ON CERTAIN RESTRICTED INTEGER PARTITIONS

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Declaration

This dissertation titled 'On Certain Restricted Integer Partitions' was carried out by Molatelo Olgar Rapudi under the supervision of Dr. D. Nyirenda of the School of Mathematics, at the University of the Witwatersrand, Johannesburg. It is being submitted for the Degree of Master of Science in Mathematics at the University of Witwatersrand, Johannesburg.

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Abstract

In this dissertation, we study some restricted integer partition functions. We first give a survey of some aspects of integer partition theory that will form a foundation of our work. These include but are not limited to, the Jacobi's triple product identity and its consequences, known recurrences such as MacMahon's recurrence for the partition function and Frobenius partitions. Our contributions to partition theory is a generalisation of one of the theorems of Sylvester, a non-trivial extension of a version of Alladi-Schur's theorem and a derivation of several parity and recurrence formulas of Sylvester-related partition functions and Frobenius partitions.

Dedication

In loving memories of

my brother

Moyahabo Alpheus Rapudi (1978-2004)

and my uncle

Phuti Simon Moremi(1968-2013).

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Chapter 1

Introduction

The study of integer partitions started around 1674. According to Andrews in [2], the discovery of integer partitions was due to Gottfried Leibniz. During that time, Leibniz wrote a letter to Johann Bernoulli asking if he had ever thought of 'how many ways a positive integer n can be partitioned'. Leonhard Euler (1707 - 1783) got interested in that question and was the first person to work on the problem of partitioning positive integers and other related problems.

Definition 1.1.1. *A partition of a positive integer n is a representation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\sum_{i=1}^{\ell} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_\ell \geq 1$.*

The integer n is called the weight of a partition, denoted by $|\lambda|$, and λ_i (for each $i = 1, 2, \ldots, \ell$) is called a part of the partition.

There are several ways of presenting a partition λ. Alternatively, λ could be presented as $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_\ell$, or $(r^{m_r}, (r-1)^{m_{r-1}}, \ldots 3^{m_3}, 2^{m_2}, 1^{m_1})$ for some r, where m_i is the multiplicity of i. Given partitions α and β, we define the $\alpha \cup \beta$ to be the partition found by taking the multiset union of α and β (in this case α and β are treated as multisets). In the notation $(r^{m_r}, (r-1)^{m_{r-1}}, \ldots 3^{m_3}, 2^{m_2}, 1^{m_1})$, if $m_i = 0$, then i does not appear in λ since it is not a part. We write $\lambda \vdash n$ to mean that λ is a partition of weight n.

For instance, $7 + 5 + 5 + 3 + 3 + 3 + 1 + 1 + 1 + 1$ can be written as $(7, 5, 5, 3, 3, 3, 1, 1, 1, 1)$ or $(7^1 5^2 3^3 1^4)$.

One may wonder, how many partitions does an integer n have? We denote

the number of all partitions of n by $p(n)$. Usually, $p(n)$ is referred to as the *unrestricted* partition function because there is no restriction on the parts. We define $p(0)$ to be 1 and $p(n) = 0$ for all negative integers n.

Example 1.1.1.

n	**partitions of** n	$p(n)$
1	(1)	1
2	$(2), (1^2)$	2
3	$(3), (2,1), (1^3)$	3
4	$(4), (3,1), (2^2), (2,1^2), (1^4)$	5
5	$(5), (4,1), (3,2), (3,1^2), (2^2,1), (2,1^3), (1^5)$	7
6	$(6), (5,1), (4,2), (4,1^2), (3^2),$ $(3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)$	11

However, a partition function that enumerates partitions in which a restriction on parts is imposed is called a *restricted* partition function. For example, the number of partitions into distinct parts is one such out of many that can be formulated. We will denote the number of partitions of n into distinct parts by $d(n)$.

Example 1.1.2. *Note that* $d(5) = 3$ *and the* 3 *partitions of* 5 *into distinct parts are;*

$$(5), (4,1), (3,2).$$

In the same way as $p(n)$, we set $d(n) := 0$ for all negative integers n and $d(0) := 1$. This partition function has been widely studied and a lot can be said about it. It is known that $d(n)$ is equal to the number of partitions of n into odd parts [11]. There is a bijective proof for this identity as well as a generating function proof, (see page 7). In terms of generating functions, we have

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}. \tag{1.1.1}$$

The identity is due to Leonard Euler, no wonder it is named after him. Identities that conform to (1.1.1) are called *identities of Euler type*. More explicitly, one side of the identity describes partitions in terms of multiplicities, and the other side describes partitions in terms of what parts are

allowed to appear. One bijective proof of this identity was found by J.W.L Glaisher [14] in 1883.

Not only is $d(n)$ connected to the number of partitions into odd parts but also, its parity can be deduced directly from Legendre's partition theoretic interpretation of the Euler's Pentagonal Number Theorem:

$$d_e(n) - d_o(n) = \begin{cases} (-1)^j, & n = \frac{j(3j\pm 1)}{2},\ j = 0, 1, 2, \dots \\ 0, & \text{otherwise,} \end{cases} \tag{1.1.2}$$

where $d_e(n)$ (resp. $d_o(n)$) is the number of $d(n)$-partitions of n into even (resp. odd) number of parts. We shall call an identity of type (1.1.2) an *identity of Euler's pentagonal type.*
There has been progress in the discovery of more identities of Euler type as well as those of the Euler's pentagonal type. The good news is that if parity of some partition function, $f(n)$ say, cannot be deduced easily, but the difference $f_e(n) - f_o(n)$ can, then we automatically know the parity.

The Euler's Pentagonal Number Theorem (EPNT) whose Legendre's interpretation is given in (1.1.2) is the equation

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty}(-1)^n(q^{\frac{n(3n+1)}{2}} + q^{\frac{n(3n-1)}{2}})$$

as long as $|q| < 1$ [5].

In the subsequent chapter, we lay down prelimnary tools. In Chapter 3, we present Andrew's elementary proof of Jacobi's triple product identity. As it will be shown, the EPNT follows from Jacobi's identity as a corollary. In Chapter 4, we discuss Legendre's partition-theoretic intepretation of EPNT. MacMahon's recurrence for $p(n)$ is also presented besides a recurrence formula for $d(n)$. Chapters 3 and 4 are based on [6] and [3].

A little amount of research in partition theory was done after Euler and later on, J. J. Sylvester started to get interested in the study of integer partitions an area in which he made great contributions. One of the partition identities discovered by Sylvester is: *the number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n* [19].

Our goal is threefold. Firstly, to generalise the aforementioned Sylvester's identity and give an explicit bijection for the generalisation which we present in Chapter 4. Secondly, to derive new parity and recurrence formulas for certain restricted partition functions and study some combinatorial consequences of Glaisher's bijection. Thirdly, to derive new partition identities involving certain restricted Frobenius partitions and geometrically interpret the bijective proof of our generalisation of Sylvester's identity.

Chapter 2

Preliminaries

2.1 Notation

For $a, q \in \mathbb{C}$, we have

$$(a;q)_n = \begin{cases} \prod_{i=0}^{n-1}(1-aq^i), & \text{for } n \geq 1 \\ 1, & n = 0. \end{cases}$$

Then,

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ldots = \prod_{n=0}^{\infty}(1-aq^n) = \lim_{n\to\infty}(a;q)_n.$$

Note: Throughout this dissertation, we generally use an abbreviation $(a)_n$ instead of $(a;q)_n$ and $(a)_\infty$ instead of $(a;q)_\infty$ whenever the base q is understood.
For all $n \in \mathbb{N}$, we may write

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}.$$

The following are some important q-products we are using in this work;

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n),$$

$$(-q)_\infty = \prod_{n=1}^{\infty}(1+q^n),$$

$$(q;q^2)_\infty = \prod_{n=1}^{\infty}(1-q^{2n-1}),$$

$$(q^2;q^2)_\infty = \prod_{n=1}^{\infty}(1-q^{2n}).$$

We denote the coefficient of q^j in the series expansion of $f(q)$ by

$$[q^j](f(q)).$$

Note: We denote the sets of partitions by capital letters. For instance, the number of all partitions on n is denoted by $P(n)$.

2.2 Background

The following theorem gives the generating function for the sequence $p(0)$, $p(1), p(2), \ldots$

Theorem 2.2.1. *For $|q| < 1$, we have*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.2.1}$$

Proof. Every partition is such that 1 occurs 0 times or once or twice or

thrice, etc., 2 occurs 0 times, once, twice, etc., and so on. Hence

$$\sum_{n=0}^{\infty} p(n)q^n = (1+q^1+q^{1+1}+q^{1+1+1}+\ldots)(1+q^2+q^{2+2}+q^{2+2+2}+\ldots)$$

$$\times (1+q^3+q^{3+3}+\ldots)(1+q^4+q^{4+4}+\ldots)\ldots,\ldots$$

$$= \frac{1}{(1-q)}\frac{1}{(1-q^2)}\frac{1}{(1-q^3)}\cdots$$

$$= \prod_{n=1}^{\infty}\frac{1}{1-q^n}.$$

□

Every partition enumerated by $d(n)$ is such that 1 occurs 0 times or once, 2 occurs 0 times or once, and so on. Thus for $|q|<1$,

$$\sum_{n=0}^{\infty} d(n)q^n = (1+q)(1+q^2)(1+q^3)\ldots \quad = \quad \prod_{n=1}^{\infty}(1+q^n).$$

Let $o(n)$ denote the number of partitions of n into odd parts. It turns out that $o(n)$ is actually equal to $d(n)$, an identity called Euler's partition identity [11]. This is stated in Theorem 2.2.2

Theorem 2.2.2. *For any nonnegative integer n,*

$$d(n) = o(n).$$

Proof. In terms of generating functions, we have

$$\begin{aligned}\sum_{n=0}^{\infty} d(n)q^n &= \prod_{n=1}^{\infty}(1+q^n)\\ &= \prod_{n=1}^{\infty}(1+q^n)\cdot\frac{(1-q^n)}{(1-q^n)}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n})(1-q^{2n-1})}\end{aligned}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}$$
$$= \sum_{n=0}^{\infty} o(n) q^n.$$

Thus

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}. \tag{2.2.2}$$

□

However, Theorem 2.2.2 has been generalised by L.W.J. Glaisher [14].

Theorem 2.2.3 (Glaisher, [14]). *Let k be a positive integer. Then the number of partitions of n where no part appears more than $k-1$ times equals the number of partitions of n in which parts are not divisible by k.*

We describe Glaisher's bijection for Theorem 2.2.3.

Glaisher's bijection

We denote Glaisher's bijection from the set of partitions of n in which parts are $\equiv r \pmod{k}$, $r \in \{1, 2, 3, \ldots, k-1\}$ onto the set of partitions of n where each part occurs at most $k-1$ times by ϕ. This map ϕ is described as follows:

Consider a partition $\lambda = (\lambda_1^{t_1}, \lambda_2^{t_2}, \lambda_3^{t_3}, \ldots, \lambda_\ell^{t_\ell})$ of n with ℓ different parts and each part is not divisible by k. We write each multiplicity t_i in its unique k-array expansion: $t_i = \sum_{j=0}^{m_i} a_{i,j} k^j$, $a_{i,j} \in \{0, 1, 2, \ldots, k-1\}$.

We define $\mu = \bigcup_{i=1}^{l} \bigcup_{j=0}^{m_i} (k^j \lambda_i)^{a_{i,j}}$ to be a partition of n whose parts are $k^j \lambda_i$ with multiplicities $a_{i,j}$ for all i, i.e., each part occurs at most $k-1$ times. Thus $\mu = \phi(\lambda)$.

Conversely, consider a partition $\mu = \left(\mu_1^{f_1}, \mu_2^{f_2}, \mu_3^{f_3}, \ldots, \mu_g^{f_g}\right)$ of n where no part appears more than $k-1$ times, that is, $f_i \in \{1, 2, 3, 4, \ldots, k-1\}$ for all $1 \leq i \leq g$. For each μ_i we find the highest power of k, say k^{j_i}, such that $\mu_i = k^{j_i} a_i$ where k does not divide a_i. Observe that

$$\lambda = \bigcup_{i=1}^{g} (a_i)^{k^{j_i} f_i}$$

is a partition of n whose parts are not divisible by k. Thus $\lambda = \phi^{-1}(\mu)$.

Further restrictions can be put on odd parts in Theorem 2.2.2, where odd parts are allowed to occur at most twice, and one observes Theorem 2.2.4, which has been recorded in [9].

Theorem 2.2.4 (Alladi-Shur, [9]). *Let $d_3(n)$ denote the number of partitions of n into distinct parts not divisible by 3 and $o_3(n)$ be the number of partitions of n into odd parts occurring not more than 2 times. Then for $n \geq 0$, we have*

$$d_3(n) = o_3(n).$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} o_3(n)q^n &= \prod_{n=1}^{\infty}(1+q^{2n-1}+q^{2(2n-1)}) \\
&= \prod_{n=1}^{\infty}\frac{1-q^{3(2n-1)}}{1-q^{2n-1}} \\
&= \prod_{n=1}^{\infty}\frac{1+q^n}{1+q^{3n}} \qquad \text{(by (2.2.2))} \\
&= \prod_{n=1}^{\infty}\frac{(1+q^{3n})(1+q^{3n-1})(1+q^{3n-2})}{(1+q^{3n})} \\
&= \prod_{n=1}^{\infty}(1+q^{3n-1})(1+q^{3n-2}) \\
&= \sum_{n=0}^{\infty} d_3(n)q^n.
\end{aligned}$$

□

2.3 Q-series

A q-series, also known as *Eulerian series*, is a series which in general contains various products of the form $(a;q)_n$ in its summands. In this section we look at one of the most useful theorems in the theory of q-series, the q-binomial theorems and discuss some important corollaries.

