$$\mu_1'(x_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\,\frac{K_{\gamma+1}(u)}{K_\gamma(u)}\,, \tag{2.5.30}$$

$$\mu_1'(x_2) = \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\,\frac{K_{\gamma+1}(u)}{K_\gamma(u)}\,, \tag{2.5.31}$$

$$\mu_2(x_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}} \tag{2.5.32}$$

$$\cdot\left[\frac{K_{\gamma+1}(u)}{K_\gamma(u)} + \frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_1+1}{\alpha_1+\alpha_2+1}\,\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right]\right],$$

$$\mu_2(x_2) = \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}} \tag{2.5.33}$$

$$\cdot\left[\frac{K_{\gamma+1}(u)}{K_\gamma(u)} + \frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_2+1}{\alpha_1+\alpha_2+1}\,\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right]\right],$$

$$\mathrm{cov}(x_1,x_2) = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+1)}\left[\frac{\alpha\theta}{2\sqrt{1-\theta}}\right]^2 .$$

$$\cdot\left[(\alpha_1+\alpha_2)\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - (\alpha_1+\alpha_2+1)\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right] \quad \text{and} \tag{2.5.34}$$

$$\rho(x_1,x_2) = \frac{\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[(\alpha_1+\alpha_2)\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - (\alpha_1+\alpha_2+1)\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right]$$

$$\cdot\,[g_1(\alpha_1,\alpha_2,\alpha,\gamma,\theta)\; g_2(\alpha_1,\alpha_2,\alpha,\gamma,\theta)]^{-1/2}\quad, \tag{2.5.35}$$

where

$$g_1(\alpha_1,\alpha_2,\alpha,\gamma,\theta) = \left[\frac{K_{\gamma+1}(u)}{K_\gamma(u)} + \frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_1+1}{\alpha_1+\alpha_2+1}\,\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right]\right]$$

and

$$g_2(\alpha_1,\alpha_2,\alpha,\gamma,\theta) = \left[\frac{K_{\gamma+1}(u)}{K_\gamma(u)} + \frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_2+1}{\alpha_1+\alpha_2+1}\,\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{K^2_{\gamma+1}(u)}{K^2_\gamma(u)}\right]\right].$$

2.5.4 The Beta-Binomial distribution for the ascending diagonal arrays and the Inverse Gaussian Poisson distribution for the third marginal

If we have the BBD on the ascending diagonal array and the IGP for the third marginal the joint distribution function of x_1 and x_2 is:

$$P(x_1,x_2) = \sqrt{\frac{2\alpha}{\pi}}\, \frac{e^{\alpha\sqrt{1-\theta}}}{B(\alpha_1,\alpha_2)}\, \frac{B(x_1+\alpha_1,x_2+\alpha_2)}{x_1!x_2!}\, (\alpha\theta/2)^{x_1+x_2}\, K_{x_1+x_2-1/2}(\alpha) \tag{2.5.36}$$

The marginal distributions for x_1 and x_2 are:

$$P_{x_1}(x_1) = \sqrt{\frac{2\alpha}{\pi}}\, \frac{e^{\alpha\sqrt{1-\theta}}}{B(\alpha_1,\alpha_2)}\, \frac{\left[\frac{\alpha\theta}{2}\right]^{x_1} \Gamma(x_1+\alpha_1)}{\Gamma(x_1+1)}$$

$$\cdot \sum_{x_2} \frac{\Gamma(x_2+\alpha_2)}{\Gamma(x_1+x_2+\alpha_1+\alpha_2)\Gamma(x_2+1)} \left[\frac{\alpha\theta}{2}\right]^{x_2} K_{x_1+x_2-1/2}(\alpha) \quad \text{and}$$

$$P_{x_2}(x_2) = \sqrt{\frac{2\alpha}{\pi}}\, \frac{e^{\alpha\sqrt{1-\theta}}}{B(\alpha_1,\alpha_2)}\, \frac{\left[\frac{\alpha\theta}{2}\right]^{x_2} \Gamma(x_2+\alpha_2)}{\Gamma(x_2+1)}$$

$$\cdot \sum_{x_1} \frac{\Gamma(x_1+\alpha_1)}{\Gamma(x_1+x_2+\alpha_1+\alpha_2)\Gamma(x_1+1)} \left[\frac{\alpha\theta}{2}\right]^{x_1} K_{x_1+x_2-1/2}(\alpha) \tag{2.5.37}$$

Further we use the Bessel function identity

$$K_{-\gamma}(z) = K_{\gamma}(z) \tag{2.5.38}$$

and the recurrence formula

$$K_{\gamma+1}(z) = \frac{2\gamma}{\alpha} K_{\gamma}(z) + K_{\gamma-1}(z)\ . \tag{2.5.39}$$

The substitution of $\gamma = -1/2$ into the formulae (2.5.30) to (2.5.35) leads to the following results:

$$\mu_1'(x_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\,, \tag{2.5.40}$$

$$\mu_1'(x_2) = \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\,, \tag{2.5.41}$$

$$\mu_2(x_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[1+\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_1+1}{\alpha_1+\alpha_2+1}\left[\frac{1}{\alpha\sqrt{1-\theta}}+1\right]-\frac{\alpha_1}{\alpha_1+\alpha_2}\right]\right], \tag{2.5.42}$$

$$\mu_2(x_2) = \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[1+\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_2+1}{\alpha_1+\alpha_2+1}\left[\frac{1}{\alpha\sqrt{1-\theta}}+1\right]-\frac{\alpha_2}{\alpha_1+\alpha_2}\right]\right], \tag{2.5.43}$$

$$\mathrm{cov}(x_1,x_2) = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+1)}\left[\frac{\alpha\theta}{2\sqrt{1-\theta}}\right]^2\left[\frac{\alpha_1+\alpha_2}{\alpha\sqrt{1-\theta}}-1\right] \tag{2.5.44}$$

and

$$\rho(x_1,x_2) = \frac{\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}\,\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_1+\alpha_2}{\alpha\sqrt{1-\theta}}-1\right] h_1(\alpha_1,\alpha_2,\alpha,\theta)\; h_2(\alpha_1,\alpha_2,\alpha,\theta),$$

where (2.5.45)

$$h_1(\alpha_1,\alpha_2,\alpha,\theta) = \left[1+\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_1+1}{\alpha_1+\alpha_2+1}\left[\frac{1}{\alpha\sqrt{1-\theta}}+1\right]-\frac{\alpha_1}{\alpha_1+\alpha_2}\right]\right]^{-1/2}$$

and

$$h_2(\alpha_1,\alpha_2,\alpha,\theta) = \left[1+\frac{\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{\alpha_2+1}{\alpha_1+\alpha_2+1}\left[\frac{1}{\alpha\sqrt{1-\theta}}+1\right]-\frac{\alpha_2}{\alpha_1+\alpha_2}\right]\right]^{-1/2}.$$

2.5.5 The Beta-Binomial distribution for the ascending diagonal arrays and the Negative Binomial distribution for the third marginal

The bivariate distribution generated from the NBD as the third marginal and the BBD for the ascending diagonal arrays is given by

$$P_{x_1x_2}(x_1,x_2) = \frac{(1-\theta)^{\gamma-\alpha_1-\alpha_2}\,\Gamma(\alpha_1+\alpha_2)}{\Gamma(\gamma)}\;\frac{\Gamma(x_1+x_2+\gamma)}{\Gamma(x_1+x_2+\alpha_1+\alpha_2)}$$

$$\cdot\left[\frac{(1-\theta)^{\alpha_1}\,\Gamma(x_1+\alpha_1)\theta^{x_1}}{\Gamma(\alpha_1)\,\Gamma(x_1+1)}\right]\left[\frac{(1-\theta)^{\alpha_2}\,\Gamma(x_2+\alpha_2)\theta^{x_2}}{\Gamma(\alpha_2)\,\Gamma(x_2+1)}\right], \qquad (2.5.46)$$

where $x_1 = 0,1,\ldots$ and $x_2 = 0,1,2,\ldots$

For the special case of $\gamma = \alpha_1+\alpha_2$, equation (2.5.46) becomes an independent bivariate NBD:

$$P_{x_1x_2}(x_1,x_2) = \left[\frac{(1-\theta)^{\alpha_1}\,\Gamma(x_1+\alpha_1)\theta^{x_1}}{\Gamma(\alpha_1)\,\Gamma(x_1+1)}\right]\left[\frac{(1-\theta)^{\alpha_2}\,\Gamma(x_2+\alpha_2)\theta^{x_2}}{\Gamma(\alpha_2)\,\Gamma(x_2+1)}\right]. \qquad (2.5.47)$$

By allowing the parameters in equation (2.5.46) to vary, a large number of interesting bivariate discrete distributions are obtained. For example, if one takes $\gamma = \alpha_1+\alpha_2-1$ and $\alpha_1 = \alpha_2 = 1$ (i.e. $\gamma = 1$), it leads to a very simple bivariate distribution given by:

$$P_{x_1x_2}(x_1,x_2) = \frac{(1-\theta)\theta^{x_1+x_2}}{x_1+x_2+1}, \quad \text{where} \qquad (2.5.48)$$

$x_1 = 0,1,2,\ldots$, $x_2 = 0,1,2,\ldots$ and $0 < \theta < 1$.

