

Non-Fermi Liquid Fixed Point in a Wilsonian Theory of Quantum Critical Metals

Master of Science (Dissertation)

Teflon Phumudzo Rabambi
(457990@students.wits.ac.za)
University of The Witwatersrand

Supervised by: Professor Robert de Mello Koch
University of The Witwatersrand

February 2015

Submitted to the Faculty of Science Wits University



Declaration

I declare that the work contained in this thesis is my work, any work done previously by others or by myself has been acknowledged and put to references. This is being submitted for the degree of Masters of Science in the University of the Witwatersrand, Johannesburg and it has not been submitted before in any other tertiary institution.

Student name: Phumudzo Teflon Rabambi

A handwritten signature in black ink, appearing to read 'Phumudzo', with a long, sweeping horizontal line extending to the right.

Student signature

Date: 19 February 2015

Acknowledgements

I would like to thank NRF for the funding they provided me with in order to do my Masters, and I would like to thank everyone close to me including my Supervisor for helping me through my Masters.

Abstract

Recently there has been significant interest in new types of metals called non-Fermi liquids, which cannot be described by Landau Fermi liquid theory. Landau Fermi liquid theory is a theoretical model used to describe low energy interacting fermions or quasiparticles. There is a growing interest in constructing an effective field theory for these types of metals. One of the paradigms to understand these metals is by the use of Wilsonian renormalization group (RG) to study a theoretical toy model consisting of fermions coupled to a gapless order parameter field. Here we will study fermions coupled to gapless bosons (order parameter) below the upper critical dimension ($d = 3$). We will treat both fermions and bosons on equal footing and construct an effective field theory which only integrates out high momentum modes. Then we compute the one-loop RG flows for the Yukawa coupling and four-Fermi interaction. We will discuss $\log^2$ and $\log^3$ subtleties associated with the one loop RG flows for the four-Fermi interaction and how they can be circumvented.

Contents

1	Introduction	1
2	Basic Formalism for Fermion and Scalar Fields	8
2.0.1	Deriving the equation of motion for a scalar field in the presence of a source.	8
2.0.2	Construction of the Green's function	9
2.0.3	Computing the generating function for bosons	11
2.0.4	Quantum harmonic oscillator and the scalar field	15
2.0.5	Generating function for Fermions	17
2.0.6	Computing a two point function for fermions	23
2.1	Chapter Summary	23
3	Renormalization at one loop	24
3.1	Renormalization group flow	24
3.2	Scaling of parameters for finite density systems	27
3.3	Renormalization at one loop	34
3.4	Chapter Summary	47
4	Renormalization group Flows For Finite density QFT	49
4.1	Finite density without four fermi coupling	49
4.1.1	Renormalization of the Fermions	61
4.1.2	Non-Fermi liquid and Landau Damping energy regimes	75
4.2	Chapter summary	77
5	Double and Triple logs at one loop	80
5.1	Non-Fermi liquid toy model with a quartic Fermion coupling	80
5.1.1	UV-IR Mixing	81
5.1.2	Divergences and RG at tree level	82
5.1.3	Ward Identity in the vicinity of the Fermi-surface	91
5.1.4	Renormalization for the Yukawa vertex	91
5.1.5	Renormalization of the Fermion Propagator	93
5.2	Chapter Summary	94
6		96
6.1	Conclusion	96
A		98
A.1	Deriving Feynman Rules	98
A.1.1	Zero Dimension Toy Model	98
A.1.2	Fermion Lagrangian Feynman rules	104
	References	116

List of Figures

1.1	Nodes of a particle inside a box.	2
1.2	One-to-one mapping of states from non-interacting to interacting system.	3
2.1	The sign of $\tau - \tau'$ determines how contours are closed.	10
3.1	Two dimensional slice through the energy-momentum shells of modes.	26
3.2	Scaling of bosons and fermions.	28
3.3	How energy is measured for Fermions	28
3.4	Interaction with momentum transfer perpendicular to the Fermi-surface	31
3.5	Interaction with generic momentum transfer.	32
3.6	Particle-hole interaction close to the Fermi-surface	33
3.7	Slices of frequency-momentum modes in two dimensions.	34
4.1	Feynman rules in momentum space.	52
5.1	One loop Feynman diagrams for 4-Fermi interaction.	85
A.1	Vacuum Feynman diagrams with interaction term.	101
A.2	Two point function Feynman diagrams with interaction term.	102
A.3	Four point function Feynman diagrams.	103
A.4	Feynman Rules	104
A.5	Feynman rules for the complex scalar.	107
A.6	Diagrams for zero, two point and four point function.	108
A.7	Zero point Feynman diagrams with interaction term.	108
A.8	Two point function with interaction term.	109
A.9	Four point function with interaction term.	110
A.10	Zero, two and four point function Feynman diagrams for specific combinations of fields.	111
A.11	Zero point function diagrams with interaction.	111
A.12	Two point Feynman diagrams.	112
A.13	Four point function Feynman diagrams.	112
A.14	Feynman rules for complex scalar field.	113
A.15	Yukawa coupling diagram.	114

Chapter 1

Introduction

Our modern day understanding of how metals behave electronically is based on Landau Fermi liquid theory, which is a standard and well accepted theoretical model to describe the physics of low energy interacting fermions especially in metals. However recently there has been an emergence of new metals, called non-Fermi liquids, which cannot be explained by Landau Fermi liquid theory. A key challenge remains to understand and construct effective field theories for these non-Fermi liquids. A fruitful and promising approach to understand non-Fermi liquids is to use the renormalization group (RG) whereby we study the fixed points of the non-Fermi liquids in limits in which the physics is under analytic control and then we try to extend the analysis beyond the range of our understanding. The problem with many modern condensed matter problems is they are a mixture of strong interaction and finite density, which makes their study challenging. The finite density complication in particular, is not usually considered in relativistic QFT. As has been appreciated for some time, non-relativistic theories contain less symmetry than relativistic theories, which makes the study of non-Fermi liquids more challenging. Example of physical systems which fail to be described by Landau Fermi liquid theory are:

- Metals close to a quantum critical point, when a phase transition takes place at absolute zero temperature. The quasiparticles scatter strongly in a way that they no longer conform to the Landau Fermi liquid description [1]
- Luttinger liquids (Metals in one dimension). In one dimensional metals, electrons are unstable and they decay into two particles, a spinon which carries a spin and holon which carries a charge [1].
- Two-channel Kondo models. When two independent electrons scatter from a magnetic impurity they leave behind half an electron [1].
- Disordered Kondo models. In this case the scattering of two electrons from the magnetic impurities is too strong to allow the Fermi quasi-particles to form [1].

One simple reason why these physical systems cannot be described by Landau Fermi liquid theory is because they don't possess well defined quasiparticles. Before we go any further we need to have a brief description of what the Landau theory entails.

Landau-Fermi liquid

One could ask why the Landau theory is such a good description for metals? Well, the application of quantum mechanics to this problem resulted in two key problems. The first problem was that each electron should contribute $\frac{3}{2}k_B$ to the specific heat capacity of a metal, which is far more than what is measured experimentally. The second problem was that the electron has a magnetic moment, but the magnetic susceptibility does not show the Curie temperature dependence for free moments ($\chi = \frac{1}{T}$) [1]. These failures led to the development of Landau Fermi liquid theory,

which is able to address these two problems, by adopting the Fermi statistics for the electron and enforcing the Pauli-exclusion principle which states that no two electrons can occupy the same state. We will begin by elaborating some properties of Fermi-liquid theory and we will indicate where it breaks down. We will start by describing a single non-interacting system called a Fermi-gas, which could be a gas of electrons inside a container or box. We obtain the many particle states by filling the single particle energy eigenstates always respecting the Pauli-exclusion principle rule. The ground states of the system is obtained by filling all states inside a sphere in momentum space with the radius k_F . The system will contain a surface with a locus of points $|\vec{k}| = k_F$ called a Fermi-surface. Low energy excitations are formed by filling a state close to the Fermi-surface. The state can be occupied by a hole (positive charge) or an electron (negative charge), and these excitations are gapless [2]. These excitations, at a quantum theoretical level, manifest themselves as poles of the Green's function in the complex frequency plane [2]. An electron's Green's function in momentum space has the form

$$G(w, \vec{k}) = \frac{1}{iw - \mu + \epsilon(\vec{k})}, \quad (1.1)$$

where $\epsilon(\vec{k})$ is the energy of the electron and μ is the Fermi-surface energy. Once we switch on interactions the notion of a single particle state doesn't make sense, but the general idea of the non-interacting gas should still apply for weakly interacting systems. It is not clear what happens at strong interaction. Landau Fermi liquid theory postulates that the above qualitative picture of a Fermi-gas should persist for a generic fermionic system by appealing to adiabatic continuity [3]. Landau theory assumes that the ground states of an interacting fermionic system is characterized by a surface in momentum space at $k = k_F$ and regardless of strong interaction, the low energy excitations near the Fermi-surface behave as weakly interacting particles [2]. In elementary quantum mechanics we know that the eigenstate of a particle inside a box consists of standing sine waves with nodes at the well walls. The eigenstate of the particle in the box system can be labelled by the number of additional nodes in the wave-function, with the energy increasing with the node number as indicated in the figure below.

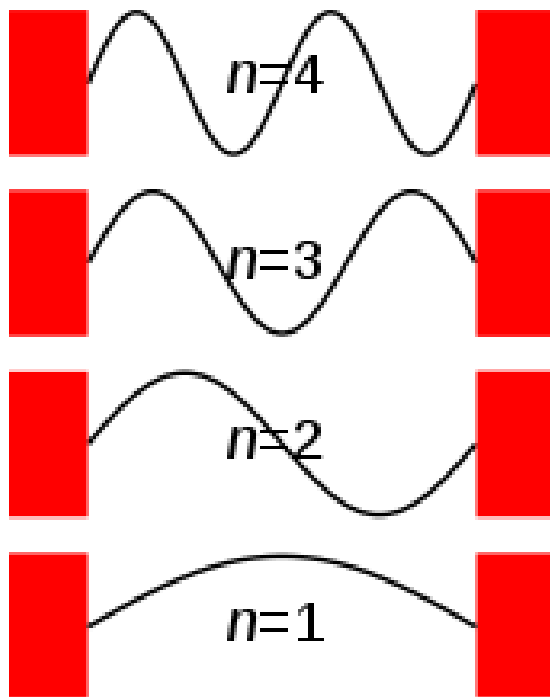


Figure 1.1: Nodes of a particle inside a box.

If we switch on a weak quadratic potential, the new eigenstates are no longer simple sine waves as in the figure above. They now involve a mixing of all the eigenstates of the original unperturbed problem, but the number of nodes still remain as good labels of the eigenstates for the interacting system [1]. This is where the importance of adiabatic continuity comes into play. The idea behind Landau Fermi liquid theory is that we imagine taking a Fermi gas and turning on the interactions between electrons slowly and observing, how the eigenstates of the Fermi gas evolve [4, 5, 6]. Landau Fermi liquid theory postulates that there will be a one-to-one mapping of the low energy eigenstates of interacting electrons with those of non-interacting Fermi gas as indicated in the figure below.

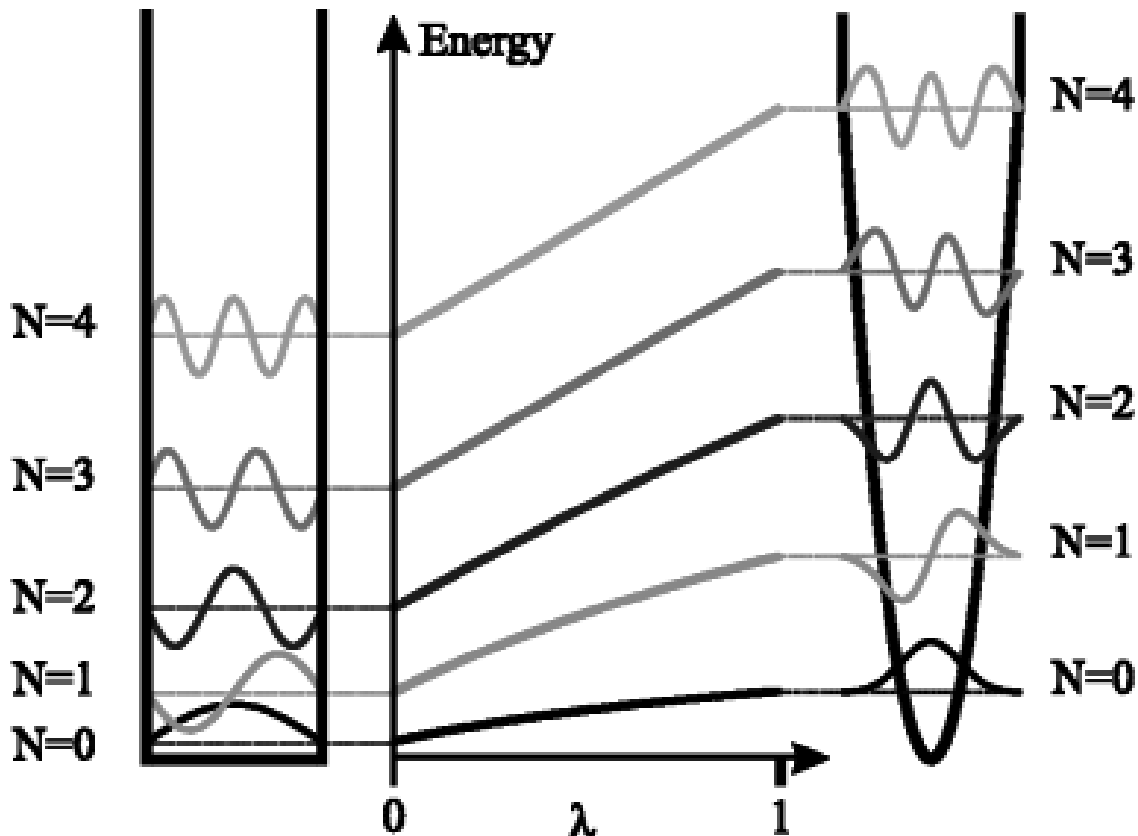


Figure 1.2: One-to-one mapping of states from non-interacting to interacting system.

We therefore retain the picture of the Fermi gas system with the Fermi surface and particle-hole excitations carrying the same quantum numbers as the non-interacting system. An important difference is that now these quantum labels are not associated with the electrons, but with quasiparticles. It is the concept of quasiparticles that is the central point of Landau Fermi liquid theory. With this we are able to address the two key problems we encountered when we applied quantum mechanics to metals, since only a small fraction of the total number of quasiparticles can take part in the process of contributing to the specific heat and magnetic susceptibility of the metal. The majority of the quasiparticles lie so far away from the Fermi surface that they are energetically unable to find an unoccupied quantum state required to magnetize them or carry excess heat. Only quasiparticles within $k_B T$ of the Fermi surface can contribute to the specific heat, so the specific heat grows linearly with temperature [7]. Only quasiparticles within $\mu_B B$ of the Fermi surface can magnetize with a moment $\sim \mu_B$ leading to a temperature dependent Pauli susceptibility instead of a Curie temperature [1]. Both thermodynamic quantities are dependent on the density of states at the Fermi surface.

Retaining the quantum labels of the non-interacting system means that the configurational entropy of the system hasn't changed for the interacting system, since the distribution function of the system remains unchanged. This means that each particle contributes additively to the total entropy of the system but the same cannot be said about the total energy of a system. The total energy is not additive. To figure out the total energy of a system, we need to first consider that when a quasiparticle moves, there will be a backflow in the filled Fermi sea as the quasiparticle pushes the ground state out. This changes the inertial mass of the quasiparticle from $m \rightarrow m^*$, where m^* is the effective mass which incorporates the effect of the electron interacting with the crystal lattice. The second thing to consider is that the quasiparticle energy also depends on the distribution of other quasiparticles, with the distribution described via the f function. The total energy of the interacting system is [1]

$$E = \sum_{k,\sigma} \frac{k_F}{m^*} (\hbar k - k_F) \delta_{n\sigma} + \frac{1}{2} \sum_{k,k',\sigma,\sigma'} f_{k\sigma k'\sigma'} \delta n_{k\sigma} \delta n_{k'\sigma'}, \quad (1.2)$$

for an isotropic system. From here we can compute thermodynamic equilibrium properties such as the specific heat and Pauli susceptibility. We can estimate the lifetime of these approximate eigenstates by considering the decay rate of quasiparticles with energy ϵ just above the Fermi surface, at absolute zero temperature using Fermi's golden rule[8],

$$\frac{1}{\tau_\epsilon} = \frac{2\pi}{\hbar} \sum_f |V_{if}|^2 \delta(\epsilon - \epsilon_F) \quad (1.3)$$

where τ_ϵ is the life-time, $|V_{if}|$ are the interaction matrix elements and the sum is over the final states. At absolute zero the scattering allowed by the Pauli principle only lowers the quasiparticle energy by w , making a particle-hole pair in the Fermi sea. Only occupied states within w of the Fermi surface can absorb this energy by making a particle state above the Fermi surface. Therefore our sum over final states become [1]

$$\frac{1}{\tau_\epsilon} \sim \frac{2\pi}{\hbar} |V|^2 \int_0^\epsilon g_F dw \int_0^w d\epsilon' \int_{-\infty}^\infty \delta(\epsilon - w - \epsilon' + \epsilon'') g_F d\epsilon'', \quad (1.4)$$

where g_F is the density of states at the Fermi surface. Close to the Fermi surface where the energy of a quasiparticle ϵ is small, the quasiparticle is always defined and its decay rate ϵ^2 is much smaller than its excitation energy ϵ . Far from the Fermi surface at large ϵ the adiabatic continuity breaks, since the quasiparticle will decay before the interaction is completely turned on [1].

In a nutshell, it is the concept of a fermion quasiparticle that is at the heart of Fermi liquid theory. The theory is able to account for the temperature dependent heat capacities and Pauli susceptibility of many metals, since these thermodynamic properties only require a well defined Fermi surface and they are not sensitive to whether we are dealing with electrons or quasiparticles. The problem with non-Fermi liquids is that they don't possess quasiparticles since they decay at a very much faster rate before the system can display its particle like properties. So we need to have a different approach when studying these physical systems. A good way to start is by studying a non-Fermi liquid system with gapless quantum critical boson coupled to a Fermi surface. The coupling of a Fermi-surface to a gapless boson leads to novel phenomena such as superconducting and charge density wave formation. The core reason of why non-Fermi liquid systems behave differently from the ordinary metals is because non-Fermi liquid systems do not contain quasiparticles. In fact the quasi-particle life span is very short, so quasi-particles decay before they can display their particle like properties. This may affect the transport and

thermodynamic properties of a material.

A key goal in this study is to review how to achieve non-Fermi liquid fixed points, in the low energy theory of interacting fermions coupled to a gapless order parameter. These fixed points can be studied using perturbation theory.

Understanding the behaviour of finite density quantum field theory is a central problem in theoretical physics, and is particularly important for condensed matter physics. Some of the most challenging open issues in condensed matter physics involve understanding quantum continuous phase transitions in metals. To obtain an understanding of what a quantum phase transition is, it is useful to contrast it with a classical phase transition. A classical phase transition (also called a thermal phase transition), is basically a reorganization of the particles, due to heat gained or lost in a system. A simple example is the melting of ice, describing a phase transition from solid to liquid. This phase transition is discontinuous if the order parameter (which could be the density here) exhibits a discontinuity in the phase diagram. Contrary to classical phase transitions, quantum phase transitions can only be accessed by varying a physical parameter, such as the magnetic field or pressure at absolute zero temperature. The transition happens abruptly, showing a sudden change in the ground state of a many body system and it is driven by quantum fluctuations. This kind of quantum phase transition can be a second-order phase transition for which there is no discontinuity in the order parameter. The defining difference between a classical phase transition and a quantum phase transition is that classical phase transitions are driven by thermal fluctuations while quantum phase transitions are driven by quantum fluctuations. Phase transitions often involve a symmetry breaking process, for example, cooling of a liquid/fluid into a crystalline solid breaks continuous translation symmetry.

There are two main categories of phase transitions, which are a first-order transition and a second-order transition. In the first-order phase transition, the state of a system changes discontinuously, as in melting of a solid, whereas in the second-order phase transition, the state changes continuously. First order phase transitions are basically classical phase transitions. Second-order phase transitions are marked by the growth of fluctuations on ever-longer length-scales. The second-order phase transition happens at the critical point where the critical fluctuations are scale invariant and extend over the entire physical system. At the quantum critical point, the fluctuations are quantum mechanical in nature and the characteristic energy of the quantum fluctuations are bigger than the characteristic Boltzmann thermal energy $k_B T$. The fluctuations at a critical point exhibit scale invariance in both space and in time, unlike classical critical points, where the critical fluctuations are limited to a narrow region around the phase transition. Quantum critical points arise when the susceptibility of a material diverges at zero temperature.

Quantum field theories of metals are less understood as compared to relativistic field theories. This is due to the reduced amount of symmetry and extensive gapless modes that need to be kept in low energy theories. A standard paradigmatic example of finite density field theory in a metal is given by a Fermi surface interacting with a gapless boson. This system underlines a wide range of systems in high energy and condensed matter physics. Non-Fermi liquid behaviour can arise when the fermion Fermi surface is coupled to a gapless boson field or order parameter field. In condensed matter an order parameter is a measure of the degree of order across the boundaries of a phase transition. A simple example of an order parameter is the magnetization in a ferromagnetic system undergoing a phase transition. A boson can be made gapless by fine tuning the microscopic parameters of the non-Fermi liquid phases within an extended region in the parameter space. In Fermi liquid metals [5], quasi particles provide a single-particle basis in which the low energy

field theories can be diagonalized [9][10]. In contrast to this, in non-Fermi liquid states, there exists no such single-particle basis in which low energy field theories can be diagonalized, because quasi-particles have a short life span as indicated above. Since there is no such single-particle basis, the low energy physics for non-Fermi liquids has to be described by genuine interacting quantum field theories, which could be fermions coupled to gapless bosons. The interaction of the Fermi surface coupled to a gapless boson could also lead to a parametric enhancement of superconductivity and formation of other new phases. While there has been promising progress over the last decade, in our understanding of non-Fermi liquids and finite density QFT in general, there is still no general definite understanding of long distance (low energy) finite density QFT. In particular the possible phases that can occur and the physics at low energy is not well understood.

The focus of this work will be on an important class of quantum phase transitions in metals known as Pomeranchuk instabilities, where the order parameter preserves the lattice translation symmetry. Examples of this phase transition include ferromagnetism and nematic order. In the case of Pomeranchuk continuous phase transitions, the order parameter field condenses to zero momentum and its fluctuations couple to the fermions everywhere on the Fermi surface. There is now substantial growing experimental evidence suggesting that such these transitions occur in highly correlated metals. Pomeranchuk instabilities can be developed by deforming the Fermi surface and studying the resulting energy gained, particularly the energy of quasi particles. The energy of an excited state quasi particle as a functional of the change in the equilibrium distribution function can be written as [11]

$$E = \int d^2k (\epsilon(k) - \mu) \delta n(k) + \frac{1}{2} \int d^2k \int d^2k' f(k, k') \delta n(k) \delta n(k'),$$

where $\delta n(k)$ is the change in the distribution function $n(k)$, and we have assumed that only two-particle interactions are present. The interaction function $f(k', k)$ can be related to the low energy limit of the two particle vertex. Low energy excited states with negative energy E are allowed. When the energy E is negative, this signals an instability and a breakdown of Landau Fermi liquid.

The standard paradigm for studying zero-temperature phase transitions in metals was introduced by Hertz [12] and Mills [13]. Their approach involved integrating out the fermionic modes, including those that lie on the Fermi surface. This left only the bosonic order parameter in the action, and the dynamics of the order parameter was analysed using the renormalization group. This kind of approach is dangerous when applied in metals, since integrating out gapless fermionic modes on the Fermi surface leads to non-local and singular interaction among the bosons. A much more systematic and tractable approach is to treat both the fermionic and bosonic modes on equal footing. In this approach high energy excitation modes are integrated out, for both fermions and bosons, and the low effective action comprises of just low energy modes. However, due to the difference in kinematical structure between the fermions and bosons (the low energy modes for the fermions are close to the Fermi surface and for the boson are close to the origin), constructing an effective field theory is challenging.

There has been a general consensus among field theorists that the anomalous power laws present in non-Fermi liquids cannot be fully understood without a controlled theory of the deep infrared behaviour of the fermion-boson interaction [14][15]. However, at finite temperature or frequency scales, a metal in the vicinity of a quantum critical point exhibits crossovers, not transitions from Fermi liquid to non-Fermi liquid. There is a chance, that starting from a weakly coupled metal, one can understand the intermediate asymptotic behaviour by either treating the coupling between the boson's and fermions perturbatively or by using large N limits. But theories of this sort can breakdown below a certain energy scale, reflecting the non-analyticities associated either with

the quantum critical point itself, or the instabilities (superconductivity or Landau damping) that are hidden in quantum effects. However this kind of instability occurs at arbitrary low energy. At high energy the theory is devoid of these non-analyticities (instabilities).

In this work, we will study the coupled field theory of a massless scalar field and a finite density of fermions. The interaction between the fermions and the bosons will be mediated by a Yukawa interaction $L \subset g\phi\bar{\psi}\psi$. We perform a perturbation study in a controlled weak coupling expansion around the critical space-time dimension $d = 4 - \epsilon$. The objective is to perform a systematic analysis of the quantum effects in the theory, and determine the renormalization group equations for the theory. With the renormalization group equations, we will determine the possible low energy phases. There are several phenomenological and theoretical motivations behind this strategic approach. One of the motivations is that there is growing experimental evidence that high- T_c superconductivity (SC) and non-Fermi liquid (NFL) behaviour happen at the vicinity of quantum phase transitions, where the interaction between bosonic excitations and finite density fermions becomes important. The second motivation is to understand the strong dynamics present in many condensed matter systems of interest. Other non-perturbative techniques have been used to study the theory, including large N expansions, field theory dualities and some promising work exploiting Holography. Regardless of these promising non-perturbative techniques, renormalization group flow remains as the most straight forward and profound way to study finite density QFT. We will work near the critical point in $(3 + 1)$ dimensions, where the theory is under perturbative control. This kind of approach has been successful in other areas of critical phenomena [16] and, we expect this approach to be useful in helping to understand low energy strongly coupled system. The work produced in this Thesis is a thorough review of the work of A.L. Fitzpatrick and S. Kachru, reported in [17, 18, 19, 20].

The dissertation is organised as follows: In the second chapter we will describe the basics necessary for one to study relativistic or non-relativistic QFT. The basics include computing the fermion and boson propagators, understanding Grassmann algebra which is the algebra used to describe fermions and the Feynman rules for theory. The third chapter will describe the idea of Renormalization Group flows, which is one of the central ideas in studying finite density non-relativistic fermions. From there we will study some flows using Feynman diagrams. In the fourth chapter we implement a proper renormalization strategy to renormalize the theory and then we compute the beta functions. In the last chapter we address the subtleties including the $\log^2$ and $\log^3$ divergent Feynman diagrams and we complete the renormalization of our theory. This allows us to achieve our primary goal of exhibiting fixed points that correspond to non-Fermi liquids.

Chapter 2

Basic Formalism for Fermion and Scalar Fields

In this chapter we are going to derive the equation of motion for both the fermions and scalar fields. Then we will derive the Green functions for both fields. The objective here is to derive the generating function of the fields, in order to use them in the following chapters. We will also study a simple quantum harmonic oscillator problem in $1 + 1$ dimension and $d + 1$ dimensions to demonstrate the fact that a quantum field is an assembly of harmonic oscillators. When we discuss the theory for fermions we first discuss Grassmann number algebra, since fermions are described by this algebra. We will begin by describing the theory of scalar fields.

2.0.1 Deriving the equation of motion for a scalar field in the presence of a source.

Consider the scalar Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2}(m^2\phi^2 + (\partial_\tau\phi)^2 + c^2\vec{\nabla}\phi \cdot \vec{\nabla}\phi), \quad (2.1)$$

with the action

$$S_\phi = \frac{1}{2} \int d\tau \int d^d x (m^2\phi^2 + (\partial_\tau\phi)^2 + c^2\vec{\nabla}\phi \cdot \vec{\nabla}\phi). \quad (2.2)$$

We add a source term $j\phi$ to the Lagrangian. To get the equation of motion with a source term $j\phi$, we have to evaluate

$$\frac{\delta}{\delta\phi(\tau', \vec{x}')} \left(-S_\phi + \int d\tau \int d^d x j(\tau, \vec{x})\phi(\tau, \vec{x}) \right) = 0. \quad (2.3)$$

$$(2.4)$$

This equation can be simplified as

$$0 = - \frac{\delta}{\delta\phi(\tau', \vec{x}')} \left[\int d\tau \int d^d x \left(\frac{1}{2}(m^2\phi^2 + (\partial_\tau\phi)^2 + c^2\vec{\nabla}\phi \cdot \vec{\nabla}\phi) - j\phi \right) \right] \quad (2.5)$$

$$\begin{aligned} &= - \int d\tau \int d^d x (m^2\phi\delta(\tau - \tau', \vec{x} - \vec{x}') + \partial_\tau\delta(\tau - \tau', \vec{x} - \vec{x}')\partial_\tau\phi \\ &\quad + c^2\vec{\nabla}\delta(\tau - \tau', \vec{x} - \vec{x}') \cdot \vec{\nabla}\phi - j\delta(\tau - \tau', \vec{x} - \vec{x}')). \end{aligned} \quad (2.6)$$

Use integration by parts on the second and third term of the integrand to get,

$$\begin{aligned}
0 = & - \int d\tau \int d^d x \left(m^2 \phi \delta(\tau - \tau', \vec{x} - \vec{x}') + \partial_\tau (\delta(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau \phi) \right. \\
& - \delta(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau^2 \phi + c^2 \vec{\nabla} \cdot (\delta(\tau - \tau', \vec{x} - \vec{x}') \nabla \phi) \\
& \left. - c^2 \delta(\tau - \tau', \vec{x} - \vec{x}') \vec{\nabla} \cdot \vec{\nabla} \phi - j \delta(\tau - \tau', \vec{x} - \vec{x}') \right). \tag{2.7}
\end{aligned}$$

The 2nd and 4th terms are surface terms and equal to zero, since

$$\begin{aligned}
& \int d\tau \int d^d x \left(\partial_\tau \delta(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau \phi + c^2 \vec{\nabla} \delta(\tau - \tau', \vec{x} - \vec{x}') \cdot \vec{\nabla} \phi \right) \\
& = \int_{\partial V} dA \left(\delta(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau \phi + c^2 \delta(\tau - \tau', \vec{x} - \vec{x}') \hat{n} \cdot \vec{\nabla} \phi \right) \\
& \text{if } \tau', \vec{x}' \notin \partial V \quad \text{then} \quad \delta(\tau - \tau', \vec{x} - \vec{x}') = 0, \\
& \text{if } \tau', \vec{x}' \in \partial V \quad \text{then} \quad \partial_\tau \phi = \vec{\nabla} \phi = 0 \quad \text{since the field vanishes at the surface.}
\end{aligned} \tag{2.8}$$

Then we are left with

$$\begin{aligned}
0 = & \int d\tau \int d^d x \left(m^2 \phi \delta(\tau - \tau', \vec{x} - \vec{x}') \right. \\
& \left. - \delta(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau^2 \phi - c^2 \delta(\tau - \tau', \vec{x} - \vec{x}') \vec{\nabla} \cdot \vec{\nabla} \phi - j \delta(\tau - \tau', \vec{x} - \vec{x}') \right), \tag{2.9}
\end{aligned}$$

which simplifies to

$$(-\partial_\tau^2 - c^2 \vec{\nabla} \cdot \vec{\nabla} + m^2) \phi = j. \tag{2.10}$$

2.0.2 Construction of the Green's function

The equation

$$(-\partial_\tau^2 - \vec{\nabla} \cdot \vec{\nabla} + m^2) \phi = j, \tag{2.11}$$

can be solved using the Green's function

$$(-\partial_\tau^2 - \vec{\nabla} \cdot \vec{\nabla} + m^2) G(\tau, \vec{x}) = \delta(\tau) \delta^{(d)}(x), \tag{2.12}$$

where ϕ has a solution of the form

$$\phi(\tau, \vec{x}) = \int d\tau' \int d^d x' G(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}'). \tag{2.13}$$

Recall the identities :

$$\delta^{(d)}(x) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} \tag{2.14}$$

$$\delta(\tau) = \int \frac{dw}{2\pi} e^{iw\tau} \tag{2.15}$$

and

$$G(\tau, \vec{x}) = \int \frac{d^d k}{(2\pi)^d} \frac{dw}{2\pi} e^{i\vec{k} \cdot \vec{x}} e^{iw\tau} G(w, \vec{k}), \tag{2.16}$$

where w is the time component of $\vec{k}$. If we plug equations (2.14), (2.15) and (2.16) back into equation (2.12) we get

$$(-\partial_\tau^2 - \vec{\nabla} \cdot \vec{\nabla} + m^2) \int \frac{d^d k}{(2\pi)^d} \int \frac{dw}{(2\pi)} e^{iw(\tau-\tau') + i\vec{k} \cdot (\vec{x}-\vec{x}')} G(w, \vec{k}) = \int \frac{d^d k}{(2\pi)^d} \int \frac{dw}{2\pi} e^{iw(\tau-\tau') + i\vec{k} \cdot (\vec{x}-\vec{x}')} \quad (2.17)$$

$$\int \frac{d^{d+1} k}{(2\pi)^{d+1}} e^{ik \cdot (x-x')} ((w)^2 + \vec{k}^2 + m^2) G(w, \vec{k}) = \int \frac{d^{d+1} k}{(2\pi)^{d+1}} e^{ik \cdot (x-x')}.$$

Note that $\frac{d^{d+1} k}{(2\pi)^{d+1}} = \frac{d^d k}{(2\pi)^d} \frac{dw}{2\pi}$ and $k \cdot (x - x') = w(\tau - \tau') + \vec{k} \cdot (\vec{x} - \vec{x}')$. For the LHS to be equal to RHS, $G(w, \vec{k})$ must be equal to

$$G(w, \vec{k}) = \frac{1}{(w)^2 + \vec{k} \cdot \vec{k} + m^2} = \frac{1}{k^2 + m^2} \quad (2.18)$$

where $k^2 = w^2 + \vec{k} \cdot \vec{k}$. Hence

$$G(\tau - \tau', \vec{x} - \vec{x}') = \int \frac{d^{d+1} k}{(2\pi)^{d+1}} \frac{1}{w^2 + \vec{k} \cdot \vec{k} + m^2} e^{ik \cdot (x-x')}. \quad (2.19)$$

This function has poles at $w = \pm i\sqrt{\vec{k} \cdot \vec{k} + m^2} = \pm iw_k$

$$G(\tau - \tau', \vec{x} - \vec{x}') = \int \frac{dw d^d \vec{k}}{(2\pi)^{d+1}} \frac{1}{2w} \left(\frac{1}{w - iw_k} + \frac{1}{w + iw_k} \right) e^{iw(\tau-\tau') + i\vec{k} \cdot (\vec{x}-\vec{x}')} \quad (2.20)$$

We can evaluate the dw integral using contour integration and the residue theorem. First we will write the time component w in the exponential in terms of real and imaginary parts

$$G(\tau - \tau', \vec{x} - \vec{x}') = \int \frac{dw d^d \vec{k}}{(2\pi)^{d+1}} \frac{1}{2w} \left(\frac{1}{w - iw_k} + \frac{1}{w + iw_k} \right) e^{i(w_R + iw_I)(\tau-\tau') + i\vec{k} \cdot (\vec{x}-\vec{x}')} \quad (2.21)$$

where w_R and w_I are the real and imaginary components of the frequency. How the contour is closed is determined by the sign of $\tau - \tau'$.

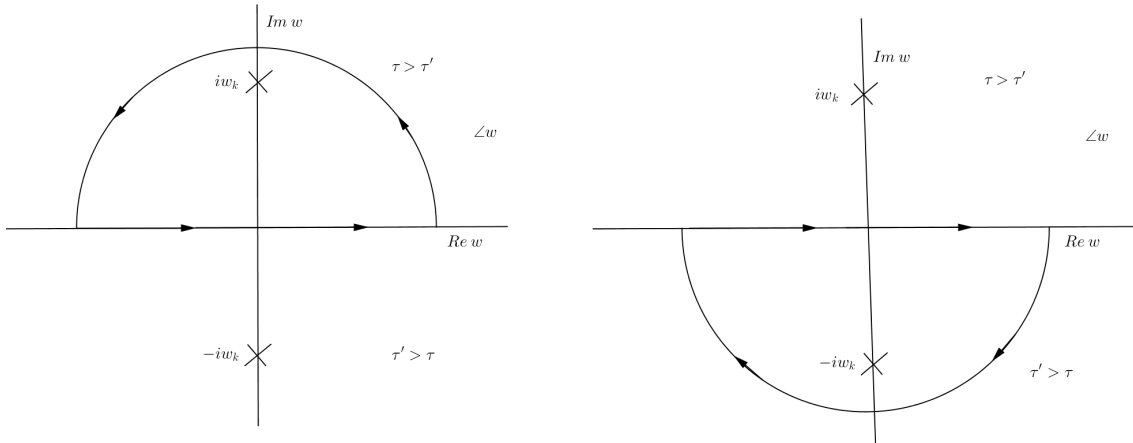


Figure 2.1: The sign of $\tau - \tau'$ determines how contours are closed.

When $\tau > \tau'$, to get a decaying exponential when $|w_k| \rightarrow \infty$ we close the contour anti-clockwise in the upper plane as shown in the first diagram of the figure above. In this case the

portion of the contour at infinity does not contribute to the integral. Closing the contour in the upper plane we pick up the pole iw_k and the Green's function becomes

$$G(\tau - \tau', \vec{x} - \vec{x}') = 2\pi i \int \frac{d^d \vec{k}}{(2\pi)^{d+1} 2iw_k} e^{-w_k(\tau - \tau') + i\vec{k} \cdot (\vec{x} - \vec{x}')}.$$

When $\tau < \tau'$, to get a decaying exponential we close the contour clockwise in the lower plane, to capture the pole $-iw_k$ as shown in the second diagram of the figure above, so that

$$G(\tau - \tau', \vec{x} - \vec{x}') = -2\pi i \int \frac{d^d \vec{k}}{(2\pi i)^{d+1} 2(-iw_k)} e^{-w_k(\tau' - \tau) + i\vec{k} \cdot (\vec{x} - \vec{x}')}.$$

Hence the choice of contour yields a time ordered Green's function

$$\begin{aligned} G(\tau - \tau', \vec{x} - \vec{x}') &= \theta(\tau - \tau') \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(\tau - \tau') + i\vec{k} \cdot (\vec{x} - \vec{x}')} \\ &\quad + \theta(\tau' - \tau) \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(\tau' - \tau) + i\vec{k} \cdot (\vec{x} - \vec{x}')} \end{aligned} \quad (2.22)$$

2.0.3 Computing the generating function for bosons

The generating function for a spin zero field is [21],

$$\begin{aligned} Z[j] &= \mathcal{N} \int D\phi e^{-S_\phi + \int d^{d+1}x j\phi} \\ &= \mathcal{N} \int D\phi e^{-\frac{1}{2} \int d^{d+1}x \left((\partial_\tau \phi)^2 + \vec{\nabla} \phi \vec{\nabla} \phi + m^2 \phi^2 \right) + \int d^{d+1}x j\phi} \\ &= \mathcal{N} \int D\phi e^{-\frac{1}{2} \int d^{d+1}x \int d^{d+1}x' \phi(\tau', \vec{x}') \delta(\tau - \tau') \delta^d(\vec{x} - \vec{x}') \left(-\partial_\tau^2 - \nabla^2 + m^2 \right) \phi(\tau, \vec{x}) + \int d^{d+1}x j\phi(\tau, \vec{x})} \end{aligned}$$

where $\mathcal{N}$ is a normalising constant. We will fix this constant by requiring $Z[j = 0] = 1$. To evaluate the integral, we discretise the field ϕ by transforming Euclidean space into a lattice with spacing distance a .

$$\phi(\tau, \vec{x}) \rightarrow \phi_{i\vec{j}},$$

where the indices $i, \vec{j}$ indicates the field at “time” i and position $\vec{j}$. We incorporate both indices $i, \vec{j}$ into just one index I , so that $\phi_{i\vec{j}} = \phi_I$. By discretising the space, the generating function becomes

$$Z[j]|_{j=0} = \mathcal{N} \int \prod_{M=1}^N d\phi_M e^{-\frac{1}{2} \sum_{I,J} \phi_I M_{IJ} \phi_J}, \quad (2.23)$$

where $M_{IJ} = \delta_{IJ}(-\Delta_i^2 - \Delta_j^2 + m^2)$. The symbol Δ_j^2 is the Laplacian operator for the discrete system. It acts on $\phi_{i\vec{j}}$ in d-dimension space as

$$\begin{aligned} \Delta_j^2 \phi_{i\vec{j}} &= \frac{\phi_{i,j_1+1,j_2,\dots,j_d} - 2\phi_{i,j_1,j_2,\dots,j_d} + \phi_{i,j_1-1,j_2,\dots,j_d}}{a^2} + \frac{\phi_{i,j_1,j_2+1,\dots,j_d} - 2\phi_{i,j_1,j_2,\dots,j_d} + \phi_{i,j_1,j_2-1,\dots,j_d}}{a^2} \\ &\quad + \dots + \frac{\phi_{i,j_1,j_2,\dots,j_{d+1}} - 2\phi_{i,j_1,j_2,\dots,j_d} + \phi_{i,j_1,j_2,\dots,j_{d-1}}}{a^2}. \end{aligned} \quad (2.24)$$

The symbol Δ_i^2 is the second order "time" derivative in the discrete system. Its action on $\phi_{i\vec{j}}$ is defined as

$$\Delta_i^2 \phi_{i\vec{j}} = \frac{\phi_{i+1,\vec{j}} - 2\phi_{i,\vec{j}} + \phi_{i-1,\vec{j}}}{a^2}.$$

We are using the same lattice spacing for "time" and space. Assume M_{IJ} is a symmetric matrix. We can diagonalize matrix M_{IJ} with U_{IL} an orthogonal matrix

$$U_{IL}^T \tilde{M}_{LP} U_{PJ} = M_{IJ},$$

where $\tilde{M}_{LP} = \delta_{LP} \lambda_P$ is a diagonal matrix. We make the transformation

$$U_{PJ} \phi_J = \tilde{\phi}_P.$$

The Jacobian of the transformation is

$$J = \left| \frac{d\tilde{\phi}_P}{d\phi_I} \right| = \left| \frac{d(U_{PJ} \phi_J)}{d\phi_I} \right| = |U_{PI}| = 1.$$

Therefore, the integration measure is invariant

$$\int \prod_{M=1}^N d\phi_M \rightarrow \int \prod_{K=1}^N d\tilde{\phi}_K.$$

The integral (2.23) now becomes

$$\begin{aligned} Z[j]|_{j=0} &= \mathcal{N} \int \prod_{K=1}^N d\tilde{\phi}_K e^{-\frac{1}{2} \sum_{I,L,J,P} \phi_I U_{IL}^T \tilde{M}_{LP} U_{PJ} \phi_J} \\ &= \mathcal{N} \int \prod_{K=1}^N d\tilde{\phi}_K e^{-\frac{1}{2} \sum_{L,P} \tilde{\phi}_L \tilde{M}_{LP} \tilde{\phi}_P}, \\ &= \mathcal{N} \int \prod_{K=1}^N d\tilde{\phi}_K e^{-\frac{1}{2} \sum_{L,P} \tilde{\phi}_L \delta_{LP} \lambda_P \tilde{\phi}_P} \\ &= \mathcal{N} \int \prod_{K=1}^N d\tilde{\phi}_K e^{-\frac{1}{2} \sum_P \tilde{\phi}_P^2 \lambda_P} \\ &= \mathcal{N} \prod_{K=1}^N \frac{(2\pi)^{\frac{1}{2}}}{\sqrt{\lambda_K}} \\ &= \mathcal{N} \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det M}}, \end{aligned}$$

where $(-\Delta_i^2 - \Delta_j^2 + m^2)\phi_K = \lambda_K \phi_K$. We can redefine the integration measure

$$\int \prod_{K=1}^N d\tilde{\phi}_K \rightarrow \int \prod_{K=1}^N \frac{d\tilde{\phi}_K}{(2\pi)^{N/2}},$$

so that we can eliminate the term $(2\pi)^{N/2}$, for convenience. When we redefine the integral and take $N \rightarrow \infty$ we recover the continuum description

$$i \rightarrow \tau, \quad \vec{j} \rightarrow \vec{x} \quad \text{and} \quad \phi_{i\vec{j}} \rightarrow \phi(\tau, \vec{x}).$$

After redefining the integral and taking $N \rightarrow \infty$, the generating function becomes

$$Z[j]|_{j=0} = \frac{\mathcal{N}}{\sqrt{\det M}}, \quad (2.25)$$

where $M = \delta(\tau - \tau')\delta^d(\vec{x} - \vec{x}')(-\partial_\tau^2 - \vec{\nabla}^2 + m^2)$. Recall that we normalise the generating function by requiring $Z[j]|_{j=0} = 1$

$$\begin{aligned} Z[j]|_{j=0} &= \frac{\mathcal{N}}{\sqrt{\det M}} = \langle 0|0 \rangle = 1 \\ &\Rightarrow \mathcal{N} = \sqrt{\det M}. \end{aligned} \quad (2.26)$$

Therefore

$$Z[j] = e^{\int d^{d+1}x \int d^{d+1}x' j(\tau, \vec{x}) G(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')}.$$

The term $\det M$ is related to the vacuum energy of the field. To see this, recall that

$$\begin{aligned} Z[j] &= \mathcal{N} \int D\phi e^{-S_\phi + \int d^{d+1}x j\phi} \\ \Rightarrow Z[j]|_{j=0} &= \mathcal{N} \int D\phi e^{-S_\phi} \\ &= \mathcal{N} \left\langle 0 \left| T \left(e^{-\int_{-\frac{T}{2}}^{\frac{T}{2}} H dt} \right) \right| 0 \right\rangle \\ &= \mathcal{N} e^{-E_0 T}, \end{aligned} \quad (2.27)$$

where H is the Hamiltonian of the theory. In making the above identification, we have used the well known translation between moments of the path integral and time ordered correlation functions in the operator approach to quantum field theory, correctly continued to Euclidean space [22]

$$\int D\phi e^{-S_\phi} \phi(x_1) \cdots \phi(x_n) = \mathcal{N} \left\langle 0 \left| T \left(e^{-\int_{-\frac{T}{2}}^{\frac{T}{2}} H dt} \phi(x_1) \cdots \phi(x_n) \right) \right| 0 \right\rangle. \quad (2.28)$$

From (2.25) we have

$$Z[j]|_{j=0} = \frac{\mathcal{N}}{\sqrt{\det M}},$$

which implies

$$\begin{aligned} e^{-E_0 T} &= \frac{1}{\sqrt{\det M}} \\ E_0 T &= \log(\sqrt{\det M}) \\ E_0 &= \frac{1}{T} \log(\sqrt{\det M}). \end{aligned} \quad (2.29)$$

We can further try to understand how $\det(M)$ relates to the vacuum energy of the system, by studying a simple quantum system, the Euclidean harmonic oscillator defined by the Lagrangian

$$\mathcal{L} = \frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 + \frac{1}{2} m \omega^2 x^2(t). \quad (2.30)$$

The corresponding action is

$$S = \frac{1}{2} \int dt \left(m \left(\frac{dx(t)}{dt} \right)^2 + w^2 m x^2(t) \right) \quad (2.31)$$

$$= \frac{1}{2} \int dt \left(-m \frac{x d^2 x}{dt^2} + w^2 m x^2 \right) \quad (2.32)$$

$$= \frac{m}{2} \int dt x \left(-\frac{d^2}{dt^2} + w^2 \right) x \quad (2.33)$$

$$= \frac{m}{2} \int dt x M x \quad (2.34)$$

where $M = -\frac{d^2}{dt^2} + w^2$. In line (2.32) we used integration by parts. For simplicity, we will now set $m = 2$. This system has ground state energy $E_0 = w$. The boundary conditions of the system are

$$\psi(0) = \psi(T) = 0.$$

Consider now the eigenproblem of M

$$M\psi = \lambda\psi \quad (2.35)$$

$$\left(-\frac{d^2}{dt^2} + w^2 \right) \psi = \lambda\psi \quad (2.36)$$

$$\frac{d^2\psi}{dt^2} = -(\lambda - w^2)\psi. \quad (2.37)$$

The solution for ψ is

$$\psi(x(t)) = A \cos\left(\sqrt{\lambda^2 - w^2}x(t)\right) + B \sin\left(\sqrt{\lambda^2 - w^2}x(t)\right). \quad (2.38)$$

Applying the boundary conditions

$$\psi(0) = 0 \Rightarrow A = 0 \quad \text{and}$$

$$\psi(T) = 0 \Rightarrow \sqrt{\lambda^2 - w^2}T = k\pi \quad \text{for } k=0,1,2,\dots,n$$

$$\lambda_k^2 - w^2 = \left(\frac{k\pi}{T}\right)^2$$

$$\lambda_k = \left(\frac{k\pi}{T}\right)^2 + w^2.$$

From the definition of the determinant

$$\prod_{k=1}^{\infty} \lambda_k = \det M,$$

we find

$$\begin{aligned} \det M &= \prod_{k=0}^{\infty} \left(\left(\frac{k\pi}{T}\right)^2 + w^2 \right) = \prod_{k=0}^{\infty} \left| w + \frac{\pi i k}{T} \right|^2 \\ &= \left| \prod_{k=0}^{\infty} \left(w + \frac{\pi i k}{T} \right) \right|^2 \\ &= |\sinh(wT)|^2. \end{aligned} \quad (2.39)$$

Thus,

$$\begin{aligned}\sqrt{\det(M)} &= \sinh(wT) \\ &= \frac{1}{2}(e^{wT} - e^{-wT}).\end{aligned}\tag{2.40}$$

In the limit $T \rightarrow \infty$

$$\begin{aligned}\sqrt{\det(M)} &= \frac{1}{2}e^{wT} \\ E_0 &= \frac{1}{T} \log(\sqrt{\det M}) \\ &= w,\end{aligned}$$

which is the correct result.