Recall that for all $n \in \mathbb{N}$, we have

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}. \tag{2.3.1}$$

For a prime power $k = p^r,\ r = 1, 2, 3, \ldots$, we have

$$(q;q)_n^k \equiv (q^k;q^k)_n \pmod{p}. \tag{2.3.2}$$

Proof. Observe that expanding $(q;q)_n^k$ using the binomial theorem we obtain,

$$\begin{aligned}(q;q)_n^k &= \left[\prod_{i=1}^{n}(1-q^i)\right]^k \\ &= \prod_{i=1}^{n}\left(1 + \sum_{m=1}^{k-1}\binom{k}{m}(-q^i)^{(k-m)} + (-q^i)^k\right).\end{aligned}$$

But $\binom{k}{m} \equiv 0 \pmod{p},\ \forall\ 1 \leq m \leq k-1.$

We have

$$(q;q)_n^k \equiv \prod_{i=1}^{n}(1 + (-1)^k q^{ik}) \pmod{p}.$$

If p is even, then $p = 2$. i.e., $k = 2^r$ and is even.
Then,

$$(-1)^k = 1 \equiv \pm 1 \pmod{p}.$$

Also, if p is odd, then $k = p^r$ is odd.
Thus

$$(-1)^k = -1.$$

In either case (p even or odd), $(-1)^k \pmod{p}$ becomes $-1 \pmod{p}$,

i.e.,

$$(q;q)_n^k \equiv \prod_{i=1}^{n}(1-q^{ik}) \pmod{p} \ = \ (q^k;q^k)_n \pmod{p}.$$

□

Theorem 2.3.1 (q-binomial theorem, [6])**.** *For* $|q|$, $|t| < 1$,

$$\sum_{n=0}^{\infty} \frac{(1-u)(1-uq)\cdots(1-uq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-utq^n)}{(1-tq^n)}. \tag{2.3.3}$$

Proof. Let

$$F(t) = \prod_{n=0}^{\infty} \frac{1-utq^n}{1-tq^n}.$$

It turns out that $F(t)$ converges uniformly on compact subsets of $|t| < 1$, see [6]. Hence it represents an analytic function on $|t| < 1$.
So, we may write

$$F(t) = \prod_{n=0}^{\infty} \frac{1-utq^n}{1-tq^n} = \sum_{n=0}^{\infty} A_n t^n, \quad |t| < 1.$$

Clearly,

$$\left[t^0\right] \left(\prod_{n=0}^{\infty} \frac{1-utq^n}{1-tq^n} \right) = 1. \tag{2.3.4}$$

So,

$$\begin{aligned}
(1-t)F(t) &= (1-ut) \prod_{n=1}^{\infty} \frac{(1-utq^n)}{(1-tq^n)} \\
&= (1-ut) \prod_{n=0}^{\infty} \frac{(1-utq^{n+1})}{(1-tq^{n+1})} \\
&= (1-ut)F(tq).
\end{aligned}$$

We have

$$\begin{aligned}
(1-t)\sum_{n=0}^{\infty} A_n t^n &= (1-ut)\sum_{n=0}^{\infty} A_n q^n t^n, \\
\sum_{n=0}^{\infty} A_n t^n - \sum_{n=0}^{\infty} A_n t^{n+1} &= \sum_{n=0}^{\infty} A_n q^n t^n - \sum_{n=0}^{\infty} u A_n q^n t^{n+1}, \\
\left(A_0 + \sum_{n=1}^{\infty} A_n t^n \right) - \sum_{n=1}^{\infty} A_{n-1} t^n &= \left(A_0 + \sum_{n=1}^{\infty} A_n q^n t^n \right) - \sum_{n=1}^{\infty} u A_{n-1} q^{n-1} t^n, \\
\sum_{n=1}^{\infty} (A_n - A_{n-1}) t^n &= \sum_{n=1}^{\infty} (A_n q^n - u A_{n-1} q^{n-1}) t^n.
\end{aligned} \tag{2.3.5}$$

Equating coefficients of t^n in (2.3.5), we find that for a fixed n

$$A_n - A_{n-1} = q^n A_n - uq^{n-1} A_{n-1},$$

or

$$A_n = \frac{(1-uq^{n-1})}{(1-q^n)} A_{n-1}.$$

Iteration of this recursion reveals that for all $n \geq 0$,

$$A_n = \frac{(1-u)(1-uq)\cdots(1-uq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} A_0.$$

But by $F(t) = \sum_{n=0}^{\infty} A_n t^n$, we have

$$A_0 = \left[t^0\right] F(t) = 1. \qquad \text{(by (2.3.4))}$$

So,

$$A_n = \frac{(1-u)(1-uq)\cdots(1-uq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Substituting this value for A_n in (2.3.4), we obtain the desired result. □

We now state two useful corollaries of Theorem 2.3.1 found by Euler.

Corollary 2.3.1. *(Euler) For $|q|$, $|t| < 1$,*

$$\sum_{n=0}^{\infty} \frac{t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-tq^n)}, \tag{2.3.6}$$

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1+tq^n). \tag{2.3.7}$$

Proof. Equation (2.3.6) is obtained from Theorem 2.3.1 by setting $u = 0$. That is, for $|q|$, $|t| < 1$,

$$\sum_{n=0}^{\infty} \frac{(1-0)(1-0q)\ldots(1-0q^{n-1})t^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-0tq^n)}{(1-tq^n)},$$

$$\sum_{n=0}^{\infty} \frac{t^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-tq^n)}.$$

Replacing u by u/v and t by vt in (2.3.3) we obtain that for $|q|, |vt| < 1$, we have

$$\sum_{n=0}^{\infty} \frac{(1-u/v)(1-(u/v)q)\dots(1-(u/v)q^{n-1})(vt)^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-utq^n)}{(1-vtq^n)}.$$

But the summation above equals

$$\sum_{n=0}^{\infty} \frac{\frac{1}{v}(v-u)\frac{1}{v}(v-uq)\dots\frac{1}{v}(v-uq^{n-1})v^n t^n}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= \sum_{n=0}^{\infty} \frac{(v-u)(v-uq)\dots(v-uq^{n-1})t^n}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Hence we have

$$\sum_{n=0}^{\infty} \frac{(v-u)(v-uq)\dots(v-uq^{n-1})t^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-utq^n)}{(1-vtq^n)}. \tag{2.3.8}$$

Now Equation (2.3.7) is obtained by setting $v = 0$ and $u = -1$ in (2.3.8), that is

$$\sum_{n=0}^{\infty} \frac{q \cdot q^2 \cdot q^3 \dots q^{n-1} t^n}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1+tq^n)}{(1)},$$

and thus

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} (1+tq^n).$$

□

However, if we set $t := q$ in (2.3.6), we get

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{n+1})}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

$$= \sum_{n=0}^{\infty} p(n)q^n. \qquad \text{(by (2.2.1))}$$

That is, for $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^n. \tag{2.3.9}$$

And also if we set $t := q$ in (2.3.7), we get

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty}(1+q^n)$$

$$= \sum_{n=0}^{\infty} d(n)q^n. \qquad \text{(by (2.2.2))}$$

That is, for $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)} = \sum_{n=0}^{\infty} d(n)q^n. \tag{2.3.10}$$

We now look at Heine's fundamental transformation as a corollary of the q-binomial Theorem.

Corollary 2.3.2 (Heine). *For $|q|, |t|, |v| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(u)_n (v)_n t^n}{(q)_n (w)_n} = \frac{(v)_\infty (ut)_\infty}{(w)_\infty (t)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(\frac{w}{v})_n (t)_n v^n}{(q)_n (ut)_n}.$$

Proof.

$$\sum_{n=0}^{\infty} \frac{(u)_n (v)_n t^n}{(q_n)(w)_n} = \sum_{n=0}^{\infty} \frac{(u)_n t^n}{(q)_n} \cdot \frac{(v)_\infty}{(vq^n)_\infty} \cdot \frac{(wq^n)_\infty}{(w)_\infty} \qquad \text{(by (2.3.1))}$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{n=0}^{\infty} \frac{(u)_n t^n}{(q)_n} \cdot \frac{(wq^n)_\infty}{(vq^n)_\infty}$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{n=0}^{\infty} \frac{(u)_n t^n}{(q)_n} \cdot \sum_{m=0}^{\infty} \frac{(w/v)_m (vq^n)^m}{(q)_m} \qquad \text{(by (2.3.3), } t = v \text{ and } u = w/v)$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(u)_n (tq^m)^n}{(q)_n} \right) \cdot \frac{(w/v)_m v^m}{(q)_m}$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{m=0}^{\infty} \left(\frac{(utq^m)_\infty}{(tq^m)_\infty} \right) \cdot \frac{(w/v)_m v^m}{(q)_m} \qquad \text{(by (2.3.3))}$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{m=0}^{\infty} \frac{(w/v)_m v^m}{(q)_m} \cdot (utq^m)_\infty \cdot \frac{1}{(tq^m)_\infty}$$

$$= \frac{(v)_\infty}{(w)_\infty} \sum_{m=0}^{\infty} \frac{(w/v)_m v^m}{(q)_m} \cdot \frac{(ut)_\infty}{(ut)_m} \cdot \frac{(t)_m}{(t)_\infty} \qquad \text{(by (2.3.1))}$$

$$= \frac{(v)_\infty (ut)_\infty}{(w)_\infty (t)_\infty} \cdot \sum_{m=0}^{\infty} \frac{(w/v)_m (t)_m v^m}{(q)_m (ut)_m}.$$

□

The following corollary follows as a corollary of Corollary 2.3.2.

Corollary 2.3.3 (Heine). *If $|w| < |uv|$, then for $|q| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(u)_n (v)_n (w/(uv))^n}{(q)_n (w)_n} = \frac{(w/u)_\infty (w/v)_\infty}{(w)_\infty (w/(uv))_\infty}.$$

Proof. If we replace t with $w/(uv)$ in Corollary 2.3.2, we get that, for $|q| < 1$, $|\frac{w}{uv}| = \frac{|w|}{|uv|} < 1$, (i.e. $|w| < |uv|$),

$$\sum_{n=0}^{\infty} \frac{(u)_n (v)_n (w/(uv))^n}{(q)_n (w)_n} = \frac{(v)_\infty (u \cdot w/(uv))_\infty}{(w)_\infty (w/(uv))_\infty} \cdot \sum_{n=0}^{\infty} \frac{(w/v)_n (w/(uv))_n v^n}{(q)_n (u \cdot w/(uv))_n}$$

$$= \frac{(v)_\infty (w/v)_\infty}{(w)_\infty (w/(uv))_\infty} \cdot \sum_{n=0}^{\infty} \frac{(w/(uv))_n v^n}{(q)_n}$$

$$= \frac{(v)_\infty (w/v)_\infty}{(w)_\infty (w/(uv))_\infty} \cdot \frac{(w/u)_\infty}{(v)_\infty} \qquad \text{(by (2.3.3))}$$

$$= \frac{(w/u)_\infty (w/v)_\infty}{(w)_\infty (w/(uv))_\infty}.$$

□

Corollary 2.3.4 (q-analog of Kummer's theorem)**.**

We first look at the q-analog of Kummer's theorem due to Cauchy.

If $|q| < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-z)(1-zq)(1-zq^2)\cdots(1-zq^{n-1})}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-zq^n)}. \tag{2.3.11}$$

Proof. In Corollary 2.3.3, we set $u = u^{-1}$, $v = v^{-1}$ and $w = z$. Then, for $|q| < 1$, $|z| < \frac{1}{|uv|}$ or $|zuv| < 1$,

$$\sum_{n=0}^{\infty} \frac{\frac{1}{u^n}(u-1)(u-q)\dots(u-q^{n-1})\frac{1}{v^n}(v-1)(v-q)\dots(v-q^{n-1})u^n v^n z^n}{(q)_n (z)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(u-1)(u-q)\dots(u-q^{n-1})(v-1)(v-q)(v-q^2)\dots(v-q^{n-1})z^n}{(q)_n (z)_n}$$

$$= \frac{(zu)_\infty (zv)_\infty}{(z)_\infty (zuv)_\infty}.$$

Setting $u = v = 0$ in this equality we get that, for $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{[(-1)(-q)(-q^2)\dots(-q^{n-1})]^2 z^n}{(q)_n (z)_n} = \frac{(z(0))_\infty (z(0))_\infty}{(z)_\infty (z(0))_\infty};$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{q^{2+4+6+\dots 2n-2} z^n}{(q)_n (z)_n} = \frac{1}{(z)_\infty};$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n}{(q)_n (z)_n} = \frac{1}{(z)_\infty}.$$

□

We now look at the other q-analog of Kummer's theorem due to Euler.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{2.3.12}$$

Proof. Equation 2.3.12 follows from Equation 2.3.11 by setting $z = q$. That is,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)} q^n}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{n+1})}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.$$

□

2.4 Graphical representation of partitions

Partitions can be represented via diagrams. One of the representations is a *Ferrers graph*. The Ferrers graph of a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a left-aligned array of dots with k rows and λ_i dots in the i^{th} row.
The Ferrers graph of $(7, 5, 3, 3, 1)$ is shown in Figure 2.1.

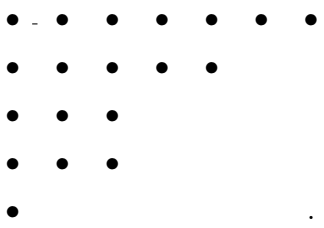
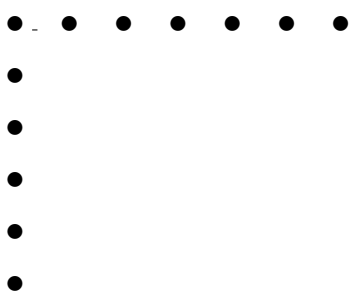


Figure 2.1: Ferrers graph of (7,5,3,3,1)

We call each diagram of a Ferrers graph given in the following picture



a *hook* of a Ferrers graph.

Definition 2.4.1. *A conjugate partition of a partition λ is a partition whose Ferrers graph is obtained by interchanging the rows and the columns of the Ferrers graph of λ. We denote a conjugate of λ by λ'.*

Alternatively, given a partition $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_\ell^{m_\ell})$ where $\lambda_1 > \lambda_2 > \lambda_3 > \ldots \lambda_\ell \geq 1$, the conjugate partition of λ is given by
$\lambda' = \left(\left(\sum_{i=1}^{\ell} m_i \right)^{\lambda_\ell}, \left(\sum_{i=1}^{\ell-1} m_i \right)^{\lambda_{\ell-1}-\lambda_\ell}, \ldots m_1^{\lambda_1-\lambda_2} \right)$.
If $\lambda = \lambda'$, then λ is called a *self-conjugate* partition.
Self-conjugate partitions are related to partitions into distinct odd parts. J. J. Sylvester gave the following correspondence.

Theorem 2.4.1 (Sylvester, [19])**.** *The number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.*

Proof. Let λ be a self-conjugate partition of n. Then,

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_s, \lambda_{s+1}, \lambda_{s+2}, \ldots)$$

where $(\lambda_{s+1}, \lambda_{s+2}, \lambda_{s+3} \ldots) = (s^{\lambda_s - s}, (s-1)^{\lambda_{s-1}-\lambda_s}, \ldots, 1^{\lambda_1-\lambda_2})$,
and $s = \max\{i : \lambda_i \geq i\}$.

Now, the partition β of n defined as

$$\beta = (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 2\lambda_s - 2s + 1)$$

is a partition into distinct odd parts.

This map $\lambda \mapsto \beta$ is a bijection. This map has a known geometric interpretation via Frobenius symbols. □

Example 2.4.1.

Consider the partitions of 14. Then the 3 partitions of 14 into distinct odd parts are;

$$(13, 1), \quad (11, 3), \quad (9, 5).$$

And the 3 self-conjugate partitions of 14 are;

$$(7, 2, 1^5), \quad (6, 3, 2, 1^3), \quad (5, 4, 2^2, 1).$$

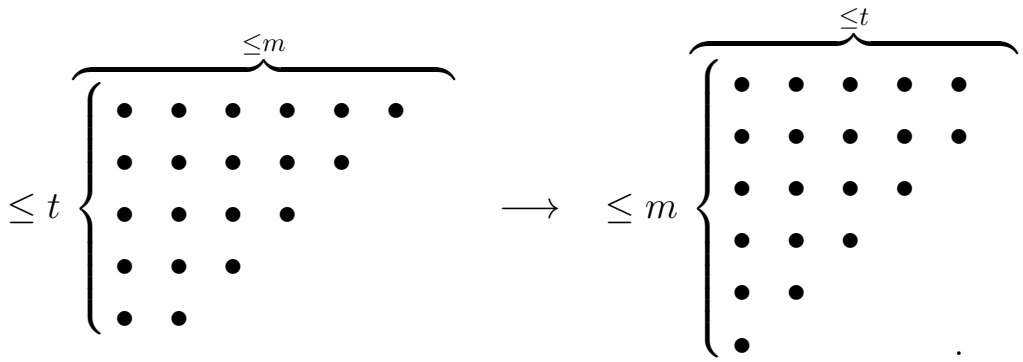
Using the map given above, observe that;

$$(7, 2, 1^5) \mapsto (13, 1),$$
$$(6, 3, 2, 1^3) \mapsto (11, 3),$$
$$(5, 4, 2^2, 1) \mapsto (9, 5).$$

The following theorem connects ordinary partitions in which part size and the number of parts are restricted.

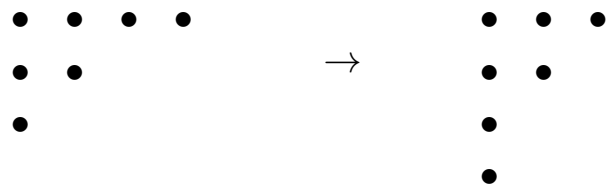
Theorem 2.4.2 ([7]). *For positive integers m and t, the number of partitions of n with at most m parts in which the largest part is at most t equals the number of partitions of n with at most t parts in which the largest part is at most m.*

Proof. Consider the Ferrers graph of a partition of n with at most m parts and the largest part at most t and then we take its conjugate;



We map each partition from the first class onto its conjugate partition. Clearly, this mapping is bijective and by looking at the Ferrers graph we observe that *the number of parts* $\leq m$ is transformed into *the largest part* $\leq m$, vice versa.
Thus there is a one-to-one correspondence between the set of partitions of n with at most m parts where the largest part is at most t and the set of partitions of n with at most t parts where the largest part is at most m.
□

Example 2.4.2. *For example, take the Ferrers graph of a partition* $(4, 2, 1)$ *of* 7, $m = 3$ *and* $t = 5$ *and conjugate it;*



Thus $(4,2,1) \mapsto (3,2,1,1)$.