Again, if one sets $\gamma = \alpha_1+\alpha_2+1$, one obtains:

$$P_{x_1x_2}(x_1,x_2) = \frac{1-\theta}{\alpha_1+\alpha_2}(x_1+x_2+\alpha_1+\alpha_2)$$

$$\cdot\left[\frac{(1-\theta)^{\alpha_1}\,\Gamma(x_1+\alpha_1)\theta^{x_1}}{\Gamma(\alpha_1)\,\Gamma(x_1+1)}\right]\left[\frac{(1-\theta)^{\alpha_2}\,\Gamma(x_2+\alpha_2)\theta^{x_2}}{\Gamma(\alpha_2)\,\Gamma(x_2+1)}\right], \quad \text{where} \qquad (2.5.49)$$

$x_1 = 0,1,\ldots$ and $x_2 = 0,1,2,\ldots$.

With the classical approach we obtain the marginal distributions of x_1 and x_2:

$$P_{x_1}(x_1) = \left[\left[\frac{1-\theta}{\alpha_1+\alpha_2}\right]x_1 + \left[1 - \frac{\alpha_1\theta}{\alpha_1+\alpha_2}\right]\right]\left[\frac{(1-\theta)^{\alpha_1}\,\Gamma(x_1+\alpha_1)\theta^{x_1}}{\Gamma(\alpha_1)\,\Gamma(x_1+1)}\right]$$

and

$$P_{x_2}(x_2) = \left[\left[\frac{1-\theta}{\alpha_1+\alpha_2}\right]x_2 + \left[1 - \frac{\alpha_2\theta}{\alpha_1+\alpha_2}\right]\right]\left[\frac{(1-\theta)^{\alpha_2}\,\Gamma(x_2+\alpha_2)\theta^{x_2}}{\Gamma(\alpha_2)\,\Gamma(x_2+1)}\right].$$

(2.5.50)

One will find great difficulty in using the classical approach when $\gamma \neq \alpha_1+\alpha_2$. Then it is not possible to obtain the marginal distributions and their moments for either integer or non-integer values of γ, α_1, α_2.

By substituting the expressions for the first two moments around the origin of the third marginal into the formulae (2.5.3) to (2.5.8) we obtain the following results for the particular case when $\gamma = \alpha_1+\alpha_2+1$.

$$\mu_1'(x_1) = \frac{\gamma}{\gamma-1}\,\frac{\alpha_1\theta}{1-\theta}, \qquad (2.5.51)$$

$$\mu_1'(x_2) = \frac{\gamma}{\gamma-1}\,\frac{\alpha_2\theta}{1-\theta}\,, \tag{2.5.52}$$

$$\mu_2(x_1) = \frac{\alpha_1\gamma\theta}{(\gamma-1)(1-\theta)}\left[\frac{\alpha_2\theta}{(\gamma-1)(1-\theta)} + \frac{\alpha_1+1}{\gamma(1-\theta)} + \frac{\alpha_2}{\gamma}\right], \tag{2.5.53}$$

$$\mu_2(x_2) = \frac{\alpha_2\gamma\theta}{(\gamma-1)(1-\theta)}\left[\frac{\alpha_1\theta}{(\gamma-1)(1-\theta)} + \frac{\alpha_2+1}{\gamma(1-\theta)} + \frac{\alpha_1}{\gamma}\right], \tag{2.5.54}$$

$$\mathrm{cov}(x_1,x_2) = -\frac{\alpha_1\alpha_2}{(\gamma-1)^2}\left[\frac{\theta}{1-\theta}\right]^2 \quad \text{and} \tag{2.5.55}$$

$$\rho(x_1,x_2) = -\frac{\theta}{\gamma-1}\left[\left[1+\frac{\theta}{\gamma-1}+\frac{\alpha_2+1}{\alpha_1}\right]\left[1+\frac{\theta}{\gamma-1}+\frac{\alpha_1+1}{\alpha_2}\right]\right]^{-1/2}. \tag{2.5.56}$$

In the general case for any arbitrary values of γ,α_1 and α_2 the results of the model are as follows:

$$\mu_1'(x_1) = \frac{\alpha_1\gamma\theta}{(\alpha_1+\alpha_2)(1-\theta)}\,, \tag{2.5.57}$$

$$\mu_1'(x_2) = \frac{\alpha_2\gamma\theta}{(\alpha_1+\alpha_2)(1-\theta)}\,, \tag{2.5.58}$$

$$\mu_2(x_1)=\frac{\alpha_1\gamma\theta}{(\alpha_1+\alpha_2)(1-\theta)}\left[\frac{\alpha_2\gamma\theta}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)(1-\theta)}+\frac{\alpha_1+1}{(\alpha_1+\alpha_2+1)(1-\theta)}+\frac{\alpha_2}{\alpha_1+\alpha_2+1}\right], \tag{2.5.59}$$

$$\mu_2(x_2)=\frac{\alpha_2\gamma\theta}{(\alpha_1+\alpha_2)(1-\theta)}\left[\frac{\alpha_1\gamma\theta}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)(1-\theta)}+\frac{\alpha_2+1}{(\alpha_1+\alpha_2+1)(1-\theta)}+\frac{\alpha_1}{\alpha_1+\alpha_2+1}\right], \tag{2.5.60}$$

$$\mathrm{cov}(x_1,x_2) = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2)^2}\,\frac{\gamma\theta^2}{(1-\theta)^2}\,(\alpha_1+\alpha_2-\gamma) \quad \text{and} \tag{2.5.61}$$

$$\rho(x_1,x_2) = v\left[\left[1 - v + \frac{\alpha_2+1}{\alpha_1}\right]\left[1 - v + \frac{\alpha_1+1}{\alpha_2}\right]\right]^{-1/2}, \quad \text{where} \qquad (2.5.62)$$

$$v = \theta\left[1 - \frac{\gamma}{\alpha_1+\alpha_2}\right].$$

If $\gamma = \alpha_1+\alpha_2$ in formula (2.5.62), then the correlation coefficient becomes $\rho = 0$ and the variables x_1 and x_2 are therefore uncorrelated and independent as shown in equation (2.5.47).

For $\gamma \neq \alpha_1+\alpha_2$ a wide range of correlations is obtained. Increases in θ and $\alpha_1+\alpha_2 >> \gamma$ or $\alpha_1+\alpha_2 << \gamma$ will result in high positive or negative correlations respectively.

2.5.6 <u>The Beta-Binomial distribution for the ascending diagonal arrays and for the third marginal</u>

The joint distribution of x_1 and x_2 generated from the BBD for the third marginal as a function of parameters m, β_1 and β_2 and the BBD for the ascending diagonal arrays as a function of S, α_1 and α_2 is

$$P(x_1,x_2) = \frac{m!}{B(\alpha_1,\alpha_2)B(\beta_1,\beta_2)} \; \frac{B(x_1+\alpha_1,x_2+\alpha_2)B(x_1+x_2+\beta_1,m-x_1-x_2+\beta_2)}{(m-x_1-x_2)!\; x_1!\; x_2!},$$

where (2.5.63)

$x_1 = 0,1,\ldots,m,$ and $x_2 = 0,1,\ldots,m.$

The marginals can be derived from the joint distribution (2.5.63) using the classical approach, but they can not be expressed in a closed form.

The first two moments around the origin for the random variable on the ascending diagonal array are

$$\mu_1'(x_1|S) = \frac{S\alpha_1}{\alpha_1+\alpha_2} \quad \text{and} \tag{2.5.64}$$

$$\mu_2'(x_1|S) = \frac{S\alpha_1[S(\alpha_1+1)+\alpha_2]}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)} . \tag{2.5.65}$$

The mean and the variance for the random variable on the third marginal are

$$\mu_1'(S) = \frac{m\beta_1}{\beta_1+\beta_2} \quad \text{and} \tag{2.5.66}$$

$$\mu_2(S) = \frac{m\beta_1\beta_2(m+\beta_1+\beta_2)}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)} . \tag{2.5.67}$$

We now apply the theorems proven in Section 2.1 to obtain the moments of the model. Theorem 2.1 gives the following means for the marginals:

$$\mu_1'(x_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{m\beta_1}{\beta_1+\beta_2} \quad \text{and} \tag{2.5.68}$$

$$\mu_1'(x_2) = \frac{\alpha_2}{\alpha_1+\alpha_2}\,\frac{m\beta_1}{\beta_1+\beta_2} . \tag{2.5.69}$$

