



UNIVERSITY OF THE WITWATERSRAND

A Master of Science dissertation

Vector-like description of $SU(2)$ matrix-valued quantum field theories

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May 2015

Declaration of Authorship

I, Celeste JOHNSON (0303028K), declare that this Dissertation titled, 'Vector-like description of $SU(2)$ matrix-valued quantum field theories', is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed: 

Date: 25 March 2015

"...but Jewel also cried out: 'Don't stop. Further up and further in! Take it in your stride.'"

C.S. Lewis, *The Last Battle*

UNIVERSITY OF THE WITWATERSRAND

Abstract

Faculty of Science

School of Physics

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Vector-like description of SU(2) matrix-valued quantum field theories

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The AdS/CFT correspondence asserts a duality between non-Abelian gauge theories and quantum theories of gravity, established by the value of the gauge coupling λ . Gerard t'Hooft found that the large N' limit in non-Abelian Yang-Mills gauge theories results in a planar diagram simplification of the topological expansion.

In this dissertation, SU(2) gauge theories are written in terms of vector models (making use of collective field theory to obtain an expression for the Jacobian), a saddle point analysis is performed, and the large N limit taken. Initially this procedure is done for gauge theories dimensionally reduced on T^4 and $\mathbb{R} \times T^3$, and then attempted for the full field theory (without dimensional reduction). In each case this results in an expression for the non-perturbative propagator. A finite volume must be imposed to obtain a gap equation for the full field theory; directives for possible solutions to this difficulty are discussed.

To my family

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Chapter 1

Introduction

1.1 AdS/CFT

1.1.1 Quantum chromodynamics and strings

String theory was initially conceived in the 1960's as a mechanism to explain the great number of particles being discovered in terms of different oscillatory modes of a smaller subset of fundamental strings [1]; it was thought that these fundamental strings, with length of the order of the size of the nucleus, could explain the strong interaction. Veneziano developed a theory of dual models (which would now be called a 26-dimensional bosonic string theory) where scattering amplitudes were calculated using approximations to the Euler gamma and Euler beta functions and this theory was later shown to have a string interpretation. For example, Veneziano model's predicts that particles with greatest spin for a particular mass follow Regge trajectories given by the formula $\alpha' m^2 = J + constant$ [2], so that α' is the slope of these Regge (and daughter) trajectories (as seen in the Chew-Frautschi plot in figure 1.1); in string theory, string rotations explain the resonances along the Regge trajectories while vibrational modes give the daughter trajectories; α' emerges as the inverse string tension. The agreement between string S-matrix scattering amplitudes and the results of meson scattering amplitudes of the time gave strong support to the theory.

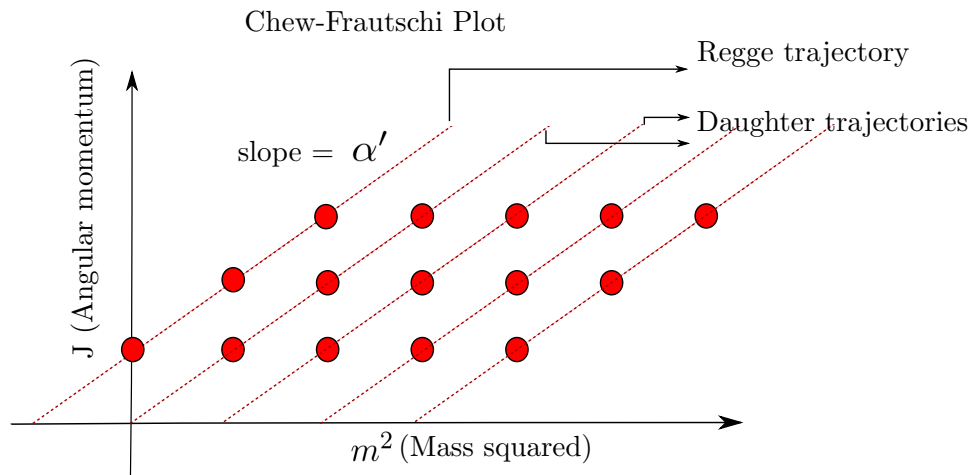


FIGURE 1.1: Chew-Frautschi plot indicating resonance poles in Veneziano’s model, resulting in the uppermost Regge trajectory, and lower daughter trajectories, with slope α' which is also the inverse string tension in string theory; one of the early driving forces for string theory was that a large number of massive particles could be explained as different oscillatory modes of a smaller subset of fundamental strings

However, this theory which predicted the high-energy, fixed scattering amplitude falling off exponentially with s (squared sum of incoming momenta) was published at about the same time as the publication of the first experimental evidence [3] of the parton-like behaviour of the strong interaction (the scattering amplitude actually falls off according to a power law in these processes) [4]. There were also concerns over the large number of extra dimensions required for the theory to be consistent and the massless particles apart from spin-1 gluon, that the theory predicted (the open-string spectrum predicted massless vector particles and the closed-string spectrum predicted massless tensor particles [3]). Furthermore, a particle known as the tachyon, with its undesirable property of violating causality, resulted in infrared divergences in the loop diagrams in these ‘consistent dual models’. By 1973/1974 it had been accepted that the strong interaction could be explained using an $SU(3)$ colour gauge theory QCD (Quantum Chromodynamics), and this was backed up well by experiment, including the parton-like behaviour resulting from asymptotic freedom [5]. $SU(3)$ colour gauge theory emerged as the successful theory of the strong interaction.

String theory, however, was found to have highly desirable features as a possible candidate for unification (the string length in this theory is then of the order the Planck length ℓ_P). For example, non-renormalisable amplitudes and ultraviolet divergences

make it difficult for other quantum field theories to incorporate gravity, whereas the massless spin-2 particle (having a Regge intercept of 2) is understandable as a graviton that is present in any string theory. String theories naturally incorporate gravitons and massless vector particles. The low energy dynamics of the massless sector of closed (super)strings yields the Einstein-Hilbert action, which, when taking into account the variational principle, gives rise to the Einstein equation (see Appendix A.1)

$$\begin{aligned}\mathcal{S}_{\text{EH}} &= \frac{c^4}{16\pi G_{\text{N}}} \int d^4x \sqrt{-\det|g|} (R - 2\Lambda) + \int d^4x \sqrt{-\det|g|} \mathcal{L}_{\text{M}} \\ \Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} &= \frac{8\pi G_{\text{N}}}{c^4} T_{\mu\nu}.\end{aligned}$$

In the 1980s, it was established that SO(32) type I superstrings are anomaly free[6]: 5 superstring theories are consistent, each of which are defined in 10 dimensions. These are Type I, non-chiral IIA and chiral IIB, and heterotic SO(32) and $E_8 \times E_8$; in compactifying from a 9+1 spacetime to a 3+1 spacetime, string dynamics require the resulting 6-dimensional manifolds are Calabi-Yau spaces which display similar features to the Standard Model [7]. The 1990's then gave rise to the discovery that all of the 5 10-dimensional superstring theories are perturbative expansions (consistent at every finite order) of one unique (although not well understood) underlying 11-dimensional theory. This 11-dimensional theory is referred to as M-theory, the low energy limit of which is 11-dimensional supergravity. Ref. [8, 9] gives a good overview of these milestones.

1.1.2 Gravity and the holographic principle

A well-known result obtained by Stephen Hawking after studying Einstein's equations is that the entropy of a black hole grows as the area of its event horizon and not as its volume as intuition may have suggested. Surface or boundary conditions therefore determine everything within the boundary. In string theory, this notion establishes itself in the following way: the actual strings move around within a 'bulk', whereas the string endpoints are confined to moving about certain boundaries known as p -dimensional Dp -branes. For example, open string / closed string duality states that a closed string moving from one D-brane to another is dual to an open string

between two D-branes whose endpoints perform a single circuit in the same direction on their respective D-branes. The holographic principle therefore gives rise to a notion of duality.

Furthermore, considering the near horizon geometry of N parallel Dp -branes of type IIB string theory and string coupling g in the large N limit generally gives a supergravity solution $AdS_{p+2} \times \mathcal{M}$ where AdS describes a space with constant negative curvature, Anti-de-Sitter space, and five-dimensional Anti-de-Sitter space (or AdS_5) is the maximally symmetric solution to Einstein's equations for a negatively curved space; \mathcal{M} is typically a compact manifold. For example, the supergravity solution related to D3-branes is given by $AdS_5 \times S^5$. The curvature of the combined space $AdS_{p+2} \times \mathcal{M}$ (in Planck units) is a positive power of $1/N$ [10], so that, in the large N limit it gives the required asymptotically flat universe. For $AdS_5 \times S^5$, the radius of the Anti-deSitter space is the same as that of the five-sphere and are both proportional to $N^{1/4}$. Moreover, in the large- N limit, this results in a conformal field theory (CFT) on the brane given by $\mathcal{N} = 4$ Super Yang-Mills (SYM).

The presence of the graviton together with the holographic principle leads to the assertion that non-Abelian gauge theories correspond to quantum theories of gravity and that they are related by the value of λ . In particular $\mathcal{N} = 4SYM$ is equivalent to type IIB strings in $AdS_5 \times S^5$, where the coupling constants are related as $g_{YM}^2 = g_s$ and the radius of both S_5 and AdS_5 is given by R where $g_{YM}^2 N = \lambda \sim R^4$ [2]. The duality is established by the value of λ . Large λ , corresponding to a non-perturbative regime on strongly coupled $\mathcal{N} = 4$ SYM, corresponds to a large nucleus of curvature R , or weakly coupled gravity. This duality is referred to as the AdS/CFT correspondence

1.2 Non-Abelian gauge theories - Yang-Mills theory

The low energy description of the world volume of d-branes is given by $\mathcal{N}=4$ SYM, which (on the boundary) is realised in terms of special unitary $SU(N')$ gauge fields which are spacetime dependent matrix-valued fields; this is also true for the bosonic sector which is of interest in this dissertation. Such non-Abelian gauge fields are fundamental in our description of particle interactions; for example, the internal symmetries of quarks are described by 3-dimensional vectors due to their 3 colours, and these components transform or mix via an $SU(3)$ matrix.

Emmy Noether found that every continuous global symmetry of the action of a physical system is associated with a conservation law. A symmetry of an action $S = \int d^4x \mathcal{L} \rightarrow \int d^4x \mathcal{L} + \int d^4x \delta \mathcal{L}$ implies $\delta S = 0$ and hence that, *for an internal symmetry*, the fields are independent of position so that $\delta \mathcal{L} = 0$. Using the Euler-Lagrange equations for the variation in $\mathcal{L}(\psi, \partial_\mu \psi)$,

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \partial_\mu \psi \\ &= \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\mu \delta \psi \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \psi \right) \\ &= 0. \end{aligned}$$

Hence $\partial_\mu J^\mu = 0$ where the conserved current is

$$J^\mu \propto \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \psi,$$

and the associated conserved charge is given by

$$Q(t) = \int d^3x J^0(x^\mu).$$

Vectors transform as

$$\psi^m(x) \rightarrow U^{mn} \psi^n(x),$$

where the transformation matrix U is an x -independent $SU(N')$ matrix which multiplies the vector $\psi(x)$, and there is implied summation over the repeated index n . The convention followed in this dissertation is that N' refers to the dimension of the fundamental representation, while N refers to that of the adjoint representation; in other words, $N = N'^2 - 1$. These transformations are a symmetry of actions involving bilinears since

$$\begin{aligned}\psi^\dagger\psi &\rightarrow (U\psi)^\dagger(U\psi) = \psi^\dagger U^\dagger U\psi = \psi^\dagger\psi \\ \partial_\mu\psi^\dagger\partial^\mu\psi &\rightarrow \partial_\mu(U\psi)^\dagger\partial^\mu(U\psi) = \partial_\mu\psi^\dagger U^\dagger U\partial^\mu\psi = \partial_\mu\psi^\dagger\partial^\mu\psi.\end{aligned}$$

The transformation $\psi \rightarrow U\psi$ thus corresponds to a global symmetry together with a conservation law. Extending this global symmetry to a local/ gauge symmetry,

$$\psi^m(x) \rightarrow U^{mn}(x)\psi^n(x), \quad (1.1)$$

one finds that $\psi^\dagger\psi$ is still invariant. Since $(\partial_\mu\psi)^\dagger(\partial_\mu\psi)$ is no longer invariant, a covariant derivative is required,

$$D_\mu^{mn} \equiv \delta^{mn}\partial_\mu + igA_\mu^{mn}, \quad (1.2)$$

which transforms in such a way that (omitting matrix indices)

$$D_\mu\psi \rightarrow D'_\mu\psi' = D'_\mu U\psi = UD_\mu\psi. \quad (1.3)$$

Hence

$$\begin{aligned}(\partial_\mu + igA'_\mu)U(x)\psi(x) &= U(x)(\partial_\mu + igA_\mu)\psi(x) \\ &\Rightarrow \partial_\mu U + igA'_\mu U = igUA_\mu \\ &\Rightarrow A'_\mu = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger.\end{aligned} \quad (1.4)$$

Elements in the Lie algebra may be rewritten in terms of the (matrix) basis T^m of the Lie algebra

$$A_{ij} = \frac{1}{2}a^m T_{ij}^m. \quad (1.5)$$

The coefficients a^m have the form of a vector. Since infinitesimal transformations may be expressed as $U(x) = e^{-i\omega(x)} = 1 - i\omega(x)$, (1.4) becomes

$$\begin{aligned}
A'_\mu &= (1 - i\omega(x)) A_\mu (1 + i\omega(x)) + \frac{i}{g} [\partial_\mu (1 - i\omega(x)) (1 + i\omega(x))] \\
&= A_\mu - i[\omega, A_\mu] + \frac{1}{g} \partial_\mu \omega(x) \\
\Rightarrow \frac{1}{2} a'^m_\mu T^m &= \frac{1}{2} a^m_\mu T^m - \frac{1}{4} i \omega^n a^p_\mu [T^n, T^p] + \frac{1}{2} \frac{1}{g} T^m \partial_\mu \omega^m(x) \\
\Rightarrow a'^m_\mu &= a^m_\mu + \frac{1}{2} f^{n\mu p} \omega^n a^p_\mu + \frac{1}{g} \partial_\mu \omega^m(x),
\end{aligned} \tag{1.6}$$

since A_μ and ω are elements of the Lie algebra. The generators of the Lie algebra obey the commutation relation $[T^m, T^n] = 2if^{mnp}T^p$ and $\text{Tr}(T^m T^n) = 2\delta^{mn}$. Useful identities for $\text{su}(N')$ can be found in Appendix A.4. Now consider in equation (1.4) that if the usual field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ were used, the interaction term of the Yang-Mills Lagrangian $-\frac{1}{4}\text{Tr}(F_{\mu\nu} F^{\mu\nu})$ would not be invariant because of the x dependence of the U matrices; hence $F_{\mu\nu}$ needs to be adjusted such that $F_{\mu\nu} \rightarrow F'_{\mu\nu} = UF_{\mu\nu}U^\dagger$ which will clearly leave the Lagrangian invariant. Consider then the inclusion of a term $ig[A_\mu, A_\nu]$: Under transformation one would hope that there would emerge a term $igU[A_\mu, A_\nu]U^\dagger$, and that the remaining terms would cancel with the problematic terms in the original expression (see Appendix A.2 which details how one can derive that $ig[A_\mu, A_\nu]$ is the expression that ought to be added, together with an illustration of this from representation theory). This is indeed what happens and one obtains

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \rightarrow \\
F'_{\mu\nu} &= U[\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]]U^\dagger.
\end{aligned} \tag{1.7}$$

Then

$$\begin{aligned}
-4\mathcal{L} &= \text{Tr}F_{\mu\nu}(x)F^{\mu\nu}(x) \\
&= \text{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu])(x)(\partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu])(x).
\end{aligned}$$

This Lagrangian splits up into three pieces as follows:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \quad (1.8)$$

where

$$\begin{aligned} \mathcal{L}_2 &= -\frac{1}{4} \text{Tr} [(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))] \\ &= -\frac{1}{2} \text{Tr} [\partial_\mu A_\nu(x) (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))] \\ &= -\frac{1}{2} \text{Tr} \left[\frac{1}{2} T^m \partial_\mu a_\nu^m(x) \frac{1}{2} T^n (\partial^\mu a^{\nu n}(x) - \partial^\nu a^{\mu n}(x)) \right] \\ &= -\frac{1}{8} \partial_\mu a_\nu^m(x) (\partial^\mu a^{\nu n}(x) - \partial^\nu a^{\mu n}(x)) \text{Tr}(T^m T^n) \\ &= -\frac{1}{8} \partial_\mu a_\nu^m(x) (\partial^\mu a^{\nu n}(x) - \partial^\nu a^{\mu n}(x)) (2\delta^{mn}) \\ &= -\frac{1}{4} (\partial_\mu a_\nu(x) \cdot \partial^\mu a^\nu(x) - \partial_\mu a_\nu(x) \cdot \partial^\nu a^\mu(x)) \end{aligned} \quad (1.9)$$

$$\begin{aligned} \mathcal{L}_3 &= -\frac{1}{4} \cdot 2ig \text{Tr} ((\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) [A^\mu(x), A^\nu(x)]) \\ &= -ig \text{Tr} (\partial_\mu A_\nu(x) [A^\mu(x), A^\nu(x)]) \\ &= -ig \text{Tr} \left(T^m (\partial_\mu \frac{1}{2} a_\nu^m(x)) \frac{1}{2} a^{\mu n}(x) \frac{1}{2} a^{\nu p}(x) [T^n, T^p] \right) \\ &= -\frac{1}{8} ig \text{Tr} (T^m (\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) (2i f^{npr} T^r)) \\ &= \frac{1}{4} g (\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) f^{npr} \text{Tr}(T^m T^r) \\ &= \frac{1}{4} g (\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) f^{npr} (2\delta^{mr}) \\ &= \frac{1}{2} g f^{npm} (\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) \end{aligned} \quad (1.10)$$

$$\begin{aligned} \mathcal{L}_4 &= \frac{1}{4} g^2 \text{Tr} ([A_\mu(x), A_\nu(x)] [A^\mu(x), A^\nu(x)]) \\ &= \frac{1}{4} g^2 \text{Tr} \left(\frac{1}{2} a_\mu^m(x) \frac{1}{2} a_\nu^n(x) [T^m, T^n] \frac{1}{2} a^{\mu p}(x) \frac{1}{2} a^{\nu r}(x) [T^p, T^r] \right) \\ &= \frac{1}{64} g^2 a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x) \text{Tr} ([T^m, T^n] [T^p, T^r]) \\ &= \frac{1}{64} g^2 a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x) \text{Tr} (2i f^{mns} T^s 2i f^{prt} T^t) \\ &= -\frac{1}{16} g^2 a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x) f^{mns} f^{prt} \text{Tr} (T^s T^t) \\ &= -\frac{1}{8} g^2 a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x) f^{mns} f^{prs}. \end{aligned} \quad (1.11)$$

1.3 't Hooft's large N expansion for matrix-valued fields

In 1973/1974, Gerard 't Hooft showed that the Lagrangian $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$, derived in the previous section for a non-Abelian Yang-Mills theory (in general a matrix theory with trace structure), gives rise to a topological expansion; this result will be derived here.

Consider the Feynman rules associated with this Lagrangian. The term \mathcal{L}_2 describes a propagator, the term \mathcal{L}_3 incorporating a factor g describes a cubic interaction, and the term \mathcal{L}_4 incorporating a factor g^2 describes a quartic interaction, where g is the coupling constant of the theory. A graph describing the interactions of this theory would then involve edges, 3-vertices and 4-vertices respectively.

Hence, the Feynman diagrams in this theory will be associated with the factor [11] (assuming there are no quark loops)

$$r = g^{V_3+2V_4} N'^F, \quad (1.12)$$

where V_3 is the number of cubic vertices, V_4 is the number of quartic vertices and $\delta_i^i = N'$. A famous formula discovered first by Leonhard Euler describes a topological invariant for polyhedra. This topological invariant is called the Euler characteristic and is given by the formula

$$\chi(H) = F - P + V = 2 - 2H, \quad (1.13)$$

where F is the number of faces, V the number of vertices, and P the number of edges / "propagators". H is called the genus of the topological surface and counts the number of holes in the topology, for example a doughnut has one hole and so its topology, together with any other one-hole object, would be described by an $H = 1$ topology. A derivation of this formula may be found in Appendix A.3.

