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Classical Lie Point Symmetry Analysis of Models Arising in Contaminant Transport Theory

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfillment of the requirements for the degree of Master of Science.

September 13, 2013

Declaration

I declare that this project is my own, unaided work. It is being submitted as fulfillment of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

Zwelithini Fanelo Mkhonta

September 13, 2013

Abstract

Groundwater contamination and soil salinisation are a major environmental problem worldwide. Living organisms depend largely on groundwater for their survival and its pollution is of course of major concern. It therefore goes without saying that remedial processes and understanding of the mathematical models that describe contaminant transport is of great importance. The theory of contaminant transport requires understanding of the water flow even at the microscopic level. In this study we focus on macroscopic deterministic models based on differential equations. Here contaminant will refer to nonreactive contaminant. We aim to calculate Lie point symmetries of the one-dimensional Advection-diffusion equation (ADE) for various forms of the diffusion coefficient and transport velocity. We aim to employ classical Lie symmetry techniques. Furthermore, reductions will be carried out using the elements of the optimal systems. In concluding, the ADE is analyzed for selected forms of the the diffusion coefficient and transport velocity via the potential symmetry method. For the potential symmetries obtained, we investigate the associated invariant solutions.

Dedications

I dedicate this work to both my late parents; my two late brothers Mahambayedvwa and Sibane. May their souls rest in peace.

Acknowledgments

I wish to express my sincere gratitude to Professor R. J. Moitsheki for his supervisory work, expertise, availability, exceptional advice and otherwise. He always found a way to accommodate me in his busy schedule. His introductory lectures on symmetries of differential equations laid the solid foundation for my work. The lack of confidence I had in the early days of my enrollment was soon a thing of the past, thanks to his exceptional motivational skills. He also made it easy for me to approach him by bridging the student-supervisor gap.

Special thanks should also go to the administrative staff of the School of Computational and Applied Mathematics (CAM) for their hospitality and assistance in matters related. I also send my appreciation to postgraduate fellows for their unlimited help in so many ways, especially in introducing me to computer software programs I had to use throughout my work. In many ways this work is a product of team work than just my efforts.

I would also like to thank the Head of School, Professor E. Momoniat as well as Dr J.S. Mathunjwa from the University of Swaziland who, in a number of ways, facilitated my preparations to enroll at the University of the Witwatersrand. I also acknowledge the school of CAM and the Financial Aid and Scholarships office for the financial support I enjoyed during my study.

Special thanks is also forwarded to my better half Nossy for her undying support in many ways. She also demonstrated understanding even when I expected the least from her. She is my pillar of strength.

Above all I thank God for everything because it was His wish that things shaped out the way they did.

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Chapter 1

Introduction

1.1 Background information

Groundwater is a term often used to denote all the waters that are found beneath ground surface, as part of the hydrological cycle. However, this definition is often reserved for the term subsurface water, while the term groundwater is associated primarily with that part of the water in the hydrological cycle that occurs only in the zone of saturation [1]. A groundwater contaminant or pollutant is any substance that, when it reaches an aquifer, makes the water unclean or otherwise unsuitable for a particular purpose. Sometimes such a substance may be a manufactured chemical or even a microbial contaminant. Contamination can also arise from naturally occuring minerals and metallic deposits [2].

1.1.1 Contamination of subsurface water

We may wish to know how groundwater gets contaminated. According to [1], liquid contaminants are sometimes spilled at ground surface, by intent or by accident. Once released, a liquid contaminant will percolate downward through the unsaturated zone, until it reaches the underlying water table. Upon reaching the aquifer, the liquid contaminant will be transported through the aquifer and later to its outlets (rivers, lakes, springs and so on). Along its way to these outlets, the concentration of the contaminant decreases gradually as a result of various processes. In many cases such contaminants may render groundwater useless for most purposes. Hence, management of aquifers must cater for both quantity and quality.

1.1.2 Sources of groundwater contamination

The authors in [1] see contaminants in groundwater assuming one of the following forms:

- *Pathogenic contaminants*, e.g. originating from poorly constructed septic tanks, or from improper disposal of waste from hospitals.

- *Inorganic contaminants*, e.g. increased levels of the ions chloride, sulphate, nitrate, and sodium, originating from landfills of domestic waste.

- *Organic contaminants*, which originate mainly from industrial waste spills. Examples include chlorinated hydrocarbons like chloroform, carbon tetrachloride, tetrachloroethylene as well as aromatic hydrocarbons like benzene and naphthalene.

1.1.3 Classification of contamination sources

The Office of Technology Assessment (OTA) of the United States Congress, has categorized sources known to have contaminated groundwater, according to the nature of their release characteristics [3].

* *Sources designed to release substances* - This category includes septic tanks, injection wells for disposal of hazardous and nonhazardous materials and substance application e.g. disposal of wastewater by surface irrigation, or dumping of waste-water sludge. Another example is solution mining.

* *Sources designed to store, treat or dispose of substances, as well as discharge resulting from unplanned release* - Examples are landfills for municipal and industrial waste, open waste dumps, surface impoundments of hazardous and nonhazardous liquid waste, animal burial sites, above-ground storage tanks for waste materials, containers for all kinds of non-waste materials, and disposal sites for radioactive materials.

* *Sources that retain substances during transportation or transmission* - Pipelines and material transportation and transfer operations of hazardous and nonhazardous materials, as well as non-waste materials are included in this category.

* *Sources that discharge substances to the environment as part of various planned activities* -Examples include irrigation practices, pesticide and fertilizer applications, animal feeding operations, urban runoff, surface and subsurface mining, and mine-drainage operations.

* *Wells and construction excavations* -Included here are oil and gas

production wells, geothermal and heat recovery wells, water supply wells, etc.

* *Naturally occurring sources whose discharge is created by human activities* -This category includes water infiltrating from precipitation and carrying atmospheric pollutants, natural leaching, saltwater intrusion, and encroachment of poor quality water as a result of man-made changes in the flow regime in an aquifer.

The OTA [3] also classifies contaminant sources according to their geometry. A contaminant source may be classified as a point source. "Point" here, means "of small areal extent", relative to the subsurface domain under consideration. For example, a large landfill may be considered a "point source" relative to an underlying aquifer contaminated by the leachate from the landfill, once the plume has reached a distance which is much larger than the dimensions of the landfill itself. On the other hand there are also non-point sources of contaminants. These are also called distributed sources or diffuse sources. Here a source extends over a large area relative to that of the contaminant subsurface. Examples include the application of pesticides, herbicides and fertilizers in agriculture.

1.1.4 The health effects of groundwater contamination

Groundwater contamination can have severe side effects on human health. Acute side effects include nausea and vomiting, diarrhea, headaches, eye irritation, and nose irritation. Chronic side effects include cancer, liver damage, kidney damage, anemia, nervous system problems, circulatory problems, bone disease, hair loss and reproductive difficulties.

1.1.5 Cleaning up contaminated groundwater

The various ways to respond to site contamination can be classified into the following categories [2],

- containing the contaminants to prevent them from migrating from their source;
- removing the contaminants from the aquifer;
- remediating the aquifer by either immobilizing or detoxifying the contaminants while they are still in the aquifer;
- treating the groundwater at its point of use; and
- abandoning the use of the aquifer and finding an alternative source of water.

1.2 Literature Review

The common processes by which contaminants that are dissolved in the subsurface are transported are *advection* (*convection*), *mechanical dispersion*, and *molecular diffusion*. The authors in [4] give descriptions of these processes in the context of contaminant flow. In advection, the contaminant is carried along with the flow of the subsurface water. This process is a result of the large-scale gradients in the fluid energy and it is the most significant mass transport process. Mechanical dispersion is the main process that causes the contaminant to spread out and become diluted. The contaminant spreads due to the spatial variation of flow paths and the variation of velocity in the groundwater movement. Molecular diffusion is the process in which the contaminants

move from areas of high concentration to those of low concentration due to concentration gradients. It is worth noting that molecular diffusion can take place in the absence of groundwater flow. All the three processes make the contaminant spread in the direction of flow both longitudinally and transversally.

Proper assessment of contamination problems as well as the design of reliable techniques to try to solve them requires one's capability to predict how chemical contaminants behave in groundwater. Quantitative predictions of contaminant movement can be made only through the understanding of processes controlling transport, advection, dispersion, chemical, physical, and biological reactions that affect solute concentrations in water [4]. To predict the behavior of contaminants, a model or group of models that represents the effects of these influences is needed.

Several researchers have worked on the solutions of the advection-diffusion equation (ADE) both analytically and numerically. The authors in [4] investigated the finite difference solution of the one-dimensional non-reactive advection-dispersion equation (ADE) in a means to find the spill history of the contaminant. They found the solution to the ADE for both forward and backward simulations. One other piece of work on numerical solutions of the ADE is the paper by Gurarslan et al. [5]. In this paper they have produced numerical solutions of the one-dimensional ADE using a sixth-order compact difference scheme in space. They also have solutions using a fourth-order Runge-Kutta scheme in time.

Kumar et al (2009) have worked out the analytic solutions for the one-dimensional ADE with variable coefficients in a longitudinal domain which

is initially solute free. They focused on two problems. With the first they considered the one-dimensional ADE for a temporally dependent solute dispersion of uniform initial concentration. In the second problem they considered the one-dimensional ADE in an inhomogeneous domain. The flow velocity was taken to be $u = u_0(1 + ax)$ where a is a parameter taking care of the inhomogeneity of the medium. The dispersion was considered to vary as the square of the velocity. In a subsequent paper, Kumar et al explain that in a one-dimensional ADE with temporally dependent coefficients, there can be three cases: solute dispersion dependent on time while the flow velocity of the solute particles is uniform, solute dispersion is uniform while the flow velocity is time dependent and for the last case both physical quantities are dependent on time. In this paper (2011), they obtained analytic solutions for the one-dimensional ADE when both the dispersion parameter and velocity of the solutes are time dependent. This was an extension of the their previous work. For details see [6] and [7].

Park and Baik [8] have obtained analytic solutions of the ADE for a ground-level area source that is finite in both the longitudinal and lateral directions. Luce et al have added on already existing theory of using temperature time series to make estimates of streambed water fluxes [9].

The importance of particle migration through porous media was addressed by [10] by reviewing literature in subject areas where this is a consideration. They mentioned that small particles in porous media can affect permeability by clogging the pores and hence it is important to understand the migration behavior of particles. One area they discussed is the filtration mechanisms that limit particle migration. For particles comparable in size to or larger than the pores of the porous media, penetration into the media does not occur at all.

As a result there is accumulation of the particles and this leads to decrease in permeability. When the particles are small enough to enter the porous media they at times get trapped inside the porous medium and they can be removed by straining in smaller pore spaces. For very small particles relative to the porous media, the grain size can be removed from solution by physical and chemical forces between particles and the media.

A similar study to that of [10] was done by [11]. Their paper discusses the movement and restrictions of particulate and colloidal contaminants as well as the mechanisms controlling them. In their paper they state that there appears to be two main types of physical capture mechanisms that restrict contaminant movement in porous media and these are straining (where the physical size of the pore is smaller than the particle , as a result the particle is unable to pass) and filtration (which covers mechanisms by which particles are captured in pores with larger dimensions than those of the particles).

The author in [12] investigated the the spatial and temporal distribution of contaminant concentrations in soil column. A model was developed which approximates contaminant concentrations in groundwater under water flow conditions where microbial activities and plant growth were present. The model assumed the flow and corresponding moisture content as well as the concentration of the contaminant to be functions dependent on both space and time. According to [12], the bulk motion of the fluid controls contaminant transport through the soil column by the processes of molecular diffusion and mechanical dispersion. The transport of the contaminants selenium, nitrogen, and pesticide were modeled. The model was applied to simulate contaminants under steady state and transient conditions. The model was successful in

predicting the concentrations of different contaminants in groundwater. The simulation results showed that as the time increased from the start to the end of the simulation, the contaminant concentration approached measured values.

The symmetry analysts got their attention drawn to classification of the one-dimensional Fokker Planck equation (see for example [13] and [14]).

1.3 Research Objectives

The objectives of this research work are:

- To carry out group classification of the one-dimensional advection-diffusion equation (ADE).
- To determine the forms of the diffusion coefficient of the one-dimensional ADE such that the principal Lie algebra is extended.
- to perform symmetry reductions using elements of the one-dimensional optimal system and obtain group invariant solutions.
- To find non-local symmetries of the one-dimensional ADE for specific forms of the diffusion coefficient and the transport velocity and obtain group invariant solutions using such symmetries.
- To produce plots and interpret them for a selected few of the realistic solutions obtained.

1.4 Outline of the dissertation

Chapter 2 is devoted to the ADE. We review the process of diffusion and explain the aspects that lead to the derivation of the ADE.

In Chapter 3 we focus on the the algebraic techniques for symmetry reductions. We try to review as much of the concepts on Lie's theory of analysis of differential equations. We also briefly explain the essentials of the potential symmetry method. This chapter generally covers the methods which are implemented in Chapters 4 through 6.

In Chapter 4 direct group classification of the ADE is carried out. Symmetry analysis of the ADE using the forms of the diffusion coefficient that lead to the extension of the principal Lie algebra is done and where possible, group invariant solutions are obtained.

Chapter 5 focuses on symmetry analysis of the ADE under various forms of the the diffusion coefficient and transport velocity. Reductions are carried out and invariant solutions are obtained where possible. This chapter constitutes the bulk of the work covered in this dissertation.

Chapter 6 deals with non-local symmetry analysis of the ADE. We investigate potential symmetries of the ADE for two chosen cases of the the diffusion coefficient and transport velocity. We also construct the associated invariant solutions.

In Chapter 7 we wrap up the entire work done in this dissertation.

Chapter 2

Formulae and theory

2.1 Introduction

Naturally, transport in fluids occurs through the combination of advection and diffusion. We wish to derive a mathematical model that describes contaminant transport and this model should incorporate both these processes. We begin by looking at the process of diffusion and work our way through to the diffusion-advection equation.

2.2 Mathematical models

2.2.1 Molecular diffusion

As mentioned earlier, diffusion is one fundamental transport process in environmental fluid mechanics. There are two things to note about diffusion: it is a process that occurs randomly and particles move from where they are highly concentrated to where they are lowly concentrated.

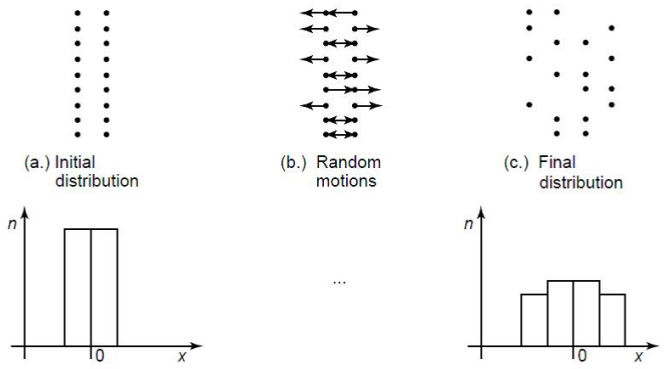

Figure 2.1: Schematic of the one-dimensional molecular (Brownian) motion of a group of molecules illustrating the Fickian diffusion model. The upper part of the figure shows the particles themselves; the lower part of the figure gives the corresponding histogram of particle location, which is analogous to concentration.

2.2.2 Fickian diffusion

Since in diffusion molecules spread out from regions of high concentration into regions of low concentration, we wish to find a mathematical model predicting this process, and we will adopt the argument by Fischer et al.[15].

To come up with a diffusive flux equation, consider two rows of molecules side-by-side and centred at $x = 0$, as shown in Figure 1(a). In response to the temperature (Brownian motion), each molecule moves about randomly. We only consider one component of their three-dimensional motion: motion right or left along the x-axis. We define the mass of particles on the left and right as M_l and M_r respectively. We also define as λ, the probability that a particle moves across $x = 0$, where λ is in units T^{-1}.