Theorem 2.2 gives the following variances for the marginals:

$$\mu_2(x_1) = \frac{\alpha_1(\alpha_1+1)}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}\left[\frac{m\beta_1[m(\beta_1+1)+\beta_2]}{(\beta_1+\beta_2)(\beta_1+\beta_2+1)}\right] +$$

$$+ \frac{m\alpha_1\alpha_2\beta_1}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)(\beta_1+\beta_2)} - \left[\frac{m\alpha_1\beta_1}{(\alpha_1+\alpha_2)(\beta_1+\beta_2)}\right]^2 \quad \text{and} \tag{2.5.70}$$

$$\mu_2(x_2) = \mu_2(x_1) + \frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\,\frac{m\beta_1\beta_2(m+\beta_1+\beta_2)}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)} . \tag{2.5.71}$$

Theorem 2.3 gives the covariance between x_1 and x_2:

$$\mathrm{cov}(x_1,x_2) = \frac{m\alpha_1\alpha_2\beta_1}{(\alpha_1+\alpha_2)(\beta_1+\beta_2)}\left[\frac{(m-1)(\beta_1+1)}{(\alpha_1+\alpha_2+1)(\beta_1+\beta_2+1)} - \frac{m\beta_1}{(\alpha_1+\alpha_2)(\beta_1+\beta_2)}\right]. \tag{2.5.72}$$

Using the definition of the correlation coefficient we obtain the correlation between x_1 and x_2. The sign of the correlation coefficient depends on the sign of the covariance between x_1 and x_2. This sign is a function of the relation between the parameters α_1 and α_2 of the ascending diagonal array and the parameters m, β_1 and β_2 of the third marginal.

The covariance will receive different values according to the value of $\alpha_1+\alpha_2$, as shown below:

$$\mathrm{cov}(x_1,x_2) > 0 \quad \text{if} \quad \alpha_1+\alpha_2 > \frac{m\beta_1(\beta_1+\beta_2+1)}{(m-\beta_1-1)(\beta_1+\beta_2)-m\beta_1}, \tag{2.5.73}$$

$$\mathrm{cov}(x_1,x_2) < 0 \quad \text{if} \quad \alpha_1+\alpha_2 < \frac{m\beta_1(\beta_1+\beta_2+1)}{(m-\beta_1-1)(\beta_1+\beta_2)-m\beta_1} \text{ and} \tag{2.5.74}$$

$$\mathrm{cov}(x_1,x_2) = 0 \quad \text{if} \quad \alpha_1+\alpha_2 = \frac{m\beta_1(\beta_1+\beta_2+1)}{(m-\beta_1-1)(\beta_1+\beta_2)-m\beta_1}. \tag{2.5.75}$$

From the above relations between the parameters $m,\alpha_1,\alpha_2,\beta_1,\beta_2$ we conclude that the variables x_1 and x_2 can be either positively or negatively correlated or uncorrelated to each other.

Let us investigate a simpler distribution for a particular case where $\beta_1 = \alpha_1+\alpha_2$. In this case the variances of x_1 and x_2, the covariance and the correlation between x_1 and x_2 can be expressed in a simple form as follows:

$$\mu_2(x_1) = \frac{m\alpha_1(\alpha_2+\beta_2)(m+\alpha_1+\alpha_2+\beta_2)}{(\alpha_1+\alpha_2+\beta_2+1)(\alpha_1+\alpha_2+\beta_2)^2}, \tag{2.5.76}$$

$$\mu_2(x_2) = \frac{m\alpha_2(\alpha_1+\beta_2)(m+\alpha_1+\alpha_2+\beta_2)}{(\alpha_1+\alpha_2+\beta_2+1)(\alpha_1+\alpha_2+\beta_2)^2}, \tag{2.5.77}$$

$$\mathrm{cov}(x_1,x_2) = -\frac{m\alpha_1\alpha_2(m+\alpha_1+\alpha_2+\beta_2)}{(\alpha_1+\alpha_2+\beta_2)^2(\alpha_1+\alpha_2+\beta_2+1)} \quad \text{and} \tag{2.5.78}$$

$$\rho(x_1,x_2) = -\left[\left[1+\frac{\beta_2}{\alpha_1}\right]\left[1+\frac{\beta_2}{\alpha_2}\right]\right]^{-1/2}. \tag{2.5.79}$$

2.5.7 <u>The Hypergeometric distribution for the ascending diagonal arrays and the Beta-Binomial distribution for the third marginal</u>

Let us assume that the variable for the ascending diagonal array is distributed in accordance with the Hypergeometric distribution, which is a particular case of the Beta-Binomial distribution. Without loss of generality let us further assume that this variable follows the same symmetric Hypergeometric distribution, which is a function of the same parameter α_1 only, as $\alpha_2 = \alpha_1$.

The moments of this distribution are:

$$\mu_1'(x_1|S) = S/2\ , \tag{2.5.80}$$

$$\mu_2'(x_1|S) = \frac{\alpha_1 S+(\alpha_1-1)S^2}{2(2\alpha_1-1)} \quad \text{and} \tag{2.5.81}$$

$$\mu_2(x_1|S) = \frac{(2\alpha_1-S)S}{4(2\alpha_1-1)}\ . \tag{2.5.82}$$

The random variable S for the third marginal is distributed as BBD with the parameters β_1, β_2 and $2\alpha_1$ and the moments

$$\mu_1'(S) = \frac{2\alpha_1\beta_1}{\beta_1+\beta_2}\,, \tag{2.5.83}$$

$$\mu_2'(S) = \frac{2\alpha_1\beta_1[2\alpha_1(\beta_1+1)+\beta_2]}{(\beta_1+\beta_2)(\beta_1+\beta_2+1)} \quad \text{and} \tag{2.5.84}$$

$$\mu_2(S) = \frac{2\alpha_1\beta_1\beta_2(2\alpha_1+\beta_1+\beta_2)}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)}\,. \tag{2.5.85}$$

The joint distribution is:

$$P_{x_1x_2}(x_1,x_2) = \begin{bmatrix}\alpha_1\\x_1\end{bmatrix}\begin{bmatrix}\alpha_1\\x_2\end{bmatrix}\frac{B(x_1+x_2+\beta_1,\ 2\alpha_1+\beta_2-x_1-x_2)}{B(\beta_1,\beta_2)}\,, \quad \text{where}$$

$$x_1 = 0,1,\ldots,\alpha_1,\quad x_2 = 0,1,\ldots,\alpha_2, \tag{2.5.86}$$

$\alpha_1 = 1,2,\ldots,\quad \alpha_2 = 1,2,\ldots,\quad \beta_1 > 0$ and $\beta_2 > 0$.

This distribution is the classical bivariate Beta-Binomial for which the marginals x_1 and x_2 are also distributed in accordance with the BBD model.

The marginal distributions are:

$$P_{x_1}(x_1) = \begin{bmatrix}\alpha_1\\x_1\end{bmatrix}\frac{B(x_1+\beta_1,\ \alpha_1-x_1+\beta_2)}{B(\beta_1,\ \beta_2)}\,, \quad \text{where} \tag{2.5.87}$$

$x_1 = 0,1,\ldots,\alpha_1,\quad \alpha_1 = 1,2,\ldots,\quad \beta_1 > 0$ and $\beta_2 > 0$; and

$$P_{x_2}(x_2) = \begin{bmatrix}\alpha_1\\x_2\end{bmatrix}\frac{B(x_2+\beta_1,\ \alpha_1-x_2+\beta_2)}{B(\beta_1,\ \beta_2)}\,, \quad \text{where} \tag{2.5.88}$$

$x_2 = 0,1,\ldots,\alpha_1,\quad \alpha_1 = 1,2,\ldots,\quad \beta_1 > 0$ and $\beta_2 > 0$.

The random variable S for the third marginal is distributed as BBD with the parameters β_1, β_2 and $2\alpha_1$ and the moments:

$$\mu_1'(S) = \frac{2\alpha_1\beta_1}{\beta_1+\beta_2}, \tag{2.5.83}$$

$$\mu_2'(S) = \frac{2\alpha_1\beta_1[2\alpha_1(\beta_1+1)+\beta_2]}{(\beta_1+\beta_2)(\beta_1+\beta_2+1)} \quad \text{and} \tag{2.5.84}$$

$$\mu_2(S) = \frac{2\alpha_1\beta_1\beta_2(2\alpha_1+\beta_1+\beta_2)}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)}. \tag{2.5.85}$$

The joint distribution is:

$$P_{x_1x_2}(x_1,x_2) = \begin{bmatrix}\alpha_1\\x_1\end{bmatrix}\begin{bmatrix}\alpha_1\\x_2\end{bmatrix}\frac{B(x_1+x_2+\beta_1,\ 2\alpha_1+\beta_2-x_1-x_2)}{B(\beta_1,\beta_2)}, \text{ where}$$

$$x_1 = 0,1,\ldots,\alpha_1,\quad x_2 = 0,1,\ldots,\alpha_2, \tag{2.5.86}$$

$$\alpha_1 = 1,2,\ldots,\quad \alpha_2 = 1,2,\ldots,\quad \beta_1 > 0 \text{ and } \beta_2 > 0.$$

This distribution is the classical bivariate Beta-Binomial for which the marginals x_1 and x_2 are also distributed in accordance with the BBD model.

