Teaching of division in Grade 3 and Grade 4: Disruptions to coherence.

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ABSTRACT

In South Africa, Reddy (2005) and Schollar (2004) have investigated learners' work on division tasks. While these studies showed that learners struggle with the solving of division tasks at the primary school level, these studies did not investigate how sequences of signifiers were assembled by teachers when they work with sharing and grouping situations and bald calculations. This focus came to the centre in this study because of evidence of different degrees of coherence in the signifier sequences produced by teachers. The focus on coherence was coupled with attention to issues identified in earlier research in the field of primary school mathematics: a lack of progression in signifier modes of representation from more concrete to more abstract forms, limited progression in the sophistication of division calculation strategies and limited number range during the teaching and learning of division. Given previous research findings that teaching division is difficult, this study sought to investigate the nature of disruptions to coherence in the teaching of division and how division situations, signifiers, division calculation strategies and number range featured in teachers' solving of division tasks.

Three Grade 3 and three Grade 4 mathematics teachers participated in this study. A total of 22 lessons on the teaching of division were observed and video recorded. Grounded theory framed the analysis of these lessons and led to the development of a framework that combined a focus on sequences of signifiers and a look at progression in signifier use, division calculation strategies and number range, considered in relation to coherence. Theoretically, the contribution of the study lies in its conceptualization of the nature of coherence in the context of evidence from lesson episodes ranged along a continuum from coherent to incoherent within the signification pathways framework. A coherent categorization of instruction relating to a division task episode indicated a pathway involving links between signifiers that were mathematically justifiable at all points within the episode. Two thirds of the coded episodes in this study played out coherently. In "coherent with limitations" episodes (14/119 episodes), while teachers' work with signifiers showed a coherent chaining across the pathways with new signifiers connecting with previous signifiers in the pathway and linking with the signified situations, some signifiers in the chain applied in localised ways to the episode, rather than in generalizable ways. In episodes coded as containing "some ambiguity" (20/119), signification pathways involved instances where signifiers linked to more than one signified in contradictory ways. In incoherent episodes (2/119), teachers work with signifiers showed mathematical errors.

These findings point to relatively widespread occurrence of episodes in division teaching in the early grades in South Africa, with the research design also pointing to more limited range across division task types in the more disadvantaged schools. Bald calculations were used commonly in both grades, and featured more often than division-situation based episodes in the coherent category. Regardless, of the initial situation (sharing or grouping) or bald calculations, tasks were more frequently enacted with group-based distributions. This language of description has been useful for identifying and developing more coherent teaching.

Keywords: coherence, division, sharing situations, grouping situations, progression of signifiers, division calculation strategies and number range, sequence of signifiers.

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Corin Mathews

25th June in the year 2021

DEDICATION

To Jesus Christ who looks beyond our brokenness and becomes our hope.

PUBLICATIONS EMANATING FROM THIS RESEARCH

Book chapter

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Mathews, C., Venkat, H., & Askew, M. (2018). Primary Teachers' Semiotic Praxis: Windows into the Handling of Division Tasks. In *Signs of Signification* (pp. 257-274). Springer, Cham.

Conference papers

Mathews, C. (2016). Division means less: Chains of signification in a South African classroom. Oral presentation: 13th International Congress on Mathematical Education 2016 (ICME-13), University of Hamburg, Germany.

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Everytime I thought I was being rejected from something good,
I was actually being re-directed to something better.
Dr. Steven Maraboli

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LIST OF ABBREVIATIONS

G1, Grade 1

G2, Grade 2

G3, Grade 3

G4, Grade 4

G5, Grade 5

G6, Grade 6

FFL, Foundations for Learning

RNCS, Revised National Curriculum Statement

NL, Number line

MPM, Mediating Primary Mathematics

MLA, Monitoring Learning Achievement

SACMEQ, *The Southern and East Africa Consortium for Monitoring Educational Quality*

ANA, Annual National Assessments

MDI's, Mathematical discourses in instruction

DNL, Double Number Line

T- table, ratio table

CCU, Complete composite units-based distributions

GBD, Grouped Based Distribution

T1, Term 1

T2, Term 2

T3, Term 3

T4, Term 4

SS, side-steps

Sr, signifier

Sd, signified

SP, Signification pathway

OHP, overhead projector

WB, White board

L1, Lesson 1

L2, Lesson 2 etc.

E1, Episode 1

E2, Episode 2 etc.

n, total number

1.1. Background to the study

This study is about coherence in the teaching of division in a South African context within three primary schools with a focus on Grade 3 (G3) and Grade 4 (G4). Coherence in the teaching of division was not, though, the initial emphasis of this research. The initial driver of the study emerged from an earlier research project (WSOE, 2009) conducted in Gauteng (one of the seven provinces in South Africa) primary schools in 2009. The focus of that research project was to evaluate the implementation of the then current South African (DoE, 2003) national curriculum, referred to as the Revised National Curriculum Statement (RNCS). The RNCS was a new curriculum at that time, which was intended, amongst other issues, to correct the lack of conceptual development and the lack of consistent assessment of learners evident in the previous curriculum. In the study, which focused on the implementation of the RNCS in Gauteng, one of the critical findings was that teachers continued to experience difficulties in assessing learners' mathematical competency.

While G4 teachers were not officially involved in the research, in informal meetings the G4 teachers noted that G3 learners were not coping mathematically when they got to G4. The G4 teachers complained that G3 teachers were "mis-assessing" learners' mathematical performance and that their assessment of learners' performance was inaccurate. According to the G4 teachers, G4 learners were significantly weaker than the reported G3 assessment outcomes indicated. Given that both the G3 and G4 teachers' judgments of learners' mathematical performance were based on their classroom mathematical tasks, this raised questions about differences in the teaching, and the nature and range of mathematical tasks provided to learners in G3 and in G4.

The G4 teachers further singled out division as a topic in which learners performed particularly poorly. This aligned with research at a national level where Reddy (2006) and Schollar (2004) had shown that division presented significant difficulties to learners at primary school level. It also aligned with international research which indicated that learners struggled with the concept of division more than the other operations (Vergnaud, 1985).

I therefore selected division as the mathematical concept on which my study would focus, with the understanding that this would allow me to gain insights into possible ways of improving alignments of learners' mathematical performance at G3 and G4 level. Consequently, the initial focus of the study looked at the nature and range, of division tasks presented to learners across G3 to G4 by teachers in three Johannesburg primary schools.

A shift from this initial focus occurred in my observations of the enactment of these tasks in classrooms. I had assumed that the teachers’ teaching of division tasks would be mathematically coherent but as the teachers taught and solved the division tasks, I began to observe differences in the extent of coherence in parts of what they said, did, and wrote. These differences were located in ambiguities and breakdowns in connection between parts of the instructional explanations offered or accepted by teachers as they worked with various division tasks. Following similar observations in several lessons, I shifted the focus of the study from the teaching, and nature and range of division tasks, to investigating coherence in the teaching of division.

This led to questions about how to study coherence in mathematical instruction, which led, in turn, to semiotic theory. I introduce the key tenets of my use of semiotic theory in the next section, with elaborated discussion offered in Chapter 3.

1.2. Semiotic theory use in this study

Many researchers have noted that one of the ways of making sense of mathematical discourse is to focus on semiotic studies of mathematical signs. Semiotics is the "study or doctrine of signs" (Colapietro, 1993, p. 179). De Saussure (1959) a linguist, proposed that a sign is produced through the linking of two elements, namely the signifier and the signified (Presmeg, Radford, Roth, & Kadunz, 2017). According to Chandler (1994, p. 20) "contemporary commentators tend to describe the signifier as the form that the sign takes and the signified as the concept to which it refers". He further mentions that "the signifier is commonly interpreted as the material (or physical) form of the sign; it is something which can be seen, heard, touched, smelled or tasted" (Chandler, 1994, p. 21). Chandler (1994, p. 21) further states "the relationship between signifier and the signified is referred to as signification". In mathematics education researchers have used these elements (signifier/signified) to describe a mathematical sign. A mathematical sign is produced when a mathematical relation links a signifier with a signified. A broad literature base suggests that mathematical discourse is constructed through a chain of signifiers (Presmeg, 1997; Walkerdine, 1988). These signifiers are linked to particular signifieds, which represent the underpinning mathematical concepts. Important to note, that I would employ semiotics as an analytical tool for investigating the teaching of division.

Several researchers in mathematics education have analysed mathematical signs and described them within what they termed "chains of signification" (Presmeg, 1985; Walkerdine, 1988; Whitson, 1997). These researchers noted that "chains of signification" became visible as teachers (coherently, in these cases) solved various mathematical tasks through sequences of signs which produce associations between signifiers. The analysis of these "chains of signification" within teachers' lessons became a possible way of me making sense of South African G3 and G4 teachers' ways of working with division.

Coherent ways of working with division, from this theoretical position, involve the setting up of mathematically defensible chains of signification. Understanding the mathematics of division, and some of the difficulties that have been described in the teaching of division in the literature, was therefore important. I introduce this literature base below, with further detail in Chapter 2. I also outline the presentation of division in the curricular context at the time of my data collection (2011-2012), as these were the requirements set out by the South African National Department of Education (DoE, 2008) for the teachers I was working with.

1.3 Division

1.3.1. The mathematics of division from a pedagogical perspective

In the case of division, the key concepts in focus in this study are division based in either "partitive" or "quotitive" situations. Briefly, division, as a mathematical concept, refers to two different kinds of situations, namely partitive and quotitive situations. In partitive situations, the number of groups to distribute between is known, and the most fundamental solution action involves the one–by–one distribution of items across the groups, with moves to more efficient solution processes effected over time. In quotitive situations, the number within a group, or the size of each group is known, and the fundamental action involves the distribution (or putting together or taking out) of groups from the original quantity until the whole quantity has been dealt with (Squire & Bryant, 2002). Carpenter, Fennema, Franke, Levi, and Empson (1999) emphasize the importance of the progression from direct modelling involving these basic one-by-one counting actions, towards truncated counts based on distributing more than one unit at a time to sharing of groups in sharing situations, and moves towards the use of "skip-counting" in multiples in quotitive, or grouping situations. These truncations, in their trajectory, lead over time to a bank of immediately recalled multiplication and division facts.

A key complexity of division is that, as a mathematical operation that links a dividend, a divisor and a quotient, it can refer either to a partitive or quotitive understanding of division, but moves between the reference to partition and quotition need to be carefully and explicitly handled to ensure coherence. In thinking about this idea in relation to semiotic theory, partitive and quotitive situations are therefore the fundamental "signified" entities that a range of division signifiers (such as the symbolic division sign ÷, or an array diagram, or a spoken or gestural sharing or grouping action) can refer to. What is observable within any working with division, or any mathematical task more generally, is a sequence of signifiers.

1.3.2. The teaching of division

The mathematical complexities of division referred to above feed into division being pedagogically difficult to teach. In this section, I highlight the major features that need to be taken into consideration when teaching division at G3 and G4 level.

A widely held position in early grades' mathematics teaching is that teachers need to take into consideration that learners have usually experienced the concept of division informally before entering school. There is the accepted view that young children develop informal knowledge of division through play and everyday experiences involving "sharing equally" (Booker, Briggs, Davey, & Nisbet, 1992). Sharing equally is linked to partitive division. Other researchers have also found that learners are likely also to have experience before school of division as grouping (Carpenter, Answell, Franke, Fennema, & Weisbeck, 1993). A key part of early division learning therefore is to support learners to shift from an informal knowledge of division based in sharing and grouping actions to unifying these two actions and their underlying situations under the umbrella of division as a mathematical concept.

In the school context, both teachers and learners need to make sense of the type of division situation that a problem is located in – sharing, grouping, or a "bald" symbolic calculation that can be attached to actions and language related to either situation. This "making sense" occurs in pathways of appropriate signifiers, and these, therefore, are crucial in the teaching of division in the primary grades. Particularly important for sense making are actions (the actions of sharing or grouping on concrete objects), linked to a particular division situation, with Presmeg, Radford, Roth, and Kadunz (2018, p. 10) noting that "gestures may indeed pave the way for linguistic and conceptual development".

These gestures or actions arise in the context of working with sharing or grouping problem situations. Researchers in the field of mathematics describe problems set in real-world or "realistic" situations as either word problems or situations (Gravemeijer, 1997, p. 394). In this study, I use the term "word problems" to describe these real-life/realistic problems as I have used the term "situations" to refer specifically to grouping or sharing contexts. The actions and talk linked to division word problems assist learners in making sense of sharing and grouping.

Evidence from children’s work with division problems led Verschaffel, Greer, and De Corte (2002) to suggest that teachers should move away from focusing on identifying division as the relevant operation and then carrying out the calculation, to spending more time with the appropriate sharing or grouping actions and language needed to produce solutions. These authors have argued that teachers too often teach learners to look for clue words that point to division operations without making sense of the division situation denoted in the word problem.

This "making sense" is made difficult in early division by complexities of language. Linguistic signifiers incorporate all spoken or written words or gestures within an episode. (Anghileri, 1995a, p. 2) notes that division word problems predominantly consist of linguistic signifiers (words), which are frequently mis-interpreted, for example, words such as "12 sweets shared by 3 people" and "12 sweets shared into threes" where both phrases contain the word "share" but the first instance implies sharing and the second instance implies grouping. In the teaching of division, the teacher has to ensure coherence between the division situation (where a grouping or sharing situation might be presented or invoked in working with the task) and the pathway of signifiers presented or accepted in the teaching of how to solve the given task.

The literature points to a number of signifiers that are useful in making sense of division situations. Linguistic word problem formats as signifiers with sharing and grouping actions of division situations have already been mentioned. Diagrammatic forms often begin with iconic images based on one-by-one distribution of objects/construction of equal groups that essentially re-play the physical actions of sharing/grouping. Another, way in which a diagrammatical signifier is experienced in the solving of division problems is the number line (NL), which allows for the use of skip counting. Evidence from teaching division has suggested the array as an important diagrammatical signifier. The work of Booker et al. (1992) and Anghileri (2006) with division notes that the array is important and useful in showing the validity of the commutative rule for multiplication and for linking multiplication and division within multiplicative reasoning. In the teaching of division, there is also a move into the use of symbolic signifiers involving numeral and symbols. In early work with division, these signifiers are often used in collaboration with the signifiers outlined above. A symbolic signifier that links the division action of repeated groups with symbolic signifiers is the number line (NL). Anghileri (2007) also describes how the number line as a diagrammatical is useful in capturing repeated sets in the form of equal jumps when solving division problems. The use of the Double Number Line (DHL) hold the relationship between two quantities constant (Kuc̈hemann, Hodgen, & Brown, 2011).

As the number ranges of dividends/divisors increase and sharing or grouping actions become increasingly cumbersome, division calculation approaches come to the fore. These calculation approaches shift from skip counting to truncations of skip counting. Knowledge of traditional short and long division algorithmic procedures, and alternative procedures, is also highlighted as an important aspect related to the mastery of division. Booker et al. (1992) mention that the effective and efficient use of addition, subtraction and multiplication is required for mastery of traditional short and long division algorithms. While the short and long division algorithms rest on quotitive models (i.e. "how many divisor-sized groups are in the dividend"), symbolic division signifiers linked to the short and long division algorithms may be linked, linguistically, to either sharing or grouping situations. Following from the Verschaffel, Greer and de Corte critique above, there is evidence that teachers frequently sideline any discussion of the underlying situations by simply focusing on teaching the algorithmic procedures (Verschaffel et al., 2002). Verification of the outcomes of division calculations has also been described as important within division teaching, and represents one avenue for emphasizing the inverse relationship between multiplication and division (Haylock, 2006).

Having introduced the key mathematical and pedagogical aspects of division, I go on now to outline the South African curricular context relating to division. In this section, I include some of the evidence about division learning that added to the motivation for conducting this study on division in Grades 3 and 4.

1.4. The South African curricular context

The primary school context in South Africa at the time of this study ran from Grade 1 to Grade 7. Three phases of schooling are encompassed across these grades: G1-3 – Foundation Phase (age appropriate children are aged 7-9, but the system does include some grade retention for what are described as "failing learners"); G4-6 – Intermediate Phase (age appropriate children aged 10-12); and G7 – part of the Senior Phase, with G8 and G9 in this phase being taught within secondary schools.

An important part of the shift from G3 to G4 that is noted in the curriculum documents and in broader writing, is that the phase transition from Foundation to Intermediate Phase involves a shift from contexts where concrete working is possible (and encouraged) to abstract thinking as the number range increases and types of number being worked with expands. A key problem that has been highlighted in South African writing is that South African learners are highly reliant on concrete strategies for solving problems at primary school level. Ensor et al (2009. p 26) concurs thus:

> we found that while shifts from concrete to abstract modes of reasoning were evident in all classrooms we observed, this did not happen at the pace or at the depth that students require in order to do more complicated arithmetic operations in the intermediate phase.

The lack of support for shifting learners from concrete to abstract ways of thinking creates difficulties for learners in the solving of division problems. Schollar (2008) provides evidence of ongoing use of inefficient strategies. In his work on establishing levels of mathematics performance, he noted that the national scores in South Africa were much higher for addition and subtraction as opposed to multiplication and division. Describing the reasons for a learner who solves the division calculation 36÷4=

with the answer 31, Schollar (2004) commented that the answer indicated a learner with no idea of conventional division methods or knowledge of timetables, or of the number sense that would suggest that the answer could not possibly be 31.

As mentioned earlier, the data collection period in this study was located in the context of a curriculum presented in the Foundations for Learning (FFL) policy. Towards the end of the last decade, ongoing low performance in primary school mathematics forced the South African government to relook at supporting their curriculum implementation. The decision to relook at supporting curriculum was based on international, national and regional studies which indicated a "crisis" in performance of standardised tests in mathematics across primary schools in South Africa (Ensor et al., 2009, p. 26). What these tests showed was that learners were performing at far lower levels than their peers internationally. A key aspect of addressing these backlogs was the presentation of the FFL curriculum. This curriculum increased the specification of coverage by noting the content (knowledge, values and skills) to be taught each term, rather than at the annual level as the previous Revised National Curriculum Statement (RNCS) had done.

Specifying progression in more detail and with more prescription was a key tenet of the FFL curriculum, with a section entitled "trajectory", looking at how word problems progressed conceptually from G3 to G4, and also the progression of number ranges and problem-solving approaches. On division, this included both sharing and grouping situations across G3 and G4. However, in both G3 and G4, the FFL suggested that alongside sharing situations a greater emphasis be placed on grouping situations. Trajectories in terms of problem solving approaches were also made visible in the FFL. Across G3 and G4 learners, the need to solve word problems and "practical" problems (the FFL does not define what practical problems are) through appropriate symbols was mentioned. Across both grades, the FFL suggested that teachers needed to use addition, repeated addition, place value partitioning, sharing by groups, repeated subtraction, ratios, rounding off, doubling and halving, and working with fractions when solving division problems. Encouraging the teachers to make use of the reciprocal relationship between multiplication and division was only stated for G4.

Progression in the number range was separated according to whether division number sentences and word problems were being worked with. In G3, the number range for solving division number sentences shifted from two-digit numbers divided by one-digit numbers to three-digit numbers divided by one-digit numbers. In G4, the FFL extended the number range from three-digit numbers divided by one-digit numbers to three-digit numbers divided by two-digit numbers for solving division number sentences. The number range related to word problems in G3 focused predominantly on two-digit numbers divided by one digit numbers, and, towards the end of the year, three-digit numbers up to 1000 divided by a one-digit number. In G4, the dividend number range started at 1000 and ended at 10 000 for division tasks related to word problems and real life contexts. Thus, the trajectories within the FFL presented progression with regard to the types of word problems and the number range and explicit approaches in which division problems needed to be solved.

Clearly, the focus on sharing and grouping as important to the development of division from G3 to G4 aligns with international research on what is important in the teaching and learning of division (Neuman, 1999). International literature argues that division tasks should incorporate various signifiers (linguistic, diagrammatical symbolic) together with, if necessary, their appropriate actions on concrete objects. However, the FFL at G3 and G4 emphasized the use of word problems with no reference to sharing/grouping actions and with limited emphasis on how diagrams and no acknowledgement of the progressive truncations of counting should be used in ways that include building up and breaking down. Yet, within my study teachers widely used concrete artifacts in G3 with division actions visibly used by both G3 and G4 teachers. Although there was a limited reference of diagrams (number charts, number lines, etc.), the use of concrete artifacts and division actions are absent across the G3 and G4 specifications. For example, when solving these division situations through diagrammatical signifiers such as the number line the teacher needs to use precise mathematical language. At a broader level, the number range increased from G3 to G4 with reference to word problems and symbolic number calculation. When using symbolic signifiers research notes that teachers need to be aware that the division symbolic signifier calculations may create ambiguity for learners. Traditional short and long division algorithms did not feature within the FFL documents in G3 and G4, but within my study some teachers in G3 and G4 made use of short and long division algorithms. Finally, the role of place value was implicitly referred to in the FFL contrasting with research that indicates that there needs to be an explicit relationship between teaching and learning of division and place value (Thompson, 2000).

In summary, the FFL curriculum recommendations contained some elements that stood at odds with the literature on the teaching of division such as, the lack of actions on objects, lack in progression from signified division situation and actions to symbolic signifiers particularly the absence of particular diagrammatical signifiers, and some gaps in relation to progression in signifiers, sharing actions (distribution of one-by-one and complete composite units), grouping actions, from skip counting to truncations of skip counting, and the from short division algorithms to long division algorithms with chunking. This was of interest, as this curriculum represented the background that was supposed to inform G3 and G4 teachers' selection and teaching of mathematical content.

1.5. Coherence within teachers' mathematical discourse

In South Africa, researchers in mathematics education have investigated teachers' mathematical discourse within South African classrooms. Venkat and Adler (2012) coined the term MDI (Mathematical Discourse in Instruction) to refer to the "mathematical aspects of what teachers say, do and write as they interact with learners in mathematical classrooms" (Venkat & Adler, 2012, p. 1). They noted a lack of mathematical coherence and connections within MDI across a range of primary and secondary mathematics instruction episodes. Instructional discourse was then further studied within a primary school context (Venkat & Askew, 2018). In taking this work forward into primary mathematics teaching, Venkat and Askew (2018) referred to the idea of teachers "Mediating Primary Mathematics" (MPM). MPM studies have looked at teachers' instructional discourse by focusing on tasks, artifacts, inscriptions and talk. Across these mediating strands, they noted that episodes of incoherence and poor connections were evident on the ground within South Africa.

Ambiguity as a notion also has the capacity to negatively influence coherence. Barwell (2005, p. 119) mentions that "ambiguity arises where more than one meaning can reasonably be attributed to a particular word or phrase". Norton (1975), working in developmental psychology, also identified "multiple meanings" as a key cause of ambiguity. Amit and Gilat (2012) argue that the use of ambiguity in a structured approach can develop learners' mathematical problem solving skills. However, for ambiguity to be effectively used as a resource, they argue that teachers needed to intentionally incorporate attention to possible ambiguity while providing clear explanations of how to solve the mathematical problem. A structured approach to identifying ambiguities with clear explanations allows learners to identify and solve all the different meanings located in one mathematical problem. For example, a teacher could write the following number sentence $10 \div 2=$. The teacher could suggest that two methods be used, in the first method two boys are drawn and the sweets are shared one at a time between them resulting in each boy getting 5. The second method could be making groups of two and then counting the number of groups to provide the answer 5. The ambiguity was made explicit through the use of the two methods in producing the same answer 5. However, in the absence of clearly mediating ambiguity as a technique for problem solving, the presence of ambiguity results in learners experiencing confusion with regard to a mathematical concept.

Researchers in semiotics note that working with signs allows ambiguity in discourse to be made visible. A key ambiguity in division working is seen in my study when a specific signifier can be related to several incompatible signified entities – most fundamentally sharing and grouping situations. For example, while a symbolic division number sentence can be linked to a sharing or grouping situation, what is crucial is that instruction ensures that coherence is achieved through careful linking, and intentional discussion of ambiguity if this is present. What is problematic for coherence is when ambiguity occurs with no acknowledgement of a shift to a different signified situation, for example, a shift, mid–explanation, from a sharing situation to a grouping action. Key markers of coherence therefore, are when a consistent division situation is visible throughout the solving of a division problem or when a shift from one division situation type to another is made with acknowledgement or discussion of the reasons for the shift.

A key contribution of my study was the development of a signification pathway framework with four categories based on the nature and extent of coherence as discussed above. The framework was developed from evidence of disruptions to coherence that created difficulties for learners to grasp the distinction between division situations. While the framework is introduced and discussed in Chapter 3, here I provide an overview of the four categories of coherence within it. In category one, coherence is seen when a sequence of signifiers are mathematically justifiable at all points in an episode. For example, the teacher may provide a sharing word problem – "Share 10 sweets between two learners" (linguistic - signifier) together with ten blocks (concrete signifiers) and two pictures of people which she sticks on the board. She then could continue to solve the word problem by a sharing action on 10 sweets (blocks) between two pictures on the board. The teacher in the solving of the sharing word problem has used an action (one-by-one distribution) of the blocks between the two pictures on the board. The teacher counts the number of blocks next to each picture of a person on the board and produces an answer of five. In the second category, although coherence is seen between the sequences of signifiers, the signification pathway was mathematically justifiable in a localised way without being generally mathematically applicable. For example, if the example above concluded with the teacher making mention of "division always making smaller", she then localised the notion of division to the episode in ways that are not mathematically generalizable, even though the pathway to that point was coherent. In the third category, signification pathways indicate instances of ambiguity. For example, a symbolic signifier 8 ÷ 2 = could be solved in two ways either as a sharing situation or a grouping situation. In a sharing situation, the sharing action on blocks might distribute one block at a time between two people producing four blocks next to each person and then counting the blocks, producing the answer 4. If a grouping situation is applied to 8 ÷ 2 = the teacher would be grouping two blocks until all eight blocks are used, producing 4 groups. While both of these actions are consistent with the division number sentence, ambiguity is seen when the teacher uses 8 ÷ 2 = and suggests that she wants to use groups of two and then proceeds to demonstrate a sharing action on the blocks with no acknowledgement of the shift between the division situations. In the last category, the signification pathways are mathematically incoherent. An example in this category consisted of the teacher asking learners to solve 20 ÷ 5 = __. Learners provided the answer 5. The teacher accepted the answer and did not offer or elicit a correct answer. The incoherence is the acceptance of an incorrect answer without further questioning or follow-up.

Like Venkat and Askew, my focus is on instructional discourse. Ideas in their MPM theorisation relating to incoherence, ambiguity and "localised" discourse were useful in this study. In relation to localisation, Venkat and Askew (2018, p. 85) describe teachers’ solving of examples as "localised" within their MPM (Mediating Primary

Mathematics) framework when the teacher provides "a method that can generate/validate answers beyond the particular example space" but does not have the ability to be generalised (Venkat & Askew, 2018, p. 85). Askew (2018) then applied the framework to two teachers lessons from 2011 to 2014. He noted that at some points in the lesson in 2011 one of the teachers was prone to use localised methods/validations, but in 2014, that the teacher’s explanations shifted away from localised explanations to more generalized explanations. Likewise, in my study, the notion of "localization" is seen in claims about division and methods of solving division problems that are valid within the particular episode and can be applied to other episodes but are not mathematically "general" in the way that is asserted. For example, as stated above after a teacher has produced an answer for a division problem, the teacher would mention that "division always makes smaller" (Tirosh & Graeber, 1991) which, while valid within certain episodes containing particular examples, does not hold as a general rule.

Coherence is further impacted when there is a disruption between signifiers within a sequence of signifiers within an episode. For example, the teacher writes the following word problem on the board: "John has 10 apples; he has 2 apples in each bag. How many bags does he have?" She then proceeds to solve the division word problem by sharing the ten apples (represented by blocks) using one–by-one distribution between two bags resulting in each bag having five items. The sign produced through this sharing action is that the five signifies the number of apples in a bag as opposed to the "two apples in each bag" suggested in the initial problem situation. This sharing action therefore produces a disruption in the sequence of signifiers, that makes it harder for learners to see mathematics as "sensible" with logical links between the steps in how division problems are solved.

With the focus on coherence, the study moved from the analysis of the nature and range of division tasks across G3 and G4, to focusing on sequences of signifiers in instruction. Thus, the emergence of the Signification Pathway Framework was critical as a way of understanding the variance in coherence while teaching division in G3 and G4 classrooms.

1.6. Signification Pathway Framework

Semiotics formed the theoretical basis for this study. Semiotics is a general theory of signs or a theory of how signs signify, that is, a theory of signification. The analysis of the relationship between signifiers within a sequence is one of the lenses that provide insights into mathematical coherence. Cobb (1997) uses the term "chains of signification" to refer to a sequence of signifiers in which there is no mathematical disruption anywhere along the chain and a straightforward link from the initial signifier to the last signifier in the focal episode. Both Presmeg (1997) and Cobb (1997), in their explanation of these chains of signification, illustrate chains of signification that are mathematically justifiable and consistent with regard to the signified entity. However, in my empirical research sites in the South African context, there were frequent instances of instructional discourses where teachers "side-stepped" to focus on a related, or alternative, signified entity in the midst of their working, as well as the disruptions to coherence noted above. As a result, I coined the phrase "*signification pathways*" to refer to pathways in which the sequence of signifiers produced in the context of teaching of a specific task could be studied. As noted already, these signifiers did not necessarily always link in mathematically coherent ways with previous signifiers within teachers' mathematical discourse in a focal episode related to teachers' work with tasks. In signification pathways I looked at the relation between signifiers within a sequence, with the analysis of the linking between signifiers highlighting the nature and extent of mathematical coherence depending on the mathematical appropriateness of signifiers within a sequence.

The focus of the signification pathway framework was to indicate a range of categories related to coherence in the teaching of division. After analysing each episode, I organised them in the four categories outlined:

a) The first category points to a signification pathway involving a sequence of signifiers which are mathematically justifiable at all points in the episode, indicating a *mathematically coherent* pathway.
b) In the second category, although coherence is visible between the sequences of signifiers within the episode, the signification pathway was mathematically justifiable in a localized way without being generally mathematically applicable. This pathway is *coherent with limitations*.
c) The third category highlights signification pathways that involve instances of ambiguity. This pathway is *coherent with some ambiguity*.
d) The last category considers signification pathways that are *mathematically incoherent*.

1.7. The Purpose of this Study

Although division is an important mathematical notion with an extensive literature base, and while coherence has been an emergent theme in South African mathematics education research for the last few years, there has been no emphasis in the literature on understanding the nature and extent of coherence in the teaching of division specifically in South Africa. As noted already, this purpose came into view when I started to examine for mathematical coherence between sequences of signifiers within the teaching of division. Studying coherence provided a route into considering differences in access to division ideas across three government primary schools in Johannesburg. I focused on division teaching in Grades 3 and 4 in each of these schools, with three teachers in each grade. In order to understand the mathematical relation between the sequence of signifiers with a signification pathway, research questions were designed to guide the analysis of lessons taught in G3 and G4 classrooms in South Africa, thus providing insights into the nature of division teaching in South Africa, in ways that can contribute to improving low division performance (Reddy, 2006; Schollar, 2004).

1.8. Research Questions

The following questions underpin this research:

a) What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G3 division teaching in the three schools?

b) What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G4 division teaching, in the three schools?

c) What do any differences seen between G3 and G4 in patterns of working across the signification pathway framework categories suggest for improving G3 and G4 teaching of division in coherent and well-aligned ways?

1.9. Methodology

There are two traditional research paradigms used for the interpretation and collection of data. The two paradigms are referred to as quantitative research and qualitative research. Both these approaches assist researchers in the collection and interpretation of data.

This study employed a qualitative approach including some quantitative summarising of the number of episodes in each category as an indicator of the nature of coherence of each lesson. Qualitative research explores as much detail as possible, of a small number of instances in "depth" rather than "breadth" (Patton, 1980). Qualitative researchers record the experiences or practices of people as closely as possible: how they think, what they say, why they say it and analyse emergent insights, in this case, how division is taught across three schools with six teachers. Qualitative research together with some elements of quantitative research has allowed me the opportunity to reflect in depth on signs and the relation between signifiers within signification pathways as a route into understanding the nature of coherent division teaching in South Africa.

In this study, some of the features of qualitative research that I drew on were video-recorded classroom observations (Merriam & Simpson, 1984) and document analysis (Hopkins, 2008). I observed teachers' classroom activities but also looked at curriculum documents as a way of understanding teachers' reasons for selecting division examples and concepts. This resulted in me video-taping six teachers of mathematics (who taught division). The observation of video recordings of teachers' sequencing of signifiers within a signification pathway became the focus of the research. Classroom observations, which were video-taped and then transcribed, allowed me to observe the transformation of signifiers across the whole class teaching episodes in each of the observed lessons. Categories and patterns emerged from the analysis of video recordings of teachers teaching division. I investigated teaching activities in detail. The detailed focus on the teachers allowed me to constitute each teacher as a case study and to compare these G3 and G4 teachers' teaching of division, which is another feature of qualitative research. In case studies, the scale of the research tends to be small and the results are documented in a natural language (Merriam, 1998).

My singular approach to data collection focused on a crucial sample of teachers and schools. The sampling strategy in this inquiry was *opportunistic* (Suri, 2011). This research was conducted in three primary schools in Gauteng. In each school the participants were two teachers who taught numeracy and mathematics: one in G3 and one in G4. I observed these teachers over a period of one year (six months in G3 and six months in G4 during lessons where division was taught). The criteria for selection of teachers were pegged on G3 and G4 teachers' years of experience in teaching numeracy and mathematics and their willingness to participate in the study. The sample of teachers' division lessons that were video recorded came from three different schools.

One of the schools selected was historically previously advantaged pre-1994, with learners who come from advantaged homes. A second school selected also categorized as historically advantaged pre-1994, but learners from this school came from socio-economically disadvantaged background. The last school selected can be categorized as being previously disadvantaged with their learners coming from socio-economically disadvantaged backgrounds. In a South African context, the idea of being previously disadvantaged refers to people who have been racially, socially and economically discriminated against based on skin colour. The idea of being previously advantaged pre-1994 in South Africa essentially refers to white people who were not discriminated against racially, socially or economically based on their skin colour.

The emergent phenomena of incoherence and ambiguity in the instructional discourse led me into a grounded analysis of the nature and extent of coherence in each episode, guided by the literature on division and the semiotic theorisation of ambiguity and extents of coherence in relation to the sequencing of signifiers. Grounded theory is one of the many qualitative approaches that assist in identifying important categories and patterns within data. Lincoln and Guba (1985) stated that "Grounded Theory is induced from the data rather than preceding [it]". The originators of grounded theory have different approaches in which they used Grounded Theory methodology. In the first approach, some researchers suggest that some knowledge of literature should not be considered when analysing data and only when an emergent theory is observed can more literature be used. In the second approach, theorists suggest that the analysis of data should rely on extensive literature and experience. I used the latter approach to interpret the data by allowing categories and patterns to emerge while keeping important features of the division literature and my prior experiences in focus. I employed grounded theory to underpin the analysis of the data. Grounded theory allowed me to develop the signification pathway framework, which assisted me in identifying the nature of episodes with regard to various categories of coherence, which included incoherence. In making sense of the data, each analytical chapter highlighted different aspects of ways of looking at coherence in the teaching of division.

1.10. Chapter Summaries

In this section, I show structurally how this report is organised; I also give a brief summary of each chapter.

Chapter One:

In the introduction to the study. I layout the following: background to the study, the importance of coherent teaching of division, theory, the purpose of this study, the research questions, methodology and chapter summaries of the study.

Chapter Two:

In this chapter, I present a review of the literature in which the study is embedded. The literature review looked at the mathematics performance of learners in primary schools in South Africa. Thereafter, the literature includes discussions on elements within division that inform primary school teachers' mathematical choices of signifiers. The literature review also looked at issues related to the South African curriculum at the time of the study that was introduced in this chapter. I also looked at literature on teaching of division. I then discussed the implications for my study from the literature base for the teaching of division. This allowed me to look at lessons with a lens on what aspects of division according to literature were seen or absent when division tasks are taught.

Chapter Three:

This chapter discusses the theoretical framework that underpins this study. The first idea is the importance of semiotics in mathematics education. This section also discusses theorists that have studied semiotics such as Pierce, de Saussure, Vygotsky, and Walkerdine among others. I also looked at the importance of chains of signification. Thereafter, I discussed the importance of signification pathways and teaching and learning of division in relation to semiotic systems. At the core of this chapter are concepts and approaches to understand how South African primary school teachers assemble and sequence signifiers related to division in mathematics because this is at the heart of how they teach division.

Chapter Four:

This chapter presents the methodology I used to answer the research questions of the study. I discuss the research paradigm, the importance of case studies for this study, the role of grounded theory in the categorisation and coding of the data, research sample and procedure, methods of data collection validity and generalizability, and ethical considerations.

Chapter Five:

This chapter presents the analysis of the teaching of division in G3 and G4 using the signification pathways framework. I focus on how teachers across the grade taught sharing/grouping/bald calculations through their use of initial signifier with number range, linking of signifiers in a sequence, the division calculation strategies and extent of coherence across each teacher's focal episodes. The analysis of episodes was based on the work of the three G3 and G4 teachers: Cathy, Mary, and Hazel (all pseudonyms, as are all names, of teachers and learners across this study) and Asha, Joy, and Sam (pseudonyms) who were identified as case studies.