Hence the 6 partitions of 7 with at most 3 parts and the largest part ≤ 5 are;

$$\begin{array}{ccc} (5,2), & (5,1,1) & (4,3), \\ (4,2,1), & (3,3,1), & (3,2,2). \end{array}$$

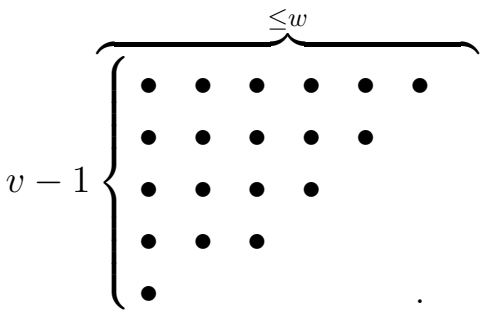
and the 6 partitions of 7 with at most 5 parts and the largest part ≤ 3 are;

$$\begin{array}{ccc} (3,3,1), & (3,2,2) & (3,2,1,1), \\ (3,1,1,1,1), & (2,2,2,1), & (2,2,1,1,1). \end{array}$$

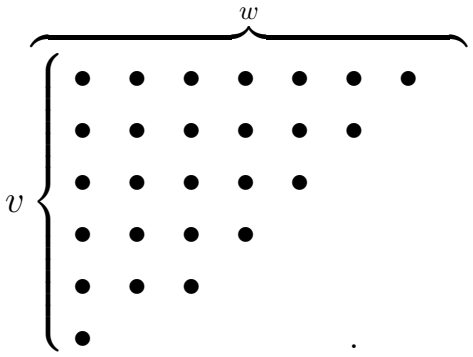
Theorem 2.4.3 ([7]). *Let $p(u-w,v)$ denote the number of partitions of $u-w$ into exactly $v-1$ parts, none exceeding w and $p(u-v,w)$ denote the number of partitions of $u-v$ into exactly $w-1$ parts, none exceeding v. Then*

$$p(u-w,v) = p(u-v,w).$$

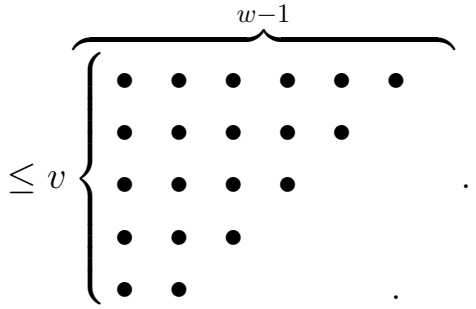
Proof. Consider the Ferrers graph of a partition of $u-w$ enumerated by $p(u-w,v)$.



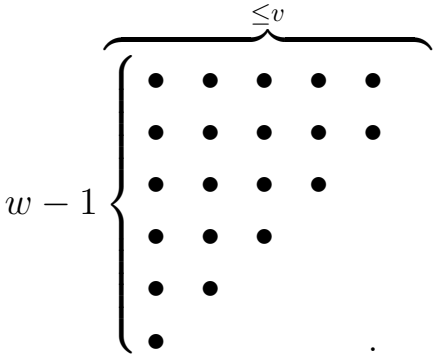
First we add a top row of w nodes to the the Ferrers graph above, so we have



Note that the above diagram is the Ferrers graph of a partition of $u - w + w = u$. We now delete the first column (which has v nodes) from the Ferrers graph above, so we have a partition of $u - v$,



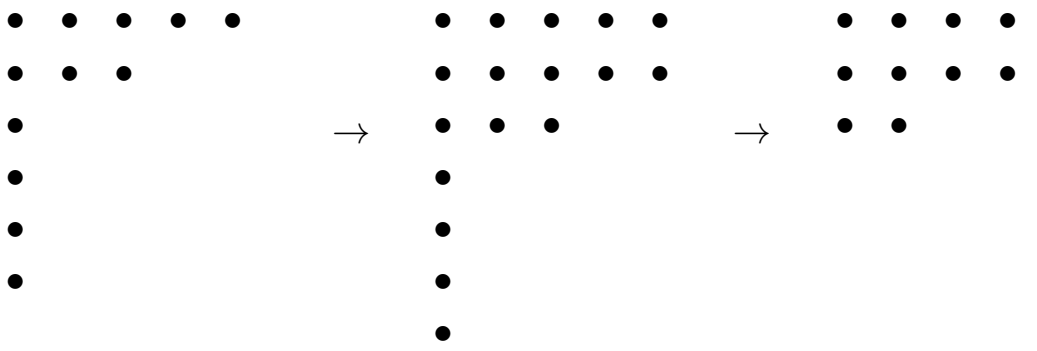
Lastly, we take the conjugate of the partition above, so we have



Thus we have the Ferrers graph of a partition of $u - v$ into exactly $w - 1$ parts and the largest part is at most v.
Note that all of these transformations are reversible. Thus we have a one-to-one correspondence between the set of partitions enumerated by $p(u - w, v)$ and the set of partitions enumerated by $p(u - v, w)$. □

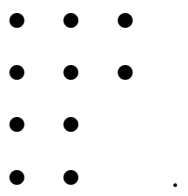
Example 2.4.3. *Let $u = 17$, $v = 7$ and $w = 5$. Take the Ferrers graph of a partition $(5, 3, 1^4)$ of $u - w = 12$ into exactly $v - 1 = 6$ parts and the largest part at most 5.*

Take the Ferrers graph of a partition $(5, 3, 1^4)$ of 12 and add a top row of 5 nodes to it, then delete the first column, which now has 7 nodes;



The Ferrers graph of $(5, 3, 1^4)$ A partition of $u - w + w$. A partition of $u - v$

We now take the conjugate of the above partition of $u - v$;



Conjugate partition of $u - v$

Hence we obtain a partition $(3, 3, 2, 2)$ of $u - v = 10$ into exactly $w - 1 = 4$ parts, none exceeding 7.

Now the 9 partitions of $u - w = 12$ into exactly $v - 1 = 6$ parts with the largest part ≤ 5 are;

$$\begin{array}{lll} (5,3,1,1,1,1), & (5,2,2,1,1,1), & (4,4,1,1,1,1), \\ (4,3,2,1,1,1), & (4,2,2,2,1,1), & (3,3,3,1,1,1), \\ (3,3,2,2,1,1), & (3,2,2,2,2,1), & (2,2,2,2,2,2). \end{array}$$

and the 9 partitions of $u - v = 10$ into exactly $w - 1 = 4$ parts with the largest part ≤ 7 are;

$$\begin{array}{lll} (7,1,1,1), & (6,2,1,1), & (5,3,1,1), \\ (5,2,2,1), & (4,4,1,1), & (4,3,2,1), \\ (4,2,2,2), & (3,3,3,1), & (3,3,2,2). \end{array}$$

Associated with the Ferrers graph of a partition is a square called the *Durfee square*. The Durfee square of a partition is largest possible square

contained in the Ferrers graph. The Durfee square for (7,5,3,3,1), whose dots are framed in circles, is shown in Figure 2.2.

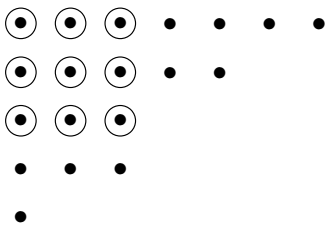


Figure 2.2: The Durfee square for (7,5,3,3,1)

Definition 2.4.2. *For any partition λ of n, we define the order of λ to be* $\max\{i : \lambda_i \geq i\}$.

We shall denote the order of λ by $s(\lambda)$. Actually, the order is simply the length of the Durfee square.

By counting the number of dots to the right of the diagonal of the Durfee square, as well as those below the diagonal, we can construct a structure called the Frobenius symbol of a partition. However, we look at its definition that is independent of a partition, and then will connect the concept with partitions.

Definition 2.4.3. *A Frobenius symbol is a two-rowed array of non-negative integers*

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_s \\ b_1 & b_2 & b_3 & \dots & b_s \end{pmatrix}$$

where $a_1 > a_2 > a_3 > \ldots > a_s \geq 0$ *and* $b_1 > b_2 > b_3 > \ldots > b_s \geq 0$.

Now consider the partition in Figure 2.2. Deleting the diagonal of the Durfee square leaves us with two subset partitions. If we let a_i be the number of dots in the i^{th} row to the right of the deleted diagonal and b_i be the number of dots in the i^{th} column beneath the deleted diagonal, then we automatically have a Frobenius symbol with a_i's and b_i's defined in that way. This is generally true for any partition. In the case of the partition in Figure 2.2, we have $a_1 = 6, a_2 = 3, a_3 = 0$ and $b_1 = 4, b_2 = 2, b_3 = 1$.

Thus the Frobenius symbol

$$\begin{pmatrix} 6 & 3 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

is a *Frobenius symbol of (7,5,3,3,1).*
It should be noted that every Frobenius symbol is a Frobenius symbol of a partition of some positive integer n. For a Frobenius symbol to be the Frobenius symbol of a partition of n, it has to satisfy the extra condition

$$n = \sum_{i=1}^{s} a_i + \sum_{i=1}^{s} b_i + s.$$

Thus a Frobenius symbol of a partition λ of n, denoted by f_λ, is given by

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_s \\ b_1 & b_2 & b_3 & \dots & b_s \end{pmatrix}$$

where $n = \sum_{i=1}^{s} a_i + \sum_{i=1}^{s} b_i + s$.
On the other hand, given a Frobenius symbol

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_s \\ b_1 & b_2 & b_3 & \dots & b_s \end{pmatrix}, \; n = \sum_{i=1}^{s} a_i + \sum_{i=1}^{s} b_i + s, \tag{2.4.1}$$

one can find a partition λ of n whose Frobenius symbol is f.
Let $f(n)$ be the number of Frobenius symbols of the form (2.4.1). This map $\psi_1 : P(n) \mapsto F(n)$ defined by $\lambda \mapsto f_\lambda$ is a bijection.

Since there is a one-to-one correspondence between $P(n)$ and $F(n)$, it follows that

$$p(n) = f(n).$$

In terms of generating functions, one can also see that

$$\sum_{n=0}^{\infty} f(n)q^n = \sum_{n=0}^{\infty} p(n)q^n.$$

That is,

$$\sum_{n\geq 0} f(n)q^n = \sum_{s\geq 0,\, a_1>a_2>\ldots>a_s\geq 0,\, b_1>b_2>\ldots>b_s\geq 0} q^s \cdot q^{a_1+a_2+\ldots+a_s} \cdot q^{b_1+b_2+\ldots+b_s}. \tag{2.4.2}$$

But,

$$\begin{aligned} a_s \geq 0 \quad &\Rightarrow a_s = n_1,\ n_1 \geq 0 \\ a_{s-1} > a_s \quad &\Rightarrow a_{s-1} = n_1 + n_2 + 1,\ n_1, n_2 \geq 0 \\ &\vdots \\ a_1 > a_2 \quad &\Rightarrow a_1 = n_1 + n_2 + \cdots + n_s + s - 1, \qquad \forall\, n_i \geq 0. \end{aligned}$$

So, (2.4.2) becomes

$$\sum_{s\geq 0,\, n_1\geq 0,\, n_2\geq 0} \cdots \sum_{n_s\geq 0,\, m_1\geq 0,\, m_2\geq 0} \cdots \sum_{m_s\geq 0} q^s \cdot q^{(s)n_1+(s-1)n_2+\ldots+n_s} \cdot q^{1+2+\ldots+(s-1)}$$

$$\times\, q^{(s)m_1+(s-1)m_2+\ldots+m_s} \cdot q^{1+2+\ldots+(s-1)}$$

$$= \sum_{s\geq 0} q^s \cdot q^{s(s-1)} \cdot \sum_{n_1\geq 0} q^{(s)n_1} \cdot \sum_{n_2\geq 0} q^{(s-1)n_2} \cdots \sum_{n_s\geq 0} q^{n_s} \cdot \sum_{m_1\geq 0} q^{(s)m_1} \cdots \sum_{m_s\geq 0} q^{m_s}$$

$$= \sum_{s\geq 0} q^{s^2} \cdot \frac{1}{(1-q^s)^2} \cdot \frac{1}{(1-q^{s-1})^2} \cdots \frac{1}{(1-q^2)^2} \cdot \frac{1}{(1-q)^2}$$

$$= \sum_{s\geq 0} \frac{q^{s^2}}{(1-q)^2(1-q^2)^2(1-q^3)^2\cdots(1-q^s)^2}$$

$$= \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s^2}.$$

We have

$$\sum_{n\geq 0} f(n)q^n = \sum_{s=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}.$$

Setting $z := q$ in (2.3.11) , we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}q^n}{(q;q)_s(q;q)_s} \quad = \quad \prod_{n=0}^{\infty} \frac{1}{1-q\cdot q^n},$$

i.e.,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} \quad = \quad \prod_{n=0}^{\infty} \frac{1}{1-q^{n+1}}, \tag{2.4.3}$$

so that

$$\sum_{n=0}^{\infty} f(n) q^n \quad = \quad \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.4.4}$$

Chapter 3

In this chapter, we look into the structure of the Frobenius symbols and provide a detailed proof of a well known and very useful theorem (Jacobi's triple product identity). We also explain some useful corollaries of this identity. One such corollary is a well known theorem called the Euler's Pentagonal Number Theorem (EPNT). The partition-theoretic interpretation of EPNT due to Legendre is presented. Furthermore, we derive some known recurrence formulas, such as the MacMahon's recurrence for the unrestricted partition function and a recurrence for the number of partitions into distinct parts.

3.1 The Jacobi triple product identity

We first present a proof for the Jacobi's triple product identity. Proposition 3.1.1 reveals that the generating function for the number of Frobenius symbols of the form (2.4.1) actually provides a simple proof to the Jacobi triple identity.