The total number of vertices is given by

$$V = V_3 + V_4$$

and the number of outgoing propagators from each vertex, given by $3V_3+4V_4$ double-counts the actual number of propagators so that

$$P = \frac{1}{2} (3V_3 + 4V_4).$$

Hence

$$P - V = \frac{1}{2} V_3 + V_4,$$

so that (1.13) gives

$$\begin{aligned} F &= P - V + 2 - 2H \\ &= \left(\frac{1}{2} V_3 + V_4 \right) + (2 - 2H). \end{aligned}$$

Finally, substituting this in (1.12) gives

$$r = (g^2 N')^{\frac{1}{2}V_3+V_4} N'^{2-2H}. \quad (1.14)$$

Hence, clearly this factor admits an infinite sum over topologies which one equates to the partition function \mathcal{Z}

$$\begin{aligned} \log \mathcal{Z} &= \sum_{h=0}^{\infty} N'^{2-2H} f_H (g_{YM}^2 N') \\ &= N'^2 \left(f_0(\lambda) + \frac{1}{N'^2} f_1(\lambda) + \frac{1}{N'^4} f_2(\lambda) + \dots \right), \end{aligned} \quad (1.15)$$

for which, in the limit where $N' \rightarrow \infty$ and $\lambda = g_{YM}^2 N'$ is constant (referred to as 't Hooft's coupling constant), all but the planar Feynman diagrams drop out giving rise to an unexpected simplification of the calculation [11]. This is a topological expansion which is the signature of a string theory (since a string moving in spacetime gives rise to a world sheet), with string coupling given by $g_s^2 = \frac{1}{N'^2}$, or $g_s = \frac{1}{N'}$. An example of this is the correspondence between type IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4SYM$ referred to earlier.

1.4 Vector-valued fields and their large N limit

Up until now the, large limit of matrix-valued fields in the adjoint representation with index N' has been considered, resulting in a systematic $\frac{1}{N'}$ expansion which, and this led to simplifications of a topological nature. It is also known that the large limit of the fundamental representation index N is amenable to a systematic $\frac{1}{N}$ expansion, giving rise to a vector model in terms of $O(N)$ invariants. In this section it will be shown that this expansion in the large N limit leads to what is known as the bubble approximation and ultimately the gap equation for dressed propagators.

In order to make the N -dependence of the theory clear, consider Lagrangian density of the form [12]:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu a^m(x) \partial^\mu a^m(x) - \frac{1}{2} \mu_0^2 a^m(x) a^m(x) \\ & - \frac{1}{8} \frac{g}{N} (a^m(x) a^m(x)) (a^n(x) a^n(x)) \end{aligned}$$

where N is the dimension of the colour index a , and there is implicit summation over repeated indices; furthermore, \mathcal{L} scales with N as

$$\mathcal{L}(Na^m(x) a^m(x)) = N\mathcal{L}(a^m(x) a^m(x)), N = 1).$$

The 2-point propagator $\langle a^{m_1}(x_1) a^{m_2}(x_2) \rangle$ will then be given by

$$\begin{aligned} & \mathcal{N} \left(\langle a^{m_1}(x_1) a^{m_2}(x_2) \rangle - \frac{g}{8N} i \right. \\ & \left. \left\langle \int d^4x_{k_1} (a^m(x_{k_1}) a^m(x_{k_1})) (a^n(x_{k_1}) a^n(x_{k_1})) a^{m_1}(x_1) a^{m_2}(x_2) \right\rangle \right. \\ & \left. - \frac{g^2}{128N^2} \left\langle \int d^4x_{k_1} \int d^4x_{k_2} (a^m(x_{k_1}) a^m(x_{k_1})) (a^n(x_{k_1}) a^n(x_{k_1})) \right. \right. \\ & \left. \left. (a^p(x_{k_2}) a^p(x_{k_2})) (a^q(x_{k_2}) a^q(x_{k_2})) a^{m_1}(x_1) a^{m_2}(x_2) \right\rangle \right) + o(g^3). \quad (1.16) \end{aligned}$$

This corresponds to incorporating an additional 4-vertex with 2 pairs of matching indices for every additional power of g . The Feynman diagrams that emerge are then

$$\begin{aligned}
& a_1 \text{---} a_2 \\
& -\frac{g}{2} i \frac{\textcircled{n\ n}}{m_1\ m\ m} m_2 \quad -\frac{g}{N} i \frac{\textcircled{m\ n}}{m_1\ m\ n} m_2 \\
& -\frac{g^2}{4m_1} \frac{\textcircled{n\ n} \textcircled{q\ q}}{m\ m\ p\ p} m_2 -\frac{g^2}{2N} \frac{\textcircled{n\ m} \textcircled{q\ q}}{n\ m\ p\ p} m_2 +\frac{g^2}{2N} \frac{\textcircled{n\ m} \textcircled{q\ q}}{n\ m\ p\ p} m_2 +\frac{g^2}{N^2} \frac{\textcircled{n\ m} \textcircled{q\ p}}{n\ m\ q\ p} m_2 \\
& +\frac{g^2}{4} \frac{\textcircled{q\ q}}{p\ n\ n} p \quad +\frac{g^2}{2N} \frac{\textcircled{p\ q}}{p\ n\ n} q \quad +\frac{g^2}{2N} \frac{\textcircled{q\ q}}{p\ n\ m} p \quad +\frac{g^2}{N^2} \frac{\textcircled{p\ q}}{p\ n\ m} q \\
& \quad +\frac{g^2}{2N} m_1 \frac{\textcircled{n\ m} \textcircled{p\ q}}{m\ q} m_2 \quad +\frac{g^2}{N^2} m_1 \frac{\textcircled{n\ m} \textcircled{q\ p}}{m\ p} m_2 ,
\end{aligned}$$

where $\delta_{m_1 m_2}$ multiplies all these diagrams. In the large N limit, the first column of these corrections prevail over the rest; these diagrams correspond to maximally matching indices and are known as tadpole or bubble diagrams. Therefore the dressed 2-point function to leading order in N is given by

$$\begin{aligned}
\text{====} &= \text{---} -\frac{g}{2} i \textcircled{\quad} -\frac{g^2}{4} \textcircled{\quad} \textcircled{\quad} -\frac{g^2}{4} \textcircled{\quad} + \mathcal{O}(g^3) \\
&= \text{---} \left(\frac{1}{1 + \frac{g}{2} i \textcircled{\quad} + \frac{g^2}{4} \textcircled{\quad} \textcircled{\quad}} \right) \\
&= \text{---} \frac{1}{\left(\text{---}^{-1} + \frac{g}{2} i \textcircled{\quad} + \frac{g^2}{4} \textcircled{\quad} \textcircled{\quad} \right) \text{---}} \\
&= \frac{1}{-i(p^2 - m^2) + \frac{g}{2} i \textcircled{\quad}}
\end{aligned}$$

since

$$\text{---} = G^0(p) = \frac{i}{p^2 - m^2} .$$

This gives rise to the full propagator expression

$$G(p) = \frac{1}{-i(p^2 - m^2) + i\frac{g}{2} \int \frac{d^d p_1}{(2\pi)^d} G(p_1)}.$$

In general, this expression diverges and so it is necessary to introduce a cutoff Λ ,

$$G^0(p) = \frac{1}{-i(p^2 - m^2) + i\frac{g}{2} \int^\Lambda \frac{d^d p_1}{(2\pi)^d} G(p_1)}.$$

Integrating this then gives

$$\int^\Lambda \frac{d^d p}{(2\pi)^d} G(p) = \int^\Lambda \frac{d^d p}{(2\pi)^d} \frac{1}{-i(p^2 - m^2) + i\frac{g}{2} \int^\Lambda \frac{d^d p_1}{(2\pi)^d} G(p_1)}.$$

After making the identification

$$\sigma(\Lambda) = \int^\Lambda \frac{d^d p}{(2\pi)^d} G(p),$$

this becomes

$$\sigma(\Lambda) = \int^\Lambda \frac{d^d p}{(2\pi)^d} \frac{1}{-i(p^2 - m^2) + \frac{g}{2}\sigma(\Lambda)}, \quad (1.17)$$

which is called the Gap equation. It is of interest that, although the terms in equation (1.16) are proportional to $\frac{1}{p^2 - m^2}$ and hence massless particles display infrared divergence to every order. However, equation (1.17) does not display this infrared divergence when $m^2 = 0$ due to the presence of the $\frac{g}{2}\sigma$ term which acts as a mass term.

1.5 SU(2) gauge theories as SO(3) vector theories

In subsection 1.2, the Lagrangian in terms of SU(N') gauge fields was found to be

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4,$$

where these pieces are given by equations (1.9), (1.10) and (1.11):

$$\begin{aligned} \mathcal{L}_2 &= -\frac{1}{8}\partial_\mu a_\nu^m(x) (\partial^\mu a^{\nu n}(x) - \partial^\nu a^{\mu n}(x)) \text{Tr}(T^m T^n) \\ \mathcal{L}_3 &= \frac{1}{4}g(\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) f^{npr} \text{Tr}(T^m T^r) \\ \mathcal{L}_4 &= -\frac{1}{16}g^2 a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x) f^{mns} f^{prt} \text{Tr}(T^s T^t). \end{aligned}$$

In the particular case where N'=2 and SU(2) gauge fields are being considered, the generators T^i become Pauli matrices σ^i and thus

$$\text{Tr}(T^m T^n) = \text{Tr}(\sigma^m \sigma^n) = 2\delta^{mn}.$$

The structure constants f^{abc} become ϵ^{abc} and therefore these pieces become

$$\mathcal{L}_2 = -\frac{1}{4}\partial_\mu a_\nu^m(x) (\partial^\mu a^{\nu m}(x) - \partial^\nu a^{\mu m}(x)) \quad (1.18)$$

$$= -\frac{1}{4}((\partial_\mu a_\nu(x)) \cdot (\partial^\mu a^\nu(x)) - (\partial_\mu a_\nu(x)) \cdot (\partial^\nu a^\mu(x)))$$

$$\mathcal{L}_3 = \frac{1}{2}g\epsilon^{mnp}(\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) \quad (1.19)$$

$$\mathcal{L}_4 = -\frac{1}{8}g^2\epsilon^{smn}\epsilon^{spr} a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x)$$

$$= -\frac{1}{8}g^2(\delta^{mp}\delta^{nr} - \delta^{mr}\delta^{np}) a_\mu^m(x) a_\nu^n(x) a^{\mu p}(x) a^{\nu r}(x)$$

$$= -\frac{1}{8}g^2[(a_\mu(x) \cdot a^\mu(x))(a_\nu(x) \cdot a^\nu(x)) - (a_\mu(x) \cdot a^\nu(x))(a_\nu(x) \cdot a^\mu(x))]. \quad (1.20)$$

Thus \mathcal{L}_2 and \mathcal{L}_4 are O(3) invariant, and \mathcal{L}_3 is SO(3) invariant due to the presence of ϵ .

In general, if correlators can be shown to be O(3) invariant, an O(N) expansion can be considered ; this is one of the aims of this dissertation.

Interestingly, Vasiliev was able to construct interacting theories of massless higher spin fields in AdS_4 . (For flat space, this construction is independent of an S-matrix and therefore circumvents the Coleman-Mandula Theorem which states that for fields with spin greater than or equal to 2, there exists no interacting theory in asymptotically flat space. Whether such an argument exists in curved AdS_4 space-time is unsettled.) In particular, AdS_4 was found to be dual to CFT_3 and in general it is assumed that an AdS space in the bulk, which relates to gravity, defines at its boundary a CFT. In particular, the $O(N)$ invariant sector of the large N limit of vector models [13] corresponds to higher spin theories [14] ($\text{AdS}_4 / \text{CFT}_3$). Although not of a topological nature, the large N limit of vector models can be studied non-perturbatively and, as such, this setting of the AdS/CFT may enable a better understanding of this correspondence.

1.6 Purpose of the dissertation

The main focus of this dissertation involves investigating the large N $O(N)$ saddle point approximation to an $SU(2)$ gauge theory. This will be done as follows:

- *Collective field theory*: The method which will be used consists of a change of variables to $O(N)$ invariants. The Jacobian is calculated using hermiticity arguments for the kinetic energy.
- *Dimensional reduction of $SU(2)$* : The matrix form of $SU(2)$ gauge theories reduces to vector models, and, upon dimensional reduction, these gauge theories may be written in terms of $O(3)$ invariants. The collective field theory is then used to obtain the Jacobian for this change of variables to $O(3)$ invariants. A large N' saddle point analysis is then performed. Initially this will be done for $SU(2)$ gauge theories dimensionally reduced on T^4 (here all space-time dimensions are dimensionally reduced and thus derivative terms in the Lagrangian are negligible compared with the non-derivative terms, and thus the Lagrangian reduces to \mathcal{L}_4). Here a path integral approach will be followed, where the effective action (including the log of the Jacobian) is extremised. Thereafter this will be done for $SU(2)$ gauge theories dimensionally reduced on $\mathbb{R} \times T^3$ (here spatial derivative terms in the Lagrangian are similarly neglected and, by making use of an Hamiltonian approach, the time derivative terms in the Lagrangian fall out as a consequence of gauge-fixing). In this case an effective potential (incorporating \mathcal{L}_4 and a Jacobian term) is minimised.
- *$SU(2)$ gauge theories in terms of vector-valued fields*: Making use of the tools developed in the build-up to this chapter, the Jacobian previously obtained is generalised to bilocals. A large- N' saddle point analysis is then performed on the full field theory, and a gap equation is obtained.
- *Discussion*: The relevance of the exact gap equation obtained will be discussed, together with a consideration of the result at low and high energy.

Chapter 2

Collective field theory

Collective field theory[15] is based on a non-trivial change of variables from the original variables of a theory to the set of invariant variables appropriate to the description of the theory's large N limit and thus reducing the number of degrees of freedom for the theory. Below is an overview of this method, which uses the Hamiltonian of the system to obtain an expression for the Jacobian.

A change of variables is performed in an invariant Hilbert space, from original variables x^a to invariant variables ϕ_α (where Greek letters α, β etc. are used to label these invariant variables). The kinetic energy portion of the Hamiltonian is proportional to ∇^2 which can be written as follows using the chain rule:

$$\nabla^2 = \omega_\alpha \partial_\alpha + \Omega_{\alpha\beta} \partial_\alpha \partial_\beta \tag{2.1}$$

$$\begin{aligned} \partial_\alpha &\equiv \frac{\partial}{\partial \phi_\alpha} \\ \omega_\alpha &= \sum_{a=1}^N \frac{\partial^2 \phi_\alpha}{\partial (x^a)^2} \\ \Omega_{\alpha\beta} &= \sum_{a=1}^N \frac{\partial \phi_\alpha}{\partial x^a} \frac{\partial \phi_\beta}{\partial x^a}. \end{aligned}$$

Here α is implicitly summed over and may take on matrix-type indices, e.g. $\alpha \equiv i, j$ (in such cases this will be carefully dealt with). The inner product is invariant in

Hilbert space and so

$$\begin{aligned}\int \psi^*(x) \psi(x) &= \int J \Phi^*(\phi) \Phi(\phi) \\ \int \psi^*(x) \partial_\alpha \psi(x) &= \int J \Phi^*(\phi) \partial_\alpha \Phi(\phi),\end{aligned}$$

where J is the Jacobian of the transformation. Equivalently, we could absorb the transformation into the individual wavefunctions and so obtain

$$\begin{aligned}\int \psi^*(x) \psi(x) &= \int J^{\frac{1}{2}} \Phi^*(\phi) J^{\frac{1}{2}} \Phi(\phi) \\ \int \psi^*(x) \partial_\alpha \psi(x) &= \int J^{\frac{1}{2}} \Phi^*(\phi) \left(J^{\frac{1}{2}} \partial_\alpha J^{-\frac{1}{2}} \right) J^{\frac{1}{2}} \Phi(\phi).\end{aligned}$$

Hence

$$\begin{aligned}\partial_\alpha f &\rightarrow J^{\frac{1}{2}} \partial_\alpha J^{-\frac{1}{2}} f = J^{\frac{1}{2}} \left(J^{-\frac{1}{2}} \partial_\alpha - \frac{1}{2} J^{-\frac{3}{2}} \partial_\alpha J \right) f = \left(\partial_\alpha - \frac{1}{2} \partial_\alpha \ln J \right) f \\ &\Rightarrow \partial_\alpha \rightarrow \partial_\alpha - \frac{1}{2} \partial_\alpha \ln J.\end{aligned}\tag{2.2}$$

Using (2.2) in (2.1) results in,

$$\begin{aligned}\nabla^2 &= \Omega_{\alpha\beta} \left(\partial_\alpha - \frac{1}{2} \partial_\alpha \ln J \right) \left(\partial_\beta - \frac{1}{2} \partial_\beta \ln J \right) + \omega_\alpha \left(\partial_\alpha - \frac{1}{2} \partial_\alpha \ln J \right) \\ &= \Omega_{\alpha\beta} \partial_\alpha \partial_\beta - \frac{1}{2} \Omega_{\alpha\beta} \partial_\alpha (\partial_\beta \ln J) - \frac{1}{2} \Omega_{\alpha\beta} (\partial_\alpha \ln J) \partial_\beta \\ &\quad + \frac{1}{4} \Omega_{\alpha\beta} (\partial_\alpha \ln J) (\partial_\beta \ln J) + \omega_\alpha \partial_\alpha - \frac{1}{2} \omega_\alpha (\partial_\alpha \ln J) \\ &= \partial_\alpha \Omega_{\alpha\beta} \partial_\beta - (\partial_\alpha \Omega_{\alpha\beta}) \partial_\beta - \frac{1}{2} \Omega_{\alpha\beta} (\partial_\alpha \partial_\beta \ln J) - \frac{1}{2} \Omega_{\alpha\beta} (\partial_\beta \ln J) \partial_\alpha \\ &\quad - \frac{1}{2} \Omega_{\alpha\beta} (\partial_\alpha \ln J) \partial_\beta + \frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) + \omega_\alpha \partial_\alpha \\ &\quad - \frac{1}{2} \omega_\alpha (\partial_\alpha \ln J) \\ &= \partial_\alpha \Omega_{\alpha\beta} \partial_\beta - (\partial_\alpha \Omega_{\alpha\beta}) \partial_\beta - \Omega_{\alpha\beta} (\partial_\alpha \ln J) \partial_\beta - \frac{1}{2} \Omega_{\alpha\beta} (\partial_\alpha \partial_\beta \ln J) \\ &\quad + \frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) + \omega_\beta \partial_\beta - \frac{1}{2} \omega_\alpha (\partial_\alpha \ln J).\end{aligned}\tag{2.3}$$

Where the last step was possible because $\Omega_{\alpha\beta}$ is symmetric. Now since kinetic energy is an observable, its operator must be hermitian and hence (2.3) must be hermitian. Operators linear in a derivative must be accompanied by the imaginery i to be hermitian (for example the momentum operator) unlike scalar terms and terms quadratic in the derivative. Hence the real terms linear in ∂_β in (2.3) must be set to zero, forming an important constraint which can be written in two useful forms due to the symmetry of $\Omega_{\alpha\beta}$,

$$\partial_\alpha \Omega_{\alpha\beta} = \omega_\beta - \Omega_{\alpha\beta} (\partial_\alpha \ln J) \quad (2.4)$$

$$\Omega_{\alpha\beta} (\partial_\beta \ln J) = \omega_\alpha - \partial_\beta \Omega_{\alpha\beta}. \quad (2.5)$$

Hence (2.1) becomes

$$\begin{aligned} \nabla^2 &= -\frac{1}{2}\omega_\alpha (\partial_\alpha \ln J) - \frac{1}{2}\Omega_{\alpha\beta} (\partial_\alpha \partial_\beta \ln J) \\ &\quad + \left[\partial_\alpha \Omega_{\alpha\beta} \partial_\beta + \frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) \right]. \end{aligned} \quad (2.6)$$

Now, using equations (2.4) and (2.5) the first line in equation (2.6) simplifies to

$$\begin{aligned} \nabla_{(1)}^2 &= -\frac{1}{2}\omega_\beta (\partial_\beta \ln J) - \frac{1}{2}\partial_\alpha (\Omega_{\alpha\beta} \partial_\beta \ln J) + \frac{1}{2} (\partial_\alpha \Omega_{\alpha\beta}) (\partial_\beta \ln J) \\ &= -\frac{1}{2} [\omega_\beta - (\partial_\alpha \Omega_{\alpha\beta})] (\partial_\beta \ln J) - \frac{1}{2} \partial_\alpha (\Omega_{\alpha\beta} \partial_\beta \ln J) \\ &= -\frac{1}{2} [\Omega_{\alpha\beta} (\partial_\alpha \ln J)] (\partial_\beta \ln J) - \frac{1}{2} \partial_\alpha (\omega_\alpha - \partial_\beta \Omega_{\alpha\beta}). \end{aligned}$$

Hence, combining with the second line in equation (2.6) results in

$$\nabla^2 = \partial_\alpha \Omega_{\alpha\beta} \partial_\beta - \frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) - \frac{1}{2} \partial_\alpha \omega_\alpha + \frac{1}{2} \partial_\alpha \partial_\beta \Omega_{\alpha\beta}. \quad (2.7)$$

Except for the derivative terms, it is evident that this expression should be incorporated with the potential to give rise to an effective potential, necessary in determining the ground state (for an example of this, see [16]).

