* is the same thing you want to change it.
258	Palesa:	You saying, ja, he was explaining that there's a difference in the way we (unclear), but I'm saying how do we identify that, cos now we have used the same letter, but we are saying there's a difference. Okay there's a difference in the formula, but how do we identify that difference? There's only one difference and we are saying (unclear), how do we identify that difference?
259	Trevor:	Okay, if you say difference in formula, I can see that.
260	Palesa:	So it's the *P*, it's the *P* and the, what's that the, *S* helping us, not the (unclear).
261	Trevor:	Yes. Ja. That's one thing.
262	Palesa:	But then remember we said, …
263	Trevor:	The *P* is helping us,
264	Palesa:	Ja the *P* and the *S*. It's the one that is functioning. But the *n* in this formula … *(carries on but Craig moves away).*

The discussion between Palesa and Trevor was picked up by my lapel microphone and so I only have a record of the discussion while I was standing at their table. Palesa was concerned about how one identifies that the interpretation is different when the same letter has been used in two different

[42] Subsequent to this incident, I became aware of Sethole's (1999) study of pre-service teachers' conceptions of T_n, and the dominant incorrect interpretation of T_n as the number of terms in a sequence.

symbols at the same time. This did not fit with her understanding of mathematical conventions when using letters.

I invited feedback from two groups on their discussions. The transcript below contains the responses of Elsie and Attiyah[43], who spoke on behalf of their respective groups.

Line	Speaker	Utterance
278	Elsie:	Okay, Craig, uh, we felt that like the way *n* is being used in the first formula, P_n, like this formula, it's like you're using the day number, so it's like you're substituting the value of *n* with day number. Like for day number four, for P_n like your *n* is going to be four, right, but then still even for, for S_n, there, you are going to put four but it's going to be in a, in a, in a format of a day, number of days ... in the sum of number of days but then for P_n you're just using it as a day number but still using four, you going to be using the, that number of a day.
279	Palesa:	So the *n* is not constant.
280	Elsie:	No, it is ...
281	Rachel:	No.
282	Elsie:	It, it, it is, but then it is! *(Elsie laughs and class laughs with her at her insistence that n is being used consistently)*
283	Palesa:	(Unclear) it's one day but there it's days, but you still have one *n*. It's changing.
284	Various students at same time:	But the formulas are different. The formulas are different. The definitions are different.
285	Elsie:	(unclear) the formulas are different!
286	Various students at same time:	Of course. Yes. We've assigned different variables.
287	Craig:	*(Jokingly to the class)* Is Palesa the only one with the problem?
288	Many students:	Yes. *(laughing)*
289	Craig:	Attiyah? *(her hand had been raised to volunteer a response)*
290	Attiyah:	Okay, well the way we thought that *n* does remain constant but in both the formulas you're going to define the *P* and your *S* to say what it is. So *n* does remain constant. So for the first formula where P_n, we'll say the salary on that day, so *P* defines the "on" and in the other formula it's the salary up to that day, so it's the, for example the 13th day so *S* will define the "up to", but the *n* stays constant.
291	Many students:	Yes. Yes. *(Clapping)*

In reporting on her group's discussion, Elsie's response suggests her group did not share Palesa's concern [278]. Elsie chose an example of $n = 4$, substituted it into both formulae and got the correct answers. So for her n was being used in the same way, that is direct substitution, and it had the same value in both calculations. Thus she did not see any inconsistency in its use. Palesa reiterated her point – in one case n was being used for one day and in the other case for four days. The comments from other students, most of which could be termed "audible mumblings", suggest that for them the difference lay in the formulae. Attiyah gave a very clear explanation [290] that the meaning of n is determined by the P and the S in the respective formulae, which reflected the way Zwaii had initially suggested we use *P* and *S* [156]. When she had finished her explanation several students applauded[44]. I got the sense that the majority of the class was satisfied with her explanation and so I did not elaborate on it, even though I felt that Palesa's concern had not been fully addressed. Earlier, as noted above, Palesa and Trevor had come to a similar conclusion and I felt this had helped Palesa to resolve the matter to some extent although she still actively challenged Elsie's argument [283].

[43] In this session "Attiyah's group" does not refer to the focus group.

[44] Attiyah's reference to "up to" day *n* should have been "up to and including" day *n*. This matter was resolved a few minutes later.

In raising her concern, Palesa challenged the conventional use of mathematical notation and how we can know that we were referring to a single value of n in the case of P_n and to multiple values of n in the case of S_n. I pursued the matter because I was interested to know whether other students shared her concern but also because subscript notation is first encountered in school mathematics and so teachers need to know its associated conventions well. In the mathematics community, the notation is taken for granted: by definition P_n refers to only one n and S_n refers to multiple n's. Palesa was not challenging the formulae *per se*. She knew they were correct. She was challenging the definition of n and the way we were using n in each formula.

7.4 What opportunities emerge for learning MfT of compound interest?

In the vignette described above, revisiting of school maths afforded students the opportunity to reflect on defining of variables and on conventions in mathematical notation – both key elements of mathematical practices. Furthermore, the issues raised by the students provided experiences of responding to peers' conceptions in ways that might resemble similar practices with learners in schools. In this section I focus on two aspects of the MfT framework: *mathematical practices* and *learners' conceptions*.

7.4.1 Mathematical practices

Revisiting school maths provided opportunity to reflect on defining of variables. It began with Mpho's query as to whether the phrases "day number" and "number of days" were equivalent, and was followed by Palesa's concern about inconsistent use of n in the two formulae. In both cases the students' focus was not on the answers. If they had been struggling to obtain answers, it is unlikely they would have been able to pay attention to the notation and definition of the variables in the ways they did.

While it may be argued that Palesa's concern focuses on the mathematical conventions of subscript notation, she might have had the same concern with function notation. For example, if we had defined $f(n) =$ wage on day $n = 2^{n-1}$ and $g(n) =$ cumulative wage on day $n = 2^n - 1$, we would still have been referring to a single value of n in f but to several values for g. The problem is that one symbol (n) simultaneously has multiple meanings. Nevertheless, it should be noted that in the closed form formula (e.g. $S_n = 2^n - 1$), we substitute a single value of n to determine the cumulative amount. However, when working iteratively, each possible value of n is used. In Lave and Wenger's (1991) terms Palesa made the invisible visible by challenging a readily-accepted mathematical convention. The conventions did not satisfy her concern about the use of a single n versus multiple n's at the same time in the same task. She had been using this notation since high school in the "invisible" sense – looking through it to focus on the mathematical meaning of the formula being defined. In this incident she brought the notation itself into focus, thus making it visible, and claimed it to be used inconsistently. This became an obstacle for her that needed resolution so that the notation could once again become invisible, and so that she could continue to use it. In requiring the class to deal with Palesa's concern about the notation, the convention was made visible for the other students too.

Based on Palesa's discussion with Trevor, it appears she accepted the fact that n was being used differently but she asked "how do we identify that difference?" This raises two important questions for mathematics teacher education: (1) how are students given access to mathematical conventions and ways of working; and (2) how are students sensitised to the conventions of notation that they readily and regularly use themselves and yet which may be unfamiliar to their learners? One obvious strategy

is to pay deliberate attention to students' use of mathematical language and notation in their talk, and to use this as an opportunity to increase their awareness of the need for precision and accuracy in their talk and use of notation. I draw briefly on two instances from the same class session to illustrate this.

As already noted, Palesa referred to the use of n being "constant" when she meant "consistent". I did not correct her until her concern had been dealt with. Then I raised her use of "constant" with the class[45]. At the same time I also dealt with a slip in Attiyah's explanation. In [290] Attiyah had referred to S_n representing the salary *up to* day n when she needed to include day n as well.

Line	Speaker	Utterance
292	Craig:	… I want to pick up on two language things. We're saying *n* stays constant. Okay? And then we know what we, it's not constant cos it's changing, but we know what Palesa was meaning. It's having a different meaning in the *P*-formula versus the *S*- formula. So we, let's be careful about how we talk about it, okay? And Attiyah, just picking up on your point you said, 'salary up to' ...?
293	Rachel:	… and including.
294	Craig:	… that day.
295	Attiyah:	Including that day.
296	Craig:	So S thirteen (S_{13}) would be, including?
297	Rehana, Attiyah and Ronald:	Including. Including. Including everything.
298	Craig:	Okay, so the language, we need to be tight on, "does it include that one" and certainly when we're working with time, it's really important to know are we including today. We'll constantly have to think about this in this course. Are we including today, or this month or this year or does it go up to just before? So we need to be careful about our language and we need to be careful about thinking about that question all the time.

It is interesting to note that Attiyah and the other students she had worked with (e.g. Rachel and Rehana) immediately corrected the timeframe to include the salary on day n [297], thus confirming that it was a slip in her talk and not an error (Olivier, 1989).

A similar issue arose with notation in Sakhile's boardwork later in the session. He was illustrating Palesa's concern about the use of multiple values for n (in his example n takes on the value of 1 and 2). He used T_2 to refer to "total two" representing the sum of the first two day's wages as shown below.

$$T_2 = (2^{1-1}) + (2^{2-1})$$
$$T_2 = 3$$

While his use of *T* to represent *Total* was sensible in the context, the conventional meaning of T_2 is "term 2". Ironically in this context this might represent the wage for day 2 alone, and not a partial sum. This incident prompted a brief discussion about the tension between a student's personal conventions and accepted mathematical conventions in notation.

Line	Speaker	Utterance
316	Craig:	(*Referring to Sakhile's boardwork*) … I think what you've done there helps us to think about if we take it day by day then we're using more than one value of *n*, and so maybe that could be confusing for people. I just want to challenge you on the use of the *T*. You wrote, what did you mean by *T*?
317	Sakhile:	The *T*, I'm just saying the total, total, total amount.
318	Craig:	And that makes sense. *T* for total. (*To class*) When you were in grade twelve, and you saw T- two (T_2) …?
319	Several students:	Term two.
320	Craig:	It was term two. Okay, so in fact what you doing here is adding. *T* makes sense for 'total', in this context, but mathematically we are used to seeing T-two (T_2) as being term two and so this could be confusing. So we need to, um, we need to bear those things in mind because the notation we use can confuse.

[45] With hindsight, perhaps I should have dealt with the issue immediately because it may have helped students to focus on the substance of Palesa's concern rather than focus on whether or not n was "constant".

7.4.2 Learners' conceptions

A participative pedagogy in any mathematics course provides opportunities for participants to engage with each other's mathematical ideas. In the context of revisiting school mathematics within a pre-service teacher education programme, some issues raised by the pre-service teachers (i.e. students) may be similar those that learners in school might raise. This provides opportunity for students to experience first-hand some of the obstacles that learners might face in learning the concepts being taught. Palesa's concern provided such an opportunity for students to engage with the ideas of a peer and thus to be exposed to different opinions and struggles. Her incorrect use of "constant" when she meant "consistent" posed another challenge in that students needed to attend to the substance of her concern and look beyond the actual words she was using. Elsie's response suggests she and her group may not have been able to this. By contrast, Trevor and Sakhile's responses show evidence that they had understood Palesa's concern, and sought to deal with its substance. This reflects both a disposition to value learners' contributions and an attempt to draw on appropriate knowledge to respond. Sakhile's choice of example illustrated that two values of n were being used in a single calculation. This contrasted with Elsie's example which suggested they were being dismissive of Palesa's claim.

7.5 Conclusion and reflection

While a participative pedagogy provides opportunity to engage with peers' ideas, one unintended consequence may come in the form of reduced pace and coverage. At the beginning of the chapter I described the three questions of the revisiting task. In the session we spent much more time on the first question than intended. This was partly due to students taking longer than expected to produce the formulae but also because of the time allocated to deal in depth with Palesa's concern. The result was that less time was spent on question two than intended and question three was abandoned in favour of converting the wage doubling question to a percentage increase task (although this still achieved the main goal of the original question 3).

In terms of the MfT framework, essential features of the exponential function received less attention than intended. I dealt with the impact of a starting wage of 3c and then dealt briefly with the structure of $f(x) = a.b^{x+c} + d$. I focused mainly on the parameter, a, with little attention to the base, b, and no discussion of parameters c and d.

All this highlights the multiple and sometimes competing goals of revisiting a concept of school mathematics within pre-service mathematics teacher education. On the one hand, there is a need for students to know the essential features of the concepts and the relationships between these features. Students also require an overview of the structure of the concepts in relation to other aspects of mathematics as described in chapters 2 and 6. Such knowledge is structured by the discipline and needs to be presented in a coherent and systematic manner. On the other hand, students need opportunity to grapple with the concepts, to seek new connections, and to be challenged by the emerging ideas of their peers, and by partially correct ideas of learners in school. Creating and maintaining this balance is an ongoing challenge. I shall return to this issue later in the thesis.

Aspects of MfT in focus in this chapter

Relationship to other mathematics

- Exponential growth in the form of doubling

Modelling and applications

- Applications of known mathematics to hypothetical contextual problem

Mathematical practices

- Knowledge of conventions of mathematical notation
- Defining of variables (links to modelling and applications)

Learners' conceptions

- Struggles of peers may give insight into obstacles that learners will face

CHAPTER 8

Revisiting school mathematics - case 2

Opportunity for making connections with other mathematics and for further mathematical inquiry

8.1 Introduction

The second case of revisiting school mathematics provides an example of a student's self-initiated attempt to make new connections between various aspects of exponential growth. It also exemplifies opportunities for students to engage in further mathematical inquiry that is directly related to school maths. The chapter follows a similar structure to chapter 7. I end the chapter by reflecting on possible reasons for the students' difficulties in engaging with the ideas that were shared.

8.2 Background to the vignette

In class session 1, the students had worked on converting the first question of the Wage Doubling Task to a percentage change problem. The doubling referred to a worker's wage which was doubled each working day, starting with 1c on the first day. While there was general consensus that the percentage increase was 100% per day, one group suggested it was 200% per day. They presented a compelling argument, and many students were not immediately able to identify the logical error in the reasoning. I asked the class to study the group's argument carefully in preparation for the following session. In class session 2, the logical error was identified and there was agreement that 200% represented the ratio of the amount earned each day to the amount earned the previous day, and not to the *increase* from the previous day. For example, if the worker earned 8c yesterday, he would earn 16c today. The increase is 8c and the percentage increase is 100%. The (unsimplified) ratio is $\frac{16}{8}$ or 200%.

Following from this, Sakhile had initiated his own investigation into the relationship between the doubling formula $y = 2^{n-1}$, the compound interest/growth formula, $A = P(1 + i)^n$, and drawing on what I shall call the 200%-formula, $200\% = x + 100\%\, x$. He had hit an obstacle and was seeking help in resolving the impasse. The essence of his struggle was that he could manipulate the bases of the doubling formula and the compound growth formula to be the same, but he had an exponent of $n - 1$ for the doubling formula and an exponent of n for the compound growth formula. He was unable to account for the difference in the exponents.

Sakhile's investigation is not a direct response to a task given in the course, but rather a self-initiated inquiry into the connections between different formulae. Consequently the distinctions between task, goals and resources are less clear than for a task I had assigned, and so I shall discuss all three aspects together.

8.3 Task, goals and resources

As noted above, Sakhile set up his own task – to explore the connections between the three formulae. From a modelling perspective, it might be argued that he was attempting to formulate different

mathematical models for the same situation. But he was also seeking to give explicit attention to the links between the different formulae. Once he had established that the bases of the doubling formula and the compound growth formula could be shown to be the same, he sought to explain why the exponents were different. The resources he drew on were the forms of the three formulae, his knowledge of percentage, and a strategy of template- matching (Sfard, 2000) whereby he attempted to map various elements of the formulae onto each other[46].

8.4 Vignette – Sakhile's attempt to make connections between formulae

Sakhile did not explicitly ask to share his work with the class. It flowed from a comment he made about links between 100% and 200% which was difficult to follow from his verbal description, and so I invited him to write on the board. This progressed to the relationship between the three formulae. His boardwork is shown below (fig. 8.1a and fig. 8.1b). The layout of his boardwork did not fully support his explanation, and so I have introduced a different layout (fig. 8.1c) in an attempt to show the links and his reasoning more clearly. The lines in the new layout are labelled A – E. I refer to lines using the same conventions I have used with transcripts, e.g. [C] is line C.

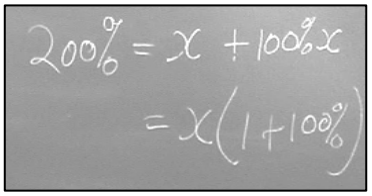


Fig. 8.1a Links between 100% and 200%

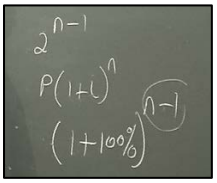


Fig. 8.1b Links between exponential form and compound interest formula

	200% formula		Doubling formula		Compound interest formula
A	$200\% = x + 100\%\, x$	C	2^{n-1}	D	$P(1+i)^n$
B	$= x\,(1 + 100\%)$			E	$(1 + 100\%)^{n-1}$

Fig. 8.1c Reformatted version of Sakhile's boardwork

Sakhile began by showing that 200% was obtained from adding the initial salary, x, and a hundred percent of the salary [A] (fig. 8.1a) (I return to the lack of referent for 200% later in this section). He then factorised the expression, producing [B], which he noted had similarities with the form of the compound interest formula. He then proceeded to the doubling formula as captured in the transcript that follows. Although his explanation was continuous, the transcript has been sub-divided for purposes of readability, with new line numbers generated specifically for this format. The *comments* column provides reference to the boardwork and some interpretations of his utterances. At the point where the transcript begins, he had written only [A] and [B] on the board.

[46] Sfard (2000) suggests that initially learners come to know the meaning of symbols through reference to templates they already have. A template is a collection of associations that accumulate from various experiences of a particular stimulus, and which occur in clusters. In financial mathematics template-matching may explain students' struggles to accept that n in the annuities formulae does not represent time but number of payments. This may stem from similarities in the formulae since both contain the expression $(1 + i)^n$. Consequently, it seems reasonable to assume that $(1 + i)^n$ has the same meaning in both formulae.

Line	Sakhile's utterances	Comment
1	I tried to take the exponential thing, into a compound … like … formula. Since last time I remember we got like something like two to the power *n* minus one	Writes 2^{n-1}
2	and then finding the hundred percent, I said okay, the compound, it's sort of like *P* into one plus *i* to the power *n.*	Writes $P(1+i)^n$
3	But I said, okay, since this one, the *n* is *n* minus one, so I said okay it has to be …	Recognises the exponents are not the same. "the n" refers to entire exponent $n-1$
4	I know the first payment for that day is one cent so it will be the first, the principal	Intends to substitute $P=1$
5	So it will be into one, and then I tried to look at this inside bracket and relate it to the two.	Writes (1 Wants to relate $(1+i)$ to 2
6	So for me to get the two I must say one plus hundred percent of …hundred percent. So the interest it means, cos the *i* stand for the interest into the hundred percent and then hundred percent is one, cos it's one hundred over one hundred and then you can see one.	Recognises that $1+\frac{100}{100}$ will give 2 in the bracket Writes $(1+100\%)$
7	So it will be one plus one and it will give me two here. So that is why I said okay, this means hundred percent, but here it will be to the power of *n* minus one. See.	Does not write 2 Copies exponent $n-1$ from 2^{n-1}
8	So if I look at these two, it's just that here I don't have, I think this represents the number of days … I'm not sure about the negative one, cos maybe you can tell me why negative one, but I know *n* it can represent the number of that day, you want to find it.	In the exponent n represents number of days Can't account for the "-1" and asks class for help
9	And then here I haven't talk about the number of days. I just, so if I look what's inside the bracket and this information, I think it's something that's the same thing.	Referring to [B] Notes similarity in the brackets, i.e. $(1+100\%)$

In the discussion that follows I refer to line numbers on the chalkboard (letters), and line numbers from the transcript. Sakhile began by recalling that the formula for daily wage on day n was given by 2^{n-1} [C] [1]. He then wrote the "right hand side" of the compound interest formula as shown in [D] [2]. Having written down all three forms, his intention was to describe and justify the links he had established, and to seek help with the exponent.

He appeared to be treating [C] and [D] as partial templates for [E]. He first substituted $P=1$ because the starting wage was 1c [4]. Then he focused on the need for the bracket in [D] to have a value of 2 [5]. This would occur if he substituted 100% for i [6]. He did not refer explicitly to the 100% from [B] but had made the connection previously in dialogue not reported here. He then appeared to treat [C] as the template and copied the exponent $(n-1)$ from [C] [7]. However, he could not explain why the exponent in [E] was different to the exponent in [C], which he took to indicate the numbers of days [8]. He pointed out the similarities between [B] and [E] [9] but acknowledged that he could not explain the exponent of $n-1$, and this is where he hoped the class could help him [8].

Based on students' initial comments to Sakhile, there was evidence that many had not followed his reasoning, and were more concerned with *what* he had written than his reasoning. Below I provide four examples of three different issues that students raised: the lack of a referent for percentages, the exponent in [E], and rejecting Sakhile's logic. I present these issues in the same order in which they were raised in the session

Lack of referent for 100% : Mpho was the first to respond when Sakhile finished his explanation. He questioned whether they could add together 1 and 100% without there being a referent for 100% [128]:

Line	Speaker	Utterance
128	Mpho:	Okay, I don't know, maybe I'm, I'm lost. Can you actually say 'one plus hundred percent' or do you have to say 'one plus one hundred percent of something' or do you just say 'one plus one hundred percent, one plus two hundred percent', can you actually understand?
129	Sakhile:	Okay, like, like, you see here, the way I see it, it's like *x* plus hundred percent of something, but when I try to factorise it, take out the like terms, it becomes like x into one hundred percent.
130	Mpho:	Please clarify, okay I think I understand, but I mean there (*pointing to right section of board*) where you say one plus hundred percent.
131	Sakhile:	Here, I know, one hundred percent, uh, it's one, so I try to, that's how I try to take, fit, like, make this information like, link it to the two. Link it to the two. I know the one is there, but what is it that I must add, which is a percentage so that I can get the two. That's what I tried to look at, from that, from the general formula, the one is there … (unclear).

Sakhile referred back to [B] where he had factorised the expression, producing a factor of $1 + 100\%$. He noted that there was a referent, x, but this had been factorised [129] as a common factor (and previously he had explained that $x = 1c$ hence there appeared to be no referent for the percentage). He then repeated his explanation given earlier regarding the 100% [131].

Concern about exponent of $n - 1$: Sizwe questioned why Sakhile was "subtracting one from the *n*" [133] which indicated that he had not heard Sakhile's own concern about the exponent. Sakhile then attempted to explain once more.

Line	Speaker	Utterance
133	Sizwe:	Eish. I don't understand the, why you are subtracting one from the *n. (referring to the exponent)*
134	Sakhile:	Subtracting what? Here? You mean here?
135	Sizwe:	Ja. Ja.
136	Sakhile:	Okay, like I said, I tried to take this, sort of the formula that we had *(pointing to* 2^{n-1}*)*, we got about the story, Ja, the story about the salary, Ja, relate it to the compound formula, so I'm, I'm shifting from this, sort of an exponential to, to the compound formula *(pointing to* $P(1+i)^n$*)* that we know in textbooks. So I tried to change, so I know this *(pointing to* 2^{n-1}*),* since this *(pointing to (1 + 100 %))*, it's here *(pointing to the 2)* and then what about this part *(pointing to the exponent in* 2^{n-1}*)*, where is it here from that formula *(referring to the compound growth formula)*, then I tried, okay it belongs (*pointing to exponent in* $(1 + 100\%)^{n-1}$*)*… that's how I, that's how I worked.
137	Craig:	And you said you're not sure why (unclear).
138	Sakhile:	Ja, the negative one I didn't, like, why it's negative one, cos the compound is only *n* (*i.e. the exponent is n*).
139	Craig:	So you're trying to make some links and some of it is not working so nicely, but you're just ignoring that for the moment.
140	Sakhile:	Ja.
141	Craig:	And that's fine, I mean it, that's how a lot of us work. We can see there's some link, there's some bit that's not really making sense for now and we just leave it and work on the part that is making sense. Sometimes the part that does not make sense just sorts itself out and sometimes it never ever does. Okay. I don't want us to get bogged down in that *n* minus one at this stage but we're gonna need to see if it sorts itself out. But can you see what Sakhile's trying to do in terms of shifting from two to the power of something which is pretty much standard exponential stuff to thinking about it in the form of the compound interest formula, which clearly must play some part in this course, right? Okay, (*no response from students*) You all asleep? There was some hand up over here?

In [136-138] Sakhile explained again how he was trying to make links between the doubling formula and the compound growth formula. His explanations and reasoning were consistent with his earlier explanation. Despite repeating the explanation of his thinking, it was clear that students were still

struggling to make sense of his ideas, and that they did not appear to be able to help him resolve the exponent issue. I attempted to shift students' focus away from the immediate problem to foreground the way in which Sakhile was working [141]: that he was drawing on his intuition that there are links between the two formulae, and that he had shown that some links are easily established, but that there was still uncertainty about others.

Rejecting Sakhile's logic: Vusi still did not follow the reasoning about the 100% and suggested Sakhile was forcing links between the formulae that were not logically derived.

Vusi: Sakhile, Ja, I don't get the story around the hundred percent, how, how, how, can the … (unclear), dividing the hundred over the hundred percent, how does that relate it to the two *n* there? … Because I can see, it's like you are cooking it, so that it can come together, but how does it relate?

It is clear from this comment that Vusi was not able to follow the links that Sakhile had already made. I requested a volunteer, other than Sakhile, to explain Sakhile's approach to Vusi.

Further concern about referents: Later Hailey drew Sakhile's attention back to $200\% = x + 100\%\, x$ and used an example to illustrate the problem with the lack of referent for x.

Line	Speaker	Utterance
182	Hailey:	But what, even about that top line? How is that equal to two hundred percent?
183	Sakhile:	You mean this one?
184	Hailey:	Mmm.
185	Sakhile:	So ...
186	Hailey:	... Cos pretend, you were saying earlier that *x* is eight, right?
187	Sakhile:	Ja?
188	Hailey:	So let's say *x* is eight, and then we say a hundred percent of eight is eight, eight plus eight is sixteen.
189	Sakhile:	Sixteen.
190	Hailey:	Is that equal to two hundred percent?
191	Sakhile:	Yes, cos it will take from, for, for that day that's the total, I said the *x* is the amount from the previous day.
192	Hailey:	That's not equal. It's equal to two hundred percent of *x*.

Hailey showed how Sakhile's equation [A] would produce the "equation": $200\% = 16$, where the left-hand side was not equal to the right-hand side. She then emphasised the need to include the referent on the left-hand side, i.e. $200\%\, x$. Sakhile could not see her problem at this point [191]. I suggest that he was using 200% as a *label* and was not intending to express some kind of equivalence in [A].

The students' responses to Sakhile reflect a lack of shared goals in the activity. While he sought help to resolve the differences in the exponents, many in the class were focusing on his use of notation. It is not clear whether the students' concerns with his notation hindered them from engaging with his primary concern, or whether they were not able to provide help on the exponent issue and therefore resorted to focus on his use of notation. Although they were unable to help him move forward, the discussion concerning his use of percentage notation led to a teachable moment (Havighurst, 1972) which I discuss briefly in the final chapter of part 1.

8.5 What opportunities emerge for learning MfT of compound interest?

The classroom episode captured in the vignette raises many issues with regard to revisiting school mathematics in pre-service maths teacher education. Sakhile exemplifies a motivated student engaging

in *personal mathematical inquiry*. That he had opportunity to share his investigation publicly was a function of the pedagogy of the course, and of his confidence to make public his incomplete mathematical work. His investigation provided an example of the potential for engaging in further mathematical inquiry that may emerge from revisiting school mathematics. In his particular case, the outcome of his activity would be an increase in the breadth of his knowledge through establishing n*ew connections between known pieces of school mathematics* (Ma, 1999).

In this section I focus on three aspects of the MfT framework that are prominent in the vignette: modelling and applications, essential features and mathematical practices. I describe the mathematical work I did to resolve Sakhile's problem for myself, and then argue that this provides a potential opportunity for learning an essential feature of compound growth. I thus use my learning, rather than Sakhile's, to make this claim.

8.5.1 Modelling and applications

Sakhile was concerned with reconciling the two formulae, and how they modelled the wage doubling scenario. Table 8.1 shows my attempt to model the Wage Doubling problem using the form 2^n as expected of learners. I also modelled it using the compound growth formula, $A = P(1 + i)^n$. In both cases I focused on the wage at the end of day *n*. By working inductively, one can see that the pattern in the exponent is always one less than the day number, and thus both models should have the same exponent of $n - 1$.

End of day	Wage	Exponential form	Modelling wage doubling with the compound interest formula
1	1	2^0	$P = 1$
2	2	2^1	$A = 1(1 + 100\%)^1$
3	4	2^2	$A = 1(1 + 100\%)^2$
4	8	2^3	$A = 1(1 + 100\%)^3$
n		2^{n-1}	$A = 1(1 + 100\%)^{n-1}$

Table 8.1 Modelling Wage Doubling problem using exponential form and compound interest formula

However, this does not yet fully resolve Sakhile's concern since I must still consider what was wrong with his reasoning.

In the context of compound interest, the present value is the value at the *beginning* of the first period, and the future value gives the accumulated amount at the *end* of n periods. In terms of the model, interest begins to accumulate by the end of the first period. Consequently the exponent will match the number of periods. However, in the wage doubling scenario, (or any similar scenario) the present value is the wage at the *end* of day 1. Consequently, when using the compound interest formula to model the doubling scenario, the exponent will be $n - 1$ because there is no compounding in the first period. Another way of thinking about it is as follows: the exponent, n, represents the value after n compoundings. Since no compounding has taken place by the end of day 1, the exponent will always be one less than the day number.

It was only in working with the compound growth formula for a problem-type where it is not typically used that I became explicitly aware of the different ways in which different problem contexts (such as compounding of interest, wage doubling, bacteria growth etc.) work with different starting conditions,

and hence are modelled by slightly different formulae. Knowledge of these subtle differences increases the breadth of teachers' knowledge.

8.5.2 Essential features

At a personal level, it was the use of the compound growth formula outside its usual scope of use that foregrounded for me the obvious meaning of present value, which I had taken for granted: that present value indicates the value at the *beginning* of the first period. This meaning of present value is so tacit when using the formula in its usual context that I was unable to draw on it immediately to resolve Sakhile's concern. Thus through use of the formula in an unusual context, the invisible became visible (Lave & Wenger, 1991). This led to a new level of awareness about present value. I had never been forced to question its position in time and, when reflecting on it subsequently, it is entirely obvious that it refers to the beginning of the first period. However, through working with the formula in the salary-doubling context, I saw in a new way something I had always "known" tacitly. Thus the present value became *transparent* for me – previously I could use it tacitly (or invisibly), now I could also draw on my new awareness of it to make it visible, and to harness the meaning of it which I had not fully appreciated.

This suggests that *using familiar mathematics in unfamiliar ways* may be a productive strategy for revisiting school mathematics. In geometry and trigonometry this strategy might include using general formulae for special cases, for example using the formula for area of a trapezium to determine the area of a rectangle, or using the cosine and area rules to determine various measurements in right-angled triangles.

8.5.3 Mathematical practices

Sakhile's struggle to solve a problem he had initiated provides an example of mathematical practices that are not generally made visible in undergraduate or school mathematics (Burton, 2004; Watson, 2008). Stacey (2008) and Cooney and Wiegel (2003) argue that pre-service mathematics teachers should engage in such practices for themselves, including posing their own questions and engaging in mathematical inquiry.

In the vignette we see an example of mathematical work that is "in progress", and an opportunity where a student asks his peers for assistance in solving the problem he is working on. Thus mathematics is shown to be a collaborative enterprise to solve a student-initiated inquiry, and not merely the providing of answers to questions posed by the teacher. With hindsight, I missed an opportunity to capitalise on this aspect of working mathematically since I did not actively encourage the students to engage with Sakhile's problem in class. This was a pedagogic decision based on students' inability to respond in the moment to the substance of his work coupled with a concern to continue with the work planned for that session. I could have asked Sakhile to write up his explanation and then give it to the class to work on and respond. Although I included a similar task in a session the following week, it was differently framed and did not carry the same sense of peer collaboration on a mathematical problem because in that instance it had become a task initiated by the lecturer.

As noted above, the vignette serves as an example of revisiting school mathematics that provides opportunity for pursuing further mathematical inquiry related to school mathematics. I propose that a general goal of revisiting tasks is for students to explore new links and to ask "what if" questions. The

connections Sakhile explored do not require advanced mathematics yet they were connections that neither I nor the students had made.

8.6 Students' difficulties in engaging with a peer's mathematical explanation

The discussion of the episode would be incomplete without some attention to the difficulties students experienced in following Sakhile's argument and his inability, despite several attempts, to make his concern clearer for them. It could be argued that the students did not share Sakhile's concern sufficiently to engage deeply with it, and therefore focused on relatively superficial aspects rather than engaging with the mathematical substance of his work. However, it could also be argued that the quality of his explanation and accompanying written text negatively influenced students' ability to engage with the substance of his concern. His lack of attention to notation led to queries concerning a referent for the percentages, and the layout of his boardwork was problematic in that it did not clearly reflect the separate pieces he was working with.

Sakhile's explanation contained elements of self- and peer-explanation (Leinhardt, 1997). This is reflected in his "casual" use of notation and language, and his poorly-planned layout. For example he said "the n is n minus one" [3] meaning the *exponent* is $n-1$. The purpose of his explanation was to *share his mathematical work* and not to *teach* his peers. However, precise use of notation and logical layout of written work are important features of mathematical explanation in all settings. This suggests the importance of paying attention to verbal and written forms of mathematical explanation in teacher education – not just for future purposes in the school classroom but also for the teacher education classroom. It is clear from this and several other incidents in the course, involving revisiting of school maths and learning of new maths, that explicit attention must be given to the key features of mathematical explanations, and that students should be given opportunity to practice producing explanations, and then to reflect on their efforts and those of their peers.

8.7 Conclusion

In this chapter I have discussed a student's self-initiated investigation that stemmed from another task in the course. In sharing his work with the class, he sought their help to resolve the obstacle he encountered with the exponents of the doubling formula and the compound growth formula. However, the students struggled to make sense of his reasoning. Instead, several students focused on his careless use of notation, and thus the importance of precision in the use of mathematical notation was highlighted once again. Despite the failure to resolve the student's problem, the incident provides insight into the opportunities that revisiting school mathematics might lever up for learning MfT. These include opportunities for engaging in further mathematical inquiry related to aspects of school mathematics, and for students to share their investigations publicly. However, this is dependent on a pedagogy and a course design that is sufficiently flexible to incorporate students' mathematical explorations into the course programme, to make them available to other students, and then to engage with them in a public forum. As noted in chapter 7, such practices require additional time and may lead to challenges in pacing and coverage.

Aspects of MfT in focus in this chapter

Essential features

- Present value linked to *beginning* of first period

Relationship to other mathematics

- Connections between exponential growth and compound interest formula

Modelling and applications

- Using different formulae to establish equivalent models for the same scenario

Mathematical practices

- Further mathematical inquiry to establish new connections between known pieces of school mathematics
- Communicating own mathematical investigation to peers
- Mathematical investigation as a social (rather than individual) enterprise

Learners' conceptions

- The difficulties experienced by peers may give insight into obstacles that learners will face

CHAPTER 9

Revisiting school mathematics - case 3

Opportunity to engage with learners' conceptions

9.1 Introduction and background

The third case of revisiting school mathematics deals with learners' conceptions of compound growth and students' struggles to clarify interpretation of timeframes. This case is based on a task I have called the *Computer Operator's Salary,* which included the test question below and the responses of four learners.

Test question: "A computer operator earns R96 000 a year. Her salary increases by 6% per year. What will her salary be after 3 years?"

Solution in memo: $\text{Salary} = 96\,000\left(1+\frac{6}{100}\right)^3 = R\ 114\ 337.54$

In this chapter I shall show how a revisiting task containing particular kinds of learners' responses provided opportunity to engage not only with learners' conceptions but also with *essential features* of compound growth, *modelling and applications*, and *knowledge of context*. I refer to compound *growth* rather than compound *interest* because we do not speak of gaining interest on a salary, but rather of "salary increase". Similarly, I refer to *knowledge of context* rather than *contextual knowledge of finance* because it is largely fortuitous that the context chosen for this task is broadly financial.

As before, I begin with a discussion of task, goals and resources in relation to the revisiting task. I then discuss students' responses to the learners' work as captured in their journal entries. Thereafter I draw on a vignette that reflects students' difficulties in clarifying the timeframe of the computer operator's salary. I then consider the opportunities for learning MfT of compound growth that emerge from the classroom episode.

9.2 How is revisiting different from learning the content at school?

9.2.1 Task

The learner task involved answering a test question on compound growth. Based on the solution in the memo shown above, learners were expected to apply the formula for compound growth/interest and to substitute the given values. However, a closer reading of the question suggests that the wording is ambiguous and therefore open to different interpretations of the timeframes involved.

The revisiting task contained the test question (without memo) and the responses of four learners. The learner task was framed as a typical class task, and not an assessment task. Students were required to analyse the learners' responses, to identify errors and to suggest ways of helping one particular learner (see Appendix C2). The learner responses had been purposefully chosen – none of the learners had used the compound growth formula, and some had not interpreted the context as one involving

compound growth, although we cannot know whether this was intentional. In addition, the logic and strategies of some of the learners were not immediately obvious, and their responses included incorrect use of symbols and notation, such as the equals sign and percentage notation. Two examples of learner responses are discussed below.

The response of Learner A is given in fig. 9.1. The learner increased R96 000 by 6%, giving an amount of R101 760, and then multiplied this amount by 3, giving R305 280. The increase of 6% was applied once only. In applying the increase and then multiplying by 3, the learner modelled a scenario where the initial increase takes place at the beginning of the first year. The final answer of R305 280 may be considered as the cumulative earnings at the end of a three-year period where there has only been one salary-increase. Note the use of "100%" as a label rather than a quantity in the last line on the left-hand side. This is similar to Sakhile's use of "200%" as discussed in the previous chapter. There is also inappropriate use of the equals sign (R101 760 = R101 760 × 3) in the second last line on the right-hand side.

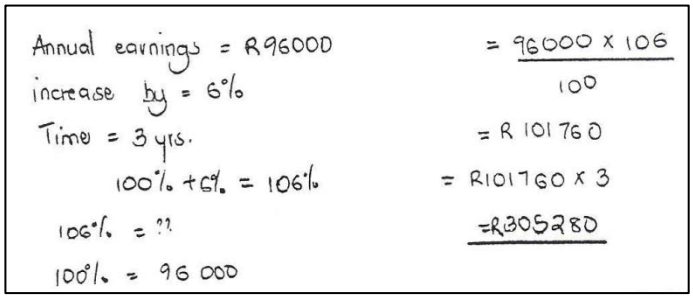


Fig. 9.1 Learner A's response

Learner B (see fig. 9.2) determined that 6% of 96 000 is R5 760 and added this to R96 000, giving R101 760 for the salary for year 1. The learner then added R5 760 twice more getting R107 520 for year 2 and R113 280 for year 3. Thus the learner modelled the increase at the beginning of the first year, and a simple growth scenario for the following two years since each salary-increase was calculated on the base amount of R96 000. This does not reflect the typical method of calculating salary-increases in the workplace.

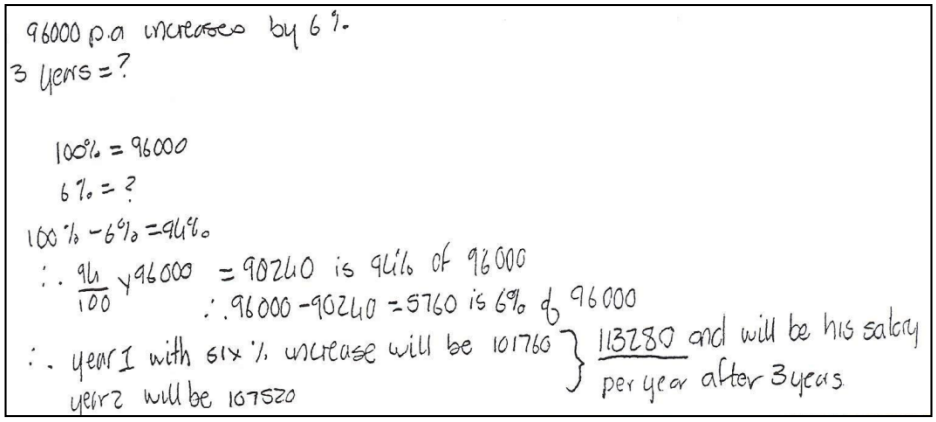


Fig. 9.2 Learner B's response

9.2.2 Goals

In terms of the MfT framework, the revisiting task explicitly required students to engage with *learners' conceptions*. However, the deliberate selection of unanticipated learner responses was intended to challenge students' own knowledge of compound growth and the use of the compound growth formula. Based on previous experiences of using a similar version of the task, I anticipated that students would have to engage with two other aspects of MfT for compound growth: *essential features* and *modelling and applications*. For example, students may need to confirm that the answer given by

the formula is an annual salary (rather than a cumulative amount), and then to check whether it is the annual salary earned during the third year or whether it will be the starting salary for the fourth year. Furthermore, students may need to check whether Learner A's multi-step approach produces the same result as the compound growth formula, and if not, how they might describe the learners' strategies, and identify errors, thus going beyond merely suggesting that the learner should have used the compound growth formula.

9.2.3 Resources

The revisiting task is clearly different from the original learner task since the students are working with learners' responses. The learners' responses may be considered as resources that are brought to bear on the learner task, pushing the pre-service teachers to consider their own interpretations of the original question more carefully. As will be seen below, students introduced Tn notation as an additional resource, in an attempt to resolve a concern about timeframes. This notation is only introduced at Grade 12 level in South Africa and so would not be available to Grade 10 learners attempting the learner task. It is also worth noting that some students were not familiar with the context of annual salary-increase, and therefore were unable to draw on this assumed everyday knowledge as a resource for interpreting the learner task and the learners' responses. It follows then that Grade 10 learners, who have less experience of employment scenarios, may have even more difficulty interpreting the context of the task.

9.3 Opportunity to engage with learners' conceptions

The revisiting task focused on analysing and interpreting learners' responses. This required careful analysis since the learners did not make use of the compound growth formula, and provided partially correct multi-step responses where their solution strategies were not always easy to follow. In making sense of the learners' responses, students had to recognise a range of different errors, including adding a constant amount each year rather than calculating 6% on the latest salary, and the adding of amounts of money to percentages (see Learner C's response in Appendix C2). Later students were required to complete a journal entry in response to the following prompt:

> Think about the responses of the 4 learners, and the teaching of simple and compound growth. What have you learned from these responses that will impact how you teach simple and compound growth?

The journal entries of 29 students were analysed. I focus here primarily on students' references to the learners' responses (rather than teaching issues) with particular attention to learners' interpretation of the question, sources of learners' difficulties, use of formulae, and emphasis on meaning.

9.3.1 Learners' interpretations of the question

Several students made explicit reference to the fact that learners interpreted the question in different ways:

Lebo: I have learned that learners interpret statements differently. Every learner had a different answer.

Sifiso: As I was trying to analyse the errors these learners did, it occurred to me that some of these learners understand the concept in their unique ways but tend to mix up what they are doing with what they already know and understand. They tend not to answer the question posed but, according to their responses, understand the question in a different way.

Gift: All four learners have different ways of looking at the problem and all have receive (sic) the concept on different levels.

In recognising learners' different interpretations of the question, some students were prompted to argue for the importance of unambiguous wording of questions, and others were prompted to suggest that learners need to know how compound interest models a salary-increase scenario.

9.3.2 Sources of learners' difficulties

While students identified a range of sources of learners' difficulties, I focus here only on the most frequently cited sources[47]. Eighteen students (62%) suggested that the difficulties arose because learners did not (or could not) distinguish between simple and compound interest. Percentage and percentage increase were cited as the main difficulty by thirteen students (45%), and nine students (31%) suggested knowledge of the salary-increase context was a main source of learners' difficulties.

9.3.3 Emphasis on meaning

There was a strong emphasis on the need for learners to establish meaning in relation to simple and compound interest. This took a variety of forms in students' responses, ranging from knowing the origins of the formulae to personal financial literacy.

Jenny: I would make this section of work very application based … if they are able to see the importance of this topic and understood this topic, it will help them with organising and managing their own money.

Hailey: Problems for each (i.e. simple and compound interest) must be done, in which both are worded similar except for one or two key words to distinguish between SI and CI, so that learners can become aware of which vocab is associated with each situation.

Joseph: When it comes to teaching simple and compound growth I think the best thing to do is to work together with the learners in getting the understanding and the difference between compound and simple growth.

Vingin: Learner must understand where formula come from. Must understand even without a formula.

9.3.4 Use of formulae

The inefficient strategies used by learners stood in contrast to the elegance and efficiency of the compound growth formula. Through the revisiting task, the pre-service teachers noticed that learners do not necessarily make use of efficient strategies. They also recognised that the question could be correctly answered without use of a formula.

Ronald : From what I have learnt from these learners I can finally say that learner's learning abilities are not the same and so the teaching strategies should not be the same, so to accommodate every learner in class and that some learners can manipulate their calculations and come up with the correct answer, just like learner A and B for the first year answer they got it correct, and so as a teacher I should not concentrate on using formulars (sic) only, also general knowledge if it works then it can be used.

In her journal entry, Palesa had commented that "formulas are just the simple ways of getting the answers". I followed up on this comment in an interview with her mid-way through the course.

Palesa: They kind of all had different approaches to the, the question. But it was different from how would I do it because I would think about the formula. If I want the interest it's *Prt* or what, but they did it in, they put <u>their</u> understanding in the thing like the maths understanding like if x of this is how much, and how will I find it? So it was not about the formula of compound interest or the simple interest. They just used their thinking and that's how they find it …

… the first time I just look at it, I'm like 'yo' what is this? This just looks wrong. But now you try to deal with it and you try to find what is the <u>meaning</u> of this because it can't just be wrong. What's wrong with it? What's right with it? What does it mean? And you find there are some of the answers are correct but the person did it, you wouldn't even <u>think</u> about doing it that way. So it

[47] In some cases students suggested more than one difficulty thus the numbers quoted here do not total to 29.

really shows, you would be doing it maybe using the formula but the person did it without using the formula but is finding the right answer and that's why I said it's just the simplest way, formulas are just simplest ways of finding the answers.

Although Palesa did not appear to recall that the learners' responses were generally incorrect, she showed an appreciation for and acknowledged the need to make sense of their strategies. She implied that the learners had to think carefully to produce their answers and that their responses should be taken seriously. She acknowledged that initially this was not easy. At the same time she acknowledged the efficiency of using formulae.

9.4 Problematising the learner task

I now move to focus on a classroom episode where a concern arose about the timeframes of the learner task. Students had completed the Computer Operator's Salary task and were discussing their responses to the learners' work in small groups.

Sizwe raised a concern about how to interpret the timeframe in the learner task. His concern was sparked by disagreements in his group which were a result of engaging with the learners' responses.

Sizwe: Eh, what, eish, they say, this, this computer operator earns ninety-six thousand per year, so what I don't understand is if you start counting from this year, two thousand and eight (2008) to two thousand and nine (2009), that is the first year, how much is the computer operator earning in that year and after three years, what is "after three years"? What, which year is that? … I don't know if I'm clear.

Before proceeding to the students' responses to his concern, it is necessary to take a deeper look at Sizwe's questions and the implications for interpreting the scenario. He was posing two related questions: (1) "how much is the computer operator earning in that (i.e. the first) year?" and (2) "what is 'after three years'?" The combination of these questions leads to four possible outcomes. However, as I shall show, three of the outcomes produce the same answer for the salary.

Sizwe's first question concerns whether the 6% increase is applied at the beginning or end of the year. For purposes of illustration I shall take 2008 to be the first year. Table 9.1 shows the salaries at the end of each year, when the increase takes effect at the beginning of the year (model 1), and at the end of the year (model 2).

Model 1 6% increase applied at beginning of each year	
End of year	Salary
2008	101 760.00
2009	107 865.60
2010	114 337.54
2011	121 197.79

Model 2 6% increase applied at end of each year	
End of year	Salary
2008	96 000.00
2009	101 760.00
2010	107 865.60
2011	114 337.54

Table 9.1 Two models for salary at end of each year

Sizwe's second question seeks a referent for the preposition "after". It could refer to the timing of the increase, or it could refer to "now", which I shall take to be the beginning of 2008. The four outcomes are shown in fig. 9.3. The salary figures are taken from table 9.1.

Timing of first increase	Referent for "after"	Salary	Case
beginning of 2008	from first salary-increase	R114 337.54	①
	from beginning of 2008	R114 337.54	②
end of 2008	from first salary-increase	R114 337.54	③
	from beginning of 2008	R107 865.60	④

Fig. 9.3 Four cases relating "timing of increase" with "referent for 'after'"

As can be seen, cases 1 and 2 reduce to a single case covering the same period because the increase takes place at the beginning of 2008. Case 3 also produces the same answer but covers a different time period. If we consider the stated salary for the year, then the end of this period is the beginning of 2011 (since the third increase will take place at the end of 2010). However, if we consider salary as the amount actually earned, then the end of the period is the end of 2011. The key issue is that the salary must have been increased three times. Case 4 involves only two salary-increases since the first increase takes place at the end of 2008 which is already one year into the three-year period.

9.5 The vignette – interpreting the timeframes in the task

Based on this analysis of Sizwe's questions, we are now in a position to interpret the responses of students to his question. However, at this point I wish to note that the discussion and analysis which follows in the vignette is the result of repeated viewing of the video footage and sustained interrogation of the transcript over a long period. As will be seen, I terminated the class discussion because I was aware that many students were struggling to follow it. Given my own struggles to interpret and analyse the data, I now appreciate more fully the extent of the students' struggles.

The vignette begins with Sizwe's interaction with Belinda. The two students were seated in diagonally opposite "corners" of the room and so their conversation was easily audible by the whole class.

Line	Speaker	Utterance
131	Craig:	… [H]ow are we interpreting the question? Belinda?
132	Belinda:	I would say, I'd just look at that last sentence, what do you, what will your salary be after three years. I'd say after three years of what? It'd be after the three years of when, from when the interest, the increase is applied. It's no point in just saying 'after three years' when it says that she's had an increase of six percent, so now there's an increase of, after three years of that increase I would assume. I would, that's how I would interpret the wording.
133	Craig:	So is it twenty-ten (2010) or twenty-eleven (2011) assuming it starts in two thousand and eight (2008)?
134	Belinda:	Tw-, Twenty-ten. At the end of twenty-ten (2010).
135	Sizwe:	So, so how many times are you gonna add this?
136	Belinda:	Three. I'd say, let's say just say today I come in to work and my boss says you getting a six percent increase, so I would assume that my salary is ninety-six thousand rand split over this year so you'll have to take away the January's earnings and calculate it accordingly and on the balance that's left over, I'd receive a six percent increase till the end of the year.
137	Sizwe:	Until which year is she earning this ninety-six thousand?
138	Belinda:	No, no, like today I get an increase, it's an instant thing, I don't think your boss is going to say to you, you getting an increase but it's only applying next year.
139	Sizwe:	But, still, I don't understand, which year is she earning …
140	Belinda:	Let's say it's this year.
141	Sizwe:	Starting from this year.
142	Belinda:	Let's say – well it would be my previous salary, it would have been …
143	Sizwe:	So this ninety-six thousand you can say is ….
144	Belinda:	last year's annual salary *(nods her head).*

Belinda responded by proposing a referent for "after", suggesting it would be the point at which the increase takes effect. So for her "after 3 years" referred to 36 months after the increase was given [132]. If the increase had occurred at the start of 2008, she argued "after 3 years" would refer to the end of 2010 [134]. This provided her answer to Sizwe's second question. He then asked "… how many times are you gonna add this?" [135] which I take as a way of asking whether the increase takes effect at the beginning or end of the year. Belinda replied that the salary would be increased three times, and proceeded to give an example but worked with partial years which may have obscured the issue for Sizwe [136]. He then pushed her to be explicit about the year in which the computer operator earns R96 000 [137-143] to which she responded that it's "last year's annual salary" [144]. This finally clarified Belinda's interpretation. In terms of the earlier analysis of the possible cases, she was assuming the increase took place at the beginning of 2008, based on a salary of R96 000, and hence that the salary after three years would be R114 337.54.

Although Belinda had provided answers to both Sizwe's questions, there was still some uncertainty for him. He shifted to introduce sequence notation [145], suggesting that T_1 is 96 000 and that they want to know T_3. Jenny responded that they needed to find T_4, not T_3, because "it's like three jumps" [146]. Sizwe then referred to work done earlier in the session and implied that moving from T_1 to T_4 spanned a four-year period [148] but this was immediately rejected by Jenny and Attiyah [149]. At this point Sizwe appeared to be confusing discrete points in time with the intervals between them. So while terms T_1 to T_4 represent four discrete points, there are only three time periods.

Line	Speaker	Utterance
145	Sizwe:	But, eish, when we were doing the thing of sequences and series, like here I can say this ninety-six thousand is term one and you want to know what is term three. Eish, but that's what I'm looking at this thing.
146	Jenny:	It will be term four you want to know because you have to add it three times onto term one. So it's like three jumps.
147	Attiyah:	Because it's after the ninety-six thousand.
148	Sizwe:	So we are doing what Palesa did there? (*referring to work written on board earlier in session*). There (unclear), multiplying by four and now it's four years, not three years.
149	Jenny & Attiyah:	No!

Belinda suggested T_1 is a starting point but not necessarily the first year [150]. Harry joined the discussion, linking the terms to actual calendar years [153] to which Belinda responded that for her T_1 refers to "an opening amount" and not to time [154]. This was in line with her earlier comment [150] that T_1 is simply a starting amount and also with Sizwe's initial idea that T_1 is 96 000 [145]. However, this introduced additional ambiguity as to whether the terms referred to time or to amounts of money.

Jenny then proposed a new index, suggesting that T_1 can be equated with year zero (Y_0) and she treated this as a starting point. She then proceeded to link the indices of terms and years where $T_1 = Y_0$ … $T_4 = Y_3$ [158-162]. Harry responded by suggesting 2007 as a base year for the indices [163].

Line	Speaker	Utterance
150	Belinda:	But term one is not necessarily year one, it's the starting amount.
151	Sizwe:	Ja. And that's year one.
152	Belinda:	Not necessarily.
153	Harry:	So if you're saying that, if you're looking for term four, that means three years will be two thousand and eleven, not two thousand and ten.
154	Belinda:	No, I, I wouldn't say that term one refers to a year. I'd say it refers to an opening amount.
155	Harry:	To an opening amount?
156	Belinda:	Yes.

Line	Speaker	Utterance
157	Harry:	Which is …
158	Jenny:	Term one can be like year zero.
159	Harry:	Year zero.
160	Jenny:	Your starting ….
161	Harry:	So if term one is year zero, when, after three years you …
162	Jenny:	So, say term one is year zero, then term two is year one, term three is year two and term four is year three.
163	Harry:	So let's take, term one is year zero which is last year?
164	Jenny:	Yes.

Harry then confirmed the link that if Y_0 is 2007, then Y_3 is 2010. He also adjusted the base year by one for Sizwe's initial question, in which case Y_0 would be 2008 and Y_3 would be 2011. This appeared to concern Belinda [166] but it is not clear what her concern was. Harry was confident in his reasoning about the timeframes but did not deal with the salary aspect.

Line	Speaker	Utterance
165	Harry:	So three years will be twenty-ten. So from Sizwe's context what he's saying if term one, year zero is this year, meaning three years will be twenty-eleven (2011).
166	Belinda:	No, but that, the ninety-six thousand hasn't, um, hasn't, you haven't um incurred that increase of six percent yet, so…
167	Harry:	But then you saying term one was last year which was two thousand and seven (2007). Then, 'after three years' will be two thousand and ten (2010). That holds. So if year zero is two thousand and eight (2008), 'after three years' will be two thousand and eleven (2011).
168	Jenny:	Ja, but now you've shifted it one up.
169	Harry:	Ja, because that was Sizwe's original question.
170	Jenny:	But then it's still three years.
171	Harry:	Three years.
172	Jenny:	Now you've shifted it one year up so you saying if the previous year was two thousand and eight (2008), so that means in two thousand and (unclear).
173	Harry:	No, cos when Craig …
174	Craig:	(*To Harry*) No, hang on let's just finish.
175	Harry:	Okay.
176	Jenny:	That means in two thousand and eight you were earning ninety-six thousand, you have to add another three years and you will get two thousand and eleven (2011). It's not the same as that cos we shifted the previous year up one.
177	Craig:	Okay, it seems to me that maybe we need to find representations that we can work with, that are gonna help us think about this …

I stopped the discussion because it was not moving forward [177], and I was concerned that many students were no longer following the thread of the discussion. I then moved to propose a timeline to represent the situation, distinguishing between discrete points on the timeline and the intervals between them. My focus was on representing the salary problem on the timeline rather than on the general features of the timeline.

The whole-class interaction described above provides an example of the students' difficulties in trying to interpret each other's reasoning. This was partly because Sizwe struggled to make explicit what he wanted to know, but also because the students were not explicit in framing their discussion of time in relation to the beginning and end of periods. Furthermore, although Belinda attempted to use a realistic example [136] to explain her interpretation of "after" and her interpretation of the salary-increase context, the complexity of initially working with partial years did not appear to help Sizwe resolve his concern.

9.6 What opportunities emerge for learning MfT of compound interest?

I have already discussed the opportunities for engaging with learners' conceptions. In this section I focus on opportunities to engage with three other aspects of the framework: *essential features*, *modelling and applications,* and *knowledge of context.* I discuss each of these below.

9.6.1 Essential features

An essential feature of working with compound growth is the need for precision in references to time. This does not mean that the task should provide precise indicators of time but rather that students be precise in the ways they work with time as they engage with the task. Sizwe's concern foregrounded the students' difficulties in communicating precisely about timeframes. His concern emanated from the ambiguous wording of the learner task, and extended to the class discussions as he tried to make sense of Belinda and Jenny's comments. One of the sources of difficulty was that he did not distinguish between the beginning and end of a year, and he also appeared to muddle discrete points in time with the intervals between these points. Belinda and Jenny worked consistently with the end of periods. However, with the exception of Belinda's comment in [134], they were not explicit about this in their talk. Even when Jenny explicitly linked time to the terms of the sequence, she did not refer explicitly to beginning or end of the period: "so, say term one *is* year zero, then term two *is* year one, term three *is* year two and term four *is* year three". Her repeated use of "is" leaves the timeframe ambiguous although from her other contributions it is clear that she was referring to the end of each year.

Jenny's talk about time was not transparent (Lave & Wenger, 1991). It may be that her use of time was still too tacit for her to recognise how she was talking about it, and thus she could not be more explicit than she had attempted to be. In order to communicate more effectively with others in the class, she needed to become more aware of *her* talk about time, in particular her references to the beginning and end of the period – to make it visible – so that she could foreground it for her peers and make it explicit. Then she would need to background it again so that it did not become an obstacle, obscuring her reasoning by being too much in focus and not being able to see through it to the problems she wanted to solve.

The students' struggle to resolve the timeframe issue led me to (re)-introduce timelines as a means of representing time. While I had assumed they had been introduced to timelines in their first year course, only one pair of students appeared to have made use of a timeline in attempting to deal with Sizwe's concern. All this suggests that revisiting compound growth provides a necessary opportunity to emphasise the importance of explicit and precise references to time. It also provides motivation for attending to the details of key representations such as timelines.

9.6.2 Knowledge of context

In the MfT framework I have referred to this aspect as *contextual knowledge of finance*. However, in the context of this task, I prefer to talk about "knowledge of context". There is a clear relationship between knowledge of context and knowledge of *modelling and applications*, which follows in the next section.

The learner task assumes knowledge of the salary context. Firstly, it assumes that a salary-increase, with a constant rate of increase, can be modelled by the compound growth formula. Secondly, it assumes that salary is taken to be the amount earned in a twelve-month period. The responses from

both the learners and the student teachers suggest that knowledge of the context of salary-increase cannot be taken for granted. For example, one needs to know that "salary" generally refers to how much one earns in a (calendar) year and does not refer to cumulative earnings over a period longer than a year.

On the other hand, it might be argued that the learner task does not require knowledge of the context of salary-increase. The learner task resembles a typical text book word problem on introductory financial maths at Grade 9 or 10 level. Word problems in text books typically give only the necessary information to answer the question correctly, and are stated in ways that reduce ambiguity of interpretation. (It might therefore be argued that text books would replace "after" in the initial learner task with a phrase like "at the end of" to avoid the ambiguity.) Sometimes knowledge of the context in which the problem is located is necessary but some would argue (e.g. Sethole, 2004) that frequently the context is merely a veneer that may be ignored. In the case of this task, knowledge of the context of salary-increase may be helpful although it may not be essential to calculate the correct answer. Thus a learner who knows the relevant mathematical concepts and the genre of word problems may complete the learner task successfully with little knowledge of the salary context. It may be that the need for knowledge of the salary context increases if one takes a multi-step approach or attempts to unpack how the compound growth formula models the salary-increase.