Proposition 3.1.1 (Andrews, [3])**.**

$$\sum_{n=0}^{\infty} f(n)q^n = \left[w^0\right] \left(\prod_{n=0}^{\infty} (1+wq^{n+1})(1+w^{-1}q^n) \right).$$

Proof. We examine the coefficient of w^0 in

$$\prod_{n=0}^{\infty} (1+wq^{n+1})(1+w^{-1}q^n) = \prod_{n=0}^{\infty} (1+wq^{n+1}) \prod_{n=0}^{\infty} (1+w^{-1}q^n)$$

$$= \{(1+wq)(1+wq^2)(1+wq^3)(1+wq^4)\ldots\}$$

$$\times \{(1+w^{-1}q^0)(1+w^{-1}q)(1+w^{-1}q^2)\ldots\}$$

$$= \{(1+(wq)q^0)(1+(wq)q^1)(1+(wq)q^2)\ldots\}$$

$$\times \{(1+w^{-1}q^0)(1+w^{-1}q^1)(1+w^{-1}q^2)\ldots\}.$$

After multiplication and careful permutation, one finds that terms in the coefficient of z^0, are of the form

$$q^s \cdot q^{a_1+a_2+\ldots+a_s} \cdot q^{b_1+b_2+\ldots+b_s},$$

where $a_1 > a_2 > a_3 > \ldots > a_s \geq 0$ and $b_1 > b_2 > b_3 > \ldots > b_s \geq 0$.
Hence the coefficient of w^0 in the product is actually the generating function for the number of Frobenius symbols, $f(n)$.
Indeed

$$\sum_{n=0}^{\infty} f(n)q^n = \left[w^0\right] \left(\prod_{n=0}^{\infty} (1+wq^{n+1})(1+w^{-1}q^n) \right).$$

□

Using (2.4.4), we have

$$\left[w^0\right] \left(\prod_{n=0}^{\infty} (1+wq^{n+1})(1+w^{-1}q^n) \right) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Let $J(w) = \prod_{n=1}^{\infty} (1+wq^n)(1+w^{-1}q^{n-1})$. Then

$$\left[w^0\right] J(w) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{3.1.1}$$

Theorem 3.1.1 (Jacobi triple product identity, [3]). *For $|q| < 1, w \neq 0$,*

$$\sum_{n=-\infty}^{\infty} w^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} (1-q^n)(1+wq^n)(1+w^{-1}q^{n-1}). \tag{3.1.2}$$

Proof. Recall that $J(w) = \prod_{n=1}^{\infty} (1+wq^n)(1+w^{-1}q^{n-1})$.
We may expand $J(w)$ in a Laurent series around $w = 0$ so that

$$J(w) = \sum_{n=-\infty}^{\infty} A_n w^n,$$

where $A_n = A_n(q)$, a function of q. Moreover,

$$\begin{aligned} J(wq) &= \prod_{n=1}^{\infty}(1+wq^{n+1})(1+w^{-1}q^{n-2}) \\ &= \prod_{n=2}^{\infty}(1+wq^n)\prod_{n=0}^{\infty}(1+w^{-1}q^{n-1}) \\ &= \frac{1}{1+wq}\prod_{n=1}^{\infty}(1+wq^n)(1+w^{-1}q^{-1})\prod_{n=1}^{\infty}(1+w^{-1}q^{n-1}) \end{aligned}$$

$$\begin{aligned} &= \frac{(1+wq)(w^{-1}q^{-1})}{1+wq}\prod_{n=1}^{\infty}(1+wq^n)\prod_{n=1}^{\infty}(1+w^{-1}q^{n-1}) \\ &= w^{-1}q^{-1}J(w). \end{aligned}$$

Hence

$$J(wq) = w^{-1}q^{-1}J(w). \tag{3.1.3}$$

Substituting wq for w in

$$J(w) = \sum_{n=-\infty}^{\infty} A_n w^n,$$

we have

$$J(wq) = \sum_{n=-\infty}^{\infty} A_n q^n w^n. \tag{3.1.4}$$

On the other hand, the right-hand side of (3.1.3) is

$$\begin{aligned} w^{-1}q^{-1}J(w) &= w^{-1}q^{-1}\sum_{n=-\infty}^{\infty} A_n w^n \\ &= \sum_{n=-\infty}^{\infty} A_n q^{-1} w^{n-1} \\ &= \sum_{n=-\infty}^{\infty} A_{n+1} q^{-1} w^n. \end{aligned}$$

Thus (3.1.3) becomes

$$\sum_{n=-\infty}^{\infty} A_n (wq)^n = \sum_{n=-\infty}^{\infty} A_{n+1} q^{-1} w^n.$$

Comparing the coefficients of w^n on both sides, we note that

$$A_n(q)^n = A_{n+1}q^{-1}$$

which implies that

$$A_n = A_{n-1}q^n.$$

Iteration of this recursion reveals that for all $n \geq 0$,

$$A_n = A_0 q^{\frac{n(n+1)}{2}}.$$

But by $J(w) = \sum_{n=-\infty}^{\infty} A_n w^n$, we have

$$A_0 = \left[w^0\right] J(w) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \qquad \text{(by (3.1.1))}$$

So,

$$J(w) = A_0 \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} w^n$$

which yields

$$\prod_{n=1}^{\infty}(1+wq^n)(1+w^{-1}q^{n-1}) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n=-\infty}^{\infty} w^n q^{\frac{n(n+1)}{2}}.$$

Hence

$$\sum_{n=-\infty}^{\infty} w^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1+wq^n)(1+w^{-1}q^{n-1}).$$

□

Corollary 3.1.1. *For $w \neq 0$, $|q| < 1$, we have*

$$\sum_{n=-\infty}^{\infty} w^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+wq^{2n+1})(1+w^{-1}q^{2n+1}). \qquad (3.1.5)$$

Proof. Setting $w := wq^{-1}$ and $q := q^2$ in Theorem 3.1.1 we obtain the

desired result. That is, for $|q| < 1, w \neq 0$,

$$\sum_{n=-\infty}^{\infty} w^n q^{n(n+1)-n} = \prod_{n=1}^{\infty}(1-q^{2n})(1+wq^{-1}q^{2n})(1+w^{-1}qq^{2n-2})$$

$$\sum_{n=-\infty}^{\infty} w^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1+wq^{2n-1})(1+w^{-1}q^{2n-1})$$

$$= \prod_{n=0}^{\infty}(1-q^{2n+2})(1+wq^{2n+1})(1+w^{-1}q^{2n+1}).$$

□

Corollary 3.1.2 (Jacobi's identity)**.** *For* $|q| < 1$,

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}. \tag{3.1.6}$$

Proof. Setting $w := -w$ in (3.1.5) we obtain

$$\sum_{n=-\infty}^{\infty}(-1)^n w^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1-wq^{2n-1})(1-w^{-1}q^{2n-1})$$

$$= \prod_{n=0}^{\infty}(1-wq^{2n+1})\prod_{n=1}^{\infty}(1-q^{2n})(1-w^{-1}q^{2n-1})$$

$$= (1-wq)\prod_{n=1}^{\infty}(1-wq^{2n+1})\prod_{n=1}^{\infty}(1-q^{2n})(1-w^{-1}q^{2n-1}).$$

Thus we have

$$(1-wq)^{-1}\sum_{n=-\infty}^{\infty}(-1)^n w^n q^{n^2} = \prod_{n=1}^{\infty}(1-w^{-1}q^{2n-1})(1-wq^{2n+1})(1-q^{2n}). \tag{3.1.7}$$

Now,

$$\begin{aligned}
&\frac{1}{(1-wq)}\sum_{n=-\infty}^{\infty}(-1)^n w^n q^{n^2} \\
&= (1+wq+w^2q^2+\ldots)\left(1+\sum_{n=1}^{\infty}(-1)^n w^n q^{n^2}+\sum_{n=1}^{\infty}(-1)^{-n} w^{-n} q^{n^2}\right) \\
&= \sum_{j=0}^{\infty} w^j q^j \left(1+\sum_{n=1}^{\infty}((-1)^n w^n+(-1)^n w^{-n})q^{n^2}\right) \\
&= 1+\sum_{j=1}^{\infty} w^j q^j + \sum_{j=0}^{\infty}\sum_{n=1}^{\infty}((-1)^n w^n+(-1)^n w^{-n}) w^j q^{n^2+j}.
\end{aligned}$$

We now let $j' = n^2 + j$, then $j = j' - n^2$. Furthermore, $j' - n^2 \geq 0$, that is $n \leq \lfloor \sqrt{j'} \rfloor$ but $n \geq 1 \Rightarrow \lfloor \sqrt{j'} \rfloor \geq 1$. We let $m_{j'} = \lfloor \sqrt{j'} \rfloor$.

Now we have

$$1+\sum_{j=1}^{\infty} w^j q^j + \sum_{j'=1}^{\infty} q^{j'} \sum_{n=1}^{m_{j'}}((-1)^n w^n+(-1)^n w^{-n}) w^{j'-n^2}$$

$$= 1+\sum_{j=1}^{\infty} w^j q^j + \sum_{j=1}^{\infty} q^{j} \sum_{n=1}^{m_{j}}((-1)^n w^n+(-1)^n w^{-n}) w^{j-n^2}$$

$$= 1+\sum_{j=1}^{\infty} q^j \left(w^j + \sum_{n=1}^{m_j}(-1)^n w^n+(-1)^n w^{-n}) w^{j-n^2}\right)$$

$$= 1+\sum_{j=1}^{\infty} q^j \left(w^j + \sum_{n=1}^{m_j}(-1)^n w^{j-n(n-1)} + \sum_{n=1}^{m_j}(-1)^n w^{j-n(n+1)}\right)$$

$$= 1+\sum_{j=1}^{\infty} q^j \left(w^j - w^j + \sum_{n=2}^{m_j}(-1)^n w^{j-n(n-1)} + \sum_{n=1}^{m_j}(-1)^n w^{j-n(n+1)}\right)$$

$$= 1 + \sum_{j=1}^{\infty} q^j \left(\sum_{n=1}^{m_j - 1} (-1)^{n+1} w^{j-n(n+1)} + \sum_{n=1}^{m_j} (-1)^n w^{j-n(n+1)} \right)$$

$$= 1 + \sum_{j=1}^{\infty} q^j (-1)^{m_j} w^{j-m_j(m_j+1)}.$$

Hence we have

$$\prod_{n=0}^{\infty}(1 - wq^{2n+1}) \prod_{n=1}^{\infty}(1 - q^{2n})(1 - w^{-1}q^{2n-1}) = 1 + \sum_{j=1}^{\infty} q^j (-1)^{m_j} w^{j-m_j(m_j+1)}.$$

Replacing w with q^{-1}, we get

$$1 + \sum_{j=1}^{\infty} q^j (-1)^{m_j} q^{-j+m_j(m_j+1)}$$

$$= 1 + \sum_{j=1}^{\infty} (-1)^{m_j} q^{m_j(m_j+1)}$$

$$= 1 + \sum_{j=1}^{\infty} a_{f(j)} \ , \qquad a_{f(j)} = (-1)^{f(j)} q^{f(j)(f(j)+1)}, \ \ f(j) = \lfloor \sqrt{j} \rfloor.$$

$$= 1 + \sum_{k=1}^{\infty} a_k |\{r \geq 1 : \lfloor \sqrt{r} \rfloor = k\}|, \quad a_k = (-1)^k q^{k(k+1)}, k = \lfloor \sqrt{r} \rfloor \text{ for some } r.$$

$$= 1 + \sum_{k=1}^{\infty} a_k |\{r \geq 1 : k^2 \leq r < (k+1)^2\}|$$

$$= 1 + \sum_{k=1}^{\infty} a_k |\{r \geq 1 : k^2 \leq r \leq (k+1)^2 - 1\}|$$

$$= 1 + \sum_{k=1}^{\infty} a_k (((k+1)^2 - 1) - k^2 + 1)$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k q^{k(k+1)} (2k+1),$$

and also,

$$\prod_{n=1}^{\infty}(1-q\cdot q^{2n-1})(1-q^{-1}\cdot q^{2n+1})(1-q^{2n}) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n})(1-q^{2n})$$
$$= \prod_{n=1}^{\infty}(1-q^{2n})^3.$$

Thus we have

$$1+\sum_{k=1}^{\infty}(-1)^k(2k+1)q^{k(k+1)} = \prod_{n=1}^{\infty}(1-q^{2n})^3. \qquad \text{(by (3.1.7))}$$

Replacing q by $q^{1/2}$, we obtain the desired result. □

3.2 The Euler's Pentagonal Number Theorem

In this section, we prove the EPNT and look at some of its implications.

Theorem 3.2.1 (Euler's Pentagonal Number Theorem, [5])**.**

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{n(3n+1)}{2}}+q^{\frac{n(3n-1)}{2}}\right) \tag{3.2.1}$$

Proof. Setting $q := q^3$ and $w := -q^{-1}$ in Theorem 3.1.1, we obtain

$$\sum_{n=-\infty}^{\infty}(-q^{-1})^n q^{\frac{3n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^{3n})(1+(-q)^{-1}q^{3n})(1+((-q)^{-1})^{-1}q^{3(n-1)}),$$

i.e.

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{-n}q^{\frac{3n^2+3n}{2}} = \prod_{n=1}^{\infty}(1-q^{3n})(1-q^{3n-1})(1-q^{3n-2})$$
$$= \prod_{n=1}^{\infty}(1-q^n).$$

The left-hand side is

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{-n} q^{\frac{3n^2+3n}{2}} &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} \\ &= \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}} + \sum_{n=0}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}} + \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n (q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}}).\end{aligned}$$

□

Remark 3.2.1. *Numbers of the form* $\frac{n(3n\pm1)}{2}$, $n \geq 0$ *are called pentagonal numbers.*

EPNT has a partition-theoretic interpretation. We discuss Legendre's interpretation in the subsequent section.

3.3 Partitition-theoretic interpretation of EPNT

Recall that $d_e(n)$ (resp. $d_o(n)$) is the number of $d(n)$-partitions of n into an even (resp. odd) number of parts. Legendre gave the following interpretation of EPNT.

Theorem 3.3.1. *For* $n \geq 0$, *we have*

$$d_e(n) - d_o(n) = \begin{cases}(-1)^j, & n = \frac{j(3j\pm1)}{2}, j = 0, 1, 2, \ldots \\ 0, & \textit{otherwise.}\end{cases}$$

Proof. We first show that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=0}^{\infty}(d_e(n) - d_o(n))q^n. \tag{3.3.1}$$

We may write

$$\begin{aligned}1 - q^n &= (-1)^0 (q^n)^0 + (-1)^1 (q^n)^1 \\ &= \sum_{i=0}^{1}(-1)^i (q^n)^i,\end{aligned}$$

so that

$$\prod_{n=1}^{\infty}(1-q^n) = \prod_{n=1}^{\infty}\sum_{i=0}^{1}(-1)^i(q^n)^i$$

$$= \left(\sum_{m_1=0}^{1}(-1)^{m_1}q^{m_1}\right)\left(\sum_{m_2=0}^{1}(-1)^{m_2}q^{2m_2}\right)\left(\sum_{m_3=0}^{1}(-1)^{m_3}q^{3m_3}\right)\ldots$$

$$= \sum_{m_1=0}^{1}\sum_{m_2=0}^{1}\sum_{m_3=0}^{1}\ldots\left((-1)^{m_1+m_2+m_3\cdots}q^{m_1+2m_2+3m_3\cdots}\right).$$

Consider the exponent of q, $m_1+2m_2+3m_3+\ldots$ and $m_1+m_2+m_3+\ldots$ where $m_i = 0$ or 1 for all i. This exponent $m_1+2m_2+3m_3+\ldots$ can be viewed as a partition into distinct parts. If the number of parts of the partition, i.e. $m_1+m_2+m_3+\ldots$ is even, then the coefficient is +1. Otherwise, the coefficient is -1. This means that the expansion is indeed the generating function for $\sum_{n=0}^{\infty}(d_e(n)-d_o(n))q^n$.

Finally, we look at the sum

$$1+\sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2-n}{2}}+q^{\frac{3n^2+n}{2}}).$$

Set $1+\sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2-n}{2}}+q^{\frac{3n^2+n}{2}}) = \sum_{n=0}^{\infty}a_nq^n$.