Chapters Six:

In this chapter I consider empirical findings that emerge from the analysis conducted in this study. I note the theoretical contributions of the study to knowledge on how teachers teach division at primary level in South Africa, and what this contribution suggests for pre - and in - service primary mathematics teacher education. Lastly, I present my reflection on my journey through this study over the years.

2.1. Introduction

This chapter begins with a brief outline of the broad context of primary school learners' mathematics performance in South Africa and of the nature of primary mathematics teaching. This outline provides a context for focusing on the coherence of teachers' teaching of division in Grades 3 and 4. In the body of this chapter, I discuss how division as a mathematical topic is described in the literature, with the emphasis on aspects of this topic that are considered as foundational in the early grades in order to support children to understand division ideas and problem-solving processes. I then go on to present findings from international research on the nature of division teaching at primary level, and what these findings have led researchers to suggest about what the teaching of division ideas should look like in terms of sequencing and how teachers should work with these ideas within instructional explanations. The chapter concludes with a discussion focused on FFL curriculum documents related to the teaching of division.

2.2. Overview of mathematics performance in South Africa

South African primary school learners' poor performance in mathematics was one of the reasons for locating this study within primary schools in South Africa. National, regional, and international studies in recent years have highlighted a "crisis" in learners' performance on standardized tests in mathematics across primary and secondary schools in South Africa. For example, the cross-national Monitoring Learning Achievement (MLA) study, conducted in the 1990s with a focus on G4, established that the average numeracy score was at 30% (Reddy, 2005). Also, the Third International Mathematics and Science Study (Mullis et al., 2000) and the Trends in International Mathematics and Science (Reddy, 2006), focused on learners in Grade 8, found again, that of all the participating countries, in both of these tests South Africa showed the lowest average score in both mathematics and science (Mullis et al., 2000). Additionally, *The Southern and East Africa Consortium for Monitoring Educational Quality* (SACMEQ) studies, focused on Grade 6 learners, showed that of the 3119 randomly selected South African Grade 6 learners, more than half did not reach the basic numeracy level (Moloi & Strauss, 2005). These studies highlighted a crisis in primary schools and suggested that the problems of numeracy begin in the lower grades of primary schooling, motivating the focus of this study on mathematics in primary schools.

With reference to the mathematical concept of "division", Schollar (2004, p. 33) notes that the scores nationally for addition and subtraction are relatively much higher than those for multiplication and division because of the greater ease of using simple unit counting methods to solve them. He exemplifies the kinds of problems that learners' work indicates when asked to solve the following division calculation 1420÷20= and 36÷4=. In the first division calculation, a Grade 7 learner used repeated addition of 20 to solve the problem, thus showing a lack of awareness of place value relationships and their role in equivalence relationships in division. In the second calculation the learner produced an answer of 31. Schollar (2004, p. 12) reports that "the child clearly has no idea how to actually use conventional division methods, has no knowledge of times table and, perhaps most significantly has no number sense in that 36 divided by four times simply cannot be 31" (See Figure 1). These examples also point to struggles in using multiplication facts to assist in solving division problems. Moving beyond whole number division, Reddy (2006) reports that in international studies, individual test item analysis of South African learners' responses to tasks point to particularly poor performance on division problems. She notes that on division problems involving the division of a whole number by a unit fraction, only 7 out of every 100

South African Grade 8 learners got full credit compared to learners in countries like Singapore in which 78 out of a 100 got full credit.

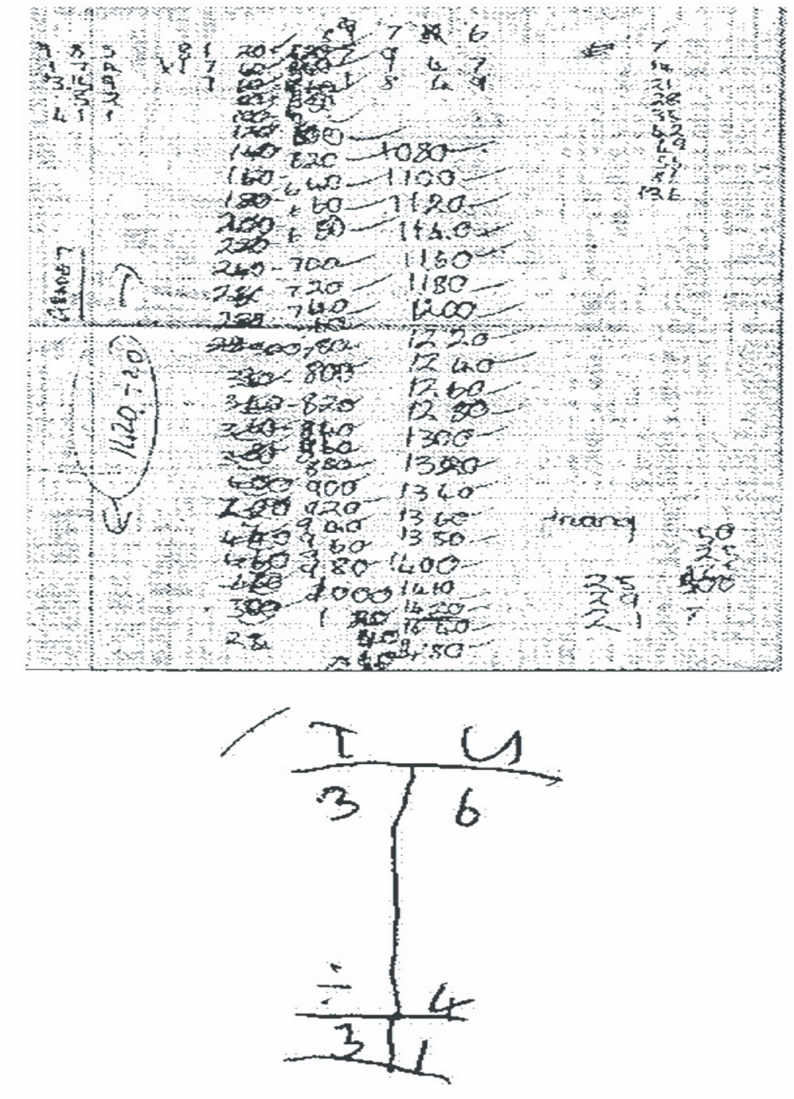


Figure 1: Examples of how division is solved by South African G7 learners (Schollar (2004, p 12).

A few years later, in the context of reporting on the "Annual National Assessments" (ANA) Kanjee and Moloi (2014) note that the ANA results indicate that learners in Grade 4 had particular problems in solving division tasks. These observations by Schollar (2004), Reddy (2006) and by Kanjee and Moloi (2014) indicate that mathematics in general, and division in particular present many difficulties for South African learners at the primary school level. Because learners struggle to solve division problems, I decided that the teaching of division in primary schools would be the focus of this study.

Other research further emphasizes the difficulties of teaching division in South African classrooms. Roberts (2012) found that when teachers attempt to assist learners to become mathematically proficient, particularly in relation to division, many difficulties become evident in their teaching. One of the difficulties she notes is that some teachers only understand division as sharing, which creates limitations for them when using algorithms. Together with their limited knowledge of division situations, Roberts found that they demonstrated ineffective use of recalled facts and inefficient ways of working with traditional long division algorithms. Her research also points to overlaps between South African learners' and teachers' experiences of the complexities linked to understanding and solving of division problems, and those of their international counterparts, subsequently described by Takker and Subramaniam (2018).

A further problem that has been noted in South Africa, both in division teaching and beyond, relates to coherence in the ways instructional explanations and representations are orchestrated in the context of particular tasks and examples (Venkat & Adler, 2012). Across a range of topics and grades, these authors point to instances of talk that are not well matched either to the problem being worked with, or to the sequence of transformations of representations that the teacher produces. In the context of division, in earlier work from the empirical dataset used in this study, I have noted similar instances of teacher explanations that tend not to cohere well with the task and/or the representations used to solve the task (Mathews, 2014).

2.3. The importance of coherence in teaching division

The idea that the coherence of the teaching plays a vital role in distinguishing between the pedagogies used in contexts of good and bad mathematical performances is gaining traction in the international literature (Chen & Li, 2010). Sekiguchi (2006) describes how Japanese mathematics instruction shows high coherence which is achieved through connections not only within a single lesson but also between lessons. In the teaching of simultaneous equations coherence manifested either as the explicit or implicit interrelations of all mathematical components in that lesson, with teachers providing learners with a particular *coherent vision of mathematical knowledge* (Sekiguchi, 2006). In Japanese mathematics lessons, skilled teachers incorporate coherent discourse reinforced by a *traditional script* (ibid). At the same time, international research also notes that there is a disruption to coherence when ambiguity exists within instructional explanations. Within my study coherence is evident when the signified division situations link to a particular division action, irrespective of which signifiers are used.

In recent years, researchers in South Africa have begun to investigate the importance of coherence within a South African landscape. Looking across four episodes drawn from South African primary and secondary mathematics teaching, Venkat and Adler (2012) examine teachers' mathematical discourses in instruction (MDI's) to establish the nature of coherence and connections in their pedagogy. The instructional discourse included examining problems, teachers' selected representations, and subsequent transformations, explanations, the justification for the representations selected and transformation performed. They note that "one expects MDI's to be both coherent and connected, and to provide some of the rationales for the representations selected and transformation activity that is enacted" (Venkat & Adler, 2012, p. 6). Of the four episodes observed in Venkat and Adler’s (2012) study, only one was coherent and connected. In South Africa, this finding suggests that providing coherence within explanations of mathematical examples may be difficult for many teachers.

Venkat and Askew (2018) have investigated how coherence could be understood within primary school settings in South Africa and to do this they designed a framework referred to as MPM (Mediating Primary Mathematics). Within this MPM framework Venkat and Askew (2018, p. 85) describe one aspect of teachers solving of examples as "localised method/validation". This localised method/validation provides "a method that can generate/validate answers beyond the particular example space" but cannot be generalized (Venkat & Askew, 2018, p. 85).

Askew (2018) then applied the framework to two teachers' lessons videotaped in 2011 and 2014. He noted that at some point in the lesson in 2011, one of the teachers was prone to use "localised methods/validations". However, in the 2014 lesson, Askew (2018) noticed that one of the teacher's explanations shifted away from localised explanations to more generalized explanations. Similarly, in my study, although coherent, "localization" is seen when ideas and methods are valid within particular episodes and can be applied to other episodes, but cannot be generalized. For example, after a teacher, has produced an answer for a division problem, the teacher would mention that "in a division sum the biggest number always goes first" which could be valid within specific episodes containing particular examples, but not when learners have to solve division calculations with negative numbers. In my study, the idea of "localization" is important for analysing episodes in the teaching of division in G3 and G4 lessons.

Ambiguities can also negatively influence coherence. Barwell (2005, p. 119) notes that "ambiguity arises where more than one meaning can reasonably be attributed to a particular word or phrase". Amit and Gilat (2012) refer to the work of Lesh and Doerr (2003) in which the latter adapted the work of Norton (1975) on ambiguity to inform their work on assisting learners with unpacking mathematical activities. The four ambiguities that Lesh and Doerr (2003) used from Norton's work were (1) multiple meaning (2) vagueness (3) probability and (4) lack of information (Amit & Gilat, 2012, p. 21). Norton (1975 p. 608) describes multiple meanings as having at "least two meanings, whether the person was aware or unaware of the multiple meanings, or clear or unclear about them".

In the work of Amit and Gilat (2012) and Barwell (2005) ambiguity is considered to be a strength that helps learners in developing ways of solving problems. However, all the researchers note that for ambiguity to be effectively used as a resource, the teacher needs to use it intentionally while providing clear ways of solving the mathematical problem through clear explanations. In the absence of clearly mediating ambiguity as a technique for problem-solving, the presence of ambiguity within the teaching of division results in learners experiencing confusion. In the case of my research, I focused on ambiguity, which was exhibited in various ways.

One kind of ambiguity was evident when one signifier could have multiple interpretations. Another ambiguity was evident when an episode would begin with one situation (grouping) and then another situation (sharing) would be introduced without the teacher acknowledging the shift. Another ambiguity occurred when no overt signifier was given at some point in the episode for a signified but then was subsequently corrected. The last ambiguity observed was evident when an incorrect signifier was linked to a signified and then again subsequently corrected. These examples illustrate how ambiguity can disrupt coherence.

Many teachers do not realize how easily conceptual shifts occur during their explanations while solving division problems, thus creating ambiguity. These shifts then produce disruptions to coherence in their explanation. Rowland (2007) investigated a novice teacher's way of solving a division problem in the context of a broader study which focused on primary school teachers' mathematical subject knowledge. The novice teacher (Maire) taught a lesson on whole number division within a "Harry Potter" context. For her lesson, Maire wanted to solve two quotitive word problems. After her explanations and reflection on her lesson, she realized that at certain points while solving these problems, she had shifted from a quotitive situation to a partitive situation. Rowland (2007, p. 21) notes that the difficulty is that "in explaining how to resolve the problem, Maire drew on the language and concepts of partition". In doing that she had inadvertently directed the learners to the incorrect division structure while producing the correct answer, thus, creating a disruption in the coherence of the explanation. This finding resonated with some of the findings in the initial phase of this study (Mathews, 2014). I concluded that to be coherent, a particular division situation must link with particular actions and language throughout the explanation. In other words, a quotitive situation should illustrate grouping actions with appropriate language such as 'Can you tell me the "number of groups”?' The teaching of division is complex and nuanced, and a teacher's lack of awareness of such complexity and nuance is likely to have a direct impact on the coherence embedded within the explanation, leading to ambiguity. Within this study, the concept of coherence is crucial for making sense of the teachers' instructional discourse.

Insights gleaned from previous work conducted in and beyond South Africa on division and overall mathematical performance informed my decision to observe six teachers' work with division in three primary schools in South Africa. By "unpacking" the teaching of division, I aimed to gain insights which could be used to improve division instruction and through this, improve learner performance. Before observing these teachers, I reviewed pertinent international literature to identify what was considered to be essential when teaching the concept of division, (including its connections to multiplication), beginning with writing in education on division as a mathematical concept considered from the perspective of teaching and learning, as this provided a lens through which I could consider division instruction. Given that progression from unit counting approaches to efficient division calculation has been noted as a key problem in South African classrooms, this chapter then considers how progression in ways of working with division problems has been described in the literature.

2.4. Division as a mathematical concept

Division, as a mathematical concept, sits within a broad network of ideas in mathematics. In this introductory overview of division, I deal with aspects of this network that are relevant to early whole number division, as this is the mathematical focus in Grades 3 and 4, the grades in which the study is located.

In the early grades, division can be understood in two key, inter-related ways. Firstly, division involves actions of splitting a quantity into equal-sized parts or equal-sized groups. Secondly, division is the inverse of multiplication, which involves the replication of equal-sized groups. These two ideas point to a number of properties, representations, and related ideas that are important to consider and include in teaching division for understanding. In the following paragraphs, I summarize these key ideas with reference to mathematics literature.

The two ways of understanding the meaning of division mean that the notion of "equal-sized groups" is core to the understanding of both multiplication and division. Equal-sized groups require learners to work with groups of items as one thing. For example, a learner must view four items as "one four" with the unit "four" treated as a countable composite unit (Anghileri, 1989). For the understanding of multiplication and division, the learner needs, eventually, to coordinate groups of equal-sized groups and recognize patterns relating to composites of composites (e.g., 4 sixes is the same as 8 threes, and equals 24). Mulligan and Mitchelmore (1997) have noted that difficulties faced by older learners can be related to their lack of working with "equal-sized groups" in their early concept formation of multiplication and division.

Booker, Briggs, Davey, and Nisbet (1992) have noted that the process of sharing (making equal-sized parts or partitioning) and the process of repeated subtraction (making equal-sized groups or quotitioning) are both related to multiplication, given that repeated addition can feature within multiplication and repeated subtraction can feature within division. Carruthers and Worthington (2006) have argued that mastery of the additive relationship allows for a better grasp of multiplication and in turn, division. This reflects the order in which many curricula (including the South African curriculum) deal with additive, followed by multiplicative, working. However, the centrality of the inverse relationships between addition and subtraction, and between multiplication and division, have led many researchers to argue that multiplication and division (and addition and subtraction) should be dealt with together rather than separated in instruction (Carpenter et al., 1999).

The inverse link between multiplication and division means that this relationship between the two operations is critical to attend to in instruction. Anghileri (2006) notes that expressing the same multiplicative relationship between three quantities in different ways can help develop awareness of the central links between multiplication and division. For example, asking children to work out what 3 x 2 makes by communicating that this can involve making three groups of two objects, can be followed by asking "How many groups of two make six?" or "If three equal groups contain six objects, how many in each group?" Anghileri (2007) stresses that the use of questions like 'How many threes in 12' relating to the multiplication fact "4 threes makes 12" are crucial for developing effective and efficient ways of doing calculations. A consequence of the inverse relationship is that, in higher number ranges where division algorithms are used, the use of multiplication provides a means of verifying the answer.

Teachers' limited notions about multiplication and division can create difficulty for learners in solving division problems effectively. Kouba (1989) notes that learners in G1 may develop an initial understanding of multiplication as a group of sets containing the same number of objects and then by G3 these same learners may view multiplication as only repeated addition. Fischbein, Deri, Nello, and Marino (1985) argue that this limited view of multiplication does not extend well to fractions and decimals, and results in such misconceptions as "multiplication always makes bigger". Tirosh and Graeber (1991) work with pre-service teachers also highlighted the limited view of division presented to these students, noting that most of the examples used with them were division problems with whole numbers. As a result, when interviewed, many of the students commented that "division always makes smaller" (Tirosh & Graeber, 1991, p. 160). Teachers need to ensure that in the teaching of multiplicative problems that they do not reinforce these limited notions.

Array images have been described as particularly useful for representing and showing the link between multiplication and division situations (Young-Loveridge, 2005). Given that array diagrams are useful also for showing the commutative and distributive properties of multiplication (Barmby, Harries, Higgins, & Suggate, 2009), with Anghileri (2006) pointing out shortcomings when children are taught that multiplication solely involves repeated addition, these images / diagrams became a key representation to look out for when observing division teaching.

The reinforcing of the inverse relationship between multiplication and division in a more abstract form occurs in the use of the Double Number Line (DNL) . Orrill and Brown (2012, p. 382) note that the "DNL builds from ideas of multiplication stretching or shrinking the number line to include two quantities". They further maintain that DNLs are useful as they hold the relationship between two numbers constant. A more compressed representation that is useful in multiplicative reasoning is the T- Table. A T- table (ratio table) is a tool that assists in looking at the relationship between two quantities. Dole (2008, p. 21) notes that "the ratio tables are tools for determining proportional situations. From the ratio table, students can be guided to explore number patterns and relationships that are occurring in the table, and to consider why the relationship between quantities is multiplicative". These diagrammatic signifiers provide learners with more opportunities of working with division problems in abstract ways.

To return to the two kinds of splitting actions that convey the meaning of division, Lampert (1992) points out that the concept division involves understanding the relationship between the three values represented by the dividend, the divisor, and the quotient. In a division problem, a learner must understand that the dividend must be divided by the divisor to produce the quotient. For example, in the division sum 32÷8=4, the dividend (32) is divided by the divisor (8). The quotient (4) is the result of this division.

Real or realistic division situations can also involve remainders and considering how to deal with them is important for sense-making. Remainders occur in some division situations and not in others. In making sense of division situations, part of the learning is deciding how to deal with the remainder if one occurs. Anghileri (2006, p. 88) notes that "in their experiences children will find that some numbers will not relate quite so neatly to equal groupings, for example, seven objects cannot be grouped exactly in twos and will involve objects that are left over". Carpenter et.al., (1999) state that the use of the remainder finds expression in four different ways in division situations. The first way requires accommodating the remainder within a situation. For example, if 20 people are going to the market and 6 people can ride in each car, how many cars are needed to get 20 people to the market? In this situation, an extra car must be included for the two left over. In a second situation, in which it takes 4 eggs to make a cake and the question is how many cakes can you make with 19 eggs, the remainder is simply left over and not taken into consideration. In a third situation in which a shop has 25 soccer balls, which they want to pack into boxes so that there are 3 balls in each box, and the question asks if they fill as many boxes as possible, how many balls will be left over, the remainder is the answer to the problem. In the last situation in which Ms. Jacobs has 17 cupcakes, wants to share them equally among her 3 children so that no one gets more than anyone else but also wants to use all the cupcakes up, the question is if she gives each child an equal amount, how many cupcakes will each child get and the answer includes a fractional part.

The use of remainders finds expression within all forms of division problems. Haylock (2006) has noted that in problems, we sometimes round up and sometimes we round down. Rounding up or rounding down is dependent on the situation that gives rise to numbers. In early division, Haylock (2006, p. 113) points to the importance of learners being offered examples of both kinds to ensure that they "experience a range of situations where the context require both rounding up and rounding down".

As number ranges get larger, the link with a base of known basic multiplication facts and relationships becomes particularly important, and a range of further features come into play. Thompson (2010, p. 169) highlights that "division can be solved by using a known multiplication fact (multiplicative inverse)". In other words, learners may solve division problems by relying on the recalled facts that they have established through their work with the concrete signified situations/actions, linguistic signifiers, diagrammatic signifiers, and symbolic signifiers. For Carpenter et.al., (1993) the shift from informal towards more formal knowledge of division occurs when learners exhibit the use of skip counting, the use of derived facts, doubling and halving and solving division problems which include a remainder.

Anghileri (2007, p. 72) also points out the importance of deriving new facts from known facts: "finding new facts from old can be very effective for showing connections among multiplication and division facts and simplifying some calculations". Also, the literature indicates that deriving new facts goes broader than the inverse relationship. It allows for "extensions", e.g., knowing that 60÷6=10, can be used to derive the answer for 66÷6=11.

Facility with doubling and halving relationships has also been described as useful for efficient working with larger numbers. Literature notes that it is important that learners are aware that "÷ 2" implies halving, while "x 2" implies doubling. Beyond this, Anghileri (2007) points to complexities involving doubling in division: whereas in multiplication, doubling of the multiplicand or the multiplier has the same effect on the product, this is not the case with division. When doubling is used on the first number (the dividend) the effect is different from using doubling on the second number (the divisor). For example, comparing the answer to the calculation 24 ÷ 4 = and how it relates to 48 ÷ 4 = and to 24 ÷ 8=, when doubling the first number, the total (the quotient) is doubled, when doubling the second number, the quotient halves. Awareness of these relationships can be used to make further calculations simpler: e.g. 350 ÷ 25 is double 350 ÷ 50.

Awareness of how place value understandings can feed into efficient division calculation also becomes important as number ranges increase (Booker et al., 1992). In division, there are two place value partitioning routes that are commonly used. Thompson (2000, p. 294) argues that when solving a short division problem like 81÷3=, partitioning 81 into 60 + 21 is useful given that both these numbers are divisible by 3: 60 ÷ 3 is 20 and 21 ÷3 is 7, producing the answer 27. In this instance, the learner has made use of a *quantity value* approach to short division, with the quantity 81 broken up into base-ten related quantities linked to the divisor. By contrast, the traditional division algorithm works instead with *digit values:* the number of 3s in 8 written in first, the remainder tagged onto the next digit, with the number of threes in this composite number then worked out. Given that the quantity approach pays greater attention to number relationships between the dividend and divisor, the research suggests that quantity value approaches are better linked to number sense, with advocates suggesting that algorithms are deferred (Anghileri & Beishuizen, 1998). Later in the chapter, I overview the position evident on this issue in the South African mathematics curriculum.

2.5. Progression within early division

Progression in early division occurs, as it does in early number more broadly, through two inter-linked routes. Firstly, there are representational truncations – moves from concrete situations and physical actions within them, to replays of these situations through the use of linguistic/gestural, diagrammatic and symbolic signifiers. Across this trajectory, signifiers become more compact, and therefore more efficient as signifiers of division. Secondly, there are truncations towards increasingly efficient calculation strategies. In this section, I provide an overview of both of these strands of progression, noting progressions within and across signifier categories.

2.5.1. Progression in terms of signifiers of division

2.5.1.1. Division situations

Basic signified situations/actions in division involve concrete sharing and grouping. The literature on division distinguishes between quotitive (grouping) and partitive (sharing) situations. In quotitive division, the divisor refers to the number in each part or the size of each part, while in partitive division, the divisor refers to the number of parts. Vergnaud (1994) notes that a key distinction between the two ideas relates to whether we need to consider the relations within one or between two measurement variables. Whereas quotitive division is associated with one measurement variable, partitive division is associated with two measurement variables. In the quotitive problem: "How many learners can get 5 apples if there are 30 apples?" - the dividend, as well as the divisor, concerns apples. Therefore, by using 5 apples as a unit of measurement, the 30 apples can be "measured". In contrast to quotitive division, in partitive problems such as: "If five learners have 30 apples to share equally, how many apples does each child get?" - the dividend concerns apples, but the divisor concerns learners. In other words, there is a relation between two variables, namely apples and learners. Thus, each learner receives a set of six apples. It is not possible to use 5 learners as a unit to "measure" 30 apples. In partitive division situations, an object or collection of objects is divided into several equal-sized groups by sharing. In quotitive division situations, the focus is on how many times the divisor quantity appears in the dividend quantity. Sharing and grouping situations form the fundamental everyday situations signified by division. In thinking about the implications of the fundamental nature of these situations from a semiotic perspective, this means that for children to make sense of the meaning of division as a mathematical operation, the teacher needs to teach in ways that coherently link these signified grouping and sharing situations to increasingly abstract signifiers in order to produce mathematical signs. For teachers, these two situations play a pivotal role in helping learners to make sense of formal knowledge of division by grounding understandings in informal, or everyday sharing/grouping situations and actions.

According to Anghileri (2007, p. 81) learners’ understanding of division is manifest in their actions during sharing and grouping activities. In the case of division situations, the nature of the division actions on concrete objects becomes a pointer to the type of division situation signified (partitive or quotitive). Carpenter et al. (1999) conducted a study in which they investigated kindergarten learners' problem-solving skills. In assessing how several learners solved problems based on various operations including both partitive and quotitive word problems, they concluded that learners in kindergarten were capable of finding correct answers to these division problems. They noted that learners possessed an informal knowledge of division that was evident in their direct modelling of situations and actions for both partitive and quotitive word problems. For Carpenter et al. (1993) the shift from informal towards more formal approaches to division is evident when learners exhibit moves from a direct replay of the situation using concrete materials and one-by-one distribution actions to symbolic number-based working involving the use of skip counting, the use of recalled facts to derive further facts, doubling and halving. More formal notions of division are also evident when learners are encouraged to use symbolic expressions to record or describe division word problems. Along the route to being able to work with division in the form of symbolic expressions, formalizing of division concepts is also evident when learners can express their thinking using diagrammatic forms as they solve division problems. Formalized knowledge of division facilitates work with larger numbers that require the use of short and long division algorithms for efficient handling.

From their empirical study of children’s work with division, Fosnot and Dolk (2001) note that grouping word problems were more accessible for solving than partitive word problems. They suggest that this is probably because in quotitive problems, learners can see the size of the group specified in the problem. They note that partitive division involves figuring out how many are in the group when the number of groups is known. The common use of building–up strategies, focused on the group and the whole simultaneously, makes quotitive problems easier and partitive problems often more difficult to solve. Quotitive problems can therefore be solved through "skip" counting in multiples while solving partitive problems involves initial trial and error or one-by-one distributing actions. Fosnot and Dolk (2001) note from this, that an early aspect of the shift from informal knowledge of division towards more formal knowledge occurs when learners solve partitive problems through replacing the use of "trial and error" with a "dealing out strategy". The "dealing out strategy" is the ability to distribute objects to different groups, resulting in an equal number of groups. A "dealing out strategy" is developed through learners' exposure to many partitive situations while reflecting on their actions and the results. In other words, another indication of shifting from informal everyday knowledge of division to formal understanding of division is when learners realize that every time they distribute items in a partitive situation they are able to form equal sized groups.

In a later study, Squire and Bryant (2002) worked with 5 to 8-year-olds to observe how they worked informally with partitive division problems. Like Carpenter et al. (1993), they noticed that learners at the age of five had informal knowledge of sharing in that they could successfully share quantities (objects) equally without being taught how to do this. In other words, they could directly model the sharing problem through sharing actions. One indicator, for Squire and Bryant of moves towards more formal notions occurred when the word "division" replaced the word "sharing". Across the work of Carpenter et al. (1993), Fosnot & Dolk (2001) and Squire & Bryant (2002), there is evidence of young children's ability to make informal sense of partitive and quotitive division situations, and to move towards a) diagrammatic replays of distributing actions, b) more truncated actions and c) the use of more formal language in the context of distributing actions in these situations. These indicators of moves towards more efficient formal work with division are useful as they provide lenses for investigating whether such moves were presented or elicited in instruction in the lessons observed for this study.

2.5.1.2. Linguistic signifiers associated with division

Communication between teachers and learners forms an essential part of their day-to-day learning activities. Within communication, linguistic signifiers play a pivotal role in the presentation, explanation and solving of division problems. The literature argues that these linguistic signifiers can take three forms — verbal, written, or gestural (non–verbal). Both Kaput (1992) and Meira (2002) believe that in thinking, talking, and demonstrating, mathematical "meaning" emerges.

Presmeg, Radford, Roth, and Kadunz (2018) classify gestures as part of linguistic signifiers. These authors argue that gestures and language form one system with a communicative function. For Arzarello, Paola, Robutti, and Sabena (2009, p. 19) "gestures are often a prelude to verbal expressions, mainly in the case of absence of pre-existing verbal statements". The use of gestures can provide a bridge between concrete actions and linguistic signifiers. Like the researchers above, I have clustered gestures under linguistic signifiers in this study.

Partitive and quotitive situations are often presented in verbal form in everyday language: e.g. "Corin shared 12 sweets equally among his three children". However, complexities arise in the context of both everyday language and in the move to the more formal language of division. Anghileri (1995b) and Ball (1990) have observed that the term "division" can indicate both grouping and sharing situations. While each situation involves particular actions as mentioned above, the literature notes that the same sets of everyday words can signify different situations. For example, "12 sweets shared by 3 people" implies a sharing procedure, while, "12 sweets shared into threes" infers a "grouping or repeated subtraction approach" (Anghileri, 1995a, p. 11). Both these phrases can be represented by the symbolic expression 12 ÷ 3 =. Researchers note that the use of words as linguistic signifiers in division illustrates a key part of the complexity involved in teaching division in that "division" as a term and terms such as "share" in the above example, can have different meanings. Furthermore, Cummins, Kintsch, Reusser, and Weimer (1988) note that when solving word problems learners experience more difficulties than when doing bald calculations and if they have a choice, will veer away from word problems due to complexity of their linguistic structure.

Similarly, the phrases "*how many groups*" and "*how many in each group*"- both containing the word "group" point to quotitive division in the first case and partitive division in the second case. In the teaching of division, communicating the need to pay careful attention to similar phrases which point to different signified situations, is therefore of key importance, Beckmann (2011) draws explicit attention to how teaching can focus on these variations in the context of dealing with the division calculation: "eight divided by two". She provides two images, four circles with two dots in each to illustrate the ideas of "*how many groups*" and then two circles with four dots in each, as an illustration of "*how many in each group*" (See Figure 2).

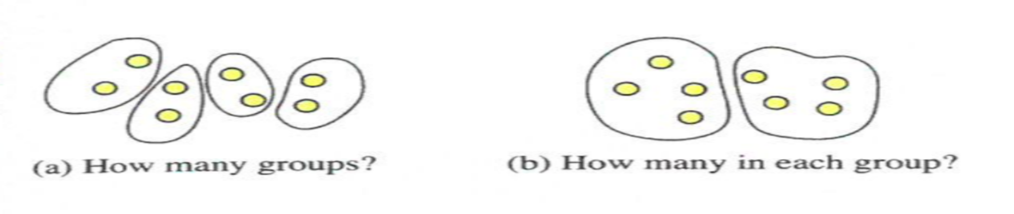


Figure 2: Examples of diagrams illustrating "How many groups?" and "How many in a group?" (Beckman 2011)

In the first image, Beckmann (2011) notes that interpretation of the phrase "*how many groups*" points to a measurement model or quotitive model of division. The divisor is the number of objects in each group, and the quotient is the number of groups. The phrase "*how many in each group*" in the second image points to a partitive model or sharing model of division (ibid). Here the divisor is the number of groups, and the quotient is the number of objects in each group. However, Squire and Bryant (2003, p. 358) note that if learners base their answer (Grouping-by-Quotient) on the provided quotient then the division model would be a grouping problem. The message here is that differences in signified situations, in the context of overlaps in linguistics signifiers, can be pointed out when a teacher makes use of diagrammatic signifiers to make sense of the division problem. Linguistic signifiers provide common entry points for describing division situations in classrooms, and the literature points to the need for care and clarity in attending coherently to overlaps in word use underpinned by different signified situations. This is particularly important in South African classrooms where many teachers are teaching in a language that might not be their own first language.

2.5.1.3. Diagrammatic signifiers associated with division

As introduced above, the construction of diagrammatic signifiers plays an important role in pointing to the differences between sharing and grouping situations. In my research, several teachers used a range of diagrams or pictures in their explanation of division problems. The ability to use images serves two purposes: it supports the cognitive process, and it mediates communication (Thompson, 1996). Indeed, in a classroom, both the teacher and learner communicate what they think or say through selected diagrams.

The signs that are made available in the division situation influence the selection of diagrammatic signifiers. Beckmann's diagrams above (Figure 2) that can be associated with 8÷2= across grouping and sharing situations, are examples of iconic diagrams that replay the signified situation. The construction of the diagrammatic form is important in this replay, because it is here that the distributing action of eight individual items is made visible. In a grouping situation, a group of two objects are drawn and circled, and this distribution of two drawn objects are repeated until the dividend value is reached resulting in four circles with two drawn objects in each one. The answer for the grouping situation is found when the number of groups are counted. A sharing situation occurs when two circles are drawn and objects are distributed one at a time between the two circles until the dividend value is reached, resulting in two circles with four drawn objects in each. The answer for the sharing situation is found when the objects in each circle are counted. As learners become more familiar with the diagrammatic replay of the distributing action and with symbolic number forms, number symbols start to be used to stand for the number of items in a group in grouping situations, and for distributed quantities in sharing situations.

The literature argues that the array is one of the most useful diagrammatic signifiers in the teaching of multiplicative reasoning. Multiplication and division exist as inverse operations that relate to the same number triple. Arrays as diagram become a meaningful form of recording division (Booker et al., 1992). Anghileri (2006, p. 91) notes that "one of the ways that the commutative rule can be investigated as a more abstract principle is by using an *array* and providing children with opportunity to arrange and rearrange sets of objects" (*Figure 3*). The diagram of an array shows the commutative rule through arranging groups vertically in a row and horizontally in columns. Arrays are therefore particularly useful for illustrating the commutative properties of multiplication. In sharing division situations where learners are asked to share 5000 ÷ 5 =, knowledge of the array enables learners to imagine an array where 5 groups of 1000 make 5000 to produce the answer of 1000 (Anghileri, 1995b). The same array diagram can refer also to 5000 ÷ 5 =, when seen as a grouping situation. Within the FFL, there is no explicit mention of the use of the array as a diagram that should be used to solve division calculations. However, in the G3 lessons in my study, diagrams did include arrays alongside iconic pictures of situations involving equal groups on flashcards. Teachers also made occasional reference to arrays in G4.

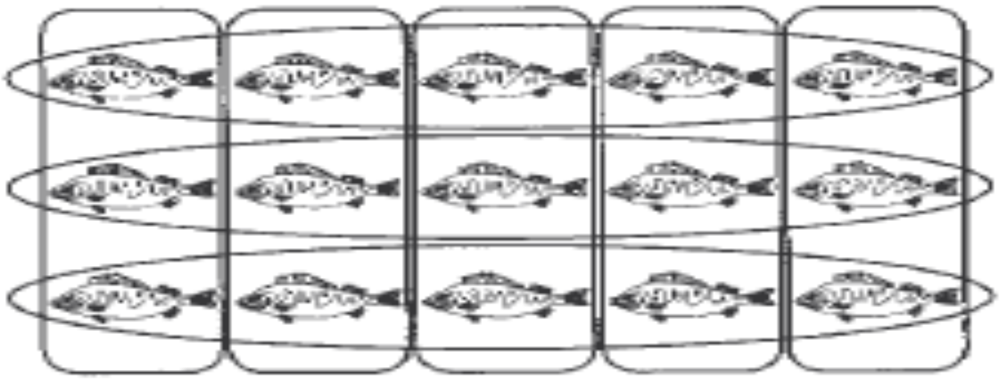


Figure 3: Illustration of the Commutative Rule.

The number line as a diagrammatic signifier assists in developing a particular approach to the relationship between division and multiplication. Anghileri (2007, p. 42) points out that "repeat sets that are identified with equal jumps on a number line will initially provide an image that matches verbal skip counting". Furthermore, by placing another NL counting the number of skips, on top of the existing NL showing the length of the jumps, the multiplicative relationship is brought into focus. These paired number lines are referred to as a "Double Number Line" (DNL). According to Küchemann et al. (2011, p. 328) "the double number line provide[s] a neat way of representing (or indeed embodying) multiplicative relations". The use of the DNL is particularly useful in presenting an image that can show multiplication as scaling. Orrill and Brown (2012, p. 383) note that the DNL provides: "a visual way of coordinating the two values in each ratio but also including visual support for magnitude as each length within a single line represents the same quantity". The DNL also provides openings for bringing the idea of the constant relationship between the two quantities to the fore.