2.0.4 Quantum harmonic oscillator and the scalar field

The scalar field in $(d+1)$ -dimensional field theory is a system of quantum harmonic oscillators. To show the relationships between the two systems, consider the harmonic oscillator Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^2(t) - \frac{1}{2}w^2x^2(t)\tag{2.41}$$

where w is the angular frequency of the system, and consider free field scalar Lagrangian [21]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.\tag{2.42}$$

The corresponding action is

$$S = \int dt d^d x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right),\tag{2.43}$$

in $(d+1)$ -dimensional Minkowski space-time. We can Fourier transform $\phi(x, t)$ as [21]

$$\phi(\vec{x}, t) = \int \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}}.\tag{2.44}$$

Plug the Fourier transform of $\phi(x, t)$ into (2.43) to obtain

$$\begin{aligned}S &= \int dt d^d x \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d \vec{k}'}{(2\pi)^d} \left(\frac{1}{2}\dot{\varphi}(\vec{k}, t)\dot{\varphi}(\vec{k}', t) + \frac{1}{2}\vec{k} \cdot \vec{k}' \varphi(\vec{k}, t)\varphi(\vec{k}', t) \right. \\ &\quad \left. - \frac{1}{2}m^2 \varphi(\vec{k}, t)\varphi(\vec{k}', t) \right) e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}.\end{aligned}\tag{2.45}$$

Using the identity

$$\delta(\vec{k} + \vec{k}') = \int \frac{d^d x}{(2\pi)^d} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}},\tag{2.46}$$

we obtain

$$\begin{aligned}S &= \int dt \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\dot{\varphi}(\vec{k}, t)\dot{\varphi}(-\vec{k}, t) - \frac{1}{2}\vec{k} \cdot \vec{k} \varphi(\vec{k}, t)\varphi(-\vec{k}, t) - \frac{1}{2}m^2 \varphi(\vec{k}, t)\varphi(-\vec{k}, t) \right) \\ &= \int dt \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\dot{\varphi}(\vec{k}, t)\dot{\varphi}(-\vec{k}, t) - \frac{1}{2}w^2 \varphi(\vec{k}, t)\varphi(-\vec{k}, t) \right)\end{aligned}\tag{2.47}$$

where $w^2(\vec{k}) = \vec{k} \cdot \vec{k} + m^2$. Comparing to the Lagrangian we gave above for the harmonic oscillator, we see that free scalar field is an assembly of harmonic oscillators, one for each value of the wave number $\vec{k}$.

Comparison between the quantum harmonic oscillator and the scalar propagator in Minkowski space

To evaluate the propagator for a quantum harmonic oscillator, we first evaluate the Green's function

$$\left(-\frac{d^2}{dt^2} - w^2\right)G_H(t-t') = i\delta(t-t') \quad (2.48)$$

where

$$G_H(t-t') = \int_{-\infty}^{\infty} \frac{dE}{(2\pi)} G_H(E) e^{iE(t-t')}. \quad (2.49)$$

Plugging (2.49) back into (2.48) we find that

$$G_H(E) = \frac{i}{E^2 - w^2 + i\epsilon} \quad (2.50)$$

where we have taken $E^2 \rightarrow E^2 + i\epsilon$, which is the correct implementation of the boundary conditions, which determine how to treat the poles when evaluating the contour integral. Evaluating the contour integral

$$\begin{aligned} G_H(t-t') &= \int_{-\infty}^{\infty} \frac{dE}{(2\pi)} \frac{i}{E^2 - w^2 + i\epsilon} e^{iE(t-t')} \\ &= \int_{-\infty}^{\infty} \frac{dE}{(2\pi)} \frac{i}{2w} \left(\frac{1}{E+w} - \frac{1}{E-w} \right) e^{iE(t-t')} \\ &= \theta(t-t') \frac{1}{(2\pi)2E} e^{iE(t-t')} + \theta(t'-t) \frac{1}{(2\pi)2E} e^{iE(t'-t)}. \end{aligned}$$

The scalar particle propagator in Minkowski space is easily deduced from the scalar propagator evaluated in $(d+1)$ -dimension Euclidean space earlier. In Euclidean space we found the scalar propagator is

$$\begin{aligned} G(\tau - \tau', \vec{x} - \vec{x}') &= \theta(\tau - \tau') \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(\tau - \tau') + i\vec{k} \cdot (\vec{x} - \vec{x}')} \\ &\quad + \theta(\tau' - \tau) \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(\tau' - \tau) + i\vec{k} \cdot (\vec{x} - \vec{x}')}. \end{aligned} \quad (2.51)$$

In momentum space, this position space Green's function is

$$G(w, \vec{k}) = \frac{1}{w^2 + \vec{k} \cdot \vec{k} + m^2}. \quad (2.52)$$

To get the propagators in Minkowski space-time, we multiply them with i and replace the $+ve$ signs in front of $\vec{k} \cdot \vec{k}$ and m^2 with $-ve$ signs. In Minkowski space the scalar propagator is

$$G(t-t', \vec{x} - \vec{x}') = \theta(t-t') \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(t-t') + i\vec{k} \cdot (\vec{x} - \vec{x}')} + \theta(t'-t) \int \frac{d^d \vec{k}}{(2\pi)^d 2w_k} e^{-w_k(t'-t) + i\vec{k} \cdot (\vec{x} - \vec{x}')}. \quad (2.53)$$

In momentum space we have

$$G(w, \vec{k}) = \frac{i}{w^2 - \vec{k} \cdot \vec{k} - m^2 + i\epsilon}. \quad (2.54)$$

Comparing to

$$G_H(t - t') = \theta(t - t') \frac{1}{(2\pi)2E} e^{iE(t-t')} + \theta(t' - t) \frac{1}{(2\pi)2E} e^{iE(t'-t)} \quad (2.55)$$

and

$$G_H(E) = \frac{i}{E^2 - w^2 + i\epsilon}, \quad (2.56)$$

which are the propagators for the quantum harmonic oscillator, we again see that the scalar field is an assembly of harmonic oscillators.

2.0.5 Generating function for Fermions

Unlike scalar fields which are described by ordinary variables, fermions are described by Grassmann variables. The algebra of Grassmann variables is quite different from the algebra of ordinary variables. It is a good idea to state the algebra and properties of Grassmann variables before we proceed to compute the generating function for fermions.

Grassmann Variables

Grassmann variables satisfy the anti-commutation relation [21]

$$[\psi_i, \psi_j]_+ = \psi_i \psi_j + \psi_j \psi_i = 0 \quad (2.57)$$

$$\Rightarrow \psi_i \psi_j = -\psi_j \psi_i$$

$$\text{for } i = j \Rightarrow \psi_i^2 = 0 \quad \text{nilpotent}, \quad (2.58)$$

where ψ_i, ψ_j are Grassmann numbers and $i, j = 1, 2, \dots, n$. From the anti-commutation relation above we find the most general function of a single Grassman variable is

$$f(\psi) = a + b\psi, \quad (2.59)$$

where a and b could be ordinary or Grassmann numbers. This expansion is completely determined by the fact that higher order terms of ψ are nilpotent ($\psi^2 = 0$). Integration of Grassmann variables also differs from that of ordinary variables. We can derive the integration theory of Grassmann variables by requiring that the integrals respect translation invariance and linearity:

Grassmann integrals must be translational invariant under the translation $\psi \rightarrow \psi + c$

$$\begin{aligned} \int d\psi f(\psi) &= \int d\psi f(\psi + c) \\ &= \int d\psi (a + bc + b\psi) = \int d\psi (a + b\psi) + \int d\psi bc \\ &= \int d\psi f(\psi), \end{aligned} \quad (2.60)$$

where c is a Grassmann variable. For (2.60) to be satisfied

$$\begin{aligned}\int d\psi(a + b\psi) &= \int d\psi(a + b(\psi + c)) \\ &= \int d\psi(a + b\psi) + bc \int d\psi \\ &\Rightarrow \int d\psi = 0.\end{aligned}\tag{2.61}$$

Grassmann integrals should satisfy linearity [22]

$$\int d\psi(f(\psi) + g(\psi)) = \int d\psi f(\psi) + \int d\psi g(\psi).\tag{2.62}$$

Suppose $f(\psi) = a + b\psi$ and $g(\psi) = c + k\psi$, then

$$\int d\psi(a + b\psi + c + k\psi) = \int d\psi(a + b\psi) + \int d\psi(c + k\psi),$$

From (2.61) we know that $\int d\psi = 0$, therefore

$$\int d\psi(b\psi + k\psi) = \int d\psi b\psi + \int d\psi k\psi,\tag{2.63}$$

linearity is satisfied if $\int d\psi\psi = \text{constant}$. We take

$$\int d\psi\psi = 1\tag{2.64}$$

as a simple choice. Differentiation of Grassmann variables obey Leibnitz anti-commutation

$$\frac{\partial}{\partial\psi_k}(\psi_i\psi_j) = \frac{\partial\psi_i}{\partial\psi_k}\psi_j - \psi_i\frac{\partial\psi_j}{\partial\psi_k}\tag{2.65}$$

$$= \delta_{ik}\psi_j - \psi_i\delta_{jk}.\tag{2.66}$$

Note that integration and differentiation of Grassmann variables are equal

$$\frac{\partial}{\partial\psi}f(\psi) = \int d\psi f(\psi) = b \quad \text{for} \quad f(\psi) = a + b\psi\tag{2.67}$$

since $\int d\psi = 0$ and $\int d\psi\psi = 1$. The results $\int d\psi = 0$ and $\int d\psi\psi = 1$ also imply the property

$$\int d\psi \frac{\partial f(\psi)}{\partial\psi} = 0.\tag{2.68}$$

Gaussian Grassmann Integrals

We required that Grassmann integrals obey linearity and translation invariance. By requiring linearity and translation invariance we established the properties

$$\int d\psi = 0 \quad \text{and} \quad \int d\psi\psi = 1.\tag{2.69}$$

We can use these properties to compute a simple Gaussian Grassmann integral of the form

$$\int d\bar{\psi}d\psi e^{a\bar{\psi}\psi},\tag{2.70}$$

where a is an ordinary variable, ψ is a complex Grassmann variable and $\bar{\psi}$ is the complex conjugate of ψ . The integral can be evaluated by first expanding the exponential and applying the properties from (2.69). We find

$$\int d\psi d\bar{\psi} e^{a\bar{\psi}\psi} = \int d\psi d\bar{\psi} (1 + a\bar{\psi}\psi) = a.$$

Computation of a generating function for Fermions

Consider the action for spinless fermions

$$S_\psi = \int d\tau \int d^d x \bar{\psi} (\partial_\tau + \mu - \epsilon(i\vec{\nabla})) \psi, \quad (2.71)$$

where $\epsilon(i\vec{\nabla})$ is the quasiparticle kinetic energy. For a free particle of mass m we have

$$\epsilon(\vec{k}) = \frac{1}{2m} \vec{k} \cdot \vec{k}, \quad (2.72)$$

where $\vec{k}$ is the momentum of the fermionic particle. The generating function for fermions with Grassmann sources $\bar{j}$ and j , is

$$Z[j, \bar{j}] = \mathcal{A} \int D\bar{\psi} \int D\psi e^{-S_\psi + \int d\tau d^d x (\bar{\psi}(\tau, \vec{x}) j(\tau, \vec{x}) + \bar{j}(\tau, \vec{x}) \psi(\tau, \vec{x}))}, \quad (2.73)$$

where $\mathcal{A}$ is a normalisation constant. To compute this integral, we will determine the shift in ψ and $\bar{\psi}$ needed to complete the square. We do this by setting the functional derivative with respect to ψ and $\bar{\psi}$ of the exponential argument equal to zero. Concretely we have

$$0 = \frac{\delta}{\delta \bar{\psi}(\tau', \vec{x}')} \left(-S_\psi + \int d\tau d^d x \bar{\psi}(\tau, \vec{x}) j(\tau, \vec{x}) + \bar{j}(\tau, \vec{x}) \psi(\tau, \vec{x}) \right) \quad (2.74)$$

$$= \frac{\delta}{\delta \bar{\psi}(\tau', \vec{x}')} \int d\tau \int d^d x \left(-\bar{\psi} \partial_\tau \psi - \bar{\psi} \mu \psi + \bar{\psi} \epsilon(i\vec{\nabla}) \psi + \bar{\psi} j + \bar{j} \psi \right) \quad (2.75)$$

$$= \int d\tau \int d^d x \left(-\delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau \psi - \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \mu \psi + \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \epsilon(i\vec{\nabla}) \psi + \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') j \right) \quad (2.76)$$

$$= -\partial_\tau \psi(\tau, \vec{x}) - \mu \psi(\tau, \vec{x}) + \epsilon(i\vec{\nabla}) \psi(\tau, \vec{x}) + j(\tau, \vec{x}) \quad (2.77)$$

$$\Rightarrow j = \partial_\tau \psi + \mu \psi - \epsilon(i\vec{\nabla}) \psi. \quad (2.78)$$

and

$$0 = -\frac{\delta}{\delta \psi(\tau', \vec{x}')} \int d\tau \int d^d x (\bar{\psi} \partial_\tau \psi + \bar{\psi} \mu \psi - \bar{\psi} \epsilon(i\vec{\nabla}) \psi + \bar{\psi} j) \quad (2.79)$$

$$= \frac{\delta}{\delta \psi(\tau', \vec{x}')} \int d\tau \int d^d x \left(-\psi \partial_\tau \bar{\psi} - \partial_\tau (\psi \bar{\psi}) + \psi \mu \bar{\psi} - \psi \epsilon(i\vec{\nabla}) \bar{\psi} + \psi \bar{j} \right) \quad (2.80)$$

$$= \int d\tau \int d^d x \left(-\delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \partial_\tau \bar{\psi} + \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \mu \bar{\psi} - \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \epsilon(i\vec{\nabla}) \bar{\psi} + \delta^{(d+1)}(\tau - \tau', \vec{x} - \vec{x}') \bar{j} \right) \quad (2.81)$$

$$= -\partial_\tau \bar{\psi} + \mu \bar{\psi} - \epsilon(i\vec{\nabla}) \bar{\psi} + \bar{j} \quad (2.82)$$

$$\Rightarrow \bar{j} = \partial_\tau \bar{\psi} - \mu \bar{\psi} + \epsilon(i\vec{\nabla}) \bar{\psi}. \quad (2.83)$$

On line (2.80) we have used integration by parts on the first term and the anti-commutation relation on the second, third and fourth terms of the integrand. The second term on line (2.80) is a surface term which vanishes, since the field falls off to zero.

Equ.(2.78) and equ.(2.83) can be solved by using the Green function

$$(\partial_\tau + \mu - \epsilon(i\vec{\nabla}))G_F(\tau, \vec{x}) = \delta(\tau)\delta^{(d)}(\vec{x}). \quad (2.84)$$

The solution is

$$\psi(\tau, \vec{x}) = \int d\tau' \int d^d x' G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}'). \quad (2.85)$$

We make the ansatz that

$$G_F(\tau, \vec{x}) = \frac{1}{(2\pi)^{d+1}} \int dw \int d^d k G_F(w, \vec{k}) e^{iw\tau + i\vec{k}\cdot\vec{x}}. \quad (2.86)$$

Recall the identities

$$\delta(\tau) = \int \frac{dw}{2\pi} e^{iw\tau} \quad (2.87)$$

and

$$\delta^{(d)}(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}}. \quad (2.88)$$

When we plug (2.86) back into (2.84) we find

$$G_F(w, \vec{k}) = \frac{1}{iw + \mu - \epsilon(\vec{k})}. \quad (2.89)$$

Hence

$$G_F(\tau - \tau', \vec{x} - \vec{x}') = \int \frac{dw d^d \vec{k}}{(2\pi)^{d+1}} \frac{1}{iw + \mu - \epsilon(\vec{k})} e^{iw(\tau - \tau') + i\vec{k}\cdot(\vec{x} - \vec{x}')}. \quad (2.90)$$

This function has a pole at $w = i(\mu - \epsilon(\vec{k})) = iw_\epsilon$. We will again evaluate the w integral using the residue theorem. Write

$$G_F(\tau - \tau', \vec{x} - \vec{x}') = \int \frac{dw d^d \vec{k}}{(2\pi)^{d+1}} \frac{1}{iw + \mu - \epsilon(\vec{k})} e^{i(w_R + iw_I)(\tau - \tau') + i\vec{k}\cdot(\vec{x} - \vec{x}')}$$

where w_R and w_I are real and imaginary components of the frequency. The sign of $\tau - \tau'$ determines in which plane we close the contour. When $\tau > \tau'$ we close the contour anticlockwise in the upper plane to get a decaying exponential. We thus pick up a contribution from the pole at iw_ϵ . The Green function becomes

$$G_F(\tau - \tau', \vec{x} - \vec{x}') = i \int \frac{d^d \vec{k}}{(2\pi)^d} e^{-w_\epsilon(\tau - \tau') + i\vec{k}\cdot(\vec{x} - \vec{x}')}. \quad (2.91)$$

When $\tau < \tau'$ we close the contour clockwise in the lower plane, and we don't pick up any pole. Therefore the Green function is zero. Hence, our time ordered Green function for Fermions is

$$G_F(\tau - \tau', \vec{x} - \vec{x}') = \theta(\tau - \tau') i \int \frac{d^d \vec{k}}{(2\pi)^d} e^{-w_\epsilon(\tau - \tau') + i\vec{k}\cdot(\vec{x} - \vec{x}')}. \quad (2.90)$$

The second thing to do is to manipulate the term

$$-S_\psi + \int d\tau d^d x (\bar{\psi}(\tau, \vec{x}) j(\tau, \vec{x}) + \bar{j}(\tau, \vec{x}) \psi(\tau, \vec{x})) \quad (2.91)$$

by shifting the functions $\psi \rightarrow \psi + \psi_0$ and $\bar{\psi} \rightarrow \bar{\psi} + \bar{\psi}_0$. Therefore

$$\begin{aligned} -S_\psi + \int d\tau \int d^d x (\bar{\psi} j + \bar{j} \psi) &= - \int d\tau \int d^d x (\bar{\psi} + \bar{\psi}_0)(\partial_\tau + \mu - \epsilon(i\vec{\nabla}))(\psi + \psi_0) \\ &+ \int d\tau \int d^d x ((\bar{\psi} + \bar{\psi}_0)j + \bar{j}(\psi + \psi_0)). \end{aligned} \quad (2.92)$$

The RHS terms can be simplified as follows

$$\begin{aligned} RHS &= \int d\tau \int d^d x \left((-\bar{\psi} \partial_\tau \psi - \bar{\psi} \mu \psi + \bar{\psi} \epsilon(i\vec{\nabla}) \psi) + (-\bar{\psi} \partial_\tau \psi_0 - \bar{\psi} \mu \psi_0 + \bar{\psi} \epsilon(i\vec{\nabla}) \psi_0 + \bar{\psi} j) \right. \\ &+ (-\bar{\psi}_0 \partial_\tau \psi - \bar{\psi}_0 \mu \psi + \bar{\psi}_0 \epsilon(i\vec{\nabla}) \psi + \bar{j} \psi) - \bar{\psi}_0 \partial_\tau \psi_0 - \bar{\psi}_0 \mu \psi_0 \\ &\left. + \bar{\psi}_0 \epsilon(i\vec{\nabla}) \psi_0 + \bar{j} \psi_0 + \bar{\psi}_0 j \right). \end{aligned} \quad (2.93)$$

The terms in the first parenthesis are equal to the original $-S_\psi$. The second and third terms in the parenthesis are equal to zero. The terms now left on the RHS are

$$RHS = -S_\psi + \int d\tau \int d^d x (-\bar{\psi}_0 \partial_\tau \psi_0 - \bar{\psi}_0 \mu \psi_0 + \bar{\psi}_0 \epsilon(i\vec{\nabla}) \psi_0 + \bar{j} \psi_0 + \bar{\psi}_0 j). \quad (2.94)$$

We can integrate the term $\bar{\psi}_0 \partial_\tau \psi_0$ by parts to get

$$RHS = -S_\psi + \int d\tau \int d^d x (\psi_0 \partial_\tau \bar{\psi}_0 - \partial_\tau (\psi_0 \bar{\psi}_0) - \bar{\psi}_0 \mu \psi_0 + \bar{\psi}_0 \epsilon(i\vec{\nabla}) \psi_0 - \psi_0 \bar{j} + \bar{\psi}_0 j).$$

The term $\partial_\tau (\psi_0 \bar{\psi}_0)$ is a surface term and its integral is zero, as the field vanishes at the boundary. Therefore

$$\begin{aligned} -S_\psi + \int d\tau \int d^d x (\bar{\psi} j + \bar{j} \psi) &= -S_\psi + \int d\tau \int d^d x (-\bar{\psi}_0 \partial_\tau \psi_0 - \bar{\psi}_0 \mu \psi_0 + \bar{\psi}_0 \epsilon(i\vec{\nabla}) \psi_0 + \bar{j} \psi_0 + \bar{\psi}_0 j) \\ &= -S_\psi + \int d\tau \int d^d x \int d\tau' \int d^d x' \bar{j}(\tau, \vec{x}) G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}'). \end{aligned}$$

We now obtain

$$Z[j, \bar{j}] = \mathcal{A} \int D\bar{\psi} \int D\psi e^{-S_\psi + \int d\tau d^d x (\bar{\psi}(\tau, \vec{x}) j(\tau, \vec{x}) + \bar{j}(\tau, \vec{x}) \psi(\tau, \vec{x}))} \quad (2.95)$$

$$= \mathcal{A} \int D\bar{\psi} \int D\psi e^{-S_\psi + \int d\tau \int d^d x \int d\tau' \int d^d x' \bar{j}(\tau, \vec{x}) G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')} \quad (2.96)$$

$$= \mathcal{A} \int D\bar{\psi} \int D\psi e^{-\int d\tau \int d\tau' \int d^d x \int d^d x' (\bar{\psi}' \delta(\tau - \tau') \delta^d(\vec{x} - \vec{x}') (\partial_\tau + \mu - \epsilon(i\vec{\nabla})) \psi + \bar{j}(\tau, \vec{x}) G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}'))} \quad (2.97)$$

$$= \mathcal{A} \int D\bar{\psi} \int D\psi e^{-\int d\tau \int d\tau' \int d^d x \int d^d x' \bar{\psi}' M \psi + \int d\tau \int d^d x \int d\tau' \int d^d x' \bar{j}(\tau, \vec{x}) G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')} \quad (2.98)$$

where $\psi' = \psi(\tau', \vec{x}')$ and $M = \delta(\tau - \tau')\delta^d(\vec{x} - \vec{x}')(\partial_\tau + \mu - \epsilon(i\vec{\nabla}))$. The integral can be evaluated using the same method used to evaluate the spin zero generating function. We discretize the field by transforming the Euclidean space into a lattice with spacing distance a . Therefore

$$Z[j, \bar{j}]|_{j=\bar{j}=0} = \int \prod_{k=1}^N d\bar{\psi}_K d\psi_K e^{-\sum_{I,J} \bar{\psi}_I M_{IJ} \psi_J},$$

where $M_{IJ} = \delta_{IJ}(\Delta_i + \mu + \epsilon(i\vec{\nabla}))$ and Δ_i is a first order “time” derivative. Its action on $\psi_{i,\vec{j}}$ is defined as

$$\Delta_i \psi_{i,\vec{j}} = \frac{\psi_{i+1,\vec{j}} - \psi_{i,\vec{j}}}{a}.$$

We can diagonalize the symmetric matrix M with an Unitary matrix, U

$$U_{IK}^\dagger \tilde{M}_{KL} U_{LJ} = M_{IJ},$$

where $\tilde{M}_{KL} = \delta_{KL} \lambda_L$ is a diagonal matrix. Make the transformation

$$U_{LJ} \psi_J = \psi'_L.$$

The integration measure transforms as follows

$$\int \prod_{K=1}^N d\bar{\psi}_K d\psi_K \rightarrow \int \prod_{K=1}^N d\bar{\psi}'_K d\psi'_K.$$

Since the Jacobian matrix is

$$J = \left| \frac{d\psi'_L}{d\psi_P} \right| = \left| \frac{d(U_{LJ} \psi_J)}{d\psi_P} \right| = |U_{LP}| = 1,$$

the measure is invariant, and we find

$$\begin{aligned} Z[j, \bar{j}]|_{j=\bar{j}=0} &= \mathcal{A} \int \prod_{P=1}^N d\bar{\psi}'_P d\psi'_P e^{-\sum_{I,J,K,L} \bar{\psi}'_I U_{IK}^\dagger \tilde{M}_{KL} U_{LJ} \psi'_J} \\ &= \mathcal{A} \int \prod_{P=1}^N d\bar{\psi}'_P d\psi'_P e^{-\sum_{K,L} \bar{\psi}'_K \tilde{M}_{KL} \psi'_L} \\ &= \mathcal{A} \int \prod_{P=1}^N d\bar{\psi}'_P d\psi'_P e^{-\sum_{K,L} \delta_{KL} \lambda_L \bar{\psi}'_K \psi'_L} \\ &= -\mathcal{A} \int \prod_{P=1}^N d\bar{\psi}'_P d\psi'_P \prod_{L=1}^N (1 + \lambda_L \bar{\psi}'_L \psi'_L) \\ &= \mathcal{A} \prod_{P=1}^N \lambda_P \\ &= \mathcal{A} \det M, \end{aligned}$$

where λ_P is an eigenvalue of matrix M_{IJ} . We can normalise the generating function by requiring $Z[j, \bar{j}]|_{j=\bar{j}=0} = 1$. Therefore

$$Z[j, \bar{j}]|_{j=\bar{j}=0} = \langle 0|0 \rangle = 1 \quad (2.99)$$

$$\Rightarrow \det M = \mathcal{A}^{-1}. \quad (2.100)$$

The generating function simplifies to

$$Z[j, \bar{j}] = e^{\int d\tau \int d\tau' \int d^d x \int d^d x' \bar{j}(\tau, \vec{x}) G_F(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')}. \quad (2.101)$$

2.0.6 Computing a two point function for fermions

The two point function for fermions in terms of the generating function can be written as

$$\langle \bar{\psi}(\tau_1, \vec{x}_1) \psi(\tau_2, \vec{x}_2) \rangle = \frac{\delta}{\delta j(\tau_1, \vec{x}_1)} \frac{\delta}{\delta \bar{j}(\tau_2, \vec{x}_2)} Z[j, \bar{j}] \Big|_{j=\bar{j}=0},$$

which is evaluated to be

$$\langle \bar{\psi}(\tau_1, \vec{x}_1) \psi(\tau_2, \vec{x}_2) \rangle = \frac{\delta}{\delta j(\tau_1, \vec{x}_1)} \frac{\delta}{\delta \bar{j}(\tau_2, \vec{x}_2)} Z[j, \bar{j}] \Big|_{j=\bar{j}=0} \quad (2.102)$$

$$= \frac{\delta}{\delta j(\tau_1, \vec{x}_1)} \frac{\delta}{\delta \bar{j}(\tau_2, \vec{x}_2)} \int D\bar{\psi} \int D\psi e^{-S_\psi + \int d\tau \int \tau' \int d^d x \int d^d x' \bar{j}(\tau, \vec{x}) G(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')} \quad (2.103)$$

$$= \frac{\delta}{\delta j(\tau_1, \vec{x}_1)} \frac{\delta}{\delta \bar{j}(\tau_2, \vec{x}_2)} e^{\int d\tau \int \tau' \int d^d x \int d^d x' \bar{j}(\tau, \vec{x}) G(\tau - \tau', \vec{x} - \vec{x}') j(\tau', \vec{x}')} \quad (2.104)$$

$$= G_F(\tau_2 - \tau_1, \vec{x}_2 - \vec{x}_1), \quad (2.105)$$

$$= i\theta(\tau_2 - \tau_1) \int \frac{d^d \vec{k}}{(2\pi)^d} e^{-w_\varepsilon(\tau_2 - \tau_1) - i\vec{k} \cdot (\vec{x}_2 - \vec{x}_1)}.$$

2.1 Chapter Summary

In this chapter we were able to derive the Green's functions and generating functions for both the scalar free field and fermion free field theory. We studied the harmonic oscillator in $1 + 1$ dimension and $d + 1$ dimension and we demonstrated that a quantum field is just an assembly of harmonic oscillators in $1 + 1$ dimensions. By translating between moments of the path integral and time ordered correlation functions in the operator approach to quantum field theory we learned that, the determinant $\det M$ of a scalar field theory obtained from a path integral when $Z[j]_{j=0}$, is related to the vacuum energy of the field.

So far we have focused only on the free theory. There is not much interesting physics we can learn from the free theory. Most physical systems in nature are not free of interactions, so in the next chapter we will study a toy model physical system of non-Fermi liquids in the low energy limit. We will study the properties of the low energy limit using Wilson's renormalization group flows.

Chapter 3

Renormalization at one loop

We are going to study renormalization, at one loop, for non-relativistic finite density fermions, using Wilson's theory of renormalization. Renormalization for non-relativistic quantum field theory is rather different from the renormalization of relativistic quantum field theory essentially because low energy fermionic states in non-relativistic QFT are states in the vicinity of the Fermi-surface. We will start by giving a basic description of how Wilson's renormalization can be implemented. We will then apply it to our non-relativistic finite density toy model. After that we will compare the renormalization strategy used in [17] with Wilson's renormalization. Then we will draw the Feynman diagrams at one loop for our field theory and evaluate the loop diagrams using the renormalization adopted in [17].

3.1 Renormalization group flow

The strategy employed in renormalization group flow analysis is to integrate over high energy-momenta modes, shell by shell, to obtain the Lagrangian describing the low energy physics. The shells are in momentum space so that for very thin shells, every mode in a given shell has the same energy. We will start with a rather conventional description of how the renormalization group flow is constructed in relativistic quantum field theories that are close to the free field fixed point. In later sections of this chapter we will describe how the renormalization group flow is constructed at finite density, again close to the free field fixed point.

We begin by introducing a physical cut-off Λ for the system we are interested in. It is a scale that tells us the domain of validity of effective field theory. Consider a system described by a field $\phi(k)$, and separate the field into two pieces [2]

$$\phi(k) = \phi_H(k) + \phi_L(k) \tag{3.1}$$

where ϕ_L contains low energy modes with $0 \leq |\vec{k}| < e^{-t}\Lambda$ and ϕ_H contains high energy modes with $e^{-t}\Lambda \leq |\vec{k}| \leq \Lambda$. The fields are defined as follows, in momentum space

$$\phi_L(k) = \begin{cases} \phi(k) & \text{for } 0 \leq |\vec{k}| \leq e^{-t}\Lambda \quad \text{or equivalently} \quad 0 \leq |\vec{k}| \leq (1-t)\Lambda \\ 0 & \text{for } e^{-t}\Lambda \leq |\vec{k}| \leq \Lambda \quad \text{or equivalently} \quad (1-t)\Lambda \leq |\vec{k}| \leq \Lambda \end{cases}$$

and

$$\phi_H(k) = \begin{cases} \phi(k) & \text{for } e^{-t}\Lambda \leq |\vec{k}| \leq \Lambda \quad \text{or equivalently} \quad (1-t)\Lambda \leq |\vec{k}| \leq \Lambda \\ 0 & \text{for } 0 \leq |\vec{k}| \leq e^{-t}\Lambda \quad \text{or equivalently} \quad 0 \leq |\vec{k}| \leq (1-t)\Lambda, \end{cases}$$

and we always assume that $t \ll 1$. In terms of these momentum space fields we can define

$$\phi_L(x) = \int \frac{d^D k}{(2\pi)^D} \phi_L(k) e^{-ik \cdot x} \quad \text{and} \quad \phi_H(x) = \int \frac{d^D k}{(2\pi)^D} \phi_H(k) e^{-ik \cdot x}. \quad (3.2)$$

in D-dimensional Euclidean space. We work in Euclidean space so that the complete set of modes is easily organized into collections of equal energy modes, by spherical momentum shells. This simple organization does not work in Minkowski spacetime. The action of the field becomes

$$S(\phi) = S(\phi_L + \phi_H), \quad (3.3)$$

and the path integral of the system is

$$Z = \int [D\phi]_{\Lambda} e^{-S(\phi)} \quad (3.4)$$

$$= \int [D\phi_L][D\phi_H] e^{-S(\phi_L + \phi_H)} \quad (3.5)$$

$$= \int D\phi_L e^{-S_{eff}(\phi_L)}, \quad (3.6)$$

where

$$\begin{aligned} [D\phi_H] &= \prod_{(1-t)\Lambda < |\vec{k}| \leq \Lambda} \phi(k), \quad [D\phi_L] = \prod_{|\vec{k}| < (1-t)\Lambda} \phi(k) \quad [D\phi]_{\Lambda} = \prod_{k \cdot k < \Lambda^2} d\phi(k) \quad \text{and} \\ e^{-S_{eff}} &= \int [D\phi_H] e^{-S(\phi_L + \phi_H)}. \end{aligned} \quad (3.7)$$

To obtain the low energy effective action S_{eff} , we integrate out the field ϕ_H leaving a path integral for the field ϕ_L . The cut-off constraint $|\vec{k}| < \Lambda$ selects a ball of radius $|\vec{k}| < \Lambda$ in the Euclidean momentum space, and for $t \ll 1$ we are able to break the ball into shells of energy-momenta modes as shown in the Figure below. Each time we integrate over a shell we rescale the parameters of the action. As we integrate over the high energy-momenta modes to obtain dynamics of low energy-momenta modes, the parameters of the action flow. Some parameters flow to zero or decay, and we call these parameters irrelevant. Those that grow or flow to infinity are relevant and those that do not grow or decay, are marginal.

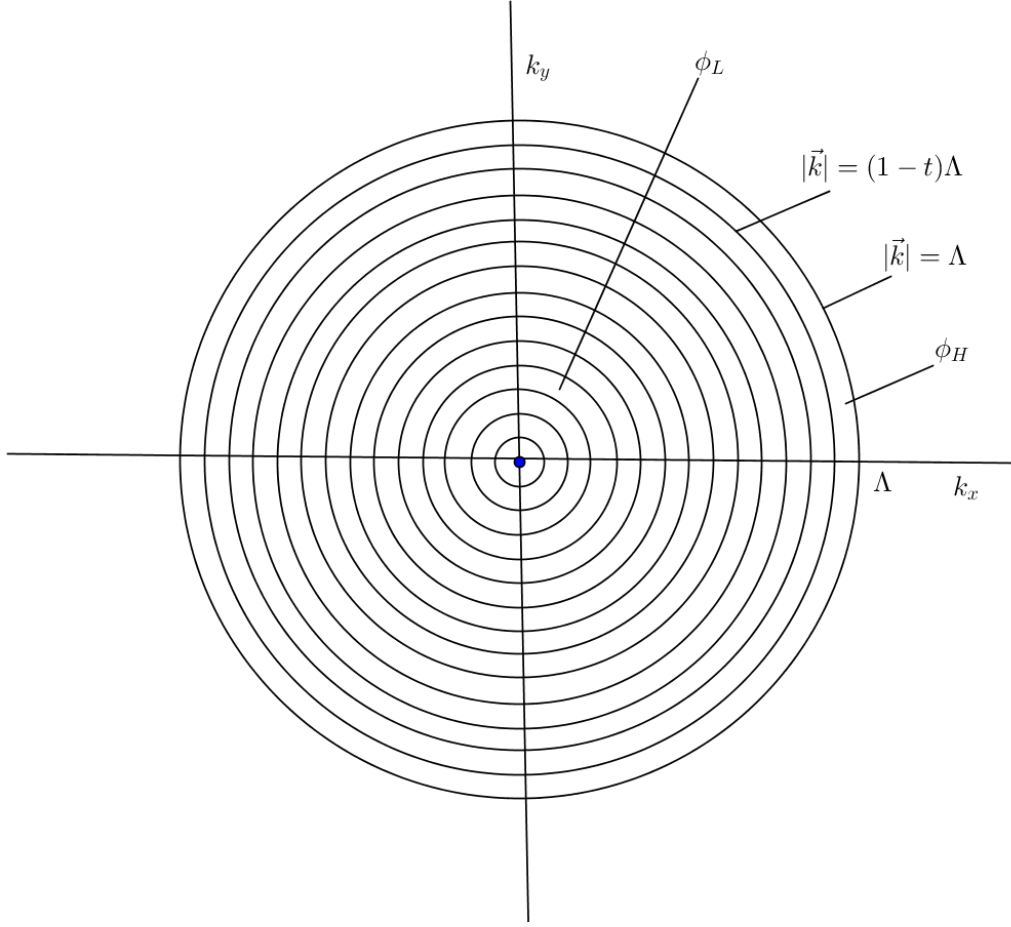


Figure 3.1: Two dimensional slice through the energy-momentum shells of modes.

S_{eff} contains infinitely many interaction terms. We can write it as [2]

$$S_{eff} = \int d^D x \sum_i g_i A_i \quad (3.8)$$

where g_i is an interaction strength, A_i is a local operator and the sum runs over all operators that can be constructed respecting the symmetries of the system. A renormalization group flow from high to low energy is accomplished by iterating three basic steps:

- a) We separate the outer shell of modes as explained above.
- b) We integrate over the modes in the outer shell.
- c) We rescale the momentum and fields.

Close to the free field fixed point, since all couplings are weak, the effect of (c) is so much larger than (b) that we can neglect step (b). In this case the rescaling is responsible for the flow and the marginal, relevant and irrelevant terms of the action can be determined using dimensional and scaling analysis. First we set $c = \hbar = 1$. With this choice of units we know that the dimensions of momentum and mass are $[p] = [m] = [E]^{-1} = L$ where p , m , E , L are momentum, mass, energy and length respectively. Each time we integrate over an energy mode $\vec{k}$ of ϕ , we have to rescale the momentum $\vec{k}$ in order for the original S_{eff} and the new integrated action to be defined on the same integration domain. It is only after this rescaling that we can meaningfully

compare them. We rescale k and x as

$$k' = \frac{k}{(1-t)} \quad \text{and} \quad x' = x(1-t) \quad (3.9)$$

where $t \ll 1$. Let us consider the scalar field kinetic term. We determine the scaling of the scalar field ϕ by requiring the kinetic term is invariant under scaling

$$\frac{1}{2} \int d^D x (\partial_\mu \phi)^2 = \frac{1}{2} \int d^D x' (\partial'_\mu \phi')^2 \quad (3.10)$$

$$= \frac{1}{2} \frac{(1-t)^D}{(1-t)^2} \int d^D x (\partial_\mu \phi')^2 \quad (3.11)$$

$$= \frac{1}{2} \int d^D x (\partial_\mu \phi)^2 \quad (3.12)$$

$$\Rightarrow \phi' = (1-t)^{\frac{2-D}{2}} \phi. \quad (3.13)$$

This will ensure that the free field action is a fixed point of the renormalization group flow. We call it the free field fixed point. Therefore, for any operator with M derivatives and N fields, we find

$$\int d^D x \partial^M \phi^N = (1-t)^{-D+M+\frac{N(D-2)}{2}} \int d^D x' \partial'^M \phi'^N \quad (3.14)$$

$$= (1-t)^\Delta \int d^D x' \partial'^M \phi'^N. \quad (3.15)$$

The above transformation has a natural explanation following from simple dimensional analysis. Indeed, note that this term has dimension

$$\left[\int d^D x \partial^M \phi^N \right] = L^{D-M-\frac{N(D-2)}{2}} = L^{-\Delta}, \quad (3.16)$$

where $\Delta = -D + M + \frac{N(D-2)}{2}$. The scaling of the coefficient coupling of this operator

$$\lambda \int d^D x \partial^M \phi^N \rightarrow \lambda' \int d^D x \partial^M \phi^N, \quad (3.17)$$

is thus $\lambda' = \lambda(1-t)^\Delta$ and is called the effective coupling. If $\Delta > 0$ the effective coupling grows as we integrate over high energy modes and the term is called relevant. If $\Delta = 0$ the term is called marginal as it remains invariant under the renormalization group flow and if $\Delta < 0$ the term flows to zero or decays and we call it irrelevant.

3.2 Scaling of parameters for finite density systems

The scaling that we adopt for the bosons and fermions is dictated by their dispersion relations. The bosons and fermions dispersion relationships are $k_0^2 = c^2 \vec{k}^2 + m_\phi^2$ and $k_0 = \epsilon(\vec{k}) - \mu$ respectively. From the dispersion relationship we see that the bosons have low energy modes close to the origin as indicated in the figure below.

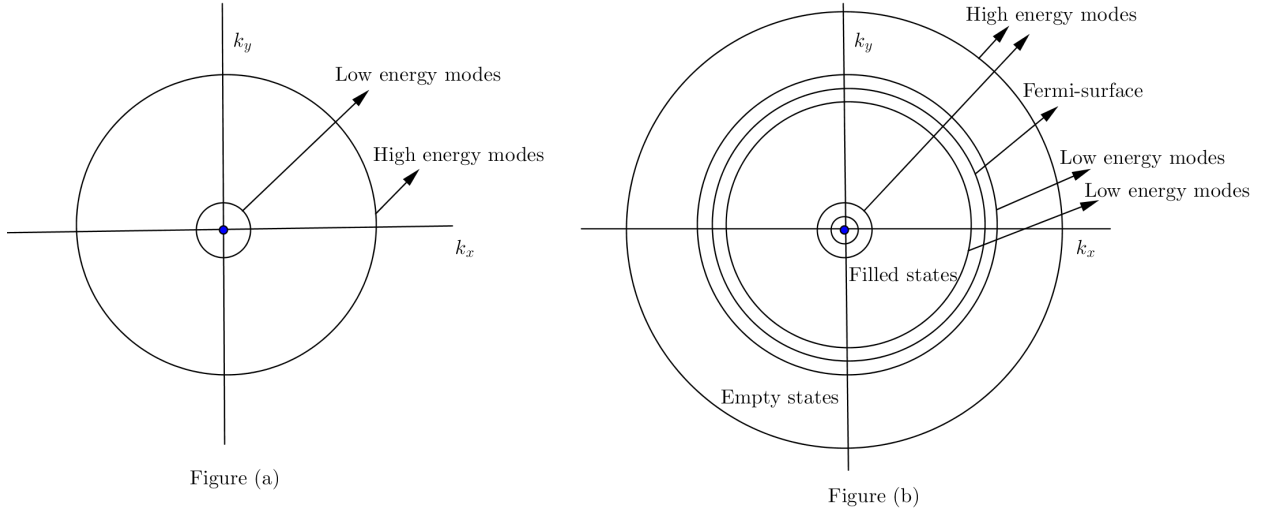


Figure 3.2: Scaling of bosons and fermions.

The fermions low energy modes are located near the Fermi-surface as shown in Figure 3.2 (b). For a rotationally invariant Fermi-surface we parametrize the fermion momentum as

$$\vec{k} = \hat{\Omega}(k_F + l) = \vec{k}_F + \vec{l},$$

where $\vec{k}_F = k_F \hat{\Omega}$ is a point on the Fermi-surface, $\hat{\Omega}$ is an angular direction and l is a normal displacement from the Fermi-surface to the excited particle with momentum $\vec{k}$ as indicated in the figure below.