Chapter 3

Dimensional reduction of SU(2) gauge theories

3.1 Collective field theory applied to vector models

Consider now space time independent fields a_i^m where $m = 1, \dots, N$ and $i = 1, \dots, d$. These then correspond to d N -dimensional vectors with $O(N)$ invariants given by $\phi_{ij} = \vec{a}_i \cdot \vec{a}_j \equiv a_i^m a_j^m$. The collective field theory can be used to find an expression for the Jacobian for the change of variables $a_i^m \rightarrow \phi_{ij}$ [17]. This procedure is reviewed here.

To ensure that these invariant fields are independent, ϕ_{ij} is defined to only be non-zero when $j \geq i$. This set of independent and invariant $O(N)$ fields may be written

in matrix form as

$$\phi_{ij} \equiv \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_3 & \cdot & \cdot & \vec{a}_1 \cdot \vec{a}_{d-1} & \vec{a}_1 \cdot \vec{a}_d \\ 0 & \vec{a}_2 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_3 & \cdot & \cdot & \vec{a}_2 \cdot \vec{a}_{d-1} & \vec{a}_2 \cdot \vec{a}_d \\ 0 & 0 & \vec{a}_3 \cdot \vec{a}_3 & \cdot & \cdot & \vec{a}_3 \cdot \vec{a}_{d-1} & \vec{a}_3 \cdot \vec{a}_d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \vec{a}_{d-1} \cdot \vec{a}_{d-1} & \vec{a}_{d-1} \cdot \vec{a}_d \\ 0 & 0 & 0 & \cdot & \cdot & 0 & \vec{a}_d \cdot \vec{a}_d \end{bmatrix}.$$

In order to determine the values of ω_α and $\Omega_{\alpha\beta}$ in equation (2.5) it proves useful to define the vector

$$\phi_\alpha \equiv \phi_{ij} = [a_1^m a_1^m, a_1^m a_2^m, \dots, a_1^m a_d^m, 0, a_2^m a_2^m, \dots, a_2^m a_d^m, 0, 0, a_3^m a_3^m, \dots, a_d^m a_d^m].$$

In other words there are $\frac{d(d+1)}{2}$ non-zero terms and there is implied summation over each separate pair of indices. For calculation purposes it is sometimes also convenient to define the symmetric matrix Φ_{ij} as follows

$$\Phi_{ij} \equiv \begin{bmatrix} \phi_{ij} & \text{for } j \geq i \\ \phi_{ji} & \text{for } j < i \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdot & \cdot & \cdot & \vec{a}_1 \cdot \vec{a}_d \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdot & \cdot & \cdot & \vec{a}_2 \cdot \vec{a}_d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vec{a}_d \cdot \vec{a}_1 & \vec{a}_d \cdot \vec{a}_2 & \cdot & \cdot & \cdot & \vec{a}_d \cdot \vec{a}_d \end{bmatrix}.$$

Substituting into (2.1) one sees that ω_α is then given by

$$\begin{aligned}
 \omega_\alpha &= \sum_{a=1}^N \frac{\partial^2 \phi_\alpha}{\partial (x_k^a)^2} \\
 \iff \omega_{ij} &= \sum_{a=1}^N \frac{\partial^2 \phi_{ij}}{\partial (x_k^a)^2} = \sum_{a=1}^N \frac{\partial^2 x_i^b x_j^b}{\partial (x_k^a)^2} \\
 &= \sum_{a=1}^N \frac{\partial}{\partial x_k^a} \left(\delta_{ik} \delta^{ab} x_j^b + \delta_{jk} \delta^{ab} x_i^b \right) \\
 &= \sum_{a=1}^N \left(\delta_{ik} \delta^{ab} \delta^{ab} \delta_{jk} + \delta_{jk} \delta^{ab} \delta^{ab} \delta_{ik} \right) \\
 &= 2N \delta_{ij}, \tag{3.1}
 \end{aligned}$$

and, for α and β mapped to (ij) and (mn) respectively, that $\Omega_{\alpha\beta}$ is given by

$$\begin{aligned}
 \Omega_{\alpha\beta} &= \sum_{a=1}^N \frac{\partial \phi_\alpha}{\partial x_k^a} \frac{\partial \phi_\beta}{\partial x_k^a} \\
 \iff \Omega_{ij,mn} &= \sum_{a=1}^N \frac{\partial \phi_{ij}}{\partial x_k^a} \frac{\partial \phi_{mn}}{\partial x_k^a} = \sum_{a=1}^N \frac{\partial x_i^b x_j^b}{\partial x_k^a} \frac{\partial x_m^c x_n^c}{\partial x_k^a} \\
 &= \sum_{a=1}^N \left(\delta^{ab} \delta_{ik} x_j^b + \delta^{ab} \delta_{jk} x_i^b \right) \left(\delta^{ac} \delta_{mk} x_n^c + \delta^{ac} \delta_{nk} x_m^c \right) \\
 &= \sum_{a=1}^N \left(\delta_{ik} x_j^a + \delta_{jk} x_i^a \right) \left(\delta_{mk} x_n^a + \delta_{im} x_m^a \right) \\
 &= \delta_{im} \phi_{jn} + \delta_{in} \phi_{jm} + \delta_{jm} \phi_{in} + \delta_{jn} \phi_{im}. \tag{3.2}
 \end{aligned}$$

Furthermore $\partial_\beta \Omega_{\alpha\beta}$ may be calculated as follows

$$\begin{aligned}
 \partial_\beta \Omega_{\alpha\beta} &= \sum_{\beta} \frac{\partial \Omega_{\alpha\beta}}{\partial \phi_\beta} \\
 &= \sum_{n \geq m} \frac{\partial}{\partial \phi_{mn}} (\delta_{im} \phi_{jn} + \delta_{in} \phi_{jm} + \delta_{jm} \phi_{in} + \delta_{jn} \phi_{im}) \\
 &= \sum_{n,m} \delta_{im} \delta_{jm} \delta_{nn} + \delta_{in} \delta_{jm} \delta_{mn} + \delta_{jm} \delta_{im} \delta_{nn} + \delta_{jn} \delta_{im} \delta_{mn} \\
 &= \delta_{ij} d + \delta_{ij} + \delta_{ij} d + \delta_{ij} \\
 &= (2 + 2d) \delta_{ij}, \tag{3.3}
 \end{aligned}$$

where the differentiation takes into account the independence of variables in ϕ_{ab} . The expressions in equations (3.1), (3.2) and (3.3) may now be used to obtain an expression for the log of the determinant in equation (2.5),

$$\begin{aligned}
 \Omega_{\alpha\beta} (\partial_\beta \ln J) &= \omega_\alpha - \partial_\beta \Omega_{\alpha\beta} \\
 &= 2N \delta_{ij} - (2 + 2d) \delta_{ij} \\
 &= 2(N - (d + 1)) \delta_{ij}. \tag{3.4}
 \end{aligned}$$

Now consider a similar calculation for $\det \Phi$, the determinant of the symmetric matrix, as was derived for J above,

$$\begin{aligned}
 \Omega_{\alpha\beta} (\partial_\beta \ln \det (\Phi)) &= \sum_{n \geq m} \Omega_{ij,mn} \frac{\partial}{\partial \phi_{mn}} \ln \det (\Phi) \\
 &= \left(\sum_{n > m} \Omega_{ij,mn} \frac{\partial}{\partial \phi_{mn}} + \sum_{m=n} \Omega_{ij,mn} \frac{\partial}{\partial \phi_{mn}} \right) \ln \det (\Phi) \tag{3.5}
 \end{aligned}$$

Now, since the determinant of a matrix is equivalent to the product of the matrix's eigenvalues λ_i which make up a diagonal eigenvalue matrix Λ ,

$$\ln \det \Phi = \ln \prod_i \lambda_i = \sum_i \ln \lambda_i = \text{Tr} \ln \lambda_i = \text{Tr} \ln \Phi.$$

Using a Taylor expansion it can be shown that $\frac{\partial \text{Tr} \ln \Phi}{\partial \Phi_{ik}} = (\Phi^{-1})_{ki}$ (see Appendix A.8). Hence equation (3.5) becomes

$$\begin{aligned} \Omega_{\alpha\beta} \partial_\beta \ln \det(\Phi) &= \sum_{m,n} \Omega_{ij,mn} (\Phi^{-1})_{nm} \\ &= 2 \sum_{n>m} \Omega_{ij,mn} (\phi^{-1})_{nm} + \sum_m \Omega_{ij,mm} (\phi^{-1})_{mm} \\ &= \sum_{m,n} \Omega_{ij,mn} (\phi^{-1})_{mn} \end{aligned}$$

by symmetry of Φ .

Hence, making use of equation (3.2) to substitute in for $\Omega_{ab,cd}$, this becomes

$$\begin{aligned} \Omega_{\alpha\beta} \partial_\beta \ln \det(\Phi) &= \sum_{m,n} (\delta_{im} \phi_{jn} + \delta_{in} \phi_{jm} + \delta_{jm} \phi_{in} + \delta_{jn} \phi_{im}) (\phi^{-1})_{mn} \\ &= \sum_n (\phi_{jn} (\phi^{-1})_{in} + \phi_{jm} (\phi^{-1})_{mi} + \phi_{in} (\phi^{-1})_{jn} + \phi_{im} (\phi^{-1})_{mj}) \\ &= \delta_{ij} + \delta_{ij} + \delta_{ij} + \delta_{ij} \\ &= 4\delta_{ij}. \end{aligned} \tag{3.6}$$

Hence, comparing equations (3.4) and (3.6), it becomes clear that

$$\begin{aligned} J &= (\det(\phi))^{\frac{N-(d+1)}{2}} \\ \Rightarrow \ln J &= \frac{N-(d+1)}{2} \text{Tr} [\ln \phi]. \end{aligned} \tag{3.7}$$

3.1.1 Determining an effective potential

The Hamiltonian may now be written out,

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} (-i\nabla)^2 + V \\ &= -\frac{1}{2} \nabla^2 + V. \end{aligned}$$

An expression was found for ∇^2 in equation (2.7),

$$\nabla^2 = -\frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) - \frac{1}{2} \partial_\alpha \omega_\alpha + \frac{1}{2} \partial_\alpha \partial_\beta \Omega_{\alpha\beta} + \partial_\alpha \Omega_{\alpha\beta} \partial_\beta.$$

From equation (3.1), $\partial_\alpha \omega_\alpha = 0$ and from equation (3.3), $\partial_\alpha \partial_\beta \Omega_{\alpha\beta} = 0$. Furthermore $\partial_\alpha \Omega_{\alpha\beta} \partial_\beta$ represents a kinetic term, whereas $-\frac{1}{4} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J)$ may be combined with V to form an effective potential which can be simplified by making use of equation (3.5)

$$\ln J = \frac{N - (d + 1)}{2} \text{Tr} [\ln \phi]$$

and (3.7):

$$\begin{aligned} \Omega_{\alpha\beta} (\partial_\beta \ln J) &= \frac{N - (d + 1)}{2} \Omega_{\alpha\beta} (\partial_\beta \ln \det \Phi) \\ &= \frac{N - (d + 1)}{2} 4 \delta_{ij} \\ &= 2 (N - (d + 1)) \delta_{ij}. \end{aligned}$$

Therefore the effective potential is given by

$$\begin{aligned} V_{\text{eff}} &= \frac{1}{8} (\partial_\alpha \ln J) \Omega_{\alpha\beta} (\partial_\beta \ln J) + V \\ &= \frac{1}{8} \partial_\alpha \left(\frac{N - (d + 1)}{2} \text{Tr} [\ln \phi] \right) 2 (N - (d + 1)) \delta_{ij} + V \\ &= \frac{1}{8} \phi_{ji}^{-1} [N - (d + 1)]^2 \delta_{ij} + V \\ &= \frac{1}{8} \text{Tr} (\phi^{-1}) [N - (d + 1)]^2 + V. \end{aligned} \tag{3.8}$$

3.2 Dimensional reduction of SU(2) gauge theories on T^4 and the vector system integral

Upon dimensional reduction of SU(2) gauge theories on the torus T^4 , all space-time dimensions are neglected and thus fields in the Lagrangian are space-time independent. The Lagrangian thus reduces to \mathcal{L}_4 ; for this system, the path integral will be calculated.

It is sometimes convenient to work in Euclidean space where the time coordinate is treated the same as spatial dimensions (and calculations produce equivalent results). This involves a transformation of the time coordinate to

$$t = x^0 \equiv -ix^4.$$

Therefore,

$$\begin{aligned} e^{i \int dt d^3x \mathcal{L}} &= e^{i \int (-i dx_E) d^3x \mathcal{L}} \\ &= e^{\int d^4x_E \mathcal{L}} \\ &= e^{-\int d^4x_E \mathcal{L}_E}. \end{aligned}$$

Thus, since this Lagrangian comprises no mass term, one can write $\mathcal{L}_E = -\mathcal{L}$ where \mathcal{L} is the Minkowski Lagrangian with $\eta^{\mu\nu}$ replaced by $\delta^{\mu\nu}$. This is necessitated by the fact that, under a transformation to Euclidean coordinates, $x_\mu x^\mu = (x^0)^2 - \vec{x} \cdot \vec{x} \rightarrow -(x^0)^2 - \vec{x} \cdot \vec{x} = -(x_\mu x_\mu)_E$. Under a change of coordinates to O(3) invariants $\phi_{ij} = \vec{a}_i \cdot \vec{a}_j$, then, the path integral becomes rewritten as

$$\int [d\vec{a}] e^{-\mathcal{S}_E(\phi)} = \int [d\phi] J e^{-\mathcal{S}_E(\phi)} = \int [d\phi] e^{-\mathcal{S}_{\text{eff}}(\phi)},$$

where J is the Jacobian associated with this change of variables. The effective action is then given by

$$\mathcal{S}_{\text{eff}} = \mathcal{S}_E - \ln J \tag{3.9}$$

$$= [\mathcal{S}_4]_E - \ln J. \tag{3.10}$$

In the large N limit, calculating the path integral is equivalent to performing a saddle point analysis on this effective action.

3.2.1 Saddle point analysis

According to equation (3.7), the Jacobian term is given by

$$\ln J = \frac{N-(d+1)}{2} \text{Tr} [\ln \phi];$$

$N \gg d + 1$ implies

$$\ln J \rightarrow \frac{N}{2} \text{Tr} [\ln \phi].$$

Under the dimensional reduction of $SU(2)$ gauge theories on the torus T^4 , space-time fields $a_\mu^m(x) \rightarrow a_\mu^m$, so they become space-time independent, and therefore \mathcal{L} reduces to \mathcal{L}_4 . Thus equation (1.11) becomes

$$\begin{aligned} \mathcal{L}_4 &= -\frac{1}{8}g^2 a_\mu^m a_\nu^n a^{\mu p} a^{\nu r} \epsilon^{smn} \epsilon^{spr} \\ &= -\frac{1}{8}g^2 a_\mu^m a_\nu^n a^{\mu p} a^{\nu r} (\delta^{mp} \delta^{nr} - \delta^{mr} \delta^{np}) \\ &= -\frac{1}{8}g^2 (a_\mu \cdot a^\mu a_\nu \cdot a^\nu - a_\mu \cdot a^\nu a_\nu \cdot a^\mu) \\ &= -\frac{1}{8}g^2 (\phi_\mu^\mu \phi_\nu^\nu - \phi_\mu^\nu \phi_\nu^\mu). \\ \Rightarrow [\mathcal{L}_4]_E &= \frac{1}{8}g^2 (\phi_{\mu\mu} \phi_{\nu\nu} - \phi_{\nu\mu} \phi_{\mu\nu}). \end{aligned}$$

Hence, including a mass term (which replaces time components of ϕ), and transforming to Euclidean coordinates, the effective Euclidean action becomes

$$\begin{aligned} \mathcal{S}_{\text{eff}} &= \frac{1}{4}\omega^2 \sum_{i=1}^d a_i \cdot a_i + [\mathcal{S}_4]_E - \ln J \\ &\rightarrow \frac{1}{4}\omega^2 \sum_{i=1}^d \phi_{ii} + \frac{g^2 V}{8} \sum_{i \neq j} [\phi_{ii} \phi_{jj} - (\phi_{ij})^2] - \frac{N}{2} \text{Tr} \ln \phi, \end{aligned} \quad (3.11)$$

where V is the space-time volume that factors out of the integral of the space-time independent action; define a new coupling strength $\tilde{g}^2 \equiv g^2 V$. It will be investigated whether the $\omega^2 \rightarrow 0$ limit can be consistently defined. Performing a saddle point

analysis on equation (3.11) results in

$$\begin{aligned}
 \frac{\partial S_{\text{eff}}}{\phi_{kl}} = 0 &= \frac{1}{4}\omega^2 \sum_{i=1}^d \delta_{ik}\delta_{il} + \frac{\tilde{g}^2}{8} \sum_{i \neq j} [2\delta_{ik}\delta_{il}\phi_{jj} - 2\delta_{ik}\delta_{jl}\phi_{ij}] - \frac{N}{2}\phi_{lk}^{-1} \\
 &= \frac{\omega^2}{4}\delta_{kl} + \frac{\tilde{g}^2}{8} \left[\sum_{j=1}^d 2\delta_{kl}\phi_{jj} - 2\phi_{kl} \right] - \frac{N}{2}\phi_{lk}^{-1} \\
 &= \frac{\omega^2}{4}\delta_{kl} + \frac{\tilde{g}^2}{4} \left[\delta_{kl} \sum_{j=1}^d \phi_{jj} - \phi_{kl} \right] - \frac{N}{2}\phi_{kl}^{-1}.
 \end{aligned}$$

Once this is multiplied through by ϕ_{pk} and summed over k this becomes

$$\frac{N}{2}\delta_{pl} = \frac{\omega^2}{4}\phi_{pl} + \frac{\tilde{g}^2}{4} \sum_{j=1}^d (\phi_{pl}\phi_{jj}) - \phi_{pl}^2 \quad (3.12)$$

$$\Rightarrow \phi_{pl} = \frac{2N}{\omega^2}\delta_{pl} + \frac{\tilde{g}^2}{\omega^2}A_{pl}, \quad (3.13)$$

where $A_{pl} = \sum_{j=1}^d \phi_{pl}\phi_{jj} - \phi_{pl}^2$. For $\tilde{g}^2 = 0$ it is noticed that

$$\phi_{pl} = \frac{2N}{\omega^2}\delta_{pl},$$

which agrees with the perturbative result A.6, where

$$\begin{aligned}
 \langle a_i^\alpha a_j^\beta \rangle &= \frac{2}{\omega^2}\delta_{\alpha\beta}\delta_{ij} \\
 \Rightarrow \langle \phi_{pl} \rangle &= \langle a_p^\alpha a_l^\alpha \rangle = \frac{2N}{\omega^2}\delta_{pl}.
 \end{aligned} \quad (3.14)$$

Similarly, a perturbative solution to first order in g^2 may be obtained by substituting the expression for ϕ_{pl} in equation (3.13) into equation (3.12) to obtain

$$\frac{N}{2}\delta_{pl} = \frac{\omega^2}{4} \left(\frac{2N}{\omega^2}\delta_{pl} + \frac{\tilde{g}^2}{\omega^2}A_{pl}^{(1)} \right) + \frac{\tilde{g}^2}{4} \sum_{j=1}^d \left(\frac{2N}{\omega^2}\delta_{pl} \frac{2N}{\omega^2}\delta_{jj} \right) - \left(\frac{2N}{\omega^2} \right)^2 \delta_{pl}.$$