The marginal distributions are:

$$P_{x_1}(x_1) = \begin{bmatrix}\alpha_1\\x_1\end{bmatrix}\frac{B(x_1+\beta_1,\ \alpha_1-x_1+\beta_2)}{B(\beta_1,\ \beta_2)}, \text{ where} \tag{2.5.87}$$

$$x_1 = 0,1,\ldots,\alpha_1,\quad \alpha_1 = 1,2,\ldots,\quad \beta_1 > 0 \text{ and } \beta_2 > 0\ ; \quad \text{and}$$

$$P_{x_2}(x_2) = \begin{bmatrix}\alpha_1\\x_2\end{bmatrix}\frac{B(x_2+\beta_1,\ \alpha_1-x_2+\beta_2)}{B(\beta_1,\ \beta_2)}, \text{ where} \tag{2.5.88}$$

$$x_2 = 0,1,\ldots,\alpha_1,\quad \alpha_1 = 1,2,\ldots,\quad \beta_1 > 0 \text{ and } \beta_2 > 0\ .$$

By applying Theorem 2.4 to this model we obtain the following results:

$$\mu_1'(x_1) = \mu_1'(x_2) = \frac{\alpha_1\beta_1}{\beta_1+\beta_2}\ , \tag{2.5.89}$$

$$\mu_2(x_1) = \mu_2(x_2) = \frac{\alpha_1\beta_1\beta_2(\alpha_1+\beta_1+\beta_2)}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)}\ , \tag{2.5.90}$$

$$\operatorname{cov}(x_1,x_2) = \frac{\alpha_1^2\ \beta_1\beta_2}{(\beta_1+\beta_2)^2(\beta_1+\beta_2+1)} \quad \text{and} \tag{2.5.91}$$

$$\rho(x_1,x_2) = \left[1 + \frac{\beta_1+\beta_2}{\alpha_1}\right]^{-1} . \tag{2.5.92}$$

In this case it is possible to work out the same result by using the conventional method.

2.5.8 <u>The Binomial distribution for the ascending diagonal arrays and the Beta-Binomial distribution for the third marginal</u>

As previously mentioned the Beta-Binomial distribution approaches the Binomial distribution for very large values of its parameters. Let us assume that the distribution for the ascending diagonal arrays is the Binomial determined by the parameters S and p. The random variable on the third marginal is distributed following the probability law of the BBD with parameters m, β_1 and β_2.

The first two moments around the origin for the random variable for the ascending diagonal arrays are:

$$\mu_1'(x_1|S) = Sp \quad \text{and} \tag{2.5.93}$$

$$\mu_2'(x_1|S) = Sp(1-p) + S^2p^2\ . \tag{2.5.94}$$

By using the formulae (2.5.66) and (2.5.67) and the moments above and by applying Theorems 2.1, 2.2 and 2.3 we obtain the following results:

$$\mu_1'(x_1) = pm\, b_1 \ , \tag{2.5.95}$$

$$\mu_1'(x_2) = (1-p)m\, b_1 \ , \tag{2.5.96}$$

$$\mu_2(x_1) = pm\, b_1 \left[(1-p) + p\, b_2\right] , \tag{2.5.97}$$

$$\mu_2(x_2) = (1-p)m\, b_1 \left[p + (1-p)\, b_2\right] , \tag{2.5.98}$$

$$\mathrm{cov}(x_1,x_2) = p(1-p)m\, b_1(b_2-1) \quad \text{and} \tag{2.5.99}$$

$$\rho(x_1,x_2) = (b_2-1) \left[\left[\frac{p}{1-p} + b_2\right]\left[\frac{1-p}{p} + b_2\right]\right] , \text{ where} \tag{2.5.100}$$

$$b_1 = \frac{\beta_1}{\beta_1+\beta_2} \quad \text{and} \quad b_2 = \frac{\beta_2(\beta_1+\beta_2+m)}{(\beta_1+\beta_2)(\beta_1+\beta_2+1)} .$$

For the particular case where we have the same symmetrical Binomial distribution with $p = 1/2$ for the ascending diagonal arrays the formulae (2.5.95) to (2.5.100) become much simpler. Due to the symmetry on the ascending diagonal arrays the marginals x_1 and x_2 have the same mean and variance. By substituting $p = 1/2$ in the expressions above we obtain the following results:

$$\mu_1'(x_1) = \mu_1'(x_2) = mb_1/2 \ , \tag{2.5.101}$$

$$\mu_2(x_1) = \mu_2(x_2) = mb_1(1+b_2)/4 \ , \tag{2.5.102}$$

$$\mathrm{cov}(x_1,x_2) = mb_1(b_2-1)/4 \quad \text{and} \tag{2.5.103}$$

$$\rho(x_1,x_2) = \frac{b_2-1}{b_2+1} . \tag{2.5.104}$$

From the foregoing mathematical development we conclude that for any parameter p of the Binomial distribution for the ascending diagonal arrays the sign of the correlation coefficient does not depend on the values of p. The sign of the correlation coefficient between the variables x_1 and x_2 is only a function of the parameters of the third marginal distribution, i.e. m, β_1 and β_2. Thus:

$$\rho(x_1,x_2) > 0 \quad \text{if} \quad m > \frac{(\beta_1+\beta_2)(\beta_1+1)}{\beta_2} ,$$

$$\rho(x_1,x_2) < 0 \quad \text{if} \quad m < \frac{(\beta_1+\beta_2)(\beta_1+1)}{\beta_2} \quad \text{and}$$

$$\rho(x_1,x_2) = 0 \quad \text{if} \quad m = \frac{(\beta_1+\beta_2)(\beta_1+1)}{\beta_2} .$$

2.5.9 <u>The Binomial distribution for the ascending diagonal arrays and for the third marginal</u>

Let us consider for all the ascending diagonal arrays the Binomial distribution with the parameters S and p. The two moments around the origin for the Binomial distribution are:

$$\mu_1'(x_1|S) = Sp \quad \text{and} \tag{2.5.105}$$

$$\mu_2'(x_1|S) = Sp(1-p) + S^2p^2 . \tag{2.5.106}$$

For the third marginal we take the Binomial distribution with parameters n and θ with the moments given as:

$$\mu_1'(S) = n\theta \quad \text{and} \tag{2.5.107}$$

$$\mu_2(S) = n\theta(1-\theta) . \tag{2.5.108}$$

The joint distribution of x_1 and x_2 is

$$P_{x_1x_2}(x_1,x_2) = \frac{n!}{x_1!x_2!(n-x_1-x_2)!}(p\theta)^{x_1}[(1-p)\theta]^{x_2}(1-\theta)^{n-x_1-x_2},$$

where $0 \leq p \leq 1$ and $0 \leq \theta \leq 1$. (2.5.109)

This is the classical trinomial distribution, which is bound by the main ascending diagonal array.

The marginal distributions for x_1 and x_2 are distributed Binomial with parameters n, $p\theta$ and n, $(1-p)\theta$ respectively. Their probability laws are:

$$P_{x_1}(x_1) = \begin{bmatrix} n \\ x_1 \end{bmatrix} (p\theta)^{x_1} (1-p\theta)^{n-x_1} \quad \text{and} \tag{2.5.110}$$

$$P_{x_2}(x_2) = \begin{bmatrix} n \\ x_2 \end{bmatrix} [(1-p)\theta]^{x_2} \Big[1-(1-p)\theta\Big]^{n-x_2} . \tag{2.5.111}$$

By applying Theorems 2.1, 2.2 and 2.3 to the joint distribution given by (2.5.109) one obtains the following results:

$$\mu_1'(x_1) = n\theta p , \tag{2.5.112}$$

$$\mu_1'(x_2) = n\theta(1-p) , \tag{2.5.113}$$

$$\mu_2(x_1) = n\theta p(1-\theta p) , \tag{2.5.114}$$

$$\mu_2(x_2) = n\theta(1-p)(1-\theta+\theta_p) , \tag{2.5.115}$$

$$\text{cov}(x_1,x_2) = -n\theta^2 p(1-p) \quad \text{and} \tag{2.5.116}$$

$$\rho(x_1,x_2) = -\theta\left[\left[\frac{1}{p} - \theta\right]\left[\frac{1}{1-p} -\theta\right]\right]^{-1/2} . \tag{2.5.117}$$

The same results could be obtained by substituting the third marginal moments into equations (2.5.18) to (2.5.23).

2.5.10 The Beta-Binomial distribution with α_1 and α_2 as functions of S for the ascending diagonal arrays and any distribution for the third marginal

Let us consider for the ascending diagonal arrays the BBD with parameters S, α_1 and α_2, where α_1 and α_2 are functions of S as explained in Section 2.3.3. For the third marginal we assume any distribution with the first two moments around the origin as $\mu_1'(S)$ and $\mu_2'(S)$.