Anghileri (2007) notes that the DNL can be a useful representation for problems where the quotient may include a remainder. For example, in the calculation 19 ÷ 4=, the DNL allows for jumps to a multiple of 4 that is close to 19, so that the whole number of jumps can be calculated first and then used to find the remainder. In this case 16 (close number) would be 4 lots of 4, so the remainder is shown as 3 units. Anghileri points out that it is important to highlight the closest number *within* the dividend on the DNL which contains a multiple of 4 - in this case, 16, the multiple number before the dividend (19). An error she notes in an instance of G4 teaching is where a teacher using the word "nearest", focuses on a multiple that lies *beyond* the dividend, in the case of 19 ÷ 4 =, this number would be 20. This error is evident when the teachers' mathematical language does not focus on the number closest to the dividend which includes the multiple seen in the divisor (Anghileri, 2007, p. 80).

The DNL acts as a precursor to the ratio table, with key differences being that the former is usually presented horizontally and the latter vertically, and the DNL often indicates all possible ratio pairs to the dividend (Küchemann, Hodgen & Brown 2011). Ratio tables or T-Tables involve more abstraction when working with multiplicative relationships. The T-Table encourages some compression of pairs (e.g., through doubling or through multiples of powers of ten going down with the column), which both Streefland (1985) and Vergnaud (1994) have argued, is particularly useful for emphasizing multiplicative relationships. For simple proportions, the T-Table's spatial arrangement helps to make explicit the scalar relationship within quantities (reading down the columns of the table) and the functional relationship between quantities (reading across the rows) (Vergnaud, 1994, p. 6). It is this spatial format that leads me to include T-Tables in the diagrammatic signifier category, in spite of the number relationship involved working largely in symbolic format. While the literature shows that the T-Table promotes more abstractions, the use of T-Tables did not feature within my study.

2.5.1.4. Symbolic signifiers associated with division

Symbolic working in mathematics signifies meaning through the use of more efficient mathematical expressions. Fosnot and Dolk (2001, p. 11) observe that "operational symbols like × and ÷ represent the actions of iterating and portioning equivalent-sized groups". In order to support sense-making of symbolic referents, children need coherent access to the situations and actions that these symbols signify, in order for the symbols to become tools and mental images to think with. A key challenge for learners when doing division problems is to recognize that one symbolic calculation can be connected to a confusing variety of situations (Anghileri, 1995b). As noted already, a division calculation like 6÷2=3 lends itself to two interpretations according to the way the divisor is interpreted. If six items are distributed equally between two people, a sharing situation is indicated. If two items at a time are distributed to each person until all six items are distributed, a grouping interpretation of the symbolic signifier is evident. The same symbolic calculation can be interpreted into two equally acceptable division situations. At the same time, analysis of the data in this study indicated confusions in instruction when moves between situations were effected in the course of work with symbolic signifiers, with no explicit acknowledgement of these moves.

Anghileri (2007) argues that in symbolic division problems, "chunking" methods based on repeated subtraction of groups (grouping) as opposed to sharing (Anghileri, 2007) tend to be efficient and meaningful. Chunking is the ability to break the dividend down into sizable chunks, and then subtract these chunks from the total. "Chunking down" is a method that does not demand that children follow a prescribed set of steps in a specific order. Instead, children with weaker number fluencies can find the answer by subtracting small chunks, whereas the more confident can subtract larger chunks (Thompson, 2010, p. 217). However, Thompson further notes that developing approaches for division that depend on easier operations such as addition and multiplication, which is known as "chunking up" or "complementary multiplication" are also useful. These approaches involve initially creating chunks based on doubling, halving, and/or multiplying the divisor by 10. Adding the "chunks" that are established by multiplying by 10 or doubling or halving of the divisor until reaching the amount indicated by the dividend, or an amount close to this quantity, completes the "chunking up" method (Thompson, 2010, p. 217). Thompson (2000) has also pointed out that chunking procedures, used at different levels of sophistication, demand good estimation skills, and the ability to use knowledge of addition, subtraction and multiplication effectively.

The efficiency of working with symbolic signifiers of division becomes more evident as the number range increases. With larger numbers, traditional instruction has focused on teaching the short and long division algorithms. Threlfall (1996) has observed that "user friendly algorithms for short and long division are dependent on the interpretation that division operates as grouping", which links to Anghileri's emphasis on grouping when working even with early symbolic signifier division calculations. Successful solving of division algorithms by learners who have had access to and practice with the underlying notions of division enables mastery of the procedure (Van Putten, Van den Brom-Snijders, & Beishuizen, 2005). Clearly, the mastery of algorithms relies on learners' shifting from the use of concrete actions to more abstract calculating strategies. Carruthers and Worthington (2006) note that division needs to begin with concrete sharing/measurement models, and only then to move into formal calculation methods, that is to move from concrete operations to abstract operations.

Long division problems can be solved through the use of chunking or with standard algorithms. However, there is dispute over the inclusion of short division, with some researchers holding the view that short division should be left out entirely in favor of long division (Watson 1992). Alternatively, for many years researchers in mathematics education have discussed whether there it is necessary for the long division algorithm to be used. The importance of solving both short and long division algorithms is supported by Millet (2004) who notes that the processes involved in doing short and long division are essential features of division problem-solving which also highlight progression in the development of division. In recent years, researchers have argued for working with long division via chunking-based approaches rather than only with the traditional standard algorithm. Anghileri and Beishuizen (1998, p. 4) support for chunking approaches is based on their argument that "it is possible that progressive steps to greater efficiency are being inhibited by an algorithmic approach that does not reflect pupils' naïve meanings for division and does not build on their understanding of the numbers involved". Although there is literature that argues that the short division algorithm and traditional long division algorithm are essential, the absence of short and long division algorithms within the FFL suggests South African curricular support for developing more efficient mental strategies, as more necessary for the development of the division concept. However, in my study, some teachers demonstrated how short and long division algorithms should be solved, so awareness of the literature on approaches to symbolic division is important. In the next section, I consider calculating strategies that run concurrently within the use of various signifiers.

2.5.2. Progression in terms of division calculation strategies

The literature notes progression within the division actions linked to different division calculation strategies (Carpenter et.al, 1999). The fundamental action linked to a partitive interpretation of division is a "sharing action". Sharing actions can refer to the distribution of single items or to a distribution of sets of items. The most basic type of sharing action is the distribution of one item at a time between people or groups. Wright, Martland, and Stafford (2010) refer to this type of sharing action as "share by ones". "Share by ones" can occur in two concrete sharing approaches. In the first approach the learner could count out the total number (dividend) and then share objects among the sets through a one-by-one distribution. A more advanced approach would be to mentally keep track of the total number of items while dealing out objects one-by-one (Carpenter et al., 1999; Kouba, 1989).

However, Carpenter et al. (1999) note that many learners proceed quite quickly beyond systematically dealing the counters out one by one. So, a second type of sharing action occurs when a learner starts placing more than one object in a set at a time, then adding or removing objects from each set as necessary to work with the given dividend quantity. For example, in sharing 15 cookies equally among 3 people, a learner might use this approach: She counts out 15 counters. She places 4 counters in one pile, 4 counters in another pile, 4 counters in a third pile. At this moment she sees that she has not used up all the counters, so she adds 1 counter to each pile. Then she counts the counters in one of the piles and answers "5". Here, the sharing involves distribution by "partial composite" units, involving more than one item being distributed at a time.

In certain sharing situations the answer is produced through a grouping strategy. Brown (1992) presented learners with six types of division situations focused on three partitive and measurement word problems. These six division word problems included problems in which the remainder is zero, the remainder is non-zero but is not used in the solution, and the remainder is non-zero and is used in the solution. She did not only note who could provide the correct answer but also observed the strategies that learners used. Brown (1992, p. 95) states that "in partitive division, however, the children can use either the sharing strategy or the technique of removing equal-sized groups to obtain a correct answer to the problem". The use of groups to produce the answer in sharing situations was evident in some of the episodes in my study and resulted in my coining of the phrase sharing through "Group Based Distribution" (GBD). In other words, GBD refers to distribution action in sharing situations where the divisor is used to form groups up to the dividend value.

The fourth type of distribution is sharing by what I call the "complete composite unit" (CCU) linked to a sharing division situation. Here, the quotient value is distributed to each set. Zweng (1964, p. 553) has noted different ways learners solve partitive problems and recalls a particular partitive problem solved by learners in which learners needed to "assign 18 objects to three groups, the child assigned six elements to each of the groups on the first try, this method of solution was called grouping". In my study, I describe this more efficient strategy for distribution of sets of items as sharing by "complete composite" units. The answer either has to be known prior to problem–solving or the answer may be already available from an offer earlier in the episode. I distinguish CCU from the strategy I refer to as sharing by "partial composite" units. In other words, distribution by "complete composite" units refers to distributing of items by the quotient value immediately, either as an estimation followed by a count of the number of items in the group to determine the answer, or as a known fact to a known number of people or things. One of the difficulties for early grades teachers is to avoid jumping to "complete composite" unit sharing based on their own knowledge of the answer before children have progressed to knowing this answer. This phenomenon was observed in the empirical data set and is discussed in Chapter 5.

Grouping situations (quotitive division) can also be associated with increasingly efficient calculating actions. In initial direct modelling, learners construct a set containing the number of objects specified by the divisor with a one-by-one construction (Grouping through 1 by 1 distribution), and then replicate this set until the dividend quantity is reached. They then count how many sets/groups there are. In this instance, grouping through 1 by 1 distribution is based on the divisor. Squire and Bryant (2003) note that grouping through 1 by 1 distribution can also occur based on the quotient value which they refer to as "Grouping-by-quotient". There are at least two ways of finding the answer, based on whether the dividend number is counted out at the start or built up as a cumulative total. In relation to 28 counters needing to be packed into bags containing 7 counters each, in the first case, a learner can count out 28 counters (making of groups based on the divisor – the dividend value is first drawn and then groups are made) and then separate (through some form of one – by - one counting) them into groups of seven counters (Carpenter et al., 1999). In the next case, the learner does not count out 28 objects but instead forms groups of seven until s/he has 28 altogether. Kouba (1989, p. 152) refers to this second strategy of grouping as "double count". "Double count" in the case of grouping problems refers to keeping a running count of the cumulative number of objects in the groups while also counting out the objects to form groups (Kouba, 1989). When the cumulative value reaches the desired dividend, the learner stops and reports the number of groups as the answer. For sharing problems, the double count occurs while dealing objects out one by one; for example, with 15 marbles shared among 3 children, once one marble has been distributed to each child, the learner might say: "3 marbles shared, 1 each", and after the next round of distribution: "6 marbles shared, 2 each" and so on. Kouba (1989) notes that learners doing grouping division problems employ a repeated building up strategy, e.g. with 15 marbles placed into bags containing 3 marbles each, the double counting would involve words like: "3 marbles, 1 bag; 6 marbles, 2 bags" and so on as evidence of double counting. The subtle differences in wording that come with the units associated with the quantities are important markers of whether the problem-solving has involved sense-making of the situation, rather than jumping directly to operating with the numbers, as noted in the writing of Greer (1997).

Kouba (1989) describes a grouping strategy that is more efficient than "double count" which she refers to as "transitional counting". "Transitional counting" occurs when learners calculate the answer to the problem by using counting sequences based on multiples of a factor in the problem. Kouba (1989) terms counting by multiples, or skip counting, "transitional counting" because although it relies on knowledge of a counting sequence, it is related to multiplication in a more fundamental way than are basic grouping strategies. "Transitional counting" is used as a repeated building up strategy to solve grouping problems. Initially, learners tend to be more proficient in skip counting in multiples involving two, three and five, than in a number like seven. While in a grouping situation, learners skip-count using the number of objects in each group, not by the number of groups, they keep track of the number of groups on their fingers or in some other way. Carpenter et al. (1999) note the limitation of skip counting within sharing situations. In sharing, skip counting is more difficult in that the number of objects in each group is the unknown. The difficulty is that learners need to use a trial and error approach that reflects the earlier description of what can happen in concrete sharing problem-solving strategies, incorporating skip counting when attempting to solve sharing situations. Therefore, verbal skip-counting for sharing situations is significantly more difficult than when solving grouping problems (Carpenter et al., 1999). In my study, in some instances, verification of the correctness of the number of groups or counting of the number in a group was done through skip counting.

Repeated "additive or subtractive" strategies also involve skip counting (Kouba, 1989). In an additive or subtractive strategy, learners use repeated addition and subtraction as a calculation strategy. Starting with the dividend, repeated subtraction occurs until the number in each group is repeatedly taken away, and the number of groups are counted as the answer. Fosnot and Dolk (2001, p. 34) consider skip counting as slightly more advanced than repeated addition if the calculation strategy is focused on object-based counting in the first instance and number relations in the second. In the following example "How many cookies are there in total in three bags of cookies that hold six cookies each?" - the learner produces three sets of six cubes. Skip counting is evident when the learner focuses on the *objects* and says: "I went six, twelve, eighteen". Skip counting is done in multiples, making it easier to keep count of the number of groups. Using the same three sets of objects another learner might translate the objects into a *numerical additive relationship*: "I added six plus six. That is twelve. Six more is eighteen" this learner has used "repeated addition". The focus on numerical relationships rather than object-based multiple-counts can be considered more advanced in terms of abstraction. The move from adding of composite units, as seen in repeated addition or skip counting, to producing an immediate answer is the precursor to multiplicative thinking. For example, being able to say immediately that 3+3+3+3 is 12, can then be interpreted multiplicatively as 4 x 3 = 12. This trajectory points to work with progression on division problem-solving being marked both by abstraction in terms of moves to working with number relationships rather than with objects, and by increasingly efficient counting strategies moving from unit counting, to skip counting or repeated addition, to known multiplication facts, that can then be used to derive further results.

Kouba (1989, p. 153) notes that an even more efficient strategy related to groups is the use of *recalled facts* as a strategy. While solving division problems some learners obtain the answer by remembering the appropriate multiplication and division combinations. Included in this category are derived facts, facts that learners could not readily recall but could calculate from a related *recalled fact*. As stated above the use of "chunking" (up or down) depends on the use of *recalled facts*. At the most efficient level of problem-solving strategy, learners use combinations of recalled and derived multiplication and division facts to mentally calculate answers to problems.

These progressions, sometimes seen simultaneously in the use of signifiers and calculation strategies, are described by Fosnot & Dolk (2001) in their descriptions of learner strategies when solving problems. For example, in the quotitive problem: "81 parents are coming to the school for a parent meeting, 6 parents need to sit around a table, how many tables should be set up?" they describe the solutions provided by four learners. One learner completed the problem by drawing every chair and counting by ones. Another learner drew a table with six chairs first, then proceeded to represent each table symbolically as a rectangle with the numeral 6 as he progressed through his solution. The third learner drew each table and chair and solved the problem by counting on as he drew the table and marked it 6, 12, 18, and so on. The use of multiples of six allowed him to keep track of the number of people seated as he worked through his solution. The last learner used the distributive property, grouping first 10 × 6 and then 4 × 6 and wrote down 14 tables under the symbolic expression. Across this sequence of solutions, the progression in the use of signifiers was from drawing each chair to using no diagram, thus showing increasingly truncated use of diagrammatic signifiers. The progression in counting strategy was from unit counts to using distributing properties with chunking. As noted already, either chunking or division algorithms in symbolic forms of the problem, are required when working with division problems with larger number ranges.

Given this last point, the increase in the number range is also an indication of progression in that it necessitates progressions in signifiers and strategies. Mulligan and Mitchelmore (1997) show that increasing number range influences how learners solve division situations. Specifically, they show that progression of number range creates an opportunity for learners to solve division problems through the use of increasingly abstract signifiers and strategies.

2.6. What needs to be advocated in the teaching of division?

In this section, I consider the significance of progressions in the use of signifiers and division calculation strategies to be advocated in the explanations when solving division problems within my study, in a South African context.

2.6.1. Teaching division for progression

When solving a division problem, it is crucial for a teacher to take note of what particular division situation (signified) links with what particular signifiers (Carpenter, Fennema, Franke, Levi, & Empson, 1999) in order to produce a specific mathematical relation (signs), together with all related division supporting and underlying mathematical ideas (e.g., rounding off, inverse relationship between multiplication and division, etc.).

After identifying a sharing or grouping situation, the next step is for the teacher to enact a particular action on concrete objects. The enacted sharing or grouping action should assist that teacher to produce a clear explanation as s/he solves a division problem. However, literature has pointed out that these division actions are more nuanced than just distributing by ones or grouping of items (Carpenter et al., 1999). These nuances, located within actions on concrete objects, are not always visible to teachers and not always taught to learners.

So, the teaching of division is underpinned by how division situation problems are solved. In most cases, linguistics signifiers are accessed by teachers and learners when solving division situation problems. Linguistic signifiers play a pivotal role in the transition from informal knowledge of division to formal knowledge. Linguistic signifiers linked to division incorporate three types: spoken words, written words, and gestures (Presmeg, et. al., 2018). These linguistic signifiers are often prone to instances of ambiguity and thus researchers argue that teachers should teach division signifieds (sharing and grouping situations) separately and explicitly as they are linked to linguistic signifiers (Beckmann, 2011).

To provide more clarity in their solving of division situations, teachers can make use of diagrammatic signifiers. These diagrammatic signifiers are dependent on how equal groups are drawn and explained. Also, the international research suggests that one of the most useful diagrammatic signifiers in the early grades, to support the solving of division, is the array (Thompson, 2010). The array as a diagrammatic signifier is an essential artefact in highlighting the inverse relationship between multiplication and division. The knowledge of multiplication facts embedded in the array can be used to find the answer for a division calculation, and thus, the answer is produced more efficiently (Anghileri, 2006). Therefore, teachers need to select diagrammatic signifiers that support multiplicative reasoning. Another diagrammatic signifier that a teacher has access to is the DNL. Researchers have argued that the DNL is also a useful diagrammatic signifier in solving division problems. Within my study, one of the teachers made extensive use of the constant relationship between two set of numbers linked to using a DNL approach in solving some of her division problems.

In solving various division situation problems, the symbolic signifiers appear to be the most consistent across different division problems. Again, the presence of ambiguity is evident within the use of division symbolic number sentence signifiers. In view of such possible ambiguity, researchers suggest that when not requiring an immediate answer teachers should clarify the interpretation of the symbolic signifier (division calculation) as either a sharing or grouping situation. Teachers need to be aware that symbolic division calculation signifiers are prone to ambiguity and that they need to select which division situation they would like to link to a particular division calculation in order to prevent the ambiguity. However, there is broad consensus in the field of primary mathematics education that learners should first solve division calculations without a remainder (Anghileri, 1995b; Correa, Nunes, & Bryant, 1998; Fischbein et al., 1985; Mulligan & Mitchelmore, 1997). As the learners make sense of the division situation problems with smaller number ranges, a progression in number range can be introduced. As the number range increases, an opportunity arises for the use of algorithms and chunking. Again, progression is evident when there is a shift from assisting learners in first using short and then in some cases, long division algorithms to establishing answers for either sharing or grouping situations. The teacher needs to make explicit to learners the notion that an algorithm allows them to solve division situations more effectively and efficiently. In addition, teachers should encourage the use of chunking to solve division calculations with larger numbers and then expose learners to using chunking within standard long division algorithms, in particular, when the dividend of a division situation problem has two digits or more. Important to note is that, increasingly abstract signifiers do not "replace" each other sequentially. Instead, what is required is an extended "parallel" working where situations are maintained alongside the increasingly abstract forms.

Teachers in the South African context needs to understand the importance of progression towards more efficient division strategies. Such progression is evident through sharing actions from the distribution of one-by-one objects (Wright, Martland, & Stafford, 2010) to the distribution of "complete composite units" (CCU) (Zweng, 1964) within partitive situations. The CCU is determined when a sharing situation is foregrounded and then the answer is provided through immediate recall and the quotient value (answer) is shared between people or circles. Another sharing action, "Grouped Based Distribution" (GBD) is based on a sharing problem, in which a sharing action involves groups being made or drawn until the dividend value is reached and the number of groups are counted to produce the answer. South African teachers also need to be aware that there is progression in the distribution of items in groups when solving quotitive situations. Progression is also evident in the use of division calculation strategies in which there is a move from skip counting to solve and verify the answer to a division problem, to the truncation of skip counting. Lastly, as the number range increases, more efficient division calculations are required. The progression in signifiers, together with the use of efficient division strategies needs to be linked consistently with the two signified situations.

These instructional explanations related to the solving of division tasks produces particular chains or pathways of signifiers, which either creates clarity or disruptions in the mathematical coherence. In the solving of division problems, the relationship between signifiers and signified needs to be thoughtfully selected and enacted within the teacher's instructional explanations together with the mathematical association that is produced between signifiers in a sequence.

Having considered the interlinked progressions involved in division teaching and learning, I now provide an overview of the FFL numeracy and mathematics curriculum – the curriculum in use at the time of this study - and consider whether, and if so how, these progressions were incorporated in it. I do this in order to understand the curricular background to some of the difficulties with coherence and connection evident in the signification pathways observed in the lesson data in this study.

2.7. Foundations for Learning (FFL) curriculum

In 2008, in response to evidence of lack of improved learner performance, the national Department of Education introduced the FFL curriculum to support primary school teachers in their selection, pacing, and sequencing of grade-appropriate content to deal with identified problems (DoE, 2008). The FFL document provided termly curricular "milestones" and "assessment tasks" to encourage curriculum "coverage". At the time when this study started (2010), teachers were using these FFL curriculum guidelines.

With reference to the literature that suggests two basic division situations (sharing and grouping), with remainders possible, and progression in relation to signifiers, calculation strategies and number ranges, I examined the FFL curriculum guidelines to understand the kinds of guidance in relation to division concepts and their teaching that were provided for the teachers in this study. Given the South African evidence of ongoing use of inefficient concrete counting or skip counting, I also looked at the ways progression was configured across the early grades, first overviewing the Grade 1 and 2 specifications and then considering the Grade 3 and 4 specifications in more detail as these were the focal grades in my study.

The FFL document presented termly "milestones" and assessment tasks, and subsequently, exemplar tasks for each grade. In this overview and discussion of the FFL curriculum, I consider both of these sections.

2.7.1. Division in Grades 1 and 2, FFL

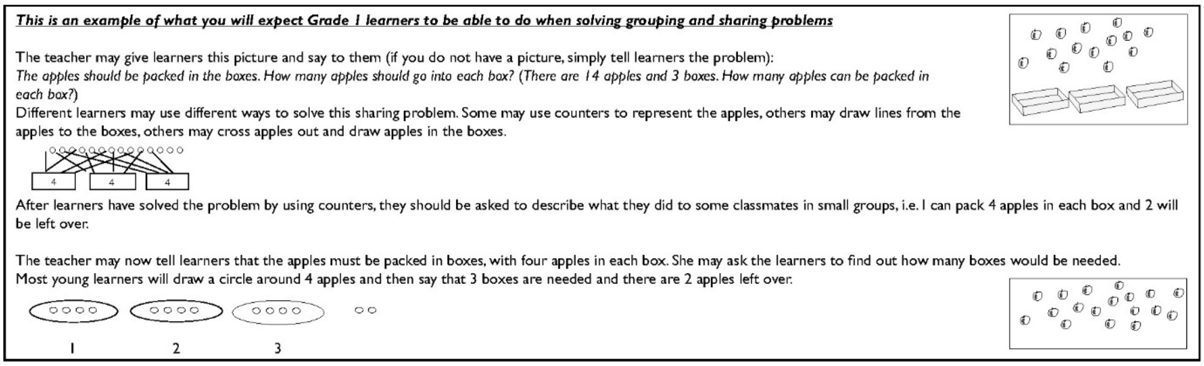
2.7.1.1. Division in Grade 1 FFL

Table 1 Milestones and Assessment Tasks in G1 FFL (DoE, 2010a, p. 8)

	T1	T2	T3	T4
G1	Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers to at least 10 and with solutions that include **remainders**, by using concrete objects and drawings	Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers to at least 20 and with solutions that include **remainders**, by using concrete objects and drawings	Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers to at least 34 and with solutions that include **remainders**, by using concrete objects and drawings	Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers to at least 34 and with solutions that include **remainders**, by using concrete objects and drawings
		Doubles and halves even numbers to 20	Doubles and halves even numbers 1 to 20	Doubles and halves numbers 1 to 34
		Ass: Solve problems involving sharing and grouping within the 1-10 number range, using concrete objects?		Ass: Solve problems involving sharing and grouping within the 1-34 number range, using concrete objects, drawings, numbers?

In line with the recommendations in the mathematics education literature, the FFL foregrounds division ideas through situations – described as "practical problems" in the specification, with the guidance specifying in G1, Term 1, the inclusion of both grouping and sharing situations. This follows what is advocated in the literature. Actions involving concrete objects to replay the problem are suggested as the means through which grouping and sharing problems should be solved. Across grouping and sharing situations, the number range of the dividend is increased gradually across the terms from 10 in Term 1 to 34 in Term 4. In this section, progression is seen only in terms of number range, and not in terms of either signifiers or calculation strategies. Aligning with the literature-based recommendation that the knowledge of doubling and halving plays an important role in developing the idea of division as well as using them as recalled facts, the FFL encourages the use of doubles and halves from Term 2 and 3 starting from a number range of 1-20 and moving to a range of 1- 34 in Term 4 (See Table 1 above).

However, the illustration of these two situations in the FFL raises a number of issues that were subsequently seen as relevant when analysing the lesson episodes in the study for coherence. The following sharing and – later - grouping situations are illustrated in the document (DoE, 2010a, p. 8):


This is an example of what you will expect Grade 1 learners to be able to do when solving grouping and sharing problems

The teacher may give learners this picture and say to them (if you do not have a picture, simply tell learners the problem):
The apples should be packed in the boxes. How many apples should go into each box? (There are 14 apples and 3 boxes. How many apples can be packed in each box?)
Different learners may use different ways to solve this sharing problem. Some may use counters to represent the apples, others may draw lines from the apples to the boxes, others may cross apples out and draw apples in the boxes.

After learners have solved the problem by using counters, they should be asked to describe what they did to some classmates in small groups, i.e. I can pack 4 apples in each box and 2 will be left over.

The teacher may now tell learners that the apples must be packed in boxes, with four apples in each box. She may ask the learners to find out how many boxes would be needed.
Most young learners will draw a circle around 4 apples and then say that 3 boxes are needed and there are 2 apples left over.

Figure 4 Example of tasks in G1 (DoE, 2010a, p. 8)

A situation with 14 apples needing to be packed into three boxes is introduced. Of interest is that there is no mention of packing the apples equally across the boxes (See Figure 4 above). The notion of multiplicative situations fundamentally involving "equal groups", as Anghileri (1989) notes, is therefore not made explicit. Concrete and diagrammatic replays of this situation (presumed as a sharing situation) are noted as likely ways that learners will solve the problem, with one diagrammatic option involving one-by-one sharing illustrated. The importance of expressing outcomes of these sharing actions in language as advocated in the literature (Carpenter et al., 1999), are incorporated.

The second situation presented is of interest. A grouping situation involving 14 apples to be packed four apples to a box is introduced, with a suggestion of the kind of diagram that might result from replaying this situation. Of interest here is the juxtaposition of a sharing situation that can be represented by the number sentence 14 ÷ 3 and a diagram representing this sharing action, with a grouping situation that would be represented by the number sentence 14 ÷ 4. The grouping diagram emanating from this grouping action matches the diagram shown for the 14 ÷ 3 sharing action, but there is no discussion pointing to the fact that the same diagrammatic representation might follow from two different starting situations. This absence in the curricular documentation is of particular interest, given that this kind of slippage was relatively common in the lessons observed for this study. The illustrations above suggest that it is concrete replays or iconic records of the situation that are expected across G1, with verbal descriptions of the outcomes.

2.7.1.2. Division in Grade 2 FFL

Table 2. Milestones and Assessment Tasks in G2 FFL (DoE, 2010a, p. 12)

G2	**Term 1**	**Term 2**	**Term 3**	**Term 4**
	Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers to at least 34 and with solutions that include **remainders**, by using concrete objects and drawings	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions that include **remainders** or fractions, by using drawings or other schematic representations	Solves and explains solutions to problems that involve equal sharing and grouping with solutions where the **remainder** becomes a fraction	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions where the **remainder** becomes a fraction.
	Doubles and halves numbers 1 to 34	Doubles and halves numbers 1 to 50	Doubles and halves two-digit numbers to 99	Doubles and halves of both odd and even numbers from 1 to 100
	Counts forwards and backwards in multiples of 2, 5 and 10 from 0-100 using concrete objects and number grids.	Counts forwards and backwards in multiples of 2, 5 and 10 from 0-150 using a number grid or number line.	Counts forwards and backwards in multiples of 2, 5 and 10 from 0-200 using a number grid or number line.	Counts forwards and backwards in multiples of 2, 5 and 10 from 0-200 using a number grid or number line.
	Completes repeated addition of 2, 5 and 10	Completes repeated addition of 2, 5 and 10	Uses repeated addition leading to multiplication with answers up to 50.	
		Ass: Solve word problems involving all four operations (+, -, x, ÷) with numbers up to 50, using any or all of the following: *concrete objects, drawings, number squares, numerals, number sentences .*		

In line with G1, the G2 curriculum document is aligned with the recommendations in the mathematics education literature that equal sharing and grouping should be the focus in the teaching of division (See Table 2 above). However, from Term 2 onwards the document introduces working with remainders in fraction form and the different forms in which remainders can be expressed when solving a problem (Carpenter et al., 1999). As in G1, the use of doubles and halves is evident with a number range for doubling and halving expanding to 100 by the end of G2. In G2, skip counting in numbers 2,5,10 is promoted from 100 in Term 1 moving to 200 in Term 3 (Anghileri, 2007). Repeated addition is introduced in Term 1 and Term 2 with an explicit link made between repeated addition as leading to multiplication in Term 3 (Carruthers & Worthington, 2006). Progression in the number range of the dividend continues in G2.

In G2, the range of signifiers suggested for use in solving word problems expands beyond the concrete, and iconic replays suggested in G1 to more indexical diagrammatic forms (e.g., number grids and number lines) and to symbolic replays of these situations. While the curriculum specification does not explain (as recommended in the literature) how these signifiers should be used in relation to the type of division situation and how to encourage moves to more truncated strategies as the number range increases, the exemplar tasks provide some details on this (See Figure 5 below). Also, of interest is that while there is evidence of expansion from concrete actions and iconic replays into indexical number and symbolic forms, there is no reference to the representations noted as particularly useful for multiplication and division in the literature – arrays, and T-tables. This "gap" in the signifier range in the curriculum document is relevant to the analysis of lesson episodes in this study.

<table>
<tr><th colspan="4">Numeracy Milestones (per Term): Grade 2 (continued)</th></tr>
<tr><th>Term 1</th><th>Term 2</th><th>Term 3</th><th>Term 4</th></tr>
<tr><td colspan="4">This is an example of what you will expect Grade 2 learners to be able to do when solving grouping and sharing problems

Solving problems that involve grouping and sharing. Understand sharing and grouping situations, can do the calculations needed to solve them, and can judge whether the result makes sense. (NB. This should include solutions with a remainder and/or solutions leading to fractions that are halves and quarters)
At some stage, the educator should challenge learners to represent their thinking with numbers and symbols. They may do this in various ways. Once all learners have actively engaged with some problems and produced symbolic representations of their work, the teacher should support them in using more elegant representations of their thinking, for example:
Dad wants to fix some chairs. He has 30 nails and uses 4 in each chair. How many chairs could Dad fix?
$4 + 4 = 8 \quad 8 + 4 = 12 \quad 12 + 4 = 16 \quad 16 + 4 = 20 \quad 20 + 4 = 24 \quad 24 + 4 = 28$
$4 + 4 \rightarrow 8 + 4 \rightarrow 12 + 4 \rightarrow 16 + 4 \rightarrow 20 + 4 \rightarrow 24 + 4 \rightarrow 28$
The fours are then counted to establish that 7 chairs can be fixed. Some learners may use doubling to quicken the process:
$4 + 4 \rightarrow 8 + 8 \rightarrow 16 + 8 \rightarrow 24 + 4 \rightarrow 28$
2 2 2 1 Dad could fix 7 chairs.
For sharing problems, learners may think in a different way than for grouping problems. For example, in the problem R34 is to be shared equally among 6 children. How much money did each child get? They may think of handing out R1 at a time and represent it somewhat like this, to conclude that each learner should get R5 and there will be R4 left:
| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
R1 each   R1 each   R1 each   R1 each   R1 each   R1 each
R5   R5   R5   R5   R5   R5   R4 left over</td></tr>
</table>

Figure 5: Example of tasks in G2 (DoE, 2010a, p. 13)

In line with recommendations in the mathematics education literature, the FFL exemplars of what learners should do when problem-solving draw attention to making sense of division situations while incorporating remainders and fractions (Greer, 1997). There is direct advocacy in G2 of teachers directing learners towards the use of "number and symbols" reflecting the expansions beyond concrete and iconic working mentioned in the G2 milestones. After that, the teacher is encouraged to use more "elegant representations" pointing towards more efficient strategies (Kouba, 1989). The curriculum document proceeds to show more efficient strategies for a grouping situation involving repeated addition in two different recording formats. The sharing situation (with equal sharing now made explicit) continues to involve a one-by-one distributing action, though combined now with some symbolic summarizing (DoE, 2010a). Looking at the two situations, the key point here is that while grouping situations are linked to more abstract signifiers and strategies, sharing situations continue to be represented more iconically and calculated using one-by-one distribution.

2.7.2. Division in Grades 3 and 4, FFL

2.7.2.1. Division in Grade 3 FFL

Table 3 Milestones and Assessment Tasks in G3 (DoE, 2010a, pp. 16-18)

G3	T1	T2	T3	T4
		Calculates division of 2-digit numbers by 1-digit numbers, e.g., 20÷5=		Calculates division of 2-digit numbers by 1-digit numbers, e.g., 75÷5=
	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions where there is a remainder, or where the remainder becomes a fraction Ass: Solve word problems, using all four operations and numbers up to 200, and explain reasoning?	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions where there is a remainder, or where the remainder becomes a fraction Ass: Solve word problems using the four operations with numbers to 500, and explain how the problem was solved?	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions where there is a remainder becomes a fraction (e.g., a third, two thirds, etc.) Ass: Solves word problems, including all four operations using numbers to 750, and explains how the solution was reached?	Solves and explains solutions to practical problems that involve equal sharing and grouping with solutions where there is a remainder becomes a fraction (e.g., a third, two thirds, etc.) Ass: Solve word problems using the four operations with numbers to 1000 (using number charts, number lines, etc. if needed) and explain how the problem was solved?
	Counts forward and backwards in multiples of 2 and 20 to 200, 5 and 50 to 500, 10 and 100 to 1000	Counts forward and backwards in multiples of 2 and 20 to 200, 5 and 50 to 500, 10 and 100 to 1000	Counts forward and backwards in multiples of 2 and 20 to 200, 5 and 50 to 500, 10 and 100 to 1000	Counts forward and backwards in multiples of 2 and 20 to 200, 5 and 50 to 500, 10 and 100 to 1000.
		Completes number sentences using repeated addition of 20, 25, 50 to 500 e.g. 150+50+50=		
		Ass: Count in multiples of 2, 5, 9, 10 and 11 and answers questions? E.g. If 4x5 is 20, how much is 5x5? (one more five) 3x5? (one less five) 8x5? (double 4 times five)		
		Ass: Correctly complete number sentences using repeated subtraction of 20, 25, 50 and 100? E.g. 400-100-100-100=?		
			Doubling and halving - number lines.	
			Solve different types of	Solve different types of

			problems and explain solutions to problems including money problems with whole numbers to at least 750, involving addition, subtraction, multiplication and division using appropriate symbols and the techniques listed below -building up and breaking down numbers -doubling and halving - number lines	problems and explain solutions to problems including money problems with whole numbers to at least 1 000, involving addition, subtraction, multiplication and division using appropriate symbols and the techniques listed below -building up and breaking down numbers - doubling and halving - number lines
				Rounding off to 10

In G3, the curriculum document introduces symbolic division calculations for the first time. Repeated subtraction is also introduced. Although repeated addition is referred to in G1 and G2, in G3, repeated addition is encouraged by adding from a larger multiple, for example, "150 + 50 + 50". In G3, solving of a range of division problems "through the use of appropriate symbols" is emphasized with a new technique described as "building up and breaking down numbers" (See Table 3 above). Although doubling and halving are mentioned in G1 and G2, in G3 number lines are suggested for use with doubling and halving, but the curriculum document does not illustrate how the NL might be used to show such doubling and halving. Also, for the first time, the curriculum document suggests the "rounding off to 10". So, although the curriculum document lists "rounding off" to 10 it does not explicitly describe how such rounding off should link to division. Haylock (2006) notes that the ability to round off is critical in developing estimation skills within division problems; therefore, the curriculum document is aligned in this regard with international literature. According to Faulk (1962) the skill of "rounding off" to 10,100,1000 develops the ability to estimate. Estimating as an ability is also evident when using "chunking" as an approach to finding the answer to division problems. Furthermore, Anghileri (2007, p. 54) notes that on a number line "rounding a number to the nearest ten" points to identifying the next decade number which is important when working with "close numbers". Ability to work with "close numbers" involves knowing that on a number line 39 is one less than 40 which provides shortcuts in calculating and enables learners to operate at a higher conceptual level. The use of division with a large number as a dividend is evident in term 3 and term 4 with a jump from 750 in term 3 to 1000 in term 4. Anghileri (2007) notes that a jump like this in number range supports progression from concrete signifiers to symbolic signifiers (traditional short and long division algorithms) and strategies (sharing from basic one-by-one to sharing of complete composite and grouping from basic grouping of sets to the use of recalled facts).