Then it is clear that

$$a_n := \begin{cases}(-1)^j, & n=\frac{j(3j\pm1)}{2},\ j\geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{3.3.2}$$

Using Theorem 3.2.1, the theorem follows. □

3.4 MacMahon's recurrence for $p(n)$

Clearly

$$\prod_{n=1}^{\infty}(1-q^n)\cdot\frac{1}{\prod_{m=1}^{\infty}(1-q^m)} = 1$$

implies that

$$\sum_{n=0}^{\infty} a_n q^n \sum_{n=0}^{\infty} p(n) q^n = 1$$

where a_n is defined in (3.3.2). By Cauchy's multiplication, we have

$$\sum_{n=0}^{\infty} a_n q^n \sum_{n=0}^{\infty} p(n) q^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k p(n-k) q^n.$$

Hence

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k p(n-k) q^n = 1 + 0q^1 + 0q^2 + \dots.$$

Comparing the coefficients of q^n yields

$$p(0) = 1 \text{ and } \sum_{k=0}^{n} a_k p(n-k) = 0.$$

We have

$$0 = \sum_{k=0,\, k \text{ pentagonal}}^{n} a_k p(n-k)$$

$$= p(n) \; + \sum_{k=1, k \text{ pentagonal}}^{n} a_k p(n-k)$$

$$= p(n) \; + \sum_{j=1}^{n} (-1)^j \left(p(n - \frac{j(3j+1)}{2}) + p(n - \frac{j(3j-1)}{2}) \right).$$

Thus we obtain the following recurrence, due to P. A. MacMahon:

$$p(n) = \sum_{j=1}^{n} (-1)^{j+1} \left(p(n - \frac{j(3j-1)}{2}) + p(n - \frac{j(3j+1)}{2}) \right)$$

$$= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) \dots .$$

Example 3.4.1. *Computing $p(n)$ for $n = 1, 2, 3, 4$ using MacMahon's recurrence.*

$$\begin{aligned} p(1) &= p(1-2) + p(1-1) - p(1-7) + ... \\ &= p(-1) + p(0) - p(-6) + ... \\ &= 0 + 1 - 0 + 0... \\ &= 1. \end{aligned}$$

$$\begin{aligned} p(2) &= p(2-2) + p(2-1) - p(2-7) - p(2-5) + ... \\ &= p(0) + p(1) - p(-5) - p(-3) + ... \\ &= 1 + 1 - 0 - 0 + ... \\ &= 2. \end{aligned}$$

$$\begin{aligned} p(3) &= p(3-2) + p(3-1) - p(3-7) + ... \\ &= p(1) + p(2) - p(-4) + ... \\ &= 1 + 2 - 0 + 0 ... \\ &= 3. \end{aligned}$$

$$\begin{aligned} p(4) &= p(4-2) + p(4-1) - p(4-7) - p(4-5) + ... \\ &= p(2) + p(3) - p(-3) - p(-1) + ... \\ &= 2 + 3 - 0 - 0 + 0 ... \\ &= 5. \end{aligned}$$

3.5 A recurrence for the function $d(n)$

In this section, we derive a known recurrence for the restricted partition function $d(n)$: the number of partitions of n into distinct parts [1]. We know that, (see page 7),

$$\sum_{n=0}^{\infty} d(n)q^n = \prod_{n=1}^{\infty}(1+q^n).$$

Using the fact that

$$\prod_{n=1}^{\infty}(1+q^n)(1-q^n) = \prod_{n=1}^{\infty}(1-q^{2n}),$$

we have

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}d(n)q^n=\prod_{n=1}^{\infty}(1-q^{2n})$$

which implies

$$\left(\sum_{n=0}^{\infty}a_nq^n\right)\left(\sum_{n=0}^{\infty}d(n)q^n\right)=\sum_{n=0}^{\infty}a_nq^{2n} \quad = \quad \sum_{n=0}^{\infty}\bar{a}_nq^n$$

where a_n is defined in (3.3.2) and $\bar{a}_n$ is defined as follows

$$\bar{a}_n=\begin{cases}(-1)^t, & n=t(3t\pm1),\ t\geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{3.5.1}$$

We then have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_kd(n-k)q^n=\sum_{n=0}^{\infty}\bar{a}_nq^n$$

which upon comparing the coefficients of q^n, gives

$$\sum_{k=0}^{n}a_kd(n-k)=\bar{a}_n.$$

So,

$$\sum_{k=0,\ k=\frac{i(3i\pm1)}{2}}^{n}a_kd(n-k) \qquad + \sum_{k=0,\ k\neq\frac{i(3i\pm1)}{2}}^{n}a_kd(n-k)$$

$$= \begin{cases}(-1)^t, & n=t(3t\pm1),\ t\geq 0\\ \\ 0, & \text{otherwise,}\end{cases}$$

i.e.

$$d(n)+\sum_{i=1}^{n}(-1)^i d\left(n-\frac{i(3i\pm1)}{2}\right)=\begin{cases}(-1)^t, & n=t(3t\pm1),\ t\geq 0\\ \\ 0, & \text{otherwise.}\end{cases}$$

Thus we have

$$d(n)=\begin{cases}(-1)^t+\sum_{i=1}^{n}(-1)^{i+1}d\left(n-\frac{i(3i\pm1)}{2}\right), & n=t(3t\pm1),\ t\geq 0\\ \sum_{i=1}^{n}(-1)^{i+1}d\left(n-\frac{i(3i\pm1)}{2}\right), & \text{otherwise,}\end{cases}$$

where $d(0) = 1$.

Example 3.5.1. *For $n = 7$ we have*

$$\begin{aligned} d(7) &= d(6) + d(5) - d(2) - d(0) \\ &= [d(5) + d(4) - d(1)] + d(5) - d(2) - d(0) \\ &= [2(d(4) + d(3) - d(0))] + d(4) - d(2) - d(1) - d(0) \\ &= [3(d(3) + d(2) - 1)] + 2d(3) - d(2) - d(1) - 3d(0) \\ &= [5(d(2) + d(1))] + 2d(2) - d(1) - 3d(0) - 3 \\ &= [7(d(1) + d(0) - 1)] + 4d(1) - 3d(0) - 3 \\ &= [11d(0)] + 4d(0) - 10 \\ &= 15(1) - 10 \\ &= 5. \end{aligned}$$

The 5 partitions of 7 into distinct parts are $(7), (6, 1), (5, 2), (4, 3), (4, 2, 1)$.

In Chapters 4 and 5, we present our contributions to partition theory. We believe that the results in these chapters have not appeared anywhere in the literature. In Chapter 4, we generalise Sylvester's identity (Theorem 2.4.1) and give its explicit bijection. We also generalise Theorem 2.2.4, present combinatorial consequences of Glaisher's bijection, and give parity and recurrence formulas for related partition functions. In Chapter 5, we present new partition identities involving certain restricted Frobenius partitions. We then conclude in Chapter 6.

Chapter 4

Contributions I

In Section 4.1, we generalise Theorem 2.2.4. In Section 4.2, we analyse some special case and give a generalisation of Theorem 2.4.1 and in Section 4.3, we provide combinatorial consequences of Glaisher's bijection. In Section 4.4, we give parity and recurrence formulas for related functions when certain restrictions are imposed on odd or even parts.

4.1 A generalisation of Theorem 2.2.4

For $k \geq 2$, let $d_k(n)$ denote the number of partitions of n into distinct parts not divisible by k, and denote by $o_k(n)$ the number of partitions of n into odd parts occurring not more than $k-1$ times. Then we have the following theorem.

Theorem 4.1.1. *For $n \geq 0$, we have*

$$d_k(n) = o_k(n).$$

Observe that if $k = 3$, we obtain Theorem 2.2.4.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} o_k(n)q^n &= \prod_{n=1}^{\infty}(1 + q^{2n-1} + q^{2(2n-1)} + \ldots + q^{(k-1)(2n-1)}) \\
&= \prod_{n=1}^{\infty} \frac{1-q^{k(2n-1)}}{1-q^{2n-1}} \\
&= \prod_{n=1}^{\infty} \frac{1+q^n}{1+q^{kn}}. \qquad \text{(by (2.2.2))}
\end{aligned}$$

We have

$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{(1+q^{kn})(1+q^{kn-1})(1+q^{kn-2})\dots(1+q^{kn-k+1})}{(1+q^{kn})} \\
= &\prod_{n=1}^{\infty}(1+q^{kn-1})(1+q^{kn-2})(1+q^{kn-3})\dots(1+q^{kn-k+1}) \\
= &\sum_{n=0}^{\infty} d_k(n)q^n.
\end{aligned}$$

□

If we set $k = 2$ in Theorem 4.1.1 we obtain partitions with distinct odd parts which we saw in Section 2.4. These partitions are related to self-conjugate partitions. We analyse this case in the following section.

4.2 The case $k = 2$

It is clear that $d_2(n)$ and $o_2(n)$ are one and the same in descriptions. However partitions enumerated by $o_2(n)$ or $d_2(n)$ are related to self-conjugate partitions in Theorem 2.4.1, which we generalise in this section. We begin by introducing a new class of partitions which we shall call symmetric partitions.

Definition 4.2.1. *For any integer $\gamma \geq 0$, a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$ for some r, is said to be γ-symmetric if*

$$(\lambda_{s+1}, \lambda_{s+2}, \lambda_{s+3}, \dots, \lambda_r) = (s^{\lambda_s - s + \gamma}, (s-1)^{\lambda_{s-1}-\lambda_s}, \dots, 1^{\lambda_1 - \lambda_2})$$

where $s = s(\lambda)$.

We denote the number of γ-symmetric partitions of n by $g(n, \gamma)$ and denote the set by $G(n, \gamma)$.

The γ-symmetric partitions are related to partitions with parts in certain residue class modulo 2. The following theorem demonstrates the relationship.

Theorem 4.2.1. *For $n \geq 0$, $g(n, \gamma)$ is equal to the number of partitions of n into distinct parts congruent to $1 + \gamma \pmod 2$ and the smallest part is at least $1 + \gamma$.*

Proof of Theorem 4.2.1.

$$\begin{aligned}
&\sum_{n=0}^{\infty} g(n,\gamma)q^n \\
&= \sum_{s\geq 0,\ \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_s\geq s} q^{(\lambda_1+\lambda_2+\ldots+\lambda_s+s(\lambda_s-s+\gamma)+(s-1)(\lambda_{s-1}-\lambda_s)+\ldots+1(\lambda_1-\lambda_2)} \\
&= \sum_{s\geq 0,\ \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_s\geq s} q^{\left(\sum_{i=1}^{s}\lambda_i\right)+s\lambda_s-s^2+s\gamma+\sum_{j=1}^{s-1} j(\lambda_j-\lambda_{j+1})} \\
&= \sum_{s\geq 0,\ \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_s\geq s} q^{\left(\sum_{i=1}^{s}\lambda_i\right)-s^2+s\gamma+\sum_{j=1}^{s}\lambda_j} \\
&= \sum_{s\geq 0,\ \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_s\geq s} q^{\left(2\sum_{i=1}^{s}\lambda_i\right)-s^2+s\gamma}.
\end{aligned}$$

But,

$$\begin{aligned}
\lambda_s \geq s \quad &\Rightarrow \lambda_s = s+n_1,\ n_1\geq 0 \\
\lambda_{s-1}\geq\lambda_s \quad &\Rightarrow \lambda_{s-1} = s+n_1+n_2,\ n_1,n_2\geq 0 \\
&\vdots \\
\lambda_1\geq\lambda_2 \quad &\Rightarrow \lambda_1 = s+n_1+n_2+\cdots+n_s, \qquad \forall\, n_i\geq 0.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\sum_{s\geq 0}\sum_{n_s\geq 0,}\sum_{n_{s-1}\geq 0}\cdots\sum_{n_1\geq 0} q^{2s^2}\cdot(q^{2s})^{n_1}\cdot(q^{2(s-1)})^{n_2}\ldots(q^4)^{n_{s-1}}\cdot(q^2)^{n_s} \\
&\qquad \times q^{-s^2+s\gamma} \\
&= \sum_{s=0}^{\infty}\frac{q^{2s^2-s^2+s\gamma}}{(1-q^{2s})(1-q^{2(s-1)})\ldots(1-q^{22})(1-q^2)} \\
&= \sum_{s=0}^{\infty}\frac{q^{s(s+\gamma)}}{(q^2,q^2)_s}.
\end{aligned}$$

We have

$$\sum_{n=0}^{\infty} g(n,\gamma)q^n \quad = \quad \sum_{n=0}^{\infty} \frac{q^{n(n+\gamma)}}{(q^2,q^2)_n}. \tag{4.2.1}$$

Replacing t by $q^{\gamma+1}$ and q by q^2 in (2.3.7) we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)+n(\gamma+1)}}{(q^2,q^2)_n} = \prod_{n=0}^{\infty}(1+q^{2n+(\gamma+1)}).$$

□

4.2.1 A bijection for Theorem 4.2.1

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a γ-symmetric partition of n. Then the mapping

$$\lambda \mapsto \beta = (2\lambda_1 + \gamma - 1, 2\lambda_2 + \gamma - 3, \ldots, 2\lambda_s + \gamma - 2s + 1) \tag{4.2.2}$$

is an explicit bijection from the set of γ-symmetric partitions of n onto the set of partitions of n with parts congruent to $1+\gamma \pmod 2$ and the smallest part is greater than or equal to $1+\gamma$. We claim that the map preserves weight, i.e. $|\lambda| = |\beta|$. To see this, suppose $|\lambda| = n$. Thus

$$\begin{aligned}
&n \\
&= (\lambda_1 + \ldots + \lambda_s + s(\lambda_s - s + \gamma) + (s-1)(\lambda_{s-1} - \lambda_s) + \ldots + 1(\lambda_1 - \lambda_2)) \\
&= \sum_{i=1}^{s} \lambda_i + s(\lambda_s - s + \gamma) + \sum_{j=1}^{s-1} j(\lambda_j - \lambda_{j+1}) \\
&= \sum_{i=1}^{s} \lambda_i + s\lambda_s - s^2 + s\gamma + \sum_{j=1}^{s} \lambda_j - s\lambda_s \\
&= 2\sum_{i=1}^{s} \lambda_i - s^2 + s\gamma \\
&= \sum_{i=1}^{s} (2\lambda_i - (2i-1) + \gamma) \\
&= |\beta|.
\end{aligned}$$

Theorem 4.2.1 is a generalisation of Theorem 2.4.1, and the explicit bijection given above generalises the explicit bijection of Theorem 2.4.1. This

is because self-conjugate partitions are precisely those 0-symmetric partitions. Observe that a 0-symmetric partition λ is such that

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_{s-1}, \lambda_s, s^{(\lambda_s - s)}, (s-1)^{(\lambda_{s-1}-\lambda_s)}, \ldots, 1^{(\lambda_1 - \lambda_2)})$$

and λ_s may be equal to s. If we conjugate λ, we get

$$\begin{aligned}\lambda' = \Bigg(& (s + \lambda_s - s + \sum_{i=1}^{s-1}(\lambda_i - \lambda_{i+1}))^1, (s + \lambda_s - s + \sum_{i=2}^{s-1}(\lambda_i - \lambda_{i+1}))^{2-1}, \ldots \\ & \ldots, \ldots, (s + \lambda_s - s)^{s-(s-1)}, (s)^{\lambda_s - s}, (s-1)^{\lambda_{s-1}-\lambda_s}, \ldots, 1^{\lambda_1 - \lambda_2} \Bigg) \\ = & (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_{s-1}, \lambda_s, s^{(\lambda_s - s)}(s-1)^{(\lambda_{s-1}-\lambda_s)}, \ldots, 1^{(\lambda_1 - \lambda_2)}),\end{aligned}$$

i.e. $\lambda = \lambda'$.

On the other hand, those parts congruent to $1 + \gamma \pmod 2$ and occurring at most once become distinct odd parts. Thus Sylvester's theorem.

Example 4.2.1. *Consider $n = 10$, $\gamma = 1$.*

The following table shows the correspondence for this case

Partitions of 10 with parts $\equiv 0 \pmod 2$	$\mapsto$	1-symmetric partitions of 10.
(10)	$\mapsto$	(1^{10})
$(8, 2)$	$\mapsto$	$(4, 2^2, 1^2)$
$(6, 4)$	$\mapsto$	$(3^2, 2^2)$

4.3 Combinatorial consequences of Glaisher's bijection

In this section we prove that Glaisher's bijection provides a bijective proof of Theorem 4.1.1 where k is a power of 2. Various formulas of partition functions arising from the theorem are derived.

4.3.1 The case $k = 2^r$ where $r \in \mathbb{N}$

Consider a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_l)$ enumerated by $d_k(n)$. We write $\lambda_i = 2^{t_i} \cdot \alpha_i$ where $2 \nmid \alpha_i$ for $i = 1, 2, \ldots, l$. Let

$$\mu = \bigcup_{i=1}^{\ell} \alpha_i^{2^{t_i}}.$$

Note that μ is a partition with parts α_i with multiplicity 2^{t_i}. It is clear that α_i's are odd and if we sum the parts of μ, we get the same weight as of λ, because each part of μ corresponds to the product $2^{t_i} \cdot \alpha_i = \lambda_i$.

We want to show that the multiplicity of α_i is at most $k-1$ times. i.e., $2^{t_i} \leq k-1$.