After sometime δt an average of half the particles have taken steps on either

side of their original position, as shown in Figure 1(b) and (c). The particle histograms in Figure 1 also suggest that maximum concentrations decrease, while the total region occupied by the particles increases. Mathematically, the average flux of particles from the left to the right is λM_l, and that from the right to the left is $-\lambda M_r$. Here the minus sign distinguishes the direction. Thus, the net flux of particles q_x is

$$q_x = \lambda(M_l - M_r). \tag{2.1}$$

For the one-dimensional case we can write (2.1) in terms of concentrations using

$$C_l = \frac{M_l}{\delta x \delta y \delta z}, \tag{2.2}$$

$$C_r = \frac{M_r}{\delta x \delta y \delta z}, \tag{2.3}$$

where δx is the width, δy is the breadth, and δz is the height of each column. Physically, δx is the average step along the x-axis taken by a molecule in the time δt. For the one-dimensional case, we want q_x to represent the flux in the x-direction per unit area perpendicular to x; hence we will consider $\delta y \delta z = 1$. Next, we note that a finite difference approximation for $\frac{dC}{dx}$ is

$$\begin{aligned} \frac{dC}{dx} &= \frac{C_r - C_l}{x_r - x_l} \\ &= \frac{M_r - M_l}{\delta x(x_r - x_l)} \end{aligned} \tag{2.4}$$

which gives us a second expression for $(M_l - M_r)$, namely,

$$M_l - M_r = -\delta x(x_r - x_l)\frac{dC}{dx}. \tag{2.5}$$

Taking $\delta x = (x_r - x_l)$ and substituting (2.5) into (2.1) yields

$$q_x = -\lambda(\delta x)^2\frac{dC}{dx}. \tag{2.6}$$

Equation (2.6) contains two unknowns, λ and δx. We take $-\lambda(\delta x)^2$ to be the diffusion coefficient D. Now (2.6) becomes

$$q_x = -D\frac{dC}{dx}. \tag{2.7}$$

It is important to note that diffusive flux is a vector and, since concentration is expressed in units of $[M/L^3]$, it has units of $[M/L^2T]$. The diffusion coefficient may be allowed to be time, spatial and concentration dependent. To compute the total mass flux rate $\dot{m}$, in units $[M/T]$, the diffusive flux should be integrated over a surface area. For the one-dimensional case we would have $\dot{m} = Aq_x$, where $A = \delta y \delta z$. Generalizing to the three dimensions, we can write the diffusive flux vector at a point by adding the other two dimensions, yielding

$$\begin{aligned} q &= -D\left(\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial z}\right) \\ &= -D\nabla C \\ &= -D\frac{\partial C}{\partial x_i}. \end{aligned} \tag{2.8}$$

Diffusion processes that obey this relationship are called Fickian diffusion, and (2.8) is called Fick's law. To obtain the total mass flux rate we integrate the normal component of q over a surface area, as in

$$\dot{m} = \int\int_A \mathbf{q} \cdot \mathbf{n} \, dA, \tag{2.9}$$

where $\mathbf{n}$ is the unit vector normal to the surface area A.

2.2.3 The advection-diffusion equation

The derivation of the advection-diffusion equation relies on the principle that the two processes are linearly independent. As a result they can be added together. The only way that advection and diffusion can be dependent is, according to [16], if one process feeds back on the other. We recall that diffusion is a random process due to molecular motion.

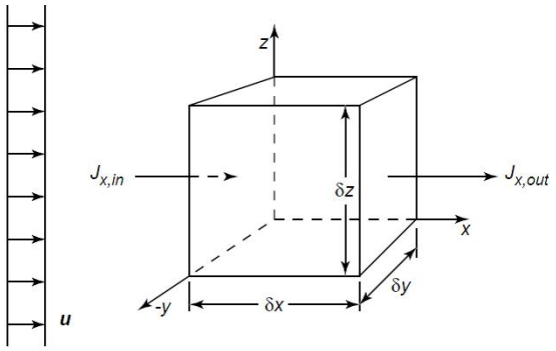

Figure 2.2: Schematic of a control volume with crossflow.

As a result of diffusion, each molecule in the time δt moves either one step leftwards or one step rightwards (i.e. $\pm\delta x$). Now, how does the molecule move with respect to advection? Advection makes the molecule move $u\delta t$ in the direction of the flow of the fluid. It should be clear that the two processes are additive and independent. Movement of the molecule in the cross-flow direction does not hinder the molecule's movement to the left or to the right

as a result of diffusion. The net movement of the molecule should be $u\delta t \pm \delta x$. Then the total flux in the x-direction J_x, including the advective transport and a Fickian diffusion term, must be

$$\begin{aligned} J_x &= uC + q_x \\ &= uC - D\frac{\partial C}{\partial x} \end{aligned} \tag{2.10}$$

where uC is the advective term. To derive the advective diffusion equation (ADE), we use the flux and the law of conservation of mass. Consider the schematic diagram of a control volume with cross-flow velocity, $\mathbf{u} = (u, v, w)$, as shown in Figure 2. We shall follow the derivation as given in [15]. From the conservation of mass, the net flux through the control volume (CV) is

$$\frac{\partial M}{\partial t} = \Sigma \dot{m}_{in} - \Sigma \dot{m}_{out}, \tag{2.11}$$

and for the x-direction we have

$$\delta \dot{m}|_x = \left(uC - D\frac{\partial C}{\partial x} \right)\Big|_1 \delta y \delta z - \left(uC - D\frac{\partial C}{\partial x} \right)\Big|_2 \delta y \delta z. \tag{2.12}$$

The Taylor series expansion is used to combine the two flux terms, giving

$$\begin{aligned} uC\big|_1 - uC\big|_2 &= uC\big|_1 - \left(uC\big|_1 + \frac{\partial (uC)}{\partial x}\Big|_1 \delta x \right) \\ &= -\frac{\partial (uC)}{\partial x}\delta x \end{aligned}$$

and

$$-D\frac{\partial C}{\partial x}\bigg|_1 + D\frac{\partial C}{\partial x}\bigg|_2 = -D\frac{\partial C}{\partial x}\bigg|_1 + \left(D\frac{\partial C}{\partial x}\bigg|_1 + \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right)\bigg|_1 \delta x\right)$$

$$= \frac{\partial}{\partial x}\left[D\frac{\partial C}{\partial x}\right]\delta x.$$

Thus, for the x-direction

$$\delta\dot{m}\big|_x = -\frac{\partial(uC)}{\partial x}\delta x\delta y\delta z + \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right)\delta x\delta y\delta z. \tag{2.13}$$

The y- and z-directions are similar but with v and w for the velocity components, giving

$$\delta\dot{m}\big|_y = -\frac{\partial(vC)}{\partial y}\delta y\delta x\delta z + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right)\delta y\delta x\delta z. \tag{2.14}$$

$$\delta\dot{m}\big|_z = -\frac{\partial(wC)}{\partial z}\delta z\delta x\delta y + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right)\delta z\delta x\delta y. \tag{2.15}$$

Substituting these results into (2.9) and recalling that $M = C\delta x\delta y\delta z$, we get

$$\frac{\partial C}{\partial t} + \nabla\cdot(\mathbf{u}C) = \nabla\cdot(D\nabla C). \tag{2.16}$$

We now have our governing equation in 3D, which, when written in Einstenian notation, reads

$$\frac{\partial C}{\partial t} + \frac{\partial(uC)}{\partial x_i} = \frac{\partial}{\partial x_i}\left(D\frac{\partial C}{\partial x_i}\right), \tag{2.17}$$

where C = contaminant concentration,

D = diffusion coefficient depending on t, x and c,

x_i = spatial variable,

u = transport velocity depending on t, x and c,

t = time.

For the purposes of this study we only consider the one dimensional non-reactive contaminant advection-dispersion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial C}{\partial x}\right] - \frac{\partial}{\partial x}\left[u(x)C\right], \tag{2.18}$$

where $D(x)$ is the diffusion coefficient and $u(x)$ is the velocity function. This equation describes contaminant flow with the assumption that the flow of the contaminant is due to the processes of advection and dispersion. The relevant initial and boundary conditions are by

$$\left.\begin{aligned} &C(x, t=0) = C_0(x); \\ &x = L \quad C = C_L(t); \\ &\left.\frac{\partial C}{\partial x}\right|_{x=L} = 0. \end{aligned}\right\} \tag{2.19}$$

2.3 Method of solution

The group-invariant solutions of the one-dimensional ADE are sought. the follwing method will be followed:

1. The direct method of group classification will be followed.
2. We find the forms of the diffusion coefficient for which the principal Lie algebra extends.

3. Once the Lie algebra has been found, we construct the one-dimensional optimal system which we use to carry out symmetry reductions.

4. We seek potential symmetries of equation (2.18) for selected forms of the diffusion coefficient and transport velocity and use these symmetries to search for invariant solutions.

Chapter 3

Algebraic techniques for symmetry reductions

3.1 Local symmetries

It is essential to give a brief preview on some of the concepts on Lie groups of transformations since this work is on analysis of differential equations from a symmetry point of view. We aim to employ Lie symmetry techniques to nonlinear advection-diffusion models. The main idea behind Lie's theory of analysis of differential equations is that the differential equations remain invariant under a transformation of dependent and independent variables. The Lie symmetry theory is well documented (see for example [17], [18], [19], [20], [21], [22] and [23]).

3.1.1 Lie point symmetries and group classification

Group classification: direct methods

Most of the problems in real life are mathematically represented in the form of PDEs. These PDEs may contain functions depending on either the dependent or independent variables. As a result it becomes difficult to calculate the symmetries admitted by such equations. The direct method of group classification may be used in such instances [17] and [18]. One may determine the forms of the arbitrary functions for which the equation in question admits extra symmetries. In advanced group classification, one would first determine the equivalence transformation (transformations which leave the equation and the contained arbitrary functions invariant) (see for example [24]).

Calculations of Lie point symmetries

A prerequisite to studying symmetries of differential equations is the understanding of infinitesimal transformations, in particular, one-parameter group of transformations and infinitesimal generators. Given a second order PDE

$$\triangle(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = 0, \tag{3.1}$$

we seek transformations of the form

$$\begin{aligned} \bar{t} &= t + \epsilon\xi^1(t, x, u) + O(\epsilon^2), \\ \bar{x} &= x + \epsilon\xi^2(t, x, u) + O(\epsilon^2), \\ \bar{u} &= u + \epsilon\eta(t, x, u) + O(\epsilon^2), \end{aligned} \tag{3.2}$$

generated by the base vector

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.3}$$

in order. In calculating the symmetries of a second order PDE, we prolong (3.3) to second order since the equation of second order. The corresponding prolongation formula is

$$X^{[2]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}, \tag{3.4}$$

where ζ_t, ζ_x, ζ_{tt}, ζ_{xx} and ζ_{tx} are defined as follows:

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2),\\
\zeta_{tx} = \zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_t(\xi^1) - u_{xx} D_x(\xi^2).
\end{aligned} \tag{3.5}$$

The invariance surface criterion

$$X^{[2]} \triangle \Big|_{\triangle=0} = 0, \tag{3.6}$$

leads to to a system of over-determined linear partial differential equations in terms of the infinitesimals ξ^i and η^k which can be solved by hand or interactively by computer software programs such as YaLie and SYM. Once obtained, one may construct one-dimensional optimal systems of sub-algebras and derive wherever possible group-invariant solutions.

The use of Lie point symmetries

Symmetries play an important role in the study of differential equations. We may classify these into three:

- *Reduction of number of variables*: It is mostly PDEs that describe many real-life applications models as opposed to ODEs. However, solving PDEs directly is often quite a handful if not an impossible task. The beauty of symmetries is that when we know the symmetries admitted by the PDE, such symmetries can be used to convert the PDE to an ODE. The resultant ODE might be less difficult to solve than the original PDE since it has fewer independent variables.

- *Reduction of order*: When symmetries reduce the number of independent variables, the order of the resultant ODE is usually the same as that of the PDE. In such a reduction process it is possible to get an ODE which still proves difficult to solve. In such situations it is usually helpful symmetries admitted by the ODE. When the ODE admits only one symmetry, its first prolongation may lead to an ODE of order one less than the original ODE. the new ODE, which is in new variables may be easier to solve.

- *Constructing invariant solutions*: The transformation of a PDE to an ODE of fewer variables may help us in that we may find the exact solutions of the ODE. By making the proper substitutions we can write the solution in terms of the original variables, hence find the solution of the PDE. The same can be said even when symmetries are used to reduce the order of ODEs.

Optimal system and symmetry reductions

For a PDE that admits an n dimensional Lie algebra spanned by the symmetry generators $(X_1, X_2, ..., X_n)$, it is possible to reduce the number of independent variables using the operator

$$X = a_1X_1 + a_2X_2 + ... + a_nX_n. \tag{3.7}$$

In order to ensure minimal set of reductions that are not connected by any transformation, one constructs an optimal system. In constructing the optimal system of the symmetry group, we follow the procedure as illustrated in [19] and [24]. Once the optimal system has been constructed, one may then classify the group invariant solutions according to the elements of the optimal system. Determining the optimal system is a standard procedure of ensuring we construct only non-equivalent groups of invariant solutions.

Given the symmetry generator (3.3), the basis of invariants are obtained by solving the characteristic equations

$$\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du}{\eta}. \tag{3.8}$$

3.2 Non-local symmetries

3.2.1 Potential symmetries and reduction

The reduction of a partial differential equation (PDE) relies in the use of a similarity variable. It is this similarity variable that allows for the construction of solutions of the original PDE through the integration of an ordinary differential equation (ODE). This approach, however, does not give all the possible solutions of the original PDE. To get those solutions of the PDE that

the classical Lie algorithm fails to reveal, some additions and modifications of the original Lie group method have been suggested.

Bluman et al. were the first authors to introduce a method to calculate a new class of symmetries for a PDE, provided such a PDE is written in conservative form [25]. They analyzed the Lie point symmetries of the associated auxiliary system $\Im(x,t,u,v)$ which is obtained by a potential $v(x,t)$ as a further unknown function. Any group of the Lie transformations for the system $\Im$ induces a symmetry for the PDE. Should at least one of the generators which correspond to the variables t, x and u depend explicitly on the potential $v(t,x)$, then the local symmetry of $\Im$ is called a *potential symmetry* of the PDE. This potential symmetry is neither a point nor a generalized symmetry. Such symmetries have infinitesimals that act on a different space than the space of independent, dependent variables, and their derivatives of the system $\Im$.

We mention here that potential symmetries of a PDE , being point symmetries of the auxiliary system can be used to find invariant solutions of the PDE which, however, cannot be found via the use of classical symmetries. This is one of many advantages of the potential symmetry approach over the classical symmetry approach. In addition, there are much fewer determining equations to deal with for this method than the classical one. By using the potential symmetry method, the symmetry group available is much bigger, hence it is possible to find more explicit solutions by the same reduction method. These advantages of the potential symmetry method have attracted more and more researchers to this method [26]. Potential symmetries are quite useful in carrying out reductions in searching for solutions that are invariant under local symmetries, classical and weak [19, 27].

There is already a lot of work that has been reported in the last 2-3 decades

which has been carried out via the use of potential symmetries. Sophocleous investigated the potential symmetries of the nonlinear diffusion-convection equations and hence the corresponding similarity solutions [28]. The potential symmetries and new explicit solutions of the Burgers equation are reported in [26]. Earlier on, similar work had been done on the generalized Fokker-Planck equation as reported in [29] and [30]. In [31] similar work was done except that the Fokker-Planck equation was considered in cylindrical coordinates related to magnetic field diffusion in magneto-hydrodynamics. In [32] Senthilvelan and Torrisi investigated the potential symmetries and new solutions of the simplified model for reacting mixtures. Khater et al extended their work in [29] by doing a similar paper on the inhomogeneous nonlinear diffusion equation [33].

Pucci and Saccomandi have outlined the necessary conditions a PDE should meet in order to have potential symmetries [27]. Because it is possible for for PDE to be written in several conservative forms, it is important to specify the chosen conservative form when one looks for potential symmetries.

3.2.2 Potential symmetries and reduction

Determination of potential symmetries

In order to find the potential symmetries for a PDE we write it in its conservative form. A PDE of order n is said to be written in conservative form if takes the form

$$D_t(P) - D_x(Q) = 0. \tag{3.9}$$

In equation (3.9) P and Q are functions of $(x, t, u_1, ..., u_{n-1})$ and u_k is the set of kth order derivatives of u. D_t and D_x are the operators of total derivatives with respect to t and x.

By considering $v(x,t)$ as the auxiliary unknown function, associated with (3.9) is the auxiliary system $\Im(x,t,u,v)$:

$$v_x = P, \quad v_t = Q. \tag{3.10}$$

A point symmetry group for $\Im$ is defined by the equations [18]:

$$\begin{aligned}
\bar{x} &= x + \epsilon\xi(x,t,u,v) + O(\epsilon^2) \\
\bar{t} &= t + \epsilon\tau(x,t,u,v) + O(\epsilon^2) \\
\bar{u} &= u + \epsilon\eta(x,t,u,v) + O(\epsilon^2) \\
\bar{v} &= v + \epsilon\phi(x,t,u,v) + O(\epsilon^2),
\end{aligned}$$

and it is completely determined by the infinitesimals τ, ξ, η and ϕ. We state here that point symmetries that satisfy $\xi_v^2 + \tau_v^2 + \eta_v^2 = 0$ correspond to point symmetries of equation (3.9). Instead, potential symmetries of equation (3.9) are obtained if $\xi_v^2 + \tau_v^2 + \eta_v^2 \neq 0$.