The term on the left cancels with the first term on the right and therefore

$$\begin{aligned} 0 &= \frac{\tilde{g}^2}{4} A_{pl}^{(1)} + \frac{\tilde{g}^2}{4} \left(\frac{4N^2}{\omega^4} \delta_{pl} \right) (d-1) \\ \Rightarrow A_{pl}^{(1)} &= -\frac{4N^2(d-1)}{\omega^4} \delta_{pl}. \end{aligned}$$

Hence

$$\begin{aligned} \langle \phi_{pl} \rangle &= \frac{2N}{\omega^2} \delta_{pl} + \frac{\tilde{g}^2}{\omega^2} A_{pl}^{(1)} + \mathcal{O}(\tilde{g}^4) \\ &= \frac{2N}{\omega^2} \delta_{pl} - \frac{\tilde{g}^2}{\omega^2} \frac{4N^2(d-1)}{\omega^4} \delta_{pl} + \mathcal{O}(\tilde{g}^4) \\ &= \frac{2N}{\omega^2} \left(1 - \frac{2\lambda}{\omega^4} (d-1) \right) \delta_{pl} + \mathcal{O}(\tilde{g}^4), \end{aligned} \quad (3.15)$$

where $\lambda = \tilde{g}^2 N$ is 't Hooft's coupling constant. This procedure of iterative substitution may be continued indefinitely to give a perturbative solution in g^2 . A non-perturbative solution may be obtained as follows: consider that the left-hand side of equation (3.12) is diagonal, making feasible an ansatz that $\phi_{pl} = b_p N \delta_{pl}$. However, by spherical symmetry, no direction is preferred and hence, for some constant B ,

$$\phi_{pl} = BN \delta_{pl}. \quad (3.16)$$

Substituting equation (3.16) into equation (3.12) (and once again substituting in $\lambda = \tilde{g}^2 N$), one obtains

$$\begin{aligned} \frac{N}{2} \delta_{pl} &= \frac{\omega^2}{4} BN \delta_{pl} + \frac{g^2}{4} \sum_{j=1}^d (BN \delta_{pl} BN \delta_{jj}) - (BN)^2 \delta_{pl} \\ &= \left(\frac{\omega^2}{4} B + \frac{g^2 N}{4} B^2 (d-1) \right) N \delta_{pl} \\ &= \left(\frac{\omega^2}{4} B + \frac{\lambda}{4} B^2 (d-1) \right) N \delta_{pl} \\ \Rightarrow 1 &= \frac{\omega^2}{2} B + \frac{\lambda}{2} B^2 (d-1). \end{aligned}$$

A quadratic equation in B is thus obtained,

$$B^2 + \frac{\omega^2}{\lambda(d-1)} B - \frac{2}{\lambda(d-1)} = 0. \quad (3.17)$$

Perturbatively in λ ,

$$\begin{aligned}
 B &= \frac{1}{2} \left[-\frac{\omega^2}{\lambda(d-1)} \pm \sqrt{\left(\frac{\omega^2}{\lambda(d-1)}\right)^2 + \frac{8}{\lambda(d-1)}} \right] \\
 &= \frac{\omega^2}{2\lambda(d-1)} \left[-1 \pm \sqrt{1 + \frac{8\lambda(d-1)}{\omega^4}} \right] \\
 &= \frac{\omega^2}{2\lambda(d-1)} \left[-1 \pm \left(1 + \frac{1}{2} \frac{8\lambda(d-1)}{\omega^4} + \frac{1}{2} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(\frac{8\lambda(d-1)}{\omega^4}\right)^2 + \dots \right) \right].
 \end{aligned} \tag{3.18}$$

Notice now that the previous perturbative result for the first term (equation (3.14) where the factor of N has been absorbed) may be retrieved when the positive root is taken and only terms to first order in λ are taken into account. Hence a solution for B is given by

$$B = \frac{2}{\omega^2} - \frac{4\lambda(d-1)}{\omega^6} + \dots, \tag{3.19}$$

which agrees with the leading term in equation (3.15). The strongly coupled solution to equation (3.18) where $\lambda \gg \omega^4$ and hence where $\frac{8\lambda(d-1)}{\omega^4} \gg 1$ is given by

$$\begin{aligned}
 B &\approx \frac{\omega^2}{2\lambda(d-1)} \left[\sqrt{\frac{8\lambda(d-1)}{\omega^4}} \right] \\
 &= \frac{1}{2} \sqrt{\frac{8}{\lambda(d-1)}}.
 \end{aligned}$$

This could be directly read off equation (3.17) with $\omega^2 = 0$. It is of interest that this expression for the propagator is independent of the mass term ω^2 and is infrared free.

3.3 Dimensional reduction of SU(2) gauge theories on $\mathbb{R} \times T^3$ and the Hamiltonian system

3.3.1 Hamiltonian formalism

The Lagrangian for SU(2) gauge fields is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\text{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu])(\partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]) \\ &\equiv -\frac{1}{2}\text{Tr}(F_{0i}F^{0i}) - \frac{1}{4}\text{Tr}(F_{ij}F^{ij}).\end{aligned}$$

Defining $B_{mn}^i = -\epsilon_{ijk}F_{mn}^{jk} \iff F_{mn}^{ij} = -\epsilon^{ijk}B_{mn}^k$ and $E_{mn}^i = F_{mn}^{i0}$, the Lagrangian may be equivalently rewritten as

$$\mathcal{L} = \frac{1}{2}\text{Tr}(E^i \cdot E^i) - \frac{1}{2}\text{Tr}(B^i \cdot B^i).$$

The conjugate momentum is given by

$$(\Pi^\mu)_{mn} = \frac{\partial \mathcal{L}}{\partial(\partial_0(A^\mu)_{nm})}.$$

Π_{mn}^0 is then identically zero and $\Pi_{mn}^i = F_{mn}^{i0} = E_{mn}^i$. By noting that

$$\begin{aligned}F_{mn}^{i0} = E_{mn}^i &= \partial^i A_{mn}^0 - \partial^0 A_{mn}^i + ig[A^i, A^0]_{mn} \\ \iff \dot{A}_{mn}^i &= \partial^i A_{mn}^0 - E_{mn}^i + ig[A^i, A^0]_{mn},\end{aligned}$$

the Hamiltonian density reduces to

$$\begin{aligned}\mathcal{H} &= (\Pi_\mu)_{mn} \dot{A}_{nm}^\mu - \mathcal{L} \\ &= (E_i)_{mn}(\partial^i A_{nm}^0 - E_{nm}^i + ig[A^i, A^0]_{nm}) - \left(\frac{1}{2}\text{Tr}(E^i \cdot E^i) - \frac{1}{2}\text{Tr}(B^i \cdot B^i)\right) \\ &= \frac{1}{2}\text{Tr}(E^i \cdot E^i + B^i \cdot B^i) - \text{Tr}(E_i(\partial^i A^0 + ig[A^i, A^0])).\end{aligned}$$

This second part of this expression can be written as follows using integration by parts for the first term and the cyclicity of trace for the second term:

$$\begin{aligned}\text{Tr}(A^0 \partial^i E_i - ig(A^0 E_i A^i - A^0 A^i E_i)) &= \text{Tr}[A^0(\partial^i E_i + ig[A^i, E_i])] \\ &\equiv (A^0)_{mn} \hat{G}_{nm},\end{aligned}$$

where

$$\hat{G}_{nm} = (\partial^i E_i + ig [A^i, E_i])_{nm}.$$

The Hamiltonian density is therefore given by

$$\mathcal{H} = \frac{1}{2} \text{Tr} (E^i \cdot E^i + B^i \cdot B^i) + (A^0)_{mn} \hat{G}_{nm}.$$

The Euler-Lagrange equations of motion become

$$\frac{\partial \mathcal{L}}{\partial A^0} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A^0)} \right) = 0.$$

Then,

$$0 = \frac{\partial \mathcal{H}}{\partial A_{mn}^0} = \hat{G}_{nm}.$$

Hence the Hamiltonian density comprises a term

$$\mathcal{H} = \frac{1}{2} \text{Tr} (E^i \cdot E^i + B^i \cdot B^i),$$

together with a constraint, $\hat{G}_{nm} = 0$, which will act on physical fields. For an infinitesimal traceless transformation parameter $\omega_{ba} \in SU(2)$, the physical fields will be obtained by considering

$$\omega_{mn} [\hat{G}_{nm}, A_{pq}^k] \equiv \omega_{mn} (\hat{G}_{nm}, A_{pq}^k).$$

where the brackets in $\omega_{mn} (\hat{G}_{nm}, A_{pq}^k)$ are intended to imply that the \hat{G}_{nm} acts on A_{pq} and no further. The quantisation condition is given by

$$\begin{aligned} (\Pi_i)_{mn} (= (E_i)_{mn}) &= -i \frac{\partial}{\partial (A^i)_{nm}} \\ [(\Pi_i)_{mn}, (A^j)_{pq}] &= \left((\Pi_i)_{mn} (A^j)_{pq} \right) = -i \delta_{np} \delta_{mq} \delta_i^j, \end{aligned}$$

Hence,

$$\begin{aligned}
 [A^i, E_i]_{nm} &= [A^i, \Pi_i]_{nm} \\
 &= (A^i)_{nr} (\Pi_i)_{rm} - (\Pi_i)_{nr} (A^i)_{rm} \\
 &= (A^i)_{nr} (\Pi_i)_{rm} - ((\Pi_i)_{nr} (A^i)_{rm}) - (A^i)_{rm} (\Pi_i)_{nr} \\
 &= (A^i)_{nr} (\Pi_i)_{rm} + i\delta_i^i \delta_{rr} \delta_{mn} - (A^i)_{rm} (\Pi_i)_{nr},
 \end{aligned}$$

and using the fact that ω is traceless,

$$\begin{aligned}
 \omega_{mn} \left([A^i, E_i]_{nm} A_{pq}^k \right) &= \omega_{mn} \left(((A^i)_{nr} (\Pi_i)_{rm} - (A^i)_{rm} (\Pi_i)_{nr}) A_{pq}^k \right) \\
 &= \omega_{mn} \left(-i (A^i)_{nr} \delta_i^k \delta_{mp} \delta_{rq} + i (A^i)_{rm} \delta_i^k \delta_{rp} \delta_{nq} \right) \\
 &= -i \omega_{pn} (A^k)_{nq} + i (A^k)_{pm} \omega_{mq} \\
 &= -i [\omega, A^k]_{pq}.
 \end{aligned}$$

Therefore infinitesimal transformations will be given by

$$\begin{aligned}
 \frac{1}{ig} \omega_{mn} [\hat{G}_{nm}, A_{pq}^k] &= \frac{1}{ig} \omega_{mn} \left((\partial_i E^i + ig [A^i, E_i])_{nm} A_{pq}^k \right) \\
 &= \frac{1}{ig} \omega_{mn} \left(\partial_i \left(-i \delta_i^k \delta_{mp} \delta_{nq} \right) + ig \left(-i [\omega, A^k]_{pq} \right) \right) \\
 &= \left(\frac{1}{g} \partial_k \omega - i [\omega, A^k] \right)_{pq} \tag{3.20}
 \end{aligned}$$

which is exactly the same form as infinitesimal transformations for unitary, non-abelian transformations as given in equation (1.6).

3.3.2 Dimensional reduction of Hamiltonian on $\mathbb{R} \times T^3$

When the SU(2) gauge fields are dimensionally reduced on $\mathbb{R} \times T^3$ where all but the time component in the action are neglected, then $\partial_k \omega$ will be identically zero. The infinitesimal transformations will therefore be given by

$$\frac{1}{ig} \omega_{mn} [\hat{G}_{nm}, A_{pq}^k] = -i [\omega, A^k]_{pq} \tag{3.21}$$

so that

$$A^k \rightarrow A'^k = A^k - i \left[\omega, A^k \right]_{pq} \quad (3.22)$$

which, by equation (1.4), is the infinitesimal form of the transformation

$$A^{k'} = U A^k U^\dagger \quad (3.23)$$

The vector form of infinitesimal transformation (3.22) is given by

$$\begin{aligned} \frac{1}{2} a_\ell^k \sigma_\ell &\rightarrow \frac{1}{2} a_\ell^k \sigma_\ell - i^2 \frac{1}{2} \epsilon_{\ell mn} \sigma_\ell \omega_m a_n^k \\ \Rightarrow a_\ell^k &\rightarrow a_\ell^k + \epsilon_{\ell mn} \omega_m a_n^k \end{aligned}$$

which describes an O(3) rotation, with O(3) invariants given by $a_\ell^k \cdot a_\ell^j$. Furthermore,

$$\begin{aligned} \frac{1}{ig} \omega_{mn} \left[\hat{G}_{nm}, a_\ell^k a_\ell^j \right] &= \frac{1}{ig} \omega_{mn} \left((ig [A^i, E_i])_{nm} A_{pq}^k A_{qp}^j \right) \\ &= \omega_{mn} \left((A_{nr}^i (\Pi_i)_{rm} - A_{rm}^i (\Pi_i)_{nr}) A_{pq}^k A_{qp}^j \right) \\ &= \omega_{mn} \left(A_{nr}^i \left(\delta_i^k \delta_{mp} \delta_{rq} A_{qp}^j + A_{pq}^k \delta_i^j \delta_{mq} \delta_{rp} \right) \right. \\ &\quad \left. - A_{rm}^i \left(\delta_i^k \delta_{rp} \delta_{nq} A_{qp}^j + A_{pq}^k \delta_i^j \delta_{rq} \delta_{np} \right) \right) \\ &= \omega_{mn} \left(A_{nr}^k A_{rm}^j + A_{nr}^j A_{rm}^k - A_{pm}^k A_{np}^j - A_{qm}^j A_{nq}^k \right) \\ &= 0 \end{aligned}$$

This shows that for $SU(2)$, the physical fields of the system are given by O(3) invariants.

3.3.3 Saddle point analysis using the Hamiltonian approach

Making use of the collective field method it was found in equation (3.8) that the Hamiltonian comprised an effective potential,

$$\mathcal{H} = V_{eff} = \frac{1}{8} \text{Tr} (\phi^{-1}) [N - (d + 1)]^2 + V,$$

where V is the Yang-mills potential comprising only spatial fields as explained in the previous section (due to the dimensional reduction of $SU(2)$ gauge theories on $\mathbb{R} \times T^3$). This potential is given by

$$V = \frac{1}{4} \text{Tr} (F_{ij} F_{ij}) + \frac{1}{2} \text{Tr} (B_i \cdot B_i),$$

which depends only on magnetic field components. That is

$$\begin{aligned} V &= \frac{g^2}{8} \sum_{i \neq j} [(\vec{a}_i \cdot \vec{a}_i)(\vec{a}_j \cdot \vec{a}_j) - (\vec{a}_i \cdot \vec{a}_j)(\vec{a}_i \cdot \vec{a}_j)] \\ &= \frac{g^2}{8} \sum_{i \neq j} [\phi_{ii} \phi_{jj} - (\phi_{ij})^2]. \end{aligned}$$

Note that, unlike in subsection 3.2.1, there is no mass term.

Performing a saddle point analysis on the Hamiltonian requires calculation of $\frac{\partial \text{Tr}(\phi^{-1})}{\partial \phi_{ab}}$:

$$\begin{aligned} A^{-1} A &= 1 \\ \Rightarrow \frac{\partial A^{-1}}{\partial \phi_{ab}} A + A^{-1} \frac{\partial A}{\partial \phi_{ab}} &= 0 \\ \Rightarrow \frac{\partial A^{-1}}{\partial \phi_{ab}} &= -A^{-1} \frac{\partial A}{\partial \phi_{ab}} A^{-1}. \end{aligned}$$

Since $\text{Tr} \frac{\partial A}{\partial \phi_{ab}} = \frac{\partial \text{Tr} A}{\partial \phi_{ab}}$,

$$\begin{aligned} \frac{\partial \text{Tr} \phi^{-1}}{\partial \phi_{ab}} &= \text{Tr} \left(\frac{\partial \phi^{-1}}{\partial \phi_{ab}} \right) \\ &= -\text{Tr} \left(\phi^{-1} \frac{\partial \phi}{\partial \phi_{ab}} \phi^{-1} \right) \\ &= -\phi_{ij}^{-1} \frac{\partial \phi_{jk}}{\partial \phi_{ab}} \phi_{ki}^{-1} \\ &= -\phi_{ij}^{-1} \delta_{ja} \delta_{kb} \phi_{ki}^{-1} \\ &= -\phi_{ia}^{-1} \phi_{bi}^{-1} \\ &= -\phi_{ba}^{-2}. \end{aligned} \tag{3.24}$$

Hence the large N saddle point analysis is given by

$$\begin{aligned}
 V_{eff} &= \frac{1}{8} \text{Tr} (\phi^{-1}) [N - (d + 1)]^2 + \frac{g^2}{8} \sum_{i \neq j} [\phi_{ii} \phi_{jj} - (\phi_{ij})^2] \\
 &\rightarrow \frac{1}{8} \text{Tr} (\phi^{-1}) N^2 + \frac{g^2}{8} \sum_{i \neq j} [\phi_{ii} \phi_{jj} - (\phi_{ij})^2] \\
 \Rightarrow \frac{\partial V_{eff}}{\partial \phi_{ab}} &= -\frac{N^2}{8} \phi_{ba}^{-2} + \frac{g^2}{8} \sum_{i \neq j} [2\phi_{ii} \delta_{ja} \delta_{jb} - 2\phi_{ij} \delta_{ib} \delta_{ja}] \\
 &= -\frac{N^2}{8} \phi_{ba}^{-2} + \frac{g^2}{4} \sum_{i \neq j} [\phi_{ii} \delta_{ab} - \phi_{ba}] = 0.
 \end{aligned}$$

Multiplying through by ϕ_{ac} and summing over a , one obtains

$$-\frac{N^2}{8} \phi_{bc}^{-1} + \frac{g^2}{4} \sum_{i \neq j} [\phi_{ii} \phi_{bc} - \phi_{bc}^2] = 0,$$

and multiplying through once again by ϕ_{cd} and summing over c gives

$$\frac{N^2}{8} \delta_{bd} + \frac{g^2}{4} \sum_{i \neq j} [\phi_{ii} \phi_{bd}^2 - \phi_{bd}^3] = 0. \quad (3.25)$$

Now, since the first term is diagonal, a logical ansatz for ϕ would be

$$\phi_{ab} = BN \delta_{ab}.$$

Substituting this ansatz into equation (3.25) results in

$$\begin{aligned}
 -\frac{N^2}{8} \delta_{bd} + \frac{g^2}{4} \sum_{i \neq j} [BN \delta_{ii} (BN)^2 \delta_{bd} - (BN)^3 \delta_{bd}] &= 0 \\
 \Rightarrow -\frac{N^2}{8} \delta_{bd} + \frac{g^2}{4} B^3 N^3 [d - 1] \delta_{bd} &= 0.
 \end{aligned}$$

Hence, considering a particular element $b = d = p$, one obtains

$$\begin{aligned}
 B^3 &= \frac{N^2}{2N^3 g^2 [d-1]} \\
 &= \frac{1}{2N g^2 [d-1]} \\
 &= \frac{1}{2\lambda [d-1]},
 \end{aligned}$$

where $\lambda = g^2 N$. Thus,

$$B = \sqrt[3]{\frac{1}{2\lambda[d-1]}}.$$

Hence a solution exists, without any need for an infrared regulator.