9.6.3 Knowledge of modelling and applications

The learners' and student teachers' responses confirm that the original question contained some level of ambiguity with regard to timeframes. While the single solution in the memo suggests that this ambiguity was unintended in the assessment task, modelling tasks typically contain some degree of ambiguity as a design feature of the task. This requires the student to make his/her assumptions explicit as part of the modelling process. For example, one needs to be explicit about whether the 6% increase is assumed to take place at the beginning or the end of the year. Table 8.1 indicated these two possibilities.

It can be shown that the compound growth formula models (annual) salary-increase, with a fixed rate of increase, r, for n years. Thus in the formula, $A = P(1 + r)^2$, the amount A gives the new salary at the end of the second year, where the increase has been effected twice. We can view salary in two subtly different ways. Consider the example of a salary of R96 000. This means that R96 000 is both the amount earned by the *end* of the year *and* the amount on which each month's salary is calculated from the *beginning* of the year. Put another way, it is the "whole" that is subdivided into 12 equal monthly portions. It is thus acted on from the beginning of the year, giving a monthly salary of R8 000. Thus 96 000 is a number linked to the end of a year but also applied throughout the year.

It is worth noting that the salary-increase context may be considered both different from, and similar to the typical compound interest scenario. It may be considered different because interest is compounded at the *end* of the period whereas the salary-increase takes effect at the *beginning* of the period. However, if one considers the reality of daily interest calculations (discussed in chapter 6) the scenarios are not different because interest is accumulating from the first day of the month, in a similar way to the salary-increase taking effect from the first month of the year.

9.7 Conclusion

In this chapter I have shown how a revisiting task that includes learners' responses to a school maths task provides a springboard for learning other aspects of MfT for compound growth such as essential features, modelling and applications and knowledge of context. I am not claiming that students in the course learned these aspects. I am suggesting that the Computer Operator's Salary Task, and how it played out in the course, provides the potential for students to learn aspects of MfT beyond the aspect of learners' conceptions.

However, it is not the use of learners' work *per se* that opens up these opportunities. It is dependent on the *kinds* of learner work that is selected. The mathematical demands of making sense of learners' inefficient, multi-step and partially correct responses are far greater than checking whether a learner has substituted correctly into the compound growth formula. It is the decompressed nature of the learners' responses that opens opportunities for students to reconsider their own knowledge of compound growth, of the mathematics that underpins the formula, the process to obtain the formula and thus the way in which the formula models compound growth in different contexts. This requires them to shift between operational and structural conceptions of compound interest. Furthermore, working with carefully-selected learner responses may prompt consideration of issues that would not arise when pre-service teachers simply work though school maths tasks to produce answers for themselves. For example, Learner A's response introduced the notion of cumulative earnings. It is unlikely that this would arise when pre-service teachers do the task. However, the learner's response requires that the student teachers consider why cumulative salary is incorrect. It may even prompt them to explore a suitable formula for cumulative salary, and hence an opportunity for further mathematical inquiry.

Aspects of MfT in focus in this chapter

Essential features

- Precision in references to time (which led to re-introduction of timeline to clarify timeframes)

Modelling and applications

- Knowledge of how modelling of compound interest scenarios differs from modelling of other exponential growth scenarios, even if the same formula is used

Mathematical practices

- Students' struggles to communicate their mathematical thinking to each other

Learners' conceptions

- Different interpretations of timeframes in learner task
- Inefficient strategies and partially correct learner responses
- No use of compound growth formula

Knowledge of context / contextual knowledge of finance

- Knowledge of salary-increase context could not be taken for granted

CHAPTER 10

Revisiting school mathematics - case 4

Opportunities for explaining essential features

10.1 Introduction and background

Mathematics teachers spend a great deal of their time providing explanations to learners. While the course did not focus on producing explanations for learners, students had many opportunities to explain their thinking to their peers. In this chapter I draw on data from the fourth week of the course where students revisited the compound interest formula. I focus particularly on two examples of explanations – one given to me, and another given to the class. I begin by describing the revisiting task and how it differs from learning the compound interest formula at school. I then describe and analyse each of the explanations, identifying specific content issues that are local to the explanation, and then stepping back to reflect on what can be learned from the students' explanations, more globally, in relation to developing MfT of compound interest.

10.2 How is revisiting different from learning the content at school?

10.2.1 The task

The revisiting task consisted of three questions. I focus here only on the first question which forms the basis of this chapter. Students were asked:

In the formula, $FV = PV(1+i)^n$, where does the 1 (one) come from?

Students would not be likely to encounter this question at school. It required them to consider the relationship between 1, *i*, *n* and *PV*. The structure of the formula means one can easily be seduced into thinking the 1 is simply multiplied by the *PV*, particularly since this is the case in the simple interest formula where $FV = PV(1 + i \times n)$. I expected students' responses would ultimately lead them to deriving the formula, even if they did not begin with a derivation. (See Appendix C3 for the whole revisiting task.)

10.2.2 Goals and resources

My goal was that students should revisit the derivation of the compound interest formula. However, I did not want to ask them directly for a derivation because this might have led to a reproduction from a text book or the web, neither of which would have achieved my purpose for revisiting. When I introduced the first question to the class, I attempted to frame it as if they, as teachers, were required to provide a response to a Grade 10 or 11 learner. However, the video footage shows that the framing is not clear. At one point I referred to responding to a learner's question but later I implied they were answering the question for themselves or for their peers. This ambiguity meant that students positioned themselves differently in relation to the task – some as teacher, some as learner. Regardless

of the way the students positioned themselves, my goal was that they should revisit the derivation of the formula.

I now move to discuss two critical instances of explaining. In both cases the students are attempting to explain "where the one comes from". The first instance involves two students explaining their thinking to me. I do not consider this as an example of unpacking since I have defined unpacking only in relation to self, peers and learners, and not in relation to someone who knows more than the unpacker. The content analysis of their explanation reveals two errors in their thinking alongside powerful use of the properties of the numbers they are dealing with.

The second instance focuses on a student's explanation to the class, and highlights the cognitive demands of his explanation on the listener/learner. His explanation may be considered as an amalgam of unpacking for peers and unpacking for learners since his peers knew the formula and had been working on the same task, and yet some had difficulty in making sense of his explanation. No doubt the different audiences implied different pedagogic demands for the students as they produced their explanations but these differences are not in focus here.

10.3 Vingin and Naasiha's explanation to me

Two students, Vingin and Naasiha worked together regularly. They frequently finished tasks before the other students but often had to be pushed to think more carefully about their responses. After working on the task for about two minutes, they said "one comes from a hundred percent of your principal value ... so it's a hundred over a hundred which is one". I pointed out that they had not accounted for the relationship between the 1 and the exponent but acknowledged their argument that their reasoning was appropriate for the simple interest formula. Later, when I returned to their table, they had made progress but were unsure whether they had achieved what was required. They were focusing on $(1 + i)^n$, which I have previously referred to as the *growth factor*, and were treating the 1 and i separately. The transcript below reflects our interaction. It is not easy to follow because the explanation is disjoint, they frequently interrupted each other, and there are several mathematical slips (Olivier, 1989) in their talk.

Line	Speaker	Utterance
42	Naasiha:	… one to any power will still give you one.
43	Craig:	Umhmm.
44	Naasiha:	Now if you're taking a percentage it's gonna be a decimal. So like …
45	Vingin:	Less than one.
46	Naasiha:	So like zero comma two five (0.25) …
47	Craig:	Umhmm
48	Naasiha:	So if you raise that power (unclear) what was the example we took? One point five … what did we take? (unclear)
49	Craig:	I dunno what you're saying because you didn't show me.
50	Naasiha:	Okay, if you take like umm a percentage increase of five percent which is zero comma five and we add that, and we take that to the, to the power four. (0.5^4)
51	Craig:	What? Zero point zero five to the four? (i.e. 0.05^4)
52	Naasiha:	One point, ja zero point five, and you're adding the one there and you're raising it to the power four, four years … (i.e. 1.05^4)
53	Craig:	Why? Hang on, I'm not clear. Is it one point zero five or … zero point five? Or zero point <u>zero</u> five?
54	Naasiha:	Zero point zero five and then we're adding the one to that. (i.e. 1.05)
55	Craig:	Okay.
56	Naasiha:	And when we raise that to any power we're still gonna get one point something.
57	Vingin:	one point something.
58	Craig:	Ja.

Line	Speaker	Utterance
59	Vingin:	And then we now, we are saying present value timesing it, that one will still uhh, you (unclear) what we were saying to you before, that it's you know this one (i.e. the 1) and then whatever that was the percentage, you know, the interest, and then but we try use other bigger number, we got like really big number
60	Craig:	What do you mean by "other bigger number"?
61	Naasiha:	Like one point five (1.5), it didn't work because the number became like twenty-three or …
62	Craig:	So one point five, what percentage increase is that?
63	Naasiha, Vingin:	Fifty percent.
64	Craig:	Okay.
65	Naasiha:	Ja. And then we raised to any other power and then we got like <u>whole</u> numbers we didn't get one (unclear) …
66	Craig:	Does it matter?
67	Vingin:	(*To Naasiha*) Ja (unclear) he asked right in the beginning because after we work out everything then that wasn't the one … (*To Craig*) so are we correct?
68	Craig:	Ah, well I'm not sure … So you said if you put one point five to the power of something you got <u>more</u> than one, I mean you got, in fact, more than two …
69	Naasiha:	Ja, (nodding)
70	Vingin:	Ja, it's five point (*trails off*) ($1.5^4 = 5.0625$)
71	Craig:	Because you got more than one point something, right?
72	Naasiha:	Ja
73	Craig:	And you think that that was wrong? Am I …?
74	Naasiha:	It didn't make, quite make sense compared to the one comma zero five.
75	Craig:	Cos you were expecting one point something …
76	Naasiha:	Ja, one point something.
77	Craig:	… is that right?
78	Naasiha:	So it's like your whole plus your percentage increase, like your interest on that (*trails off*)
79	Craig:	And so you don't think it should get beyond two …
80	Naasiha:	Ja.
81	Craig:	Or it shouldn't reach two, basically is what you're saying?
82	Naasiha:	Ja, we're finding it hard to understand that part.

Naasiha said that 1 raised to any (natural) power is 1 [42], and since the *i* is a percentage, it would "become a decimal" and, when raised to any (natural) power, would be less than 1 [44-45]. Therefore the sum $(1 + i)$ would lie between 1 and 2 [50-57]. She illustrated this by means of an example they had tested earlier where they used 5% p.a. compounded annually for four years, giving $1.05^4 = 1.21550625$. This supported their assumption that the answer lay between 1 and 2. Naasiha distinguished between "the whole" (i.e. the 1) and "the percentage increase" (i.e. the decimals) [78] so when they got an answer of "one point something" this fitted with her expectations. Vingin explained that when they multiplied their answer by the present value, the 1 would give the original amount and the decimal part (0.2155...) would give the amount of interest that accumulates [59]. However, when they substituted a bigger decimal, such as 0.5 (i.e. 50%), the growth factor was much bigger than 2 which challenged their assumption that it lay in the interval (1,2], and they were unable to reconcile this immediately [68-82].

However, a few seconds later, Naasiha suddenly interrupted to say that the growth factor could get bigger than 2 if the money accumulated interest for long enough [84-101]. For example, a quarterly interest rate of 5% compounded quarterly for five years would give a growth factor of 5.0625 [93-95]. Vingin immediately agreed with this reasoning and contributed to the explanation [88, 92, 98].

Line	Speaker	Utterance
83	Craig:	Okay, maybe let the, mull over than and move on to the next one because when we get people to come up I'm gonna ask, I'm gonna mention it, either by name, or say some people are anticipating that this, this thing in the bracket shouldn't get, it shouldn't reach two. It should always be one point …
84	Naasiha:	(*interrupting*) Ah, it makes sense!

Line	Speaker	Utterance
85	Craig:	Does it make sense?
86	Naasiha:	Yeesss, because obviously if you gonna accumulate it over the years …
87	Craig:	Ja?
88	Vingin:	(*Nodding*) and then
89	Naasiha:	So like if you …
90	Vingin:	(unclear) and then it's over …
91	Naasiha:	Ja.
92	Vingin:	(unclear) four times the thing.
93	Naasiha:	So like it's one comma zero five.
94	Craig:	Ja.
95	Naasiha:	Five times, for your five years … (i.e. 1.05^{20})
96	Craig:	Mmm (*i.e. expressing agreement*)
97	Naasiha:	multiplied, so that's when we got like twenty-three because it was that, now there
98	Vingin:	(unclear) not twenty-three otherwise (unclear)
99	Naasiha:	Okay not twenty-three, ja. (*To Craig*) Do you understand?
100	Craig:	Um, not completely. So, and I think partly it's cos you still trying to get it right in your head (*Naasiha nodding in agreement*) but I think you've got beyond where you were when I arrived here.
101	Vingin:	The next time you come we might get it…

10.3.1 A content analysis of their explanation

The students focused on individual elements of the formula, and then considered the relationship between them. Their initial response – that 1 represents a hundred percent of the present value – suggests their focus was on the growth factor, and multiplying it by the present value. They knew this worked in the case of the simple interest formula but here they were ignoring the exponent and thus not making appropriate connections between 1 and *n*.

Interpreting the follow-up discussion is more complex. It is clear they were drawing on their experience of compound interest calculations and their knowledge of the range of typical values that arise in that context. They appeared to reason inductively from typical numerical examples, and then attempted to justify their findings based on the properties of the numbers. This led to a conjecture. However, when they then extended the range of test cases, they found a disconfirming case and could not account for it. When they realised the answer from the disconfirming case was viable in the financial context, they did not reconsider their conjecture. There appear to be two errors in their thinking. Firstly, they assume that the growth factor should be less than 2. Secondly it appears that they are implicitly assuming the relationship $(x + y)^n = x^n + y^n$.

I propose that their reasoning, based on the properties of the numbers, may have resembled the following shown in table 10.1. Line numbers in the transcript are included for reference:

Claim	Line no/s.
Based on our experience, the growth factor is usually less than two, i.e. $(1 + i)^n < 2$	
We know that $1^n = 1$	42
When the interest rate is converted to decimal it is less than 1, e.g. $5\% = 0.05$	44, 45
When you raise a decimal value to a (natural) exponent the number gets smaller, e.g. $(0.05)^4 = 0.00000625$	
So the growth factor is "one point something"	54-57

Table 10.1 Vingin and Naasiha's reasoning about $(1+i)^n$

Thus the students worked separately with the two components within the brackets and linked each of them to the exponent. Hence it appears they were "distributing the exponent into the bracket", i.e. $(x + y)^n = x^n + y^n$, and this led them to the incorrect conclusion that:

$$(1 + decimal)^n = (1)^n + (decimal)^n = 1 + (decimal)^n$$

However, as can be seen from the transcript, they did not state this relationship explicitly, and it is likely that this reasoning was implicit for them. It is important to note that this error was not exposed by their calculations because when they worked numerically, they simplified $(1 + i)$ before raising it to the given exponent, and so there was no error in their calculations. The error was only exposed in their attempt to generalise and explain their numeric results.

It appears that in order to focus on the "one" in the formula, they separated the whole from the percentage increase, and reasoned that the whole preserved the present value, and the decimal portion represented the interest [78]. For example: $(1 + 0.05)^4 = 1.21550625$, so 21.550625 is the percentage increase on the present value. Mathematically it is correct to think of the 1 as preserving the present value and so it seems reasonable to assume the decimal part represents the interest. However, this thinking was challenged by their own example because a 50% increase produced a growth factor of 5.0625 which is clearly greater than 2, and so the conjecture that the whole number part was the original amount and the decimal was the interest no longer held. At that point, they did not see that the 1 preserved the present value and the *rest* (i.e. 4.0625) represented the interest. There is no evidence to suggest that the students revised their conjecture when they recognised that it was possible for the growth factor to be greater than 2.

10.3.2 What can be learned about *explanations*?

Vingin and Naasiha's explanation contained many elements of a peer and self-explanation. It lacked a systematic coherent trace in their reasoning, and it assumed knowledge of properties of percentages, decimals and exponents. It also assumed a shared knowledge of the compound interest context although there are only two mentions of the word "interest" [59, 78]. They paid little attention to the interest rate, working implicitly with an effective rate per period since they did not refer to adjusting the rate for quarterly compounding, for example. This was not a focus of their explanation and it did not provide an obstacle for me. However, I was aware that I had not fully understood their explanation at the time, and as noted above, it took very careful analysis to produce the interpretation I have provided. This suggests the importance of making peer explanation an object of attention in pre-service maths teacher education courses.

In addition, there are further insights that can be gained from Vingin and Naasiha's explanation. I discuss each of these below.

Students' explanations provide a window into their MfT

Vingin and Naasiha's attempts to explain the compound interest formula exposed incorrect thinking which may not easily be revealed through typical learner tasks. In most cases learners are given a numeric value for the interest rate and so will simplify what is inside the bracket before raising it to the exponent. In the case where they are required to determine the interest rate, the most efficient strategy will be to take roots on both sides, and so it is unlikely they will attempt to "distribute the exponent into the bracket". In this particular case the students themselves may not have been aware

that their thinking was premised on the overgeneralisation $(1+y)^n = 1+y^n$. In part 2, I shall show further evidence of incorrect student thinking that is not exposed through their written work but only through their explanations. This suggests that it is important to provide opportunities for students to explain their thinking regarding school mathematics. It also highlights the importance of carefully designed tasks for mathematics teacher education that access potential misconceptions with regard to the mathematics in focus.

The value of a "feeling for the numbers"

Despite the error in their thinking, Vingin and Naasiha displayed a "feel" for the magnitudes of the numbers that are typically worked with in compound interest calculations in the school context. They recognised that the growth factor is usually less than 2, which is a useful observation, because if $1+i=2$ then the interest rate would have to be a hundred percent which is not realistic. This awareness enabled them to generalise their reasoning about the relationships. Such situated, experience-based knowledge comes from doing many examples and becoming aware of the relationships between quantities. Noss and Hoyles (1996b) found a similar awareness in their studies in financial institutions where the employees knew the range of likely values based on their daily experience, and could extract trends in growth based on their investigations with the software. This kind of "intuitive" knowledge is valuable for teachers when dealing with learners' productions. Teachers may not have time to engage with the details of learners' calculations but if they have a sense of the magnitudes that result from various calculations at various steps in a procedure, and if they can draw on this knowledge in the act of teaching, then they may be able to help learners identify their errors without having to deal with the detail of the calculations.

Explaining may lead to opportunities for further mathematical inquiry

Unpacking for oneself or for others may lead to further opportunities for mathematical inquiry related to school mathematics. This may result from asking a "what if" question, from looking for new links (as Sakhile did in chapter 8) or it may stem from incorrect thinking that needs addressing. Consider the following example from Vingin and Naasiha's work. They discovered that an interest rate of 50% was one case where the growth factor exceeded 2, and this challenged their conjecture about the magnitude of the growth factor. Following from this, it would be useful for them to explore the following question: "What decimal value, when added to 1 and then raised to the relevant power, will give an answer bigger than 2?" For example, assuming a scenario of annual compounding over 4 years, they would need to solve for x where: $\left(1+\frac{x}{100}\right)^4 > 2 \Rightarrow \frac{x}{100} > \sqrt[4]{2}-1 \Rightarrow x > 18.9207\ldots\%$ which translates to an interest rate of approximately 18.92% p.a.

This is not mathematics that teachers will teach their learners, nor is it essential knowledge they need to teach compound interest well. However, they could expand their knowledge base for teaching compound interest by recognising that the "tipping point" is dependent on the value of *n* and the number of compounding periods per year, and then to have some sense of the magnitudes at which this happens, and thus to extend their feeling for the numbers.

10.4 Zwaii's explanation to the class

Zwaii volunteered to share his strategy with the class. He started by separating the original amount from the interest gained in the first year, writing: (*Amount* × 6%) + (*Amount*), as shown in the reproduction of his boardwork in fig. 10.1. He indicated that the expression in the first bracket represented the interest gained, and then proceeded to factorise the expression, writing $A(1+6\%)$,

and saying he could factorise because the value of *Amount* is the same in each term in the first line. (Note that he abbreviated *Amount* with *A* in the second line). For him, this showed where the 1 came from in the formula. He then explained how the exponent, *n*, arises. He spoke of having multiplied *A* by $(1 + 6\%)$ in the first year and said they must now multiply $A(1 + 6\%)$ by a further $(1 + 6\%)$ to get the amount for the second year. This is indicated by the large curly bracket below line 2. The new product is multiplied by $(1 + 6\%)$ again for the third year. As can be seen from his boardwork, he did not simplify the expression each time to show the exponent, nor did he explicitly show the relationship with *FV* (i.e. FV =) although he does refer to future value as shown in the transcript below.

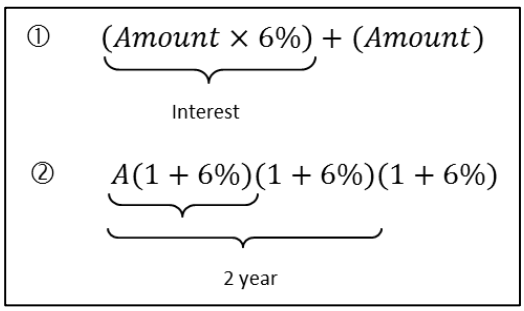


Fig. 10.1 Reproduction of Zwaii's boardwork

In the transcript several references are made to his gestures in order to clarify what he is referring to. Although his explanation is continuous, the text has been subdivided for purposes of readability. Line numbers have been inserted for this layout. I distinguish between line 1 (①) and line 2 (②) of his boardwork. In the first line of his boardwork I distinguish between '*Amount* on the left' and '*Amount* on the right'.

Line	Utterance
	(*Writes on board*: *(Amount x 6%) + (Amount)*)
1	Ok the first thing that I did I said the amount that we've been given is this one (*points to* Amount *on left*).
2	This is the increase, interest (*draws curly bracket below (Amount x 6%) and writes 'interest'*)
3	So this I have said is the interest. So this interest we have to add to this amount here (*points to* Amount *on right)* to get the amount, the future amount, the future value.
	(*about to write A(1+6%)*)
4	So what I did is, I said because of the amount that I have multiplied with the 6%, that I have added, is the same amount so I can just take out that amount.
5	So I've got the same amount so I'm having my one (*writes A(1)*] so this is my one from here (*points to* amount *on right)* plus this (*writes +*) if I take the amount here I've got 6% (*writes 6%*). So this is how I got the one, for the first year.
6	So to continue with showing the *n*, I'm not sure whether do I have to continue with showing the *n* how I thought of, why did I represent *n* as the exponent?
7	What I did, I've said because of this is the first year amount, it's the first year salary, so this first year salary (*draws curly bracket below A(1 +6%) in second line*) to get the salary for the next year I just have to multiply this whole thing by 1 plus 6%.
8	So this is the amount (referring to *A(1 +6%))*, first amount (*referring to A)* we've multiplied by this (*1 +6%*) for the first year.
9	So now I'm saying this is the first year amount (*A(1 +6%))* I'm multiplying it by this (*referring to* (*1 + 6%))* so this is again, this whole thing (*indicated by large curly bracket below line 2*) will give me for second year.
	(*He inserted the third factor of (1+6%) in line 2 a short while later to indicate gaining interest for the third year*)

Zwaii started from the first year and built up the expression for each successive year. He drew on the compound interest context as well as the computer operator's salary context[48] in terms of meaning [3,7]. He also referred to the mathematical structure when factorising the first line [4]. Zwaii provides an example of a peer explanation. He began systematically with the first period, and then proceeded to deal with the second and third periods. His boardwork was incomplete with a lack of attention to separating the calculations for each year. At one stage he inserted numbers in the position of exponents to show the factors for each of the first three years:

[48] The "computer operator's salary" refers to the learner task discussed in the previous chapter. Zwaii acknowledges this a short while later when asked about his choice of 6% as the interest rate.

$$A(1+6\%)^1\ (1+6\%)^2\ (1+6\%)^3.$$

Seemingly his assumption was that his audience could see three separate periods within the expression, i.e. $A(1+6\%)$ represents the first period, $A(1+6\%)(1+6\%)$ represents the second period and $A(1+6\%)(1+6\%)(1+6\%)$ represents the third period. However, one of the students immediately suggested he remove the numbers "because they look like exponents". Zwaii also shifted from $Amount$ in ① to A in ②, although this was not problematic. He used a mixture of generalised terms (e.g. A) and specific values (e.g. 6%). When questioned on this he referred to the computer operator's salary. He thus drew on the shared knowledge of the class which is an element typical of instructional explanations (Leinhardt, 1997).

10.4.1 What can be learned about *explanations*?

Zwaii's attempt to unpack the formula is interesting because of the way in which he shifted from an operational to a structural view of the compounding process. He began with an additive approach, explicitly separating the principal amount from the interest gained, and then factorised the expression [3-4]. In Sfard's (1991) terms this reflects an operational view since it models the process of calculating interest and then adding it to the principal amount. However, after the first iteration he shifted to a multiplicative approach, viewing the expression $A(1+6\%)$ as an object to be multiplied by a growth factor of $(1+6\%)$ [7]. He was no longer separating the "new" principal amount from the interest. In Sfard's terms this reflects a structural view where the expression $A(1+6\%)$ is no longer treated as the result of factorisation but as a single entity that can be operated with and on. He continued this approach for the following year, treating $A(1+6\%)(1+6\%)$ as a new object to be operated on again. Thus the compounding process for each year produced a new object which included an additional factor of $(1+6\%)$. This is indicated by the horizontal curly brackets in his boardwork. The "2 year" annotation indicates the expression that represents the future value for the second year.

A key insight for explaining, based on this incident, relates to the demands placed on the audience to make sense of Zwaii's explanation. This is illustrated in the transcript below by Sizwe's challenge to Zwaii [39] which refers to line ②. Sizwe asked why Zwaii had multiplied $A(1+6\%)$ by $(1+6\%)$, and not by $\boldsymbol{A}(1+6\%)$. It is not clear whether Sizwe was asking this for himself or whether he was provoking as a learner might do. Nevertheless, Zwaii's response [49] suggests he was assuming Sizwe was asking for himself.

Line	Speaker	Utterance
39	Sizwe:	My question is, why do we have to multiply by 1+i instead of multiplying by *A* into 1+i (i.e. (A(1+i))
40	A student:	Repeat your question please.
41	Sizwe:	Okay, first it was *A* into the bracket of 1+i and for the second year, you multiplied by 1+i. Why do you have to do that instead of multiplying by the whole thing, *A* into (*trails off*)
42	Craig:	I think that's a question that as teachers you are going to be asked by grade 12s or grade 10s or grade 11s if you're trying to sort this formula out. I think the 1+6% and the 1+i is right, but I think this question is an important one and I think some of you are skirting the issue. Why is it that you multiply by 1+6% or 1+i? Because I think there is a whole lot of detail that is hidden in those little brackets …
…		
47	Zwaii:	Okay, Sizwe …
48	Craig:	Not just Sizwe, South Africa basically ... just that 1+6%.
49	Zwaii:	Okay, South Africa we cannot, we cannot because this is the amount that have to increase, that has to increase at 6% so this whole thing is the amount, there is this interest plus that amount so we cannot say again 'amount', we cannot say this multiplied again by that. This is the amount in the first year, in the first year so that amount in the first year because look at what happened, this amount was multiplied by this to get the sum, this amount was multiplied by this to get the sum. So now we want to get that, what will this whole amount will be equal to in the next coming year? So we take the whole the thing multiplied by this like we did when multiplying this first amount.
50	Craig:	So are you saying the old amount, plus it's interest becomes the new amount …
51	Zwaii:	Yes.

Line	Speaker	Utterance
52	Craig:	... that you are multiplying?
53	Zwaii:	Yes.
54	Craig:	And that is why there is only ever one *A*?
55	Zwaii:	Yes.

In his response [49], Zwaii identified A as the starting amount which had been multiplied by $(1 + 6\%)$ to get the final amount for the first year. He then treated the expression $A(1 + 6\%)$ as the object representing the new amount for the second year and said it needed to grow by 6%. For him this explained why we multiply only by $(1 + 6\%)$ and not also by A. It is important to note that Zwaii's initial explanation of his first line made use of an additive approach [4-5 in first transcript]. In responding to Sizwe, he used a multiplicative approach: "this amount was multiplied by this to get the sum" which was not the way he had initially described the process. This suggests he may not have been aware of the shifts he was making between operational and structural thinking.

It should be noted that Sizwe's question may reflect a typical confusion between additive and multiplicative approaches. For example, using an additive approach, the future value at the end of the second period could be calculated as follows: $FV_2 = A(1 + 6\%) + A(1 + 6\%) \times 6\%$,
whereas Zwaii had written: $A(1 + 6\%)(1 + 6\%)$
It may be that Sizwe (or the learner he was pretending to be) was expecting $A(1 + 6\%)$ to appear twice in the expression since it appears twice in the additive formulation. While the two expressions are equivalent, they require different thinking which reflects the essential distinction between operational and structural views of the compounding process.

Zwaii's explanation thus required his audience to shift from an operational to a structural view. Sizwe's question reflects the difficulty of doing this. Zwaii did not make his own shift explicit, nor emphasise the different views of the algebraic representation (perhaps he was not aware of different views). If a student is unable to make the shift from an operational to a structural view, s/he may not be able to make sense of Zwaii's explanation. This raises the importance of audience in relation to explanations. The teacher needs to calibrate explanations in such a way that they are accessible to the audience. A typical Grade 10 audience may not be able to follow an approach that requires a structural view.

10.5 Conclusion

My goal for the revisiting task was that students should work on the derivation of the compound interest formula by requiring them to explain the origins of an aspect of the formula that is taken for granted. In that respect my goal was not achieved. The variety of student responses and strategies came as a surprise to me. Many students did not provide evidence of being able to derive the formula although this did not mean they could not do so – it may simply have reflected their interpretation of the requirements of the task. Despite this, the students' work provided insight into the ways they attempted to make sense of the formula.

Vingin and Naasiha drew on the properties of the numbers and their feel for the magnitudes of the quantities in order to unpack the formula. This provides an example of powerful reasoning based on knowledge of the typical magnitudes that occur in compound interest calculations. However, their reasoning contained two errors. Zwaii's explanation provides evidence of the relationship between explanations and learners' conceptions. His explanation focused on the multiplicative relationship

which has been shown to be difficult to grasp (Hoyles, et al., 2010), and required a structural conception which may not be appropriate at Grade 10 or 11 level. While it is both powerful and efficient to represent compound growth multiplicatively using a growth factor, it is not as easy for learners to comprehend as an additive approach. Teachers need to recognise that these are not simply two alternative ways of approaching compound interest. The conceptual demands on learners are substantially different and thus choices need to be made relative to the audience.

These examples of explanations point to the need for attention to the production *and* explanation of derivations of key formulae in school mathematics, *and* to doing so in ways that are accessible to school learners. This may require pre-service teachers to begin by unpacking the formula for themselves and each other. More generally, the examples point to the importance of paying explicit attention to the production of explanations, both for peers and for learners in school. Furthermore, Zwaii's explanation reflects the importance of knowledge of operational and structural thinking as elements of MfT. This was not included in the MfT framework proposed in chapter 2. I shall return to this issue in part 2 and in the concluding chapter.

Aspects of MfT in focus in this chapter

Essential features

- Focus of task is the relationship between 1 and the exponent n, in $FV = PV(1+i)^n$

Modelling and applications

- Use of salary context to explain components of formula
- Use of typical numerical quantities to explain meaning of components of formula

Mathematical practices

- Use of "feeling" for magnitudes of numerical quantities based on experience

Explanations

- Students' explanations may reveal errors in their thinking that are not revealed by their written work
- Different explanations make different cognitive demands on learners (e.g. structural conceptions of concept)

Learners' conceptions

- Struggles of peers may give insight into obstacles that learners will face

CHAPTER 11

Concluding Part 1

Learning MfT through revisiting school mathematics

11.1 Introduction

In chapter 3, I noted that calls for revisiting of secondary school mathematics are frequently done from the position of reform mathematics, and the assumption is that students have had traditional, procedural experiences of school mathematics and that revisiting, done from a reform, conceptual perspective will address the inadequacy of school and university mathematics in preparing future secondary mathematics teachers to teach in reform ways. Such calls tend to assume that students possess the necessary procedural fluency (Kilpatrick, Swafford, & Findell, 2001). However, I also noted that in South Africa many pre-service mathematics teachers have had poor experiences of school mathematics to the extent that they may not even have been taught some of the content that is assessed in the final year of schooling. Thus, *for these students revisiting may actually be a first encounter with some sections of secondary school mathematics*. The students in this study had all encountered simple and compound interest previously – either at school or in a mathematical literacy course in the first year of their degree. Not surprisingly, however, there was evidence that many students lacked depth in their knowledge of compound interest.

As noted in chapter 3, the notion of *revisiting* – something more than metaphor and jargon – only emerged for me through the analysis. Yet, as I now complete the write up of the thesis, it is difficult to think about the first part of the course without thinking in terms of revisiting school mathematics. While I had always intended to deal with mathematical content covered at school, such as percentage change, and simple and compound interest, revisiting as I have come to define it was not a design feature of the study. Consequently, I cannot focus on whether "it worked", nor bemoan the opportunities I may now consider as "missed", from the perspective of revisiting I now hold. My concern is with the opportunities that emerged, and potential opportunities that might be levered up through revisiting school mathematics in teacher education, and thus to propose what might be dimensions of revisiting to be taken up in further research and practice.

In this chapter I summarise the opportunities and potential opportunities for learning MfT that were discussed in chapters 7 to 10. I do so through the lens of the MfT framework. Thereafter I reflect on the framework itself, with particular attention to the aspects of *modelling and applications* and *explanations*. This is followed by a reflection on the structure of the framework and the relationships between clusters of aspects. Finally, I reflect on the "revisiting section" of the course and identify four issues that emerge for revisiting in pre-service teacher education.

11.2 Opportunities and potential opportunities for learning MfT through revisiting

In the four cases of revisiting I identified several opportunities for learning MfT that emerged from revisiting compound growth. I also discussed potential opportunities that might have been created in relation to what transpired in the session. In most cases the insights into potential opportunities emerged through the analysis of the data. Both sets of opportunities are summarised below.

	Case 1 Wage doubling problem	Case 2 Links between the doubling & compound interest formulae	Case 3 Computer operator's salary	Case 4 Where does the 1 come from?
What mathematics is being revisited?	Exponential growth	Exponential growth expressed in different forms	Compound growth	Compound interest formula
How is revisiting different from learning for the first time/at school level?	Applying known mathematics to model exponential growth, generalising exponential relationship	Student-initiated task, unlikely to be considered at school level	Working with learners' responses to task	Unfamiliar question about familiar formula
Essential features of compound interest/ growth	Impact of a on $f(x) = a.b^{x+c} + d$ (Not directly compound interest)	Present value refers to *beginning* of first period	Precision in working with time	Deriving formula, relationship between 1 and exponent
Relationship to other mathematics	Sequence and series (T_n, S_n) notation	Percentage change, exponential growth	Sequence (T_n) notation, percentage change	Binomial expansion, percentage change
Modelling & applications	Defining variables in context, applying known maths to contextual problem	Two different models for same problem	Modelling time, modelling of salary-increase scenario is similar to and different from modelling of compound interest scenario, index numbers	Use of context to explain meaning of components of formula
Mathematical practices	Defining, conventions of notation, precision in mathematical talk	Sharing personal work-in-progress, requesting assistance of class community to resolve a problem, use of notation	Power of formulae for efficient computation, shared interpretation of mathematical problem	Intuitive feel for quantities based on experience, inductive reasoning to derive formula
Basic repertoire				
Different teaching approaches and sequences				
Explanations	Use of personal notation vs accepted mathematical notation	Insufficient attention to written forms in peer explanation	Peer explanations didn't make timeframes explicit	Cognitive demands of explanation - what constitutes a level-appropriate explanation (for peers & for school level), explaining reveals students' mathematical errors
Learners' conceptions *School learners*			Unanticipated, inefficient, partially correct responses, errors in foundational work such as percentage and use of notation	Awareness of cognitive demand of explanation on learner/peer
Peers	Listening for substance of peer's conception vs distracted by inappropriate word	Distracted by sloppy notation, making sense of partially complete work, identifying error	Confusion between discrete points in time vs intervals	
Contextual knowledge of finance			Knowledge of salary-increase context	

Table 11.1 Summary of chapters 7, 8, 9, and 10 in relation to MfT framework

Table 11.1 provides an interpretation of cases, read through my MfT framework. The events themselves were inevitably a co-construction involving the students and myself, as both teacher and researcher. There were multiple factors involved in what came to constitute the course and the cases selected. These include the revisiting tasks and the goals of those tasks, mediation, the resources brought to bear, and students' resulting activity, particularly the extended periods of time given in class for working on the tasks. Other factors include what students offered and how I chose to respond to these contributions. In some cases, such as Palesa, Sizwe and Sakhile, I chose to pursue the ideas. There are many instances, not reported here, of contributions that I chose not to pursue. Had I made different decisions regarding these contributions, the consequences for the course and for the findings of the study may have been different.

The most notable absences in table 11.1 relate to *different teaching sequences and approaches* and *basic repertoire*. Neither of these aspects was intended as a focus in this part of the course. It could be argued that, by dealing early in the course with exponential growth and percentage, I was modelling a different sequence leading to compound growth. However, this was not made explicit to the students. Another aspect that is relatively absent in the table is *contextual knowledge of finance*. In case 3, contextual knowledge related specifically to the salary-increase context. While attention could have been given to various socio-economic issues such as whether the salary was appropriate for the job, whether 6% is a fair increase in relation to inflation, and the extent to which a constant salary increase over three a three-year period is realistic, this was not the focus of the task and so these issues were not problematized.

11.2.1 Essential features

While it may be tempting to assume students know the essential features of compound growth well since it is school mathematics, this was not the case. For example, some students struggled to produce a coherent derivation of the compound interest formula without help, and very few students made use of timelines.

Opportunities for dealing with essential features arose in cases 2, 3 and 4. In dealing with the "exponent problem" in case 2, the role of the present value in modelling the doubling process comes into focus. In case 3, the nature of the learners' responses provided opportunity to reconsider one's own knowledge of how the simple and compound interest formulae arise, and how they model salary increase. For example, an inefficient and partially correct learner response may require one to engage with the learners' thinking, and prompt one to reconsider what has been compressed in the formula. This unpacking for self may simply constitute a reminder to self of "how the formula works" but it may also provide opportunity to gain deeper insight into those elements that are generally taken for granted. For example, I described earlier how, through unpacking the formula for myself, I gained a new awareness of the (embarrassingly) obvious fact that present value is the amount at the beginning of the first period.

The findings from the four cases suggest that *revisiting of compound growth/interest is necessary to deal with the aspects that students might be assumed to know,* but also holds the *potential for students to gain further insight into mathematics they already know*. This potentially leads to what Ma (1999) referred to as "thoroughness" of knowledge.

11.2.2 Relationship to other mathematics

All four cases provide opportunity to explore the relationship of compound interest to other mathematics – at both lower and higher levels. In cases 2, 3 and 4, links were present to the underpinning concept of *percentage increase*, with particular attention to the way in which the formula is induced from multiple percentage calculations.

In case 4 there is potential opportunity to consider links to the *binomial expansion* of $(1+i)^n$. This had not occurred to me prior to analysing the data from Vingin and Naasiha, where they implicitly worked with the idea that $(1+i)^n = 1^n + i^n = 1 + i^n$, which is a typical error seen in learners' algebraic work from Grade 9 onwards. The binomial expansion provides an alternate algebraic expression for compound interest, for example $FV = PV(1+i)^2 = PV(1+2i+i^2)$. This provides opportunity for teachers to create new connections with typical algebraic forms, and thus to broaden and deepen their knowledge of compound interest and its connections to other mathematics. This could be pursued further with regard to modelling, where they might consider what $2PVi$ and PVi^2 represent in the compounding process. The students explored similar algebraic manipulation in their attempts to derive an annuities formula. This is dealt with in part 2 of the thesis.

In cases 1 and 3, the students introduced *notation from sequences and series*. This illustrates a typical way in which revisiting may be different from learning the mathematical content for the first time – students bring to bear new resources, learned in higher grades, which therefore would not have been known when they first learned exponential growth or compound interest.

11.2.3 Modelling and applications

Opportunities for learning about modelling and applications arose in all four cases, although in different ways. In case 1, students *applied their knowledge of known mathematics* to solve a hypothetical contextual problem. In case 2, Sakhile *began with the situation and attempted to model it* with two different exponential models. By contrast, in case 4, Zwaii began with the model (formula) and *used the context of salary increase to explain the model*. My own investigations for case 3 revealed the important connection between knowledge of modelling and knowledge of the salary-increase context. In chapter 9 I discussed in detail the way in which the salary increase is modelled by the compound growth formula, and the assumptions that are embedded in the use of the formula in this context.

11.2.4 Mathematical practices

While mathematical practices are clearly present in all work, my focus here is on instances where aspects of mathematical practice became the object of attention. In case 1, explicit and extended attention was given to the *defining of variables* and to the *conventions of mathematical notation*, specifically the meaning and use of subscripts. Across the four cases there were several opportunities to focus on *precision in mathematical communication* – in both spoken and written forms. The most obvious of these occurred in case 3 in students' talk about timeframes. Sakhile's "sloppy" boardwork in case 2 prompted concerns from students but Zwaii's boardwork in case 4 did not, despite his lack of attention to detail. There were also other opportunities in case 3 relating to learners' inappropriate use of the equals sign and the percentage symbol. In case 4, Vingin and Naasiha exemplified *intuitive ways of working with number* specifically in relation to magnitudes of quantities in compound interest calculations. Although their intuitive ideas were not fully correct, the ideas provided a useful starting point from which they could reason. Given the importance ascribed to intuition by mathematicians (e.g. Burton, 1999, 2004), there is opportunity in learning compound interest and annuities to exploit

this “feel for quantities” and for teachers to pay more attention to the magnitudes of numbers that are produced at various stages of the calculation process.

In case 3 the *power of formulae for efficient computation* was highlighted in contrast to learners’ inefficient multi-step strategies where they “just used their thinking” (as Palesa commented).
Case 2 provided an opportunity for students to experience some of the messiness of mathematical work-in-progress as Sakhile shared his findings and his problem in relating the exponents of two different formulae. There was potential here for students to take up the challenge of *further mathematical inquiry*, had I dealt with the situation differently. Vingin and Naasiha’s work also provided potential further inquiry as noted in chapter 10.

11.2.5 Explanations

In chapter 3 I noted that learning to produce explanations was not given attention during the twelve-day period of revisiting school mathematics. Nevertheless, examples of students’ peer explanations were evident in all four cases. In cases 1, 2 and 3, I referred to isolated instances where I addressed the *lack of precision in students’ talk or written representations* but this was not done within a larger framework of learning about explanations and learning to explain. In case 4, two important issues arose regarding explanations. Firstly, Zwaii’s explanation that required a structural view of the compound interest formula highlights the importance of *level-appropriate explanations.* Some of the pre-service teachers struggled initially to make sense of his explanation, and Grade 10 and 11 learners may experience even more difficulty and may not be able to adopt a structural view at all. Secondly, Vingin and Naasiha’s explanations revealed two errors in their thinking. They explicitly stated that in a growth factors such as $(1 + 0.12)^5 = 1.76234$ ..., the 1 represents the principal amount and the decimal portion represents the interest gained. They also appeared to be reasoning implicitly that $(1 + y)^n = 1 + y^n$. I discussed these errors in detail in chapter 10 and argued that both *errors were only exposed through their talk and not in their written work*. In part 2, I shall discuss further instances where students’ errors are revealed through their talk but not through their written work.

11.2.6 Learners’ conceptions

There was no explicit focus on learning about learners’ conceptions of compound interest in weeks 2 to 4, although authentic learner work was included in case 3, and hypothetical learner work was included in part of the task for case 4 (although this was not discussed in chapter 10). Consequently students dealt with learners’ productions but no specific resources were provided through which to interpret learners’ conceptions.

In case 4, the explanation provided by Zwaii required a structural conception. As noted above, students need to be aware of the demands placed on learners by the explanations they give and how these will impact the extent to which learners can make sense of their explanations. This implies that *students need resources to think and talk about learners’ conceptions*, which suggests the importance of introducing students to theoretical constructs such as Sfard’s (1991) operational/structural distinction.

I would also argue that in revisiting school maths though a participative pedagogy, and with many pre-service teachers who have poor backgrounds in school maths, there may be several *similarities between the struggles of the pre-service teachers and those of learners* in schools. However, there will also be obstacles which learners face that the pre-service teachers may not anticipate. This can be seen

from Palesa's comments on the learners' responses in case 3. In chapter 9, I argued that *the nature of learners' responses may have the potential to shift pre-service teachers' attention from the learners' conceptions to the essential features* of the concept, and then to return to the learners' work to engage more deeply with it.

11.3 Reflecting on the MfT framework

In reflecting on the MfT framework, I begin with issues relating to two individual aspects of the framework: *modelling and applications* and *explanations*. Thereafter I consider possible clusters of relationships between the aspects.

11.3.1 Individual aspects

Modelling and applications

In chapter 2 I noted that *modelling and applications* may be considered an element of *mathematical practices* (e.g. Watson, 2008). I chose to separate the two aspects because of the prevalence of modelling in financial maths. Based on the four cases, I am further convinced of the value of the separation. Various mathematical practices, apart from modelling, were in focus in all four cases. This may not have been as easy to identify if the two aspects had been combined.

Explanations

The aspect of *explanations* has proved to be a slippery one in the analysis. If it is restricted to teachers' *instructional explanations* to learners, then there would be little to say about it based on the data of the four cases. However, there were explanations in all four cases, and these highlighted various important aspects for pre-service teachers' mathematical knowledge, including knowledge of explanations. For example, in chapter 10 I discussed the close relationship between explanations and *learners' conceptions* in terms of the cognitive demands that an instructional explanation places on a learner. In case 3 Belinda drew on *knowledge of context* in an attempt to explain her interpretation of the timeframes. In case 4 Vingin and Naasiha's explanation drew on typical magnitudes of numbers in the growth factor, thus making a link with *modelling and applications*. More generally, mathematical communication is a key element of *mathematical practices*, and so explanations feature strongly in this aspect too. It can therefore be seen that the aspect of *explanation* cuts across several other aspects of the framework, which suggests it may be *a different kind of aspect* of teachers' mathematical knowledge for teaching.

Several frameworks and studies treat explanations as a *task of teaching* (e.g. Ball, et al., 2004; Ball, et al., 2008; Ferrini-Mundy, et al., 2006; Kazima & Adler, 2006). I included explanations as an aspect of teachers' knowledge following Huillet (2007). I am now questioning that decision. While I agree that *giving* explanations is a task of teaching, what is the knowledge base on which teachers draw in order to produce "good" explanations? (i.e. explanations that are mathematically sound, use appropriate representations, and are pitched at an appropriate level for the audience). Based on the four cases, it appears that teachers draw on many aspects of the MfT framework in producing explanations.

Despite this concern, I shall not revise the framework at this stage. I shall track the concern through part 2 and reflect further on it at the end of the thesis.

11.3.2 Structure of the framework

Table 11.2 summarises the aspects of MfT that were in focus in each case, as listed at the end of the previous chapters. While this table reflects some similar trends to table 11.1, my focus here is on exploring possible connections between aspects of the framework.

	Aspect of MfT	Case 1	Case 2	Case 3	Case 4
Mainly mathematical	Essential features of compound interest/ growth		x	x	x
	Relationship to other mathematics	x	x		
	Modelling & applications	x	x	x	x
	Mathematical practices	x	x	x	x
Mainly pedagogical	Basic repertoire				
	Different teaching sequences and approaches				
	Explanations				x
	Learners' conceptions - school learners			x	
	- peers	x	x	x	x
Contextual	Contextual knowledge of finance			x	

Table 11.2 Aspects of MfT framework in focus in each case

From table 11.2 we can make the obvious point that not all aspects of MfT were in focus in all four cases. It is not possible to attend to all aspects of MfT all the time. However, it can be seen that *mathematical practices* and *modelling and applications* were in focus in all four cases. The table also shows that *learners' conceptions* in the form of the pre-service teachers' own conceptions were in focus in all four cases. Taken together, this reflects learning as both acquisition and participation. As noted above *explanations* were present in all four cases but were only in focus *in the analysis* of case 4. In the other cases, explanation was a means to an end – it was not the object of attention, and therefore not in focus in the analysis. Another important trend is that in all four cases there are aspects in focus from both the *mainly mathematical* and *mainly pedagogical* categories. It could be argued that this only occurs because I have expanded the aspect of learners' conceptions to include peer conceptions. However, I argue that it is precisely because the students encountered obstacles with various aspects of the work that many of the other aspects came into focus. For example, if Palesa and Sizwe had not raised their concerns about notation or timeframes, the attention to mathematical practices and precision in references to time (essential features) would not have occurred. This serves to emphasise the point that what came into focus was a co-construction by the students and me, enabled by a participative pedagogy. Furthermore, the opportunity to engage with peers' struggles is an important component of teachers' mathematical learning as it requires them to become aware of "the other" in relation to mathematics and to learning.

11.4 Reflections on revisiting school mathematics in pre-service maths teacher education

In this section I reflect on revisiting of school maths in the Financial Maths course, and consider four issues that emerge for revisiting within pre-service teacher education more generally. As noted at the beginning of this chapter, any discussions of revisiting can only be made in relation to the teaching and learning context in which the revisiting took place. The issues are: teachable moments, challenges of pace and coverage, classroom culture, and distinguishing between revisiting school maths and preparing to teach the tasks in school.

11.4.1 Teachable moments

There are times when, as a teacher, one becomes aware of an unusual silence and senses a deeper level of engagement of the class, when students seem to be leaning forward to listen, ignoring the usual distractions, focused on what the teacher has to say. These are instances when the teacher has something specific to teach, and students are attentive and open to hear it. Several of these *teachable moments* (Havighurst, 1972) arose in the sessions on compound growth. In most cases, they were opportunities for me to clarify an issue or introduce a new idea. The teachable moments tended to emerge (at least from my perspective) when class discussion had reached an impasse on an issue. I discuss one example below.

In case 2, many students were concerned about Sakhile's use of percentage notation without a referent. He had written $200\% = x + 100\%x$ and, not surprisingly, students were concerned that 200% had no referent. However, after several minutes of interaction, it became clear to me that he was using 200% as a *label* rather than a quantity. For example, he could have replaced "200%" with "double wage". Consequently, he saw nothing wrong in what he had written. By contrast, the rest of the class saw it as an equation with a missing referent on the left hand side (see his interaction with Hailey in chapter 8 [182-192]). I took the opportunity to contrast his meaning with the class's interpretation, and then proceeded to talk about the need for teachers to pay careful attention to what they write on the board.

After the above incident I had a sense that the input I had given was well-timed, and that it resolved an immediate issue for the students. There had been several previous discussions about the importance of paying attention to teachers' boardwork, but here there appeared to be a heightened receptiveness to what I was saying. This may be a key element of revisiting – *actively seeking out opportunities to emphasise key issues in mathematics and teaching at points where students appear to be particularly receptive*. Of course, the idea of planning a teachable moment seems like a paradox. So it may not be about deliberately planning a particular teachable moment, but rather actively seeking to capitalise on opportunities as they arise. I am also well aware that the above discussion comes from my perspective alone, and that the students' reading of the situation may have been different from mine. I base my claims of a teachable moment on the unusual silence in the room, and what I perceived to be increased attention of the students to what I was saying and writing on the board.

11.4.2 Pace and coverage

While undergraduate secondary mathematics teacher education programmes should require students to revise the secondary school mathematics curriculum over the duration of the programme, much of this revision can be done as independent study. The goal of such revision would be to (re)-establish familiarity with school mathematics and to identify areas that may need additional attention. This is distinct from revisiting as the notion is used in this study.

Revisiting is characterised by in-depth and deliberate engagement with carefully-chosen mathematical content to provide future teachers with opportunities to grapple with the mathematical content they will teach. However, such depth of engagement should lever up knowledge beyond the content area of study and enable students to develop a range of the aspects of MfT, particularly knowledge of the essential features and connections with other mathematics. In addition, through the process of revisiting, students have opportunity to engage further with the disciplinary practices of mathematics in ways initially envisaged by Shulman (1986). This provides opportunities for mathematical inquiry

and for dealing in depth with issues particularly pertinent to school mathematics such as defining, use of notation, reasoning and proving.

Despite this, a concern in all four cases of revisiting is the *small amount of content covered and the slow pace*. No doubt there will always be a tension between depth of engagement, and pace and coverage. In the first four weeks of the course, my decisions about pace and coverage were substantially and deliberately influenced by the students' responses to tasks and their contributions to discussions. For example, as the Wage Doubling session unfolded, I made conscious decisions to give more time and attention to some of the issues students were raising. This was partly informed by my concerns as the researcher. For example, I wanted to explore the extent to which Palesa's concern was shared by other students. My decisions were also informed by my concerns as the lecturer that students should have opportunity to make sense of the mathematical content through interacting with each other's ideas. However, for some students the pace was too slow, as reflected by Attiyah in her final interview:

> Attiyah: The first two topics of the salary and the … wages … the fact that a lot of the lectures we stuck on one problem the whole time. That was frustrating for me. Sometimes I just felt that 'we got it, let's just move on' or do something else. That was frustrating, some of, like the first few topics I felt we spent too much time on them.

The consequence of the time spent on some aspects was that less time was spent on others, and we did not deal with some aspects at all. For example, the general structure of exponential functions was dealt with very quickly, and the similarities and differences between the Wage Doubling Problem and the percentage change problem from the previous day were not considered at all in class.

At a more general level, pacing and coverage is a concern for all teachers – whether in schools or in universities. In the course, the tension tended to emerge when students raised issues I had not anticipated. The dilemma which the lecturer faces is whether to pursue the students' concerns or to continue with the plans for the session. Such dilemmas cannot be resolved but must be managed (Adler, 1999).

11.4.3 Classroom culture

The pedagogy throughout the course, but particularly in the first nine weeks, represents what some might consider a typical reform-oriented mathematics environment at university level (Speer & Wagner, 2009). Students were familiar with this practice from several of their other undergraduate mathematics courses. This pedagogy gives rise to a classroom culture that is different from that of many undergraduate university mathematics classes (Alsina, 2001; Burton, 2004; Watson, 2008), and I argue that *it was this culture that produced and sustained the kinds of interactions described in the four cases* and hence impacts the opportunities for learning MfT. For example, on separate occasions both Palesa and Sakhile forged a space in the class proceedings to raise issues that ultimately provoked extended discussion. Palesa believed the space was sufficiently safe to challenge what was being taken for granted. Similarly, Sakhile felt safe to admit publicly that he could not solve a problem he was working on, and requested assistance from the class. The issues they both raised suggest a disposition to make sense of the mathematics under consideration, and to know it more deeply than just being satisfied with an answer.

I am well aware that Palesa and Sakhile's actions could be interpreted as merely behaving in the ways expected within my class. However, I argue that they were not simply "playing the part" but that they had recognised the privileged forms of participation across the undergraduate maths programme and had taken up these practices. The quote below from Palesa came in response to a discussion about teaching and learning formulae, in the mid-course interview. The quote from Sakhile came from his mid-course interview where I asked him to compare the Financial Maths course with the other maths courses in the undergraduate programme.

Palesa: You know at school it was not important to understand a formula. It, it's just a formula there, but after we came here at Wits and we start questioning where does the formula come from, why are we doing this? And now it kind of, some sort of light shining now because you start asking yourself 'where does this formula come from?' And the minute you understand where does it come from, why are we using it in this situation, it makes it more … for you to understand it more, I think. Because now just having a formula without understanding why are you using it, where did it come from and why is this happening to this, I mean, it's really important I think.

Sakhile: I mean like the way the discussions in class, and maybe like we are given, still given a chance to present - those things are the same and then still like, that open environment for everyone is still there, since from, since from first year. Okay, but okay, like from second year to this year it's, it's the same, like I don't see a different (sic). Only from first year because we were too many in class and we, we didn't like, it was the first time for all of us to get, we were not used to the environment. But I got used to it in second year and still I'm working in that process from second year.

Both students indicate a change from learning maths at school to learning it in the undergraduate teacher education programme. Palesa spoke of the emphasis on meaning and the importance of connecting mathematical ideas. Sakhile focused on the forms of participation. In both cases they imply that *the transition to different emphases in learning mathematics and to different forms of participation is not an easy one to make*.

11.4.4 Revisiting is not about learning to use the task in one's own teaching

The use of school-level tasks in revisiting mathematics should not be treated as simultaneously preparing the students to use the tasks in their own classrooms. No doubt students may gain insights into details of the task beyond the mathematical content, and benefit from the way in which the task is implemented in their course. However, revisiting school mathematics and focusing on task selection and implementation are different responsibilities of teacher education and, for the most part, should be separated. The latter is not in focus in this study.

Consider the Wage Doubling Problem: the original learner task focused on extracting the exponential relationship in the scenario. However, the student teachers did not focus on exponential growth but rather on defining and notation. While both these aspects should be given attention at Grade 9 level, they are not the focus of the original task. In addition, the use of subscripts is not part of the Grade 9 curriculum. Thus, the notation that was proposed as a means for resolving the debate about *day number* versus *number of days* may not be "usable knowledge" in a Grade 9 classroom. So, while student teachers had opportunity to broaden their mathematical knowledge by making new connections, this new knowledge may not necessarily help them to enact the original task with a Grade 9 class.

Similarly, Zwaii's explanation of the compound interest formula required a structural conception that few Grade 10 or 11 learners are likely to have. Thus *a mathematical explanation that is acceptable and meaningful for peers in the context of revisiting school mathematics may not be appropriate for a*

school-level audience. This is not a problem for revisiting school mathematics *per se*. It is only a problem if students assume their explanations are suitable for *both* audiences.

11.5 Conclusion

In part 1 of this thesis I have focused on revisiting school mathematics with particular emphasis on compound interest. I have shown that revisiting is not merely an opportunity for students to top up their knowledge of school mathematics but an opportunity to deepen and broaden their knowledge of school mathematics and to learn several aspects of MfT. For example, I have exemplified opportunities and potential opportunities to engage with the *essential features* of compound interest, to explore *relationships with other mathematics*, and to attend to a variety of *mathematical practices* which students may not be able to attend to when the school content is learned for the first time. I have shown evidence of the need to promote *movement between compressed and decompressed mathematical forms*, and hence the importance of students' becoming aware of operational and structural conceptions – for their own learning and that of their future learners in school.

INTERLUDE

Introducing Part 2 – Learning new mathematics

The second part of the thesis is built on the assumption that the obstacles experienced by pre-service mathematics teachers, i.e. their concerns, difficulties and errors, provide insight into what might constitute mathematics-for-teaching annuities. I work with the assumption that the concerns, difficulties and errors that arose amongst the pre-service teachers are similar to those that might arise with learners in school. I acknowledge that these are all constituted by and constitutive of the Financial Maths course itself, the tasks and the pedagogy, and these elements will differ at school level. Nevertheless, I believe there is sufficient commonality to learn from the pre-service students in ways that inform teachers' knowledge and teaching.

Part 2 consists of five chapters. Chapter 12 provides a *reference landscape* for annuities for teaching, similar to that provided for compound interest in chapter 6. The chapter emerged from initial analysis of the data and was prompted by the lack of literature on conceptions of annuities. In the chapter I propose an initial response to the question "what is mathematics-for-teaching annuities?" Chapters 13, 14 and 15 deal with three separate but related issues concerning the teaching and learning of annuities:

- *Introduction to an individual payment approach* – I set out to explore the extent to which students make use of an *individual payment* (IP) approach when first introduced to annuities. Linked to this is their take-up of an IP approach over the eight-day period.
- *Talk about time and use of timelines* – As I analysed the data to explore my question about take-up of an IP approach, I became increasingly aware of varied levels of attention to time in students' talk and their use of timelines. I therefore pursued this issue further, drawing on a variety of data across the eight-day period.
- *Use of spreadsheets for annuities* – The two focus groups provide powerful examples of how spreadsheets might be used to engage the mathematics of annuities at a deeper level than formulae and calculations. I deal with each focus group's use of the triangular spreadsheet in different group tutorials, and introduce the notion of *spreadsheet thinking*, which I argue holds potential to harness insights gained in working with spreadsheets even in the absence of technology.

The above chapters are not structured in the same way as those in part 1, although at the end of each chapter I identify the aspects of the MfT framework that were in focus as I did in part 1. I conclude part 2 by extracting commonalities across the chapters, drawing them together in relation to the MfT framework, and then reflecting once again on the framework itself.

Group tutorial 3 (GT3) and group tutorial 4 (GT4) marked the beginning and end of the eight-day period respectively. I describe these two tutorial tasks below and give the rationale behind their design. This is important background information since much of the analysis in the next three chapters concerns these two tasks. The handouts for the tutorials can be found in Appendices C4 and C5.

Group tutorial 3 – The task for GT3 was structured around an annuity-based savings product from one of the big South African banks. The first part of the handout included details of the minimum opening

deposit, monthly deposits, and maximum length of the investment. It was explicitly stated that interest was calculated on the daily balance and paid on maturity. Interest rates were tiered, depending on the balance. In addition, bonus interest was offered on maturity to clients who had made all payments on time. Students were first required to determine how much money would accumulate if they made monthly deposits of R250 at the end of each month for one year, at 6% p.a. compounded monthly. They were then asked to produce a formula to generalize their calculations. Two further questions focused on the bonus rates and the quoted tiered rates.