It is known that the system which characterizes the generators is obtained from

$$X^{(1)}\left(v_x - P\right)\big|_{\Im} = 0, \quad X^{(1)}\left(v_t - Q\right)\big|_{\Im} = 0, \tag{3.11}$$

which must hold identically. In (3.11) $X^{(1)}$ is the first prolongation of

$$\begin{aligned} X &= \tau(x,t,u,v)\frac{\partial}{\partial t} + \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta + \xi(x,t,u,v)\frac{\partial}{\partial u} \\ &\quad +\phi(x,t,u,v)\frac{\partial}{\partial v}, \end{aligned} \tag{3.12}$$

given by

$$X^{(1)} = X + \eta_1^{(1)}\frac{\partial}{\partial u_t} + \eta_2^{(1)}\frac{\partial}{\partial u_x} + \phi_1^{(1)}\frac{\partial}{\partial v_t} + \phi_2^{(1)}\frac{\partial}{\partial v_x}; \tag{3.13}$$

$$\begin{aligned} \eta_1^{(1)} &= \eta_t + (\eta_u - \tau_t)u_t - \tau_u u_t^2 - \xi_t u_x - \xi_u u_t u_x + \eta_v v_t - \tau_v u_t v_t - \xi_v u_x v_t, \\ \eta_2^{(1)} &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \tau_u u_x u_t - \xi_u u_x^2 + \eta_v v_x - \tau_v u_t v_x - \xi_v u_x v_x, \\ \phi_1^{(1)} &= \phi_t + (\phi_v - \tau_t)v_t - \tau_v v_t^2 + \phi_u u_t - \tau_u u_t v_t - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x, \\ \phi_2^{(1)} &= \phi_x + (\phi_v - \xi_x)v_x - \xi_v v_x^2 + \phi_u u_x - \tau_u u_x v_t - \tau_x v_t - \tau_v v_x v_t - \xi_u u_x v_x. \end{aligned}$$

Invariant solutions

Given a point symmetry for $\Im$, the invariant surface conditions are [27]:

$$\begin{aligned} \xi(x,t,u,v)u_x + \tau(x,t,u,v)u_t - \eta(x,t,u,v) &= 0 \\ \xi(x,t,u,v)v_x + \tau(x,t,u,v)v_t - \phi(x,t,u,v) &= 0, \end{aligned}$$

and the characteristic equations for $\Im$ are

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta} = \frac{dv}{\phi}.$$

The solutions of this characteristic system are given by the three independent integrals:

$$\left.\begin{aligned} J_1 &= \omega_1(x,t,u,v) \\ J_2 &= \omega_2(x,t,u,v) \\ J_3 &= \omega_3(x,t,u,v) \end{aligned}\right\}, \tag{3.14}$$

where J_1, J_2 and J_3 are constants of integration. We define the similarity variable $J_1 = z$ with $J_2 = h_1(z)$ and $J_3 = h_2(z)$ as the similarity functions. Now we can write the system (3.14) as

$$\left.\begin{aligned} u &= P(x,t,u,v,h_1(z),h_2(z)) \\ v &= Q(x,t,u,v,h_1(z),h_2(z)) \\ R(x,t,u,v,h_1(z),h_2(z)) &= 0 \end{aligned}\right\}, \tag{3.15}$$

where P, Q and R are arbitrary functions. The system (3.15) gives the invariant solutions of the auxiliary system $\Im$ where $h_1(z)$ and $h_2(z)$ are the solutions of the ordinary system Θ, determined by substitution into the system $\Im$. We may refer as ψ to the family of solutions of $\Im$. We obtain the family of solutions ψ^* of the PDE by direct substitution of $(3.15)_1$ and $(3.15)_3$ into the original PDE.

Concluding remarks

In this chapter we have reviewed some of the basic theory behind the symmetry analysis of differential equations. We have looked at both local and non-local symmetries respectively. We have seen that a number of researchers have used these approaches in their study and analysis of various models. In the next two chapters the use of local symmetries in carrying out reductions and finding invariant solutions is demonstrated. The non-local symmetry aproach is used in chapter 6.

Chapter 4

Direct group classification and invariant solution of the advection-dispersion equation

4.1 Introduction

In the previous chapter we previewed some theory on Lie groups of transformations. Before we saw the derivation of our governing equation (2.18). In this chapter we consider equation (2.18) when D is a function of only the contaminant concentration C and we also let $u(x) = 1$. Classical Lie symmetry methods were applied to two-dimensional ADE [34, 35]. Non-classical symmetry techniques have been used to analyze the one-dimensional problem in [36]. We carry out direct group classification of equation (2.18) under these conditions and find the forms of D that extend the principal Lie algebra. With these forms of D we find, where possible, the invariant solutions.

4.2 Direct group classification

Our equation now reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\frac{\partial C}{\partial x}\right) - \frac{\partial C}{\partial x}. \tag{4.1}$$

We seek to find the Lie point symmetries admitted by (4.1). Group classification of a class of nonlinear heat equation has been considered and carried out by Dorodnitsyn (see chapter 10 in [37]). Also, classical Lie symmetry analysis of a subclass of Dorodnitsyn class was considered by Moitsheki and Makinde [38], and Mhlongo et al [39]. Here we focus on equation (4.1) describing contaminant transport, another subclass of a class in chapter 10 of [37]. The invarience criterion is given by

$$X^{[2]}\Big(C_t - (D(C)C_x)_x + C_x\Big)\Big|_{C_t=(D(C)C_x)_x - C_x} = 0, \tag{4.2}$$

where $X^{[2]}$ is the second prolongation described as before.

The equation (4.2) leads to the following determining equations

$$\begin{aligned}
\xi^1_c &= 0, && (4.3)\\
\xi^2_c &= 0, && (4.4)\\
\xi^1_x &= 0, && (4.5)\\
\eta_x - D\eta_{xx} + \eta_t &= 0, && (4.6)\\
-D'\eta + 2D\xi^2_x - D\xi^1_t &= 0, && (4.7)\\
-D''\eta - D'\eta_C - D\eta_{CC} + 2D'\xi^2_x - D'\xi^1_t &= 0, && (4.8)\\
-2D'\eta_x - \xi^2_x - 2D\eta_{xC} + D\xi^2_{xx} + \xi^1_t - \xi^2_t &= 0. && (4.9)
\end{aligned}$$

In the above system (4.3 - 4.9) and in subsequent calculations D' denotes differentiation with respect to C. The solution to the system (4.3 - 4.9) when

D is arbitrary is

$$\left.\begin{aligned} \eta &= 0, \\ \xi^1 &= 2m_1 t + m_3, \\ \xi^2 &= m_1(x+t) + m_2, \end{aligned}\right\} \tag{4.10}$$

where the m_i are arbitrary constants.

Thus equation (4.1) admits a 3D Lie algebra spanned by

$$\left.\begin{aligned} X_1 &= 2t\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial t}. \end{aligned}\right\} \tag{4.11}$$

Such Lie point symmetries span the principal Lie algebra (symmetries obtained for arbitrary functions).Our next task is to find the forms of D for which the principal Lie algebra extends.

From equation (4.7)

$$\eta = \frac{D}{D'}\left(2\xi^1_x - \xi^1_t\right). \tag{4.12}$$

From equation (4.12) we find η_C and η_{CC} to be

$$\eta_C = m - m\frac{DD''^{\,2}}{D'}\,, \tag{4.13}$$

$$\eta_{CC} = -m\left(\frac{(D'D'' + DD''')D' - 2D(D'')^2}{(D')^3}\right). \tag{4.14}$$

Substituting (4.14) into (4.8) it follows that D satisfies the ODE

$$DD'D''' - 2D(D'')^2 + D''(D')^2 = 0 \tag{4.15}$$

The equation (4.15), when solved, gives the following forms of the diffusivity D for which the principal Lie algebra is extended. Such expressions of D are of the form:

$D = (\alpha + \beta C)^m$ or $\quad D = e^{mC}$ or $\quad D = k$

where k, m, α and β are arbitrary constants.

4.3 Case 1 $D = k$

For simplicity we take the value of k to be unity. The equation then becomes

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}. \tag{4.16}$$

In this case equation (4.1) admits a six-dimensional Lie algebra spanned by

$$\left.\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + C(x-t)\frac{\partial}{\partial C}, \\
X_3 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + C(2tx - x^2 - 2t - t^2)\frac{\partial}{\partial C}, \\
X_4 &= \frac{\partial}{\partial x}, \\
X_5 &= 2t\frac{\partial}{\partial x} + C(t-x)\frac{\partial}{\partial C}, \\
X_6 &= C\frac{\partial}{\partial C}.
\end{aligned}\right\} \tag{4.17}$$

4.3.1 One-dimensional optimal system

In the next section we construct the one-dimensional optimal system sub-algebras of the algebra spanned by vectors in (4.17). The maximum number of symmetries admitted by the equation allows us to construct a commutator table where the commutator of two symmetries X_i and X_j is given by

$$[X_i, X_j] = X_i X_j - X_j X_i, \tag{4.18}$$

where the subscripts i and j take values from 1 to 6. Using the Lie point symmetries above we only demonstrate the use of(4.18) with calculation of the commutator $[X_1, X_2]$.

$$\begin{aligned}
[X_1, X_2] &= X_1 X_2 - X_2 X_1 \\
&= \left(\frac{\partial}{\partial t}\right)\left(4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + C(x-t)\frac{\partial}{\partial C}\right) - \left(4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + C(x-t)\frac{\partial}{\partial C}\right)\left(\frac{\partial}{\partial t}\right) \\
&= \left(4\frac{\partial}{\partial t} - C\frac{\partial}{\partial C}\right) - 0 \\
&= 4X_1 - X_6.
\end{aligned}$$

Worth noting in these calculations is that the commutator $[X_i, X_j]$ is zero if $i = j$ and $[X_i, X_j] = -[X_j, X_i]$. The following table is a summary of the commutators of the six-dimensional Lie algebra.

Using Table 4.1 of commutators and the formula

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] - ..., \tag{4.19}$$

Table 4.1: The commutator table of the sub-algebras

X_i, X_j	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$4X_1 - X_6$	$X_2 - X_6$	0	$2X_4 + X_6$	0
X_2	$X_6 - 4X_1$	0	$4X_3$	$-(2X_4 + X_6)$	$2X_5$	0
X_3	$X_6 - X_2$	$-4X_3$	0	$-2X_5$	0	0
X_4	0	$2X_4 + X_6$	$2X_5$	0	$-X_6$	0
X_5	$-2(X_4 + X_6)$	$-2X_5$	0	X_6	0	0
X_6	0	0	0	0	0	0

where i and j take values from 1 to 6, we work out the adjoint representation of X_i. We demonstrate calculations with two examples:

$$\begin{aligned} \mathrm{Ad}(e^{\epsilon X_1})X_5 &= X_5 - \epsilon[X_1, X_5] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_5]] - ..., \\ &= X_5 - \epsilon(2X_4 + X_6) + \frac{1}{2!}\epsilon^2[[X_1, 2X_4] + [X_1, X_6]] - ..., \\ &= X_5 - (2X_4 + X_6). \end{aligned}$$

$$\begin{aligned} \mathrm{Ad}(e^{\epsilon X_2})X_3 &= X_3 - \epsilon[X_2, X_3] + \frac{1}{2!}\epsilon^2[X_2, [X_2, X_3]] - ..., \\ &= X_3 - 4\epsilon X_3 + \frac{4^2\epsilon^2}{2!}X_3 - ..., \\ &= e^{-4\epsilon}X_3. \end{aligned}$$

Table 4.2 shows a summary of the entire results.

We now calculate the one-dimensional optimal system of the symmetry subgroups. Given the nonzero operator

$$X = a_1X_1 + a_2X_2 + ... + a_6X_6, \tag{4.20}$$

we aim to simplify as many as possible the coefficients a_i by careful application

Table 4.2: The adjoint table of sub-algebras

Adj.	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	$X_2 - 4\epsilon X_1 + \epsilon X_6$	$X_3 - \epsilon(X_2 - X_6)$ $+\frac{\epsilon^2}{2}(4X_1 - X_6)$	X_4	$X_5 - \epsilon(2X_4 + X_6)$	X_6
X_2	$X_1 - \epsilon(X_6 - 4X_1)$ $+2\epsilon^2(4X_1 - X_6)$	X_2	$e^{-4\epsilon}X_3$	$X_4 + \epsilon(2X_4 + X_6)$ $\epsilon^2(2X_4 + X_6)$	$e^{-2\epsilon}X_5$	X_6
X_3	$X_1 - \epsilon(X_6 - X_2)$ $+2\epsilon^2 X_3$	$X_2 + 4\epsilon X_3$	X_3	$X_4 + 2\epsilon X_5$	X_5	X_6
X_4	X_1	$X_2 - \epsilon(2X_4 + X_6)$	$X_3 - 2\epsilon X_5 - \epsilon^2 X_6$	X_4	$X_5 + \epsilon X6$	X_6
X_5	$X_1 + \epsilon(2X_4 + X_6)$ $-\epsilon^2 X_6$	$X_2 + 2\epsilon X_5$	X_3	$X_4 - \epsilon X_6$	X_5	X_6
X_6	X_1	X_2	X_3	X_4	X_5	X_6

of the adjoint maps to the operator X. From the adjoint table, acting on X by $\mathrm{Ad}(\exp(\alpha X_1)$, one obtains

$$\begin{aligned} X' &= (a_1 - 4\alpha a_2 + 2\alpha^2 a_3)X_1 + (a_2 - \alpha a_3)X_2 + a_3 X_3 + (a_4 - 2\alpha a_5)X_4 \\ &\quad + a_5 X_5 + \left(a_6 + \alpha(a_2 + a_3 - a_5 - \frac{\alpha}{2}a_3)\right) X_6. \end{aligned}$$

Acting on X' by $\mathrm{Ad}(\exp(\beta X_3))$ yields

$$\begin{aligned} X'' &= (a_1 - 4\alpha a_2 + 2\alpha^2 a_3)X_1 + \left[(a_2 - \alpha a_3) + \beta(a_1 - 4\alpha a_2 + 2\alpha^2 a_3)\right] X_2 \\ &\quad + \left[2\beta^2(a_1 - 4\alpha a_2 + 2\alpha^2 a_3) + 4\beta(a_2 - \alpha a_3) + a_3\right] X_3 \\ &\quad + (a_4 - 2\alpha a_5)X_4 + \left[a_5 + 2\beta(a_4 - 2\alpha a_5)\right] X_5 \\ &\quad + \left[a_6 + \alpha(a_2 + a_3 - a_5 - \frac{\alpha}{2}a_3) - \beta(a_1 - 4\alpha a_2 + 2\alpha^2 a_3)\right] X_6 \end{aligned}$$

To simplify the coefficients a_i we concentrate only on the coefficients $a_i, i = 1, 2, 3$ namely:

$$\left.\begin{aligned} \bar{a_1} &= a_1 - 4\alpha a_2 + 2\alpha^2 a_3 \\ \bar{a_2} &= a_2 - \alpha a_3 + \beta(a_1 - 4\alpha a_2 + 2\alpha^2 a_3) \\ \bar{a_3} &= 2\beta^2(a_1 - 4\alpha a_2 + 2\alpha^2 a_3) + 4\beta(a_2 - \alpha a_3) + a_3) + a_3 \end{aligned}\right\} \tag{4.21}$$

It is important to observe that the function $\eta = 4a_2^2 - 2a_3 a_1$ is the invariant of the full adjoint action. The nature of the invariant function determines how far we go as we try to simplify the operator X.

Three cases arise depending on the sign of η:

Case 1 For $\eta > 0$, we choose α to be the real root of $a_1 - 4\alpha a_2 + 2\alpha^2 a_3 = 0$, so that $\bar{a_1} = 0$. We also let $\beta = \dfrac{a_3}{4(\alpha a_3 - a_2)}$ so that $\bar{a_3} = 0$ as well.

Then $\bar{a_1} = \bar{a_3} = 0$ while $\bar{a_2} \neq 0$. In fact $\bar{a_2} = 3a_2 \pm \sqrt{4a_2^2 - 2a_3 a_1}$. Hence X is equivalent to the scalar multiple of

$$X' = X_2 + a_4X_4 + a_5X_5 + a_6X_6$$

where the coefficient of X_2 has been re-scaled to 1.

To eliminate a_4X_4 we act on X' by $\mathrm{Ad}(\exp(\frac{a_4}{2}X_4))$ and obtain

$$X'' = X_2 + a_5X_5 + \bar{a_6}X_6$$

for some $\bar{a_6} = a_6 + \frac{a_4a_5}{2} - \frac{a_4}{2}$.

To eliminate the coefficient of X_5 we apply $\mathrm{Ad}(\exp(\frac{-a_5}{2}X_5))$ to X''. In so doing we obtain

$$X''' = X_2 + \bar{a_6}X_6$$

It is not possible to simplify this further using entries from the adjoint table.

Case 2 For $\eta < 0$, we set $\alpha = 0$ so that $\bar{a_1} = a_1 = 0$ and choose $\beta = -a_2/a_1$. This makes $\bar{a_2} = 0$. In this case the nonzero operator X becomes

$$X = X_1 + X_3 + a_4X_4 + a_5X_5 + a_6X_6$$

where the coefficient of X_3 has been re-scaled to 1.

To eliminate the coefficient of X_5 we act on X by $\mathrm{Ad}(\exp(\frac{a_5}{2}X_4))$ and obtain

$$X' = X_1 + X_3 + a_4X_4 + a_6'X_6$$

where $a_6' = a_6 + a_5^2/4$. To eliminate the coefficient of X_4 we act on X' by $\mathrm{Ad}(\exp(\frac{-a_4}{2}X_5))$ and obtain

$$X'' = X_1 + X_3 + \bar{a_6}X_6$$

where $\bar{a_6} = a_6' - a_4/4$.

We observe that there are no entries in the adjoint table for further simplification.