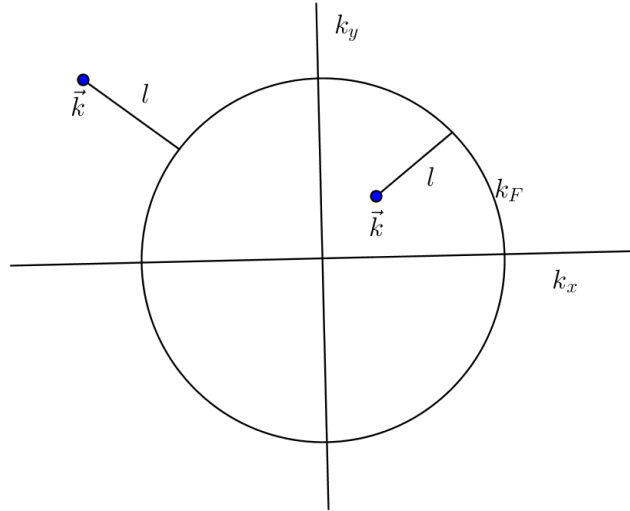


Figure 3.3: How energy is measured for Fermions

The Fermi-surface is defined as $\epsilon(\vec{k}) = \mu$. The Fermi-sea occupies all states with $\epsilon(\vec{k}) < \mu$ which constitutes the ground state. From the figure above we can conclude that the l for the low energy modes is much smaller than that of high energy modes since low energy modes are found near the Fermi-surface and high energy modes are found further away from the Fermi-surface. Low energy excitations can be formed by removing an electron inside but close to the Fermi-surface (creating a particle hole), or placing an electron just above the Fermi-surface [2].

As we integrate over the high energy modes we have to scale the fermion momenta towards the Fermi-surface and boson momenta towards the origin. To accomplish this we adopt the scaling

$$w' = e^t w, \quad k'_F = k_F \quad \text{and} \quad l' = e^t l, \quad (3.18)$$

for fermions. We don't scale k_F because we want the Fermi-surface to be invariant. We want the Fermi-surface to remain invariant because the fermions energy and momentum is measured with respect to the Fermi-surface. The bosons scaling is

$$w' = e^t w \quad \text{and} \quad \vec{k}' = e^t \vec{k}.$$

Scaling of the Scalar couplings

From the scalar action

$$S_\phi = \int d\tau d^d x \left(\frac{1}{2} ((\partial_\tau \phi)^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2) + \frac{\lambda_\phi}{4!} \phi^4 \right),$$

the differentials and partial derivatives scale as

$$d\tau \rightarrow e^{-t} d\tau, \quad dx \rightarrow e^{-t} dx, \quad \partial_\tau \rightarrow e^t \partial_\tau \quad \text{and} \quad \vec{\nabla} \rightarrow e^t \vec{\nabla}. \quad (3.19)$$

To find out how the scalar field should be scaled, we use the free field fixed point scaling which requires the kinetic term of the action is invariant. Fourier transforming the action to momentum space we obtain

$$\frac{1}{2} \int \frac{dw d^d k}{(2\pi)^{d+1}} w^2 \phi(w, \vec{k}) \phi(-w, -\vec{k}) = \frac{1}{2} \int \frac{dw' d^d k'}{(2\pi)^{d+1}} w'^2 \phi'(w', \vec{k}') \phi'(-w', -\vec{k}') \quad (3.20)$$

$$= \frac{1}{2} e^{(1+d)t+2t} \int \frac{dw d^d k}{(2\pi)^{d+1}} w^2 \phi'(w', \vec{k}') \phi'(-w', -\vec{k}') \quad (3.21)$$

$$\Rightarrow \phi'(w', \vec{k}') = e^{-\frac{1}{2}(d+3)t} \phi(w, \vec{k}), \quad (3.22)$$

for the kinetic term of the action to be invariant. We will now determine the scaling of the quartic scalar interaction, using the free field fixed point scaling. Consider

$$\int \prod_{i=1}^4 \frac{d^{d+1} k'_i}{(2\pi)^{d+1}} \delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) \frac{\lambda'_\phi}{4!} \phi'(k'_1) \phi'(k'_2) \phi'(k'_3) \phi'(k'_4). \quad (3.23)$$

The delta function scales as

$$\delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) = e^{-(1+d)t} \delta^{d+1}(k_1 + k_2 + k_3 + k_4). \quad (3.24)$$

Therefore

$$e^{4(d+1)t-(1+d)t-2(d+3)t} \int \prod_{i=1}^4 \frac{d^{d+1} k_i}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \frac{\lambda'_\phi}{4!} \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \quad (3.25)$$

$$\begin{aligned} &= e^{(d-3)t} \int \prod_{i=1}^4 \frac{d^{d+1} k_i}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \frac{\lambda'_\phi}{4!} \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \\ &= \int \prod_{i=1}^4 \frac{d^{d+1} k_i}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \frac{\lambda_\phi}{4!} \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \end{aligned} \quad (3.26)$$

which implies

$$\lambda'_\phi = e^{(3-d)t} \lambda_\phi. \quad (3.27)$$

For $d = 3$ the interaction is marginal, $d < 3$ the interaction is relevant and $d > 3$ the interaction is irrelevant.

Scaling of the Fermion coupling

The action for fermions in momentum space is

$$S_\psi = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-k) (iw + \mu - \epsilon(\vec{k})) \psi_\sigma(k) \quad (3.28)$$

$$+ \int \prod_i^4 \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \lambda_\psi \bar{\psi}_\sigma(k_1) \bar{\psi}_{\sigma'}(k_2) \psi_{\sigma'}(k_3) \psi_\sigma(k_4)$$

$$= \int \frac{dw dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-k) (iw + \mu - \epsilon(\vec{k})) \psi_\sigma(k) \quad (3.29)$$

$$+ \int \prod_i^4 \frac{dw_i dl_i d^{d-1}k_{||}}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \lambda_\psi \bar{\psi}_\sigma(k_1) \bar{\psi}_{\sigma'}(k_2) \psi_{\sigma'}(k_3) \psi_\sigma(k_4)$$

$$= \int \frac{dw dl d^{d-1} \hat{\Omega} (k_F + l)^{d-1}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-k) (iw + \mu - \epsilon(\vec{k})) \psi_\sigma(k) \quad (3.30)$$

$$+ \int \prod_i^4 \frac{dw_i dl_i d^{d-1} \hat{\Omega} (k_F + l)^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \lambda_\psi \bar{\psi}_\sigma(k_1) \bar{\psi}_{\sigma'}(k_2) \psi_{\sigma'}(k_3) \psi_\sigma(k_4),$$

in the UV region $\vec{k}_F \gg \vec{l}$, therefore

$$S_\psi = \int \frac{dw dl d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-k) (iw + \mu - \epsilon(\vec{k})) \psi_\sigma(k) \quad (3.31)$$

$$+ \int \prod_i^4 \frac{dw_i dl_i d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \lambda_\psi \bar{\psi}_\sigma(k_1) \bar{\psi}_{\sigma'}(k_2) \psi_{\sigma'}(k_3) \psi_\sigma(k_4),$$

where σ = spin up or spin down. Requiring that the kinetic term is invariant, we find that the fermion field operators at the free field fixed point scale as

$$\int \frac{dw dl d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-w, -\vec{k}) w \psi_\sigma(w, \vec{k}) = \int \frac{dw' dl' d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \bar{\psi}'_\sigma(-w', -\vec{k}') w' \psi'_\sigma(w', \vec{k}') \quad (3.32)$$

$$= e^{3t} \int \frac{dw dl d^{d-1} \hat{\Omega} k_F^{d-1}}{(\pi)^{d+1}} \bar{\psi}'_\sigma(-w', -\vec{k}') w \psi'_\sigma(w', \vec{k}')$$

$$= \int \frac{dw dl d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \bar{\psi}_\sigma(-w, -\vec{k}) w \psi_\sigma(w, \vec{k}) \quad (3.33)$$

$$\Rightarrow \psi'_\sigma(w', \vec{k}') = e^{-\frac{3}{2}t} \psi_\sigma(w, \vec{k}). \quad (3.34)$$

The fermion field operator for spinless fermions scales in the same as the fermion field operator with spin, so that

$$\psi' = e^{-\frac{3}{2}t} \psi. \quad (3.35)$$

Notice that the scaling of the fermion field does not depend on the space dimensionality d . This is a direct consequence of the fact that we deal with a system with finite density, so we scale towards the Fermi surface. Consequently, for any d it is only w and l that are scaled. We will use the same method as for the bosons above, to evaluate the scaling of the fermion coupling. Consider

$$\int \prod_{i=1}^4 \frac{dw'_i dl'_i d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) \lambda'_\psi \bar{\psi}'_\sigma \bar{\psi}'_{\sigma'} \psi'_{\sigma'} \psi'_\sigma \quad (3.36)$$

where k'_1, k'_2 are the incoming energy-momenta of two quasi-particles and k'_3, k'_4 are the out-going energy-momenta of the quasi-particles. The momentum conserving delta function for fermions can be expanded as

$$\delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) = \delta^d(\vec{k}'_{F1} - \vec{l}'_1 + \vec{k}'_{F2} - \vec{l}'_2 + \vec{k}'_{F3} - \vec{l}'_3 + \vec{k}'_{F4} - \vec{l}'_4) \delta(w'_1 + w'_2 + w'_3 + w'_4). \quad (3.37)$$

For interactions which are perpendicular to the Fermi-surface such that, $|\vec{k}'_{F1} + \vec{k}'_{F2} + \vec{k}'_{F3} + \vec{k}'_{F4}| \ll |\vec{l}'_1 + \vec{l}'_2 + \vec{l}'_3 + \vec{l}'_4|$ as indicated in figure 3.4 below, the delta function scales as

$$\delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) = \delta^d(\vec{k}'_{F1} - \vec{l}'_1 + \vec{k}'_{F2} - \vec{l}'_2 + \vec{k}'_{F3} - \vec{l}'_3 + \vec{k}'_{F4} - \vec{l}'_4) \delta(w'_1 + w'_2 + w'_3 + w'_4). \quad (3.38)$$

Now, as a consequence of the kinematical configuration we are studying, one component of the delta function only constrains the $\vec{l}'$ s. This component, as well as the delta function constraining the energy scale, so that

$$\delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) = e^{-2t} \delta^{d+1}(k_1 + k_2 + k_3 + k_4). \quad (3.39)$$

In this case the fermion coupling scales as

$$\begin{aligned} e^{8t-2t-6t} \int \prod_{i=1}^4 \frac{dw_i dl_i d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) \lambda'_\psi \bar{\psi}_\sigma \bar{\psi}_{\sigma'} \psi_{\sigma'} \psi_\sigma \\ = \int \prod_{i=1}^4 \frac{dw_i dl_i d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) \lambda'_\psi \bar{\psi}_\sigma \bar{\psi}_{\sigma'} \psi_{\sigma'} \psi_\sigma \\ = \int \prod_{i=1}^4 \frac{dw_i dl_i d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) \lambda_\psi \bar{\psi}_\sigma \bar{\psi}_{\sigma'} \psi_{\sigma'} \psi_\sigma \\ \Rightarrow \lambda'_\psi = \lambda_\psi. \end{aligned} \quad (3.40)$$

The quartic fermion interaction scales in the same way for spinless fermions. The above kinematical set up corresponds to forward scattering. The fermion quartic coupling for forward scattering is thus marginal, since the coupling will remain the same as we integrate over high energy modes to obtain the coupling for low energy modes.

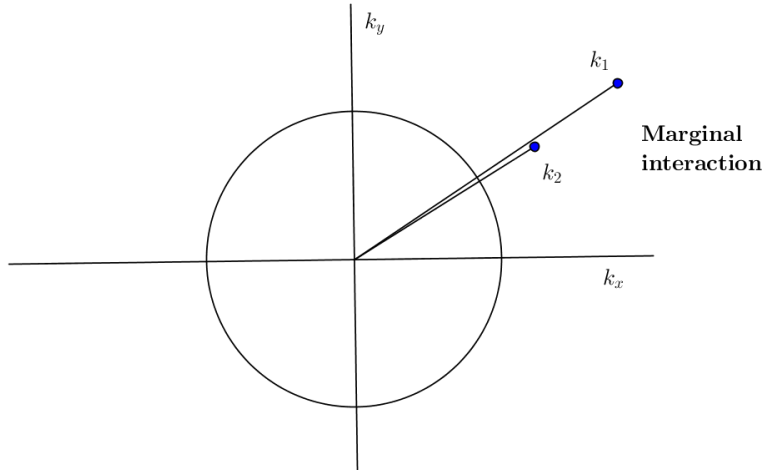


Figure 3.4: Interaction with momentum transfer perpendicular to the Fermi-surface

Given the above discussion, we find the complete scaling of the four fermion interaction as follows. In figure 3.4 the kinematics relevant to forward scattering is shown. It is clear that the momentum transfer ($= \vec{k}_1 - \vec{k}_2$) is perpendicular to the Fermi-surface and is thus scaled. The generic scattering process is shown in figure 3.5. There is no component of the delta function constraining a momentum perpendicular to the Fermi-surface [2]. The delta function scales as

$$\delta^{d+1}(k'_1 + k'_2 + k'_3 + k'_4) = e^{-t} \delta^{d+1}(k_1 + k_2 + k_3 + k_4). \quad (3.41)$$

and the fermion quartic coupling scales as

$$\lambda'_\psi = e^{-t} \lambda_\psi. \quad (3.42)$$

This is an irrelevant interaction, since the fermion quartic coupling will decay as we integrate out the high energy modes to obtain the interactions of the low energy modes.

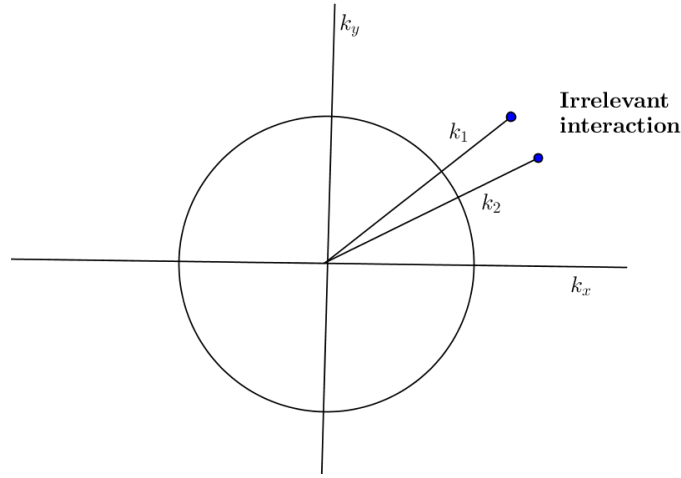


Figure 3.5: Interaction with generic momentum transfer.

Figure 3.5 shows interaction of two quasi-particles which have momentum transfer that is not normal to the Fermi-surface. These type of interactions are irrelevant at low energy since they flow to zero. The difference between the boson and fermion scaling originates from their dispersion relations and the fact that for bosons we scale to zero but for fermions we scale to the Fermi-surface.

Scaling of the Yukawa coupling

The Yukawa coupling which would be relevant in this scaling at low energy, is the one which connects particle-hole states separated by small momenta near the Fermi-surface as depicted in the picture below.

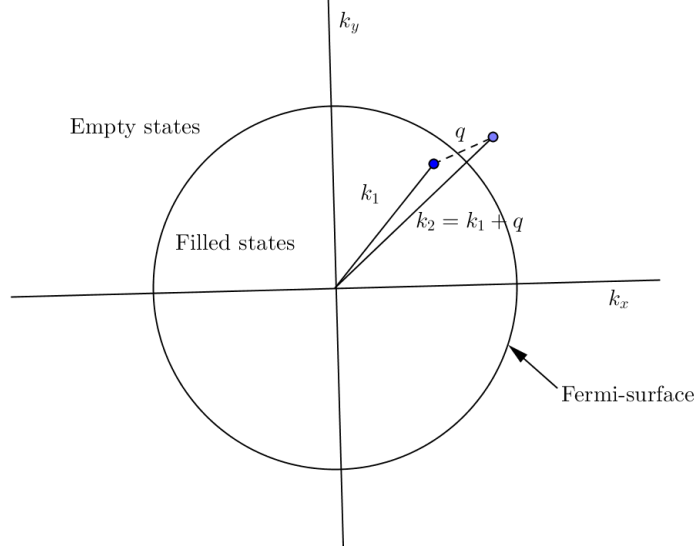


Figure 3.6: Particle-hole interaction close to the Fermi-surface

The scaling of the Yukawa coupling for particle hole-states separated by small momenta q , with momentum transfer due to boson exchange near the Fermi-surface, is derived as follows

$$\begin{aligned}
& \int \frac{d^{d+1}k_1 d^{d+1}k_2 d^{d+1}q}{(2\pi)^{3(d+1)}} g(k_1, q) \bar{\psi}(k_1) \psi(k_2) \phi(q) \delta^d(\vec{k}_{F1} - \vec{k}_{F2} + \vec{l}_1 - \vec{l}_2 - \vec{q}) \delta(w_1 - w_2 - q_0) \\
&= \int \frac{d^{d+1}k'_1 d^{d+1}k'_2 d^{d+1}q'}{(2\pi)^{3(d+1)}} g'(k'_1, q') \bar{\psi}'(k'_1) \psi'(k'_2) \phi'(q') \delta^d(\vec{k}'_{F1} - \vec{k}'_{F2} + \vec{l}'_1 - \vec{l}'_2 - \vec{q}') \delta(w'_1 - w'_2 - q'_0) \\
&= e^{4t + (1+d)t - \frac{1}{2}(d+3)t - \frac{3}{2}t - \frac{3}{2}t} \int \frac{d^{d+1}k_1 d^{d+1}k_2 d^{d+1}q}{(\pi)^{3(d+1)}} g'(k'_1, q') \bar{\psi}(k_1) \psi(k_2) \phi(q) \times \\
& \quad \delta^d(\vec{k}_{F1} - \vec{k}_{F2} + \vec{l}_1 - \vec{l}_2 - \vec{q}') \delta(w'_1 - w'_2 - q'_0). \tag{3.43}
\end{aligned}$$

When $|\vec{l}'_1|, |\vec{l}'_2| \ll |\vec{k}'_{F1} - \vec{k}'_{F2}|$ the delta function $\delta^d(\vec{k}'_{F1} - \vec{k}'_{F2} + \vec{l}'_1 - \vec{l}'_2 - \vec{q}') \approx \delta^d(\vec{k}'_{F1} - \vec{k}'_{F2} - \vec{q}')$ does not scale. Only $\delta(w'_1 - w'_2 - q'_0)$ scales as

$$\delta(w'_1 - w'_2 - q'_0) = e^{-t} \delta(w_1 - w_2 - q_0). \tag{3.44}$$

This means the Yukawa coupling scales as

$$g'(\vec{k}', \vec{q}) = e^{\frac{1}{2}(1-d)t} g(\vec{k}, \vec{q}), \tag{3.45}$$

which is irrelevant for all $d > 1$. When $\vec{k}'_{F1} \approx \vec{k}'_{F2}$ both delta functions scale as

$$\begin{aligned}
& \delta^d(\vec{k}'_{F1} - \vec{k}'_{F2} + \vec{l}'_1 - \vec{l}'_2 - \vec{q}') \delta(w'_1 - w'_2 - q'_0) \approx \delta^d(\vec{l}'_1 - \vec{l}'_2 - \vec{q}') \delta(w'_1 - w'_2 - q'_0). \\
&= e^{-2t} \delta^d(\vec{k}_{F1} - \vec{k}_{F2} + \vec{l}_1 - \vec{l}_2 - \vec{q}) \delta(w_1 - w_2 - q_0), \tag{3.46}
\end{aligned}$$

Now the Yukawa coupling scales as

$$g'(\vec{k}', \vec{q}) = e^{\frac{1}{2}(3-d)t} g(\vec{k}, \vec{q}). \tag{3.47}$$

In this case the Yukawa coupling is marginal at $d = 3$, relevant when $d < 3$ and irrelevant when $d > 3$. From the scaling of the above parameters we see that the fermion interaction term describing forward scattering is marginal at tree level, for all $d > 0$. The scalar interaction term is marginal at $d = 3$, irrelevant for $d > 3$ and relevant for $d < 3$. Therefore the upper critical dimension for the scalar fields is $d = 3$. The upper critical dimension is where the Yukawa coupling is marginal. Below the critical dimension the Yukawa coupling is relevant and above it the coupling is irrelevant.

Renormalization group strategy for the Boson and Fermion employed here

The renormalization group structure used in [17] and which we have here differs from standard treatments in relativistic quantum field theory. We will introduce a cut-off scale Λ only for the frequency modes, and we integrate over the frequency modes w between the intervals

$$e^{-t}\Lambda < w < \Lambda. \quad (3.48)$$

We don't put a momenta cut-off scale. Instead we let the modes in the shell take all possible momenta values, and we integrate over all these momenta modes

$$0 \leq |\vec{k}| \leq \infty \quad \text{and} \quad -\infty < \vec{l} < \infty. \quad (3.49)$$

The renormalization strategy employed here is to partition the momentum space into thin layers of frequency modes $(1-t)\Lambda < w < \Lambda$ with any $\vec{k}$ for $t \ll 1$, as in the figure below.

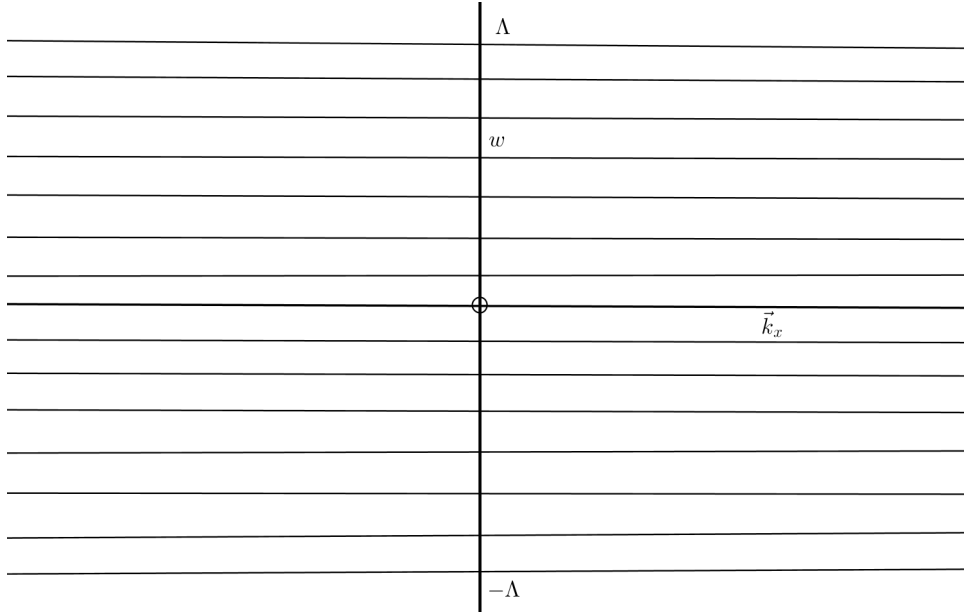


Figure 3.7: Slices of frequency-momentum modes in two dimensions.

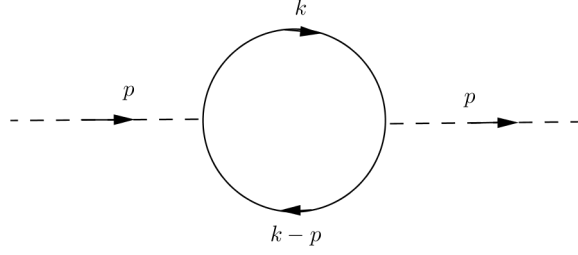
The problem with integrating over all possible momenta is that we will get divergences. This kind of renormalization strategy is in contrast with the Wilson's renormalization introduced earlier. In Wilson's renormalization we introduced a scale cut-off Λ for both the frequency and momenta modes, and we decimated shells with frequency and momenta modes between

$$e^{-t}\Lambda < w < \Lambda \quad , \quad e^{-t}\Lambda < |\vec{k}| < \Lambda \quad \text{and} \quad e^{-t}\Lambda < |\vec{l}| < \Lambda. \quad (3.50)$$

Wilson's procedure does not integrate over all possible momenta. In this case we don't get divergent loop integrals.

3.3 Renormalization at one loop

Boson self-energy diagram



In our notation a solid line represents a fermion propagator and a dash line represents a boson propagator. We compute $\Pi(p)$, the self-energy of a boson. To do this we need to evaluate the loop integral

$$\Pi(p) = g^2 \int_{e^{-t}\Lambda}^{\Lambda} \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{iw + \mu - \epsilon(\vec{k})} \frac{1}{i(w - p_0) + \mu - \epsilon(\vec{k} - \vec{p})}. \quad (3.51)$$

We integrate modes in frequency shells and over all spatial momenta as reviewed above

$$e^{-t}\Lambda < w < \Lambda \quad \text{and} \quad 0 \leq |\vec{k}| < \infty, \quad (3.52)$$

where Λ is the ultra violet (UV) cut-off scale and the external propagators have frequency modes $w < e^{-t}\Lambda$. For small $t = \delta$ we expand the exponential as $e^{-\delta} = 1 - \delta$. Then

$$\Pi(p) = g^2 \int_{(1-\delta)\Lambda}^{\Lambda} \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{iw + \mu - \epsilon(\vec{k})} \frac{1}{i(w - p_0) + \mu - \epsilon(\vec{k} - \vec{p})}. \quad (3.53)$$

Since $\epsilon(\vec{k}) = \frac{\vec{k} \cdot \vec{k}}{2m}$ and $\vec{k} = \hat{\Omega}(k_F + l)$, the first denominator of the integral can be simplified as follows

$$iw + \mu - \epsilon(\vec{k}) = iw + \mu - \frac{\hat{\Omega}(k_F + l) \cdot \hat{\Omega}(k_F + l)}{2m} \quad (3.54)$$

$$= iw + \mu - \frac{k_F^2 + 2k_F l + l^2}{2m}. \quad (3.55)$$

The chemical potential energy $\mu = \frac{k_F^2}{2m}$ and for an excitation close to a Fermi-surface, since $k_F \gg l$, we neglect the l^2 term in (3.55). Hence

$$iw + \mu - \epsilon(\vec{k}) = iw - v_F l, \quad (3.56)$$

where $v_F = \frac{k_F}{m}$. The second denominator can be simplified as follows

$$i(w - p_0) + \mu - \epsilon(\vec{k} + \vec{p}) = i(w - p_0) + \mu - \frac{(\vec{k} - \vec{p}) \cdot (\vec{k} - \vec{p})}{2m} \quad (3.57)$$

$$= i(w - p_0) + \mu - \frac{\vec{k} \cdot \vec{k} - 2\vec{k} \cdot \vec{p} + \vec{p} \cdot \vec{p}}{2m} \quad (3.58)$$

$$= i(w - p_0) + \mu - \frac{\hat{\Omega}(k_F + l) \cdot \hat{\Omega}(k_F + l) - 2\hat{\Omega}(k_F + l) \cdot \vec{p} + \vec{p} \cdot \vec{p}}{2m} \quad (3.59)$$

We are working in a region where the boson energy and momenta $(p_0, \vec{p})$ are very small compared to the Fermi-surface momentum k_F . We can drop the terms $\vec{p}^2$, $l\hat{\Omega} \cdot \vec{p}$ and l^2 since they are very small compared to the remaining terms. Then

$$i(w - p_0) + \mu - \epsilon(\vec{k} - \vec{p}) = i(w - p_0) + \mu - \frac{k_F^2}{2m} - \frac{k_F l}{m} + \frac{k_F \hat{\Omega} \cdot \vec{p}}{m} \quad (3.60)$$

$$= i(w - p_0) - v_F l + v_F \hat{\Omega} \cdot \vec{p}. \quad (3.61)$$

Note that $\int d^{d+1}k = \int dw \int d^{d-1}\hat{\Omega} \int d(k_F + l)(k_F + l)^{d-1}$. Therefore

$$\Pi(p) = g^2 \int_{(1-\delta)\Lambda}^{\Lambda} dw \int_{-\infty}^{\infty} dl \int \frac{d^{d-1}\hat{\Omega}}{(2\pi)^{d+1}} \frac{(k_F + l)^{d-1}}{iw - v_F l} \frac{1}{i(w - p_0) - v_F l + v_F \hat{\Omega} \cdot \vec{p}}. \quad (3.62)$$

since $k_F \gg l$ we can make an approximation

$$\Pi(p) \approx g^2 \int_{(1-\delta)\Lambda}^{\Lambda} dw \int_{-\infty}^{\infty} dl \int \frac{d^{d-1}\hat{\Omega}}{(2\pi)^{d+1}} \frac{k_F^{d-1}}{iw - v_F l} \frac{1}{i(w - p_0) - v_F l + v_F \hat{\Omega} \cdot \vec{p}}. \quad (3.63)$$

Assuming the dw interval of integration is small compared to the scale over which $f(x)$ varies, we use

$$\int_{(1-\delta)\Lambda}^{\Lambda} f(x) dx = \delta \Lambda f(\Lambda), \quad (3.64)$$

to evaluate the dw integral. We find that

$$\Pi(p) = g^2 \delta \Lambda \int \frac{dl d^{d-1}\hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \frac{1}{i\Lambda - v_F l} \frac{1}{i(\Lambda - p_0) - v_F l + v_F \hat{\Omega} \cdot \vec{p}}. \quad (3.65)$$

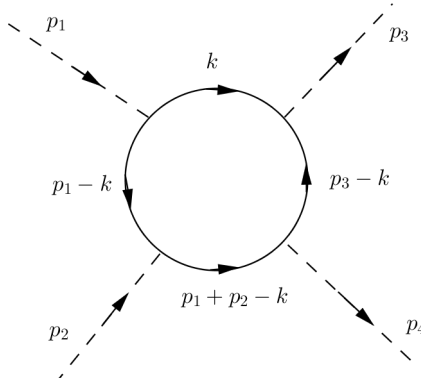
The integral has poles at $l = i\frac{\Lambda}{v_F}$ and $l = \frac{i(\Lambda - p_0)}{v_F} + \hat{\Omega} \cdot \vec{p}$. For $\Lambda > |p_0|$ both poles are on the same side of the real axis. Closing the contour anti-clockwise in the upper l complex plane we obtain

$$\Pi(p) = g^2 2\pi i \delta \Lambda \int \frac{d^{d-1}\hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \left(\frac{1}{i\Lambda - i(\Lambda - p_0) + v_F \hat{\Omega} \cdot \vec{p}} + \frac{1}{i(\Lambda - p_0) - i\Lambda - v_F \hat{\Omega} \cdot \vec{p}} \right) \quad (3.66)$$

$$\begin{aligned} &= g^2 2\pi i \delta \Lambda \int \frac{d^{d-1}\hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} (0) \\ &= 0. \end{aligned} \quad (3.67)$$

The dl integral evaluates to zero since when both poles are on the same side of the real axis, we sum all the residues which gives zero.

Fermion loop dressing of Bosonic interaction



To calculate the one loop renormalization of the quartic coupling λ_ϕ due to the fermion loop we first amputate the external legs as before and then compute

$$\delta\lambda_{\phi_1} = g^4 \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{-iw + \mu - \epsilon(\vec{k})} \frac{1}{i(w_1 - w) + \mu - \epsilon(\vec{p}_1 - \vec{k})} \times \frac{1}{i(w_3 - w) + \mu - \epsilon(\vec{p}_3 - \vec{k})} \frac{1}{i(w_1 + w_2 - w) + \mu - \epsilon(\vec{p}_1 + \vec{p}_2 - \vec{k})}. \quad (3.68)$$

The UV divergences are insensitive to the external momenta, so we set them to zero. Then

$$\delta\lambda_{\phi_1} = g^4 \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{(iw + \mu - \epsilon(\vec{k}))^4}. \quad (3.69)$$

$$(3.70)$$

Note that $\int d^{d+1}k = \int dw \int d^{d-1}\hat{\Omega} \int d(k_F + l)(k_F + l)^{d-1}$ so that we find

$$\delta\lambda_{\phi_1} = \frac{g^4}{(2\pi)^{d+1}} \int_{(1-\delta)\Lambda}^{\Lambda} dw \int d^{d-1}\hat{\Omega} \int_{-\infty}^{\infty} dl \frac{(k_F + l)^{d-1}}{(iw - v_F l)^4} \quad (3.71)$$

$$\approx \frac{g^4}{(2\pi)^{d+1}} \int_{(1-\delta)\Lambda}^{\Lambda} dw \int k_F^{d-1} d^{d-1}\hat{\Omega} \int_{-\infty}^{\infty} dl \frac{1}{(iw - v_F l)^4}. \quad (3.72)$$

We will again assume $\delta\Lambda$ is small enough that

$$\int_{(1-\delta)\Lambda}^{\Lambda} f(x) dx = \delta\Lambda f(\Lambda). \quad (3.73)$$

After we evaluate the dw integral we find

$$\delta\lambda_{\phi_1} = \frac{g^4}{(2\pi)^{d+1}} \delta\Lambda \int k_F^{d-1} d^{d-1}\hat{\Omega} \int dl \frac{1}{(i\Lambda - v_F l)^4}. \quad (3.74)$$

For $d = 3$ we have

$$\delta\lambda_{\phi_1} = \frac{g^4}{(2\pi)^4} \delta\Lambda \int k_F^2 d^2\hat{\Omega} \int dl \frac{1}{(i\Lambda - v_F l)^4}. \quad (3.75)$$

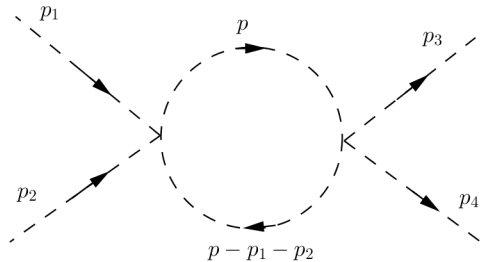
Adding the negative thin shells energy $-\Lambda < w < -\Lambda(1 - \delta)$ to the $\delta\lambda_{\phi_1}$, we find that

$$\delta\lambda_{\phi_1} = \frac{g^4}{(2\pi)^4} \delta\Lambda \int k_F^2 d^2\hat{\Omega} dl \frac{1}{(i\Lambda - v_F l)^4} + (\Lambda \rightarrow -\Lambda) \quad (3.76)$$

$$= 2 \frac{g^4}{(2\pi)^4} \delta\Lambda \int k_F^2 d^2\hat{\Omega} dl \frac{1}{(i\Lambda - v_F l)^4}. \quad (3.77)$$

The integral has a pole of order 4 at $l = \frac{i\Lambda}{v_F}$. The dl integral evaluates to zero since the integral has all the poles on one side of the real axis.

Boson loop dressing of Bosonic interaction



$$\delta\lambda_{\phi_2} = \frac{\lambda_\phi^2}{2} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{p_0^2 + \vec{p} \cdot \vec{p} + m^2} \frac{1}{(p_0 - p_{01} - p_{02})^2 + (\vec{p} - \vec{p}_1 - \vec{p}_2)^2 + m^2}. \quad (3.78)$$

The UV divergences are insensitive to the external momenta, so we again neglect or set them to zero, along with m_ϕ the mass of the bosons. For $d = 3$ the integral becomes

$$\delta\lambda_{\phi_2} = \frac{\lambda_\phi^2}{2} \int_{(1-\delta)\Lambda}^\Lambda dp_0 \int \frac{d^3p}{(2\pi)^4} \frac{1}{(p_0^2 + \vec{p}^2)^2} \quad (3.79)$$

$$= \frac{\lambda_\phi^2}{2} \int \frac{d^3p}{(2\pi)^4} \frac{1}{(\Lambda^2 + \vec{p}^2)^2} \delta\Lambda. \quad (3.80)$$

Since the integral is rotationally invariant, we find

$$\delta\lambda_{\phi_2} = 2 \frac{\lambda_\phi^2}{2} \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^2}{(\Lambda^2 + p^2)} \delta\Lambda. \quad (3.81)$$

The factor of 2 in front of the integral comes from integrating both the positive and negative energy shells. The integral has a double pole at $p = \pm i\Lambda$. This loop integral has two contributions, one contribution comes from the positive energy shell and the other contribution from the negative energy shell. We will evaluate the integral over p by extending the integral from $p = -\infty$ to $p = \infty$ and then closing in the upper half plane. The residues of the integral can be evaluated using the general formula.

$$Res[f, z_0] = \frac{1}{(k-1)!} \lim_{z \rightarrow z_0} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)] \quad (3.82)$$

where z_0 is the pole of the function f and k is the order of the pole. The positive energy shell pole at $p = i\Lambda$ contributes when we close the contour anticlockwise to obtain

$$2\pi i Res[\delta\lambda_{\phi_2}, i\Lambda], \quad (3.83)$$

where

$$Res[\delta\lambda_{\phi_2}, i\Lambda] = \lim_{p \rightarrow i\Lambda} \frac{d}{dp} \left[(p - i\Lambda)^2 \frac{p^2}{(p - i\Lambda)^2 (p + i\Lambda)^2} \right] \quad (3.84)$$

$$= \lim_{p \rightarrow i\Lambda} \left(\frac{2p(p + i\Lambda)^2 - 2p^2(p + i\Lambda)}{(p + i\Lambda)^4} \right) \quad (3.85)$$

$$= -\frac{i}{4\Lambda}. \quad (3.86)$$

For the negative energy shell, we also close the pole at $p = i\Lambda$ anticlockwise, to obtain

$$2\pi i Res[\delta\lambda_{\phi_2}, i\Lambda], \quad (3.87)$$

where

$$Res[\delta\lambda_{\phi_2}, -i\Lambda] = \frac{-i}{4\Lambda}. \quad (3.88)$$

Adding the positive and negative energy shell contributions, we find that

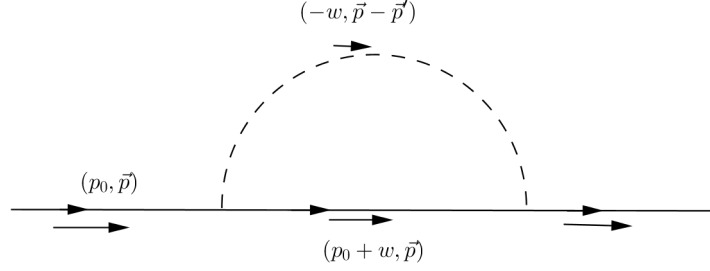
$$\delta\lambda_{\phi_2} = \left(\underbrace{-2\pi i \frac{\lambda_\phi^2}{2(2\pi)^3} \left(\frac{i}{4\Lambda} \right)}_{\text{from negative energy shell}} + \underbrace{2\pi i \frac{\lambda_\phi^2}{2(2\pi)^3} \left(\frac{-i}{4\Lambda} \right)}_{\text{from positive energy shell}} \right) \delta\Lambda \quad (3.89)$$

$$= 2 \frac{\lambda_\phi^2}{2(16\pi^2)} \frac{\delta\Lambda}{\Lambda}. \quad (3.90)$$

Adding the two other similar diagrams which come from the s- and t-channels we get

$$\delta\lambda_{\phi_2} = \frac{3\lambda_\phi^2}{(16\pi^2)} \frac{\delta\Lambda}{\Lambda}. \quad (3.91)$$

Fermion Wavefunction renormalization at one loop



Consider a fermion with frequency p_0 and momentum $\vec{p} = \hat{\Omega}(k_F + \tilde{l})$. We let the internal fermion momenta have frequency $p_0 + w$ and momentum $\vec{p}' = \hat{\Omega}'(k_F + l)$. Amputating the external legs, the loop integral is

$$\Sigma(p_0, \tilde{l}) = -g^2 \int \frac{dw dl d^{d-1} \Omega' k_F^{d-1}}{(2\pi)^{d+1}} \frac{1}{w^2 + c^2(\vec{p} - \vec{p}')^2 + m_\phi^2} \frac{1}{i(w + p_0) - v_F l}. \quad (3.92)$$

We will neglect the boson mass. The boson momentum squared is

$$\begin{aligned} (\vec{p} - \vec{p}') \cdot (\vec{p} - \vec{p}') &= \vec{p} \cdot \vec{p} - |\vec{p}| |\vec{p}'| \cos \theta + \vec{p}' \cdot \vec{p}' \\ &= (k_F + \tilde{l})^2 - 2(k_F + \tilde{l})(k_F + l) \cos \theta + (k_F + l)^2 \\ &= k_F^2 + 2k_F \tilde{l} + \tilde{l}^2 - 2(k_F^2 + k_F l + k_F \tilde{l} + l \tilde{l}) \cos \theta + k_F^2 + 2k_F l + l^2 \\ &= 2k_F^2(1 - \cos \theta) + (l - \tilde{l})^2 + (2k_F(l + \tilde{l}) + 2l\tilde{l})(1 - \cos \theta). \end{aligned} \quad (3.93)$$

For a small angle between the boson and fermion the first term above is very large and will suppress contributions which dress the fermions. The only contributions come from small angles so that the last term above is very small $1 - \cos \theta \sim \frac{1}{k_F^2}$. Therefore the contribution of the last term in (3.94) is $\frac{1}{k_F}$ suppressed and we can safely neglect it [17]. We shift the integration variable $l \rightarrow l + \tilde{l}$ to get

$$\Sigma(p_0, \tilde{l}) = -g^2 \int \frac{dw dl d^{d-1} \hat{\Omega} k_F^{d-1}}{(2\pi)^{d+1}} \frac{1}{w^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{i(w + p_0) - v_F(l + \tilde{l})}. \quad (3.94)$$

Define $m = ip_0 - v_F \tilde{l}$ to obtain

$$\Sigma(m) = -g^2 \int \frac{dw dl k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{w^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{iw - v_F l + m}. \quad (3.95)$$

Integrate with respect to w over the interval $(1 - \delta)\Lambda \leq w < \Lambda$ at $d = 3$ to obtain

$$\begin{aligned}
\Sigma(m) &= -g^2 \delta \Lambda \int \frac{dk_F^2 d^2 \hat{\Omega}}{(2\pi)^4} \frac{1}{\Lambda^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{i\Lambda - v_F l + m} \\
&= -g^2 \delta \Lambda \int \frac{dk_F^2 d\phi \sin \theta d\theta}{(2\pi)^4} \frac{1}{\Lambda^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{i\Lambda - v_F l + m}.
\end{aligned} \tag{3.96}$$

Performing the $d\phi$ integral from 0 to 2π we get

$$\Sigma(m) = -g^2 \delta \Lambda \int \frac{dk_F^2 \sin \theta d\theta}{(2\pi)^3} \frac{1}{\Lambda^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{i\Lambda - v_F l + m}. \tag{3.97}$$

Using the identities

$$\int_{-\pi}^{\pi} \sin \theta d\theta = -2 \int_{-\pi}^{\pi} d \sin^2 \frac{\theta}{2}, \tag{3.98}$$

and

$$(1 - \cos \theta) = 2 \sin^2 \frac{\theta}{2}, \tag{3.99}$$

we find

$$\begin{aligned}
\Sigma(m) &= 2g^2 \delta \Lambda \int \frac{dk_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\Lambda^2 + c^2 l^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} \frac{1}{i\Lambda - v_F l + m} \\
&= 2g^2 \delta \Lambda \int \frac{dk_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{c^2 \left(l + i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} \right)} \frac{1}{\left(l - i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} \right)} \frac{1}{i\Lambda - v_F l + m}.
\end{aligned} \tag{3.100}$$

The integral has poles at $l = \pm i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}$ and $l = \frac{i\Lambda + m}{v_F}$. The loop integral has two contributions, one coming from the positive energy shell and the other coming from the negative energy shell. Closing the contour anti-clockwise in the upper half complex plane, we obtain

$$2\pi i \text{Res}(\Sigma(m)) \tag{3.101}$$

where

$$\begin{aligned}
Res(\Sigma(m)) &= 2g^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{2ic^2\sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{v_F \left(-i\sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} + \frac{i\Lambda}{v_F} + \frac{m}{v_F} \right)} \\
&\quad - 2g^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{v_F \left(\Lambda^2 + c^2 \left(\frac{i\Lambda+m}{v_F} \right)^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2} \right)} \\
&= 2g^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{i\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{(-i\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} + ic\Lambda + cm)} \\
&\quad - 2g^2\delta \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{v_F}{ic\Lambda + cm + iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}} \times \\
&\quad \frac{1}{ic\Lambda + cm - iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}} \\
&= -ig^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} (ic\Lambda + cm + iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}})} \times \\
&\quad \frac{ic\Lambda + cm + iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} - 2iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}}{(ic\Lambda + cm - iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}})} \\
&= -ig^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} (ic\Lambda + cm + iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}})}.
\end{aligned} \tag{3.102}$$

This is the contribution from the positive energy shell $\Lambda(1-\delta) < w < \Lambda$. For the negative energy shell $-\Lambda(1-\delta) < w < -\Lambda$, the poles are at $l = \pm i\sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}$ and $l = \frac{m-i\Lambda}{v_F}$. We close the contour line in the same way as in positive energy shell. We obtain

$$Res(\Sigma(m)) = -2g^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{2ic^2\sqrt{\frac{\Lambda^2}{c^2} + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{v_F \left(-i\sqrt{\frac{\Lambda^2}{c^2} + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} - \frac{i\Lambda}{v_F} + \frac{m}{v_F} \right)}. \tag{3.103}$$

Therefore the sum from both the positive and negative contribution are therefore

$$\begin{aligned}
Res(\Sigma(m)) &= -ig^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} (ic\Lambda + cm + iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}})} \\
&\quad + ig^2\delta\Lambda \int \frac{k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} (-ic\Lambda + cm - iv_F\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}})}.
\end{aligned} \tag{3.104}$$

The first integral above is evaluated as follows

$$\text{let } u = \sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} \tag{3.105}$$

$$\Rightarrow \frac{du}{d\sin^2 \frac{\theta}{2}} = \frac{2c^2 k_F^2}{\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}}. \tag{3.106}$$

substituting this to the integral we get

$$\begin{aligned} & i \frac{g^2 \delta \Lambda}{2c^2} \int_0^1 \frac{du}{(2\pi)^3} \frac{1}{ic\Lambda + cm + iv_F u} \\ &= \frac{g^2 \delta \Lambda}{2(2\pi)^3 c^2 v_F} \ln \left(\frac{iv_F u + ic\Lambda + cm}{ic\Lambda + cm} \right) \Big|_0^1, \end{aligned} \quad (3.107)$$

substituting back we get

$$\frac{g^2 \delta \Lambda}{2(2\pi)^3 c^2 v_F} \ln \left(\frac{icv_F \sqrt{\Lambda^2 + 4c^2 k_F^2} + ic\Lambda + cm}{iv_F \Lambda + ic\Lambda + cm} \right). \quad (3.108)$$

The second integral of (3.104) is evaluated as

$$\frac{g^2 \delta \Lambda}{2(2\pi)^3 c^2 v_F} \ln \left(\frac{-icv_F \sqrt{\Lambda^2 + 4c^2 k_F^2} - ic\Lambda + cm}{-iv_F \Lambda - ic\Lambda + cm} \right). \quad (3.109)$$