A point of interest in my study is that there was only one episode in a G3 classroom in which the number range exceeded 150. This observation suggests that teachers were reluctant to solve division calculations, including dividends with large numbers despite the guidelines provided by the curriculum documents. The G3 curriculum document emphasizes the use of symbolic signifiers with larger number ranges, to enable learners to work with division in more abstract ways (Fosnot & Dolk, 2001). Furthermore, building up, breaking down, doubling and halving point to the curriculum document endorsing the use of more efficient calculation strategies. However, there is broad agreement across the mathematics education literature that techniques should not just be listed, but how they relate to division and how they progress should be explained. Anghileri (2007) has observed that when a division problem includes a remainder, the teacher has to be explicit in how the jumps should happen on the NL. Although an increase in number range is evident in the curriculum document, the specifications do not include the array as a useful diagrammatic signifier that can provide a bridge between unit counting and counting in multiples to support a move to the DNL.

In G3, the following examples are illustrated in the document (DoE, 2010a) (see Table 4 – a and b).

Table 4a. Example of tasks in G3 (DoE, 2010a, p. 17)

Numeracy Milestones (per Term): Grade 3 (continued)			
Term 1	Term 2	Term 3	Term 4

This is an example of what you will expect Grade 3 learners to be able to do when solving grouping and sharing problems

Solving problems that involve grouping and sharing. Understand sharing and grouping situations, can do the calculations needed to solve them, and can judge whether the result makes sense. (NB.This should include solutions with a remainder and/or solutions leading to fractions that are halves and quarters)
At some stage, the educator should challenge learners to represent their thinking with numbers and symbols. They may do this in various ways. Once all learners have actively engaged with some problems and produced symbolic representations of their work, the teacher should support them in using more elegant representations of their thinking, for example:

Sharing:

- *R84 is to be shared between 5 children. How much money can be given to each child?*

Learners may be solving problems in this way:	The teacher may at some stage suggest to learners that their thinking may be represented as follows:
- *R10 for each child is R50, then there is R34 left.* - *Another R5 for each child is R25, then there is R9 left.* - *Another R1 for each child is R5, then there is R4 left.*	*5 × R10 = R50* *Each child gets R10+R5+R1=R16* *5 × R5 = R25* *There is R4 left, so that is 400 cents* *5 × R1 = R5* *5x80c=400c Each child gets R16 and 80cents.*

Grouping

- *There are 84 apples. How many bags can be filled with 5 apples each?*

When learners are challenged to look for a shorter way in which this problem can be solved than the way in which they did it in Grade 2, they may come up with the thought that 10 bags of 5 apples each would be 50 apples, and another 5 bags of 5 apples each would be 25 apples (or 6 bags of 5 would be 30 apples). In their minds they would know that the 50 was obtained by filling 10 bags with 5 apples each, and the 25 by filling 5 apples each	
The teacher may suggest that their thinking can be represented in two lines like this: *50 + 25 → 75 + 5 → 80* *10 × 5 + 5 × 5 → 75 + 1 × 5 = 80*	The teacher may point out that this thinking can also be represented like this: *5 × 10 = 50* *5 × 5 = 25* *50 + 25 = 75* *75+5=80 so there are 4 apples over* *5 × 1 = 5* *10+5+1=16 bags can be filled and there will be 4 apples left over.*

- *Miriam has R90 available to buy cakes for a birthday party. The cakes cost R27 each. How many cakes can she buy?*

This may be easily done by repeated addition:	Some learners may do it like this:
27 + 27 → 54 + 27 → 81 and there is R9 left. In this case the written work to do the additions can be shown separately. *27 + 27 + 27 is the same as 20 + 7 + 20 + 7 + 20 + 7, which is the same as (20 + 20 + 20)+ (7 + 7 + 7), which is 60 + 21 = 81 with R9 left*	*27 + 27 is the same as (20 + 7) + (20 + 7), which is 40 + 14 = 54* *54 + 27 is the same as (50 + 4) + (20 + 7), which is the same as (50 + 20) + (4 + 7), which is 70 + 11 = 81. R9 is left*

Table 4b Example of tasks in G3 (DoE, 2010a, p. 18)

Numeracy Milestones (per Term): Grade 3 (continued)			
Term 1	Term 2	Term 3	Term 4

In Grade 3, learners should be led to notice that although sharing and grouping problems are different, one may use the same method of computation for both cases	
You have R95 to buy bread for a picnic. One loaf costs R7. How many loaves can you buy? ***10 loaves cost R70.*** ***2 loaves cost R14, so that is R84.*** ***1 loaf cost R7, so that is R91.***	*95 learners have to sit at 7 tables. How many at each table?* ***If 10 learners sit at one table, it makes 70 learners.*** ***2 more learners can sit at each table, then 84 learners are seated.*** ***1 more learner can sit at each table, then 91 learners are seated.***

This thinking may be represented like this:	***So if 13 learners sit at each table, it is 91 learners.***
10 × 7 = 70 ***2 × 7 = 14*** ***70 + 14 = 84*** ***1 × 7 = 7*** ***84 + 7 = 91*** ***There is R4 left.***	***There are 4 learners left, so 4 tables should each get one more learner.*** ***So there will be 4 tables with 14 learners each, and 3 tables with 7 learners each.*** **This thinking may be represented like this:** ***10 × 7 = 70*** ***2 × 7 = 14*** ***70 + 14 = 84*** ***1 × 7 = 7*** ***84 + 7 = 91*** ***At 3 of the tables there should be 13 learners each, and at the other 4 tables there should be 14 learners each.***

These exemplars in Table 4.a indicate the use of more efficient division calculation strategies across sharing and grouping, involving working with "chunks" based on multiples of ten, rather than sharing by ones. In the first example, the solution builds up towards the dividend (Kouba, 1989). Symbolic signifiers are used in a sharing process that uses "partial composite units" of Rands (R10, R5) and then ones (R1) to find the answer. Thus, the development of more advanced mental strategies is encouraged within the curriculum document as suggested in the literature, but this is accompanied by a "jump" from iconic signifiers into symbolic working, with limited detail on how interim diagrammatic signifiers like arrays and DNLs might be used.

The grouping situation, presented as a word problem about placing apples in bags in Table 4a, is similarly solved using symbolic working through the use of "chunking up" and involves the use of "recalled facts" for solving the first grouping situation (Anghileri, 2006). The use of recalled facts to produce immediate answers to division calculations is evident in this section of the curriculum document, pointing primarily to a shift to progressively truncated strategies – chunking up to make the dividend, involving the use of a growing body of recalled multiplication facts, which allows for building up with larger (and therefore fewer) chunks. Anghileri (2006) notes that the use of multiplication facts to solve division word problems is aligned with a large body of work which encourages the use of this technique. Overall, the examples in the sharing and grouping situations show more efficient strategies as well as a progression in the number range evident within the dividend. In other words, in G3, the curriculum document focuses on how symbolic signifiers should be used in relation to the type of division situation and includes the use of more truncated strategies as the number range increases.

Moving to Table 4b, the point that both grouping and sharing situations can be solved by the same numerical working is highlighted. Knowing this means that once a particular situation is represented as a "division" situation, then a division calculation method can be applied to the numbers without worrying about whether a grouping or a sharing action underpins the calculation. In the examples presented in Table 4 b, a coherent way of talking about solution processes is offered in both cases, leading into numerical representations that are consistent with the situation. However, this kind of coherence between the "talk" – between the linguistic and the diagrammatic/symbolic signifiers was not always apparent in the instructional discourse analysed in this study. In the curriculum document for G3, the possibility of using the same "method of computation" is noted for solving different kinds of word problems. However, following the point made earlier that there is no emphasis on similar diagrammatic signifiers being possible for different division situations in either G1 or G2, nor of different distributing actions leading to the same end diagram, questions about the lack of guidance in the curriculum documents for presenting coherent explanations remain. Once again, there is a progression in the number range as the dividend has increased together with remainders now worked with through quantities and/or a variety of fractions. Furthermore, "chunking up" is visible in the use of multiplying by 10, doubling and then adding (Thompson, 2005).

2.7.2.2. Division ideas in the Grade 4 FFL

Table 5 Milestones and Assessment Tasks in G4 FFL (DoE, 2010b, p. 10)

G4	T1	T2	T3	T 4
	In the number range 1 to 1 000	In the number range 1 to 2 000	In the number range 1 to 5 000	In the number range 1 to 10 000
	Solves problems involving equal sharing and measurement, involving fractions including halves, thirds, quarters, fifths, eighths and tenths and mixed numbers involving these fractions, expressed in • words • diagrammatic form • common fraction notation • decimal notation for tenths	Solves problems involving equal sharing and measurement, involving fractions including halves, thirds, quarters, fifths, sixths, sevenths, eighths, ninths and tenths and mixed numbers involving these fractions, expressed in • words • diagrammatic form • common fraction notation • decimal notation for tenths	Solves problems involving equal sharing and measurement, involving fractions including halves, thirds, quarters, fifths, sixths, sevenths, eighths, ninths and tenths and mixed numbers involving these fractions, expressed in • words • diagrammatic form • common fraction notation • decimal notation for tenths	Solves problems involving equal sharing and measurement, involving fractions including halves, thirds, quarters, fifths, sixths, sevenths, eighths, ninths and tenths and mixed numbers involving these fractions, expressed in • words • diagrammatic form • common fraction notation • decimal notation for tenths
			Assessment Task 1: Estimates and problems that involve: Rounding off	***Assessment Task 1:*** Recognizes, describes and uses: The reciprocal

			to the nearest 10, 100 or 1 000 Equal sharing with **remainders** calculates by selecting and using operations appropriate to solve	relationship between multiplication and division
			Assessment Task 2: Division of at least whole 3-digit by 2-digit numbers.	***Assessment Task 2:*** Division of at least whole 3-digit by 1-digit numbers. Equal sharing with **remainders**.

The G4 specifications are also aligned with recommendations in the mathematics education literature to include equal sharing and grouping in the teaching of division (See Table 5 above). The number range continues to be extended but is also expanded with mention of a range of fractions to be dealt with. The progression in number range is aligned with international literature that shows that an increase in the number range supports the use of progression in strategies and the use of abstract signifiers linked to division such as short, long division algorithms and chunking (Thompson, 2010). Interestingly, there is no explicit mention of traditional short and long division algorithms in this increased number range.

Unlike the previous grades, various replays of signifiers for equal sharing and grouping problems express a range of signifiers from linguistic signifiers to diagrammatic signifiers to symbolic signifiers. As in previous grades, the curriculum document does not stipulate which signifiers might be best suited for the increased number range to 10 000 nor how to use signifiers or translate between them. In the G4 lesson episodes in my study, the number range did not exceed three-digit number contextual problems and bare number calculations. While the curriculum documents do not suggest the teaching of algorithms, in my study some teachers were observed solving traditional short and long division calculations using algorithms.

The concept of "rounding off" continues at the G4 level, with the exemplar tasks in Table 6 indicating moves into the use of multiples of the divisor that range from 10 to 100 to 1000. In G4, the dividend of the symbolic division calculation number range increases from two-digit to three-digit, with moves to working with two-digit divisors (the mention of 1-digit divisors in Term 4 was probably a typographical error).

Table 6 Example of tasks in G4 FFL (DoE, 2010b, pp. 9-16)

• In the number range 1 to 1 000	• In the number range 1 to 2 000	• In the number range 1 to 5 000	• In the number range 1 to 10 000
• **Solves problems that involve grouping and sharing:** Understands sharing and grouping situations, can do the calculations needed to solve them, and can judge whether the result makes sense. Examples: - *Tebogo buys chairs at R115 each. How many chairs can he buy for R1000?* 5 × R115 = R575 Many learners may do this in parts, e.g. 5 × R100 = R500 and 5 × R10 = R50 and 5 × R5 = R25 R575 + R230 = R805 1 × R115 = R115 R805 + R115 = R925 He gets 5+2+1= 8 chairs and has R75.			

Mathematics Milestones (per Term): Grade 4 (continued)			
Term 1	**Term 2**	**Term 3**	**Term 4**
- *Jaamiah buys 7 chairs. She pays with nine R100 notes and she gets R4 change. How much does each chair cost?* She actually pays R896, that is at least R100 for each chair. 7 × R100 = R700 7 × R20 = R140, that is R840 7 × R5 = R35, that is R875 7 × R2 = R14, that is R889 7 × R1 = R7, that is R896 One chair is R100 + R20 + R5 + R1 = R128.			

As in G3, as the number range increases, more abstract signifiers with truncated strategies are advocated. At the G4 level, in producing the answer, "chunking up" is stipulated used solving the suggested division situation.

2.7.3. Concluding comments on the FFL curriculum in relation to mathematics education literature

In conclusion, G1 to G4 include both equal sharing and grouping situation division problems, with moves over time into symbolic bare number calculations as the literature recommends (Vergnaud, 1985). Across these grades, the FFL includes attention to solving division problems with remainders, including remainders in fraction form – once again aligned with the literature (Carpenter et al., 1999). Progression in strategies accompanies increases in the number range of the dividend, from term 1, in G1 from 10, to 10 000 in term 4 in G4, with moves to using repeated addition and counting in multiples, and then chunking using multiples and powers of ten to build up or break down larger chunks.

While the exemplars do indicate moves over time to more abstract numerical signifiers linked to division, the document does not make explicit to teachers how signifiers can be used to support moves towards truncated strategies. A central progression is from linguistic, to concrete, to diagrammatic, to symbolic replays of sharing and grouping situations. However, three types of signifiers that literature recognizes as relevant to the teaching of division are absent from the FFL. These are two diagrammatic signifiers: arrays and T- tables; and symbolic signifiers in the form of traditional short and long division algorithms as the number range increases.

Across the four years, the solutions for sharing problems shift from basic one -by- one distribution to sharing of "partial composite" units. The use of known facts is incorporated into some of the exemplar tasks, with these known facts related to single digit multiples of the divisor, and multiples of 10, 100, 1000, etc. of the divisor. Yet, across the four years there are no sharing division solutions which include sharing by "complete composite" units, either through the use of known facts, or as the New Zealand curriculum mentions, through the use of symmetry whereby the complete composite group can be seen or easily constructed (Young-Loveridge, 2005). Similarly, solutions for grouping problems shift from the distribution of groups to more truncated strategies for solving the grouping division problems. Approaches involving doubles and halves, skip counting, repeated addition, repeated subtraction, building up and down, chunking up are advocated as more truncated strategies in different grades.

So, although the curriculum document does not use short and long division standard algorithms, it seems as if the development of more advanced mental strategies is being encouraged within the curriculum document. However, the curriculum document does not make an explicit link between how these approaches relate to an increased number range, together within the replay of situations with the progression from linguistic signifiers to symbolic signifiers and the use of more advanced division calculating strategies. Similarly, there is no exemplification of situations leading to fractional remainders.

The literature reviewed, together with the analysis of the FFL curriculum have highlighted which mathematical concepts are considered important for the teaching and learning of division internationally and within South Africa. The literature review also noted the importance of coherence in the teaching of division. Both the literature review and the curriculum analysis have highlighted the importance of progression when using division calculation strategies and the importance of signifiers. I argue that semiotic theory is useful for theorizing South African teachers' modes of assembling and sequencing signifiers related to division in mathematics.

2.8. Concluding remarks

There is a broad consensus in the national and international literature on many aspects of the enactment of coherence during instructional exaplantions in the teaching of division such as:

- The importance of terminology linked to division,
- The importance of understanding the relationship between division concepts and multiple mathematical concepts when solving division problems,
- The importance of identifying the two division situations signified as distinct ideas and understanding how they link with other division signifiers to produce signs,
- The importance of progression from signified division situations to abstract symbolic signifiers,
- The importance of progression in the use of efficient division calculation strategies within the teaching and learning of division, together with the progression in number range.

In Chapter Three, I discuss the theoretical framework that I have employed to explain and understand the phenomena underlying teachers' choices of signification pathways in teaching division.

3.1. Introduction

From the review of mathematical literature on division in the previous chapter, two fundamental division situations stand out as being foundational to all work on this topic: sharing and grouping situations, with these situations marked concretely by sharing and grouping actions. These situations and actions can then be described using a range of diagrams, such as grouping and sharing pictures, array diagrams, and number lines. Division situations and actions can also be represented by division number sentences and other symbolic representations such as short and long division algorithms.

The literature on the teaching of division points to its complexity given that both diagrammatic and symbolic representations can often be linked to sharing or grouping situations, leading to possibilities of slippage between the two situations in teachers' and learners' problem-solving work.

The literature on division also identifies two key avenues of progression: the first related to the use of increasingly truncated and abstract signifiers (the form the sign takes) and the second to the use of increasingly efficient division calculation strategies. Analysis of the South African FFL curriculum has revealed that this document includes some aspects of these two avenues of progression, but fails to give attention to the complexities of coherence in relation to situations, actions and signifiers. There are also gaps in the guidance given on key signifiers and how to incorporate them.

As explained earlier, in this study coherence is centrally about the nature and extent of mathematically defensible connections in teachers' work with learners on division tasks. The study focuses on the signification pathways teachers choose in presenting division tasks and on coherence within these. In this chapter I discuss the semiotic theory which underpins the study, and explain my reasons for its choice, and for the choice of particular theoretical positions from within the semiotics literature.

Sharing and grouping situations and actions can be considered, from a semiotic perspective, as the two key situations signified by the various signifiers of division mentioned above. In this chapter, I introduce and discuss key concepts in the work of several semioticians and then focus on the work of Valerie Walkerdine (1988), who articulated the shifts in discourse involved in moving along a chain of signification, the application of sign combinations and the formation of chains of signification as described by Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997). In the latter part of the chapter I discuss the relevance of semiotics to the formation of what I have come to term "signification pathways" which enable understanding of various ways in which teachers teach division. The chapter concludes with examples of lesson excerpts in which differences in the signification pathways identifiable in each one.

3.2. Semiotics in Mathematics Education

For the last three decades semiotic approaches have been evident in research focused on the learning and teaching of mathematics (Presmeg, Radford, Kadunz, & Roth, 2017). Semiotics has its origins in the concept of "semiosis". Peirce used the term "semiosis" to refer to any sign action, or sign process: in general, the activity of a sign. A sign is "*something that stands for something else*" (Charles S Peirce, 1992). The idea of "standing for" expresses a process in which some newly introduced entity becomes part of a new sign containing traces of the prior sign (Sáenz-Ludlow, 2003). Presmeg et al. (2018) note that some researchers argue that "sign vehicle" should be used for the representamen /signifier in order to avoid confusion.

Semiotics is described as the "study or doctrine of signs" (Presmeg, Radford, Kadunz, et al., 2017, p. 1). Two people are primarily associated with its origins: de Saussure, a Swiss linguist whose ideas inspired structuralism and Peirce, an American philosopher known as the originator of pragmatism. Other theorists of semiotics have focused on semiotic mediation Vygotsky (1978), social semiotics (Halliday (1978) and semiotics in relation to key mathematical representations in different conceptual fields Vergnaud (1985). More recently, Radford (2009) and Roth (2007) have used semiotics in theories of embodiment that include gestures and the body as modes of signification. Presmeg, Radford, Kadunz, et al. (2017) have used semiotics in mathematics and emphasize that "the significance of semiotics for mathematics education lies in the use of signs; this use is ubiquitous in every branch of mathematics" (p. 1).

3.2.1. De Saussure

De Saussure (1959) worked on semiology within a structural theory of general linguistics. According to de Saussure (1959), a linguistic sign consists of two elements, namely a concept (signified) and an acoustic image (signifier). De Saussure used a dyad schematic to represent the relationship between the signifier and the signified. Two examples using de Saussure's notation are shown in Figure 6:

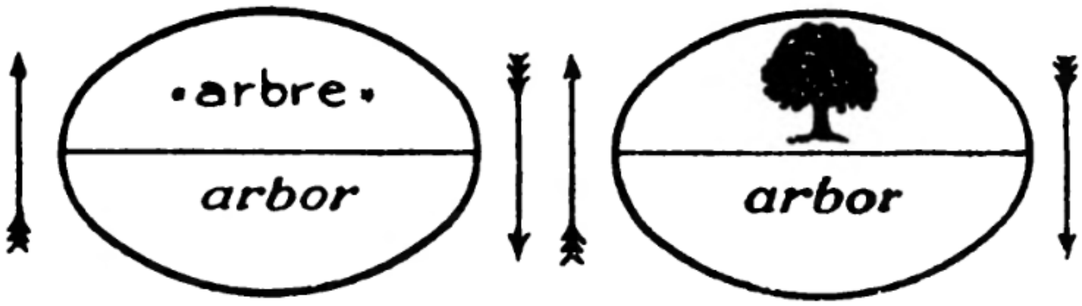


Figure 6: De Saussure's diagrammatic representation of a sign

In the first diagram, for a French-speaking child learning Latin, the Latin word "*arbor*" (tree) acts as signifier at the bottom with the French word "*arbre*" (tree) on top indicating what the new Latin word signifies. In the second diagram, the signifier retains the word "*arbor*" and a drawing of a tree on top takes the place of the signified. The signifier remains the same while the signified is changed, emphasising that signifiers are arbitrary and are socially determined. For Whitson (1994), having the signified on top suggests the dominance of the signified over the signifier. Having the signified on the top prevents the emergence of a chain of signifiers in which a signifier in a previous sign combination becomes the signified in a new sign combination (Presmeg, 2006).

3.2.2. Lacan

By contrast, Lacan (1966) inverted the relationship in the structure by placing the signified at the bottom and the signifier on top. In inverting these two elements, Lacan (1966) provides a critique of the Saussurian idea of the importance of the signified, and asserts instead, the primacy of signifiers. While Ross (2002) acknowledges that the connection between the signifier and the signified is arbitrary, his focus is on the stability obtained if a signifier is "habitually associated with a particular signified retaining its claim through a process of differentiation not from other signifieds but from other signifiers" (p. 12). Ross (2002) also argues that, for Lacan, the function of a signifier is to stop the otherwise endless movement of signification to produce "the necessary illusion of a fixed meaning". Evans (2006) notes that "Lacan uses the term 'signifying chain' to refer to a series of signifiers which are linked together" (p. 90). Therefore, a signifying chain can never be complete, since it is always possible to add another signifier to it ad infinitum, and signification is not present at any one point in the chain, but rather "insists" on the movement from one signifier to another (Ross, 2002). For Roth (2012), placing the signified under the signifier suggests that "the signified (below) is never attainable in itself, but only mediated through an ever-expanding sequence of signifiers" (p. 462). An example is having four identical doors (signifieds) being differentiated by different words (signifiers—such as *gentlemen, women, family,* and *disabled*). Roth (2012), in his explanation, notes that "signifiers can become inscribed in the signified world such that—in the current example—otherwise identical doors come to be differentiated" (p. 462; see Figure 7 below). So by swapping the signifier and signified, Lacan creates "a chain of signifiers that never really attain the signified" (Presmeg, Radford, Kadunz, et al., 2017, p. 6). The Lacanian interpretation of the sign then identifies the signifier as the physical form of the sign, while the signified refers to the concept embedded within the sign (Chandler, 1994). For my study on division, I will be using Lacan's interpretation, which focuses on a chain of signifiers as a tool for providing insights into the teaching of division. My focus is on the relation made at each stage between signifiers (linguistic, diagrammatic, symbolic) and whether they point back logically to the same sharing or grouping division situation (signified) and, if there are shifts in the signified situation, whether or not these shifts are made explicit. Important to note, that I would employ semiotics as an analytical tool for investigating the teaching of division.

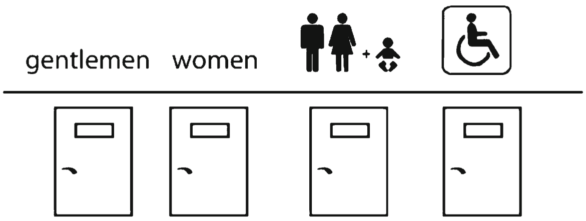


Figure 7: Chains of signification according to Lacan

3.2.3. Peirce

Peirce uses a model that incorporates three elements in the form of a triad. The triad includes an *object*, a *representamen* (sign vehicle) that stands for the object in some way and an *interpretant*. It is important to note that the interpretant is not the interpreter, but rather, the effect of the sign vehicle on the mind of the interpreter. In Peirce's writing, the word "sign" sometimes designates his whole triad, but at other times designates the representamen only, as stated by Presmeg et al. (2018).

Peirce explains the differences between iconic, indexical and symbolic sign vehicles. The iconic sign vehicle is evident when "there are likeness, or icons which serve to convey ideas of the things they represent by imitating them" (1992, p.6). For instance, in my study, a teacher drew a stick figure of a person to "stand in" for her brothers who were receiving sweets. Charles Sanders Peirce (1998, p. 6) states that indexical sign vehicles involve inscriptions with a physical connection to a real-life object, for example, smoke invoking the interpretation that there is a fire, or a sign-post pointing to a road. Symbolic sign vehicles are underpinned by conventions rather than any overt similarity to the object—for example the division symbol (÷). Different people can look at the same sign vehicle and interpret it in different ways (iconic, indexical, or symbolic) depending on its relationship to the object. Peirce (1992) argues that these subtle tensions differences need consideration when communicating using these sign vehicles. The need for expression (*representations or signification)* or communication drives Peirce's model.

While Peirce's sign vehicle forms refer primarily but not exclusively to inscriptions, the literature also points to another form with which Peirce would agree—embodied gestures—as a critical signifier form that is important (Roth, 2007). Given that the literature on division includes sharing and grouping actions as the basic gestures linked to sharing and grouping situations, and also that this study is based in relatively early grades, gestural signifiers are important to include in the current study when looking at the procession of signifiers incorporated in instruction.

Peirce's work has been used by (Presmeg, 1985) to extend her earlier research (1985) on visualization in high school mathematics. She uses the formula for the roots of a quadratic equation as an example of how a signifier could be *iconic*, *indexical*¸ or *symbolic,* depending on a person's interpretation. Presmeg has moved from a Lacanian approach in her earlier work to a Peircean approach, in order to stress the importance of interpretation as formulated in the *interpretant,* in processes of signification. My focus in this study is more on the associations between signifiers offered in instruction and the extent to which these associations are defensible from a mathematical perspective for division. Thus, while I end with some claims about the interpretants of teachers in the study, the teachers' meaning making concerning division is not my central focus; instead my focus is on their chaining of signifiers as pointed to in Lacan's approach.

However, Peirce's attention to different sign vehicle forms has been useful in this study given that transitions between these forms have been highlighted as an important part of progression. A further feature that makes Peirce's work relevant to my study is Presmeg's (2006) noting that mathematical working between Peirce's sign vehicle forms involves the emergence of a "nested model" of sign vehicles—a "*web* of signs", rather than linear semiotic chains Presmeg (2006, p. 179). A web of signs points to the notion of signs that connect or link to each other in complex ways as mathematical explanations unfold. This idea is useful for my conceptualisation of "*signification pathways*" which is discussed later in the chapter.

3.2.4. Walkerdine

Walkerdine (1988) has considered the value of analysing mathematical discourse for making sense of work done by learners. Like de Saussure previously, she argues that the relationship between the signifier and signified is arbitrary, but goes on to argue that the uniting of the signifier with a particular signified produces a sign, thus transforming the previous signifier/signified relationship in a chain. Walkerdine (1988) analysis is useful in that it illustrates the movement of signs across signifiers in mathematical learning situations. She states that "the signified only forms a sign out of fusion with the signifier" (p. 206), within the explanation of mathematical ideas. Walkerdine (1988) used a conversation between a mother and daughter as they planned the number of glasses needed for a picnic to exemplify shifting fusions: initially, the children's names (signifiers) were linked by the child with the actual people (signifieds); then the daughter showed five fingers for the number of children; then the fingers dropped to the level of signifieds in becoming united with the new signifier: the spoken word five. Additionally, Walkerdine notes that repetition of a particular union between signifier and signified reduces the arbitrariness of the relationship between them.

Signifiers have the potential to produce multiple meanings. Because the relationship is arbitrary, there is the possibility of signs being used in competing discourses because, according to Walkerdine (1988, p. 30), "signifiers do not cover fixed meaning" (p. 30). The multiplicity of possible meanings led Walkerdine (1988) to discuss what signifiers "signify" and "produce" rather than what they represent. This conceptual shift is useful for my study because it has enabled me to observe whether signifiers have maintained a consistent association to a signified situation or whether there is a shift in the signified situation, and therefore, a shift in the production of meaning through the establishment of associations between signifiers.

Following Lacan, for Walkerdine, the primacy of the signifier over the signified allows for a focus on the associations between signifiers. This focus on chains of signifiers is central to my study, and therefore, I discuss the methodology and applications of chains of signification in more detail, and my adaptations of this theorisation.

3.3. Chains of signification

Mathematical discourse can be thought of as the construction of signifying chains when solving mathematical tasks. Walkerdine (1988) describes the prising apart of the signifiers and signifieds that makes visible the production of a particular chain of relations of signification: "prising apart of relations of signification and rearticulating them [that] produces new signs" (Walkerdine, 1988, p. 119).

Whitson (1994, p. 110) use of Lacan's restructuring of the schematic unification of the signifier and signified, points to the autonomous nature of the signifier that allows for the formation of a chaining process. This chaining process is evident when the signifying term (signifier^1) in a preceding sign combination serves as a signified term (signified^2) in a succeeding sign combination. In such a "chaining of signifiers", the preceding signifieds and sign combination are sometimes described as "sliding under" the succeeding signifier. Walkerdine (1988) states that the meaning of the original sign combination is masked by succeeding signifiers because a chain of signification is constituted by the replacement of original and earlier signs by later signifiers. Walkerdine's theorisation of the chaining process of signifiers is valuable in my study for making sense of the teachers' explanations of division problems.

Cobb et al. (1997) illustrate a chain of signification produced over a few lessons in tasks that were designed to assist learners in identifying ten as a single entity numerical composite referring to a collection of 10 units of one. After placing 100 interlocking cubes in a bag, the teacher develops a narrative about working in a candy shop and wanting to shorten the process of counting out candies individually by packing 10 candies into a roll. Cobb et al. (1997) describe an instructional sequence where learners first used unifix cubes to re-enact the packing of cookies. Through this activity, a bar of ten unifix cubes came to signify a numerical composite of 10 for most learners. In the second phase the learners accompanied their concrete composite units with verbal answers indicating "taken as shared" understandings of what the teacher had emphasized in phase one. Evidencing what was "taken as shared" the teacher placed pictures on the white board of two rolls of ten with three loose candies and then three rolls of ten and two candies. In both cases, learners provided correct totals. Some learners' ability to make use of composite units of ten to produce answers extended to more complex instances involving, for example, three rolls of ten with 13 loose candies in groups of 4, 5 and 4. Subsequently, tasks involved different ways of describing pictured quantities, for example, a picture of 4 rolls of ten with three loose ones. As the learners provided answers the teacher drew pictures of their different responses on the board (see Figure 8 below).

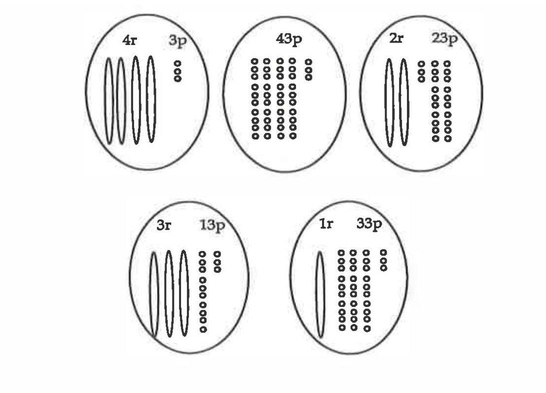


Figure 8 Recorded images of different ways the number 43 was represented on the white board (Cobb et al., 1997, p. 188)

Cobb et al. (1997) claim that semiotics provides a way of making sense of whether instructional discourse is clear and coherent. For them, coherence is observed when the preceding sign combinations become the signified of the succeeding sign combination within a chain of signification and "the meaning of the sign combination evolves as the chain of signification is constituted" (Cobb et al., 1997, p. 192).

From the use of unifix cubes into the drawing of the pictures by the teacher, to the learners working with verbal enumerations of tens and units, there is a clear and coherent chaining of signifiers as presented by Cobb et al. in Figure 9 below. Cobb et al. (1997) note that the interlocking cubes were used to simulate the activity of packing candies. Furthermore, the teacher ensured that she had provided signifiers which allowed development from identifying composite units of 10 to using composite units as ways of forming different numbers.

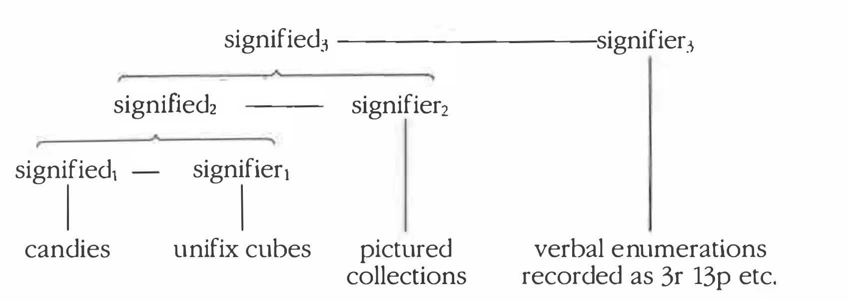


Figure 9: Cobb's chain of signification—the preceding sign combination becomes the signified of the succeeding sign combination (Cobb et al., 1997, p. 192)

More broadly, some researchers using a semiotic approach have focused on signification chains related to a single mathematical object (e.g., Sfard, 2008; Cobb, 1977). In my dataset, there are frequent "side-steps" to sub-elements of the initial signified entity, which meant that the approaches presented in the work above could not be directly taken up. I have used the term "signification pathways" because it has allowed me to deal with these "side-steps" (SS) into sub-parts of the signified situation. Thus, signification pathways in my study are comprised of a single chain, or of an initial chain with a single or many "side-steps".

3.4. The relevance of semiotics for this study

This study, drawing from Lacan's approach and aspects of others' theorisations of semiotics as detailed above, offers and uses a categorization of South African teachers' modes of assembling and sequencing signifiers related to division. Division is replete with mathematical and pedagogical complexities. In mathematical terms, and at first glance, division relations are based on a simple triad of quantities: the dividend, the divisor, and the quotient. Mathematically, as noted in Chapter 2, two basic phenomenological situations lead to the need for division: partitioning or sharing situations, and quotitioning or measuring situations. In pedagogical terms, complexity is evident in relation to several elements in the teaching and learning situation. Among these are similarities in the phonology of words that relate to different dividing actions and diagrammatic forms, and the "one-many" nature of the relationship between division as a mathematical operation and the situation types that this operation indexes. This already complex picture is further compounded in South African research into the teaching of mathematics, by the concerns raised about coherence and connections in the presentation of mathematical ideas (Venkat & Naidoo 2012; Askew et al., 2012). Given semiotics' close attention to the connections between signifiers, this theoretical frame is valuable for providing insights into teachers' connection-making when working with division.

A Lacanian approach is best suited to this study in that the inversion of the signified and signifier produces chains of signifiers (Lacan, 1966). The formation of the chain of signifiers enables me to unpack the teachers' instruction while they endeavour to develop learners' mathematical concepts, in this case concerning division. Lacan points out that the result of the signified sliding underneath the signifier creates more signifiers (a chain of signifiers), which designate meaning, and thereby produce new signifieds. The teachers in my study selected different signifiers in their solving and explanations within division tasks; the selection of these signifiers pointed towards whether division could be interpreted as sharing or grouping. However, although these signifiers have the ability to connect to each other as the chain develops, Lacan states that that within a chain of signifiers the fundamental property of the signifier is to be able to cancel itself out (Lacan, Miller, & Grigg, 1993, p. 122). Even though signifiers have the ability to cancel each other out, Lacan acknowledges that within a chain of signifiers, signifiers are connected to other signifiers creating a network of signifiers (Lacan, 1966, p. 119). The connection of signifiers to other signifiers provides an opportunity for analysing how coherence is maintained throughout a teacher's explanation. Although these chains of signifiers have the ability to continue, Lacan observes that "there is always a moving towards a closure of meaning" (Lacan et al., 1993, p. 137). In other words, a chain of signifiers only makes sense in retrospect once you have indicated that you have reached the end by punctuating it. In my study the chains of signifiers only made sense when the teacher had solved the division task or had explained the division concept before moving to the next episode.

In this respect, my interest goes beyond the idiosyncratic chains of signifiers used by individuals; instead, I have sought to unpack how signifiers are produced in instruction in ways that help to broaden teachers' semiotic practices in developmental ways, from a mathematical standpoint in which powerful ways of working include moves towards increasing generality. This exploration has led to the construction of a key contribution of this study—the signification pathways framework, in which the categories are based on the nature and extent of coherence of signification in relation to division as a mathematical network of signifiers.