Chapter 4

SU(2) gauge theories in terms of vector-valued fields

4.1 Collective field theory for space-time dependent fields

In the rest of this dissertation, the possibility of developing an $O(N)$ systematic approximation for dimensionally unreduced SU(2) gauge theories will be explored; that is, SU(2) gauge theories with space-time dependent fields. In this case, the invariant fields are bilocals

$$\phi(x, y) \equiv \sum_m a^m(x) a^m(y)$$

The Jacobian obtained in section 3.1 is now extended to incorporate space-time dependence, where a single field is considered. Thus equation (2.5) becomes

$$\Omega_{\alpha\beta}(\partial_\beta \ln J) = \omega_\alpha - \partial_\beta \Omega_{\alpha\beta}$$

with

$$\begin{aligned}
\omega_\alpha \equiv \omega(x, y) &= \int d^d z \sum_{m=1}^N \frac{\partial^2 \phi(x, y)}{\partial [a^m(z)]^2} \\
&= \int d^d z \sum_{m=1}^N \frac{\partial}{\partial [a^m(z)]} \left(\frac{\partial \phi(x, y)}{\partial [a^m(z)]} \right) \\
&= \int d^d z \sum_{m=1}^N \frac{\partial}{\partial [a^m(z)]} \left(a^m(x) \delta^d(y-z) + a^m(y) \delta^d(x-z) \right) \\
&= 2N \delta^d(x-y)
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{\alpha\beta} \equiv \Omega(x, y; x', y') &= \int d^d z \sum_{m=1}^N \frac{\partial \phi(x, y)}{\partial a^m(z)} \frac{\partial \phi(x', y')}{\partial a^m(z)} \\
&= \int d^d z \sum_{m=1}^N (a^m(x) \delta(y-z) + a^m(y) \delta(x-z)) \\
&\quad (a^m(x') \delta(z-y') + a^m(y') \delta(z-x')) \quad (4.1) \\
&= \phi(x', x) \delta^d(y-y') + \phi(x, y') \delta^d(y-x') \\
&\quad + \phi(y, x') \delta^d(x-y') + \phi(y, y') \delta^d(x-x').
\end{aligned}$$

Therefore $\frac{\partial \Omega(x, y; x', y')}{\partial \phi(x', y')}$ is given by

$$\begin{aligned}
&\int dx' \int dy' \delta^d(x' - x') \delta^d(x - y') \delta^d(y - y') + \delta^d(x' - x) \delta^d(y' - y') \delta^d(y - x') \\
&+ \delta^d(x' - y) \delta^d(y' - x') \delta^d(x - y') + \delta^d(x' - y) \delta^d(y' - y') \delta^d(x - x') \\
&= V \delta^d(0) \delta^d(x - y) + V \delta^d(0) \delta^d(x - y) + \delta^d(x - y) + \delta^d(x - y) \\
&= (2 + 2V \delta^d(0)) \delta^d(x - y).
\end{aligned}$$

Thus the right-hand side of the hermiticity relation in equation (2.5) becomes

$$2 \left(N - \left(1 + V \delta^d(0) \right) \right) \delta^d(x - y).$$

The left-hand side of equation (2.5) is given by $\Omega_{x,y;x',y'} \frac{\partial \ln J}{\partial \phi(x',y')}$ so consider

$$\begin{aligned}
\Omega_{x,y;x',y'} \frac{\partial \ln \det \Phi}{\partial \Phi(x',y')} &= \Omega_{x,y;x',y'} \Phi^{-1}(y',x') \\
\Rightarrow \Omega_{x,y;x',y'} \frac{\partial \ln \det \phi}{\partial \phi(x',y')} &= \int d^d x' \int_{y' > x'} d^d y' \Omega_{x,y;x',y'} \Phi^{-1}(y',x') \\
&= \int d^d x' \int d^d y' \phi^{-1}(y',x') \\
&\quad \phi(x',x) \delta^d(y-y') + \phi(x,y') \delta^d(y-x') \\
&\quad + \phi(y,x') \delta^d(x-y') + \phi(y,y') \delta^d(x-x') \\
&= 4\delta^d(x-y).
\end{aligned}$$

Comparing the two expressions

$$\begin{aligned}
\Omega_{x,y;x',y'} \frac{\partial \ln J}{\partial \phi(x',y')} &= 2 \left(N - \left(1 + V\delta^d(0) \right) \right) \delta^d(x-y) \\
\Omega_{x,y;x',y'} \frac{\partial \ln \det \phi}{\partial \phi(x',y')} &= 4\delta^d(x-y),
\end{aligned}$$

results in

$$\ln J = \frac{1}{2} \left[N - \left(1 + V\delta^d(0) \right) \right] \ln \det \phi = \frac{1}{2} \left[N - \left(1 + V\delta^d(0) \right) \right] \text{Tr} \ln \phi \quad (4.2)$$

which is an extension of equation (3.7) to field-valued theories. Here the trace is functional over coordinates x and y . In the large N limit, then, the determinant is given by

$$\ln J \rightarrow \frac{N}{2} \text{Tr} \ln \phi.$$

With d original vector fields $a_i^m(x)$, $i = 1, \dots, d$, the invariant bilocals are

$$\phi_{ij}(x,y) \equiv \sum_m a_i^m(x) a_j^m(y),$$

the Jacobian in the large N limit becomes

$$\ln J = \frac{N}{2} \text{Tr} \text{Tr} \ln \phi.$$

where the additional trace is taken with respect to vector indices i and j .

4.2 The full field theory

In this section, the full (not dimensionally reduced) field theory for $SU(2)$ gauge theories is considered, where none of the fields are independent of space-time. Note that the bilocals $\phi_{\mu\nu}(x, y) = \vec{a}_\mu(x) \cdot \vec{a}_\nu(y)$ no longer span the set of gauge invariant fields. However, one can introduce a mass term that would break the local gauge symmetry to a global one, and then, once a gap equation is obtained, the limit as the mass goes to zero would be considered. This is the case that will be assumed. The Lagrangian was found in equation (1.8) to be

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} \\ &= -\text{Tr} \frac{1}{4} [(\partial_\mu A_\nu - \partial_\nu A_\mu) + ig [A_\mu, A_\nu]] [(\partial^\mu A^\nu - \partial^\nu A^\mu) + ig [A^\mu, A^\nu]] \\ &\equiv \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4. \end{aligned}$$

In Euclidean coordinates, \mathcal{L}_2 in equation (1.18) reduces to the following for $SU(2)$:

$$\begin{aligned} \mathcal{L}_2 &= -\frac{1}{4} (\partial_\mu a_\nu(x) \cdot \partial^\mu a^\nu(x) - \partial_\mu a_\nu(x) \cdot \partial^\nu a^\mu(x)) \\ &\rightarrow -\frac{1}{4} (\partial_\mu a_\nu(x) \cdot \partial_\mu a_\nu(x) - \partial_\mu a_\nu(x) \cdot \partial_\nu a_\mu(x)). \\ \Rightarrow (\mathcal{L}_2)_E &= \frac{1}{4} (\partial_\mu a_\nu(x) \cdot \partial_\mu a_\nu(x) - \partial_\mu a_\nu(x) \cdot \partial_\nu a_\mu(x)) \end{aligned}$$

The accompanying action for $(\mathcal{L}_2)_E$ is

$$\begin{aligned} (\mathcal{S}_2)_E &= \frac{1}{4} \int d^4x d^4x' \delta(x - x') [\partial_\mu \bar{\partial}_\mu a_\nu(x) a_\nu(x') - \partial_\mu \bar{\partial}_\nu a_\nu(x) a_\mu(x')] \\ &= \frac{1}{4} \int d^4x d^4x' \delta(x - x') [\partial_\mu \bar{\partial}_\mu \phi_{\nu\nu}(x, x') - \partial_\mu \bar{\partial}_\nu \phi_{\nu\mu}(x, x')], \end{aligned}$$

where ∂_μ is the derivative with respect to x and $\bar{\partial}_\mu$ is the derivative with respect to x' . Similarly \mathcal{L}_4 in equation (1.20) becomes

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{8}g^2 [(a_\mu(x) \cdot a^\mu(x)) (a_\nu(x) \cdot a^\nu(x)) - (a_\mu(x) \cdot a^\nu(x)) (a_\nu(x) \cdot a^\mu(x))] \\ &\rightarrow -\frac{1}{8}g^2 [(a_\mu(x) \cdot a_\mu(x)) (a_\nu(x) \cdot a_\nu(x)) - (a_\mu(x) \cdot a_\nu(x)) (a_\nu(x) \cdot a_\mu(x))] \\ \Rightarrow (\mathcal{L}_4)_E &= \frac{1}{8}g^2 [(a_\mu(x) \cdot a_\mu(x)) (a_\nu(x) \cdot a_\nu(x)) - (a_\mu(x) \cdot a_\nu(x)) (a_\nu(x) \cdot a_\mu(x))],\end{aligned}$$

and hence the accompanying action for \mathcal{L}_4 is

$$(\mathcal{S}_4)_E = \frac{1}{8}g^2 \int d^4x [\phi_{\mu\mu}(x, x) \phi_{\nu\nu}(x, x) - \phi_{\mu\nu}(x, x) \phi_{\nu\mu}(x, x)].$$

Last of all, \mathcal{L}_3 from equation (1.10) simplifies as follows:

$$\begin{aligned}\mathcal{L}_3 &= \frac{1}{2}g\epsilon^{mnp} (\partial_\mu a_\nu^m(x)) a^{\mu n}(x) a^{\nu p}(x) \\ &\rightarrow \frac{1}{2}g\epsilon^{mnp} (\partial_\mu a_\nu^m(x)) a_\mu^n(x) a_\nu^p(x) \\ \Rightarrow (\mathcal{L}_3)_E &= -\frac{1}{2}g\epsilon^{mnp} (\partial_\mu a_\nu^m(x)) a_\mu^n(x) a_\nu^p(x).\end{aligned}$$

with accompanying action

$$(\mathcal{S}_3)_E = \frac{1}{2}g \int d^4x \epsilon^{mnp} (\partial_\mu a_\nu^m(x)) a_\mu^n(x) a_\nu^p(x)$$

Unlike in previous calculations, this term is $SO(3)$ invariant due to the presence of the ϵ^{mnp} term, and a change of coordinates may not be automatically done. The possibility of generalising the approach followed in Ref. [17] will therefore be explored.

While $(\mathcal{S}_2)_E$ and $(\mathcal{S}_4)_E$ are both even, leading to $O(3)$ invariants, $(\mathcal{S}_3)_E$ is odd leading to $SO(3)$ invariants. Consider for instance the partition function. Expanding $e^{\mathcal{S}_3} = e^{\int d^4x \mathcal{L}_3}$, one notes that the expectation value of odd expressions vanish, so

that the following is obtained:

$$\begin{aligned} \left\langle e^{\int d^4x (\mathcal{L}_3(x))_{\text{E}}} \right\rangle &= \left\langle 1 + \int d^4x (\mathcal{L}_3(x))_{\text{E}} + \frac{1}{2!} \int d^4x d^4x' (\mathcal{L}_3(x))_{\text{E}} (\mathcal{L}_3(x'))_{\text{E}} - \dots \right\rangle \\ &= \left\langle 1 + \frac{1}{2!} \int d^4x d^4x' (\mathcal{L}_3(x))_{\text{E}} (\mathcal{L}_3(x'))_{\text{E}} + \frac{1}{4!} \int d^4x d^4x' d^4x'' d^4x''' \right. \\ &\quad \left. (\mathcal{L}_3(x))_{\text{E}} (\mathcal{L}_3(x'))_{\text{E}} (\mathcal{L}_3(x''))_{\text{E}} (\mathcal{L}_3(x'''))_{\text{E}} + \dots \right\rangle \end{aligned}$$

Now define

$$\begin{aligned} \Delta &= \int d^4x d^4x' (\mathcal{L}_3(x))_{\text{E}} (\mathcal{L}_3(x'))_{\text{E}} \\ &= \frac{g^2}{4} \int d^4x d^4x' \epsilon^{abc} (\partial_\mu a_\nu^a)(x) a_\mu^b(x) a_\nu^c(x) \epsilon^{a'b'c'} \left(\bar{\partial}_{\mu'} a_{\nu'}^{a'}(x') \right) a_{\mu'}^{b'}(x') a_{\nu'}^{c'}(x') \end{aligned}$$

Note that contractions of $\epsilon^{abc}\epsilon^{a'b'c'}$ will result in bilocals of the form $\phi_{\mu\mu'} \equiv a_\mu^a \cdot a_{\mu'}^a(x, x')$. The following (with implicit dependence on (\tilde{x}, \tilde{x}')) will therefore be obtained:

$$\begin{aligned} \Delta &= \frac{g^2}{4} \int d^4x d^4x' (\partial_\mu \bar{\partial}_{\mu'} \phi_{\nu\nu'}) \phi_{\mu\mu'} \phi_{\nu\nu'} + (\partial_\mu \phi_{\nu\mu'}) \phi_{\mu\nu'} (\bar{\partial}_{\mu'} \phi_{\nu\nu'}) \\ &\quad + (\partial_\mu \phi_{\nu\nu'}) (\bar{\partial}_{\mu'} \phi_{\mu\nu'}) \phi_{\nu\mu'} - (\partial_\mu \bar{\partial}_{\mu'} \phi_{\nu\nu'}) \phi_{\mu\nu'} \phi_{\nu\mu'} \\ &\quad - (\partial_\mu \phi_{\nu\nu'}) \phi_{\mu\mu'} (\bar{\partial}_{\mu'} \phi_{\nu\nu'}) - (\partial_\mu \phi_{\nu\mu'}) (\bar{\partial}_{\mu'} \phi_{\mu\nu'}) \phi_{\nu\nu'}. \end{aligned}$$

The Taylor expansion of $\cosh x$ is given by

$$\begin{aligned} \cosh x &= \frac{1}{2} [e^x + e^{-x}] \\ &= \frac{1}{2} \left[\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \right) + \left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \dots \right) \right] \\ &= \frac{1}{2} \left[2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \dots \right] \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \end{aligned}$$

and hence the partition function is given by

$$\begin{aligned}
\mathcal{Z} &= \int [dA] e^{-((\mathcal{S}_2)_E + (\mathcal{S}_4)_E)} e^{(\mathcal{S}_3)_E} \\
&= \int [dA] e^{-((\mathcal{S}_2)_E + (\mathcal{S}_4)_E)} \cosh(\sqrt{\Delta}) \\
&= \frac{1}{2} \int [dA] e^{-((\mathcal{S}_2)_E + (\mathcal{S}_4)_E)} (e^{\sqrt{\Delta}} + e^{-\sqrt{\Delta}}) \\
&= \frac{1}{2} \int [d\phi] e^{-((\mathcal{S}_2)_E + (\mathcal{S}_4)_E) + \ln J} (e^{\sqrt{\Delta}} + e^{-\sqrt{\Delta}}) \\
&= \frac{1}{2} (\mathcal{Z}^- + \mathcal{Z}^+).
\end{aligned}$$

Two effective actions will therefore be obtained (equation (3.9)):

$$\begin{aligned}
\mathcal{S}_{\text{eff}} &= \mathcal{S}_E - \ln J \\
\Rightarrow \mathcal{S}_{\text{eff}}^\pm &= \mathcal{S}_2 + \mathcal{S}_4 - \ln J \pm \sqrt{\Delta} \\
&= \frac{1}{4} \int d^4x d^4x' \delta(x - x') [\partial_\mu \bar{\partial}_\mu \phi_{\nu\nu}(x, x') - \partial_\mu \bar{\partial}_\nu \phi_{\nu\mu}(x, x')] \\
&\quad \frac{1}{8} g^2 \int d^4x [\phi_{\mu\mu}(x, x) \phi_{\nu\nu}(x, x) - \phi_{\mu\nu}(x, x) \phi_{\nu\mu}(x, x)] \\
&\quad - \frac{N}{2} \text{TrTr} [\ln \phi] \pm \sqrt{\Delta}.
\end{aligned}$$

Making use of a translational invariant ansatz, which is applicable for the leading large N saddle point, the fields may be written in terms of their Fourier transforms,

resulting in

$$\begin{aligned}
\mathcal{S}_{\text{eff}}^{\pm} &= \frac{1}{4} \int d^4x d^4x' \delta(x-x') \int \frac{d^4p}{(2\pi)^4} \left[\partial_{\mu} \bar{\partial}_{\mu} \phi_{\nu\nu}(p) e^{ip \cdot (x-x')} \right. \\
&\quad \left. - \partial_{\mu} \bar{\partial}_{\nu} e^{ip \cdot (x-x')} \phi_{\nu\mu}(p) \right] + \frac{g^2}{8} \int d^4x \int \frac{d^4p^1}{(2\pi)^4} \int \frac{d^4p^2}{(2\pi)^4} \\
&\quad \left[e^{ip^1 \cdot (x-x)} \phi_{\mu\mu}(p^1) e^{ip^2 \cdot (x-x)} \phi_{\nu\nu}(p^2) - e^{ip^1 \cdot (x-x)} \phi_{\mu\nu}(p^1) e^{ip^2 \cdot (x-x)} \phi_{\nu\mu}(p^2) \right] \\
&\quad - \frac{N}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-x)} \text{Tr} [\ln \phi(p)] \pm \sqrt{\Delta_{\text{FT}}} \\
&= \frac{1}{4} V \int \frac{d^4p}{(2\pi)^4} [p^2 \phi_{\nu\nu}(p) - p_{\mu} p_{\nu} \phi_{\nu\mu}(p)] \\
&\quad + \frac{g^2}{8} V \int \frac{d^4p^1}{(2\pi)^4} \int \frac{d^4p^2}{(2\pi)^4} [\phi_{\mu\mu}(p^1) \phi_{\nu\nu}(p^2) - \phi_{\mu\nu}(p^1) \phi_{\nu\mu}(p^2)] \\
&\quad - \frac{N}{2} V \int \frac{d^4p}{(2\pi)^4} [\ln \phi(p)] \pm \sqrt{\Delta_{\text{FT}}}, \tag{4.3}
\end{aligned}$$

where Δ_{FT} is the Fourier transform of Δ , given by

$$\begin{aligned}
\Delta_{\text{FT}} &= \frac{g^2}{4} \int d^4x \int d^4x' \int \frac{d^4p^1}{(2\pi)^4} \int \frac{d^4p^2}{(2\pi)^4} \int \frac{d^4p^3}{(2\pi)^4} \\
&\quad \left(\partial_{\mu} \bar{\partial}_{\mu'} e^{ip^1 \cdot (x-x')} \phi_{\nu\nu'}(p^1) \right) e^{ip^2 \cdot (x-x')} \phi_{\mu\mu'}(p^2) e^{ip^3 \cdot (x-x')} \phi_{\nu\nu'}(p^3) \\
&\quad + \left(\partial_{\mu} e^{ip^1 \cdot (x-x')} \phi_{\nu\mu'}(p^1) \right) e^{ip^2 \cdot (x-x')} \phi_{\mu\nu'}(p^2) \left(\bar{\partial}_{\mu'} e^{ip^3 \cdot (x-x')} \phi_{\nu\nu'}(p^3) \right) \\
&\quad + \left(\partial_{\mu} e^{ip^1 \cdot (x-x')} \phi_{\nu\nu'}(p^1) \right) \left(\bar{\partial}_{\mu'} e^{ip^2 \cdot (x-x')} \phi_{\mu\nu'}(p^2) \right) e^{ip^3 \cdot (x-x')} \phi_{\nu\mu'}(p^3) \\
&\quad - \left(\partial_{\mu} \bar{\partial}_{\mu'} e^{ip^1 \cdot (x-x')} \phi_{\nu\nu'}(p^1) \right) e^{ip^2 \cdot (x-x')} \phi_{\mu\nu'}(p^2) e^{ip^3 \cdot (x-x')} \phi_{\nu\mu'}(p^3) \\
&\quad - \left(\partial_{\mu} e^{ip^1 \cdot (x-x')} \phi_{\nu\nu'}(p^1) \right) e^{ip^2 \cdot (x-x')} \phi_{\mu\mu'}(p^2) \left(\bar{\partial}_{\mu'} e^{ip^3 \cdot (x-x')} \phi_{\nu\nu'}(p^3) \right) \\
&\quad - \left(\partial_{\mu} e^{ip^1 \cdot (x-x')} \phi_{\nu\mu'}(p^1) \right) \left(\bar{\partial}_{\mu'} e^{ip^2 \cdot (x-x')} \phi_{\mu\nu'}(p^2) \right) e^{ip^3 \cdot (x-x')} \phi_{\nu\nu'}(p^3) \\
&= \frac{g^2}{4} \int d^4x \int d^4x' \int \frac{d^4p^1}{(2\pi)^4} \int \frac{d^4p^2}{(2\pi)^4} \int \frac{d^4p^3}{(2\pi)^4} e^{i(p^1+p^2+p^3) \cdot (x-x')} \\
&\quad p_{\mu}^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) + p_{\mu}^1 p_{\mu'}^3 \phi_{\nu\mu'}(p^1) \phi_{\mu\nu'}(p^2) \phi_{\nu\nu'}(p^3) \\
&\quad + p_{\mu}^1 p_{\mu'}^2 \phi_{\nu\nu'}(p^1) \phi_{\mu\nu'}(p^2) \phi_{\nu\mu'}(p^3) - p_{\mu}^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\nu'}(p^2) \phi_{\nu\mu'}(p^3) \\
&\quad - p_{\mu}^1 p_{\mu'}^3 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) - p_{\mu}^1 p_{\mu'}^2 \phi_{\nu\mu'}(p^1) \phi_{\mu\nu'}(p^2) \phi_{\nu\nu'}(p^3).
\end{aligned}$$

Setting $s = x - x'$ and $t = x + x'$ implies that $x = \frac{1}{2}(s + t)$ and $x' = \frac{1}{2}(t - s)$. Then the Jacobian will be given by $|\frac{\partial x, x'}{\partial s, t}| = \frac{1}{2}$. Thus Δ_{FT} becomes

$$\begin{aligned}
\Delta_{FT} &= \frac{1}{2} \frac{g^2}{4} \int d^4 s \int d^4 t \int \frac{d^4 p^1}{(2\pi)^4} \int \frac{d^4 p^2}{(2\pi)^4} \int \frac{d^4 p^3}{(2\pi)^4} e^{i(p^1 + p^2 + p^3) \cdot s} \\
&\quad p_\mu^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) + p_\mu^1 p_{\mu'}^3 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) \\
&\quad + p_\mu^1 p_{\mu'}^2 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\mu'}(p^3) - p_\mu^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\mu'}(p^3) \\
&\quad - p_\mu^1 p_{\mu'}^3 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) - p_\mu^1 p_{\mu'}^2 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) \\
&= \frac{1}{2} \frac{g^2}{4} V \int \frac{d^4 p^1}{(2\pi)^4} \int \frac{d^4 p^2}{(2\pi)^4} \int \frac{d^4 p^3}{(2\pi)^4} (2\pi)^4 \delta(p^1 + p^2 + p^3) \\
&\quad p_\mu^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) + p_\mu^1 p_{\mu'}^3 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) \\
&\quad + p_\mu^1 p_{\mu'}^2 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\mu'}(p^3) - p_\mu^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\mu'}(p^3) \\
&\quad - p_\mu^1 p_{\mu'}^3 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) - p_\mu^1 p_{\mu'}^2 \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(p^3) \\
&= \frac{1}{2} \frac{g^2}{4} V \int \frac{d^4 p^1}{(2\pi)^4} \int \frac{d^4 p^2}{(2\pi)^4} \\
&\quad p_\mu^1 p_{\mu'}^1 \phi_{\nu\nu'}(p^1) [\phi_{\mu\mu'}(p^2) \phi_{\nu\nu'}(-p^1 - p^2) - \phi_{\mu\nu'}(p^2) \phi_{\nu\mu'}(-p^1 - p^2)] \\
&\quad + p_\mu^1 (-p_{\mu'}^1 - p_{\mu'}^2) \phi_{\nu\nu'}(-p^1 - p^2) [\phi_{\nu\mu'}(p^1) \phi_{\mu\mu'}(p^2) - \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p^2)] \\
&\quad + p_\mu^1 p_{\mu'}^2 \phi_{\mu\mu'}(p^2) [\phi_{\nu\nu'}(p^1) \phi_{\nu\mu'}(-p^1 - p^2) - \phi_{\nu\mu'}(p^1) \phi_{\nu\nu'}(-p^1 - p^2)].
\end{aligned}$$

In the context of vector models, it was suggested in Ref. [17] that one should be able to obtain the large N saddle point configurations for \mathcal{Z}^\mp and, of the two, choose the one with lowest energy, with the second saddle point configuration contributing exponentially small correlators.