My intention was that the task should focus on developing a formula for the future value of an ordinary annuity. I included the calculations about bonus interest and tiered rates to increase the authenticity of the task in relation to personal investment, and to extend the task for students who might finish questions 1 and 2 quickly. However, I did not make this explicit to the students, with the result that a few groups abandoned question 2 to pursue questions 3 and 4, which were not the intended focus of the task. I deliberately did not use the language of annuities in the handout because I wanted to see whether students would recognise the situation as one involving annuities. I was concerned that the use of annuities language might direct their attention to the formulae, which they might recall from a previous course or possibly "google" on their cell phones.

Group tutorial 4 – The task for GT4 was designed after completing class session B. While I wanted to reinforce the use of the future value of annuities formula that had been derived in the class session, I also wanted students to work with the process that had led to it, focusing on individual payments. I therefore decided to combine an annuity due scenario with a missed payment. The annuity due scenario meant that students would have to adapt the ordinary annuity formula to deal with payments at the beginning of the month. The missed payment component would provide opportunity to investigate students' ability to deal with individual payments and to conceptualise the problem using an IP approach. As with GT3, the task was based on an authentic bank product which consisted of an initial lump sum payment followed by a series of equal monthly payments. Students were asked to determine the total amount accumulated over a particular period. They were then asked to determine the effect on the accumulated balance of missing a particular payment. Finally, they were required to determine whether doubling the payment immediately following the missed payment would make up the deficit. They were not explicitly asked to produce formulae for the situation.

The task wording was deliberately ambiguous, for example I used phrases such as "you do this *for* 18 months" and "how much will you accumulate *over* the period". I expected students to notice the lack of clarity, and that this would prompt them to pay attention to the time aspects of the task. I then expected them to choose an interpretation and to develop their model according to the interpretation.

As in part 1, I make extensive use of transcripts in part 2. The conventions used in the transcripts are explained in the introductory interlude to part 1.

CHAPTER 12
A reference landscape for annuities for teaching

12.1 Introduction

In the first chapter of the thesis I described how each time I taught the Financial Maths course, I became increasingly aware of the students, their interpretations of the mathematical ideas, and their struggles. The increased awareness of the students and their learning forced me to think more deeply about the concept of annuities, and how I might help them to engage more meaningfully with the concept, going beyond definitions and calculations. However, this still did not provide the depth of insight nor the kinds of tools I needed to analyse the data. In my initial analysis, I struggled to explain and account for some of the difficulties that students appeared to experience. As I grappled with the data, I became increasingly aware of the need to elaborate further the notion of annuities. I did not want to simply provide a set of definitive statements about the nature of annuities. I wanted and needed to focus on conceptions of annuities in a similar way to the work by Sfard and Linchevski (1994) on conceptions of algebra, and by Thompson (1994) on rates of change. This chapter is the product of my initial theorising about conceptions of annuities. Elements of the chapter are extended in later chapters as I grappled again with new issues that arose through further analysis of the data. I could have included the later developments in this chapter but I chose instead to locate them alongside the data that gave rise to them. I do not seek to produce an epistemological genesis of the annuity concept from a mathematical perspective. My focus is largely on its use in school mathematics, and in the financial and everyday world.

The literature does not provide answers to questions like: "what does it take to learn annuities?" or "what difficulties do students experience in making sense of annuities?" In the light of this gap, I attempt in this chapter to build a conception of annuities that focuses on aspects that need particular attention for learning, and then to extrapolate from this to knowledge for teaching annuities. I am not attempting to produce an exhaustive list of all that it takes to make sense of annuities. The aspects discussed below are drawn partly from experience of teaching financial maths, but mainly from initial analysis of the data. As I grappled with the errors that students made, and tried to account for these errors and their struggles, I began to think more carefully about what might be the key elements of learning annuities. In the absence of prior research, I use the context of the course and previous experiences of teaching financial maths to produce my own construction of learning of annuities. In the chapters that follow, I extend these ideas to include issues that arose after the initial analysis of the data.

I begin this chapter with a discussion of annuities in the South African school mathematics curriculum, followed by a summary of the small amount of research on conceptions of annuities. Thereafter I propose a hierarchical network of annuities-related concepts. I then discuss briefly the need for an expanded view of the compound interest formula, followed by a detailed discussion of timelines for representing annuities situations. This is followed by the introduction of two key constructs which I

use to distinguish two approaches to annuities: the *account balance approach* and the *individual payment approach*. I draw on these distinctions to discuss the derivation of the annuities formulae with specific focus on future value of annuity due. I introduce two of the spreadsheets that were used in the course in preparation for chapter 15. I conclude the chapter by relating all these aspects to my framework for MfT, and answering the question "what is mathematics-for-teaching annuities?"

12.2 Annuities in the South African school curriculum

In the South African school curriculum for Mathematics (Department of Education (DoE), 2003) annuities are treated as an application of geometric series and therefore only introduced at Grade 12 level, once the topic of sequences and series has been completed[49]. Learners are expected to "[a]pply knowledge of geometric series to solving annuity, bond repayment and sinking fund problems, with or without the use of the formulae" (DoE, 2003, p. 19). Only two formulae are given – future value of an annuity due and present value of an annuity:

$$FV = pmt\left[\frac{(1+i)^n - 1}{i}\right] \qquad PV = pmt\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Learners are also required to "[c]ritically analyse investment and loan options and make informed decisions as to the best option(s) (including pyramid and micro-lenders' schemes)" and to "solve non-routine, unseen problems" (p. 21) involving financial mathematics. This appears to include problems involving outstanding balance although neither the term *outstanding balance* nor a suitable formula appears in the curriculum document.

Although learners are not required to make use of the annuities formulae, the revisions to the curriculum recommend that teachers should show learners the derivations of the formulae using the geometric series formula (Department of Basic Education (DBE), 2011). This presumes that the formulae will be used. The document also provides an example requiring the calculation of outstanding balance (p. 42) but still no formula for outstanding balance is provided.

Two text book questions involving annuities are given below.

Question 1 Thabo starts his own band and buys sound equipment for R30 000. He pays R5 000 in cash and takes out a bank loan for the balance. Calculate the monthly repayments if he repays the loan over three years. The interest rate is 8.2% p.a. compounded monthly.

(Laridon, et al., 2007, p. 86)

Question 2 On his 18th birthday Steve decides to bank R20, and then R20 every week thereafter. If the account he chooses pays 5.75% per annum compounded weekly,

a) How much will he have on his 30th birthday?
b) How much will he have on his 40th birthday?

(van der Lith, 2007, p. 50)

Question 1 involves a typical present value scenario, although, unlike many loan repayment problems in text books, it does not state explicitly when the repayments begin. However, learners are likely to assume that payments begin one month after the loan is taken since this is how the present value formula models such situations. Question 2 involves a future value scenario. The most noticeable

[49] I acknowledge that in the Advanced Programme Mathematics (AP Maths) curriculum, annuities are introduced at Grade 11 level. I have not included the AP Maths curriculum in my discussion.

aspect is the unusual compounding period – weekly. This is necessary because payments are made weekly and the curriculum deals only with simple annuity scenarios (where payment period matches the compounding period). A closer look at the problem reveals the assumptions that payments are made on the same day each week but nothing is said about the time gap between the payment and compounding. This impacts whether the problem is modelled as an ordinary annuity situation or an annuity due situation. Similarly, the question is not explicit about whether the payments on the 30^{th} and 40^{th} birthdays are included. From a modelling perspective, the lack of explicit attention to timeframes may be considered a strength of each question. However, given that text book and assessment questions are generally explicit about timeframes, learners may not pay attention to the assumptions they are making, and simply substitute the given values into a formula.

The introduction of annuities at Grade 12 level is too late, for at least two reasons: an uneven progression across the curriculum, and because it takes time time to grasp some of the underlying concepts. I discuss each of these below.

With regard to curriculum progression, an analysis of the South African financial maths curriculum (Pournara, 2007) and discussions with teachers suggests that one of the problems with the financial maths trajectory is too much repetition and too little progression up to Grade 11 level. By contrast the Grade 12 section is overloaded and insufficient time is allocated to cover the new content. For example, in the work programme provided by the Department of Basic Education (DBE, 2011) only 2 weeks (which amounts to less than 7% of teaching time for mathematics) are allocated for financial maths and yet learners are expected to grasp future and present value of annuities, outstanding balance and sinking funds. They are also expected to critique investment and loan choices, and to work with logs to determine time periods in compound growth and decay scenarios. This constitutes much more content, and more complex content, than in the previous grades.

My second point is that learners need time to make sense of the important underlying ideas of annuities, such as the time-value of money. Research (e.g. Stuebs, 2011) shows that this is not an easy concept to grasp, and yet in the current curriculum learners are expected to do so within two weeks. However, there is a dilemma here that is not easily resolved: if one chooses to approach annuities as an application of geometric series, then one has to delay the introduction until geometric series has been completed. Since geometric series is only covered in Grade 12, annuities cannot be introduced earlier.

In chapter 13, I shall show that approaching annuities as an application of geometric series is not the only approach. I shall also provide evidence that most pre-service secondary teachers in the study did not make use of geometric series when first attempting to derive a formula for future value of an ordinary annuity. In addition, I shall argue that conceiving of annuities situations in terms of geometric sequences requires a shift in thinking that focuses on individual payments rather than the far more intuitive focus on account balance. This suggests there is opportunity to rethink the introduction of annuities in the school curriculum. I shall take up this issue again in the final chapter.

In the current curriculum, from Grade 8 to Grade 11the focus is almost exclusively on single payment scenarios, although some text books (e.g. Laridon, et al., 2006, p. 36) introduce tasks with more than one payment at Grade 11 level. I argue that learners should be introduced to the idea of accumulating

money, or repaying debt, by means of systematic multiple payments by Grade 11 level. This can be done through a series of iterative compound interest calculations for a small number of payments, and thus not require the annuities formulae. It has potential to lay the foundation for annuities and the important idea that multiple payments/withdrawals deposited/withdrawn at different points in time cannot be modelled with the compound interest formula.

At this point it is important to pause and consider the study in relation to curriculum and teachers' institutional contexts. Financial maths is a small section of the curriculum at Grade 12 level, being allocated less than 7% of teaching time, and approximately 6% of the marks in the Grade 12 national assessments. This time allocation places pressure on teachers to cover the section in a short period of time. The low weighting in terms of mark allocation shows that financial maths may be considered a relatively unimportant section of the curriculum in comparison to topics such as functions, calculus, trigonometry and geometry. Consequently, Grade 12 teachers may choose to deal only with the essential aspects, to show learners how to substitute values into the annuities formulae, and to practise a range of questions that will prepare learners appropriately for their final assessment. This may be a wise decision if learners are struggling with other topics in the curriculum that are more critical for further study of mathematics such as algebra, functions and calculus.

While these curriculum issues are important aspects of teachers' mathematical knowledge for teaching, they are not my concern in this study. Simply put, I shall assume there is endless time to help *all* learners understand *all* aspects of financial maths in depth. My concern in this study is that future secondary mathematics teachers have the necessary knowledge to teach financial maths in ways that enable their learners both to master the relevant routines and to make sense of the mathematics and its impact on their personal lives and on the wider society.

12.3 Research on conceptions of annuities

As noted in chapter 6, there is little research on students' conceptions of annuities at school or university level. There is some literature on approaches to teaching annuities at university level (e.g. Dempsey, 2003; Eddy & Swanson, 1996; Gardner, 2004; Jalbert, et al., 2004) but, with the exception of Dempsey (2003), these are largely anecdotal. The work by Hoyles, Noss and their colleagues (e.g. Bakker, et al., 2006; Hoyles, et al., 2010) focuses on the workplace and provides the only research-based evidence on thinking about annuities.

As part of their study on techno-mathematical literacies in the workplace, Hoyles, Noss and their colleagues (Hoyles, et al., 2010) focused on two pensions companies and a mortgage company, all of which involved annuities-based scenarios. Through their job-studies they identified a range of knowledge they perceived as critical for successfully performing various jobs, yet which sales and service employees appeared to lack. The range of knowledge included an appreciation of the mathematical models that underpinned the documents produced by the IT-system, e.g. pension statements; an understanding of the key variables and their relationships in mortgage scenarios; an understanding of the growth of money and the notion of present value of money; the ability to interpret visual representations; and the ability to make estimates and predictions of costs for individual customers. Since employees lacked this knowledge, they were unable to deal with non-standard customer queries. I include one specific example for illustrative purposes.

The specialist mortgage company sold current account mortgages where all debt is combined into a single account and the mortgage debt can be offset by the borrower's savings and income, so that interest is paid only on the difference between the outstanding mortgage and the positive balance in the account. However, employees were unable to explain how and why this provided benefits to customers. For example, if customers referred to the monthly interest rate currently being charged on their credit cards (say 1.9%), the employee was unable to show how this translated into an annual rate of approximately 24% p.a. whereas the mortgage loan rate being offered was approximately 6% p.a. Employees also struggled to interpret graphical representations of outstanding balance that compared a standard mortgage product with their own product. One of the underlying reasons for their inability to do this was that the graphical representation did not make explicit the key inputs in the model (interest rate and monthly payment), and the employees did not know how these variables were interrelated and how changing one or more of the variables would impact the output.

An important difference between this work and mine is that Hoyles et al (2010) sought to avoid exposing employees to the underlying algebraic formulae and structures, while I expected students to produce and manipulate the algebraic forms in order to explore, model and justify relationships. An interesting similarity between my study and the workplace study is the importance of gaining knowledge of annuities for the purpose of helping others to make sense of annuities. It is beyond the scope of this study to explore this issue further.

12.4 A hierarchy of annuities concepts

I propose the following diagrammatic representation (fig. 12.1) of a hierarchical network of links between key concepts relating to annuities. Concepts that are higher up in the diagram build on those that are lower down. In the discussion below I use small-caps font (SMALL CAPS) to identify the nodes in the diagram.

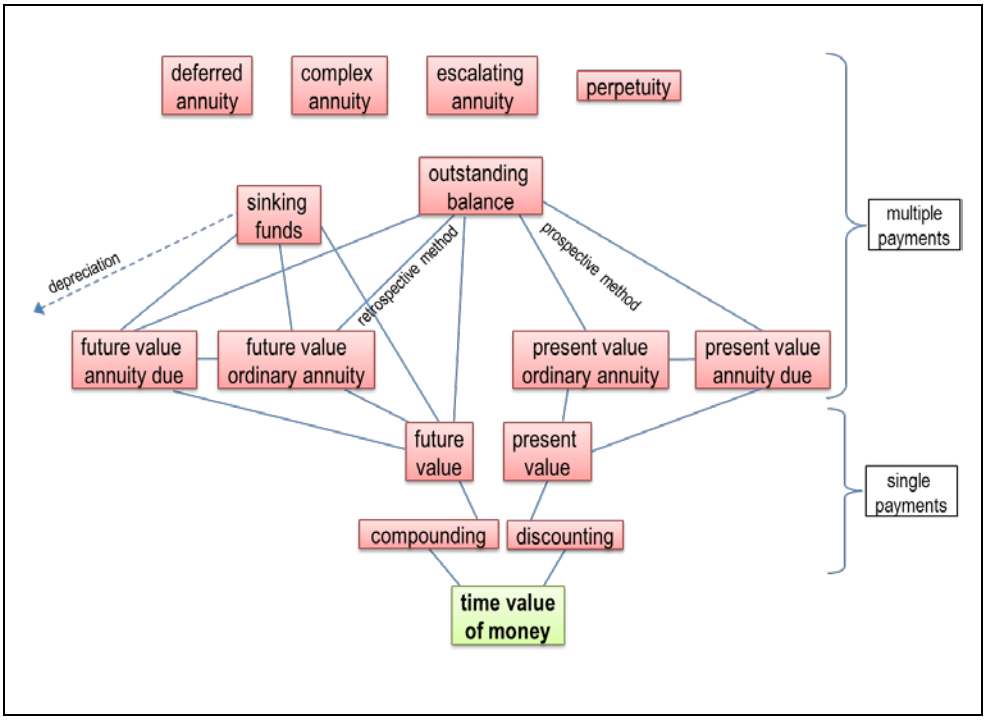


Fig. 12.1 A hierarchy of annuities concepts

The notion of the TIME-VALUE OF MONEY underpins all aspects of annuities. The operation of COMPOUNDING and its reverse, the operation of DISCOUNTING, provide the mechanisms by which an amount of money is moved forwards or backwards in time. This is effected by multiplying the present value, or dividing the future value, by the unit growth factor $(1 \pm i)$. (See chapter 6 for detailed discussion of growth factor and unit growth factor).

In the diagram, the operations of COMPOUNDING and DISCOUNTING are linked to PRESENT VALUE and FUTURE VALUE of single payments respectively, and then to multiple payments in the four ANNUITIES scenarios. In chapter 15, I will show the importance of these two operations when working with spreadsheets.

In locating multiple payments in the diagram, I begin with simple annuities where the frequency of the payments corresponds with the frequency of compounding interest. Here I include FUTURE VALUE OF ORDINARY ANNUITY, FUTURE VALUE OF ANNUITY DUE, PRESENT VALUE OF ORDINARY ANNUITY and PRESENT VALUE OF ANNUITY DUE, thus distinguishing payments in advance (annuity due) from payment in arrears (ordinary annuity) for both present value and future value. I include SINKING FUNDS and OUTSTANDING BALANCE since they are included in the Grade 12 curriculum. I refer here only to sinking funds that involve setting up a fund to make provision for the replacement of an asset at some time in the future, since this is the scope of sinking funds studied at school level.

FUTURE VALUE OF AN ANNUITY is linked to SINKING FUNDS because it is used to determine the value of the regular payments at some point in the future when the new asset will be purchased. Similarly FUTURE VALUE (of a single payment) is linked to SINKING FUNDS because it determines the depreciated value of the asset that will be replaced. For this reason, a link with DEPRECIATION (see fig. 6.1 in chapter 6) is also indicated.

OUTSTANDING BALANCE at some time, T_k , can be calculated in two ways. In the retrospective method the loan is moved forward T_k thus calculating the interest on the loan at T_k as if no repayments have been made. Each of the k repayments is also moved forward to T_k and the outstanding balance is the difference between the future value of the loan amount, and the sum of the future values of each repayment, as given by the formula: Outstanding balance = $PV(1+i)^k - pymt\frac{(1+i)^k-1}{i}$. In the diagram, the link from FUTURE VALUE (of a single payment) indicates the loan, and the links from FUTURE VALUE OF ANNUITY indicate the k repayments.

In the prospective method the OUTSTANDING BALANCE at some time, T_k , is the sum of the present values of all the repayments that have not yet been made. The outstanding payments are therefore moved back in time to T_k as shown by the formula: Outstanding balance = $pymt\frac{1-(1+i)^{-m}}{i}$ where m = no. of outstanding payments. This is reflected by the links between PRESENT VALUE OF ANNUITY and OUTSTANDING BALANCE.

The concepts of DEFERRED ANNUITY, COMPLEX ANNUITY, ESCALATING ANNUITY and PERPETUITY are included for completeness. Although deferred annuity is not specifically mentioned in the curriculum, it is possible to include it at Grade 12 level since it only requires a simple adjustment for time. The other three annuity concepts are beyond the scope of the Grade 12 curriculum. I have not indicated

links from PRESENT VALUE and FUTURE VALUE OF ANNUITY to these concepts. Neither have I included other mathematical concepts that relate to annuities in this diagram. These are mentioned in relation to the MfT framework later in the chapter.

12.5 Expanding the view of the compound interest formula

In chapter 6, I argued for an expanded view of the compound interest formula. I distinguished between an *accumulation* view and an *adjustment* view. An accumulation view focuses on how much interest has been gained over time. It is a static view where the nominal value of the principal amount is largely considered apart from time. The only purpose for considering time is to determine the number of compoundings. By contrast, an adjustment view is a dynamic view that focuses on the *value* of an amount relative to its position in time. In this case the compound interest formula is used as a mechanism to move the amount left and right along the timeline. As a result, the value of the amount is a function of its position on the timeline (and the interest rate) and is thus inextricably linked to time.

The accumulation view of the compound interest formula is emphasised throughout the secondary school maths curriculum, but an adjustment view is necessary in the context of annuities, to grasp the notion of the time-value of money. This involves moving individual payments forward or backward in time to determine their separate contributions to a loan, outstanding balance or projected savings. Since annuities are only introduced in Grade 12, the need for an adjustment view comes very late. However, an adjustment view is the one used most often in the financial sector as captured in the typical statement 'discounting a payment back to T_0' (i.e. to the point at which a loan is taken or an annuity is purchased). This implies that the dominant view within the financial sector is not promoted at school level, thus reflecting different practices within different institutions as noted in chapter 3.

12.6 Issues relating to time

The importance of attending to time arose in several chapters in part 1 (e.g. chapters 6 and 9). These instances highlighted the need for precision in talk about time and ways of representing time. In annuity scenarios it is important to differentiate between payments at the beginning of a period and payments at the end of a period. It therefore becomes even more problematic to speak of paying *in a month* or to speak of the *March payment* because neither of these indicates whether the payment is made at the beginning or end of the month. Timelines play an important role in scenarios involving multiple payments, hence their importance for annuities. In this section I discuss in detail the conventions associated with timeline representations. Thereafter I briefly discuss the paradox that students may encounter when modelling annuity payments at the end of a period.

12.6.1 The timeline as a representation of events over time

Timelines are introduced in most financial maths texts as a way to represent key events, such as deposits, withdrawals and interest rate changes. In this section I discuss the importance of the conventions in the timeline representation, and some of the struggles that students may face as they begin to make use of timelines for annuities. In chapter 14, I focus in detail on several instances where timelines were used in the course.

While timelines may appear simple to those already familiar with them, there are several instances in the data where students' use of the timeline was problematic. Parramore (2011) notes that students have difficulty in distinguishing points in time from intervals of time. This suggests the need for

students to become familiar with the conventions of the timeline representation and for these conventions to be explicitly taught. In the section that follows I discuss the ways in which three financial maths texts introduce the timeline, and I highlight some important differences in their presentations. In this section I deliberately refer to *period* rather than *month* to acknowledge the generality of the timeline with regard to units of time.

The financial maths text by Young (1993) was a prescribed text for many years for the first year general mathematics course for commerce students in our University. It foregrounds that T_n represents *the end of period n*, although the notation is not explicitly defined in the text. Thus T_1 is the end of period 1 (see fig. 12.2a and 12.2b). T_0 may then be considered the start of period 1 or "now" and hence the point for calculating present value. Since T_0 is included on the timeline, the portion of the line between T_0 and T_1 is the interval of period 1. By convention, the P_t indicated below any T_n is the payment made at the end of that particular T_n. The use of T-notation reflects the discourse of the financial industry, and also emphasises points in time rather than intervals. Young does not deal with the future value of annuity due and so does not deal with the distinction between representing payments in advance and payments in arrears on the timeline.

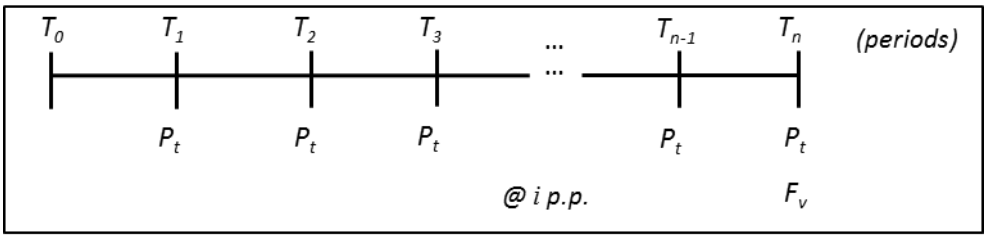


Fig. 12.2a Timeline for future value of an ordinary annuity (reproduced from Young (1993))

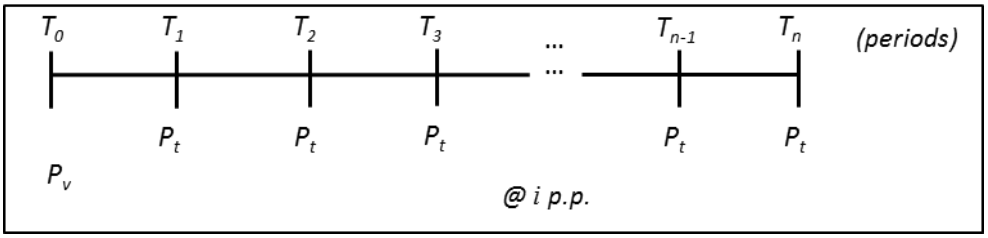


Fig. 12.2b Timeline for present value of an annuity (reproduced from Young (1993))

I have included the text by Smal (2004) to contrast his approach with that of Young. This text is written mainly for practitioners, particularly those in the real estate sector and so it is deliberately designed to be less technical and more accessible. Smal indicates explicitly that the convention is for payments to be made at the end of interest periods and juxtaposes this directly with payments made at the start of an interest period. As can be seen in fig. 12.3a and 12.3b, there is no use of T-notation and hence no attention to T_0.

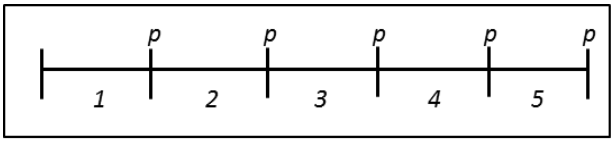


Fig 12.3a Timeline for future value of an ordinary annuity (reproduced from Smal (2004))

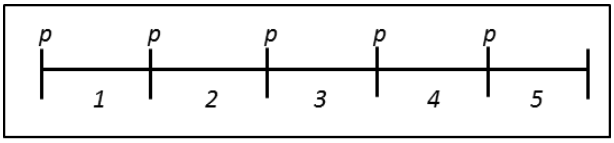


Fig 12.3b Timeline for present value of an annuity (reproduced from Smal (2004))

By placing the numbers *between* the divisions of the timeline, the above diagrams foreground the *intervals of time* rather than particular *points in time*. In placing *p* above the division lines, he indicates that payments are made at particular points in time. However, it is not clear whether these payments are made at the beginning or end of the period until one considers the position of the first and last *p*'s. In the first timeline (fig. 12.3a), there is no *p* at the beginning of the timeline but there is a *p* at the end. By convention, this positioning of the *p*'s is taken to indicate payments at the end of the interest periods, which corresponds with the convention in Young. In fig. 12.3b the first *p* appears at the beginning of the timeline and there is no *p* at the end. This indicates that payments are made at the beginning of the period. If one does not pay attention to both ends of the timelines, one cannot tell whether payments are being made at the beginning or end of the period. Thus this convention requires that the beginning and end of the entire timeframe be represented in order to communicate the timing of payments to the reader.

The text by Basson et al (2005) was written as a supportive and extended text for undergraduate commerce students taking a first course in mathematics. It resembles a school level maths text book although it is explicitly directed at beginning students in higher education. In this text there is explicit distinction between timelines for payments in arrears and payments in advance.

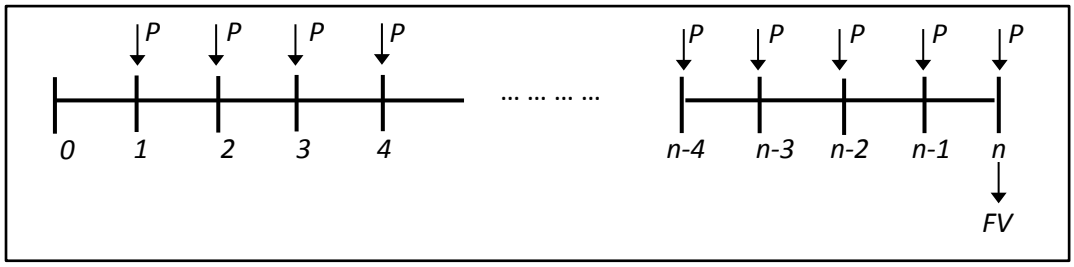


Fig 12.4a Timeline for future value of an ordinary annuity (reproduced from Basson et al (2005))

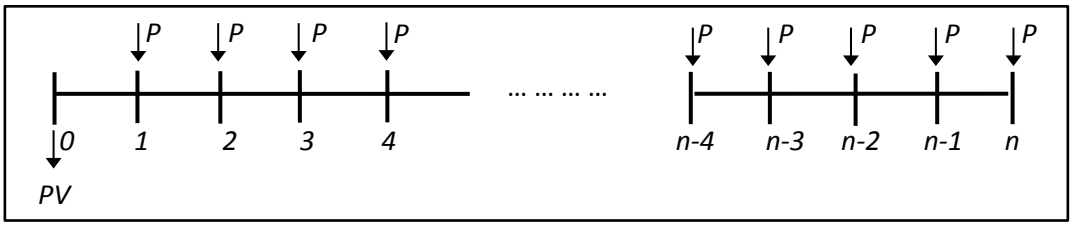


Fig 12.4b Timeline for present value of an annuity (reproduced from Basson et al (2005))

As with the Young text, these diagrams focus on points in time rather than intervals. Thus numbers and payments are written directly below/above the markers on the timeline. The representation of payments in arrears reflects the same positioning as in Smal with no payment at T_0 and a payment at T_n. The representation of payment in advance indicates a payment at T_0 but no payment at T_n. It is interesting to note that the text does not use T-notation despite it being a standard notation for time in the financial sector. One other difference worth noting is that the two academic texts (Basson, et al., 2005; Young, 1993) work with a general form of the timeline with reference to n, $n-1$, etc. Smal, being a practitioner's text, works with a specific numeric example of five periods.

As can be seen from the above discussion, a key aspect of working with timelines is to pay attention to the extremities of the timeline. This is particularly important when representing long periods of time

where not all periods are represented. However, even if one does attend to the extremes of the timeline, the timing of payments will only be clear if one is aware of the conventions. Consider, for example, the scenario with first payment at T_0. This implies payment at the beginning of period 1, payment at the beginning of period 2, etc. But this configuration might be read as payment at *beginning* of period 1, payment at *end* of period 1, payment at *end* of period 2 and so forth until payment at the end of period $n-1$, which still constitutes n payments. Because the end of any period k coincides with the beginning of period $k+1$, the difference is not distinguishable on the timeline. There is evidence in my data that when students were first attempting to derive a formula for future value of an ordinary annuity, some students worked with a payment at the beginning and end of the first month. This does not necessarily violate the definition of an annuity because in the students' minds the two payments are still made a full period (month) apart. Furthermore, if the end of month k marks the beginning of month $k+1$, then the difference in time is seemingly irrelevant. However, the difference is important because of the way annuities are modelled.

If students are introduced to future value of ordinary annuities first, they begin by representing payments at the end of the period, for example they begin their timeline at T_1 and end it at T_n. They do not state explicitly that they are focusing on the ends of the periods and there seems little need to do so at this point. Once greater attention is paid to time, the introduction of T_0 on the timeline is a sensible addition because it then clarifies that the first payment is made at the end of the first period. If students are representing P_1, P_2, etc. for payments, they are satisfied that the introduction of T_0 does not disrupt the relationship that any payment P_k is made at T_k.

However, a problem may emerge when students are required to think about payments at the *beginning* of the period. Some are satisfied to use the same timeline, starting at T_1 and ending at T_n, and may or may not state explicitly that they are now focusing on the beginning of the period. For those who notice the ambiguity of their representation, there is a sense of unease – how do you know whether it is the start or the end of the period? Thus the need for conventions and precision arises. In my experience, if the notion of T_0 is introduced at this point, students are generally satisfied that the first payment is made at T_0 and thus the beginning of the first period. They also appear to accept the need to represent the whole of the last period on the timeline, but recognise that no payment is made at the "end" of the timeline. Students representing P_1, P_2, etc. have to recognise that any payment P_k is made at T_{k-1}. While this may be disconcerting at first, it helps in decoding questions that refer to a particular period or payment by its ordinal value. For example, the third payment is made at T_2 on the timeline which represents the beginning of the third period.

A related problem is that students need to learn when to work with the idea that the end of period k is the start of period $k+1$, and when to ignore it. For example, when talking about the closing balance for March and the opening balance for April, it is obvious that these amounts are the same. But on the timeline it may be problematic to ignore the distinction. Since we define T_k as the end of period k, any payment that is written opposite T_k on the timeline suggests that the payment is made at the end of the period. But when dealing with payments in advance, a payment indicated on the timeline at T_4 represents the fifth payment made at the *beginning* of the fifth period. It is not paid at the end of the fourth period. In the data there is evidence of students wanting to indicate this payment slightly to the right of the T-marker on the timeline in order to indicate explicitly that it is made at the beginning of the period.

While timelines are a useful representation for solving annuities problems, explicit attention needs to be given to learning the conventions of the representation. Anecdotal evidence suggests that when timelines are introduced with problems that are so simple there is no need for a timeline, learners tend to ignore the timeline. Consequently, when the problems increase in difficulty learners may not have the necessary skills to make effective use of timelines to deal with the complexity of the problem. There is evidence in the data that the students tended not to make use of timelines when the problems were simple. However, they produced detailed timelines when the problems became more complex.

12.6.2 Modelling time in annuities scenarios

In the world of banking, an annuity payment can be made at any time of the month, and interest is calculated daily and compounded monthly. However, the typical mathematical models for simple annuities remove much of this complexity by considering only two scenarios – *payment in advance* where the payment is made at the beginning of the period, or *payment in arrears* where the payment is made at the end of the period. Furthermore, the beginning and end of the month are defined very specifically. I will focus on paying at the end of the month to illustrate my point. From an everyday perspective, payment at end of month might be seen as paying *near* the end of the month, possibly not even on the last day of the month. In the model, the end of the month means exactly midnight on the last day of the month. This leads to a paradox: that payment happens at the end of the month and that interest is also compounded at the end of the month. Thus there are two processes happening at the same time. The convention is that a payment made at the end of the month does not earn interest in the month in which is it paid. So, if interest is capitalised at the end of the month, and the payment is not included in this interest calculation, then one could reason that the payment must be happening *after the end of the month*, in other words at the beginning of the next month. Thus it would be feasible to argue that payment at the end of month n is really payment at the start of month $n + 1$, but this is not the convention. Students need to accept that payment at the end of the month takes place after the interest is capitalised for the month but before the beginning of the next month. This is discussed in more detail in the next chapter.

12.7 Two approaches to annuities

Based on my previous teaching of annuities, I noticed that when students first encountered annuity-based tasks, many students did not focus on each payment and make use of a geometric progression. Rather they tracked the account balance over time. From this observation I have distinguished two different approaches to annuity situations.

The first approach I call the *account balance* (AB) approach because it focuses on tracking the account balance. This approach mirrors what goes on in the bank each month (although the detail of daily interest calculations is ignored). In the case of annuity-based savings: a deposit is made, it is added to the account balance, interest accumulates and the closing balance is calculated at the end of each month. In the case of a loan: the loan is granted, interest accumulates for the first month, a repayment is made and deducted from the capital balance, and this process is repeated until the loan is repaid. This approach is easy to make sense of and appears to be the approach students adopt when initially attempting to model an annuity-based scenario. The unit of focus[50] is account balance against time. It

[50] I use the term "unit of focus" to refer to the part of the calculations/process that students most pay attention to. It concerns the component/s that students are tracking to monitor change over time.

depends on simple iterative calculations, but when there is a departure from the perfect payment plan, all balances need to be recalculated. This makes it an inefficient and cumbersome approach.

The second approach focuses on the behaviour of each individual payment over time, and so I refer to it as the *individual payment* (IP) approach. In this approach each payment is disaggregated from the whole and its contribution to the overall balance is modelled by moving it forward (or backward) in time by means of compound interest calculations. In the case of an investment, each deposit is moved forward to the end of the investment period so one can see the contribution it has made to the final amount. In the case of a loan, each payment is moved back to the point when the loan is granted. The unit of focus is individual payment against time.

The IP approach is mathematically more powerful than the AB approach because it gives access to the geometric progression which reflects the mathematical structure of annuity-based scenarios. It is also a more efficient approach: when there is a departure from the perfect payment plan, only the changed payments need to be considered when recalculating balances. For both these reasons, it is not surprising that maths text books adopt this approach from the outset. However, based on previous experiences of teaching annuities, where I posed a problem and gave opportunity for students to approach it in their own way, the majority of students did not adopt an IP approach. I therefore conjectured that an IP approach is not intuitive for students (Fischbein, 1987, 1999). The investigation of this conjecture forms the focus of the next chapter.

In Fischbein's terms the AB approach has qualities of self-evidence and immediacy. In previous courses, most students approached the annuities task by calculating monthly account balances – beginning with the first payment, adding monthly interest, determining the closing balance and then repeating this process for each monthly payment. This approach is based on life experience since it reflects, at least in general terms, the way in which the money accumulates in a bank account. It is also a sensible approach because it reflects the monthly process of making payments and gaining interest, and readily provides useful information such as the amount of money accumulated at a particular point in time. It may therefore be considered a *primary intuition*. By contrast the IP approach is not a primary intuition. There is nothing self-evident or "obvious" in the task that cues one to think in terms of individual payments running in isolation from each other until the end of the annuity. The IP approach does not reflect the monthly process of making payments and gaining interest. Rather, it is an analytical approach that projects the growth of money into the future (or back to T_0). However, as I will show in chapter 13, once students had accepted the IP approach, they used it readily in future tasks. Therefore an IP approach may be considered a *secondary intuition* since it results from instruction and, based on students' take up of the approach in the course, becomes the typical approach to "familiar" annuities tasks. I use the qualifier "familiar" because there is evidence that when an annuities task involves some departure from the perfect payment plan, for example a missed payment, students may resort to AB approaches to deal with the unfamiliar component of the task.

The constructs of *account balance* and *individual payment* emerged early in the analysis and have framed much of the analysis in this part of the thesis as will be seen in the chapters that follow. I shall use the following terms interchangeably with reference to the two approaches: AB/IP *approach*, AB/IP *thinking*, AB/IP *conception(s)*, and AB/IP *framework*.

12.7.1 Deriving a formula for future value of annuity due using an Account Balance approach

As mentioned above, the AB approach broadly models what happens in banks on a monthly basis. In order to make sense of the real world problem, students need to understand how this works. By doing the iterative calculations for a few months, they will see the patterns in the actions that are modelled by the calculations. From this inductive process it is possible to derive the annuities formulae. However, the route to do this may be fraught with algebraic temptations that seduce one down blind alleys of frustrating symbolic manipulation. I use the example of the future value of annuity due to illustrate this since I have examples of student work that exemplify the blind alleys.

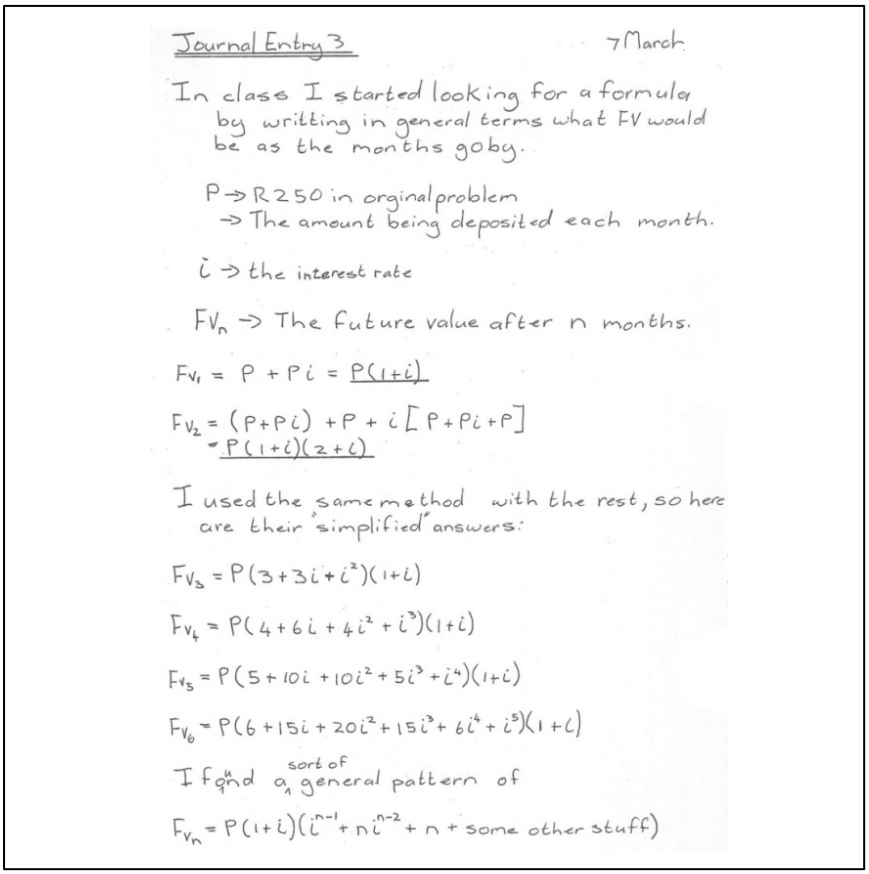
Table 12.1 provides general expressions for the calculations that are done at the end of every period. Thus each row (lines 1 to 6) represents a period. The middle column indicates that payment is made at the beginning of the period and that interest is gained at the end of the period. The right-hand column gives the *strategically factorised expression*. I call it *strategically* factorised because it is possible (and tempting) to expand the expressions in the square bracket and then collect like terms which may or may not lead to some form of factorisation. The factorised form shown below preserves the unit growth factor $(1 + i)$ which ultimately produces the geometric series. In the middle column, the factor $(1 + i)$ is multiplied by each term in the "expanding bracket". Thus in each line the expression in the square bracket expands but the emerging pattern in exponents is easily seen.

Line	End of period	Expression for process of making new payment and gaining interest	Strategically factorised expression
1	1	$FV_1 = P + Pi$	$FV_1 = P\,[1+i]$
2	2	$FV_2 = (P\,[1+i] + P)\,(1+i)$	$FV_2 = P\,[(1+i)^2 + (1+i)]$
3	3	$FV_3 = (P\,[(1+i)^2 + (1+i)] + P)\,(1+i)$	$FV_3 = P\,[(1+i)^3 + (1+i)^2 + (1+i)]$
4	4	$FV_4 = (P\,[(1+i)^3 + (1+i)^2 + (1+i)] + P)\,(1+i)$	$FV_4 = P\,[(1+i)^4 + (1+i)^3 + (1+i)^2 + (1+i)]$
5	...		
6	n	$FV_n = (P\,[(1+i)^{n-1} + \ldots + (1+i)^3 + (1+i)^2 + (1+i)])\,(1+i)$	$FV_n = P\,[(1+i)^n + (1+i)^{n-1} + \ldots + (1+i)^3 + (1+i)^2 + (1+i)]$

Table 12.1 Account balance approach to generate series for future value of annuity due

Obviously a strategic substitution of $k = 1 + i$ would substantially improve the readability of the expressions and also show more clearly the geometric progression embedded in the expanding expression. However, the elegance of the expressions above is not easily revealed in the struggle to derive the formula. For example, in fig. 12.5 and fig. 12.6, I show the attempts and observations of Hailey and Lina as they attempted to derive this formula using an AB approach.

Hayley has compressed many lines of algebraic manipulation into this summary. She notes the common factor of $P(1 + i)$ and then the patterns in the highest two powers of the exponent and the constant, and refers to the remaining terms as "some other stuff". She calls this "a sort of general pattern". She does not recognise elements of the binominal expansion in her work.



Journal Entry 3 — 7 March

In class I started looking for a formula by writting in general terms what FV would be as the months go by.

P → R250 in orginal problem
→ The amount being deposited each month.

i → the interest rate

FV_n → The future value after n months.

$FV_1 = P + Pi = \underline{P(1+i)}$

$FV_2 = (P+Pi) + P + i[P+Pi+P]$
$= \underline{P(1+i)(2+i)}$

I used the same method with the rest, so here are their "simplified" answers:

$FV_3 = P(3+3i+i^2)(1+i)$

$FV_4 = P(4+6i+4i^2+i^3)(1+i)$

$FV_5 = P(5+10i+10i^2+5i^3+i^4)(1+i)$

$FV_6 = P(6+15i+20i^2+15i^3+6i^4+i^5)(1+i)$

I found a sort of general pattern of

$FV_n = P(1+i)(i^{n-1}+ni^{n-2}+n+$ some other stuff$)$

Fig 12.5 Hailey's attempt to simplify algebraic expressions for future value of annuity

In her journal entry Lina provided evidence of three attempts to simplify the algebraic expressions. The third attempt is included here. She begins with an error which gets carried through her algebraic work (she writes $P + i$ instead of $P + Pi$ at the end of month 1). Despite this error, she produces a systematic pattern where she notices several trends: the coefficient of P and the highest power of i in each expression are the month number, each expression has a term i, and the first order differences between the coefficients of Pi are linked to the month number (e.g. in month 5, $14Pi - 9Pi = 5Pi$). However, with or without errors, the structures that Hailey and Lina identified did not lead them to the required expression.

Yet, even if Hailey and Lina's attempts at structuring the situation had produced the standard formula for future value of an annuity due, their thinking does not reflect the thinking that will be most useful in working with annuity-based problems. Rather, they need to shift to an IP approach since the AB approach is limited in providing a useful and flexible model to deal with annuity-based scenarios.

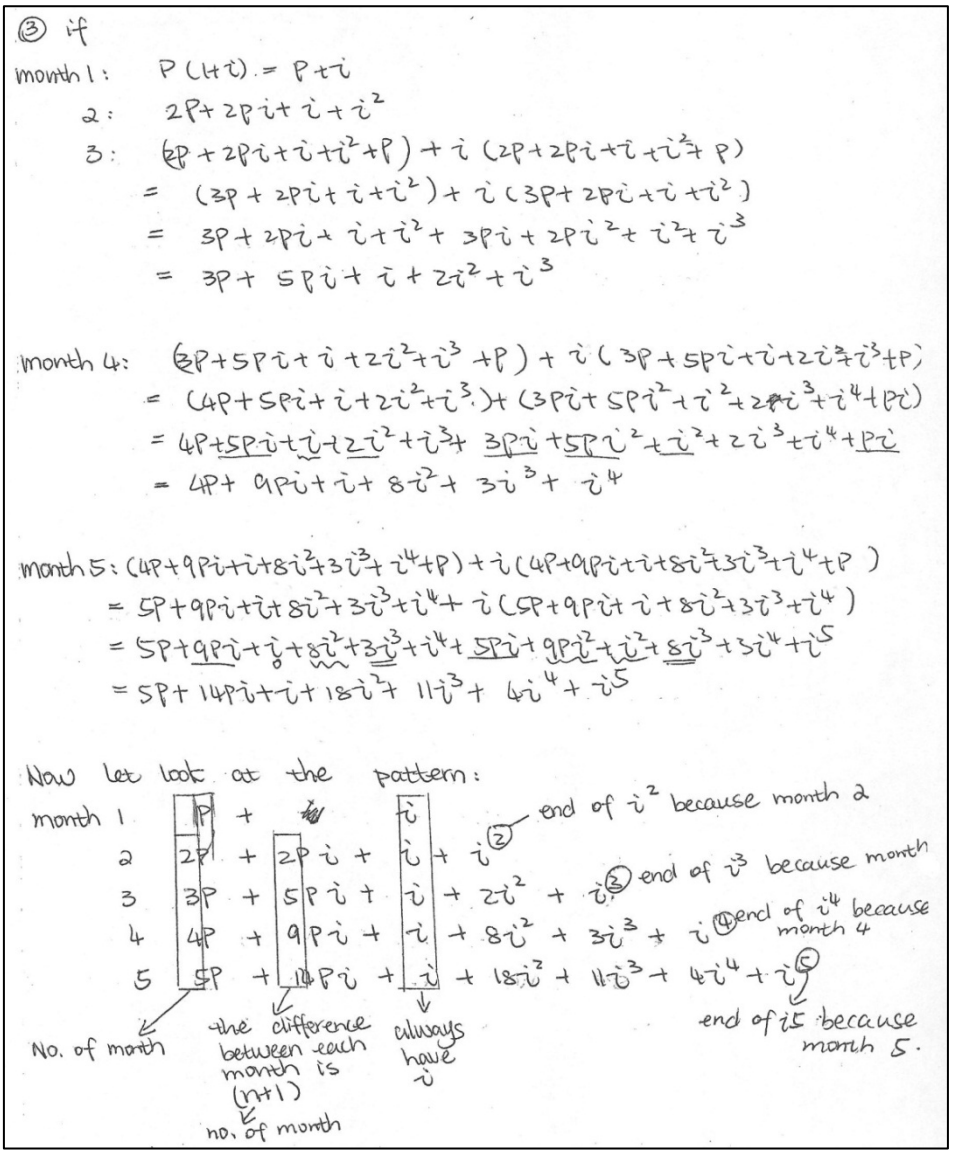


Fig 12.6 Lina's attempt to simplify algebraic expressions for future value of annuity

12.7.2 Deriving a formula for future value of annuity due using an Individual Payment approach

The following table shows the elegance of the IP approach for the future value of an annuity due, assuming 12 equal monthly payments. Here each row (from line 1 to 6) represents the future value of a particular payment at the end of period 12. The terms generated are clearly recognisable as those of a geometric progression.

Line	Value of payment at end of period 12 (T_{12})	Strategically factorised expression
1	$FV_1 = P(1+i)^{12}$	These terms are then added together and a common factor of P is factorised FV of annuity = $P(1+i)^{12} + P(1+i)^{11} + \ldots + P(1+i)^3 + P(1+i)^2 + P(1+i)$ $= P[(1+i)^{12} + (1+i)^{11} + \ldots + (1+i)^3 + (1+i)^2 + (1+i)]$
2	$FV_2 = P(1+i)^{11}$	
3	$FV_3 = P(1+i)^{10}$	
4	...	
5	$FV_{11} = P(1+i)^2$	
6	$FV_{12} = P(1+i)$	

Table 12.2 Individual payment approach to generate series for future value of annuity due

While this method produces the same final result as the AB approach, the underlying thinking is substantially different. This approach does not model the monthly process in the bank. Each payment

requires one to run through the timeframe from the point at which the payment is made to the end of the investment. To move from this form to the standard formula for future value of annuity due, students have two options: either they must recognise the geometric progression (which may or may not include common factors), identify the first term, common ratio and then substitute appropriately into the formula for a geometric series "sum to n terms", or they must use the elimination method to remove all "middle" terms and then complete the necessary algebraic manipulation.

In chapter 13, I shall provide evidence to support my conjecture that the IP approach is not intuitive, and does not appear to be the obvious starting point for many. It requires a substantial shift in thinking to conceive of an annuity situation in this way. However, once this shift is made, students appear to have little difficulty in adopting an IP approach. I shall argue that, in order for students to succeed with annuities, they need to adopt an IP approach.

12.8 Spreadsheets for learning annuities

Spreadsheets were used for a variety of topics in the course. In a later chapter I focus on students' use of spreadsheets as resources for learning annuities. At this point I will merely introduce two different kinds of spreadsheets that were used for annuities. Consider the scenario of making 12 monthly deposits of R250 at the end of each month with an interest rate of 6% p.a.

Spreadsheet 1 (fig. 12.7) tracks the balance at the end of every period, thus reflecting an AB approach. I refer to this as the *column spreadsheet* because it takes the typical spreadsheet form of columns of values. The box at the top of the frame indicates monthly compounding since the user has chosen 12 for the number of compounding periods in a year. Since each row focuses on a particular period, as one moves from left to right across each row, one moves from the beginning to the end of the month. The spreadsheet shows explicitly that payment is made after interest has been calculated. The closing balance at the end of one month is the opening balance for the following month. When payments are made at the end of the month, the column "balance on which interest is calculated" is the same as the opening balance. However, when payments are made at the beginning of the month, then this balance will include the deposit made in the current month.

	A	B	C	D	E	F
1	**Investments (Future Value of an annuity)**					
2	Payments made at end of period					
3						
4			**Rate**	**6**	**% p.a.**	
5			**No. of compounding periods in 1 year**	**12**		
6			**Payment**	**250.00**		
7						
8						
9						
10	**Period**	**Opening balance**	**Balance on which interest is calculated**	**Interest**	**Payment**	**Capital balance**
11	1	0.00	0.00	0.00	250.00	250.00
12	2	250.00	250.00	1.25	250.00	501.25
13	3	501.25	501.25	2.51	250.00	753.76
14	4	753.76	753.76	3.77	250.00	1 007.53
15	5	1 007.53	1 007.53	5.04	250.00	1 262.56
16	6	1 262.56	1 262.56	6.31	250.00	1 518.88
17	7	1 518.88	1 518.88	7.59	250.00	1 776.47
18	8	1 776.47	1 776.47	8.88	250.00	2 035.35
19	9	2 035.35	2 035.35	10.18	250.00	2 295.53
20	10	2 295.53	2 295.53	11.48	250.00	2 557.01
21	11	2 557.01	2 557.01	12.79	250.00	2 819.79
22	12	2 819.79	2 819.79	14.10	250.00	3 083.89

Fig 12.7 Spreadsheet 1 – Focusing on balance in account

The column may be redundant for payments at the end of the month but it is important for payments made at the beginning of the month because it distinguishes the balance before interest from the capital balance at the end of the month.

Spreadsheet 2 (fig. 12.8) foregrounds the growth of individual payments, thus reflecting an IP approach. Since it has an approximately triangular shape, I refer to it as the *triangular spreadsheet*.

It contains 12 rows (from 9 to 20) representing the 12 separate payments made at the end of each month from January to December. Each 250 represents the value of the payment at the time it is made. Thereafter each payment grows at 0.5% per month. Each row constitutes a geometric progression. As a result, the same pattern appears in each row but less and less of the pattern can be seen as one moves from left to right across the columns. The balance in the account at the end of any month can be calculated by summing the column for that month. For example, the balance at the end of July is R1776.47, and the total amount accumulated at the end of December (R3083.89) can be found in cell M22. The values in the columns form the same geometric progression as the rows, and each entry in a column indicates the value that a particular payment contributes to the account balance at the end of that month.

It is worth noting that neither the compound interest formula nor the annuities formulae are used in either spreadsheet. The spreadsheets contain only basic arithmetic calculations. Thus, in order to build the spreadsheets one needs to understand the low-level calculations that ultimately combine to produce the annuities formulae. In the case of spreadsheet 2, each payment is moved forward by means of a compounding calculation. In chapter 15, payments will be moved back in time by means of a discounting calculation.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Investments with monthly payments at end of period**												
2													
3	Monthly Payment	250											
4	Rate	6	% p.a. compounded monthly										
5													
6		End of month											
7		Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
8		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
9		250.00	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48	262.79	264.10
10			250.00	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48	262.79
11				250.00	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18	261.48
12					250.00	251.25	252.51	253.77	255.04	256.31	257.59	258.88	260.18
13						250.00	251.25	252.51	253.77	255.04	256.31	257.59	258.88
14							250.00	251.25	252.51	253.77	255.04	256.31	257.59
15								250.00	251.25	252.51	253.77	255.04	256.31
16									250.00	251.25	252.51	253.77	255.04
17										250.00	251.25	252.51	253.77
18											250.00	251.25	252.51
19												250.00	251.25
20													250.00
22	Account balance	250.00	501.25	753.76	1007.53	1262.56	1518.88	1776.47	2035.35	2295.53	2557.01	2819.79	3083.89

Fig 12.8 Spreadsheet 2 – Growth of individual payments

In chapter 13, I will discuss how spreadsheet 2 was used to introduce the derivation of the formula for future value of an ordinary annuity. Students also worked with similar spreadsheets for present value of annuities. This is discussed in chapter 13, where I shall discuss links that students made between the annuities formulae and the spreadsheets.

12.9 What is Maths-for-Teaching annuities at school?

In this section I draw together the issues discussed above, and link them into the MfT framework in order to begin to answer the question "what is maths for teaching annuities at school?" I include the phrase "at school" to indicate the institutional context that I am presuming. At the beginning of the chapter I conceded that annuities is a very small section of the school curriculum and so it is understandable that teachers might not consider this as a section worthy of investment (pun intended) in terms of teaching time. However, given the importance of a knowledge of annuities at the level of personal finance, I contend that teachers will benefit personally and professionally from a deeper and broader knowledge of the mathematics of annuities, with its implications for their personal finances as well as preparing their learners better to deal with financial issues beyond school. As with the earlier discussion of mathematics-for-teaching compound interest, I am not claiming that teachers should know all the aspects discussed below in order to teach annuities successfully.

12.9.1 Essential features

Identifying the concept of an annuity – Knowledge of what constitutes the annuity concept includes definitions and formulae for present value and future value of ordinary annuity and annuity due, as well as related concepts such as sinking funds and outstanding balance. It also includes knowledge of the meaning of each symbol in the formulae. Given that mathematical concepts can only be accessed through their representations, the essential features include knowledge of the following representations: tables of values, algebraic forms (particularly formulae), graphs, timelines, and spreadsheets. In chapter 6 I introduced the key notions of *growth factor, unit growth factor* and *rate per period*. Each of these underpins annuities calculations as they apply to the operations of compounding and discounting. The ability to identify a concept includes knowledge of what is not an instance of the concept. In the case of annuities, it includes distinguishing annuity scenarios from single-payment compound interest scenarios. It also includes the ability to distinguish payment in advance from payment in arrears.

I note here the "angle notation" that is common in financial texts, and which is a compressed symbolic form used to abbreviate the formulae for present value and future value of an annuity respectively as follows:

$a_{n\neg i}$ read as "a angle n at i" and representing the factor $\left[\frac{1-(1+i)^{-n}}{i}\right]$ so PV = pmt · $a_{n\neg i}$

$s_{n\neg i}$ read as "s angle n at i" and representing the factor $\left[\frac{(1+i)^{n}-1}{i}\right]$ so FV = pmt · $s_{n\neg i}$

This notation was not used in the course since it is not required in the school curriculum. Also, students did not make use of financial calculators which make use of angle notation.

Ways of working with annuities – In order to manipulate the entity, teachers require knowledge of and the ability to execute a range of routines related to the concept of annuity, such as: deriving the formulae for present and future value; substitution into formulae; manipulating inputs so that they are in the same unit of time and have been adjusted to the same point in time (i.e. time-value), where necessary; and manipulating formulae to solve for any unknown, including techniques such as Newton's method to determine the interest rate. The essential features also include knowledge of different approaches to problems such as the prospective and retrospective methods of determining the outstanding balance on a loan/mortgage. Spreadsheets provide another means of working with

annuities as reflected in the two examples discussed earlier in the chapter. In chapter 15, I shall show how students made links between the formulae and values in particular cells/sections of a spreadsheet. Given that the AB and IP approaches constitute ways of working with annuities, it seems appropriate to include knowledge of these approaches under essential features. However, they are also included under *modelling and applications* since they can be considered as ways of modelling annuities.

12.9.2 Relationship to other mathematics

In considering the relationship of annuities to other mathematics, it is difficult to make hard distinctions between financial mathematics and "pure" mathematics.

Drawing again on Ma's (1999) notions of depth, there are three key concepts at a lower level: geometric progressions, exponential growth and compound growth (see fig. 6.1 in chapter 6). I distinguish compound growth as a specific form of exponential growth given its central role in financial mathematics.

Aspects at higher levels are more complex mathematical concepts. In this particular case, they include calculus tools and techniques which are necessary to model the mathematics of change, particularly continuous change such as continuous cash flows. This includes natural logarithms, e, limits and integrals.

There is also an important relationship to Newton's method which is used to determine the interest rate in annuities scenarios, but which can also be used more broadly to find roots of any real function that is differentiable over the relevant interval. In chapter 6, I identified several connections between compound interest and other aspects of mathematics which are also relevant to annuities but which I do not repeat here.

12.9.3 Modelling and applications

Since an annuity is a concept of financial mathematics rather than pure mathematics, knowledge of modelling and applications of mathematics is a key component of MfT annuities. This knowledge includes knowing that the future value of an annuity models investments, whereas the present value of an annuity models loans and regular payments (withdrawals) from a lump sum. It is important to distinguish between between these two scenarios modelled by the present value is important because when the scenarios are stated as contextual problems, students do not easily recognise that the same mathematical model is appropriate for both scenarios. Teachers also require knowledge of the ways in which the real world situation is simplified and mathematized. For example, the daily interest processes of banks are ignored. There are two other conventions of the modelling process that are important: payments are assumed to be made only at the beginning or end of the month, and a payment made at the end of the month will not gain interest in the month in which it is deposited.

As noted above, teachers should be able to model annuities using both AB and IP approaches. They should also know how to make adjustments for departures from the perfect payment plan, such as missed or delayed payments, as well as paying an amount different to the regular amount, such as doubling a particular payment. In the case of complex annuities, it is important to know how to make the appropriate adjustments in order to deal with the fact that payments are made at a different frequency to the compounding of interest. This would provide a meaningful extension to the second

text book question given earlier where *weekly* compounding could be replaced with *monthly* compounding.