Therefore

$$\begin{aligned} \Sigma(m) = & i \frac{g^2 \delta \Lambda}{2(2\pi)^2 c^2 v_F} \ln \left(\frac{icv_F \sqrt{\Lambda^2 + 4c^2 k_F^2} + ic\Lambda + cm}{iv_F \Lambda + ic\Lambda + cm} \right) \\ & + i \frac{g^2 \delta \Lambda}{2(2\pi)^2 c^2 v_F} \ln \left(\frac{-icv_F \sqrt{\Lambda^2 + 4c^2 k_F^2} - ic\Lambda + cm}{-iv_F \Lambda - ic\Lambda + cm} \right). \end{aligned} \quad (3.110)$$

The wave function renormalization will depend on the term linear in m . For convenience we will evaluate $\Sigma'(0)$, since this expression will be useful in calculations that appear later. Starting from (3.95) we find

$$\left. \frac{d\Sigma(m)}{dm} \right|_{m=0} = g^2 \delta \Lambda \int \frac{dl 2k_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\Lambda^2 + c^2 l^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} \frac{1}{(i\Lambda - v_F l)^2} = g^2 \mathcal{I} \quad (3.111)$$

where

$$\begin{aligned} \mathcal{I} = & \delta \Lambda \int \frac{dl 2k_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{\Lambda^2 + c^2 l^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}} \frac{1}{(i\Lambda - v_F l)^2} \\ = & \delta \Lambda \int \frac{dl k_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{c^2 \left(l + i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} \right)} \frac{1}{\left(l - i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} \right)} \frac{1}{\left(i\Lambda - v_F l \right)^2}. \end{aligned} \quad (3.112)$$

The integral $\mathcal{I}$ has poles at $l = \pm i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}$ and a double pole at $l = \frac{i\Lambda}{v_F}$. Closing the contour of integration clockwise in the lower half plane, we obtain

$$2\pi i \text{Res}(\Sigma'(0)) \quad (3.113)$$

where

$$\text{Res}(\Sigma'(0)) = \delta \Lambda \int \frac{2k_F^2 d \sin^2 \frac{\theta}{2}}{(2\pi)^3} \frac{1}{2ic^2 \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{\left(i\Lambda - v_F i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}} \right)^2}. \quad (3.114)$$

This is a contribution from the positive energy shell. For the negative energy shell we close the contour line the same way and we obtain the same answer. Combining both contributions we get

$$\begin{aligned}
\mathcal{I} &= \delta\Lambda \int \frac{2k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^2} \frac{1}{2c^2 \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{\left(i\Lambda - v_F i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}\right)^2} \\
&+ \delta\Lambda \int \frac{2k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^2} \frac{1}{2c^2 \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{\left(i\Lambda - v_F i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}\right)^2} \\
&= \delta\Lambda \int \frac{2k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^2} \frac{1}{c^2 \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}} \frac{1}{\left(i\Lambda - v_F i \sqrt{\frac{\Lambda^2}{c^2} + 4k_F^2 \sin^2 \frac{\theta}{2}}\right)^2}.
\end{aligned} \tag{3.115}$$

Multiply the right hand side by $\frac{c\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}}{c\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}}$ we get

$$\mathcal{I} = -\delta\Lambda \int \frac{2k_F^2 d\sin^2 \frac{\theta}{2}}{(2\pi)^2} \frac{c\sqrt{\Lambda^2 + 4c^2 k_F^2 \sin^2 \frac{\theta}{2}}}{\left(4c^2 k_F^2 \sin^2 \frac{\theta}{2} v_F + v_F \Lambda^2 + c\Lambda \sqrt{4c^2 k_F^2 \sin^2 \frac{\theta}{2} + \Lambda^2}\right)^2}. \tag{3.116}$$

This integral can be evaluated with a change of variables. Let

$$u = \sqrt{4c^2 k_F^2 \sin^2 \frac{\theta}{2} + \Lambda^2}. \tag{3.117}$$

Then

$$\frac{du}{d\sin^2 \frac{\theta}{2}} = \frac{2c^2 k_F^2}{\sqrt{4c^2 k_F^2 \sin^2 \frac{\theta}{2} + \Lambda^2}}. \tag{3.118}$$

Consequently

$$\begin{aligned}
-\delta\Lambda \int_0^1 \frac{du}{(2\pi)^2} \frac{u^2}{c((u^2 - \Lambda^2)v_F + v_F \Lambda^2 + c\Lambda u)^2} &= -\delta\Lambda \int_0^1 \frac{du}{(2\pi)^2} \frac{u^2}{c(u^2 v_F + c\Lambda u)^2} \\
&= -\delta\Lambda \int_0^1 \frac{du}{(2\pi)^2} \frac{1}{c(uv_F + c\Lambda)^2} \\
&= \frac{\delta\Lambda}{(2\pi)^2 c v_F} \frac{1}{(uv_F + c\Lambda)} \Big|_0^1.
\end{aligned} \tag{3.119}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{I} &= \frac{\delta\Lambda}{(2\pi)^2 c v_F} \frac{1}{(v_F \sqrt{4c^2 k_F^2 \sin^2 \frac{\theta}{2} + \Lambda^2} + c\Lambda)} \Big|_0^1 \\
&= \frac{\delta\Lambda}{(2\pi)^2 c v_F} \left[\frac{1}{(v_F \sqrt{4c^2 k_F^2 + \Lambda^2} + c\Lambda)} - \frac{1}{v_F \Lambda + c\Lambda} \right].
\end{aligned} \tag{3.120}$$

When $ck_F \gg \Lambda$, the first function is very small compared to the second one. Therefore we neglect it. Then

$$\mathcal{I} = -\frac{1}{4\pi^2 c v_F} \left(\frac{1}{v_F + c} \right) \frac{\delta\Lambda}{\Lambda}, \tag{3.121}$$

and

$$\begin{aligned}\Sigma'(0) &= g^2 \mathcal{I} \\ &= -\frac{g^2}{4\pi^2 c v_F} \left(\frac{1}{v_F + c} \right) \frac{\delta \Lambda}{\Lambda}.\end{aligned}\tag{3.122}$$

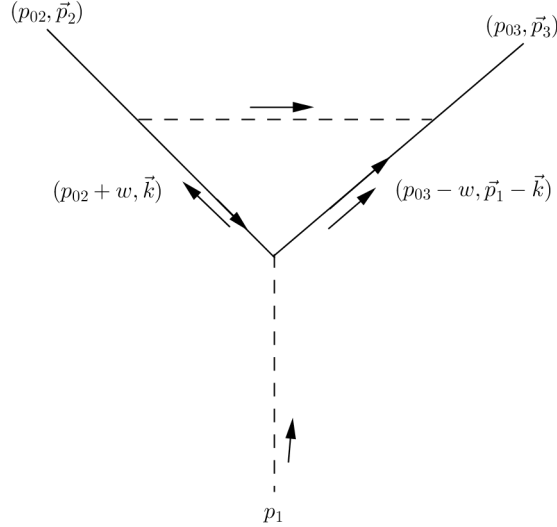
We will define

$$a_g = \frac{1}{4\pi^2 c v_F (c + v_F)}.\tag{3.123}$$

Note that if we take (3.110) and compute the derivative with respect to m in the limit $ck_F \gg \Lambda$ we obtain the same results as in (3.122),

$$\left. \frac{d\Sigma(m)}{dm} \right|_0 = -\frac{g^2}{4\pi^2 c v_F} \left(\frac{1}{v_F + c} \right) \frac{\delta \Lambda}{\Lambda}.\tag{3.124}$$

One loop Yukawa vertex correction



Consider an incoming boson with momenta splitting into a fermion and anti-fermion with energy momenta $(p_{02} + w, \vec{k})$ and $(p_{03} - w, \vec{p}_1 - \vec{k})$ respectively. The fermion and anti-fermion exchange a boson with $(w, \vec{p}_2 - \vec{k})$ where $\vec{p}_2 = \hat{\Omega}'(k_F + l)$ and $\vec{k} = \hat{\Omega}(k_F + l)$, the boson momentum squared is

$$\begin{aligned}(\vec{p}_2 - \vec{k})^2 &= \vec{p}_2^2 - 2\vec{p}_2 \cdot \vec{k} + \vec{k}^2 \\ &= (k_F + \tilde{l})(k_F + \tilde{l}) - 2(k_F + \tilde{l})(k_F + l) \cos \theta + (k_F + l)(k_F + l) \\ &= 2k_F^2(1 - \cos \theta) + (l - \tilde{l})^2 + (2k_F(l + \tilde{l}) + 2\tilde{l}l)(1 - \cos \theta).\end{aligned}\tag{3.125}$$

We make the same argument as before, that $1 - \cos \theta \sim \frac{1}{k_F^2}$ if the diagram is not suppressed, so the last term is $\frac{1}{k_F^2}$ suppressed and we neglect it. We shift the variable $\rightarrow l + \tilde{l}$, so that the one loop correction to the Yukawa coupling is

$$\begin{aligned}\delta g &= g^3 \int \frac{dw dl k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{w^2 + c^2 l^2 + 2c^2 k_F^2 (1 - \cos \theta)} \frac{1}{i(w + p_{02}) + \mu - \epsilon(\vec{k})} \times \\ &\quad \frac{1}{i(-w + p_{03}) + \mu - \epsilon(\vec{p}_1 - \vec{k})}.\end{aligned}\tag{3.126}$$

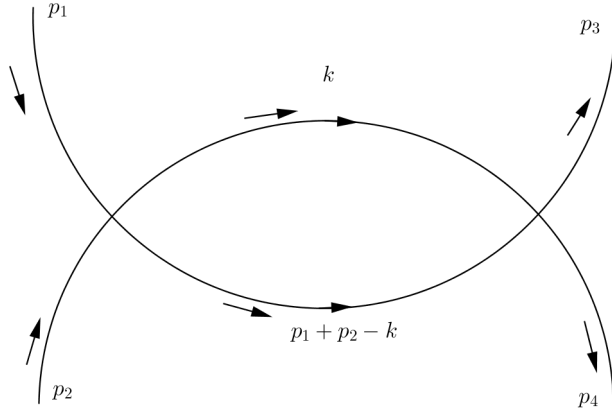
The divergent part this integral is independent of the external momenta, so we set the external momenta zero, to obtain

$$\delta g = g^3 \int \frac{dw dl k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{w^2 + c^2 l^2 + 2c^2(1 - \cos \theta)} \frac{1}{(iw - v_F l)^2}. \quad (3.127)$$

This integral has already been considered in (3.116). From that analysis we find

$$\begin{aligned} \delta g &= g^3 \mathcal{I} \\ &= - \frac{g^3}{4\pi^2 c v_F (c + v_F)} \frac{\delta \Lambda}{\Lambda}. \end{aligned} \quad (3.128)$$

The four Fermion one loop vertex correction to the BCS coupling



The one loop integral is

$$\delta \lambda_\psi = \lambda_\psi^2 \int \frac{dl dw k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{(iw - v_F l)(-iw - v_F l)}. \quad (3.129)$$

We are again only interested in the divergent part of this diagram since this is all that is required to determine the renormalization group flow. Consequently we again set the external momenta to zero. Evaluating the dw integral with range $(1 - \delta)\Lambda < w < \Lambda$ at $d = 3$, we find

$$\delta \lambda_\psi = \lambda_\psi^2 \delta \Lambda \int \frac{dl k_F^2 d^2 \hat{\Omega}}{(2\pi)^4} \frac{1}{(i\Lambda - v_F l)} \frac{1}{(-i\Lambda - v_F l)}, \quad (3.130)$$

the integral has poles at $l = \pm \frac{i\Lambda}{v_F}$. Closing the contour in the upper half plane we get

$$-2\pi i \text{Res} \left(\delta \lambda_\psi, \frac{i\Lambda}{v_F} \right), \quad (3.131)$$

thus

$$-2\pi i \lambda_\psi^2 \int \frac{k_F^2 d^2 \hat{\Omega}}{(2\pi)^4} \left(\frac{\delta \Lambda}{-2i v_F \Lambda} \right). \quad (3.132)$$

The negative energy shell gives an identical contribution. Consequently

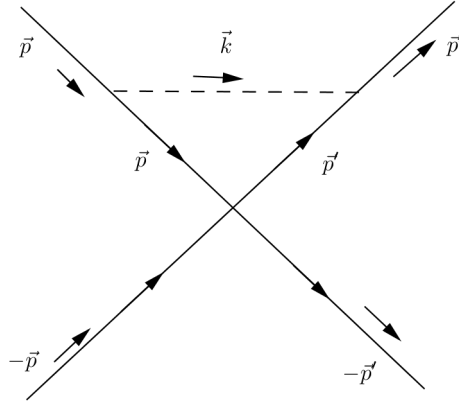
$$\begin{aligned}
\delta\lambda_\psi &= \lambda_\psi^2 \int \frac{k_F^2 d^2\hat{\Omega}}{(2\pi)^3} \left(\frac{\delta\Lambda}{2v_F\Lambda} \right) + \lambda_\psi^2 \int \frac{k_F^2 d^2\hat{\Omega}}{(2\pi)^3} \left(\frac{\delta\Lambda}{2v_F\Lambda} \right) \\
&= \lambda_\psi^2 \int \frac{k_F^2 d^2\hat{\Omega}}{(2\pi)^3} \left(\frac{\delta\Lambda}{v_F\Lambda} \right).
\end{aligned} \tag{3.133}$$

We will now evaluate the angular integral

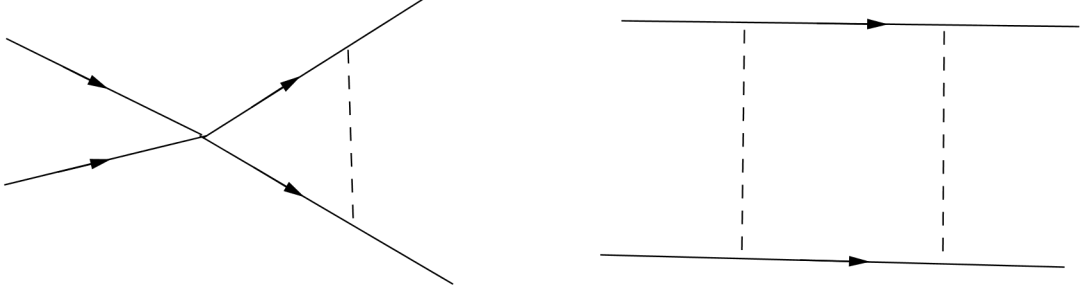
$$\begin{aligned}
\delta\lambda_\psi &= \lambda_\psi^2 k_F^2 \int_0^{2\pi} d\phi \int_0^\pi \frac{d\theta \sin\theta}{(2\pi)^3} \frac{\delta\Lambda}{v_F\Lambda} \\
&= \frac{\lambda_\psi^2 k_F^2}{2\pi^2 v_F} \frac{\delta\Lambda}{\Lambda}.
\end{aligned} \tag{3.134}$$

Note that spinless fermions have an additional symmetry factor of $\frac{1}{2}$.

Four Fermion vertex correction diagram



This diagram gives no divergence because of the kinematic constraints of the Fermi-surface. The reason is that the four-fermion interaction is marginal if the external momenta are such that the momenta of two lines in a four-fermion vertex sum to zero. The diagram above will only lead to a marginal interaction if the internal fermions have momenta $\vec{p}$ and $\vec{p}'$ such that $\vec{p} - \vec{p} = \vec{p}' - \vec{p}' = 0$ at the vertex. Since the external fermion momenta are $\vec{p}$ and $\vec{p}'$, the transferred boson is constrained to have momentum $\vec{k} = 0$. Since the boson momentum transferred is $\vec{k} = 0$, this constrains the angular integration measure to zero. Another way to view it is: the four-fermion momenta connects antipodal points of the Fermi-surface in the BCS interaction and the internal fermions must be $\vec{p}$ and $\vec{p}'$ as indicated in the figure above. The interaction being renormalized also connects the antipodal points, so the external fermions at the top of the diagram will also have momenta $\vec{p}$ and $\vec{p}'$. This constrains the transferred boson momentum to zero. Having said this, note that the previous vertex correction diagram is irrelevant since there are no two vertex lines with momenta summing to zero. The other four fermion vertex correction diagrams at one loop are



The loop integrals of these diagrams produce $\log^2$ and $\log^3$ divergences respectively. The study of these diagrams is beyond the scope of the study of this chapter. One thing to note though is that the first diagram is irrelevant by the argument stated above. The $\log^2$ and $\log^3$ divergences will be addressed properly in chapter 5.

3.4 Chapter Summary

Basically the methodology of renormalization group flow involves dividing up the complete set of energy modes of a field into distinct shells of modes of similar energy. After that we integrate over the shells which correspond to the high energy modes to obtain an effective theory for the low energy modes. In doing so, some parameters in the action begin to flow. Marginal parameters remain constant, irrelevant parameters flow to zero and relevant parameters flow to infinity as we flow to low energy. It is important to fine tune relevant terms (so that they disappear from the low energy theory) in the action to avoid infinities creeping into the theory. The renormalization group flow for non-relativistic finite density systems is different from the relativistic zero density theory because in the non-relativistic theory states which define the behaviour of a system are close to the Fermi surface. This makes both the mathematical and geometrical setup of a system different from the ordinary relativistic theory. Also, Lorentz invariance is broken in non-relativistic finite density systems. As we have seen in Euclidean space, low energy modes for fermions are located close to the Fermi surface and high energy modes are located further from the Fermi surface. For the bosons, high energy modes are located away from the origin and low energy modes close to the origin. Any interaction involving momenta that are tangential to the Fermi surface are marginal for the four Fermi coupling in $d = 3$ dimensions. Further, only interactions between fermions close to the Fermi-surface, with small boson momentum transfer, are relevant below $d = 3$ dimensions and are marginal in $d = 3$. The same applies for scalar self-coupling: the interaction is marginal in $d = 3$ and relevant below $d = 3$. We adopted the renormalization strategy used in [17], where we rescale the fermions towards the Fermi surface and the bosons towards the origin. We decimated the shell modes in a following manner

$$0 \leq |\vec{k}| \leq \infty \quad \text{and} \quad -\infty < l < \infty$$

for both the fermions and bosons. We computed the Feynman diagrams in the $d = 3$ dimensions, since the scalar self-coupling, Yukawa coupling and four Fermi coupling are marginal in this dimension. Below the critical dimension the Yukawa and scalar self-coupling are relevant, which implies that even weak interactions will produce large effects when one includes contributions from the loop diagrams. We computed the one loop quantum corrections of these couplings following the approach adopted in [17]. We have found a number of deficiencies with the approach of [17]. Since we continue to integrate over an infinite number of modes this decimation will not remove all infinities from the theory. Further, in Wilson's renormalization theory all physical theories come equipped with a cut off, above which the theory ceases to be valid. Thus, in contrast to the approach of [17], we expect any sensible approach to have a large momentum cut off. In the next section we will come up with a better approach to renormalize our theory.

Evaluating the loop integrals, we see that the boson self energy loop integral (3.135)

$$\Pi(p) = g^2 \int_{e^{-t}\Lambda}^{\Lambda} \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{iw + \mu - \epsilon(\vec{k})} \frac{1}{i(w - p_0) + \mu - \epsilon(\vec{k} - \vec{p})}. \quad (3.135)$$

is zero, which implies that there is no wave-function renormalization for the bosons at one loop. The fermions do not renormalize the bosons, but the bosons do play a role in the wave-function renormalization of the fermions. Actually the fermions do not contribute to the renormalization of the scalars since their loop integrals vanishes. In the next chapter we will see that fermion loop integrals do not necessarily vanish when momentum integrals are restricted to a finite range. This too, shows that the renormalization strategy employed [17] is misleading at times.

Chapter 4

Renormalization group Flows For Finite density QFT

4.1 Finite density without four fermi coupling

The objective in this chapter is to study QFT of fermions coupled to a gapless order parameter (given by a bosonic field). Our procedure will be different from the previous chapter. In this chapter we will use the renormalized perturbation theory formalism. To do this we write down our toy model action in terms of physical parameters and introduce the counter terms in the action to renormalize the theory. We write down the Feynman rules for the renormalized perturbation theory. We will use these rules to study the effective low energy dynamics of the theory. We will use the marriage between Wilson's idea of renormalization and Feynman, Tomanaga and Swinger idea of renormalization, which is captured by the Callan-Symanzik equation. The Callan-Symanzik equation will be used to evaluate the renormalization group flow properties of the theory such as the anomalous dimension of the fields, fixed points and beta-function. Lets consider a field theory model with the action

$$S = \int d\tau d^d x \left(\frac{1}{2} [(\partial_\tau \phi_0)^2 + c^2 (\vec{\nabla} \phi_0)^2 + \frac{\lambda_{\phi_0}}{4!} \phi_0^4] + \bar{\psi}_0 (\partial_\tau + \mu - \epsilon(i\vec{\nabla})) \psi_0 + g_0 \phi_0 \bar{\psi}_0 \psi_0 \right), \quad (4.1)$$

with the fermion quartic coupling $\lambda_\psi = 0$ and the boson mass $m_\phi = 0$. The theory has a physical UV cut-off scale $\Lambda \ll k_F$. Close to the Fermi-surface, the dispersion relation is

$$i\omega = \mu - \epsilon(i\vec{\nabla}) = \mu - \left(\epsilon(\vec{k}) \Big|_{\vec{k}=k_F} + \vec{l} \cdot \frac{\partial \epsilon(\vec{k})}{\partial \vec{k}} \Big|_{\vec{k}=k_F} + \dots \right) \quad (4.2)$$

$$= l v_0 + \dots \quad (4.3)$$

where

$$v_0 = \frac{\partial \epsilon(k)}{\partial k} = \epsilon'(\vec{k}). \quad (4.4)$$

We will treat the quadratic plus higher order corrections to the dispersion relation as an interaction arising from Fermi surface curvature effects. The action for the theory becomes

$$S = \int d\tau d^d x \left(\frac{1}{2} [(\partial_\tau \phi_0)^2 + c^2 (\vec{\nabla} \phi_0)^2 + \frac{\lambda_{\phi_0}}{4!} \phi_0^4] + \psi_0 (\partial_\tau + l v_0) \psi_0 + g_0 \phi_0 \bar{\psi}_0 \psi_0 \right), \quad (4.5)$$

and we set $c = 1$ for convenience. The action is written in terms of bare parameters ϕ_0 , ψ_0 , λ_{ϕ_0} , g_0 and v_0 . The bare fields will be replaced by the physical fields ϕ , $\bar{\psi}$, ψ by using the relations

$$\phi_0 = \sqrt{Z_\phi}\phi \quad \text{and} \quad \psi_0 = \sqrt{Z_\psi}\psi, \quad (4.6)$$

where Z_ϕ and Z_ψ are the probability of creating a state with one particle. We now write the Lagrangian in momentum space as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(p_0^2 + \vec{p}^2)\phi(p)\phi(-p) + \tilde{\lambda}_\phi \int \prod_i^3 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3)\phi(p)\phi(p_1)\phi(p_2)\phi(p_3) \\ & + \bar{\psi}(-p)(ip_0 - lv)\psi(p) + \tilde{g} \int \prod_i^2 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p)\phi(p)\bar{\psi}(p_1)\psi(p_2) \\ & + \frac{1}{2}(p_0^2\delta Z_\phi + \vec{p}^2\delta Z_\phi)\phi(p)\phi(-p) + \bar{\psi}(-p)(ip_0\delta Z_\psi - l\delta v)\psi(p) \\ & + \tilde{\lambda}_\phi\delta\lambda_\phi \int \prod_i^3 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_2 + p_3 + p_4)\phi(p)\phi(p_2)\phi(p_3)\phi(p_4) \\ & + \tilde{g}\delta g \int \prod_i^2 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p)\phi(p)\bar{\psi}(p_1)\psi(p_2) \end{aligned} \quad (4.7)$$

with counter terms defined as

$$\begin{aligned} \delta Z_\phi &= Z_\phi - 1, & \tilde{\lambda}_\phi\delta\lambda_\phi &= \lambda_{\phi_0}Z_\phi^2 - \tilde{\lambda}_\phi, & \delta Z_\psi &= Z_\psi - 1, \\ \delta v &= v_0Z_\psi - v & \text{and} & & \tilde{g}\delta g &= g_0Z_\phi^{\frac{1}{2}}Z_\psi - \tilde{g}. \end{aligned}$$

In $D = d + 1$ dimensions, the length dimension of the scalar field is

$$[\phi] = L^{-\frac{1}{2}(D-2)} \quad \text{so that} \quad [\phi^4] = L^{-2(D-2)}.$$

Therefore the dimension of λ_ϕ is

$$\begin{aligned} [\lambda_\phi] &= L^{-(4-D)} \\ &= L^{-\epsilon} = M^\epsilon \end{aligned}$$

where $\epsilon = 4 - D$ and M is the renormalization scale. We can write $\tilde{\lambda}_\phi$ as $\tilde{\lambda}_\phi = M^\epsilon\lambda_\phi$, where λ_ϕ is a dimensionless coupling. Also the dimensions of fermionic fields ψ and $\bar{\psi}$ are

$$[\psi] = [\bar{\psi}] = L^{-\frac{1}{2}(D-1)},$$

which means the dimension of the Yukawa coupling is

$$\begin{aligned} [g] &= L^{-\frac{1}{2}(4-D)} \\ &= L^{-\frac{\epsilon}{2}} = M^{\frac{\epsilon}{2}}. \end{aligned}$$

We can thus write $\tilde{g}$ as

$$\tilde{g} = M^{\frac{\epsilon}{2}}g,$$

where g is a dimensionless constant. Now, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}(p_0^2 + \vec{p}^2)\phi(p)\phi(-p) + M^\epsilon \lambda_\phi \int \prod_i^3 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \phi(p)\phi(p_1)\phi(p_2)\phi(p_3) \\
& + \bar{\psi}(-p)(ip_0 - lv)\psi(p) + M^{\frac{\epsilon}{2}}g \int \prod_i^2 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p) \phi(p)\bar{\psi}(p_1)\psi(p_2) \\
& + \frac{1}{2}(p_0^2 \delta Z_\phi + \vec{p}^2 \delta Z_\phi)\phi(p)\phi(-p) + \bar{\psi}(-p)(ip_0 \delta Z_\psi - l\delta v)\psi(p) \\
& + M^\epsilon \lambda_\phi \delta \lambda_\phi \int \prod_i^3 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \phi(p)\phi(p_1)\phi(p_2)\phi(p_3) \\
& + M^{\frac{\epsilon}{2}}g \delta g \int \prod_i^2 \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p) \phi(p)\bar{\psi}(p_1)\psi(p_2)
\end{aligned}$$

where the counter terms are defined as

$$\begin{aligned}
\delta Z_\phi &= Z_\phi - 1, & \delta Z_\psi &= Z_\psi - 1, & \delta v &= v_0 Z_\psi - v & (4.8) \\
M^\epsilon \lambda_\phi \delta \lambda_\phi &= \lambda_{\phi_0} Z_\phi^2 - M^\epsilon \lambda_\phi \Rightarrow \lambda_{\phi_0} = \frac{M^{-\epsilon} Z_\lambda}{Z_\phi^2} \lambda_\phi & \text{ and } & M^{\frac{\epsilon}{2}} g \delta g &= g_0 Z_\psi Z_\phi^{\frac{1}{2}} - M^{\frac{\epsilon}{2}} g \Rightarrow g_0 = \frac{M^{-\frac{\epsilon}{2}} Z_g}{Z_\psi Z_\phi^{\frac{1}{2}}} g,
\end{aligned}$$

where $Z_\lambda = \delta \lambda_\phi + 1$ and $Z_g = \delta g + 1$. The Feynman rules for the theory in momentum space are:

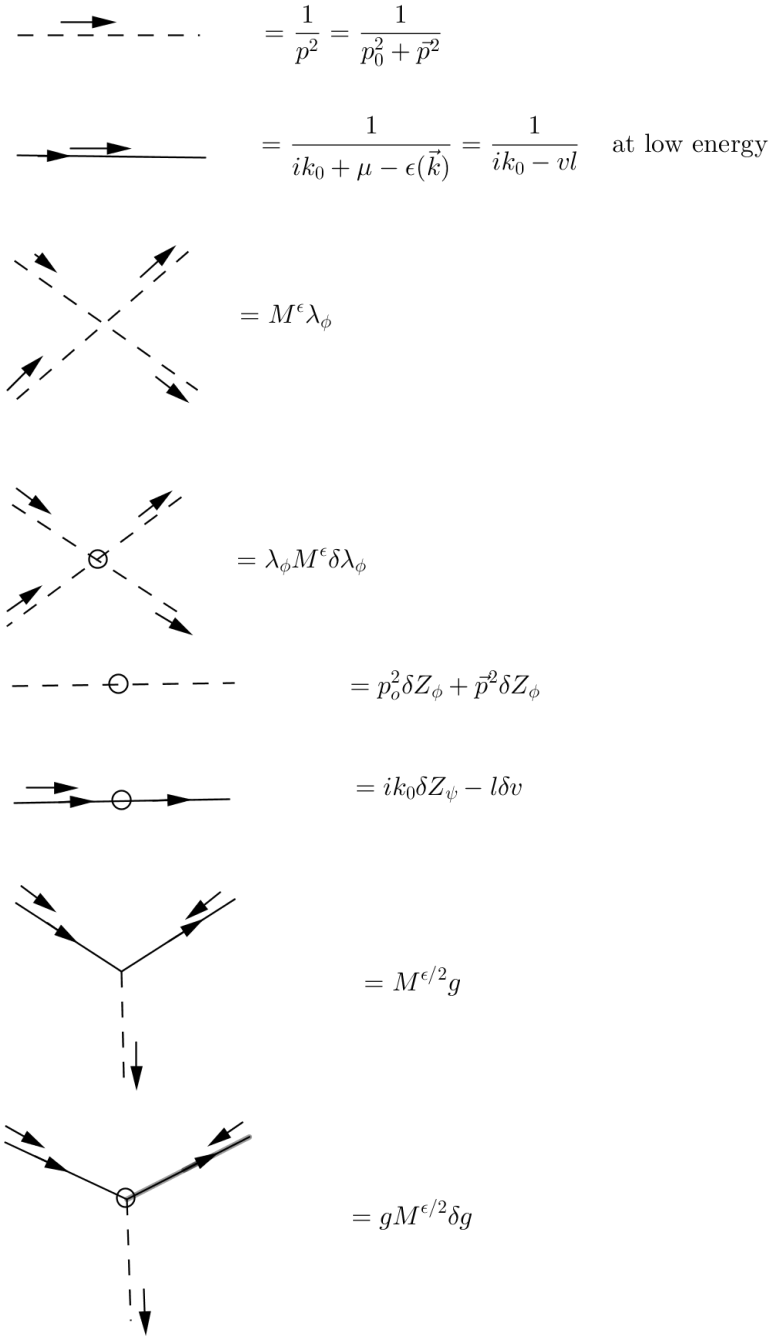


Figure 4.1: Feynman rules in momentum space.

Before we state the renormalization conditions for the bosons, define

$$- - - - \bigcirc \text{1PI} - - - - = \tilde{\Pi}(p^2)$$

where $\tilde{\Pi}(p^2)$ is the sum of 1-particle irreducible (1PI) diagrams and the full boson two point function is

$$\begin{aligned}
\text{Full propagator} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots \\
&= \frac{1}{p^2 + \tilde{\Pi}(p^2)} = \frac{1}{p_o^2 + \vec{p}^2 + \tilde{\Pi}(p_o, \vec{p})}
\end{aligned}$$

The renormalization conditions for the bosons are

$$\tilde{\Pi}(p^2) = 0 \quad \text{at} \quad p^2 = -M^2 \quad (4.9)$$

$$\left. \frac{d\tilde{\Pi}(p^2)}{dp^2} \right|_{p^2 = -M^2} = 0 \quad (4.10)$$

$$\begin{aligned}
&\text{Amputated} \\
&= \lambda_\phi M^\epsilon \delta \lambda_\phi \quad \text{at} \quad p^2 = -M^2
\end{aligned}$$

where $(p_1 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2 = p^2$. We define the sum of 1-particle irreducible diagrams for the fermions as

$$\text{---} \text{---} \text{---} \text{---} \text{---} = \tilde{\Sigma}(k_0, l)$$

The full fermion propagator is

$$\begin{aligned}
\text{Full propagator} &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \dots \\
&= \frac{1}{ik_o + \mu - \epsilon(\vec{k}) + \tilde{\Sigma}(k_o, l)}
\end{aligned}$$

The renormalization conditions for the fermions are

$$\tilde{\Sigma}(k_0, l) = 0 \quad \text{at} \quad k_0 = -M \quad \text{and} \quad (4.11)$$

$$\left. \frac{d\tilde{\Sigma}(k_0, l)}{dk_0} \right|_{k_0 = -M} = 0 \quad (4.12)$$

and the renormalization condition for the Yukawa vertex is

$$= gM^{\epsilon/2}\delta g \quad \text{at } p_i^2 = -M^2 \quad \text{for the boson}$$

and $p_{0i} = -M$ for the fermion

Renormalization of the Bosons

The sum of the 1-particle irreducible (1PI) Feynman diagrams for the boson 2-point function at one loop is

$$= g^2 M^\epsilon \int \frac{dldk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ik_0 + \mu - \epsilon(\vec{k})} \frac{1}{i(k_0 - p_0) + \mu - \epsilon(\vec{k} - \vec{p})} + p_0^2 \delta Z_\phi + \vec{p}^2 \delta Z_\phi$$

$$\tilde{\Pi}(p_0, l) = g^2 M^\epsilon \int \frac{dldk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ik_0 + \mu - \epsilon(\vec{k})} \frac{1}{i(k_0 + p_0) + \mu - \epsilon(\vec{k} + \vec{p})} + p_0^2 \delta Z_\phi + \vec{p}^2 \delta Z_\phi. \quad (4.13)$$

The first 1PI Feynman diagram on the left is the boson self-energy, sometimes referred to as the vacuum polarization diagram. This diagram is responsible for Landau damping of the bosons. This diagram was evaluated in (3.135) in the previous chapter and it was equal to zero. With a more careful analysis we will see that is not necessarily zero. We evaluate the loop integral of the boson self energy as follows

$$\begin{aligned} \Pi(p_0, \vec{p}) &= g^2 M^\epsilon \int \frac{dk_0 d^d k}{(2\pi)^{d+1}} \frac{1}{ik_0 + \mu - \epsilon(\vec{k})} \frac{1}{i(k_0 - p_0) - \epsilon(\vec{k} - \vec{p}) + \mu} \\ &= g^2 M^\epsilon \int \frac{dldk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ik_0 - vl} \frac{1}{i(k_0 - p_0) - vl + v|\vec{p}| \cos \theta} \\ &= g^2 M^\epsilon \int \frac{dldk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ik_0 - vl} - \frac{1}{i(k_0 - p_0) - vl + v|\vec{p}| \cos \theta} \right). \end{aligned} \quad (4.14)$$

The integral has poles at $k_0 = \frac{vl}{i}$ and $k_0 = \frac{vl - v|\vec{p}| \cos \theta}{i} + p_0$. If l is very large both poles will be on the same side of the real line axes and the integral will vanish. The integral will not vanish if the poles are on different side of the real axis. This happens when l runs over a finite range which is

$0 \leq l \leq |\vec{p}| \cos \theta$. Integrating over this finite dl over a finite range we get

$$\begin{aligned} \Pi(p_0, \vec{p}) &= g^2 M^\epsilon \int \frac{dk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ik_0 - vl} - \frac{1}{i(k_0 + p_0) - vl + v|\vec{p}| \cos \theta} \right) l \Big|_{l=0}^{l=|\vec{p}| \cos \theta} \\ &= g^2 M^\epsilon \int \frac{dk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{|\vec{p}| \cos \theta}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ik_0 - v|\vec{p}| \cos \theta} - \frac{1}{i(k_0 - p_0)} \right). \end{aligned}$$

Now evaluating the dk_0 integral using the residue theorem we obtain

$$\Pi(p_0, \vec{p}) = ig^2 M^\epsilon \int \frac{k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^d} \frac{|\vec{p}| \cos \theta}{ip_0 - v|\vec{p}| \cos \theta}.$$

For $d = 3$ we finally obtain

$$\Pi(p_0, \vec{p}) = \frac{ig^2 k_F^2}{4\pi^2} \int_{-1}^1 d(\cos \theta) \frac{|\vec{p}| \cos \theta}{ip_0 - v|\vec{p}| \cos \theta}.$$

Using long division to simplify the integrand we get

$$\begin{aligned} \Pi(p_0, \vec{p}) &= \frac{ig^2 k_F^2}{4\pi} \int_{-1}^1 d(\cos \theta) \left(\frac{1}{iv} - \frac{p_0}{iv(p_0 + iv|\vec{p}| \cos \theta)} \right) \\ &= \frac{ig^2 k_F^2}{4\pi} \left(\frac{\cos \theta}{iv} - \frac{p_0}{(iv)^2 |\vec{p}|} \ln(p_0 + iv|\vec{p}| \cos \theta) \right) \Big|_{-1}^1 \\ &= \frac{ig^2 k_F^2}{4\pi} \left(\frac{2}{iv} + \frac{p_0}{v^2 |\vec{p}|} \ln \left(\frac{p_0 + iv|\vec{p}|}{p_0 - iv|\vec{p}|} \right) \right) \\ &= \frac{ig^2 k_F^2}{4\pi} \left(\frac{2}{iv} + \frac{p_0}{v^2 |\vec{p}|} \ln \left(\frac{re^{i\zeta}}{re^{-i\zeta}} \right) \right) \\ &= \frac{ig^2 k_F^2}{4\pi} \left(\frac{2}{iv} + \frac{p_0}{v^2 |\vec{p}|} (2i\zeta) \right) = \frac{g^2 k_F^2}{2\pi^2 v} \left(1 - \frac{p_0 \zeta}{v|\vec{p}|} \right) \\ &= \frac{g^2 k_F^2}{2\pi^2 v} \left(1 - \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right) \right), \end{aligned} \tag{4.15}$$

where $p_0 + iv|\vec{p}| = re^{i\zeta}$ and $\zeta = \tan^{-1}(\frac{v|\vec{p}|}{p_0})$. This integral is finite but there is a caveat. Later when we calculate the renormalization group flows we will find that there is a fixed point where $v \rightarrow 0$ at low energy and when $v \rightarrow 0$ this loop integral vanishes. This can be shown by using the general Taylor expansion

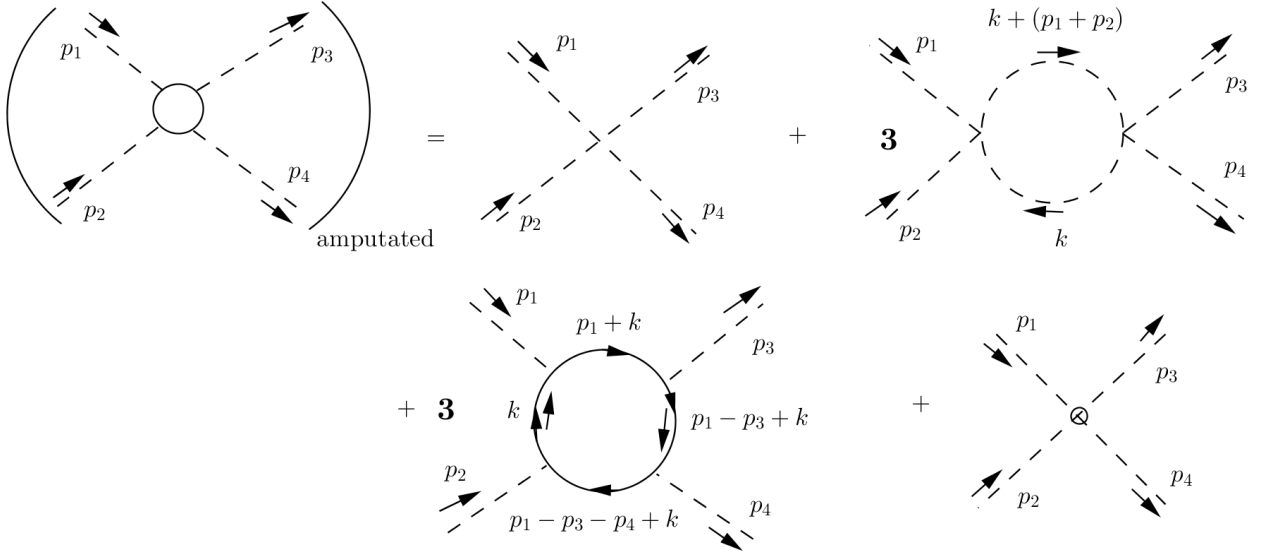
$$\tan^{-1} \left(\frac{x}{a} \right) = \frac{x}{a} - \frac{1}{3} \left(\frac{x}{a} \right)^3 + \frac{1}{5} \left(\frac{x}{a} \right)^5 + \dots,$$

to $\Pi(p_0, \vec{p})$

$$\Pi(p_0, \vec{p}) = \frac{g^2 k_F^2}{2\pi^2 v} \left(1 - \frac{p_0}{v|\vec{p}|} \left(\frac{v|\vec{p}|}{p_0} - \frac{1}{3} \left(\frac{v|\vec{p}|}{p_0} \right)^3 + \dots \right) \right) \tag{4.16}$$

$$\approx \frac{g^2 k_F^2 v |\vec{p}|^2}{6\pi^2 p_0^2} + \mathcal{O}(v^3). \tag{4.17}$$

From here we can see that as $v \rightarrow 0$ the self energy vanishes. We will assume that our UV theory starts in the basin of attraction of the fixed point $v = 0$ which sets the boson self energy integral to zero [20], and therefore the counter terms δZ_ψ and δv will be zero. The physics as $v \rightarrow 0$ will be explained in detail later, because higher order curvature terms will now be comparable to vl . The 1PI 4-point function for the bosons is



The constant 3 in front of the second and third diagrams on the left comes from adding the s-t-u channels, assuming that momentum is conserved at the vertex and the external legs carry the same momentum. Therefore

$$\left(G_B^{(4)}\right)_{\text{amputated}} = M^\epsilon \lambda_\phi + \Gamma_1^{(4)} + \Gamma_2^{(4)} + \lambda_\phi M^\epsilon \delta \lambda_\phi,$$

where $\Gamma_1^{(4)}$ and $\Gamma_2^{(4)}$ are the loop integrals for the second and third Feynman diagrams. We have to first evaluate $\Gamma_1^{(4)}$ and $\Gamma_2^{(4)}$ and then use the renormalization condition to evaluate the counter term $\delta \lambda_\phi$. $\Gamma_1^{(4)}$ is evaluated as follows:

$$\Gamma_1^{(4)} = \frac{3\lambda_\phi^2 M^{2\epsilon}}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{k^2} \frac{1}{(k+p)^2}$$

where $p = p_1 + p_2$. Introduce the Feynman parameters using the formula

$$\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n \frac{\delta(\sum x_i - 1) \prod x_i^{m_i-1} \Gamma(\sum(m_i))}{(\sum x_i A_i)^{\sum m_i} \prod \Gamma(m_i)}.$$

The loop integral becomes

$$\begin{aligned} \Gamma_1^{(4)} &= \frac{3\lambda_\phi^2 M^{2\epsilon}}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_0^1 dx dy \frac{\delta(x+y-1)}{(k^2 x + (k+p)^2 y)^2} \\ &= \frac{3\lambda_\phi^2 M^{2\epsilon}}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_0^1 dx dy \frac{\delta(x+y-1)}{(k^2 y + k^2 x + 2k \cdot px + p^2 x)^2}. \end{aligned}$$

Performing the $\int dy$ integral with the delta function we get

$$\Gamma_1^{(4)} = \frac{3\lambda_\phi^2 M^{2\epsilon}}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_0^1 dx \frac{1}{(k^2 + 2k \cdot px + p^2 x)^2}.$$

We can complete the square in the denominator by shifting

$$k \rightarrow k - p \cdot x,$$

so that

$$\Gamma_1^{(4)} = \frac{3\lambda_\phi^2 M^{2\epsilon}}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_0^1 dx \frac{1}{(k^2 - p^2 x^2 + p^2 x)^2}.$$

Using dimensional regularization to evaluate the integral we get

$$\Gamma_1^{(4)} = \frac{3\lambda_\phi^2}{2} \frac{M^{2\epsilon}}{(4\pi)^{\frac{d+1}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d+1}{2})}{\Gamma(2)((1-x)xp^2)^{2-\frac{d+1}{2}}}.$$

Expand the integrand at $\epsilon = 3 - d$ when $d \rightarrow 3$ to find

$$\begin{aligned} \Gamma_1^{(4)} &= \frac{3\lambda_\phi^2}{2} \frac{M^{2\epsilon}}{(4\pi)^{\frac{1}{2}(4-\epsilon)}} \int_0^1 dx \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)((1-x)xp^2)^{\frac{\epsilon}{2}}} \\ &= \frac{3\lambda_\phi^2}{2(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \log((1-x)xp^2) + \log(4\pi) + \mathcal{O}(\epsilon) + 2\log(M^2) \right), \end{aligned}$$

where

$$\Gamma(\frac{\epsilon}{2}) = \frac{\epsilon}{2} - \gamma \quad \text{and} \quad \gamma \quad \text{is the Euler Macheroni constant}$$

and we have used the expansion

$$\lim_{x \rightarrow 0} a^x = 1 + x \log(a) + \frac{x^2}{2!} \log(a) + \dots$$

The $\Gamma_2^{(4)}$ integral is as follows

$$\begin{aligned} \Gamma_2^{(4)} &= g^4 M^{2\epsilon} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{ik_0 + \mu - \epsilon(\vec{k})} \frac{1}{i(k_0 + p_{01}) + \mu - \epsilon(\vec{k} + \vec{p}_1)} \times \\ &\quad \frac{1}{i(k_0 + p_{01} - p_{03}) + \mu - \epsilon(\vec{k} + \vec{p}_1 - \vec{p}_3)} \times \\ &\quad \frac{1}{i(k_0 + p_{01} - p_{03} - p_{04}) + \mu - \epsilon(\vec{k} + \vec{p}_1 - \vec{p}_3 - \vec{p}_4)}. \end{aligned}$$