Note that Δ_{FT} is proportional to the volume, and hence $\sqrt{\Delta_{FT}}$ is proportional to \sqrt{V} ; one of the terms in $\mathcal{S}_{\text{eff}}^\pm$ is therefore not extensive. This requires further study. In the following, V will be kept finite.

$$\begin{aligned}
\frac{\partial \Delta_{\text{FT}}}{\partial \phi_{\alpha\beta}(p_0)} &= \frac{1}{2} \frac{g^2}{4} V \int \frac{d^4 p^1}{(2\pi)^4} \\
& p_{0\mu} p_{0\mu'} [\phi_{\mu\mu'}(p^1) \phi_{\alpha\beta}(-p^1 - p_0) - \phi_{\mu\beta}(p^1) \phi_{\alpha\mu'}(-p^1 - p_0)] \\
& + p_{\alpha}^1 p_{\beta}^1 \phi_{\nu\nu'}(p^1) \phi_{\nu\nu'}(-p^1 - p_0) + p_{\mu}^1 p_{\mu'}^1 \phi_{\alpha\beta}(p^1) \phi_{\mu\mu'}(-p_0 - p^1) \\
& - p_{\alpha}^1 p_{\mu'}^1 \phi_{\nu\beta}(p^1) \phi_{\nu\mu'}(-p^1 - p_0) - p_{\mu}^1 p_{\beta}^1 \phi_{\alpha\nu'}(p^1) \phi_{\mu\nu'}(-p_0 - p^1) \\
& + p_{\mu}^1 p_{0\mu'} [\phi_{\alpha\mu'}(p^1) \phi_{\mu\beta}(-p^1 - p_0) - \phi_{\alpha\beta}(p^1) \phi_{\mu\mu'}(-p^1 - p_0)] \\
& + p_{0\mu} (-p_{0\beta} - p_{\beta}^1) \phi_{\alpha\nu'}(-p_0 - p^1) \phi_{\mu\nu'}(p^1) + p_{\alpha}^1 (-p_{\mu'}^1 - p_{0\mu'}) \\
& \phi_{\nu\beta}(-p^1 - p_0) \phi_{\nu\mu'}(p^1) \\
& - p_{0\mu} (-p_{0\mu'} - p_{\mu'}^1) \phi_{\alpha\beta}(-p_0 - p^1) \phi_{\mu\mu'}(p^1) - p_{\alpha}^1 (-p_{\beta}^1 - p_{0\mu'}) \\
& \phi_{\nu\nu'}(-p^1 - p_0) \phi_{\nu\nu'}(p^1) \phi_{\mu\mu'}(p) \\
& + p_{\alpha}^1 p_{0\mu'} [\phi_{\nu\beta}(p^1) \phi_{\nu\mu'}(-p^1 - p_0) - \phi_{\nu\mu'}(p^1) \phi_{\nu\beta}(-p^1 - p_0)] \\
& + p_{0\mu} p_{\mu'}^1 \phi_{\mu\beta}(p^1) \phi_{\alpha\mu'}(-p_0 - p^1) + p_{\mu}^1 (-p_{\beta}^1 - p_{0\beta}) \phi_{\mu\nu'}(-p^1 - p_0) \\
& \phi_{\alpha\nu'}(p^1) \\
& - p_{0\mu} p_{\beta}^1 \phi_{\mu\nu'}(p^1) \phi_{\alpha\nu'}(-p_0 - p^1) - p_{\mu}^1 (-p_{0\mu'} - p_{\mu'}^1) \phi_{\mu\beta}(-p_0 - p^1) \\
& \phi_{\alpha\mu'}(p^1) \\
& = \frac{1}{2} \frac{g^2}{4} V \int \frac{d^4 p^1}{(2\pi)^4} \\
& 2p_{0\mu} p_{0\mu'} \phi_{\mu\mu'}(p^1) \phi_{\alpha\beta}(-p^1 - p_0) - p_{0\mu} p_{0\mu'} \phi_{\mu\beta}(p^1) \phi_{\alpha\mu'}(-p^1 - p_0) \\
& + 2p_{\alpha}^1 p_{\beta}^1 \phi_{\nu\nu'}(p^1) \phi_{\nu\nu'}(-p^1 - p_0) + p_{\mu}^1 p_{\mu'}^1 \phi_{\alpha\beta}(p^1) \phi_{\mu\mu'}(-p_0 - p^1) \\
& - p_{\alpha}^1 p_{\mu'}^1 \phi_{\nu\beta}(p^1) \phi_{\nu\mu'}(-p^1 - p_0) - 2p_{\mu}^1 p_{\beta}^1 \phi_{\alpha\nu'}(p^1) \phi_{\mu\nu'}(-p_0 - p^1) \\
& + 2p_{\mu}^1 p_{0\mu'} \phi_{\alpha\mu'}(p^1) \phi_{\mu\beta}(-p^1 - p_0) - p_{\mu}^1 p_{0\mu'} \phi_{\alpha\beta}(p^1) \phi_{\mu\mu'}(-p^1 - p_0) \\
& - p_{0\mu} p_{0\beta} \phi_{\alpha\nu'}(-p_0 - p^1) \phi_{\mu\nu'}(p^1) + p_{\alpha}^1 (-p_{\mu'}^1 - p_{0\mu'}) \phi_{\nu\beta}(-p^1 - p_0) \\
& \phi_{\nu\mu'}(p^1) \\
& + p_{0\mu} p_{\mu'}^1 \phi_{\alpha\beta}(-p_0 - p^1) \phi_{\mu\mu'}(p^1) + p_{\alpha}^1 p_{0\beta} \phi_{\nu\nu'}(-p^1 - p_0) \phi_{\nu\nu'}(p^1) \\
& + p_{\alpha}^1 p_{0\mu'} [\phi_{\nu\beta}(p^1) \phi_{\nu\mu'}(-p^1 - p_0) - \phi_{\nu\mu'}(p^1) \phi_{\nu\beta}(-p^1 - p_0)] \\
& + p_{0\mu} p_{\mu'}^1 \phi_{\mu\beta}(p^1) \phi_{\alpha\mu'}(-p_0 - p^1) - p_{\mu}^1 p_{0\beta} \phi_{\mu\nu'}(-p^1 - p_0) \phi_{\alpha\nu'}(p^1) \\
& - 2p_{0\mu} p_{\beta}^1 \phi_{\mu\nu'}(p^1) \phi_{\alpha\nu'}(-p_0 - p^1) + p_{\mu}^1 p_{\mu'}^1 \phi_{\mu\beta}(-p_0 - p^1) \phi_{\alpha\mu'}(p^1).
\end{aligned}$$

Giving an expression for $\frac{\partial \Delta_{\text{FT}}}{\partial \phi_{\alpha\beta}(p_0)}$ of

$$\begin{aligned} & \frac{1}{2} \frac{g^2}{4} V \int \frac{d^4 p^1}{(2\pi)^4} \\ & (2p_{0\mu} p_{0\mu'} + p_{0\mu} p_{\mu'}^1) \phi_{\mu\mu'}(p^1) \phi_{\alpha\beta}(-p^1 - p_0) \\ & + (p_{\mu}^1 p_{\mu'}^1 - p_{\mu}^1 p_{0\mu'}) \phi_{\alpha\beta}(p^1) \phi_{\mu\mu'}(-p^1 - p_0) \\ & + (2p_{\alpha}^1 p_{\beta}^1 + p_{\alpha}^1 p_{0\beta}) \phi_{\nu\nu'}(p^1) \phi_{\nu\nu'}(-p^1 - p_0) \\ & - (2p_{\alpha}^1 p_{0\mu'} + p_{\alpha}^1 p_{\mu'}^1) \phi_{\nu\beta}(-p^1 - p_0) \phi_{\nu\mu'}(p^1) \\ & - (p_{\alpha}^1 p_{\mu'}^1 - p_{\alpha}^1 p_{0\mu'}) \phi_{\nu\beta}(p^1) \phi_{\nu\mu'}(-p^1 - p_0) \\ & + (2p_{\mu}^1 p_{0\mu'} + p_{\mu}^1 p_{\mu'}^1) \phi_{\alpha\mu'}(p^1) \phi_{\mu\beta}(-p^1 - p_0) \\ & + (p_{0\mu} p_{\mu'}^1 - p_{0\mu} p_{0\mu'}) \phi_{\mu\beta}(p^1) \phi_{\alpha\mu'}(-p^1 - p_0) \\ & - (2p_{0\mu} p_{\beta}^1 + p_{0\mu} p_{0\beta}) \phi_{\mu\nu'}(p^1) \phi_{\alpha\nu'}(-p_0 - p^1) \\ & - (2p_{\mu}^1 p_{\beta}^1 + p_{\mu}^1 p_{0\beta}) \phi_{\alpha\nu'}(p^1) \phi_{\mu\nu'}(-p_0 - p^1). \end{aligned}$$

Performing a saddle point analysis then results in

$$\begin{aligned} 0 &= \frac{\partial \mathcal{S}_{\text{eff}}^{\pm}}{\partial \phi_{\alpha\beta}(p_0)} \\ &= \frac{1}{4} V [p_0^2 \delta_{\alpha\nu} \delta_{\beta\nu} - p_{0\mu} p_{0\nu} \delta_{\alpha\nu} \delta_{\beta\mu}] + \frac{g^2}{8} V \int \frac{d^4 p^1}{(2\pi)^4} \\ & \quad [\delta_{\alpha\mu} \delta_{\beta\mu} \phi_{\nu\nu'}(p^1) + \phi_{\mu\mu}(p^1) \delta_{\alpha\nu} \delta_{\beta\nu} - \delta_{\alpha\mu} \delta_{\beta\nu} \phi_{\nu\mu}(p^1) - \phi_{\mu\nu}(p^1) \delta_{\alpha\nu} \delta_{\beta\mu}] \\ & \quad - \frac{N}{2} V [\phi(p_0)]^{-1} \pm \frac{\partial \sqrt{\Delta_{\text{FT}}}}{\partial \phi_{\alpha\beta}(p_0)} \\ &= \frac{1}{4} V [p_0^2 \delta_{\alpha\beta} - p_{0\beta} p_{0\alpha}] + \frac{g^2}{8} V \int \frac{d^4 p^1}{(2\pi)^4} [\delta_{\alpha\beta} \phi_{\mu\mu}(p^1) + \phi_{\mu\mu}(p^1) \delta_{\alpha\beta} \\ & \quad - \phi_{\beta\alpha}(p^1) - \phi_{\beta\alpha}(p^1)] - \frac{N}{2} V [\phi(p_0)]^{-1} \pm \frac{\partial \sqrt{\Delta_{\text{FT}}}}{\partial \phi_{\alpha\beta}(p_0)}, \end{aligned}$$

and scaling the field $\phi \rightarrow N\phi$ ($\Delta_{\text{FT}} \rightarrow N^2 \Delta_{\text{FT}}$ keeping $\lambda = g^2 N$ finite) in this analysis gives

$$\begin{aligned} 0 &= \frac{1}{4} V [p_0^2 \delta_{\alpha\beta} - p_{0\beta} p_{0\alpha}] + \frac{\lambda}{8} V \int \frac{d^4 p^1}{(2\pi)^4} [2\delta_{\alpha\beta} \phi_{\mu\mu}(p^1) - 2\phi_{\beta\alpha}(p^1)] \\ & \quad - \frac{V}{2} [\phi(p_0)]^{-1} \pm \frac{1}{2\sqrt{\Delta_{\text{FT}}}} \frac{\partial \Delta_{\text{FT}}}{\partial \phi_{\alpha\beta}(p_0)}. \end{aligned}$$

The factors of N therefore drop out of this equation as before. Factors of V , however, do not drop out, as can be seen in the resulting expression for the propagator:

$$\phi^0(p_0) = \frac{2}{p_0^2 \delta_{\alpha\beta} - p_{0\beta} p_{0\alpha} + \lambda \int \frac{d^4 p^1}{(2\pi)^4} [\delta_{\alpha\beta} \phi_{\mu\mu}(p^1) - \phi_{\beta\alpha}(p^1)] \pm \frac{4}{\sqrt{\Delta_{\text{FT}}}} \left(\frac{1}{V} \frac{\partial \Delta_{\text{FT}}}{\partial \phi_{\alpha\beta}(p_0)} \right)}.$$

Even for finite V , the gap equation is unlikely to yield a closed solution as there is no longer an expectation that $\phi_{\alpha\beta} \propto \delta_{\alpha\beta}$. This can be remedied by introducing the gauge condition $\partial_\mu A^\mu = 0$, which introduces ghosts, and they would have to be generalised. It should be noted that the Jacobian of the transformation to fermionic bilocals is known [18]. This is beyond the scope of this dissertation.

Chapter 5

Conclusion

Non-Abelian gauge theories are of interest in Physics as they describe particle interactions on the one hand and the low energy dynamics of the world volume of d-branes on the other. $SU(2)$ gauge theories are therefore of interest and were the subject of this dissertation. $SU(2)$ gauge theories were written in terms of vector models and when these comprised solely $O(3)$ invariants, a change of variables was possible using a Jacobian determined via the collective field theory. A large- N saddle point analysis ensued, giving rise to an expression for the propagator. The following results were obtained:

- When $SU(2)$ gauge theories were dimensionally reduced on T^4 , a mass term and a large N Jacobian term were incorporated into a purely quartic effective action. The saddle point analysis performed on this effective action yielded a quadratic equation and an expression for the propagator obtained in the strongly coupled (and large N) limit was obtained. It is free of infrared divergences.
- When $SU(2)$ gauge theories were dimensionally reduced on $\mathbb{R} \times T^3$, an Hamiltonian formalism was applied, in which it was argued that the physical fields of the $SU(2)$ system are those given by $O(3)$ invariants. The Hamiltonian was derived using the collective field theory, and the effective potential thereof once again reduced to the quartic interaction as a result of the specified dimensional reduction and canonical quantisation; no mass term was added. The large N limit gave rise to a solution without any need for an infrared regulator.

- In the case where $SU(2)$ gauge theories were not dimensionally reduced at all and the full field theory was considered, collective field theory was used to obtain a new space-time dependent expression for the Jacobian. The cubic contribution to the effective action was treated with care and the Fourier transform was taken. Upon performing the large N saddle point analysis, an expression for the propagator was obtained; it was, however, not independent of volume, necessitating that a finite volume condition be imposed. It was argued that the complexity of the gap equation could be simplified using a gauge condition, which introduces ghosts (and for which the Jacobian of transformation is known), but which lies outside the scope of this dissertation. This may form the premise for future work.

Appendix A

Derivations and additions

A.1 A derivation of Einstein's equation from the Einstein-Hilbert action

Making use of the variational principle, Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu} \quad (\text{A.1})$$

will here be derived from the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{-\det|g|} (R - 2\Lambda) + \int d^4x \sqrt{-\det|g|} \mathcal{L}_M, \quad (\text{A.2})$$

where $g \equiv \det|g| \equiv \det_{a,b}(g_{ab}(x^k))$ (using Roman numerals a, b etc. in intermediate calculations, to differentiate them from the μ and ν indices which are expected to appear in the final expression); ref. [19] provides useful insight in solving this problem. The following results / identities are important before progressing further:

- (a) A determinant may be written in terms of adjoints, expanding about any row (or column). Expanding about the a^{th} row of the matrix g_{ab} results in

$$\begin{aligned} g_{(a)} &= \sum_b g_{ab} (\text{adj}g)^{ab} \\ \Rightarrow \sum_a g^{ca} g &= \sum_a g^{ca} \sum_b g_{ab} (\text{adj}g)^{ab} = (\text{adj}g)^{ac}. \end{aligned}$$

Hence

$$\begin{aligned}
 \partial_c g &= \sum_{a,b} \frac{\partial g}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial x^c} \\
 &= \sum_{a,b} \frac{\partial \sum_f g_{df} (\text{adj}g)^{df}}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial x^c} \\
 &= \sum_f (\text{adj}g)^{df} \frac{\partial g_{df}}{\partial x^c} \\
 &= \sum_f \sum_d g^{fd} g \frac{\partial g_{df}}{\partial x^c} \\
 \Rightarrow \quad \delta g &= g g^{ab} \delta g_{ba}
 \end{aligned}$$

with implicit summation over a and b . Therefore

$$\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} (-\delta g) = \frac{1}{2} \sqrt{-g} g^{ab} \delta g_{ba},$$

and finally

$$\delta \left(\sqrt{-g} g^{cd} \right) = \frac{1}{2} \sqrt{-g} g^{ab} \delta g_{ba} g^{cd} + \sqrt{-g} \delta g^{cd}$$

(b) A small variation in the Ricci tensor is given by

$$\begin{aligned}
 \delta R_{ab} &= \delta R_{acb}^c \\
 &= \delta \left(\partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{cd}^c \Gamma_{ab}^d - \Gamma_{bd}^c \Gamma_{ac}^d \right) \\
 &= \partial_c \delta \Gamma_{ab}^c - \partial_b \delta \Gamma_{ac}^c + \delta \Gamma_{cd}^c \Gamma_{ab}^d + \Gamma_{cd}^c \delta \Gamma_{ab}^d - \delta \Gamma_{bd}^c \Gamma_{ac}^d - \Gamma_{bd}^c \delta \Gamma_{ac}^d \\
 &= \left(\partial_c (\delta \Gamma_{ab}^c) + \Gamma_{cd}^c (\delta \Gamma_{ab}^d) - \Gamma_{ac}^d (\delta \Gamma_{db}^c) - \Gamma_{bc}^d (\delta \Gamma_{da}^c) \right) \\
 &\quad - \left(\partial_b (\delta \Gamma_{ac}^c) + \Gamma_{bd}^c (\delta \Gamma_{ac}^d) - \Gamma_{ab}^d (\delta \Gamma_{dc}^c) - \Gamma_{bc}^d (\delta \Gamma_{da}^c) \right) \\
 &= \nabla_c (\delta \Gamma_{ab}^c) - \nabla_b (\delta \Gamma_{ac}^c).
 \end{aligned}$$

This is Palatini's theorem, expressing the Ricci tensor as an absolute derivative.