Case 3 For $\eta = 0$, two different be dealt with-either all the coefficients a_1, a_2 and a_3 vanish or we choose α and β in (4.21) such that $\bar{a_1} \neq 0$ but $\bar{a_2} = \bar{a_3} = 0$.

In the later case we have

$$X = X_1 + a_4X_4 + a_5X_5 + a_6X_6.$$

We eliminate the coefficient of X_4 by acting on X by $\mathrm{Ad}(\exp(\frac{a_4}{2a_5}X_1))$ to obtain

$$X' = X_1 + a_5X_5 + a_6'X_6,\ a_5 \neq 0,.$$

Here, $a_6' = a_6 - (a_4a_5)/2$. Next we act on X' by $\mathrm{Ad}(\exp(\frac{-a_6'}{a_5}X_4))$ to get rid of the coefficient of X_6 and obtain

$$X'' = X_1 + a_5X_5,\ a_5 \neq 0.$$

Now suppose $a_5 = 0$. The operator X becomes

$$X = X_1 + a_4X_4 + a_6X_6.$$

To eliminate a_4X_4, we act on X by $\mathrm{Ad}(\exp(\frac{-a_4}{2}X_5))$ and obtain

$$X' = X_1 + \bar{a_6}X_6$$

where $\bar{a_6} = \frac{a_4^2}{4} - \frac{a_4}{4}$.

When $\bar{a_1} = \bar{a_2} = \bar{a_3} = 0$,

$$X = X_4 + a_5X_5 + a_6X_6$$

after re-scaling the coefficient of X_4 to be 1.

To eliminate a_6X_6, we act on X by $\mathrm{Ad}(\exp(a_6X_5))$ and obtain

$$X' = X_4 + a_5X_5$$

We can eliminate a_5X_5 by applying $\mathrm{Ad}(\exp(\frac{-a_5}{2}X_3))$ to X'. This gives

$$X'' = X_4.$$

The complete one-dimensional optimal system of sub-algebras is

$$\{X_4,\ X_1 + aX_5,\ X_1 + aX_6,\ X_2 + aX_6,\ X_1 + X_3 + aX_6\}, \tag{4.22}$$

for $a \in \mathbb{R}$.

4.3.2 Symmetry reductions and invariant solutions

We perform symmetry reductions using the elements of the optimal system (4.22). Reduction under space translation is left out. We only show calculations for reduction under $X_2 + X_6$ ($a = 1$). The remaining cases are summarized in Table 4.3.

For invariance under $X_2 + X_6$, we solve the characteristic equations

$$\frac{dt}{4t} = \frac{dx}{2x} = \frac{dC}{C(x - t + 1)}. \tag{4.23}$$

Solving (4.23) leads to

$$C(x,t) = t^{\frac{1}{4}} \exp\left\{\frac{x}{2} - \frac{t}{4}\right\} F\left(\frac{x}{\sqrt{t}}\right). \tag{4.24}$$

Substituting into the original PDE one obtains the ODE

$$4F'' - 2\gamma F' - F = 0, \quad \gamma = \frac{x}{\sqrt{t}}. \tag{4.25}$$

The ODE (4.25) admits the symmetry

$$X = F\frac{\partial}{\partial F}. \tag{4.26}$$

The first prolongation of X is given by

$$X^{[1]} = 0\frac{\partial}{\partial \gamma} + F\frac{\partial}{\partial F} + F'\frac{\partial}{\partial F'} \tag{4.27}$$

and from this we obtain the invariants

$$u = \gamma, \quad v = \frac{F}{F'}.$$

By writing $v = v(u)$ and from the definitions of u and v and the chain rule, one arrives at the first order ODE

$$4v' + v - 2u - 4 = 0. \tag{4.28}$$

The solution to (4.25) can be expressed as

$$F = B\exp\left\{\int \frac{d\gamma}{-4 + 2\gamma + Ae^{\frac{-\gamma}{4}}}\right\}, \tag{4.29}$$

where A and B are arbitrary constants. Hence $C(x,t)$ is found to be

$$C(x,t) = Bt^{\frac{1}{4}}\exp\left\{\frac{x}{2} - \frac{t}{4}\right\}\exp\left\{\int \frac{d\gamma}{-4 + 2\gamma + Ae^{\frac{-\gamma}{4}}}\right\}, \tag{4.30}$$

where $\gamma = \gamma(x,t)$ as defined before. Table 4.3 shows a summary of reduction by the entire optimal system.

Table 4.3: Summary of symmetry reductions and invariant solutions given constant D

Symmetry	Reduced Equation	Invariant solution in terms of orginal variables
$X_1 + X_5$	$F'' + F' - \gamma F = 0,\ \gamma = t^2 - x$	$C(x,t) = A\ e^{\frac{x-t^2}{2}}\text{Ai}\left(\frac{1}{4} + t^2 - x\right)$ $+B\ e^{\frac{x-t^2}{2}}\text{Bi}\left(\frac{1}{4} + t^2 - x\right)$
$X_1 + X_6$	$F'' - F' - F = 0,\ F = F(x)$	$C(t,x) = e^t\left(A\ e^{\frac{1+\sqrt{5}}{2}x} + B\ e^{\frac{1-\sqrt{5}}{2}x}\right)$
$X_2 + X_6$	$4F'' + 2\gamma F' - F = 0,\ \gamma = \frac{x}{\sqrt{t}}$	$C(x,t) = Bt^{\frac{1}{4}}\exp\left\{\frac{x}{2} - \frac{t}{4}\right\}$ $\exp\left\{\int \frac{d\gamma}{-4 + 2\gamma + Ae^{\frac{-\gamma}{4}}}\right\}$
$X_1 - X_6$	$F'' - F' + F = 0,\ F = F(x)$	$C(x,t) = e^{-t}\left[A\ e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)\right.$ $\left.+Be^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$
$X_1 + X_3 + aX_6$	$16\gamma^4 F'' + 24\gamma^3 F'$ $+(4 - 5\gamma)F = 0,\ \gamma = \frac{4t^2+1}{x^2}$	$C(x,t) = \frac{1}{\sqrt{x}}\exp\left[\frac{5}{8}\tan^{-1}(2t) - \right.$ $\left.\frac{x^2 t}{4t^2+1} - \frac{t}{4} + \frac{x}{2}\right]$ $\times\left[A\ M\left(\frac{5i}{16}; \frac{1}{4}; \frac{ix^2}{4t^2+1}\right)\right.$ $\left.+B\ W\left(\frac{5i}{16}; \frac{1}{4}; \frac{ix^2}{4t^2+1}\right)\right]$

From Table 4.3, the reduction by $X_1 - X_6$ leads to the solution

$$C(x,t) = e^{\frac{x}{2}-t}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\cos\left(\frac{\sqrt{3}}{2}x\right)\right]. \tag{4.31}$$

The initial condition $C(x_0 = 0,\ t_0 = 0) = 1$ leads to $A = 1$. To find the value of the arbitrary constant B, suppose that at an instant, say $t_1 = 5$, we have $x_1 = L_1 = 10$. Then the boundary condition

$$\left.\frac{dC}{dx}\right|_{x_1=L_1} = 0$$

leads to $B = -3.44$.

Figures 4.1-4.3 are typical concentration profiles for solution (4.31).

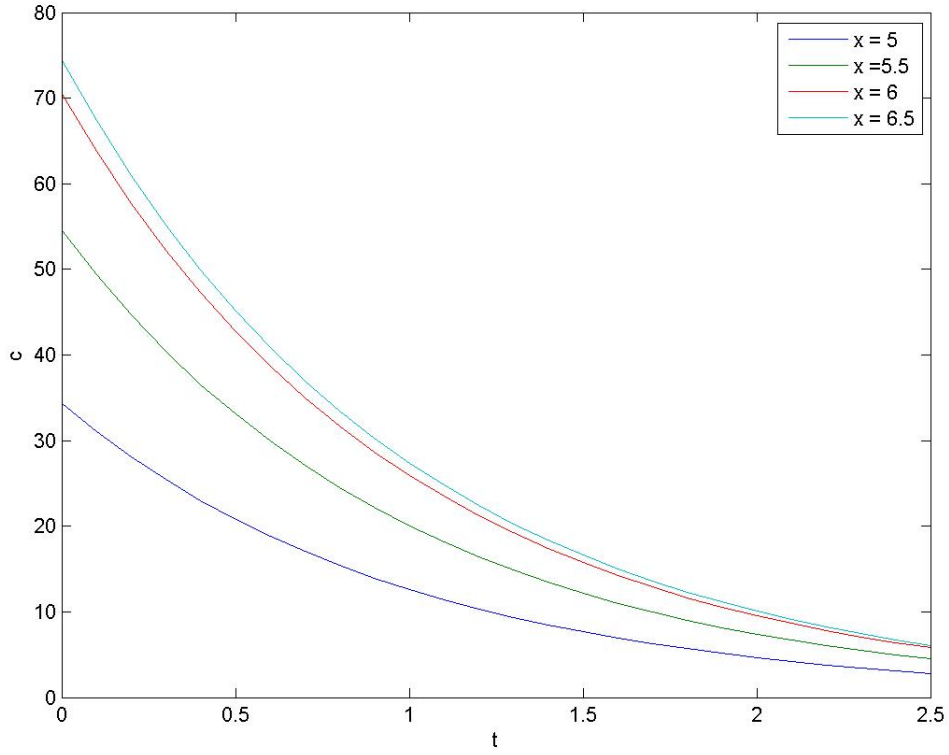

Figure 4.1: Graph of concentration with time given varying values of the spatial variable x for solution (4.31).

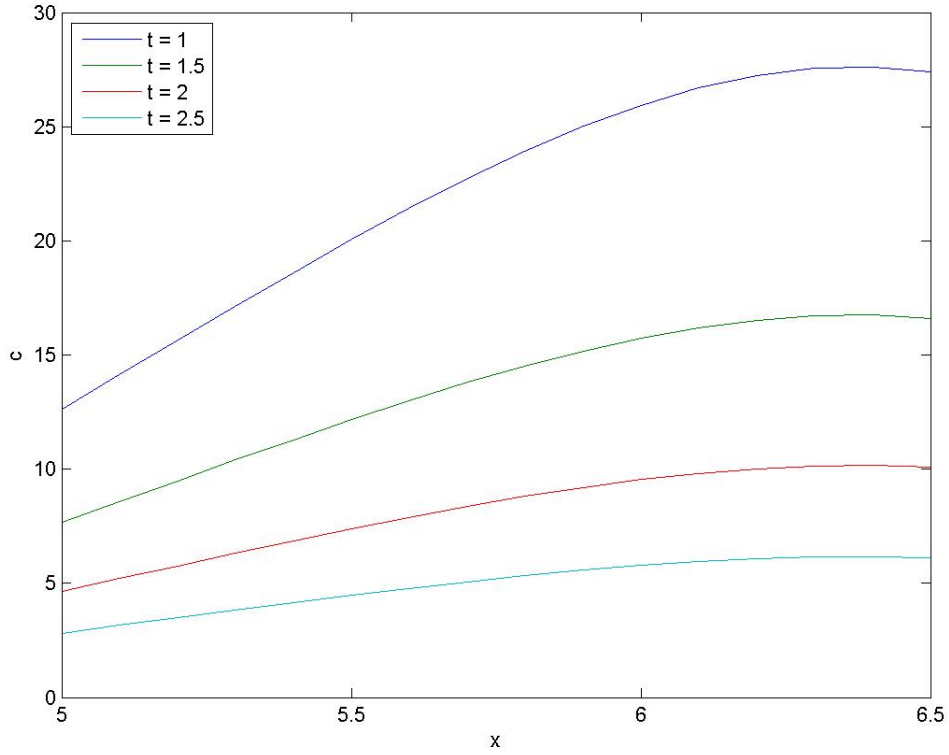

Figure 4.2: Graph of concentration vs space given varying values of the variable t for solution (4.31).

4.4 Case 2 $D = (\alpha + \beta C)^m$

In this case our governing equation reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[(\alpha + \beta C)^m \frac{\partial C}{\partial x}\right] - \frac{\partial C}{\partial x}. \tag{4.32}$$

and it admits a four-dimensional Lie algebra spanned by the vector fields

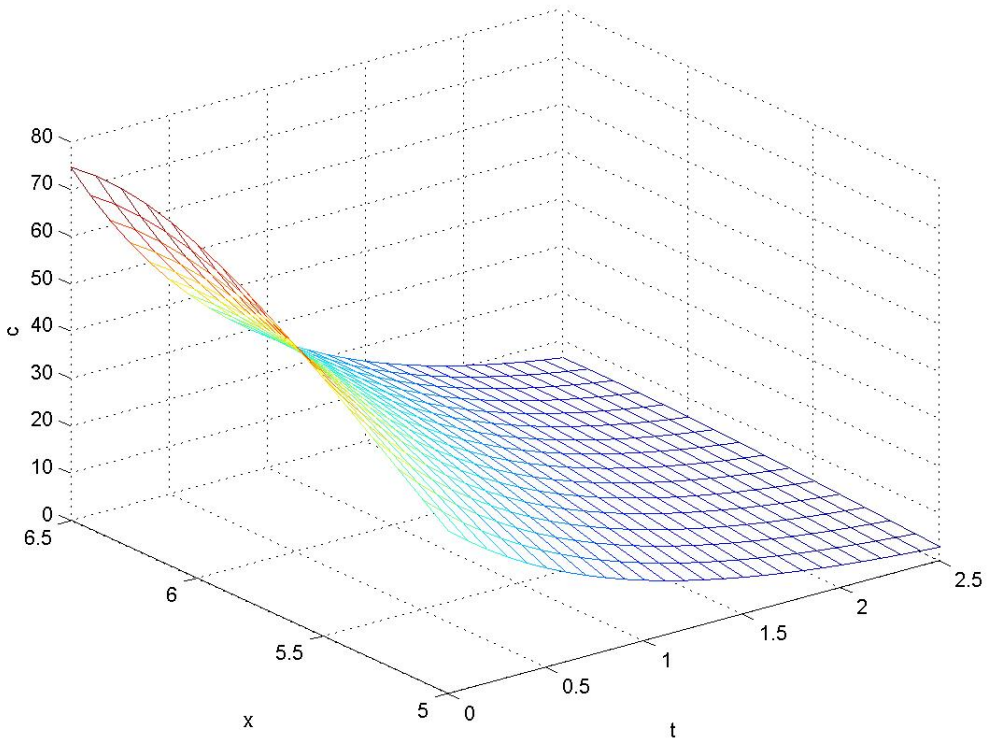

Figure 4.3: Surface plot of concentration with time and space for solution (4.31).

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial x}, \\ X_3 &= -\frac{t}{\beta}\frac{\partial}{\partial t} - \frac{t}{\beta}\frac{\partial}{\partial x} + \frac{\alpha+\beta C}{m\beta^2}\frac{\partial}{\partial C}, \\ X_4 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{4.33}$$

4.4.1 One-dimensional optimal system

Following are the corresponding commutator and adjoint representation tables. Calculations have been explained before.

Table 4.4: The commutator table of the sub-algebras

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$2X_1 + X_4$	$-\frac{1}{\beta}(X_1 + X_4)$	0
X_2	$-(2X_1 + X_4)$	0	0	$-X_4$
X_3	$\frac{1}{\beta}(X_1 + X_4)$	0	0	0
X_4	0	X_4	0	0

Table 4.5: The adjoint table of the sub-algebras

Adjoint	X_1	X_2	X_3	X_4
X_1	X_1	$X_2 - \epsilon(2X_1 + X_4)$	$X_3 + \frac{\epsilon}{\beta}(X_1 + X_4)$	X_4
X_2	$e^{2\epsilon}X_1 + \epsilon X_4 + 3\frac{\epsilon^2}{2}X_4$	X_2	X_3	$e^{\epsilon}X_4$
X_3	$e^{-\frac{\epsilon}{\beta}}X_1 - \frac{\epsilon}{\beta}X_4 + \frac{\epsilon^2}{2\beta^2}X_4$	X_2	X_3	X_4
X_4	X_1	$X_2 - \epsilon X_4$	X_3	X_4

To construct the one-dimensional optimal system of the symmetry group (4.37) we use the operator

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 \tag{4.34}$$

where the a_i are arbitrary constants. As before we aim to simplify as much as we can these constants so as to have a much simpler (4.34).

We begin by assuming $a_2 = 1$. From (4.34) we have

$$X = X_2 + a_1X_1 + a_3X_3 + a_4X_4.$$

With reference to Table 4.5 we act on X by $\mathrm{Ad}(\exp(-\frac{a_1\beta}{a_3-2\beta}X_1))$ to eliminate a_1X_1. This gives

$$X' = X_2 + a_3X_3 + \bar{a_4}X_4.$$

for some constant $\bar{a_4}$. We can eliminate $\bar{a_4}X_4$ by applying $\mathrm{Ad}(\exp(\bar{a_4}X_4))$, on X' to get

$$X'' = X_2 + a_3X_3.$$

We cannot simplify this further.