3.5. Background to the Signification Pathways framework

This study is an investigation of how three Grade 3 teachers and three Grade 4 teachers taught division to the same cohort of learners across the 2011 and 2012 school years. As I observed and video-recorded the teaching, I focused on how teachers worked with division tasks. To recapitulate, in the sharing model of division, the fundamental action is the unit distribution of items (Neuman, 1999), which, over time, can be truncated by distributions of "complete composite" units (Carpenter et al., 1993). In a grouping model of division, the fundamental action is the distribution of groups. Linguistic signifiers vary in subtle but important ways across these two situations, as does the construction of diagrammatic signifiers. In symbolic signifiers, either situation can be signified. Moves towards increasingly abstract signifiers and increasingly truncated division calculation strategies represent key routes of progression.

Initially, I was interested in exploring how teachers attended to both sharing and grouping models, given the literature-based evidence of the predominance in the early grades of sharing interpretations of division (Booker et al., 1992). However, several segments of transcript of teachers’ work with tasks did not "fit" tidily into these two categories, with "mid-flow" shifts sometimes evident between grouping and sharing models. In retrospect, given the evidence of incoherence and disconnection in primary mathematics teaching in South Africa (Mathews, 2014), this finding of lack of coherent working with division is unsurprising. While few segments were straightforwardly incoherent, ambiguity and localized ways of working were relatively common and necessitated the development of an alternative framework for analysis of lesson episodes.

As a result, the focus of the investigation shifted to an examination of the extent of coherence in the teachers' signification pathways. Writing about South African classrooms, Venkat & Adler (2012), discuss coherence in terms of links between tasks and the instructional discourse around them at the episode level, with episodes focused on instruction around particular tasks, rather than at the lesson (or cross-lessons) level, more commonly seen in the international literature (Sekiguchi, 2006). Following Venkat and Adler's example, in this study, I focus on task-based episodes (episodes are explained further in the Methodology chapter that follows) to understand the nature and extent of coherence evident in the teaching of division in three G3 and three G4 classrooms.

3.6. The Signification Pathways framework

Taking a grounded approach to looking for similarities and differences between lesson episodes led to a focus on the nature and extent of coherence in the signification pathway. The analysis of each episode concentrated on the sequence of signifiers produced and the consideration of the mathematical implications of significations in these pathways. Where signifiers produced in instructional talk/inscriptions did not cohere with the mathematical implications of earlier signifiers in some ways, these were categorised using the descriptors "limitations", " some ambiguities", and "errors". Where subsequent signifiers maintained a connection with earlier significations, the episode was termed "coherent". In Table 7 below, each of these pathways is described and subsequently, in this chapter, illustrated and discussed through exemplar episodes.

Table 7: The Signification Pathways framework

Signification Pathways	
Category	**Description**
Coherent	*This signification pathway involves links between signifiers that are mathematically justifiable at all points within the episode. There is either no shift in the division model worked with, or the shift is acknowledged in the discourse.*
Coherent with limitation	*This signification pathway is coherent in terms of the signifiers produced. However, some signifiers are not applicable generally across examples. Instead, some parts of the signification pathway are localized to a particular sub-set of examples that includes those presented within the pathway, but without this limitation being articulated.*
Some ambiguities	*This signification pathway involves instances of ambiguity. Ambiguity relates to one of the following:* *a) An incorrect signifier for a signified, with no acknowledgment of the incorrect signifier, but with this ambiguity subsequently corrected,* b) *A signified comes to be associated with two contradictory signifiers, but along the pathway, each signifier can be linked back to the previous signifier,* c) *Signifiers shift between division models or actions with no acknowledgment of the shift.*
Incoherent	*This signification pathway involves mathematical incoherence. Incoherence occurs when:* *a) the teacher's answer is incorrect,* *b) no answer is provided,* *c) involves incorrect signifiers at some points, and these incorrect signifiers are not subsequently corrected.*

3.7. Illustration and discussion of the Signification Pathways categories

In this section, empirical episodes from the dataset are used to illustrate and discuss the central features of each pathway in the framework, and I also note the empirical range seen within each of the pathways.

3.7.1. Coherent pathways

In coherent episodes, teachers' work with signifiers showed a coherent chaining across the pathways with new signifiers connecting with previous signifiers in the pathway and linking with the same signified situation. If a change in the signified situation occurred, this was explicitly acknowledged. The excerpt below is drawn from the teaching of one of the teachers in the sample—Cathy (pseudonym), a Grade 3 teacher at a previously advantaged school. In her first lesson Cathy provides an illustration of a coherent pathway episode:

> *Cathy asked her class for a number sentence that could be used to "do equal sharing" with some cut out stars that she had on her desk. "Six divided by two" was offered by a learner and accepted by Cathy, and she asked the learner to demonstrate to the class how to represent the problem using six stars, using the phrase: "Can you group them?"*
>
> *The learner proceeded to stick one column of three stars on the board and then a second column of three stars alongside, resulting in an array formation. Below the stars, the learner wrote 6 ÷ 2 = 3.*
>
> *The teacher commented thus: "Six divided by two equals three, if you look at the stars it's two groups of three" (gesturing down one vertical group and then the next).*

The instructional pathway in this episode could be represented diagrammatically as follows (see Figure 10). "Sr" is used to denote a signifier, and "Sd" denotes the signified entity, with numbering indicating the sequence of signifiers and signifieds in the pathway. Signification pathway (SP) diagrams like this were constructed for every episode in the empirical dataset and then analysed for the nature and extent of coherence.

"Six divided by two" (Sr 1)	Six stars produced as two columns of three stars (Sr 2)	6 ÷ 2 = 3 (Sr 3)	"Six divided by two equals three" (Sr 4, repeat of Sr 1 linked with 3 as answer)	"it's two groups of three". (Sr 5)
"Equal sharing" (Sd 1)	"Six divided by two" (Sd 2)	Six stars produced as two columns of three stars (Sd 3)	6 ÷ 2 = 3 (Sd 4)	"Six divided by two equals three". (Sd 5)

Figure 10: SP diagram—Cathy, G3, Lesson 1, Episode 7

In this pathway, an equal sharing situation is requested, produced and described through a coherently connected sequence of signifiers—an appropriate oral number sentence, an array arrangement of stars produced through what I have described in Chapter 2 as sharing by "complete composite" units, and then a number sentence that includes the outcome that is elaborated further in words as a situation involving "two groups of three". Mathematically, all the signifier-signified pairings are mathematically justifiable in this pathway, and all signifiers connect coherently with an equal sharing situation linked to 6 ÷ 2 =. Episodes like this can be described as "coherent" in their form.

The next excerpt below is drawn from the teaching of the same teacher in her second lesson and provides an illustration of a coherent pathway episode:

> *In the following episode, the teacher placed a transparency on the OHP (Over Head projector). The teacher also ensured that each learner received counters. Initially, the word problem written on the OHP was incorrect; however, the teacher corrected it. The transparency read as follows "Jade shares 21 beads with (among) three friends. How many beads will each one get, if shared equally?"*
>
> *A learner proceeded to write 21 ÷ 3 = on the whiteboard (WB) and drew three circles. After counting 21 counters, the learners distributed the cubes one at a time among the three circles resulting in seven objects in each circle.*

After the distribution of counters into the three circles, the learner wrote "7" as the answer for 21 ÷ 3 =, which the teacher accepted.

"Jade shares 21 beads with (between) three friends? How many beads will each one get, if shared equally?" (Sr 1)	Teacher writes 21 ÷ 3 = on the WB. (Sr 2)	Learner drew three circles. (Sr 3)	Distributes 21 counters one at a time between three circles. (Sr 4)	"21 ÷ 3 = 7" (Sr 5 , repeats for Sr 2 linked with 7 as answer)
"Equal sharing". (Sd 1)	"Jade shares 21 beads with (between) three friends? How many beads will each one get, if shared equally?" (Sd 2)	Teacher writes 21 ÷ 3 = on the WB. (Sd 3)	Learner drew three circles. (Sd 4)	Distributes 21 counters one at a time between three circles. (Sd 5)

Figure 11: SP diagram—Cathy, G3, Lesson 2, Episode 11

In this pathway, an equal sharing situation is requested, produced as described through a coherently correct sequence of signifiers: an appropriate written word problem, together with the production of a symbolic signifier of the word problem, three circles which point to a sharing interpretation of division (number of groups), 21 counters distributed one at a time between three circles through what I have discussed in Chapter 2 as the fundamental division sharing action of one-by-one distribution and a number sentence that includes the answer 7. In this episode the counting out of 21 counters first, when sharing, is an example of the less advanced division calculating strategy described in the literature (Carpenter et al., 1999).

Mathematically all the signifier-signified pairings are mathematically justifiable in the pathway. Moreover, all signifiers connect coherently with an equal sharing situation linked to a word problem "*Jade shares 21 beads with (among) three friends. How many beads will each one get, if shared equally?*"

The next excerpt below, drawn from the teaching of the same teacher in her third lesson, provides an illustration of a coherent pathway episode:

The teacher worked with some learners on the mat. She instructed them to count out thirteen counters, and to place them on a WB, in "groups of four".

While walking around, the teacher observed a learner placing four counters into three groups with one remainder.

The teacher reinforced that a group of four was required by circling four counters with a whiteboard pen. The teacher then asked the learners to provide a number sentence.

Two offers were made, the first, "thirteen divided by three", and other learners responded, "thirteen divided by four". The teacher accepted the second offer and wrote "13 ÷ 4=" on the board.

Above the symbol "4" she wrote the words "in a group". She then continued to complete the symbolic inscription by inserting "3 rem 1" as the answer, noting that the "remainder" was the "left over".

Teacher instruction "make groups of four". (Sr 1)		Learners made three groups with four counters in each. (Sr 2)		Number sentence "thirteen divided by four" – provided. (Sr 3)		*13* ÷ 4=. (Sr 4)		Above the 4 she wrote in "a group". (Sr 5)
"Grouping". (Sd 1)		Teacher instruction "make groups of four". (Sd 2)		Learners made three groups with four counters in each. (Sd 3)		Number sentence "thirteen divided by four" – provided. (Sd 4)		*13* ÷ 4=. (Sd 5)

	"13 ÷ 4= 3 rem 1". (Sr 6 repeat of Sr 4 with 3 rem 1 as answer)		The remainder is the left over. (Sr 7)
	Above the 4 she wrote in "a group". (Sd 6)		"*13* ÷ *4= 3 rem 1*". (Sd 7)

Figure 12: SP diagram—Cathy, G3, Lesson 3, Episode 4

In this pathway, a grouping situation was requested, produced and described through a coherently connected sequence of signifiers—appropriate groups were formed of three groups with four counters in each through what I have described in Chapter 2 as "grouping" and then producing a number sentence with the words "in a group" written on top of the four, completing the number sentence by including the answer "3 rem 1", and then elaborating further in words that the "remainder is the left over". As in the previous example, a less advanced grouping strategy was observed when the learners first counted out thirteen counters as opposed to making groups of four immediately.

Mathematically all the signifier - signified pairings are mathematically justifiable in the pathway. Moreover, all signifiers connect coherently with a grouping situation linked to the grouping action on counters in which thirteen counters were placed into groups of 4 with 1 rem.

The next excerpt below, drawn from the teaching of the same teacher in the same lesson provides an illustration of a coherent pathway episode:

> *The teacher wrote 19 ÷ 5 = on the board and provided the learner with WB and counters. She instructed learners to put the counters into groups of five. Some learners immediately made three groups of five with a remainder.*
>
> *However, one of the learners made four groups of five in which the teacher asked the learner to redo. The learner counted out nineteen counters and made three groups of five with a remainder 4.*
>
> *The teacher then inserted the answer 3 4 rem on the board.*

"19 ÷ 5 =" (Sr 1)	Teacher's instruction "put them (counters) into groups of five". (Sr 2)	With counters learners made three groups of five with remainder. (Sr 3)	19 ÷ 5 = 3 rem 4 (Sr 4 repeat of Sr 1 with 3 rem 4 as answer)
"Grouping" (Sd 1)	"19 ÷ 5 =" (Sd 2)	Teacher's instruction "put them into groups of five". (Sd 3)	With counters learners made three groups of five with remainder 4. (Sd 4)

Figure 13: SP diagram—Cathy, G3, Lesson 3, Episode 6

In this pathway, a grouping situation was requested, produced, and described through a coherent connected sequence of signifiers—an appropriate grouping of counters was accepted (three groups of five with 4 as the remainder), through what I have described in Chapter 2 as "grouping". A few learners used a more sophisticated strategy by making groups of five while other learners showed a less sophisticated strategy by first counting out 19 counters then making groups of five. Then a number sentence included the outcome "3 rem 4".

Mathematically all the signifier-signified pairings are mathematically justifiable in the pathway. Furthermore, all signifiers connect coherently with a grouping situation linked to the grouping action on counters in which 19 counters were placed into groups of 3 with a remainder of 2.

The next excerpt below is drawn from the teaching of another teacher in the sample—Joy (pseudonym), a Grade 4 teacher at a previously advantaged school. This excerpt from her first lesson, provides an illustration of a coherent pathway episode:

> *The teacher wrote the following number sentence on the board 63 ÷ 9 =.*
>
> *She called a learner to the front of the class to solve the division calculation and said: "show us where (we are) going to circle the number 63". On the NL, the learner placed a circle around the number 63.*

The teacher asked the learner to show her how to jump in nines till 63. The learner used his finger; he jumped on the NL five times from 0 to 45. He said, "One, two three, four, [and] five". Also, using the chalk, he jumped one more time from 45 to 54 and said "six".

The learner made another jump on the NL from 54 to 63, and as he drew the jump he said "seven".

The teacher asked, "what is the answer?" The learner responded "seven" and wrote 7 on top of the last jump.

"63 ÷ 9 = " (Sr 1)	Teacher instruction: "Show us where are you going to circle the number 63". (Sr 2)	The learner circled the number 63 on the NL. (Sr 3)	The learner illustrated how to jump on the number line in nines from 0 to 54 and said: "Six". (Sr 4)	The learner jumped from 54 to 63 on the NL and on the last jump wrote the number "7". (Sr 5)
"Grouping" (Sd 1)	"63 ÷ 9 = " (Sd 2)	Teacher instruction: "Show us where are you going to circle the number 63". (Sd 3)	The learner circled the number 63 on the NL. (Sd 4)	The learner illustrated how to jump on the number line in nines from 0 to 54 and said: "Six". (Sd 5)

The learner jumped from 0 to 63 on the NL and on the last jump wrote the number " 7 ". (Sr 6)	"63 ÷ 9 = 7" (Sr 7 repeat of Sr 1 with 7 as answer)
The learner jumped from 54 to 63 on the NL and on the last jump wrote the number "7". (Sd 6)	The learner jumped from 0 to 63 on the NL and on the last jump wrote the number "7". (Sd 7)

Figure 14: SP diagram—Joy, G3, Lesson 1, Episode 14

In this pathway, a grouping situation was requested, produced and described through a coherently connected sequence of signifiers—an appropriate use of the NL, the number of jumps in nines produced through what I have described as in Chapter 2 as grouping through skip counting (jumping in multiples of 9), and then producing a number sentence $63 \div 9 = 7$ that includes the outcome of "7" pointing to seven jumps in nine. Mathematically all the signifier-signified pairings are mathematically justifiable in the pathway. Moreover, all signifiers connect coherently with a grouping situation linked $63 \div 9 = 7$. Episodes like this can be described as coherent in their form.

Overall, the coherent category episodes illustrated above involve three types of division actions. The first two fundamental actions are examples of sharing, which is seen in a one-by-one distribution action (Wright, Martland, & Stafford, 2010). The second type of sharing action is identified as "sharing by composite" units (CCU) in which the total number of items shared is through a grouping action based on the answer (Zweng, 1964). Another fundamental action evident in these examples is a grouping action (Carpenter et al., 1999). The three examples include signification pathways incorporating a symbolic signifier, a diagrammatic signifier, and a linguistic signifier. Within two of the episodes, more sophisticated ways of solving division calculations are evident. The first is an example of solving a sharing situation by "complete composite" units. Secondly, a more advanced strategy is evident when learners or teachers distribute one-by-one or grouped items without counting the initial number of objects indicated by the dividend. When it comes to either sharing or grouping actions, a less advanced strategy when working with sharing and grouping actions occurs when items are counted first and then shared one at a time or through groups (Fosnot & Dolk, 2001). Mathematically all the signifier - signified pairings are mathematically justifiable in the pathway with all the signifiers connected coherently throughout the pathway.

3.7.2. Coherent with limitation pathways

In coherent with limitation episodes, teachers' work with signifiers showed a coherent chaining across the pathways with new signifiers connecting with previous signifiers in the pathway and linking with the same signified situation. However, the selection of some signifiers in the chaining process is localized only to the episode and not generalizable. If a change in the signified situation occurred, this was explicitly acknowledged.

The excerpt below is drawn from the teaching of one of the teacher sample—Joy (pseudonym), a Grade 4 teacher at a previously advantaged school with previously disadvantaged learners. The excerpt from her second lesson provides an illustration of a "coherent with limitation pathway" episode:

> *The teacher had an NL on the board with multiples of nine from 0 to 99, which she counted. The teacher then wrote 28 ÷ 9 = $\square$ r $\square$ on the board. The teacher then said, "that she will look at twenty - eight on the NL" and pointed at 28 in the number sentence.*
>
> *The teacher then asked whether twenty-eight appears on the NL. The learners responded, "no". The teacher accepted that 28 did not appear on the NL and then asked, "which is the nearest to twenty – eight?". The learners responded, "twenty – seven", which the teacher accepted.*
>
> *The teacher proceeded to make an asterisk beneath the 27 and stated, "how many nines". She then said, "which means I have three sets of nine" and wrote 3 above the last jump. She then noted that although she wrote 3, she did not go up to twenty – eight but twenty – seven. She then added 3 into the number sentence, she asked the learners how much she must jump to get to 28, and the learner stated "one".*
>
> *The teacher then mentioned that the 1 was the remainder and the teacher wrote 1 into the second block 28 ÷ 9 = 3 r 1*

28 ÷ 9 = □r □ (Sr 1)	"Show me where 28 and 9 on the NL". (Sr 2)	"What is number nearest to 28?" (Sr 3)	"Twenty -seven". (Sr 4)	"How many sets, how many nines?" (Sr 5)
"Grouping" (Sd 1)	28 ÷ 9 = □r □ (Sd 2)	"Show me where 28 and 9 on the NL". (Sd 3)	"What is number nearest to 28?" (Sd 4)	"Twenty - seven". (Sd 5)

"3 sets of nine and wrote three on top of the third jump". (Sr 6)	"28 ÷ 9 = 3" (Sr 7 repeat of Sr 1 with 3 as answer)	"How much must you jump to get to 28?" (Sr 8)	Learner said: "One". (Sr 9)	"28 ÷ 9 = 3 r 1" (Sr 10 repeat of Sr 6 with 3 r 1 as answer)
"How many sets, how many nines?" (Sr 6)	"3 sets of nine and wrote three on top of the third jump". (Sd 7)	"28 ÷ 9 = 3" (Sd 8)	"How much must you jump to get to 28?" (Sd 9)	Learner said: "One". (Sd 10)

Figure 15: SP diagram—Joy, G4, Lesson 2, Episode 2

In this pathway a grouping situation was implicitly requested, produced and described through connected sequences of signifiers with limitations—an appropriate written number sentence and NL produced through what is described in Chapter 2 as skip counting, and then a number sentence that included the outcome with a remainder. Mathematically all the signifier - signified pairings are mathematically justifiable in this pathway with a limitation that arises from the method of producing the answer that works only in the division task or episode the teacher provided in the lesson. For example, if the division task were *26 ÷ 9 =*, the *nearest* multiple of nine would be "27", and therefore, the answer would be incorrect. Instead, the teacher should have asked what multiple of nine on the NL is smaller than the dividend "28" in order for the explanation to be generalizable. In other words, the imprecise mathematical language showed the limitation. Episodes like this can be described as "coherent with limitations" in their form.

The excerpt below is drawn from the teaching of one of the teacher sample—Hazel (pseudonym), a Grade 3 teacher at a previously disadvantaged school with previously disadvantaged learners. This excerpt from her third lesson provides an illustration of a coherent with limitation pathway episode:

> *The teacher wrote the following multiplication calculation on the board 6x2=12, 7x2=14, 8x2=16. The teacher then asked, "what is the opposite of multiply", and the learners responded "divided". She then asked, "how do we write these multiplication calculations as division calculations and the learners responded 12 ÷ 2 = 6, 14 ÷ 2 = 7 and 16 ÷ 2 = 8".*
>
> *The teacher then asked, "can you build another division number sentence with the same division calculation? What is the first rule I taught you?"*
>
> *The learners responded, "the biggest number goes first". The teacher converts 12 ÷ 2 = 6, to 12 ÷ 6 = 2, 14 ÷ 2 = 7 to 14 ÷ 7 = 2 and 16 ÷ 2 = 8 or 16 ÷ 8 = 2. The teacher said, "all the same number sentences, it is about our arrangement and pattern of numbers". She said that "times is the opposite of divide" and that "to check the answers of the divided sums, we use opposite operation".*
>
> *The teacher stated, "what is tricky is the number that must be used". The rule is when we do division, our answer is always smaller; when we do multiplication, the answer is bigger".*
>
> *The teacher referred to 14 ÷ 2 = 7, 14 ÷ 7 = 2 and 16 ÷ 2 = 8, and 16÷ 8 = 2 to highlight how the quotient is multiplied by the divisor to produce the dividend.*

6x2=12 7x2=14 8x2=16 (Sr 1)	Teachers' utterance: "the rule is when we do multiplication". (Sr 2)	Teacher's utterance: "the answer is bigger". (Sr 3)
"Rules of multiplication" (Sd 1)	6x2=12 7x2=14 8x2=16 (Sd 2)	Teachers' utterance: "the rule is when we do multiplication". (Sd 3)

6x2=12 7x2=14 8x2=16 (Sr 1)	Teacher's utterance: "What is the opposite of multiplication". (Sr 2)	Teacher wrote "divided". (Sr 3)
"Multiplication calculations" (Sd 1)	6x2=12 7x2=14 8x2=16 (Sd 2)	Teacher's utterance: "What is the opposite of multiplication". (Sd 3)

12÷ 2= 6 and 12 ÷ 6 = 2, 14 ÷ 2= 7 and 14 ÷ 7 = 2 16 ÷ 2 = 8 and 16 ÷ 8 = 2 (Sr 1)	Teachers' utterance: "the rule is when we do division". (Sr 2)	Teacher's answered: "the answer is always smaller". (Sr 3)
"Rules of division" (Sd 1)	12÷ 2= 6 and 12 ÷ 6 = 2, 14 ÷ 2= 7 and 14 ÷ 7 = 2 16 ÷ 2 = 8 and 16 ÷ 8 = 2 (Sd 2)	Teachers' utterance: "the rule is when we do division". (Sd 3)

Figure 16: SP diagram—Hazel, G3, Lesson 3, Episode 2

In this pathway, one initial chain and three side-steps are produced highlighting three ideas: multiplication as the inverse of division, the rules for multiplication and the rules for division ideas, through a locally coherently connected sequence of signifiers with some limitations—number sentences were produced that showed the inverse relationship between multiplication and division and also showed that the same three numbers could produce different division calculations. Mathematically, all the signified-signifier pairings are mathematically justifiable in this pathway, with limitations. All these rules are limitations, even though they are locally coherent. The limitation is that division does not always make smaller and multiplication does not always make bigger. According to Tirosh and Graeber (1991), the idea that "division makes smaller" emanates from restrictions in tasks in the early grades to provide integer divisors. All the signifiers are locally connected coherently, with localization. Episodes like this can be described as "coherent with limitations" in their form.

Overall then, coherent categories with limitation pathways involve two types of limitations. The first example of a limitation is evident in the solving of a division calculation on a NL in that the mathematical language contains limitations. The second example is a limitation in how multiplication and division are understood. The limitation that the teacher illustrated was that in multiplication, the answer is always bigger, and in division, the answer is always smaller (Tirosh & Graeber, 1991).

3.7.3. Coherent with some ambiguity pathways

In coherent episodes with ambiguity, teachers' work with signifiers showed a coherent chaining across the pathways with new signifiers connecting with previous signifiers in the pathways. However, these signification pathways involve instances of some ambiguity.

The excerpt below, drawn from the first lesson taught by Joy, a Grade 4 teacher at a previously advantaged school, provides an illustration of a coherent pathway episode with ambiguity:

> *On the board, an NL in multiples of 9 from 0 to 63 was visible. The teacher placed the following flashcard with the word problem "Mary bought 27 sweets for her 9 brothers? How many sweets did each one get?" on the board.*
>
> *To illustrate the word problem, the teacher drew a figurine and named it "Mary" with a circle with dots showing "twenty-seven sweets". She then continues to mention that Mary has 9 brothers by pointing to the 9 on the flashcards.*

The teacher then wrote the numbers 27 and 9 with a square in between the two numbers. To illustrate the number of brothers, the teacher drew three figurines.

The teacher then reread the word problem on the flashcard and proceeded to insert a division symbol within the square between the numbers 27 and 9 making it 27÷ 9 =.

She then stated that the interpretation of the division symbol should be sharing. The teacher asked, "to share twenty-seven sweets among nine boys, how many sweets [does] each get? Ignoring an incorrect answer, the teacher jumped three times on the NL from 0 to 27 and counted with figures, the teacher accepts the answer "three".

She then proceeded to write 3 on top of the figurines and 3 as the answer to the division symbolic calculation. At this point, the teacher reverted but to the idea that the "three boys" should be "nine boys" by saying "three, three, three". She then stated that the answer should be "three".

Written problem "Mary bought 27 sweets for her 9 brothers. How many does each one get?" (Sr 1)	A figurine of Mary with a circle with twenty-seven sweets with nine brothers was drawn. (Sr 2)	27 $\square$ 9 = (Sr 3)	Teachers' utterance: "how many brothers?". (Sr 4)	The teacher drew three figurines. (Sr 5)
'Equal sharing' (Sd 1)	Written problem "Mary bought 27 sweets for her 9 brothers, How many does each one get?" (Sd 2)	A figurine of Mary with a circle with twenty-seven sweets with nine brothers was drawn. (Sd 3)	27 $\square$ 9 = (Sd 4)	Teachers utterance: "how many brothers?" (Sd 5)

Teacher inserts the division symbol in between the 27 and 9. (Sr 6)	27 ÷ 9 = (Sr 7)	The division symbol must be interpreted as sharing. (Sr 8)	The teacher wrote 3 on top of the figurines and then placed the answer into the symbolic calculation 21 ÷ 9 = 3. (Sr 9)	Teacher restates that it should be nine boys and not three boys. (Sr 10)
The teacher drew three figurines. (Sd 6)	Teacher inserts the division symbol in between the 27 and 9. (Sd 7)	27 ÷ 9 = (Sd 8)	The division symbol must be interpreted as sharing. (Sd 9)	The teacher wrote 3 on top of the figurines and then placed the answer into the symbolic calculation 21 ÷ 9 = 3. (Sd 10)

Figure 17: SP diagram—Joy, G3, Lesson 1, Episode 15

In this pathway, a sharing situation is requested, produced and described through a coherent sequence of signifiers with some ambiguity—an appropriate word problem, a diagrammatic signifier of the word problem produced with the use of a NL, through what I have described as more progressive use of signifiers, and then the completion of a number sentence. Mathematically, most of the signifiers – signified pairings - are mathematically justifiable in this pathway. However, an incorrect signifier was produced for the signified, with no acknowledgment of an incorrect signifier, but with the ambiguity subsequently corrected and filled in. In other words, in drawing "three men" as opposed to "nine men", the teacher introduced a disconnect between the new signified three brothers and the initially signified "nine boys". However, towards the end of the signification pathway, the teacher changes the signification from "three boys" to "nine boys". The ambiguity became evident because the teacher initially accepted an incorrect signifier; however, as the episode developed, she corrected the incorrect signifier.

The excerpt below, drawn from the teacher Cathy's third lesson, provides an illustration of a coherent pathway episode with ambiguity:

> *The teacher placed a transparency on the overhead projector bearing an image of four squares with two biscuits in each square and one biscuit on the outside of these squares.*
>
> *The following written questions are beneath the image. Between how many dogs are the biscuits divided? How many biscuits does each dog get? How many biscuits are there in total?*
>
> *Initially, the teacher asked the learners, "between how many dogs were the biscuits divided?" The learners answered, "four".*
>
> *The teacher then asked again, "between how many dogs were the biscuits divided?" and another learner stated "two" and the teacher also accepted the answer.*
>
> *The teacher asked, "in how many groups have we put them?". Learners responded, "four groups". The teacher then asked, "who can give me a number sentence?" Counting in multiples of two and added on one, the teacher and learner established that there were nine biscuits altogether.*
>
> *The teacher then asked, "nine divided by two equals" and the learners answer "four one remainder".*

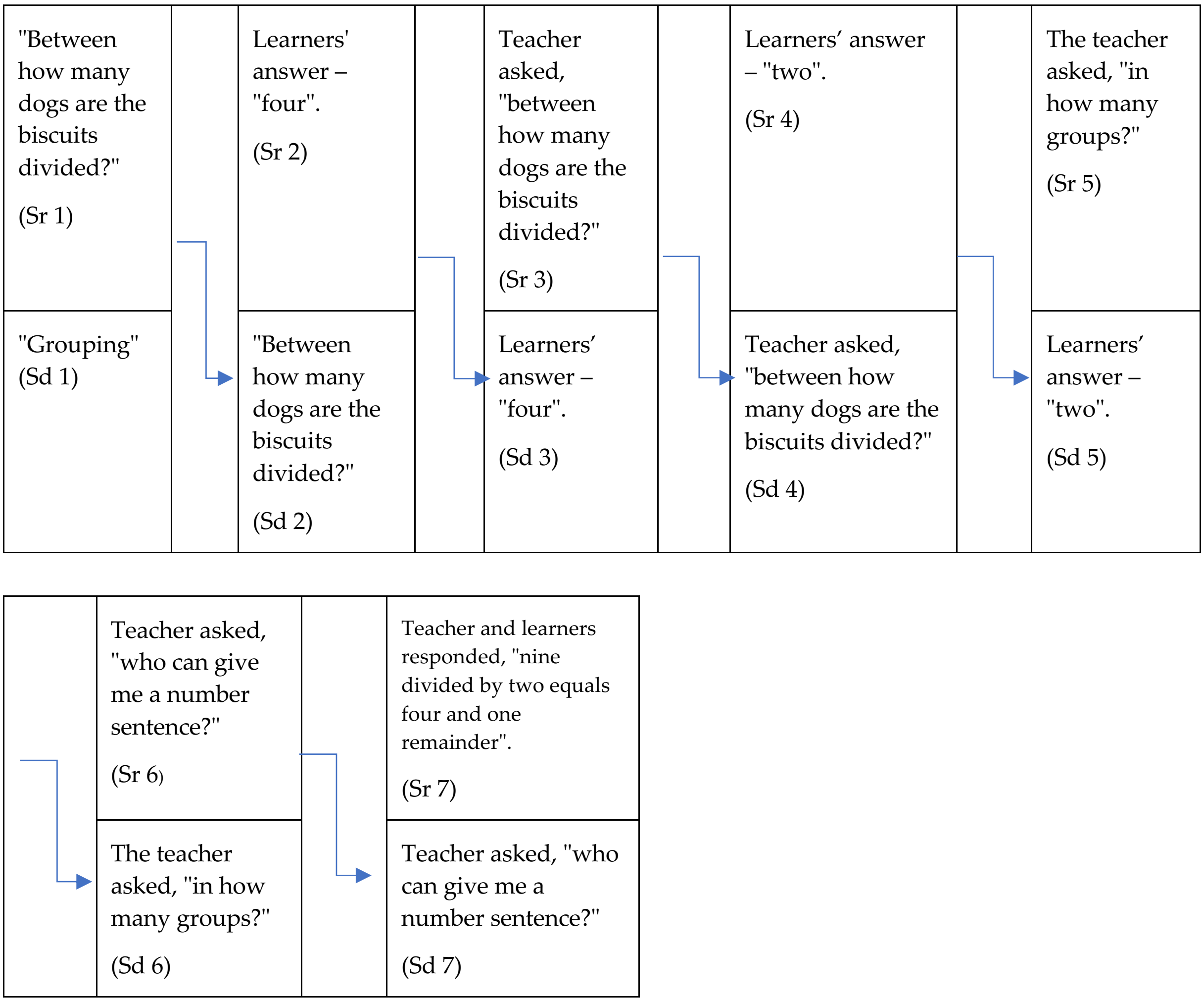


Figure 18: SP diagram—Cathy, G3, Lesson 3, Episode 8

In this pathway, a grouping situation is requested, produced and, described through a coherently connected sequence of signifiers with ambiguity—an appropriate written word problem with a diagrammatic signifier was used as a more sophisticated signifier (Thompson, 2005) and then produced a number sentence that includes the outcome that nine biscuits are divided by four dogs with a remainder of one.

Mathematically, most of the signifier-signified pairings are mathematically justifiable in this pathway, and most of the signifiers connect coherently with a grouping situation linked to "between how many dogs are the biscuits divided". The ambiguity is seen in the teacher producing two signifiers for the images inside the squares—as "dogs" (signifier) and also as "biscuits" (signifier). Within the signification pathway of this division task different signifiers for the same signified led to multiple meanings and associations and possible ambiguity for learners. Therefore, this episode was placed in category three of the signification pathway framework. However, it is evident from the image that both interpretations are viable. Episodes like this can be described as "coherent with some ambiguity" in their form.

The excerpt below, drawn from the teacher Mary's first lesson, provides an illustration of a coherent pathway episode with ambiguity:

> *The teacher placed the flashcard on the board with the symbolic number inscription 21 ÷ 3 = . She then affirmed the solving of the symbolic number sentence inscriptions should occur through sharing.*
>
> *The teacher then asked, "twenty – one divided by three equals to?"*
>
> *Five learners responded, "seven", and the teacher added "7" to the symbolic division number inscription on the flashcard.*
>
> *The teacher asked, "how many big circles can I use to get the answer?" The learners responded and said, "seven big circles".*
>
> *The teacher then drew seven big circles in a row as the learners counted them. She then asked, "and into the big circles, how many sweets must I put". The learners answered "three sweets".*
>
> *The teacher proceeded to add three small circles at a time into each of the seven circles and counted the number of groups of three through multiples of three.*
>
> *She then asked what the answer would be, and the learners responded, "seven", and the teacher counted the big circles.*

$21 \div 3 =$ (Sr 1)	Teacher's utterance: "the number sum must be solved through sharing". (Sr 2)	Teacher says, "twenty - one divided by three equals?" (Sr 3)	$21 \div 3 = 7$ (S4 repeat of Sr 1 with *7* as answer)	The teacher asked, "how many big circles can I use to get the answer?" (Sr 5)
"Equal sharing" (Sd 1)	$21 \div 3 =$ (Sd 2)	Teacher's utterance: "the number sum must be solved through sharing". (Sd 3)	Teacher says, "twenty - one divided by three equals?" (Sd 4)	$21 \div 3 = 7$ (Sd 5)

Learners' answer, "seven big circles". (Sr 6)	Teacher drew seven big circles. (Sr 7)	Teacher asked, "and in the big circles how many sweets must I put". (Sr 8)	The learners answered, "three small circles". (Sr 9)
The teacher asked, "how many big circles can I use to get the answer?" (Sd 6)	Learners answered – "seven big circles". (Sd 7)	Teacher drew seven big circles. (Sd 8)	Teacher asked, "and in the big circles how many sweets must I put". (Sd 9)

The teacher counted all the small circles using multiples of three. (Sr 10)	Teacher asked, "what is answer?" (Sr 11)	Learners' answer, "seven". (Sr 12)	Teacher counted the seven big circles. (Sr 13)
The learners asked, "three small circles". (Sd 10)	The teacher counted all the small circles using multiples of three. (Sd 11)	Teacher asked, "what is answer?" (Sd 12)	Learners' answer, "seven". (Sd 13)

Figure 19: SP diagram—Mary, G3, Lesson 1, Episode 5

In this pathway, an initial equal sharing situation is requested, produced and described through a coherently connected sequence of signifiers with some ambiguity —an appropriate written number sentence with an answer, a row of circles produced through what I have described in Chapter 2 as grouping action, and then a completed number sentence.

Mathematically, most of the signifier-signified pairings are mathematically justifiable in this pathway, and all signifiers initially connected coherently with an equal sharing situation linked to $21 \div 3 =$. However, as soon as the action was enacted, a grouping interpretation of the symbolic number sentence produced a shift from a sharing model of division to a grouping model of division. The ambiguity occurred because the shift between the division models was not acknowledged and thus this episode belongs in Category Three. Episodes like this can be described as "coherent with some ambiguity" in their form.

Overall, coherent categories with ambiguity pathways are generally coherent, with some instances of ambiguity. In the first example of ambiguity (Barwell, 2005) the teacher initially accepted an incorrect signifier; however, as the episodes developed, she corrected the incorrect signifier. In the second, the ambiguity resulted from one signified having two competing signifiers. In the last example of ambiguity observed within these signification pathways there was a shift between division models without acknowledgement of the shift in division models.

3.7.4. Incoherent pathways

In incoherent episodes, teachers' work with signifiers showed, in some instances, incoherence in the chaining of signifiers across the pathways. Incoherence occurs when the teacher's answer is incorrect or when a signification pathway involves incorrect signifiers at some points, and these incorrect signifiers are not subsequently corrected.