Case 1: Suppose $\alpha_i \neq \alpha_j$ in μ for $i \neq j$. We assume to the contrary that $2^{t_i} > k-1$. Then $2^{t_i} \geq k$, i.e. $2^{t_i} \geq 2^r$ which implies that $t_i \geq r$.
Therefore $\quad t_i = r+n, \;\; n \geq 0$.
But

$$\begin{aligned} \lambda_i &= \alpha \cdot 2^{t_i} \\ &= \alpha \cdot 2^{r+n_i} \\ &= \alpha \cdot 2^r \cdot 2^{n_i} \end{aligned}$$

which yields, $2^r = k \mid \lambda_i$, a contradiction. Therefore $2^{t_i} \leq k-1$.

Case 2: Suppose that $\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,m}$ in μ are such that $\lambda_{i,j} = 2^{t_{i,j}} \cdot \alpha \;\; \forall j$.

We claim that $\sum_{j=1}^{m} 2^{t_{i,1}} \leq k-1$.
To see that, let $v = \max\{t_{i,j}\}$. By Case 1, $2^{t_{i,j}} < k = 2^r$, i.e. $t_{i,j} < r$.
So $v < r$. Then

$$\begin{aligned} \sum_{j=1}^{m} 2^{t_{i,j}} = \quad & 2^{t_{i,1}} + 2^{t_{i,2}} + \ldots + 2^{t_{i,m}} \\ \leq \quad & 2^v + 2^{v-1} + \ldots + 2^0 \\ = \quad & 2^v - 1 \\ < \quad & 2^v \\ \leq \quad & k-1 \end{aligned}$$

To invert the map, consider a partition of n into odd parts and each part occurs at most $k-1$ times. For each odd part $(2m_i - 1)$, let t_i denote its multiplicity. Thus

$$\beta = t_1(2m_1 - 1) + t_2(2m_2 - 1) + \ldots + t_l(2m_l - l).$$

We write each t_i in its unique binary expansion: $t_i = \sum_{j=1}^{s_i} 2^{r_{i,j}}$.

Let $\lambda = \bigcup_{i=1}^{l} \bigcup_{j=1}^{s_i} (2^{r_{i,j}}(2m_i - 1))$. Clearly $|\lambda| = n$.

A new partition λ is a partition of n with parts $(2^{r_{i,j}}(2m_i-1))$'s. Each part $(2^{r_{i,j}}(2m_i - 1))$ is not divisible by $k = 2^r$ since each $2^{r_{i,j}} \leq k - 1$ because each $t_i \leq k - 1$.
Indeed, the Glaisher bijection provides a bijective proof of Theorem 4.1.1 when k is a power of 2.

Note: If k is not a power of 2, Glaisher's bijection does not necessarily work. See Example 4.3.1.

Example 4.3.1.

Let $k = 5$. Then a partition $(9, 5^3, 3^4, 1^2)$ maps to $(12, 10, 9, 5, 2)$.
But $10, 5$ are divisible by 5.

Corollary 4.3.1. *For $t \geq 2$, let $f(n,t)$ denote the number of partitions of n in which even parts occur exactly t times and odd parts occur at most $2t-1$. Then for $n \geq 0$, we have*

$$f(n,t) = d(n).$$

First proof of Corollary 4.3.1. In terms of generating functions, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n,t)q^n = & \prod_{n=1}^{\infty}(1+q^{2tn})\left(1+q^{(2n-1)}+q^{2(2n-1)}+q^{3(2n-1)}+\right. \\
& \left. \ldots + q^{(2t-1)(2n-1)}\right) \\
= & \prod_{n=1}^{\infty}(1+q^{2tn})\frac{(1-q^{2t(2n-1)})}{(1-q^{2n-1})} \\
= & \prod_{n=1}^{\infty}(1+q^{2tn})\frac{(1+q^n)}{(1+q^{2tn})} \\
= & \prod_{n=1}^{\infty}(1+q^n) \quad = \quad \sum_{n=0}^{\infty} d(n)q^n.
\end{aligned}$$

□

Bijective proof. We want to show a bijection from the set of partitions λ of n enumerated by $f(n,t)$ onto the set of partitions of n enumerated by $d(n)$ in the case when t is a power of 2. Recall Glaisher's bijection ϕ.

Let λ be a partition of n enumerated by $f(n,t)$. Decompose λ as $\lambda = (\lambda_o, \lambda_e)$ where λ_o is the subpartition of λ consisting of all odd parts and λ_e is that subpartition of λ with even parts.

Note that $\lambda_e = (\lambda_1^t, \lambda_2^t, \lambda_3^t, \ldots, \lambda_l^t)$ where $\lambda_i = 2m_i$ for some $m_i \geq 1$ and $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_l$. Then the map

$$\lambda \mapsto \mu = \phi^{-1}(\lambda_o) \cup \left(\bigcup_{i=1}^{l} t\lambda_i \right)$$

is a bijection.

To invert the bijection above, let μ be a partition of n enumerated by $d(n)$. Decompose μ as $\mu = (\mu_1, \mu_2)$ where μ_1 is the subpartition of μ consisting of distinct parts congruent to 0 (mod $2t$) and μ_2 is the subpartition of μ with distinct parts not divisible by $2t$.

Suppose that $\mu_1 = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$. Then the map

$$\mu \mapsto \beta = \phi(\mu_2) \cup \left(\bigcup_{i=1}^{r} \left(\frac{\alpha_i}{t} \right)^t \right)$$

is the inverse.

□

Example 4.3.2. *If $n = 10$ and $t = 2$, then $f(n,t) = 10 = d(n)$.*

Hence the 10 partitions of n enumerated by $f(10,2)$ are;

$$(9,1), \quad (7,3), \quad (7,1^3), \quad (5^2), \quad (5,3,1^2),$$

$$(5,2^2,1), \quad (4^2,1^2), \quad (3^3,1), \quad (3^2,2^2), \quad (3,2^2,1^3).$$

and the 10 partitions of n into distinct parts are;

$$(10), \quad (9,1), \quad (8,2), \quad (7,3), \quad (7,2,1),$$
$$(6,4), \quad (6,3,1), \quad (5,4,1), \quad (5,3,2), \quad (4,3,2,1).$$

4.4 Parity and recurrence formulas

In Theorem 2.2.2, we looked only at partitions with restrictions on odd parts. In this section we investigate what happens if we include certain restrictions on the even parts. We also provide parity and recurrence formulas in some cases.

Let $c(n)$ denote the number of partitions of n in which even parts occur not more than 2 times, and odd parts occur at least 2 times.

Theorem 4.4.1. *For $n \geq 0$, we have*

$$c(n) \equiv \begin{cases} 1 \pmod 2, & n = \frac{3j(3j\pm 1)}{2},\ j \geq 1, \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} c(n)q^n &= \prod_{n=1}^{\infty} \left(1 + q^{2n} + q^{2(2n)}\right)\left(1 + q^{2(2n-1)} + q^{3(2n-1)} + \ldots\right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1-q^{6n}}{1-q^{2n}}\right)\left(1 + \frac{q^{2(2n-1)}}{1-q^{2n-1}}\right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1-q^{6n}}{1-q^{2n}}\right)\left(\frac{1-q^{2n-1}+q^{2(2n-1)}}{1-q^{2n-1}}\right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1-q^{6n}}{1-q^{2n}}\right)\left(\frac{1+q^{3(2n-1)}}{1-q^{2(2n-1)}}\right) \\
&\equiv \prod_{n=1}^{\infty} \left(\frac{(1-q^{3n})^2}{1-q^{2n}}\right)\left(\frac{1-q^{3(2n-1)}}{1-q^{2(2n-1)}}\right) \pmod 2 \quad \text{(by (2.3.2))}
\end{aligned}$$

$$= \prod_{n=1}^{\infty} \left(\frac{1-q^{3n})^2}{1-q^{2n}}\right)\left(\frac{1}{1+q^{3n}}\right)(1+q^{2n}) \pmod 2 \qquad \text{(by (2.2.2))}$$

$$\equiv \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2(1-q^{2n})}{(1-q^{2n})(1-q^{3n})} \pmod 2$$

$$= \prod_{n=1}^{\infty}(1-q^{3n}) \pmod 2$$

$$\equiv \sum_{n=0}^{\infty} q^{\frac{3n(3n\pm 1)}{2}} \pmod 2. \qquad \text{(by (3.2.1) with } q := q^3)$$

Then the result follows. □

Example 4.4.1.

$$\begin{aligned}
c(1) = 0 &\equiv 0 \pmod 2, \quad 1 \neq 3j(3j \pm 1)/2.\\
c(2) = 2 &\equiv 0 \pmod 2.\\
c(3) = 1 &\equiv 1 \pmod 2, \quad 3 = 3(1)(3(1)-1)/2.\\
c(4) = 4 &\equiv 0 \pmod 2.\\
c(5) = 2 &\equiv 0 \pmod 2.\\
c(6) = 7 &\equiv 1 \pmod 2, \quad 6 = 3(1)(3(1)+1)/2.\\
c(7) = 4 &\equiv 0 \pmod 2.
\end{aligned}$$

Interchanging the multiplicity roles of even and odd parts in $c(n)$, we get $\bar{c}(n)$ which denotes the number of partitions of n in which odd parts occur at most 2 times, and even parts occur at least 2 times. Then we find a recurrence formula for parity of $\bar{c}(n)$. Theorem 4.4.2 reveals this.

Theorem 4.4.2. *For $n \geq 0$, we have*

$$\bar{c}(n) \equiv \begin{cases} 1 + \sum_{j=1,}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} \bar{c}(n - \frac{j(j+1)}{2}) \pmod 2, & n = \frac{3r(3r\pm 1)}{2},\ r \geq 1, \\ \sum_{j=1,}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} \bar{c}(n - \frac{j(j+1)}{2}) \pmod 2, & \textit{otherwise.} \end{cases}$$

We define $\bar{c}(0) := 1$.

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{c}(n) q^n &= \prod_{n=1}^{\infty} \left(1 + q^{2(2n)} + q^{3(2n)} + \ldots\right) \left(1 + q^{2n-1} + q^{2(2n-1)}\right) \\ &= \prod_{n=1}^{\infty} \left(1 + \frac{q^{4n}}{1-q^{2n}}\right) \left(\frac{1-q^{6n-3}}{1-q^{2n-1}}\right) \\ &= \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}+q^{4n}}{1-q^{2n}}\right) \left(\frac{1-q^{6n-3}}{1-q^{2n-1}}\right) \\ &= \prod_{n=1}^{\infty} \left(\frac{1+q^{6n}}{1-q^{4n}}\right) \left(\frac{1}{1+q^{3n}}\right) (1+q^n) \\ &\equiv \prod_{n=1}^{\infty} \left(\frac{1-q^{6n}}{1-q^{4n}}\right) \left(\frac{1-q^n}{1-q^{3n}}\right) \pmod 2 \\ &\equiv \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2 (1-q^n)}{(1-q^n)^4 (1-q^{3n})} \pmod 2 \\ &= \prod_{n=1}^{\infty} \frac{1-q^{3n}}{(1-q^n)^3} \pmod 2. \end{aligned}$$

$$\text{Thus} \quad \prod_{n=1}^{\infty} (1-q^n)^3 \cdot \sum_{n=0}^{\infty} \bar{c}(n) q^n \equiv \prod_{n=1}^{\infty} (1-q^{3n}) \pmod 2,$$

$$\text{and hence} \quad \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \cdot \sum_{n=0}^{\infty} \bar{c}(n) q^n \equiv \sum_{n=0}^{\infty} q^{\frac{3n(3n\pm 1)}{2}} \pmod 2.$$

(by (3.1.6)) and (by (3.2.1) with $q := q^3$)

We have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\bar{c}(n-k)q^n \equiv \sum_{n=0}^{\infty} q^{\frac{3n(3n\pm1)}{2}} \pmod 2,$$

$$\text{where} \qquad a_k \equiv \begin{cases} 1 \pmod 2, & k=\frac{j(j+1)}{2},\ j\geq 1, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

$$\text{Thus} \qquad \sum_{k=0,\ k=\frac{j(j+1)}{2}}^{n} a_k\bar{c}(n-k) + \sum_{k=0,\ k\neq\frac{j(j+1)}{2}}^{n} a_k\bar{c}(n-k)$$

$$\equiv \begin{cases} 1 \pmod 2, & n=\frac{3r(3r\pm1)}{2},\ r\geq 1, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

We have

$$\bar{c}(n) + \sum_{j=1,}^{\lfloor\frac{\sqrt{8n+1}-1}{2}\rfloor} \bar{c}(n-\frac{j(j+1)}{2}) \equiv \begin{cases} 1 \pmod 2, & n=\frac{3r(3r\pm1)}{2},\ r\geq 1, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

□

Example 4.4.2. *For $n=4$, $t=1$*

$$\begin{aligned} \bar{c}(4) &\equiv \bar{c}(3) + \bar{c}(1) \pmod 2 \\ &\equiv 1 + \bar{c}(2) + \bar{c}(0) + \bar{c}(1) \pmod 2 \\ &\equiv 1 + \bar{c}(1) + \bar{c}(1) + \bar{c}(0) \pmod 2 \\ &\equiv 1 + 2\bar{c}(1) + \bar{c}(0) \pmod 2 \\ &\equiv 1 + \bar{c}(0) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

But $\bar{c}(4)=2$ because the partitions of $\bar{c}(4)$ are $(3,1),(2,2)$.

Let $s(n)$ denote the number of partitions of n in which odd parts occur at most 3 times and even parts are distinct.

Theorem 4.4.3. *For $n \geq 0$, we have*

$$p(n) \equiv s(n) \pmod 2.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} s(n)q^n &= \prod_{n=1}^{\infty} \left(1 + q^{2n-1} + q^{2(2n-1)} + q^{3(2n-1)}\right)(1+q^{2n}) \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{4(2n-1)})}{(1-q^{2n-1})}(1+q^{2n}) \\
&= \prod_{n=1}^{\infty} \frac{1}{(1+q^{4n})}(1+q^n)(1+q^{2n}) \\
&\equiv \prod_{n=1}^{\infty} \frac{(1+q^n)(1+q^n)^2}{(1+q^n)^4} \pmod 2 \\
&= \prod_{n=1}^{\infty} \frac{1}{1+q^n} \pmod 2 \\
&\equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} \pmod 2 \\
&= \sum_{n=0}^{\infty} p(n)q^n \pmod 2. \qquad \text{(by (2.2.1))}
\end{aligned}$$

□

Corollary 4.4.1. *The number of partitions of n in which at least one even part is repeated or there is an odd part appearing at least 4 times is always even.*

Proof. Let $t(n)$ denote the number of partitions of n in which at least one even part is repeated or there is an odd part occurring at least 4 times. Thus

$$\begin{aligned}
t(n) &= p(n) - s(n) \\
&\equiv 0 \pmod 2.
\end{aligned}$$

□

Theorem 4.4.4. *For $n \geq 1$, we have*

$$s(n) = \begin{cases} 2(-1)^t + \sum_{j=1,}^{\lfloor \frac{\sqrt{24n+1}+1}{6} \rfloor} (-1)^{j+1} s(n - \frac{j(3j\pm 1)}{2}), & n = 4t^2,\ t \geq 1, \\ \sum_{j=1,}^{\lfloor \frac{\sqrt{24n+1}+1}{6} \rfloor} (-1)^{j+1} s(n - \frac{j(3j\pm 1)}{2}), & \textit{otherwise.} \end{cases}$$

We define $s(n) := 0$ for all negative integers n and $s(0) := 1$.