Evaluating this integral is not trivial, but luckily we know that for $\vec{p}_3 = \vec{p}_4 = 0$ and $p_{03} = p_{04} = 0$ we can relate the $\Gamma_2^{(4)}$ integral to the boson self-energy integral $\Pi(p_0, \vec{p})$ in (4.15) as in [23]

$$\Gamma_2^{(4)}(p_{01}, \vec{p}_1) = -\frac{g^2}{2} \frac{d^2}{dp_{01}^2} \Pi(p_{01}, \vec{p}_1) \quad (4.18)$$

$$= -\frac{g^2}{2} \frac{d^2}{dp_{01}^2} \left[\frac{g^2 k_F^2}{2\pi^2 v} \left(1 - \frac{p_{01}}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_{01}} \right) \right) \right] \quad (4.19)$$

$$= \frac{g^2 k_F^2}{4\pi v} \left(\frac{d}{dp_{01}} \left(\frac{1}{v|p_{01}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_{01}} \right) - \frac{p_{01}}{v|\vec{p}|} \frac{1}{p_{01}^2} \frac{1}{(1 + (\frac{v|\vec{p}|}{p_{01}})^2)} \right) \right) \quad (4.20)$$

$$= \frac{g^2 k_F^2}{4\pi v} \left(-\frac{1}{p_{01}} \frac{1}{(1 + (\frac{v|\vec{p}|}{p_{01}})^2)} + -\frac{1}{p_{01}} \frac{1}{(1 + (\frac{v|\vec{p}|}{p_{01}})^2)} + \frac{2}{p_{01}^2} \frac{(\frac{v|\vec{p}|}{p_{01}})^2}{(1 + (\frac{v|\vec{p}|}{p_{01}})^2)} \right) \quad (4.21)$$

$$= -\frac{g^2 k_F^2}{2\pi v} \left(\frac{v^2 |\vec{p}|^2}{(p_{01}^2 + v^2 |\vec{p}|^2)^2} \right), \quad (4.22)$$

$\Gamma_2^{(4)}(p_{01}, \vec{p}_1)$ vanishes since

$$\lim_{p_{01} \rightarrow 0} \lim_{|\vec{p}| \rightarrow 0} \Gamma_2^{(4)}(p_{01}, \vec{p}_1) = 0.$$

Because the Yukawa coupling connects particle-hole states near the Fermi-surface with small boson momenta we must consider the limit $p_{01} \rightarrow 0$ and $|\vec{p}| \rightarrow 0$. Now, using the renormalization condition for the boson, we find the counter term

$$\begin{aligned} \lambda_\phi \delta \lambda_\phi &= -\Gamma_1^{(4)}(p_{01}, \vec{p}_1) \\ &= -\frac{3\lambda_\phi^2}{2(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \log(-(1-x)xM^2) + \log(4\pi) + \log(M^2) + \mathcal{O}(\epsilon) \right). \end{aligned}$$

Then the renormalized boson 4-point vertex function is

$$\left(G_B^{(4)} \right)_{\text{amputated}} = \lambda_\phi + \frac{3\lambda_\phi^2}{2(4\pi)^2} \log \left(\frac{-M^2}{p^2} \right) \quad \text{where} \quad p^2 = (p_1 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2$$

and the boson 4-point function is

$$G_B^{(4)} = \frac{1}{p_1^2} \frac{1}{p_2^2} \frac{1}{p_3^2} \frac{1}{p_4^2} \left(\lambda_\phi + \frac{3\lambda_\phi^2}{2(4\pi)^2} \log \left(\frac{-M^2}{p^2} \right) \right).$$

We can find the anomalous dimension γ_ϕ for the boson propagator and β_λ -function for the running coupling λ_ϕ in two ways. Either by deriving γ_ϕ and β_λ using the counter-terms or using the Callan-Symanzik equations. We will use both methods to calculate λ_ϕ and β_λ . We will start with the counter-term method and then use the Callan-Symanzik equation. The relationship between the bare scalar field and the physical field is

$$\phi_0 = \sqrt{Z_\phi} \phi \quad \text{and} \quad Z_\phi = \delta Z_\phi + 1.$$

Taking a derivative with respect the renormalization scale M we have

$$M \frac{d\phi_0}{dM} = M \frac{d}{dM} (Z_\phi^{\frac{1}{2}} \phi).$$

The bare parameters are independent of the RG scale, so the LHS is zero, and we find

$$\begin{aligned} 0 &= \frac{\phi}{2} \frac{M}{Z_\phi} \frac{dZ_\phi}{dM} + M Z_\phi^{\frac{1}{2}} \frac{d\phi}{dM} \\ &\Rightarrow -M \frac{d\phi}{dM} = \frac{\phi}{2} \frac{M}{Z_\phi} \frac{d\delta Z_\phi}{dM} \\ &\frac{d\phi}{d(-\log(\frac{P}{M}))} = \phi \left(\frac{1}{2} \frac{M}{Z_\phi} \frac{d\delta Z_\phi}{dM} \right). \end{aligned}$$

Finally, recall that

$$\frac{d\phi_\phi}{dt} = \gamma_\phi \phi$$

where $t = -\log(\frac{P}{M})$ is the RG time and $\gamma_\phi = \frac{1}{2} \frac{M}{Z_\phi} \frac{d\delta Z_\phi}{dM}$. We flow from scale M to scale P , the scale of the low energy effective field theory. We find

$$\gamma_\phi = 0,$$

since we found $\delta Z_\phi = 0$ from the renormalization condition. The β_λ -function for the running coupling is computed as follows: we start from the counter term

$$\begin{aligned} M^\epsilon \lambda_\phi \delta \lambda_\phi &= Z_\phi^2 \lambda_{\phi_0} - M^\epsilon \lambda_\phi \\ \Rightarrow \lambda_{\phi_0} &= \frac{M^\epsilon \lambda_\phi (\delta \lambda_\phi + 1)}{Z_\phi^2} \end{aligned}$$

taking a derivative with respect to the scale M we find

$$\frac{d\lambda_{\phi_0}}{dM} = \frac{\epsilon M^{\epsilon-1} \lambda_\phi (\delta \lambda_\phi + 1)}{Z_\phi^2} + \frac{M^\epsilon (\delta \lambda_\phi + 1)}{Z_\phi^2} \frac{d\lambda_\phi}{dM} - \frac{M^\epsilon \lambda_\phi (\delta \lambda_\phi + 1)}{Z_\phi^3} \frac{dZ_\phi}{dM} + \frac{M^\epsilon \lambda_\phi}{Z_\phi^2} \frac{d\delta \lambda_\phi}{dM}.$$

Note that $\frac{d\lambda_{\phi_0}}{dM} = 0$ since the bare coupling λ_{ϕ_0} is independent of the renormalization scale. Therefore

$$-\frac{M^\epsilon (\delta \lambda_\phi + 1)}{Z_\phi^2} \frac{d\lambda_\phi}{dM} = \frac{\epsilon M^{\epsilon-1} \lambda_\phi (\delta \lambda_\phi + 1)}{Z_\phi^2} - \frac{M^\epsilon \lambda_\phi (\delta \lambda_\phi + 1)}{Z_\phi^3} \frac{dZ_\phi}{dM} + \frac{M^\epsilon \lambda_\phi}{Z_\phi^2} \frac{d\delta \lambda_\phi}{dM} \quad (4.23)$$

$$-M \frac{d\lambda_\phi}{dM} = \left(\epsilon - \frac{M}{Z_\phi} \frac{dZ_\phi}{dM} + \frac{M}{(\delta \lambda_\phi + 1)} \frac{d\delta \lambda_\phi}{dM} \right) \lambda_\phi \quad (4.24)$$

$$\beta_\lambda = \frac{d\lambda_\phi}{dt} = \left(\epsilon - 2\gamma_\phi + \frac{M}{Z_\lambda} \frac{d\delta \lambda_\phi}{dM} \right) \lambda_\phi, \quad (4.25)$$

where $t = -\log\left(\frac{M}{P}\right)$ is the renormalization group time and $Z_\lambda = \delta \lambda + 1$. Since $\gamma_\phi = 0$, β_λ reduces to

$$\beta_\lambda = \left(\epsilon + \frac{M}{Z_\lambda} \frac{d\delta \lambda}{dM} \right) \lambda_\phi.$$

Using the renormalization conditions for the boson 4-point function, we find

$$\delta \lambda_\phi = -\frac{3\lambda_\phi}{2(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \log(-(1-x)xM^2) + \log(4\pi) \right).$$

We can further simplify β_λ by assuming that $\delta \lambda_\phi \ll 1$ to obtain

$$\beta_\lambda = \left(\epsilon + \frac{M}{Z_\lambda} \frac{d\delta \lambda}{dM} \right) \lambda_\phi \quad (4.26)$$

$$= \left(\epsilon + \frac{M}{1 + \delta \lambda_\phi} \frac{d\delta \lambda}{dM} \right) \lambda_\phi \quad (4.27)$$

$$= \left(\epsilon + M(1 - \delta \lambda_\phi) \frac{d\delta \lambda}{dM} \right) \lambda_\phi \quad (4.28)$$

$$= \left(\epsilon + M \frac{d\delta \lambda_\phi}{dM} \right) \lambda_\phi \quad (4.29)$$

where on line (4.29) we have used the binomial theorem and in the last line we have neglected the $\delta \lambda_\phi \frac{d\delta \lambda_\phi}{dM}$ since it will give us a term in $\mathcal{O}(\lambda_\phi^3)$ which is only captured correctly by a higher loop analysis. Plugging $\delta \lambda_\phi$ into β_λ we get

$$\beta_\lambda = \frac{d\lambda_\phi}{dt} = \left(\epsilon - \frac{3\lambda_\phi}{2(4\pi)^2} \right) \lambda_\phi.$$

We will now reproduce these results using the the Callan-Symanzik equation [21]

$$\left(M \frac{\partial}{\partial M} + \beta_\phi \frac{\partial}{\partial \lambda_\phi} + n\gamma_\phi\right) G_B^{(n)} = 0,$$

to evaluate β_λ and γ_ϕ . Recall that the renormalized two point function for the boson is

$$G_B^{(2)} = \frac{1}{p_0^2 + \vec{p}^2}.$$

Plugging the two point function into the Callan-Symanzik equation

$$\left(M \frac{\partial}{\partial M} + \beta_\lambda \frac{\partial}{\partial \lambda_\phi} + 2\gamma_\phi\right) G_B^{(2)} = 0 \quad (4.30)$$

for a two point function, we find that $\gamma_\phi = 0$. This means that at one loop there is no wave-function renormalization for the boson propagator. The physical meaning of this is that at one loop the physical field and the field we integrate over in the path integral are equal. The Callan-Symanzik equation for the 4-point function is,

$$\left(M \frac{\partial}{\partial M} + \beta_\lambda \frac{\partial}{\partial \lambda_\phi} - 4\gamma_\phi\right) G_B^{(4)} = 0. \quad (4.31)$$

Since $\gamma_\phi = 0$ we only have to compute

$$M \frac{\partial}{\partial M} G_B^{(4)} = M \frac{\partial}{\partial M} \left(\lambda_\phi + \frac{3\lambda_\phi^2}{32\pi^2} \log\left(\frac{-M^2}{p^2}\right) \right), \quad (4.32)$$

recall that $\lambda_\phi = M^{-\epsilon} \tilde{\lambda}_\phi$, therefore

$$\begin{aligned} M \frac{\partial}{\partial M} G_B^{(4)} &= -\epsilon \tilde{\lambda}_\phi M^{-\epsilon} + \frac{3\tilde{\lambda}_\phi^2 M^{2\epsilon}}{16\pi^2} + \frac{3\epsilon \tilde{\lambda}_\phi^2 M^{-2\epsilon}}{16\pi^2} \log\left(\frac{-M^2}{p^2}\right) \\ &= \epsilon \lambda_\phi + \frac{3\lambda_\phi^2}{16\pi^2} - \frac{3\epsilon \lambda_\phi^2 M}{8\pi^2} \log\left(\frac{-M^2}{p^2}\right) \\ &= -\epsilon \lambda_\phi + \frac{\lambda_\phi^2}{16\pi^2} + \mathcal{O}(\epsilon \lambda_\phi^2) \end{aligned} \quad (4.33)$$

We neglect the last term in the previous line because terms of this order are only treated correctly at higher loops. Plugging this result back into to the Callan-Symanzik equation, we find β_λ to be

$$\beta_\lambda = \epsilon \lambda_\phi - 3 \frac{\lambda_\phi^2}{16\pi^2} \quad (4.34)$$

$$\frac{d\lambda_\phi}{d \log\left(\frac{P}{M}\right)} = \epsilon \lambda_\phi - 3 \frac{\lambda_\phi^2}{16\pi^2} \quad (4.35)$$

$$-\frac{d\lambda_\phi}{d \log\left(\frac{M}{P}\right)} = \epsilon \lambda_\phi - 3 \frac{\lambda_\phi^2}{16\pi^2} \quad (4.36)$$

$$\frac{d\lambda_\phi}{dt} = \epsilon \lambda_\phi - 3 \frac{\lambda_\phi^2}{16\pi^2} \quad (4.37)$$

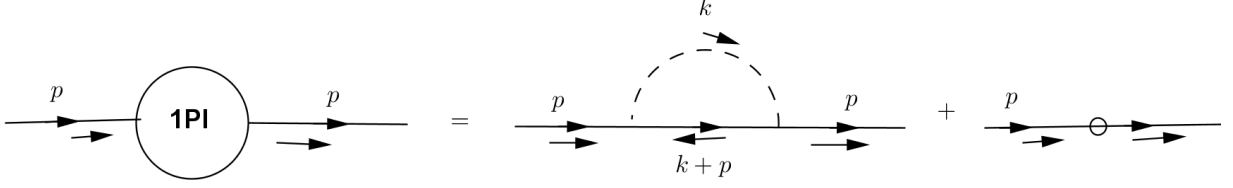
where $t = -\log\left(\frac{M}{P}\right)$. This demonstrates that we get the same results from both methods. The fixed points are determined by $\beta_\lambda = 0$, which are at

$$\lambda_\phi^* = \frac{16\pi^2\epsilon}{3} = \frac{\epsilon}{a_\lambda} \quad \text{and} \quad \lambda_\phi^* = 0,$$

where $a_\lambda = \frac{3}{16\pi^2}$. The fixed point $\lambda_\phi^* = \frac{\epsilon}{a_\lambda}$ is called the Wilson-Fisher fixed point.

4.1.1 Renormalization of the Fermions

The 1-particle irreducible Feynman diagrams for the fermion 2-point function at one loop are



$$\tilde{\Sigma}(p_0, \tilde{l}) = -\frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \int dk_0 dl d^{d-1} k_{||} \frac{1}{k_0^2 + c^2(l^2 + k_{||}^2)} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} + ip_0 \delta Z_\psi - \tilde{l} \delta v \quad (4.38)$$

The Fermion self energy is evaluated as follows

$$\begin{aligned} \Sigma(p_0, \tilde{l}) &= -\frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \int dk_0 dl d^{d-1} k_{||} \frac{1}{k_0^2 + c^2(l^2 + k_{||}^2)} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} \\ &= \frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \int dk_0 dl d^{d-1} k_{||} \frac{1}{k_0^2 + c^2(l^2 + k_{||}^2)^2} \frac{i(k_0 + p_0) + v(l + \tilde{l})}{(k_0 + p_0)^2 + v^2(l + \tilde{l})^2} \end{aligned} \quad (4.39)$$

we have decomposed the boson momentum as $\vec{k} = \hat{\Omega}l + k_{||}$, where $k_{||}$ is tangential to the Fermi-surface. Note that we have replaced c^2 in the boson propagator for dimensional reasons. Using the result[23]

$$\int d^{d-1} k_{||} \frac{1}{A + k_{||}^2} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{\pi}{2 \sin(\frac{\pi}{2}(d-1))} A^{(d-3)/2}, \quad (4.40)$$

where $\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the surface area of the $d-1$ unit sphere, we find

$$\Sigma(p_0, \tilde{l}) = -\frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{\pi}{2c^{2-\epsilon} \sin(\frac{\pi}{2}(d-1))} \int dk_0 dl \frac{1}{(k_0^2 + c^2 l^2)^{\epsilon/2}} \frac{i(k_0 + p_0) + v(l + \tilde{l})}{(k_0 + p_0)^2 + v^2(l + \tilde{l})^2}$$

where $\epsilon = d-3$, and $\sin(\frac{1}{2}\pi(d-1))$ in terms of $\epsilon = d-3$ is

$$\sin(\pi/2(d-1)) = \sin(\pi\epsilon/2).$$

Introducing Feynman parameters to rewrite the loop integral

$$\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n \frac{\delta(\sum x_i - 1) \prod x_i^{m_i-1} \Gamma(\sum(m_i))}{(\sum x_i A_i)^{\sum m_i} \prod \Gamma(m_i)},$$

the loop integral becomes

$$\begin{aligned} \Sigma(p_0, \tilde{l}) &= \frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{\pi \Gamma(1 + \frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2}) 2c^{2-\epsilon} \sin(\frac{\pi}{2}\epsilon)} \int dk_0 dl \int_0^1 dx dy \times \\ &\quad \frac{\delta(x+y-1) x^{\epsilon/2-1} (i(k_0 + p_0) + v(l + \tilde{l}))}{((k_0^2 + c^2 l^2)x + ((k_0 + p_0)^2 + v^2(l + \tilde{l})^2)y)^{1+\epsilon/2}}. \end{aligned}$$

Before we proceed to evaluate this integral, let us simplify the factor

$$\frac{g^2 M^\epsilon}{(2\pi)^{d+1}} \frac{2\pi^{d/2}}{\Gamma(\frac{3-\epsilon}{2})} \frac{\pi\Gamma(1+\frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})2c^{2-\epsilon}\sin(\frac{\pi}{2}\epsilon)}.$$

First, consider the factor

$$\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})}.$$

Using $\Gamma(1+x) = x\Gamma(x)$ we find

$$\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})} = \frac{\epsilon}{2}. \quad (4.41)$$

Next we will simplify the factor

$$\frac{1}{\Gamma(\frac{3-\epsilon}{2})\sin(\frac{\pi}{2}\epsilon)} = \frac{\csc(\pi\epsilon/2)}{(\frac{1}{2}-\frac{\epsilon}{2})\Gamma(\frac{1}{2}-\frac{\epsilon}{2})}.$$

Use the definition

$$\Gamma(z+\frac{1}{2}) = \frac{2^{1-2z}\sqrt{\pi}\Gamma(2z)}{\Gamma(z)}$$

to expand the denominator of the Γ function and power series

$$\csc(x) = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \dots$$

to expand the $\csc(\pi\epsilon/2)$ function as $\epsilon \rightarrow 0$. Hence

$$\begin{aligned} \frac{1}{\Gamma(\frac{3-\epsilon}{2})\sin(\frac{\pi}{2}\epsilon)} &= \frac{\Gamma(-\frac{\epsilon}{2})\csc(\pi\epsilon/2)}{(\frac{1}{2}-\frac{\epsilon}{2})2^{1+2(\frac{\epsilon}{2})}\sqrt{\pi}\Gamma(-\epsilon)} \\ &= \frac{\Gamma(-\frac{\epsilon}{2})\csc(\pi\epsilon/2)}{(1-\epsilon)2^\epsilon\sqrt{\pi}\Gamma(-\epsilon)} \\ &= \frac{(1+\epsilon)\Gamma(\frac{\epsilon}{2})2^{-\epsilon}\csc(\pi\epsilon/2)}{\sqrt{\pi}\Gamma(-\epsilon)}, \end{aligned}$$

where we used the binomial theorem to bring the denominator term $(1-\epsilon)$ up to the numerator.

Expanding the numerator we get

$$\begin{aligned} \frac{\Gamma(-\frac{\epsilon}{2})\csc(\pi\epsilon/2)}{(\frac{1}{2}-\frac{\epsilon}{2})2^{1+2(\frac{\epsilon}{2})}\sqrt{\pi}\Gamma(-\epsilon)} &= \frac{(1+\epsilon)(\frac{2}{\epsilon}-\gamma+\dots)(1-\epsilon\log(2)+\dots)(\frac{2}{\pi\epsilon}+\frac{\pi}{12}\epsilon+\dots)}{\sqrt{\pi}\Gamma(-\frac{\epsilon}{2})} \\ &= \frac{\frac{2}{\pi\epsilon^2}+\frac{2}{\pi\epsilon}+\mathcal{O}(\epsilon^0)}{\sqrt{\pi}\Gamma(-\frac{\epsilon}{2})}. \end{aligned}$$

Expanding the denominator we get

$$\begin{aligned} \frac{\Gamma(-\frac{\epsilon}{2})\csc(\pi\epsilon/2)}{(\frac{1}{2}-\frac{\epsilon}{2})2^{1+2(\frac{\epsilon}{2})}\sqrt{\pi}\Gamma(-\epsilon)} &= \frac{\frac{2}{\pi\epsilon^2}+\frac{2}{\pi\epsilon}+\mathcal{O}(\epsilon^0)}{\sqrt{\pi}}(-\epsilon+\gamma(-\epsilon)^2+\dots) \\ &\approx -\frac{2}{\epsilon\pi\sqrt{\pi}}+\mathcal{O}(\epsilon^0). \end{aligned} \quad (4.42)$$

Using the results that we found in (4.41) and (4.42) the loop integral simplifies to

$$\begin{aligned}\Sigma(p_0, \tilde{l}) &= \frac{g^2 M^\epsilon 2\pi^{3/2-\epsilon/2}}{(2\pi)^{4-\epsilon} c^{2-\epsilon} \pi \sqrt{\pi}} \int dk_0 dl \int_0^1 dx dy \frac{\delta(x+y-1) x^{\epsilon/2-1} (i(k_0+p_0) - v(l+\tilde{l}))}{((k_0^2 + c^2 k^2)x + ((k_0+p_0)^2 + v^2(l+\tilde{l})^2)y)^{1+\epsilon/2}} \\ &= \frac{g^2 M^\epsilon 2(c(2\pi))^\epsilon \pi^{-\epsilon/2}}{(2\pi)^4 c^2} \int dk_0 dl \int_0^1 dx dy \times \\ &\quad \frac{\delta(x+y-1) x^{\epsilon/2-1} (i(k_0+p_0) - v(l+\tilde{l}))}{(k_0^2(x+y) + c^2 l^2 x + 2k_0 p_0 y + p_0^2 y + v^2 l^2 y + 2v^2 \tilde{l} y + v^2 \tilde{l}^2 y)^{1+\epsilon/2}}.\end{aligned}$$

Applying the constraint $x^2 + y^2 = 1$ from the delta function to the denominator term $k_0^2(x+y) = k_0^2$, we now have

$$\Sigma(p_0, \tilde{l}) = \frac{g^2 M^\epsilon 2(c(2\pi))^\epsilon \pi^{-\epsilon/2}}{(2\pi)^4 c^2} \int_0^1 dx dy \frac{\delta(x+y-1) x^{\epsilon/2-1} i(k_0+p_0) - v(l+\tilde{l})}{(k_0^2 + c^2 l^2 x + 2k_0 p_0 y + p_0^2 y + v^2 l^2 y + 2v^2 \tilde{l} y + v^2 \tilde{l}^2 y)^{1+\epsilon/2}}.$$

Integrating over dy , we find

$$\begin{aligned}\Sigma(p_0, \tilde{l}) &= \sigma \int_0^1 dx \times \\ &\quad \frac{x^{\epsilon/2-1} i(k_0+p_0) - v(l+\tilde{l})}{(k_0^2 + c^2 l^2 x + 2k_0 p_0(1-x) + p_0^2(1-x) + v^2 l^2(1-x) + 2v^2 \tilde{l}(1-x) + v^2 \tilde{l}^2(1-x))^{1+\epsilon/2}}\end{aligned}$$

where $\sigma = \frac{g^2 M^\epsilon 2(c(2\pi))^\epsilon \pi^{-\epsilon/2}}{(2\pi)^4 c^2}$. Shift

$$k_0 \rightarrow k_0 - p_0(1-x)$$

and

$$l \rightarrow l - \frac{v^2 \tilde{l}(1-x)}{(c^2 x + v^2(1-x))}$$

to obtain

$$\begin{aligned}\Sigma(p_0, \tilde{l}) &= \sigma \int dk_0 dl \int dx \times \\ &\quad \frac{x^{\epsilon/2-1} \left[(ik_0 - ip_0(1-x) + ip_0) + v(l - \frac{v^2 \tilde{l}(1-x)}{c^2 x + v^2(1-x)} + \tilde{l}) \right]}{(k_0^2 + p_0^2(1-x)^2 + (c^2 x + v^2(1-x))\tilde{l}^2 - \frac{v^4 \tilde{l}(1-x)^2}{c^2 x + v^2(1-x)} + v^2 \tilde{l}^2(1-x) + p_0^2(1-x))^{1+\epsilon/2}}.\end{aligned}$$

Since the denominator only contains terms even in l^2 and k_0^2 , we can drop the k_0 and l terms in the numerator. Finally shift $x \rightarrow x - 1$ to obtain

$$\Sigma(p_0, \tilde{l}) = \sigma \int dk_0 dl \int dx \frac{x(x-1)^{\epsilon/2-1} \left[ip_0(1-x) + v\tilde{l}(1 - \frac{v^2 x}{c^2(1-x)+v^2 x}) \right]}{(k_0^2 + p_0^2 x^2 + (c^2(1-x) + v^2 x)\tilde{l}^2 - \frac{v^4 \tilde{l} x^2}{c^2(1-x)+v^2 x} + v^2 \tilde{l} + p_0^2 x)^{1+\epsilon/2}}.$$

We evaluate the dl and dk_0 integral using the using formula[23]

$$\int \frac{dad b}{(a^2 + cb^2 + y)^z} = \frac{\pi y^{1-z}}{c^{1/2}(z-1)}. \quad (4.43)$$

Then the integral becomes

$$\begin{aligned}
& \sigma \pi \int dx \frac{(x-1)^{\epsilon/2-1} \left(ip_0(1-x) + v\tilde{l} \left(1 - \frac{v^2 x}{c^2(1-x)+v^2 x} \right) \right)}{\sqrt{c^2(1-x)+v^2 x} (1 + \frac{\epsilon}{2} - 1)} \times \\
& \quad \left(p_0^2 x^2 - \frac{v^4 \tilde{l} x^2}{c^2(1-x)+v^2} - \frac{v^4 \tilde{l}^2 x^2}{c^2(1-x)+v^2} + v^2 \tilde{l}^2 + p_0^2 x \right)^{-\epsilon/2} \\
&= \frac{2\sigma\pi}{\epsilon} \int dx \frac{ip_0(1-x)(xv^2 + c^2(1-x)) + v\tilde{l}(xv^2 + c^2(1-x) - v\tilde{l}v^2 x)}{(xv^2 + c^2(1-x))\sqrt{c^2(1-x)+v^2 x}} \times \\
& \quad \left(\frac{xv^2 \tilde{l}^2(xv^2 + c^2(1-x)) - v^4 \tilde{l}^2 x^2 + x(1-x)p_0^2(xv^2 + c^2(1-x))}{(xv^2 + c^2(1-x))} \right)^{-\epsilon/2} \\
&= \frac{2\sigma\pi}{\epsilon} \int dx \frac{(x-1)^{\epsilon/2-1} (x-1)(ip_0(xv^2 + c^2(1-x)) + v\tilde{l}c^2)}{(xv^2 + c^2(1-x))^{\epsilon/2}} \times \\
& \quad \left(\frac{(x-1)(c^2 xv^2 \tilde{l}^2 + xp_0^2(xv^2 + c^2(1-x)))}{xv^2 + c^2(1-x)} \right)^{-\epsilon/2},
\end{aligned}$$

performing an $\epsilon \rightarrow 0$ expansion, including also the terms inside the σ , we find

$$\sigma(p_0, \tilde{l}) = \frac{g^2 4\pi^2}{(2\pi)^4 c^2} \int dx \left(\frac{ip_0}{(xv^2 + c^2(1-x))^{\epsilon/2}} + \frac{v\tilde{l}c^2}{(xv^2 + c^2(1-x))^{3/2}} \right) \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0).$$

We evaluate the first term in the integral

$$\int_0^1 dx \frac{ip_0}{(v^2 - c^2)x + c^2}^{1/2},$$

as follows: Let

$$\alpha = (v^2 - c^2)x + c^2 \quad \Rightarrow \quad d\alpha = (v^2 - c^2)dx$$

so that

$$\begin{aligned}
\int_0^1 dx \frac{ip_0}{(v^2 - c^2)x + c^2}^{1/2} &= \frac{ip_0}{(v^2 - c^2)} \int_0^1 dx \frac{d\alpha}{\alpha^{1/2}} \\
&= \frac{2ip_0}{v^2 - c^2} \alpha \Big|_0^1 = \frac{2ip_0}{v^2 - c^2} (\sqrt{v^2} - \sqrt{c^2}) \\
&= \frac{2ip_0(|v| - c)}{(|v| - c)(|v| + c)} \\
&= \frac{2ip_0}{(|v| + c)}.
\end{aligned}$$

The second term in the integral

$$\int dx \frac{v\tilde{l}c^2}{((v^2 - c^2)x + c^2)^{3/2}}$$

is evaluated as follows: Let

$$\alpha = (v^2 - c^2)x + c^2 \quad \Rightarrow \quad d\alpha = (v^2 - c^2)dx$$

then

$$\begin{aligned}
\int dx \frac{v\tilde{l}c^2}{((v^2 - c^2)x + c^2)^{3/2}} &= \frac{-2v\tilde{l}c^2}{v^2 - u^2} \frac{1}{\sqrt{\alpha}} \Big|_0^1 \\
&= -\frac{2v\tilde{l}c^2}{v^2 - u^2} \left(\frac{1}{\sqrt{v^2}} - \frac{1}{\sqrt{c^2}} \right) \\
&= -\frac{2v\tilde{l}c^2}{v^2 - u^2} \left(\frac{1}{|v|} - \frac{1}{c} \right) \\
&= -\frac{2v\tilde{l}c^2}{(|v| - c)(|v| + c)} \frac{(c - |v|)}{|v|c} \\
&= 2 \frac{\tilde{l}\text{sign}(v)c^2}{(c + |v|)},
\end{aligned}$$

where

$$\text{sign}(v) = \frac{v}{|v|}.$$

Hence

$$\Sigma(p_0, \tilde{l}) = \frac{g^2}{\epsilon} \frac{ip_0 + \text{sign}(v)c\tilde{l}}{4\pi^2 c^2(c + v^2)}.$$

Now using the renormalization conditions

$$\begin{aligned}
\tilde{\Sigma}(p_0, \tilde{l}) &= 0 \quad \text{at} \quad k_0 = -M \quad \text{and} \\
\frac{d\tilde{\Sigma}(p_0, \tilde{l})}{dp_0} \Big|_{p_0 = -M} &= 0,
\end{aligned}$$

we find the counter terms

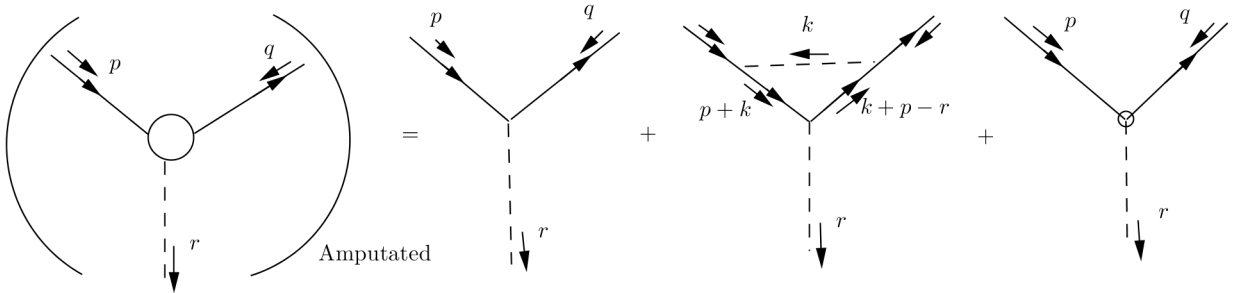
$$\delta Z_\psi = -\frac{g^2}{4\pi^2 c^2(c + |\vec{v}|)} \frac{1}{\epsilon} \quad (4.44)$$

and

$$\tilde{l}\delta v = \frac{g^2 \tilde{l}\text{sign}(v)c}{4\pi^2 c^2(c + |\vec{v}|)} \frac{1}{\epsilon}. \quad (4.45)$$

Renormalization of the Yukawa vertex

The 1-particle irreducible Feynman diagrams relevant for the Yukawa vertex to one loop are



$$\left(G_{\phi\psi\psi}^{(3)}\right)_{\text{amputated}} = M^{\epsilon/2}g + \Gamma^{(3)} + gM^{\epsilon/2}g\delta g \quad (4.46)$$

We decompose the internal fermion momenta into a tangential and perpendicular component $\vec{p} = \hat{\Omega}(k_F + \tilde{l}) = \vec{k}_F + \tilde{l}$ and note that the boson's momentum implies it is not a low energy mode. This is a clear example of UV-IR mixing. We will decompose the boson components as $\vec{r} = \hat{\Omega}(k_F + l') = k_{||} + \vec{l}'$. We do the same for the internal boson momentum $\vec{k} = \hat{\Omega}(k + l) = k_{||} + \vec{l}$ where $\vec{l}$ is the perpendicular component and $k_{||}$ are components parallel to the angular sphere $\hat{\Omega}$. The vertex loop integral $\Gamma^{(3)}$ can be evaluated as follows: Let

$$\Gamma^{(3)} = \frac{g^3 M^{3\epsilon/2}}{(2\pi)^{d+1}} \int d^{d+1}k \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} \frac{1}{i(k_0 + p_0 - r_0) - v(l + \tilde{l} - l')}.$$

To simplify the integral we will assume that $p_0 - r_0 \ll k_F$ and $\tilde{l} - l' \ll l$. The above integral becomes

$$\Gamma^{(3)} = \frac{g^3 M^{3\epsilon/2}}{(2\pi)^{d+1}} \int d^{d+1}k \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} \frac{1}{ik_0 - vl}. \quad (4.47)$$

Multiplying the integrand by

$$\frac{i(k_0 + p_0) + v(l + \tilde{l})}{i(k_0 + p_0) + v(l + \tilde{l})} \frac{ik_0 + vl}{ik_0 + vl} = 1$$

we find

$$\Gamma^{(3)} = g^3 M^{3\epsilon/2} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{i(k_0 + p_0) + v(l + \tilde{l})}{(k_0 + p_0)^2 + v^2(l + \tilde{l})^2} \frac{ik_0 + vl}{k_0^2 + v^2 l^2}.$$

Introducing the Feynman Parameters to rewrite the loop integral we obtain

$$\Gamma^{(3)} = g^3 M^{3\epsilon/2} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \int_0^1 dx dy dz \frac{\delta(x + y + z - 1)(3-1)!(i(k_0 + p_0) + v(l + \tilde{l}))(ik_0 + vl)}{[(k_0^2 + l^2 + k_{||}^2)x + ((k_0 + p_0)^2 + v^2(l + \tilde{l})^2)y + (k_0^2 + v^2 l^2)z]^3}.$$

Evaluating the integral over dz using the $\delta(x + y + z - 1)$ function we get

$$\Gamma^{(3)} = g^3 M^{3\epsilon/2} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \int_0^1 dx \int_0^{1-x} dy \times \frac{2(i(k_0 + p_0) + v(l + \tilde{l}))(ik_0 + vl)}{[k_0^2 + 2k_0 p_0 y + ((1 - v^2)x + v^2)l^2 + 2v^2 l \tilde{l} y + k_{||}^2 x + p_0^2 y + v^2 \tilde{l}^2 y]^3}.$$

To complete the square in the denominator, we shift

$$k_0 \rightarrow k_0 - p_0 y \quad \text{and} \quad l \rightarrow l - \frac{v^2 \tilde{l} y}{((1 - v^2)x + v^2)}$$

then

$$\Gamma^{(3)} = g^3 M^{3\epsilon/2} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \int_0^1 dx \int_0^{1-x} dy \times \frac{2(i(k_0 - p_0 y + p_0) + v(l - \frac{v^2 \tilde{l} y}{((1 - v^2)x + v^2)} + \tilde{l}))(i(k_0 - p_0 y) + vl - \frac{v^3 \tilde{l} y}{((1 - v^2)x + v^2)})}{[k_0^2 + ((1 - v^2)x + v^2)l^2 + k_{||}^2 x + p_0^2 y + F(x, y)]^3}.$$

where

$$F(x, y) = p_0^2 y + v^2 \tilde{l}^2 y - p_0^2 y^2 - \frac{v^4 \tilde{l}^2 y^2}{((1-v^2)x + v^2)}.$$

We will drop odd terms in the numerator since the denominator contains even terms, therefore

$$\begin{aligned} \Gamma^{(3)} = & g^3 M^{3\epsilon/2} \int \frac{dk_0 dl d^{d-1} k_{||}}{(2\pi)^{d+1}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{-k_0^2 + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x + v^2)}}{[k_0^2 + ((1-v^2)x + v^2)l^2 + k_{||}^2 x + p_0^2 y + F(x, y)]^3} \right. \\ & \left. + \frac{v^2 l^2 + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x + v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1-v^2)x + v^2)}}{[k_0^2 + ((1-v^2)x + v^2)l^2 + k_{||}^2 x + p_0^2 y + F(x, y)]^3} \right). \end{aligned}$$

Using the general integrals below to evaluate the dl and $d^{d-1}k_{||}$ loop integral

$$\int dk_0 dl d^{d-1} k_{||} \frac{k_0^2}{(Ak_0^2 + Bl^2 + Ck_{||}^2 + D)^n} = \frac{\pi^{\frac{d+1}{2}} \Gamma(\frac{2n-d-3}{2})}{2\Gamma(n)} \frac{1}{\sqrt{A^3 B C^{d-1} D^{2n-d-3}}} \quad (4.48)$$

and

$$\int dk_0 dl d^{d-1} k_{||} \frac{1}{(Ak_0^2 + Bl^2 + Ck_{||}^2 + D)^n} = \frac{\pi^{\frac{d+1}{2}} \Gamma(\frac{2n-d-1}{2})}{2\Gamma(n)} \frac{1}{\sqrt{A B C^{d-1} D^{2n-d-1}}}, \quad (4.49)$$

we obtain

$$\begin{aligned} \Gamma^{(3)} = & \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \times \\ & \left(\frac{2v^2}{\sqrt{((1-v^2)x + v^2)^3 x^{2-\epsilon} F(x, y)^\epsilon}} - \frac{2}{\sqrt{((1-v^2)x + v^2) x^{2-\epsilon} F(x, y)^\epsilon}} \right) \\ & + \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{1-\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x + v^2)} \right. \\ & \left. + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x + v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1-v^2)x + v^2)} \right) \times \\ & \frac{1}{\sqrt{((1-v^2)x + v^2) x^{2-\epsilon} F(x, y)^{2-\epsilon}}} \\ = & I_1 + I_2 \end{aligned}$$

where $d = 3 - \epsilon$ and

$$\begin{aligned} I_1 = & \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{2v^2}{\sqrt{((1-v^2)x + v^2)^3 x^{2-\epsilon} F(x, y)^{-\epsilon}}} \right) \\ & - \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{2}{\sqrt{((1-v^2)x + v^2) x^{2-\epsilon} F(x, y)^{-\epsilon}}} \right) \end{aligned} \quad (4.50)$$

and

$$\begin{aligned} I_2 = & \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{1-\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x + v^2)} \right. \\ & \left. + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x + v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x + v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1-v^2)x + v^2)} \right) \times \\ & \frac{1}{\sqrt{((1-v^2)x + v^2) x^{2-\epsilon} F(x, y)^{2-\epsilon}}} \end{aligned}$$

Evaluate I_1 first

$$\begin{aligned} I_1 &= \frac{\pi^{2-\epsilon/2}\Gamma(\frac{\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \frac{2v^2 - 2((1-v^2)x + v^2)}{((1-v^2)x + v^2)^{3/2} x^{(2-\epsilon)/2} F^{\epsilon/2}} \\ &= \frac{\pi^{\frac{4-\epsilon}{2}}\Gamma(\frac{\epsilon}{2})}{\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \frac{x(1-v^2)}{((1-v^2)x + v^2)^{3/2} x^{1-\epsilon/2} F^{\epsilon/2}}. \end{aligned}$$

Expanding $F(x, y)$ and x as $\epsilon \rightarrow 0$ when $d \rightarrow 3$ and after that evaluating the dy integral we get

$$I_1 = \frac{\pi^{(4-\epsilon)/2}\Gamma(\frac{\epsilon}{2})}{\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \frac{(1-v^2)(1-x)}{((1-v^2)x + v^2)^{3/2}} + \mathcal{O}(\epsilon^0).$$

Using the formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (4.51)$$

we get

$$I_1 = \frac{\pi^{(4-\epsilon)/2}}{2(2\pi)^{4-\epsilon}} \frac{\pi}{\Gamma(1-\frac{\epsilon}{2})\sin(\frac{\pi\epsilon}{2})} \int_0^1 dx \frac{(1-v^2)(1-x)}{((1-v^2)x + v^2)^{3/2}} + \mathcal{O}(\epsilon^0) \quad (4.52)$$

$$= \frac{\pi^{(4-\epsilon)/2}}{2(2\pi)^{4-\epsilon}} \frac{\pi \csc(\frac{\pi\epsilon}{2})}{\Gamma(1-\frac{\epsilon}{2})} \int_0^1 dx \frac{(1-v^2)(1-x)}{((1-v^2)x + v^2)^{3/2}} + \mathcal{O}(\epsilon^0). \quad (4.53)$$

Expanding in $\epsilon \rightarrow 0$ as $d \rightarrow 0$ we obtain

$$I_1 = \frac{\pi^2}{(2\pi)^2} \int_0^1 dx \frac{(1-v^2)(1-x)}{((1-v^2)x + v^2)^{3/2}}. \quad (4.54)$$

We will evaluate the integral

$$\int_0^1 dx \frac{(1-v^2)(1-x)}{((1-v^2)x + v^2)^{3/2}} \quad (4.55)$$

as follows: Consider the first piece

$$\int_0^1 dx \frac{(1-v^2)}{((1-v^2)x + v^2)^{3/2}}. \quad (4.56)$$

Let $\alpha = (1-v^2)x + v^2$ then $d\alpha = (1-v^2)dx$. Thus

$$\int_0^1 d\alpha \frac{1}{\alpha^{3/2}} = -2 \frac{1}{\alpha^{1/2}} \Big|_0^1. \quad (4.57)$$

Substituting back we obtain

$$\begin{aligned} \frac{-2}{((1-v^2)x + v^2)^{1/2}} \Big|_0^1 &= -2 \left(1 - \frac{1}{|v|}\right) \\ &= -2 \frac{(|v| - 1)}{|v|}. \end{aligned} \quad (4.58)$$

We now evaluate the second piece

$$- \int_0^1 \frac{(1-v^2)x}{((1-v^2)x + v^2)^{3/2}}. \quad (4.59)$$

Let $\alpha = (1 - v^2)x + v^2)^{3/2}$ and

$$\begin{aligned} d\alpha &= \frac{3}{2}((1 - v^2)x + v^2)^{1/2}(1 - v^2)dx \\ &= \frac{3}{2}\alpha^{1/3}(1 - v^2)dx \quad \text{and} \\ x &= \frac{\alpha^{2/3} - v^2}{1 - v^2}. \end{aligned} \tag{4.60}$$

Thus the integral becomes

$$-\frac{2}{3(1 - v^2)} \int \frac{d\alpha}{\alpha^{1/3}} \frac{\alpha^{2/3} - v^2}{\alpha} \tag{4.61}$$

$$= -\frac{2}{3(1 - v^2)} \int d\alpha \alpha^{-4/3} (\alpha^{2/3} - v^2) \tag{4.62}$$

$$= -\frac{2}{3(1 - v^2)} \left(3\alpha^{1/3} + 3\alpha^{-1/3} \right) \Big|_0^1. \tag{4.63}$$

Substituting back we obtain

$$\begin{aligned} & -\frac{2}{(1 - v^2)}(1 - 2|v| + v^2) \\ &= -2 \frac{(1 - |v|)(1 - |v|)}{(1 - |v|)(1 + |v|)} \\ &= -2 \frac{1 - |v|}{1 + |v|}. \end{aligned} \tag{4.64}$$

Adding both pieces of the integral we get

$$\begin{aligned} & -2 \left(\frac{v^2 - 1}{|v|(1 + |v|)} + \frac{1 - |v|}{1 + |v|} \right) \\ &= -\frac{2(1 - |v|)}{|v|(1 + |v|)}. \end{aligned} \tag{4.65}$$

Therefore

$$I_1 = \frac{1}{8\pi^2\epsilon} \left(\frac{1 - |v|}{|v|(1 + |v|)} \right). \tag{4.66}$$

Now lets evaluate I_2

$$\begin{aligned} I_2 &= \frac{\pi^{\frac{4-\epsilon}{2}} \Gamma(\frac{1-\epsilon}{2})}{2\Gamma(3)(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^{1-x} dy \left(\frac{ip_0 v^3 \tilde{l} y^2}{((1 - v^2)x + v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1 - v^2)x + v^2)} \right. \\ &\quad \left. + \frac{ip_0 v^3 \tilde{l} y^2}{((1 - v^2)x + v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1 - v^2)x + v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1 - v^2)x + v^2)} \right) \times \\ &\quad \frac{1}{\sqrt{((1 - v^2)x + v^2)x^{2-\epsilon} F(x, y)^{2-\epsilon}}}. \end{aligned} \tag{4.67}$$

Using the formula

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{4.68}$$

we get

$$\begin{aligned}
I_2 &= \frac{\pi^{(4-\epsilon)/2}}{2(2\pi)^{4-\epsilon}} \frac{\pi \csc(\frac{\pi\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})} \int_0^1 dx \int_0^{1-x} dy \left(\frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x+v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x+v^2)} \right. \\
&\quad \left. + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x+v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x+v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1-v^2)x+v^2)} \right) \times \\
&\quad \frac{1}{\sqrt{((1-v^2)x+v^2)x^{2-\epsilon} F(x,y)^{2-\epsilon}}} \\
&= \frac{\pi^{(4-\epsilon)/2}}{2(2\pi)^{4-\epsilon}} \frac{\pi \csc(\frac{\pi\epsilon}{2})}{2} \int_0^1 dx \int_0^{1-x} dy \left(\frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x+v^2)} - p_0^2 y^2 + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x+v^2)} \right. \\
&\quad \left. + \frac{ip_0 v^3 \tilde{l} y^2}{((1-v^2)x+v^2)} + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x+v^2)^2} - ip_0 v \tilde{l} y - \frac{v^2 \tilde{l}^2 y}{((1-v^2)x+v^2)} \right) \times \\
&\quad \frac{1}{\sqrt{((1-v^2)x+v^2)x^{2-\epsilon} F(x,y)^{2-\epsilon}}} \epsilon.
\end{aligned} \tag{4.69}$$