The excerpt below, drawn from teacher Mary's first lesson, provides an illustration of an incoherent pathway episode with ambiguity:

The teacher placed 15 ÷ 3 = on flashcard and places on the white board.

Teacher told learners to make groups of three with counters.

She then asked: "What is the answer?"

Learners answered: "Five!"

Teacher then said: "How many big circles?"

Learners said: "Five".

Teacher then drew four circles with three dots in each.

15 ÷ 3 = (Sr 1)		Teacher's asked, "Make groups of three?" (Sr 2)		Teacher asked, "What is the answer?" (Sr 3)		Learners answered, "five". (Sr 4)		Teacher then asked: "How many big circles?" (Sr 5)
Sharing problem (Sd 1)		15 ÷ 3 = (Sd 2)		Teacher's asked, "Make groups of three?" (Sd 3)		Teacher asked, "What is the answer?" (Sd 4)		Learners answered, 'five'. (Sd 5)

	Learners answered, "Five". (Sr 6)		Teacher then drew four circles with three dots in each. (Sr 7)
	Teacher then asked: "How many big circles?" (Sd 6)		Learners answered, "Five". (Sd 7)

Figure 20: SP diagram—Mary, G3, Lesson 1, Episode 8

In this excerpt, a bald calculation to be solved is requested, and it is described as incoherent because an incorrect signifier is not corrected. So, all the parts of producing the answer are correctly described, as explained in Chapter 2 when working with symbolic signifiers and diagrams. The only problem is that the drawing of the diagrams was incorrect to signifier the answer. The answer should be five circles with three dots in each. Mathematically, most of the signifier-signified pairings are mathematically justifiable in the pathway, however, incoherence is evident in that one incorrect signifier was not corrected by the end of the signification pathway. Episodes like this could be described as "incoherent" in their form.

Overall, incoherent signification pathways are identified when a teacher produces an incorrect signifier and this is not corrected by the end of the signification pathway.

3.8. Conclusion

In this chapter I have argued that key concepts from semiotics can be used to make sense of teachers' mathematical discourse when they are explaining division tasks. As illustrated in the analysis of excerpts from teaching episodes presented in this chapter, analysis of teacher's sequencing of signifiers and signifieds can lead to identification of mathematically coherent explanations and to identification of explanations in which there are limitations, ambiguity or errors. In analysing the sequence of signifiers in a teaching episode it is important to determine what mathematical associations are produced for the learners. The signification pathway framework developed during this research is useful for this purpose.

In the next chapter, I discuss the methodology employed for collecting and analysing data from six teachers as they taught division in their respective classrooms

4.1. Introduction

In Chapter Three I discussed several key concepts in semiotic theory which have informed the signification pathway framework developed for this study and which enable analysis of a series of teaching episodes. Identification of signification pathways enabled me to understand the nature and extent of coherence in the teaching of division and also to notice whether or not there were instances of progression, for example in calculation strategies and increasing number ranges. As the focus of this chapter is the methodological choices made for the study, it begins with a restatement of the research questions:

a) What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G3 division teaching in the three schools?

b) What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G4 division teaching in the three schools?

c) What do any differences seen between G3 and G4 in patterns of working across the signification pathway framework categories suggest for improving G3 and G4 teaching of division in coherent and well-aligned ways?

The chapter begins with a general overview of research paradigms, of features of qualitative research and of case studies. The sampling procedures and the methodological approach chosen for the study are then detailed. The grounded theory approach to data analysis that led to the construction of the signification pathways framework is then outlined, before I turn to an elaboration of the four categories within this framework: (i) coherence, (ii) coherence with limitations, (iii) coherence with some ambiguity, (iv) incoherence.

4.2. Research paradigms

Historically quantitative and qualitative paradigms have framed methodological approaches to research that were seen as largely incompatible and not used within the same research project. Quantitative researchers typically focus on what factors or variables might cause certain results and they carry out tests to either support or reject their hypotheses (Anderson & Arsenault, 2005); Leedy and Ormrod (1993, p. 143) note that "the whole process is cold, calculating, deductive logic—from the positing of a hypothesis to the supporting or not supporting it". By contrast, qualitative researchers have traditionally emphasized process and meaning with their main focus being on insight, discovery, and interpretation rather than hypothesis testing (Noor, 2008). However, there has been increasing use of combinations of qualitative and quantitative methods in contemporary research. Dowling and Brown (2010, p. 89) note that "studies that combine or mix qualitative and quantitative research techniques fall into a class of research that is appropriately called mixed methods research or mixed research". This study is located within a qualitative research paradigm, with some quantification included as a means for understanding the prevalence of particular kinds of phenomena. Some of the key features of qualitative research are discussed briefly in the next section.

4.2.1. Qualitative research

Qualitative research enables the investigator to explore, in detail, a small number of instances or examples that require depth of exploration rather than breadth (Merriam, 1998). In such investigations, qualitative researchers aim to record the experiences or practices of people in as much detail as possible. These may include how they think, what they say or do, and why they say or do these things. Such approaches permit the researcher to examine the participants' particular practices or experiences, and issues related to these, and to open possibilities for understanding these issues in different, or expanded, ways from those already available. Ogegbo, Gaigher, and Salagaram (2019) have noted that qualitative research approaches are particularly useful for studying classroom instruction in South Africa because these approaches are able to identify teaching and learning gaps which could be used in further planning and improvement of classroom instruction. Given that the qualitative paradigm usually depends on insights gathered from a small number of cases, I next outline the nature of case study research and then describe the cases selected for this study.

4.2.2. Case study research

Lincoln and Guba (1985) define a case study as an in-depth study of interactions of a single instance in an enclosed system. They suggest that the focus of a case study is a real situation, with real people in an environment often familiar to the researcher. Silverman (2004) describes case study research as involving a method in which one designs and collects data using procedures related to the study. Yin (1994) suggests that case study research can strengthen the credibility of the process of data collection, while Opie (2004, p. 71) notes the importance of the ethics of data gathering being made explicit to all involved in the case study. The data must also be presented transparently in ways that enable ready re-analysis, with this being particularly important in case studies where "negative instances" or aspects of practices or experiences may be reported. The researcher also has to endeavour to acknowledge their biases within the data collection and analysis of data, making the relationship between the claims and the supporting evidence explicit. Opie also emphasizes the importance of separating the interpretation of the data from its description.

Case studies can involve multiple cases of the phenomenon of interest – in this study, the nature and extent of coherence in the teaching of division. Multiple case—studies involve collecting and analysing data from several cases. Merriam (1998, p. 40) notes that "the more cases included in a study, and the greater the variation across the cases, the more compelling an interpretation is likely to be". The comparison of multiple case—studies focused on a particular research interest can create more precise, more valid, and stable findings. Noor (2008, p. 1604) notes that "the development of consistent findings, over multiple cases, can then be considered a very robust finding".

4.2.3. The multiple case study

The multiple "cases" in this study of the teaching of division in South African primary schools are those of six participating teachers drawn from three Johannesburg government primary schools situated in a range of socio–economic locations. One Grade 3 and one Grade 4 teacher was selected in each of the schools on the basis of willingness to participate in the study following initial discussion of the study's aims. These schools are titled Schools A, B and C, to protect anonymity. During informal visits preceding this study, teachers in Schools B and C had indicated concerns about disjunctures between numeracy performance in Grade 3 and mathematics performance in Grade 4, with a "Grade 4 dip" being widely reported in the context of school and annual national assessments in the broader South African context (DoE, 2014). School A had not reported this kind of Grade 4 dip. This choice of schools as the sample provided potential for investigating possible reasons for the differences in mathematics performance in Grade 4 between School A and Schools B and C.

While all three schools were identified as "Quintile 5" or relatively advantaged fee-charging government schools in the South African context at the time of data collection, two of the three schools (Schools B and C) were labelled as "underperforming" by the province in terms of learning outcomes in language and mathematics. Fee levels provide a useful proxy indicator of the three schools' socio-economic background. School A served a mix of "black", "coloured", "indian" and white learners and charged R 983.00 per month. School B served an almost entirely black African learner population and charged R375 per month in fees. Schools A and B were located in historically white suburbs. School C, located in a historically "coloured" township, charged R78.00 per month for school fees at the time of data collection. In School A fee collection levels stood at 80 % of the learner body in the same period. In Schools B and C, fee collection levels stood at around 50% of the learner body at the time of the study.

Each of the six teachers' instruction related to division constitutes a "case" within the overall multiple-case study.

4.2.4. Sampling of teachers within the three schools

The second level of sampling in this study was related to the selection of the teachers whose division instruction was studied within each of the schools. Having selected the school sample as outlined above, I focused on teachers who were familiar with teaching Grades 3 and 4 Mathematics. Drawing from the more experienced teacher pool in each grade, a G3 and a G4 teacher from each school was approached to ask if they would be willing to participate in the study, with its focus on the observation of their teaching of division. All teachers approached gave their informed consent for participation. Thus, three G3 teachers, with work located within the South African Foundation Phase and three G4 teachers, with work located in the Intermediate Phase became the opportunistic teacher sample for this study. From my side, the selection of the six teachers was based on their familiarity with the teaching of mathematics at their grade level. The teacher selection was also based on their willingness to participate in the study and them allowing me to observe them teaching division at different times of the year.

Based on the criteria listed above, three Foundation Phase (FP, Grades 1-3, grade-appropriate learners aged 7-9 years) teachers (Cathy, School A; Mary, School B; Hazel, School C), and three Intermediate Phase (IP, Grades 4-6, grade-appropriate learners aged 10-12 years) teachers (Asha, School A; Joy, School B; Sam, School C) were involved in this study. All six teachers had taught for over twenty years in their respective grades. In Table 8, the school and Grade of teaching of all six teachers is summarised.

Table 8: Background characteristics of the sample teachers and schools

Schools	Participants	Gender	Teaching Grade	Number of lessons observed	Year lessons were observed	Type of school	Fees/month
School A	Cathy	Female	Grade 3	4 lessons	2011	Historically advantaged school with white, black, coloured and indian advantage learners	R 983.00
	Asha	Female	Grade 4	3 lessons	2012		
School B	Mary	Female	Grade 3	4 lessons	2011	Historically advantaged school with only black learners.	R 375.00
	Joy	Female	Grade 4	3 lessons	2012		
School C	Hazel	Female	Grade 3	5 lessons	2011	Historically dis - advantaged school with black and coloured learners	R 78.00
	Sam	Female	Grade 4	3 lessons	2012		

The observation period covered two years. In 2011, I observed G3 division teaching in all three schools. In 2012, I observed G4 division teaching in all three schools. I followed the same cohort of learners with the aim of exploring whether teachers in G4 would build on what was taught about division in G3.

4.3. Data sources and data collection methods

Qualitative inquiry requires in-depth data collection in relation to participants' experiences or practices (Merriam, 1998). The key source of data in this study was lesson observation. Data from lesson observation of the division lessons taught by the six case study teachers were collected in the following way:

- Video-recordings of instruction in lessons that were then observed several times to create verbatim transcriptions that included teachers' words, division actions and gestures, their inscriptions and division calculation strategies and number ranges.

Observations are a technique used to observe people in a particular context. A researcher observes settings or physical environments, social interaction, physical activities, non–verbal communications, planned and unplanned activities, and interactions (Hopkins, 2008). Such observation usually entails making detailed notes of behaviour, words, and actions. Video recording can capture the richness and complexity of classrooms for later analysis (Andrews, 2009). Borko, Jacobs, Eiteljorg, and Pittman (2008) note that the use of video for educational purposes brings new and imaginative perspectives to almost any subject matter. Santagata et al. (2011) note that because video can be played over and over and accessed digitally, it allows for a depth of reflection and analysis not possible during live observations. Teaching is a cultural activity, and cultural routines are more easily unveiled when the teaching process is slowed down and critically analysed. Santagata, Zannoni, and Stigler (2007) also emphasize the need to be clear about whether the excerpts drawn from video data are typical or unusual excerpts.

I worked in a non-participant observation role that allowed for full focus on observation activity during the lessons (Merriam, 1998). The videotaped lesson observation data source detailed above allowed me to create a faithful record of the signifiers and explanations connected with the signifiers produced by each teacher. This provided a rich data base of teachers' work with division tasks that enabled me to capture and analyse the signification pathways incorporated in each teacher's division instruction. The video-recording captured details of what the teacher was pointing at as she spoke and also her gestures, in a degree of detail that could not be achieved by observation notes alone.

Limitations of lesson observation have been noted in the literature – for example, dangers of bias in "subjective" viewing, curiosity about the researcher as a "newcomer" in the classroom, and the possibilities of the researcher presence influencing the activity in the classroom (Merriam, 1998). However, given this study's focus on instruction, lesson observation data was essential. Through communicating my interests openly and building respectful and non-judgemental relationships with the teachers, and with my presence across a number of lessons with each teacher, these dangers were addressed in the research processes in ways that led to data that I felt was a fair reflection of the participating teachers' normal repertoire of practices related to division instruction – addressing the point made by Sangatata et al (2007) about typicality. Grounded approaches to working with the data (detailed in Section 4.5.1) also led to "coherence" as an emergent lens through which to view the data. The development of the categories of the signification pathways framework, with indicators for each category, also helped to maintain consistency and transparency in the ways in which the data was subsequently viewed.

In addition to the lesson observations, I also used the FFL curriculum document as an important background data source. The overview analysis of this curriculum's presentation of division, detailed in Chapter 2, provided a picture of the curricular support context available to the teachers in this study. As noted already, the instances of ambiguity in this document observed in the shift between sharing and grouping situations without acknowledgement of this shift provided some forewarning of potential difficulties with division instruction for the teachers.

Hopkins, Burton, Brundrett, and Jones (2008, p. 84) describe working with documents such as curricula as "documentary evidence". They also state that "the main use of documents in classrooms research is that they provide a context for understanding a particular curriculum or teaching method". Documents associated with curriculum or other educational concerns can illuminate the rationales and purposes that may underlie teachers' actions. Therefore, study of these materials can provide background information and understanding of issues that would not otherwise be available. Researchers often engage in document-based research to investigate the kinds of meaning constructed through texts (Lanshear & Knoebel, 2004). This study examined curriculum documents (FFL) to identify specifications that could shed light on how teachers in G3 and G4 selected sequences of signifiers, incorporated division calculations strategies, chose number ranges within their division tasks and explained division tasks in particular ways. The FFL curriculum document served as another lens through which to look at the six case studies as teachers produced their division instructional work, providing an avenue for some triangulation (Leedy & Ormrod, 1993) of the video-observation data.

4.4. Data processing approaches

The first stage in making sense of the data was to produce transcripts of the video-recorded lesson episodes. For this to happen, I transcribed verbatim what I saw in the video. The literature had highlighted certain key areas that are critical in the teaching of division, and in particular, the roles and progressions of signifiers: linguistic signifiers expressed through all the spoken and written words of both the teachers and the learners; action-based signifiers in the form of teachers' gestures and actions related to sharing and grouping; diagrammatic signifiers used in the various division tasks; and lastly, symbolic signifiers. Initially, I created verbatim transcripts that included details of all these signifier forms – teacher talk and teacher-learner interactive talk and which recorded teachers' use of actions/gestures and manipulatives, and their written inscriptions in the form of words, diagrams and symbolic language. The tasks focused on were also transcribed incorporating the number ranges that each teacher worked with, and the transcript included details of the calculation strategies that were used. Tasks were often explicitly stated by teachers but in some instances, had to be inferred.

The chaining of signifiers occurred in the context of tasks, so I then proceeded to break up each lesson transcript into "chunks" based on division tasks. I came to view the start of a "new" division task as marked by the teacher drawing the learners' attention to a new focus related to division. These "new" foci occurred in a number of different forms – for example: division problems set either as word problems or as division number sentences; asking or telling learners about the names of parts of a division number sentence (e.g., dividend); working with division diagrams; skip counting as a fluency needed for working with division. I came to refer to each of these chunks as an episode. I therefore ended up with a set of transcripts for each lesson taught by each teacher, divided up into episodes based on instruction related to different tasks in the teacher's task sequence. In episodes containing the same example (Asha L1: E7 – E9) the teacher dealt with tasks as separate, and therefore they were demarcated as separate episodes. Instructional work occurred in whole-class teaching episodes and in "mat-work" episodes where the teacher worked with a sub-group of learners. In some lessons, there were segments of individual seatwork or learner group work with no input from the teacher such as reading division facts from a division sheet or reciting jumps in multiples on a NL. In the Grade 4 classes, there was also often a "stand-alone" mental maths segment at the start of the lesson focused on a range of different topics. Given my focus on division instruction, these segments of working were noted, but omitted from the analysis. In addition, when working with small groups, the teacher would repeat the same example and explanation with a new group of learners and these episodes were also left out. In certain instances, multiple symbolic calculations were described as "one episode". The teacher provided an immediate answer without elaborating/ explaining how they arrived at the answer. In a different instance, another episode can be described as "one episode". This occurs when a teacher uses another episode to create clarity for the initial episode. As noted already, episodes involving counting in multiples were also marked, but given that they were not focused directly on division, these episodes were also omitted from the analysis. The outcome of this process was that it gave me a "thick description" (Geertz, 1973) of the division instruction in each lesson taught by each teacher in episode format. Each episode of division instruction constituted a unit of analysis.

The processed data across all six teachers led to the generation of 180 episodes of which 61 were uncoded for the reasons outlined above. The breakdown of episodes across the six teachers, with detail of task foci and the nature and extent of coherence, is presented in the analysis chapters that follow.

4.5. Data analysis

While the literature pointed to key ideas in division, and approaches to teaching division that took progression into account, my early lesson observations and subsequent reading of the episode-based transcripts pointed to instances of ambiguity and incoherence in the ways in which teachers' talk, inscriptions and explanations unfolded. These unexpected phenomena led me towards more "grounded" approaches to examining the data, rather than being led solely by the literature-based categories. The background to grounded theory, and the ways in which I followed grounded theory-based approaches in this study are detailed in the next section.

4.5.1. Grounded theory-based approaches

Grounded theory is defined as a theory developed to fit the situation under study. By fit, I mean that the categories must be readily (not forcibly) applicable to and indicated by the data under study. "Fit" also indicates that categories must be meaningfully relevant and be able to provide insight into the instructional practices under study (Andrews, 2009).

Grounded theory has been used in both qualitative and quantitative settings (Glaser, 2010, p. 1). Important debates feature within grounded theory related to research methodology. On sampling, some grounded theorists argue that the emerging theory should drive sampling, while others suggest that a purposive sampling may rest on some pre-existing theory or literature (Glaser, 2010). The sampling of schools in this study was purposive in the inclusion of data across Grades 3 and 4 based on discussions of disjuncture in the context of an earlier study, and purposive also in terms of schools with different socio-economic profiles, as this indicator pointed to the likelihood of variation in the extent of disjuncture initially, but also – subsequently – of coherence in instruction given the association between socio-economic background and South African primary school teachers' mathematical knowledge (Venkat & Spaull, 2015). Some grounded theorists also submit that the researcher should not conduct a literature review before the data collection and analysis (Cutcliffe, 2000). However, other theorists have argued that a general reading of literature is important for providing researchers with a sound background understanding of the subject area (Glaser, 2010). I did work with the literature on division and its teaching ahead of data collection but supplemented this reading with a range of additional literature as findings related to coherence began to emerge. Thus my research lens has been informed by the emerging literature on coherence in the field with the data from this study feeding into identifying the nuances of the phenomena being studied.

As in other qualitative research, the data for a grounded theory analysis can come from various sources. Grounded theory works with specific procedures for data collection and analysis, although there is flexibility and latitude within these (Cutcliffe, 2000, p. 1477). Data collection and analysis are interrelated processes in that the data collection process, and initial analysis are undertaken in overlapping fashion to shed light on emerging ideas. In my study, "coherence" emerged as a central feature of interest in the data initially collected. While this emergent theme did not affect my subsequent collection of data, the variation among cases in the dataset from the lesson observations, broken down into episodes, helped me to identify patterns related to the nature and extent of coherence. Therefore, coherence as a concept related to the teaching of division became visible through the process of revisiting the data, and subsequently, became central to the development of a framework for analysing the data.

Grounded approaches have been used in other studies in which semiotic theory informs the determination of analytic categories (Smyrnaiou, Sotiriou, Sotiriou, &

Georgakopoulou, 2017). The focus of this study employed semiotics as an analytical tool for investigating the teaching of division. From the time at which the initial data was collected, I used a grounded approach to look at some of the similarities and differences between episodes, which drew my attention to the nature and extent of coherence in the signification pathways. Cobb et al. (1997) note that semiotics provided them with a way to make sense of whether the instructional discourse was clear and coherent. The "grounded approach" was influential in the formation of the "signification pathways framework" in this study, in which the categories are based on the nature and extent of coherence of signification in relation to teaching the topic of division, which is viewed as entailing a mathematical network of signifiers.

Grounded theory provides an opportunity for a system of categorization to emerge. A system of categorization can be broken down into stages, phases, or steps. Corbin and Strauss (1990) note that the capturing and categorization of these stages, phases, or steps as a phenomenon is referred to as "memos". These "memos" assist in the formulation of, and revision of, theory during the research process. They suggest that categories should be continuously revisited until these categories identified can be proven valid for all the evidence concerning the phenomena under study, as gathered in repeated observations or documents.

Coding is pivotal in some approaches to data analysis. In taking a grounded theory approach to data analysis, there are three basic types of coding: open, axial, and selective. Open coding is a way of interpreting the data by breaking the data down analytically (Corbin & Strauss, 1990). The purpose of open coding is to provide new insights through thinking about the phenomenon reflected in the data. Axial coding focuses on putting together data in new ways by making connections between categories, in other words, looking at the patterns that emerge. Selective coding refers to the process of unifying all categories around the "core" category – "extent of coherence" in this study.

Open coding develops categories of information until the additional data stops offering new insights about the phenomenon. After each teaching episode was identified, I started to look at what I have termed signification pathways (Chapter 3). In other words, I looked at the sequence of signifiers produced by the teacher. In looking at this sequence within each episode I began to observe disruptions to coherence in the instructional discourse. I then began to cluster episodes on the basis of the ways in which they were similar and different to each other in terms of coherence. This method of looking at ways in which episodes are similar to and different from each other is referred to as the "constant comparison method" (Glaser, 1965). After revisiting episodes several times and clustering them based on similarities related to the extent of coherence, I arrived at four clusters. I had reached saturation point which is an indication of the end of the open coding phase.

Axial coding came to the fore at this point as I was noticing connections within and between different clusters of episodes. I started to make connections about how coherence was affected by different situations and division actions. I then observed how teachers worked with different signifiers and division calculations within episodes. I also looked at how different number ranges informed the selection of signifiers and division calculation strategies and how links between these elements, all highlighted in the literature, could be expressed in terms of extent of coherence. In making the connections within and across different clusters of episodes I began to notice the emergence of "core" categories.

Selective coding refers to the core categories within the study that are usually understood at a later stage of the research. The core category represents the central phenomenon of the study. Corbin and Strauss (1990, p.14) note that "the core category might emerge from among the categories already identified or a more abstract term may be needed to explain the main phenomenon.The other categories will always stand in relationship to the core category as conditions, action/interactional strategies, or consequences". In this study, the core category was identified as the "extent of coherence".

The sub-categories of extent of coherence that became core to my study, as detailed in Chapter 3 (see page 91) were coherent, coherent with limitation, some ambiguity, and incoherent. Following the detailed work of tracing the signifiers produced in instruction, and then coding episodes into categories, I turned my attention to ways to present the data in summary form for the purpose of disseminating findings related to the signification pathways map for each teacher, and then using these to compare instruction between the G3 and G4 teachers and also between the more advantaged Schools A and B and School C.

Following the analysis of the transcripts and the identification of the episodes and categories, I decided to present the analysis in one chapter with a focus on G3 teachers, then G4 teachers across the three schools. I present the analysis in this way for clarity since G3 and G4 teachers taught division in quite dissimilar ways.

Table 9 (which follows) shows how each research question points to data sources, data processing and analysis techniques and theoretical approaches.

Research questions	Data source	Data processing technique/ Analysis	Theoretical Framework
a. What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G3 division teaching in the three schools?	Video recording of Grade 3 teachers' instruction during the teaching of division in three schools. G1-3 and G4 FFL specifications focused on division.	Video recording was transcribed and separated into task-based episodes within each lesson. The curriculum document was analysed across grades 1,2,3 and 4 looking at progressions in signifiers, division calculations strategies and number range.	Semiotics with a particular focus on how teachers' select sequences/chains of signifiers in the teaching of division in G3.
b. What are the similarities and the differences in the nature and extent of coherence in the signification pathway/s in G4 division teaching in the three schools?	Video recording of Grade 4 teachers' instruction during the teaching of division in three schools.	Video recording was transcribed and separated into task-based episodes within each lesson.	Semiotics with a particular focus on how teachers' select sequences/chains of signifiers teaching of division in G4.

	G1-3 and G4 FFL specifications focused on division.	The curriculum document was analysed across grades 1,2,3 and 4 looking at progressions in signifiers, division calculations strategies and number range.	
c. What do any differences seen between G3 and G4 and between socioeconomic school clusters in patterns of working across the signification pathway framework categories suggest for improving G3 and G4 teaching of division in coherent and well-aligned ways?	Outcomes of above analyses form the database for this research question, with data considered now in terms of grade and socio-economic clusters	Further work with "constant comparison" approach on the basis of clustered grade and socio-economic cluster data	Grounded "constant comparison" method using the outcomes of analysis using semiotic theory.

4.6. Data Presentation

In chapter 3, I detailed and discussed the ways in which signification pathways were developed and then used to understand issues of coherence. In this chapter, my focus is on the outcomes of the analyses of extent of coherence using semiotic approaches as described in the previous chapter, and ways of producing summary versions of episodes that can be used to present the overview "story" of coherence in instruction for each of the teachers in this study. Doing this has required some summarising of the level of detail in the transcripts and in the signification pathways elaborated in Chapter 3, while retaining sufficient information to give the reader a sense of the reasons for coding episodes into particular coherence categories. In some instances, in which an episode contains two different codes, I have coded the episode based on the overarching idea. In this section, I detail the ways in which I chose to work with summary presentations, and the reasons for my choices.

Each case study, begins with a description of the socio-economic context of the school, the number of learners in a class, fees per month and quintile (the number given to a school to indicate how resources with be allocated) and the number of teaching episodes. Thereafter, I present a table which shows, across all the taught episodes, the weighting of the number of instances of sharing situations, grouping situations, bald calculations and other division related foci. In line with using a teaching episode as the unit of analysis, the summary of each episode includes the following details: the task, the initial signifier and a paraphrased description of how instruction played out in the episode which incorporates the signifiers worked with and accompanying explanations. I then analyse whether a grouping or sharing situation, bald calculation or some other aspect (e.g., definition of terms) was in focus, any division action that followed, the calculation strategy used and the number range involved. The paraphrased summary with the noting of the initial signifier, and associated actions and explanations in terms of subsequent signifiers has enabled me to interpret the analysis for extent of coherence in the episode, while also noting any other salient points. Specifically, I found that initial symbolic signifiers were more often associated with coherent pathways than situational signifiers presented in oral or written form and diagrammatic signifiers. Following the description of each episode, I include other tables that either focus on sharing or grouping situations or bald calculations with the following categories: all the initial signifiers with number range, signifier sequence and the calculation strategies referred to as "enactments" of each category (sharing/grouping/bald calculations), with appropriation discussions. Lastly, I include a final "coherence table" which captures the analysis of the episodes in terms of the signification framework This table includes the number of lessons, each category, episodes not analysed and the total number of episodes per lesson. Also, the table shows the number of episodes per category, followed by a discussion. This process reflects Wolcott's (1994) suggestion to work with description, followed by analysis that remains relatively "close" to the data, and then to bring theory and literature into play at the interpretation stage. Below, I illustrate this way of working with an example drawn from my data set for each of the categories from the signification pathways framework.

Table 10: Category 1: Coherence - School A, Term 2, Tr. Joy L1: E1

Description	Analysis	Interpretation
Task: Six sweets to be shared between three children Summary	Situation: Sharing Action: Sharing by two counters per learner	Signification Pathway Category: 1 – Coherent

Explaining that her mother was a vendor with six sweets to share between three children, the teacher gestured three people, drew "6" in the air and showed six fingers before asking: 'How many sweets does each child get?' A learner responded: "Two". Taking out six counters, the teacher asked a learner to: "share them equally between three learners". One learner came up and distributed two sweets to each of three learners.	Initial signifier: Oral word problem in the task Division calculation strategy: Immediate answer suggested a known fact for the learner, with concrete replay involving sharing by complete composite units (CCU) Division number range: Single digit dividend and divisor	All signifiers in the pathway can be traced back to previous signifiers in ways that are mathematically justifiable at all points. Other comments: The number range is much lower than that specified in FFL, possibly playing into the correct answer being stated prior to any division action or calculation.

In this episode, the flow of signifiers occurs smoothly, with the initial orally presented sharing situation producing an immediate correct answer from a learner, with the teacher then proceeding to offer counters that allow for a concrete replay, which a learner executes – again smoothly – using the "complete composite" unit sharing described in the literature review. Thus, this episode was coded as a Category 1 – Coherent example.

Table 11: Category 2: Coherent with limitations - School B, Term 2, Tr. Hazel L1: E3

Description	Analysis	Interpretation
Task: Distinguishing between multiplication and division. Summary The teacher wrote 12 x2 = on the board and asked for the answer and learners stated: "24". Teacher asked for a division symbolic calculation. The learners offered the division calculation 24÷2=12. Noting 24 as the answer to the multiplication, she then added: "Whenever you do a divided sum the biggest of the numbers always goes first".	Situation: No reference to a division situation. Action: No reference to a division action. Initial signifier: Symbolic number sentence: 12 x2 =24 converted to 24÷2=12 Division calculation strategy: Immediate answer derived from multiplication calculation. Division number range: Two-digit dividend and single digit divisor, but once again, calculation is not the direct focus.	Signification Pathway Category: 2 – Coherent with limitations. While signifiers within the signification pathway are coherent in terms of "match" with the specific example used, the phrase: that the biggest number always goes first is not applicable universally across examples. Other comments: The number range, once again, is much lower than that specified in FFL, as noted above.

In this episode, the flow of signifiers occurs smoothly, with division symbolic number sentences derived from a multiplication calculation. However, in elaborating on the placement of numbers in moving from the multiplication sentence to the division sentence, the teacher's statement of "the biggest number goes first" in relation to multiplication and division respectively, is limited in its applicability. While these statements hold true for the examples in the episode (and would hold for almost all examples that teachers would work with in Grade 3), the statement is not universally true in a mathematical sense, as described earlier in the literature. While, mathematically, this enactment could simply be read as incorrect, the fact that this statement holds true for so many of the multiplication and division examples that primary teachers deal with, including the examples in this episode, led to the coding of this (and similar) episodes as Category 2 – coherent with limitations.

Table 12: Category 3: Some ambiguity - School B, Term 2, Tr: Mary L1: E7

Description	Analysis	Interpretation
Task: Solve "20 ÷ 2 =" Summary Teacher placed a flashcard on the white board (WB) with the division calculation: 20 ÷ 2 =. She then instructed learners to make groups of two with counters. Teacher asked: "How many groups of two?" and a learner stated: "Twenty divided by two equals ten". Teacher stated: "Can you see if you are sharing with your counters it is easier". Thereafter the teacher drew ten circles on the WB and asked: "In the ten big circles how many sweets must I put?" Learners responded: "Two sweets" and the teacher drew two dots in each circle. Teacher then instructed learners to count in multiples of two and they counted in twos from two to twenty. Teacher asked: "How many big circles?" and the learners shouted out: "Ten big circles". The teacher then wrote in "10" as the answer on the flashcard.	Situation: Ambiguous Action: Gestures making groups of two with counters. Initial signifier: Symbolic signifier – bald calculation. Division calculation strategy: CCU and Skip counting to determine quantity value. Division number range: Single digit dividend and divisor	Signification Pathway Category: 3 – Some ambiguity The signification pathway involves instances of ambiguity. The bald calculation is enacted through a grouping action then linked to a sharing problem and then demonstrated in a grouping action by drawing the number of groups based on the answer and then a sharing action is enacted without acknowledgement of any distinction between sharing and grouping. Other comments: The number range is much lower than that specified in FFL.

In this episode the bald calculation (the initial signifier) could have been interpreted as both sharing and grouping at the same time. Initially, the bald calculation is enacted through a grouping action which was linked to a sharing problem. However, as the explanation unfolded a grouping problem was enacted as the number of groups were based on quotient, and then a sharing action was enacted again, with no acknowledgement of a shift from one division situation to a different division action, which literature suggests creates disruptions to coherence. This episode was therefore coded as Category 3 – coherent with some ambiguity.

Table 13: Category 4: Incoherence - School C, Term 2, Tr: Mary L1: E8

Description	Analysis	Interpretation
Task: Solving a bald calculation 15÷ 3= Summary The teacher placed a flashcard with the following bald calculation 15÷ 3= on the WB. The teacher stated: "How many big circles?" as she pointed to the "3" in" bald calculation. The learners responded: "Three". She instructed the learners to solve the bald calculation. Learners are seen making five groups of three. The teacher asked: "What would the answer be?" The learners answered: "Five". The teacher does not write five in as the answer. The teacher then asked: "And how many big circles?" The learners answered: "Five" and the teacher proceeded to draw four circles. Together with the learners the teacher counted the four circles as she pointed to each one. A learner shouts out: "Five" which the teacher did not accept. She then said: "Sweets inside the circles". The learners shouted out: "Three". She then reiterated: "Three". The teacher then drew in three small circles in each of the circles.	Situation: Action: Drawing in three circles in a big circle. Initial signifier: Bald calculation Division calculation strategy: making groups of three. Division number range: Double digit dividend and divisor	Signification Pathway Category: 4 – Incoherent The signification pathway involves mathematical incoherence. Incoherence occurred in this pathway in that an incorrect signifier offered "four circles" is not subsequently corrected. Other comments: The number range is much lower than that specified in FFL.

In this episode, an incorrect signifier was produced that was not subsequently corrected. At the beginning of the signification pathway the teacher wanted the learners to solve a bald calculation 15÷3= through making groups of three. After learners made five groups of three the teacher then accepted a learner's verbal offer of "5" as the answer for the bald calculation which she did not insert. She then proceeded to enact the solving of the bald calculation. However, it is in the solving of the bald calculation on the WB that an error appears, in that "four circles" are draw instead of five circles. The teacher then continued to draw three small circles in each of the four circles. This incorrect diagrammatic signifier is not corrected by the end of the signification pathway. This episode was coded as a Category 4 – incoherent.

4.6.1. Uncoded episodes

Uncoded episodes are episodes that were video-taped but not placed into categories 1 to 4 based on the absence of instruction – e.g. group work activities not teacher-led, mental maths and counting in multiples not linked to division or individual seatwork episodes or reading division facts from division sheet (section 4.4).

The episodes discussed above are typical of the four categories described. These categories provide insights into the nature and extent of coherence.

4.7. Validity and generalizability

Qualitative researchers aim to gain insight from people who will enrich and illuminate their understanding of actions, concepts, discourse, activities, and practices of a phenomenon. The validity of the research findings is based on the integrity and transparency of the methods of collecting, analysing and interpreting data. Validity is an issue that concerns not only qualitative research but also quantitative research. In quantitative research, generally speaking, "validity" refers to the accuracy of measurement (Lesser & Blake, 2007). Validity concerns are also linked to the "reliability" and consistency of measurement (Hammersley, 1990, p. 55), and whether results can be generalized to a larger population (Hammersley, 1990, p. 332). In qualitative research, unique situations cannot be recreated. Even in the case of the most exact replication of research methods, the research may fail to provide identical results (Bickman & Rog, 1998, p. 134). Le Compte and Preissle (1993) argue that in qualitative research, a single case or small non-random sample is chosen, where the researcher emphasizes "the particular in-depth, not to find what is generally true of the many". Validity has thus been a more significant concern for qualitative research than reliability or generalizability. Hammersley, cited in Merriam (1998, p. 208), defines validity in qualitative research based on accuracy of accounts: "an account is valid or true if it represents accurately those features of the phenomena, which … is intended to described, explained or theorize".

The "low stakes" nature of my data collection, without consequence for the teachers, coupled with observation of multiple lessons with attention to prior discussion with the teachers of the study goals helped me to promote validity in the sense of data that I believe does represent accurately the phenomenon of the division instruction activity of these teachers.

Triangulation, as stated above, can also be used to promote the validity of the research, and involves using multiple investigators, multiple sources of data, or multiple methods to confirm the emerging findings (Merriam & Tisdell, 2015). As stated already, in this study some triangulation was promoted by using video recording together with curriculum document analysis which provided detail on the policy context of teachers' work as an additional contextual feature, highlighting some of the inconsistencies that were subsequently seen in instruction.

4.8. Ethical considerations

Qualitative research often investigates the role(s) people play in a particular context. Since this research focuses on people, the researcher needs to understand the ethical issues that may impact on participants during the study (Creswell, 1998, p. 202). The rights of the participants must be protected and acknowledged across all stages of the research process (McMillan & Schumacher, 2006). For this study participants were selected based on informed consent, following ethical clearance having been obtained from the University (2011ECE010C), and permission were also gained from the provincial Education Department that the study was located in. Confidentiality, anonymity, privacy, and fairness characterized all stages of the research process, with pseudonyms used in all written reports and data kept confidential at all times (Wilson, 2009).