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} s(n) q^n &= \prod_{n=1}^{\infty} \left(1 + q^{2n-1} + q^{2(2n-1)} + q^{3(2n-1)}\right)(1 + q^{2n}) \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{4(2n-1)})(1 + q^{2n})}{(1 - q^{2n-1})} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{4(2n-1)})(1 + q^{2n})}{(1 - q^{2n-1})} \frac{(1 - q^{2n})}{(1 - q^{2n})} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{(8n-4)})(1 - q^{4n})}{(1 - q^{n})}. \end{aligned}$$

We have

$$\prod_{n=1}^{\infty}(1 - q^n) \sum_{n=0}^{\infty} s(n) q^n = \prod_{n=1}^{\infty}(1 - q^{8n})(1 - q^{8n-4})(1 - q^{8n-4}),$$

$$\text{i.e.,} \quad \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(3n\pm 1)}{2}} \cdot \sum_{n=0}^{\infty} s(n) q^n = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{4n^2}.$$

(by (3.1.6)) and (by (3.1.2) with $w : -q^{-4}$ and $q := q^8$)

Hence we have

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n} a_k s(n-k) q^n = 2\sum_{n=1}^{\infty}(-1)^n q^{4n^2},\ a_k = \begin{cases} (-1)^j, & k = \frac{j(3j\pm 1)}{2},\ j \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{k=0,\ k=\frac{j(3j\pm 1)}{2}}^{n} a_k s(n-k) + \sum_{k=0,\ k\neq\frac{j(3j\pm 1)}{2}}^{n} a_k s(n-k) = \begin{cases} 2(-1)^t, & n = 4t^2,\ t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$s(n) + \sum_{j=1,}^{\lfloor \frac{\sqrt{24n+1}+1}{6} \rfloor} (-1)^j s(n - \frac{j(3j \pm 1)}{2}) = \begin{cases} 2(-1)^t, & n = 4t^2,\ t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

□

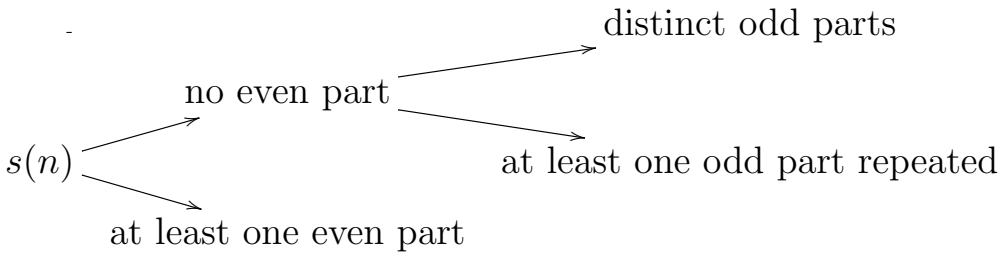
Example 4.4.3. *For $n = 5$ and $t = 1$, we have*

$$\begin{aligned} s(5) &= s(4) + s(3) - s(0) \\ &= [-2 + s(3) + s(2)] + s(3) - s(0) \\ &= -2 + [s(2) + s(1)] + s(2) + [s(2) + s(1)] - s(0) \\ &= -2 + [3(s(1) + s(0))] + 2s(1) - s(0) \\ &= -2 + 3(s(0) + s(0)) + 2s(0) - s(0) \\ &= 5. \end{aligned}$$

And the 5 partitions of $s(5)$ are $(5), (4,1), (3,2), (3,1^2), (2,1^3)$.

4.4.1 Certain $s(n)$-partitions

In this section we look at certain subsets of the set of partitions of n enumerated by $s(n)$ and we refer to such partitions as $s(n)$-partitions of n. We observe that $s(n)$ may be split in such a way that 'no even part occurs, but either distinct odd parts occur or at least one odd part is repeated' and 'at least an even part occurs'. The diagram below illustrates the splitting. In the diagram, the arrow points towards a subset of partitions with the described condition.



If we let $a_1(n)$ denote the number of $s(n)$-partitions of n in which no even part occurs and at least one odd part is repeated, and $a_2(n)$ be the number

of $s(n)$-partitions of n containing at least an even part, we observe that

$$\sum_{n=0}^{\infty} s(n)q^n = \prod_{n=1}^{\infty}(1+q^{2n-1}) + \sum_{n=0}^{\infty} a_1(n)q^n + \sum_{n=0}^{\infty} a_2(n)q^n. \tag{4.4.1}$$

Theorem 4.4.5. *For $n \geq 1$, we have*

$$a_1(n) \equiv \begin{cases} 1 + \sum\limits_{j=1,}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} a_1(n - \frac{j(j+1)}{2}) \pmod 2, & n = \frac{t(3t\pm 1)}{2},\ t \geq 1, \\ \sum\limits_{j=1,}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} a_1(n - \frac{j(j+1)}{2}) \pmod 2, & \textit{otherwise.} \end{cases}$$

Where $a_1(0) \equiv 0 \pmod 2$.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_1(n)q^n &= \prod_{n=1}^{\infty}\left(1+q^{2n-1}+q^{2(2n-1)}+q^{3(2n-1)}\right) - \prod_{n=1}^{\infty}(1+q^{2n-1}) \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{4(2n-1)})}{(1-q^{2n-1})} - \prod_{n=1}^{\infty}(1+q^{2n-1}) \\
&\equiv \prod_{n=1}^{\infty}\frac{(1-q^{n})}{(1-q^{4n})} - \prod_{n=1}^{\infty}(1-q^{2n-1}) \pmod 2 \\
&\equiv \prod_{n=1}^{\infty}\frac{1}{(1-q^{n})^3} - \prod_{n=1}^{\infty}\frac{1}{(1+q^{n})} \pmod 2 \qquad \text{(by (2.2.2))} \\
&\equiv \prod_{n=1}^{\infty}\frac{1}{(1-q^{n})^3} + \prod_{n=1}^{\infty}\frac{1}{(1-q^{n})} \pmod 2 \\
&\equiv \frac{1+\prod\limits_{n=1}^{\infty}(1-q^n)^2}{\prod\limits_{n=1}^{\infty}(1-q^n)^3} \pmod 2.
\end{aligned}$$

Hence we have

$$\prod_{n=1}^{\infty}(1-q^n)^3 \cdot \sum_{n=0}^{\infty} a_1(n)q^n \equiv 1 + \prod_{n=1}^{\infty}(1-q^{2n}) \pmod 2,$$

i.e., $$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \cdot \sum_{n=0}^{\infty} a_1(n)q^n \equiv \sum_{n=1}^{\infty} q^{n(3n\pm 1)} \pmod 2.$$

(by (3.1.6)) and (by (3.2.1) with $q := q^2$)

Thus

$$a_1(0) + \sum_{n=1}^{\infty}\sum_{k=0}^{n} a_k \cdot a_1(n-k)q^n \equiv \sum_{n=1}^{\infty} q^{n(3n\pm 1)} \pmod 2,$$

$$a_k \equiv \begin{cases} 1 \pmod 2, & k = \frac{j(j+1)}{2},\ j \geq 1, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

We have $$\sum_{k=0,\, k=\frac{j(j+1)}{2}}^{n} a_k \cdot a_1(n-k) + \sum_{k=0,\, k\neq\frac{j(j+1)}{2}}^{n} a_k \cdot a_1(n-k)$$

$$\equiv \begin{cases} 1 \pmod 2, & n = t(3t \pm 1),\ t \geq 1, \\ \\ 0 \pmod 2, & \text{otherwise.} \end{cases}, \quad a_1(0) \equiv 0 \pmod 2.$$

Thus

$$a_1(n) + \sum_{j=1,}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} a_1\left(n - \frac{j(j+1)}{2}\right) \equiv \begin{cases} 1 \pmod 2, & n = \frac{t(3t\pm 1)}{2} \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

□

Theorem 4.4.6. *For $n \geq 1$, we have*

$$a_2(n) \equiv a_1(n) \pmod 2.$$

Proof. Note that

$$\begin{aligned} \sum_{n=0}^{\infty} s(n)q^n &= \prod_{n=1}^{\infty}(1+q^{2n-1}) + \sum_{n=0}^{\infty} a_1(n)q^n + \sum_{n=0}^{\infty} a_2(n)q^n \quad \text{(by (4.4.1))} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1+q^n)} + \sum_{n=0}^{\infty} a_1(n)q^n + \sum_{n=0}^{\infty} a_2(n)q^n \\ &\equiv \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} + \sum_{n=0}^{\infty} a_1(n)q^n + \sum_{n=0}^{\infty} a_2(n)q^n \pmod 2 \\ &\equiv \sum_{n=0}^{\infty} p(n)q^n + \sum_{n=0}^{\infty} a_1(n)q^n + \sum_{n=0}^{\infty} a_2(n)q^n \pmod 2. \end{aligned}$$

$$\text{We have} \qquad \sum_{n=0}^{\infty}(a_1(n)+a_2(n))q^n \equiv \sum_{n=0}^{\infty}(s(n)-p(n))q^n \pmod 2,$$

$$\text{i.e.} \qquad a_1(n)+a_2(n) \equiv 0 \pmod 2 \qquad \text{(by Theorem 4.4.3)},$$

$$a_1(n) \equiv a_2(n) \pmod 2.$$

□

Interchanging the multiplicity roles of even parts and odd parts in $s(n)$, we get $\bar{s}(n)$, which is the number of partitions of n in which even parts occur at most three times and odd parts are distinct.

Theorem 4.4.7. *For $n \geq 0$, we have*

$$\bar{s}(n) = \begin{cases} (-1)^t + \sum\limits_{j=1,}^{\lfloor \frac{\sqrt{24n+1}+1}{6} \rfloor} (-1)^{j+1}\bar{s}(n-\frac{j(3j\pm 1)}{2}), & n = 2t(2t\pm 1),\ t \geq 1, \\ \sum\limits_{j=1,}^{\lfloor \frac{\sqrt{24n+1}+1}{6} \rfloor} (-1)^{j+1}\bar{s}(n-\frac{j(3j\pm 1)}{2}), & \textit{otherwise.} \end{cases}$$

We define $\bar{s}(0) = 1$.

Proof.

$$\sum_{n=0}^{\infty}\bar{s}(n)q^n = \prod_{n=1}^{\infty}\left(1+q^{2n}+q^{2(2n)}+q^{3(2n)}\right)(1+q^{2n-1})$$

$$= \prod_{n=1}^{\infty}\frac{(1-q^{4(2n)})}{(1-q^{2n})}(1+q^{2n-1})$$

$$= \prod_{n=1}^{\infty}\frac{(1-q^{8n})(1-q^{4n-2})}{(1-q^n)}.$$

We have

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\bar{s}(n)q^n = \prod_{n=1}^{\infty}(1-q^{8n})(1-q^{8n-2})(1-q^{8n-6})$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(3n\pm1)}{2}}\cdot\sum_{n=0}^{\infty}\bar{s}(n)q^n = \sum_{n=0}^{\infty}(-1)^n q^{2n(2n\pm1)}.$$

(by Theorem (3.2.1)) and (by (3.1.2) with $w : -q^{-2}$ and $q := q^8$)

Now we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k\bar{s}(n-k)q^n = \sum_{n=0}^{\infty}(-1)^n q^{2n(2n\pm1)},$$

$$\text{where} \qquad a_k = \begin{cases}(-1)^j, & k=\frac{j(3j\pm1)}{2},\ j\geq 1,\\ \\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\sum_{k=0,\ k=\frac{j(3j\pm1)}{2}}^{n} a_k\bar{s}(n-k) \quad + \sum_{k=0,\ k\neq\frac{j(3j\pm1)}{2}}^{n} a_k\bar{s}(n-k)$$

$$= \begin{cases}(-1)^t, & n=2t(2t\pm1),\ t\geq 1,\\ 0, & \text{otherwise,}\end{cases}$$

i.e.,

$$\bar{s}(n)+\sum_{j=1,}^{\lfloor\frac{\sqrt{24n+1}+1}{6}\rfloor}(-1)^j\bar{s}(n-\frac{j(3j\pm1)}{2}) = \begin{cases}(-1)^j t & n=2t(2t\pm1),\ t\geq 1,\\ 0, & \text{otherwise.}\end{cases}$$

□

Example 4.4.4. *Consider* $n = 6$.

In this case, $t = 1$. We have

$$\begin{aligned}
\bar{s}(6) &= (-1) + \bar{s}(5) + \bar{s}(4) - \bar{s}(1) \\
&= -1 + [\bar{s}(4) + \bar{s}(3) - \bar{s}(0)] + \bar{s}(4) - \bar{s}(1) \\
&= -1 + [2(\bar{s}(3) + \bar{s}(2))] + \bar{s}(3) - \bar{s}(1) - \bar{s}(0) \\
&= -1 + [3(\bar{s}(2) + \bar{s}(1))] + 2\bar{s}(2) - \bar{s}(1) - \bar{s}(0) \\
&= -1 + [5(-1 + \bar{s}(1) + \bar{s}(0))] + 2\bar{s}(1) - \bar{s}(0) \\
&= -6 + 11\bar{s}(0) \\
&= 5.
\end{aligned}$$

And the 5 partitions of $\bar{s}(6)$ are; $(6), (5,1), (4,2), (3,2,1), (2^3)$.

Chapter 5

Contributions II

Recall that

$$F(n) = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_s \end{pmatrix} : a_s, b_s \geq 0, a_i > a_{i+1},\ b_i > b_{i+1},\ \forall i, \right.$$
$$\left. n = s + \sum_{i=1}^{s} (a_i + b_i) \right\},$$

and $f(n) = |F(n)|$.

Similarly, we introduce the following notation:

$$F_{e,e}(n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_s \\ b_1 & b_2 & b_3 & \dots & b_s \end{pmatrix} :\ a_i, b_i \equiv 0 \pmod 2\ \forall i \right\} \subseteq F(n),$$

and $f_{e,e}(n) = |F_{e,e}(n)|$.

$$F_{o,o}(n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_s \\ b_1 & b_2 & b_3 & \dots & b_s \end{pmatrix} :\ a_i, b_i \equiv 1 \pmod 2\ \forall i \right\} \subseteq F(n),$$

and $f_{o,o}(n) = |F_{o,o}(n)|$.

$$F_{o,e}(n) = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_s \end{pmatrix} :\ a_i \equiv 1 \pmod 2,\ b_i \equiv 0 \pmod 2\ \forall i \right\} \subseteq F(n),$$

and $f_{o,e}(n) = |F_{o,e}(n)|$.

Note: $f_{o,e}(n) = f_{e,o}(n)$, since the mapping $\eta : F_{o,e}(n) \mapsto F_{e,o}(n)$ is defined by

$$\begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_s \end{pmatrix} \mapsto \begin{pmatrix} b_1 & b_2 & \dots & b_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}.$$

We denote the number of partitions of n into distinct parts $\equiv 1 \pmod 4$ by $d^1(n,4)$. Hence

$$\sum_{n=0}^{\infty} d^1(n,4)q^n = \prod_{n=0}^{\infty}(1+q^{4n+1}).$$

We denote the number of partitions of n into distinct parts $\equiv 3 \pmod 4$ by $d^3(n,4)$. Hence

$$\sum_{n=0}^{\infty} d^3(n,4)q^n = \prod_{n=0}^{\infty}(1+q^{4n+3}).$$

5.1 F-partitions

Motivated by the properties of Frobenius partitions, we impose restrictions on parts for such partitions and derive new partition identities. We shall call the Frobenius partitions the F-partitions. We also give a geometric interpretation of the given explicit bijection of Theorem 4.2.1, which is an extension of the known geometric interpretation of Sylvester's theorem.

Theorem 5.1.1. *For $n \geq 0$, we have*

$$f_{e,o}(2n) = p(n).$$

First proof of Theorem 5.1.1. We first give a proof in terms of generating functions.