12.9.4 Mathematical practices

The mathematical practices most typical of working with annuities in the school curriculum are the use of algebraic formulae and substitution to obtain numerical answers. Here, as with compound interest, the importance of accuracy in numeric work can be illustrated. If spreadsheets are utilised, then the use of numerical approaches and numerical representations increases. Furthermore, the impact of small changes in the interest rate or changes in the monthly (re-)payment can be powerfully illustrated without the burden of multiple manual calculations. There is little, if any, attention to generalisation and proof.

Annuities provide a powerful context to deal with *numerical methods* such as Newton's method, to determine the interest rate. However, this is beyond the scope of the school curriculum. In chapter 4, I explained that I chose to include Newton's method in the course in order to expose students to numerical methods, and thus a new way of working mathematically. The use of Newton's method for annuities leads to "messy" polynomial functions such as:

$$f\,(x) = 250000x\,(1+x)^{240} - 2120(1+x)^{240} + 2120 \quad \text{and}$$
$$f'(x) = 250000(1\ +\ x)^{240} + 60000000x(1+x)^{239} - 508800\,(1\ +\ x)^{239}$$

The large coefficients and exponents increase the level of complexity of the manipulations when compared with simpler functions such as $g(x) = x^5 - 3x + 1$ which are more typical of the examples used to introduce Newton's method.

12.9.5 Different teaching approaches and sequences

As with teaching of compound interest, there are different approaches to teaching annuities. Teachers require knowledge of the different approaches, and their implications, so they can make appropriate choices for their context. The first is an *applications* approach as promoted by the school curriculum (Department of Education (DoE), 2003) where annuities is introduced as an application of geometric series. Earlier in this chapter I discussed some of the implications of this approach in relation to overloading of the Grade 12 financial maths curriculum. A second approach is a *modelling* approach where learners are given realistic tasks and are expected to extract the relevant mathematics through the modelling process. In such a situation, learners may not choose to work with geometric series at first and may instead choose an AB approach. In the course I chose a modelling approach. However, based on students' struggles in the course (which are discussed in chapter 13), this approach may be too demanding at school level and beyond. Nevertheless, if learners take an AB approach and work numerically, they may at least appreciate that the compound interest formula is not an appropriate model for an annuity situation. This provides a rationale for introducing a new formula (or model) for annuities.

Irrespective of the teaching approach adopted, it is possible to vary the sequence in which different aspects of annuities are introduced. For example Young (1993) and Basson et al (2005) both introduce present value of an annuity before future value of an annuity, while Laridon et al (2007) introduce future value of an annuity first. In the course I dealt with future value first because, based on experience, students appear to grasp future value more easily than present value. It seems that moving

money forward in time is more intuitive (Fischbein, 1999) than moving it backward in time (i.e. discounting), as required in present value scenarios.

12.9.6 Basic repertoire

A basic repertoire of examples is connected to *essential features*, *learners' conceptions* and *explanations*. These components of teachers' knowledge need to be coordinated in order to identify and select (or produce) basic examples and more complex tasks that focus on aspects such as payments in advance and in arrears, present and future value, and outstanding balance. The examples and tasks should illustrate the different algorithms for determining the values of each unknown in the formulae, and the knowledge of what new tools may be required for finding each unknown. This will include the use of logs to determine the number of payments (n), and the use of Newton's method to determine the interest rate. The choice of time periods is an important aspect of the basic repertoire. It is sufficient to work with a 12-month period when focusing on the key ideas (see discussion of spreadsheets in chapter 15). However, in order to appreciate the power of compounding, other examples need to cover much longer periods such as twenty years. This will also enable learners to engage with the socio-economic and personal consequences of changes in interest rates and missed payments, as noted below. A basic repertoire for annuities also includes appropriate choices of interest rates as discussed in chapter 6.

12.9.7 Explanations

Knowledge of how to explain the mathematics of annuities includes demonstrations of how to use the various annuities formulae, and how the formulae model the timing of payments. This will likely include demonstrating the use of timelines, which become an increasingly important representation when dealing with multiple payments. It is likely that particular attention will need to be given to discounting of payments when working with present value scenarios. As with compound interest, attention must be given to the calculation of a rate per period. Explanations are also likely to include attention to calculator usage since the formulae are more "complex" than most other formulae encountered in school mathematics. In chapter 15, I shall argue that the column and triangular spreadsheets provide tools for explaining aspects of annuities that are not easily accessed in other representations.

12.9.8 Learners' conceptions

Knowledge of learners' conceptions and difficulties is acquired from research literature and from practice. However, given the paucity of research on conceptions of annuities, the empirical findings regarding the obstacles that students experienced form a central contribution of this thesis. I have suggested that an expanded view of the compound interest formula – being able to view it as both accumulation and adjustment – is an important component of students' conceptions of annuities.

In chapter 13, I shall show that most students did not initially choose an IP approach to work on an annuities task, and some questioned the validity of the IP approach as a model of the situation. This is important knowledge for teachers because it runs contrary to the approaches taken in most introductory financial maths texts, including school text books. Earlier in this chapter I also provided some evidence of students' initial difficulties in deriving an annuities formula from an AB approach.

There is a range of evidence in the data to suggest that students struggle with the meaning of n in the annuities formulae. While the derivation of the formulae shows that n represents the number of terms

which therefore represents the number of payments, there is evidence to suggest that students relate n to number of compounding periods. This is not surprising since the exponent in the compound interest formula represents the number of compoundings and students have worked a great with this formula by the time they begin with annuities. It must be noted that in some cases, such as future value of annuity due, the number of terms corresponds with the number of compounding periods but this is not the case in the future value of an ordinary annuity.

Students raised concerns about the final payment at the end of the month in a future value scenario. Some students questioned the need to include the payment in the calculation since it would not gain any interest, and thus, from a practical point of view, there would be no sense in making the deposit.

I suggested that there may be evidence of students equating equal *percentages* of interest per period with equal *amounts* of interest per period. This may reflect an over-dominance of linear thinking will is well-documented in the research literature (e.g. De Bock, et al., 2002; Esteley, et al., 2010).

12.9.9 Contextual issues of finance

Financial concepts and conventions – In regard to this sub-component, I focus briefly on financial concepts at lower levels and higher levels (see fig. 6.1 and 12.1). At a lower level one finds knowledge of compounding, discounting and compound growth. This requires an adjustment view of the compound interest formula and thus an appreciation for the time-value of money. At a higher level, there are some aspects that may be considered more general and complex cases of simple annuities such as *complex annuities, growing annuities, perpetuities,* and *uneven cash flows.* On the other hand concepts such as *net present value* and *internal rate of return* are associated with annuities (Feng & Kwan, 2011; Sugden & Miller, 2010) and provide the teacher with a broader financial picture. Complex annuities was the only one of these concepts dealt with in the course.

Socio-economic issues and financial literacy – Knowledge of socio-economic and financial issues impacts annuity-based scenarios in several ways. I have already proposed a moral imperative for mathematics teachers to increase learners' levels of financial literacy and thus help them appreciate the power – both positive and negative – of compound interest. This reflects an expectation that teachers pay attention to both *content and values* in their teaching. While mathematics teachers cannot be expected to possess extensive knowledge of finance and economics, a rudimentary knowledge of socio-economic and financial issues will strengthen their teaching and help learners to see connections between the mathematics they are learning and its application in their personal lives and in society more broadly. It is beyond the scope of this study to identify the extent of such rudimentary knowledge but it might include knowledge of inflation, buying power, hire purchase, vehicle finance, and the National Credit Act (Government Gazette, 2006).

12.10 Conclusion

In this chapter I have set up a reference landscape for the concept of annuities, to complement the reference landscape for compound interest in chapter 6. In both cases the landscape spans school financial mathematics, introductory financial mathematics at university level, mathematics teacher education, and (aspects of) the world of finance. I have proposed a network of concepts related to annuities, and discussed aspects such as timelines and spreadsheets which I consider important within the landscape, and which are central to my data analysis. I also considered each aspect of the MfT framework in relation to teachers' knowledge for teaching annuities. The AB and IP approaches to

annuities generate a link between three aspects of the framework: *essential features*, *modelling and applications* and *different approaches to the concept*. Further connections between the different aspects of the framework will be explored in the analysis chapters that follow.

Aspects of MfT in focus in this chapter

Essential features

- Network of annuities-related concepts building from *compounding* and *discounting* which in turn build on *growth factor*, *unit growth factor* and *rate per period*
- Key representations: formulae, timelines (and associated conventions) and spreadsheets
- *Adjustment* views of the compound interest formula

Modelling and applications

- Modelling of payments at beginning and end of period only
- Convention that payment made at end of period does not gain interest in that period
- Account balance (AB) and individual payment (IP) approaches

Mathematical practices

- Numeric methods exemplified by Newton's method

Different teaching approaches

- Modelling of banking scenario vs application of geometric series

CHAPTER 13

First encounter with *individual payment* approach

13.1 Introduction

The following annuities question appears in a widely-used Grade 12 text book in South Africa:

> "Jabu opens a savings account with R5000. He deposits R2000 two years later, and a further R1500 five years after the account was opened. The interest rate is calculated at 11% p.a. compound annually. Calculate the amount of money in the account at the end of eight years." (Laridon, et al., 2007, pp. 2-3).

It is followed by a worked solution where the authors provide a timeline indicating the positions of the three payments as well as T_8. They then state "There are two ways of calculating the answer. The first method is perhaps more logical, but it is more cumbersome and not recommended" (p. 73).

The "first method" is, in my terms, an *account balance* (AB) approach where each deposit is added to the balance, and interest accumulates for the appropriate period until the next deposit is made. The second method uses what I have called an *individual payment* (IP) approach where all three payments are moved forward to T_8, and then their values are added together. The authors state that they will use this approach throughout the chapter.

In many financial mathematics texts, annuities are introduced by focusing on the behaviour of each payment using an IP approach. This produces a geometric progression, which is then summed and leads to the annuities formula. A fundamental assumption made in these texts is that a focus on individual payments is an obvious way to model annuities problems, and by implication, that this approach makes sense to students. While the text referred to above acknowledges that an AB approach may be more logical, this approach is rejected because it is "cumbersome". Although there is no doubt that an IP approach is more elegant and efficient than an AB approach (see chapter 12), an IP approach may not be an intuitive way for students to approach annuities problems at first.

Prior to the doctoral study, and based on my previous teaching of financial maths to pre-service secondary maths teachers, I had become increasingly convinced that an IP approach to solving annuities problems is not intuitive for students. In fact, in previous courses, when students were first given an annuities-based problem and left to choose an approach, they tended to choose an AB approach. Eddy and Swanson (1996) report similar anecdotal findings. Since I had only anecdotal evidence to support this observation, I sought to gather empirical evidence to explore my conjecture as part of the study.

In this chapter I show that there is evidence to support my conjecture that an IP approach is not a primary intuition. Furthermore, there is some evidence to suggest that the approach may not be immediately accepted by all students as a valid model of an annuity scenario. Despite this, my findings

suggest that once students accept an IP approach, they are soon able to reason about annuity-based scenarios using IP thinking, and thus an IP approach may be considered a secondary intuition (Fischbein, 1999). While it seems redundant to claim that teachers require knowledge of both approaches, I shall argue that working between the two approaches may be a powerful way of strengthening teachers' knowledge for teaching annuities, and for learning annuities. With respect to MfT more generally, I provide evidence of students' errors that are only revealed in their verbal explanations and not in their written work. I discuss the implication of this for teacher education.

As noted in chapter 5, the data I draw on comes from four consecutive two-hour sessions over an eight-day period. The research questions that guide this chapter are:

- To what extent do students make use of an IP approach in their initial attempts to solve annuities-based tasks?
- What obstacles do students experience in their initial engagement with an IP approach?
- What insights may be gained into mathematics-for-teaching annuities by investigating students' initial difficulties and errors when engaging with an IP approach?

My focus throughout the chapter is on students' thinking. I also include students' interactions that dealt with an IP approach, and on the contrast between AB and IP approaches. I do not deal with the different approaches students adopted to find a formula in group tutorial 3 (GT3), nor their problem-solving strategies, nor their talk about time and use of timelines, nor their use of spreadsheets. The latter two issues are dealt with in subsequent chapters. I do not attend to issues such as the nature of students' interactions, nor to whose ideas came to be privileged in the interactions.

This chapter is structured as follows: I begin by arguing that the eight-day period constitutes students' first substantial encounter with annuities and an IP approach, despite the fact that most students had been introduced to annuities in the first year of their studies. Thereafter I identify and elaborate six obstacles that various students encountered over the period in relation to an IP approach. This is followed by the discussion of a vignette from which I argue that the ability to move easily between AB and IP approaches is an important component of mathematics for teaching annuities. Finally I reflect on the implications of these findings for MfT of annuities, and teachers' mathematical knowledge more generally.

13.2 Students' first substantial engagement with an IP approach

The eight-day period in focus constitutes the first substantial engagement with annuities and an IP approach for the majority of students in the course - what Chevallard might term a "first encounter" (Barbe, et al., 2005). Technically speaking, this was not the students' first encounter with annuities. All students had some experience of annuities in their mathematical literacy course in first year. However, there was almost no evidence of students' drawing on annuities-type thinking in their responses to the following annuities question in the questionnaire:

> Say you take a loan of R4000 at a rate of 12% p.a. compounded monthly. You will pay off the loan in 2 years if you pay about R188 each month. How much must you pay each month if you want to pay off the loan in 18 months? Explain how you got your answer.

Based on responses from 35 students, there were only two who displayed some form of annuity thinking evidenced by adjusting the value of individual payments made at different times. The most common response was to move R4000 forward for 18 months at 12% p.a. compound interest, and then divide the final amount by 18. Another common response was based on proportional reasoning, and is best described as follows: "if 24 payments of R188 will pay off the loan in 2 years, then if there are only 18 payments, how much should they each be?" Neither of these approaches reflects an appreciation of the time-value of money.

13.3 Initial use of an IP approach

Based on the data from GT3[51], there were four groups that worked with an IP approach in some way in GT3. Two of these were Nosisi's group and Rachel's group. This is not surprising since Rachel had done financial maths as part of a maths course in another faculty before she changed to teaching, and Nosisi was repeating the course.

A third group completed the first question of GT3 using an AB approach and then using an IP approach, and showed that both methods produced the same answer. However, they were not able to use either method to produce the required formula. The fourth group to make use of IP thinking was Attiyah's group. Shaun initiated the possibility of thinking about each payment separately, and worked with Attiyah to explore this further. They abandoned the approach when they were unable to simplify the geometric series they had produced. This is discussed in more detail later in this chapter.

There were two other groups that produced geometric progressions but this was not an indication of an IP approach. Based on their reports, both groups used an AB approach, and the geometric progressions were the result of algebraic manipulation. Hailey's group was one of these groups and thus there is a video-record of their tutorial work. In the video footage there is no evidence to suggest that they worked with an IP approach at any stage. Their work is discussed in more detail later in the chapter.

With regard to my conjecture that IP approaches are not initially intuitive: I do not take Nosisi and Rachel's groups as disconfirming evidence of my conjecture, given their prior exposure to IP approaches and the assumption that this influenced the approach of their groups. However, the evidence from the other two groups that used an IP approach suggests that, for some students, IP approaches may be more intuitive than I had appreciated. Taken together, this data supports my claim that IP approaches may be considered a secondary intuition. The remaining two groups show clearly that the use of geometric progressions should not necessarily be considered evidence of IP thinking.

13.4 Obstacles and an IP approach

I identified six obstacles that students encountered with regard to an IP approach. In chapter 3, I distinguished different kinds of obstacles as *concerns*, *difficulties* and *errors*. In some cases it is difficult to distinguish between difficulties and errors because a difficulty may lead to an error or be manifested in an error. Also, a difficulty for one student may be an error for another. In such cases I have referred to the obstacle as a difficulty. The order of the obstacles in the list below reflects the order in which they are discussed in this chapter.

[51] See Appendix C4 for the handout for GT3.

1. *Error*: Use of compound interest formula to model an annuities scenario.
   Indicators: Use of compound interest formula, verbal or written evidence questioning what values to substitute for principal amount and number of compounding periods.

2. *Difficulty*: Modelling the timing of payments and compounding of interest in annuities scenarios.
   Indicators: Oral or written evidence, frequently manifested in calculations, that reflect an incorrect or contradictory order in the sequence of deposit and interest, for example, adding the deposit to the account balance before accumulating interest when payments are made at the end of the month; incorrect interpretations of the convention that payments made at the end of the month don't receive interest in that month.

3. *Difficulty*: Trying to make sense of an IP approach from the perspective of an AB approach.
   Indicators: Oral or written evidence, including calculations, containing contradictory aspects of AB and IP thinking such as choosing $n = 1$ to calculate the future value of the first payment on maturity.

4. *Difficulty*: Not recognising the need to apply the summation formula for a geometric progression.
   Indicators: Oral or written evidence, possibly indicating a geometric progression and then various inappropriate and/or unsuccessful attempts to simplify progression, or abandoning progression due to lack of strategy to simplify it.

5. *Concern*: That an IP approach does not provide a valid model for an annuities scenario, and that aspects of the model do not make sense from an everyday perspective.
   Indicators: Written or oral evidence directly questioning how an IP approach makes provision for some aspect of modelling the future value scenario, such as accumulating interest.

6. *Difficulty*: Meaning and use of the exponent, n, in the future value of annuities formulae – typically concerning whether n represents number of compounding periods or number of payments.
   Indicators: Oral or written evidence questioning the meaning of n or what value to use for n, incorrect statements about what n represents, incorrectly adjusting the value of n to deal with a missed payment.

The discussion of the obstacles is linked to a single vignette which contains the threads of several of the obstacles. The discussion is supplemented with further analysis of supporting evidence that either broadens the discussion or amplifies the obstacle. In each case the analysis is based on a transcript of students' interactions, with particular focus on their utterances. Following the discussion of the obstacles, I contrast the demands of working with AB and IP approaches based on vignette of H-group working on GT4.

13.5 Introducing an IP approach

The first vignette begins with Nosisi's explanation of her group's IP approach and the class's response. It constitutes the first substantial engagement with an IP approach for most students in the class. The students' responses to Nosisi and her group provide insight into their initial interpretations and concerns about the IP approach. Several of the issues raised by students

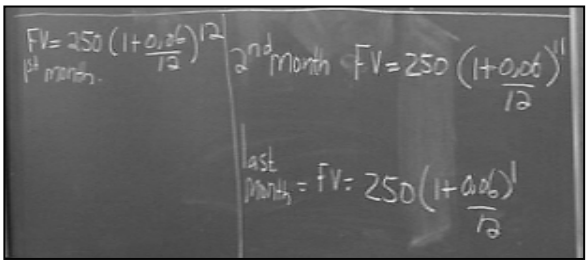


Fig. 13.1 Nosisi's boardwork

recurred over the eight-day period and, in some cases, further into the course.

Nosisi began by writing separate calculations for the first, second and last months, as shown alongside. The calculations represented the future value at the end of month 12 of each of the three payments. She stated that each payment was being treated separately; that each payment had a different number of compoundings as reflected by the different exponents; and that the twelve totals must be added together to get a total of R3099.43. In all three calculations the exponents are one unit too big but her group did not realise this at the time, and I did not correct them. The transcript below includes her initial explanation as well as the interaction that followed immediately thereafter.

Line	Speaker	Utterance
2	Nosisi:	Okay, what we decided, we decided to teat, treat each and every two-fifty (250) differently. We started with the first two-fifty (250) and which we said ok 'how many times is it gonna be compounded?' then the first two-fifty (250) is gonna be compounded twelve times and then we said the second two-fifty is gonna be compounded eleven times and so, and … and then the third two-fifty is gonna be compounded ten times until the last one that is gonna be compounded once. Then I didn't write all the story cos it's gonna be a long (trails off) … cos the last one is gonna be to the exponent one, that shows that it's compounded once, then we add all those things and we get (trails off).
3	Belinda:	What was your total?
4	Nosisi:	Three, zero, nine, nine, point four three (3099.43).
5	Craig:	(*At the back of the room*) Won't you just write it there.
6	Nosisi:	Okay, can I write it here… *(Writes on the board*: 'Total = 3099.43').
7	Craig:	Three zero nine nine point three (3099.3).
8	Jenny:	(*Correcting Craig*) Point four three. (i.e. 3099.43)
9	Craig:	Did other groups get, I mean, I know some of you got it. *(To himself*) 'did you get that?' I marked your work. (*To class*), um some other groups did get that, some other groups didn't get that. Okay. Has anybody got questions about this method that you want to clarify? *(waits approx. 20 seconds).* If this approach is different to yours have you written it down? Hint hint.
10	Joe:	Can we (unclear), the, the formula down? (unclear).
11	Craig:	Can you have your formula down? Not yet. Vusi, your hand was up?
12	Vusi:	Yes I wanted to ask um some questions. Guys what was your logic when you were working, work this (trails off)
13	A student:	Huh?
14	Vusi:	What was your logic, your starting point? How did you (trails off)
15	Joseph:	No, since like the, the first two-fifty (250) that you gonna put, during the thirty-first January is going to compound twelve times at the end of the year. So that's what we can see (unclear) and the last two-fifty (250) will compound just only once.
16	Lizbeth:	And one other thing that we did is like we found that each and every two-fifty (250) that we invest is getting an interest so that's why we treated, treated this thing for like each and every month.
17	Nosisi:	And the last, the last thing is that you deposited the two-fifty first and the other two-fifty is coming it cannot be compounded. It cannot have the compound, same compounding period as the first one, and this is where we had a problem. We cannot combine all these things because the other, the, the first one was two-fifty (250) and another two-fifty (250) is coming, then they are not equal (unclear) is taken on the same period (unclear) and we decided to use the exponent.
18	Craig:	Sifiso?
19	Sifiso:	(*In a questioning tone*)The other two-fifty (250) is for the next month or is for the same month that you, that you add …
20	Joseph:	No it …
21	Sifiso:	… cos … cos as far as I can see you deposit two-fifty (250) right? Then we get interest on the two-fifty (250). On the next month you add another two-fifty (250) on top of the interest that you already have.
22	Joseph:	No, which means for the first two-fifty (250) that is compounded for twelve months, it's money on, on one side. Then you put one, another two-fifty (250), compounded for another period which is eleven months and then you take it, you put it aside, and then other one is coming as well.
23	Sifiso:	Oh, so the two-fifty (250) compounds for twelve months, another two-fifty (250) for eleven months, ten, nine, eight.
24	Joseph:	Ja.
25	Lizbeth:	And then you get (unclear) the interest.

Line	Speaker	Utterance
26	Jefferson:	So what's the overall, the n, the n, when you add like? Because you're treating each and every, er, er, P differently. So when you add eleven plus twelve plus nine (11 + 12 + 9) plus … do you add them, all of them to, to, to get the total because that's what we did.
27	Lizbeth:	I think the overall n is twelve, but if we say for eleven, that, that n for eleven meaning that's the month … it stands for the month.
28	Jenny:	But it's not the eleventh month.
29	Jefferson:	Ja, it's not the eleventh month.
30	Various mumbling:	Ja. Eh. It's the second month. It's the second month
31	Joseph:	Ja, it's the second month.
32	Lizbeth:	Ja, it's the second month.
33	Jenny:	So then it doesn't stand for the month.

For many students, Nosisi's input constituted their first (substantial) encounter with an IP approach. It was not a faultless explanation – she made no explicit reference to the time of month that the payments were made, and her board work was ambiguous. For example, in writing *1st month*, *2nd month* and *last month*, she did not make clear that she was referring to the payments made in those months, and that FV referred to the future value of the payment at the end of the investment period and not to the account balance at the end of each of the specified months. Joseph's reference to making the first payment on the 31st of January [15] suggests they were aware that the question dealt with payments at the end of the month although their calculations model payments at the *beginning* of the month[52].

Despite these errors, the ideas presented by Nosisi and her group generated an extended class discussion, part of which is included in the above transcript. The extract contains the threads of five obstacles that emerged over the eight-day period: (1) attempting to use the compound interest formula to model annuities situations; (2) difficulties in working with the conventions of modelling the timing of payments and interest in annuities scenarios; (3) difficulties in making sense of IP thinking from an AB perspective; (4) not recognising the need to apply the summation formula for a geometric progression; and (5) concerns about the value and meaning of n. I discuss the first four issues below. I deal with the meaning and value of n later in the chapter.

13.6 Obstacle 1 – Attempting to use the compound interest formula as a model for annuities

In [17] Nosisi referred to different payments having different compounding periods "and this is where we had a problem. We cannot combine all these things …." I take this as recognition that the compound interest formula alone cannot model the annuity scenario. The video records of the two focus groups in GT3 indicate that Hailey and Shaun were the only members in their respective groups who understood the scenario sufficiently well to recognise from the outset that the compound interest formula could not be used to determine the total accumulated amount because payments were not made at the same time. Other students initially made use of the compound interest formula, although in slightly different ways. For example Attiyah combined all 12 payments of R250 and calculated the future value using the compound interest formula (i.e. R3000 $(1.005)^{12}$ = R3185.03). She assumed this represented the accumulated amount at the end of the year. Palesa compounded R250 for 12 months, and then wondered whether she could simply multiply this future value (R265.42) by 12 to

[52] A more careful analysis of Nosisi's talk and that of fellow group members, Joseph and Lizbeth, shows almost no explicit attention to time in their talk. In chapter 14, I contrast this with Rachel's attention to time in her explanation later in class session A.

represent the sum of the 12 separate payments. This is the equivalent calculation to Attiyah's but reflects subtle differences in reasoning. Lina initially calculated one month's interest on R250 and then multiplied this by 12, thus modelling a simple interest scenario. Therefore, *an important first step for students was to realise that the situation could not be modelled with the compound interest formula*.

While Shaun recognised that the compound interest formula was not a suitable model, he was not yet able to compare the results obtained from the compound interest formula with those from the iterative calculations for the 12 payments, without actually doing the calculations. He suggested that the iterative calculations might produce a larger answer than the total Attiyah had obtained:

> Shaun: I think it's gonna be more than if we just work out the compound interest for, d'ya know what I mean? If we say all the payments and work out the compound interest? I think it's going to be more if we work out the compound per month plus the two hundred and fifty, the compound for that month plus two hundred and fifty. I mean maybe it would be interesting to see if it does make a difference.

Shaun's comment reflects that he is not thinking about accumulating individual payments. This lends further support to my claim that this was students' first substantial encounter with an IP approach. Attiyah's calculation models 12 payments all gaining interest for 12 months. In the annuity scenario that Shaun is modelling by means of iterative calculations, there is only one payment that gains interest for 12 months, and each payment thereafter gains interest for one month less. Once one is familiar with IP thinking, it is difficult to imagine that students cannot see that Attiyah's answer will be larger than Shaun's. However, at this point the students were not thinking in this way.

13.7 Obstacle 2 - Modelling the timing of payments and compounding of interest in annuity scenarios

Initially I was astounded that in GT3 the calculations of nine of the ten groups modelled a scenario where payment is made at the *beginning* of the month, yet the task explicitly stated that payments were made at the *end* of the month. Some groups had written "end of month", but their calculations did not model this. In the transcript above, Joseph referred to making a payment on the "thirty-first of January" [15] and "compound[ing] twelve times at the end of the year", thus confirming that he was aware that payments are being made at the end of the month for a full year. Yet the calculations the group had done implied payment at the beginning of the month based on the exponents they had chosen.

After encountering this contradiction in several data sources, I came to the realisation that the problem was related to students' lack of knowledge of the ways in which simple annuities are modelled. In simple annuity models only two scenarios are considered: *payment in advance* modelled by the future value of an annuity due where the payment is made at the beginning of the period, and *payment in arrears* modelled by future value of an ordinary annuity where the payment is made at the end of the period. In the former case, the payment gains interest at the end of the period in which it is deposited while in the latter case the payment does not gain interest in the period in which it is deposited.

Although students did not appreciate these subtleties at the beginning of the eight-day period, their attention to time in their GT4 reports reflects that most students had come to appreciate them by the end of the eight-day period.

13.7.1 The different timings of payments

There is only a subtle difference in the calculations to distinguish payment at the beginning of the month from payment at the end of the month. The difference is in the starting point and how time is framed around the calculations, as shown in fig. 13.2.

Fig 13.2a shows a section of a time-strip sequence of alternating deposits and interest calculations. I shall assume this sequence of events is infinite in both directions in time. Assume a deposit of R250 is added to the account balance and interest is compounded on the new balance. Then another deposit is added, interest is compounded, and so on. This sequence is typical of the way in which students performed their iterative calculations in GT3, and reflects the underlying sequence of an AB approach. While students may have been aware that payments were made at the end of the month, their calculations modelled payments at the beginning of the month based on the way in which they imposed timeframes on their calculations.

Fig. 13.2 Different timings of payments

… deposit R250			… deposit R250				… deposit R250	
interest compounded			interest compounded			Month 1	interest compounded	0.00
deposit R250		Month 1	deposit R250	250.00			deposit R250	250.00
interest compounded			interest compounded	251.25		Month 2	interest compounded	251.25
deposit R250		Month 2	deposit R250	501.25			deposit R250	501.25
interest compounded			interest compounded	503.76		Month 3	interest compounded	503.76
deposit R250		Month 3	deposit R250	753.76			deposit R250	753.76
interest compounded			interest compounded	757.53		Month 4	interest compounded	757.53
deposit R250		Month 4	deposit R250	1007.53			deposit R250	1007.53
interest compounded			interest compounded	1012.57		Month 5	interest compounded	1012.57
deposit R250		Month 5	deposit R250	1262.57			deposit R250	1262.57
interest compounded			interest compounded	1268.88			interest compounded	1268.88
deposit R250			deposit R250				deposit R250	
interest compounded …			interest compounded …				interest compounded …	

Fig 13.2a Fig 13.2b Payment at beginning of month Fig 13.2c Payment at end of month

In fig. 13.2b and fig. 13.2c I have overlaid timeframes onto the time-strip of fig. 13.2a, and randomly chosen a five-month period. I have also included numeric quantities which represent regular monthly payments of R250 at an interest rate of 6% p.a. compounded monthly. The sequence of numbers in each figure represents the changing balance in the account. Bold numbers indicate the balance in the account at the end of each month.

In fig. 13.2b I have demarcated a *deposit-then-interest* unit in each month. Thus payment is made before interest is compounded in the month. This represents payment at the beginning of the month. By convention of the model, the order of deposit followed by interest implies that the deposit is made at the *beginning* of the month and interest is compounded at the *end* of the month. The column of numbers shows that each payment is added to the balance at the beginning of the month and interest accumulates on the new balance at the end of the month.

In fig. 13.2c I have demarcated an *interest-then-deposit* unit in each month. Thus payment is made after interest is compounded in the month. This represents payment made at the end of the month. However, in contrast to fig. 13.2b, both events take place at the *end* of the month. This may be a source of confusion for at least two reasons. Firstly, students may interpret the order to mean that

compounding of interest takes place at the *beginning* of the month since the first event in a payment-in-advance scenario (fig. 13.2b) takes place at the beginning of the month. However, a more likely problem is created by the paradox that two events are taking place simultaneously at the end of the month, which is clearly not possible. Given that banks compound interest at midnight on the last day of the month, any event following this in time must, by definition, take place at the beginning of the following month. Conversely, any deposit preceding the compounding of interest, cannot take place at the end of the month, in which case the money is in the account when interest is compounded. If the deposit is included in the compound interest calculation, then it models a situation where the payment was made at the *beginning* of the month, and yet the scenario being modelled is a payment at the end of the month.

The only way to overcome the paradox is by a convention that *in the model* payments made at the end of a period do not gain interest in that period. This does not resolve the paradox, it simply provides a standard by which financial institutions operate when modelling payments at the end of a period. It is important to note that the problem arises because of the models that are used. It does not exist in the daily reality of banking because banks calculate interest daily and compound it monthly. So a payment made on the last day of the month will accumulate interest for one day of that month. If the payment is made just after midnight at "month end", then it will accumulate interest from the first day of the next month.

Returning to fig. 13.2b and fig. 13.2c, it can be seen that the right-most column in both cases contains the same sequence of numbers but these numbers have different meanings in the two scenarios because of their positions in time. Consider month 2: In fig. 13.2b the opening balance for month 2 is R251.25, then R250 is deposited and the balance on which interest is calculated at the end of the month is R501.25. Thus the closing balance at the end of month 2 is R503.76.

In fig. 13.2c the opening balance for month 2 is R250.00. At the end of the month this is the balance that accrues interest, thus yielding R251.25. Then R250 is deposited and so the closing balance at the end of month 2 is R501.25.

13.7.2 Jefferson's interpretation of month-end activity

Jefferson showed evidence of struggling with the timing of interest payments described above. He expressed this in a journal entry and also during the group interview. The following journal entry was written on the same day as class session A, after the IP approach had been introduced:

> I think that what is happening is that from Jan when you put the deposit it will only accumulate the interest at the end of Jan which will be the last day of Jan at 12:01 am which is the beginning of Feb.
>
> (Jefferson, journal entry, 7/3/2008)

In the interview three days after class session A, he said:

> Jefferson: At the end of January, if you put the deposit on January, the end of January, the interest that will accumulate will be on the first of February, that's what I said and then actually when you, because they say that the interest will be starting from twelve o' clock, that will be the end of January, then the interest that will accumulate will be on the one past twelve (12:01am), that's when two-hundred-and-fifty-one-point-two-five (251.25) will be in your account on the first of February ... Does that make sense? ... Yes? No?

In his journal entry he did not state explicitly when the payment was being made. In the interview he was explicit about payment at the end of the month. However, he stated that interest on a payment made at the end of January accrues at the *beginning* of February. While this is true in reality, it is not

so in the model. Based on this comment, he appears to be aware that a deposit made at the end of a month first gains interest in the following month, but he assumed that the full month's interest was paid at the beginning of that month (12:01am). So, while he is aware that the January payment does not gain interest in January, the time gap he is working with is a few seconds on either side of midnight on the last day of the month. This suggests he had not recalled that interest is dependent on the number of midnights that an amount "lives" in the bank and which is modelled by a single interest calculation at the end of the month.

This error in his thinking appears to stem from trying to make sense of modelling the payment and the compounding of interest at the end of the month. Ironically, the error in his thinking may not be revealed in his calculations because by the end of month 2, the first deposit would still have a value of R251.25, irrespective of whether he added interest at the beginning or the end of the third month. So while his calculation might model payment at the *end* of the second month, he appears to be thinking that interest is compounded at the *beginning* of the second month. This will be problematic when he is explaining the process to others because his understanding of the model is incorrect, even if his calculations, by convention, may not reflect the error. Furthermore, fig. 13.2c may not be useful in helping Jefferson overcome the error in his thinking, and may even reinforce the error because the first event for each month is "compound interest". Jefferson needs to accept the convention that interest is always compounded at the end of a month, and that in fig. 13.2c both events take place at the end of the month whereas in fig. 13.2b, the deposit takes place at the beginning of the month and compounding at the end of the month.

13.8 Obstacle 3 - Difficulties to make sense of IP thinking from an AB perspective

Apart from difficulties with the conventions of timing in annuity models, I conjecture that the main difficulty students experience in making sense of an IP approach is attempting to do so from the perspective of an AB approach. While students should not abandon AB approaches, they need to recognise the distinct nature of each approach and therefore they need to break with the AB approach in order to make sense of an IP approach. The key shift they must make concerns the unit of focus - *from focusing on the account balance at the end of each period, to focusing on the value of each payment at the end of the term of the investment* (or at T_0 in the case of a present value scenario).

13.8.1 Evidence from the class's response to Nosisi

There were two instances of conflating the two approaches following immediately from Nosisi's explanation. The first is reflected in Sifiso's query in [19] when he questioned whether "you add another two-fifty on top of the interest that you already have". This suggests he is referring to an AB approach. The rest of his description fits an AB approach although it also makes sense in terms of what Nosisi had written on the board. Her first calculation (FV_1) indicated the first deposit and its interest. The second calculation (FV_2) referred to the second deposit and its interest. In his response, Joseph clarified the need to treat each payment separately when he referred to "[i]t's money on, on one side" [22]. Sifiso's response [23] suggests he then recognised they were separating the monthly payments and the compounding of those payments. This interaction suggests that it was not difficult for Sifiso to follow the reasoning behind an IP approach, but it required breaking from his AB thinking.

Later in the same session Patrick's interpretation of Nosisi's boardwork suggested he was also viewing her calculations from an AB perspective. He appeared to consider the answer to the first calculation as the future value (or account balance) at the end of first month. In the extract below he refers to Nosisi's calculation for the payment made in the first month, i.e. $FV = 250\left(1 + \frac{0{,}06}{1}\right)^{12}$

Patrick: *(In response to another student)*
Let me repeat, I'm saying that future value, the first one, the one that's underneath the first month *(points to work on board)*, if they take that future value as the total value for one month, but not for twelve months, they're right. But if that total balance, they say it's for, it's for all the twelve months, then they are wrong... *(class mumbling)*. Okay that twelve months doesn't mean that the balance they get there, is for twelve months, the balance they get there is for only one month, even if you put twelve there, you don't work out a twelve as the total balance for twelve months.

Patrick appears to consider the answer to the first calculation as the total for the first month despite the fact that Nosisi had used an exponent of 12. He is not seeing it as the future value of the first annuity payment at the end of 12 months. This may be based on template thinking (Sfard, 2000) from iterative calculations of compound interest and an AB approach to annuities where typical reasoning might be: "if the first calculation relates to the first month then it should have an exponent of one to indicate compounding for one month". From the context of a typical compound interest calculation, Nosisi's calculation covers a period of 12 months because n = 12 (and compounding is monthly). However, she has written "1st month" next to the calculation. Based on this, Patrick appears to be treating the calculation as the account balance at the end of the first month. He says "the balance they get is for only one month, even if you put twelve there". So, for him, the use of 12 in the exponent does not imply a future value after 12 months. He may be thinking that the 12 relates to the total number of payments in the scenario although we cannot be sure.

Patrick's errors may stem partly from Nosisi's boardwork which did not include some key aspects she emphasised verbally [2, 17]. For example, when writing "1st month", "2nd month" and "last month", she did not indicate in writing that she was referring to the deposit made in the respective month and to its growth. It appears that Patrick was not yet able to break from his AB thinking and so he appeared to be forcing a calculation based on an IP approach to fit with his AB thinking. This suggests evidence of template matching (Sfard, 2000) where he is attempting to force the new approach into an existing template.

13.8.2 Evidence from Attiyah and Shaun's work

In GT3 Shaun and Attiyah spent a short period investigating the behaviour of individual payments but they abandoned the approach when they could not produce an explicit formula. This incident shows that even though the two students had produced a geometric progression, they were unable to recruit knowledge of school mathematics, specifically geometric series, to take the next step. The incident also reflects an error in Attiyah's thinking, and I shall argue that this error stems from her use of an AB-framework to think about the separate payments.

After completing the iterative calculations, Shaun suggested that the formula might be connected to the fact that the first deposit got 12 month's interest, the second deposit got 11 month's interest, and so on. This showed that he could think about the payments separately. He explored this approach for a three-month scenario and got the same answer as the equivalent period for his AB approach. He then shared this idea with Attiyah, who worked on factorising the expressions. In their report they showed evidence of this work, as follows:

$$FV_2 = R250(1.005)[(1.005) + 1]$$
$$FV_3 = R250(1.005)[(1.005)^2 + (1.005) + 1]$$
$$FV_4 = R250(1.005)[(1.005)^3 + (1.005)^2 + (1.005) + 1]$$

(A-group, GT3 report)

The sequence shows the same common factor in each expression, followed by a square bracket with terms of the form 1.005^n. They recognised that in each month it was only the square bracket that was changing, and this appeared to involve the adding of a new term with a base of 1.005 and a higher exponent. They did not know how to simplify the square bracket.

When they started with this approach, they both recognised that larger exponents indicated that the payment would gain interest for more months. However, when working with the factorised version, Attiyah did not interpret the algebraic representation correctly. She made her thinking explicit during the group interview, as captured in the transcript below.

Attiyah had noticed that each month a new term was added to the square bracket. This term had a higher power which was one unit less than the month number. Her error was that she saw this term or "bracket of interest" [41] as the growth factor of the payment for the *new* month whereas it represented the interest on the *first* deposit and it was increasing each month because it had been in the account for a month longer. This thinking is revealed in the transcript below. In the right hand column I have indicated the written work she was referring to, and also provided some interpretation of her talk to assist the reader.

Line	Speaker	Utterance	Interpretation of Attiyah's talk
31	Attiyah:	… we said the future-value-three was future-value-one plus [future-value] two. No this was supposed to be four. So it was future-value-three plus two plus one.	$FV_3 = FV_1 + FV_2$ (+ FV_3 but doesn't say this loud) then changes to $FV_4 = FV_3 + FV_2 + FV_1$ (later changes back to FV_3)
32	Craig:	(to others) Just stop her if you don't get it or if you're not sure, okay?	
33	Attiyah:	This is supposed to be FV-four	
34	Craig:	So that's supposed to be FV-four, okay.	
35	Attiyah:	So it's three plus two plus one. *(she hesitates)*	Meaning $FV_4 = FV_3 + FV_2 + FV_1$
36	Craig:	Carry on. If nobody stops you just keep going 'cos I'll stop you.	
37	Attiyah:	No, actually it was right. It was three. I'm wrong. It was future-value-three because you get the interest for the third month, second month and first month. Then I noticed, you take the two-fifty (250) out common and the interest part common and you got left with that (*shows what she had written in her book*).	Reverts to $FV_3 = FV_1 + FV_2 + FV_3$ Sees each term above representing a full month where FV_n represents n^{th} month $250(1.005)[1.005^2 + 1.005^1 + 1]$
38	Craig:	So this is? What's this now? Two, what's that a two?	Referring to exponents of 1.005
39	Attiyah:	Two, one and then that would be plus one or it would be one point zero zero five (1.005) to the power of zero, which would give you one as well. Then we noticed, I noticed, with Shaun, that this was fine because you got that one standard, but then for each consecutive month you kept adding another bracket on. So it would be...	Refers to first 2 exponents $1.005^0 = 1$ "Bracket" seems to refer to (1+i) i.e. 1.005 something
40	Craig:	What kind of bracket?	
41	Attiyah:	In here it would be one comma zero zero five (1,005) to the power of three, two, one, zero. Four, three, two, one, zero. It would keep carrying on like that. You'd have to add that one, one more bracket of interest in, with the ascending power. So that couldn't get, and then we tried to condense it and it didn't condense any further.	For n=4 the square bracket will be: $1.005^3 + 1.005^2 + 1.005^1 + 1$ For n=5 the square bracket will be: $1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1$ Sees new term as the one with the highest power, not 1.005

At first Attiyah explained in terms of future values and her explanation was correct [31-37]. Technically her use of FV_n on both sides of the equality was problematic and it did not reflect her meaning, for example $FV_3 = FV_3 + FV_2 + FV_1$ but she did not recognise this and no-one else objected to it. She saw each future value as representing the amount gained in a particular month. For example FV_2 represented the second payment with its interest [37]. However, once this was represented in factorised algebraic form, she appeared to divorce it from the original process. Her interpretation of the algebraic form revealed her error. She was not seeing that because another month had passed, each payment had gained another month's interest and so the exponent of each term in the square bracket increased by one unit. Instead, she appeared to interpret the symbols at a visual level – focusing on the syntactic changes to each line (i.e. the introduction of a new term with a higher exponent). She therefore interpreted this new term to represent all the increase for that month. It is, in fact, the "1" in the square bracket that represents the new payment (because there is a common factor $(1 + i)$). In [41] she refers to "one more bracket of interest" which suggests she is still working with the process of multiplying the large bracket by $(1 + i)$. Shaun did not see her error during the tutorial and no-one else in the group commented on it in the interview.

Attiyah's error was typical of someone shifting between AB and IP thinking. She had not shifted her unit of focus from *account balance at month end*, to *value of payment at time n*. Given that she had been working from an AB perspective earlier in the tutorial, it was not surprising that she assumed the new term was the one with the highest exponent and reflected the amount gained in the month. It is also important to note that her error would not be picked up in her algebraic work because the algebra is correct and will produce the correct answer. The error is only visible in her talk. I shall return to this issue later in the chapter.

13.9 Obstacle 4 - Application of the summation formulae

The difficulties students experienced in producing an annuities formula was substantially impacted by the fact that they did not recognise the need to apply their school mathematics knowledge of summing a geometric progression. In [41] Attiyah referred to the fourth and fifth partial sums of the geometric series they had set up. But she and Shaun were unable to "condense" it further. There were three other groups that also identified geometric progressions in their work on GT3 but none of them recognised that the sum of a geometric series was the tool they needed to produce the kind of formula they sought.

In the interview with Hailey's group, Sakhile reflected on the work he had done during GT3 and the geometric series he had derived which is reproduced in fig. 13.3.

Let n = months

$\therefore$ 1 $\quad Pk = Pk^n$

2 $\quad Pk^n + Pk^{n-1}$

3 $\quad Pk^n + Pk^{n-1} + Pk^{n-2}$

…

12 $\quad P(k^n + k^{n-1} + k^{n-2} + k^{n-3} + k^{n-4} + k^{n-5} + k^{n-6} + k^{n-7} + k^{n-8} + k^{n-9} + k^{n-10} + k^{n-11})$

Fig. 13.3 Sakhile's work for finding annuities formula after GT3

Sakhile: … if you want me to find for a hundred month, I will tell you what will happen but it's gonna be like too many, like, so it's going to be a line that will take you forever to reach. So, but I was worried about the general formula, but now I had an idea that the pattern. I have sort of the behaviour of the pattern in doing this. But I was worried that this will not be twelve months, but what if it's twenty four months …

Sakhile had identified the general form of the geometric series and said he could describe the pattern for any number of terms ("for a hundred month"), but he was concerned that his expression for the series got longer as n increased, and he did not know how to reduce the expression.

It may be surprising that students did not recognise the sum of a geometric series as the next important step in solving their problem. This may suggest that their knowledge of the formula was not sufficiently usable to recruit in solving the problem they faced. For example after I had derived the formula for the future value of an ordinary annuity in class session B, Sandile commented:

Sandile: That formula like we know it, neh, … eh from high school but we didn't like know it, where does it come from, what does it, the meaning of it. Okay, we know that it is the sum of the all terms but we don't know where does it come from.

Sandile's comment reinforces my suggestion in the opening chapter that the experience of school mathematics for many of the students was one characterised by "rules without reasons" (Skemp, 1987). Students may have been given the formula without derivation and without discussion of the conditions under which it can be used. However, it is also important to point out that the students had not done any modules on sequences and series that typically form part of an introductory university algebra course. In fact, they took a course containing these modules in the semester following the Financial Maths course. If they had dealt with sequences and series earlier in their degree programme, their experiences and levels of success in GT3 may have been substantially different. The same may be true if I had intervened during GT3 and suggested the use of the summation formula.

13.10 Obstacle 5 - Concerns about the IP approach as a model

I discuss two different concerns raised by students. The first relates to the validity of the model and their confidence in it. The second involves a challenge to the model from the perspective of daily reality.

13.10.1 Challenging the validity of the IP approach as a valid model

Shortly after the section captured in the transcript of Nosisi's explanation, some students raised concerns about the validity of the IP approach as an appropriate model for the situation. This issue stems from students comparing it with an AB approach and so it is appropriate to discuss it here.

Jenny raised the initial concern about whether the approach provides for interest being compounded each month.

Jenny: I just want to know does this approach, I'm not sure, does it take into consideration, that, interest from the previous month has also been compounded, cos I don't think it is. Cos if you, if it's, if for month one that future value gives a total of two-hundred-and-fifty- rand (R250) at the end of the year and for month two it gives a two-hundred-and-fifty-rand (R250) compounded for eleven months it's not, for me, it's not taking into consideration that interest is compounded, that interest, of interest compounded.

Jenny's concern about the validity of the model most likely arose as part of her own struggle to make sense of an IP approach. She could not see evidence of the compounding of interest on the account balance each month and suspected it was not accounted for by the separate calculations. When working with an AB approach, one sees the interest accumulating on the entire account balance at the end of each month. But with an IP approach the total accumulating interest is split across the separate

payments, and their accumulating interest combines to give the total accumulating interest on the account.

Linked to Jenny's concern, Mpho suggested they should treat each payment as a separate investment and open a new account for each deposit. In this way it would be easy to track the interest on each deposit and to be clear on the value of n. This showed that he understood how the IP approach worked, recognised the problem in determining what value of n to use, but did not see the need to accumulate the money into one account.

The concern about an IP approach as a valid model remained an issue for two groups the following week during GT4. Both groups used the appropriate formulae to determine the future value of the lump sum and the annuity component of the investment. But it seems their thinking was still dominated by an AB approach because they were not convinced that the lump sum could be separated from the annuity. One group wrote the following comment in their report:

> "So now the question is how does the monthly diposit (sic) of 300 affect the initial deposit of 1000 so we decided to calculate interest accumulated in each and every months because it did not make sense to us that 1000 is compounde[d] alone and the 300's are compounded alone. **The 1000 had an effect on the interest accumulated at the end of each and every month.**"
>
> (Group 5, GT3 report)

The sentence in bold suggests they were using AB thinking. If one calculates the closing account balance at the end of each month, then the growing interest on the R1000 lump sum payment will impact every balance from the first month onwards. However, if the lump sum and the annuities components are separated, then the R1000 does not impact the growth of the R300 payments. This is what they were struggling to accept, and may suggest their lack of confidence in the model at this stage.

It appears that the essence of the students' difficulty is that *multiple* payments are being made into the *same* account at *different* times **but** the students want to focus on the overall balance at the end of each month. When one works with an AB approach, all the payments must be merged together to calculate the interest on the accumulated balance. So there are separate streams that flow into a single combined stream before interest can be calculated. Thus each payment impacts the overall interest calculation. With an IP approach, there is no merging process. Each deposit is treated as a separate stream flowing in parallel with the other streams. Each stream carries its own interest and thus the combined total consists of adding individual entries that bring their interest with them. This group wanted to account for the impact of the R1000 on the overall balance and did not appear to think about the situation in terms of parallel streams.

13.10.2 Challenging the model from the perspective of daily reality

A second concern regarding the model came from Jefferson after class session A, and appeared to be based on my explanation of the triangular spreadsheet where I emphasised that payments did not get interest in the month in which they were deposited. He questioned the purpose of including in the calculation the payment at the end of the last month since it would not accumulate any interest. In his journal entry he wrote:

> I think that the R250 which is placed on T_{12} (December) does not make sense because I think that the increase is done at the end of the month and when we looked at the account balance I found that we are adding R250 which is not part of the month because at the end of the 12 month that's when you are going to withdraw the money but not to deposit it, so that is why I thought that we only have eleven months.
>
> (Jefferson, journal entry, 7/3/2008)

Jefferson was challenging the model from the perspective of everyday reality. In the context of personal investments, one would not make a payment and immediately withdraw it. However, in modelling the annuity, one needs to work with full periods and full repetitions of cycles. For example, in a scenario where monthly deposits are made at the end of the month for 12 months, one must include the entire first month, the entire last month and all 12 payments. Jefferson's argument makes sense from an everyday perspective. However, from a modelling perspective, he was not appreciating the need for full cycles, i.e. 12 months of paying at the end of the month. He appeared to be including the period of time up to the point at which interest was compounded in the twelfth month, but excluding the twelfth payment. He concluded by saying that he thought there were only eleven months to consider, most likely because he was proposing that only eleven payments should be considered. However, the period still spans 12 months. Since payment is being made at the end of the month, time is measured from T_0 at the beginning of the first month. In this way, each payment P_n is made at the end of period, T_n. Thus P_{12} is made at the end of T_{12}. However, if P_{12} is not made for the reasons mentioned above, one still needs to consider the whole of the twelfth month because that is the period during which all other payments gain their last portion of interest.

13.11 Obstacle 6 - The meaning of n in the annuities formulae

One of the difficulties in making sense of an IP approach concerns a "shift" in the meaning of n. This appears to be one of the most persistent obstacles related to the annuities concept. In compound interest calculations, n represents the number of compounding periods whereas in the annuities formulae n represents the number of payments. Students have difficulty appreciating this shift because the annuities formulae are built from compound interest calculations. There is evidence that students also have difficulty in adjusting the value of n when there is a disruption of the perfect payment plan, such as a missed payment.

Students' concerns with the meaning of n emerged several times over the eight-day period. I begin with three examples, all of which relate to the meaning of n. I then discuss in depth possible reasons for students' difficulty in accepting that n represents number of payments rather than number of compounding periods. Later in the chapter I provide an example of student difficulties in adjusting the value of n for changes in the payment schedule.

13.11.1 Examples of students' difficulties with the meaning of n

The first example comes from Jefferson's inquiry to Nosisi's group about the "overall" value of n [26]. At that point the annuities formulae had not yet been derived but Nosisi had referred to 12 separate compound interest calculations each with a different value of n. Lizbeth appeared to be unsure. She thought the overall value of n was 12 but also suggested that n related to the months [27]. It is possible that she was focusing on the number of interest-gaining periods. Jenny immediately pointed out that an exponent of 11 was linked to the second month and so the exponent could not represent the month [28]. The class was not able to resolve the situation immediately and I did not address the issue of an "overall" value of n at this point but attempted to clarify the meaning of n in the individual calculations, explaining that it was linked to the number of months remaining until the end of the investment term, which was in turn linked to the number of opportunities for compounding.

The second example comes from Attiyah's error which was discussed earlier. Her error stemmed from her focus on the exponent. For her a term with a higher exponent was associated with the most recent payment. This likely stemmed from compound interest work where the month number is the same as

the exponent, as discussed earlier in this section. In her exploratory work with an IP approach, the exponent did indeed represent number of compoundings of each payment. But she appeared to be viewing the exponent from an AB perspective and was not yet thinking in terms of each payment being moved forward in time.

The last example is drawn from the beginning of class session B. Trevor raised the issue again as I talked with his group. He began by contrasting the compounding of a single payment with the multiple deposits of an annuity scenario.

Line	Speaker	Utterance
1	Trevor:	How can I put this? This is different to all the other compounding stuff we did previously.
2	Craig:	How's it different?
3	Trevor:	Here we are investing an amount, the same amount every month, for the same period and with other stuff, with the spreadsheets we invested an amount once and then that amount gained interest like … (*moves hands to show payment gaining interest several times*). So now when I look at the formula here, I want to try now and see what that *n* will be. If I put an *n* here *(in the formula)*, will it still mean number of compounding periods or will it mean something different 'cos if I put, like when I put eleven there I get this total.
4	Craig:	Okay.
5	Trevor:	When I put eleven there I get this total, and I know from (*pause*) and that's telling me that that *n* can't mean number of compounding periods but number of times the two-fifty (250) is being paid in. So I've been, I've just been asking them (*referring to the group*).

Trevor had substituted eleven into the annuities formula and got the answer he expected. This suggested to him that n did not represent the number of compounding periods but rather the number of deposits made. His group were not able to help him resolve his concern at that stage and I did not confirm that he was correct because the issue would arise in deriving the formula later in the session.

The meaning of n remained a source of difficulty for several students throughout the course. For example, in an interview six months after the course finished, Shaun, who had obtained one of highest marks for the course, was still unsure whether n represented number of payments or number of compounding periods.

13.11.2 Tracing the shift in the meaning of n

In this section I focus on the way in which the meaning of n changes from the compound interest formula to the annuities formulae. I have chosen the future value of an annuity due as a specific case. In tracing the way in which students initially use the compound interest formula, and then use it to derive the annuities formulae, I show why it is reasonable for students to think that n is related to compounding periods in both formulae. While this section might have been included in the previous chapter, I include it here since I did this work at a much later stage in the analysis process than the issues discussed in the previous chapter. This is work that I had to do for myself in attempting to account for students' difficulties regarding the meaning of n. It represents my attempts to unpack and connect the compound interest and annuity formulae in order to appreciate the source of the students' difficulties.

Consider the scenario of a single payment, P, gaining compound interest monthly at a monthly rate, i, for three months. Table 13.1 shows two different expressions for the future value at the end of each month. The exponent in the right-hand column represents the number of times the principal amount

has been compounded. In the simplified expression the month number, subscript and exponent all have the same value for any particular month.

End of month	Expression for process of compounding	Simplified expression
1	$FV_1 = P + Pi$	$FV_1 = P(1+i)$
2	$FV_2 = P(1+i) + P(1+i)\,i = P(1+i)[1+i]$	$FV_2 = P(1+i)^2$
3	$FV_3 = P(1+i)^2 + P(1+i)^2\,i = P(1+i)^2[1+i]$	$FV_3 = P(1+i)^3$

Table 13.1 Compounding of single payment

A similar pattern arises when working with annuities using an AB approach. Consider a scenario with three equal payments, P, made at the beginning of each month with monthly compounding of interest, again at a monthly rate, i. Table 13.2 shows the pattern that the subscript has the same value as the highest exponent in the factorised expression. This reinforces the association of n with number of compoundings because at the end of each month interest is compounded on the balance. Of course n also indicates the number of terms in the factorised expression but this is less obvious than the relationship between the subscript and the highest exponent.

End of month	Expression for making new payment and gaining interest	Factorised expression
1	$FV_1 = P + Pi$	$FV_1 = P[1+i]$
2	$FV_2 = (P[1+i] + P)(1+i)$	$FV_2 = P[(1+i)^2 + (1+i)]$
3	$FV_3 = (P[(1+i)^2 + (1+i)] + P)(1+i)$	$FV_3 = P[(1+i)^3 + (1+i)^2 + (1+i)]$ or $FV_3 = P(1+i)^3 + P(1+i)^2 + P(1+i)$

Table 13.2 Multiple payment using AB approach

Now consider the same annuities scenario with an IP approach. Table 13.3 indicates the accumulating interest on all three payments separately. Each future value represents the future value of a single payment as in table 13.1, but in contrast with table 13.2. All three payments are then combined to determine the future value of the annuity at the end of month 3. The symbol FV_3 has not been used again since it would be ambiguous.

Payment number	Value of payment at end of month 3 (T_3)
1	$FV_1 = P(1+i)^3$
2	$FV_2 = P(1+i)^2$
3	$FV_3 = P(1+i)$
	$FV_{ann} = P(1+i)^3 + P(1+i)^2 + P(1+i)$

Table 13.3 Multiple payment using IP approach

There is a subtle shift in the meaning of the exponent across the three tables. In the compound interest calculations in table 13.1, the exponent represents the number of times a payment *has gained interest since it was deposited*. In the IP calculations in table 13.3, the exponent represents the number of times the payment *will gain interest by the end of the term of the investment*, in this case the end of the third month. While the exponent refers to the number of compoundings in both cases, there is a subtle difference in what it represents. There is also a difference in the ordering of the exponents. In table 13.1 the exponents are ascending, while in table 13.3 they are descending. However, the sums of the three terms are equal.

In the AB approach in table 13.2, each exponent represents the number of times each payment has gained interest since being deposited – as in the compound interest formula. Yet, if this is factorised as shown in the table, the resulting expression is the same as that in table 13.3. However, students may

not consider the exponent to represent anything in particular. They may simply see the expression as the result of their algebraic manipulation. This is a subtle distinction which may not be immediately obvious. Even if they can see the individual payments, they will not have used IP thinking to generate their expressions for each period.

The shift comes when the annuities formulae are introduced. All simple annuity formulae develop from geometric series where each term represents the future value of an individual payment at some point T_n. Thus the expression for the future value of an annuity due at T_n is given by:

$$FV_n = P(1+i) + P(1+i)^2 + P(1+i)^3 + \cdots + P(1+i)^{n-1} + P(1+i)^n$$

In this expression all the exponents represent the number of times a particular payment will gain interest by time T_n. Thus, in this expression, n still represents number of compoundings. However, once the expression is manipulated to produce the formula, $FV_n = \frac{pmt[(1+i)^n-1)]}{i}(1+i)$, the meaning of n shifts. It then represents the number of payments since the annuities formula has emerged from the formula for the n^{th} partial sum of a geometric series where n represents the number of terms in the series. But this explanation is inadequate because it does not explain *how* the meaning of n shifts from compounding periods to number of payments. The line of argument depends on knowledge that the exponent in the formula for the sum of a geometric series, $S_n = \frac{a(r^n-1)}{r-1}$ $r \neq 1$ represents the number of terms in the series.

Ironically, even if one works from first principles to derive the above formula, it may be no more convincing that n represents number of terms. The typical elimination method is shown in fig. 13.4.

1	$FV_n = P(1+i) + P(1+i)^2 + P(1+i)^3 + \cdots + P(1+i)^{n-1} + P(1+i)^n$ …①
2	$(1+i)FV_n = P(1+i)^2 + P(1+i)^3 + \cdots + P(1+i)^{n-1} + P(1+i)^n + P(1+i)^{n+1}$ …②
3	Subtracting ① from ② produces: $((1+i)-1)FV_n = P(1+i)^{n+1} - P(1+i)$
4	Which leads to: $FV_n = \frac{P(1+i)[(1+i)^n-1)]}{i}$

Fig. 13.4 Elimination method to derive formula for future value of an annuity due

In lines 1 and 2 the exponents represent the number of times each payment will gain interest by time T_n. Lines 3 and 4 involve algebraic manipulation and the n which remains comes from the last term in line 2. Thus n is still associated with compoundings. However, in line 4 it must be accepted that n now represents number of terms. I argue that there is no reason to accept that the meaning of n in line 4 should be any different from its meaning in lines 1, 2 and 3.