Ethical working was maintained throughout the research. Trust plays an essential feature in qualitative research (Roth, 2007). I was aware of the power that I could bring into the local school that could lead to a lack of trust. It was necessary for me to be transparent with the teachers, open to their suggestions, and work with them in a professional and non-threatening way. In doing this, I reinforced ethical interactions and built relationships that have continued for the duration of the study and beyond.

4.9. Conclusion

In this chapter, I have provided a description of the study's research design, given reasons for the choices I made in the case study design and participants in the study. I have offered reasons for the use of grounded theory for the analysis of the data collected, and described how a grounded theory approach to data collection and analysis was used in the study. Issues of ethics, validity and reliability have been briefly discussed.

In the following chapter, I present the findings that came from analysing episodes across the lessons of G3 and G4 teachers, placing episodes into categories-based sharing or grouping situations or into bald calculations or episodes which highlight other division related ideas. These categories have been further split into the initial signifier with number range, signifier sequence and division calculation strategies. Finally, episodes from each teacher have been categorised according to the signification pathway framework in order to examine the nature and extent of coherence in their teaching of division.

5.1. Introduction

In Chapter Three, I presented the main signification pathways with exemplifications. In this chapter, I present six case studies based on the sample of three Grade 3 and three Grade 4 teachers' explanations of division tasks involving sequences of signifiers. The aim of this analysis is to describe and discuss how these six teachers, drawn from three schools with varying socio-economic profiles, taught division.

In each case study, I provide a series of tables, for all the episodes across all the lessons taught, which include content and activities identified from the literature review as critical in the teaching of division. For each teacher, the analysis begins with an overview of the division foci and sequence across these episodes and lessons. Given this study's focus on division instruction related to division calculations, all "other" episodes within the observed division lessons - for example, episodes focused on individual work, lead-in work to division via a multiplication focus (counting in multiples or revision of multiples or reciting jumps in multiples on a NL) etc. as listed in Chapter 4.4 - were not coded. These episodes are detailed but "greyed out" in the chronological overview tables of episodes for each teacher (see Appendices A - F) to indicate their falling outside of the analysis.

For each teacher, the first summary analysis within this chapter indicates the spread across sharing and grouping situations, bald calculations and other division related concepts, with a brief discussion on the weighting of episodes. The second analysis deals with each of these division foci in turn, and how they were taught by each teacher. I include a table with extracted episodes related to sharing/grouping/bald calculations before each analysis of either sharing or grouping or bald calculations episodes. In the analysis I focus on the initial signifiers and the signifier sequences, and calculation strategies that were enacted, taking number ranges into account. Within each of the signifier sequences, I use the symbol " > ", to indicate "followed by".

For example, word problem > number sentence > answer, should read as word problem "followed by" number sentence "followed by" answer. The arrangement of the signifier sequence within tables heading enactments is based on a combination of increasing coherence and the progression of calculation strategies. The categorized episodes start with episodes:

- containing incoherence/errors in calculated answers,
- then ambiguity in calculated answers,
- followed by limitations in calculated answers,
- afterwards fundamental sharing and grouping actions on objects,
- followed by iconic sharing and grouping,
- next distribution through partial composite units,
- Group Based Distributions (GBD),
- "complete composite unit" (CCU) distribution or repeated subtraction,
- then array – based group distribution,
- followed short or long division algorithms and finally immediate recalled facts as answers.

Not all of these calculation strategies were evident in all the initial signifier sequences at the same time, so the arrangement of categorised episodes depends on how the sequence of signifiers were produced. Thereafter, I provide an overall summary of the teacher's work with division related concepts, the progression in the use of signifiers, division calculation strategies and number ranges as she enacted each episode. Each case study concludes with what I have termed a "table of coherence" which summarises the coding of episodes in relation to the signification pathways introduced and discussed in Chapter 3: coherent, coherent with limitations, episodes with some ambiguities or incoherent episodes across the lessons of each teacher.

At the end of the chapter, I synthesize all six case studies to present overall findings in terms of coherence, signifier use and calculation strategies. In this synthesis, there is particular emphasis on the differences and similarities in division foci and coherence between G3 and G4 teachers' teaching of division and consideration of how the different socio economic backgrounds of these schools may have impacted on the coherent teaching of division.

5.2. Cathy, G3: School A

Cathy taught a G3 cohort of middle-class learners in a previously advantaged school. The school, which charged fees of R983.00 per month at the time of data collection, was identified by the South African government as a quintile 5 school (the wealthiest quintile) and had between 23 and 30 learners in the class during the observed lessons. Data for the study came from four lessons taught by Cathy in Terms 2 and 3 of 2011. Her lessons, as with all the lessons observed in this study, were videotaped, transcribed and chunked into episodes that were then summarised for the division focus or situation in evidence and the number range of the problem in focus in the episode (for episodes involving a problem). The signifier sequence was then analysed and coded using the signification framework. Cathy's four division lessons were broken down into 64 episodes (see Appendix A), with these tables providing an overview of the division foci and sequence across these episodes and lessons. In Table 14, I provide a summary of the weighting of the teaching across different division situations and bald calculations across Cathy's four lessons. Sections 5.2.1-5.2.3 provide an analysis of Cathy's teaching across sharing situations, grouping situations and bald calculations respectively. Section 5.2.4 summarizes key ideas related to sharing/grouping/ bald calculations as applicable. Finally, in Section 5.2.5, I discuss all the episodes categorized within the table of coherence. The same structure of tables and comments is applied to the other five case studies.

Table 14 Weighting of episode types across division foci in Cathy's lessons

Sharing situations	Grouping situations	Bald calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
17	16	16	15

The overview of episodes across Cathy's division lessons is presented according to the weighting given to each division foci. The balance achieved is summarized in Table 14, and points to a relatively even weighting across sharing situations, grouping situations, bald calculations and other division-related foci in the 64 episodes. The number ranges were low in relation to the FFL curriculum specifications outlined in Chapter 2, with dividends being less than 30 in 46/49 episodes in which dividends were included. Dividends leading to remainders in relation to the divisor were included in 25 episodes. Across sections 5.2.1- 5.2.3, I describe and discuss the predominant ways in which each of the division focus categories in Table 14 (sharing/grouping/bald calculations) were addressed in Cathy's instruction.

5.2.1. Sharing situations

All the sharing focused episodes from Cathy's four division lessons were collated in order to examine how she dealt with sharing. A summary of Cathy's enactment of sharing situations in terms of number range, signifiers and calculation strategies is presented in Table 15. Analysis relating to coherence is deferred to Section 5.2.5. Table 16 then presents an analysis of Cathy's instruction in the sharing episodes, categorised on the basis of the initial signifier in her presentation of sharing situations across lessons 1- 4, because this choice subsequently emerged as important in relation to the coherence of the signification pathway. Lesson, episode and dividend values are detailed for each episode in the sharing situations category. Different patterns of developing understanding of the initial signifier are then detailed, along with the calculation strategies used. This summary model is also used across subsequent division foci for each teacher. By way of reminder for the reader, sharing by CCU refers to the distributing of items immediately, based on the quotient value in sharing situations. In these cases, the answer is either known prior to problem–solving or the answer may already be available from an offer earlier in the episode or through visual inspection of an offered diagram. In contrast, GBD refers to a distributing action in sharing situations where the divisor is used to form groups up to the dividend value.

This distinction relates back to the writings of Brown (1992) and Zweng (1964) on progression within calculation strategies detailed in the literature review, and is used in this chapter in relation to both calculation strategies and coherence.

Table 15 Description of sharing situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (SHARING SITUATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
16-May-11	L1	E4	On transparency: 7 column by 2 row array of mice. T asked: "If I give 14 mice to 2 cats. How many mice will each cat get?" T accepted answer "7" pointing to each row of 7 mice.	Sharing situation. Limitation: Answer tied to presence of array.	2
	L1	E6	Verbal task: "We have 12 flowers and three girls. How many flowers can we give to each of the girls?". Ls: "Four". Asked to prove this, Ls distributed 4 flowers at a time to each of 3 girls. T wrote 12÷ 3 = 4 on board.	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L1	E7	T: "Can we do equal sharing with stars?". She requested a number sentence with the dividend being smaller than 9 which needed to be divided by 2. L offers: "6 divided by 2" and places a set of 3 stars into 1 column and another set of 3 stars into a 2nd column. L then wrote: 6÷ 2 =3 below on the board.	Sharing situation, based on request for equal sharing worked with through complete composite unit sharing action. Coherent.	1
	L1	E9	On transparency: 5 column by 3 row array of chocolate bars. T offered story situation: "Share these bars of chocolates amongst 3 Ls equally with nothing left". T asked for the total number of chocolates and only to provide a number sentence. T accepted 15 and "15 divided by 3" as the answers.	Array linked to sharing situation, matched to verbal number sentence. Coherent.	1
	L1	E10	On transparency: 5 branches with 6 blue gum leaves on each branch and 5 koala bears. T accepted 6 as answer for number of blue gum leaves each koala bear eats. She then accepted: "Thirty divided by five" as number sentence for the situation. As T distributed 6 leaves to each koala bear, Ls asked to check answer by counting in 6s. T stated: "We said that your answer is definitely going to be smaller". The word "smaller" pointed to on the board. 30 ÷ 5 = 6 is written.	Sharing situation, with answer checked through counting in multiples of 6. Limitation: division answer reiterated as "definitely ... smaller".	2
	L1	E11	T told a story about John having 8 sweets to share among himself and 3 friends. T asked: "How many did each one get?" A L, given 8 sweets to distribute between himself and 3 Ls, deals them out one-by-one. T asked for number sentence: "8 divided by 4" is offered. T: "Why 4?" Ls: "Because it's him and his 3 friends". T: "How many did each one get?" T accepted the answer 2.	Sharing situation, worked with through a one by one distribution sharing action. Coherent.	1
17-May-11	L2	E2	T stated focus on "equal sharing" between 3 people. She took 6 sweets and distributed 2 sweets at a time to each of 3 Ls and asked for number sentence. T stated number sentence as 6 ÷ 3 = 2.	Sharing situation: action interpretable as complete composite unit sharing action. Coherent.	1
	L2	E6	Poster with 3 columns. Each column contains 3 rabbits holding groups of Easter eggs: column 1, 3 rabbits each holding 4 Easter eggs; column 2, 3 rabbits each holding 3 Easter eggs; column 3: 3 rabbits holding 2 Easter eggs each. T showed number sentences: 12÷3=, 9÷3= and 6÷3= on flashcards. Ls asked to match number sentences with correct column on poster.	Sharing situation, matching diagrams with written number sentences. Coherent.	1
	L2	E9	On transparency: 7 column by 3 row array. T: "Three girls eat 21 smarties. If shared equally, how many smarties did each one get?" T instructed Ls to count out 21 counters and to provide a number sentence. A L provided the number sentence "21÷ 3 =", then drew 3 circles, distributed the counters one at a time between the three circles, counted the counters in one circle and wrote "7" as the answer for "21÷ 3 =".	Sharing situation, empty array present but not used. Situation worked with through one by one distribution sharing action. Coherent.	1
	L2	E11	On transparency: "Jared shared 21 beads with him and 2 friends. How many beads will each 1 get, if shared equally?" L wrote "21÷3=" and drew 3 circles on the WB (White board). L distributed counters 1 at a time between the 3 circles and counted the counters in one circle, stated: "7" and wrote in "7" as the answer.	Sharing situation, worked with through one by one distribution sharing action. Coherent	1
01-Aug-11	L3	E7	Stating focus on sharing, T showed 2 large circles with 4 sweets in each, and 3 sweets outside of these circles. T: "Between how many people were the sweets divided? How many sweets did each person get? How many sweets are there in total?" T stated that the sweets should be shared between 2 people, each would receive 4 sweets with a remainder of 3. She then counted the 11 sweets pointing to each sweet in turn. Then, pointing to 1st and 2nd group of 4 sweets and then the 3 loose sweets, T stated that because of the remainder "3", "11 divided by 2" would not be appropriate. Circling each group of sweets, she accepted 4 as answer to: "How many did each one get?" Stating that she wanted to prove that 11 sweets could not have been divided between 2 people she wrote "11÷ 2=" on board. Accepting "5" as answer, T stated: "Each one will get 5" with "one remainder". "5 rem 1" was recorded as answer. T then stated that the correct number sentence for the diagram was: "11 divided by 4 equals 2 and 3 remainder" and wrote "11÷ 4= 2 rem 3". Telling Ls to work with groups, T "over-circled" the 2 groups of 4 sweets and asked: "11 divided by 4, each one will get?" "2" accepted. T stated 3 as remainder (pointing to the 3 loose sweets). She asked:	Sharing situation with remainder initially indicated. The four objects in each large circle come to be associated with two contradictory signifiers: as a "group of four sweets" and as "four people". Ambiguity: one object associated with two contradictory signifiers. Limitation also flagged in instruction to interpret divisors as "numbers in a group".	3

			"Between how many people were the sweets divided (pointing to the 4 sweets in each circle)?". "4" accepted. T stated: "11 divided by 4, each one will get a group of 2 (the teacher pointed at 2 circles), what will the remainder be?" Ls: "3". T then drew 5 groups with 2 circles in each with 1 single dot outside. Listing 11÷ 5=, 11÷ 2=,11÷ 3= , T indicated divisors as the number of items in a group.		
	L3	E8	Stating focus on problems about sharing, T showed 4 squares with 2 dog biscuits in each and 1 dog biscuit on the outside with questions: Between how many dogs were the biscuits divided? How many biscuits did each dog get? How many biscuits are there in total? T: "Between how many dogs were the biscuits divided?" Ls offered: "2" and "4". T accepted: "2" (pointing to the 2 biscuits in each square). She then stated: "Into how many groups did we put it?" (pointing to 4 squares). Ls: "4" which T accepted. T: "Who can give me a number sentence?" (pointing across all 4 squares). T accepted: "9 divided by 2 equals 4 with 1 remainder".	Sharing situation with remainder with matching diagram. Verbal number sentence requested and accepted. Ambiguity in T producing two signifiers for the objects inside the square as "dogs" and as "biscuits". Ambiguity: one object associated with two contradictory signifiers.	3
	L3	E9	Diagram: 3 squares each containing 3 slices of bread and 1 slice on the outside with questions: "Between how many boys was the sliced bread divided? How many slices of bread did each boy get? How many slices of bread are there in total?" For the number of boys involved, T accepted the answer "3". For the number sentence, she accepted: "9 divided by 3 equals 3 and 1 remainder". T went on to distribute 10 counters (slices) one at a time between 3 boys. She pointed at each boy and noted that each boy received 3 slices. She then stated: "9 divided by 3 equals 3 and this is my remainder (pointing to the 1 slice). But you can also do this 9 divided by 3, I take my 9 slices, I will put them into groups of 3 and I know 3 times 3 is 9". A L corrected the T's number sentence with T's agreement to "10 divided by 3 equals 3 and 1 remainder".	Sharing situation with remainder based on word problem. Situation worked with through one-by-one unit sharing action, with acknowledgement of the "slip" in the dividend value. Shift to group based distribution is also acknowledged explicitly as an alternative option here for the orally stated number sentence. Coherent	1
	L3	E10	On transparency: "Share 24 biscuits among 5 children?" T: "Share 24 among 5. You can count in 5". T wrote "24÷5=" on board and added: "in a group" on top of the 5. She listed more division calculations (16÷3=,14÷4=,13÷2=) and pointed to the divisors as the number of objects in a group. Returning to "24÷5=" she drew 5 dots and repeated the groups of 5, 4 times. Then circled the groups in five and counted in fives to make 20 and drew a group of 4 dots and inserted that the answer was 4 remainder 4.	Sharing situation with remainder. Grouping model acknowledged with "you can count in five" and solved by group based distribution. Limitation also flagged in instruction to interpret divisors as "numbers in a group".	2
03-Aug-11	L4	E2	Holding 9 sweets, T stated: "Let's share 9 sweets. How many groups can she make?" Ls responded, "3". T counted 3 groups of 3 sweets. Calling 3 Ls to the front, T gave each one 3 sweets and stated: "9 divided by 3 and each child will get 3". T took out 2 more sweets in her hand and asked: "If I have 11 sweets what do I call the 2 sweets". Ls responded: "11 divided by 3 equals 3 with 2 remainder". T wrote "9÷3=3", and then corrected this to "11÷3=3 2 rem".	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L4	E3	T: "If you have 13 sweets and there are 3 children can you group it?" Ls told to write a number sentence for "13 divided by 3" and calculate answer by making groups of 3 with counters. T wrote "13÷3=" on board and proceeded to draw a group of 3 dots at a time and circle them while counting in multiples of 3 resulting in 4 groups of 3. She then drew in one dot to show the remainder and added "4 rem 1" as the answer.	Sharing situation with remainder, linked to a number sentence, with diagram produced through group based distribution. T produces two signifiers for "3": as "3 children" and as "3 sweets". Ambiguity: one object associated with two contradictory signifiers.	3
	L4	E10	Diagram: 2 circles, 4 sweets in each and 3 sweets on the outside with the following questions: Between how many people were the sweets divided? How many sweets did each person get? How many sweets are there in total? T stated: "You see all these sweets" (points to all 11 sweets). Without pointing to the diagram again T asked: "How many sweets are there altogether?" Ls: "11". T: "How many in a group?" Ls: "4". T: "How many groups of 4?" Ls: "2". T: "How many left over?". Ls: "3". T: "Between how many people were the sweets divided?" Ls: "2". T: "How many sweets did each one get?" Ls responded: "4". T asked: "What is your remainder?" Ls replied: "3". T then asked: "How many sweets are there in total?" Ls replied: "11".	Situation repeats sharing situation with remainders in L3 E7, but sweets and children remain unambiguously pointed to throughout the signification pathway here. Remaining in the situation leads to coding as: Coherent.	1

Table 16 Enactments of sharing situations in Cathy's lessons

	Sharing situations (n=17/64 episodes)	
Initial signifier [dividend]	Oral/written linguistic situations (9/17 episodes) L1 - E6 [12], E7 [6], E11 [8] L2 - E2 [6], E9 [21], E11 [21] L3 – E10 [24] L4 – E2 [11], E3 [13]	Diagrammatic signifiers (8/17 episodes) Array diagrams (3/17 episodes: L1 – E4 [14], E9 [15], E10 [30]) Iconic equal groups diagrams (5/17 episodes: L2 – E6 [12]; L3 - E7 [11], E8 [9], E9 [10]; L4 – E10 [11])
Signifier sequence	Word problem > Divisor as # in a group > Written number sentence > Sharing through GBD diagram with rem > Answer (L4 E3 – Some ambiguity) Word problem > Number sentence > Sharing through 1 by 1 distribution > Written number sentence with answer (L2 E9 & E11) Word problem > Sharing through 1 by 1 distribution > Number sentence > Divisor as # of groups > Answer (L1 E11) Word problem > Number sentence > Divisor as # in a group > Sharing through GBD diagram > written answer (L3 E10 - Limitation) Equal share context > Verbal number sentence with answer known > Sharing by CCU > Written number sentence with answer (L1 E7) Equal share context > Answer known > Sharing by CCU > verbal number sentence with answer (L2 E2) Word problem >Answer known > Sharing by CCU > Number sentence with answer > Extra objects added in > Oral and written number sentence with rem (L4 E2) Word problem > immediate verbal answer > Sharing by CCU > Written number sentence with answer (L1 E6)	Equal groups with loose items diagram > total number, number in a group, number of groups, left over, number of people, number each get, remainder, total number of objects identified (L3 E7 – Some ambiguity) Equal groups with loose items diagram > Divisor as # of items in a group > verbal number sentence with answer (L3 E8 - Some ambiguity) Array > Word problem > Verbal Answer (L1 E4 & L1 E10 – Limitation) Equal groups with loose items diagram > Number sentence > Sharing through 1 by 1 distribution > Number sentence > Sharing by GBD > Verbal number sentence with remainder (L3 E9) Equal groups with loose items diagram > Total number with divisor as # in each group, quotient as number of groups, remainder as left over > Divisor as # of items in a group reasserted (L4 E10) Multiple equal groups diagram > Written number sentences (matching) (L2 E6) Array > Word problem > Total number > Verbal Number sentence (L1 E9)
Calculation strategy	Sharing through 1 by 1 distribution (3/9 episodes - L1 E11; L2 E9 & E11) Sharing through GBD (2/9 episodes - L3 E10; L4 E3) Sharing by CCU (4/9 episodes - L1 E6 & E7; L2 E2; L4 E2)	No calculation strategies (4/8 episodes - L1 E9; L2 E6; L3 E8; L4 E10) Sharing through 1 by 1 distribution and share through GBD (1/8 episodes - L3:E9) CCU share by inspection (3/8 episodes – L1 E4 & E10; L3 E7

In the 9/17 episodes beginning with oral/written sharing situations, 24 is the highest dividend value, well below the FFL specifications for division word problems where dividend values up to 500 are suggested. Signifier sequences played through via a translation into concrete and/or iconic distributions of counters to produce the result, which was usually then written as a symbolic number sentence. Such moves between signifiers reflect recommendations made in the literature (Anghileri, 2007). In these episodes beginning with a sharing word problem, sharing through one-by-one distribution was accepted in 3/9 episodes, including one episode involving a single digit dividend. Sharing by "complete composite unit" (CCU) occurred in 4/9 episodes (in these episodes, the answer either has to be known prior to problem-solving, or the answer may be already available from an offer earlier in the episode). In 2/9 episodes, group-based distribution (GBD) was used to find the answer, with instruction in some episodes directing learners to treat the divisor as the "number in a group", in spite of the task being set within a sharing situation, and in spite of this being contradicted in other episodes where divisor as "number of groups" was accepted. The shift from a sharing situation to a grouping strategy was acknowledged in one episode but not in the other.

Of interest is that sharing word problems were solved by sharing actions at the two extremes of the progression in sharing strategies noted in the literature – one-by-one distribution or CCU distribution (based on knowing the answer at the outset). There are no instances involving encouragement of moves towards knowing the answer via partial composite distribution in the ways that are described in the literature (Kouba, 1989). As noted above, a grouping strategy was also indicated in two episodes.

Moving to the sharing problems set in initial diagrammatic signifiers (n=8/17 episodes), two types of diagrams were evident: iconic equal groups diagrams and array diagrams. Predictably, matching to number sentence activities predominated in this group given that both diagram types provide visual evidence of the answers to word problems, alleviating the need for calculation. Interestingly, there was no overt acknowledgement of "visibility" of answers to calculations being available through inspection of either array or equal groups diagrams.

Taken together, Cathy's sharing episodes show extensive attention to moving between signifier categories. There is limited evidence of progression towards more abstract signifiers over time, although number sentences matching word problems and diagrams are widely incorporated. Teacher talk suggests "fragmented" knowledge (diSessa, 2018) related to divisors interpretable as both the number of groups or the number in a group, and limited awareness of how to move between these interpretations smoothly. There are also some indications of awareness of the division property that remainders have to be smaller than the divisor, but no reasoning or signification discussing the reasons for this rule. The 7/17 sharing episodes overall involving CCU sharing of some sort underpinned by knowledge, or use of the answer, raise questions about the appropriateness of tasks and/or the teaching of division calculation strategies: if the answers are already known, then calculation strategies are not required in the context of these problems. If the answers are not known, then accepting or offering CCU-based distribution strategies is problematic.

5.2.2. Grouping situations

16/64 of the episodes in Cathy's lessons involved the grouping situations collated in Table 17. Using initial signifiers, signifier sequences and calculation strategies as before, Table 18 presents a breakdown of the enactment of grouping episodes.

Table 17 Description of grouping situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (GROUPING SITUATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
16-May-11	L1	E5	Transparency: 4-2=2 and 2-2=0 in 1st column, with 2 mice per row alongside each subtraction in the next column. Columns 3 and 4 showed repeated subtraction of 2 starting from 6, and columns 5 & 6, starting from 8. T talked through each repeated subtraction as: "a cat eats 2 mice at a time". Asking for the division number sentence matching each column of repeated subtractions, T initially says 6 ÷ 2 = 3 and then 6 ÷ 3 = 2 for column 4. She then states 8 ÷ 4 = 2 for column 6.	Grouping situation: inferred from repeated subtraction & initial teacher talk of "two mice" eaten "at a time". Final statement of number sentences as 6 ÷ 3 = 2 and 8 ÷ 4 = 2 instead of 6 ÷ 2 = 3 and 8 ÷ 2 = 4, breaks with initial situation. Ambiguity: unacknowledged move from grouping to sharing situation.	3
	L1	E8	On a transparency: "Eric works at a farm stall. He ties carrots into bundles of 4. If there are 20 carrots in a box, he makes __ bundles. What will the number sentence be?" Ls' response: "20 divided by 4" is accepted.	Grouping situation matched to verbal number sentence. Coherent.	1
17-May-11	L2	E3	Based on a 3 column by 3 row array of dots (three dots seen in vertical circles). T asked "Can you give me the division sum for these groups?" "9 divided by 3 equals 3" is offered and accepted.	Grouping situation inferred through circles within array. Coherent.	1
	L2	E4	Based on a 3 column by 2 row array of dots (three dots seen in horizontal circles). T asked: "I want a division sum for this grouping?" "6 divided by 2 equals 3" is offered and accepted as the answer.	Grouping situation inferred through circles within array. Coherent.	1
	L2	E5	Based on 6 column by 3 row array of dots (three dots seen in vertical circles). T asked: "Who can give me the number sentence?" "18 divided by 6" is offered and accepted as the answer. T pointed to the 6 vertical groups of 3 dots.	Grouping situation inferred through circling of groups within an array. Coherent.	1
	L2	E7	On a flashcard 7 groups of 3 Easter eggs shown. Ls counted the 21 Easter eggs and instructed to identify the groups of 3 resulting in 7 groups. T asked for number sentence and accepted: "21 divided by 7". She then stated: "We have 7 groups of 3 because if you count in 3s 7 times you will get 21". She then asked for a story sum. A L provided the following story sum: "There are 21 Easter eggs and 7 children how many will each one get?"	Grouping situation inferred from instruction to make groups of three. Situation matched to a number sentence. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from grouping action to sharing oriented number sentence.	3
	L2	E8	On a transparency: "How many sets of 2 in 18?" with a 6 column by 3 row array. T asked for a number sentence. "18 divided by 2" was offered and accepted. Ls proceeded to write "18÷2=9" on their whiteboards. T then stated: "Remember how you share 1 by 1". L then drew 2 circles and distributed counters 1 at a time between the 2 circles. T asked: "What is 18 divided by 2?" L responded: "9".	Grouping situation inferred from task presentation. In final section, a sharing action is offered and accepted. Ambiguity: unacknowledged move from grouping to sharing situation. No use of array.	3
	L2	E10	On a transparency: "How many 5's in 15? Write a number sentence as quick as you can?" L stated: "15 divided by 5 equals 3".	Grouping situation matched to verbal number sentence. Coherent.	1
01-Aug-11	L3	E2	T asked Ls to place 9 counters in groups of 3. Ls made three groups with 3 counters in each. T asked: "Are there remainders?" Ls answered: "No". T asked for, accepted and wrote up "9 ÷ 3 = 3" as the number sentence. She then asked for a story sum for the number sentence. A L offers: "I share 9 marbles with 2 friends" (gesturing herself as well as part of the group) is accepted.	Grouping situation: 1 by 1 distribution into groups of 3. Towards end of episode T introduces a sharing situation. Ambiguity: unacknowledged move from grouping to sharing situation.	3
	L3	E3	Diagram showing 11 dolls. T stated: "So we are going to share. 11 divided by 3". T then wrote "11 ÷ 3 =" on the board and counted in 3s. As they counted in 3s the T made circles around groups of 3 dolls. She then asked: "Do we have any left?" Ls replied: "2 remainders". T then reiterated that they made 3 groups of 3 with 2 remainders. She then added "3 rem 2" as the answer.	Grouping situation with remainder played through by grouping through 1 by 1 distribution. Number sentence with answer produced. While the word "share" is used in opening part of this episode, the	1

				enactment within the episode is coherent. Coherent	
	L3	E4	With 13 counters, T asked Ls to make groups of 4. A L made 3 groups of 4 counters and the T circled each group and instructed the L to indicate that the 1 counter is the remainder. T asked for a number sentence and accepted: "13 divided by 4" and wrote "13 ÷ 4=" on the board and wrote on top of the 4 "in a group". She then asked: "How many groups did you make?" A L stated: "3". T wrote "3" in the answer. She then asked: "Is that it?" L stated: "And 1 remainder" T wrote in "rem 1".	Grouping situation with a remainder; enacting a concrete experience and a number sentence played through by grouping through 1 by 1 distribution. Coherent	1
	L3	E5	With 14 counters, T instructed the Ls to make groups of 3. A L made 4 groups of 3 counters with 2 as the remainder. She then asked for a number sentence. A L stated: "14 divided by 3 equals 4 and remainder 2" and then wrote "14 ÷ 3 = 4 2 rem". T asked for a story sum and accepted: "I have 14 apples and 3 friends, how many will each one get?"	Grouping situation with remainder played through by grouping through 1 by 1 distribution. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from grouping to sharing situation.	3
03-Aug-11	L4	E9	T asked class to draw 5 groups of 3 and to write down a division written number sentence with answer. 15 ÷ 3 = 5 written down by Ls. T then instructed Ls to add 2 counters and stated that the 2 counters were called the remainder. She said: "When we divide our answer is always smaller". Finally, she told the Ls to change the initial division calculation of 15 ÷ 3= 5 to 17 ÷3= 5 rem 2.	Grouping situation with remainder – limitation: answer to division "always smaller". Ambiguity in what is in focus: "answer is always smaller" could refer to either quotient integer or remainder or both together. Ambiguity: Set of words associated with two or more contradictory signifiers.	3
	L4	E14	T placed a chart with 3 rows of teddy bears in groups. In the first row there are 11 teddy bears in a row made up of four groups of teddy bears. In the first 3 groups there were 3 teddy bears in the last group there were 2 teddy bears. She asked: "How many teddy bears in a group?" She then established with a re-created diagram of circles and dots that the first 3 groups had "3" teddy bears, the last group had 2 teddy bears.	Grouping situation with a remainder matched to diagram of circles and dots. Coherent.	1
	L4	E19	On transparency: "Divide 19 sweets in groups of 3 __ left __". T instructed the Ls to make groups of 3 using 19 counters, Ls ended with 6 groups of 3 with one remainder, some Ls wrote "6 1 rem" as the answer.	Grouping situation with a remainder played through by grouping through 1 by 1 distribution. Coherent.	1
	L4	E21	T placed a flashcard with 3 rows of teddy bears - 1st row, 11 bears in 4 groups, 3 teddy bears in the first 3 groups & 2 teddy bears in the last group; 2nd row, 7 bears in 3 groups, 2 groups of 2 bears and 1 more bear; 3rd row: 7 bears in 2 groups, one with 4 teddy bears and another group of 3 teddy bears. T asked for number sentences for each row. As number sentences were produced, she circled groups and wrote the number sentences: 11÷3= 3 rem 2 for 1st row, 7÷2= 3 rem 1 for 2nd row, and 7÷4= 1 rem 3 for last row.	Grouping situation with a remainder played through by grouping through 1 by 1 distribution. Coherent.	1

Table 18: Enactments of grouping situations in Cathy's lessons

	Grouping situations (n=16/64 episodes)	
Initial signifier (dividend in brackets)	Oral/written word problem situations (7/16 episodes): L1 – E8 [20] L2 – E8 [18], E10 [15] L3 – E2 [9], E4 [13], E5 [14] L4 – E19 [19]	Diagrammatic signifier (9/ 16 episodes) Iconic scattered dividend quantity - L3-E3 [11] Iconic equal groups diagrams - L2- E7 [21], L4- E9 [19], E14 [11], E21 [11] Array diagrams - L1-E5 [8], L2 – E3 [9], E4 [6], E5 [6]
Signifier sequence	Grouping context > grouping through 1 by 1 distribution > verbal or written number sentence with answer (without or with remainder) > sharing word problem (L3 E2 & L3 E5 - Some ambiguity) Word problem with array > verbal written number sentence and answer > situation replayed with counters involving sharing through 1 by 1 distribution to produce an answer (L2 E8 - Some Ambiguity) Word problem > verbal number sentence without answer (L1 E8) Grouping context > grouping through 1 by 1 distribution > verbal and written number sentence with answer with remainder (L3:E4) OR directly to answer with remainder (L4 E19) Word problem > correct immediate verbal answer and number sentence is accepted. (L2 E10)	Equal groups diagram > grouping through 1 by 1 distribution > written number sentence and answer > two loose items added to the equal groups diagram reconstructing the equal groups diagram. The two loose items are identified as the remainders > rewriting of the initial written number sentence with answer includes the remainder (L4 E9 - Some Ambiguity) Equal groups diagram > counting the number of equal groups through "over circling" > verbal number sentence > skip counting to confirm the dividend > verbal sharing word problem provided (L2 E7 - Some Ambiguity) Pre-structured array alongside written repeated subtraction sums where a row is subtracted each time > grouping situation > sharing symbolic number sentence (L1 E5- Some Ambiguity) Diagrams showing equal groups with loose items > either through grouping through 1 by 1 distribution by drawing of groups or "over-circling" groups to emphasize them > written number sentences with answers or no answer provided (L4 E14 & E21) Scattered dividend quantity diagram > written number sentence > circling of groups to produce written answer with a remainder (L3 E3) Pre-structured array > written number sentence with and without answer (L2 E3, E4,E5)
Calculation strategy	Matching signifiers without calculating answer episode. (1/ 7 episodes - L1:E8) Sharing through 1 by 1 distributions (1/7 episodes - L2:E8) Grouping through 1 by 1 distribution (4/7 episodes – L3:E2, E4, E5 & L4: E19) Recalled number facts (1/7 episodes - L2:E10)	No calculation strategies (3/9 episodes – L2:E3, E4 & E5) Grouping through 1 by 1 distribution (1/9 episodes – L4: E9) Over – circling of groups (2/9 episodes – L3:E3 & L2: E7) Skip counting – (1/9 episodes – L2: E7 repeated episode) Recalled number facts - (1/9 episodes -L1:E5)

Table 18 summarises the enactments of grouping situations in terms of signifiers and calculation strategies. In relation to signifier sequences, episodes beginning with oral/ written word problems, grouping situations (n=7/16 episodes) played through via a translation into concrete and/or iconic distributions of counters. The results were written as symbolic number sentences. Distributing actions occurred in 5/7 of these episodes in one of two ways: either there was a one-by-one sharing distribution (1/7 episodes, and in spite of the grouping situation), or there was grouping through 1 by 1 distribution where groups of counters were formed or drawn, based on the quantity indicated by the divisor (4/7 episodes). Of interest, once again, is the lack of distinction made between sharing actions and grouping actions (Carpenter et al., 1993). As in the sharing situations category, calculation using one-by-one distribution predominated as the key route to producing the answer, with 1/9 episodes indicating the use of recalled facts, and no evidence of progression in the intermediate calculation strategies.

Moving to grouping problems sets in initial diagrammatic signifiers (9/16 episodes), three types of diagram were visible: (i) iconic scattered dividend quantity diagrams, (ii) equal groups and (iii) array diagrams. Grouping approaches to solving the tasks predominated in this category together with solving through visual inspection of arrays.

Taken together, grouping episodes in Cathy's lessons show extensive attention to moving between signifier categories. There are instances of number sentences with answers produced by visual inspection of the structure located in the array (Young-Loveridge, 2005). 9/16 grouping episodes overall involved group based approaches, particularly grouping through "over circling" of groups in the solving of the problems. Reliance on these fundamental grouping actions points to limitations when the number range increases. As in the sharing episodes, there were some unacknowledged shifts between division models, creating disconnections between the initial grouping model and the sharing actions or situations that subsequently emerged.