By applying Theorems 2.1, 2.2 and 2.3 the following results are obtained:

$$\mu_1'(x_1) = \frac{1}{c+1}\,\mu_1'(S)\ , \tag{2.5.118}$$

$$\mu_1'(x_2) = \frac{c}{c+1}\,\mu_1'(S)\ , \tag{2.5.119}$$

$$\mu_2(x_1) = \frac{\mu_2(S)}{(c+1)^2} + b\,\mu_2'(S) + a\,\mu_1'(S)\ , \tag{2.5.120}$$

$$\mu_2(x_2) = \left[\frac{c}{c+1}\right]^2 \mu_2(S) + b\,\mu_2'(S) + a\,\mu_1'(S)\ , \tag{2.5.121}$$

$$E(x_1x_2) = \left[\frac{c}{(c+1)^2} - b\right]\mu_2'(S) - a\,\mu_1'(S)\ , \tag{2.5.122}$$

$$\operatorname{cov}(x_1,x_2) = \frac{c}{(c+1)^2}\,\mu_2(S) - b\,\mu_1'(S) - a\,\mu_1'(S) \quad \text{and} \tag{2.5.123}$$

$$\rho(x_1,x_2) = \frac{1 - \frac{(c+1)^2}{c}\,z}{\sqrt{\left[1 + (c+1)^2\,z\right]\left[1 + \left[\frac{c+1}{c}\right]^2 z\right]}}\ , \quad \text{where} \tag{2.5.124}$$

$$z = \frac{b\,\mu_2'(S) + a\,\mu_1'(S)}{\mu_2(S)}\ . \tag{2.5.125}$$

The correlation coefficient $\rho(x_1,x_2)$ depends on c and z, where z actually depends on the moments of the distribution for the third marginal and on the relation between the variances on the ascending diagonal arrays.

For $z > \frac{c}{(c+1)^2}$ $\rho(x_1,x_2) < 0$,

for $z = \frac{c}{(c+1)^2}$ $\rho(x_1,x_2) = 0$ and

for $z < \frac{c}{(c+1)^2}$ $\rho(x_1,x_2) > 0$.

If we assume that for each ascending diagonal array the variable is distributed symmetrically according to the BBD model and that the parameters α_1 and α_2 are still functions of S, then in equation (2.3.21) we take c = 1. In the symmetrical case for the ascending diagonal arrays we obtain the following results:

$$\mu_1'(x_1) = \mu_1'(x_2) = \mu_1'(S)/2 \ , \tag{2.5.126}$$

$$\mu_2(x_1) = \mu_2(x_2) = \mu_2(S)/4 + b\mu_2'(S) + a\mu_1'(S) \ , \tag{2.5.127}$$

$$E(x_1x_2) = \mu_2'(S)/4 - a\mu_1'(S) \ , \tag{2.5.128}$$

$$cov(x_1,x_2) = \mu_2(S)/4 - b\mu_1'(S) - a\mu_1'(S) \quad \text{and} \tag{2.5.129}$$

$$\rho(x_1,x_2) = \frac{1-4z}{1+4z} \ . \tag{2.5.130}$$

If we further assume that for the variances on the ascending diagonal arrays we can fit

$$\mu_2(x_1|S) = b\ S^2 \ , \quad \text{then} \tag{2.5.131}$$

$$\rho(x_1,x_2) = \frac{1-bu}{1+bu} \ , \quad \text{where} \tag{2.5.132}$$

$$u = 4\ \mu_1'(S)/\mu_2(S) \ . \tag{2.5.133}$$

The correlation coefficient $\rho(x_1,x_2)$ depends on b and u:

For $b > \frac{1}{u}$ $\rho(x_1,x_2) < 0$,

for $b = \frac{1}{u}$ $\rho(x_1,x_2) = 0$ and

for $b < \frac{1}{u}$ $\rho(x_1,x_2) > 0$.

As illustrated in this specific case we can still apply the model for different functions fitted for the variances on the ascending diagonal array, as long as the functions are of a polynomial type.

If the functions are not polynomials one can use the Taylor series and an approximation is obtained. It will then probably be necessary to calculate higher order moments around the origin for the third marginal. In such a case we may encounter difficulties in calculating the high order moments. The model will therefore be not so easy to apply.

2.6 Regression

The knowledge accumulated so far raises the major question whether it is possible to derive the regression curves from the developed model. The answer is positive. Unfortunately it is not possible to develop general mathematical equations for this purpose, because the marginal distribution must be known first. Each single situation must be treated individually and one can probably obtain a closed form for the regression curves only for certain values of the parameters.

In this section we discuss the bivariate distribution developed in Section 2.5.5, which assumes the Negative Binomial distribution with parameters γ and θ for the third marginal and the Beta-Binomial distribution with the same parameters α_1 and α_2 for all the ascending diagonal arrays.

In what follows we show that for certain values of α_1, α_2 and γ one obtains a closed form function for the regression curve $E(x_2|x_1)$. We consider five different cases which can be used by interpolation for the fitting of other regression curves.

<u>Case 1</u>

For the given values of the parameters

$\alpha_1 = n$, $\alpha_2 = 1$ and $\gamma = n$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{n(1-\theta)^n}{(n-1)!}(x_1+1)\dots(x_1+n-1)\frac{\theta^{x_1+x_2}}{x_1+x_2+n}, \text{ where} \quad (2.6.1)$$

$x_1 = 0,1,\dots$ and $x_2 = 0,1,\dots$.

The marginal distribution of x_1 is given by

$$P_{x_1}(x_1) = \frac{n}{(n-1)!}\left[\frac{1-\theta}{\theta}\right]^n (x_1+1)\dots(x_1+n-1)\ \lambda\left[{}_n x_1\right], \quad (2.6.2)$$

where $x_1 = 0,1,\dots$,

and the probability law of x_2 given x_1 is

$$P(x_2|x_1) = \frac{\theta^{x_1+x_2+n}}{(x_1+x_2+n)\lambda\left[{}_n x_1\right]}, \text{ where} \quad (2.6.3)$$

$x_2 = 0,1,\dots$.

Therefore the regression curve can be written as

$$E(x_2|x_1) = \frac{\theta^{x_1+n}}{(1-\theta)\lambda\left[{}_n x_1\right]} - (x_1+n) . \tag{2.6.4}$$

The expressions $\lambda\left[{}_n 0\right]$ and $\lambda\left[{}_n x_1\right]$ above are defined as:

$$\lambda\left[{}_n 0\right] = \sum_{j=n}^{\infty} \frac{\theta^j}{j} = -\left[\ln(1-\theta) - \sum_{j=1}^{n-1} \frac{\theta^j}{j}\right] \quad \text{and} \tag{2.6.5}$$

$$\lambda\left[{}_n x_1\right] = \lambda\left[{}_{n+1}(x_1-1)\right] - \frac{\theta^{x_1+n-1}}{x_1+n-1} . \tag{2.6.6}$$

Let us consider a particular situation of Case 1 when the values of the parameters are

$\alpha_1 = 1$, $\alpha_2 = 1$ **and** $\gamma = 1$.

The joint probability function of x_1 **and** x_2 **is**

$$P(x_1,x_2) = (1-\theta) \frac{\theta^{x_1+x_2}}{x_1+x_2+1} , \quad \text{where} \tag{2.6.7}$$

$x_1 = 0,1,\ldots$ **and** $x_2 = 0,1,\ldots$.

The marginal distribution of x_1 **is**

$$P_{x_1}(x_1) = \frac{1-\theta}{\theta} \lambda\left[{}_1 x_1\right] , \quad \text{where} \tag{2.6.8}$$

$x_1 = 0,1,\ldots$.

The conditional distribution of x_2 **given** x_1 **is**

$$P(x_2|x_1) = \frac{\theta^{x_1+x_2+1}}{(x_1+x_2+1)\ \lambda\left[{}_1 x_1\right]} , \quad \text{where} \tag{2.6.9}$$

$x_2 = 0,1,\ldots$.