By expanding the Gamma function in the denominator, the integral is suppressed by ϵ . However in the limit $x \rightarrow 0$, the integral of the numerator is order ϵ , so that the would be logarithmic singularity is replaced by $\frac{1}{\epsilon}$ which cancels the ϵ suppression in front. Therefore

$$\begin{aligned}
I_2 &= \frac{\pi^3 \csc(\frac{\pi\epsilon}{2})}{2(2\pi)^4} \left(\lim_{x \rightarrow 0} \int_0^1 dx \int_0^{1-x} dy \left(p_0^2 y^2 + \frac{ip_0 v y^2}{((1-v^2)x+v^2)} + p_0^2 y - \frac{ip_0 v^3 \tilde{l} y}{((1-v^2)x+v^2)^2} \right. \right. \\
&\quad \left. \left. + iv \tilde{l} p_0 y + \frac{v^6 \tilde{l}^2 y^2}{((1-v^2)x+v^2)^2} + \frac{v^4 \tilde{l}^2 y}{((1-v^2)x+v^2)} \right) \frac{1}{\sqrt{((1-v^2)x+v^2)x^{2-\epsilon} F(x,y)^{2-\epsilon}}} \epsilon \right).
\end{aligned} \tag{4.70}$$

We obtain

$$I_2 = \frac{\pi^3 \csc(\frac{\pi\epsilon}{2})}{2(2\pi^2)} \int_0^1 dy \frac{2y(y-1)(ip_0 + v\tilde{l})^2}{|v|F(0,y)} + \mathcal{O}(\epsilon^0) \tag{4.72}$$

$$= \frac{\pi^3 \csc(\frac{\pi\epsilon}{2})}{(2\pi^2)} \int_0^1 dy \frac{2y(y-1)(ip_0 + v\tilde{l})^2}{y(1-y)(ip_0^2 + v^2 \tilde{l}^2)|v|} + \mathcal{O}(\epsilon^0). \tag{4.73}$$

The integral is independent of the y variable, so that

$$I_2 = -\frac{2\pi^2}{(2\pi)^4} \frac{1}{\epsilon} \frac{ip_0 + v\tilde{l}}{p_0^2 + v^2 \tilde{l}^2}. \tag{4.74}$$

Adding I_1 and I_2 we get

$$\Gamma^{(3)}(p_0, \tilde{l}) = \frac{1}{\epsilon} \frac{g^3}{4\pi^2(1+|v|)} \frac{ip_0 + \text{sgn}(v)\tilde{l}}{ip_0 - v\tilde{l}}. \tag{4.75}$$

The coefficient of $\frac{1}{\epsilon}$ has a momentum dependence which is strange. This makes renormalization of the Yukawa vertex complicated, since we will need a momentum dependent counter term to cancel the divergence. With a momentum dependent counter term we will need an infinite number of conditions to be able to renormalize the theory, which calls the existence of the low energy theory into question. For example, imagine having a divergent term of the form

$$\frac{1}{\epsilon} \left((\vec{p} \cdot \vec{p})^2 + (\vec{p} \cdot \vec{p})^3 \right), \tag{4.76}$$

in our theory, which means we have to add two terms of the form

$$(\vec{p} \cdot \vec{p})^2 \phi(p+q) \bar{\psi}(p) \psi(q) + (\vec{p} \cdot \vec{p})^3 \phi(p+q) \bar{\psi}(p) \psi(q), \quad (4.77)$$

in our action to renormalize the theory. In position space we will add

$$\int d^{d+1}x \phi(x) ((\vec{\nabla} \cdot \vec{\nabla})^2 \bar{\psi}) \psi + \int d^{d+1}x \phi(x) ((\vec{\nabla} \cdot \vec{\nabla})^3 \bar{\psi}) \psi. \quad (4.78)$$

This means we need two additional renormalization conditions to renormalize the theory. In our case, our divergent term is

$$\frac{1}{\epsilon} \frac{g^3}{4\pi^2(1+|v|)} \frac{ip_0 + \text{sgn}(v)\tilde{l}}{ip_0 - v\tilde{l}}. \quad (4.79)$$

Expanding the denominator in power series we will have an infinite number of terms, which means we will need an infinite number of renormalizing conditions to renormalize the theory. We interpret $\Gamma^{(3)}$ as a sum of two contributions, a local term that renormalizes the Yukawa interaction and a second piece whose significance will only be appreciated in the next chapter[24]. Therefore

$$\Gamma^{(3)}(p_0, \tilde{l}) = -gi \frac{\partial}{\partial p_0} \Sigma(p_0, \tilde{l}) + \Gamma_{oper}^{(3)}(p_0, \tilde{l}) \quad (4.80)$$

$$\begin{aligned} \Rightarrow \Gamma_{oper}^{(3)}(p_0, \tilde{l}) &= ig \frac{\partial}{\partial p_0} \Sigma(p_0, \tilde{l}) - \Gamma^{(3)}(p_0, \tilde{l}) \\ &= -\frac{g^3}{4\pi^2(1+|v|)} \frac{1}{\epsilon} + \frac{g^3}{4\pi^2(1+|v|)} \frac{1p_0 + \text{sgn}(v)}{ip_0 - v\tilde{l}} \frac{1}{\epsilon} \\ &= \frac{v\tilde{l} + \text{sgn}(v)\tilde{l}}{4\pi^2(1+|v|)(ip_0 - v\tilde{l})} \frac{1}{\epsilon} \\ &= \frac{\text{sgn}(v)}{ip_0 - v\tilde{l}} \frac{1}{\epsilon}. \end{aligned} \quad (4.81)$$

We will ignore $\Gamma_{oper}^{(3)}$ in our renormalization since it makes our theory non-renormalizable. We will consider $\Gamma_{oper}^{(3)}$ in the next section when we switch on the four-fermi interaction. Using the renormalization condition for the Yukawa vertex, we find the counter term δg is

$$\delta g = \frac{g^2}{\epsilon} \frac{1}{4\pi^2 c^2(c+|v|)}. \quad (4.82)$$

The reason why the one loop Yukawa interaction diagram has a nonlocal term is because of the kinematic constraint imposed by the Fermi-surface. The Fermi-surface has a delta function peak. This will become clearer in the next section. From here we can use the Callan-Symanzik equation to find the β -functions for the running Yukawa coupling and the Fermi-velocity. It is easier to find the expression for the β -functions from the bare parameters and counter terms than the Callan-Symanzik equation, because in the Callan-Symanzik equation we have to first compute the renormalized two point function and four point function, and then we have to solve the equations simultaneously to get the solution for the β -functions which is tedious. Adopting the counter term method, first we will evaluate γ_ψ from the expression $\psi_0 = Z_\psi \psi$ and $\delta Z_\psi = Z_\psi - 1$ as follows

$$M \frac{d\psi_0}{dM} = M \frac{d}{dM} \left(Z_\psi^{\frac{1}{2}} \psi \right),$$

where M is the renormalization scale. Since the bare parameters are independent of RG scale M the LHS is zero, and we find

$$\begin{aligned}
0 &= \frac{\psi}{2} \frac{M}{Z_\psi^{\frac{1}{2}}} \frac{dZ_\psi}{dM} + M Z_\psi^{\frac{1}{2}} \frac{d\psi}{dM} \\
&\Rightarrow -M \frac{d\psi}{dM} = \frac{\psi}{2} \frac{M}{Z_\psi} \frac{dZ_\psi}{dM} \\
&\frac{d\psi}{d(-\log(M))} = \psi \left(\frac{1}{2} \frac{M}{Z_\psi} \frac{dZ_\psi}{dM} \right) \\
&\frac{d\psi}{d(-\log(\frac{M}{P}))} = \psi \gamma_\psi
\end{aligned}$$

where

$$\gamma_\psi = \frac{1}{2} \frac{M}{Z_\psi} \frac{dZ_\psi}{dM}.$$

Note that the arbitrary scale P is introduced to make the log dimensionless. The β_g -function for the running coupling constant is computed as follows. Recall the counter term

$$\begin{aligned}
M^{\frac{\epsilon}{2}} g \delta g &= g_0 Z_\psi Z_\phi^{\frac{1}{2}} - M^{\frac{\epsilon}{2}} g \\
&\Rightarrow g_0 = \frac{M^{-\frac{\epsilon}{2}} Z_g}{Z_\psi Z_\phi^{\frac{1}{2}}} g
\end{aligned} \tag{4.83}$$

and

$$\begin{aligned}
M \frac{dg_0}{dM} &= M \frac{d}{dM} \left(\frac{M^{\epsilon/2} g (\delta g + 1)}{Z_\psi Z_\phi^{1/2}} \right) \\
0 &= \frac{\epsilon}{2} \frac{M^{\epsilon/2} g (\delta g + 1)}{Z_\psi Z_\phi^{1/2}} + \frac{M^{\epsilon/2+1} (\delta g + 1)}{Z_\psi Z_\phi^{1/2}} \frac{dg}{dM} + \frac{M^{\epsilon/2+1} g}{Z_\psi Z_\phi^{1/2}} \frac{d\delta g}{dM} - \frac{M^{\epsilon/2+1} (\delta g + 1)}{Z_\psi^2 Z_\phi^{1/2}} \frac{dZ_\psi}{dM} \\
&\quad - \frac{M^{\epsilon/2+1} (\delta g + 1)}{2 Z_\psi Z_\phi^{3/2}} \frac{dZ_\phi}{dM} \\
&\Rightarrow M \frac{dg}{dM} = -\frac{\epsilon}{2} g - \frac{M g}{\delta g + 1} \frac{d\delta g}{dM} + \frac{M}{Z_\psi} \frac{dZ_\psi}{dM} + \frac{M}{2 Z_\phi} \frac{dZ_\phi}{dM} \\
&= \left(-\frac{\epsilon}{2} - \frac{1}{\delta g + 1} \frac{d\delta g}{d \log M} + 2\gamma_\psi + \gamma_\phi \right) g
\end{aligned}$$

where $2\gamma_\psi = \frac{M}{Z_\psi} \frac{dZ_\psi}{dM}$ and $\gamma_\phi = \frac{M}{2Z_\phi} \frac{dZ_\phi}{dM}$. From our one loop calculation for the boson self energy $\gamma_\phi = 0$. Hence

$$\beta_g = \frac{dg}{dt} = \left(\frac{\epsilon}{2} - 2\gamma_\psi + M \frac{d\delta g}{dM} \right) g,$$

where $t = -\log(\frac{P}{M})$. The β_v -function for the Fermi-velocity is derived as follows,

$$\delta v = Z_\psi Z_\phi^{1/2} v_0 - v. \tag{4.84}$$

Since $Z_\phi = 1$ at one loop we find

$$v_0 = \frac{\delta v + v}{Z_\psi}.$$

Taking a derivative with respect to M .

$$\begin{aligned} M \frac{dv_0}{dM} &= M \frac{d}{dM} \left(\frac{\delta v + v}{Z_\psi} \right) = 0 \\ 0 &= \frac{M}{Z_\psi} \frac{d\delta v}{dM} + \frac{M}{Z_\psi} \frac{dv}{dM} - (\delta v + v) \frac{M}{Z_\psi^2} \frac{dZ_\psi}{dM} \\ &\Rightarrow -M \frac{dv}{dM} = -M \frac{d\delta v}{dM} + 2\gamma_\psi + \delta v \gamma_\psi, \end{aligned}$$

we will neglect the last term on the RHS because it will give us a $\mathcal{O}(g^3)$ term, which is only treated correctly at higher loops. Thus

$$\beta_v = \frac{dv}{dt} = -M \frac{d\delta v}{dM} + 2\gamma_\psi, \quad (4.85)$$

where $\beta_v = \frac{dv}{dt}$. Now we can compute the anomalous dimension, β_v and β_g explicitly. We will start with the anomalous dimension

$$\gamma_\psi = \frac{1}{2} \frac{M}{Z_\psi} \frac{dZ_\psi}{dg} \frac{dg}{dM}$$

where

$$\begin{aligned} M \frac{dg}{dM} &= M \frac{d}{dM} (M^{-\epsilon/2} \tilde{g}) \\ &= -\frac{\epsilon}{2} g. \end{aligned}$$

We found $\delta Z_\psi = -\frac{g^2}{4\pi^2 c^2 (c^2 + |v|)} \frac{1}{\epsilon}$ which implies

$$\frac{1}{Z_\psi} \frac{dZ_\psi}{dg} = -\frac{g^2}{2\pi^2 c^2 (c^2 + |v|)} \frac{1}{\epsilon}. \quad (4.86)$$

Therefore

$$\begin{aligned} \gamma_\psi &= \frac{1}{2} \frac{M}{Z_\psi} \frac{dZ_\psi}{dg} \frac{dg}{dM} \\ \gamma_\psi &= \frac{g^2}{8\pi^2 c^2 (c + |v|)} \frac{1}{(1 + \delta Z_\psi)} \\ &= \frac{g^2}{8\pi^2 c^2 (c + |v|)} (1 - \delta Z_\psi). \end{aligned}$$

Above we used the binomial theorem, $\frac{1}{1+\delta Z_\psi} \approx 1 - \delta Z_\psi$ which is valid to one loop by. Consequently

$$\gamma_\psi = \frac{g^2}{8\pi^2 c^2 (c + |v|)} + \mathcal{O}(g^4). \quad (4.87)$$

The β_v -function for the Fermi-velocity is

$$\begin{aligned}\beta_v &= -M \frac{d\delta v}{dM} + 2v\gamma_\psi \\ &= -M \frac{dg}{dM} \frac{d\delta v}{dg} + 2v\gamma_\psi.\end{aligned}$$

Earlier we found

$$\delta v = \frac{g^2 \text{sgn}(v)}{4\pi^2 c(c + |v|)} \frac{1}{\epsilon},$$

So that

$$\beta_v = \frac{g^2 \text{sign}(v)}{4\pi^2 c(c + |v|)} + \frac{vg^2}{4\pi^2 c^2(c + |v|)} \quad (4.88)$$

$$\begin{aligned}&= \frac{g^2(\text{sign}(v)c + v)}{4\pi^2 c^2(c + |v|)} \\ &= \frac{g^2(\text{sign}(v)c + |v|\text{sign}(v))}{4\pi^2 c^2(c + |v|)} \\ &= \frac{g^2 \text{sign}(v)}{4\pi^2 c^2}.\end{aligned} \quad (4.89)$$

The β_g -function for the Yukawa coupling is

$$\begin{aligned}\beta_g &= \left(\frac{\epsilon}{2} + 2\gamma_\psi - M \frac{d\delta g}{dM} \right) g \\ &= \frac{\epsilon}{2} g.\end{aligned} \quad (4.90)$$

We choose to completely ignore the Yukawa interaction counter term δg since the renormalization of the Yukawa vertex is complicated. This approximation is consistent because we switched off the four-fermi interaction which will renormalize the Yukawa interaction. We will return to this point in the next chapter. Thus

$$\begin{aligned}\beta_g &= \left(\frac{\epsilon}{2} + 2\gamma_\psi \right) g \\ &= \frac{\epsilon}{2} g - \frac{g^3}{4\pi^2 c^2(c + |v|)}.\end{aligned} \quad (4.91)$$

The Yukawa coupling has an interacting fixed point at

$$\begin{aligned}g^* &= \sqrt{2\pi^2 c^2(c + |v|)\epsilon} \\ g^* &\sim \mathcal{O}(\sqrt{\epsilon}),\end{aligned}$$

and a trivial fixed point at

$$g^* = 0.$$

The anomalous dimension γ_ψ at fixed point $g^* = \sqrt{2\pi^2 c^2(c + |v|)\epsilon}$ is

$$\begin{aligned}\gamma_\psi &= \frac{g^2}{8\pi^2 c^2(c + |v|)} \Big|_{g=g^*} \\ &= \frac{\epsilon}{4}.\end{aligned} \quad (4.92)$$

At this point the fermion 2-point function has a scale invariant form

$$G_F^2(p_0, l) = \frac{1}{(ip_0 - vl)^{1-\gamma_\psi}} F(p_0, l). \quad (4.93)$$

Recall that we claimed that in the limit $v \rightarrow 0$ as we approach a fixed point. In this limit

$$\begin{aligned} G_F^2(p_0) &= \frac{1}{(ip_0)^{1-\epsilon/4}} \\ &\sim \frac{1}{(p_0)^{1-\epsilon/4}}. \end{aligned}$$

A system behaves like a non-Fermi liquid when the fermion propagator

$$G_F^2(p_0, l) = \frac{1}{(ip_0 - vl)^{1-\gamma_\psi}} F(p_0, l)$$

has anomalous dimension γ_ψ . So at the fixed point we find a non-Fermi liquid (NFL) fixed point in which gapless bosons are coupled to a NFL. The β -functions for the Yukawa coupling and running velocity satisfy the Callan-Symanzik two point function

$$\left(M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial g} + \beta_v \frac{\partial}{\partial v} + 2\gamma_\psi \right) G_F^{(2)} = 0 \quad (4.94)$$

4.1.2 Non-Fermi liquid and Landau Damping energy regimes

In this section we will try to get the estimates of scales at which the physical phenomenon such as non-Fermi liquid and Landau damping start to emerge using the corresponding β -functions. Here we will set the scale $M = \Lambda$ where Λ is the cut off scale as usual, v_0 and g_0 will be the bare parameters at the cut off scale. We will begin by evaluating the scale at which the Fermi-velocity starts to vanish.

The scale at which the Fermi-velocity starts to vanish is evaluated as follows

$$\begin{aligned} \beta_v &= \frac{g^2}{4\pi^2 c^2} \text{sign}(v) \\ -\frac{dv}{d \log\left(\frac{P}{M}\right)} &= \frac{g^2}{4\pi^2 c^2} \text{sign}(v) \\ \Rightarrow \int_{v_0}^v dv &= \int \frac{g^2}{4\pi^2 c^2} d \log\left(\frac{P}{M}\right) \\ v - v_0 &= \frac{g^2}{4\pi^2 c^2} \log\left(\frac{P}{M}\right). \end{aligned} \quad (4.95)$$

When the Fermi velocity $v = 0$, with $M = \Lambda$ we have

$$P_{v=0} = e^{\frac{-4\pi^2 c^2 v_0}{g^2}} \Lambda.$$

The scale where non-Fermi liquid behaviour starts to emerge is determined as follows

$$\begin{aligned} \frac{dg}{d(-\log(\frac{P}{\Lambda}))} &= \frac{g^3}{8\pi^2(1+|v|)} \\ \Rightarrow -\int_{g_0}^g \frac{dg}{g^3} &= \int \frac{d \log(\frac{M}{\Lambda})}{8\pi^2 c^2(c+|v|)} \\ \frac{1}{2} \left(\frac{1}{g^2} - \frac{1}{g_0^2} \right) &= \frac{\log(\frac{M}{\Lambda})}{8\pi^2 c^2(c+|v|)}. \end{aligned}$$

If we set $g = g^* = 0$ since this is the non-Fermi liquid fixed point $g^* = \sqrt{2\pi^2 c^2 (c + |v|)} \epsilon$ we find

$$P_{NFL} = e^{-\frac{4\pi^2 c^2 (c + |v|)}{g_0^2}} \Lambda,$$

where P_{NFL} is the energy scale where non-Fermi liquid behaviour starts to emerge. The effect of Landau damping emanates from the from the boson self energy $\Pi(p_{01}, \vec{p})$ when this correction becomes comparable to the tree term.

$$\Pi(p_{01}, \vec{p}) = \frac{g^2 k_F^2}{2\pi^2 v} \left(A - \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right) \right)$$

where A is equal to 1 or 0 depending on the order of integration. This ambiguity arises because the integral is singular. We can fix ambiguity by applying the Ward identity to the integral which will be explained in the next chapter. If we integrate the perpendicular component l first then $A = 1$ as we found above. If we integrate the frequency component k_0 first then A would be zero. Choose to integrate

$$\Pi(p_0, \vec{p}) = g^2 M^\epsilon \int \frac{dl dk_0 k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ik_0 - vl} - \frac{1}{i(k_0 + p_0) - vl + v|\vec{p}| \cos \theta} \right)$$

by integrating k_0 first. We have to integrate k_0 over a finite range $0 \leq k_0 \leq p_0$ so that the integral doesn't vanish. Hence

$$\begin{aligned} \Pi(p_0, \vec{p}) &= g^2 M^\epsilon \int \frac{dl k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{1}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ik_0 - vl} - \frac{1}{i(k_0 - p_0) - vl + v|\vec{p}| \cos \theta} \right) k_0 \Big|_{k_0=0}^{k_0=p_0} \\ &= g^2 M^\epsilon \int \frac{dl k_F^{d-1} d^{d-1} \hat{\Omega}}{(2\pi)^{d+1}} \frac{p_0}{ip_0 - v|\vec{p}| \cos \theta} \left(\frac{1}{ip_0 - vl} - \frac{1}{-vl + v|\vec{p}| \cos \theta} \right). \end{aligned}$$

Evaluating the dl integral using the residues we get

$$\begin{aligned} \Pi(p_0, \vec{p}) &= i g^2 \int \frac{k_F^2 d^2 \hat{\Omega}}{(2\pi)^3} \frac{p_0}{ip_0 - v|\vec{p}| \cos \theta} \\ &= \frac{i g^2 k_F^2}{(4\pi)} \int_{-1}^1 d(\cos \theta) \frac{p_0}{ip_0 - v|\vec{p}| \cos \theta} \\ &= - \frac{i g^2 k_F^2}{(4\pi)} \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right) \\ &= M_D^2 \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right), \end{aligned}$$

where $M_D^2 = \frac{i g^2 k_F^2}{(4\pi)}$ is the Debye scale. To study Landau damping effects it is better to use the case where $A = 0$, since the case $A = 1$ makes the boson self energy look like it has a mass term. Define the scale

$$P_{LD}^2 = p_0^2 + \vec{p}^2 \quad (4.96)$$

where $p_0^2 + \vec{p}^2$ is the boson frequency momentum for the tree level propagator. The scale where the Landau damping effects start to be important is when the boson self-energy contribution starts to be comparable to the boson tree level propagator. From nearly on-shell bosons where $p_0 \sim |\vec{p}|$, this happens at

$$\begin{aligned} P_{LD}^2 &\sim \Pi(p_0, \vec{p}) \\ P_{LD}^2 &\sim M_D^2 \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right). \end{aligned} \quad (4.97)$$

At this scale when we calculate the boson amplitude we need to include the boson self energy term in the boson propagator to get the correct physics.

4.2 Chapter summary

We renormalized the low energy theory using the marriage between Wilson's idea of renormalization and Feynman, Tomanaga and Schwinger's idea of renormalization. It involves adding counter terms to the action, writing down the Feynman rules for the theory from the action and deriving a finite number of renormalization conditions, that we can use to fix the counter terms. We computed the one loop Feynman diagrams and used the renormalization conditions to fix the counter terms. The computation used a controlled weak coupling expansion around $d = 3 - \epsilon$ dimensions, using dimensional regularization. Quantum corrections to the correlation functions had poles as $\epsilon \rightarrow 0$. Applying the renormalization conditions resulted in the counter terms being dependent on ϵ poles. The dependence of counter terms on ϵ was used to obtain beta functions of the theory.

In the previous chapter when we integrated over all possible momenta modes the vacuum polarization loop integral vanished, which is an indication that at one loop there is no wave-function renormalization and consequently Landau damping is absent. In this chapter we have integrated the vacuum polarization loop integral over a finite momentum range in which case we find it doesn't vanish and there is Landau damping. The boson self energy is

$$\Pi(p_0, \vec{p}) = \frac{g^2 k_F^2}{4\pi v} \left(1 - \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right) \right).$$

We can think of this term as a non-local "mass like" term. Had this term been independent of the momentum p and Fermi velocity v parameters, the physics of this term would be well understood as a mass term that can be cancelled by a counter term at criticality. Since we know that below $d = 3$ dimensions the Yukawa and scalar self couplings are relevant, any weak interactions will still have a large effect in the low energy physics. Conversely this implies that the theory enjoys a weakly coupled UV fixed point. In the high energy regime the theory is just a Landau Fermi liquid nearly decoupled from a critical scalar boson. In this energy regime we treat the weak interaction perturbatively. As we move from the UV to IR regions the coupling flows to non trivial values. At $d = 3 - \epsilon$ where ϵ is small, we can follow these coupling flows to low energy (IR) by computing logarithmic loop divergent Feynman diagrams of the theory and thereby obtain the renormalization group flows equations for the couplings. However the Landau damping effect from the boson self energy above, creates some obstacles as we flow to low energy, because it starts to behave like a relevant operator (that is, it starts to increase at low energy). This can be seen by taking $p_0 \rightarrow 0$ while keeping the other terms constant in $\Pi(p_0, \vec{p})$. As $p_0 \rightarrow 0$ the trigonometric function

$$\tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right),$$

will remain constant. On the other hand the RG flow for the four Fermi velocity takes $v \rightarrow 0$ at low energy. The physics at low energy when $v \rightarrow 0$ is characterised by the fermions becoming so heavy (the effective mass of the fermions diverges, since $v = \frac{k_F}{m}$) that they can no longer be created (even in virtual loops), and are therefore similar to fermions in relativistic zero density case. Consequently the fermions loops vanish for v exactly equal to zero, and the finite density fermions do not contribute to the scalar coupling flow β_λ . These terms can thus be set to zero and the fact that they could have been relevant and hence spoiled the low energy effective theory is not realized.

The fermion self energy

$$\Sigma(p_0, \tilde{l}) = \frac{g^2}{\epsilon} \frac{ip_0 + \text{sign}(v)c\tilde{l}}{4\pi^2 c^2(c + v^2)},$$

exhibit a non-zero anomalous dimension. This is a smoking gun signal of non-Fermi liquid (NFL) behaviour and a running of Fermi velocity. An interesting property of the fermion self-energy that we should note, is that, it is not proportional to the tree level kinetic term $ip_0 - v\tilde{l}$. This is an indication that there is both fermion wave-function renormalization and a quantum correction to the fermion Fermi velocity. The term

$$\frac{ig^2 p_0}{4\pi c^2(c+v)} \frac{1}{\epsilon},$$

is responsible for the wave-function renormalization for the fermions, and the other term which is

$$\frac{\text{sign}(v)c\tilde{l}}{4\pi c^2(c+v^2)} \frac{1}{\epsilon},$$

is responsible for the renormalization of the non-relativistic velocity of fermions, which gives quantum corrections to the classical velocity of the fermions. We evaluated the one loop Yukawa vertex

$$\Gamma^{(3)} = \frac{g^3 M^{3\epsilon/2}}{(2\pi)^{d+1}} \int d^{d+1}k \frac{1}{k_0^2 + l^2 + k_\parallel^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} \frac{1}{i(k_0 + p_0 - r_0) - v(l + \tilde{l} - l')}.$$

To do this we decompose the internal fermion momenta into a tangential and perpendicular component $\vec{p} = \hat{\Omega}(k_F + \tilde{l}) = \vec{k}_F + \tilde{l}$ and note that the boson's momentum implies it is not a low energy mode. This is a clear example of UV-IR mixing. Here extracting the pole for $\Gamma^{(3)}$ is not trivial as compared to the fermion self energy. The reason, being, there are two contributions to the pole; one coming from the UV region and another from the IR region where the two fermions poles approach the real axis. The Yukawa loop integral was evaluated, with the result

$$\Gamma^{(3)}(p_0, \tilde{l}) = \frac{1}{\epsilon} \frac{g^3}{4\pi^2(1+|v|)} \frac{ip_0 + \text{sgn}(v)\tilde{l}}{ip_0 - v\tilde{l}}.$$

This shows that the Fermi surface interacting with the gapless boson has a three point correlation function with a ϵ pole that depends on $(p_0, \vec{p})$. The momentum dependence, comes from integrating out the light degrees of freedom near the Fermi surface. Physically this correspond to virtual low momenta particle-holes. Momentum dependent divergences are signals that locality of the quantum field theory has been lost. This poses a dilemma in computing the Yukawa coupling flow and renormalising the theory in general, because we will need an infinite number of renormalising conditions. The conundrum of the momentum dependent term for one loop Yukawa vertex will be properly addressed in the next chapter. Here we extract a local contribution, proportional to the derivative of the self energy,

$$ig \frac{\partial}{\partial p_0} \Sigma(p_0, \tilde{l}) = \frac{g^3}{4\pi^2(1+|v|)} \frac{1}{\epsilon}$$

and we will use it to renormalize the theory and the coupling flow. In this case, the one loop correction contribution to δg and δZ_ψ to the beta function of g precisely cancel, and the Yukawa coupling flow becomes

$$\beta_g = \frac{\epsilon}{2} g,$$

this shows that β_g is proportional to the classical dimension of the coupling. This implies that the theory in $d = 3$ does not admit one loop fixed points. If we totally discard the Yukawa quantum correction, the Yukawa coupling beta function becomes

$$\beta_g = \frac{\epsilon}{2} g - \frac{g^3}{4\pi^2 c^2(c+|v|)},$$

The scalar coupling beta function (β_λ) is negative in (4.34). A negative sign for coupling flow means that the coupling strength λ_ϕ will be strong at high energy and weak at low energy. If the sign is positive, the coupling strength will be strong at low energy and weak at high energy. Thus the scalar coupling will vanish at low energy. The fixed points of the β_g -function are at $g = 0$ and $g_* = \frac{16\pi^2}{3}\epsilon$ (this is the famous Wilson-Fischer fixed point). At the fixed points the theory is conformally invariant. The Yukawa interaction induces non-Fermi liquid behaviour through wavefunction renormalization of the fermions. At the non-trivial Yukawa fixed point, the anomalous dimension of the fermions is $\gamma = \frac{\epsilon}{4}$. At this fixed point, corresponding to the energy scale

$$P_{NFL} = e^{-\frac{4\pi^2 c^2 (c+|v|)}{g_0^2}} \Lambda,$$

the system behaves as a non-Fermi liquid. The physics and transport properties of the system can no longer be described by the Landau theory. The construction of a theory to explain the physics in this energy regime, where the system behaves as a non-Fermi liquid, is an open challenge.

The effect of Landau damping becomes important when the term for the vacuum polarization becomes comparable to the tree level diagram term. This happens at an energy scale

$$P_{LD}^2 \sim M_D^2 \frac{p_0}{v|\vec{p}|} \tan^{-1} \left(\frac{v|\vec{p}|}{p_0} \right).$$

Comparing these scales we see that the Landau damping phase has a loop suppression proportional to $g^2/4\pi^2 v = M_D^2$, while the non-Fermi liquid has an exponential suppression of factor $4\pi^2(1+|v|)g^2$, and the Fermi liquid has an exponential suppression of factor $4\pi^2 c^2 v_0/g_0^2$. The last important scale not computed here, is the energy scale region where the superconductivity instabilities takes place and it is of order [24]

$$P_{SC} = e^{-\gamma \pi^2 v_0^{\frac{1}{2}}/g_0} \Lambda,$$

where the subscript SC stand for superconductivity, and $\gamma \sim \mathcal{O}(1)$. If we assume that at weak coupling $g_0^2 \ll 1$ and $v_0 \gg 1$, so that $\frac{g_0^2}{v} \ll 1$. Comparing the energy regimes of these phases, they can be ranked as follows

$$P_{LD} > P_{SC} > P_{NFL} > P_{v=0}.$$

This means that non-Fermi liquid is always preceded by the superconducting phase, and both of them are preceded by a Landau damping phase.

Chapter 5

Double and Triple logs at one loop

5.1 Non-Fermi liquid toy model with a quartic Fermion coupling

In our previous study of the finite density non-relativistic QFT for non-Fermi liquids we decided to set the four-Fermi interaction strength to zero ($\lambda_\psi = 0$). In this section we will relax this. The reason why we didn't include it is because the one loop correction to the 4-Fermi interaction gives rise to $\log^2$ and $\log^3$ divergences which is contrary to relativistic QFT where the $\log^2$ and $\log^3$ start to appear in two loop and three loop orders. In this section we will try to explain why divergences pose a problem to the RG flows, how these divergences appear and how we can handle them. Our toy model action for the theory is

$$S = \int d\tau d^d x \left(\frac{1}{2} [(\partial_\tau \phi_0)^2 + c^2 (\vec{\nabla} \phi_0)^2 + \frac{\lambda_{\phi_0}}{4!} \phi_0^4] + \bar{\psi}_0 (\partial_\tau + \mu - \epsilon(i\vec{\nabla})) \psi_0 + g_0 \phi_0 \bar{\psi}_0 \psi_0 + \lambda_{\psi_0} (\bar{\psi}_0 \psi_0)^2 \right). \quad (5.1)$$

Note that the action is similar to the one we considered in the last section. We have just added the fermion quartic interaction term. In terms of physical parameters the theory is

$$\mathcal{L} = \frac{1}{2} (p_0^2 + \vec{p}^2) \phi(p) \phi(-p) + M^\epsilon \lambda_\phi \int \prod_i^3 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \phi(p) \phi(p_1) \phi(p_2) \phi(p_3) \quad (5.2)$$

$$\begin{aligned} & + \bar{\psi}(-p) (ip_0 - lv) \psi(p) + M^{\frac{\epsilon}{2}} g \int \prod_i^2 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p) \phi(p) \bar{\psi}(p_1) \psi(p_2) \\ & + \frac{1}{2} (p_0^2 \delta Z_\phi + \vec{p}^2 \delta Z_\phi) \phi(p) \phi(-p) + \bar{\psi}(-p) (ip_0 \delta Z_\psi - l \delta v) \psi(p) \\ & + M^\epsilon \lambda_\phi \delta \lambda_\phi \int \prod_i^3 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \phi(p) \phi(p_1) \phi(p_2) \phi(p_3) \\ & + M^{\frac{\epsilon}{2}} g \delta g \int \prod_i^2 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1 + p_2 + p) \phi(p) \bar{\psi}(p_1) \psi(p_2) \\ & + \lambda_\psi \int \prod_i^3 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \bar{\psi}(p) \bar{\psi}'(p_1) \psi(p_2) \psi'(p_3) \\ & + \lambda_\psi \delta \lambda_\psi \int \prod_i^3 \frac{d^{d+1} p_i}{(2\pi)^{d+1}} \delta^{d+1}(p + p_1 + p_2 + p_3) \bar{\psi}(p) \bar{\psi}'(p_1) \psi(p_2) \psi'(p_3). \end{aligned} \quad (5.3)$$

All the counter terms are the same as in (4.8). We only have a new counter term for the fermion coupling

$$\lambda_{\psi_0} = \frac{(\lambda_\psi + \lambda_\psi \delta\lambda_\psi)}{Z_\psi^2}. \quad (5.4)$$

It will be wise to first evaluate the one loop Feynman diagrams for the fermion coupling term before we state the renormalization conditions for the theory. Here we will only consider the BCS interaction. In order to exhibit the $\log^2$ and $\log^3$ divergences from the Feynman diagrams, it is best to use a hard cut-off method for this computation. Later we will switch back to the usual dimensional regularization. If a diagram is $g^2 \log(\Lambda)$ divergent in a hard cut off method it is proportional to g^2/ϵ in dimensional regularization. Thus not much physics is lost with the use of either regularization method, since it is simple to switch between them. We only employ the hard cut off here to illustrate the $\log^2$ and $\log^3$ divergences. Even though the kinematics of the bosons and fermions are different, it is still sensible to define the cutoff scale to be $p^2 < \Lambda^2$ and $|\epsilon(p)| < \Lambda$ for both the bosons and fermions and we will still be working in the region with scale $ck_F \gg \Lambda$.

5.1.1 UV-IR Mixing

In the previous section when we tried to renormalize the Yukawa coupling (boson-fermion cubic coupling), we ran into trouble with (4.47)

$$\Gamma^{(3)} = \frac{g^3}{4\pi^2(1 + |v|)\epsilon} \frac{ip_0 - \text{sgn}(v)p_\perp}{ip_0 - vp_\perp}.$$

Note we have changed notation $\tilde{l} \rightarrow p_\perp$. There is a divergence which we did not account for, and which we will denote Γ_{sing} . Γ_{sing} is a contribution from the delta function peak on the Fermi surface, arising when we have a product of two fermion propagators. This singular momentum dependent term is a consequence of enhanced quantum effects by a Fermi-surface of low energy excitations. One should note that close to the Fermi-surface, the product of two fermion propagators with momenta p and $p + q$ has the structure [18]

$$G(p)G(p + q) \approx G(p)^2 + \frac{2\pi \text{sgn}(v)q_\perp}{iq_0 - vq_\perp} \delta(p_0) \delta(p_\perp) \quad (5.5)$$

with the limit $q \rightarrow 0$ understood. This can be verified by integrating over p_0 and $p_\perp$ on both sides of the above equation as follows

$$\int dp_0 dp_\perp G(p)G(p + q) \sim \int dp_0 dp_\perp \left(G(p)^2 + 2\pi \frac{\text{sgn}(v)q_\perp}{iq_0 - vq_\perp} \right). \quad (5.6)$$

Therefore

$$\int dp_0 dp_\perp \frac{1}{ip_0 - vp_\perp} \frac{1}{i(p_0 + q_0) - v(p_\perp + q_\perp)} \sim \int dp_0 dp_\perp \left(\frac{1}{(ip_0 - vp_\perp)^2} + 2\pi \frac{\text{sgn}(v)q_\perp}{iq_0 - vq_\perp} \right), \quad (5.7)$$

the first term on the RHS is equal to zero since both poles sit on the same side of the real line axis and the term on the LHS simplifies as follows

$$\begin{aligned}
\int dp_0 dp_\perp \frac{1}{ip_0 - vp_\perp} \frac{1}{i(p_0 + q_0) - v(p_\perp + q_\perp)} &= \int dp_0 dp_\perp \left(\frac{1}{ip_0 - vp_\perp} - \frac{1}{i(p_0 + q_0) - v(p_\perp + q_\perp)} \right) \times \\
&\quad \frac{1}{iq_0 - vq_\perp} \\
&= 2\pi i \int dp_\perp \left(i\Theta(p_\perp) - i\Theta(p_\perp + q_\perp) \right) \frac{\text{sgn}(v)}{iq_0 - vq_\perp} \\
&= 2\pi \frac{\text{sgn}(v)q_\perp}{iq_0 - vq_\perp}, \tag{5.9}
\end{aligned}$$

where $\text{sgn}(v)$ determines whether the pole is situated above or below the real line axis. In Fermi liquids without gapless bosons, the singular second term is not included. Since there is a gap in the spectrum, there are no very low energy modes that would be sensitive to the singular term and it does not contribute. However, when a system is coupled to a gapless boson, such a term appears in the product of fermion propagators and it leads to logarithmic enhancement from the exchange of virtual bosons. An example is the one loop correction to the cubic vertex we considered in (4.47). As a simple illustration of how this singular momentum dependent term arises, consider multiplying (5.5) with a boson propagator $D(p)$

$$\begin{aligned}
\int dp_0 \int dp_\perp \int d^2 p_\parallel D(p) \frac{q_\perp}{iq - vq_\perp} \delta(p_0) \delta(p_\perp) &\sim \frac{q_\perp}{iq_0 - vq_\perp} \int \frac{d^2 p_\parallel}{p_\parallel^2} \\
&= \pi \frac{q_\perp}{iq_0 - vq_\perp} \int \frac{dp_\parallel^2}{p_\parallel^2}, \tag{5.10}
\end{aligned}$$

which produces an extra log divergences. This is a UV-IR mixing where low energy fermionic excitations can exchange high energy bosonic modes. Note that in dimensional regularization[18]

$$G(p)G(p+q) \approx G(p)^2 + \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \delta(p_0) \delta(p_\perp). \tag{5.11}$$

Then

$$\int dp_0 \int dp_\perp \int d^2 p_\parallel D(p) \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \delta(p_0) \delta(p_\perp) \sim \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \int \frac{d^2 p_\parallel}{p_\parallel^2} \tag{5.12}$$

which produces the Γ_{sing} term.

5.1.2 Divergences and RG at tree level

Recall that we can write the 4-Fermi interaction as[18]

$$S \supset - \int \prod_i d^{d+1} \lambda_\psi(k_1 \cdots k_4) \delta^{d+1}(k_1 + k_2 + k_3 + k_4) \psi(\bar{k}_1) \psi(k_2) \psi(\bar{k}_3) \psi(k_4). \tag{5.13}$$

Only two kinematic configurations are marginal, due to the constraint imposed by the Fermi-surface. These are the forward scattering (FS) and BCS. In FS the angle between the incoming pair of quasi-particles is the same as the out going and in BCS the incoming angles of a quasi-particle are opposite. Therefore in FS $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$ or a permutation of these and for BCS $\theta_1 = -\theta_2$ and $\theta_3 = -\theta_4$. So we can write the Fermi coupling as a function of angles $\lambda(\theta_i - \theta_j)$

to distinguish BCS or FS. It will be convenient to expand the function in terms of spherical harmonics as[18]

$$\lambda_L = \int_0^{2\pi} \frac{d\theta}{\sqrt{2\pi}} e^{iL\theta} \lambda_\psi(\theta) \quad (5.14)$$

In $d = 3$ space dimensions we use the Legendre polynomials and $\cos \theta_{ij} = \frac{\vec{k}_i \cdot \vec{k}_j}{k_F^2}$. The effect of integrating out a high momentum boson mode generates an effective $(\bar{\psi}\psi)^2$ interaction

$$S \supset -\frac{g^2}{2} \int_{p,p_1,p_2} \frac{\Theta(q^2/\Lambda^2)}{q_0^2 + \vec{q}^2} \bar{\psi}(p) \psi(p_1 + p) \bar{\psi}(p_2 - p) \psi(p_1), \quad (5.15)$$

where the Heaviside function Θ has the effect of restricting q to a shell of size $\delta\Lambda$. In the notation of (5.13) this generates a 4-Fermi coupling

$$\lambda_\psi(p, p_1 + p, p_2 - p, p_1) = -\frac{g^2}{2} \frac{\Theta(q^2/\Lambda^2)}{q_0^2 + \vec{q}^2}. \quad (5.16)$$

We can absorb this momentum dependent vertex into a renormalization of coupling constants. This can be achieved by writing the coupling in the angular momentum basis

$$\delta\lambda_L = \frac{g^2}{2} \int \frac{d^2 q_{||}}{(\sqrt{2\pi})^2} \frac{e^{i\vec{L} \cdot \vec{q}/k_F^2}}{q_0^2 + q_{||}^2 + q_{\perp}^2} \Theta\left(\frac{q^2}{\Lambda^2}\right). \quad (5.17)$$

Adopting Wilson's idea of the renormalization group we can evaluate $\delta\lambda_L$ as follows

$$\delta\lambda_L = \frac{g^2}{2} \int \frac{d\theta d(q_{||}^2)}{2\pi} \frac{e^{i\vec{L} \cdot \vec{q}/k_F^2}}{q_0^2 + q_{||}^2 + q_{\perp}^2} \Theta\left(\frac{q^2}{\Lambda^2}\right). \quad (5.18)$$

Let

$$y^2 = q_0^2 + q_{\perp}^2 + q_{||}^2 \quad \text{and} \quad d(y^2) = d(q_{||}^2).$$

Then

$$\delta\lambda_L = \frac{g^2}{2} \int \frac{d\theta d(y^2)}{2\pi} \frac{e^{i|L|\sqrt{y^2 - q_0^2 - q_{\perp}^2} \cos \theta/k_F}}{y^2} \Theta(q^2/\Lambda^2). \quad (5.19)$$

Integrating momentum shell modes between $(\Lambda - d\Lambda)^2 < y^2 < \Lambda^2$, we obtain

$$\delta\lambda_L = \frac{g^2}{2} \int_0^{2\pi} d\theta \int_{(\Lambda-d\Lambda)^2}^{\Lambda^2} \frac{d(y^2)}{2\pi} \frac{e^{i|L|\sqrt{y^2 - q_0^2 - q_{\perp}^2} \cos \theta/k_F}}{y^2} \Theta(q^2/\Lambda^2). \quad (5.20)$$

Using the result

$$\int_{(\Lambda-d\Lambda)}^{\Lambda} f(x) dx = f(\Lambda) d\Lambda,$$

we find

$$\delta\lambda_L \simeq g^2 \frac{d\Lambda}{\Lambda} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i|L|\Lambda \cos \theta/k_F}$$

where we have made the approximation $\Lambda^2 - q_0^2 - q_\perp^2 \approx \Lambda^2$ for $\Lambda^2 \gg q_0^2 + q_\perp^2 \approx \Lambda^2$. Now, using the general formula

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{iz \cos \theta}$$

where J_0 is the Bessel function, we find

$$\delta\lambda_L \simeq g^2 J_0(|L|\Lambda/k_F) \frac{d\Lambda}{\Lambda}.$$

The Bessel function decays and oscillates for large $|L|$, which means the logarithmic running will effectively be turned off exponentially quickly for $\Lambda|L| \gg k_F$. Already at tree level we need to add a counter term for the 4-Fermi coupling in order to cancel the divergence. We will write the bare 4-Fermi coupling in terms of the physical coupling plus a counter term

$$\lambda_{0,L} = \lambda_L + \delta\lambda_L. \quad (5.21)$$

In dimensional regularization the counter term is

$$\delta\lambda_L = \frac{g^2}{\epsilon} \quad \text{at renormalization scale } M \quad (5.22)$$

The renormalization condition will be as before, but with renormalization scale M at $p_0 = -xM$ and $p_\perp = -M$ as in [18] where x is arbitrary. We have a tree level RG function

$$\begin{aligned} \beta_{\lambda L} &= -M \frac{d\delta\lambda_L}{dM} \\ &= -M \frac{d\delta\lambda_L}{dg} \frac{dg}{dM} \\ &= g^2, \end{aligned} \quad (5.23)$$

since $\frac{dg}{dM} = -\frac{\epsilon}{2}g$ from (4.86). A more concrete way to understand the tree level running of λ_L , is to study the Feynman diagrams dressing the 4-Fermi vertex. The one loop vertex correction Feynman diagrams, including the tree level diagrams for the fermion interaction, are

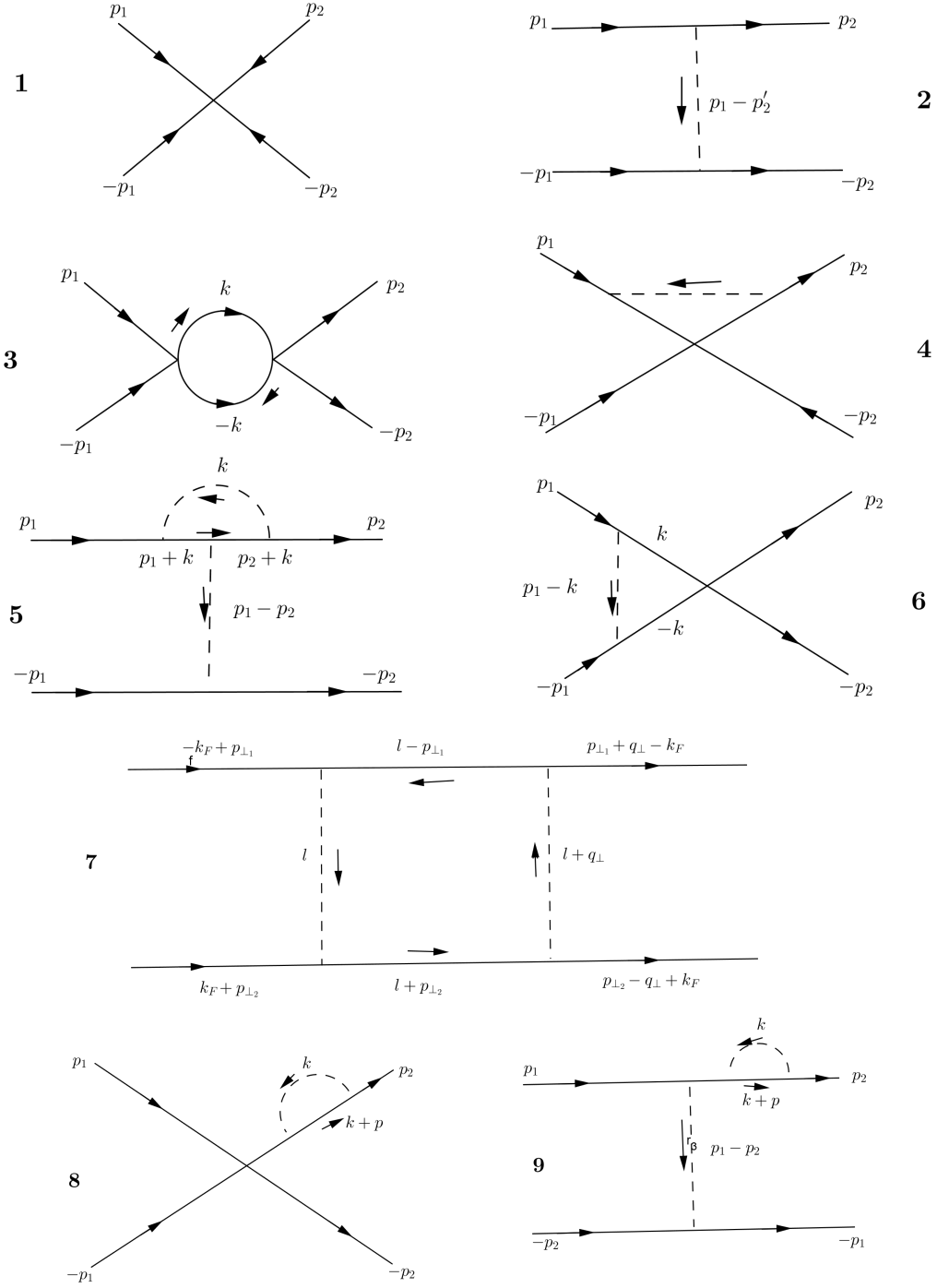


Figure 5.1: One loop Feynman diagrams for 4-Fermi interaction.