Next we assume $a_2 = 0$ and $a_3 \neq 0$. We take $a_3 = 1$ and the operator (4.34) becomes

$$X = X_3 + a_1X_1 + a_4X_4.$$

Acting on X by $\mathrm{Ad}(\exp(-a_1\beta X_1))$ eliminates the coefficient of X_1 to obtain

$$X' = X_3 + a_4 X_4.$$

Lastly we choose $a_2 = a_3 = 0$ and this simplifies the operator to

$$X = a_1 X_1 + a_4 X_4$$

under which $a_1 \neq 0$ gives $X = X_1$. Otherwise, $X = X_4$. Thus the one-dimensional optimal system of sub-algebras is

$\{X_1,\ X_4,\ X_3 + aX_4,\ X_2 + aX_3\}$

where $a \in \mathbb{R}$.

4.4.2 Symmetry reductions and invariant solutions

(a) *Reduction by* $X_3 + aX_4$

For $a = 1$ we solve the characteristic equations

$$-\frac{\beta}{t}dt = \frac{\beta}{\beta - t}dx = \frac{m\beta^2}{\alpha + \beta C}dC,$$

and find that C takes the form

$$C = \frac{1}{\beta}\left(F^{\frac{1}{m}}t^{-\frac{1}{m}} - \alpha\right).$$

The function F satisfies the nonlinear ODE

$$m\gamma^2 FF'' + \gamma(\gamma F' + mF - m\beta)F' + mF = 0, \tag{4.35}$$

where $\gamma = t^\beta e^{x-t}$.

(b) *Reduction by* $X_2 + aX_3$

For $a = 1$ we solve the characteristic equations

$$\frac{\beta}{t(2\beta - 1)}dt = \frac{\beta}{\beta x + t(\beta - 1)}dx = \frac{m\beta^2}{\alpha + \beta C}dC,$$

and find that C takes the form

$$C = \frac{1}{\beta}\left[F^{\frac{1}{m}}t^{\frac{1}{m(2\beta-1)}} - \alpha\right].$$

Reduction of equation (4.32) leads to the nonlinear ODE

$$(1 - 2\beta)(mFF'' + F'^2) - m\beta\gamma F' + mF = 0, \tag{4.36}$$

where $\gamma = t^{\frac{\beta}{1-2\beta}}(x - t)$.

We are unable to find any exact solutions of equation (4.36).

We mention here that for the special case $m = -4/3$, equation (4.32) extends its Lie algebra to a five-dimensional one spanned by the vectors

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial x}, \\ X_3 &= \frac{t}{3\beta}\frac{\partial}{\partial t} + \frac{t}{3\beta}\frac{\partial}{\partial x} + \frac{\alpha+\beta C}{4\beta^2}\frac{\partial}{\partial C}, \\ X_4 &= \frac{1}{12\beta^2}\left\{-\beta(x^2+2tx+t^2)\frac{\partial}{\partial x} + [3x(\alpha+\beta C) \right. \\ &\qquad \left. -3t(\alpha+\beta C)]\frac{\partial}{\partial C}\right\}, \\ X_5 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{4.37}$$

4.5 Case 3 $D = e^{mC}$

In this case equation (4.1) admits four Lie point symmetries spanned by

$$\left.\begin{aligned} X_1 &= m(x-t)\frac{\partial}{\partial x} + 2\frac{\partial}{\partial C}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= mt\frac{\partial}{\partial t} + mt\frac{\partial}{\partial x} - \frac{\partial}{\partial C}, \\ X_4 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{4.38}$$

4.5.1 The one-dimensional optimal system

In order to construct the one-dimensional optimal system one requires the commutators and adjoint representation of this symmetry group in Tables 4.6 and 4.7, respectively.

Table 4.6: The commutator table of the sub-algebras

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	mX_4	0	$-mX_4$
X_2	$-mX_4$	0	$m(X_2 + X_4)$	0
X_3	0	$-m(X_2 + X_4)$	0	0
X_4	mX_4	0	0	0

Table 4.7: The adjoint table of the sub-algebras

Adjoint	X_1	X_2	X_3	X_4
X_1	X_1	$X_2 - m\epsilon X_4$ $-m^2\frac{\epsilon^2}{2!}X_4$	X_3	$e^{m\epsilon}X_4$
X_2	$X_1 + m\epsilon X_4$	X_2	$X_3 - m\epsilon(X_2 + X_4)$	X_4
X_3	X_1	$X_2 + m\epsilon(X_2 + X_4)$ $+m^2\frac{\epsilon^2}{2!}(X_2 + X_4)$	X_3	X_4
X_4	$X_1 - m\epsilon X_4$	X_2	X_3	X_4

The one-dimensional optimal system of sub-algebras is worked out to be $\{X_1,\ X_2,\ ,X_1 + X_3,\ X_3 + X_4\}$.

4.5.2 Symmetry reduction and invariant solutions

We show reduction by $X_3 + X_4$ and only present the results for the reduction using the elements X_1 and $X_1 + X_3$. Reduction by $X_3 + X_4$ yields the ODE

$$\gamma^2 FF'' + \gamma FF' - \gamma F' + F = 0. \tag{4.39}$$

The ODE (4.39) is obtained when $m = 1$. The same is done for the other two reductions. It is found that (4.39) admits

$$X = \gamma \frac{\partial}{\partial \gamma} \tag{4.40}$$

as the only symmetry algebra. By employing the method of differential invariants, we obtain the invariants

$$u = F, \quad v = \gamma F'.$$

By writing $v = v(u)$ and using the chain rule the ODE (4.39) reduces to the first order ODE

$$v' = \frac{v - u}{uv}. \tag{4.41}$$

We are unable to go further than this point. Table 4.8 summarizes the results of the reduction and invariant solutions by the elements of the optimal system.

Table 4.8: Symmetry reduction and invariant solutions given $D = e^{mc}$

Symmetry	Reduced Equation	Invariant solution in terms of original variables
X_1	$F' - 2F^2 + 2F = 0,$ $F = F(t)$	$c(x,t) = \ln\left(\frac{x-t}{1+Ae^{2t}}\right)$
$X_1 + X_3$	$\gamma^2 F^2 F'' + (2\gamma F^2 - \gamma F + F)F'$ $-F^2 = 0,\ \gamma = \frac{x}{t}$	No solution
$X_3 + X_4$	$\gamma^2 FF'' + \gamma FF' - \gamma F' + F = 0,$ $\gamma = te^{t-x}$	$c(x,t) = \ln\left(\frac{F(te^{t-x})}{t}\right)$

Some discussion

Figures 4.1-4.3 depict the particular solution (4.31) of equation (4.16). We note that the concentration increases with increasing distance for a fixed time t while it drops significantly as time evolves for a fixed distance x. We also note that as time evolves the concentration tends to remain constant. This behavior explains the phenomenon of evaporation of a substance from the base to the surface in porous media. Here $x = 0$ is at the base and the initial concentration is maximum. At the surface $x = L$ and the concentration is initially zero.

Concluding remarks

In this chapter we have successfully carried out direct group classification of equation (2.18) when the diffusion coefficient is defined by only the contaminant concentration and the transport velocity is unity. In this case we found the principal Lie algebra to be three-dimensional. We also found the forms of the diffusion coefficient for which the principal Lie algebra extends.

With these new algebra, we have carried out symmetry reductions and, where possible, constructed the associated group invariant solutions.

Chapter 5

Invariant solutions of the Advection-Dispersion Equation for solute transport with non-constant water velocity

In this chapter we extend our work to finding invariant solutions when the diffusion coefficient and transport velocity assume different functions of the spatial variable. We observe that direct group classification of equation (2.18) will be a major task. However, we focus on the following realistic cases.

5.1 Symmetry analysis and reductions

Case 1: Given $D(x) = e^{mx}$ and constant u

In this case the equation admits a four-dimensional Lie algebra spanned by

the vectors

$$\left.\begin{aligned} X_1 &= C\frac{\partial}{\partial C}, \\ X_2 &= e^{kmt}\left(\frac{\partial}{\partial t} - k\frac{\partial}{\partial x} - k^2 e^{-mx} C\frac{\partial}{\partial C}\right), \\ X_3 &= -e^{-kmt}\left(\frac{\partial}{\partial t} + k\frac{\partial}{\partial x}\right), \\ X_4 &= \frac{\partial}{\partial t}, \end{aligned}\right\} \tag{5.1}$$

for $k,\ m \in \mathbb{R}$.

The one-dimensional optimal system of the sub-algebras (5.1) is

$\{X_3,\ X_4,\ X_1 + aX_3,\ X_1 + aX_4,\ X_2 + aX_4\}$, for $a \in \mathbb{R}$.

Without loss of generality, we consider the constant a to be unity. Calculations are shown only for reduction by $X_1 + X_3$. The basis of invariants is constructed by solving the characteristic equations

$$-\frac{dt}{e^{-kmt}} = -\frac{dx}{ke^{-kmt}} = \frac{dC}{C}. \tag{5.2}$$

By solving (5.2) we obtain

$$C = \exp\left(-\frac{1}{mk}e^{kmt}\right) F(kt - x). \tag{5.3}$$

Substituting into the governing equation results to the ODE

$$F'' - mF' + e^{m\gamma}F = 0, \tag{5.4}$$

where $\gamma = kt - x$.

The solution to this ODE, in terms of the original variables, is given by

$$F(x,t) = e^{m(kt-x)}\left[A\ J_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right) + B\ Y_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right)\right], \tag{5.5}$$

where J_1 and Y_1 are Bessel functions of the first and second kind respectively. A and B are constants of integration.

Hence the solution to the governing PDE is given by

$$C(x,t) = \exp\left(-\frac{e^{kmt}}{km}\right) e^{m(kt-x)} \left[A\ J_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right) + B\ Y_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right)\right] \tag{5.6}$$

Below is a table summarizing the symmetry reductions and invariant solutions under this case.

Table 5.1: Summary of results of the reduction when $\forall\ k,\ m \neq 0$.

Symmetry	Reduced Equation	Invariant solution in terms of original variables
$X_1 + X_4$	$e^{mx}F'' + (me^{mx} - k)F' - F = 0$	$C(x,t) = \exp\left(t - mx - \frac{ke^{-mx}}{m}\right) \times \left[A\ M\left(\frac{mk+1}{mk}; 2; \frac{ke^{-mx}}{m}\right) + B\ U\left(\frac{mk+1}{mk}; 2; \frac{ke^{-mx}}{m}\right)\right]$
$X_1 + X_3$	$F'' - mF' + e^{m\gamma}F = 0,$ $\gamma = kt - x$	$C(x,t) = \exp\left(-\frac{e^{kmt}}{km}\right) e^{m(kt-x)} \times \left[A\ J_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right) + B\ Y_1\left(\frac{2e^{\frac{m}{2}(kt-x)}}{m}\right)\right]$
X_3	$F'' - mF' = 0,$ $F = F(\gamma),\ \gamma = kt - x$	$C(x,t) = \frac{A}{m}e^{m(kt-x)} + B$
$X_2 + X_4$	$m\gamma^2 F'' + (2m\gamma + k)F' = 0,$ $\gamma = e^{mx}(1 + e^{kmt})$	$C(x,t) = -\frac{Am}{k}\exp\left(\frac{k}{me^{mx}(1+e^{kmt})}\right) + B$

In the table M and U are Kummer confluent hypergeometric functions.

Case 2: Given $D(x) = e^{mx}, \quad u(x) = x^n$

Our governing equation in this case reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(e^{mx}\frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial x}\left(x^n C\right). \tag{5.7}$$

We consider analysis for chosen values of m and n for $m,\ n \in \mathbb{R}$.

sub-case 2.1: $m = 0$ and $n = -2$

In this case equation (5.7) admits only two Lie point symmetries namely

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = C\frac{\partial}{\partial C}. \tag{5.8}$$

We perform reduction by $X_1 + aX_2$ and we solve the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{aC}. \tag{5.9}$$

This leads to

$$C = e^{at}F(x). \tag{5.10}$$

Substitution of (5.10) into the governing equation produces the second order ODE

$$x^3F'' - xF' + (2 - ax^2)F = 0. \tag{5.11}$$

We are, however, unable to find any exact solutions to (5.11).

sub-case 2.2: $m = 0$ and $n = -1$

The Lie algebra extends to four Lie point symmetries in this case, and, it is spanned by the vectors

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ X_3 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - x^2C\frac{\partial}{\partial C}, \\ X_4 &= C\frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.12}$$

The corresponding one-dimensional optimal system of sub-algebras is

$\{X_3,\ X_4,\ X_1 + aX_4,\ X_2 + aX_4,\ X_3 + aX_4\}$,

for $a \in \mathbb{R}$.

Symmetry reductions are performed using each of these elements. We omit reduction by X_4. The arbitrary constant a is taken as unity.

(a) *Reduction by* X_3

We solve the characteristic equations

$$\frac{dx}{4tx} = \frac{dt}{4t^2} = -\frac{dC}{x^2C}. \tag{5.13}$$

This implies

$$C = \exp\left(-\frac{x^2}{4t}\right) F\left(\frac{x}{t}\right). \tag{5.14}$$

Substituting into the governing equation results in the ODE

$$\gamma^2 F'' - \gamma F' + F = 0, \tag{5.15}$$

where $\gamma = x/t$.

The solution to (5.15) is

$$F(\gamma) = A\ \gamma + B\ \ln\gamma \tag{5.16}$$

where A and B are arbitrary constants. Hence we find the invariant solution, in terms of the original variables, to be

$$C(x,t) = \exp\left(-\frac{x^2}{4t}\right)\left[A\frac{x}{t} + B\ \ln\left(\frac{x}{t}\right)\right]. \tag{5.17}$$

For the solution (5.17), the boundary conditions

$$C(L_1 = 0.02,\ t_1 = 0.01) = 1$$

$$\left.\frac{\partial C}{\partial x}\right|_{x=L_2=10,\ t_2=5} = 0$$

lead to $A = 0.71$ and $B = -0.59$.

Figures 4.1-4.3 are typical concentration profiles for solution (5.17).

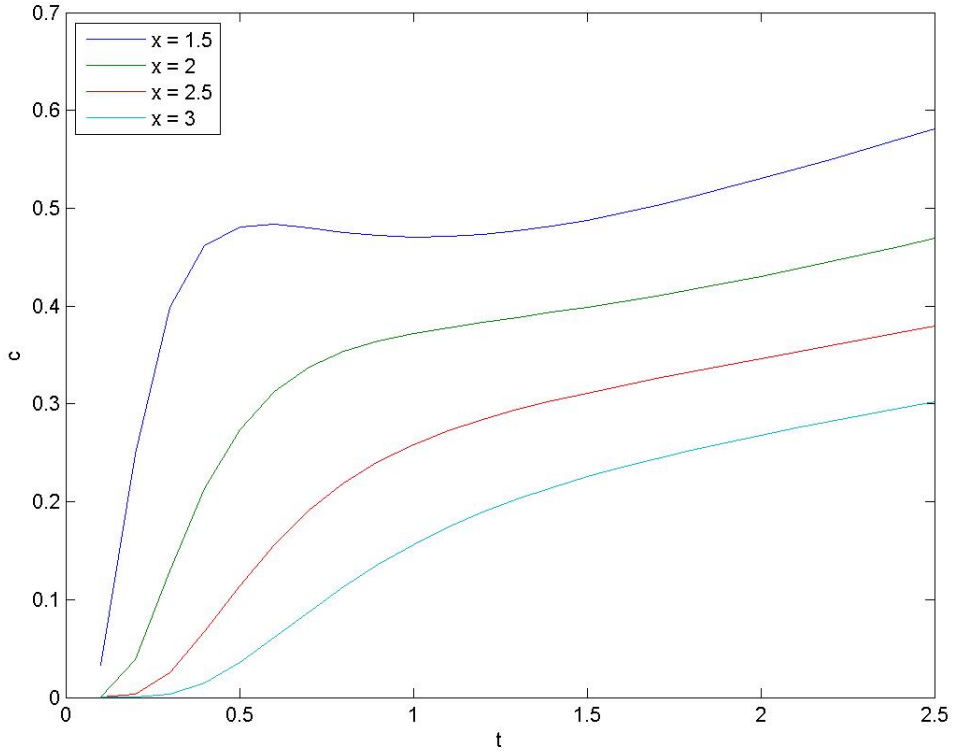

Figure 5.1: A plot of concentration with time at various spatial values for solution (5.17).