The regression curve of x_2 on x_1 is

$$E(x_2|x_1) = \frac{\theta^{x_1+1}}{(1-\theta)\lambda\left[{}_1x_1\right]} - (x_1+1) \text{ , where} \tag{2.6.10}$$

$$\lambda\left[{}_1x_1\right] = \lambda\left[{}_1(x_1-1)\right] \text{ , and} \tag{2.6.11}$$

$$\lambda\left[{}_1 0\right] = -\ln(1-\theta) \text{ .} \tag{2.6.12}$$

Case 2

For the given values of the parameters

$\alpha_1 = n$, $\alpha_2 = 1$ and $\gamma = n+2$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{n}{(n+1)!}(1-\theta)^{n+2}(x_1+1)\ldots(x_1+n-1)(x_1+x_2+n+1)\theta^{x_1+x_2}$$

where $x_1 = 0,1,\ldots$ and $x_2 = 0,1,\ldots$. (2.6.13)

The marginal distribution of x_1 is

$$P_{x_1}(x_1) = \frac{n(1-\theta)^{n+2}}{(n+1)!}(x_1+1)\ldots(x_1+n-1)\left[\frac{x_1+n+1}{1-\theta} + \frac{\theta}{(1-\theta)^2}\right]\theta^{x_1}$$

where $x_1 = 0,1,\ldots$. (2.6.14)

The conditional distribution of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)^2(x_1+x_2+n+1)\theta^{x_2}}{(x_1+n+1)(1-\theta)+\theta} \text{ .} \tag{2.6.15}$$

The regression curve of x_2 on x_1 is

$$E(x_2|x_1) = \frac{\theta}{1-\theta}\left[1 + \frac{1}{(x_1+n+1)(1-\theta)+\theta}\right] \text{ .} \tag{2.6.16}$$

Let us consider a particular situation of Case 2 when the values of the parameters are

$\alpha_1 = 1, \quad \alpha_2 = 1 \quad$ and $\quad \gamma = 3.$

The joint probability function of x_1 and x_2 is

$$P(x_1,x_2) = \frac{(1-\theta)^3}{2} (x_1+x_2+2)\, \theta^{x_1+x_2}, \quad \text{where} \tag{2.6.17}$$

$x_1 = 0,1,\ldots \quad$ and $\quad x_2 = 0,1,\ldots$.

The marginal distribution of x_1 is

$$P_{x_1}(x_1) = \frac{1-\theta}{2}\, \theta^{x_1} [(x_1+2)(1-\theta)+\theta] \ , \quad \text{where} \tag{2.6.18}$$

$x_1 = 0,1,\ldots$.

The conditional distribution of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)^2(x_1+x_2+2)\theta^{x_2}}{(x_1+2)(1-\theta)+\theta}, \quad \text{where} \tag{2.6.19}$$

$x_2 = 0,1,\ldots$.

The regression curve of x_2 on x_1 is

$$E(x_2|x_1) = \frac{\theta}{1-\theta} \left[\frac{1}{(x_1+2)(1-\theta)+\theta}\right] . \tag{2.6.20}$$

<u>Case 3</u>

For the given values of the parameters

$\alpha_1 = n, \quad \alpha_2 = 1 \quad$ and $\quad \gamma = n+3$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

Let us consider a particular situation of Case 2 when the values of the parameters are

$\alpha_1 = 1, \quad \alpha_2 = 1 \quad$ and $\quad \gamma = 3.$

The joint probability function of x_1 and x_2 is

$$P(x_1,x_2) = \frac{(1-\theta)^3}{2}(x_1+x_2+2)\,\theta^{x_1+x_2}, \text{ where} \tag{2.6.17}$$

$x_1 = 0,1,\ldots \quad$ and $\quad x_2 = 0,1,\ldots$.

The marginal distribution of x_1 is

$$P_{x_1}(x_1) = \frac{1-\theta}{2}\,\theta^{x_1}\,[(x_1+2)(1-\theta)+\theta]\ , \text{ where} \tag{2.6.18}$$

$x_1 = 0,1,\ldots$.

The conditional distribution of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)^2(x_1+x_2+2)\theta^{x_2}}{(x_1+2)(1-\theta)+\theta}\ , \text{ where} \tag{2.6.19}$$

$x_2 = 0,1,\ldots$.

The regression curve of x_2 on x_1 is

$$E(x_2|x_1) = \frac{\theta}{1-\theta}\left[\frac{1}{(x_1+2)(1-\theta)+\theta}\right] . \tag{2.6.20}$$

<u>Case 3</u>

For the given values of the parameters

$\alpha_1 = n, \quad \alpha_2 = 1 \quad$ and $\quad \gamma = n+3$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{n(1-\theta)^{n+3}}{(n+2)!}(x_1+1)\dots(x_1+n-1)(x_1+x_2+n+1)(x_1+x_2+n+2)\theta^{x_1+x_2}$$

where $x_1 = 0,1,\dots$ and $x_2 = 0,1,\dots$. (2.6.21)

The marginal distribution of x_1 is

$$P_{x_1}(x_1) = \frac{n(1-\theta)^n(x_1+1)\dots(x_1+n-1)\theta^{x_1}}{(n+2)!}$$

$$\cdot \left[\theta(1-\theta) + (2x_1+2n+3)\theta(1-\theta) + (x_1+n+1)(x_1+n+2)(1-\theta)^2\right]$$

where $x_1 = 0,1,\dots$. (2.6.22)

The conditional distribution of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)^3(x_1+x_2+n+1)(x_1+x_2+n+2)\theta^{x_2}}{\theta(1+\theta)+(2x_1+2n+3)\theta(1-\theta)+(x_1+n+1)(x_1+n+2)(1-\theta)^2}$$

where $x_2 = 0,1,\dots$. (2.6.23)

The regression curve of x_2 on x_1 is

$$E(x_2|x_1) = \frac{\theta}{1-\theta}\left[\frac{\theta^2+4\theta+1+(2x_1+2n+3)(1-\theta^2)+(x_1+n+1)(x_1+n+2)(1-\theta)^2}{\theta(1+\theta)+(2x_1+2n+3)\theta(1-\theta)+(x_1+n+1)(x_1+n+2)(1-\theta)^2}\right].$$

(2.6.24)

Case 4

For the given values of the parameters

$\alpha_1 = n$, $\alpha_2 = 1$ and $\gamma = n-1$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{n(1-\theta)^{n-1}}{(n-2)!}(x_1+1)\dots(x_1+n-1)\frac{\theta^{x_1+x_2}}{(x_1+x_2+n-1)(x_1+x_2+n)},$$

where $x_1 = 0,1,\dots$ and $x_2 = 0,1,\dots$. (2.6.25)

The marginal distribution of x_1 is given by

$$P_{x_1}(x_1) = \frac{n}{(n-2)!}\left[\frac{1-\theta}{\theta}\right]^{n-1}(x_1+1)\ldots(x_1+n-1)$$

$$\cdot\left[\lambda\left[{}_{n-1}x_1\right] - \frac{1}{\theta}\lambda\left[{}_{n}x_1\right]\right], \quad \text{where} \tag{2.6.26}$$

$x_1 = 0,1,\ldots$,

and the probability law of x_2 given x_1 is

$$P(x_2|x_1) = \frac{\theta^{x_1+x_2+n}}{\left[\lambda\left[{}_{n-1}x_1\right]\theta - \lambda\left[{}_{n}x_1\right]\right](x_1+x_2+n-1)(x_1+x_2+n)},$$

where $x_2 = 0,1,\ldots$. (2.6.27)

The regression curve can therefore be written as

$$E(x_2|x_1) = \frac{\theta\lambda\left[{}_{n-1}x_1\right]}{\theta\lambda\left[{}_{n-1}x_1\right] - \lambda\left[{}_{n}x_1\right]} - (x_1+n), \tag{2.6.28}$$

where

$$\lambda\left[{}_{n-1}0\right] = -\left[\ln(1-\theta) + \sum_{j=1}^{n-2}\frac{\theta^j}{j}\right], \tag{2.6.29}$$

$$\lambda\left[{}_{n-1}x_1\right] = \lambda\left[{}_{n-1}(x_1-1)\right] - \frac{\theta^{x_1+n-2}}{x_1+n-2}, \tag{2.6.30}$$

$$\lambda\left[{}_{n}0\right] = -\left[\ln(1-\theta) - \sum_{j=1}^{n-1}\frac{\theta^j}{j}\right] \quad \text{and} \tag{2.6.31}$$

$$\lambda\left[{}_{n}x_1\right] = \lambda\left[{}_{n}(x_1-1)\right] - \frac{\theta^{x_1+n-1}}{x_1+n-1}. \tag{2.6.32}$$

Case 5

For the given values of the parameters

$\alpha_1 = n$, $\alpha_2 = 2$ and $\gamma = n+1$, where n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{(n+1)(1-\theta)^{n+1}}{(n-1)!} \frac{(x_1+1)\dots(x_1+n-1)(x_2+1)}{x_1+x_2+n+1} \theta^{x_1+x_2},$$

where $x_1 = 0,1,\dots$ and $x_2 = 0,1,\dots$. (2.6.33)

The marginal distribution of x_1 is given by

$$P_{x_1}(x_1) = \frac{n+1}{(n-1)!} \left[\frac{1-\theta}{\theta}\right]^{n+1} (x_1+1)\dots(x_1+n-1)$$

$$\cdot \left[\frac{\theta^{x_1+n+1}}{1-\theta} - (x_1-n)\ \lambda\left[{}_{n+1}x_1\right]\right], \quad \text{where} \qquad (2.6.34)$$

$x_1 = 0,1,\dots$,

and the probability law of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)(x_2+1)\theta^{x_1+x_2+n+1}}{(x_1+x_2+n+1)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\ \lambda\left[{}_{n+1}x_1\right]\right]},$$

where $x_2 = 0,1,\dots$. (2.6.35)