Diagram 2, which is a tree level process of boson-mediated scattering in the Cooper channel has an internal propagator

$$\delta\lambda_\psi(2) = \frac{g^2}{(p_{01} - p_{02})^2 + (\vec{p}_1 - \vec{p}_2)^2}.$$

where $\vec{p}_1$ and $\vec{p}_2$ are the external fermion momenta. The number 2 inside the brackets of λ_ψ refers to diagram number in figure 5.1. There is a logarithmic divergence hidden in this tree level diagram, which contributes to the running of λ_ψ . The easiest way to see this divergence is to

consider the decomposition into angular momentum harmonics [19]

$$\delta\lambda_L(2) = \frac{1}{2} \int_{-1}^1 d(\cos\theta) \delta\lambda_\psi(2) P_L(\cos\theta),$$

where θ is the angle between $\vec{p}_1$ and $\vec{p}_2$. For simplicity we will consider only the $L = 0$ term, since we will get similar results for all other values of L . Consider

$$\delta\lambda_{L=0}(2) = \frac{g^2}{2} \int_{-1}^1 \frac{d(\cos\theta)}{(p_{01} - p_{02})^2 - (\vec{p}_1 - \vec{p}_2)^2} \quad (5.24)$$

For small angles we know that

$$q^2 = (\vec{p}_1 - \vec{p}_2)^2 \simeq 2k_F^2(1 - \cos\theta).$$

Since $|\vec{k}_F| \gg |\vec{l}|$ and

$$d(q^2) = -2k_F^2 d(\cos\theta),$$

we find

$$\delta\lambda_{L=0}(2) = -\frac{g^2}{4k_F^2} \int \frac{d(q^2)}{(p_{01} - p_{02})^2 + q^2}.$$

If we let $y = (p_{01} - p_{02})^2 + q^2$ and $dy = d(q^2)$. Then

$$\delta\lambda_{L=0}(2) = -\frac{g^2}{4k_F^2} \int \frac{dy}{y}.$$

We will evaluate this integral by integrating a shell of momentum $< \Lambda - d\Lambda < q < \Lambda$, which gives

$$\delta\lambda_{L=0}(2) = -\frac{g^2}{4k_F^2} \int_{(p_{01}-p_{02})^2 - (\Lambda-d\Lambda)^2}^{(p_{01}-p_{02})^2 - \Lambda^2} \frac{dy}{y}.$$

The result is

$$\delta\lambda_{L=0}(2) = -\frac{g^2}{2k_F^2} \frac{d\Lambda}{\Lambda}.$$

This is log divergent. This shows that even though the diagram $\delta\lambda_{L=0}(2)$ is a tree diagram, it is log divergent and the coupling λ_L is running at tree level. The log divergence is made manifest by working in the basis of angular momentum states. The presence of the k_F scale plays a crucial role in the appearance of such a divergence because it enables the conversion of an angle-dependent scattering process in the BCS channel to a momentum transfer imparted to a scalar. This in turn can be subjected to a Wilsonian mode elimination. If the scale k_F is absent there is no log divergence. Diagram 4 exhibits a log divergence, diagrams 5 and 6 exhibit $\log^2$ divergences, diagram 7 has a $\log^3$ divergence and diagrams 8 and 9 have log and $\log^2$ divergences respectively. Explicit calculation for diagram 3 shows

$$\begin{aligned} \delta\lambda_L(3) &= (\lambda_L + \delta\lambda_L)^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{ik_0 + \mu - \epsilon(\vec{k})} \frac{1}{-ik_0 - \mu - \epsilon(-\vec{k})} \\ &= (\lambda_L + \delta\lambda_L)^2 \int_{-\Lambda}^{\Lambda} dl \int_{-\Lambda}^{\Lambda} dk_0 \int \frac{k_F^2 d^2\hat{\Omega}}{(2\pi)^4} \frac{1}{ik_0 - vl} \frac{1}{-ik_0 - vl} \\ &= (\lambda_L + \delta\lambda_L)^2 \int_{-\Lambda}^{\Lambda} dl \int_{-\Lambda}^{\Lambda} dk_0 \int_{-1}^1 \frac{k_F^2 d(\cos\theta)}{(2\pi)^3} \frac{1}{k_0^2 + v^2 l^2}. \end{aligned} \quad (5.25)$$

Setting $y = vl$ we obtain

$$\delta\lambda_L(3) = (\lambda_L + \delta\lambda_L)^2 \frac{2k_F^2}{v} \int_{-\Lambda}^{\Lambda} \frac{dy dk_0}{(2\pi)^3} \frac{1}{k_0^2 + y^2}. \quad (5.26)$$

We will perform the integrals in polar coordinates, by defining $y = r \sin \Phi$ and $k = r \cos \Phi$. The radius r has a UV cut off Λ . Therefore

$$\begin{aligned} \delta\lambda_L(3) &= (\lambda_L + \delta\lambda_L)^2 \frac{2k_F^2}{(2\pi)^2 v} \int_p^{\Lambda} dr \frac{1}{r} \int_0^{2\pi} d\Phi \\ &= (\lambda_L + \delta\lambda_L)^2 \frac{2k_F^2}{(2\pi)^2 v} \log\left(\frac{\Lambda}{p}\right) \end{aligned} \quad (5.27)$$

where we can think of p (the momentum of the external propagators) as an IR regulator. We will multiply the integral by $\frac{1}{2}$ appropriate for spinless fermions, to find

$$\delta\lambda_L(3) = (\lambda_L + \delta\lambda_L)^2 \frac{k_F^2}{(2\pi)^2 v} \log\left(\frac{\Lambda}{p}\right). \quad (5.28)$$

Now we can see explicitly that diagram 3 gives a $\log$, $\log^2$ and $\log^3$ divergence. If we didn't include the tree level counter term $\delta\lambda_L = g^2 \log(\Lambda/p)$, the diagram would just be $\log$ divergent as we hinted at earlier. Explicit computation for diagram 4 gives

$$\delta\lambda_L(4) = g^2(\lambda_L + \delta\lambda_L) \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_{01}) - v(l + \tilde{l})} \frac{1}{(k_0 + p_{02}) - v(l + l')}. \quad (5.29)$$

Set the momentum $\vec{k}_2 = \hat{\Omega}(k_F + l')$ to $\vec{k}_2 = \hat{\Omega}k_F$ which implies the perpendicular momentum $l' = 0$ and the corresponding energy frequency $p_{02} = 0$. The loop integral reduces to

$$\delta\lambda_L(4) = g^2(\lambda_L + \delta\lambda_L) \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_{01}) - v(l + \tilde{l})} \frac{1}{k_0 - vl}. \quad (5.30)$$

We have already evaluated a similar loop integral using dimensional regularization in (4.47) and we found it to be $1/\epsilon$ divergent which is $\log$ divergent in a hard cut-off regularization. Comparing the integral in (4.47) with the one above we can see that

$$\delta\lambda_L(4) = \frac{g^2(\lambda_L + \delta\lambda_L)}{4\pi^2} \frac{1}{1 + |v|} \frac{\text{sgn}(v)\tilde{l}}{ip_{01} - v\tilde{l}} \log(\Lambda/p). \quad (5.31)$$

As we remarked above, this momentum dependence comes from UV-IR mixing as a consequence of exchanging high momentum bosons that induce virtual scatterings tangential to the Fermi-surface, so that modes far below the cut off can exchange a mode near the cut-off. Note that diagram 4 has a $\log$ and $\log^2$ divergent piece. Diagram 5 is related to diagram 4 by the transformation $(\lambda_L + \delta\lambda_L) \rightarrow g^2 D(p_1 - p_2)$ with D the boson propagator. Hence

$$\begin{aligned} \delta\lambda_L(5) &= \frac{g^4}{4\pi^2} \frac{1}{1 + |v|} \frac{\text{sgn}(v)\tilde{l}}{ip_{01} - v\tilde{l}} D(p_1 - p_2) \log(\Lambda/p) \\ &= \frac{g^4}{4\pi^2} \frac{1}{1 + |v|} \frac{\text{sgn}(v)\tilde{l}}{ip_{01} - v\tilde{l}} \frac{1}{(p_1 - p_2)^2} \log(\Lambda/p). \end{aligned} \quad (5.32)$$

Diagram 5 is not one particle irreducible, even though it cancels the divergence of diagram 4. Diagram 5 renormalizes diagram 4. Now, we have to integrate over the parallel components of the boson propagator when we change into the spherical harmonics basis. Therefore

$$\delta\lambda_L(5) = \frac{g^4}{4\pi^2} \frac{1}{1 + |v|} \frac{\text{sgn}(v)\tilde{l}}{ip_{01} - v\tilde{l}} \log(\Lambda/p) \int \frac{d(\cos \theta) P_L(\cos \theta)}{(p_{01} - p_{02})^2 + (\vec{p}_1 - \vec{p}_2)^2}. \quad (5.33)$$

In (5.22) we decided to set the integration over the parallel components of the boson propagator in spherical harmonics to be equal to

$$g^2 \int \frac{d(\cos \theta) P_L(\cos \theta)}{(p_{01} - p_{02})^2 + (\vec{p}_1 - \vec{p}_2)^2} = -g^2 \log(\Lambda/p) = -\delta\lambda_L. \quad (5.34)$$

Therefore

$$\delta\lambda_L(5) = -\frac{g^4}{4\pi^2} \frac{1}{1+|v|} \frac{\text{sgn}(v)\tilde{l}}{ip_{01}-v\tilde{l}} \log^2(\Lambda/p). \quad (5.35)$$

Explicit computation for diagram 6 gives

$$\delta\lambda_L(6) = g^2(\lambda_L + \delta\lambda_L) \int \frac{dk_0 d^3k}{(2\pi)^4} \frac{1}{-ik_0 - vl} \frac{1}{ik_0 - vl} \frac{1}{(p_{01} - k_0)^2 + (\vec{p}_1 - \vec{k})^2} \quad (5.36)$$

$$= g^2(\lambda_L + \delta\lambda_L) \int \frac{dk_0 2dl k_F^2 d(\cos \theta)}{(2\pi)^3} \left(\frac{1}{(p_{01} - k_0)^2 + 2k_F^2(1 - \cos \theta)} \right) \left(\frac{1}{(-ik_0 - vl)(ik_0 - vl)} \right). \quad (5.37)$$

Integrating over $\int_{-1}^1 d(\cos \theta)$ we get[19]

$$\begin{aligned} \delta\lambda_L(6) &= \frac{g^2 k_F^2 (\lambda_L + \delta\lambda_L)}{2k_F^2} \int \frac{dk_0 dl}{(2\pi)^3} \log \left[\frac{(p_{01} - k_0)^2 + 4k_F^2}{(p_{01} - k_0)^2} \right] \frac{1}{k_0^2 + v^2 l^2} \\ &\simeq g^2(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dl}{2(2\pi)^3} \frac{1}{k_0^2 + v^2 l^2} \log \left[\frac{4k_F^2}{(p_{01} - k_0)^2} \right]. \end{aligned} \quad (5.38)$$

Since $4k_F^2 \gg (p_{01} - k_0)^2$, we neglected the term $(p_{01} - k_0)^2$ in the numerator of the argument of the log. Let $y = vl$, so that

$$\delta\lambda_L(6) = g^2(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dy}{2v(2\pi)^3} \frac{1}{k_2 + y^2} \log \left[\frac{4k_F^2}{(p_{01} - k_0)^2} \right]. \quad (5.39)$$

We will use again polar coordinates to evaluate this integral. Let $y = r \sin \phi$ and $k_0 = r \cos \phi$ and r has UV cut off Λ . Thus

$$\delta\lambda_L(6) = \frac{g^2(\lambda_L + \delta\lambda_L)}{2v(2\pi)^3} \int_p^\Lambda dr \frac{1}{r} \int_0^{2\pi} d\phi \left[\frac{4k_F^2}{(p_{01} - r \cos \theta)^2} \right] \quad (5.40)$$

$$\begin{aligned} &= \frac{g^2(\lambda_L + \delta\lambda_L)}{2v(2\pi)^3} \log(\Lambda/p) \int_0^{2\pi} d\phi \log \left[\frac{4k_F^2}{\Lambda^2 \cos^2 \theta} \right] \\ &= \frac{g^2(\lambda_L + \delta\lambda_L)}{2v(2\pi)^2} \log(\Lambda/p) \log \left[\frac{\Lambda}{2k_F} \right] + \dots \end{aligned} \quad (5.41)$$

Diagram 6 thus has a $\log^2$ and $\log^3$ divergence. While diagram 4 and 6 might look the same, diagram 6 has a stronger divergence because the boson exchange happens at anti-podal points giving an enhanced UV-IR mixing. To see how the $\log^3$ divergence arises in diagram 7, consider a Cooper-pair $\hat{\Omega}(k_F + p_2, -k_F + p_1)$ scattering into $\hat{\Omega}(k_F + p_2 + q, -k_F + p_1 - q)$ and note that $\hat{\Omega} \cdot p_1 = p_{\perp 1}$, $\hat{\Omega} \cdot p_2 = p_{\perp 2}$ and $\hat{\Omega} \cdot q = q_{\perp}$ are perpendicular components. We focus on the component $q_{\perp}$ to simplify the calculation. The loop integral for the diagram is

$$\delta\lambda_L(7) = -\frac{g^4}{(2\pi)^4} \int \frac{dk_0 dl d^2 k_{\parallel}}{(k_0^2 + l^2 + k_{\parallel}^2)(k_0^2 + (l + q_{\parallel})^2 + k_{\parallel}^2)} \frac{1}{(ik_0 - v(l + p_{\perp 2}))(ik_0 + v(l - p_{\perp 1}))}. \quad (5.42)$$

We will first evaluate the k_0 integral using the Residue Theorem. To see how a $\log^3$ divergence appears it is enough to focus on one of the fermionic residues, which is $k_0 = -iv(l + p_{\perp 2})$ for $l < p_{\perp 2}$ [19]. We will multiply the integral by two to account for the contribution from the other fermionic pole.

$$\delta_L(7) \sim \frac{-2g^4}{(2\pi)^3 v} \int \frac{dl d\theta k_{||} dk_{\perp}}{(-v^2(l + p_{\perp 2})^2 + l^2 + k_{||}^2)(-v^2(l + p_{\perp 2})^2 + (l + q_{\perp})^2 + k_{||}^2)} \frac{1}{(2l + p_{\perp 2} - p_{\perp 1})} \quad (5.43)$$

$$\sim \frac{1}{2} \frac{g^4}{(2\pi)^2 v} \int dl dk_{||}^2 \left[\frac{1}{k_{||}^2 - v^2(l + p_{\perp 2})^2 + (l + q_{\perp})^2} - \frac{1}{k_{||}^2 - v^2(l + p_{\perp 2})^2 + l^2} \right] \times \quad (5.44)$$

$$\frac{1}{(2l + p_{\perp 2} - p_{\perp 1})} \frac{1}{(l + q_{\perp})^2 - l^2} \quad (5.45)$$

$$\sim \frac{g^4}{(2\pi)^2 v} \int_{p'}^{\Lambda} dl \log \left[\frac{-v^2(l + p_{\perp 2})^2 + (l + q_{\perp})^2}{k_{||}^2 + l^2 - v^2(l + p_{\perp 2})^2} \right] \frac{1}{(-2q_{\perp} l - q_{\perp}^2)(2l + p_{\perp 2} - p_{\perp 1})} \quad (5.46)$$

$$\sim F(p_{\perp 1}, p_{\perp 2}, q_{\perp})$$

The integral is UV convergent by power counting, and equals a finite answer $F(p_{\perp 1}, p_{\perp 2}, q_{\perp})$. The correction to the tree level 4-Fermi vertex $\sim \frac{1}{q^2}$, so that further log-divergences will be induced when we integrate over $q_{||}$ to change into the spherical harmonics basis. For this we expand $F(p_{\perp 1}, p_{\perp 2}, q_{\perp})$ for large q using mathematica and we find

$$F(p_{\perp 1}, p_{\perp 2}, q_{\perp}) = \frac{g^4}{4\pi^2 v} \left(\frac{\log^2 q}{q^2} + \frac{\log q}{q^2} \right). \quad (5.47)$$

Note that we have only introduced $q_{\perp}$ for the boson propagator to make our computation tractable, but $q_{||}$ will appear in the same footing as $(\frac{1}{q_{\perp}})^2$. Therefore when we integrate over $(q_{\perp})^2$ in angular momentum basis we will get a $\log^3$ divergences [19]. The loop integral for diagram 8 is

$$\delta\lambda_L(8) = g^2(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})}. \quad (5.48)$$

Similar loop integrals were evaluated using dimensional regularization in (4.39), where we found the result

$$\frac{g^2}{\epsilon} \frac{ip_0 + \text{sgn}(v)\tilde{l}}{4\pi^2(1 + |v|)}. \quad (5.49)$$

Therefore in the hard cut-off method, $\delta\lambda_L(8)$ is

$$\delta\lambda_L(8) = \frac{g^2(\lambda_L + \delta\lambda_L)}{4\pi^2} \frac{ip_0 + \text{sgn}(v)\tilde{l}}{(1 + |v|)} \log(\Lambda/p) \quad (5.50)$$

$$= \frac{g^2(\lambda_L + g^2 \log(\Lambda/p))}{4\pi^2} \left(\frac{ip_0 + \text{sgn}(v)\tilde{l}}{(1 + |v|)} \right) \log(\Lambda/p). \quad (5.51)$$

Diagram 8 is log and $\log^2$ divergent, but $\delta\lambda_L(8)$ doesn't contribute to the fermion beta function β_{λ_L} because it is cancelled by the wave function renormalization Z_{ψ} as follows

$$+ \quad = \quad 0$$

The loop integral for diagram 9 is

$$\delta\lambda_L(9) = \frac{g^4}{(p_{01} - p_{02})^2 + (\vec{p}_1 - \vec{p}_2)^2} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})}. \quad (5.52)$$

We write the boson propagator in terms of spherical harmonics as

$$\delta\lambda_L(9) = g^4 \int_{-1}^1 \frac{d(\cos\theta)}{(p_{01} - p_{02})^2 + 2k_F^2(1 - \cos\theta)} \int \frac{dk_0 dl d^{d-1}k_{||}}{(2\pi)^{d+1}} \frac{1}{k_0^2 + l^2 + k_{||}^2} \frac{1}{i(k_0 + p_0) - v(l + \tilde{l})} \quad (5.53)$$

for $L = 0$. The second integral is the same as the one we just evaluated above, so that

$$\delta\lambda_L(9) = g^4 \int_{-1}^1 \frac{d(\cos\theta)}{(p_{01} - p_{02})^2 + 2k_F^2(1 - \cos\theta)} \left(\frac{ip_0 + \text{sgn}(v)\tilde{l}}{(1 + |v|)} \right) \log(\Lambda/p) \quad (5.54)$$

and evaluating the boson integral in spherical harmonics we get

$$\begin{aligned} \delta\lambda_L(9) &= \frac{g^4}{2k_F^2} \log \left[\frac{(p_{01} - p_{02})^2 + 4k_F^2}{(p_{01} - p_{02})^2} \right] \left(\frac{ip_0 + \text{sgn}(v)\tilde{l}}{(1 + |v|)} \right) \log(\Lambda/p) \\ &\simeq \frac{g^4}{2k_F^2} \log \left[\frac{4k_F^2}{(p_{01} - p_{02})^2} \right] \left(\frac{ip_0 + \text{sgn}(v)\tilde{l}}{(1 + |v|)} \right) \log(\Lambda/p), \end{aligned} \quad (5.55)$$

demonstrating that $\delta\lambda_L(9)$ is $\log^2$ divergent. The diagram is cancelled by the wave-function renormalization as follows,

$$+ \quad = \quad 0$$

Even though double and triple logs appear at one loop level, a proper evaluation of these Feynman diagrams shows that the beta function for the Four-Fermi coupling is independent of the cut-off scale. Thus the notion of a fixed point survives. Evaluating the Feynman diagrams properly we see that all the $\log^2$ and $\log^3$ diagrams cancel. This is exactly what happens in relativistic QFT where the $\log^2$ counter term diagrams cancel the $\log^3$ divergent diagrams. The fact that non-relativistic finite density theory has double and triple log divergent diagrams at one loop level doesn't invalidate the idea of fixed points as the divergences cancel.

5.1.3 Ward Identity in the vicinity of the Fermi-surface

Ward identities are a consequence of a symmetry. Our finite density system has a conserved charge associated to the complex rotations of the fermions, which becomes a gauge symmetry if a dynamical electromagnetic field is added. The low energy expression for a conserved current is

$$J_\mu(p) = \bar{\psi}(p)V_\mu(p)\psi(p) + \mathcal{O}(p_\perp/k_F) \quad (5.56)$$

where we have introduced a 4-vector $V_\mu = (i, -v\hat{\Omega})$ such that the scalar product is

$$V \cdot p = ip_0 - v\hat{\Omega}. \quad (5.57)$$

The conserved current leads to a Ward identity relating the 3-point function $\langle J_\mu \bar{\psi}\psi \rangle$ to the fermion self-energy

$$\langle J_\mu(q) \bar{\psi}(p+q)\psi(p) \rangle = \Gamma_\mu(p,q)G(p)G(p+q) \quad (5.58)$$

and the Ward identity is

$$q_\mu \Gamma_\mu(p,q) = G^{-1}(p) - G^{-1}(p+q), \quad (5.59)$$

where G is the full propagator. It is related to the tree level propagator and fermion self energy as follows

$$G^{-1} = G_0^{-1} - \Sigma. \quad (5.60)$$

Eq (5.59) can be used to relate the Yukawa vertex correction

$$\Gamma(p,q) = -\langle \phi(q) \bar{\psi}(p+q)\psi(p) \rangle - g, \quad (5.61)$$

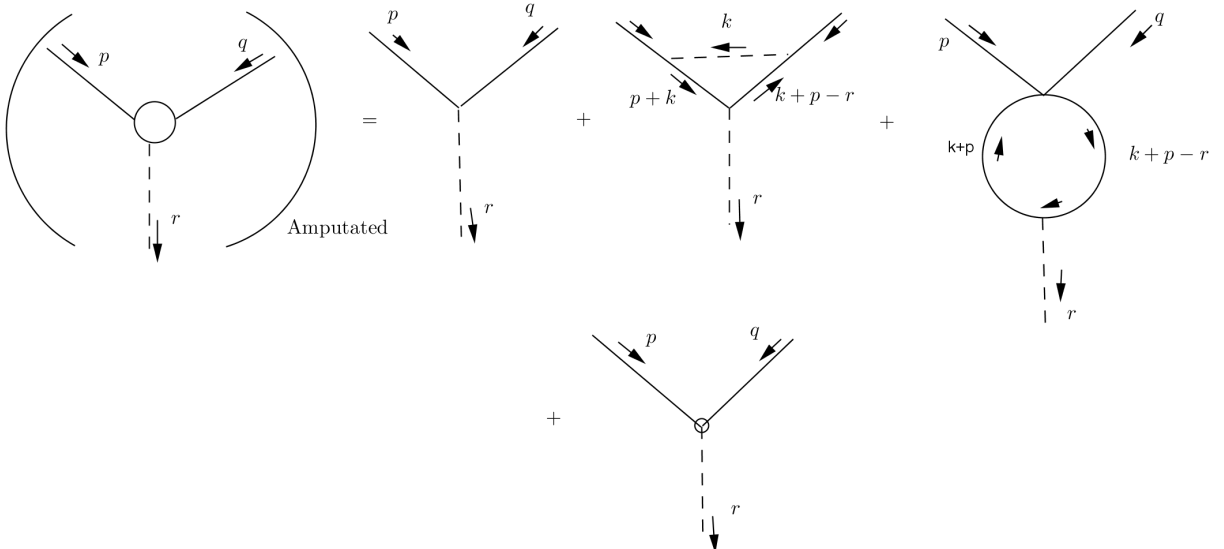
and the fermion self energy. In the approximation (5.56), we find by plugging the above expression into (5.56),

$$\Gamma(p,q) = g \frac{\Sigma(p+q) - \Sigma(p)}{V \cdot q} + \mathcal{O}(q_\perp/k_F). \quad (5.62)$$

This result will be important in what follows.

5.1.4 Renormalization for the Yukawa vertex

The one loop vertex correction diagrams are



$$\left(G_{\phi\bar{\psi}\psi}^3\right)_{\text{amputated}} = M^{\epsilon/2} + \Gamma_1^3 + \Gamma_2^3 + M^{\epsilon/2}g\delta g. \quad (5.63)$$

The renormalization condition for the Yukawa vertex is

$$\left(G_{\phi\bar{\psi}\psi}^3\right)_{\text{amputated}} = M^{\epsilon/2} \quad \text{at} \quad p^2 = q^2 = r^2 = -M^2, \quad p_0 = q_0 = r_0 = -xM \quad (5.64)$$

and $p_\perp = q_\perp = r_\perp = -M.$

For convenience we define

$$\Gamma = \Gamma_1^3 + \Gamma_2^3. \quad (5.65)$$

We have already evaluated Γ_1^3 , and we found it to be

$$\Gamma_1^3 = \frac{g^3}{4\pi(1+|v|)\epsilon} \frac{ip_0 + \text{sgn}(v)p_\perp}{ip_0 - vp_\perp}. \quad (5.66)$$

Note we have changed notation $\tilde{l} \rightarrow p_\perp$. Γ consists of a regular term and a singular term

$$\Gamma = \Gamma_{\text{sing}} + \Gamma_{\text{reg}}. \quad (5.67)$$

When we renormalized our theory in the previous section, we didn't account for Γ_{sing} . The loop integral for Γ_2^3 is

$$\Gamma_2^3 = -g(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dk_\perp d^{d-1}k_\parallel}{(2\pi)^{d+1}} \frac{1}{i(k_0 + p_0) - v(k_\perp + p_\perp)} \frac{1}{i(k_0 + p_0 - r_0) - v(k_\perp + p_\perp - r_\perp)} \quad (5.68)$$

and we make the same assumption we made in (4.47) that is we will assume $p_0 - r_0 \ll k_F$ and $p_\perp - r_\perp \ll k_\perp$. Thus

$$\Gamma_2^3 = -g(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dk_\perp d^{d-1}k_\parallel}{(2\pi)^{d+1}} \frac{1}{i(k_0 + p_0) - v(k_\perp + p_\perp)} \frac{1}{ik_0 - vk_\perp}. \quad (5.69)$$

We evaluate Γ_2^3 using the identity

$$G(p)G(p+q) \approx G(p)^2 + \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \delta(p_0)\delta(p_\perp), \quad (5.70)$$

to find

$$\Gamma_2^3(p) = -\frac{g(\lambda_L + \delta\lambda_L)}{8\pi^2} \frac{i\pi}{|v|} \frac{ip_0 + vp_\perp}{ip_0 - vp_\perp}. \quad (5.71)$$

Then

$$\Gamma = -\frac{g^2}{8\pi^2|v|\epsilon} \frac{1-|v|}{1+|v|} - \frac{g\lambda_L}{8\pi^2|v|} \frac{ip_0 + vp_\perp}{ip_0 - vp_\perp}. \quad (5.72)$$

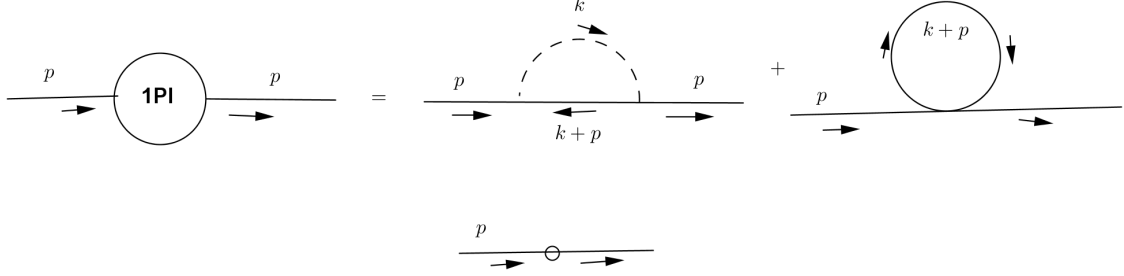
Finally, using the renormalization condition, we find

$$\delta g = -\frac{g^3}{8\pi^2} \frac{1-|v|}{|v|(1+|v|)} \frac{1}{\epsilon} - \frac{\lambda_L}{8\pi^2|v|} \frac{1-ix/v}{1+ix/v} \quad (5.73)$$

where $p_0 = -xM$ and $p_\perp = -M$.

5.1.5 Renormalization of the Fermion Propagator

The one-particle irreducible Feynman diagrams for the Fermion 2-point function are



$$\tilde{\Sigma}(p_0, p_\perp) = \Sigma_1(p_0, p_\perp) + \Sigma_2(p_0, p_\perp) + ip_0 \delta Z_\psi - p_\perp \delta v \quad (5.74)$$

Let $\Sigma = \Sigma_1 + \Sigma_2$. We have already evaluated Σ_1 and we found

$$\Sigma_1(p) = \frac{g^2}{\epsilon} \frac{ip_0 + \text{sgn}(v)p_\perp}{4\pi(1 + |v|)}. \quad (5.75)$$

The loop integral for diagram Σ_2 is

$$\Sigma_2(p) = -(\lambda_L + \delta\lambda_L) \int \frac{dk_0 dk_\perp d^2 k_\parallel}{(2\pi)^{d+1}} \frac{1}{i(p_0 + k_0) - v(p_\perp + k_\perp)}. \quad (5.76)$$

The value of this loop is ambiguous in the low energy limit as a consequence of the fact that the integral diverges. We will regularize it in a way that is consistent with the Ward identity developed in (5.62). By power counting, the loop integral Σ_2 is linearly divergent, which complicates the regularization of the subleading logarithmically divergent kinetic terms. So we compute the less divergent quantity $\Sigma_2(p) - \Sigma_2(0)$. Without any loss of generality, we have

$$\Sigma_2(p) - \Sigma_2(0) \propto -(\lambda_L + \delta\lambda_L) \int dk_0 \int dk_\perp \left(\frac{1}{i(p_0 + k_0) - v(p_\perp + k_\perp)} - \frac{1}{ik_0 - vk_\perp} \right) \quad (5.77)$$

$$= -(\lambda_L + \delta\lambda_L) \int dk_\perp (\Theta(p_\perp + k_\perp) - \Theta(k_\perp)) \text{sgn}(v) \quad (5.78)$$

$$= -(\lambda_L + \delta\lambda_L) p_\perp \text{sgn}(v) \quad (5.79)$$

where $\text{sgn}(v)$ determines whether the pole is situated above or below the real line axis. The integral has an ambiguity which can be seen by shifting the $dk_\perp$ integral of Σ_2 by $-p_\perp$ relative to $\Sigma_2(0)$, in which case the integral vanishes. This ambiguity can be parametrized by a shift in $a(k)$ in the integration variable k as follows

$$\Sigma_2(p) - \Sigma_2(0) \propto -(\lambda_L + \delta\lambda_L) \int dk_0 \int dk_\perp \left(\frac{1}{i(p_0 + k_0 + a_0(p)) - v(p_\perp + k_\perp + a_\perp(p))} - \frac{1}{ik_0 - vk_\perp} \right) \quad (5.80)$$

$$= -(\lambda_L + \delta\lambda_L) \int dk_\perp \left(\Theta(k_\perp + p_\perp + a_\perp(p)) - \Theta(k_\perp) \right) \text{sgn}(v) \quad (5.81)$$

$$= -(\lambda_L + \delta\lambda_L) \left(\Theta(p_\perp + a_\perp(p)) \right) \text{sgn}(v). \quad (5.82)$$

The parameter $a_\perp(p)$ represents an additional piece of data that must be input to the theory, either from matching to a UV theory or by the low energy constraints. When we Taylor expand $a_\perp$ in p on the RHS, we can discard the first constant piece since $\Sigma_2(p) - \Sigma_2(0) = 0$ at $p = 0$. We also discard terms of $\mathcal{O}(p_0^2, p_\perp^2)$ and higher order terms since these are irrelevant. Hence we are left with $a_\perp(p) = a_1 p_0 + a_2 p_\perp$. This gives

$$\Sigma_2(p) = -\frac{(\lambda_L + \delta\lambda_L)}{4\pi^2} \text{sgn}(v)(a_1 p_0 + a_2 p_\perp + p_\perp). \quad (5.83)$$

Our choice of a_1 and a_2 is fixed by the Ward identity. In this case a_1 and a_2 should be

$$a_1 = \frac{i}{2|v|\text{sgn}(v)} \quad \text{and} \quad a_2 = -\frac{1}{2}, \quad (5.84)$$

for the Ward identity (5.62) to be satisfied. Therefore

$$\Sigma_2(p) = \frac{-(\lambda_L + \delta\lambda_L)}{8\pi^2|v|}(ip_0 + vp_\perp), \quad (5.85)$$

and then

$$\tilde{\Sigma}(p_0, p_\perp) = -\frac{1}{8\pi^2|v|} \left(\frac{1-|v|}{1+|v|} \frac{g^2}{\epsilon} + \lambda_L \right) (ip_0 - vp_\perp) - \frac{\lambda_L}{4\pi^2} \text{sgn}(v)p_\perp + ip_0 \delta Z_\psi - p_\perp \delta v. \quad (5.86)$$

Using the renormalization condition in the previous section we find the wave function counter term is

$$\delta Z_\psi = \frac{1}{8\pi^2|v|} \left(\frac{1-|v|}{1+|v|} \frac{g^2}{\epsilon} + \lambda_L \right) \quad (5.87)$$

and the 4-Fermi velocity counter term is

$$\delta v = \frac{1}{8\pi^2|v|} \left(\frac{1-|v|}{1+|v|} \frac{g^2}{\epsilon} + \lambda_L \right) v - \frac{\lambda_L}{4\pi^2} \text{sgn}(v). \quad (5.88)$$

Using the formula derived earlier

$$\beta_g = \frac{dg}{dt} = \left(\frac{\epsilon}{2} - 2\gamma_\psi - M \frac{d\delta g}{dM} \right) g,$$

we find that β_g is

$$\beta_g = -g \frac{\epsilon}{2} + \frac{g^3}{4\pi^2|v|} \frac{1}{1+ix/v}. \quad (5.89)$$

The β_g function now depends on the parameter $x = -\frac{p_0}{M}$. It is no longer dependent on the non-local momentum term. We are to choose any value for the ratio x from 0 to ∞ . The beta function for the Yukawa coupling still has two contributions as before, the tree level term and quantum correction term.

5.2 Chapter Summary

Note that unlike zero density where Lorentz invariance is not broken, the breaking of Lorentz invariance in finite density implies that, the spatial and time components of the kinetic term can be renormalized independently. This is why we had the counter term δZ_ψ and δv . In the previous chapter we had a difficulty in renormalizing the Yukawa vertex at one loop, because

the loop integral produced a momentum dependent divergent term. We have added a four Fermi coupling into our action in this chapter. By doing so we are able to successfully renormalize the low energy theory. Computing the tree level process of boson-mediated scattering in the Cooper channel in an angular momentum basis, we discovered that the tree level diagram is log divergent, which implies that the four Fermi coupling has a tree level running. This is not clearly seen if one is not working in the angular momentum basis. This kind of divergence physically arises from low energy fermions exchanging a high energy boson with momentum tangent to the Fermi-surface. The presence of the k_F scale plays a crucial role in appearance of such divergences because it enables the conversion of an angle-dependent scattering process to a momentum transfer imparted to a scalar. Some of the one loop four Fermi vertex diagrams exhibit $\log^2$ and $\log^3$ divergences which calls into question the notion of fixed points for finite density theories. These strong divergences appearing at one loop are a consequence of the tree level running of the four Fermi coupling. One can argue that the non-local momentum dependent term found in (4.75) is not Wilsonian, but rather it comes from the delta function peak

$$G(p)G(p+q) \approx G(p)^2 + \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \delta(p_0) \delta(p_\perp),$$

from the Fermi-surface, and hence we should not include it in our RG. Ignoring the nonlocal divergent term at one loop might look fine, but it is dangerous at higher loops, because nonlocal divergence will mix with local ones, which makes it hard to ignore them. They force themselves into the theory. Writing the bare coupling in terms of the physical coupling plus the log divergent term we are able to address the problem of $\log^2$ and $\log^3$ divergence since they all cancel. This helped to verify the validity of fixed points in non-relativistic finite density.

The Ward type identity in (5.62), is not exact in our theory. At low energy, smaller than k_F , we can linearize the inverse propagator as follows

$$ip_0 - vp_\perp \approx G^{-1}(p+q) - G^{-1}(q),$$

and hence

$$(ip - vp_\perp) \Gamma^3(p_0, p_\perp) = g \left(\Sigma(p+q) - \Sigma(q) \right).$$

The Ward identity implies that generally whenever the Fermi velocity runs, such that the fermion self-energy $\Sigma(p)$ is not proportional to the kinetic term $ip_0 - vp_\perp$, the vertex term will have a singular dependence as obtained in (4.75). Even though the identity is not valid to all orders, this phenomenon is general. We emphasize again that, the running of the Fermi velocity is a direct consequence of UV-IR mixing as we discussed above. The inclusion of the four-Fermi coupling also helped us to renormalize the Yukawa vertex. The required counter term to renormalize the problematic Γ_{sing} term it is now found. The singular piece

$$\begin{aligned} \Gamma_{sing}^3(p_0, p_\perp) &= \Gamma^3(p_0, p_\perp) - \Gamma_{reg}^3 \\ &= \frac{i\pi}{|v|} \frac{iq_0 + vq_\perp}{iq_0 - vq_\perp} \delta(p_0) \delta(p_\perp) \end{aligned}$$

comes from the contribution of the Fermi-surface from the delta function peak.

Chapter 6

6.1 Conclusion

Our study of the quantum field theory of finite density systems has taught us a number of important lessons. In this final chapter we summarize the most important results we have reviewed. In chapter 4 when we were computing the one loop diagram that corrected the Yukawa interaction, we realised that the loop integral had a singular momentum dependent term. The singular momentum dependent term emanates from the constraints imposed by the Fermi-surface on the fermion propagators close to the Fermi-surface. The Yukawa interaction could not be renormalised because we required an additional infinite number of renormalization conditions to renormalize the theory. With this hurdle, it prompted us to switch on the four-Fermi interaction in chapter 5. Introducing the four-Fermi interaction we were able to renormalize the theory, but there were some features which needed to be clearly understood concerning the four-Fermi interaction one loop diagrams. Computing the loop integrals for the one loop four-Fermi diagrams, we found that diagrams exhibit $\log^2$ and $\log^3$ divergences which is unusual, since one expects logarithmic divergences from one loop Feynman diagrams as suggested by power counting. This kind of divergence is dangerous, because this invalidates the notion of fixed points, since the fixed point will depend on the history of the system. Every starting point of a system will have a unique fixed point. Indeed, a $(\log \Lambda)^2$ divergences in a coupling correction would imply a beta function

$$\beta \sim \Lambda \frac{\partial \delta g}{\partial \Lambda} \sim \log \Lambda.$$

Since this β function depends on Λ , the RG trajectory will too, and ultimately the fixed point physics itself will depend on Λ . Fixed points of a theory must be the same, regardless of where your starting point is, you must always end in that same fixed point. The problem was resolved by looking at the tree level diagrams. If we compute the tree level diagrams for boson exchange in an angular momentum basis they are logarithmic divergent, which means there was a tree level running in the four-Fermion interaction. The tree level divergence had to be cancelled by replacing the bare coupling $\lambda_{0,L}$ with $\lambda_{0,L} = \frac{\lambda_L + \lambda_L \delta \lambda_L}{Z_\psi^2}$. We found the $\log^2$ and $\log^3$ divergences cancel out and consequently the four-Fermi beta function is independent of the cut-off and therefore the notion of a fixed point survives.

In chapter 5 we were able to renormalize the Yukawa interaction by introducing a one-loop fermion diagram with the singular momentum dependent term. We changed the renormalization scale to be at $p_0 = -xM$ and $p_\perp = -M$, where x is an arbitrary parameter and could be a ratio $x = \frac{p_0}{p_\perp}$. The Yukawa beta-function is dependent on this ratio. In chapter 3 we were able to show that we do have a non-Fermi liquid fixed point at low energy and the fixed point g^* can be made small, which implies that we can use perturbation to study the theory.

There are number of ways in which this work can be extended. We can borrow some of the

recently developed computation techniques from the work [25]. In this work Dirac fermions coupled to a Chern-Simons term [25]

$$S = \frac{ik}{4\pi} \int d^3x \text{Tr} \left(AdA + \frac{2}{3} A \wedge A \wedge A \right) + \int d^3x \bar{\psi} \gamma^\mu D_\mu \psi, \quad (6.1)$$

in $(2+1)$ dimensions, where $D_\mu \psi = \partial_\mu \psi - iA^a T^a \psi$ is studied. Calculations are performed, in the light cone gauge, where

$$\begin{aligned} A_\pm &= \frac{A_1 \pm iA_2}{\sqrt{2}}, \\ x_\pm &= \frac{x_1 \pm ix_2}{\sqrt{2}}, \\ p_\pm &= \frac{p_1 \pm ip_2}{\sqrt{2}}, \end{aligned}$$

and

$$p_s^2 = p_1^2 + p_2^2 = 2p^+ p^-.$$

With a light cone gauge condition $A_- = 0$, the cubic gauge self interaction vanishes, which simplifies the computation. Ignoring the Holonomy from Chern-Simons term, exploiting the rotational invariance symmetry in x_1 and x_2 (p_1 and p_2 in momentum space) plane, they managed to compute an exact solution for the fermion gap equation. Using the exact fermion two point function, they were able to find an exact solution for the fermion vertex. These results are remarkable, since we can use them to compute exact beta functions of the theory. The other finding which is of less interest for this dissertation, is the duality between, the theory of critical scalar coupled to a Chern-Simons and the critical fermion coupled to a Chern-Simons term. We can study the theory in (6.1) and extend the results obtained there, by taking the non-relativistic limit of the theory. Then we compute the gap equation for the fermion and the fermion vertex function using the gap equation we derived for the fermions. After that, we compute the renormalization group flows and study the fixed points of the theory. Another alternative is to introduce flavours to both the fermions and gauge fields in the Chern-Simons term, compute the renormalization group flows and then take the large N limit for both the fermions and gauge fields before we study the fixed points of the theory. We leave this interesting questions for the future.

Appendix A

A.1 Deriving Feynman Rules

We are going to derive the Feynman rules for the effective low energy Euclidean action consisting of a fermionic term, a bosonic term and a Yukawa term:

$$\begin{aligned}
 S &= \int d\tau \int d^d x \mathcal{L} = S_\psi + S_\phi + S_{\psi-\phi} \\
 \mathcal{L}_\psi &= \bar{\psi}_\sigma [\partial_\tau + \mu - \epsilon(i\vec{\nabla})] \psi_\sigma + \lambda_\psi \bar{\psi}_\sigma \bar{\psi}_{\sigma'} \psi_{\sigma'} \psi_\sigma \\
 \mathcal{L}_\phi &= m_\phi^2 \phi^2 + (\partial_\tau \phi)^2 + c^2 (\vec{\nabla} \phi)^2 + \frac{\lambda_\phi}{4!} \phi^4 \\
 S_{\psi,\phi} &= \int \frac{d^{d+1}k d^{d+1}q}{(2\pi)^2(d+1)} g(k, q) \bar{\psi}(k) \psi(k+q) \phi(q),
 \end{aligned} \tag{A.1}$$

where the first Lagrangian $\mathcal{L}_\psi$ represent the Landau Fermi Liquid with weak residual self interaction and the second Lagrangian ϕ represent the scalar boson field with mass m_ϕ and moving at speed c which we will set $c = 1$ from now on. The third term $S_{\psi,\phi}$ is the Yukawa coupling action between the bosons and scalars.

We will use a zero dimensional toy model to derive the Feynman rules for the Lagrangians given above. A zero dimensional toy model helps us to derive Feynman rules for Feynman diagrams with correct combinatorics for correlation functions.