In class session B, I derived the formula for future value of an ordinary annuity, and made use of the formula for sum of a geometric progression to transform the progression to an explicit formula. In that instance I required students to accept that n represents number of terms - based on its meaning in the formula for sum of a geometric progression, and because each term represented a deposit in the initial expression. Thus in the annuities formula, n represents number of payments.

It seems that the struggle to accept a shift in the meaning of n is a result of modelling. If the exponent had no contextual meaning in either formula, the problem would not exist. But since the exponent is

initially laden with experience as the number of compounding periods, it is very difficult to conceive of it as number of payments. The derivations of the annuities formulae do not provide adequate explanation for a shift in the meaning on n. It must simply be accepted. The challenge for teachers is not to know *that* the meaning has shifted but to know *why* and *how* it has shifted, regardless of whether they will have to provide an explanation to learners. This challenge thus may arise if students are asked how they know that the n in the formula for sum of a geometric progression represents the number of terms. The challenge of the shift also arises if one derives the formula from first principles as shown above. This occurred for H-group after they had derived their formula. In the interview Sakhile concluded that n represented the number of compoundings in the formula they had derived:

> Sakhile: We could see now that err, the exponents is equal to the number of compounding periods. So now we need to say that exponent, it means, it's represented by n. So n will be like the number of compounding periods, of months, and then this was our general formula.

The above discussion suggests that whether or not students have access to the derivation of the annuities formulae, they may still struggle to accept that n represents the number of payments made, and not the number of compounding periods.

13.12 Contrasting the demands of working with AB and IP approaches

Teachers require knowledge of working with both AB and IP approaches. In the second vignette, I show evidence of how AB and IP approaches may require substantially different thinking when there is a departure from the perfect payment plan. The incident is drawn from Hailey's group as they worked on the missed payment aspect of GT4. It highlights how a subtle yet critical aspect in an AB approach does not even need to be considered in an IP approach. It also highlights students' difficulties with the meaning of n in the annuities formula.

The vignette begins with Lina explaining how she had used the future value annuity formula as part of her strategy and then resorted to an AB approach for the rest of the question. Sakhile had attempted to establish a sophisticated formula to deal with the disruption and was attempting to adjust the formula for future value of an annuity due to deal with the missed payment. Hailey was interacting with both of them.

Line	Speaker	Utterance
306	Hailey:	Okay, but let's get back to this now. Okay, so fourteen, just quickly (*reading question on handout*) "Assume you are unable to make the fourteenth payment. How will this affect the total that you accumulate?" How will it affect it?
307	Lina:	Umm, umm, Because I didn't think April or May or July …
308	Hailey:	(*interrupting and agreeing*) Ja, let's just go with …
309	Lina:	… but I think it doesn't matter, it can start any month but only fourteen months, so just I calculated the end of the thirteen month.
310	Hailey:	Ummhmm
311	Lina:	Ja I use the formula I did, we did yesterday, last time. Then I, so the period will be thirteen months. I don't know, I'm right or wrong. Then I think this is the payment.
312	Jefferson:	For the seventeenth?
313	Hailey:	That's the total after thirteen months.
314	Lina:	Thirteen months
315	Jefferson:	Is this thirteen or seventeen? (*referring to the exponent Lina has written*)
316	Lina:	Thirt… (*hesitates*)
317	Hailey:	Thirteen.
318	Lina:	No, this is n, so this is thirteen. I, I'm, don't know.
319	Sakhile:	(*to Hailey and Lina*) Which one are you answering now?
320	Hailey:	(*to Lina*) No, I agree with what you're doing. (*to Sakhile*) um, we, we number two and number

Line	Speaker	Utterance
		three, kind of go together.. .
321	Sakhile:	Okay, cos
322	Hailey:	But I mean, basically the amount's gonna decrease if you miss a, a payment.
323	Jefferson:	Why, why?
324	Sakhile:	Yes, like . . .
325	Hailey:	. . . because if you miss a payment, you're not gonna have that extra three hundred rand plus the interest that is accumulated.
326	Jefferson:	(*speaking simultaneously with Hailey*) Three hundred plus the interest yes
327	Lina:	Then for (*sighs*)… I, I'm not sure, because if we got the answer for the number two, which is the amount end of thirteen month then we miss the fourteenth month, then we double the amount in the fifteenth month, so which means we put a six hundred in fifteen month. So I just use amount plus six hundred times the interest …
328	Hailey:	… interest
329	Lina:	… then it says end of the fifteen month, then I plus . . .
330	Hailey:	(interrupts) Okay, wait you didn't calculate the fourteenth month though.
331	Lina:	Because fourteenth month is missed, is it …
332	Hailey:	But you still get interest in your fourteenth month.
333	Jefferson:	How?
334	Lina:	(Inaudible)
335	Sakhile:	I didn't get like the thirteenth month?
336	Jefferson:	How are you going to get fourteen?
337	Sakhile:	How do you calculate it?
338	Jefferson:	You going to get interest for the, for the previous . . .
339	Hailey:	Yes, cos look …
340	Lina:	Oh (*appears to recognise the problem and follows what Hailey is saying*)
341	Hailey:	(*points to a drawing of triangular spreadsheet she had made earlier in the tutorial but cannot see details because her hand obscures what she is pointing to*) Cos you're going to get interest here for that and interest here for that, but you're not going to have anything here because you haven't made a payment. Then in your fifteenth month you're going to make a payment and you're gonna have that plus interest and those are gonna get interest and those are gonna get interest. So it's almost like you skipped out a little line here, and that's all.
342	Lina:	So we still have interest for the fourteenth month but we don't have the interest for the three hundred?
343	Hailey:	Yes.
344	Lina:	Oh, ja, ja. Okay, I understand.
345	Jefferson:	Oh, ja.
346	Hailey:	So any other thoughts on it?
347	Lina:	So, so, …
348	Hailey:	(*to Lina*) Okay, so you did it manually?
349	Lina:	Ja, because I don't know the formula (*giggles*)
350	Hailey:	(*to Lina*) No, no, it's fine. (*to Jefferson*) How did you do it? How did you work with it?
351	Jefferson:	How, what, what I did, I was trying like I was trying to subtract the interest on the fourteenth, month from the seventeen …
352	Hailey:	Ummhmm.
353	Jefferson:	…and, but I've seen that, but it give me like too much amount. Maybe I didn't divide by twelve for using this …
354	Hailey:	Okay.
355	Jefferson:	… but that's why my answer is so much
356	Hailey:	But you were trying to subtract the interest.
357	Jefferson:	Ja, the interest.
358	Hailey:	Okay, (*to Sakhile*) and what were you trying to do, Sakhile?
359	Sakhile:	Okay, I just, uhhh, I changed it, the like okay let's say I was treating, like let's say, not the eighteen months now I was talking about seventeen months now, I like if you have to … I, I said okay I know the thousand rand is not affected, if it's a thousand rand for eighteen months plus, then I changed, but, what I did , I changed the, the power, the n instead of to the power seventeen, to the power sixteen.
360	Hailey:	Ummhmm. Why?
361	Sakhile:	Because I was thinking that fourteen is the one that … it affected so which means I have to, sub-, like subtract, take it out, not include it in the formula.
362	Hailey:	Okay (*hesitantly*)
363	Sakhile:	I think there's something I didn't notice here …

Line	Speaker	Utterance
364	Jefferson:	Ja, maybe you know ...
365	Sakhile:	... eh ... like ... I didn't think of fourteen ... cos fourteen here it, if it was included, it, it accumulate, I think five times. So from, from like fourteen to reach the eighteen. It's fourteen, then from fifteen, sixteen, seventeen, eighteen,(*Hailey counting with him*) so five times. So I, I had to say maybe, sort of, subtract that part, maybe subtract eh, using that compound, subtract PV into, maybe subtract three-hundred into one plus that interest to the power of five, cos that was going to be included in that.
366	Hailey:	Okay.
367	Sakhile:	Yes, but now I see that I have to sort out that error.

This discussion highlights important differences in thinking when approaching the missed and double payment problem from AB and IP approaches.

Lina used the formula for future value of an annuity due to calculate the balance at the end of the thirteenth month [311] thus separating the thirteen R300 payments that were unaffected by the changes to the perfect payment plan. She then resorted to an AB approach for the rest of the calculations because she did not have a formula to deal with the adjustments [327, 349]. Since there was no payment in the fourteenth month, she ignored it and doubled the payment for the fifteenth month thus adding $\mathrm{R}600(1+i)$. From the group's written report, it can be seen that she then proceeded with the remaining months, adding a new monthly payment of R300 each time and calculating the interest on the account balance at the end of each month.

Lina had not considered the interest that would accumulate on the account balance at the end of the fourteenth month [330]. It seems that Jefferson and Sakhile had also not spotted this error in her thinking [333-337]. Hailey's explanation [341] is difficult to follow in the transcript because she does not name explicitly the aspects of her written work that she is pointing to, and this detail cannot be seen in the video records. However, the students appear to follow her explanation [344, 345]. She does not appear to be using AB thinking because she appears to be referring to clusters of payments "those" and "these" that will get interest. What is important here is that, with an AB approach, the interest on the account balance at the end of the fourteenth month must be *added* to update the account balance. By contrast, when working with an IP approach, the fourteenth payment and its interest must be *subtracted*.

When using an AB approach, one must deal with monthly interest accumulating on the balance irrespective of whether a payment is made in that month. This problem does not arise when taking an IP approach. In this case, the initial calculation is to determine the accumulated amount for the perfect payment plan. Thereafter the fourteenth payment is deducted together with the interest it would have accumulated in the months until the end of the term of the investment. In this case, it is not necessary to consider separately the interest accumulating at the end of the fourteenth month since this is taken care of by the interest gained on each of the first 13 payments during the fourteenth month. The interest is thus contained in the separate compound interest calculations for each of those payments and then embedded in the calculation made with the formula for future value of an annuity due.

The deducting of the interest on the fourteenth payment is necessary in an IP approach because the perfect payment plan assumes that the payment has been made and moves all amounts forward to the end of the last period. By contrast, in an AB approach, the fourteenth payment is not made at all and hence there is no need to consider interest on that payment.

Another difference that does not arise in this discussion, but which did appear in some group reports, concerns the double payment. When working with an AB approach, one needs to add in a double payment of R600 as Lina did. This is because one is building up from zero. When working with an IP approach, one only adds R300 (not R600) because the original R300 of the fifteenth payment still remains from the perfect payment plan.

So, depending on whether one uses an AB or an IP approach, one may be adding rather than subtracting for the fourteenth payment and one might be adding R300 or R600 in dealing with the double payment in the fifteenth month. Teachers need to be aware of how the different approaches will impact the strategies and calculations that need to be performed. Hailey appears to be able to shift between AB and IP thinking as she listens to Lina and then to Sakhile and Jefferson, although her hesitation in [362] suggests that either she did not follow Sakhile's explanation or she did not agree with this strategy.

In referring to the missed payment, Hailey notes: "it's almost like you skipped out a little line here" [341], indicating that she recognised that missing a payment was equivalent to a blank line or a row of zeroes in the triangular spreadsheet. This is illustrated in the spreadsheets below which represent an adapted problem in order to save space. Assume the missed payment occurs in the fourth month, and the double payment follows in the fifth month. In order to represent this in fig. 13.5a, one must indicate a zero payment for month four. The *Interest* column indicates that R5.81 interest accumulates on the balance at the end of the fourth month. The double payment is indicated in the next row. The total amount accumulated at the end of the sixth month is R1838.65.

Time (T_n)	Opening balance	Payment	Balance on which interest is calculated	Interest	Capital balance
1	0.00	300.00	300.00	1.91	301.91
2	301.91	300.00	601.91	3.84	605.75
3	605.75	300.00	905.75	5.77	911.52
4	911.52	0.00	911.52	5.81	917.33
5	917.33	600.00	1 517.33	9.67	1 527.01
6	1 527.01	300.00	1 827.01	11.65	1 838.65

Fig.13.5a Spreadsheet showing AB approach for missed and double payment

Jan	Feb	Mar	Apr	May	Jun	End June
300.00	301.91	303.84	305.77	307.72	309.69	311.66
	300.00	301.91	303.84	305.77	307.72	309.69
		300.00	301.91	303.84	305.77	307.72
			0.00	0.00	0.00	0.00
				600.00	603.83	607.67
					300.00	301.91
Balance						1 838.65

Fig.13.5b Spreadsheet showing IP approach for missed and double payment

In fig 13.5b, all the numbers in the cells refer to values at the beginning of each month. Here the missed payment is also represented by inserting a zero in month four. This results in zeroes in each cell of the row because there is no deposit to gain interest. The double payment is indicated in the next row and the interest amounts in each cell will also be double. It is important to note that even though

the double payment is indicated in this spreadsheet, the additional payment will only be R300 as noted above. An extra column has been added to determine values of payments at the end of the sixth month. As can be seen, the total value of the six payments is also R1838.65.

Hailey demonstrates an ability to move easily between AB and IP approaches and to recognise what aspects change and how they change in the missed- and double-payment scenario. The juxtaposition of the AB and IP approaches may promote deeper insight into each. For example, the different ways in which the fourteenth and fifteenth payments are dealt with in the two approaches requires one to understand what differs across the two approaches and how these aspects differ, and yet produce the same final answers.

Towards the end of the transcript there is evidence that Sakhile is correctly viewing the exponent as the number of payments. However, there were errors in the way he manipulated the exponent to deal with the missed payment in GT4. In order to deal with the missed payment he reduced the value of n from 17 to 16 since there was one less payment [359-367]. The problem is that reducing the value of n does not model the missed payment appropriately. In this case, by changing n to 16 he was calculating the future value of only 16 payments at the end of the sixteenth period. It seems that through his interaction with the group, he recognised that his strategy of adjusting n would not produce the correct answer and that he would need to subtract the fourteenth payment and its interest [365,367].

13.13 Summarising - the obstacles students encountered

Much of this chapter has been devoted to the obstacles that students faced in relation to an IP approach over the eight-day period. These obstacles are both a consequence of the course and constitutive of the course. That is, they arise from decisions I made regarding tasks, activity and pedagogy. Simultaneously they partially constitute the course because many of the students' responses to tasks were incorporated into sessions and engaged with in depth. I chose to introduce annuities by means of a realistic task in GT3, where students were expected to model the given scenario and then to abstract the appropriate mathematics, ultimately leading to a formula for the future value of an annuity. I assumed students would be able to draw on the annuities work they had done in their mathematical literacy course in first year. I have shown that this assumption was not valid. While six of the ten groups worked with a geometric progression at some stage during the tutorial, none of the groups was able to produce a formula within the two-hour tutorial period, and only one group was able to derive the formula themselves after engaging further with the task. The common obstacle was an inability to find a means of moving from the geometric sequence to an explicit formula. The tool they lacked was the formula for the sum of a geometric progression. In class session two, after deriving the annuity formula, some students commented that, although they recognised the formula for the sum of a geometric progression, they did not realise it was the tool they were seeking because they did not know how and when to apply it.

Three important concerns were raised. The first was to question whether an IP approach provided a suitable model for an annuity situation. The central concern was that the students could not see how the monthly interest was accumulating on the balance. The second concern related to the final deposit in a future value scenario where payments are made at the end of the period. Jefferson questioned the purpose of making the final payment at the end of the investment period, when there was no time for it to gain interest. The third and most important concern relates to the meaning of n in the annuities

formula. Students need to recognise that in the compound interest formula, n refers to number of compounding periods whereas in the annuities formulae it represents the number of payments.

There were three main difficulties. The first involves making sense of the timing of payments and interest calculations when modelling annuities scenarios. I have discussed in detail the paradox that emerges in modelling annuities scenarios with payments at the end of the period. Students need to accept the convention that payments made at the end of the period do not gain interest in the period in which they are deposited.

The second difficulty, which tends to lead to errors, is students' attempts to make sense of an IP approach from an AB perspective. I have shown examples of students' talk which suggests they are trying to fit an IP approach into their AB thinking. A typical example of such an error involves the incorrect use of exponents in the individual compound interest calculations.

The third difficulty (which may better be described more generally as an obstacle) was students' lack of knowledge of the summation formula for geometric progressions. While they recognised the formula when it was discussed in class, they had not been aware that it was the resource they required to produce an explicit formula.

With regard to errors, I noted that there is evidence in both focus groups and in the report of one other group that some students initially attempted to use the compound interest formula to model the annuities scenario. I also reported on various errors emerging from students' difficulties such as the incorrect sequence of calculations to model payment at the end of the period. Other errors involved incorrect values for n, such as Sakhile's error in dealing with the missed payment.

13.14 What opportunities emerge for learning MfT of annuities?

In this section I return to the MfT framework to reflect on what might be learned about mathematics-for-teaching annuities from the obstacles that students encountered regarding an IP approach. I have already summarised the obstacles encountered by students and so I shall not repeat them here under *learners' conceptions*. It seems reasonable to assume that learners in schools may face similar obstacles to the pre-service teachers if they are given the opportunity to grapple with the IP approach. I focus below on the following aspects of the framework: *essential features, modelling and applications, different teaching sequences and approaches,* and *explanations*.

13.14.1 Essential features

Annuity scenarios, in their simplest form, are characterised by equal payments made at regular intervals. Typically, annuities problems involve moving individual payments forwards or backwards in time, and then summing the adjusted values of the separate payments. This is the essence of an IP approach. The underlying concept is that the nominal value of one payment cannot be added to the nominal value of another payment made at a different point in time. Their respective values must be adjusted for the effects of time by compounding or discounting. In this chapter I have not dealt with the extent to which students demonstrated explicit knowledge of the time-value of money. An ability to solve annuities problems does not necessarily reflect knowledge of the time-value of money, at least in a way that students could articulate. Rather I suggest that such knowledge is implicit and needs to be brought into focus so that students can harness it as a resource.

13.14.2 Modelling and applications

I separate the components in this discussion.

Modelling – A key element of making the shift from an AB approach to an IP approach is shifting the unit of focus from *account balance at the end of the period* to *value of payment at* T_n. I have discussed at length the problem of dealing with payments and compounding of interest at the end of the month. It is important to note that this is a consequence of modelling and does not arise in actual banking practice because in banks interest is calculated daily and compounded monthly. It is also worth noting that several students did not recognise immediately that the compound interest formula is not an appropriate model of an annuity scenario.

Applications – Students did not recognise the annuity scenario as an application of the summation formula for geometric progressions. While they appeared to know how to work with the formula to substitute appropriate values, they did not know that it would resolve their struggle. According to some students this was a consequence of not knowing how the formula arose and where it could be applied.

13.14.3 Different teaching sequences and approaches

This aspect lies at the heart of the chapter since I am challenging the wisdom of an IP approach as the initial approach to annuities. I have described how annuities were approached in the course through realistic contextualised problems. In the light of this approach, students' responses and strategies suggest that IP approaches are not primary intuitions for all students, and that AB approaches should be given attention before promoting IP approaches. However, I acknowledge that this is likely to be more time-consuming in the school mathematics context. As previously noted, if annuities are introduced strictly as an application of geometric series, then they can only be introduced at Grade 12 level within the current curriculum structure. However, if an AB approach is used, annuities can be introduced in earlier grades in preparation for an IP approach in Grade 12.

13.14.4 Explanations

Jefferson and Attiyah's explanations revealed errors in their thinking which were not visible in their written work. In Jefferson's case the error related to the timing of payments and compounding of interest. In Attiyah's case the error concerned the connection between payments and terms in the geometric series. Since all terms in the series are added, the error in her thinking is not revealed in the calculations because the sums are equal. In part 1, a similar incident arose with Vingin and Naasiha as they worked with the compound interest formula. This reinforces my earlier call for opportunities for student teachers to explain their ideas – both verbally and in written form. There is evidence now that this applies both to their knowledge of new mathematics and school mathematics.

13.14.5 Contextual knowledge of finance

Earlier in the chapter I noted that Hailey and Shaun were the only students in the focus groups who understood the scenario of GT3 sufficiently well to appreciate that the compound interest formula was not an appropriate model. They both resorted to a "first principles" approach of alternating payments and compounding interest, as shown in fig. 13.2. While one might assume that all students would have this knowledge, it was not the case. It should be noted that the payment-interest cycle is a simplified version of what takes place in banks in that it ignores the daily calculation of interest. In order to develop models for annuity scenarios it is necessary to suspend this knowledge of banking practices.

13.15 Conclusion

I set out to explore my conjecture that an IP approach is not intuitive. In this chapter I have provided evidence of students' concerns, difficulties and errors with regard to the first encounter with an IP approach. The data shows that two groups employed IP thinking, most likely the result of a group member in each case who had previous experience of the approach. A further two groups made use of AB and IP thinking without prior exposure to the latter. The remaining six groups worked only with an AB approach. There is much evidence of students' difficulties in making sense of an IP approach from an AB perspective. This suggests that while they need to see the two approaches as distinct, it may not be easy to do so at first. Furthermore, while some students had made sense of the IP approach, they did not immediately accept it because they were concerned about its validity as an appropriate model. It could be argued that these two issues emerged as a direct result of students working with an AB approach at first. Nevertheless, the data points to the fact that an IP approach may become a secondary intuition since students readily make use of it once they have accepted it as a viable model.

If one wishes to promote mathematical modelling, then the evidence presented here raises an important challenge to approaches that introduce annuities as an application of geometric series. Geometric progressions are merely mathematical models of how annuities work in the real world and thus students need to recognise that the model does not reflect the complexity of reality and, in fact, leads to a paradox when dealing with the simultaneous events of making a payment and compounding interest at the end of a period. However, I also concede that within the confines of the South African school curriculum a modelling approach may be difficult to implement. Nevertheless, teachers require knowledge of how annuities work in the financial world, the ways in which geometric progressions both simplify and distort the phenomenon that is being modelled, and ways of moving between the financial context and the mathematical model.

Aspects of MfT in focus in this chapter

Essential features
- Derivation of future value formula

Modelling and applications
- Use of AB approach (primary intuition) contrasted with IP approach (secondary intuition)
- Importance of shifting unit of focus from account balance at end of period to value of payment at T_n

Explanations
- Students' erroneous thinking revealed through their explanations but not in their written work

Different teaching sequences and approaches
- Challenging standard practice of introducing annuities as an application of geometric progressions

Learners' conceptions
- Recognising that compound interest formula is not an appropriate model
- Questioning whether an IP approach is a valid model
- Questioning the need to make last deposit at end of period if it does not accumulate interest
- Difficulties in making sense of IP approach from AB perspective
- Problem of blurring of boundary between end of period and beginning of the following period
- Meaning of n in annuities formulae
- Inadequate pre-knowledge of summing of geometric progressions

Contextual knowledge of finance
- Rudimentary knowledge of payment-interest (or interest-payment) cycles

CHAPTER 14
Use of timelines and talk about time

14.1 Introduction

An individual payment (IP) approach requires careful attention to time. When working with an account balance (AB) approach, one focuses on the alternating pattern of deposits/withdrawals and adding interest to the balance. Since simple models of annuities consider only the cases where payments are made at the beginning or the end of a period, the modelling of the timing of payments is determined by whether interest is added to the balance after or before the payment is added. Consequently, there is little need for a timeline to represent the pattern of events.

By contrast, in an IP approach each payment must be considered in terms of its starting and end points. The order of the sequence of paying interest and making a deposit/withdrawal is not a factor to be considered. Instead, one must pay attention to the number of times each deposit will gain interest, or the number of times each loan repayment must be discounted back to T_0. This requires explicit attention to time.

This chapter has its origins in the group tutorial reports. Some of my early data analysis focused on the reports of GT3 and GT4 for all ten groups[53]. In those reports I noticed a shift in students' attention to time and their use of timelines. For example, in GT3 only one group included a timeline in their report while five groups included timelines in their GT4 reports. In GT3 only four groups made explicit references to the timing of payments at the beginning/end of the period whereas seven groups did so in GT4. However, when I analysed the video records of the focus groups working on GT3 and GT4, their talk about time varied at different points in the tutorial, and this was obviously not reflected in their written reports. It became clear that the group tutorial reports were not necessarily a good indicator of students' attention to time. I therefore decided to restrict the analysis in this chapter to the two focus groups.

In this chapter I focus first on students' attention to time in their talk, with particular reference to the eight-day period during which the IP approach was introduced. In the second part of the chapter I focus specifically on the use of timelines over the same period, and I extend this analysis to students' use of timelines in one question of Test 1. The research questions that frame this chapter are:

- To what extent do students pay particular attention to time in their talk?
- To what extent do students adopt the conventions of timelines, and with what effects?
- What errors do students make when working with timelines?
- What insights for MfT of annuities can be gained from students' errors and difficulties regarding talk about time and use of timelines?

[53] See Appendix C4 and C5 for the handouts for GT3 and GT4 respectively.

14.2 Students' talk about time in the group tutorial setting

In GT3 both groups spent most of the tutorial struggling to find a formula. As a result, time aspects were seldom in focus. There is evidence of vague talk about time but also some evidence of explicit attention to time. In GT4 there is more explicit attention to time which is likely a consequence of the content of the task (e.g. the need to deal with the number of times the fourteenth payment would have accumulated interest), as well as the attention given in the class sessions to timing of payments. Below I give one typical example where students were not explicit about time, and then I discuss two examples where there is explicit attention to time in their talk.

14.2.1 Lack of attention to time in students' talk

The extract below comes from Attiyah's group working on GT3. Shaun is explaining that they cannot use the compound interest formula for the problem because a new payment is being made every month [30]. He ends by describing the sequence of calculations for the twelve month period.

Line	Speaker	Utterance
21	Attiyah:	Can I see what you did? (*looking at Shaun's work*) Two-hundred-and …..
22	Shaun:	fifty compounded once.
23	Attiyah:	Why to the exponent of one?
24	Shaun:	Because in the first month all that you gonna get will …
25	Attiyah:	(*Interrupts*) Oh! You're working out, er, monthly.
26	Shaun:	Ja.
27	Attiyah:	But now why don't you just work it out for twelve compounding periods?
28	Shaun:	Because it's gonna be more than that. It's gonna be more than that.
29	Attiyah:	No but isn't compound interest, if you're working out for the year …?
30	Shaun:	But you're putting in, plu, we're adding two-hundred-and-fifty every month. It's not just two-hundred-and-fifty in the beginning. (*reading handout*) Ja, it's not just two-hundred-and-fifty in the beginning …
31	Attiyah:	Oh yes …
32	Shaun:	… we're plus-ing….
33	Attiyah:	… you're plus, you're adding two, your two-hundred-and-fifty in each month and you're getting interest on that amount, on that …
34	Shaun:	… each month, ja …
35	Palesa:	So if you multiply by twelve would it make a difference? What would that mean?
36	Shaun:	Well that's what I want to see. Well I don't know if it would make a difference, or I'd, I'd presume it would make a difference but I don't know. That's what I was working out, that's why … (unclear)
37	Vusi:	Shaun, how did you work it?
38	Shaun:	Well I worked out the compound interest for two-hundred-and-fifty, compounded it once, then I plus-ed two-hundred-and-fifty, worked out the compound interest for that, for once, plus two-hundred-and-fifty, worked out the compound interest for that, and then plus two-hundred-and-fifty and then I just carried on for twelve months.

The language relating to time is vague, with references to "*in* the first month" [24], "*for* the year" [29] and "*in* each month" [33], and there is no reference to beginning or end of month in this part of their discussion. In [38] Shaun works with an AB approach as he describes the process of making the payments and gaining interest on the balance. This reflects the alternating sequence of payment and interest as shown in fig. 13.2 in the previous chapter. At this point the lack of explicit reference to time does not appear to be problematic for the students because their attention is on the process of making payments and gaining interest. However, the process described by Shaun assumes payment at the beginning of the month which does not fit with the scenario given in GT3.

14.2.2 Explicit reference to time in students' talk

In contrast to the above example, I now provide an example of both groups paying explicit attention to time. In GT3 Vusi struggled at first to make sense of the process of making separate payments each month and accumulating interest on the account balance. He wanted to know how Attiyah had obtained an amount of R251.25 at the end of the first month [63]. She attempted to explain the process by referring to the timing of payments and interest.

Line	Speaker	Utterance
63	Vusi:	Why you say two-hundred-and-fifty-one-comma-two-five (251,25) is the first month?
64	Attiyah:	Because if you invest two-fifty (250) in the beginning of January, at the end of January you gonna get interest so that's for your first month.
65	Vusi:	*(Seeking clarification)* If you invest two-fifty (250) at the end of …
66	Attiyah:	In the beginning of …
67	Vusi:	… January
68	Attiyah:	… January, ja so at the end of January because you calculate interest at the end of every month, so at the end of January you gonna get interest, end of Feburary, March, April, May, June, July, August (trails off) the thirty-first of December.
69	Vusi:	So meaning it's two-fifty (250) times …
70	Attiyah:	The six perc , er one plus zero point zero six divided by twelve (1 + 0.06/12) which is *(trails off and points to her written work which Vusi looks at.)* See here *(tearing page from pad to start a written explanation, and then writes on page)*
71	Vusi:	*(Pointing to her written work)* Two-fifty-one-point-two-five (251.25) plus the two-fifty (250) that makes the first year.
72	Attiyah:	First month *(correcting his reference to year)*

Attiyah's attention to time here contrasts strongly with the previous extract. There is also a marked contrast between Attiyah and Vusi in their reference to time. As with the previous extract, this is likely a function of what they were attending to. Attiyah's careful attention to time formed the basis of her explanation, yet Vusi appeared not to be focusing on references to time. In his opening comment Vusi said the amount R251.25 "*is* the first month" [63], thus not indicating whether it is an opening or closing balance. By contrast, Attiyah referred precisely to the beginning and end of January [64]. In his next utterance, Vusi sought clarification of her explanation[54] but he said "if you invest … at the *end* of January" [65]. Attiyah had just referred to investing at the beginning of January and calculating interest at the end of the month [64]. It is not clear whether he was confusing this timing within the month, or whether he had in mind the original question that referred to payments made at the end of the month. Either way, Attiyah referred to the *beginning* of January, and elaborated her previous explanation [66, 68]. She did not correct Vusi's reference to the end of the month nor did he challenge her reference to "beginning of January", and neither his body language nor his facial expressions suggested that he was concerned about her reference to the beginning of the month. His responses to her second explanation [69, 71] contain another vague reference to time in "*makes* the first year" (and she then corrected "year" to "month" [72]).

This interaction shows how one student paid careful attention to time in her explanation but another student did not appear to be noticing this emphasis despite the fact that it was the basis of her attempt to clarify what he was seeking to know.

The second example comes from Hailey's group working on GT4. Jefferson had asked Hailey how she had calculated her answer for the first question. She explained that she had used the compound interest

[54] From the video footage it is clear that Vusi is not contrasting payment at the beginning and end of the month.

formula for the lump sum and the formula for the future value of annuity due (which she called "first month formula" [46]) for the monthly payments of R300.

Line	Speaker	Utterance
46	Hailey:	Ok, I worked with the thousand rand on its own, so I calculated how much it would be after eighteen months and then for the three hundred rand I used the first month formula, but to seventeen months instead of eighteen.
47	Jefferson:	Why did you then subtract the deposit?
48	Hailey:	Hmm?
49	Jefferson:	Why did you subtract the deposit?
50	Hailey:	I didn't subtract the deposit.
51	Jefferson:	Why seventeen instead of eighteen?
52	Hailey:	Because April is my first month, but I don't make a three hundred rand deposit in April.
53	Jefferson:	You make it at the end of April?
54	Hailey:	No, I make it at the beginning of May.
55	Jefferson:	Ja, at the end of April.
56	Hailey:	No, the beginning of May. It's very different.
57	Sakhile:	At the beginning, yes.
58	Hailey:	Yes, at the beginning of May, I make this deposit.
59	Jefferson:	Oh, ja.

Jefferson assumed that Hailey had used an exponent of 17 for the annuities formula because she was subtracting the lump sum deposit and thus reducing the number of payments in the annuities component [47-51]. This may suggest he was still experiencing difficulties in working with the exponent in the annuities formulae. This was discussed in detail in chapter 13. Hailey explained that it was only 17 months because the annuities component started a month after the deposit had been made (and both gained interest until the investment matured) [52]. Hailey's attention to the number of compounding periods is evidence of attending to time issues.

When Hailey said she did not make the R300 payment in April, Jefferson asked whether she made it at the end of April and she replied that she made it at the beginning of May [52-53]. He replied "Ja, at the end of April" [55], seemingly not seeing a distinction between the two points in time. His comment suggests he had still not recognised the importance of distinguishing between payment at the end of month n and payment at the beginning of month $n+1$. Hailey immediately responded and emphasised that the beginning of May is "very different" to the end of April [56]. Sakhile supported her [57]. It is not clear whether Jefferson's response of "Oh, ja" [59] indicates an acceptance of the distinction Hailey was trying to emphasise. At one level there is no difference between end of April and beginning of May because the closing balance of one month is the opening balance of the following month. However, when modelling annuities, the distinction between payment at the end of a month and payment at the beginning of the following month will impact which formula is used – ordinary annuity or annuity due.

This incident reflects Hailey's attentiveness to precision in talking about time – both in her *own* talk and in *hearing how Jefferson referred to time*. This contrasts with Attiyah and Vusi who did not challenge each other when references were made to making payments at different times of the month. The three examples of students' talk about time taken together suggest that precision in talk about time *per se* may not be the key issue here. It may be more important to recognise *when* such precise talk is necessary, and then to be able to talk in appropriate ways, and also *to attend to specific references to time in the talk of peers and learners*. This echoes Adler's (1999) work on transparency of language in

multilingual mathematics classrooms where teachers face a dilemma in deciding when to give explicit attention to language teaching. In the context of talk about time, there are times (pun intended) when reference to time needs to be brought into focus and made visible for the purposes of dealing with aspects such as the timing of payments. However, there are also times when reference to time is peripheral to the focus of attention and therefore should remain in the background. In such cases vague references such as "*for* the month" or "*in* the month" may be adequate, and it may be counter-productive to force learners to be explicit about time.

14.3 Use of timelines

The use of timelines in the course emerged as an issue through the analysis. In this section I focus on three instances where timelines were in focus in the eight-day period. The first is Rachel's use of the timeline in class session A when she explained her group's IP approach diagrammatically. The second instance is my use of the timeline in class session B as I set up the scenario to derive the formula for future value of an ordinary annuity. The third instance involves Shaun and Palesa working together during GT4, and their attempt to resolve a concern about number of payments and number of interest periods. In each instance I focus on the extent to which the conventions of timeline representations were adhered to and made explicit, and the extent to which the representation supports the expressed purposes of the user. The first two instances deal with timelines in the whole class setting and thus show what was made available for all students to learn. The third instance provides a detailed view of two students working with the timeline. I end this section with a discussion of students' use of timelines in Test 1. This complements the in-depth focus on two students with a broader consideration of how the class worked with timelines albeit in a very different setting.

14.3.1 Using a timeline to introduce an IP approach

Rachel made use of what might be called an "elaborated timeline" to explain her group's use of an IP approach to GT3. Rachel prepared the diagram, shown alongside, on an overhead transparency and so her talk was focused on explaining the pre-drawn diagram rather than vocalising what she was writing.

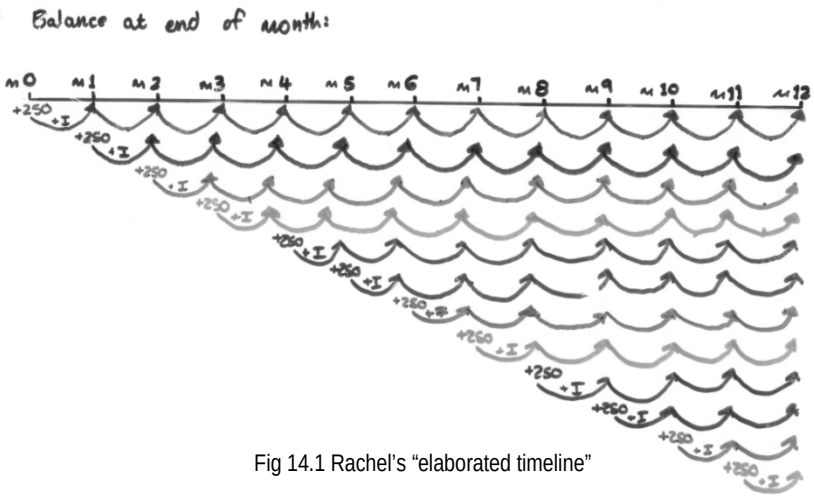


Fig 14.1 Rachel's "elaborated timeline"

She stated that she usually begins financial maths problems by drawing a timeline [89] (see transcript on next page), a practice that was likely the result of her doing a financial maths course previously as a commerce student. She then proceeded to frame her explanation in relation to time, focusing first on the heading and then the "numberline section" which indicated the time divisions. She started with what she called "month-zero" (m_0), and was explicit that the first payment, represented by "+250" below m_0, was made at the end of month zero and would gain interest, represented by "+I", for 12 months. Thereafter each payment grows for one less month than the previous payment, with the last payment accumulating interest for only one month [89, 91]. As can be seen in the section of transcript

below, Rachel continually referred to the timing of payments at month end. Palesa then asked her to explain the section of the timeline from m_0 to m_1.

Line	Speaker	Utterance
89	Rachel:	Okay. Okay, the way I saw it is, usually when I'm given these kind of problems I normally just start with a timeline. So that's pretty much my starting point with any financial maths problem. So, I started with the timeline, goes from begin, okay, it's at the end of the month, so I assumed that, um, the first two-hundred-and-fifty rand payment would be in from month (unclear), for, you would get interest for it and then that would accumulate for, you'd get interest for twelve months. Okay. And, so there's that first payment of two-hundred-and-fifty and it would gain interest over that first month and then, it would, grow, and do compounding, not simple, so it's, you gaining interest on your interest for the full twelve months …
90	Student:	Okay.
91	Rachel:	… and then at the beginning of, well at the end of the, that period *(pointing between month zero and month one)*, first period you'd add another two-hundred-and-fifty rand.
92	Craig:	Okay, let's just stop where we can see the brown only. (*referring to timeline and first row of hops*)
93	Rachel:	Okay. *(covers rest of diagram)*
94	Craig:	Are there any questions at that stage, just so that …
95	Palesa:	Yes, Rachel, there where you're assuming one, can you just um, explain that, cos that's where (unclear).
96	Rachel:	Um, okay this month zero, because I, because of the title is the balance at end of month, if I put that as at the end of first month, but it had already been there for a month so I had to like show it somewhere, that it came from somewhere, it didn't fly in, so I put it from month zero, end of th, zero month so then for one full month you get to the end of, the first month.
97	Craig:	So, time, time zero, if we don't call it by month zero. Time zero is?
98	Rachel:	Right at the beginning when the first payment is made.

Rachel was explicit about the timing of payments at the end of the month even though her timeline was ambiguous in this regard, as I shall show. She introduced the notion of "month-zero" as a mechanism for showing that the first payment had been in the account for one month when it first accrued interest. I assumed it held some connection for her to T_0 hence my query as to where time-zero would be located on her timeline [97]. Her response suggests that T_0 is the same as m_0 [98]. However, there are important differences between the two. Rachel's notion of month-zero was problematic in that it broke the conventions and extended the time period by a full month. By convention, the symbol T_n represents the *end* of month n. In referring to m_0 as the end of month zero, and all other m_k's as the end of month k, Rachel was maintaining consistency with this convention. However this then broke another convention - that T_0 represents the *beginning* of month one. So in maintaining consistency with respect to T_n in general, she was breaking the convention with respect to T_0.

She also broke a second convention with regard to representing the payments. According to convention, if the first payment is placed at m_0 on the timeline, all payments are considered to be made at the *beginning* of the month. While Rachel was explicitly intending to represent payment at the *end* of the month, her payment below m_0 could be taken to mean all payments were made at the beginning of the month. In terms of the conventions, the final payment in her diagram is made at the beginning of month 11 (m_{11}) and gains interest for the whole month. This is a sensible interpretation of the last line of her diagram but it is not what she intended to communicate. A third consequence of Rachel's use of month-zero is that all payments gained interest for an additional month because she was effectively working with a thirteen month period (beginning of month 0 to end of month 12) although she did not represent the whole of month-zero in her diagram.

When I asked Rachel to share her diagram, I was not aware of the potential problems described above. My purpose was to expose the class to a variety of representations of an IP approach and I considered her diagram to be useful for this purpose. Based on the students' responses to Rachel, it appeared that they found the representation useful. Apart from Palesa's query discussed above, there is no evidence that the other students recognised any problems with the timeline at that stage of the course. It is not surprising that they were not yet able to consider the finer details of representing time because their attention was on the new ideas associated with the IP approach.

It is worth noting that Rachel made no reference to names of months in her talk or on her diagram. This stands in contrast to the two instances discussed below.

14.3.2 Using a timeline to set up the derivation of a formula

In chapter 13 I noted that there were no explicit references to time in Nosisi's explanation of her group's work. I was intrigued by the way in which this contrasted strongly with Rachel's explanation. Consequently I decided to study my introduction of the timeline for deriving the formula for future value of an ordinary annuity in class session B. My intention in the lecture had been to emphasise time and the features of the timeline, and so I was interested to see the extent to which I had in fact done this. I focus only on the introductory part of my explanation because the later part that dealt with the algebraic manipulation and derivation of the formula did not focus primarily on time. Although I had inserted a timeline as a means to resolve a student's concern in the early weeks of the course, (see chapter 9), my explanations in this session focused on the use of the timeline in the context of annuities, and did not make assumptions about students' prior exposure to timelines. I wanted to see whether and how I had been precise in my talk about time, and had been explicit about the conventions of timelines. This contrasts with Rachel's explanation where the timeline itself was not the focus of her explanation but rather provided a basis for explaining about separate payments gaining interest, and how these accumulate to give the future value. A screenshot of my boardwork and a section of the transcript are provided below. The transcript contains additional detail of my actions as I wrote on the board, and my interactions with the written text on the board.

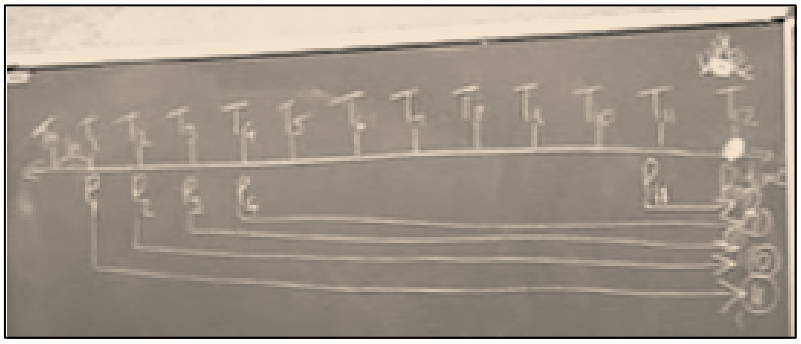


Fig 14.2 My timeline to illustrate payments for future value of an ordinary annuity

The boardwork (fig. 14.2) shows a timeline from T_0 to T_{12} with payments P_1, P_2, P_3, P_{11}, and P_{12} indicated beneath the respective T_n's. Between T_0 and T_1 I indicated that the time period was *months*. I made a large dot under T_{12} and drew an eye to indicate that this was where we would focus when calculating the future value of each payment. I indicated moving each of the above payments to T_{12} with an arrow and wrote the number of compounding periods for each payment in a circle at the end of the arrow. For example, eleven (11) is written at the end of the arrow linked to P_1.

No.	Speaker	Utterance
1	Craig:	We are focusing on a section of this thing (*points to timeline*). So the arrows are indicating time goes on in both directions. I'm gonna decide that this is time-zero (*indicates T_0 on timeline*) because that's the space I'm interested in for this problem. I can put it wherever I like in time but that for me is time-zero (*pointing to T_0*), whatever it turns out to be. Could be beginning of January, could be end of January, it's my choice. And then I wanna make sub-divisions in here (*drawing them on timeline*). So I'm gonna tell you what I do and then you can do it while I am doing it. I wanna make twelve sub-divisions on that line to represent January to December. And they need to be equal because they're equal months. So I'm not saying you need to use rulers but that's the kind of thing we wanna achieve.
		Completes drawing timeline drawn on board. Draws arc between T_0 and T_1 in different colour.
2	Craig:	I'm gonna indicate between T_0 and T_1 that I'm working in months. So that that gap here is a month and that's just for me to remember that what I'm doing here is operating in months, not quarters, not years, or any other unit. Because T_0 and T_1 can be any distance apart that I like in time. They could be one day. They could be one year, they could be six months. And I'm gonna indicate that first two-hundred-and-fifty (250) that gets made, I'm gonna make over there, that's payment-one (*writing P_1 under T_1*). Not at T-zero (T_0) because it, the task and the thing that we're working with on the spreadsheet is that it gets paid at the end of the month so where T_0 in our example would represent the beginning of January, T_1 is representing the end of January beginning of February. Okay. You with me? So clearly then payment-two, payment-three, payment-four (*writing P_2, P_3, P_4 them on board then pause as students copy down*)
		Writes P_{11} and P_{12} underneath, doesn't say anything, makes big dot on T_{12}, draws eye above T_{12} looking down. Some giggling from class.
3	Craig:	Get what that is?
4	Students:	Eye …. Eye.
5	Craig:	Okay. It's not my eye (inaudible) So the focus, what we're focusing on, is what's going on by the end of this year, and for some people the concern is that this year is actually only 11 months. But that's where we're looking (i.e. T_{12}). So we wanna see what goes on with all these payments by the time we get to T-twelve (T_{12}). Of course we could stop at T-six (T_6) and figure out what's going on as well but the point in time we want to look at is time twelve and obviously that's relative to wherever time zero was in the beginning. So what we are doing, if you think about that January payment, is we're taking it forward to T_{12} and we're looking at what's going on at T_{12}. *(draws arrow on board from below P_1 to T_{12})* . So when this (referring to P_1) went forward from T_1 to T_{12} what happened? *(Points to CI calc showing $FV_1 = 250(1.005)^{11}$)*. We compounded it … ?
6	Students:	Eleven times.
7	Craig:	Eleven times. So I'm just gonna stick on here (*end of arrow from P_1*) that was eleven compounding periods to get there. Similarly P_2, that is the February payment, was put in the bank and it grew. It was compounded … ?
8	Students:	Ten. *(Draws lines on board from P_2, P_3 and P_4 ending at T_{12})*
9	Craig:	Nine times for March, eight times for April. What happened to November?
10	Students:	Once.
11	Craig:	It gets compounded once. So diagrammatically what that is showing is the movement of all of those individual payments through to the end of the spreadsheet. (*Pulls down screen over board to project spreadsheet*)
12	Craig:	So if we just quickly go back to the spreadsheet and take a look … what we've represented on the timeline is the shifting of every payment. Okay, that January payment has grown up to the end of December. Likewise February (pointing to row on screen), March, all the rest. December's hasn't grown. Okay? No compounding because it got paid at the end of the month. So I'm not concentrating on vertical movement in the spreadsheet at all. I'm only interested right now in taking a payment and seeing its behaviour through to the end of the year. The end of the year being the end of the calendar year. You with me? Hope so.

As can be seen, and in contrast to Rachel, I made many references to the names of months in my talk [1, 2, 5, 9, 12]. However, unlike Shaun (see below) I did not write the names of months on the diagram. The transcript also shows several explicit references to the end of the month [1, 2, 5, 11, 12] but also some vague references, for example "the February payment, was put in the bank and grew" [7] and "eight times *for* April" [9]. The first example does not indicate when in the month of February the payment was made. The second phrase refers to the number of times interest compounds on the payment made at the end of April but I was not explicit about the timing of the April payment.

In addition to specific verbal references to time, I drew students' attention to time in various other ways. One of these was the use of the eye above T_{12}. This is a device that had proved useful in the past

to distinguish where the focus is, in time. For example, in a future value scenario, the eye would be over T_n while in a present value scenario it would be over T_0. I also focused on linking the timeline to the calculations I had already written on the board [5], and then to the triangular spreadsheet I had introduced in the previous class session and which had been further discussed earlier in the current session [12]. The calculations referred to the first three payments (January, February and March) and the last two payments (November and December). It was important to emphasise why the first payment only gained interest 11 times and why the last payment did not gain any interest. This contrasted with Nosisi and Rachel's explanations where they had compounded the first payment 12 times and the last payment once. In doing all this I was emphasising that payments were made at the end of the month, which explained why the exponents were one unit smaller than Nosisi and Rachel's.

With regard to conventions of the timeline, I was explicit about the following issues: using T-notation to represent points in time; choosing T_0 with reference to the context of the problem; making sub-divisions equal in length; being able to choose the unit of time between T_k and T_{k+1}, and making this unit of time explicit; indicating payments with P_n at the appropriate points on the timeline.

There were three key aspects that I did not make explicit. Firstly, in referring to T_n, I said that T_0 could represent the *beginning or end* of a period [1]. While T_0 may be considered an arbitrary point in time, the convention is that it refers to the *beginning* of the first period and I did not make the convention clear although I made explicit that I was choosing T_0 to represent the beginning of January [2]. I also did not make explicit that, by convention, T_n represents the *end* of period n, although this was implied by my use of T_n. However, I also said that T_1 represented the "end of January beginning of February". While it is useful to see T_n as a boundary between periods, I did not distinguish between the meaning of T_n by convention and an elaborated interpretation of it as a boundary. Given the potential difficulty for students in making sense of the boundary between periods, as exemplified in Jefferson's struggles in the previous chapter, this issue needed more careful attention on my part.

Secondly, I did not distinguish between essential components and elaborated components of the timeline. As can be seen from the diagram, my timeline, like Rachel's, has been substantially elaborated. I did not distinguish clearly between what would be considered necessary and sufficient elements of a timeline (such as the line itself, the demarcations of time, and the payments), and those aspects of my diagram which might be considered to be idiosyncratic uses for the purposes of teaching. Teachers need to identify and produce the key elements of the representation as part of their knowledge of the essential features. I did not model this in my teaching that day. Teachers also need to elaborate a representation to emphasise specific features and to help learners access the conventions. I modelled this implicitly.

A third convention I did not make explicit was that by placing P_1 under T_1, P_{12} under T_{12} and not having a payment below T_0, I was indicating payments at the *end* of the period. I deliberately chose not to mention this at that point. I was of the opinion that this was not important knowledge for students who were still trying to make sense of an IP approach. I intended to deal with this issue when representing present value of annuities on a timeline.

In summary, the above analysis of the transcript shows that there was explicit reference to the timing of payments at the end of the month, and that this was done through direct talk as well as by reference

to the number of times a particular payment would be compounded. With regard to conventions of timelines, some key features were made explicit but three important conventions were not made explicit although they were exemplified.

14.3.3 Using a timeline to address a fellow student's concern

In GT4 Palesa raised a concern about the numbers her group had chosen when reporting on their approach to the problem. Shaun set up a timeline in response to her concern. As with Rachel, this is an incident where a student chose to use a timeline *as part of an explanation*, rather than merely to represent information given in a problem statement. In contrast to Rachel, Shaun indicated month-names on the timeline rather than making use of T-notation. As he proceeded with his explanation, the timeline became the object of attention rather than a tool for explanation, because he struggled to represent the ideas he wanted to communicate. This situation thus provides insight into how he and Palesa worked with the timeline in a moment of breakdown. The incident reveals the importance of paying attention to the end of the last period when payments are made at the beginning of each period. It also suggests a potential problem with using month-names when labelling the timeline.

The page on which Shaun and Palesa worked was not submitted with their report, and so the timeline in fig. 14.3 is a reconstruction from the video and audio records of the tutorial. It is not possible to reproduce all the details Shaun had included on the timeline because the camera was focused on the whole group and not zoomed onto the desk where he was writing. The details that have been shown are based on his and Palesa's utterances as well as visual cues such as hand-movements from the video record.

The task for GT4 involved a lump sum payment of R1000 and 18 monthly payments of R300, all of which were paid at the beginning of the month. In the opening paragraph of their GT4 report, the group provided two different interpretations of the task scenario. One of these interpretations, proposed by Attiyah and Shaun, read as follows:

> "You pay a deposit of R1000 in April 2008, which will gain interest for 19 months, we would then start paying R300 in the beginning of May 2008 and continue for the next 17 months."
>
> (A-group, GT4 report)

They intended the second part of the sentence ('we would then ….') to imply that there were 18 monthly payments of R300 starting in May 2008. Palesa queried why they had used the numbers 17 and 19.

One interpretation of Palesa's concern is that she had focused on the numbers 19 and 17 and ignored the phrase "continue for the next 17 months". Consequently she would have missed the implication that there were a total of 18 payments of R300. This interpretation, which I initially held, suggests Palesa was confusing number of payments with number of interest-gaining periods. I became increasingly dissatisfied with this interpretation because it did not explain some of Palesa's comments, particularly towards the end of their interaction. This dissatisfaction prompted further analysis of the transcript, video and audio footage, which led me to reject my initial interpretation in favour of the following interpretation.

I shall argue that Palesa interpreted the statement regarding the task scenario to mean that the R300 payment made at the beginning of May would continue *to receive interest* for the next 17 months.

Assuming she had interpreted it in this way, her concern about the interest periods would be valid: if the lump sum of R1000 gained interest for 19 months, then the May payment of R300 should accumulate interest for 18 months. Alternatively, if the May payment accumulated interest for 17 months, then the lump sum should accumulate interest for only 18 months. My rationale for this interpretation is given towards the end of my description of the interaction.

Regardless of Palesa's intended meaning when she raised her concern, she did not make her interpretation explicit and the group members did not seek clarity. I make no claims about how the rest of the group interpreted her concern. I focus rather on Shaun's attempt to explain to her what they had meant by their interpretation of the task scenario. My discussion is based on a detailed analysis of the transcript. The interaction took place towards the end of the tutorial session. The transcript is long, and includes a commentary to assist the reader and to provide evidence of my interpretations of the students' utterances. It is important to note that the group uses *deposit* to refer to the lump sum of R1000 and *payment* to refer to the regular monthly payments of R300. When students were counting, the numbers are indicated as numerals to shorten the transcript. When a number in the list was emphasised, the full word is given.

Shaun began by drawing a timeline with 18 demarcations, which would cover the period beginning of April 2008 to beginning of September 2009. He did not make explicit that he was focusing on the beginning of each month nor did he state where the timeline ended. He indicated April 2008, January 2009 and April 2009. He likely drew markers on the timeline to indicate the other months [1, 3]. Soon he would recognise that a total of 18 markings would be insufficient because there were 19 payments in total.

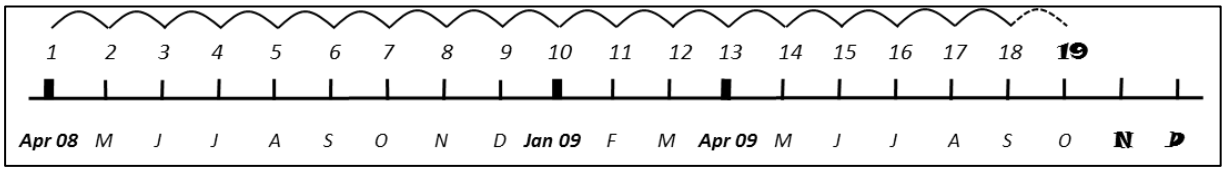


Fig 14.3 Reconstruction of Shaun's timeline for his explanation to Palesa

Shaun wanted to show that the deposit would get interest for 19 periods [5] but when checking this, by drawing hops from April 2008 to the end of his timeline, he could only fit 18 hops [5-7]. He then appeared to shift to count the number of R300 payments (but didn't state this explicitly), expecting to get 18 but there were only 17 markers on the timeline [9-11]. He realised he needed to add another marker to the end of the timeline for the eighteenth monthly payment [11-13]. He then rechecked the number of payments, having to correct Palesa when she wanted him to count from the payment for April 2008 rather than May 2008 [13-16]. He then checked the numbers of hops for the lump sum deposit again and still got 18, yet he knew there should be 19 [17-21]. At this point Palesa was becoming more confident that it should be 18 [6, 8, 18-20, 28, 31]. Attiyah attempted to help them but could not find the problem [22, 24-26]. Shaun then realised that "you take your money out at the end" but "[t]his is the beginning" [34]. He thus recognised that he was working implicitly with the beginning of each month and he needed to include the end of the last month in order to get the final interest-gaining period, which would provide the nineteenth hop. This resolved the issue for him but not yet for Palesa. In order to show the situation more clearly on the timeline, Attiyah suggested he indicate explicitly that he was representing the first of each month [37-39, 41]. He did this for April

2008, January 2009, April 2009 and November 2009 [40, 42]. He then focused on explaining the eighteen R300 payments to Palesa, starting on 1st May 2008 [46-52]. He paid particular attention to the last payment and emphasised that it had to "sit for a month" before the term of the investment ended [52-60]. This required the final period he had added. He also checked that the lump sum deposit would gain interest for 19 months [67-69]. Palesa then asked again about the payments and he explained again [70-77].

Line	Speaker	Utterance	Comment
1	Shaun:	Look here. I'm gonna draw you a picture. Ok here we go. Here's our picture(*begins to draw timeline*). We won't put this in there (*i.e with their report*). There's April 08, (*marking off the months*) May, June, July, August, September, (fades) December, January 09, (*goes back to beginning of timeline to count markings*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, (*fades*) 17, 18. Am I right?	Draws timeline with 18 markings. Indicates explicitly, April '08, Jan '09, April '09. Draws markings for the other months. (A total of 18 markings is insufficient since there are 18 payments of R300 as well as the R1000 initial deposit.)
2	Palesa:	Umhmm.	
3	Shaun:	It should be should twelve to April, counting, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Ja, okay I'm right. So that's April 09. We gonna put our deposit in over here (*April 08*). Okay.	Confirms that there are 12 markings before April '09 which indicates a 12-month period.
4	Palesa:	Umhmm.	
5	Shaun:	We're finishing over here. Okay, so it's gonna get interest for this (*drawing hops and counting gaps between markers on line*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 …	Presumably 'here' refers to Oct '09. Counts number of times the 1000 deposit will get interest and draws hops to indicate this. Counts a total of 18 hops but expects 19.
6	Palesa:	Eighteen! Yes!	A total of 18 supports Palesa's concern.
7	Shaun:	Maybe I've counted it wrong.	
8	Palesa:	No!	
9	Shaun:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, oh no but it's from here, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17.	Seems Shaun has shifted to count number of R300 payments. Likely that he counted from April '08 and then realised he should count from May '08 so he starts again, and counts 17 in total.
10	Palesa:	Seventeen.	
11	Shaun:	I need to put one more month in.	Realises he needs another marker for the last payment.
12	Palesa:	No! (*laughing*)	
13	Shaun:	I do because we're putting eighteen payments in from May. So that's one payment, two payments, three payments, four payments, five payments, six payments, (unclear)	Starts counting payments again to show Palesa.
14	Palesa:	No that's one (*pointing to first transaction in April 08*).	She refers to April '08 payment as the first one.
15	Shaun:	No but that's the deposit not the three-hundred-rand payment.	Shaun explains why R300 payments must be counted from May '08, not April '08.
16	Palesa:	Oh ja.	
Attiyah interrupts to ask for help in the wording of the report – line numbers do not include this section			
17	Shaun:	Okay. (*Counting on timeline*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18. Now I'm puzzled.	Shaun checks his counting again. Presumably he counted compounding periods based on his response to Palesa in line 19.
18	Palesa:	Yes!	
19	Shaun:	But it's nineteen.	He knows the lump sum compounds for 19 periods but his diagram does not reflect this.
20	Palesa:	No, it's … (*both laughing*)	
21	Shaun:	It is nineteen.	
22	Attiyah:	No, can I show you.	
23	Palesa:	If that one is seventeen (*referring to what Vusi is writing in report*), this one	Palesa reinforces that there can only be

Line	Speaker	Utterance	Comment
		will be eighteen.	a difference of one unit (17 to 18, or 18 to 19).
24	Attiyah:	(*moves to look at what they've written*) If you pay, if you pay (*looking at Shaun's written work*) If that's April, beginning, end of April? Beginning of April?	Attiyah tries to help them out.
25	Shaun:	Ja, it's beginning of April.	
26	Attiyah:	Beginning of April to end of April is your thousand right, that's (*counting*) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (*fades*) 16, 17, 18,	She also counts 18, presumably she counted markings on timeline from May '08 onwards which would indicate 18 payments of R300.
27	Shaun:	Eighteen (*laughs*)	
28	Palesa:	Eighteen! Where's nineteen? Uh-uh, uh-uh (*indicating her disagreement*)	Attiyah also gets 18 which Palesa feels supports claim that 19 is incorrect.
29	Shaun:	I also get …	
30	Attiyah::	Yes (*returns to her seat*)	
31	Palesa:	It's eighteen (Shaun laughing) If that one is one less than this one …	
32	Shaun:	Oh no, but, but …	
33	Palesa:	It will have to be one less …	
34	Shaun:	… but you take your money out at the end. This is the beginning. Yeeeeesss! You see.	Shaun recognises the problem. He has been focusing on the markings on the timeline which represent the beginning of the month. This means there needs to be another marker on the timeline to indicate the end of the last month.
35	Craig:	What?	
36	Shaun:	(*to Craig*, *Palesa laughing*) I'm just gonna explain (*fades*) (*to Palesa*) look here's the beginning of the month.	
37	Attiyah:	Put the dates in, put the dates in Shaun.	Attiyah appears to have recognised the solution. Suggests Shaun be more specific when indicating dates on the timeline.
38	Shaun:	I have put the dates in.	
39	Attiyah:	The first of April.	
40	Shaun:	Okay. The first, and the first of January, the first of April, so that's May, June, …	Writes 1st of the following months: April '08, Jan '09, April '09, Nov '09.
41	Attiyah:	Then it makes sense.	
42	Shaun:	… July, August, September, October, November, (*writing*) first of No-vember 09, okay.	Seemingly first of November is the last month on this timeline.
43	Palesa:	Now you're getting cheeky.	
44	Shaun:	Yes, now I'm (unclear)	
45	Palesa:	(unclear) convince me.	
46	Shaun:	Yes you right that's exactly what I'm going to do. Okay. So do you agree, this we put, here this started in May, that's the first of May, let's just look at our payments for now.	Focusing only on R300 payments.
47	Palesa:	First?	
48	Shaun:	Of May. No here this one over here is the first of May end of April.	
49	Palesa:	Ja.	
50	Shaun:	Okay, you happy with that? So we gonna put our one payment in here, (*writing*) payment-one, payment-two, payment-three, payment-four, payment-five, can I just put numbers in? (*continues to write*) 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, eighteen!	Indicates the R300 payments from 1 to 18.
51	Palesa:	Umhmm.	
52	Shaun:	Okay, so you're happy, that's eighteen payments. Okay, good. Okay, we gonna put our deposit in. (*Then shifting focus back to R300 payments*) Okay, but then we're not just gonna pay our eighteen in the beginning and take it out straight away, We're gonna leave that actually until, to gain interest for one more month, so we're gonna take, this is when we, I've drawn it wrong. So when we take that money out, it's gonna be the 1st of December. Do you agree with me?	Indicates R1000 deposit at beginning of April '08 then focuses again on 18th payment of R300. Emphasises that it remains in account for 1 month. Then he claims he's made a mistake in his drawing but it's unlikely there was an error. He adds another month onto the timeline and says that the 18th payment will be withdrawn on 1st December.