(b) *Reduction by* $X_1 + X_4$

Solving the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{C} \tag{5.18}$$

leads to

$$C = e^t F(x). \tag{5.19}$$

When substituted into the governing PDE the result (5.19) gives us the ODE

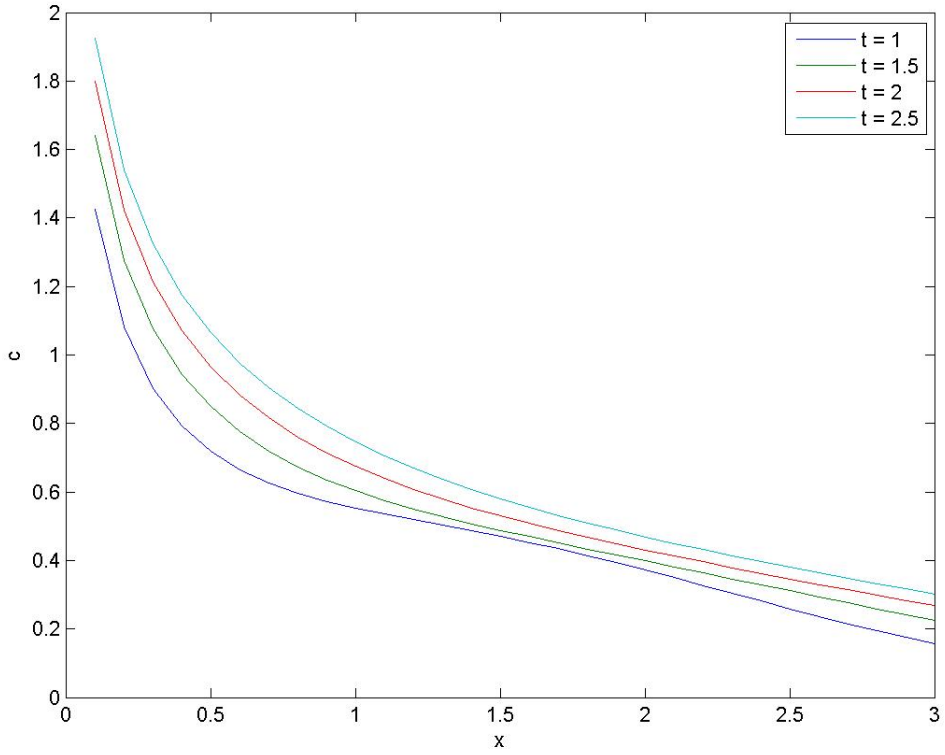

Figure 5.2: A plot of concentration with position at various times for solution (5.17).

$$x^2F'' - xF' + (1 - x^2)F = 0. \tag{5.20}$$

The equation (5.20) has the solution

$$F(x) = x[A\ I_0(x) + B\ K_0(x)], \tag{5.21}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind respectively.

Hence we can write the invariant solution as

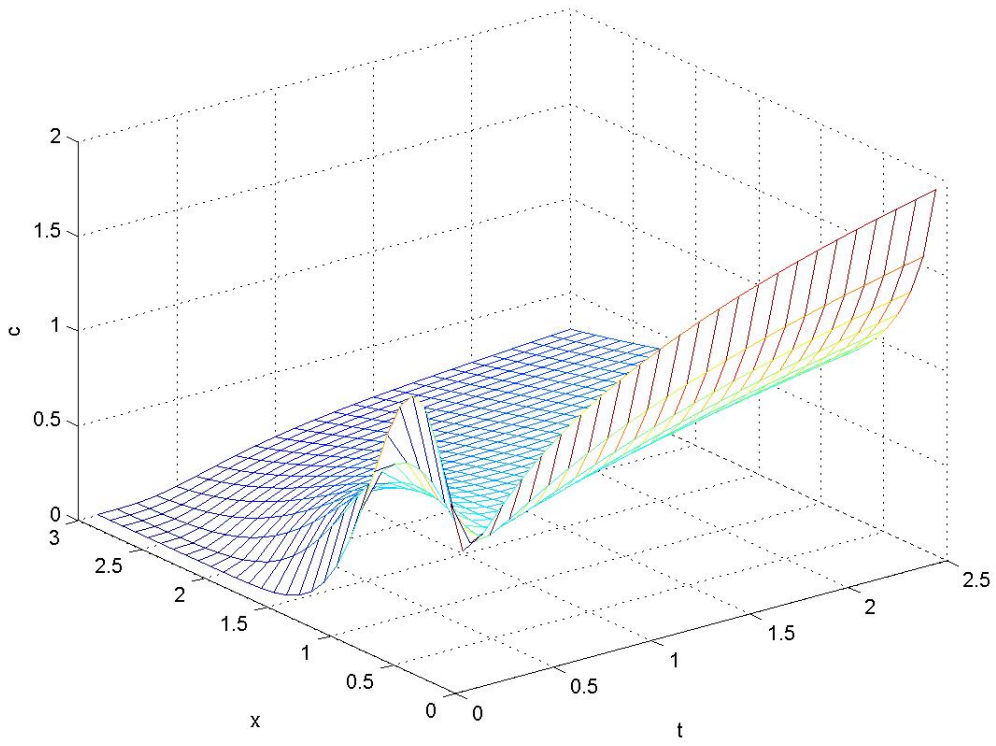

Figure 5.3: A plot of concentration with time and space for solution (5.17).

$$C(x,t) = xe^t \left[A\ I_0(x) + B\ K_0(x)\right]. \tag{5.22}$$

(c) *Reduction by* $X_2 + X_4$

Solving the characteristic equations

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{dC}{C}, \tag{5.23}$$

one obtains the result

$$C = \sqrt{t} F\left(\frac{x}{\sqrt{t}}\right). \tag{5.24}$$

By substituting into the governing PDE one gets the ODE

$$2\gamma^2 F'' + \gamma(\gamma^2 - 2)F' + (2 - \gamma^2)F = 0, \tag{5.25}$$

where $\gamma = x/\sqrt{t}$.

The solution to (5.25) is

$$F(x,t) = A\ \frac{x}{\sqrt{t}} + \frac{B}{2}\frac{x}{\sqrt{t}} Ei\left(\frac{x^2}{4t}\right), \tag{5.26}$$

where Ei is the Exponential integral function.

We write our invariant solution as

$$C(x,t) = Ax + \frac{B}{2} xEi\left(\frac{x^2}{4t}\right). \tag{5.27}$$

(d) *Reduction by* $X_3 + X_4$

We solve the characteristic equations

$$\frac{dt}{4t^2} = \frac{dx}{4tx} = \frac{dC}{C(1 - x^2)} \tag{5.28}$$

and from this we write

$$C = \exp\left(-\frac{x^2 + 1}{4t}\right) F\left(\frac{x}{t}\right). \tag{5.29}$$

Substituting into the governing equation produces the ODE

$$4\gamma^2 F'' - 4\gamma F' + (4 - \gamma^2)F = 0, \tag{5.30}$$

where $\gamma = x/t$.

The solution to (5.30), in terms of the original variables, is

$$F(x,t) = A\ \frac{x}{t} I_0\left(\frac{x}{2t}\right) + B\ \frac{x}{t} Y_0\left(\frac{x}{2t}\right), \tag{5.31}$$

where I_0 and K_0 are Bessel functions.

Hence the invariant solution is given as

$$C(x,t) = \frac{x}{t}\exp\left(-\frac{x^2+1}{4t}\right)\left[A\ I_0\left(\frac{x}{2t}\right) + B\ K_0\left(\frac{x}{2t}\right)\right]. \tag{5.32}$$

Case 3: Given $D(x) = e^{mx}$, $u(x) = e^{nx}$

Our governing equation in this case reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(e^{mx}\frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial x}\left(e^{nx}C\right), \tag{5.33}$$

where $m,\ n \in \mathbb{R}$.

Sub-case 3.1: $m \neq n$

We consider the sub-case $m = 0,\ n \neq 0$.

The equation (5.33) admits only the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = C\frac{\partial}{\partial C} \tag{5.34}$$

under this condition. Hence we perform symmetry reductions using the linear combination $X_1 + aX_2$ and solve the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{aC}. \tag{5.35}$$

This leads to

$$C = e^{at}F(x). \tag{5.36}$$

Substitution of (5.36) into the governing equation produces the second order ODE

$$F'' - e^{nx}F' - (a + ne^{nx})F = 0. \tag{5.37}$$

The solution to the ODE (5.37) is

$$\begin{aligned} F(x) &= A\exp\left(\frac{1}{2}\left[e^{nx} - \sqrt{4a + e^{2nx} + 4ne^{nx}}\right]\right) \\ &\quad + B\exp\left(\sqrt{4a + e^{2nx} + 4ne^{nx}}\right). \end{aligned} \tag{5.38}$$

Hence, under this condition we find the invariant solution to be

$$\begin{aligned} C(x,t) &= e^{at}\left\{A\exp\left(\frac{1}{2}\left[e^{nx} - \sqrt{4a + e^{2nx} + 4ne^{nx}}\right]\right)\right. \\ &\quad \left. + B\exp\left(\sqrt{4a + e^{2nx} + 4ne^{nx}}\right)\right\} \end{aligned} \tag{5.39}$$

Sub-case 3.2: $m = n,\ n \neq 0$

In this instance the PDE (5.33) admits the four Lie point symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2nt\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x} - C\frac{\partial}{\partial C}, \\ X_3 &= n^2t^2\frac{\partial}{\partial t} - 2nt\frac{\partial}{\partial x} - C(e^{-nx} + nt)\frac{\partial}{\partial C}, \\ X_4 &= C\frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.40}$$

Th complete one-dimensional optimal system of these sub-algebras is

$$\{X_2,\ X_3,\ X_2 + aX_4,\ X_3 + aX_4\}. \tag{5.41}$$

We show calculations for reduction under X_3.

Taking the constant a as unity again, we show the remaining symmetry reductions and group invariant solutions using (5.41) in Table 5.2.

To carry out reduction using X_3, we solve the characteristic equations

$$\frac{dt}{n^2t^2} = -\frac{dx}{2nt} = -\frac{dC}{C(nt + e^{-nx})}. \tag{5.42}$$

Solving these characteristic equations one obtains

$$C(x,t) = \exp\left(\frac{x}{2} - \frac{e^{-nx}}{n^2t}\right) F\left(te^{\frac{nx}{2}}\right), \tag{5.43}$$

and substitution into the original PDE yields the ODE

$$n^2\gamma^2F'' + 3n^2\gamma F' - (2n+1)F = 0, \quad \gamma = te^{\frac{nx}{2}}. \tag{5.44}$$

The solution to equation (5.44) is found to be

$$F(\gamma) = A\ \gamma^{\frac{1}{n}} + B\ \gamma^{-\frac{2n+1}{n}}. \tag{5.45}$$

Hence under this reduction we find the invariant solution, in terms of the original variables, to be

$$C(x,t) = \exp\left[\frac{x}{2} - \frac{e^{-nx}}{n^2t}\right]\left[A\ (te^{\frac{nx}{2}})^{\frac{1}{n}} + B\ (te^{\frac{nx}{2}})^{-\frac{2n+1}{n}}\right], \tag{5.46}$$

where A and B are constants.

Table 5.2 presents a summary of the complete reduction by the elements (5.41).

Table 5.2: Summary of results of the reduction when $m = n,\ n \neq 0$.

Symmetry	Reduced Equation	Invariant solution in terms of original variables
X_3	$n^2\gamma^2F'' + 3n^2\gamma F'$ $-(2n+1)F = 0,\ \gamma = te^{\frac{nx}{2}}$	$C(x,t) = \exp\left[\frac{x}{2} - \frac{e^{-nx}}{n^2t}\right]\left[A\ \left(te^{\frac{nx}{2}}\right)^{\frac{1}{n}}\right.$ $\left.+B\ (te^{\frac{nx}{2}})^{-\frac{2n+1}{n}}\right]$
$X_3 + X_4$	$n^4\gamma^4F'' + 3n^4\gamma^3F'$ $-(2n^3\gamma^2 + n^2\gamma^2 + 4)F = 0,$ $\gamma = te^{\frac{nx}{2}}$	$C(x,t) = \frac{1}{t}\exp\left[\frac{x(1-n)}{2} - \frac{e^{-nx}+1}{n^2t}\right] \times$ $\left[A\ K_\upsilon\left(\frac{2}{n^2te^{\frac{nx}{2}}}\right) + B\ I_{-\upsilon}\left(\frac{2}{n^2te^{\frac{nx}{2}}}\right)\right]$
X_2	$2n^2\gamma^2F'' + (4n^2\gamma - 1)F'$ $+(1-n)F = 0,\ \gamma = te^{nx}$	$C(x,t) = t^{-\frac{n+1}{2n}}\exp\left(-nx - \frac{1}{4n^2te^{nx}}\right) \times$ $\left[A\ I_\nu\left(\frac{1}{4n^2te^{nx}}\right) + [B\ K_\nu\left(\frac{1}{4n^2te^{nx}}\right)\right]$
$X_2 + X_4$	$n^2\gamma^2F'' + (2n^2\gamma - n\gamma - 1)F'$ $-nF = 0,\ \gamma = te^{nx}$	$C(x,t) = \frac{A}{t}\exp\left(-nx - \frac{1}{n^2te^{nx}}\right) \times$ $F_1\left(\frac{n+1}{n}, \frac{2n+1}{n}, \frac{1}{n^2te^{nx}}\right) + B\ t^{\frac{1}{n}}e^x$

We define the following variables as they appear in Table 5.2.

F_1 - Hypergeometric function,

$I_{-\upsilon}$ and I_ν are modified Bessel functions of the first kind, K_υ and K_ν are modified Bessel functions of the second kind,

$$\upsilon = \frac{n+1}{n},$$

$$\nu = \tfrac{\sqrt{n^2+2n-2}}{2n}.$$

Sub-sub-case 3.2.1: $m = n = -2$

In this instance the PDE (5.33) admits a six-dimensional Lie algebra spanned by the vectors

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \\ X_3 &= 4t^2\frac{\partial}{\partial t} + 4t\frac{\partial}{\partial x} + (2t - e^{2x})\frac{\partial}{\partial C}, \\ X_4 &= e^{-x}\frac{\partial}{\partial x} + Ce^{-x}\frac{\partial}{\partial C}, \\ X_5 &= 2te^{-x}\frac{\partial}{\partial x} + C(2te^{-x} - e^{x})\frac{\partial}{\partial C}, \\ X_6 &= C\frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.47}$$

Th complete one-dimensional optimal system of these sub-algebras is

$$\{X_4,\ X_1 + aX_5,\ X_1 + aX_6,\ X_2 + aX_6,\ X_1 + X_3 + aX_6\}.$$

We perform symmetry reductions using these elements.

(a) *Reduction by* X_4

We solve the characteristic equations

$$\frac{dx}{e^{-x}} = \frac{dt}{0} = \frac{dC}{e^{-x}C}. \tag{5.48}$$

This implies

$$C = e^x F(t). \tag{5.49}$$

Substituting into the governing equation results in the ODE

$$F' = 0 \tag{5.50}$$

with the solution

$$F = \text{constant}. \tag{5.51}$$

We write our invariant solution as

$$C = C(x) = A\ e^x, \tag{5.52}$$

where A is an arbitrary constant. Hence we observe that the concentration in this case only depends on the spatial variable.

(b) *Reduction by* $X_1 + aX_6$

We solve the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{aC} \tag{5.53}$$

and obtain

$$C = e^{at}F(x). \tag{5.54}$$

The resultant ODE

$$e^{-2x}F'' - 3e^{-2x}F' + (2e^{-2x} - a)F = 0 \tag{5.55}$$

solves to

$$F(x) = e^{x}\left[A\,\sinh(\sqrt{a}e^{x}) + B\,\cosh(\sqrt{a}e^{x})\right]. \tag{5.56}$$

Hence the original PDE has an invariant solution of the form

$$C(x,t) = e^{x+at}\left[A\,\sinh(\sqrt{a}e^{x}) + B\,\cosh(\sqrt{a}e^{x})\right]. \tag{5.57}$$

(c) *Reduction by* $X_2 + aX_6$

By solving the characteristic equations

$$\frac{dt}{2t} = \frac{dx}{1} = \frac{dC}{aC} \tag{5.58}$$

one obtains

$$C = e^{at}F(te^{-2x}). \tag{5.59}$$

By substitution of (5.59) in the governing PDE (5.33) one obtains the ODE

$$4\gamma^2F'' + [\gamma(10-a) - 1]F' + (a\gamma - 1)F = 0, \tag{5.60}$$

where $\gamma = te^{-2x}$.

We are unable to find the exact solution of (5.60).

(d) *Reduction by* $X_1 + X_5$

We solve the characteristic equations

$$\frac{dt}{2t} = \frac{dx}{1} = \frac{dC}{aC} \tag{5.61}$$

and obtain

$$C(x,t) = \exp\left(x - te^x + \frac{2}{3}t^3\right) F(t^2 - e^x). \tag{5.62}$$

This leads to the Airy equation

$$F'' - \gamma F = 0, \tag{5.63}$$

where $\gamma = t^2 - e^x$.

The solution of equation (5.85) can be written as

$$F(\gamma) = A\ \mathrm{Ai}(\gamma) + B\ \mathrm{Bi}(\gamma), \tag{5.64}$$

where Ai and Bi are the Airy functions of the first and second kind, respectively. A and B are arbitrary constants.

Hence we write the sought invariant solution as

$$C(x,t) = \exp\left(x - te^x + \frac{2}{3}t^3\right) \left[A\ \mathrm{Ai}(t^2 - e^x) + B\ \mathrm{Bi}(t^2 - e^x)\right]. \tag{5.65}$$

(e) *Reduction by* $X_1 + X_3 + X_6$

Upon solving the characteristic equations

$$\frac{dt}{4t^2+1} = \frac{dx}{4t} = \frac{dC}{C(1+2t-e^{2x})}, \tag{5.66}$$

one obtains

$$C = (4t^2+1)^{\frac{1}{4}} \exp\left(\frac{1}{2}\tan^{-1}(2t) - \frac{te^{2x}}{4t^2+1}\right) F\left(\frac{4t^2+1}{e^{2x}}\right). \tag{5.67}$$

The function F satisfies the ODE

$$4\gamma^4 F'' + 10\gamma^3 F' + (2\gamma^2 - \gamma + 1)F = 0, \tag{5.68}$$

where $\gamma = e^{-2x}(4t^2+1)$.