A.1.1 Zero Dimension Toy Model

Scalar Boson Lagrangian

A field in a zero dimension is just a map from a single point to the real numbers. This implies that the generating function of a scalar boson in zero dimension is

$$Z(b) = N \int_{-\infty}^{\infty} d\phi \left(e^{-\frac{1}{2}m_\phi^2 \phi^2 - \frac{\lambda_\phi}{4!} \phi^4 + b\phi} \right),$$

where N is a normalisation constant. Note that we dropped the derivatives from the scalar generating function (scalar Lagrangian) because in zero dimension there are no derivatives. If we set $\lambda_\phi = 0$ we find that $Z(0) = 1$. This implies that N is

$$\begin{aligned}
 Z(0) &= N \int_{-\infty}^{\infty} d\phi \left(e^{-\frac{1}{2}m_\phi^2 \phi^2} \right) = 1 \\
 \Rightarrow N &= \sqrt{\frac{m_\phi^2}{\pi}}.
 \end{aligned}$$

The relationship between the generating function and the 2n-point correlation function is

$$\begin{aligned}\langle \phi^{2n} \rangle &= \left. \frac{d^{2n} Z(b)}{db^{2n}} \right|_{b=0} \\ &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^{2n} \left(e^{-\frac{1}{2}m_\phi^2 \phi^2 - \frac{\lambda_\phi}{4!} \phi^4} \right).\end{aligned}$$

Without the interaction term, the 2n-point function is evaluated to be

$$\begin{aligned}\langle \phi^{2n} \rangle &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^{2n} e^{-\frac{1}{2}m_\phi^2 \phi^2} \\ &= \sqrt{\frac{m_\phi^2}{\pi}} (-2)^n \frac{d^{2n}}{d(m_\phi^2)^{2n}} \left(\int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m_\phi^2 \phi^2} \right) \\ &= \frac{(2n-1)!!}{(m_\phi^2)^n}.\end{aligned}\tag{A.2}$$

Using the results we have found above we can calculate the zero (n=0), two (n=1) and four point (n=2) functions respectively,

$$\begin{aligned}\langle 1 \rangle &= 1, \\ \langle \phi^2 \rangle &= \frac{1}{m_\phi^2} \quad \text{and} \\ \langle \phi^4 \rangle &= \frac{3}{(m_\phi^2)^2}.\end{aligned}$$

When we include the interaction term, we use perturbation theory to evaluate the correlation functions.

Consider a zero point function

$$\langle 1 \rangle = \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \left(e^{-\frac{1}{2}m_\phi^2 \phi^2 - \frac{\lambda_\phi}{4!} \phi^4} \right)$$

we can evaluate the correlation function if we assume that λ_ϕ is small. Expanding in λ_ϕ we find

$$\begin{aligned}\langle 1 \rangle &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m_\phi^2 \phi^2} \sum_{n=0}^{\infty} \frac{(-\lambda_\phi)^n \phi^{4n}}{(4!)^n n!} \\ &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m_\phi^2 \phi^2} \left(1 + (-\lambda_\phi) \frac{\phi^4}{4!} + (-\lambda_\phi)^2 \frac{\phi^8}{(4!)^2 (2!)} + \dots \right) \\ &= 1 + (-\lambda_\phi) \frac{3}{4!(m_\phi^2)^2} + (-\lambda_\phi)^2 \frac{105}{(4!)^2 2! (m_\phi^2)^4} + \dots\end{aligned}\tag{A.3}$$

Evaluating the two point correlation function with the interaction term, we find

$$\begin{aligned}
\langle \phi^2 \rangle &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^2 \left(e^{-\frac{1}{2}m_\phi^2 \phi^2 - \frac{\lambda_\phi}{4!} \phi^4} \right) \\
&= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^2 e^{-\frac{1}{2}m_\phi^2 \phi^2} \sum_{n=0}^{\infty} \frac{(-\lambda_\phi)^n \phi^{4n}}{(4!)^n n!} \\
&= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^2 e^{-\frac{1}{2}m_\phi^2 \phi^2} \left(1 + (-\lambda_\phi) \frac{\phi^4}{4!} + (-\lambda_\phi)^2 \frac{\phi^8}{(4!)^2 (2!)} + \dots \right) \\
&= \frac{1}{m_\phi^2} + (-\lambda_\phi) \frac{15}{4! (m_\phi^2)^3} + (-\lambda_\phi)^2 \frac{945}{(4!)^2 2! (m_\phi^2)^5} + \dots
\end{aligned} \tag{A.4}$$

The four point function with the interaction term is also easily computed

$$\begin{aligned}
\langle \phi^4 \rangle &= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^4 \left(e^{-\frac{1}{2}m_\phi^2 \phi^2 - \frac{\lambda_\phi}{4!} \phi^4} \right) \\
&= \sqrt{\frac{m_\phi^2}{\pi}} \int_{-\infty}^{\infty} d\phi \phi^4 e^{-\frac{1}{2}m_\phi^2 \phi^2} \sum_{n=0}^{\infty} \frac{(-\lambda_\phi)^n \phi^{4n}}{(4!)^n n!} \\
&= \frac{3}{(m_\phi^2)^2} + (-\lambda_\phi) \frac{105}{4! (m_\phi^2)^4} + (-\lambda_\phi)^2 \frac{10395}{(4!)^2 2! (m_\phi^2)^6} + \dots
\end{aligned} \tag{A.5}$$

We will first formulate the Feynman rules and then use them to sketch Feynman diagrams for the corresponding correlators. The combinatorics of the diagrams must give us the same values that we found above using a perturbation expansion for evaluating the correlation functions.

Feynman rules for the scalar Lagrangian

The Feynman rules for the scalar Lagrangian are:

- Draw $2n$ dots for $\langle \phi^{2n} \rangle$.
- Draw v vertices with 4 four legs for order λ_ϕ^v .
- Connect dots in pairs, using a line, in all possible ways.
- Replace each line with the factor $\frac{1}{m_\phi^2}$.
- Multiply by a factor $(-\lambda_\phi)^v$ where v is the number of vertices.
- Multiply by $S = \frac{1}{2^{m_1}} \frac{1}{m_2!} \frac{1}{m_3!} \frac{1}{m_4!}$ where S is a product of
 - * $\frac{1}{2^{m_1}}$ for m_1 closed loops
 - * $\frac{1}{m_2!}$ for m_2 equivalent lines
 - * $\frac{1}{m_3!}$ for m_3 equivalent vertices
 - * $\frac{1}{m_4!}$ for m_4 equivalent diagrams.

With the Feynman rules above, we can sketch the Feynman diagrams for the correlation functions, then replace the diagrams with their mathematical expression. The combinatorics of the diagrams must give us the same results for correlation functions that we found above using perturbation theory. The combinatoric factor of the Feynman diagrams will be indicated below each Feynman diagram. The general combinatoric factor for a Feynman diagram with P lines and V vertices is

$$C = \frac{(-\lambda_\phi)^V}{2^{m_1} m_2! m_3! m_4!} \left[\frac{1}{m_\phi^2} \right]^P,$$

where C is the combinatoric factor. To read the results below, the symmetry factor $S = \frac{1}{2m_1m_2m_3m_4}$ is quoted in the order shown.

The Feynman diagram for a 2-point correlation function, without interaction term, is

$$\langle \phi^2 \rangle = \text{---} \bullet \text{---} \bullet \text{---}$$

$$= \frac{1}{m_\phi^2}$$

and for 4-point function we have

$$\langle \phi^4 \rangle = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---}$$

$$= \frac{1}{(m_\phi^2)^2} + \frac{1}{(m_\phi^2)^2} + \frac{1}{(m_\phi^2)^2}$$

$$= \frac{3}{(m_\phi^2)^2}$$

We can see that the sum of the Feynman diagrams for two and four point correlation functions reproduce the corresponding correlators that we evaluated above. We will now include an interaction term and check whether of Feynman diagrams give us the results that we found in (A.3),(A.4) and (A.5) using perturbation theory.

$$\langle 1 \rangle = \bullet + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---}$$

$$C = \frac{1}{2^0 0! 0! 0!} = 1$$

$$C = \frac{(-\lambda_\phi)}{2^2 2! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^2 = \frac{3(-\lambda_\phi)}{4!(m_\phi^2)^2}$$

$$C = \frac{(-\lambda_\phi)^2}{2^4 2! 2! 2!} \left[\frac{1}{m_\phi^2} \right]^4 = \frac{9(-\lambda_\phi)^2}{(4!)^2 2! (m_\phi^2)^4}$$

$$C = \frac{(-\lambda_\phi)^2}{2^2 2! 2! 0!} \left[\frac{1}{m_\phi^2} \right]^4 = \frac{72(-\lambda_\phi)^2}{(4!)^2 2! (m_\phi^2)^4}$$

$$+ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---}$$

$$C = \frac{(-\lambda_\phi)^2}{2^0 4! 2! 0!} \left[\frac{1}{m_\phi^2} \right]^4 = \frac{24(-\lambda_\phi)^2}{(4!)^2 2! (m_\phi^2)^4}$$

$$= 1 + (-\lambda_\phi) \frac{3}{4!(m_\phi^2)^2} + (-\lambda_\phi)^2 \frac{105}{(4!)^2 2! (m_\phi^2)^4} + \dots$$

Figure A.1: Vacuum Feynman diagrams with interaction term.

One can see that the sum of the Feynman diagrams gives us the results we found using perturbation theory. For the two point correlation function, we get

$$\begin{aligned}
\langle \phi^2 \rangle = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \\
C = & \frac{(-\lambda_\phi)^0}{2^0 0! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^1 = \frac{1}{m_\phi^2} & C = \frac{(-\lambda_\phi)}{2^2 2! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^3 = \frac{3(-\lambda_\phi)}{4!(m_\phi^2)^3} & C = \frac{(-\lambda_\phi)}{2^1 0! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^3 = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^3} \\
& C = \frac{(-\lambda_\phi)^2}{2^4 2! 2! 0! 2!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{9(-\lambda_\phi)}{(4!)^2 2!(m_\phi^2)^5} & C = \frac{(-\lambda_\phi)^2}{2^0 4! 2! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{24(-\lambda_\phi)^2}{(4!)^2 2!(m_\phi^2)^5} \\
& + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \\
C = & \frac{(-\lambda_\phi)^2}{2^3 2! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{72(-\lambda_\phi)^2}{(4!)^2 2!(m_\phi^2)^5} & C = \frac{(-\lambda_\phi)^2}{2^1 2! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{288(-\lambda_\phi)^2}{(4!)^2 2!(m_\phi^2)^5} & C = \frac{(-\lambda_\phi)^2}{2^2 2! 2! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{72(-\lambda_\phi)^2}{(4!)^2 2!(m_\phi^2)^5} \\
& + \text{Diagram 9} + \text{Diagram 10} \\
C = & \frac{(-\lambda_\phi)^2}{2^2 0! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{288(-\lambda_\phi)^2}{(4!)^2 2!(m_\phi^2)^5} & C = \frac{(-\lambda_\phi)}{2^0 3! 0! 0!} \left[\frac{1}{m_\phi^2} \right]^5 = \frac{192(-\lambda_\phi)}{(4!)^2 2!(m_\phi^2)^5}
\end{aligned}$$

Figure A.2: Two point function Feynman diagrams with interaction term.

The sum of the diagrams is

$$\langle \phi^2 \rangle = \frac{1}{m_\phi^2} + (-\lambda_\phi) \frac{15}{4!(m_\phi^2)^3} + (-\lambda_\phi)^2 \frac{945}{(4!)^2 2!(m_\phi^2)^5}, \quad (\text{A.6})$$

which agrees with our previous result. Lets also sketch the Feynman diagrams for the 4-point function, up to λ_ϕ

$$\begin{aligned}
\langle \phi^4 \rangle = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \\
& C = \frac{1}{m_\phi^2} \quad C = \frac{1}{m_\phi^2} \quad C = \frac{1}{m_\phi^2} \quad C = \frac{3(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{3(-\lambda_\phi)}{4!(m_\phi^2)^4} \\
& + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} \\
& C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{3(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \\
& + \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13} \\
& C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{12(-\lambda_\phi)}{4!(m_\phi^2)^4} \quad C = \frac{(-\lambda_\phi)}{(m_\phi^2)^4} \\
= & \frac{3}{(m_\phi^2)^2} + (-\lambda_\phi) \frac{105}{4!(m_\phi^2)^4}
\end{aligned}$$

Figure A.3: Four point function Feynman diagrams.

This sum of the Feynman diagrams reproduces the correlation functions that we calculated before. This shows that our zero dimensional Feynman rules are correct. Since these Feynman rules were derived from a zero dimensional toy model, for the $d + 1$ dimension scalar Lagrangian

$$\mathcal{L}_\phi = m_\phi^2 \phi^2 + (\partial_\tau \phi)^2 + c^2 (\vec{\nabla} \phi)^2 + \frac{\lambda_\phi}{4!} \phi^4,$$

we can use the same combinatoric factor as for the zero-dimension case. The Feynman rules for the $d + 1$ dimensional for scalar Lagrangian in momentum space are

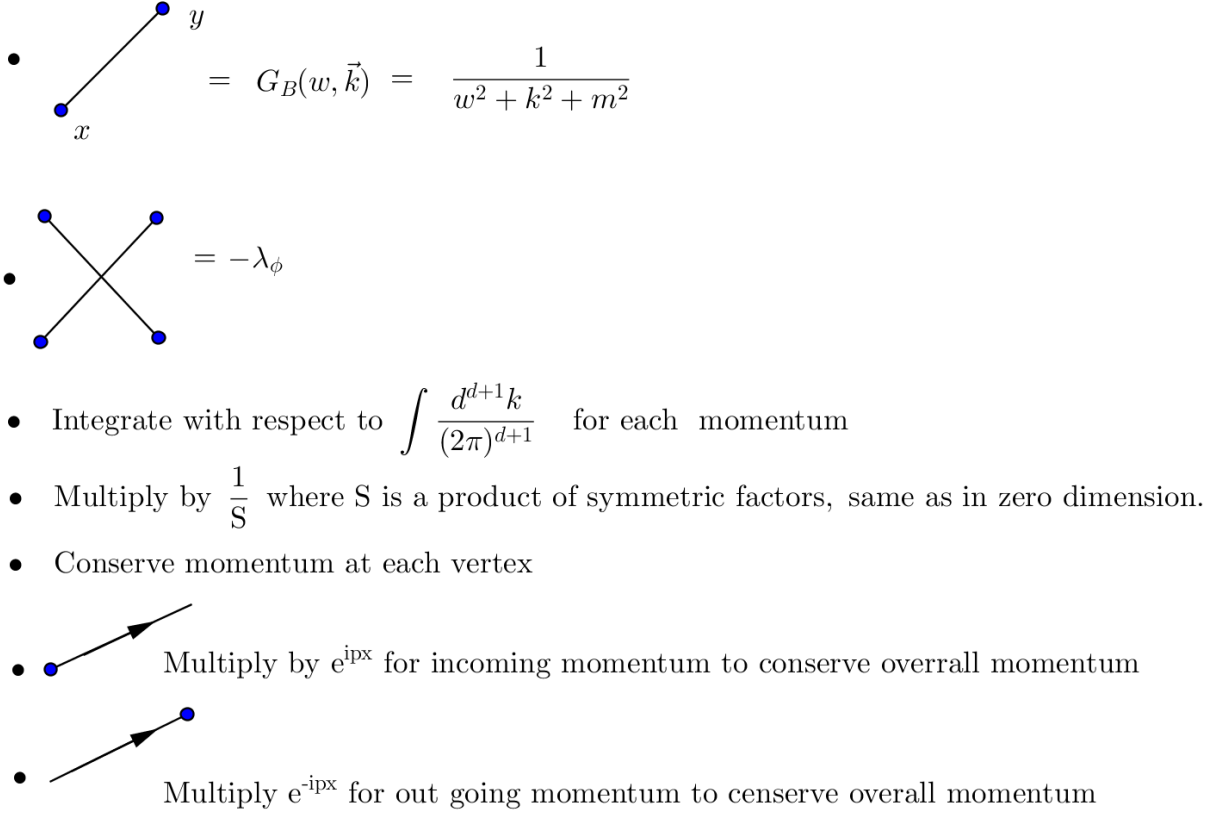


Figure A.4: Feynman Rules

A.1.2 Fermion Lagrangian Feynman rules

In the previous section we managed to derive the Feynman rules for the $d + 1$ dimension scalar Lagrangian using a zero dimensional toy model. Our next task is to derive the Feynman rules for the fermion Lagrangian

$$\mathcal{L}_\psi = \bar{\psi}_\sigma [\partial_\tau + \mu - \epsilon(i\vec{\nabla})] \psi_\sigma + \lambda_\psi \bar{\psi}_\sigma \bar{\psi}_{\sigma'} \psi_{\sigma'} \psi_\sigma.$$

Using a zero dimensional toy model for this Lagrangian to derive the Feynman rules is not that trivial. However, the complex scalar Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - g(\phi \phi^*)^2,$$

where ϕ is scalar field, m is the mass of the scalar particle and g is the interaction strength of the fields, has similar Feynman rules. The only difference is that the combinatorics factors for the complex scalar Feynman diagrams are positive while the corresponding diagrams for fermions include a negative sign for each closed fermion loop, due to the anti-commuting nature of the fermions fields. We can thus use this Lagrangian to derive the Feynman rules for the fermion fields. We can write the fields of a complex scalar field in terms of

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \text{and} \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}},$$

where ϕ_1 and ϕ_2 are real fields. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{m^2}{2} (\phi_1^2 + \phi_2^2) - \frac{g}{4} (\phi_1^4 + 2\phi_1^2 \phi_2^2 + \phi_2^4).$$

and the zero dimensional correlation function is

$$\begin{aligned}\langle (\phi\phi^*)^n \rangle &= \frac{m^2}{\pi} \int d\phi\phi^* e^{-m^2\phi\phi^* - g(\phi\phi^*)^2} \\ &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{g}{4}(\phi_1^4 + 2\phi_1^2\phi_2^2 + \phi_2^4)}.\end{aligned}$$

We will calculate the zero, two and four point correlation functions as we did before and follow the same procedure to find the Feynman rules. First lets calculate the correlation functions without the interaction term. The zero point function without interaction term is

$$\begin{aligned}\langle 1 \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \\ &= 1.\end{aligned}$$

The two point function without interaction term is

$$\begin{aligned}\langle \phi\phi^* \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \phi\phi^* e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \\ &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \frac{1}{2}(\phi_1^2 + \phi_2^2) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \\ &= \frac{1}{2} \left(\frac{1}{m^2} + \frac{1}{m^2} \right) \\ &= \frac{1}{m^2}.\end{aligned}$$

The four point function without interaction term is

$$\begin{aligned}\langle (\phi\phi^*)^2 \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 (\phi\phi^*)^2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \\ &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \frac{1}{4}(\phi_1^4 + 2\phi_1^2\phi_2^2 + \phi_2^4) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \\ &= \frac{1}{4} \left(\frac{3}{(m^2)^2} + \frac{2}{(m^2)^2} + \frac{3}{(m^2)^2} \right) \\ &= \frac{2}{(m^2)^2}\end{aligned}$$

We will now repeat these computations including the interaction term:

The zero point function with interaction term is

$$\begin{aligned}\langle 1 \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2) - \frac{g}{4}(\phi_1^4 + 2\phi_1^2\phi_2^2 + \phi_2^4)} \\ &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \sum_{n=0}^{\infty} \frac{(-g)^n (\phi_1^4)^n}{n!} \sum_{m=0}^{\infty} \frac{(-g)^m (\phi_1^4)^m}{m!} \sum_{l=0}^{\infty} \frac{(-g)^l (\phi_1^4)^l}{l!} \\ &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \left(1 - \frac{g}{4}(\phi_1^4 + 2\phi_1^2\phi_2^2 + \phi_2^4) + \dots \right) \\ &= 1 + \frac{(-g)}{4} \left(\frac{3}{(m^2)^2} + \frac{2}{(m^2)^2} + \frac{3}{(m^2)^2} \right) + \dots \\ &= 1 + 2 \frac{(-g)}{(m^2)^2} + \dots\end{aligned}$$

The two point function with interaction term is

$$\begin{aligned}
\langle \phi \phi^* \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \frac{1}{2} (\phi_1^2 + \phi_2^2) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \sum_{n=0}^{\infty} \frac{(\frac{-g}{4})^n (\phi_1^4)^n}{n!} \sum_{m=0}^{\infty} \frac{(\frac{-g}{4})^m (\phi_1^4)^m}{m!} \sum_{l=0}^{\infty} \frac{(\frac{-g}{4})^l (\phi_1^4)^l}{l!} \\
&= \frac{1}{m^2} + \frac{-g}{2} \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 (\phi_1^2 + \phi_2^2) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \left(\frac{\phi_1^4}{4} + \frac{1}{2} \phi_1^2 \phi_1^2 + \frac{\phi_2^4}{4} + \dots \right) \\
&= \frac{1}{m^2} + \frac{-g}{8} \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \left(\phi_1^6 + 2\phi_1^4 \phi_2^2 + \phi_1^2 \phi_2^4 + \phi_1^4 \phi_2^2 + 2\phi_1^2 \phi_2^4 + \phi_2^6 + \dots \right) \\
&= \frac{1}{m^2} + \frac{-g}{8} \left(\frac{15}{(m^2)^3} + 2 \frac{3}{(m^3)^3} + \frac{3}{(m^2)^3} + 2 \frac{3}{(m^3)^3} + \frac{3}{(m^2)^3} + \frac{15}{(m^2)^3} + 2 \frac{3}{(m^3)^3} \right) + \dots \\
&= \frac{1}{m^2} + \frac{6(-g)}{(m^2)^3} + \dots
\end{aligned}$$

The four point function with interaction term is

$$\begin{aligned}
\langle (\phi \phi^*)^2 \rangle &= \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \frac{1}{4} (\phi_1^4 + 2\phi_1^2 \phi_2^2 + \phi_2^4) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \sum_{n=0}^{\infty} \frac{(\frac{-g}{4})^n (\phi_1^4)^n}{n!} \times \tag{A.7} \\
&\quad \sum_{m=0}^{\infty} \frac{(\frac{-g}{4})^m (\phi_1^4)^m}{m!} \sum_{l=0}^{\infty} \frac{(\frac{-g}{4})^l (\phi_1^4)^l}{l!} \\
&= \frac{2}{(m^2)^2} + \frac{-g}{16} \frac{m^2}{\pi} \int d\phi_1 \int d\phi_2 \frac{1}{4} (\phi_1^4 + 2\phi_1^2 \phi_2^2 + \phi_2^4) e^{-\frac{m^2}{2}(\phi_1^2 + \phi_2^2)} \left(\phi_1^8 + 4\phi_1^6 \phi_1^2 \right. \\
&\quad \left. + 2\phi_1^4 \phi_2^4 + 4\phi_1^4 \phi_2^4 + 4\phi_1^2 \phi_2^6 + \phi_2^4 \phi_2^4 \right) + \dots \\
&= \frac{2}{(m^2)^2} + \frac{-g}{16} \left(\frac{105}{(m^2)^4} + 4 \frac{15}{(m^2)^3} \frac{1}{m^2} + 2 \frac{3}{(m^2)^2} \frac{3}{(m^2)^2} + 4 \frac{3}{(m^2)^2} \frac{3}{(m^2)^2} + 4 \frac{1}{m^2} \frac{15}{(m^2)^3} + \frac{105}{(m^2)^4} \right) \\
&= \frac{2}{(m^2)^2} + 24 \frac{-g}{(m^2)^4} + \dots
\end{aligned}$$

The Feynman rules for these correlators are:

- For each $(\phi\phi^*)^n$ evaluate $(\phi_1^2 + \phi_2^2)^n$ first
- After evaluation ignore the coefficients, draw $2n$ dots for ϕ_1^{2n} and $2n$ stars for ϕ_2^{2n}
- Dots can only be connected to a solid line, and stars to a broken line
- For each $g\phi_1^4$ draw 
- For each $g\phi_2^4$ draw 
- For each $g\phi_1^2\phi_2^2$ draw 
- Dots can only be connected in pairs or to a solid vertex leg, same applies to star but stars are connected to a broken line vertex leg
- Connect the dots, stars and vertices legs in all possible ways.
- Multiply by or insert $-g4!$ for each vertex 
- Multiply by or insert $-g4!$ for each vertex 
- Multiply by or insert $-g(2!2!)$ for each vertex 
- Diagrams come in pairs with vertex $g\phi_1^2\phi_2^2$ 
- Multiply by or insert $\frac{1}{S}$, where S is the product of symmetric terms as before.
- Replace both the dash and solid lines with the factor $\frac{1}{2} \frac{1}{m^2}$.

Figure A.5: Feynman rules for the complex scalar.

From the Feynman rules, the most general combinatoric factor for a Feynman diagram with P lines and vertices $((-g)\phi_1^2\phi_2^2)^Z$, $((-g)\phi_1^4)^V$ and $((-g)\phi_2^4)^Y$ is

$$C = \frac{(2!2!(-g))^Z (4!(-g))^V (4!(-g))^Y}{2^{m_1} m_2! m_3! m_4!} \left[\frac{1}{2m} \right]^P.$$

Using these Feynman rules, the diagrams for the zero, two and four point correlation functions without interaction term, are

$$\begin{aligned}
\langle 1 \rangle &= \bullet + \times \\
&= \frac{1}{2} + \frac{1}{2} = 1 \\
\langle \phi \phi^* \rangle &= \text{diagram with two solid dots connected by a solid line} + \text{diagram with two blue crosses connected by a dashed line} \\
&= \frac{1}{2} \frac{1}{m^2} + \frac{1}{2} \frac{1}{m^2} \\
&= \frac{1}{m^2} \\
\langle (\phi \phi^*)^2 \rangle &= \text{diagram with two pairs of solid dots connected by two solid lines} + \text{diagram with two pairs of solid dots connected by two solid lines in a crossed configuration} + \text{diagram with two pairs of solid dots connected by two vertical solid lines} + \text{diagram with two pairs of blue crosses connected by two horizontal dashed lines} + \text{diagram with two pairs of blue crosses connected by two horizontal dashed lines in a crossed configuration} + \text{diagram with two pairs of blue crosses connected by two vertical dashed lines} + \text{diagram with two pairs of blue crosses connected by two vertical dashed lines in a crossed configuration} + \text{diagram with two pairs of blue crosses connected by two vertical dashed lines} \\
&C = \left[\frac{1}{2m^2} \right]^2 \quad C = \left[\frac{1}{2m^2} \right]^2 \quad C = \left[\frac{1}{2m^2} \right]^2 \quad C = \left[\frac{1}{2m^2} \right]^2 \quad C = \left[\frac{1}{2m^2} \right]^2 \quad \boxed{C = \left[\frac{1}{2m^2} \right]^2} \quad C = \left[\frac{1}{2m^2} \right]^2 \quad C = \left[\frac{1}{2m^2} \right]^2 \\
&= \frac{1}{4} \left(\frac{3}{(m^2)^2} + \frac{2}{(m^2)^2} + \frac{3}{(m^2)^2} \right) \\
&= \frac{2}{(m^2)^2}
\end{aligned}$$

Figure A.6: Diagrams for zero, two point and four point function.

For convenience we will label our Feynman diagrams with a label $C_{m,n}$, where C is the usual symbol for the combinatoric factor, n is for n-point function and m indicates the diagram.

Feynman diagrams for a zero point function with an interaction term are:

$$\begin{aligned}
\langle 1 \rangle &= \bullet + \times + \text{diagram with two circles connected by a solid line} + \text{diagram with two circles connected by a dashed line} + \text{diagram with two circles connected by a dashed line} + \text{diagram with two circles connected by a dashed line} \\
&C_{0,1} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m} \right]^2 = \frac{3(-g)}{(m^2)^2} \quad C_{0,2} = \frac{2! 2! (-g)}{2^2 0! 0! 0!} \left[\frac{1}{m^2} \right]^2 = \frac{(-g)}{(m^2)^2} \quad C_{0,3} = \frac{2! 2! (-g)}{2^2 0! 0! 0!} \left[\frac{1}{m^2} \right]^2 = \frac{(-g)}{(m^2)^2} \quad C_{0,4} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m} \right]^2 = \frac{3(-g)}{(m^2)^2} \\
&= 1 + \frac{2(-g)}{(m^2)^2}
\end{aligned}$$

Figure A.7: Zero point Feynman diagrams with interaction term.

The two point function with interaction term Feynman diagrams are:

$$\begin{aligned}
\langle \phi \phi^* \rangle = & \text{---} + \text{---} \times \text{---} \times + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} \times \text{---} \times + \text{---} \text{---} \times \text{---} \times + \text{---} \text{---} \\
& C_{2,1} = \left[\frac{1}{2m^2} \right] \quad C_{2,2} = \left[\frac{1}{2m^2} \right] \quad C_{2,3} = \frac{4!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{12(-g)}{(2m^2)^3} \quad C_{2,4} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{3(-g)}{(2m^2)^3} \\
& C_{2,5} = \frac{2! 2!(-g)}{2^1 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{2(-g)}{(2m^2)^3} \quad C_{2,6} = \frac{2! 2!(-g)}{2^1 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{2(-g)}{(2m^2)^3} \quad C_{2,7} = \frac{2! 2!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{(-g)}{(2m^2)^3} \\
& + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} \times \text{---} \times + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} \times \text{---} \times \\
& C_{2,8} = \frac{2! 2!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{(-g)}{(2m^2)^3} \quad C_{2,9} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{3(-g)}{(2m^2)^3} \quad C_{2,10} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{3(-g)}{(2m^2)^3} \\
& C_{2,11} = \frac{2! 2!(-g)}{2^1 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{2(-g)}{(2m^2)^3} \quad C_{2,12} = \frac{2! 2!(-g)}{2^1 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{2(-g)}{(2m^2)^3} \quad C_{2,13} = \frac{2! 2!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{(-g)}{(2m^2)^3} \\
& + \text{---} \text{---} \times \text{---} \times + \text{---} \text{---} \times \text{---} \times + \text{---} \text{---} \times \text{---} \times \\
& C_{2,14} = \frac{2! 2!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{(-g)}{(2m^2)^3} \quad C_{2,15} = \frac{4!(-g)}{2^2 0! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{12(-g)}{(2m^2)^3} \quad C_{2,16} = \frac{4!(-g)}{2^2 2! 0! 0!} \left[\frac{1}{2m^2} \right]^3 = \frac{3(-g)}{(2m^2)^3} \\
& = \frac{1}{m^2} + \frac{6(-g)}{(m^2)^3}
\end{aligned}$$

Figure A.8: Two point function with interaction term.

For a 4-point function, we don't draw both diagrams for diagrams that come in pairs. We only draw one diagram and put $2\times$ in front of the diagram to indicate that is one of a pair. The combinatoric factor appearing below the diagram will be for one of the diagrams. The 4-point function diagrams are

Adding all the Feynman diagrams, we find

$$\langle(\phi\phi^*)^2\rangle = \frac{2}{(m^2)^2} + \frac{24(-g)}{(m^2)^4}. \quad (\text{A.8})$$

This reproduces our earlier results. This shows that the Feynman rules derived above are correct. We can improve things further by obtaining Feynman rules solely in terms of ϕ and ϕ^* . We form combinations of tree diagrams as follows

$$\begin{aligned} \langle 1 \rangle &= \bullet + \times = \circ \\ &= \frac{1}{2} + \frac{1}{2} = 1 \\ \langle \phi\phi^* \rangle &= \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \times \\ \diagdown \\ \times \end{array} = \begin{array}{c} \circ \\ \diagup \\ \circ \end{array} \\ &= \frac{1}{2} \frac{1}{m^2} + \frac{1}{2} \frac{1}{m^2} \\ &= \frac{1}{m^2} \\ \langle (\phi\phi^*)^2 \rangle &= \begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \times \quad \times \\ \hline \times \quad \times \end{array} + \begin{array}{c} \bullet \quad \times \\ | \quad | \\ \bullet \quad \times \end{array} + \begin{array}{c} \times \quad \times \\ \hline \times \quad \times \end{array} + \begin{array}{c} \times \quad \times \\ \diagdown \quad \diagup \\ \times \quad \times \end{array} + \begin{array}{c} \times \quad \times \\ | \quad | \\ \times \quad \times \end{array} \\ &= \begin{array}{c} \circ \quad \circ \\ \hline \circ \quad \circ \end{array} + \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \circ \quad \circ \end{array} \\ &= \frac{1}{(m^2)^2} + \frac{1}{(m^2)^2} = \frac{2}{(m^2)^2} \end{aligned}$$

Figure A.10: Zero, two and four point function Feynman diagrams for specific combinations of fields.

Using the combination above, the diagrams for the zero point function are

$$\begin{aligned} \langle 1 \rangle &= \circ + \begin{array}{c} \bigcirc \quad \bigcirc \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \bigcirc \quad \bigcirc \\ \diagdown \quad \diagup \end{array} \\ \tilde{C}_{0,1} &= 1 \quad \tilde{C}_{0,2} = \frac{(-g)}{(m^2)^2} \quad \tilde{C}_{0,3} = \frac{(-g)}{(m^2)^2} \\ &= 1 + \frac{2(-g)}{(m^2)^2} \end{aligned}$$

Figure A.11: Zero point function diagrams with interaction.

To get the above Feynman diagrams with their corresponding combinatorics, we had to add

the diagrams with their combinatorics factors from a zero point function as follows

$$\begin{aligned}\tilde{C}_{0,2} &= C_{0,1} + C_{0,4} \quad \text{and} \\ \tilde{C}_{0,3} &= C_{0,2} + C_{0,3}.\end{aligned}$$

To get $\tilde{C}_{0,1}$ we added a star and a dot. The new combined Feynman diagrams for the two point function are

$$\begin{aligned}\langle \phi \phi^* \rangle &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} \\ \tilde{C}_{2,1} &= \frac{1}{m^2} \quad \tilde{C}_{2,2} = \frac{2(-g)}{(m^2)^3} \quad \tilde{C}_{2,3} = \frac{2(-g)}{(m^2)^3} \quad \tilde{C}_{2,4} = \frac{(-g)}{(m^2)^3} \quad \tilde{C}_{2,5} = \frac{(-g)}{(m^2)^3} \\ &= \frac{1}{m^2} + \frac{6(-g)}{(m^2)^4}\end{aligned}$$

Figure A.12: Two point Feynman diagrams.

In figure A.12 added the diagrams and their combinatorics combine in the following way

$$\begin{aligned}\tilde{C}_{2,1} &= C_{2,1} + C_{2,2} \\ \tilde{C}_{2,2} &= C_{2,3} + C_{2,11} + C_{2,12} \\ \tilde{C}_{2,3} &= C_{2,5} + C_{2,6} + C_{2,15} \\ \tilde{C}_{2,4} &= C_{2,4} + C_{2,7} + C_{2,14} + C_{2,16} \\ \tilde{C}_{2,5} &= C_{2,8} + C_{2,9} + C_{2,10} + C_{2,13}\end{aligned}$$

The new diagrams for the 4-point function are

$$\begin{aligned}\langle (\phi \phi^*)^2 \rangle &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} \\ &+ \text{diagram 8} + \text{diagram 9} + \text{diagram 10} + \text{diagram 11} + \text{diagram 12} + \text{diagram 13} + \text{diagram 14} + \text{diagram 15} \\ \tilde{C}_{4,1} &= \frac{1}{(m^2)^2} \quad \tilde{C}_{4,2} = \frac{1}{(m^2)^2} \quad \tilde{C}_{4,3} = \frac{2!2!(-g)}{(m^2)^4} \quad \tilde{C}_{4,4} = \frac{(-g)}{(m^2)^4} \quad \tilde{C}_{4,5} = \frac{(-g)}{(m^2)^4} \\ \tilde{C}_{4,6} &= \frac{(-g)}{(m^2)^4} \quad \tilde{C}_{4,7} = \frac{(-g)}{(m^2)^4} \\ \tilde{C}_{4,8} &= \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,9} = \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,10} = \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,11} = \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,12} = \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,13} = \frac{2(-g)}{(m^2)^4} \\ \tilde{C}_{4,14} &= \frac{2(-g)}{(m^2)^4} \quad \tilde{C}_{4,15} = \frac{2(-g)}{(m^2)^4} \\ &= \frac{2}{(m^2)^2} + \frac{24(-g)}{(m^2)^4}\end{aligned}$$

Figure A.13: Four point function Feynman diagrams.

The diagrams and their combinatorics combine as follows

$$\begin{aligned}
\tilde{C}_{4,1} &= C_{4,1} + C_{4,3} + C_{4,6} + C_{4,8} \\
\tilde{C}_{4,2} &= C_{4,2} + C_{4,4} + C_{4,5} + C_{4,7} \\
\tilde{C}_{4,3} &= C_{4,13} + 2C_{4,41} + C_{4,61} \\
\tilde{C}_{4,4} &= C_{4,11} + C_{4,59} + C_{4,33} + C_{4,45} + C_{4,24} + C_{4,54} + C_{4,38} + C_{4,37} \\
\tilde{C}_{4,5} &= C_{4,26} + C_{4,46} + C_{4,34} + C_{4,44} + C_{4,24} + C_{4,54} + C_{4,37} + C_{4,38} \\
\tilde{C}_{4,6} &= C_{4,10} + C_{4,58} + C_{4,14} + C_{4,60} + C_{4,23} + C_{4,53} + C_{4,28} + C_{4,55} \\
\tilde{C}_{4,7} &= C_{4,27} + C_{4,47} + C_{4,30} + C_{4,48} + C_{4,23} + C_{4,53} + C_{4,28} + C_{4,55} \\
\tilde{C}_{4,8} &= C_{4,12} + C_{4,66} + C_{4,39} + C_{4,35} \\
\tilde{C}_{4,9} &= C_{4,20} + C_{4,50} + C_{4,32} + C_{4,42} \\
\tilde{C}_{4,10} &= C_{4,15} + C_{4,67} + C_{4,21} + C_{4,51} \\
\tilde{C}_{4,11} &= C_{4,18} + C_{4,64} + C_{4,29} + C_{4,57} \\
\tilde{C}_{4,12} &= C_{4,17} + C_{4,62} + C_{4,25} + C_{4,56} \\
\tilde{C}_{4,13} &= C_{4,22} + C_{4,52} + C_{4,16} + C_{4,63} \\
\tilde{C}_{4,14} &= C_{4,49} + C_{4,65} + C_{4,35} + C_{4,40} \\
\tilde{C}_{4,15} &= C_{4,31} + C_{4,43} + C_{4,19} + C_{4,49}
\end{aligned}$$

From the correlation functions above, we see that distinct Feynman diagrams come in pairs. More precisely, distinct loop diagrams come in pairs. The Feynman rules in terms of ϕ and ϕ^* are:

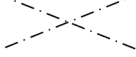
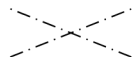
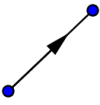
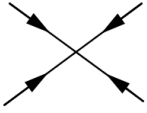
- for each product of $\phi\phi^*$, draw two small circles.
- For each g , draw a vertex 
- Connects the small circles in pairs or with vertices legs, in all possible ways
- Replace each line with a factor of $\frac{1}{m^2}$
- Replace each vertex with a factor $2!2!(-g)$,  $= 2!2!(-g)$
- Multiply by $\frac{1}{S}$, where S is the product of symmetric terms :
 - * $\frac{1}{n_1!}$, where n is the number of equivalent diagrams
 - * $\frac{1}{n_2!}$, where n is the number of equivalent lines
 - * $\frac{1}{n_3!}$, where n is the number of equivalent vertices
 - * $\frac{1}{2^{n_4}}$, where n is the number of closed loops

Figure A.14: Feynman rules for complex scalar field.

When we use these rules to evaluate the Feynman diagrams above we recover our earlier results.

Recall that our goal was to derive the Feynman rules for the fermion Lagrangian in $d+1$ dimensions using the complex scalar Lagrangian. The Feynman rules are the same up to some signs. Using the results we found from complex scalar Lagrangian, our Feynman rules for the fermion Lagrangian are:

-  $G_F(w, \vec{k}) = \frac{1}{iw + \mu - \epsilon(\vec{k})}$

-  $= 2!2!(-\lambda_\psi)$

- Assume momentum is conserved at each vertex
- integrate with respect to $\int \frac{d^{d+1}k}{(2\pi)^{d+1}}$ for each momentum appearing
- Multiply by $\frac{1}{S}$, where S is the same product of symmetric factors as before
- Multiply by e^{ikx} for incoming momentum to conserve the overall momentum
- Multiply by e^{-ikx} for incoming momentum to conserve the overall momentum
- For each closed fermion loop include a factor of -1 .

The Feynman rules of a scalar boson Lagrangian added to a fermion Lagrangian is a combination of both the scalar and fermions rules. Adding the Yukawa coupling term

$$g(\vec{k}, \vec{q})\phi\psi\bar{\psi}$$

to both the scalar and Fermion Lagrangian introduces an extra Feynman, which is

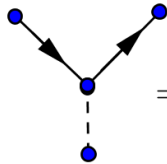
-  $= -g(\vec{k}, \vec{q})$

Figure A.15: Yukawa coupling diagram.

References

- [1] Andrew J Schofield. Non-Fermi liquids. *Contemporary Physics*, 40(2):95–115, March 1999.
- [2] Nabil Iqbal, Hong Liu, and Mark Mezei. Lectures on holographic non-Fermi liquids and quantum phase transitions. pages 707–816, 2011, 1110.3814.
- [3] P. W. Anderson. Infrared catastrophe in Fermi gases with local scattering potentials. *Phys. Rev. Lett.*, 18:1049–1051, Jun 1967.
- [4] Gordon Baym and Siu A Chin. Landau theory of relativistic Fermi liquids. *Nuclear Physics A*, 262(3):527–538, 1976.
- [5] Joseph W Serene and Dierk Rainer. The quasiclassical approach to superfluid ^3He . *Physics Reports*, 101(4):221–311, 1983.
- [6] Lev Petrovich Gorkov. Microscopic derivation of the Ginzburg-Landau equations in the theory of superconductivity. *Sov. Phys. JETP*, 9(6):1364–1367, 1959.
- [7] Ch.G. van Weert, M.C.J. Leermakers, and A.M.J. Schakel. Landau theory of Fermi liquids reformulated. *Physica A: Statistical Mechanics and its Applications*, 138(3):404 – 414, 1986.
- [8] S Babu and G.E Brown. Quasiparticle interaction in liquid ^3He . *Annals of Physics*, 78(1):1 – 38, 1973.
- [9] Achim Schwenk, Bengt Friman, and Gerald E Brown. Renormalization group approach to neutron matter: quasiparticle interactions, superfluid gaps and the equation of state. *Nuclear Physics A*, 713(1):191–216, 2003.
- [10] Rev Shankar. Renormalization-group approach to interacting fermions. *Reviews of Modern Physics*, 66(1):129, 1994.
- [11] CA Lamas, DC Cabra, and N Grandi. Fermi liquid instabilities in two-dimensional lattice models. *Physical Review B*, 78(11):115104, 2008.
- [12] John A Hertz. Quantum critical phenomena. *Physical Review B*, 14(3):1165–1184, 1976.
- [13] M Nicklas, M Brando, G Knebel, F Mayr, W Trinkl, and A Loidl. Non-fermi-liquid behavior at a ferromagnetic quantum critical point in $\text{Ni}_x\text{Pd}_{1-x}$. *Physical review letters*, 82(21):4268, 1999.
- [14] Chetan Nayak and Frank Wilczek. Renormalization group approach to low temperature properties of a non-fermi liquid metal. *Nuclear Physics B*, 430(3):534–562, 1994.
- [15] David F Mross, John McGreevy, Hong Liu, and T Senthil. Controlled expansion for certain non-fermi-liquid metals. *Physical Review B*, 82(4):045121, 2010.
- [16] Kenneth G Wilson and John Kogut. The renormalization group and the expansion. *Physics Reports*, 12(2):75–199, 1974.

- [17] A. Liam Fitzpatrick, Shamit Kachru, Jared Kaplan, and S. Raghu. Non-Fermi-liquid fixed point in a Wilsonian theory of quantum critical metals. *Phys. Rev. B*, 88:125116, Sep 2013.
- [18] A. L. Fitzpatrick, G. Torroba, and H. Wang. Aspects of Renormalization in Finite Density Field Theory. *ArXiv e-prints*, October 2014, 1410.6811.
- [19] A. L. Fitzpatrick, S. Kachru, J. Kaplan, S. Raghu, G. Torroba, and H. Wang. Enhanced Pairing of Quantum Critical Metals Near $d=3+1$. *ArXiv e-prints*, October 2014, 1410.6814.
- [20] A. L. Fitzpatrick, S. Kachru, J. Kaplan, S. A. Kivelson, and S. Raghu. Slow Fermions in Quantum Critical Metals. *ArXiv e-prints*, February 2014, 1402.5413.
- [21] Michael Edward Peskin and Daniel V. Schroeder. *An introduction to quantum field theory*. Advanced book program. Westview Press Reading (Mass.), Boulder (Colo.), 1995. Autre tirage : 1997.
- [22] Mark Srednicki. *Quantum Field Theory*. Cambridge Univ. Press, Cambridge, 2007.
- [23] Xi Dong, Samuel McCandlish, Eva Silverstein, and Gonzalo Torroba. Controlled non-Fermi liquids from spacetime dependent couplings. 2014, 1402.5965.
- [24] Gonzalo Torroba and Huajia Wang. Quantum critical metals in $4-\epsilon$ dimensions. *Phys. Rev. B*, 90:165144, Oct 2014.
- [25] Sachin Jain, Shiraz Minwalla, and Shuichi Yokoyama. Chern Simons duality with a fundamental boson and fermion. *JHEP*, 1311:037, 2013, 1305.7235.
- [26] D. T. Son. Superconductivity by long-range color magnetic interaction in high-density quark matter. *Phys. Rev. D*, 59:094019, Apr 1999.
- [27] S-W Tsai, AH Castro Neto, R Shankar, and DK Campbell. Renormalization-group approach to strong-coupled superconductors. *Physical Review B*, 72(5):054531, 2005.
- [28] DT Son. Superconductivity by long-range color magnetic interaction in high-density quark matter. *Physical Review D*, 59(9):094019, 1999.
- [29] Christoph J Halboth and Walter Metzner. d-wave superconductivity and pomeranchuk instability in the two-dimensional hubbard model. *Physical review letters*, 85(24):5162, 2000.
- [30] Johan Nilsson and AH Castro Neto. Heat bath approach to landau damping and pomeranchuk quantum critical points. *Physical Review B*, 72(19):195104, 2005.
- [31] Mario Zacharias, Peter Wölfle, and Markus Garst. Multiscale quantum criticality: Pomeranchuk instability in isotropic metals. *Physical Review B*, 80(16):165116, 2009.
- [32] T Holstein, RE Norton, and P Pincus. de haas-van alphen effect and the specific heat of an electron gas. *Physical Review B*, 8(6):2649, 1973.