Line	Speaker	Utterance	Comment
53	Palesa:	Um-hmm.	Palesa does not pick up the error.
54	Shaun:	Cos were gonna leave our eighteenth payment in for …	
55	Palesa:	For that one month.	
56	Shaun:	… for a whole month so this, over here we'll draw a big arrow cos this is when we take our money out, and that's the first of December here 09. Are you happy with that?	
57	Palesa:	Ja.	
58	Shaun:	Okay. So that's eighteen months, eighteen interest periods, you happy? You happy that we're taking our money out on the first of December?	Presumably referring to 18 interest periods for the annuity component (although he now has 19 periods from May 08).
59	Palesa:	Because we want interest, isn't it?	
60	Shaun:	Ja, because our eighteenth payment has to sit for a month.	
61	Palesa:	Ja. And we've paid it at the beginning.	
62	Shaun:	Yes.	
63	Palesa:	So at midnight somewhere…	Confirming that payment has been in account for a full month and will gain interest at midnight on the last day of the month.
64	Shaun:	Ja.	
65	Palesa:	… it will gain interest, go for (unclear)	
66	Shaun:	Ja. (*long pause*)	
67	Shaun:	Okay. So then this thousand we put in over there would also have to sit for an extra month.	
68	Palesa:	Umhmm.	
69	Shaun:	So that's the first of December and that is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, it's nineteen months of interest.	Confirms that the R1000 deposit will accumulate interest for 19 months.
70	Palesa:	And the payment?	Presumably referring to the first R300 payment.
71	Shaun:	The payment would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.	Confirms 18 compounding periods for the first deposit.
72	Palesa:	Eighteen but you said seventeen (*in the report*)	Palesa returns to her original argument about the mismatch between 17, 18 and 19.
73	Shaun:	Okay, because what she said (*referring to Attiyah*) over here is we've put one payment in.	
74	Palesa:	Umhmm.	
75	Shaun:	Okay.	
76	Attiyah:	In the beginning of May.	
77	Shaun:	In the beginning of May then we have seventeen other payments we have to put in so 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 16, 17.	Counts number of payments from June '08 to the end of the investment.
78	Palesa:	Ah seventeen payments.	
79	Shaun:	After the first one.	
80	Palesa:	After the first one. Not the interest.	Confirms that 17 additional payments are made and that they are referring to payments and not to compounding periods.
81	Shaun:	Interest.	
82	Palesa:	Here we are calculating the interest.	
83	Shaun:	Yes.	

It is the last section of the transcript [78-83] that gives me confidence in my interpretation of Palesa's meaning when she raised her concern. In [79, 80] Shaun and Palesa respectively emphasise "after" the first payment. I take this to indicate agreement that the total number of payments is 18. In [78], Palesa emphasises "payment" and in [80] she says "not the interest". I take this to mean she now recognised that the original statement which she had challenged referred to *number of payments* and not to the *number of interest-gaining periods*.

Shaun thought he had made an error in his timeline [52] and so he "corrected" it by adding another period thus ending his timeline on 1st December 2009 [52]. It is not clear why he thought this was necessary and Palesa did not challenge him [52, 56, 58]. If one counts 19 payments and 19 interest-gaining periods from the beginning of April 2008, the investment period should end on 1st November 2009.

In the analysis of this incident I focus on the following issues: the changing role of the timeline as the discussion progressed; the potential problem of using month-names on the timeline; and a tendency to shift between discrete points in time and intervals. I conclude by showing that it was only when Shaun's use of time became visible to him that he was able to resolve the impasse and proceed.

Changing role of the timeline – The role of the timeline shifted as the discussion progressed. In the beginning Shaun set up the timeline to assist in this explanation to Palesa [1-5]. At that stage the timeline was a resource to support his explanation and to deal with Palesa's concern. But then he discovered that his diagram did not produce the answers he expected. The timeline itself then became the focus of attention. It was no longer a resource to support an explanation but became the entity to be attended to and corrected. It took a while to resolve the problems with the timeline [6 – 34, 52] and for Shaun to recognise that he was implicitly working with the beginning of the month and needed to complete the last month on his timeline to get the correct number of interest-gaining periods. He then corrected the timeline and focused again on representing the necessary information on it. It was only from [67 – 83] that they returned to Palesa's original concern and resolved it, and thus the timeline became a resource for supporting the explanation once again.

Use of month-names – The second issue concerns the use of month-names rather than T-notation on the timeline. Shaun indicated specific months rather than using of T-notation. While month-names may appear to provide a stronger connection to the context of the problem, they may have been the main source of the students' difficulties. When the markers on the timeline are used to represent actual months, they may be interpreted as the *beginning* of each month, which is the way in which Shaun used them. This contrasts with the conventions of the T-notation where T_n represents the *end* of month n, and points to an ambiguity which I had not previously noticed.

Consider the two timelines in fig. 14.4 both of which represent a three-month period. In Timeline A (fig. 14.4a) time is represented using T-notation with T_0 to T_3. P_1 is written underneath T_1. In timeline B (fig, 14.4b) the T-notation is replaced by month-names: May, June, July and August, so P_1 lies beneath the June marker.

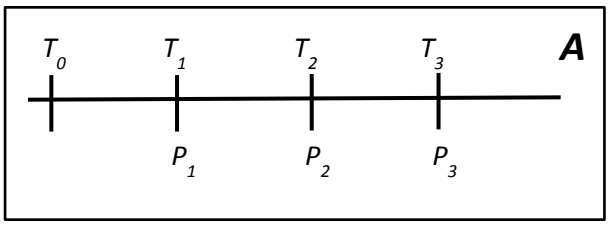


Fig 14.4a Timeline using T-notation

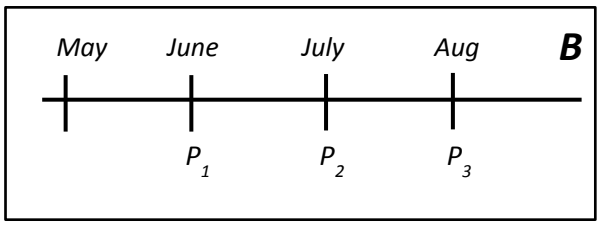


Fig 14.4b Timeline using month-names

By convention Timeline A represents payment at the *end* of T_1. But in Timeline B, the scenario is more likely to be interpreted as payment at the *beginning* of June, rather than payment at the end of May. At one level the difference is irrelevant since the closing balance at the end of May is the same

as the opening balance at the beginning of June. However, the model for payment at the end of the month is different from the model for payments at the beginning of a month. The two timelines differ only in the labels for time, yet they may imply different timing of payments. This ambiguity may become a source of students' difficulties.

The consequence for Shaun and Palesa was that the last month was not represented – their time period stopped on the first day of the last month and not the last day of that month. It was only when Shaun recognised that he was implicitly representing the beginning of each month that he recognised the need to indicate the end of the last period explicitly. This problem may be addressed by explicitly representing full periods, particularly when dealing with scenarios involving payment in advance.

For this reason it may be advisable to discourage the use of month-names to represent time on timelines because the month-names may be taken as the *beginning* of each month which does not follow the convention of representing the end of each period, as with T-notation. This is an issue for further research.

Reference to discrete points in time versus intervals – A third issue arising from the incident is the tendency to shift between *discrete points in time* on the timeline, such as the making of payments, and the *intervals between the discrete points*, which may be considered as interest-gaining periods. Parramore (2011) notes that students have difficulty distinguishing points and intervals in relation to time. Regardless of how one interprets Palesa's concerns about the wording in their report, she was comparing a number related to payments with a number related to compounding periods. Shaun counted features of the timeline – either markers or intervals – nine times [3, 5, 9, 17, 40-42, 50, 69, 71, 77] but he was not always explicit in his talk about what he was counting. In some instances his gestures indicated what he was counting, for example drawing hops to span intervals, but in other instances one can only infer what he had counted from the total number he obtained or from his response to the answer obtained. For Palesa there may not have been any confusion because the visual cues provided by his gestures may have indicated what he was counting. However, in a didactical situation, be it whole-class teaching or one-on-one interaction such as this incident, it would be advisable to be more explicit as to whether the focus is on discrete points in time or on the intervals.

One of the main problems with Shaun's use of the timeline was that initially he did not indicate the full final period. This appears to be a consequence of the fact that he was working implicitly with the timing of payments at the beginning of the month. Although Attiyah was not able to diagnose the problem herself, her intervention may have been the trigger that enabled Shaun to recognise the problem with his diagram. She asked explicitly whether the April marker indicated the beginning or end of April [24], and Shaun was clear that it represented the beginning of the month. She then referred to the first period as "beginning of April to end of April" [24-26]. Soon thereafter Shaun realised his error [34].

It seems that Attiyah's comments helped Shaun to bring into focus his representation of time periods on the timeline. Prior to that, his talk about time had not been explicit, for example there are no explicit references to time in lines 1 to 23 of the transcript – he spoke "*to* April" [3] and "*from* May" [13]. But in [34] time becomes visible for him (Lave & Wenger, 1991) so we might say that he "sees time". This is the turning point of the incident. In making time issues visible, he realised the need to

represent the end of the last month, and thereafter (apart from the problem with introducing 1st December 2009), he was able to solve his problem, to correct his timeline, and to draw on the timing of payments as a means to clarify for Palesa both the number of payments and the number of interest-gaining periods.

14.3.4 Students' use of timelines in a test

Following the insights gained from analysing the three instances described above, I was interested to see the extent to which the larger group had adopted the conventions of the timeline, the ways in which they represented payments at the beginning of the period and their attention to representing the full final period. My only access to this was through students' responses to a question in Test 1. The test question[55] was similar to GT4, involving an annuity due scenario with a missed payment. Students were only required to represent the perfect payment plan on the timeline although some students added the details of the missed payment as they proceeded through the question. I analysed only the first part of the question that required the production of a timeline and involved the perfect payment plan. I did not analyse the sub-questions dealing with the missed and double payments.

I began with a quantitative analysis of the 35 scripts, counting how many students: used T-notation and/or numbers on their timelines; indicated months with month-names; used a combination of the two. Since the question involved an annuity due situation, I counted the number of students who explicitly represented the end of the last period.

Annotation on timeline	No. of students	No. of students who indicated the end of the last month
T-notation/numbers only	24	16
Month-names only	6	1
Month-names and numbers	3	1
No timeline provided	2	-
TOTAL	35	18

Table 14.1 Summary of students' use of timelines in test

As can be seen from table 14.1, most students (68.6%) used T-notation (or some numeric representation such as fig. 14.6 below). Only 18 students explicitly indicated the end of the last period and 16 of these had used only numbers or T-notation. However, the ability to produce a complete and correct timeline did not necessarily lead to correct calculations. Only 13 students obtained the correct answer, and four of these students had not indicated clearly the end of the last period. There is a weak correlation ($r = 0.212$) between students who indicated the end of the last period and those who gave the correct answer. The students' errors included incorrect choice of formula, incorrect substitution for n in the formula, and incorrect rate per period, e.g. use of annual rate instead of monthly rate.

I then undertook a qualitative analysis of the students' timelines paying particular attention to the extent to which they made use of the conventions. I also considered the kinds of personal extensions they made to their timelines. I found that many students did not work strictly with the conventions of the timeline. They included a variety of additional elements as shown in the selection of timelines presented below. The following six examples have been selected as illustrative cases of the range of students' responses.

[55] See Appendix C7 for the test question.

The first three examples (figs. 14.5, 14.6 and 14.7) make use of the conventions of the timeline for payments at the beginning of the period, indicating the first payment at T_0 and the last payment at T_{16}. They all indicate the end of the last period. However, in fig 14.6 the student has used numbers rather than T-notation to indicate distinct points in time. In fig 14.7 the hops do not line up exactly with the markers on the horizontal line – each hop ends to the right of the marker. I take this as explicit indication that payments are made at the beginning of the month. Shaun used the same technique in one of his interviews. While this emphasises the timing of payments, it may be considered redundant in the light of the conventions (see discussion in chapter 12). But it may suggest the student is not yet comfortable to accept the conventions as sufficient indication of payment at the beginning of the month.

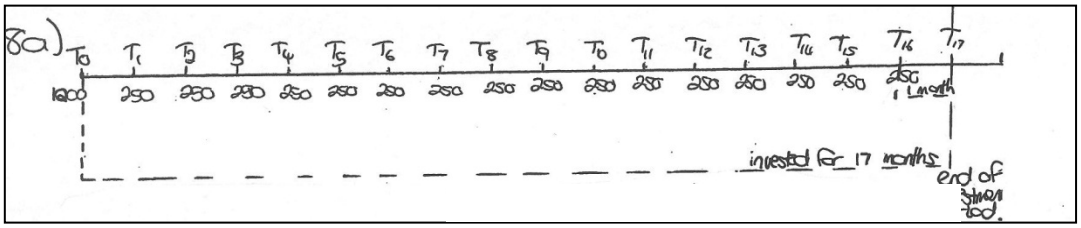


Fig 14.5 Timeline using T-notation

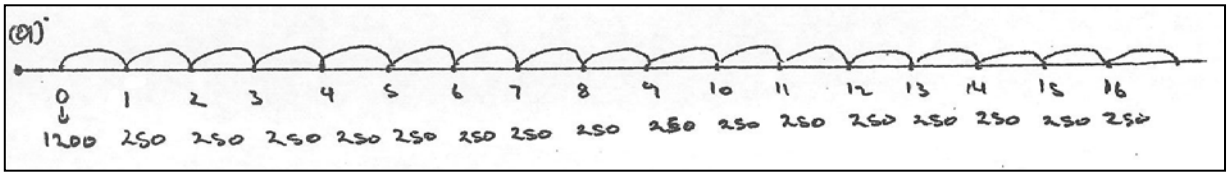


Fig 14.6 Timeline using numbers

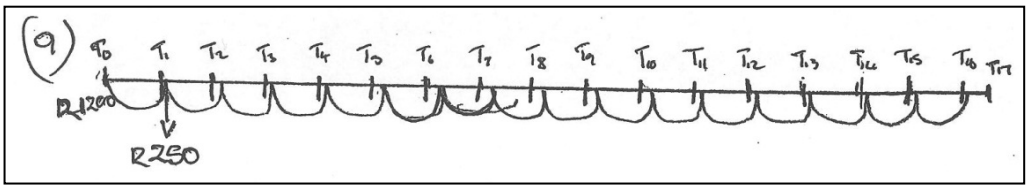


Fig 14.7 Timeline using T-notation with explicit indication of payment at beginning of month

The next three examples (figs. 14.8, 14.9 and 14.10) deviate from the conventions in various ways. Figs 14.8 and 14.9 emphasise intervals. In fig. 14.8 there is no use of T-notation and the numbers are placed between the markers thus numbering the *intervals* rather than discrete points in time. This resembles the diagrams from Smal (2004) (see fig12.3 in chapter 12). There is clear indication of the end of the last period.

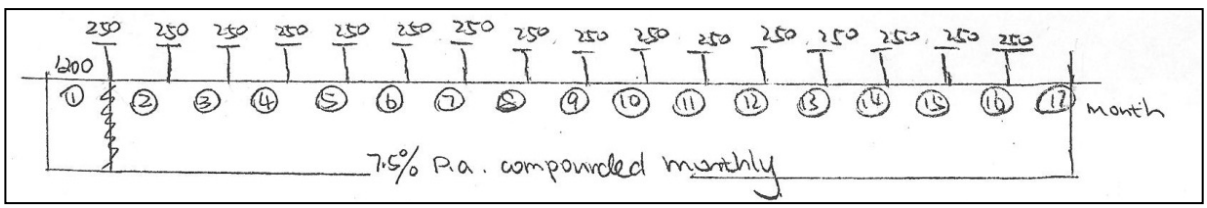


Fig 14.8 Timeline with numbering of intervals

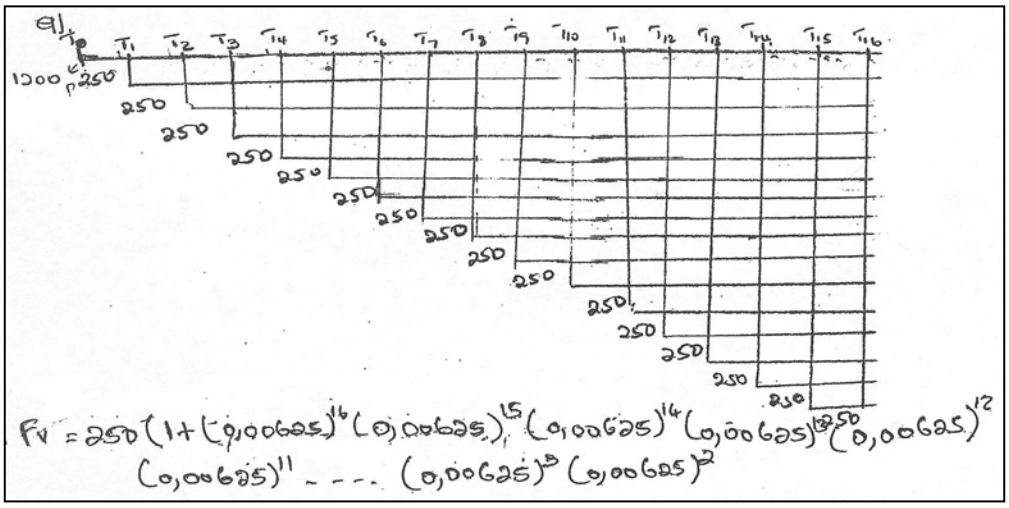


Fig 14.9 Timeline resembling triangular spreadsheet



Fig 14.10 Timeline using month-names abbreviated by letters

Fig. 14.9 resembles the layout of the triangular spreadsheet. The student has used T-notation to indicate T_0 to T_{16}. It appears that R1200 is paid at T_0. Sixteen payments of R250 are indicated but their timing and the number of periods for which they compound is ambiguous. For example, it is not clear whether the first payment of R250 is made at T_0 or T_1. Based on the calculations beneath the timeline where the highest exponent is 16, it appears the student is assuming the payment is made at T_0, and therefore simultaneously with the lump sum deposit. This would also mean that T_{16} should be taken as the end of the seventeenth period in which case the full final period has been included. However, this is not clear from the diagram alone.

Fig.14.10 represents time with letters standing for months. The student has indicated 17 months (December to April two years later) which is the same as the total number of payments. However, this student has made the same error as Shaun in his discussion with Palesa. The student has not indicated the full final month and thus has not represented 17 full months. Consequently all payments gain interest for one month less than they should. Also, the student has only indicated 15 monthly payments of R250 and 15 hops.

One has to be cautious in making claims about the timelines produced in the test. It seems reasonable to claim that attention to the conventions of the representation was not a priority for the students. However, the ability to produce an appropriate timeline was not strongly correlated with correct answers to the perfect payment plan. This may suggest that students' use of timelines does not necessarily reflect their grasp of the situation. This issue requires further research.

14.4 Summarising – talk about time and use of timelines

At the beginning of this chapter I set out to investigate four questions. At this point I summarise the findings of the first three questions by considering students' attention to talk and their use of timelines.

Students' attention to time in their talk – As noted in the focus groups' interactions there is a range in the extent to which students pay explicit attention to time in their talk. There are situations when a lack of attention is problematic and other instances when it is not. The key issue appears to be *knowing when* to be explicit about time. It may be productive to distinguish being *explicit* about references to time, from being *precise* about them. I would argue that teachers always need to be *precise* in their references to time. However, it may not always be necessary to be *explicit* because this may detract from the other issues in focus. Another important issue, particularly for teachers, is *paying attention to how others are talking about time*. I have described several instances where students were not precise in their talk about time. In chapter 9 I suggested that Jenny was implicitly working with the end of the period and was succeeding in doing so. But her lack of precision in her talk meant that this was not explicit for Sizwe. In the case of Hailey discussed above, she was precise in her own references to time. When Jefferson blurred the distinction between end of April and beginning of May, she brought the issue into focus and was explicit in noting that there was a difference for the model they were working with.

Students' use of timelines and potential for errors – Students varied in their use of timelines, and the associated conventions. These findings are no doubt influenced by what was made available in class sessions A and B. However, I do not seek to claim any causal relationships. Some students used T-notation, others used a similar numeric format; some used month-names and yet others used a combination. Within this range, some paid explicit attention to representing T_0 and the end of the last period. Several students provided a variety of elaborations that go beyond the necessary features of a timeline. The ability to represent the appropriate information correctly on the timeline was not strongly correlated with producing the correct answer for the calculation. Thus while it could be argued that there were "errors" in students' timelines because they did not follow the conventions, this did not necessarily lead to errors in their calculations. However, as I have argued, the use of month-names may generate errors. Similarly, failure to indicate the end of the last period may result in excluding the last compounding period.

14.5 What opportunities emerge for learning MfT of annuities?

I return now to the MfT framework and reflect on what might be learned about mathematics-for-teaching annuities from students' talk about time and their use of timelines. As with the introduction to the IP approach, it seems reasonable to assume some similarities between pre-service teachers and learners in schools with regard to talk about time and use of timelines. The issues discussed in this chapter pertain to the following aspects of the framework: *essential features, modelling and applications, explanations* and *learners' conceptions*. Although there is much overlap between *essential features* and *modelling and applications* with regard to timelines, I have chosen to separate the discussion of the two aspects.

14.5.1 Essential features

In chapter 12, I provided a detailed discussion of the conventions of timelines and argued for the importance of making these conventions explicit to students. Knowledge of the conventions is important to access financial maths texts. Teachers also require this knowledge to teach the representation to learners. However, as discussed above, there was a weak correlation between students' accurate use of timelines and success in the test question. This suggests that it is not essential to adhere to the conventions for "personal use" of timelines although Shaun's struggles in his explanation to Palesa were a direct consequence of not paying attention to the conventions.

14.5.2 Modelling and applications

The conventions associated with timelines are important from the point of view of modelling time-series data. One issue to consider is consistency in the use of T-notation across all problem types. For example, one could easily indicate a first annuity payment at T_1 on the timeline regardless of whether it is made at the *beginning* or the *end* of the period. However, this would not give consistent meaning to T_1. Hence the need for a convention in the meaning of T_n which, based on existing conventions, is taken to mean the *end* of period n. This in turn creates a need to represent payment at the *beginning* of the period in a different way, hence the need for T_0 to ensure that the entire first period is represented on the timeline.

The analysis discussed above suggests that there may be advantages and disadvantages of using month-names when mathematizing the contextual problem. The use of month-names provides easy movement between the model and the initial problem. The main disadvantage lies in the likely assumption that month-names may be taken to indicate the *beginning* of the month whereas T-notation represents the end of the month. Consequently learners may fail to indicate the end of the last month, particularly in a scenario where payments are made at the beginning of the month.

14.5.3 Explanations

I focus here specifically on explanations in relation to talk about time. I have argued that teachers should always pay attention to time in their talk and therefore be *precise* in their references to time, for example stating whether a payment is being made at the beginning or end of a month rather than "in" the month. However, I suggested that it may not always be appropriate to make this visible to students/learners, i.e. to be *explicit* about it. If the focus of attention is not directly on timeframes, then it may be counter-productive to shift attention to time and thus away from the focus of attention. This requires that teachers learn *when* it is appropriate and necessary to pay explicit attention to time with their learners.

14.5.4 Learners' conceptions

Teachers need to be attentive to learners' awareness of time issues in financial maths. Such awareness is manifested in learners' use of timelines and in their talk. I have discussed several potential difficulties that students encountered in their use of timelines – both in explanations and in their test scripts, where there was a lack of attention to indicating the full first period and full last period. Teachers should therefore be aware of this as a potential primary source of error when working with timelines. Teachers need to know when and how to support learners in paying attention to time. There may be instances, such as those described at the start of the chapter, where learners' attention is on something other than time and so their references to time may be vague and even largely absent, and this may not be problematic. By contrast, in situations where learners need to attend to time and do not, such as the example with Shaun and Palesa, teachers need to help them bring time to the foreground. This implies the importance of listening to learners' talk and drawing their attention to time elements when appropriate.

The increased use of timelines in GT4 suggests that when tasks are more complex, timelines provide a concise way of organising all the important information in a simple diagrammatic form. It may be that the production of a timeline serves as a trigger to pay attention to key issues such as the timing of payments in the month, the number of times different payments will gain interest, particularly with respect to the final period, and the time-gap between taking a loan and the first repayment.

14.6 Conclusion

In this chapter I have shown that there is evidence of students working implicitly with time in their talk and their use of timelines. There were instances where this was problematic such as Shaun's initial use of the timeline in his interaction with Palesa, but other instances where implicit references were not problematic. There is also evidence of students paying explicit reference to time such as Hailey's interaction with Jefferson. I have argued that being able to pay attention to time (in talk and in use of timelines) is an important resource for working with annuities. It seems obvious that teachers should always be precise in their references to time, and their use of timelines, but that they need to exercise careful judgement when drawing learners' attention to time, and only to do so when a lack of explicit reference to time is problematic for learners' progress.

Aspects of MfT in focus in this chapter

Essential features

- Precision in references to time
- Conventions of timelines, meaning of T_n and T_0

Modelling and applications

- Consistency in use of T-notation across problem types
- Importance of indicating full time periods
- Potential problem of using month-names

Explanations

- Importance of teachers being precise in talk about time but may not be appropriate always to be explicit about this

Learners' conceptions

- Use of timelines may depart from the conventions
- Use of month-names may break conventions of T-notation
- Failure to indicate beginning of first period and/or end of last period
- Confusion between discrete points and intervals on timeline
- Timelines may not be used if task is too simple
- Listening for implied references to time in peers'/learners' talk

CHAPTER 15

Spreadsheets as a resource for learning annuities

15.1 Introduction

I developed the triangular spreadsheets while teaching a financial maths course in 2006. Initially the spreadsheets were a resource to unpack for myself the time-value of individual payments, an issue which to me was hidden in the annuities formulae. I then introduced the triangular spreadsheets to my students, and so they have become a resource for teaching and learning annuities. I then became interested in the extent to which the triangular spreadsheets might influence students' ways of thinking about annuities and their ways of approaching annuities questions, *in the absence of a computer*. This is of particular interest given the limited possibilities of using computer technology for teaching and learning mathematics in the majority of South African schools. I am therefore interested in the possibility that students' use of spreadsheets in their undergraduate studies might trigger new ways of thinking about annuities which might then be exploited in their teaching, and ultimately impact their learners, regardless of access to spreadsheets in schools.

Despite this, I did not set out to investigate students' use of spreadsheets in my study. However, two separate incidents involving each of the focus groups prompted me to look more closely at spreadsheet usage in the study. In the incidents the students made use of spreadsheets in ways that may be extended beyond immediate access to technology. Also, the students' use of the spreadsheets speaks to the matter of MfT in that it exemplifies additional resources for teachers to make sense of annuities that go beyond applications of geometric progressions and the use of the annuities formulae. Thus this chapter exemplifies opportunities for learning mathematics-for-teaching annuities as well as the use of spreadsheets.

Based on my own experience of learning annuities, I argue that teachers need more than algebraic formulae, numeric calculations and timelines to access the essential features of annuities for themselves and to mediate annuities concepts for learners. Elsewhere (Pournara, 2011b) I have argued that spreadsheets enable one to "get inside" annuities, to access the processes that are hidden in the formulae, and thus to track the account balance or individual payments over time. In this chapter I extend that work by introducing the emerging notion of *spreadsheet thinking* and showing how the two focus groups made use of spreadsheet thinking to solve problems that emerged in the group tutorials. The one group worked without access to technology. The other group had access to technology but I shall argue that their reasoning is no longer dependent on the presence of the technology.

The research questions that guide this chapter are:

- In what ways does spreadsheet thinking support the learning of annuities?
- What are the implications of students' spreadsheet thinking for MfT?

I begin by elaborating the notion of spreadsheet thinking. Thereafter I review very briefly the literature on spreadsheets for learning financial maths, and then I discuss two cases drawn from each of the focus groups where students made use of spreadsheets. In the first case I investigate how A-group used spreadsheet thinking without a computer to account for unanticipated findings in their calculations in group tutorial 4 (GT4). In the second case I focus on H-group's use of spreadsheet thinking to make sense of an unusual formula for present value of an annuity. Finally, I reflect on the implications of these cases for MfT annuities.

15.2 The notion of *spreadsheet thinking*

The notion of *spreadsheet thinking* concerns students' propensity to think in ways that are based on spreadsheets and spreadsheet usage. It is a socio-cultural notion since it is predicated on the ways in which thinking is mediated by tool-use (Vygotsky, 1978). It assumes a dialectic relationship between the spreadsheet tool becoming a resource for the user, and the ways in which the user manipulates the tool as a resource for thinking (Artigue, 2002; Hoyles, Noss, & Kent, 2004). For the purposes of this study I restrict my discussion of spreadsheet thinking to the context of annuities[56].

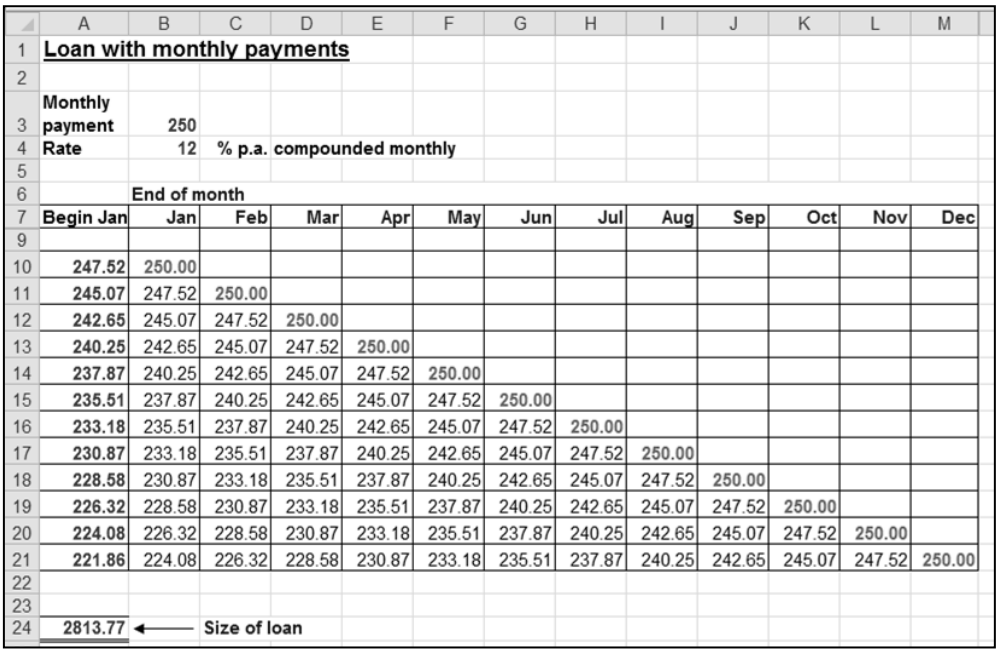
The power of a spreadsheet rests on the ways in which the user can manipulate entries in each cell by means of built-in and easily-programmable formulae, thus generating collections of values in rows and columns to meet the user's purpose. In fig 15.1 and 15.2, I provide examples of the column and triangular spreadsheets used in the course. Similar examples were provided in chapter 12.

	A	B	C	D	E	F	G	H	I	J
1	**Loans, repayments and outstanding balances**									
2										
3			Amount of loan	4 000.00						
4			Rate	14	% p.a.					
5			No. of compounding periods in 1 year	12						
6			Payment	450						
7										
8	Tn	Interest	Balance before payment	Payment	Capital balance	Capital (Amt pd in)	Cumulative interest	Amount of capital paid off	% of installmt on interest	% of installmt on capital
10	1	46.67	4 046.67	450.00	3 596.67	450.00	46.67	403.33	10%	90%
11	2	41.96	3 638.63	450.00	3 188.63	900.00	88.63	408.04	9%	91%
12	3	37.20	3 225.83	450.00	2 775.83	1 350.00	125.83	412.80	8%	92%
13	4	32.38	2 808.21	450.00	2 358.21	1 800.00	158.21	417.62	7%	93%
14	5	27.51	2 385.73	450.00	1 935.73	2 250.00	185.73	422.49	6%	94%
15	6	22.58	1 958.31	450.00	1 508.31	2 700.00	208.31	427.42	5%	95%
16	7	17.60	1 525.91	450.00	1 075.91	3 150.00	225.91	432.40	4%	96%
17	8	12.55	1 088.46	450.00	638.46	3 600.00	238.46	437.45	3%	97%
18	9	7.45	645.91	450.00	195.91	4 050.00	245.91	442.55	2%	98%
19	10	2.29	198.19	450.00	- 251.81	4 500.00	248.19	447.71	1%	99%

Fig. 15.1 Column spreadsheet showing loan repayment and outstanding balance

The column spreadsheet in fig 15.1 models a loan scenario of R4000 with payment at the end of the month. Interest of R46.67 accumulates by the end of the first month (cell B10). A payment of R450 is then made and reduces the balance to R3596.67 at the end of the first month. Columns G to J show how much of each repayment is given to capital and to interest, and the respective percentages. Moving down column E shows that the capital balance reduces each month and the loan would be paid off in 10 months with a final payment of R198.19. Moving down columns I and J shows how the percentage of the instalment given to interest decreases, and the percentage of the instalment give to capital increases.

[56] I refer to the annuities spreadsheets that were used in the course, and not to the built-in functions such as FV, IPMT and NPV in packages such as Microsoft Excel.



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Loan with monthly payments**												
2													
3	**Monthly payment**	**250**											
4	**Rate**	**12**	**% p.a. compounded monthly**										
5													
6		**End of month**											
7	**Begin Jan**	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
9													
10	**247.52**	250.00											
11	**245.07**	247.52	250.00										
12	**242.65**	245.07	247.52	250.00									
13	**240.25**	242.65	245.07	247.52	250.00								
14	**237.87**	240.25	242.65	245.07	247.52	250.00							
15	**235.51**	237.87	240.25	242.65	245.07	247.52	250.00						
16	**233.18**	235.51	237.87	240.25	242.65	245.07	247.52	250.00					
17	**230.87**	233.18	235.51	237.87	240.25	242.65	245.07	247.52	250.00				
18	**228.58**	230.87	233.18	235.51	237.87	240.25	242.65	245.07	247.52	250.00			
19	**226.32**	228.58	230.87	233.18	235.51	237.87	240.25	242.65	245.07	247.52	250.00		
20	**224.08**	226.32	228.58	230.87	233.18	235.51	237.87	240.25	242.65	245.07	247.52	250.00	
21	**221.86**	224.08	226.32	228.58	230.87	233.18	235.51	237.87	240.25	242.65	245.07	247.52	250.00
22													
23													
24	**2813.77**	←	**Size of loan**										

Fig. 15.2 Triangular spreadsheet showing loan repayment of R250 per month for 12 months

Fig 15.2 shows a triangular spreadsheet with 12 monthly payments of R250, each of which is made at the end of the month. The payments are discounted to T_0, the beginning of January, and these time-values are summed to determine the size of the loan (R2813.77).

Drawing on Sfard's (1991) operational/structural distinction, I argue that spreadsheets can be viewed as having both *operational* and *structural* characteristics. The operational component involves the processes (formulae) within the spreadsheet that are manipulated and monitored by means of the inputs and outputs. For example, in fig. 15.2 the basic inputs are *monthly repayment*, *interest rate* and *term of the loan*, which is indicated by the 12 payments from January to December. These inputs are manipulated to generate intermediate outputs which are the time-values at T_0 in column A. The intermediate outputs are then added to determine the size of the loan, which is the final output (cell A24). Thus the time-values are both inputs *and* outputs, generated by the iterative formulae that create the values in each row.

The structural component foregrounds the reified relationships between values. For example, discounting each payment back to T_0 involves multiple operations in the spreadsheet. But at a structural level these processes may be compressed into a single operation such as "move all payments back to T_0", accompanied by a visual image of the triangle where each payment and its growth are visible yet not in focus. This enables one to move away from the spreadsheet itself, and to activate the image and the embedded processes without access to technology.

Spreadsheet thinking is powerful because the spreadsheets provide *opportunity to "get inside" the annuities calculations*. The column and triangular spreadsheets used to model annuities scenarios do not make use of the actual annuities formulae, but *foreground the processes that underpin the annuities formulae*. In most cases the spreadsheets consist only of percentage-change calculations involving the *unit growth factor* (see chapter 6), sums and differences. It is the *absence* of the annuities formulae in the spreadsheets that contributes to deepening students' knowledge of annuities because they are forced to work with the processes and calculations involving "lower level" ideas such

as percentage-change, summation and compound interest (Ma, 1999). This promotes the decompression of the annuities formulae into their constituent processes. Whereas the annuities formulae mask the processes that led to them, the spreadsheet representations consist only of those processes. (See chapter 12 for discussion of hierarchy of annuities concepts.)

Spreadsheet thinking is also characterised by the *spatial images and relations* within the spreadsheet. This means that the spreadsheet image may be used as a *template* (Sfard, 2000) for further learning but also requires the *visualisation* of the spreadsheet image. I draw on Presmeg's (1986) definition of visualisation as a "mental scheme depicting visual or spatial information" (p. 42). In the column spreadsheet this involves a movement from left to right across each row, and from top to bottom down the columns. The horizontal movement reflects movement through each period (usually a month) from opening to closing balance, as well as the sequence of events within that month. The vertical movement, particularly down the closing balance column indicates the changing balance in the account. In the case of a present value scenario, looking down the closing balance column is concerned with checking that the account balance is decreasing, and then looking for an account balance of zero (or less than zero) which indicates that the loan has been repaid. In the case of the triangular spreadsheets, horizontal movement tracks the changing value of a particular payment to T_n or T_0. Vertical movement focuses on the values of each payment that will comprise the account balance at the beginning/end of the period depending on the scenario. The triangular shape of the spreadsheet foregrounds the staggering of the payments in time and thus the different number of periods that must be considered for each payment.

Presmeg (1986) argues that visualisation can occur with or without the visual entity being present. Similarly, I argue that spreadsheet thinking can occur with or without the presence of a spreadsheet – whether it be access to manipulating an electronic spreadsheet or a hardcopy printout of a spreadsheet. However, it seems reasonable to assume that spreadsheet thinking is developed in front of a computer, manipulating the spreadsheet cells and engaging with the results of that manipulation. This is similar to the ways in which *reasoning by continuity* (Cuoco, Goldenberg, & Mark, 1996) is most likely initially developed through use of dynamic geometry software but can later be applied to static geometric figures by imagining the impact on the figure of dragging a point.

An important assumption concerning spreadsheet thinking is that the thinking involved in engaging with a spreadsheet can be activated in settings where there is no access to spreadsheets but where the tasks are similar to those of the spreadsheet setting. In such settings students would reason as if the spreadsheet were present. For example they may draw on the visual form and/or structure of a particular spreadsheet to reason about the task in the new setting. Some (e.g. Billett, 1996) would argue that this constitutes evidence of *transfer*. Lave and Wenger (1991) might argue that the spreadsheet is reconstructed in the new setting/practice and becomes a resource for the user. I argue that spreadsheet thinking is transferable because the spreadsheet becomes a transparent resource for the user (Lave & Wenger, 1991) who is able to make the necessary properties of the spreadsheet visible to be harnessed and used, even when the spreadsheet is not present. In the language of Nemirovsky and Monk (2000), the user is able to *make the absent present*, and thus to engage in the same kind of thinking without the spreadsheet as s/he did when the spreadsheet was present.

15.3 Research on the use of spreadsheets for learning financial mathematics

I did not find any research studies that focus on the use of spreadsheets for learning financial mathematics at school level. Most research on the use of spreadsheets in mathematics education has focused on the learning of early algebra, particularly developing the notion of variable (e.g. Ainley, 1996; Bills, Ainley, & Wilson, 2006; Sutherland & Rojano, 1993). There is some work on spreadsheets and modelling at school level (e.g. Geiger & Goos, 1996; Goos & Geiger, 1995; Lingefjärd, 2006) although most of this work tends to treat the spreadsheet as a resource within the modelling process, and does not focus on the impact of its use on learning mathematical content.

The work by Hoyles, Noss and their colleagues (e.g. Bakker, et al., 2006; Hoyles, et al., 2010) appears to be the only published research-based work to date that focuses on the use of spreadsheets for learning financial maths. In their work on techno-mathematical literacies in the workplace, they designed several interventions with spreadsheets to provide opportunities for employees to make sense of the mathematical models that underpinned their daily work. They argue that spreadsheets gave employees access to the mathematical structure of financial instruments through the simplified model presented in the spreadsheet. By adapting and extending spreadsheets, and constructing their own spreadsheets, employees were able to make sense of the calculations which enabled them to understand the financial products and processes better.

I note too that a number of recent articles promote the use of spreadsheets for learning key aspects of annuities and for modelling financial scenarios (e.g. Feng & Kwan, 2011; Parramore, 2011; Sugden & Miller, 2010) but none of these is research-based.

15.4 Case 1 – Using spreadsheet thinking to deal with an unanticipated answer

In GT4 students worked on an annuities task involving a lump sum and an annuities component with 18 regular payments. The task contained three sub-questions: to determine the total accumulated over a particular period; to determine the impact of missing a particular payment; and to determine whether doubling the payment immediately following the missed payment would make up the deficit (see Appendix C5).

In this vignette I show how Shaun and the group made use of spreadsheet thinking to reconcile an unanticipated answer, without the presence of a computer. In so doing, he made use of the triangular spreadsheet as a template (Sfard, 2000) for his thinking. However, their reasoning contained an error which they did not recognise and which was not revealed through their spreadsheet thinking.

15.4.1 The vignette – justifying an unanticipated answer

At the beginning of the tutorial the students spent a great deal of time discussing each question and considering suitable strategies, but did not yet do any calculations. They recognised that the lump sum could be separated from the annuities component and that each annuity payment gained interest for one month less than the previous payment. They decided to work with an 18-month period for the calculations and so they assumed that the first R300 payment was made at the same time as the lump sum deposit. They recognised the need to calculate interest on the balance at the end of the fourteenth month even though there was no deposit in that month. They also recognised that the missed payment did not affect the payments from month 15 onwards.

These insights reflect a dominance of IP thinking since they were focusing on the individual payments. The only exception was the comment about needing to calculate interest at the end of the fourteenth month which reflects AB thinking since the focus was on the account balance. Surprisingly, the group chose an AB approach for all questions. This decision was partly based on the fact that they did not know what balance to use when calculating interest at the end of the fourteenth month. For question 1 they produced 18 iterative calculations with no attempt to make use of an annuities formula. For question 2 they recalculated the account balance from month 14 onwards, and did likewise from month 15 onwards for question 3. Based on these calculations, Shaun found that the overall amount lost was R1.96. In earlier discussions, Shaun and Attiyah had anticipated that the double payment would lead to approximately R2 *more interest* being gained from the double payment. When they got a *smaller* amount, the whole group agreed that the smaller answer made sense because they had lost one month's interest, and they assumed it was the amount of R1.96. When Attiyah and Palesa calculated a month's interest on R300, they got R1.91. The group ignored the 5c difference assuming it was a result of rounding in Shaun's iterative calculations.

Although they were correct in reasoning that one month's interest had been lost, they assumed the lost interest was from the *first* month (after the deposit had been made) whereas the lost interest is from the *last* month (after depositing). In this scenario it would be the interest of the fifth month which is the amount of R1.96 that they had obtained as the difference.

As they continued to justify their unanticipated answer and to "convince themselves", Shaun imagined the triangular spreadsheet and "drew in the air" to explain how he saw the situation.

Line	Speaker	Utterance
41	Shaun:	Ja. Because the way I see it is that six-hundred rand, if you had to look at it over the spreadsheet, for question two you were missing a whole row out (*draws imaginary row in the air*) …
42	Attiyah:	Ja.
43	Shaun:	… that whole fourteenth payment row was out, but in question three that six-hundred rand row would double up. So it would be three-hundred over here (*pointing to imaginary fifteenth payment*) with that interest going here going three times (*makes "bunny hop" movement with his hand in the air*) and up there (*pointing to imaginary fourteenth payment*) that three-hundred rand, do you know what I'm saying? Would fill in the row above.
44	Attiyah:	Ja.
45	Vusi:	Ja.
46	Shaun:	So that's why it would only be …
47	Attiyah:	to the four.
48	Shaun:	… to the four. Because then there'd be that one three-hundred rand block that wasn't gaining interest. Do you understand what I'm saying?
49	Attiyah:	Ja, ja, ja. For the four months.

Shaun compared the missed fourteenth payment to a "missing whole row" in the spreadsheet [41, 43]. Similarly he saw the fifteenth payment as filling up the empty fourteenth row [43]. But there would be one cell that would remain empty [48] and it would be the first cell for the fourteenth month. I take his reference to "that one three-hundred rand block" [48] to refer to the cell representing the month in which a payment is made (and which would therefore contain the number 300). Shaun also made a connection to the interest calculation (which Attiyah had mentioned previously) noting that the exponent would be four [46-48] and not five.

When Palesa said she didn't follow his explanation, Shaun produced a diagrammatic version of the spreadsheet which I shall refer to as the "spreadsheet diagram" (see fig. 15.3).

The spreadsheet diagram shows the lump sum deposit (D) as well as the 18 monthly payments (P_n). The staggered horizontal arrows indicate when each deposit is made, and the number above each arrow indicates how many months' interest will be accumulated. For example, payment 13 (P_{13}) accumulates interest for 6 months. The students assumed the lump sum and first R300 payment were paid at the same time, and so both payments accumulate interest for 18 months. (Palesa suggested Shaun redraw the line for P_2 to end in line with the previous two lines rather than start beneath them and stop "early". He did so and crossed out his initial line with the wavy line.)

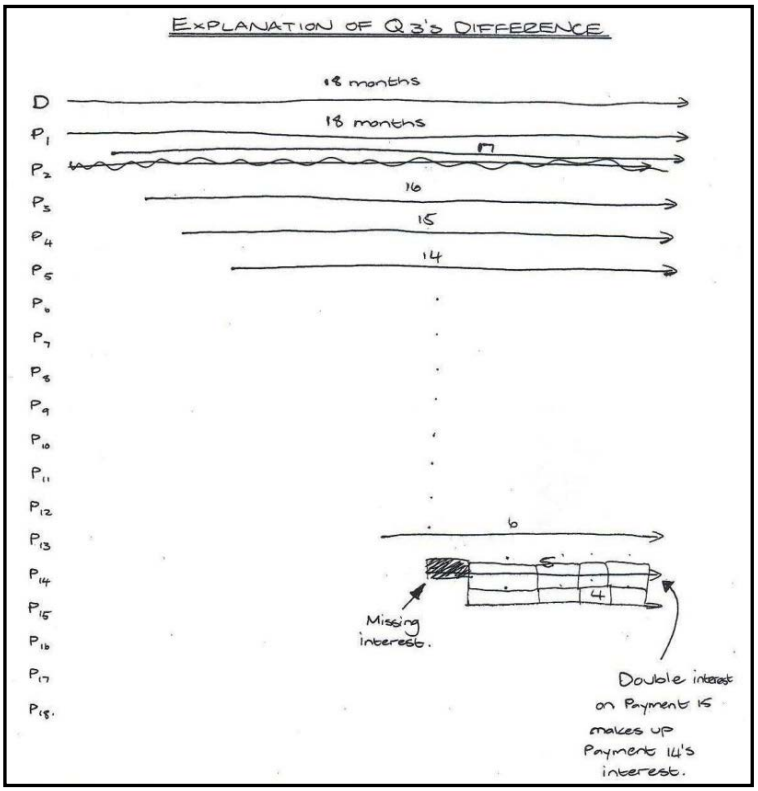


Fig. 15.3 Shaun's spreadsheet diagram

He indicated the doubling of the fifteenth payment with rectangles to indicate each month's interest. The eight "white" rectangles indicate that "double interest" is received for four months. Note that the top four rectangles are drawn over the arrow representing the original fourteenth payment. The shaded rectangle represents the one month's interest that is lost. Based on the idea of copying month 15 over month 14, it makes sense to consider the interest from the first month as the interest that is lost, because visually this is the only month that "sticks out".

Shaun: Because I mean, there would be, that amount of interest will be there and there, that there and there, that there, there, that there, there. But we missing this, okay let me, let me colour this block. That block over there we're missing and that's where that, instead of having five month's interest we're having four months interest.

Thus their error was reinforced rather than exposed by this representation. They did not recognise that the interest for each month should be shifted one month later – the extra payment starts gaining interest in the fifteenth month, and gains interest four times not five. The interest that should have accumulated at the end of the fourteenth month (R1.91) is now only accumulating at the end of the fifteenth month. And since there are only four interest-gaining periods, it does not gain the fifth interest portion of R1.96, which explains why this is the amount that is lost.

15.4.2 Making use of spreadsheet thinking

Shaun used the diagram to illustrate that the impact of departing from the perfect payment plan is the loss of only one month's interest. In drawing the diagram, he mirrored the *process* of setting up the rows of the spreadsheet, which involves multiplying each cell by the unit growth factor. This process was compressed in each line, and the lines became *objects* on which he could operate. He equated

missing a payment with the *removal* of a row in the spreadsheet or crossing out the line for P_{14}. In a similar way he equated the double payment with *copying* of a spreadsheet row, and so the interest for P_{15} was copied (by a vertical translation) over the line of P_{14}, with no horizontal adjustment. The resulting *spatial image* revealed a single empty cell (coloured black in the diagram) which they interpreted as the lost interest.

The power of their reasoning is that it ignores the specific values in individual cells and leads to a generalised explanation that can be applied to similar scenarios involving missed payments, double payments, triple payments etc. However, their diagram did not reveal the error in their reasoning. The benefits of zooming out to operate on the visual image by removing and copying rows, needed to be complemented by zooming in to focus on the values in the cells for payments 14 and 15 which would reveal the growth of the R300 payment each month. It seems reasonable to assume that if the students had access to a digital version of the spreadsheet, and had performed the actions of removing and copying the relevant rows, they may have seen the amounts with interest were shifted one cell (i.e. month) to the right. This may or may not have enabled them to identify the error in their thinking, but it may have led them to reconsider their assumption that the 5c difference in their earlier answers was merely a matter of rounding.

Presmeg (1986) argues that, although visualisation is a powerful resource for mathematical reasoning, it needs to be accompanied by rigorous analytical processes. In this situation the students did not support their visual reasoning with more careful analytic thought. So, while a strength of spreadsheet thinking is the level of generality at which it operates, it also requires some attention to the details of key cells, and thus the deeper analytic thinking to which Presmeg refers.

15.4.3 An alternate interpretation of the error

There may, however, be an alternate explanation as to why students did not recognise the error in their thinking: it may be that their underlying and implicit assumption was that interest for a given principal amount is the same each month (see chapter 6 for a detailed discussion of this issue). If this is the case, then there would be no need to distinguish which month's interest has been lost because the amount is the same. While I cannot be sure, I suspect that if the students had been asked explicitly whether the amounts of interest were the same for each month, they would have responded that they were not. But it may be that when this issue is not in focus, students resort to linear thinking (e.g. De Bock, et al., 2002; Esteley, et al., 2010) and hence work implicitly with an assumption that interest portions are equal for each month. Further support for this argument was found in the test where Attiyah showed evidence of linear thinking when calculating the missing interest in the question based on a missed payment scenario. She was, however, able to identify and correct her error in an interview soon after the test.

15.5 Case 2 – Using spreadsheet thinking to make sense of a peer's formula

The second case comes from Hailey's group where she adapted a spreadsheet to explain why the outstanding balance formula could be used to determine the present value of an ordinary annuity. The task they worked on comes from GT5. Earlier in the week the class had investigated a formula for the following scenario:

A student wants to know how much of her earnings from her year-end vacation work she must invest, at the end of January, in order to be able to make ten monthly withdrawals of R250 during the year (from February to November).

Four different formulae were proposed, all of which produced the correct numeric answer. Three of the formulae were variations of the formula for present value of an ordinary annuity, for example: $O/b = \frac{Pt\left[(1+i)^n - 1\right]}{i(1+i)^n}$ where O/b = outstanding balance, Pt = monthly payment.

The fourth formula, shown below, closely resembled the outstanding balance formula, but students were not yet aware that it was the outstanding balance formula, and so I call it the *JS-formula* after the students who produced it.

$$F_v = P_v(1+i)^n - x\left[\frac{(1+i)^n - 1}{i}\right]$$

where:
P_v = amount saved by student,
x = monthly withdrawals,

I used the four formulae to construct the task for GT5. Students were required to decide which formulae were appropriate models, what the various symbols represented, how the formulae might be derived or obtained, and possible links between them. Students were also required to determine the minimum amount of the lump sum deposit if it were made one month earlier, and then to adapt the general formula accordingly (see Appendix C6).

In order to appreciate the significance of Hailey's contribution described in the vignette below, one must recognise the situation the students found themselves in. The first three formulae were similar and, with minimal algebraic manipulation, could be shown to be equivalent. The JS-formula looked considerably different, and what may have been particularly disconcerting was that a formula for dealing with *withdrawals* contained the formula they had been using for *deposits*. In addition, the JS-formula contained the compound interest formula but it was not clear to them what its role was in the overall formula[57].

15.5.1 Vignette – Spreadsheet thinking to make sense of the JS-formula

During the tutorial the group concluded that the JS-formula worked because it produced the correct numeric value for P_v (provided that $F_v = 0$). However, they did not understand *why* it worked nor where it came from.

The students worked independently to explore the formula, as was their practice, and Hailey made use of spreadsheets. She adapted the triangular spreadsheet to look like the spreadsheet shown in fig. 15.4. This enabled her to show that the deposit made at the end of January should be R2424.37 as shown in cell B21. The original spreadsheet consisted of an "upper triangle" only (this is described in detail in chapter 12, see fig. 12.8). Hailey adapted it to deal with the relevant months, and extended it to produce a rectangle of values by discounting each payment of R250 to the end of January. The different values in each row come from the same geometric progression but have different starting values which are determined by the position of the R250 in the sequence. The different positions of the R250 indicate the month-end at which the student is withdrawing the money.

[57] The only difference between the JS-formula and the outstanding balance formula is that F_v would be replaced with "outstanding balance at T_n". Essentially the formula determines the value of the loan at T_n. It also determines the value of each (re-)payment at T_n, and then subtracts the sum of these values from the adjusted loan amount.

The 11 columns of values show the same geometric progression as the rows. In summing the January column, one is able to calculate the lump sum that must be deposited at the end of January. In the remaining columns, the totals give the balance in the account at the end of each month, assuming no withdrawals or deposits are made. One can calculate the outstanding balance in the account at the end of each month (once R250 has been withdrawn) by adding only the values in each column that lie *below* the 250. For example, at the end of September, the outstanding balance will be R495.81.

Investments with monthly payments at the end of period

Monthly payment: 250
Rate: 6.75 % p.a. compounded monthly

End of Month

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	261.47	262.94	
	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	261.47	
	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	260.01	
	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	258.56	
	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	257.11	
	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	255.67	
	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	254.24	
	239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	252.82	
	237.69	239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	251.41	
	236.36	237.69	239.03	240.37	241.73	243.09	244.45	245.83	247.21	248.60	**250.00**	
Account Balance	2424.37	2438.00	2451.72	2465.51	2479.38	2493.32	2507.35	2521.45	2535.63	2549.90	2564.24	0.00

Fig. 15.4 Hailey's spreadsheet for JS-formula

Row 21 (account balance) can be viewed in two ways. It can be seen as the sum of the values in the column. It can also be viewed independently of the other numbers in the columns. In this case, it is simply a lump sum (of R2424.37) that is accumulating interest each month until the end of November. The essence of Hailey's interpretation of the formula was that the compound interest component was represented by the rectangle, and the annuity component was represented by the original upper triangle. If the upper triangle is subtracted from the rectangle, the lower triangle remains, and this gives the present value. She attempted to explain it to the rest of her group, to see whether her interpretation of the spreadsheet and its relationship to the formula made sense to them. Below I provide sections of transcript of her explanations. She began by focusing on the compound interest component and thus the lump sum, and spoke of sharing it unequally across the ten payments at the end of January to fill up the rectangle. The figures she refers to can be seen in the spreadsheet above.

Hailey: Now if I take this amount [*i.e. 2424.37*] and I divide it into the number of, into the different payments that I need, right, to get to two-fifty (R250), but I don't know what they are. But I take that amount and I share it, okay? And I compound all of those for ten months, right? We're going to have this whole thing [*referring to the rectangle*] full of numbers, right? Are you with me so far? Are you with me about sharing this part? Okay, then it says you're taking that [*referring to Pv*] and you're compounding it ten times, right? Coz you got Pv, um to the 1 plus *i* to the *n* right? [*referring to* $Pv(1+i)^n$] So that will fill this whole table up here, so all of these here you will have numbers in right?

After a brief digression she continued, focusing on the annuity component of the formula, explaining that she thought the purpose of the subtraction was to remove the top triangle – "making this space clean" [255].

Line	Speaker	Utterance
255	Hailey:	… We need to work with their formula to explain how they got it. Okay, so do you understand what I'm saying for the first Pv plus *i,* right? Then we look at the second part, right. It's saying if I put two-fifty (R250) and I compound, and I do different payments, ten different payments, I'm going to get an amount, right, which is what should be after each of these, right? If I put two-fifty (R250) here I'm going to get something, if I put in two-fifty (R250) and it's compounded I'm going to get something so when they minus, they're making this space clean. Are you with me?
256	Sakhile:	Mmmm.
257	Hailey:	And they're saying all of this, what happens here and with that cleaning that must equal to zero - which we also want. We wanted, after this is taken out, to be equal to zero. Okay? Are you with me? Does that make sense?
258	Sakhile:	Ja.
259	Jefferson:	Okay wait a minute.
260	Hailey:	Okay so what would you guys say?

The other students struggled to make sense of Hailey's explanation and so she had to repeat herself at various stages during the tutorial. The fact that they all persisted reflects her confidence in her reasoning, and their acknowledgement that it was worth pursuing. The exchange below took place approximately thirty minutes after the previous explanation. Here she explains that the withdrawals of R250 mean that the top triangle must be removed because the amount is no longer in the account [529].

Line	Speaker	Utterance
521	Hailey:	Okay, now it's compounded ten times, right? So remember with the five-hundred, you could, you could work with each payment differently, and look at how it was compounded and if you added it together it would give you the same amount. So you all with me on that? So if I take it and I share it into these different amounts here in January right? And I compound each of them ten times, I'm going to get this, all of this here (*referring to rectangle of values*), right? So that's what they've done, they've taken this amount, they've compounded it ten times and I'm sure if we put in that value in and we compounded it ten times we should get this total. Okay. You with me on that? Then…are you with me…?
522	Jefferson:	(*Checking their earlier work*) Ja, must get that (unclear) two-five-six-four (*R2564 the total amount at end of November, see cell L21*)
523	Hailey:	Um…ja two five six four (*R2564*).
524	Jefferson:	(unclear) we must get. For … based on looking at the, the annuity one?
525	Hailey:	No, on the, um, if you take two-four-two-four (*R2424*) and you compound it ten times.
526	Sakhile, Lina:	Okay. Umhmm.
527	Hailey:	Okay. Now if you look at here, I've made these bold right? [*referring to the 250's*] This is that annu …, this part here is that annuity. If I put in two-fifty (*R250*) and I compound it ten times, or no, nine times. If I compound it [*counting number of compound-ings]* no that's ten times.
528	Jefferson:	Ten, ten.
529	Hailey:	Sorry, it's compounded ten times, nine times. Okay, if I made <u>payments</u> of two-fifty (R250) for instance. This top triangle is what I get, right? But that's not we want, because this top triangle is no more, coz when he, she takes out that two-hundred-and-fifty then from that two-hundred-and-fifty that is no longer there right? When she takes out that two-hundred-and-fifty, that is no longer….