(c) Since the environment is assumed to be torsion-free, the Christoffel symbols are symmetric in their lower indices and the following identity is

obtained:

$$\begin{aligned}
 \nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma_{ac}^d g_{db} - \Gamma_{bc}^d g_{ad} \\
 &= \partial_c g_{ab} - \frac{1}{2} g^{de} (\partial_a g_{ce} + \partial_c g_{ae} - \partial_e g_{ac}) g_{db} \\
 &\quad - \frac{1}{2} g^{de} (\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}) g_{ad} \\
 &= \partial_c g_{ab} - \frac{1}{2} \delta_b^e (\partial_a g_{ce} + \partial_c g_{ae} - \partial_e g_{ac}) \\
 &\quad - \frac{1}{2} \delta_a^e (\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}) \\
 &= \partial_c g_{ab} - \frac{1}{2} (\partial_a g_{cb} + \partial_c g_{ab} - \partial_b g_{ac}) \\
 &\quad - \frac{1}{2} (\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}) \\
 &= 0.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \partial_c g_{ab} &= \Gamma_{ac}^d g_{db} + \Gamma_{bc}^d g_{ad} \\
 \Rightarrow \partial_c g &= g g^{ab} \partial_c g_{ab} \\
 &= g g^{ab} (\Gamma_{ac}^d g_{db} + \Gamma_{bc}^d g_{ad}) \\
 &= g \delta_a^a \Gamma_{ac}^d + g \delta_b^b \Gamma_{bc}^d \\
 &= 2g \Gamma_{bc}^b,
 \end{aligned}$$

and since g is a scalar of weight $+2$,

$$\nabla_c g = \partial_c g - W g \Gamma_{bc}^b = \partial_c g - 2g \Gamma_{bc}^b = 0$$

(a proof of this may be found in arXiv:physics/9802027), resulting in

$$\nabla_c (\sqrt{-g} g^{\mu\nu}) = 0.$$

Solving the original problem now, the variation in the first part of equation (A.2) may be written as

$$\delta \int d^4x \sqrt{-g} R = \delta \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \int d^4x \delta (\sqrt{-g} g^{\mu\nu}) R_{\mu\nu} + \int d^4x \sqrt{-g} g^{\mu\nu} \delta (R_{\mu\nu}).$$

By (b) and (c), the second part of this expression is zero,

$$\begin{aligned}
 \int d^4x \sqrt{-g} g^{\mu\nu} \delta (R_{\mu\nu}) &= \int d^4x \sqrt{-g} g^{\mu\nu} (\nabla_c (\delta \Gamma_{ab}^c) - \nabla_b (\delta \Gamma_{ac}^c)) \\
 &= \int d^4x \nabla_b (\sqrt{-g} g^{\mu\nu} \delta (\Gamma_{ac}^c) - \nabla_c (\sqrt{-g} g^{\mu\nu} \delta \Gamma_{ab}^c)) \\
 &= 0.
 \end{aligned}$$

Since

$$\delta(\sqrt{-g}g^{\mu\nu}) = \sqrt{-g}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ba}g^{\mu\nu},$$

one has that

$$\begin{aligned} \int d^4x \delta(\sqrt{-g}g^{\mu\nu}) R_{\mu\nu} &= \int d^4x \delta(\sqrt{-g}g^{\mu\nu}) R_{\mu\nu} \\ &= \int d^4x \left(\sqrt{-g}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ba}g^{\mu\nu} \right) R_{\mu\nu} \\ &= \int d^4x \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) \sqrt{-g}\delta g^{\mu\nu} \end{aligned}$$

Hence the invariance of

$$\mathcal{S}_{\text{EH}} = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{-g} (R - 2\Lambda) + \int d^4x \sqrt{-g} \mathcal{L}_M,$$

leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}.$$

A.2 Derivation of the non-Abelian field strength tensor

For an Abelian field, the gauge field and field strength tensor respectively may be written as

$$\begin{aligned}
 A'_\mu &= A_\mu + \frac{i}{g} (\partial_\mu u) u^* \\
 &= A_\mu + \frac{1}{g} \partial_\mu \alpha \\
 F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
 &= F'_{\mu\nu},
 \end{aligned}$$

where $u = e^{-i\alpha(x)} \in U(1)$. In the non-Abelian case, however (since we now extend the principle to matrices, and so the transformation matrix becomes $U = e^{-i\omega(x)} \in U(N)$), the gauge field becomes

$$\begin{aligned}
 A'_\mu &= U A_\mu U^\dagger + \frac{i}{g} (\partial_\mu U) U^\dagger \\
 &= A_\mu - i [\omega, A_\mu] + \frac{1}{g} \partial_\mu \omega.
 \end{aligned}$$

In this case the field strength tensor is not quite as straight forward as in the Abelian case where the differential term cancelled due to the commutativity of differential operators. In order to shed some light on what the field strength tensor ought to look like in this non-commutative case, consider

$$\begin{aligned}
 F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + f(A_\mu, A_\nu) \\
 &\rightarrow \partial_\mu A'_\nu - \partial_\nu A'_\mu + f(A'_\mu, A'_\nu) \\
 &= U [\partial_\mu A_\nu - \partial_\nu A_\mu + f(A_\mu, A_\nu)] U^\dagger \\
 &\equiv U F_{\mu\nu}^{abelian} U^\dagger + U f(A_\mu, A_\nu) U^\dagger. \tag{A.3}
 \end{aligned}$$

Using the identity,

$$\begin{aligned}
 U.U^\dagger = 1 &\Rightarrow \partial_\mu (U.U^\dagger) = (\partial_\mu U) U^\dagger + U \partial_\mu U^\dagger = 0 \\
 &\Rightarrow \partial_\mu U^\dagger = -U^\dagger (\partial_\mu U) U^\dagger,
 \end{aligned}$$

consider, as a starting point then,

$$\begin{aligned}
 F_{\mu\nu}^{\prime\text{abelian}} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
 &= \partial_\mu \left(U A_\nu U^\dagger + \frac{i}{g} (\partial_\nu U) U^\dagger \right) - \partial_\nu \left(U A_\mu U^\dagger + \frac{i}{g} (\partial_\mu U) U^\dagger \right) \\
 &= \left[\partial_\mu (U A_\nu U^\dagger) - \partial_\nu (U A_\mu U^\dagger) \right] + \left[\partial_\mu \left(\frac{i}{g} (\partial_\nu U) U^\dagger \right) \right. \\
 &\quad \left. - \partial_\nu \left(\frac{i}{g} (\partial_\mu U) U^\dagger \right) \right].
 \end{aligned}$$

The second bracket becomes

$$\begin{aligned}
 &\frac{i}{g} (\partial_\mu \partial_\nu U) U^\dagger + \frac{i}{g} (\partial_\nu U) (\partial_\mu U^\dagger) - \frac{i}{g} (\partial_\nu \partial_\mu U) U^\dagger - \frac{i}{g} (\partial_\mu U) (\partial_\nu U^\dagger) \\
 &= \frac{i}{g} (\partial_\nu U) (\partial_\mu U^\dagger) - \frac{i}{g} (\partial_\mu U) (\partial_\nu U^\dagger) \\
 &= -ig \left[\left(\frac{i}{g} (\partial_\mu U) U^\dagger \right) \left(\frac{i}{g} (\partial_\nu U) U^\dagger \right) - \left(\frac{i}{g} (\partial_\nu U) U^\dagger \right) \left(\frac{i}{g} (\partial_\mu U) U^\dagger \right) \right] \\
 &= -ig \left[\frac{i}{g} (\partial_\mu U) U^\dagger, \frac{i}{g} (\partial_\nu U) U^\dagger \right],
 \end{aligned}$$

whilst the first bracket becomes

$$\begin{aligned}
 &((\partial_\mu U) U^\dagger) (U A_\nu U^\dagger) - ((\partial_\nu U) U^\dagger) (U A_\mu U^\dagger) \\
 &+ U (\partial_\mu A_\nu - \partial_\nu A_\mu) U^\dagger \\
 &- \left[(U A_\nu U^\dagger) ((\partial_\mu U) U^\dagger) - (U A_\mu U^\dagger) ((\partial_\nu U) U^\dagger) \right] \\
 &= U F_{\mu\nu}^{\text{Abelian}} U^\dagger - ig \left[\frac{i}{g} (\partial_\mu U) U^\dagger, U A_\nu U^\dagger \right] - ig \left[U A_\mu U^\dagger, (\partial_\nu U) U^\dagger \right]
 \end{aligned}$$

and so

$$\begin{aligned}
 F_{\mu\nu}^{\prime\text{Abelian}} &= U F_{\mu\nu}^{\text{Abelian}} U^\dagger - \\
 &\quad ig \left(\left[U A_\mu U^\dagger, \frac{i}{g} (\partial_\nu U) U^\dagger \right] + \left[\frac{i}{g} (\partial_\mu U) U^\dagger, \frac{i}{g} (\partial_\nu U) U^\dagger \right] \right. \\
 &\quad \left. + \left[\frac{i}{g} (\partial_\mu U) U^\dagger, U A_\nu U^\dagger \right] \right) \\
 &= U F_{\mu\nu}^{\text{Abelian}} U^\dagger - \\
 &\quad ig \left[U A_\mu U^\dagger + \frac{i}{g} (\partial_\mu U) U^\dagger, U A_\nu U^\dagger + \frac{i}{g} (\partial_\nu U) U^\dagger \right] + ig U [A_\mu, A_\nu] U^\dagger \\
 &= U F_{\mu\nu}^{\text{Abelian}} U^\dagger - ig [A'_\mu, A'_\nu] + ig U [A_\mu, A_\nu] U^\dagger
 \end{aligned}$$

Hence if one defines the non-Abelian field strength tensor to be

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu],$$

then

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu + ig [A'_\mu, A'_\nu] \\ &= U (\partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]) U^\dagger \\ &= U F_{\mu\nu} U^\dagger, \end{aligned}$$

which (as desired) would result in an invariant Lagrangian.

An illustration from representation theory

This invariance result goes hand-in-hand with a key result from representation theory which states that for a representation ρ where

$$\rho : G \rightarrow \text{GL}(\mathbb{V}),$$

the *character* $\chi_V : G \rightarrow \mathbb{C}$ of the representation, defined by

$$\chi_V(g) = \text{Tr}_V(\rho(g)),$$

is a complete invariant. In particular, χ_V is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g), \forall g, h \in G.$$

So one sees, highlighted in representation theory, a group element $g = F_{\mu\nu}$ and representation $\rho(g) = F_{\mu\nu} F^{\mu\nu}$, giving an invariant trace under conjugation of g by group element $h = U$, to obtain

$$\begin{aligned} \chi_V(g) = \text{Tr}(F_{\mu\nu} F^{\mu\nu}) &= \text{Tr}[U F_{\mu\nu} F^{\mu\nu} U^{-1}] \\ &= \text{Tr}[(U F_{\mu\nu} U^{-1})(U F^{\mu\nu} U^{-1})] \\ &= \text{Tr}(F'_{\mu\nu} F'^{\mu\nu}) \\ &= \chi_V(g'), \end{aligned}$$

implying an invariant Lagrangian

$$\mathcal{L} = -\frac{1}{4} \text{Tr}[F_{\mu\nu} F^{\mu\nu}].$$

A.3 A derivation of Euler's formula

Euler's formula may be proven in two steps [12]:

1. Distorting a polyhedron in any way leaves the Euler characteristic invariant.
2. Introducing a hole reduces the Euler characteristic by 2.

To prove the first point, note that a cube, for example, has Euler characteristic of $\chi = F - P + V = 6 - 12 + 8 = 2$; similarly for tetrahedrons, octahedrons, pyramids etc (polyhedra with no holes). Now consider any polyhedron: there are three fundamental ways to distort the polygonal surface:

- Any deformation (e.g. rescaling of edges) to the surface that leaves P , V and F unchanged will leave the Euler characteristic invariant.
- Shrinking an edge to a point simultaneously reduces P and V by 1, thus leaving χ unchanged and
- Shrinking an N -sided face to a point reduces F by 1, P by N and V by $N-1$, once again leaving χ unchanged.

Any deformation to the surface is given by some combination of these fundamental deformations.

To prove the second point, consider that introducing a hole requires matching two non-adjacent N -sided faces with each other and eliminating them, resulting in F decreasing by 2, P by N and V by N ; hence χ reduces by two for each hole introduced.

A.4 $su(N')$ algebra identities

The generators T^a , $a = 1 \dots N'^2 - 1$, in the fundamental representation of the $su(N')$ algebra possess certain properties regardless of their dimension.

They satisfy the following relations (adapted from [20]):

$$[T^a, T^b] = 2if^{abc}T^c \quad (\text{A.4})$$

$$\{T^a, T^b\} = \frac{4}{N'}\delta_{ab} + 2d^{abc}T^c \quad (\text{A.5})$$

$$\Rightarrow T^a T^b = \frac{2}{N'}\delta_{ab} + d^{abc}T^c + if^{abc}T^c. \quad (\text{A.6})$$

From these equations, the following normalisation is obtained:

$$\text{Tr}(T^a T^b) = 2\delta_{ab}. \quad (\text{A.7})$$

In the above, f^{abc} is antisymmetric in its indices and d^{abc} is totally symmetric in its indices and is given by:

$$d^{abc} = \frac{1}{4}\text{Tr}[\{T^a, T^b\}T^c].$$

Furthermore,

$$\begin{aligned} T^a T^b T^c &= \frac{2}{N'}\delta_{ab}T^c + (d^{abd} + if^{abd})T^d T^c \\ \Rightarrow \text{Tr}(T^a T^b T^c) &= 2(d^{abc} + if^{abc}), \end{aligned}$$

and

$$\begin{aligned} \text{Tr}(T^a T^b T^c T^d) &= \text{Tr}\left(\frac{2}{N'}\delta_{ab}\frac{2}{N'}\delta_{cd} + (d^{abf} + if^{abf})\left(d^{fcd} \frac{2}{N'}\delta_{de} + if^{fcd} \frac{2}{N'}\delta_{de}\right)\right) \\ &= \frac{4}{N'}\delta_{ab}\delta_{cd} + 2(d^{abf} + if^{abf})(d^{fcd} + if^{fcd})\delta_{de} \\ &= \frac{4}{N'}\delta_{ab}\delta_{cd} + 2(d^{abe} + if^{abe})(d^{ecd} + if^{ecd}). \end{aligned}$$

A.5 Path integral calculations

Path integrals are understood as follows. A propagator is the amplitude of finding a state at x_1 which originally had an eigenstate at coordinate x_2 [21]. This is equivalent to summing over all paths between x_1 and x_2 ,

$$G(x_1, x_2) = \langle 0 | T(a(x_1) a(x_2)) | 0 \rangle = \int [da] a(x_1) a(x_2) e^{i \int d^4x \mathcal{L}(a)},$$

where $\int d^4x \mathcal{L}$ is called the action and denoted by \mathcal{S} . By extension, the time-ordered expectation value of an operator \mathcal{O} (in units of $\hbar = 1$) is given by

$$\langle 0 | T(\mathcal{O}) | 0 \rangle = \int [da] \mathcal{O}(a) e^{i\mathcal{S}(a)}.$$

The intricacies involved in performing such calculations may be found in appendices A.5.1, A.5.2 and A.5.3.

A.5.1 Calculating propagators

Consider the moment generating function where the matrix J has the same order as that of M ,

$$\begin{aligned} Z[J] &= \mathcal{K} \int [dM] e^{-\alpha \text{Tr}(M^2) + \beta \text{Tr}(JM)} & (\text{A.8}) \\ &= \mathcal{K} \int [dM] e^{-\alpha(M_{ab}M_{ba}) + \beta(J_{ab}M_{ba})} \\ \Rightarrow \frac{d}{dJ_{kl}} [Z[J]]_{J=0} &= \mathcal{K} \int [dM] \beta M_{lk} e^{-\alpha(M_{ab}M_{ba})} \\ &= \beta \langle M_{lk} \rangle \\ \text{and } \frac{d}{dJ_{k_1 l_1} dJ_{k_2 l_2}} [Z[J]]_{J=0} &= \mathcal{K} \int [dM] \beta^2 M_{l_1 k_1} M_{l_2 k_2} e^{-\alpha(M_{ab}M_{ba})} \\ &= \beta^2 \langle M_{l_1 k_1} M_{l_2 k_2} \rangle, \end{aligned}$$

where $\mathcal{K} = \frac{1}{\int [dM] e^{-\alpha \text{Tr}(M^2)}}$ is the normalising factor. Once again, the moments of the distribution fall out as derivatives of the generating function with J set to zero.

Completing the square in equation (A.8) gives

$$\begin{aligned}
Z[J] &= \mathcal{K} \int [dM] e^{-\alpha \text{Tr}(M^2) + \beta \text{Tr}(JM)} \\
&= \mathcal{K} \left[e^{\text{Tr}\left(\alpha\left(\frac{\beta}{2\alpha}J\right)^2\right)} \int [dM] e^{-\alpha \text{Tr}\left(M - \frac{\beta}{2\alpha}J\right)^2} \right] \\
&= e^{\text{Tr}\left(\alpha\left(\frac{\beta}{2\alpha}J\right)^2\right)} \\
&= e^{\frac{\beta^2}{4\alpha} J_{ab} J_{ba}}.
\end{aligned} \tag{A.9}$$

Hence, the moments of the distribution are given by

$$\begin{aligned}
\langle M_{lk} \rangle &= \frac{1}{\beta} \frac{d}{dJ_{kl}} [Z[J]]_{J=0} \\
&= \frac{1}{\beta} \frac{2\beta^2}{4\alpha} \left[J_{lk} e^{\frac{\beta^2}{4\alpha} J_{ab} J_{ba}} \right]_{J=0} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\langle M_{l_1 k_1} M_{l_2 k_2} \rangle &= \frac{1}{\beta^2} \frac{d^2}{dJ_{k_1 l_1} dJ_{k_2 l_2}} [Z[J]]_{J=0} \\
&= \frac{1}{\beta^2} \frac{2\beta^2}{4\alpha} \frac{d}{dJ_{k_2 l_2}} \left[J_{l_1 k_1} e^{\frac{\beta^2}{4\alpha} J_{ab} J_{ba}} \right]_{J=0} \\
&= \frac{1}{\beta^2} \left[\left(\frac{2\beta^2}{4\alpha} \delta_{l_1 k_2} \delta_{l_2 k_1} + \left(\frac{2\beta^2}{4\alpha} \right)^2 J_{l_1 k_1} J_{l_2 k_2} \right) e^{\frac{\beta^2}{4\alpha} J_{ab} J_{ba}} \right]_{J=0} \\
&= \frac{1}{2\alpha} \delta_{l_1 k_2} \delta_{l_2 k_1}.
\end{aligned} \tag{A.10}$$

Furthermore, a Gaussian distribution

$$\int_{-\infty}^{\infty} [dM] e^{-\frac{1}{2}(M_{l_1 k_1})(\hat{O}_{l_1 k_1 l_2 k_2})(M_{l_2 k_2})} \tag{A.11}$$

(with indices in no particular order) reduces to

$$\int_{-\infty}^{\infty} [dM] e^{-\frac{1}{2}\alpha \text{Tr} M^2} \tag{A.12}$$

under a unitary transformation; thus the operator has the form

$$\hat{O}_{l_1 k_1 l_2 k_2} = \alpha \delta_{k_1 l_2} \delta_{k_2 l_1}. \tag{A.13}$$

Then the 2-point correlator $\langle M_{l_1 k_1} M_{l_2 k_2} \rangle$ is then inverse of this operator

$$\langle M_{l_1 k_1} M_{l_2 k_2} \rangle = \hat{O}_{l_1 k_1 l_2 k_2}^{-1} = \frac{1}{\alpha} \delta_{k_1 l_2} \delta_{k_2 l_1} \tag{A.14}$$

which is an important and general result for Gaussian distributions. 2n-point correlators may similarly be obtained, while odd correlators are identically zero.