For equation (5.68) we find the exact solution

$$\begin{aligned} F(x,t) &= \left(\frac{e^{2x}}{4t^2+1}\right)^{\frac{1}{4}} \left[A\ M\left(\frac{i}{4};\frac{1}{4};\frac{ie^{2x}}{4t^2+1}\right)\right. \\ &\left. +B\ W\left(\frac{i}{4};\frac{1}{4};\frac{ie^{2x}}{4t^2+1}\right)\right], \end{aligned}$$

where i is the imaginary number and M and W are hypergeometric functions. This implies equation (5.68) does not have a real solution.

We write the invariant solution as

$$\begin{aligned} C(x,t) &= \exp\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}(2t) - \frac{te^{2x}}{4t^2+1}\right) \left[A\ M\left(\frac{i}{4};\frac{1}{4};\frac{ie^{2x}}{4t^2+1}\right)\right. \\ &\left. +B\ W\left(\frac{i}{4};\frac{1}{4};\frac{ie^{2x}}{4t^2+1}\right)\right]. \end{aligned}$$

Case 4: Given $D(x) = x^m, \;\; u(x) = \text{constant}$

Our governing equation in this case reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(x^m \frac{\partial C}{\partial x}\right) - k\frac{\partial C}{\partial x}, \tag{5.69}$$

for $m, \; k \in \mathbb{R}$.

We investigate the group invariant solutions of equation (5.69) under the three sub-cases $k \neq 0, \; m \neq 0$; $k = 0, \; m = -2$ and $k = m = 0$.

Sub-case 4.1: $k \neq 0, \; m \neq 0$

In particular we look at the sub-case $k \neq 0, \; m = 2$. The only Lie point symmetries admitted by equation (5.69) under this condition are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = C\frac{\partial}{\partial C}.$$

Reduction by the linear combination $X_1 + aX_2$ produces the ODE

$$x^2F'' + (2x - k)F' - aF = 0. \tag{5.70}$$

The solution to equation (5.70) can be written as

$$F(x) = \frac{e^{-\frac{k}{2x}}}{\sqrt{x}}\left[A \; I_p\left(\frac{k}{2x}\right) + B \; K_p\left(\frac{k}{2x}\right)\right],$$

where I_p and K_p are modified Bessel functions and $p = \sqrt{4a+1}/2$.

We write the invariant solution of equation (5.69) as

$$C(x,t) = \frac{e^{\frac{2axt-k}{2x}}}{\sqrt{x}}\left[A \; I_p\left(\frac{k}{2x}\right) + B \; K_p\left(\frac{k}{2x}\right)\right].$$

Sub-case 4.2: $k = 0,\ m = -2$

In this case equation (5.69) admits a four-dimensional Lie algebra spanned by the vectors

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 4t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ X_3 &= 16t^2\frac{\partial}{\partial t} + 8tx\frac{\partial}{\partial x} - C(4t + x^4)\frac{\partial}{\partial C}, \\ X_4 &= C\frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.71}$$

The optimal system of the sub-algebras (5.71) works out to be

$$\{X_3,\ X_4,\ X_2 + aX_4,\ X_3 + aX_4\}. \tag{5.72}$$

By letting the constant a to be unity, we concentrate on reductions by the elements $X_3,\ X_2 + X_4,$ and $X_3 + X_4$.

(a) *Reduction by* X_3

By solving the characteristic equations

$$\frac{dt}{16t^2} = \frac{dx}{8tx} = -\frac{dC}{C(4t + x^4)},$$

one obtains

$$C = \frac{1}{t^{\frac{1}{4}}} \exp\left(-\frac{x^4}{16t}\right) F\left(\frac{x}{\sqrt{t}}\right)$$

and substituting this result in our governing PDE, one gets the ODE

$$\gamma F'' - 2F' = 0, \tag{5.73}$$

where $\gamma = x/\sqrt{t}$.

The solution to (5.73) is of the form

$$F(x,t) = A\ \frac{x^3}{3t^{\frac{3}{2}}} + B$$

where A and B are arbitrary constants.

Hence the group invariant solution in this case is given by

$$C(x,t) = \frac{1}{t^{\frac{1}{4}}} \exp\left(-\frac{x^4}{16t}\right)\left(A\ \frac{x^3}{3t^{\frac{3}{2}}} + B\right). \tag{5.74}$$

(b) *Reduction by* $X_2 + X_4$

Reduction by $X_2 + X_4$ results in the second order ODE

$$16\gamma^2 F'' + \gamma(\gamma + 19)F' - 2F = 0, \tag{5.75}$$

where $\gamma = \frac{x^4}{t}$.

The solution of equation (5.75) can be written in the form

$$\begin{aligned} F(\gamma) &= e^{-\frac{\gamma}{16}}\gamma^{\frac{1}{32}(-3+\sqrt{137})}\left[A\ U\left(\frac{35}{32} + \frac{\sqrt{137}}{32}, 1 + \frac{\sqrt{137}}{16}, \frac{\gamma}{16}\right)\right. \\ &\quad \left. + B\ L\left(-\frac{35}{32} - \frac{\sqrt{137}}{32}, \frac{\sqrt{137}}{16}, \frac{\gamma}{16}\right)\right], \end{aligned}$$

where $U(a,b,z)$ and $L(m,n,z)$ are the confluent hypergeometric functions and associated Laguerre polynomials respectively.

The invariant solution can be written, in terms of the original variables, as

$$\begin{aligned} C(x,t) &= xe^{-\frac{x^4}{16t}}\left(\frac{x^4}{t}\right)^{\frac{1}{32}(-3+\sqrt{137})}\left[A\ U\left(\frac{35}{32}+\frac{\sqrt{137}}{32},1+\frac{\sqrt{137}}{16},\frac{x^4}{16t}\right)\right. \\ &\quad \left. +\ B\ L\left(\frac{-35}{32}-\frac{\sqrt{137}}{32},\frac{\sqrt{137}}{16},\frac{x^4}{16t}\right)\right]. \end{aligned} \tag{5.76}$$

(c) *Reduction by* X_3+X_4

Reduction by X_3+X_4 produces the second order ODE

$$16\gamma F''-32F'-\gamma^3F=0, \tag{5.77}$$

for $\gamma = x/\sqrt{t}$.

The solution to equation (5.77), in terms of the original variables, is given in terms of the modified Bessel functions I and K as

$$F(x,t)=\left(\frac{x}{\sqrt{t}}\right)^{\frac{3}{2}}\left[A\ I_{\frac{3}{4}}\left(\frac{x^2}{8t}\right)+B\ K_{\frac{3}{4}}\left(\frac{x^2}{8t}\right)\right],\ t\neq 0, \tag{5.78}$$

and we write the invariant solution as

$$c(x,t)=\frac{x^{\frac{3}{2}}}{t}\exp\left(-\frac{x^4+1}{16t}\right)\left[A\ I_{\frac{3}{4}}\left(\frac{x^2}{8t}\right)+B\ K_{\frac{3}{4}}\left(\frac{x^2}{8t}\right)\right] \tag{5.79}$$

Sub-case 4.3: $k=m=0$

Equation (5.69) admits the six Lie point symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 2t\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \\ X_3 &= 4t^2\frac{\partial}{\partial t} + 4t\frac{\partial}{\partial x} + (2t - e^{2x})\frac{\partial}{\partial C}, \\ X_4 &= e^{-x}\frac{\partial}{\partial x} + Ce^{-x}\frac{\partial}{\partial C}, \\ X_5 &= 2te^{-x}\frac{\partial}{\partial x} + C(2te^{-x} - e^{x})\frac{\partial}{\partial C}, \\ X_6 &= C\frac{\partial}{\partial C} \end{aligned}\right\} \tag{5.80}$$

under this condition. The complete one dimensional optimal system of the algebra (5.80) is

$$\{X_4,\ X_1 + aX_5,\ X_1 + aX_6,\ X_2 + aX_6,\ X_1 + X_3 + aX_6\}.$$

We use this optimal system to perform reductions. We outline the calculations of reduction by $X_1 + X_3 + aX_6$. The rest of the results are summarized in Table 5.3.

We solve the characteristic equations

$$\frac{dt}{4t^2 + 1} = \frac{dx}{4tx} = \frac{dC}{C[a - (2t + x^2)]}.$$

and find that C takes the form

$$C = \frac{1}{(4t^2+1)^{\frac{1}{4}}} \exp\left(\frac{a}{2}\tan^{-1}(2t) - \frac{tx^2}{4t^2+1}\right) F\left(\frac{4t^2+1}{x^2}\right). \tag{5.81}$$

and the function F satisfies the ODE

$$4\gamma^4 F'' + 6\gamma^3 F' + (1 - a\gamma)F = 0, \tag{5.82}$$

where $\gamma = (4t^2+1)/x^2$.

The solution of equation (5.82)can be written as

$$\begin{aligned} F(x,t) &= \frac{(4t^2+1)^{\frac{1}{4}}}{\sqrt{x}} \left[A\ \mathrm{M}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right. \\ &\left. +B\ \mathrm{W}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right], \end{aligned} \tag{5.83}$$

where $\mathrm{M}(a,b,z)$ and $\mathrm{W}(a,b,z)$ are hypergeometric functions and i is the imaginary number.

We write the invariant solution as

$$\begin{aligned} C(x,t) &= \frac{1}{\sqrt{x}} \exp\left(\frac{a}{2}\tan^{-1}(2t) - \frac{tx^2}{4t^2+1}\right) \left[A\ \mathrm{M}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right. \\ &\left. +B\ \mathrm{W}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right]. \end{aligned} \tag{5.84}$$

Table 5.3: Summary of symmetry reductions and invariant solutions given constant D

Symmetry	Reduced Equation	Invariant solution in terms of orginal variables
$X_1 + aX_5$	$F'' - a\gamma F = 0,\ \gamma = at^2 - x$	$C(x,t) = A\ e^{\frac{x-t^2}{2}}\mathrm{Ai}(\frac{1}{4} + t^2 - x)$ $+B\ e^{\frac{x-t^2}{2}}\mathrm{Bi}(\frac{1}{4} + t^2 - x)$
$X_1 + aX_6$	$F'' - aF = 0,\ F = F(x)$	$C(x,t) = e^{at}\left(A\ e^{\sqrt{a}x} + B\ e^{-\sqrt{a}x}\right)$
$X_2 + aX_6$	$2F'' + \gamma F' - aF = 0,$ $\gamma = \frac{x}{\sqrt{t}}$	$C(x,t) = xt^{\frac{a-1}{2}}\exp\left(-\frac{x^2}{4t}\right)\times$ $\left[A\ \mathrm{M}\left(\frac{a+2}{2}, \frac{3}{2}, \frac{x^2}{4t}\right)\right.$ $\left.+B\ \mathrm{U}\left(\frac{a+2}{2}, \frac{3}{2}, \frac{x^2}{4t}\right)\right]$
$X_1 + X_3 + aX_6$	$4\gamma^4 F'' + 6\gamma^3 F'$ $+(1 - a\gamma)F = 0,\ \gamma = \frac{4t^2+1}{x^2}$	$C(x,t) = \frac{1}{\sqrt{x}}\exp\left(\frac{a}{2}\tan^{-1}(2t)\right.$ $\left.-\frac{tx^2}{4t^2+1}\right)\times\left[A\ \mathrm{M}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right.$ $\left.+B\ \mathrm{W}\left(\frac{ai}{4}, \frac{1}{4}, \frac{ix^2}{4t^2+1}\right)\right]$

In Table 5.3 M, U and W are hypergeometric functions.

Case 5: Given $D(x) = x^m,\ \ u(x) = x^n,\ \ m,\ n \in \mathbb{R}$

Our governing equation in this case reads

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(x^m \frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial x}\left(x^n C\right). \tag{5.85}$$

We consider the following sub-cases.

Sub-case 5.1: $m \neq n$

(a) We begin by looking at the case $m = 3 - n$.

Equation (5.85) admits only the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = C\frac{\partial}{\partial C}$$

under this condition. We perform symmetry reductions using the linear combination $X_1 + aX_2$ and solve the characteristic equations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{aC}.$$

This leads to

$$C = e^{at}F(x). \tag{5.86}$$

Substitution of (5.86) into the governing equation leads to the second order ODE

$$F'' + x^{2n-3}(3x^{2-2n} - nx^{2-2n} - 1)F' - x^{n-3}(a + nx^{n-1})F = 0. \tag{5.87}$$

We are, however, unable to find the exact solution of equation (5.87).

(b) Next we consider the case $m = n + 1$.

Under this condition equation (5.85) extends the Lie Algebra to four Lie point symmetries, namely

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= t(n-1)\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}, \\ X_3 &= t^2(n-1)^2\frac{\partial}{\partial t} - 2tx\frac{\partial}{\partial x} - x^{1-n}C\frac{\partial}{\partial C}, \\ X_4 &= \frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.88}$$

From the symmetry group (5.88) we obtain the one-dimensional optimal system

$$\{X_2,\ X_3,\ X_2 + aX_4,\ X_3 + aX_4\}. \tag{5.89}$$

Symmetry reductions are carried out using the elements in (5.89). Here we only show calculations for reduction by $X_3 + X_4$ and only show the results in Table 5.4 for the remaining elements.

The basis of invariants is constructed by solving the characteristic equations

$$\frac{dt}{t^2(n-1)^2} = -\frac{dx}{2tx(n-1)} = \frac{dC}{C(1-x^{1-n})}. \tag{5.90}$$

As a result we obtain

$$C(x,t) = \exp\left(-\frac{1+x^{1-n}}{t(n-1)^2}\right) F\left(\frac{t^2}{x^{1-n}}\right). \tag{5.91}$$

Substituting this result into (5.85) gives the ODE

$$(n-1)^4\gamma^3 F'' + 2(n-1)^4\gamma^2 F' - F = 0, \tag{5.92}$$

where $\gamma = t^2/x^{1-n}$.

The table shows a summary of the reductions and invariant solutions.

Table 5.4: Summary of results of the reduction when $m = n + 1$

Symmetry	Reduced Equation	Invariant solution in terms of original variables
X_2	$(n-1)^2\gamma^2 F'' + (2n^2\gamma - 3n\gamma + 3\gamma - 1)F' - (n\gamma + \gamma + n)F = 0,\ \gamma = \dfrac{t}{x^{1-n}}$	No solution
$X_2 + X_4$	$(n-1)^2\gamma^2 F'' + (2n^2\gamma - 6n\gamma + 4\gamma - 1)F' + 2(1-n)F = 0,\ \gamma = \dfrac{t}{x^{1-n}}$	$C(x,t) = \exp\left(-\dfrac{x^{1-n}}{(n-1)^2 t}\right) \times \left[\dfrac{A}{tx^n} F_1\left(\dfrac{n+1}{n-1},\ \dfrac{2n}{n-1},\ \dfrac{x^{1-n}}{(n-1)^2 t}\right) + B\ t^{\frac{2}{n-1}} x\right]$
X_3	$(n-1)^2\gamma^2 F'' - 2\gamma(2n-1)F' - nF = 0,\ \gamma = \dfrac{t^2}{x^{1-n}}$	$C(x,t) = \exp\left(-\frac{x^{1-n}}{(n-1)^2 t}\right) \times \left[A\ \left(\dfrac{t^2}{x^{1-n}}\right)^{\frac{p+q}{r}} + B\ \left(\dfrac{t^2}{x^{1-n}}\right)^{\frac{p-q}{r}}\right]$
$X_3 + X_4$	$(n-1)^4\gamma^3 F'' + 2(n-1)^4\gamma^2 F' - F = 0,\ \gamma = \dfrac{t^2}{x^{1-n}}$	$C(x,t) = \dfrac{x^{\frac{1}{2}(1-n)}}{t} \exp\left(-\dfrac{1+x^{1-n}}{(n-1)^2 t}\right) \times \left[A\ I_1\left(\dfrac{2x^{\frac{1}{2}(1-n)}}{(n-1)^2 t}\right) + B\ K_1\left(\dfrac{2x^{\frac{1}{2}(1-n)}}{(n-1)^2 t}\right)\right]$

In the table,

F_1 is a hypergeometric function,

I_1 and K_1 are modified Bessel functions of the first and second kind, respectively,

$p = n^2 + 2n - 1, \quad q = \sqrt{n^4 + 8n^3 - 6n^2 + 1}, \;\; r = 2(n^2 - 2n + 1).$

Sub-case 5.2: $m = n$

When $m = n = 0$ equation (5.85) reduces to equation (4.16) which we have already analyzed in the previous chapter. However, when n assumes any of the values $2/3,\ 1,\ 3/2,\ 2,\ 4$ and 6 equation (5.85) admits the two symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = C\frac{\partial}{\partial C}.$$

We only select $n = 2$, of which the reduction eventually leads to the second order ODE

$$x^2F'' + (2 - x)F' - (2x + a)F = 0. \tag{5.93}$$

Equation (5.93) does not produce any exact solutions.