Another brief digression and then she continued.

Line	Speaker	Utterance
544	Hailey:	Okay, so you get, this, if they were paying it in, we would use the <u>annuity</u> formula, right?
545	Jefferson:	Ja.
546	Hailey:	To find out this part of the triangle [*i.e. upper triangle*]. But because she's taking it out …
547	Jefferson:	Ja, every month.
548	Hailey:	… she doesn't want, …
549	Jefferson:	The top part.
550	Hailey:	… we don't want that part, so we have to subtract it out and that's why they're subtracting the annuity formula. Are you guys with me?

15.5.2 Hailey's use of spreadsheet thinking

Hailey began by focusing on the *process* of discounting all payments to T_0, thus producing a rectangle of values. From there she adopted a *structural* view of the spreadsheet, isolating two triangles within the rectangle and connecting these to the formula. The numbers in the spreadsheet were not important to her. Instead her attention was on the geometric figures, and how these related to the JS-formula. The rectangle of values represents the lump sum of R2424.37 being invested at the end of January and receiving interest until the end of November. This made it easy for her to account for the first part of the JS-formula, i.e. $Pv\,(1+i)^n$, because it represented the initial amount growing without any withdrawals or deposits. This can be seen more clearly in the "account balance" row but since this row consists of the column-sums of the rest of the spreadsheet, using a rectangle to represent the total amount is appropriate.

The upper triangle represents the future value of the annuity component, and shows each R250 withdrawal/ payment accumulating interest beyond the R250 value. This is the amount that must be subtracted because it is no longer in the account and so cannot earn interest [529, 550]. For Hailey this explained why the entire upper triangle must be subtracted. Therefore, only the lower triangle remains, representing money in the account that was gaining interest before being withdrawn.

Spatially Hailey was seeing a rectangle and the upper triangle, and she was subtracting the triangle from the rectangle (fig. 15.5). The lower triangle represents the present value of an annuity but the outstanding balance formula does not foreground this. So while the other three formulae represented only the lower triangle, the JS-formula represented "rectangle take away triangle" and Hailey had found a way of explaining why it worked.

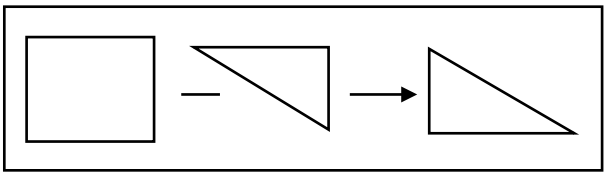


Fig. 15.5 Hailey's geometric interpretation of the spreadsheet

15.5.3 Exploring the group's take-up of Hailey's ideas

I was concerned that the other three students might be operating merely at a visual level, focusing on the rectangle and triangles, without connecting these to the relationships between the numbers in the spreadsheet. If this were the case, then they might be operating with *pseudostructural* conceptions (Sfard & Linchevski, 1994) where the geometric representation is no longer connected to the numeric progressions within the shapes. I therefore set up an additional interview with the group to investigate this further. There is also evidence of further learning during the interview, which I discuss briefly below.

Sakhile, Jefferson and Lina admitted they had struggled to make sense of Hailey's ideas. Sakhile could not see immediately how the spreadsheet would help them to understand the formula and he did not want to invest in trying to understand it because it might "confuse me worse". But once he started to get a sense of Hailey's reasoning, he pursued it. He explained how he had understood that the upper triangle needed to be removed but had struggled to link the compound interest component of the JS-formula into the spreadsheet. He said the totals (row 21) helped him to see this by focusing on compounding of R2424.37 for ten months.

There was evidence that Jefferson and Sakhile could decompress the geometric interpretation. For example, Sakhile recognised that to set up the rectangle of values, each amount of 250 must be moved backwards (discounted) an appropriate number of times. When I asked what would happen if we divided the January balance of R2424.37 by $(1 + i)$, Jefferson correctly said it would give the balance for December of the previous year. Lina did not respond to my probing questions directly and so I cannot make claims about her take-up of Hailey's ideas.

15.5.4 Further learning in the interview

There was evidence of further learning resulting from the interactions during the interview. For example, Sakhile saw that depositing a lump sum of R2424.37 was equivalent to making ten monthly deposits of R250. He saw these as two investment options that would produce the same amount. Hailey too gained new insights. Previously she had not noticed that the values of the two components in the JS-formula became the same at some point, and thus produced a difference of zero. She had thought the compound interest component should be R2424.37 (starting value) and the annuity component would be R2564.24 (value at the end of November), but then realised this would not give a difference of zero. In focusing on trying to explain why the JS-formula worked, she had imposed a geometric interpretation on the spreadsheet, and in doing so she had missed what is obvious in the algebraic representation - that the two components must produce a difference of zero and therefore must be equal. We did not pursue the fact that this would only occur for one value of n.

15.6 In what ways does spreadsheet thinking support the learning of annuities?

In this section I focus on two ways in which spreadsheet thinking supports the learning of annuities: *working with multiple representations* and *sustaining connections between processes and objects at different levels*. I also discuss briefly the ways in which the spreadsheet became a transparent resource for the students.

Working with multiple representations

Both vignettes illustrate that spreadsheet thinking involves working with multiple representations. This requires students to *make connections* between representations as they *shift between* the representations. In A-group, the students moved from a numeric representation, where they were trying to account for the differences in their answers, to a pictorial representation of the triangular spreadsheet. From there they reasoned about the missed and double payments and ultimately associated the empty rectangle in their diagram with the missing interest, thus returning to the numeric. They also connected numeric and algebraic representations: the loss of one month's interest was connected to a change in the exponent in the formula "it would only be to the four" [46, 48].

H-group demonstrated connections between algebraic, numeric, and geometric/pictorial representations as they moved between them. Hailey began with a shift from the algebraic formula to the numeric values in the spreadsheet. This led her to see geometric shapes in the clusters of numbers. She then associated the two components of the formula with the geometric shapes. Finally, the subtraction within the formula was given a spatial meaning as the "geometric subtraction" of triangle from rectangle.

Connecting processes and objects at different levels

The use of spreadsheets has the potential to reinforce links between processes and objects at different levels, where low-level processes give rise to objects which are operated on at higher levels, leading to

higher-level objects etc. (Sfard, 1991). I argue that spreadsheet thinking may promote the ability to move between the different levels, and between processes and objects, which may then reduce the possibility that reified higher-level objects become severed from their operational roots. Given the complexity of the annuities formulae, it seems possible that students and learners may lose the links between the formulae and the processes that generated them, unless they are consistently required to attend to these links.

In chapters 6 and 12, I proposed networks and hierarchies of concepts with links between *unit growth factor*, *growth factor*, the *compound interest formula* and the *annuities formulae*. Spreadsheets consist mainly of low-level percentage calculations such as $250(1 + 0.05)$, while algebraic work involves the higher-level compound interest formula and annuities formulae. Capitalising on connections between the two representations may serve to "keep alive" the links between processes and objects, and thus reduce the possibility of reifications becoming frozen and disconnected from the processes. Consider the following example which is illustrated in fig. 15.6.

We begin with two objects, a principal amount of R250 and a unit growth factor, $(1 + i)$, which are multiplied to produce the object $250(1 + i)$. This new object is then multiplied again by $(1 + i)$ to produce another, more complex object $250(1 + i)(1 + i)$, and the process continues in adjacent cells along the row. The compounding process can be compressed in the compound interest formula, yielding a new object, $FV = 250(1 + i)^m,\ m\epsilon\mathbb{N}$. This occurs on multiple rows, thus generating a future value for each payment. These future values undergo another process where they are summed, generating yet another higher level object in the annuities formula, $FV = 250\left(\frac{(1+i)^n-1}{i}\right)$.

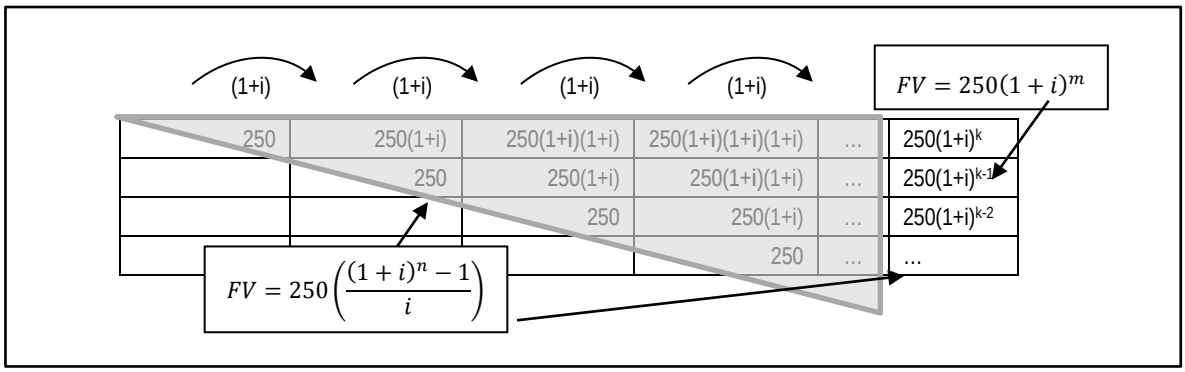


Fig. 15.6 Links between processes and objects at multiple levels in an annuities spreadsheet

The entire process described above can be executed in a right-to-left movement where amounts are discounted back to T_0, by dividing by the unit growth factor. The repetition of this process is captured in the discounting formula, $PV = FV(1 + i)^{-n}$. The present values generated in each row are then summed, which ultimately leads to the present value of annuities formula, $PV = 250\left(\frac{(1+i)^{-n}-1}{i}\right)$.

In case 2, Hailey and her group used the spreadsheet to open up the mathematical situation to the level of individual payments growing monthly. Additional entries were generated by discounting all amounts back to T_0 which created a rectangle of values. The cells were then clustered and compressed to the level of particular investments, either a lump sum or an annuity, which enabled them to collapse ranges of values into two different (investment) processes: the process of compounding a single deposit and the process of an annuity-based investment. These investment processes were reified into geometric shapes which were operated on by subtracting the annuity component from the lump sum.

15.6.1 Using spreadsheets as a transparent resource

Both groups show evidence of working with spreadsheets as a transparent resource (Lave & Wenger, 1991). In A-group the students, particularly Shaun, were able to draw on the features of the spreadsheet to make it literally and figuratively visible to solve their problem (although they did not account completely for the difference in final amounts). They were making the *absent present* (Nemirovsky & Monk, 2000) since they had no access to computers (although Shaun did refer to a hard copy of the spreadsheet after his initial explanation [41-49]).

As I have argued elsewhere (Pournara, 2009b), there were several instances for H-group where the spreadsheet became transparent. Firstly, in seeing through the spreadsheet, the students had to give a double meaning to each 250. It was simultaneously the amount to be withdrawn *and* the amount that grows but which isn't there. In this sense the 250's belong to the upper *and* lower triangles. This required the students to interpret the spreadsheet even beyond the rectangle and triangle. Secondly, by imposing these geometric shapes over the clusters of payments, Hailey made the *absent present*. Thirdly, the final row of the spreadsheet gives the account balance. If one were strictly modelling the remaining balance in the account, then the values in this row should be diminishing to zero. But since the students had added all terms in each column, the "account balance" actually represented the hypothetical account balance, assuming no withdrawals were made. In the original spreadsheet containing only the upper triangle, the column totals accurately represented the growing account balance. In the adapted spreadsheet the summing of columns enabled students to assign meaning to the first part of the JS-formula but this no longer produced the correct values for the account balance. They ignored this fact because it did not fit with the way they wanted to interpret the representation. Thus they were making the *present absent.*

15.7 What opportunities emerge for learning MfT of annuities?

The use of spreadsheets across the course provided various opportunities for learning MfT of annuities. For example, the annuities spreadsheets provided powerful insights for students into socio-economic and financial aspects because of the ease with which the spreadsheets accommodate long periods of time, and the ease with which the user can manipulate the inputs and get immediate feedback. Students were easily able to test the impact of a 50 basis-point increase in the interest rates on the terms of a loan, and the consequences of not increasing the monthly repayments when interest rates rise. This gave them insight into the negative impact of compound interest on loans over time and the consequences for the extension of the loan and the increase in the cost of the loan. Fig 15.1 shows how each instalment is distributed between repaying interest and repaying capital. In the case of a larger loan such as R200 000 at 14% p.a. compounded monthly, and repaid over approximately 20 years at R2500 per month, 93% of the first payment goes to repaying interest.

In this section I draw on the data from the two cases to consider the aspects of *essential features* and *learners' conceptions*. I also reflect briefly on the role of spreadsheet thinking with regard to explanations.

15.7.1 Essential features

I have discussed in detail the important role of percentage calculations in building the annuities spreadsheets, and how these are compressed into "virtual" compounding and discounting calculations, particularly in the triangular spreadsheet. I speak of "virtual" because the compound interest and discounting formulae are not coded into the spreadsheet. Nevertheless, students work with an

adjustment view of both these formulae as they relate them to the movement of payments along the rows of the triangular spreadsheet.

In chapter 2, I argued that representations are a component of essential features (rather than a separate component, as in Even's (1990) framework) because mathematical entities can only be accessed through representations. I identified algebraic, numeric, graphical, verbal and pictorial representations. In financial maths the timeline is an important pictorial representation that has been endorsed by the financial mathematics community.

In this chapter I have argued that spreadsheet thinking requires and enables students to *move between representations* and to *make connections between representations*. Despite this, spreadsheets are not widely endorsed as a representation, although there is increasing evidence that they are useful resources (Feng & Kwan, 2011; Hoyles, et al., 2010; Parramore, 2011; Sugden & Miller, 2010). The question to consider then is whether spreadsheets may be considered an *essential* representation or whether they are peripheral and "nice to have". The answer may lie in the fact that practices, and resources within them, change over time. For example, the number line is widely used from the early years of schooling, yet historically there was contestation over its acceptance as a valid mathematical representation (Heeffer, 2011). Perhaps in the future spreadsheets will be regarded as an essential means to represent, work with and learn annuities, particularly given the growing possibilities of online text books. However, in developing world contexts such as South Africa, this begs the question whether particular representations can be considered essential features of a concept if they cannot be readily available to all.

15.7.2 Modelling and applications

It could be argued that all the work done over the eight-day period concerned modelling and/or applications since students were working with annuity scenarios. In this chapter I have also included data from GT5 which required students to investigate four formulae proposed as models for the monthly pay-out scenario and which ultimately led to the standard formula for present value of an annuity. The students' use of spreadsheets and spreadsheet thinking came in response to modelling the missed and double payment scenario (in the case of A-group) and the monthly pay-out scenario (in the case of H-group). Both groups used and adapted the triangular spreadsheet to create spatial models of their respective problems, and connected these to the algebraic and/or numeric models.

15.7.3 Explanations

Spreadsheet thinking functions as a new knowledge resource for teachers in explaining the processes and relationships that are central to the annuities concept, and which are hidden in the formulae. Both groups were able to engage with their problems, even if not completely correctly, by making use the triangular spreadsheet which enabled them to access lower level processes and objects. From the outset of this chapter I have been concerned with students' ability to use spreadsheet thinking in the absence of technology. The reasoning displayed by both groups draws on the *structure* of the spreadsheet and the relationship to the annuities formulae and the compound interest formula. It therefore appears to be sufficiently general to apply beyond the context of the tasks on which they were working, and does not require access to technology. Although it cannot be assumed that students' use of spreadsheet thinking in the course will transfer to their teaching, the possibility for such transfer appears to exist, and requires empirical investigation.

15.7.4 Learners' conceptions

I deal here with the students' conceptions, working on the assumption that if learners are given similar access to and experiences of spreadsheets, they may be able to make use of the column and triangular spreadsheets in similar ways, and develop spreadsheet thinking.

Firstly, it must be noted that the students were familiar with the triangular spreadsheet, and were able to produce it and other spreadsheets with relative ease. This knowledge and the associated skills meant that they could access the annuity concept at the level of its constituent processes – percentage change, sums and differences – to generate the entries in each cell. The many repeated values in the spreadsheet provide access to the patterns of growth of each payment. These patterns are a powerful feature of the spreadsheets because they are not visible in lists of separate numeric calculations nor in the annuities formulae. Furthermore, while the compound interest formula is not used to generate the triangular spreadsheet, the students were able to take "short-cuts" to imagine using it to move payments backwards or forwards in time, in a single step, thus taking an adjustment view of the formulae. All these factors supported the students' spreadsheet thinking and enabled them to move between processes and objects at multiple levels as discussed above.

Nevertheless, there were two content errors in A-group's reasoning. Firstly, the students assumed that a 5c difference in their answers was due to rounding. However, the difference was due to the additional payment being made a month later than the original payment and thus accumulating interest for one month less. The second error is related to the first and concerns the possibility that students may have been working implicitly with a notion of equal interest per month. The group had correctly divided the nominal annual interest rate into equal monthly portions to yield a monthly rate but may then have associated this with equal amounts of interest per month.

15.8 Conclusion

In this chapter I have drawn on two cases of spreadsheet usage to show how the students in the two focus groups made use of spreadsheet thinking to solve problems in ways that do not require access to technology. I have suggested that the column and triangular spreadsheets as well as spreadsheet thinking provide powerful resources for teachers to "get inside" annuities and to mediate the learning of annuities, even in the absence of technology. The notion of spreadsheet thinking is an emerging one that needs to be further theorised. In addition, further research is necessary to investigate the assumption that the ability to make use of spreadsheet thinking to solve problems in an undergraduate course (or in teachers' own learning more generally) will enable teachers to employ spreadsheet thinking in their teaching where technology may or may not be present.

Aspects of MfT in focus in this chapter

Essential features

- Adjustment view of compound interest and discounting formulae
- Spreadsheet thinking is a resource that supports shifting between representations and making connections between them
- Testing formulae

Modelling and applications

- Using IP model for missed and double payment scenario
- Confirming equivalence of present value models
- Modelling a withdrawal scenario using outstanding balance formula

Explanations

- Spreadsheet thinking is a knowledge resource for explaining key processes and relationships in annuities
- Use of spreadsheet thinking to explain how peers' formula models present value scenario

Learners' conceptions

- Using spreadsheets and/or spreadsheet thinking to make sense of workings of new formula and unanticipated numeric answers
- Errors - possibility of inappropriate linear thinking, and incorrect assumption about rounding errors

CHAPTER 16
Concluding Part 2
Learning MfT through learning new mathematics

16.1 Introduction

In the previous four chapters I have argued that learning annuities is fundamentally about changing the unit of focus from *account balance at the end of the period*, to *the behaviour of individual payments over time*. In GT3, many students attempted to use the compound interest formula to model the annuities scenario. For example, Attiyah added all 12 payments together and then compounded interest for 12 months on the sum. This calculation modelled all 12 payments gaining interest for 12 months. Shaun, on the other hand, recognised that the compound interest formula could not be used and so had begun to do twelve AB-calculations. He suggested that his calculations might yield a larger total than hers:

> Shaun: I think it's gonna be more than if we just work out the compound interest for, d'ya know what I mean? If we say all the payments and work out the compound interest? I think it's going to be more if we work out the compound per month plus the two hundred and fifty, the compound for that month plus two hundred and fifty. I mean maybe it would be interesting to see if it does make a difference.

To the person who is thinking in terms of individual payments gaining interest, it may be surprising that Shaun could not predict that his answer would be smaller than Attiyah's since only one payment was gaining interest for 12 months. By contrast, a week later he was drawing on the image of a spreadsheet with individual payments, and compared a missed payment with the removal of a line in the spreadsheet. There is no doubt that he had made the necessary shift in his thinking and taken up an IP approach. I have argued that IP thinking may be considered a secondary intuition since it is dependent on instruction (Fischbein, 1999).

In this chapter I summarise the obstacles students experienced in learning annuities, as well as the opportunities for learning mathematics-for-teaching annuities that were discussed in chapters 13 to 15. As before, I do so through the lens of the MfT framework. I then reflect on the framework itself, considering again the aspects of *modelling and applications* and *explanations* as signalled in chapter 11. This is followed by a brief reflection on the structure of the framework and the relationships between clusters of aspects within the framework. I then critique an argument regarding the teaching of time-value calculations by Gardner (2004) which I discovered late in the write-up of this thesis, and which challenges the approach I took in the course. Finally, I reflect on three aspects of learning new mathematics in pre-service mathematics teacher education.

16.2 Opportunities for learning MfT through learning new mathematics

This section serves a similar purpose to the corresponding section in chapter 11, in that it draws together the previous chapters. However, there are several differences here. Firstly, part 1 of the thesis was structured around four cases, each of which emerged from critical incidents. Part 2 has focused on

students' thinking and has been structured around three themes which cut across a variety of incidents and data sources. In all three themes mathematical content has been in the foreground, and consequently different aspects of MfT came in and out of focus. This is discussed in more detail below.

	First encounter with IP approach	Talk about time and timelines	Spreadsheets and annuities
Essential features of annuities	Deriving formula for future value of ordinary annuity	Key representation in financial maths; conventions of timeline; precision in references to time	Adjustment view of compound interest and discounting formulae; moving between and connecting representations; testing formulae
Relationship to other mathematics	Geometric progressions	Compound interest	Percentage change; compound interest; generating links between processes and objects at different levels
Modelling & applications	AB vs IP models; change unit of focus from *account balance at end of month* to *value of payment at* T_n; conventions in modelling payment at end of period; application of sum of geometric progression	Consistency in use of T-notation; potential problems of using month-names on timeline; representing full time periods	Using IP model for missed and double payment scenario; confirming equivalence of present value models; modelling a withdrawal scenario using outstanding balance formula
Mathematical practices	Working intuitively; working inductively; generalising; applying known formulae	Attending to conventions of representations; precision in mathematical communication – verbal and written	Visualising; connecting representations; testing and interpreting formulae; accounting for unexpected results; generalising
Basic repertoire			
Different teaching sequences and approaches	Application of geometric progressions vs modelling of annuity scenario		Deriving present value of annuity formula through solving a realistic problem
Explanations	Erroneous thinking emerged through students' explanations (2 cases)	Learning from own explanation to peers; need for teachers to be *precise* always in their talk about time but may not always be appropriate to be *explicit* about this to learners	Spreadsheet thinking is resource for explaining key processes and relationships; use of spreadsheet thinking to explain new formula and unanticipated answer
Learners' conceptions	Recognising that compound interest formula is not an appropriate model; questioning whether IP approach is a valid model; questioning the need to include last deposit if it gets no interest; struggling to make sense of IP approach from an AB perspective; blurring boundary between end of month and beginning of next month; questioning the meaning of n in annuities formula Inadequate pre-knowledge of summation formulae for geometric series	Use of timelines may depart from the conventions; failure to indicate beginning of first period and/or end of last period; confusion between discrete points and intervals on timeline; timelines may not be used if task is too simple Listening for implied references to time in peers'/learners' talk	Access workings of new formula and unanticipated numeric answers through spreadsheet thinking; possibility of inappropriate linear thinking; incorrect assumption about rounding errors
Contextual knowledge of finance	Need to suspend knowledge of how banks deal with interest when using IP model		Spreadsheets provide efficient means to see impact of changes in interest rate/monthly payment, etc. over long periods of time.

Table 16.1 Summary of chapters 13, 14 and 15 in relation to MfT framework

Table 16.1 provides a summary of the previous three chapters, read through my MfT framework. The tasks I designed for group tutorials 3, 4 and 5, and the ways in which I chose to implement them, had substantial bearing on the activity in the tutorial groups and in class session A. As I noted in chapter 11, if I had designed different tasks for introducing annuities, or if I had implemented the same tasks in different ways, the consequences for this part of the course and for the findings of the study may have been different.

As with part 1, the most notable absence in the table is the aspect of *basic repertoire*. This aspect was not intended to be in focus when annuities were introduced. It may be argued that my introduction of the timeline in preparation for deriving the future value annuity formula constituted evidence of a basic repertoire in teaching the representation, but this was not necessarily in focus for students since they were still grappling with the basic elements of the IP approach.

16.2.1 Essential features

Across the eight-day period, the tasks and activity resulted in much attention being given to the *annuities formulae* and to *multiple representations*. In GT3, and class sessions A and B, students focused on the *derivation* of a formula for future value of an ordinary annuity. In GT5 they investigated four proposed formulae for present value of an annuity. In addition, many groups attempted to produce formulae in GT4 to model the missed and double payments, although this was not reported in the previous chapters. When working with the compound interest and discount formulae, students needed to adopt an *adjustment view* of the formulae since they were moving amounts forward or backward in time. With regard to representations, I addressed several *features of the timeline* in class session B, and in chapter 15, I argued that the use of spreadsheets required students to *move between representations* and to *make connections between representations*. In that chapter I also questioned whether spreadsheets could be considered an essential feature of annuities since they are not (yet) widely endorsed in introductory financial maths texts (e.g. Basson, et al., 2005; Young, 1993; Zima & Brown, 1996) and, in developing countries, the technology is not available to all.

16.2.2 Relationship to other mathematics

Given the focus on learning new mathematics, there was little emphasis on the relationship of annuities to other mathematics. In class session B, the future value formula was derived through the summation of a geometric progression, and students made similar use of geometric progressions when deriving a formula for present value in the lead-up to GT5. Connections to lower level concepts such as percentage increase and compound interest were evident when dealing with timelines and spreadsheets. In order to generate spreadsheets for annuities, students had to build from individual percentage calculations which combined to model compounding of payments, and these in turn generated the individual annuity payments. Thus the students generated links between processes and objects at different levels, and thus worked between operational and structural conceptions of annuities.

16.2.3 Modelling and applications

I separate the discussion on modelling and applications.

Modelling – Opportunities for learning to model annuity scenarios were present in the three previous chapters. However, the modelling process itself was backgrounded. In chapter 13, I focused on the distinction between AB and IP models of annuity scenarios, arguing that students need to make a

fundamental shift in the unit of focus – from account balance at the end of the period to growth of payments over time, particularly their value at T_n or T_0. The power of the IP model was reinforced in chapter 14 when students modelled the missed payment and double payment scenario. In chapter 15, H-group initially struggled to grasp how the JS-formula modelled a withdrawal scenario, but this was resolved through their ability to use an IP approach to deconstruct the components of the formulae and to relate these to the spreadsheet. In chapter 13, I noted that a paradox arises in the IP model because two independent processes occur simultaneously at the end of a period: a payment is made and interest is compounded but the payment does not gain interest. This paradox is resolved by the convention that a payment made at the end of the period does not gain interest in the period in which it is deposited. I noted that the paradox arises only in the model and not in reality because banks calculate interest daily, and so a payment made at the end of a month will gain one day's interest. With regard to modelling time, I discussed the convention that T_n refers to the *end* of period n but T_0 refers to the *beginning* of period 1. I also suggested that the use of month-names may be problematic because they are likely to be interpreted as indicating the *beginning* of the month whereas the convention on timelines is to indicate the *end* of the month. In the example of Shaun and Palesa, and in the timelines produced by some students in the test, I showed evidence that students sometimes fail to represent the full final period in a payment-in-advance (or annuity due) scenario, which may lead to difficulties.

Applications – As noted in chapter 13, students did not recognise the need to apply the formula for the sum of a geometric progression (or suitable techniques to derive this formula) in order to produce an explicit formula for the future value scenario. However, when working on the present value scenario, they applied the techniques and/or formula with ease.

16.2.4 Mathematical practices

While learning annuities, there were few instances when mathematical practices were the object of attention. This is reflected in the previous three chapters where *mathematical practices* were not identified as an "aspect in focus" in any of the chapters. Consequently, when I first produced table 16.1, I struggled to complete the row relating to mathematical practices. I then went back to each chapter and deliberately focused on the mathematical practices that were exemplified. I was then easily able to insert entries in the relevant cells. This does not suggest that it was difficult to identify the practices, but that mathematical practices were not in focus and hence were not part of the discussion in each chapter. For the purposes of this combined reflection on the three chapters, I distinguish mathematical activity where students *generated* mathematical entities, from activity where they *engaged with the mathematical productions of others*.

In order to generate their own mathematical entities, such as formulae for future value and present value, the students *worked inductively* from numeric examples. As noted in chapter 13, most groups did not recognise that they needed to apply the summation formula for geometric progressions in order to generate an explicit (or closed-form) formula for future value. In the same chapter I argued that many students' *intuitive approach* to the problem was to focus on account balance and not on the individual payments, and thus they did not treat the task as an *application of known mathematics* viz. geometric progressions. When working on GT4 and GT5, both focus groups *reasoned with visual images* of the triangular spreadsheet, which required them to *connect several representations*. I also discussed the work done by A-group to *deal with an unexpected answer* when the missed payment did not generate the additional amount of interest they had *intuitively* predicted. In chapter 14, I focused

on the importance of *precision in verbal and written mathematical communication* with particular focus on talk about time and use of timelines.

There were also several instances where students had opportunity to engage with the mathematical productions of others. For example, in GT5 H-group had to test and interpret the formulae produced by other students. They began by confirming that all four formulae produced the correct answer and therefore concluded that all were valid models of the present value scenario. They were able to identify similarities between three formulae, and showed how one formula could lead to another through appropriate algebraic manipulation. However, they struggled initially to interpret the JS-formula and to make sense of *how* it modelled the present value scenario. Hailey's spreadsheet work ultimately provided a convincing explanation of their interpretation of the formula.

16.2.5 Different teaching sequences and approaches

I adopted a different approach to introduce annuities, choosing to work from a realistic problem rather than the typical application of geometric progressions. Thus students' first encounter with annuities came through *deriving a formula* for future value as opposed to merely *applying a known formula*. This is discussed in detail in chapter 13. In a similar way students were later introduced to present value of annuity through solving a realistic problem involving a monthly pay-out, and were also required to derive a formula. My approach to present value differed from the standard loan repayment scenario. I chose a monthly pay-out scenario because it "works forwards" in a similar way to the future value scenario, i.e. each payment must have enough time to gain interest to reach the required amount of the pay-out. By contrast, the loan repayment "works backwards" where each repayment is discounted to T_0. Based on my previous experience, students had initially struggled to appreciate why summing of these present values determines the loan amount, whereas in the pay-out scenario they appeared to find it easier to accept that the present value is smaller than the pay-out amount.

With regard to different teaching sequences, I noted in chapter 12 that some texts deal first with present value annuity scenarios. I chose to begin with future value scenarios because my experience of previous financial maths courses suggested that students found it easier to move amounts forward in time than to move them back to T_0.

16.2.6 Explanations

As in part 1, learning to produce explanations was not a goal of the eight-day period of learning new mathematics. However, there are many examples of students' peer explanations across chapters 13, 14 and 15. These include instances where students appeared to gain new insights while providing an explanation to a peer, for example, Shaun's explanation to Palesa in chapter 14 and the importance of attending to full periods when payments are made at the beginning of the period.

In chapter 13, I discussed two instances where *students' explanations exposed errors in their thinking that were not revealed in their written work.* When working on the future value formula, Attiyah and Shaun produced factorised expressions such as:

$$FV_4 = 250[(1.005)^4 + (1.005)^3 + (1.005)^2 + (1.005)]$$

Attiyah incorrectly assumed that the term representing the fourth payment was $(1.005)^4$ whereas the term is (1.005). This is not revealed in her written work because of the terms that constitute the expression. I also discussed Jefferson's error when dealing with payments and compounding of interest at the end of a period. While he recognised that payments made at the end of the period do not

accumulate interest in that period, he appeared to assume that the interest accumulated at the *beginning* of the next month. This too was not exposed in his calculations.

In chapter 14, I discussed the varying extents to which students made explicit reference to time in their talk. I noted instances where a lack of attention was not problematic for students, and other instances where a similar lack of attention proved to be an obstacle to further progress. I concluded by suggesting that the issue is not attention to time *per se* but rather in knowing *when* to bring time into focus. In a similar way, I suggested that teachers should always be *precise* in their references to time but that *it may not always be beneficial for learners to make this explicit*, since it may distract from the focus of learners' attention. In Lave and Wenger's (1991) terms this may lead to a situation where time issues are too visible for learners, and consequently they may not be able to see through the time issues to focus on other aspects that require attention.

Students' use of spreadsheets provided opportunity to "get inside" annuities, and thus to make use of spreadsheet thinking to explain aspects of annuities through reference to processes and objects at different levels. This promoted shifts between operational and structural conceptions. For example both Hailey and Shaun referred to percentage calculations and compound interest calculations in explaining their ideas to their peers. The spreadsheet thus gave access to more features of annuities and provided a means to explain them.

16.2.7 Learners' conceptions

There was no explicit attention to learning about learners' conceptions of annuities over the eight-day period. However, I discussed in detail the obstacles that students themselves encountered, particularly when the IP approach was introduced, and I make the assumption that learners may encounter these or similar obstacles. The obstacles are listed in table 16.1 and I do not repeat them here. There is evidence to suggest that the IP approach was not intuitive for all students, but that once they had accepted it, they were able to use it flexibly as shown by the focus group students while working on GT4 and GT5. In Fischbein's terms this may be regarded as secondary intuition (Fischbein, 1999). (See chapter 3 for discussion primary and secondary intuitions)

In GT4, A-group incorrectly attributed differences in their answers to rounding error. This suggests the need for greater attention to error in modelling and applications work, particularly to the likely sources of error, and to typical magnitudes of error in a given problem context. Related to the incident in GT4, I suggested that the students may have been working implicitly with a notion of linear growth. Research suggests that over-generalisation of linear thinking is persistent even at university level (Esteley, et al., 2010). This is important knowledge for teachers as they support learners to deal with proportion and exponential growth in applied contexts.

Students' use of spreadsheets stands in contrast to the deficits in their conceptions described above. Spreadsheet thinking appears to provide a powerful means of accessing the annuity concept at a range of lower levels and thus may provide a knowledge resource for teachers to support their explanations to learners, even in the absence of technology.

16.2.8 Contextual knowledge of finance

In chapter 11, I noted that this aspect was virtually absent from table 11.1. In contrast I anticipated that contextual knowledge of finance would be prevalent in the annuities work. However, when I came to

complete table 16.1, I found a similar absence which puzzled me. On reflection I recalled that in GT3 there was evidence that students had not fully grasped the cycles of "payment followed by interest". For example, in chapter 14 Shaun explained to the group how the cycles worked, and why compounding a lump sum 12 times was not an appropriate model of the situation. This reflected some grasp of the banking process on his part. It also reflected the importance of ignoring the actual banking practice of daily interest calculations, which had been an obstacle for some groups as they attempted to include the detail in their models in GT3. In GT4 there was clear evidence that students were working with at least a rudimentary knowledge of the payment-interest cycles. For example, A-group noted that in the fourteenth month they needed to accumulate interest on the account balance even though no payment had been made in the month. This may suggest that knowledge of the payment-interest (or interest-payment) cycles is the minimum contextual knowledge of finance for working with annuity calculations.

In chapters 6 and 12, I suggested a wider range of contextual knowledge of finance which I argued is important knowledge for teachers but which may not be necessary for successfully teaching time-value calculations. In their interviews, several focus group students expressed concern that learners might ask about aspects of finance, and that they might not be able to provide suitable answers.

16.3 Reflecting further on the MfT framework

In reflecting on the MfT framework in chapter 11, I paid particular attention to the aspects of *modelling and applications* and *explanation*. I return to these aspects here and continue my earlier discussion. In addition, I reflect on *mathematical practices* in the context of learning new mathematics. Thereafter I consider again possible clusters of relationships between the aspects of the framework.

16.3.1 Individual aspects

Modelling and applications

In chapter 2, I motivated my decision to separate *modelling and applications* from *mathematical practices* because of the prevalence of modelling in financial maths. In chapter 11, I acknowledged that the separation of the two aspects had forced me to look more carefully at and for other mathematical practices that may otherwise have gone unnoticed in part 1. In part 2, *modelling and applications* has been dominant. In earlier chapters I acknowledged that other mathematical practices were not in focus in the analysis, and so it is even more likely that the mathematical practices discussed above would have remained hidden had the framework not forced me to make them visible. This further confirms the value of separating *modelling and applications* from *mathematical practices.*

Explanations

In chapter 11, I showed that *explanation* cuts across other aspects of the MfT framework, and suggested that pre-service teachers appear to draw on several of these aspects in producing explanations. I further suggested that *explanation* may be *a different kind of aspect* of teachers' mathematical knowledge for teaching. In part 2, as in part 1, knowledge of explanation has not been in focus although there are many examples of peer explanations across the previous three chapters. I have drawn on the content of these explanations to support various claims such as students' attention or lack thereof, to time; students' use of IP thinking; and students' spreadsheet thinking. My focus has thus been on the "what" of the explanation rather than on the "how". It seems inevitable that this should be the case when little is known about students' conceptions of the mathematical content under

discussion because the "what" constitutes new findings, and is not merely a confirmation of the findings of previous research about students' conceptions of a particular concept.

16.3.2 Structure of the MfT framework

In table 16.2, I summarise the aspects of MfT that were in focus in each of the previous three chapters. This table reflects similar trends to table 16.1. I have identified with "xx" those aspects that were given substantial attention in each chapter. I have used "x" to indicate aspects that were present but not in focus in the analysis.

The table shows that three common aspects were in focus in each chapter: *essential features, modelling and applications* and *learners' (or students') conceptions*. In addition, *different teaching sequences and approaches* and *contextual knowledge of finance* were given some attention in students' first encounter with an IP approach. Given the focus on applied mathematics and that students were learning annuities for the first time, it is not surprising that *essential features* and *modelling and applications* are dominant in the table. As noted above, I work with the assumption that learners may experience similar obstacles to the pre-service teachers and hence included these aspects under *learners' conceptions*. This provides the potential to learn about learners' conceptions as part of future research.

In drawing on data from the group tutorials, it is also reasonable that students' explanation should be prevalent in the data. However, as noted previously, the reference to explanation here is not to the knowledge base for producing explanations but on what might be learned about teacher explanations from the explanations given by students.

	Aspect of MfT	First encounter with IP approach	Talk about time and timelines	Spreadsheets and annuities
Mainly mathematical	Essential features of annuities	xx	xx	xx
	Relationship to other mathematics			
	Modelling & applications	xx	xx	xx
	Mathematical practices	x	x	x
Mainly pedagogical	Basic repertoire			
	Different teaching sequences & approaches	xx		
	Explanations	x	x	x
	Learners' (students') conceptions	xx	xx	xx
Contextual	Contextual knowledge of finance	x		

Table 16.2 Aspects of MfT framework in focus in chapters 13, 14 and 15

A similarity between table 16.2 and the corresponding table 11.2 related to revisiting school maths is that there are aspects in focus from both the *mainly mathematical* and *mainly pedagogical* categories across the three chapters. This supports my claim in chapter 2 that it is difficult to separate SMK from PCK and hence my preference for the notion of mathematics-for-teaching that embraces both and is not concerned with strict classifications of aspects of the framework.

With regard to learning as acquisition and participation, the analysis of part 2 has taken a strongly acquisitionist position in chapter 13. In chapters 14 and 15 a participationist view comes to the fore in the focus on students' talk, and their use of timelines and spreadsheets.

16.4 Reflecting on teaching and learning annuities

In this section I begin with a response to a journal article that challenges the approach I have taken to annuities and the modelling of time. This is followed by a brief reflection on what might constitute an appropriate appreciation of the time-value of money.

16.4.1 Gardner's challenge to my approach to annuities and the modelling of time

While completing the write-up of this thesis, I came across a short article by Gardner (2004) in a non-accredited journal (Journal of College Teaching and Learning). In this article, Gardner challenges typical practices in the teaching of time-value of money calculations. The article piqued my interest for two reasons. Firstly, Gardner rejects the emphasis on the timing of payments at the beginning/end of each period. Thus he argues against the position I have taken throughout the thesis with regard to modelling time. Secondly, I had recently attended a financial maths workshop presented by an experienced and well-respected maths teacher from a local secondary school, where there were similarities between Gardner's approach and what was presented in the workshop. In this section I discuss two of Gardner's challenges and reflect on these in relation to my own teaching of annuities, and in relation to my argument in the thesis regarding the modelling of time.

Gardner's argument is an attempt to deal with the problem that students struggle to identify the correct formula for annuities problems, and hence he seeks to reduce the number of formulae they are required to know. It is important to note the differences between Gardner's context and mine. He works with business/commerce students who make use of financial calculators, and are not expected to derive the annuities formulae, nor to work from first principles with sums of geometric progressions. By contrast, my students are located in a mathematics teacher education setting, are required to derive the formulae, to teach the derivation to others (if necessary), and do not work with financial calculators.

Gardner's first challenge concerns the meaning of n in the annuities formulae. He argues that n should be defined as "number of payments" and not "number of periods". This comes as a surprise since n *does* represent the number of payments. As shown previously, the meaning of n originates in summing the geometric progression, and thus represents the *number of terms* added, where each term represents an individual payment. However, Gardner's call comes in response to his survey of ten US text books on introductory financial management/corporate finance which he claims do not explicitly define n as "number of payments". Moreover, he claims that five texts incorrectly define n as "number of periods". This being the case, his call is both sensible and necessary. I would further support his call for explicit attention to the meaning of n in the annuities formulae since it is a source of difficulty for students, as shown in chapter 13. However, Gardner speaks of "*re*defining" n (p. 25, italics mine) which misses the point that *from the outset, n is defined as the "number of payments"* based on the derivation of the annuities formulae.

In Gardner's second and more controversial challenge, he calls for "dispens[ing] with the pointless and purely semantic discussion of whether payments occur 'at the beginning' or 'at the end' of the period" (p. 28) suggesting that this "becomes confusing and arbitrary for students" (p. 27). In response, I argue that distinguishing between the timing of annuity payments provides the *basis for justifying* the different (albeit seemingly arbitrary) positions in time to which the annuity formulae refer. For example, the formula for the future value of an ordinary annuity, consisting of k regular

payments, gives the accumulated amount *immediately after* the k^{th} payment has been made, and the formula for the present value of an annuity gives the time-value of all the payments *one full period before* the first (re-)payment is made. Gardner implies that students should merely accept the timings associated with the formulae as fact. While this may be pragmatic for business/commerce students, it is not appropriate for teachers who should be able to justify the timings to themselves and to others.
Gardner proposes that the correct answer for both future value and present value annuity calculations can be obtained if n is defined as number of payments, *irrespective of whether payment is made at the beginning or end of the period*. In response I argue that teachers should know *why* the correct answer is obtained when focusing on number of payments rather than points in time. This requires some unpacking of the derivation of the annuities formula. Consider the following example where the problem is *framed by the number of payments* and not by the time period: five monthly payments, P, are made at a nominal monthly interest rate of i (compounded monthly). Assume the first payment is made today and we want to know the future value immediately after the fifth payment is made.

As shown in fig. 16. 1, this scenario spans a four-month period but involves five payments. The first payment, P_1, accumulates interest four times, the second payment, P_2, accumulates interest three times, and so on. Thus the fifth payment, P_5, does not accumulate interest. This generates the sum of five terms of a geometric progression:
$P(1+i)^4 + P(1+i)^3 + P(1+i)^2 + P(1+i) + P$
which leads to the standard formula for future value of an ordinary annuity, and gives the accumulated amount at the point when the last payment is made.

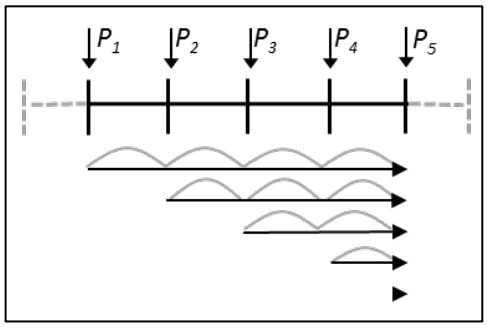


Fig. 16.1 FV scenario: 5 payments covering 4 periods

A minor adjustment to the timeline is necessary if we view this problem from a standard *ordinary annuity* perspective. In this case each P_n is made at T_n which by convention is the end of period n. In order to represent this on the timeline in fig. 16.1, we need to represent the whole of the first period which is indicated by the broken line on the left of the timeline.

A similar addition to the timeline is necessary if we view the problem as a standard *annuity due* (or payment in advance) scenario with payments at the beginning of the period. Here we need to represent the whole of the last period which is indicated by the broken line on the right of the timeline. However, the answer using Gardner's method is still the accumulated amount immediately after the fifth payment has been made, and thus each payment has gained interest for one less period than in the standard annuity due scenario. The adjustment to match an annuity due scenario would require that the future value be compounded for one more month using the compound interest formula. This generates a multi-step solution which Gardner argues is preferable to learning different formulae for ordinary annuity and annuity due scenarios. However, given the minor difference between these two formulae, the burden of learning two formulae seems minimal.

In a present value scenario, the conventional model is that all payments are made at the end of the period. This requires us to consider the full first month and by convention to mark the point T_0 as the beginning of the first month. All payments are then discounted to T_0. If we extend Gardner's approach

to a present value scenario (something that he does not do explicitly), then we are seeking to know the present value of the five payments when the first payment is made. This is represented in the fig. 16.2

There are five payments but only four discounting periods, as shown in the diagram. For example, P_3 is discounted twice. If we sum the present values of the five payments, we obtain the present value of the annuity *when the first payment is made*. This differs from the standard present value scenario, and would require that all payments are discounted one more period as indicated by the broken line on the left of the timeline in fig. 16.2, and the additional "hops". Thus Gardner's proposal appears to break down further in present value scenarios.

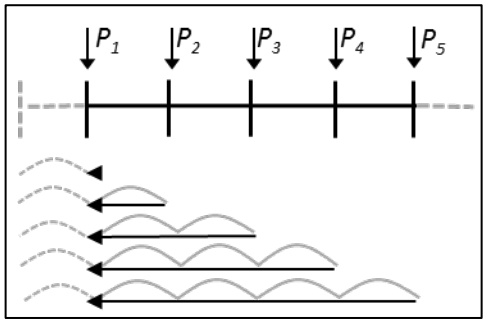


Fig. 16.2 PV scenario: 5 payments covering 4 periods

While Gardner argues that the distinction between payments at the beginning/end of a period is "pointless", "confusing" and "arbitrary", one might suggest a similar arbitrariness in his approach. I refer to the question raised by Jefferson in chapter 13: why include the final payment if it does not gain any interest? In chapter 13 my response was that the final payment was included because we are modelling full time periods. This argument does not hold for Gardner. In his method he has to include the final payment (but no further compounding time) in order to get the correct answer from the standard ordinary annuity formula.

In summary, I provide two criticisms of Gardner's approach. Firstly, Gardner disregards (full) time periods and focuses on the timeframe spanning first payment to last payment. While his approach produces the same answer as the ordinary annuity formula for future value, it does not do so for future value of an annuity due or for present value of an annuity because the time-values in his approach are given at different points in time to the standard formulae. Thus while he seeks to break with the conventions of explicitly modelling payments at the beginning/end of a period, he cannot avoid the ways in which the standard formulae model these scenarios, and how the answers from his approach may or may not need to be adjusted to fit with the standard answers. My second criticism is that in his attempts to simplify the solution process for students, Gardner departs from the conventions of modelling time-series data, particularly the attention to full time periods. For teachers, and others who require knowledge of applied mathematics beyond basic annuities calculations, it is short-sighted to ignore these conventions.

16.4.2 Students' appreciation of the time-value of money

Gardner (2004) claims that the "[t]ime value of money (TVM) is a crucial component in understanding finance. Students must become comfortable and competent in the use of TVM techniques if they are to be successful in the business school" (p. 25). Few, if any, would disagree with Gardner's sentiment that knowledge of the time-value of money is central to understanding finance, and that this applies to all economically active citizens, and is not limited to students in business schools. However, it seems to me, that in introductory financial maths/corporate finance texts (e.g. Brigham & Ehrhardt, 2008; Correira, et al., 2003; Zima & Brown, 1996) an "understanding" of the

time-value of money has largely been reduced to competence in executing time-value calculations. This is reflected in chapter titles such as "time value of money" which consist of several pages of formulae and calculations involving present and future values of single amounts and annuities.

While there is no doubt that an appreciation of the time-value of money *includes* the "competent use of the TVM techniques" (to quote Gardner above), facility with the techniques is not sufficient evidence of an adequate appreciation of the time-value of money. Such an appreciation should include knowledge of buying power and inflation, and the ability to move amounts of money forward and backward in time as part of a personal problem-solving strategy that goes beyond the standard use of the basic annuity formulae. In the course I gave explicit attention to such time-value issues in weeks 10 and 11, including buying power and inflation, but this section of the course does not form part of the doctoral study.

16.5 Reflections on learning new mathematics in pre-service mathematics teacher education

16.5.1 Opportunities for students to re-engage their own mathematical productions

In chapter 15, I discussed briefly that Sakhile and Hailey acknowledged gaining new insights into their spreadsheet work as a result of the focus group interview. This suggested that the opportunity to re-engage their work in a setting where they were required to explain and justify their thinking was a productive learning experience for them. In his final interview, Shaun expressed this sentiment convincingly:

> Shaun: I think this whole doctorate thing was probably the most helpful for me because when we came for those group interviews, we would then almost work double on stuff we did in class … I mean I'll never forget that one where we sat at this table, we sat there with that spreadsheet for I don't know how long trying to explain what was going on, on that spreadsheet and … you know, in class you do it for marks and in this group interview you do it to clarify.

Later in the interview we spoke about his experiences of participating in a focus tutorial group:

Line	Speaker	Utterance
223	Craig:	So it makes me wonder whether there is something about being under the, the microscope type of thing that made it work, umm, I don't know, I really …
224	Shaun:	But I also think it was these, these interviews afterwards …
225	Craig:	…that, ja, I mean, you could be, that, as you say, you're actually getting more time on a problem …
226	Shaun:	That's it
227	Craig:	… umm than the others.
227	Shaun:	… and, and more, I think it's more than more time, it's that you came in prepared with questions digging deep into the information that was on there (*i.e. their GT report*). So it wasn't just this surface thing anymore, it was almost, you know, the bottom of the ocean that we're, that we're digging up to get to the surface.

Shaun's comments highlight two aspects. Firstly, the opportunity to re-engage with their work provided additional time on task. He spoke of the opportunity to "almost work double on the stuff we did in class". But then he pushed it further: "it's more than more time" [227]. Here he referred to my probing in the interview, which required them to explain further and to elaborate on what they had written in their report. This suggests that the opportunity to re-engage with their own work was productive because it involved additional time on task but, more importantly, it was the nature of the re-engagement that was particularly productive. The students were required to explain their ideas

verbally, and were probed further on various aspects. Shaun's reference to "digging up to get to the surface" [227] suggests an unpacking for self, and a further interrogation of relationships they may previously have over-looked. There may be similarities here with my interactions with Vingin and Naasiha as described in chapter 10, who in explaining their thinking to me, came to recognise that the growth factor could become larger than 2 which enabled them to account for an unexpected earlier result.

Shaun's enthusiasm for the focus group interviews may need to be tempered by the fact that no other students made explicit reference to them in their interviews. It may be that students who were coping well with the course, such as he and Hailey, were in a better position to gain from the focus group interviews than students who were not coping as well. Nevertheless, further research is necessary into the learning opportunities that may be opened up for pre-service teachers through mediated re-engagement with their mathematical productions. Such mediation could be done by a lecturer or peer in a later year of study, but such a practice is both labour-intensive and time-consuming and so one would need to establish the extent of the benefit to students before proposing it as a viable mechanism for learning mathematics-for-teaching.

It would be remiss of me to ignore Shaun's comments that the group tutorial and the focus group interview were considerably different settings for him, and with different goals. Whereas in the group tutorial "you do it for marks", in the interview "you do it to clarify". His comment is interesting because in chapters 14 and 15, I provided extended transcripts of his explanations to Palesa and the rest of the group which reflected deep engagement with the task, a commitment to helping each other learn, and hence much clarification of the mathematics they were working on.

On the other hand, it could be argued that the assessment component of the group tutorial may have impacted negatively on the depth to which some students engaged with the tasks – privileging correct answers over the learning of the group. Based on evidence provided in chapters 14 and 15, this was not the case for the two focus groups. I cannot make the same claims for the other tutorial groups due to a lack of evidence. However, a fellow doctoral student who was using a similar tutorial-group design in her study, and who observed all tutorial groups working on GT5, suggested that the requirement of a report (and its assessment) appeared to push students to engage with more commitment than in her study where reports were not required.

16.5.2 Pace and coverage

In chapter 11, I noted the slow pace in weeks 2 to 4 when revisiting school maths, and the consequences for content coverage. In the eight-day period in part 2, the pace was marginally faster but the amount of content covered was also small. During both periods of the course my decisions about pace and coverage were influenced by students' responses to tasks and particular issues I wished to pursue in the research such as students' first encounter with an IP approach. In part 1, I partly justified the concerns about pace and coverage by arguing that revisiting provided opportunity not only to attend to school maths content but also to pay attention to mathematical practices such as defining, proving and the use of mathematical notation. However, the same justification cannot be used in part 2 since mathematical practices were not in focus in the teaching, as noted earlier in the chapter. In hindsight, and following the analysis of class session A, the time given for students to work further on deriving the future value formula was not well spent, with the notable exception of H-group

who succeeded in deriving a formula in that period. Many other groups made little progress in that time and would have benefited from intervention that directed them towards an IP approach. Having said this, the opportunity to listen to groups as they worked on their derivations provided further evidence that most students were not considering an IP approach, and this was of benefit to the research.

16.5.3 Learning with technology

In the South African context, as in many developing countries, there is limited access to technology for teaching and learning. In their interviews, Sakhile and Jefferson raised concerns about the fact that students in rural schools would not have access to spreadsheets. For Sakhile, this was motivation to ensure that he could solve problems with and without technology. In chapter 15, I provided evidence of the power of spreadsheet technology to access the annuities concept. Furthermore, I argued that teachers who have learned financial maths with technology may be able to provide a bridge into contexts where teaching and learning take place without technology, by employing spreadsheet thinking. In so doing, the potential may exist for the knowledge resources acquired from working with spreadsheets to be transferred to contexts where learners are learning without access to spreadsheets. This requires further research.

16.6 Conclusion

In part 2 I have dealt with the obstacles students' encountered in learning annuities. I have also overlaid this with a focus on opportunities for learning mathematics-for-teaching through learning new mathematics. I have shown the difficulty of focusing at the same time on the specifics of the new mathematical content and general aspects of mathematical practices. This struggle took place at the level of students' and my attention in the course, but also in the analysis. I have argued that spreadsheets and spreadsheet thinking may provide a means for teachers to work with annuities in ways that open up the objects and processes which are reified in the annuities formulae, and thus to enable continual shifts between operational and structural conceptions of annuities.

CHAPTER 17
Conclusions and reflections

17.1 Introduction

Mathematics-for-teaching is complex and multi-faceted, comprising some aspects that are mainly mathematical and others that are mainly pedagogical. In the case of financial maths, MfT also includes broader knowledge of finance and socio-economic issues. The mathematical content of MfT includes school mathematics and advanced mathematics. Therefore learning MfT involves revisiting school math and learning new maths. Part 1 of the thesis focused on opportunities and potential opportunities for learning MfT by revisiting school maths. In part 2, I focused on students' obstacles in learning new mathematics. I argued that the obstacles faced by the student teachers in learning annuities are likely to be similar to those encountered by learners in schools. Therefore by studying the student teachers' errors and difficulties, we gain knowledge of an aspect of MfT of annuities, i.e. *learners' conceptions*.

In chapters 11 and 16, I summarised the findings of parts 1 and 2 respectively, and so I do not repeat them here. In this chapter I summarise the contributions of the study to teaching and learning financial mathematics. I then reflect on and revise the conceptual framework for MfT, focusing on the relationships between the aspects of the framework. In the first part of the study I elaborated the notion of *revisiting*. I return to revisiting in this chapter, and I exemplify it in relation to annuities, drawing on the revised MfT framework. I then identify and discuss two issues relating to MfT that have not been resolved, and which require further research. I conclude the thesis with a brief reflection on insider research, and reflect on my own learning of financial maths as an instance of revisiting.

17.2 Contributions to introductory financial mathematics

The study makes both a theoretical and an empirical contribution to knowledge of teaching and learning introductory financial mathematics. The theoretical contribution takes the form of conceptual tools for thinking about compound interest and annuities. These are elaborated in chapters 6 and 12, and are listed below.

- A *reference landscape* for compound interest which includes the following constructs:
  - A hierarchy of interest concepts;
  - The notions of *growth factor*, *unit growth factor* and *rate per period*;
  - A network of concepts for *growth factor*;
- An expanded view of the compound interest formula, distinguishing an *accumulation view* from an *adjustment view*;
- A *reference landscape* for annuities which includes the following constructs:
  - A hierarchy of annuities concepts;
  - The notions of account balance (AB) approach and individual payment (IP) approach to annuities;
- The notion of *spreadsheet thinking.*

In addition to the notion of spreadsheet thinking, the spreadsheets developed for exploring annuities, particularly the triangular spreadsheets, (see fig. 12.8 and fig. 15.2) are a contribution at the level of teaching and learning materials.

The empirical contribution takes the form of insights into students' conceptions of annuities, their use of resources such as timelines and spreadsheets, and the obstacles they encountered in learning annuities. In addition, revisiting school maths provided insights into students' conceptions of compound interest as reported in chapters 10 and 11.

17.3 Revisiting a conceptual framework for MfT

In chapter 2, I proposed a framework for MfT, drawing on existing frameworks in the literature. In chapters 6 and 12, I noted potential concerns about this framework. In this section I reflect on the framework, and propose a revised and elaborated framework, based on the analysis reported in the thesis. I begin by identifying a problem in my interpretation of one aspect of the framework.

17.3.1 A shift in my interpretation of an aspect of the MfT framework

When I set up the MfT framework in chapter 2, I included the aspect *knowledge of different teaching sequences and approaches* which I ascribed to Ferrini-Mundy et al (2006), and not to Even's (1990) better-known aspect *knowledge of different ways of approaching a concept*. As it turned out, opportunities for learning this aspect did not arise in part 1. However, in part 2, I tended to slip in my interpretation of this aspect, at times moving towards Even's interpretation. This shift can be seen in the descriptions of this category in table 16.1. There are instances where I referred to the aspect from a pedagogical perspective, for example introducing learners to annuities by means of realistic problems versus an application of geometric series. There are also instances of referring to approaching the annuities concept using AB versus IP thinking. In this case, my interpretation tends towards Even's aspect. The distinction may be blurred by the constructs of AB and IP approaches. However, it is productive to include both "versions" of this aspect in the framework, but to separate them. I take Even's aspect to foreground *mathematical* aspects (as in her example of approaching functions from global or point-wise perspectives). I shall refer to this as *different approaches to a concept*. By contrast Ferrini-Mundy et al foreground *pedagogical* aspects in focusing on *teaching approaches and teaching sequences*. I shall refer to this as *different teaching sequences and (different teaching) approaches*. This aspect is linked to curriculum knowledge because it depends on knowledge of what must be taught at different grade levels, as well as decisions about sequencing in relation to necessary pre-knowledge, for example prior knowledge of n^{th} roots is required to determine the interest rate in a compound interest calculation. These additions to the framework are indicated in fig. 17. 2.

17.3.2 Vertical and horizontal relationships between aspects of MfT

In chapter 2, I proposed three clusters of aspects, as listed below, and represented them diagrammatically as shown in fig. 17.1.

Mainly mathematical:	Essential features, relationship to other mathematics, modelling and applications, and mathematical practices
Mainly pedagogical:	Basic repertoire, different teaching sequences and approaches, explanations, and learners' conceptions
Contextual:	Contextual aspects of finance

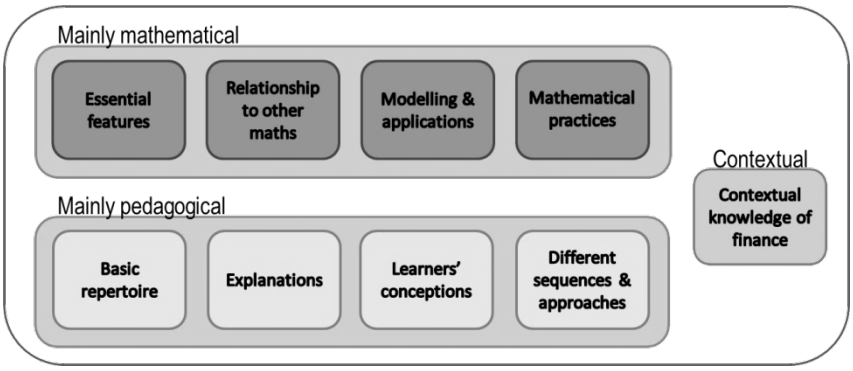


Fig. 17.1 Initial MfT framework

Based on the analysis, I now propose a revised structure of the MfT framework (fig. 17.2), including the additional aspect *different approaches to a concept*. In the revised structure I focus on the relationships between aspects. While all the aspects are interrelated to some extent, I foreground the key relationships that emerged from the analysis.