A.5.2 Green's functions

Consider the following generating function

$$\mathcal{Z}(J) = \int \mathcal{D}\phi e^{\int_{-\infty}^{\infty} d^4x' i \left(\frac{1}{2} \partial_\mu \phi^b(x') \partial^\mu \phi^b(x') - \frac{1}{2} m_0^2 \phi^b(x') \phi^b(x') \right) + J^b(x') \phi^b(x')}.$$

The term in the exponent is maximised when its derivative with respect to $\phi^a(x)$ is zero, giving an expression comparable to finding the position of the peak of a Gaussian distribution:

$$0 = \int_{-\infty}^{\infty} d^4x' i \left[(\partial_\mu \delta^4(x-x') \partial^\mu \phi^a(x')) - m^2 \delta^4(x-x') \phi^a(x') \right] + J^a(x') \delta(x-x').$$

Setting surface terms to zero, this results in

$$(\partial_\mu \partial^\mu + m^2) \phi_0^a(x) = -i J^a(x). \quad (\text{A.15})$$

The generating function may then be written in terms of a new variable ϕ'^a by setting $\phi^a = \phi_0^a + \phi'^a$; after normalising using $\mathcal{N} = \int \mathcal{D}\phi e^{i\mathcal{S}(\phi)}$, this simplifies to

$$\mathcal{Z}[J] = \mathcal{N} e^{\int d^4x \frac{\phi_0^b J^b}{2}}$$

where terms linear in ϕ' have been eliminated and it remains to find a useful expression for ϕ_0 from (A.15). Consider

$$(\partial_\mu \partial^\mu + m^2) G^{ab}(x-y) = \delta(x-y) \delta^{ab} \quad (\text{A.16})$$

$$\Rightarrow \phi_0^a(x) = -i \int d^4y G^{ab}(x-y) J^b(y)$$

$$\Rightarrow \mathcal{Z}[J] = \mathcal{N} e^{\frac{-i}{2} \int d^4x J'^a(x) G^{ab}(x-y) J'^b(y)}, \quad (\text{A.17})$$

where $G^{ab}(x-y)$ are Green's functions.

Fourier transforming $G^{ab}(x-y)$ and $\delta(x-y)$ and substituting into equation (A.16) results in

$$\begin{aligned} & \int \frac{d^4p}{(2\pi)^4} (\partial_\mu \partial^\mu + m^2) e^{ip \cdot (x-y)} G^{ab}(p) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} \delta^{ab} \\ \Rightarrow & G^{ab}(p) = \frac{-\delta^{ab}}{p_\mu p^\mu - m^2} \\ \Rightarrow & G^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-\delta^{ab} e^{ip \cdot (x-y)}}{p_\mu p^\mu - m^2}. \end{aligned}$$

Propogators (time-ordered) may then be obtained by taking successive derivatives of the generating function in equation (A.16) with respect $J'^{a_i}(x_i)$. For example,

$$\langle \phi^{a_1}(x_1) \phi^{a_2}(x_2) \rangle = \frac{\delta^2 \mathcal{Z}}{\delta J^{a_1}(x_1) \delta J^{a_2}(x_2)} \Big|_{J=0} = -i G^{a_1 a_2}(x_1 - x_2)$$

and

$$\langle \phi^{a_1}(x_1) \phi^{a_2}(x_2) \phi^{a_3}(x_3) \phi^{a_4}(x_4) \rangle = -G_{x_1, x_2}^{a_1 a_2} G_{x_3, x_4}^{a_3 a_4} - G_{x_1, x_3}^{a_1 a_3} G_{x_2, x_4}^{a_2 a_4} - G_{x_1, x_4}^{a_1 a_4} G_{x_2, x_3}^{a_2 a_3},$$

while odd correlators are identically zero; these results are a direct application of Wick's theorem (see Appendix A.5.3).

A.5.3 Wick's theorem

Wick's theorem gives an algorithm for calculating any order moment for Gaussian distributions. One see that the expectation of odd combinations of x_i will be zero (by symmetry and since odd expressions will always have a coefficient of j which will be set to zero when calculating the expectation). For even correlators, Wick's theorem gives the intuitive result that even correlators may be written as the sum over the possible groupings of 2-point correlators for which there are $(2k-1)!! = (m-1)!!$ possibilities which can be seen as follows: For $m = 2k$ even the number of pairings of $[x_{i_1}, x_{i_2}, \dots, x_{i_m}]$ is

$$\frac{{}^{2k}C_2 \cdot {}^{2k-2}C_2 \dots {}^2C_2}{k!}, \quad (\text{A.18})$$

since the number of ways one can choose the first pair is ${}^{2k}C_2 = \frac{2k(2k-1)}{2 \cdot 1}$, the number of ways one can choose the second pair is ${}^{2k-2}C_2 = \frac{(2k-2)(2k-3)}{2 \cdot 1}$ and so forth for all k

pairs, and the number of ways of ordering the k pairs is $k!$ which needs to be divided out. Expression (A.18) is a telescoping series simplifying to

$$\frac{(2k)!}{2^k k!}.$$

Even terms in the numerator cancel with factors of 2 in the denominator leaving only odd terms, resulting in

$$(2k-1)(2k-3)\dots 1 = (2k-1)!! = (m-1)!! \quad (\text{A.19})$$

possible pairing combinations.

A.6 Zeroth order perturbative calculation for comparison purposes

Consider the action used in equation (A.20) before the change of variables (so ignoring the $\ln J$ term):

$$\mathcal{S}_{eff} = \frac{1}{4}\omega^2 \sum_{i=1}^d \phi_{ii} + \frac{g^2}{8} \sum_{i \neq j} \left[\phi_{ii}\phi_{jj} - (\phi_{ij})^2 \right] - \frac{N}{2} \text{Tr} \ln \phi. \quad (\text{A.20})$$

The path integral is then given by

$$Z = \int [d\phi] e^{-\frac{1}{4}\omega^2 \sum_{i=1}^d \vec{a}_i \cdot \vec{a}_i - \frac{g^2}{8} \sum_{i \neq j} [(\vec{a}_i \cdot \vec{a}_i)(\vec{a}_j \cdot \vec{a}_j) - (\vec{a}_i \cdot \vec{a}_j)(\vec{a}_i \cdot \vec{a}_j)]}.$$

Considering the equality of the following expression

$$e^{-\frac{1}{4}\omega^2 \sum_{i=1}^d \vec{a}_i \cdot \vec{a}_i} = e^{-\frac{1}{2}a_i^\alpha \left[\frac{\omega^2}{2} \delta_{\alpha\beta} \delta_{ij} \right] a_j^\beta}$$

shows that to zeroth order in g^2 the 2-point function is given by the inverse of the operator acting on a_i^α and a_j^β and hence is equivalent to

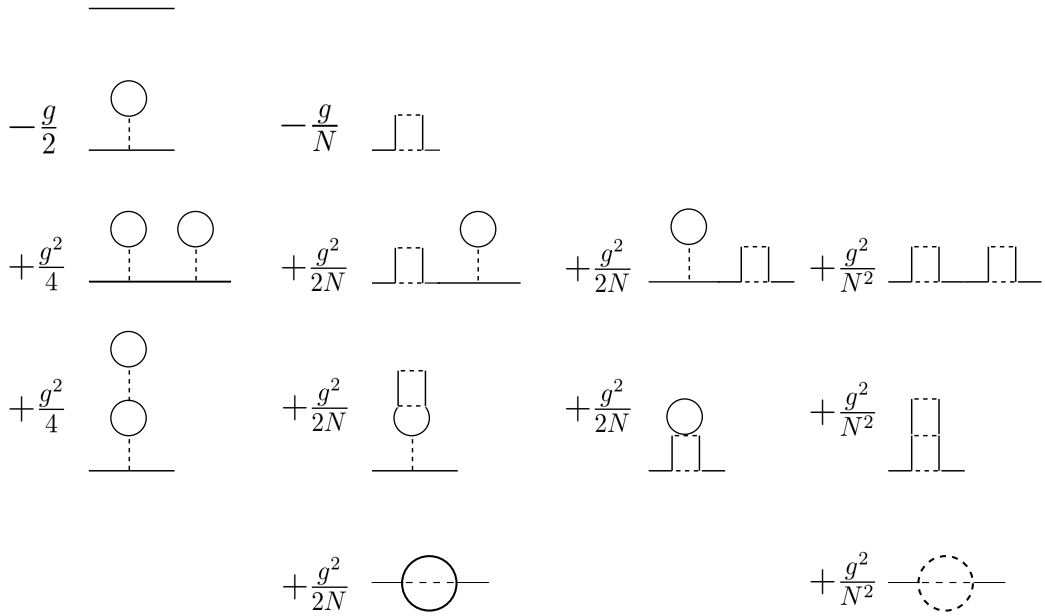
$$\langle a_i^\alpha a_j^\beta \rangle = \frac{2}{\omega^2} \delta_{\alpha\beta} \delta_{ij}. \quad (\text{A.21})$$

A.7 A formalism for bubble diagrams

A potentially more visually appealing way of visualising bubble diagrams which needs no examination of indices is made possible as follows:

$$\begin{aligned}\mathcal{L} &\rightarrow \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a - \frac{1}{8}\frac{g}{N}(\phi^a\phi^a)^2 + \frac{1}{2}\frac{N}{g}\left(\sigma - \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a\right)^2 \\ &= \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a - \frac{1}{8}\frac{g}{N}(\phi^a\phi^a)^2 + \frac{1}{2}\frac{N}{g}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a.\end{aligned}$$

The additional term doesn't change the dynamics of the theory in that it amounts to a change in the normalisation factor, and the added auxiliary field σ just places a constraint on $\frac{g}{N}\phi^a\phi^a$. Furthermore, the only non-trivial interaction term which remains in the Lagrangian density is the $\sigma\phi^a\phi^a$ term. This term allows matching index labels to be replaced by solid lines which represent ϕ lines, and non-matching index labels to be replaced by dashed lines which represent σ lines; where closed solid lines are then indicative of ϕ loops and carry a factor of N . The previously drawn Feynman diagrams then become



. To further simplify the diagrams, considering integrating over the external ϕ lines

$$e^{iS_{\text{eff}}(\sigma)} = \int \Pi_a d\phi^a e^{iS(\phi^a, \sigma)},$$

which has the effect of further reducing the Feynman diagrams above to

$$\begin{aligned}
 & -\frac{g}{2} \text{ (circle with one external line) } - \frac{g}{N} \text{ (square with one external line)} \\
 & + \frac{g^2}{4} \text{ (two circles connected by a line) } + \frac{g^2}{2N} \text{ (square and circle connected by a line) } + \frac{g^2}{2N} \text{ (circle and square connected by a line) } + \frac{g^2}{N^2} \text{ (two squares connected by a line)} \\
 & + \frac{g^2}{4} \text{ (two circles stacked vertically) } + \frac{g^2}{2N} \text{ (square and circle stacked vertically) } + \frac{g^2}{2N} \text{ (circle and square stacked vertically) } + \frac{g^2}{N^2} \text{ (two squares stacked vertically)} \\
 & + \frac{g^2}{2N} \text{ (circle with a horizontal dashed line) } + \frac{g^2}{N^2} \text{ (square with a horizontal dashed line)} \\
 & + \frac{g^2}{N^2} \text{ (circle with a horizontal dashed line and a dashed outer boundary)}
 \end{aligned}$$

. Since all terms in S_{eff} are proportional to N , the problem amounts to counting powers of N using a graph theoretic argument. Internal (ϕ) and external (σ) propagators need to be differentiated from each other and written as P_I and P_E , and the number of loop integrations is given by the number of faces of the graph. This is written as

$$F = P_I - V + 1 \Rightarrow P_I - V = F - 1.$$

The overall power of $\frac{1}{N}$ associated with a given graph is calculated by taking into account that each external/internal propagator contributes a factor of $\frac{1}{N}$ while each vertex contributes a factor N , resulting in

$$\left(\frac{1}{N}\right)^{P_I+P_E-V} = \left(\frac{1}{N}\right)^{F+P_E-1}.$$

Hence diagrams which minimise the power of $\frac{1}{N}$ for a given number of integration loops, F , are those with the smallest number of external lines, P_E .

A.8 A Taylor expansion calculation of a matrix derivative

Define $\phi_{kl} = 1 + A_{kl}$. Then $\delta\phi_{kl} = \delta A_{kl}$ and thus

$$\begin{aligned}
 \ln(1 + A_{ij}) &= A_{ij} - \frac{A_{im}A_{mj}}{2} + \frac{A_{im}A_{mn}A_{nj}}{3} - \dots \\
 \Rightarrow \frac{\partial \ln(1 + A)_{ij}}{\partial A_{kl}} &= \delta_{ik}\delta_{jl} - \frac{1}{2}[\delta_{ik}\delta_{ml}A_{mj} + A_{im}\delta_{mk}\delta_{jl}] \\
 &\quad + \frac{1}{3}[\delta_{ik}\delta_{ml}A_{mn}A_{nj} + A_{im}\delta_{mk}\delta_{nl}A_{nj} + A_{im}A_{mn}\delta_{nk}\delta_{jl}] \\
 &\quad + \dots \\
 \Rightarrow \frac{\partial \ln \text{Tr}(1 + A)}{\partial A_{kl}} &= \frac{\partial \ln(1 + A)_{ii}}{\partial A_{kl}} \\
 &= \delta_{ik}\delta_{il} - \frac{1}{2}[\delta_{ik}\delta_{ml}A_{mi} + A_{im}\delta_{mk}\delta_{il}] \\
 &\quad + \frac{1}{3}[\delta_{ik}\delta_{ml}A_{mn}A_{ni} + A_{im}\delta_{mk}\delta_{nl}A_{ni} + A_{im}A_{mn}\delta_{nk}\delta_{il}] \\
 &\quad + \dots \\
 &= \delta_{kl} - \frac{1}{2}[A_{lk} + A_{lk}] + \frac{1}{3}[A_{ln}A_{nk} + A_{ik}A_{li} + A_{lm}A_{mk}] \\
 &\quad + \dots \\
 &= \delta_{kl} - A_{lk} + A_{ln}A_{nk} + \dots \\
 &= (1 - A + A^2 - \dots)_{lk} \\
 &= \left(\frac{1}{1 + A}\right)_{lk} \\
 \Rightarrow \frac{\partial \ln(\phi)_{ii}}{\partial \phi_{kl}} &= \frac{\partial \ln(\phi)_{ii}}{\partial \phi_{kl}} = \left(\frac{1}{\phi}\right)_{lk} = (\phi^{-1})_{lk}. \tag{A.22}
 \end{aligned}$$

A.9 Alternative derivation of the space-time dependent determinant using Schwinger-Dyson equations

In equation (4.2), it was found that

$$\ln J = \frac{1}{2} \left[N - \left(1 + V\delta^d(0) \right) \right] \ln \det \phi = \frac{1}{2} \left[N - \left(1 + V\delta^d(0) \right) \right] \text{Tr} \ln \phi$$

A second method which is able to reproduce this calculation may be by making use of Schwinger-Dyson equations. An example of such a calculation may be found in Ref. [18], in which the calculation is performed for charged scalar fields.

This result may also be obtained by writing the Schwinger-Dyson equations (as shall be shown) for a and ϕ and equating them. These are given by

$$\begin{aligned} 0 &= \int \mathbf{D}a \int \mathbf{D}a \frac{\delta}{\delta a^m(x)} [a^m(y) F[\phi] e^{i\mathcal{S}}] \\ &= \int \mathbf{D}a \int \mathbf{D}a N\delta^d(x-y) + \frac{\delta}{\delta a^m(x)} [F[\phi] e^{i\mathcal{S}}] \\ &= \langle N\delta^d(x-y) \rangle + \left\langle \frac{\delta}{\delta a^m(x)} [F[\phi] e^{i\mathcal{S}}] \right\rangle \end{aligned} \quad (\text{A.23})$$

and

$$\begin{aligned} 0 &= \int \mathbf{D}\Phi \int d^d z \frac{\delta}{\delta \Phi(z,x)} [\Phi(z,y) JF[\phi] e^{i\mathcal{S}}] \\ &= \int \mathbf{D}\Phi \int d^d z \delta^d(z-z) \delta^d(x-y) + \delta^d(z-x) \delta^d(z-y) \\ &\quad + \Phi(z,y) \frac{\delta}{\delta \Phi(z,x)} [JF[\phi] e^{i\mathcal{S}}] \\ &= \int \mathbf{D}\Phi \left(V\delta^d(0) + 1 \right) \delta^d(x-y) JF[\phi] e^{i\mathcal{S}} \\ &\quad + \int \mathbf{D}\Phi \int d^d z \Phi(z,y) \frac{\delta}{\delta \Phi(z,x)} [JF[\phi] e^{i\mathcal{S}}] \\ &= \left\langle \left(V\delta^d(0) + 1 \right) \delta^d(x-y) F[\phi] \right\rangle + \left\langle \int d^d z \Phi(z,y) \frac{\delta \ln J}{\delta \Phi(z,x)} F[\phi] \right\rangle \\ &\quad + \left\langle \int d^d z \Phi(z,y) \frac{\delta F[\phi]}{\delta \Phi(z,x)} \right\rangle + i \left\langle \int d^d z \Phi(z,y) \frac{\delta \mathcal{S}}{\delta \Phi(z,x)} F[\phi] \right\rangle \end{aligned} \quad (\text{A.24})$$

where $F[\phi]$ is a time-ordered product of bilocals

$$F[\phi] = T [\prod_{i=1}^n \phi(x_i, y_i)].$$

Using the chain rule,

$$\begin{aligned}
 \frac{\delta}{\delta a^m(x)} &= \int d^d z \int d^d y \frac{\delta \phi(z, y)}{\delta a^m(x)} \frac{\delta}{\delta \phi(z, y)} \\
 &= \int d^d z \int d^d y \left[a^m(z) \delta^d(x - y) + a^m(y) \delta^d(x - z) \right] \frac{\delta}{\delta \phi(z, y)} \\
 &= \int d^d z a^m(z) \left[\frac{\delta}{\delta \phi(z, x)} + \frac{\delta}{\delta \phi(x, z)} \right] \\
 &= \int d^d z a^m(z) \frac{\delta}{\delta \Phi(x, z)} \equiv \int d^d z a^m(z) \frac{\delta}{\delta \Phi(z, x)},
 \end{aligned}$$

by the symmetry of Φ . Incorporating this into equation (A.23) gives

$$\begin{aligned}
 0 &= \left\langle N \delta^d(x - y) F[\phi] \right\rangle + \left\langle \int d^d z a^m(y) a^m(z) \frac{\delta F[\phi]}{\delta \Phi(x, z)} \right\rangle \\
 &\quad + i \left\langle \int d^d z a^a(y) a^a(z) F[\phi] \frac{\delta S}{\delta \Phi(x, z)} \right\rangle \\
 &= \left\langle N \delta^d(x - y) F[\phi] \right\rangle + \left\langle \int d^d z \Phi(y, z) \frac{\delta F[\phi]}{\delta \Phi(x, z)} \right\rangle \\
 &\quad + i \left\langle \int d^d z \Phi(y, z) F[\phi] \frac{\delta S}{\delta \Phi(x, z)} \right\rangle
 \end{aligned}$$

Equations (A.23) and (A.24) both equal zero, and hence are equal to each other

$$\left\langle \left(V \delta^d(0) + 1 \right) \delta^d(x - y) F[\phi] \right\rangle + \left\langle \int d^d z \Phi(z, y) \frac{\delta \ln J}{\delta \Phi(z, x)} F[\phi] \right\rangle = \left\langle N \delta^d(x - y) F[\phi] \right\rangle.$$

Since

$$\int d^d z \Phi(z, y) \frac{\delta}{\delta \Phi(z, x)} = 2 \int d^d z \phi(z, y) \frac{\delta}{\delta \phi(z, x)},$$

therefore,

$$\begin{aligned}
 2 \left\langle \int d^d z \phi(z, y) \frac{\delta \ln J}{\delta \phi(z, x)} F[\phi] \right\rangle &= \left\langle N \delta^d(x - y) F[\phi] \right\rangle - \left\langle \left(V \delta^d(0) + 1 \right) \delta^d(x - y) F[\phi] \right\rangle. \\
 \Rightarrow \int d^d z \phi(z, y) \frac{\delta \ln J}{\delta \phi(z, x)} &= \frac{1}{2} \left(N - \left(V \delta^d(0) + 1 \right) \right) \delta^d(x - y),
 \end{aligned}$$

in agreement with equation (4.2) as promised.

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