The regression curve can therefore be written as:

$$E(x_2|x_1) = \frac{\theta^{x_1+n+1}\left[(x_1+n+1)(1-\theta)+\theta\right] - (2x_1+2n+1)(1-\theta)\theta^{x_1+n+1}}{(1-\theta)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\ \lambda\left[{}_{n+1}x_1\right]\right]}$$

$$+ \frac{(x_1+n+1)(x_1+n)(1-\theta)\ \lambda\left[{}_{n+1}x_1\right]}{(1-\theta)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\ \lambda\left[{}_{n+1}x_1\right]\right]},$$

(2.6.36)

where

Case 5

For the given values of the parameters

$\alpha_1 = n$, $\alpha_2 = 2$ and $\gamma = n+1$, whe n is an integer,

one can write the joint probability function of x_1 and x_2 as:

$$P(x_1,x_2) = \frac{(n+1)(1-\theta)^{n+1}}{(n-1)!} \; \frac{(x_1+1)\ldots(x_1+n-1)(x_2+1)}{x_1+x_2+n+1} \; \theta^{x_1+x_2} ,$$

where $x_1 = 0,1,\ldots$ and $x_2 = 0,1,\ldots$. (2.6.33)

The marginal distribution of x_1 is given by

$$P_{x_1}(x_1) = \frac{n+1}{(n-1)!} \left[\frac{1-\theta}{\theta}\right]^{n+1} (x_1+1)\ldots(x_1+n-1)$$

$$\cdot \left[\frac{\theta^{x_1+n+1}}{1-\theta} - (x_1-n)\, \lambda\left[{}_{n+1}x_1\right]\right] , \quad \text{where} \qquad (2.6.34)$$

$x_1 = 0,1,\ldots$,

and the probability law of x_2 given x_1 is

$$P(x_2|x_1) = \frac{(1-\theta)(x_2+1)\theta^{x_1+x_2+n+1}}{(x_1+x_2+n+1)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\, \lambda\left[{}_{n+1}x_1\right]\right]} .$$

where $x_2 = 0,1,\ldots$. (2.6.35)

The regression curve can therefore be written as:

$$E(x_2|x_1) = \frac{\theta^{x_1+n+1}\left[(x_1+n+1)(1-\theta)+\theta\right] - (2x_1+2n+1)(1-\theta)\theta^{x_1+n+1}}{(1-\theta)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\, \lambda\left[{}_{n+1}x_1\right]\right]}$$

$$+ \frac{(x_1+n+1)(x_1+n)(1-\theta)\, \lambda\left[{}_{n+1}x_1\right]}{(1-\theta)\left[\theta^{x_1+n+1} - (x_1+n)(1-\theta)\, \lambda\left[{}_{n+1}x_1\right]\right]} ,$$

(2.6.36)

where

$$\lambda\left[{}_{n+1}0\right] = -\left[\ln(1-\theta) - \sum_{j=1}^{n}\frac{\theta^{j}}{j}\right] \quad \text{and} \tag{2.6.37}$$

$$\lambda\left[{}_{n+1}x_1\right] = \lambda\left[{}_{n+1}(x_1-1)\right] - \frac{\theta^{n+x_1}}{n+x_1}\ . \tag{2.6.38}$$

Notes

(a) One can develop formulae for regression curves for a large variety of parameters α_1, α_2 and γ. However, for larger α_1 and α_2 the mathematical expressions become more complex.

(b) For non-integer parameters α_1, α_2 and γ we can not find simple mathematical expressions for the regression curves because of the nature of the above recurrence formulae. For certain non-integer values of α_1, α_2 and γ we can approximate the shape of the regression curve by interpolating between the consecutive integer values of the parameters.

(c) In Section 2.5.5 we developed a general formula for the correlation between x_1 and x_2 and assumed that the distribution on the third marginal was $NBD(\theta,\gamma)$ and the distribution for the ascending diagonal arrays was $BBD(\alpha_1,\alpha_2)$ with the same parameters along all the ascending diagonal arrays. By substituting the values of the parameters α_1, α_2 and γ into formula (2.5.62) one obtains the correlation coefficient for each of the above five cases:

For case 1: $\rho(x_1,x_2) > 0$,

For case 2: $\rho(x_1,x_2) < 0$,

For case 3: $\rho(x_1,x_2) < 0$,

For case 4: $\rho(x_1,x_2) > 0$ and

For case 5: $\rho(x_1,x_2) > 0$.

(d) The asymptote of the above regression curves is $\frac{\theta}{1-\theta}$.

(e) The distribution applied to the ascending diagonal arrays determines the shape of the regression curve.

For BBD with $\alpha_1 > 0$ and $\alpha_2 > 0$ the regression curve is convex,

for GPHD with $\alpha_1 < 0$ and $\alpha_2 < 0$ the regression curve is concave and

for Binomial distribution the regression curve is linear.

2.7 The Indicator of the Distribution on the Ascending Diagonal Arrays

Let us assume that the distribution on the ascending diagonal arrays is a BBD with constant parameters α_1 and α_2. Without loss of generality we consider the symmetrical case where $\alpha_1 = \alpha_2$. Then the variance of the ascending diagonal arrays is formulated in terms of S and α_1 as follows:

$$\mu_2(x_1|S,\alpha_1) = \frac{S^2 + 2\alpha_1 S}{4(2\alpha_1+1)} . \qquad (2.7.1)$$

If $\alpha_1 \to +\infty$ or $\alpha_1 \to -\infty$, we have the limiting process where the BBD (i.e. $\alpha_1 > 0$) or the Hypergeometric distribution (i.e. $\alpha_1 < 0$)

are approaching the symmetrical Binomial distribution with the variance given by:

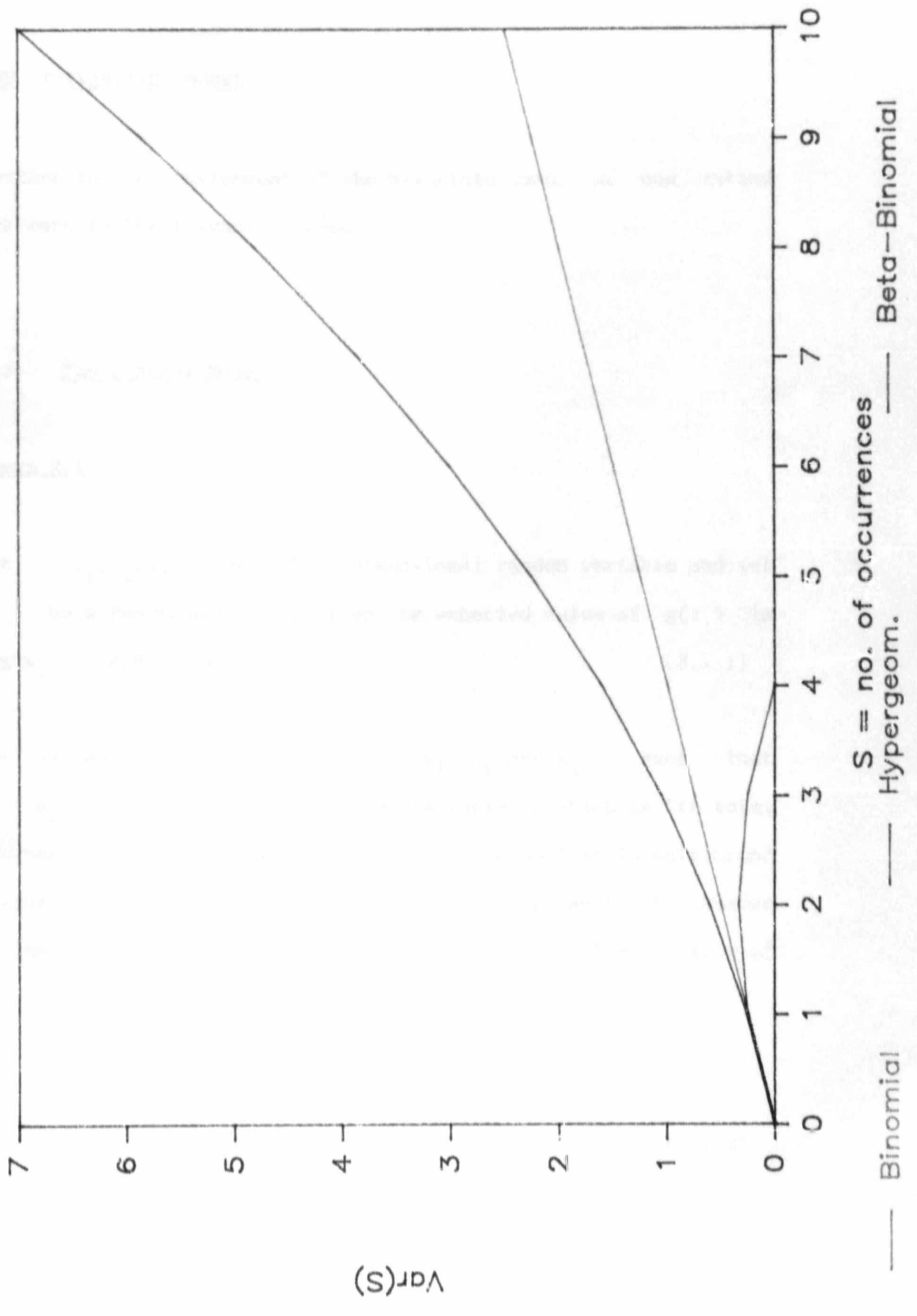
$$\lim_{\substack{\alpha_1 \to +\infty \\ \alpha_1 \to -\infty}} \mu_2(x_1|S,\alpha_1) = S/4, \tag{2.7.2}$$

and the mean given by:

$$\lim_{\substack{\alpha_1 \to +\infty \\ \alpha_1 \to -\infty}} \mu_1'(x_1|S,\alpha_1) = S/2, \tag{2.7.3}$$

The graph in Figure 2.3 displays the variance function for the Binomial distribution, for the BBD at $\alpha_1 = 2$ and for the Hypergeometric distribution at $\alpha_1 = -2$. This graph represents a tool for deciding which distribution can be fitted on the ascending diagonal arrays. On the graph one should plot the variances of the ascending diagonal arrays. If they fall above the S/4 line we have the Beta Binomial case. If the graph falls below the S/4 line the Hypergeometric distribution should be fitted.