Case 6: Given $D(x) = \text{constant}, \quad u(x) = x^n$

(a) For nonspecific values of k and n, our equation admits the symmetries $X_1 = \partial/\partial t$ and $X_2 = C\ (\partial/\partial C)$ and reduction by the linear combination $X_1 + aX_2$ yields the ODE

$$kF'' - x^nF' - (a + nx^{n-1})F = 0. \tag{5.94}$$

The ODE (5.94) admits $F(\partial/\partial F)$ as its only Lie point symmetry. The method of differential invariants leads to the invariants

$$u = x, \qquad v = \frac{F}{F'}.$$

We write $v = v(u)$ and use the chain rule to reduce equation (5.94) to the equation

$$kv' - kv^2 - u^n v - (a + nu^{n-1}) = 0,$$

which, however, does not yield any solution.

We can mention here that for $k = 0$ equation (5.94) admits the Lie point symmetry

$$\frac{1}{x^n}\frac{\partial}{\partial x} - F\left(\frac{n}{x^{n+1}} + \frac{a}{x^{2n}}\right)\frac{\partial}{\partial F}.$$

We leave out this condition.

(b) For $k \neq 0$ and $n = -2$ the same Lie algebra is admitted and reduction produces the ODE

$$kx^3F'' - xF' + (2 - ax^3)F = 0. \tag{5.95}$$

For $k = 0$ equation (5.95) admits the Lie point symmetry

$$x^2\frac{\partial}{\partial x} + (2xF - ax^4F)\frac{\partial}{\partial F}.$$

We, however, leave out this condition.

Otherwise equation (5.95) does not produce any exact solutions.

(c) However, for $k \neq 0$ and $n = -1$, the Lie algebra extends to the four Lie point symmetries

$$\left.\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 4kt\frac{\partial}{\partial t} + 2kx\frac{\partial}{\partial x} + \frac{\partial}{\partial C}, \\ X_3 &= 4kt^2\frac{\partial}{\partial t} + 4ktx\frac{\partial}{\partial x} + C(2t - 2kt - x^2)\frac{\partial}{\partial C}, \\ X_4 &= \frac{\partial}{\partial C}. \end{aligned}\right\} \tag{5.96}$$

The one-dimensional optimal system of the Lie algebra (5.96) is calculated to be

$$\{X_3,\ X_4,\ X_1 + aX_4,\ X_2 + aX_4\}. \tag{5.97}$$

We only present the results of the reductions and invariant solutions using (5.97) in Table 5.5. Reduction by the element X_4 is omitted.

Table 5.5: Summary of results of the reduction when $k \neq 0,\ n = -1$

Symmetry	Reduced Equation	Invariant solution in terms of original variables
X_3	$k\gamma^2 F'' - \gamma F' + F = 0,$ $\gamma = \frac{x}{t}$	$C(x,t) = t^{\frac{1-k}{2k}} \exp\left(-\frac{x^2}{4kt}\right)\left[A\ \left(\frac{x}{t}\right)^{\frac{1}{k}} + B\ \frac{x}{t}\right]$
$X_2 + X_4$	$2k^2\gamma^2 F'' + k\gamma(\gamma^2 - 2)F'$ $+(2k - \gamma^2)F = 0,\ \gamma = \frac{x}{\sqrt{t}}$	$C(x,t) = e^t x^{\frac{k+1}{2k}}\left[A\ J_{\frac{k-1}{2k}}\left(-\frac{ix}{\sqrt{k}}\right) + B\ Y_{\frac{k-1}{2k}}\left(-\frac{ix}{\sqrt{k}}\right)\right]$
$X_1 + X_4$	$kx^2 F'' - xF' + (1 - x^2)F = 0$	$C(x,t) = t^{\frac{1}{2k}}\left[A\ \left(\frac{x}{\sqrt{t}}\right)^{\frac{1}{k}} - \frac{B}{2^{\frac{1}{k}}}\left(\frac{x}{\sqrt{t}}\right)\left(\frac{x^2}{kt}\right)^{\frac{1-k}{2k}} \times \Gamma\left(\frac{k-1}{2k}, \frac{x^2}{4kt}\right)\right]$

In the table $J_{\frac{k-1}{2k}}$ and $Y_{\frac{k-1}{2k}}$ are Bessel functions of the first and second kind, respectively, while i is the imaginary number and $k,\ t \neq 0$.

5.2 Some discussion

Figures 5.1-5.3 depict the particular analytic solution (5.17) of equation (5.7) for the case $m = 0,\ n = -1$. Here, we observe that the concentration decreases with increasing distance for fixed time t. We can explain this in terms of diffusion and advection of contaminants as they are carried downstream by moving water molecules. For a fixed distance x, the concentration increases with increasing time. This is because the initial concentration is zero at a point $x = L$ downstream while the particles tend to pile up with time.

5.3 Concluding remarks

For the realistic cases of the forms the diffusion coefficient considered, we have succeeded in reducing the associated PDEs using elements of the optimal system. The same cannot be said, however, with the construction of the invariant solutions. For some reductions we have successfully found the group invariant solutions while we have failed with some. A number of the group invariant solutions found are given in terms of special mathematical functions such as the Bessel, Whitaker hypergeometric and Airy functions. Some solutions are complex as they are given in terms of the imaginary number i. There are only a few of the ODEs that were solved by manual integration of the ODEs. The majority of the ODEs were solved with the help of the mathematical softwares Mathematica and Maple. Overall, it is safe to say we succeeded in reducing the PDEs and constructing the group invariant solutions of the ADE for solute transport with non-constant water velocity.

Chapter 6

Non-local Symmetry Analysis

Introduction

In this chapter we investigate the potential symmetries and invariant solutions of equation (2.18) when the diffusion coefficient and velocity are defined by the exponential function. In the second section we let both these two physical quantities be defined by the power law.

6.1 The exponential law

In order to find the potential symmetries of equation (5.33), we write it in its conserved form

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[e^{mx}\frac{\partial C}{\partial x} - e^{nx}C\right]. \tag{6.1}$$

The associated auxiliary system $\Im(x, t, C, v)$ is

$$v_x = u, \quad v_t = e^{mx}C_x - e^{nx}C, \tag{6.2}$$

with $v(x,t)$ as the potential. We only consider cases for which m and n are arbitrary.

Case 1: $m \neq n$

For the system (6.2), for $m \neq n$ (n arbitrary), we obtain point symmetries with the following generators :

$$\begin{aligned} X_1 : &\quad \tau = 1, \quad \xi = \eta = \phi = 0 \\ X_2 : &\quad \tau = \xi = 0, \quad \eta = C, \quad \phi = v \end{aligned}$$

and the ∞-dimensional symmetry

$$X_\infty = H(t,x)\frac{\partial}{\partial v} + H_x\frac{\partial}{\partial C}.$$

It is clear that none of these symmetries is a potential symmetry of (6.1).

Case 2: $m = n$

However, for the case $m = n$ (n arbitrary), we obtain point symmetries with the following generators:

$$\begin{aligned} X_1 : &\quad \tau = 1, \quad \xi = \eta = \phi = 0 \\ X_2 : &\quad \tau = nt, \quad \xi = -1, \quad \eta = \phi = 0 \\ X_3 : &\quad \tau = n^2t^2, \quad \xi = -2nt \quad \eta = ne^{-nx}v - C(nt + n^2t + e^{-nx}) \\ &\quad \phi = -v(nt + n^2t + e^{-nx}) \\ X_4 : &\quad \tau = \xi = 0, \quad \eta = C, \quad \phi = v \end{aligned}$$

and the ∞-dimensional symmetry

$$X_\infty = H(t,x)\frac{\partial}{\partial v} + H_x\frac{\partial}{\partial u}.$$

Of all these symmetries, X_3 is only genuine potential symmetry of (6.1). For this potential symmetry the characteristic system related to the invariant surface conditions is

$$\left.\begin{aligned} &2ntC_x - n^2t^2C_t + ne^{-nx}v - C(nt + n^2t + e^{-nx}) = 0\\ &2ntv_x - n^2t^2v_t + v(nt + n^2t + e^{-nx}) = 0. \end{aligned}\right\} \tag{6.3}$$

From the potential symmetry X_3 we write the characteristic system

$$\frac{dt}{n^2t^2} = -\frac{dx}{2nt} = \frac{dC}{ne^{-nx}v - C(e^{-nx} + nt + n^2t)} = -\frac{dv}{v(e^{-nx} + nt + n^2t)},$$

which on solving yields the three integrals

$$\begin{aligned} &k_0 = t^2e^{nx}, \qquad k_1 = vt^{\frac{n+1}{n}}\exp\left(\frac{e^{-nx}}{n^2t}\right),\\ &k_2 = t^{\frac{n+1}{n}}\exp\left(\frac{e^{-nx}}{n^2t}\right)\left[C - \frac{ve^{-nx}}{nt}\right]. \end{aligned} \tag{6.4}$$

We let $k_0 = z$ be the similarity variable with $k_1 = h_1(z)$ and $k_2 = h_2(z)$ being the corresponding similarity functions. As a result the solutions of (6.3) are

$$\begin{aligned} &z = t^2e^{nx}, \qquad v = h_1(z)t^{-\frac{n+1}{n}}\exp\left(-\frac{e^{-nx}}{n^2t}\right),\\ &C = t^{-\frac{n+1}{n}}\exp\left(-\frac{e^{-nx}}{n^2t}\right)\left[\frac{e^{-nx}}{nt}h_1(z) + h_2(z)\right]. \end{aligned} \tag{6.5}$$

To find the family of solutions ψ^* for the PDE (6.1), we introduce equation $(6.5)_3$ in (6.1) obtaining

$$nz[nz^2h_2'' + z(2n-1)h_2' - h_2] + t(nz^2h_1'' - zh_1 + h_2) = 0, \tag{6.6}$$

which should hold for any value of t. In equation (6.6) prime denotes differentiation with respect to z. As a result of (6.6) we have the system φ of ODEs:

$$\left.\begin{aligned} nz^2h_2'' + z(2n-1)h_2' - h_2 = 0 \\ nz^2h_1'' - zh_1 + h_2 = 0, \end{aligned}\right\} \tag{6.7}$$

which on solving produces

$$\left.\begin{aligned} h_1(z) = a_1z^{\frac{1}{n}} + \frac{b_1nz^{\frac{n+1}{n}}}{n+1} - \frac{a_2}{z(2n+1)} + b_2 \\ h_2(z) = a_1z^{\frac{1}{n}} + \frac{a_2}{z}, \end{aligned}\right\} \tag{6.8}$$

where a_i, b_i are arbitrary constants.

The family of solutions ψ^* is therefore given by

$$\begin{aligned} C(x,t) &= a_1(t^2e^{nx})^{\frac{1}{n}}\left[1 + \frac{e^{-nx}}{nt}\right] + \frac{a_2}{t^2e^{nx}}\left[1 - \frac{e^{-nx}}{n(2n+1)t}\right] \\ &+ \frac{e^{-nx}}{t}\left[\frac{b_2}{n} + \frac{b_1(t^2e^{nx})^{\frac{n+1}{n}}}{n+1}\right]. \end{aligned} \tag{6.9}$$

Now, to find the family of solutions ψ which are solutions of $\Im$, we introduce equations $(6.5)_2$ and $(6.5)_3$ in the system (6.2) obtaining the system

$$\left.\begin{aligned} h_1' = 0, \\ h_2 = 0 \end{aligned}\right\} \tag{6.10}$$

which after solving yields

$$\left.\begin{aligned} h_1(z) &= a_3, \\ h_2(z) &= 0 \end{aligned}\right\}. \tag{6.11}$$

Hence the family of solutions ψ can be written as

$$C(x,t) = \frac{a_3 e^{-nx} t^{\frac{1-2n}{n}}}{n} \exp\left(-\frac{e^{-nx}}{n^2 t}\right) \tag{6.12}$$

for some arbitrary constant a_3.

6.2 The power law

As in the case of the the previous section, we express our governing equation its conserved form. Expressed in its conserved form, equation (5.85) becomes

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[x^m \frac{\partial C}{\partial x} - x^n C\right], \tag{6.13}$$

and the associated auxiliary system $\Im(x,t,C,v)$ is

$$v_x = u, \quad v_t = x^m C_x - x^n C. \tag{6.14}$$

Here $v(x,t)$ is the potential. Of the several cases which m can assume, it is only when $m = n+1$ that we have a potential symmetry. This condition yields point symmetries with the generators:

$$
\begin{aligned}
X_1 :&\quad \tau = 1, \quad \xi = \eta = \phi = 0\\
X_2 :&\quad \tau = \xi = 0, \quad \eta = C, \quad \phi = v\\
X_3 :&\quad \tau = n(n-1)t, \quad \xi = -x, \quad \eta = 0, \quad \phi = -v\\
X_4 :&\quad \tau = t^2[x^n(n^2+1) - 2nx^2], \quad \xi = 2tx^{n+1}(1-n), \quad \phi = (1-n^2)tx^n v\\
&\quad \eta = -(n-1)^2 tx^n C + (n-1)v - xC.
\end{aligned}
$$

It is clear that X_4 is the only genuine potential symmetry of equation (6.13). The associated characteristic system related to the invariant surface conditions is

$$
\left.
\begin{aligned}
&2tx^{n+1}(1-n)C_x + t^2[(n^2+1)x^n - 2nx^2]C_t + (n-1)^2 tx^n C\\
&-(n-1)v + xC = 0\\
&2tx^{n+1}(1-n)v_x + t^2[(n^2+1)x^n - 2nx^2]v_t - (1-n^2)tx^n v = 0.
\end{aligned}
\right\} \tag{6.15}
$$

For the potential symmetry X_4, we solve the characteristic system

$$
\frac{dt}{t^2(n^2x^n - 2nx^2 + x^n)} = \frac{dx}{2tx^{n+1}(1-n)} = \frac{dv}{v(tx^n - x - n^2tx^n)}
$$

$$
= \frac{dC}{tx^n(n-1)^2C + (n-1)v - xC}.
$$

On solving this system we find the integral

$$
k_0 = \frac{x^{n^2+1}}{t^{2-2n}} \exp\left(-\frac{2nx^{2-n}}{2-n}\right),
$$

and we find it a mammoth task to solve for the the other integrals k_1 and k_2. This could be attributed to the fact that the similarity variable $z = k_0$ looks rather a complex expression yet this is used in the subsequent calculations of k_1 and k_2.

6.3 Concluding remarks

We have successfully explored the potential symmetries of the ADE when the diffusion coefficient and the transport velocity terms are defined by the exponential and power law functions. In the case of the former, we found that the ADE admits a potential symmetry only when $m = n$ (n arbitrary). with this potential symmetry we were able to find the family of solutions ψ^* for (6.1). The family of solutions ψ for the auxiliary system (6.2) was also established. As much as we were able to find the potential symmetry for equation (6.13), it was quite difficult to explore the invariant solutions of the PDE (6.13) and its associated auxiliary system (6.14). This bears testimony to that as much as we can succeed in finding the potential symmetries of a PDE it is possible that we fail to find its invariant solutions.

Chapter 7

Conclusion

The three-dimensional ADE describing the movement of contaminants in porous media has been derived. We adopted the derivation by Fischer et al. [15]. However, we used the one-dimensional ADE to analyze flow in only the x direction.

We have used symmetry techniques as our tool to analyze the one-dimensional ADE. We successfully carried out direct group classification of the ADE assuming the diffusion coefficient given in terms of the concentration and the transport velocity equal to unity. The principal Lie algebra was found to be three-dimensional. The forms of the diffusion coefficient $D =$ constant, e^{mC}, and $(\alpha + \beta C)^m$ resulted in extension of the principal Lie algebra and using such algebra, group invariant solutions were constructed. It is only when the diffusion coefficient was a constant that we obtained reduced linear ODEs. The other two cases mainly produced nonlinear ODEs, which did not produce any exact solutions.

The ADE (2.18) has generally proved to be very rich in symmetries for the cases of $D(x)$ and $u(x)$ considered. This has made us search for group invariant

solutions which, in most instances, has yielded realistic solutions. A vast number of the solutions have been found to be given in terms of the special mathematical functions such as the Bessel, Hypergeometric and Airy functions. For analysis and interpretation we selected a few of the realistic solutions and imposed the boundary conditions (2.19). The mathematical software packages Mathematica and Maple have shown us that some of the combinations of diffusion coefficient and velocity result in rather complex solutions. For such solutions there is very little we can do in analyzing and interpreting them physically.

Non-local or potential symmetry analysis of the one-dimensional ADE has been carried out. To some extent we have succeeded in this task. When both the diffusion coefficient and transport velocity are given in terms of the exponential law it was possible to find the potential symmetry and construct the family of solutions ψ^* and ψ for the governing equation (6.1) and the auxiliary system (6.2), respectively. However, when these physical quantities are given as functions of the power law, we experienced difficulties in working beyond the similarity variable z.

Bluman and Kumei [25] also carried out direct group classification of the models analyzed in chapter 4. However, we have made an improvement on their work in that we have found, where possible, the associated group invariant solutions. The work covered in chapters 5 and 6 involves symmetry analysis of the one-dimensional ADE where diffusion and transport coefficients assume some specific forms. As far as we are concerned, the work in these two chapters has not been reported to date.

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