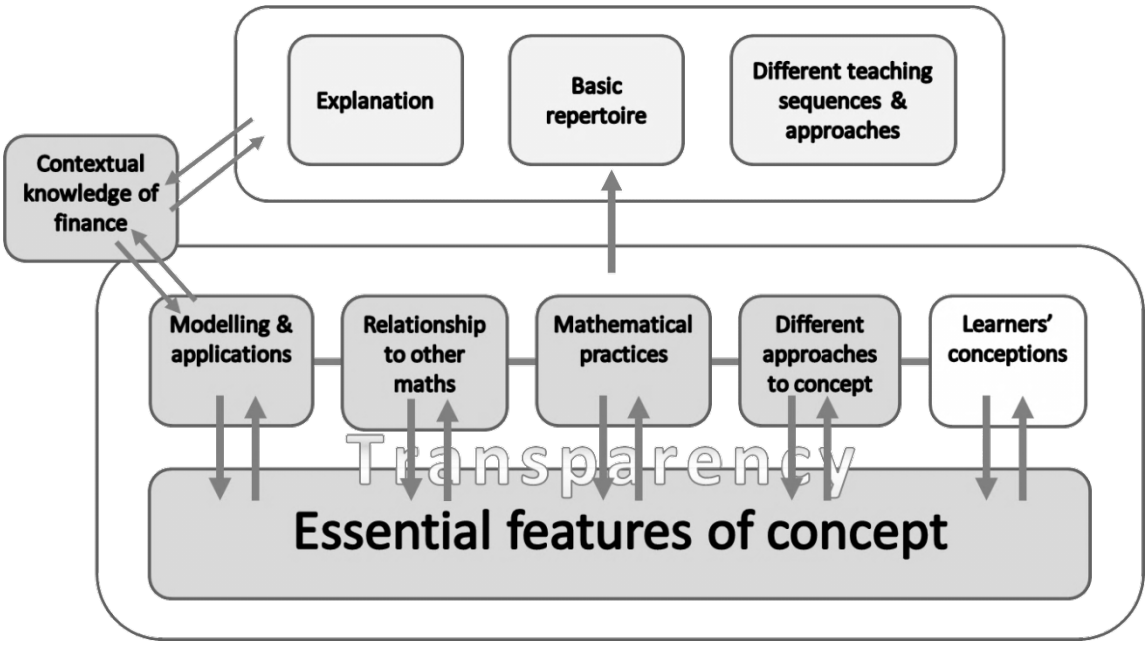


Fig. 17.2 Revised MfT framework

In the revised framework, I propose *vertical and horizontal relationships* between different aspects. These relationships are structured by ways of learning (or revisiting) the essential features of a concept. *Essential features* is a foundational (or primary) aspect, and underpins all other aspects (with the possible exception of *contextual knowledge of finance*). The five aspects located above *essential features*, which I shall refer to as *secondary aspects* of MfT, draw on the essential features, but also give further insight into essential features. This vertical relationship is indicated by the double arrows. Upward arrows indicate an aspect draws on essential features. Downward arrows indicate that the aspect provides further access to essential features. As can be seen, four aspects come from the

"mainly mathematical" cluster. The fifth aspect, *learners' conceptions*, was previously located in the "mainly pedagogical" cluster. I place it alongside the other four aspects because they all provide further insight into essential features. I illustrate the vertical and horizontal relationships by means of four examples dealing with the different aspects, and show how the relationships promote shifts between operational and structural conceptions.

Learners' conceptions – In chapter 9, I showed that *strategic use of learners' responses was productive for revisiting essential features of compound growth*. The selected learners' responses were inefficient, partially correct and did not make use of the compound growth formula. In order to make sense of responses such as these, the teacher has to reconsider: the fundamental elements of the formula, the way in which compounding occurs, and how the compounding process is compressed in the formula. In moving between these compressed and decompressed mathematical forms, teachers shift between operational and structural conceptions, as shown in fig. 3.1.

Relationship to other mathematics – In chapter 8, I reflected on using the compound interest formula to solve a typical exponential growth problem. I argued that the *unusual use of the formula* made visible for me that the present value (or principal amount) relates to the *beginning* of the first period. While this is a basic element of essential features of compound interest, it was tacit for me, and thus initially I was unable to draw on it. In reproducing the step-by-step calculations shown in table 8.1, I shifted from the compressed formula to the daily doubling process and was able to recognise what should have been obvious to me from the outset. Once again this reinforces the importance of being able to work operationally and structurally.

Mathematical practices – I have discussed several instances where mathematical practices, such as working inductively, are exemplified in the work on compound interest and annuities. The use of Newton's method to determine the interest rate of an annuity illustrates a close relationship between *essential features*, *mathematical practices* and *relationship to other mathematics*. The use of Newton's method *per se* is an instance of the relationship between annuities and other mathematics. The link to mathematical practices lies in the use of numerical methods as a *new mathematical technique* which students require to determine the interest rate. Thus annuities provides a rationale for learning the new technique.

My final example exemplifies vertical and horizontal relationships in the framework, specifically the relationships between *essential features, modelling and applications* and *different approaches to a concept*. When dealing with outstanding balance on a loan, the AB approach of repeatedly deducting the repayment and then adding interest, clearly reveals the small portion of capital that is paid off in the early stages of the loan, and the large portion of the repayment that goes to interest. While the outstanding balance formula, based on an IP approach, provides an efficient means to determine the amount still owing, it is the iterative AB calculations that show the process whereby the balance increases with interest and then decreases after a repayment. In this case the *compressed and decompressed forms provide different insights* into the loan repayment process hence the need for teachers to move between the compressed and decompressed forms in their own learning and in their teaching.

The above examples illustrate the vertical relationship between *essential features* and secondary aspects in the MfT framework, as well as horizontal relationships between the secondary aspects. In working between essential features and the other aspects, various elements of the essential features may become transparent. I list three examples drawn from the data, with the relevant aspect/s indicated in brackets:

- Transparency of mathematical notation for Palesa (relationship to other maths, chapter 7);
- Transparency of symbol in compound interest formula for me (modelling and applications, relationship to other maths, chapter 9);
- Transparency of spreadsheet for H-group in interpreting JS-formula (learners' conceptions (peer thinking), modelling and applications, chapter 15).

In each case, the mathematical resource became transparent through use and provided new insights into the essential features of the mathematical entity in focus.

17.3.3 Contextual knowledge within the MfT framework

In chapter 2, I indicated that the aspect *contextual knowledge of finance* did not form part of my initial framework for MfT. Despite this, I anticipated it would be more visible in the analysis than it has been, and I partly attribute the reduced emphasis to the two sections of the course that I chose to analyse. If, for example, I had included week 1, contextual knowledge would have been more prevalent in the analysis. Nevertheless, in fig. 17.2, I place *contextual knowledge of finance* in a position partially overlapping the larger rectangular shape to indicate that it is part of MfT of compound interest and annuities, and yet it is not closely connected to other aspects with the exception of *modelling and applications*. In the case of a concept that is not drawn from applied mathematics, such as *angle*, there may be little or no need for a contextual aspect in the framework. Similarly, it may not be necessary to separate *modelling and applications* from *mathematical practices*. In chapters 11 and 16, I argued for the importance of separating these two aspects in the study, and I discussed how the separation has enabled me to pay attention to features of mathematical practices beyond modelling and applications which are clearly also present in financial maths. I retain the separation in fig. 17.2.

17.3.4 Aspects of MfT that are "mainly pedagogical"

In the top part of fig. 17.2, I have separated three aspects which come from the "mainly pedagogical" cluster of the initial framework: *explanation*, *basic repertoire* and *different teaching sequences and approaches*. These aspects are primarily concerned with knowledge of how to connect learners with the mathematical concept in focus, and thus include: example and task selection, choice of representations, and mediation. They draw on knowledge of the other aspects of MfT, as well as curriculum knowledge and general pedagogic knowledge. For example, a basic repertoire of examples is dependent on knowledge of the content to be covered at different grade levels. Since these aspects were not in focus in the analysis, any proposals concerning them are made with caution. I propose that these aspects may be of a different kind to *essential features* and the other five secondary aspects, and that they draw on essential features and the secondary aspects. Perhaps these aspects are more closely aligned with the *tasks of teaching* proposed by Ferrini-Mundy et al (2006). (See fig. 2.2.) This requires further research. I discuss each of the three aspects briefly below and illustrate relationships with the other aspects of the framework.

Basic repertoire is derived from the relationship between *essential features* and *learners' conceptions* since a repertoire of tasks and examples must take into account both the mathematics and the learner. There may be instances where other aspects are also relevant, such as dealing with the meaning of n in the annuities formulae, where *modelling and applications* also informs a basic repertoire. In chapter 13, I showed that students tend to think that n in the annuities formulae represents the number of compounding periods because this is its meaning in the compound interest formula. If we know that students and learners find it difficult to grasp the new meaning of n, then we should anticipate two consequences. Firstly, derivations of the annuities formulae may not even make them aware that the meaning of n is different; and secondly, changing students' and learners' perception of the meaning of n may take time. This informs our selection of a basic repertoire: for instance, we are obliged to include tasks and examples that address the meaning of n in ways that will challenge students and learners to reconsider their assumptions about its meaning, based on their prior experiences with the compound interest formula.

The aspect of *explanation* is also derived from knowledge of secondary aspects and *essential features*. I am not concerned here with generic knowledge of the features of "good" mathematical explanations – this is the responsibility of mathematics methodology courses. I am concerned with topic-specific explanations that draw on the relevant knowledge of MfT of the concept (or topic). Consider the example of explaining that simple interest does not necessarily mean that the same amount of interest is accumulated each period. This is based on the definition that interest is calculated on the principal amount. However, given the limited example space (Watson & Mason, 2005) regarding simple interest that learners encounter in schools, they may make this erroneous association. To explain that simple interest does not necessarily mean equal amounts of interest each period, the teacher will need to: draw on the definition of simple interest (essential features), draw on knowledge of the daily interest calculations done in banks (contextual knowledge of finance), and produce a numerical model of the accumulated interest over several months (modelling and applications) similar to that shown in table 6.3. (It is also worth noting that in this particular instance the teacher is expanding the example space which suggests a horizontal connection between *explanation* and *basic repertoire*.)

Different teaching sequences and approaches are exemplified in part 2 with respect to AB and IP approaches to annuities. I have argued that working with an IP approach requires a shift in the unit of focus, and that students may not initially treat the payments separately. This challenges the common practice of introducing annuities as an application of geometric progressions. Earlier in the chapter I noted that AB and IP approaches give different insights into annuities. They may therefore be considered as complementary rather than competing approaches.

17.3.5 Applying the revised framework – An example of revisiting annuities

A further contribution of this study is the elaboration of the notion of revisiting, particularly in the context of school mathematics. I now draw together the notion of revisiting and the revised MfT framework. I exemplify what it might mean to revisit annuities by means of a revisiting task.

The task (see fig. 17.3) is adapted from GT3 and inspired by the interaction between the students in A-group, as discussed in the introduction to chapter 16. It targets a core issue of annuities: "how many times does each payment accumulate interest?" Kate's approach assumes all 12 payments gain interest for 12 months. Andrew's approach assumes the first payment gets interest for 12 months and

thereafter each payment gains interest for one month less. However, this is not easily seen in an AB approach. If one works with an IP approach, the question appears to be a very simple one. However, the data shows that the answer was not obvious for students when they began annuities.

Consider the following problem:

> At the beginning of every month you invest R250 in a savings account with an interest rate of 6% p.a. compounded monthly. You continue this for a year. How much will you have accumulated by the end of the year?

Kate used the compound interest formula to answer this question:

$$FV = 250 \times 12\left(1 + \frac{0.06}{12}\right)^{12} = R3185.03$$

Andrew says you cannot use the compound interest formula. He says you have to work it out the long way like this:

> End of month 1: 250 + 0.005×250 = 256.25
>
> End of month 2: (256.25 + 250)×1.005 = 508.78
>
> End of month 3: (508.78 + 250)×1.005 = …

Without doing any calculations, determine whether Andrew's answer will be larger or smaller than Kate's. Whose approach do you think is more appropriate to solve the problem, and why?

Fig. 17.3 Revisiting task for annuities

In terms of revisiting, the task differs from learning annuities for the first time in that it assumes knowledge of annuities, and juxtaposes compound interest and annuities in a single question. The goal of the task is to address the different ways in which the two models deal with the multiple payments. The resources available for the task have been restricted by removing the option of doing calculations. This requires students to draw on other representations and ways of approaching annuities that do not involve calculations. One example is the spreadsheet image used by Shaun as discussed in chapter 15. Another approach might involve unpacking Kate's approach and recognising the repeated addition of 12 payments each accumulating interest for 12 months.

The design of this task is informed by the *essential features* of compound interest and annuities, and three of the secondary aspects in the revised MfT framework: *learners' conceptions*, *modelling and applications* and *different approaches to a concept*. As noted above, the key element of essential features is that payments do not earn the same amount of interest because they are deposited at different times. So while students may recognise that Kate's use of the compound interest formula is not an appropriate *model*, they may not be able to predict which strategy will generate the larger answer. In order to deal with this issue, they need to shift the unit of focus from account balance at the end of the period, to the growth of each payment over time, moving from an AB approach to an IP approach.

17.4 Two elements of MfT that require further research

In previous chapters I pointed to various issues concerning MfT and teaching of financial maths that require further research. I do not repeat them here. However, there are two issues concerning MfT that have emerged but have not been resolved, and which deserve further attention. They relate to the role of *explanations* in learning MfT, and the place of the *operational-structural distinction* in theoretical frameworks for MfT.

17.4.1 The importance of *explanations* in learning MfT

I have acknowledged that, as an aspect of the MfT framework, *explanations* was not in focus in the study. I have also suggested that *explanation* (along with *basic repertoire* and *different teaching sequences and approaches*) may be a different kind of aspect to *essential features* and the five secondary aspects in the MfT framework. This requires further theoretical consideration and empirical investigation.

Nevertheless, throughout the thesis I have provided numerous examples of students' explanations, and analysed several of these in detail with regard to their mathematical content. I discussed three instances where errors in students' thinking were only revealed in their verbal explanations, and could not be seen in their written work. This points to the importance of providing opportunity for students to produce verbal explanations *and* for these explanations to become an object of attention.

More generally, it may be productive to consider instructional and peer explanations as windows into teachers' MfT. In order to produce a mathematical explanation, a teacher draws on various aspects of MfT, particularly *essential features*. These aspects must be coordinated in the preparation and production of the explanation, which may require managing dilemmas and tensions such as how to produce a level-appropriate definition that retains mathematical integrity. If the explanation is then viewed as the *outcome* of the coordinating and decision-making processes, it provides insight into teachers' MfT in use. I am not suggesting a direct correspondence between teachers' MfT and the quality of their explanations. I am merely suggesting that explanations give some access to how teachers enact their knowledge, acknowledging that there are multiple other factors that impact the kind of explanation a teacher provides, such as the purpose of the explanation and the time available. Nevertheless, the explanation may be studied from the perspective of mathematical correctness, as well as from a pedagogical perspective where it represents the outcome of the teachers' choices of representations and examples, as well as decisions regarding use of language and written symbolic forms[58]. Of course, teachers may not be aware of making deliberate choices about these features of their explanations. This, in itself, is an issue to be taken up in studying the explanation.

With regard to the MfT framework, explanations may be taken to reflect the depth, breadth and connectedness of teachers' mathematical knowledge. An explanation may also reflect the ways in which teachers make connections for themselves between various elements of the *essential features*, the links they make to other *aspects of mathematics*, and their knowledge of *learners' conceptions* of the concept. These issues all require further research.

[58] These are merely three components that teachers need to consider and are illustrative of the coordinating that needs to be done.

17.4.2 Locating operational-structural conceptions in an MfT framework

Throughout the thesis, but particularly in part 2, I have argued for the importance of teachers being able to move between compressed and decompressed forms of mathematics, and hence the need for tasks that require and promote such shifts. I have drawn on Sfard's notion of operational and structural conceptions to do so (Sfard, 1991). My argument is not new – it extends the work of Ball and her colleagues (e.g. Ball, et al., 2004) who suggest that *unpacking* may be the distinguishing feature of teachers' mathematical knowledge. It also builds on Ferrini-Mundy et al's framework (2006) where they refer to *decompressing*. The notions of unpacking or decompressing are considered to span the different categories of the typologies – they are not attributed to a particular category/aspect. In a similar way, it seems that the operational-structural distinction is "everywhere but nowhere" in my framework. As a construct I have been unable to find a suitable location for it within the framework. Some might argue that moving between processes and objects is a typical mathematical practice, and would thus treat *moving between compressed and decompressed forms* as a component of the aspect *mathematical practices*. However, this overlooks the difference between compressing and decompressing mathematics *for oneself* and doing so *to help others learn mathematics*. Therefore there appears to be a need to distinguish the mathematical practice from the pedagogical practice. The ability to make the shifts also reflects a flexibility in teachers' knowledge, which might be considered as an example of the ways in which teachers know (or "hold") their mathematics. This is an important issue for further research on teachers' mathematical knowledge[59].

17.5 Reflecting on insider research

In choosing to engage in first-person research, I took on the roles of designer, builder, lecturer and researcher, and as noted in chapter 5, these roles were enacted during the course and later in the analysis. While teaching the course, I had great difficulty in taking a researcher's perspective when thinking and talking about the course. Once the course was completed, I struggled to create distance – from the course and from the data. My initial "analysis" was dominated by description and narrative. This was exacerbated by my attempts to work with critical incidents, and my assumption that by increasing the level of detail in the description of an incident, I was increasing the rigour of the research. The use of analytic narrative vignettes enabled me to overcome these struggles.

I struggled to isolate foci and thus to strip away much of the detail that, as an insider, I had access to. I was continually concerned that the loss of detail might lead to distortions which might render my interpretations simplistic and inadequate. Ball (2000) argues that a key challenge for insider research is to balance distance and insight. Based on my experience, an additional challenge is that of *selection* and *simplification*.

The first step is in selecting foci from the myriad of opportunities, incidents and issues that already constitute a sub-set of the setting and the initial research focus. The second step is to simplify – to strip away those aspects that are non-essential to the selected focus. This is not an easy task, and is influenced by a range of factors including theoretical tools and personal choices. Taken together, selection and simplification enable one to establish distance by backgrounding and excluding some elements, and consequently allowing others to stand out. In my case, the elements I needed to strip away were frequently a reflection of my personal biases. I then had to accept the selections I had made

[59] Ball et al (2008) refer to ways in which teachers "hold" their mathematical knowledge (p. 400) and Barton (2009) has made some tentative proposals for future research in this regard.

and the simplifications I had engineered. While these were my choices, it was often uncomfortable to live with the consequences, and there was regular temptation to reinsert the "lost detail".

The interpretations I have produced and the story I have told do not represent the "full picture". This is the case in all research. It is merely exacerbated in first-person research because of the multiple roles of the designer-builder-researcher-teacher and thus the multiple insights that come from these different roles. On the other hand, the analysis that has been done, and the insights gained would not have been possible in the tangle of detail with which I began.

17.6 Conclusion - My journey of revisiting and insights into learning MfT

In drawing this thesis to conclusion, I refer to chapter 1 where I described my first attempt at teaching financial maths, and how it was characterised by learning the content for myself and then transmitting it to my students. For the most part, my struggles were not with the mathematical content *per se* but with the relationship between the maths and the financial context, of which I knew very little. The chapter-headings in the text books were my indicators of what was important to teach. They also suggested the sequence in which to teach the content.

As I became more familiar with the content, and particularly with the relationship between the maths and the financial context, I was able to pay more attention to the students: to their difficulties and errors, and to the ways in which their life experiences did, or did not, support their learning of financial maths. This led to the development of new resources for (my) teaching and (our) learning, such as the spreadsheets discussed in chapters 12 and 16. Later I gave increasing priority to extending students' knowledge of finance and their general levels of financial literacy, and thus to the moral and political project which has not been visible in the thesis.

I have frequently been asked what I have learned from the research that I did not (or could not) have learned from teaching the Financial Maths course. Not surprisingly, my greatest learning came through the analysis. The surprise (to me at least) is that *the most powerful learning about financial maths has been the deepening and broadening of my own knowledge of the essential features of compound interest and annuities*. This was the result of engaging with the data: provoked by students' concerns and class interactions, and driven by my efforts to make sense of their explanations and errors. In the absence of reference organisations for compound interest and annuities, I had to construct my own. I consider the references landscapes for compound interest and annuities, in chapters 6 and 12 respectively, to represent some of the most substantial work of this thesis, partly because they reflect much of my own learning. I do not propose that teachers require knowledge of all the issues dealt with in those chapters to teach compound interest and annuities "well". I did not "have" all that knowledge myself when the 2008 course commenced, nor when it finished!

The research study has thus brought me full circle. When the journey began, I focused first on the content and then on the students. But in the study it was the students' contributions that forced me to return to the mathematics, particularly the essential features, and then to the relationships between mathematics and finance. Thus the students' learning has been a trigger for my learning, which took the form of revisiting the mathematics they were learning.

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Appendix A1: Student information letter for research project

INFORMATION ABOUT PARTICIPATION IN DOCTORAL RESEARCH PROJECT

31 January 2008

Dear Student

As you know I am using the Financial Mathematics course (Applied Mathematics in Education B, EDUC4041) as a key part of the data collection for my PhD in mathematics education.

My PhD focuses on how student teachers' understanding of the time value of money grows over the duration of the course. This issue is part of my broader interest in the mathematical knowledge that teachers need in order to teach mathematics well. There is some agreement across the world that the kinds of mathematics courses that teachers should take must be different from the mathematics courses offered to students who are going into other professions. I believe this too and through my study I want to understand what those kinds of courses might look like. I am really concerned that mathematics teacher education programmes must "deliver the goods" for practising and future teachers. Too often we blame learners' poor performance on their teachers but we seldom ask how good the teacher preparation programmes are.

So there are 2 aspects to consider here:

- The financial maths course (Applied Mathematics in Education B) - You are registered for this course as part of your degree. It is a compulsory course and so you must complete it and pass it.
- My doctoral study - I am requesting you to participate in my study. I can't really *invite* you because you are already registered for the course. But I need to ask for your consent to collect and publish data from this course. You may choose to participate or not to participate in the research – either way this will not impact your participation and marks for the course.

There are two levels of involvement in the PhD research project. The first is related to the fact that you are registered for the course. This is how it will affect you:

- We are going to video-record and audio-record all the contact sessions of the course. We want to do this to capture what happens in the class. On the whole, the camera will be focused on what is going on publicly in the class. So, much of the time it will focus on me as the lecturer. We want to use the camera to capture whole-class discussions and the contributions of students who write on the board, share their ideas with the class etc.
- With your permission, we will also photograph some of the work that is written on the board and on poster sheets to ensure that we have a good copy of it. This might be work I have written up but most likely it will be work that has been written by other members of the class.
- All students will be required to keep journals in the course. I would like to copy the journals of some students (see discussion of tutorial groups). This will enable me to see how you experienced the course and how the interactions in the course may have shaped your understanding of the content.
- Three modules within the course are of particular interest to me because they relate most strongly to the time value of money. I would like to make copies of the work that you produce during these 3 modules. I will bring this to your attention when we start each of the modules. You will also be required to produce two concept maps during the course. I would like to make copies of these too. All these documents will give me insight into how your understanding of the time value of money grows over the duration of the course.

- Towards the end of the course you will be doing a project on an aspect of financial maths. I want to analyse the work done in these projects. This will give me insight into how you and your group work with the mathematical, financial and teaching issues that we have dealt with in the course.
- I would like to make copies of your tests and exam scripts and analyse your responses. This will give me some understanding of your ability to cope with the content of the course in situations where you do not have additional help and where you have prepared for the task (i.e. test or exam). This analysis will be done after the course is finished. It will not impact your results.
- You will be given questionnaires at the start and end of the course. These are not course evaluations. Your responses to the questionnaires will help me understand what knowledge and experiences you bring to the course, and the main aspects you have learned from the course.

The second level concerns a smaller group of students:

You are all going to be allocated to tutorial groups (4 students per group) for the weekly tutorial sessions. These groups will be diverse in terms of race, gender and marks obtained in your most recent maths course. Where possible groups will include 3rd and 4th year students. The study requires that work carried out in tutorials is also captured. It is not possible to do this effectively for the whole class and all groups. I will thus choose 2 of these groups to focus on - groups that reflect diversity and that facilitate data collection for the study. In addition to the aspects already mentioned, these students will also be involved in the study in the following way:

- We are going to video- and audio-record their interactions while they work as a group in the tutorial sessions on Thursday mornings. The camera will be mounted on a tripod and there will also be a microphone placed on the desk to ensure that the discussion is clearly audible. I will not be present during these discussions. These recordings will take place in a separate venue to the rest of the class in order to get good quality recordings. The video recordings will help me to see who is talking. It's very difficult to work out who is talking from an audio tape when there may be several people talking.
- The students will be interviewed as a group after the tutorials that relate to the 3 key modules mentioned above. Each interview will last approximately 1 hour. This interview will be audio- and video-recorded.
- The students will be interviewed individually 3 times during the course.
    - Interview 1 – soon after the start of the course (February 2008), focusing mainly on responses to the questionnaire, approx 30min.
    - Interview 2 – in the middle of the course (early April 2008), focusing on certain important and interesting aspects from the course at that stage, 30-45min.
    - Interview 3 – after the course is finished (June 2008), focusing on key aspects of the course, concept map, and the responses to the second questionnaire, approx 60min.
- All interviews will be arranged at a time that is mutually convenient. The interviews will be audio-recorded.
- The journals of these students will be copied.

Below I try to answer some of the questions you may have. Please ask any further questions, at any time.

1. **Question: Do I have to participate?**

   Answer: Participation in the *course* is compulsory. Participation in the *doctoral study* is voluntary. If you choose not to participate in the research components, you will still attend the course but you will not be involved in any data collection that is not part of the course itself, e.g. questionnaires, focus group, and your work will not be copied as part of the data for the study. We will need to discuss how to deal with the fact that the course is being videoed. I suggest that you sit behind the camera so that you will not be captured on video.

2. **Question: What if I change my mind about participating?**

   Answer: You may withdraw from participation in the *research project* at any time. But if you withdraw from the *course*, you will not graduate. You should be aware that this course will not be offered in 2009 but will be offered again in 2010. Please feel free to chat to me at any time if you are hesitant about participating. Also, if there are any questions during interviews or in class sessions that you prefer not to answer, you are free to say so.

3. **Question: Will my participation / non-participation in the study affect the marks I get for Applied Mathematics in Education B (EDUC4041)?**

   Answer: I will be playing the role of **lecturer** and **researcher** throughout the course. While it is not possible to separate the course from the research project in a neat way, the research component will not impact your marks. My aim in this study is not to assess you to give marks. It is to understand how your understanding of the time value of money grows over time, what aspects of the course help it to grow, what aspects hinder it etc. I suspect that if any student chooses not to participate in the study s/he may feel that s/he may be disadvantaged or that I may be biased against her/him. I will do my utmost to ensure that there is no discrimination. One of the ways in which I will try to address this is to make sure that the tests of any students who are not participating in the research are moderated by a colleague and that the exam script and project work of these students is moderated by the external examiner (Dr Bruce Brown, Rhodes University). In this way I hope to ensure that all students will be treated fairly. If at any stage you feel that you are being disadvantaged through participation in the research and that this is affecting your marks, please feel free to talk to me.

4. **Question: What if I have questions about the study?**

   Answer: I will arrange discussion sessions with all participants during the process, but you may ask questions whenever you wish. If you want to talk to someone else about your involvement, you may contact Mrs Lampen on 082 442 8589 or in her office M136. Prof Jill Adler is my supervisor and you may address questions to her too but she is very busy and will be out of the country for parts of the course.

5. **Question: Will I have an opportunity to see the results of the study and to comment on these results?**

   Answer: I will begin my analysis in July 2008 and will organize to present my initial analyses to you in late October/early November 2008. This will give you opportunity to give me feedback. My analysis will continue through 2009 and I hope to submit my dissertation in June 2010. I will organize a second feedback session in October 2009. Some of you will still be students at that stage. I hope that the current 4^{th} years will be teaching at that stage(!) but will be able to come back to give me feedback.

6. **Question: Who else will see the data and the results of the study? Will other people know that I participated in the study?**

   Answer: This is a hard question to answer, mainly because of the extensive use of video in collecting data. Below are the steps I will take to ensure the anonymity of your participation and the confidentiality of the data, wherever this is possible.

   (a) All data will be stored securely during the research process.
   (b) All persons involved in the data collection (eg. the camera persons, interviewer, the transcriber etc.) will be required to commit to ensuring confidentiality of the data.
   (c) When we copy your work, we will remove your names from the copies and replace them with codes.
   (d) The most difficult aspect regarding confidentiality relates to the video recordings. When we video class interactions and tutorial group interactions, people are talking to each other and they will refer to each other by name. We might even write “Sizwe’s method” or “Tasneem’s conjecture” on the

board in class. In this way your identity will be revealed. So the issue we have to consider is who sees the videos and how is this done. Only a few people will view the videotapes, and these people will be directly involved in the project – my supervisor, research assistants and some members of staff in the Division of Mathematics and Science Education. In the interviews I may show some video footage to those who are being interviewed. All students who view these clips will be required to commit to ensuring confidentiality of this data.

(e) The work in his study will be presented to the wider community in the following four ways (in all such reports of the study, your name will be changed to ensure anonymity).
   - While busy with the study I will present my initial work to my PhD supervisor, to my fellow PhD students and at academic conferences so that I can get feedback on my progress.
   - The results of the study will be presented in my final PhD dissertation. Your names will not be revealed in the dissertation.
   - The results of the study will also be presented to other researchers in the form of academic journal articles.
   - I also plan to present my work in journals for practising teachers, such as the Amesa journal *Learning and Teaching Mathematics*

There are four forms to complete before we begin the study. This is where you will indicate whether you are going to participate in the study or not. You may think that there is unnecessary repetition in the forms but it is important for me to ask consent for each aspect listed. You may choose to participate in some parts of the study but not others. For example, you might agree to the whole-class aspects but not to the tutorial group aspects.

- *Student consent form: Participation in PhD research project (whole class)*
- *Student consent form: Data collection and data usage for PhD project (whole class)*
- *Student consent form: Participation in PhD research project (tutorial group)*
- *Student consent form: Data collection and data usage for PhD project (tutorial group)*

Please complete and return the signed forms to me by 10h30 on Tues 5 February 2008. Place the forms in the assignment box for the course.

Feel free to visit me in my office (Room M134) if you have any further questions now, or during the study. I look forward to working with you and learning together on this study.

Appendix A2: Student consent form 1

Student consent form: Participation in PhD research project (whole class)

I, …….. (write your name) consent to participate in this research project.

I am aware that participation will involve:

- Video- and audio-recording of all the contact sessions of the course.
- Photographing of some of the work that is written on the board and on poster sheets.
- Completion of 2 questionnaires.

If you give consent, please initial. If you don't give consent,

Student initial: ……...

I am satisfied that the aims of the study and my role in the study have been explained at the beginning of the project, and that there will be ongoing discussion during the process.

Student initial: ……...

I am aware that I can withdraw from participation in the study at any time during the process.

Student initial: ……...

I understand that my participation or non-participation in the research project will not be used against me in any way, and that it will not form part of my normal assessment in the course.

Student initial: ……...

Signature of Student: …….. Date: …….

Student number: ……..

Contact telephone number: ……

Signature of Witness: …….. Date: …….

Appendix A3: Student consent form 2

Student consent form: Data collection and data usage for PhD project (whole class)

I consent to the video-recording of my participation in the contact sessions of the Applied Mathematics in Education B (EDUC4041) course.

Student initial: ……...

I consent to the audio-recording of my participation in the contact sessions of the Applied Mathematics in Education B (EDUC4041) course.

Student initial: ……...

I consent to have any work I write on the board or on poster-sheets photographed.

Student initial: ……...

I consent to the copying of my coursework for the 3 focus units on time value of money.

Student initial: ……...

I consent to the copying of my concept maps. **Student initial:** ……...

I consent to the copying of my project work. **Student initial:** ……...

I consent to the copying of my test scripts. **Student initial:** ……...

I consent to the copying of my exam scripts. **Student initial:** ……...

I consent to selected video clips of the contact sessions being used in individual or small group interviews.

Student initial: ……...

I am aware that the initial and final results of the study will be presented (anonymously) as part of the researcher's studies, at academic conferences, in journal articles and in the PhD dissertation. I consent to the results being used in this way.

Student initial: ……...

Signature of Student: …….. Date: …….

Student number: ……..

Contact telephone number: ……

Signature of Witness: …….. Date: …….

Appendix A4: Student consent form 3

Student consent form: Participation in PhD research project (tutorial group)

I, …….. (write your name) consent to participate in this research project.

If you give consent, please initial. If you don't give consent,

I am aware that participation will involve:

- Video- and audio-recording of all tutorial sessions.
- Group interviews on tutorial sessions related to the 3 focus units on time value of money
- 3 individual interviews.

Student initial: ……...

I am satisfied that the aims of the study and my role in the study have been explained at the beginning of the project, and that there will be ongoing discussion during the process.

Student initial: ……...

I am aware that I can withdraw from participation in the study at any time during the process.

Student initial: ……...

I understand that my participation or non-participation in the tutorial group component of the research project will not be used against me in any way, and that it will not form part of my normal assessment in the course.

Student initial: ……...

Signature of Student: …….. Date: …….

Student number: ……..

Contact telephone number: ……

Signature of Witness: …….. Date: …….

Appendix A5: Student consent form 4

Student consent form: Data collection and data usage for PhD project (tutorial group)

I consent to the video-recording of my participation in the tutorial sessions of the Applied Mathematics in Education B (EDUC4041) course.

Student initial: ……...

I consent to the audio-recording of my participation in the tutorial sessions of the Applied Mathematics in Education B (EDUC4041) course.

Student initial: ……...

I consent to the copying of any written work I produce in tutorial sessions.

Student initial: ……...

I consent to the video-recording of the group interviews.

Student initial: ……...

I consent to the audio-recording of the group interviews.

Student initial: ……...

I consent to the audio-recording of the 3 individual interviews.

Student initial: ……...

I consent to the copying of my journal. **Student initial:** ……...

I consent to selected video clips of the tutorial sessions being used in individual or small group interviews.

Student initial: ……...

I am aware that the initial and final results of the study will be presented (anonymously) as part of the researcher's studies, at academic conferences, in journal articles and in the PhD dissertation. I consent to the results being used in this way.

Student initial: ……...

Signature of Student: …….. Date: …….

Student number: ……..

Contact telephone number: ……

Signature of Witness: …….. Date: …….

Appendix B1: Questionnaire administered at beginning of course

Financial Mathematics
Questionnaire 1

The purpose of this questionnaire is to collect data on your previous experiences of learning and teaching financial maths, and of experiences you have of the world of finance. Note that the questionnaire is not anonymous. If you feel uncomfortable to answer any of the questions, then feel free to leave the answer blank.

Please complete this questionnaire digitally and email it to: craig.pournara@wits.ac.za by 2pm on Fri 15 Feb 2008. Save you file as surnameQ1.doc e.g. JacobsQ1.doc

Name	
Student no.	
Date	

Please answer the following questions as fully as possible. Type your answers in the boxes.

1. Did you do any work on financial mathematics as a learner at school? Please give details.

2. You did some work on financial mathematics in the *Maths in Society* course in first year.
   a. What do you remember of the content you covered in that course? Please make a list.

   b. How would you describe learning financial maths in *Maths in Society*?

3. Have you had any experience of teaching financial maths yourself? For example, on teaching experience, extra lessons for school learners, or to other B.Ed students? Please give details.

4. These questions relate to your personal experiences of finance in the real world.
   Please answer Yes/No to the following:

Do you have a bank account?	
Do you save money regularly?	
Do you have a student loan?	
Do you have another loan that is not a student loan?	
Do you have an account at a department store such as Edgars, Woolworths etc?	
Do you have a part-time job that provides income?	

5. How do you travel to campus each day? (what transport do you use?)

6. Estimate the cost of your travel to and from campus for a 5-day week.

7. Do you know people who borrow money from microlenders/mashonisa? What do you think of borrowing money in this way?

8. Do you talk about financial matters with the people you live with? If so, what aspects do you talk about? Please make a list.

9. Say you take a loan of R4000 at a rate of 12% p.a. compounded monthly. You will pay off the loan in 2 years if you pay about R188 each month. How much must you pay each month if you want to pay off the loan in 18 months? Explain how you got your answer.

CENTRE FOR LEARNING, TEACHING AND DEVELOPMENT
University of the Witwatersrand, Johannesburg

Code 50204

PROMOTIONAL_SURVEY

Course/Topic Name MATH1001 Venue

Date Y Y M M D D: 0 8 0 5 2 0

Group Number ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

THIS SURVEY IS ABSOLUTELY ANONYMOUS

PLEASE DO NOT WRITE YOUR NAME ANYWHERE ON THIS FORM

MARKING INSTRUCTIONS

1. Use only an HB-pencil, blue or black ballpoint to complete this sheet.
2. Check that only one answer per question has been marked.
3. Darken the circle completely.
4. Erase any marks completely.
5. Do not make stray marks on this form.

CORRECTLY MARKED ● INCORRECTLY MARKED ⊘ ⊗ ◒ ⊙

Read each of the statements carefully and indicate whether you
A) strongly agree
B) agree
C) are neutral
D) disagree
E) strongly disagree
by marking the option which most closely corresponds with your view.

		STRONGLY AGREE	AGREE	NEUTRAL	DISAGREE	STRONGLY DISAGREE
1	The lecturer makes the purpose of the lecture clear to you	1	(a)	(b)	(c)	(d)	(e)
2	The lecturer stimulates my interest in the subject.	2	(a)	(b)	(c)	(d)	(e)
3	The lecturer is always well prepared for classes	3	(a)	(b)	(c)	(d)	(e)
4	The lecturer is available for consultations outside of lecture periods	4	(a)	(b)	(c)	(d)	(e)
5	The lecturer encourages audience participation.	5	(a)	(b)	(c)	(d)	(e)
6	The lecturer communicates effectively.	6	(a)	(b)	(c)	(d)	(e)
7	The lecturer chooses and organises the material well.	7	(a)	(b)	(c)	(d)	(e)
8	The lecturer pitches his/her lectures at an appropriate level	8	(a)	(b)	(c)	(d)	(e)
9	I gained a good understanding of concepts and principles in this field.	9	(a)	(b)	(c)	(d)	(e)
10	The lecturer motivated me to read or do extra work related to the lectures	10	(a)	(b)	(c)	(d)	(e)
11	The lecturer is clear and understandable in his/her explanations.	11	(a)	(b)	(c)	(d)	(e)
12	The lecturer gives alternative explanations of difficult points	12	(a)	(b)	(c)	(d)	(e)
13	The lecturer shows a thorough knowledge of his/her subject.	13	(a)	(b)	(c)	(d)	(e)
14	I developed skills needed by professionals in this field	14	(a)	(b)	(c)	(d)	(e)
15	The content of the course was a worthwhile learning experience	15	(a)	(b)	(c)	(d)	(e)
16	Methods of evaluating student work were fair and appropriate	16	(a)	(b)	(c)	(d)	(e)
17	Overall, this lecturer played a significant role in my university education	17	(a)	(b)	(c)	(d)	(e)
18	The lecturer welcomes different viewpoints and independent thinking	18	(a)	(b)	(c)	(d)	(e)
19	I feel that my contribution is taken seriously by the lecturer	19	(a)	(b)	(c)	(d)	(e)
20	Working in small groups in this course was a valuable learning experience	20	(a)	(b)	(c)	(d)	(e)
21	I am able to transfer some of the skills learnt in this course to other subjects	21	(a)	(b)	(c)	(d)	(e)
22	Returned work is accompanied by helpful written comment and suggestions	22	(a)	(b)	(c)	(d)	(e)
23	The computer component was a valuable learning experience	23	(a)	(b)	(c)	(d)	(e)
24	I developed a greater sense of personal responsibility towards my studies	24	(a)	(b)	(c)	(d)	(e)
25	The lecturer has enthusiasm for his/her subject.	25	(a)	(b)	(c)	(d)	(e)

PLEASE TURN OVER

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SECTION B: OPEN-ENDED RESPONSES

(Please give your own opinion about the questions below & be as honest as possible).

DO NOT MARK THIS AREA

A Which aspects of the course were most valuable? Please be specific

B Which aspects of the course were least valuable?

C The most important thing I learned in this course is

THANK YOU FOR YOUR CO-OPERATION

Appendix B3: Sample interview schedule for mid-course interview

Mid-course Interview

Student:

Date:

Time:

Venue:

Normal text indicates common questions for all focus group students; text in italics is specific to the particular student

1. Compare this course with other maths courses you've done so far.
    a. What's the same about learning in this course?
    b. What's different about learning in this course?

2. In your opinion, what are the most important things you have learnt so far in this course?

3. *In the first journal entry I asked you to list 3 important things you want to learn in the course. You listed the following headings:*
    - *financial maths*
    - *calculus*
    - *the direction of developing maths*

    Can you say some more about these?

    You also said that you don't know "how the above case relates to maths, or applied maths or microfinance" (p1). Can you say more? What do you mean?

    You also say "the course might have an opportunity to explore the different aspects of using maths" (p3). Can you say more about what you mean?

4. *You also said you were starting to take care of your mom's car and the home loan.*
    a. Can you tell me more about this?
    b. Do you talk about these things at home?

5. In GT4 we focused on the missed payments.
    a. Let's look at what you wrote.
        i. Can you explain what you are doing for Q3?
        ii. How would you calculate the final amount if you focused on individual payments? (see Hailey's approach)

6. Tell me about your tut group.
    a. Do you feel that the group is working well?
    b. Does the camera bug you?

7. Lebo's conjecture:
    a. You said that your first feeling was that he was wrong. Then you were surprised that it seems to be right. Can you tell me more about this?

b. It seems that you were looking for the logic underlying the maths and it was not obvious what the logic was. Can you tell me more?

OR just say:

c. Tell me about the work you did on Lebo's conjecture.

8. I'd like to ask you some questions about the test.
    a. How did you prepare for the test?
    b. How do you feel you did in the test?
    c. Q1
        i. Pretend you are writing the test now. Can you explain to me what you will say in your answers?
    d. Q4
        i. Can you tell me what you wrote here and what it means?
    e. 7a
        i. Can you explain to me why this formula works for an annuity due?
    f. Q7b
        i. I am not sure that I understand what you are saying. Can you use an example to help me understand?
    g. Q9
        i. Can you explain what you did in 9c?
        ii. Can you estimate how much would Cindy have saved after 8 years?
        iii. Did you use the table in answering the questions?
        iv. You said the graph might be a straight line if we only show the data for every 3 months. What do you mean? What would happen if we show every month?

9. We have focused a lot on spreadsheets in the course so far.
    a. How do you feel about this?
    b. Have they helped you? Can you give some examples?

10. Tell me about your experience in doing the microlending assignment?

11. Have you noticed anything in the media recently that relates to the course?

    (could be inflation, petrol and oil price, exchange rates, interest rates)

12. Is there anything you'd like to ask or say?

Appendix B4: Sample interview schedule for final interview

Final Interview

Student :

Date:

Time:

Venue:

Questions

1) Think about the financial maths course.
   a) What stands out for you?
   b) Why?

2) Have you used or thought about any aspects of the course since it finished?

3) In the future you will be required to teach financial maths in schools.
   a) How do you feel about this?
   b) Say you were required to teach about *annuities due*.
      i) How would you do this?
      ii) Would you give students a formula? Which one?

   Check the following for follow up:

   - *Which version of formula:* $FV = pymt\left[\frac{(1+i)^n - 1}{i}\right](1+i)$ *or* $FV = pymt\left[\frac{(1+i)^{n+1} - 1}{i}\right] - pymt$
   - *Focus on individual payments vs account balance*

4) Think about the following parts of the course.
   - The wage doubling problem
   - The computer operator's salary
   - Future value of an annuity
   - Newton's method
   - Microlending

   a) How did you feel when you were learning each section?
   b) What stands out for you in each section?
   c) Think about the section in relation to M, F, T: what comes to mind?

   Check:
   - *Deal with all 3 aspects for each*
   - *Don't push if student doesn't recall*

5) Think about the people you worked with most in the course.
   a) Who were they?
   b) Why do you work with them?
   c) In what ways do you think this helps you?
   d) How much of your work do you do on your own?

6) Extract 1

This extract comes from a Grade 11 text book.

a) How did it feel when you were working through this extract?
b) Did anything strike you?
c) Would you use this in your Grade 11 class?
d) If you did use it, do you anticipate any difficulties for your learners?

7) Extract 2

a) How did it feel when you were doing this question?
b) Did anything strike you?
c) What aspects did you find easy/difficult?
d) Do you think this is a good question for Grade 12's?

Check for:

- *Confidence in answers*
- *Use of technology*
- *Realistic nature of the scenarios*

Appendix B5: Preparatory task for final interview

Craig's PhD
Final Interview

Hi

I am asking you to do some prep for this final interview. There are some questions I'd like you to think about – we will talk about them in the interview. I have also included extracts from two text books. Please would you try to do them and bring your work with you. We will talk about them too.

Thanks
Craig

Questions to think about

1. Think about the financial maths course. What stands out for you? Why?

2. In the future you will be required to teach financial maths in schools. How do you feel about this?

3. Think about the following parts of the course and reflect on:
    - The wage doubling problem
    - The computer operator's salary
    - Future value of an annuity
    - Newton's method
    - Microlending

    a. What stands out for you in each section?
    b. How did you feel when you were learning each section?

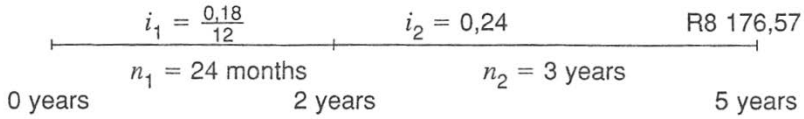
Extract 1

This extract comes from a Grade 11 text book. Work through it.

- Would you use this in your Grade 11 class?
- If you did use it, do you anticipate any difficulties for your learners?
- Any other comments on this extract?

Example 8 Tim borrowed a certain sum of money from a moneylender. The moneylender charged interest at the rate of 18% p.a., compounded monthly for the first two years, and 24% p.a. compounded annually for the next three years. If, after five years, the moneylender sends a letter of demand for R8 176,57 to Tim, how much did he borrow?

Solution: Here is the timeline:



We can reason this problem out in two ways:

(a) If we know the future value, we can find the present value using the present value formula, $P_v = F_v(1 + i)^{-n}$. Since we have the future value (R8 176,57), we reduce it to its present value over the two periods, year 5 to year 2, and then year 2 to the present. Now the problem becomes quite straightforward:

$$\begin{aligned} P_v &= F_v(1 + i_2)^{-n_2}(1 + i_1)^{-n_1} \\ &= 8\,176,57(1 + 0,24)^{-3}(1 + \tfrac{0,18}{12})^{-24} \\ &= \text{R3 000} \end{aligned}$$

(b) If we wish to use the future value formula $F_v = P_v(1 + i)^n$, then we must have the present value. But the present value is what is required. So we let the present value be x. We now find the future value of x over the two periods and equate this expression to R8 176,57:

$$\begin{aligned} \text{Future value of } x &= 8\,176,57 \\ x(1 + i_1)^{n_1}(1 + i_2)^{n_2} &= 8\,176,57 \\ x(1 + \tfrac{0,18}{12})^{24}(1 + 0,24)^3 &= 8\,176,57 \\ x &= \frac{8\,176,57}{(1 + \frac{0,18}{12})^{24}(1 + 0,24)^3} \\ x &= \text{R3 000} \end{aligned}$$

Extract 2

This extract comes from a Grade 12 text book. Try to answer the questions.

- What aspects did you find easy/difficult?
- Do you think this is a good question for Grade 12's?

3 A couple put a R20 000 down payment on a new home and arranged to pay off the rest in monthly instalments of R625 for 30 years at a monthly compounded interest rate of 8,5% per annum.

a What was the selling price of the house to the nearest R100?

b How much interest will they pay over the term of the loan?

c How much do they owe after 6 years?

After 6 years the interest rates climb by 0,9%. The couple must now extend the period of their loan in order to pay it back in full.

d How much will they still owe after the original 30-year period?

e Will they ever repay the loan at their original monthly repayment of R625?

f Calculate the new monthly repayment amount required if the couple still wish to pay off the loan in 30 years.

Financial Mathematics
Exponential growth

A WORKER'S SALARY NEGOTIATIONS

A worker was earning R1 000 a month but he was unhappy with his wage. He went to talk to his boss about a raise. The boss did not think a raise was necessary, but offered to pay the worker R200 a month more.

The worker was not happy with this offer. He sighed, and made the following proposal to the boss:

"Instead of paying me R1 200 a month, I suggest the following method of payment:

- Pay me 1 cent on the first working day of the month,
- Pay me 2 cents on the second working day of the month,
- Pay me 4 cents on the third working day of the month,
- Pay me 8 cents on the fourth working day of the month,
- Pay me 16 cents on the fifth working day of the month, and so on.

This means that on every working day you pay me double the amount of money that you gave me the day before. I promise not to complain at the end of the month, no matter how low my wage is."

The boss thought about the worker's proposal for a minute and did some calculations. He then realised that if he accepted the worker's suggestion, he would only have to pay about R10 by the end of the second week (i.e. after 10 working days)!

He thought he could save a lot of money, so he agreed to pay the worker in the way the worker had suggested.

1) Determine a formula to express the worker's wage.

2) Say the worker started with 3c per day and then doubled each day. How would this impact the formula?

3) Compare this problem with the Saturday School Rent problem
   a) What is the same?
   b) What is different?

Appendix C2: Handout for Computer Operator's Salary Task

Task 2 – Salary increase

The following question was given to a group of learners and their responses are included:

> A computer operator earns R96 000 a year. Her salary increases by 6% per year. What will her salary be after 3 years?

1) Calculate the correct answer.

2) Refer to Learner A.
   a) What does the value 101 760 represent?
   b) Describe Learner A's error.
   c) Pay attention to the learner's use of the equal sign towards the end of the calculation. Describe what the learner has done and why it is incorrect.

3) Refer to Learner B
   a) What does the value 5760 represent?
   b) Describe Learner B's errors.

4) Refer to Learner C.
   a) This learner has made two fundamental errors. Identify them.
   b) What strategies would you use to help Learner C understand the important concepts better?

5) Analyse the response of Learner D. List the aspects that are correct and the aspects that are not correct.

6) **Journal entry**: (Not for submission with the tutorial)
   Think about the responses of the 4 learners, and the teaching of simple and compound growth. What have you learnt from these responses that will impact how you teach simple and compound growth?

Learner A

Annual earnings = R96000
increase by = 6%
Time = 3 yrs.
100% + 6% = 106%
106% = ??
100% = 96 000

$= \frac{96000 \times 106}{100}$
$= R\ 101\ 760$
$= R101760 \times 3$
$= R305280$

Learner B

6. 96000 p.a increased by 6%.
3 years = ?

100% = 96000
6% = ?
100% − 6% = 94%
∴ $\frac{94}{100} \times 96000 = 90240$ is 94% of 96000
∴ 96000 − 90240 = 5760 is 6% of 96000
∴ year 1 with 6% increase will be 101760
year 2 will be 107520

} 113280 and will be his salary per year after 3 years

Learner C

End of Year 1
6% of R96 000
$\frac{6}{100} = 0,6$
R96 000 + 0,6 = ~~96 000~~ R96,6

End of Year 2.
6% of R96 000 = 0,6
R96,6 + 0,6 = 97,2

End of Year 3
6% of R96000 = 0,6
97,2 + 0,6 = ~~97,~~ 97,8

Learner D

Step 1

$\frac{106}{106}$ of $\frac{96\,000}{106}$

1% of 96000 = $\frac{96000}{106}$

100% of 96000 = $\frac{96\,000}{106} \times 100$

= R90566.03 (increase after one year)

Step 2

$\frac{106}{106}$ of $\frac{90566,03}{106}$

1% of 90566.03 = $\frac{90566.03}{106}$

100% of 90566.03 = $\frac{960566.03}{106} \times 100$

= R906194.36. (increase after 2 years)

Step 3

$\frac{106}{106}$ of $\frac{906194.36}{106}$

1% of 906194.36 = $\frac{906194.36}{106}$

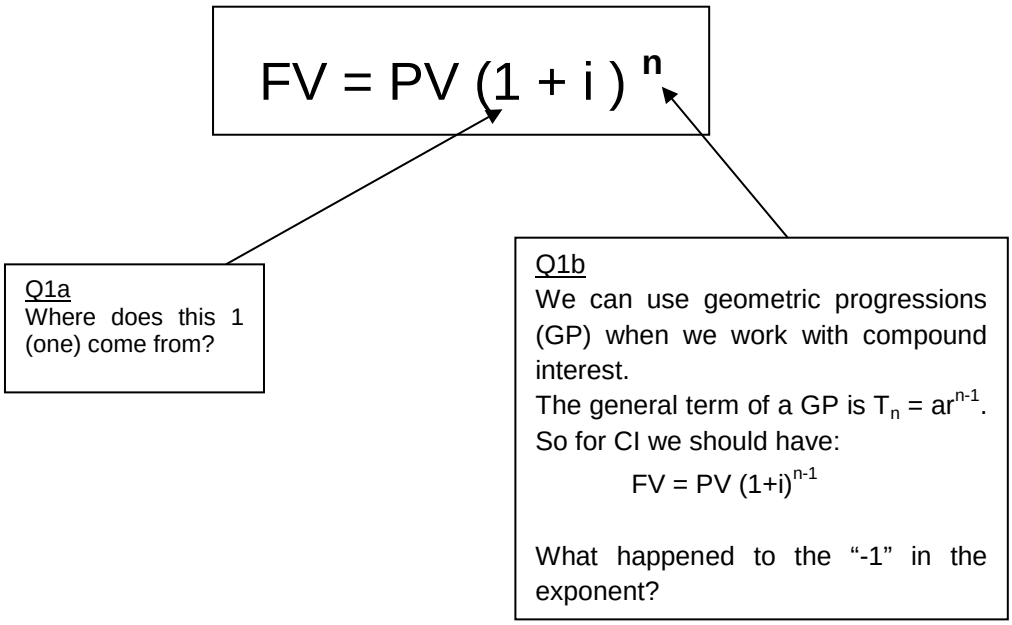
100% of 906194.36 = $\frac{906194.36}{106} \times 100$

= R854900.34 (salary after 3 yrs)

Appendix C3: Handout for "Where does the 1 come from?"

Financial Mathematics
Investigating the compound interest formula

1) We have worked with the following version of the CI formula:



2) Consider the following problem:

> I invest R5000 at 9% p.a. compounded quarterly for 6 years.
> How much will I have at the end of the 6 year period?

A learner starts her response as follows:

$$FV = 5000\left(1+\frac{0.09}{4}\right)^{6}$$

a) Why is the exponent (6) incorrect? What should it be?

b) Why does the learner divide by 4? Is this correct?

c) Change the question so that the learner's response would be an appropriate model of the situation.

Appendix C4: Handout for Group Tutorial 3

Financial Mathematics
Group Tutorial 3

In this tutorial you will investigate aspects of the *BonusPlus* savings account from *Standard Bank*. Read the following part of their advert.

BonusPlus Standard Bank

If you need a little help disciplining yourself to save, try BonusPlus - a goal-oriented savings account that requires you to make monthly investments over a fixed period.

To open a BonusPlus account visit your nearest branch.

BonusPlus offers you these features and benefits:

- You need an opening deposit of between R50 and R1 000.
- You can choose to invest over one, two or three years. Once you've chosen an investment period, you can't change it.
- The pay out date is calculated from the date of your first deposit.
- Your monthly deposits must be equal to - or more than - your first deposit.
- You can pay your deposits by stop order.
- We'll give you a bonus at the end of your investment period, as long as you haven't missed or been late with a deposit. This bonus is a percentage of the interest you receive at the end of the investment period. The longer you invest for, the higher the bonus percentage.
- The higher your balance, the higher your interest rate. Interest rates are variable, and any rate change will be effective immediately.
- Interest is calculated on your daily balance, and paid when your investment period ends.
- You can check your balance and interest rate at any Standard Bank AutoPlus or through Internet banking.

For more information on interest rates click here.



PTO

BonusPlus

These rates are effective from 7 December 2007.

Balance	Interest rate
R50 - R499	5.00%
R500 - R999	5.50%
R1 000 - R4 999	6.50%
R5 000 - R9 999	7.00%
R10 000 and above	8.00%
Bonus rate*	
1 year	5% of interest accrued
2 years	10% of interest accrued
3 years	20% of interest accrued

Interest rates are quoted as per annum rates.

*You will receive bonus interest if you've never missed or been late with a deposit. This bonus is a percentage of the interest you receive at the end of your investment term.

Reference: http://www.standardbank.co.za/SBIC/Frontdoor_02_02/0,2454,3447_17823_0,00.html

Accessed: 3 March 2008

Consider the following scenario:

At the end of every month you invest R250 in the *BonusPlus* savings account. You continue this for a year.

1. How much will you have accumulated in the year? Assume there is a fixed interest rate of 6% p.a. compounded monthly. Ignore the bonus interest aspect at this stage.
2. Can you produce a formula for calculating your answer in (1)? Show how you obtained your formula and provide evidence that it works.
3. Assume you qualify for the bonus rate. How much will you now have accumulated in the year?
4. Now work with the actual rates quoted in the table. These are based on the account balance. How much will you have accumulated in the year? If you were able to find a formula, does it still work? Does it need to be adapted?

Towards the end of the tutorial session, write up a report that captures all the work your group has done. Pay attention to the journey and your conclusions. Provide evidence of what you investigated.

To do in your own time

- Investigate similar savings/investment products offered by the other big banks in SA.
- What aspects are the same and different?
- Pay attention to differences in interest rates, minimum balances, minimum deposit amounts etc.

Appendix C5: Handout for Group Tutorial 4

Financial Mathematics

Group Tutorial 4

You open a *NedTerm* account at Nedbank in April 2008. This account enables you to make monthly deposits. When you open the account, you have to make an initial deposit of at least R1 000 and then monthly payments of R100 or more. You decide to deposit an initial amount of R1 000 and monthly deposits of R300. These payments are made at the start of every month and you do this for 18 months. The monthly payments start in May 2008. The interest rate is 7.65% p.a. compounded monthly.

1. How much money will you accumulate over the period?
2. Assume you are unable to make the 14^{th} payment. How will this affect the total that you accumulate?
3. Since you missed the 14^{th} payment, you decide to make a double payment for the 15^{th} payment. Will this enable you to accumulate the same amount as in (1) above? Provide evidence.

Writing up your findings

- Plan your report before you write it.
- Include the important contributions of each group member.
- Focus on the conclusions you have come to as a group, and give evidence for your conclusions
- Make use of appropriate representations (numeric, algebraic, graphical, timelines etc.)
- Give some details of the journey you travelled in working on this problem
    - Did you use different strategies?
    - Did they all work? If not, what was wrong with them?
    - How did you decide on good strategies?
    - Were there any surprises?
    - Did you have any disagreements?
    - Which representations did you find most helpful?
    - Are you sure of your answers? If not, what concerns do you have? If yes, what makes you sure?
- Read through your report carefully when you are finished.

Appendix C6: Handout for Group Tutorial 5

Financial Mathematics

Group Tutorial 5

In class on 3 April we were working on finding a formula for *Jeeva's savings*. She wanted to invest a lump sum in January and receive monthly payments of R250 from February to November. Some people put up formulae on the board to generalise this problem. These formulae are given below.

Formula 1: $S_n = \frac{x\left[1-\left(\frac{1}{1+i}\right)^n\right]}{i}$

Formula 2: $PV = pymt\left[\frac{1-(1+i)^{-n}}{i}\right]$

Formula 3: $O/b = \frac{Pt\left[(1+i)^n-1\right]}{i(1+i)^n}$

Formula 4: $F_v = P_v(1+i)^n - x\left[\frac{(1+i)^n-1}{i}\right]$

1. Which of these formulae work? Provide evidence to show that they work.

2. Can you explain how to obtain the formulae?

3. Think back to the original problem. Jeeva opened the account at the beginning of January. What if Jeeva opened the account at the beginning of December 2007 and made the lump sum payment when she opened the account.

    a. How will this change the amount she needs to deposit?
    b. How will this change the general formula?

Please pay attention to your timelines. Make sure it is clear to the reader what is being represented on your timelines.

Appendix C7: Test 1 – Question requiring use of timelines

8) You open a *NedTerm* account at Nedbank and you make an initial deposit of R1 200. A month later you begin making monthly payments of R250. You make 16 deposits of R250 at the *start* of every month. The interest rate is 7.5% p.a. compounded monthly.

   a) Set up a timeline to represent this situation. (3)

   b) How much money will you have saved by the *end* of the month in which you make the last deposit of R250? (6)

   c) Assume you are unable to make the 5^{th} payment. How much less will you have saved by the end? (3)

   d) You try to make up for missing the 5^{th} payment so you double the 6^{th} payment. Explain why you won't make up fully for missing the payment. You are not required to give calculations if you feel they are not necessary. (2)