Figure 2.3 The indicator function



C H A P T E R 3

THE TRIVARIATE MODEL

Further to the development of the bivariate case, we now extend our work to the trivariate case.

3.1 The General Model

Lemma 3.1

Let (x_1,x_2,x_3) be a three dimensional random variable and let $g(.)$ be a function of x_1. Then the expected value of $g(x_1)$ is

$$E[g(x_1)] = E[E[E[g(x_1)|x_2,x_3]]] \qquad (3.1.1)$$

Let us assume three variables x_1, x_2 and x_3 such that $x_1 + x_2 + x_3 = n$, and n is a random variable which is the total of variables. In case that the model is applied to multibrand buying field, the variables x_1, x_2 and x_3 represent the number of purchase occasions of the individual brands. The totality of the individual brands is the whole product field.

Let us denote the variable

$$S = x_1 + x_2 \qquad (3.1.2)$$

which is known as the third marginal of the bivariate table of x_1 and x_2. Therefore the ranges of variability of x_1, x_2 and S are:

$x_1 = 0,1,2,\ldots,n$,

$x_2 = 0,1,2,\ldots,n$,

$S = 0,1,2,\ldots,n$ and $S \leq n$. (3.1.3)

Let us denote the probability law for the ascending diagonal arrays of the bivariate table of x_1 and x_2 to be $\delta(x_1|S)$ and the probability law for the third marginal of the same bivariate table to be $\lambda(S|n)$. Let us denote the probability law for the total of variables n as f(n). The joint distribution of x_1, S and n is therefore given by

$$p(x_1,S,n) = \delta(x_1|S)\ \lambda(S|n)\ f(n) \tag{3.1.4}$$

and the conditional distribution of x_1 given S and n is

$$p(x_1|S,n) = \delta(x_1|S) \tag{3.1.5}$$

The following theorem is developed based on the above assumptions.

<u>Theorem 3.1</u>

For any distribution f(n) for the total of variables with finite moments, for any distribution $\lambda(S|n)$ for the third marginal with finite moments around the origin and for any distribution $\delta(x_1|S)$ for the ascending diagonal arrays of the bivariate table of the variables x_1 and x_2 with finite $\mu_1'(x_1|S)$ and $\mu_2'(x_1|S)$, the moments $\mu_1'(x_1)$, $\mu_1'(x_2)$, $\mu_2(x_1)$, $\mu_2(x_2)$ and the $\text{cov}(x_1,x_2)$ can be expressed as follows:

$$\mu_1'(x_1) = \sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\ , \tag{3.1.6}$$

$$\mu_1'(x_2) = \sum_n \mu_1'(S|n)\ f(n) - \sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\ , \tag{3.1.7}$$

$$\mu_2(x_1) = \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n) - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right]^2 , \tag{3.1.8}$$

$$\begin{aligned}\mu_2(x_2) = {} & \sum_n \mu_2'(S|n)\ f(n) - \left[\sum_n \mu_1'(S|n)\ f(n)\right]^2 \\ & - 2\left[\sum_n [\sum_S S\ \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n) - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right]\left[\sum_n \mu_1'(S|n)\ f(n)\right]\right] + \\ & + \sum_n [\sum_S \mu_2'(x_1|S)\lambda(S|n)]\ f(n) - \left[\sum_n [\sum_S \mu_1'(x_1|S)\lambda(S|n)]\ f(n)\right]^2\end{aligned}$$

and (3.1.9)

$$\begin{aligned}\operatorname{cov}(x_1,x_2) = {} & \sum_n [\sum_S S\ \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n) \\ & - \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n) \\ & - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right] \\ & \cdot \left[\sum_n \mu_1'(S|n)\ f(n) - \sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right]\end{aligned} \tag{3.1.10}$$

Proof

(a) Using Lemma 3.1 we can write

$$E[E[E(x_1|S,n)]] = E(x_1) \tag{3.1.11}$$

Therefore from (3.1.11) and using (3.1.4) we obtain the following relation:

$$\sum_n [\sum_S [\sum_{x_1} x_1 \delta(x_1|S)]\ \lambda(S|n)]\ f(n) = \mu_1'(x_1)\ . \tag{3.1.12}$$

In view of the fact that

$$\sum_{x_1} x_1\ \delta(x_1|S) = \mu_1'(x_1|S)$$

equation (3.1.12) becomes

$$\mu_1'(x_1) = \sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\ .$$

(b) Using the same Lemma 3.1 and the constraint (3.1.2) we can write for $\mu_1'(x_2)$ the following relation:

$$E[E[E(S-x_1|S,n)]] = E(x_2)\ . \tag{3.1.13}$$

By definition, the expected value of x_2 is:

$$\mu_1'(x_2) = \sum_n \sum_S \sum_{x_1} (S-x_1)\ p(x_1,S,n)\ . \tag{3.1.14}$$

By substituting (3.1.4) into (3.1.14) we obtain:

$$\mu_1'(x_2) = \sum_n \mu_1'(S|n)\ f(n) - \sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\ .$$

The expected value of x_2, $\mu_1'(x_2)$ can be expressed in terms of $\mu_1'(x_1)$ as well:

$$\mu_1'(x_2) = \sum_n \mu_1'(S|n)\ f(n) - \mu_1'(x_1)\ .$$

(c) Using Lemma 3.1 we can write the second moment around the origin for the marginal x_1 as follows:

$$\mu_2'(x_1) = \sum_n[\sum_S[\sum_{x_1} x_1^2\ p(x_1,S,n)]] = \sum_n [\sum_S [\sum_{x_1} x_1^2\ \delta(x_1|S)]\ \lambda(S|n)]\ f(n)$$

$$= \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n)\ .$$

From the definition of the second moment around the mean we can write $\mu_2(x_1)$ as

$$\mu_2(x_1) = \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n) - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right]^2 .$$

(d) In a similar way we can calculate the second moment around the mean for the variable x_2.

$$\mu_2'(x_2) = \sum_n [\sum_S [\sum_{x_1} (S-x_1)^2\ p(x_1,S,n)]]$$

$$= \sum_n [\sum_S [\sum_{x_1} (S-x_1)^2\ \delta(x_1|S)]\ \lambda(S|n)]\ f(n)$$

$$= \sum_n \mu_2'(S|n)\ f(n) - 2\sum_n [\sum_S S\ \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)$$

$$+ \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n) .$$

From the definition of the second moment around the mean we can write $\mu_2(x_2)$ as:

$$\mu_2(x_2) = \sum_n \mu_2'(S|n)\ f(n) - \left[\sum_n \mu_1'(S|n)\ f(n)\right]^2$$

$$+ \sum_n [\sum_S \mu_2'(x_1|S)\ \lambda(S|n)]\ f(n) - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)] f(n)\right]^2$$

$$- 2\left[\sum_n [\sum_S S\ \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right.$$

$$\left. - \left[\sum_n [\sum_S \mu_1'(x_1|S)\ \lambda(S|n)]\ f(n)\right]\left[\sum_n \mu_1'(S|n)\ f(n)\right]\right] .$$

(e) In order to calculate the covariance between x_1 and x_2 we must first calculate $E(x_1x_2)$. Using Lemma 3.1 we obtain:

$$E(x_1x_2) = \sum_n \sum_S \sum_{x_1} x_1(S-x_1)\ p(x_1,S,n)$$

$$= \sum_n [\sum_S [\sum_{x_1} x_1(S-x_1)\ \delta(x_1|S)]\ \lambda(S|n)]\ f(n)$$

$$= \sum_n [\sum_S S\mu_1'(x_1|S)\lambda(S|n)]\ f(n) - \sum_n [\sum_S \mu_2'(x_1|S)\lambda(S|n)]\ f(n) .$$

From the covariance definition it is obvious that