In a similar manner as done in the proof of (2.4.2)

$$\sum_{n\geq 0} f_{o,e}(n)q^n$$
$$= \sum_{s\geq 0,\ \alpha_1>\alpha_2>\ldots>\alpha_s\geq 0,\ \beta_1>\beta_2>\ldots>\beta_s\geq 1} q^s \cdot q^{2\alpha_1+2\alpha_2+\ldots+2\alpha_s} \cdot q^{2\beta_1+2\beta_2+\ldots+2\beta_s-s}$$

$$= \sum_{s\geq 0,\ n_1\geq 0,\ n_2\geq 0} \cdots \sum_{n_s\geq 0,\ m_1\geq 0,\ m_2\geq 0} \cdots \sum_{m_s\geq 0} q^{(2s)n_1+2(s-1)n_2+\ldots+2n_s} q^{2(1+2+\ldots+(s-1))}$$

$$\times\, q^{(2s)m_1+2(s-1)m_2+\ldots+2m_s} \cdot q^{2(1+2+\ldots+(s+1))}$$

$$= \sum_{s\geq 0} q^{s(s-1)} \cdot q^{s(s+1)} \cdot \sum_{n_1\geq 0} q^{(2s)n_1} \cdot \sum_{n_2\geq 0} q^{2(s-1)n_2} \cdots \sum_{n_s\geq 0} q^{2n_s} \cdot \sum_{m_1\geq 0} q^{(2s)m_1}$$

$$\times \sum_{2m_2\geq 0} q^{2(s-1)m_2} \cdots \sum_{m_s\geq 0} q^{2m_s}$$

$$= \sum_{s\geq 0} q^{2s^2} \cdot \frac{1}{(1-q^{2s})^2} \cdot \frac{1}{(1-q^{2(s-1)})^2} \cdots \frac{1}{(1-q^4)^2} \cdot \frac{1}{(1-q^2)^2}$$

$$= \sum_{s\geq 0} \frac{q^{2s^2}}{(1-q^2)^2(1-q^4)^2(1-q^6)^2\cdots(1-q^{2s})^2}$$

$$= \sum_{s=0}^{\infty} \frac{q^{2s^2}}{(q^2;q^2)_s^2},$$

i.e.,

$$\sum_{n=0}^{\infty} f_{o,e}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n^2}.$$

If we set $q := q^2$ in Equation (2.4.3) , we get

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n^2} = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}},$$

$$\text{i.e.,} \quad \sum_{n=0}^{\infty} f_{o,e}(n)q^n = \sum_{n=0}^{\infty} p(n)q^{2n} = \sum_{n\geq 0,\, n \text{ even}} p(n/2)q^n.$$

i.e. $f_{o,e}(n) = p(n/2)$, when n even.
Therefore, $f_{o,e}(2n) = p(n)$ for all n. (by replacing n with $2n$) □

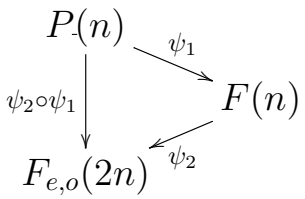
Second proof of Theorem 5.1.1. We construct a bijection between $F_{e,o}(2n)$ and $P(n)$.

Recall the bijection $\psi_1 : P(n) \mapsto F(n)$ (in Chapter 2). Let $\psi_2 : F(n) \mapsto F_{e,o}(2n)$ be defined as

$$\begin{pmatrix} a_1 & a_2 & \dots & a_s \\ b_1 & b_2 & \dots & b_s \end{pmatrix} \mapsto \begin{pmatrix} 2a_1 & 2a_2 & \dots & 2a_s \\ 2b_1+1 & 2b_2+1 & \dots & 2b_s+1 \end{pmatrix}.$$

Clearly, ψ_2 is a one-to-one correspondence.

Let $\psi = \psi_2 \circ \psi_1$. Then using ψ, we see that the sets $F_{e,o}(2n)$ and $P(n)$ are in one-to-one correspondence. The diagram below illustrates the mappings.



.

□

Theorem 5.1.2. *For $n \geq 0$, we have*

$$f_{e,e}(n) \equiv d^1(n,4) \pmod 2. \tag{5.1.1}$$

$$f_{o,o}(n) \equiv d^3(n,4) \pmod 2. \tag{5.1.2}$$

We define $f_{e,e}(0) = f_{o,o}(0) := 1$.

Proof. The generating function for $f_{e,e}(n)$ is given by

$$\sum_{n\geq 0} f_{e,e}(n)q^n$$

$$= \sum_{s\geq 0,\ \alpha_1>\alpha_2>\ldots>\alpha_s\geq 0,\ \beta_1>\beta_2>\ldots>\beta_s\geq 0} q^s \cdot q^{2\alpha_1+2\alpha_2+\ldots+2\alpha_s} \cdot q^{2\beta_1+2\beta_2+\ldots+2\beta_s}$$

$$= \sum_{s\geq 0,\ n_1\geq 0,\ n_2\geq 0} \cdots \sum_{n_s\geq 0,\ m_1\geq 0,} \cdots \sum_{m_s\geq 0} q^s q^{(2s)n_1+2(s-1)n_2+\ldots+2n_s} q^{2+4+\ldots+2(s-1)}$$

$$\times\ q^{(2s)m_1+\ldots+2m_s} \cdot q^{2+4+\ldots+2(s-1)}$$

$$= \sum_{s\geq 0} q^{2s^2-s} \cdot \frac{1}{(1-q^{2s})^2} \cdot \frac{1}{(1-q^{2(s-1)})^2} \cdots \frac{1}{(1-q^4)^2} \cdot \frac{1}{(1-q^2)^2}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q^2;q^2)_n^2}$$

$$\equiv \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q^4;q^4)_n} \pmod 2.$$

If we set $t := q$ and $q := q^4$ in Equation (2.3.7) , we get

$$\sum_{n=0}^{\infty} \frac{q^n q^{2n(n-1)}}{(q^4;q^4)_n} = \prod_{n=0}^{\infty}(1+q\cdot q^{4n}),$$

i.e.,

$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q^4;q^4)_n} = \prod_{n=0}^{\infty}(1+q^{4n+1}) = \sum_{n=0}^{\infty} d^1(n,4)q^n. \tag{5.1.3}$$

Then,

$$\sum_{n=0}^{\infty} f_{e,e}(n)q^n \equiv \sum_{n=0}^{\infty} d^1(n,4)q^n \pmod 2. \tag{5.1.4}$$

This completes a proof of Equation (5.1.1).

Similarly,

$$\sum_{n\geq 0} f_{o,o}(n)q^n$$
$$= \sum_{s\geq 0,\ \alpha_1>\alpha_2>\ldots>\alpha_s\geq 1,\ \beta_1>\beta_2>\ldots>\beta_s\geq 1} q^s \cdot q^{2\alpha_1+2\alpha_2+\ldots+2\alpha_s-s} \cdot q^{2\beta_1+2\beta_2+\ldots+2\beta_s-s}$$

$$= \sum_{s\geq 0,\ n_1\geq 0,\ n_2\geq 0} \cdots \sum_{n_s\geq 0,\ m_1\geq 0,\ m_2\geq 0} \cdots \sum_{m_s\geq 0} q^{-s} q^{(2s)n_1+2(s-1)n_2+\ldots+2n_s} q^{2+4+\ldots+2s}$$

$$\times\ q^{(2s)m_1+2(s-1)m_2+\ldots+2m_s} \cdot q^{2+4+\ldots+2s}$$

$$= \sum_{s\geq 0} \frac{q^{2s^2+s}}{(1-q^2)^2(1-q^4)^2(1-q^6)^2\cdots(1-q^{2s})^2}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^2;q^2)_n^2}$$

$$\equiv \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^4;q^4)_n} \pmod 2.$$

If we set $t := q^3$ and $q := q^4$ in Equation (2.3.7) , we get

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^4;q^4)_n} = \prod_{n=0}^{\infty}(1+q^{4n+3}) = \sum_{n=0}^{\infty} d^3(n,4)q^n. \tag{5.1.5}$$

Thus

$$\sum_{n=0}^{\infty} f_{o,o}(n)q^n \equiv \sum_{n=0}^{\infty} d^3(n,4)q^n \pmod 2. \tag{5.1.6}$$

This completes a proof of Equation (5.1.2). □

Corollary 5.1.1. *For $k, n \geq 0$, we have*

$$p(n) \equiv \sum_{k=0}^{n} f_{e,e}(k) f_{o,o}(n-k) \pmod 2.$$

Proof. Multiplying Equation (5.1.4) by (5.1.6), we get

$$\sum_{n=0}^{\infty} f_{e,e}(n)q^n \cdot \sum_{n=0}^{\infty} f_{o,o}(n)q^n = \prod_{n=1}^{\infty}(1+q^{2n-1}) \pmod 2,$$

$$\text{i.e. } \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} f_{e,e}(k)f_{o,o}(n-k)\right)q^n \equiv \prod_{n=1}^{\infty}(1-q^{2n-1}) \pmod 2$$

$$= \prod_{n=1}^{\infty}\frac{1}{1+q^n} \pmod 2 \qquad \text{(by (2.2.2))}$$

$$\equiv \prod_{n=1}^{\infty}\frac{1}{1-q^n} \pmod 2$$

$$= \sum_{n=0}^{\infty} p(n)q^n \pmod 2.$$

□

Example 5.1.1.

n	$d^1(n,4)$	$d^3(n,4)$	**F-partitions of** n	$f_{e,e}(n)$	$f_{o,o}(n)$
1	1	0	$\begin{pmatrix}0\\0\end{pmatrix}$	1	0
2	0	0	$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$	0	0
3	0	1	$\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}$	2	1
4	0	0	$\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$	0	0
5	1	0	$\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}2&0\\1&0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}1&0\\2&0\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}0\\4\end{pmatrix}$	3	2

Example 5.1.2. *Consider* $n = 5$.

$$\begin{aligned}
&\sum_{k=0}^{5} f_{e,e}(k) f_{o,o}(5-k) \\
&= f_{e,e}(0)f_{o,o}(5) + f_{e,e}(1)f_{o,o}(4) + f_{e,e}(2)f_{o,o}(3) + \ldots + f_{e,e}(5)f_{o,o}(0) \\
&= 1(2) + 1(0) + 0(1) + 2(0) + 0 + 3(1) \\
&= 5 \equiv 1 \pmod 2.
\end{aligned}$$

But $p(5) = 7 \equiv 1 \pmod 2$, and so the result is verified.

5.1.1 Geometric interpretation of the bijection for Theorem 4.2.1

In this section we give a geometric interpretation of the bijection for Theorem 4.2.1.
A γ- symmetric partition of n, i.e., $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_s, \lambda_{s+1}, \lambda_{s+2}, \ldots)$ where s is the order of λ, can be mapped to the Frobenius symbol

$$\mathrm{f}_\gamma = \begin{pmatrix} \lambda_1 - 1 & \lambda_2 - 2 & \ldots & \lambda_s - s \\ \lambda_1 - 1 + \gamma & \lambda_2 - 2 + \gamma & \ldots & \lambda_s - s + \gamma \end{pmatrix}$$

In this case, each hook in the Ferrers graph of λ is represented by a sum of $(\lambda_i - i) + 1$ plus $(\lambda_i - i + \gamma)$ for all $1 \leq i \leq s$.

It is now clear that the new partition constructed in this way is given by

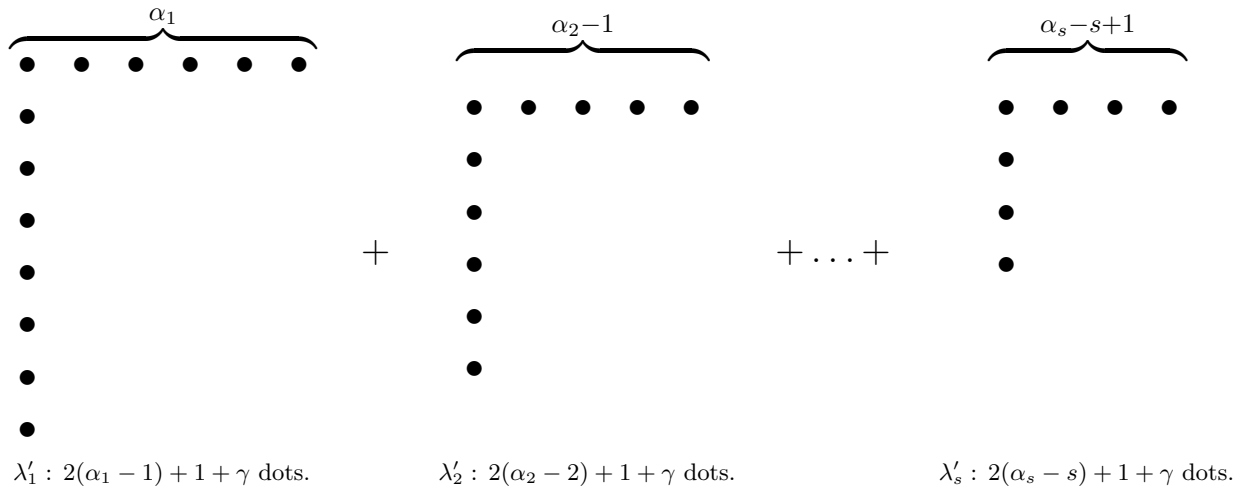
$$\beta = (2\lambda_1 + \gamma - 1,\, 2\lambda_2 + \gamma - 3, \ldots,\, 2\lambda_s + \gamma - 2s + 1)\,.$$

It is clear that β is a partition in which parts are distinct and definitely each part is congruent to $1+\gamma \pmod 2$ where the smallest part is greater than or equal to $\gamma + 1$.

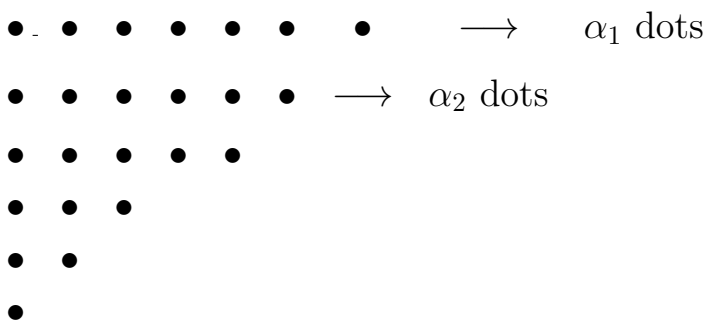
To invert the map, given a partition $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_l)$ of n enumerated by $G(n, \gamma)$ where the smallest part is greater than or equal to $1 + \gamma$, we can write each λ'_j as

$$\lambda'_j = 2\alpha_j - 2j + 1 + \gamma,$$

for some unique $\alpha_j \geq j$. Then, we construct hooks as shown in the diagram below where each horizontal section consists of $\alpha_j - j + 1$ dots and the number of dots going downwards equals to $\alpha_j - j + \gamma$.



Then, we nest these hooks together into a Ferrers graph. See the diagram below.



We then identify the partition corresponding to the above Ferrers graph. This partition has its parts as the number of dots in each row, i.e.,

$$\rho = (\alpha_1, \alpha_2, \ldots, \alpha_s, \alpha_{s+1}, \alpha_{s+2}, \ldots).$$

Clearly, this new partition is a γ-symmetric partition.

Chapter 6

Conclusion

In this dissertation, we studied some restricted integer partition functions. We first introduced our work and gave some definitions and notation we needed. In Chapter 3, we laid down some foundational tools that included proving the Jacobi's triple product identity. One important corollary of the Jacobi's triple product is the Euler's pentagonal number theorem. A discussion on a partition-theoretic interpretation of the Euler's pentagonal number theorem was presented. Furthermore, as part of the foundation, the parity of $d(n)$ was looked at by considering Legendre's partition theoretic interpretation of Euler's pentagonal number theorem.

Our contributions to partition theory were presented in Chapters 4 and 5. To the best of our knowledge, the results are new.

In Chapter 4, we gave a simple extension of a version of the so called Alladi-Schur theorem. We also generalised one of the theorems of Sylvester and gave an explicit bijection of the generalisation. We also proved that Glaisher's map provides a combinatorial proof of Theorem 4.1.1 when $k = 2^r$ where r is a positive integer. However, it does not generally work if k is not a power of 2, (see Example 4.3.1). We believe that Theorem 4.1.1 can still be generalized and this is part of our further work. Besides, we came up with several parity and recurrence formulas of some partition functions in the fashion of Sylvester.

In Chapter 5, we derived some partition identities involving Frobenius partitions by imposing parity restrictions on the entries of the Frobenius symbols. Lastly, we gave a geometric interpretation of our explicit bijection for a theorem in Chapter 4.

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