5.2.3. Bald calculations

Table 19 Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
17-May-11	L2	E12	On board: "9÷3=", "16÷4=", "6÷3=", "8÷2 =" and "4÷2=". T accepted correct answers that were offered immediately with no further elaboration.	Bald calculations answered correctly and immediately. Coherent.	1
	L2	E13	On board: "93 ÷3 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent.	1
	L2	E14	On board: "46 ÷2 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent	1
	L2	E15	On board: "63 ÷3 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent	1
01-Aug-11	L3	E6	T wrote "19÷5=" on the board. T stated: "We going to group them into 5s". The Ls made groups of 5 counters resulting in 3 groups of 5 with 4 remainders. T then wrote in the answer "3 rem 4".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E11	T wrote "16÷3=" on the board and instructed the Ls to draw groups of 3. Working with 2 Ls and their WB, she demonstrated how to draw groups of 3 and ended with 5 groups of 3 with 1 remainder.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E12	T wrote "14÷4=". Based on the task, T instructed the Ls to solve the calculation by drawing the diagrams. Ls are seen drawing 3 groups with 4 dots in each with 2 dots. T wrote in the answer "3 rem 2".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E13	T wrote "13÷2=". Ls then wrote "13÷2=" on the WB and then drew 6 groups of 2 with 1 remainder and added the answer "6 rem 1"	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
03-Aug-11	L4	E4	T wrote "9÷2=". Ls drew answer on WBs. To check the answer T took a WB with the symbolic number sentence "9÷2= 4 1 rem" with 4 groups with 2 dots in each with 1 dot on the outside. She then asked the Ls to count the dots in 2s and added the 1 loose dot as she pointed to the 9.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E5	T wrote "17÷5=" and asked: "You are going to do your grouping in?" Ls responded: "5". A L wrote "17÷5=" on their WB and then drew 3 groups of 5 with 2 dots left outside and provided the answer "3 rem 1".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E6	T wrote "14÷4=". She then accepted 3 circles with 4 dots in each with 2 dots on the outside and "14÷4= 3 2 rem". One L did not answer correctly. To assist the T offered 7 divided by 3 which produced 2 groups of 3 with 1 remainder. She then stated to the Ls that they needed to remember that: "When they do division the answer is always smaller".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Limitation: division answer always smaller.	2
	L4	E7	T wrote "17÷4=". She then stated: "How many groups of 4?" A L responded: "4". A L responded, "4 and 1 remainder". The T proceeded to write "4 1 rem" as the answer.	Bald calculation with remainder produced through recalled number facts. Coherent	1
	L4	E15	T wrote "24÷5=". Ls told to draw groups of 5. T assisted Ls by drawing circles 4 groups of 5 dots and then drawing in 4 separate dots. She ends by "4 4 rem" as the answer.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E16	T wrote "11÷2 =" Ls told to make groups of 2 using counters. Ls made groups of 2 and added the answer "5 1 rem" to "11÷2 =" on the WB.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E17	T wrote "14÷3=". She told Ls: "Remember to count, 14 sweets in groups of 3". Ls packed 4 groups of 3 with 2 counters. Two offers for remainder: "1" and "2". T affirmed that remainder was "2".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E18	T wrote "18÷5=". She told the Ls to pack out 18 counters. asked: "How many in a group?" Ls answered:"5". She then asked: "And what is the 3?" Ls responded: "Remainder". T stated: "My answer is 3 rem 3".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent.	1

Table 20: Enactments of bald calculation in Cathy's lessons

	Bald Calculations (n=16/64 episodes)
Initial signifier (dividend in brackets)	L2 – E12 [16], E13 [93], E14 [46], E15 [63] L3 – E6 [19], E11 [16], E12 [14], E13 [13] L4 – E4 [9], E5 [17], E6 [14], E7 [17], E15 [24], E16 [11], E17 [14], E18 [18]
Signifier sequence	Written symbolic signifier > drawing of dots 1 by 1 and circled into groups, with or without some loose dots or counters > written answer with remainder (L3: E11, E12, E13 and L4: E4, E5, E6 – Limitations, E15) Written symbolic signifier > 1 by 1 laying out and grouping of counters to produce the answer > written answer (L3: E6, L4: E16, E17, E18) Written symbolic signifier > short division algorithm > written answer (L2: E13, E14 and E15) Written symbolic signifier > correct immediate answer (L2: E12, L4:E7)
Calculation strategy	Grouping through 1 by 1 distribution – (11/16 episodes - L3: E6, E11, E12, E13 & L4:E4, E5, E6, E15, E16, E17, E18) Skip counting - (1/16 episodes - L4:E4 repeated episode) checking results Short division algorithm – (3/16 episodes – L2: E13, E14, E15) Recalled number facts – (2/16 episodes - L2:E12 & L4: E7)

Table 20 summarises the enactments of bald calculations in terms of signifiers and calculation strategies. In relation to signifier sequences, in the 16 episodes beginning with bald calculations, the solving of the bald calculation occurred via a translation into concrete distributions of counters or drawing of circles and dots *into* equal groups or short division algorithms to produce the result, stated either as a written or verbal number sentence (16/16 episodes).

In relation to the literature on division and its teaching, these findings raise some points of interest. Firstly, only in Lesson 2 is the number range in this set of episodes higher than required in the curricular specifications for solving bald calculations: the highest dividend value is 93 and is evident in only three episodes. Of interest is that bald calculations are solved through grouping actions in 14/16 episodes. One-by-one distribution, where groups of counters were constructed based on the divisor value, was used in 11/16 episodes and short division algorithms were used in 3 episodes. Skip counting was used in one episode to check the answer. Cathy's solving of bald calculations is therefore predominantly through group based approaches with a focus on the divisor as the number in a group. Fosnot and Dolk (2001) note from their research that learners found that working with groups was easier when solving division tasks. The majority of episodes in Cathy's lessons were worked with through one-by-one creation of groups, but there was some evidence of moves into more efficient approaches such as skip counting and use of the short division algorithm, with this last group including larger dividend values than seen in the other episodes. However, no connections were made between making groups/skip counting and the short division algorithm.

Fifteen episodes were coded as "other" division related foci. These focused on introducing division terminology, individual work or group work in which the teacher did not lead, repeated examples with the same explanation from another group (8/18 episodes), or recapping of prior work done on division based ideas (7/18 episodes). However, in three of the uncoded episodes the teacher continued to make reference to the answer being smaller in the context of division.

5.2.4. Summary

In Cathy's lessons there is evidence of an even spread of sharing problems, grouping problems and bald calculations. When sharing and grouping verbal/written word problems were used, the number range was small. In bald calculations with a smaller number range, group-based distribution predominated. Larger dividends (> 30) were only seen within one lesson in the bald calculation category and these were solved by short division algorithms. Progression towards more abstract signifiers was uneven, with more sophisticated diagrams such as arrays sometimes used with less truncated diagrams (diagrams reflecting unit counts or concrete strategies) used in subsequent lessons. The structure of the array was sometimes used to produce an immediate answer, but sometimes sidelined. Across all the episodes which focused on sharing and grouping situational problems, one-by-one grouping/sharing distribution was the most widely used calculation approach. More sophisticated strategies/recalled number facts were evident only occasionally. Across lessons, the divisor was interpreted as the number of groups or the number in a group. However, learners were not explicitly taught the difference between diagrams and actions associated with either interpretation of the divisor. Beckmann (2011) notes that in the teaching of division, learners should know the difference between the "number of groups" and the "number in each group" to determine the division situation.

5.2.5. Analysis of coherence in Cathy's lessons

Table 21 Analysis of episodes within signification pathways framework

Number of lessons	Category 1- Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	5	2	1	0	4	12
L2	12		2	0	3	17
L3	7	1	4	0	2	14
L4	12	1	2	0	6	21
Total	**36**	**4**	**9**	**0**	**15**	**64**

Table 21 summarises the episodes as categorised within the signification pathway framework. A "coherent" categorization indicates that the division task produced a pathway involving links between signifiers that were mathematically justifiable at all points within the episode. For Cathy, coherence is evident in n = 36/64 episodes, a little over half of her instructional episodes. Episodes involving her working with bald calculations make up almost half of the coherent category (15/36 episodes). In 11/36 coherent episodes, she solved sharing situational problems; in another 10/36, she solved grouping situational problems. Recalling that there were 16 episodes overall involving bald calculations, it is interesting to note that 15/16 of these episodes were coherent. The literature argues that when teachers encourage the use of symbols to think with when solving division tasks, they are shifting learners from concrete approaches to solving division tasks to more abstract ways of solving division tasks (Van Putten et al., 2005). In contrast, the proportion of coherent episodes across the sharing and grouping situations was much lower (12/17 for sharing episodes and 10/16 for grouping episodes). The imbalance in proportions of coherent episodes reflects the difficulties noted by Rowland (2007) for teachers when working with sharing and grouping situations.

Episodes classified as coherent with limitations were evident in n = 4/64 episodes. Three phenomena underpinned the limitations category: a statement that the answer is always smaller (usually in relation to dividend here, but sometimes also referring to the remainder in comparison to the divisor); the answer was tied to the presence of an array; and learners were told to interpret the divisor only as the number within a group. Reference to division making the answer smaller also occurred in the uncoded episodes which focused on vocabulary or general statements, so the message of division making smaller was more widespread than the figures in the coded episodes indicate.

9/64 episodes were classified as having "some ambiguity". Of these, five showed a shift between signifiers concerning division models or actions with no acknowledgement of the shift. In three episodes one signified or object came to be associated with two contradictory signifiers, although along the pathway at some point, each signifier could be linked back to the previous signifier. In one episode it was unclear whether the teacher referred to the quotient integer or the remainder part or both together, when making reference to the idea of smaller.

There were no episodes that indicated outright error or incoherence in Cathy’s division teaching.

5.3. Mary, G3: School B

Mary taught a G3 cohort of previously disadvantaged learners in a previously advantaged school. There were between 30 and 35 learners in the class during the observed lessons. The school fees were R 375 per month at the time of data collection, and the government classified the school as a quintile 5 school. I identified 20 episodes across Mary’s five lessons (see Appendix B).

Table 22 Weighting of episode types across division foci in Mary's lessons

Sharing situation	Grouping situations	Bald calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
1	2	8	9

The episodes from Mary's lessons were unevenly spread across sharing situations, grouping situations, bald calculations and other division-related foci in the 20 episodes. 3/20 episodes focused on division situations and 8/20 were located in bald calculations. The number ranges in sharing and grouping situation-based episodes and bald calculations were low in relation to the FFL curriculum specifications, with dividends being less than 36 in all 11 of the situation/calculation episodes.

5.3.1. Sharing situations

Table 23 Description of sharing situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (SHARING SITAUTION EPISODES)	Division focus / coherence comment	Coherence category
02 June 2011	L2	E1	T wrote: "There are 36 fish in the boat. All 4 fishermen caught the same number of fish. How many did each one get?" Ls asked to raise four fingers to show that fish will be shared amongst four people. T wrote "36 ÷ 4 = _". T: "How many big circles should we draw to share the thirty-six fish to those men?" pointing at "4". "Four" & "nine" offered. "Nine" accepted. "9" added to calculation. T drew nine circles. T: "And how many fish do we put into each of the circles?" L: "Four!" T accepted the answer. T drew four dots in each circle with Ls counting 1- 4. T: "If we count all the fish in the circles it will give us?" Ls: "Thirty-six fish". T pointed to the calculation: "Thirty-six fish are divided between four men; how many do these four men get?" L: "Nine" T: "Each man gets nine" as she pointed to the nine circles.	Sharing situation produces a number sentence. "9" accepted as answer # of circles & inserted as answer to number sentence. T's diagram uses 9 as # of groups & 4 as # in each group: a grouping interpretation of symbolic calculation. Group of dots drawn into circles. Ambiguity: unacknowledged move from sharing to a grouping interpretation of the bald calculation.	3

Table 24 Enactment of a sharing situation in Mary's lessons

	Sharing situations (n=1/20 episodes)
Initial signifier (dividend in square brackets)	Oral/written word problem situation (1/1 episode): L2 – E1 [36]
Signifier sequence	Word problem > written number sentence with answer > Grouping-based diagram > sharing problem & dividend value restated > # of groups in diagram accepted as answer (L2: E1 - Some ambiguity)
Calculation strategy	Numerical offer of number of groups accepted as answer and inscribed into diagram. Divisor then inscribed into each group using grouping through 1 by 1 distribution.

In this episode (L2 E1) an oral/written sharing situation was translated into a number sentence, an answer for the number of circles to draw (9) was written in as the answer to the number sentence. A diagram was subsequently drawn with 9 as the number of groups and an iconic distribution of 4 *dots was made* into each group – a grouping through one-by-one distribution. The move from a sharing situation to a grouping-based distribution is taken up in the discussion of coherence in Mary's lessons.

Here, in the solving of the tasks the quotient was accepted prior to a problem-solving process as the answer to the number of groups, thus making the subsequent use of diagrammatic signifiers somewhat redundant as a way of producing the answer. Grouping through one-by-one distribution based on the divisor was subsequently inscribed.

5.3.2. Grouping situations

Table 25 Description of grouping situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (GROUPING SITAUTIONS EPISODES)	Division focus/ coherence-related comment	Coherence category
13 Sept 2011	L3	E3	T wrote: "A soccer team scores 15 goals. Each scorer scores 3 goals. How many players score goals?" T: "Something is being grouped into something" L: "You make groups of three until you get to fifteen" T: "What are we grouping?" L: "Each soccer team must be grouped into three". T circled the "15" in the word problem: "How many groups (pointing to the three) can give me the fifteen?" L: "Five groups of three" T: "She is going to divide or group or share the fifteen goals into groups of three". T drew three dots and circled the three dots, she continued to draw groups of three until she had five groups of three. T: "How many groups of three am I having here?" while pointing to the five groups. L: "Five groups of three". Ls counted the five circles as told to do. T reiterated that five groups of three makes fifteen.	Grouping situation played out through 1 by 1 distribution, linked to a word problem, with diagram produced. While the word "share" is used in the middle of this episode and there is a slip-up between "soccer team" and "goals", the enactment within the episode is coherent. Coherent	1
14 Sept 2011	L4	E3	T wrote: "A teacher has a bag of 32 sweets. He gives each child 4 sweets. How many children are there?" L: "I drew thirty-two sticks and then I circled four sticks and counted the circles and got to the answer eight". T accepted the answer. T drew thirty-two sticks as Ls counted from 1 to 32. T counted four sticks and then circled them and Ls counted the circles. T drew eight circles altogether. T wrote "32 ÷ 4 = 8".	Grouping situation translated into a diagram and number sentence, and played out through a grouping 1 by 1 distribution. Coherent	1

Table 26 Enactments of grouping situations in Mary's lessons

	Grouping situations (n=2/20 episodes)
Initial signifier (dividend in brackets)	Oral/written linguistic word problem situation (2/2 episodes): L3 – E3 {15] L4 – E3 [32]
Signifier sequence	Word problem > correct immediate verbal answer > dots/sticks circled by 1-1 distribution based on the divisor with or without written answer (L3: E3 and L4: E3)
Calculation strategy	Making of groups based on the divisor – the dividend value is first drawn and then grouped (1/2 episodes - L4:E3) Grouping through 1 by 1 distribution based on the divisor (1/2 episodes - L3: E3)

In relation to signifier sequences, in these two episodes beginning with oral/written grouping situations, the playing through occurred via a translation into iconic distribution of dots or sticks into groups to produce the result. The distribution itself occurred through grouping approaches: either the dividend value was first drawn and groups were circled based on the divisor, or grouping was done through a one-by-one distribution based on the divisor in the word problem counting up to the dividend. In both cases, fundamental grouping actions involving concrete approaches to solving division tasks are evident, although Carpenter et al. (1999) note that first counting the dividend quantity and then making groups is a longer approach and shifts learners away from abstraction to a more concrete approach to solving grouping problems.

Taken together, the use of these fundamental grouping action through unit counting, indicates concrete approaches to the solving of division tasks, with "pulling back" suggested in learners being able to provide the answer before the diagram in the first episode in this category.

5.3.3. Bald calculation

Table 27: Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
1 June 2011	L1	E4	10 ÷ 2 on flashcard and a two row by five column array drawn on board. T: "I have ten sweets and then I have to share them to two people", pointing to array. T circles two dots in the array at a time and gestured that five learners would each receive two dots. T: "How many big circles must I give these children? L: "Five". T: "Five big circles" T: "Let's share the sweets to five people." Teacher drew five circles, put two dots in each and Ls said: "Two sweets" in each circle. T pointed to circles: "We have ten sweets into five big circles which means ten divided by two is?". Ls counted in twos as T pointed to each circle. T wrote the answer "5" on the board and on the flashcard.	Bald calculation interpreted as a sharing situation but divisor is then worked with as the number in a group from the array. Situation involving sharing between 5 people is then redrawn in equal groups diagram, with skip counting in 2s to 10, after which T writes in 5 as answer. Ambiguity in that 2 refers at one point to 2 Ls and later to 2 sweets: one number is associated with two contradictory signifiers.	3
	L1	E5	21 ÷ 3 = on flashcard. T: "We shared what? How many? Who can give me the answer?" L: "Seven". T wrote "7" T: "How many big circles can I make for getting the answer?" L: "Seven". T drew seven big circles. T: "Into the big circles how many sweets must I put?" L: "Three sweets". T added three dots to each big circle while Ls counted 1-3. T: "21 sweets were distributed between the seven children". Ls counted in threes from 3 to 21 as the T pointed to each big circle. T: "Which means your answer is?" Ls: "Seven".	Bald calculation interpreted as a sharing situation with immediate answer provided. Problem then worked with through grouping through 1 by 1 distribution based on this quotient. Skip counting used to count up to dividend. The quotient is used as the basis for the drawing of diagrams pointing to the number of groups and the divisor is then interpreted as the number in each group. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation.	3
	L1	E6	15 ÷ 5= on flashcard and T wrote the answer 3; T produced answer in the same way as in L1:E5	Bald calculation – Same reason as L1:E5	3
	L1	E7	20 ÷ 2 = on flashcard and T wrote in the answer 10; T produced answer in the same way as in L1:E5	Bald calculation – Same reason as L1:E5	3
	L1	E8	15 ÷ 3 = on flashcard. T told Ls to make groups of three with counters. T: "What is the answer?" Ls: "Five!" T: "How many big circles?" Ls: "Five". T drew four circles with three dots in each.	Bald calculation interpreted as a grouping problem with immediate answer which produced diagrams through grouping through 1 by 1 distribution - Instead of drawing five circles the teacher only drew four circles and did not correct the error. Incoherent: incorrect diagram with no answer provided.	4
13 Sept 2011	L3	E4	T wrote 15 ÷ 3 =. T stated: "These types of number sentences have nothing to do with grouping, they can be answered as a mental exercise using multiplication". T read the number sentence and added "5" as the answer.	Bald calculations answered correctly and immediately. Coherent	1
	L3	E5	T wrote 24 ÷ 3 = L: "Twenty-four divided by three is eight" T: "How did you get to the answer?" L: "We group twenty-four into threes". T added "8" as the answer to the symbolic calculation.	Bald calculations answered correctly and immediately. Coherent	1
	L3	E6	T wrote 30 ÷ 5 = T: "How did you get to the answer?" L: "Maybe a person could draw 30 circles and make groups into five" which T accepted. T then added the number "6" to the division symbolic calculation.	Bald calculations answered through a verbal description of grouping through 1 by 1 distribution based on the divisor to produce the answer. Coherent	1

Table 28 Enactments of bald calculations in Mary's lessons

	Bald Calculations (n=8/20 episodes)
Initial signifier (dividend in brackets)	L1 – E4[10], E5[15], E6[20], E7[15], E8[15] L3 – E4[15], E5[24], E6 [30]
Signifier sequence	Written symbolic signifier > correct verbal answer > drawing of (incorrect) # of circles > dots are drawn into each circle through grouping by 1-1 distribution (L1: E8 - Incoherent) Written symbolic signifier > verbal sharing situation and written answer > circles are drawn based on quotient indicating the # of groups > dots are drawn in each circle linked to the divisor (L1: E4, E5, E6, E7 - Some ambiguity) Written symbolic signifier > verbal replay of drawing groups of circles based on divisor > written number answer (L3: E6) Written symbolic signifier > correct immediate and written answers (L3: E4 & E5)
Calculation strategy	Over – circling of groups (1/8 episodes – L1:E4) Grouping through 1 by 1 distribution based on the divisor (1/8 episodes – L3: E6) Skip counting (3/8 episodes – L1:E4, E5, E6 repeated episodes (checking results) Recalled number facts (6/8 episodes – L1: E5, E6, E7, E8, L3: E4, E5)

Of the eight bald calculations, four can be described in terms of sharing situations. Three of the eight were solved by a grouping approach, and one by recalled multiplication facts. The linking of sharing situations and grouping language in the bald calculation category points to a more extended inclusion of division situations than the summary numbers detailed in Table 22 suggest, while also pointing to ambiguities which are discussed later. In relation to signifier sequences, in 5/8 episodes beginning with bald calculations, the solving of the bald calculation occurred via the construction of groups of circles/dots in equal groups or arrays to produce the result, which was usually then written or stated as a verbal number sentence (5/8 episodes). In spite of the occurrence of translation of bald calculations into sharing situations, only one type of distribution was observed: grouping through one-by-one distribution based on the divisor. The four kinds of calculation strategies presented or accepted involved: the use of recalled number facts (6/8 episodes); skip counting in different multiples as a way of checking the answer in 3/8 episodes, together with grouping through one-by-one distribution based on the divisor (1 episode) and "over circling groups" (1 episode).

Of interest, again in these bald calculations a grouping orientation predominates. Drawing of the groups through a one-by-one distribution and skip counting were evident in some episodes. The use of recalled facts is also seen in most of the episodes, with the answer frequently offered prior to any problem-solving process – suggesting, once again, that one-by-one distribution may actually have pulled some learners back to more rudimentary methods than needed (Kouba, 1989). Although some of the episodes were interpreted as a sharing situation, no incidence of sharing actions was evident. Mary seems to view division in terms of sharing *situations*, but also in terms of grouping-oriented *calculations*. This happens even in calculations that she interprets, initially, in sharing terms.

Nine of the twenty episodes were not analysed (9/20 episodes). Two episodes focused on group or individual work in which the teacher did not lead (2/9 episodes). In three episodes there was a focus on restating prior work on division and identifying mathematical notation (3/9 episodes). Three episodes focused on skip counting as the repetition of specific multiples and the reading of division facts from sheets with no mediation from the teacher (3/9 episodes). The last episode focused on recapping homework from the previous day (1/9 episodes).

5.3.4. Summary

Across the four lessons, 8/20 of the episodes focus on bald calculations with half of the episodes incorporating divisional situational word problems. Bald calculations were frequently translated into sharing situations and/or grouping calculation strategies. No tasks contained a remainder. Furthermore, across five episodes pointing to a sharing situation or problem, an initially accepted quotient was used to create the number of groups in a diagram as opposed to the divisor informing the number of groups to be drawn. The use of more sophisticated calculation strategies like skip counting was sporadically evident. While the other G3 and G4 teachers used skip counting at the beginning of the lesson and not within the lesson, this teacher, in some episodes made use of skip counting in the solving of division tasks.

5.3.5. Analysis of coherence in Mary's lessons

Table 29 Analysis of episodes within signification pathways framework

Number of lessons	Category 1- Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	0		4	1	3	8
L2			1	0	1	2
L3	4			0	2	6
L4	1				3	4
Total	**5**		**5**	**1**	**9**	**20**

Table 29 summarises the episodes categorised within the signification pathway framework. For Mary, coherence was evident in n = 5/20 episodes. Episodes involving her working with bald calculations made up most of the coherent category (3/5 episodes). In two coherent episodes, she solved a grouping situational problem (L3:E3 & L4:E3). In contrast to Cathy, there was no reference to the kinds of limitation-related statements, perhaps also linked to the problems being answered through recalled number facts.

Five of the episodes were classified as having "some ambiguity". In one episode, a signified came to be associated with two contradictory signifiers – as seen in some of Cathy's episodes - even though within the pathway, each signifier could be linked back to the previous signifier. In 4/5 episodes, there was a shift between signifiers concerning division models with no acknowledgement of the shift. In the one episode identified as incoherent, an error was made during the solving of the division task, that was not corrected by the end of the episode.

Taken together, the analysis indicates that 6/20 episodes have been coded as containing either ambiguity or error, and if only coded episodes are looked at, it is 6/11 episodes - a higher in proportion than evident in Cathy's lessons. Of the 11 episodes which were coded, 8 episodes began with bald calculations, a higher proportion of work with bald calculations in comparison to Cathy, but in 4 of these episodes the teacher incorporated divisional situations. Mary included two grouping situations in her teaching, but some episodes which included sharing situations and bald calculations were solved through grouping-based approaches, though these were sometimes used after the quotient was known and inscribed into her diagrams. Furthermore, the uncoded episodes also show that Mary focused extensively on "vocabulary" and general division talk separately to her work with calculations – a feature that was more evident in the lower socio-economic category schools.

5.4. Hazel, G3: School C

Hazel taught a G3 cohort of previously disadvantaged learners in a previously disadvantaged school. In her class there were between 30 and 35 learners during the observed lessons. The school fees were R78 per month, and it was a quintile 5 school. In this case study, I identified 25 episodes across five lessons (see Appendix C). Table 30 details the attention given to different division foci in Hazel's lessons. Sections 5.4.1-5.4.2 provide an analysis of Hazel's teaching across sharing situations and bald calculations respectively. Section 5.4.3 summarizes key ideas related to sharing and bald calculations. Finally, in Section 5.4.4, I discuss all the episodes categorized within the coherence table.

Table 30 Weighting of episode types across Hazel's lessons?

Sharing situation	Grouping situation	Bald Calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
1	0	13	11

As with Mary, Hazel's division teaching was unequally distributed across division, starting with signifiers, with much more use of initial bald calculations and general vocabulary, and only 1/25 episodes dealing with situations: sharing in this case. The number ranges were low in relation to the FFL curriculum specifications outlined in Chapter 2, with dividends of less than 37 in 13/25 episodes in which dividends were included. Again, I begin by extracting the sharing episode and all the bald calculation episodes and placing them in separate tables.

5.4.1. Sharing situations

Table 31 Description of sharing situation episodes across lessons

Date	Lesson	Episode	Task & its enactment	Division focus in episode/ coherence-related comment	Coherence category
13-May-11	L4	E3	T stated: "Dad has got two thousand sunflower seeds. He plants them into two rows. How many seeds will he plant in each row?" Ls responded: "One thousand". T then asked: "What kind of number sentence will I write for that sum?" A L stated: "Two thousand divided into two equals one thousand". The T wrote 2000÷2=1000 on the board. T reiterated that dividing is "to share equally".	Sharing situation with correct answer offered immediately. Coherent	1

Table 32 Enactment of a sharing situation in Hazel's lessons

	Sharing situations (n=1/25 episodes)
Initial signifier (dividend in square brackets)	Oral/written linguistic situations (1/1 episode): L4 – E3 [2000]
Signifier sequence	Word problem > correct immediate verbal number sentence with answer > written number sentence with answer (L4: E3)
Calculation strategy	Recalled number facts – (L4, E3)

In relation to signifier sequence, this sharing episode began with a verbal sharing situation, to which an answer was immediately provided, followed by a verbal number sentence with the answer, which was then written as a written number sentence.

The dividend value of 2000 in this episode is greater than specified in the curriculum, but 2000 divided by 2 is a relatively easy problem to visualize and solve mentally, with the immediate answer given by a learner suggesting this. However, this large dividend number was only seen in one episode. Overall, there was very limited use of division situational problems, in that across the five lessons, only one sharing problem was evident.

5.4.2. Bald calculations

Table 33 Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATIONS EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
10-May-11	L1	E3	T stated: "Times is the opposite of division". On a 200 number chart on the board, T covered the multiples of 2 from 2 to 24 with magnets. Writing 12x2=24 on the board, T asked for a division symbolic calculation. Ls offers the division calculation 24÷2=12. T stated: "Whenever you do a divided sum the biggest of the numbers always goes first".	Bald calculation with correct answer offered immediately – limitation: in division the biggest number always goes first.	2
	L1	E4	T wrote 24÷2=12 and added a printed three column sheet on the board. She mentioned that because the divisor was 2 she would use two columns. She asked: "How many counters do I need, 24, 2 or 12?". T rejected an offer of 12 stating 24 as the answer. She reiterated: "The biggest number ..?" Ls responded: "always goes first". T picked out 24 magnets and a L distributed these between the two columns one by one. T then counted the number of magnets in one column and stated 12 as the answer.	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution – limitation: in division the biggest number always goes first.	2
11-May-11	L2	E2	T wrote 1x2=, 2x2= and 3x2=. Ls completed these calculations correctly. Next to each multiplication calculation the T drew one/ two/three circles respectively and placed two magnets in each. She then created division calculations as follows 2÷2=1, 4÷2=2, 6÷2=3, drawing two circles next to each division calculation. Pointing to the divisors, she stated: "We are sharing equally, only share between two" pointing to the two columns on the printed three columns sheet. T then distributed one magnet at a time between the two columns or drew in one dot at time between the two circles for 2÷2=1, 4÷2=2. To solve 6÷2=3, a L placed three counters in the first column and then three counters in the second column and then proceeded to draw three dots in the first circle and then three dots in the next circle. T then wrote 8÷2= 4 on the board. A L came to the board, drew two circles and proceeded to draw a dot one at a time between the two circles until four dots appeared in each circle.	Bald division calculations with correct answer offered immediately, answer verified through 1 by 1 sharing distribution and sharing through complete composite units - Ambiguity in teacher producing two contradictory signifiers for "2": "2 magnets inside each circle" for multiplication calculations and as "2 circles" for division calculations without acknowledging this shift. Ambiguity is seen in that one number is associated with two contradictory signifiers.	3
12-May-11	L3	E2	T wrote 6x2=12, 7x2=14,8x2=16 on board, and rewrote them as 12÷2=6, 14÷2=7, 16÷2=8. Saying that more division calculations could be made from the same numbers, T wrote 12÷6=2, 14÷7=2 and 16÷8=2 on the board. She stated again that the biggest number always goes first and added: "When we divide the answer is always smaller and when you multiply your answer is always bigger".	Bald calculation with correct answer offered immediately - Limitation: the biggest number always goes first in division and the answer is always smaller. [Also, in multiplication the answer is always bigger].	2
	L3	E3	T wrote x2 and ÷2 on board and explained that when multiplying by two, doubling occurs which is repeated addition and when you divide by 2, halving happens. She wrote 2x2=4, 3x2=6 and 4x2=8 to illustrate doubling and then stated that the opposite of doubling is halving. Writing 2÷2=1, 4÷2=2,6÷2=3, she linked these calculations to halving saying: "half of two is one, half of four is two and half of eight is four". She mentioned that halving meant: "Share equally".	Bald calculations with correct answer offered immediately. Doubling highlighted as any number multiplied by two and halving is any number divided by two. Coherent	1
	L3	E4	T said that not all division calculations can be shared equally and wrote 7÷3=_. T asked for seven magnets and used a three column printed sheet. She distributed the magnets one at a time between the three columns producing two magnets in each column with one magnet outside the printed sheet. She then wrote "r" next to the outside magnet. Pointing to the two magnets in a column and the one magnet outside she wrote 2 r 1 as the answer.	Bald calculation with remainder with correct answer offered immediately, answer verified solved through sharing 1 by 1 distribution. Coherent	1
	L3	E5	T wrote 9 ÷2= and took out nine magnets to distribute between two columns stating the divisor in the division task was 2. T distributed the magnets one at a time between the two columns (labelled Tom and Dick) resulting in 5 magnets (Tom) in one column and 4 magnets (Dick) in another column. She asked: "Did I share them equally?" Ls said: "No". T asked: "Who got more than the other?" Ls responded: "Tom". T asked: "How can I make it even?" A L	Bald calculation with remainder solved through 1 by 1 sharing distribution. Coherent	1

			responded that the T should take away one from Tom which she did. Stating that there were 4 magnets in each column with one as the remainder T wrote "2 r 1" as the answer.		
	L3	E6	T wrote 36÷3=. Ls answered: "12". T stated: "36 needs to be divided by 3" and drew three circles as she pointed to 3 in the division calculation. T asked: "How much are you going divide?" Ls responded: "36". T drew in crosses one at a time between the three circles. T pointed at one circle and counted crosses in threes, then asked: "How many will each one get?' Ls responded: "Twelve". T wrote "12" as the answer.	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution and skip counting. Coherent	1
	L3	E7	T wrote 15÷5= on the board and solved the task as in L3:E6	Bald calculation – worked with as L3:E6	1
	L3	E8	Pointing to 12÷6= T asked: "How many groups would she need to draw?" Ls responded: "6". T drew six circles on the board and asked: "How much will she have to share?" Ls answered: "Twelve". Ls drew crosses one at a time between the six circles counting up to 12. T asked: "How much did each one get?" A L stated: "2". T wrote in the answer "2".	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution. Coherent	1
	L3	E10	T wrote 37÷3=. She said: "Thirty-seven beans divided by three". On printed three column mats Ls distributed 37 counters one at a time between the three columns resulting in 12 counters in each column with one counter on the outside. T asked: "How many did each one get?" Ls responded: "12" as T wrote in 12 as the answer, then L stated: "and 1 remainder" and the T added "rem 1"	Bald calculation with remainder solved through 1 by 1 sharing distribution. Coherent	1
	L5	E2	T pointed to 6 x 5= and noted that 5 x 6 is the same and wrote 30 as the answer for both. She remarked that the opposite of multiplication is division and asked: "What happens when we divide, does our answer get bigger or smaller?" Ls replied: "Smaller". T then asked Ls for a division calculation for 6 x 5=30. A L answered: "Thirty divided by five equals six". T wrote 30÷5 = 6 and asked: "How do I draw that sum? How many groups?" Ls replied: "Five groups". T drew five circles and reiterated: "I must share between five groups". T then asked: "How much must I divide?" Ls responded: "Thirty". T then asked: "Each one will get?" A L answered: "Six". T then drew in dots one at a time between the five circles resulting in six dots in each circle. "Six groups" was accepted again as the "number of groups".	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution - Limitation: in division, the answer is always smaller.	2
	L5	E3	T pointed to 9 x 3= and said: "What is the answer to nine times three?" 27 was offered and written in as the answer. T asked for a linked division calculation and accepted and wrote "27÷3=9" on the board. As in a previous episode, T accepted "3" as number of groups and "9" as number in each group. T then asked for a different division calculation. A L suggested: "Twenty-seven divided by nine equals three". T then wrote "27÷9=3". T stated: "If I had to draw that sum, it would be nine groups and I needed to share twenty-seven between the nine groups. There would be three in each group".	Bald calculation with correct answer offered immediately. Coherent	1

Table 34 Enactments of bald calculations in Hazel's lessons

	Bald Calculations (n=13/25 episodes)
Initial signifier (dividend in brackets)	L1 – E3 [24], E4 [24] L2 – E2 [6] L3 – E2 [16], E3 [4], E4 [7], E5 [9], E6 [36], E7 [15], E8 [12], E10 [37] L5 – E2 [30], E3 [27]
Signifier sequence	Written symbolic signifier in the form of a multiplication number sentence with answer > written division number sentence with answer derived from the multiplication number sentence > divisor interpreted as the # of groups > sharing though 1 by 1 distribution through drawing dots between circles or distribution with counters between columns on a chart or sharing through CCU distribution by drawing dots between circles to verify the answer (L2: E2 – Some ambiguity) Written symbolic signifier > divisor interpreted as the # of groups > sharing of counters through 1 by 1 distribution between columns to produce the verbal or written answer (L1:E4 – Limitation, L3:E4, L3:E5, L3:E10) Written symbolic signifier in the form of a multiplication number sentence with answer > written division number sentence with answer derived from the multiplication number sentence > divisor interpreted as the # of groups > sharing though 1 by 1 distribution through drawing dots between circles (L5: E2 - Limitation) Written symbolic signifier in the form of a multiplication number sentence with answer > written division number sentence based on multiplication number sentence with immediate answers (L1: E3 – Limitation & L3 E2 – Limitation) Written symbolic signifier > drawing of circles and dots to signify the divisor as # of groups > sharing through 1 by 1 distribution between circles > dots in circles are counted through skip counting to produce the written answer (L3: E6, E7, E8) Written symbolic signifier in the form of a multiplication number sentence with answer > written division number sentence with answer derived from the multiplication number sentence > divisor interpreted as the # of groups > converting the previous written division number sentence and answer into a new written division number sentence including the same values and answers produced through recalled number facts (L5: E3) Written symbolic signifiers in the form of multiplication number sentence with answer to show that when you double you multiply by 2 > written division number sentence with answer derived from the multiplication number sentence to show that when halving divide by two (L3: E3)
Calculation strategy	Sharing through 1 by 1 distribution (4/13 episodes – L3: E4, E5, E8, E10) Skip counting - (2/ 13 episodes - L3: E6 & E7 repeated episodes) checking result Recalled Facts – (8/13 episodes -L1: E3, E4, L2: E2, E6 & L3, E2 E3 & L5: E2 & E3)

In relation to signifier sequences, the solving of bald division calculations started from a symbolic multiplication calculation in 6/13 episodes, with the multiplications used to form a symbolic division calculation through the use of diagrams (as columns on a chart with counters or circles and dots inserted in them) and then moved to production of written answers or to deriving another division calculation from a previous division calculation (6/13 episodes). Anghileri (2006) notes that expressing the same multiplicative relationship between three quantities in different ways can help develop an awareness of the central links between multiplication and division. In other episodes beginning with bald division calculations, the solving of the division bald calculation occurred with division number sentences that were solved through the use of diagrams (as columns on a chart or drawing circles and inserting dots) or counters (7/13 episodes). In contrast to Cathy and Mary, Hazel worked with the divisor interpreted as the number of groups as opposed to the number in a group (Beckmann, 2011) across all of the 13 episodes.

The demonstration or acceptance of sharing through one by one distribution into a sharing action (9/13 episodes) suggests that Hazel considered this to be an acceptable approach even in low number range episodes (e.g. L2, E2 involving a dividend of 6). The use of skip counting was evident in two episodes to check that the answer was correct and there was a sharing through CCU distribution in one episode.

In 10/13 of the bald calculation episodes, learners offered the correct answer to the division calculation immediately, suggesting recalled number facts. This points, once again, to sharing through one-by-one distribution and skip counting being unnecessary for several learners, and points to the "pulling back" into less efficient working that has been noted in Venkat & Askew's (2018) writing.

11/ 25 episodes were uncoded. Of these 11 episodes which were not analysed, 5/11 focused on group work in which the teacher did not lead or involved preparation for assessments. In 4/11 episodes there was a focus on recapping prior work on division-based ideas or vocabulary. In one of the 4 episodes the teacher noted that "the biggest number goes first". The last two episodes focused on skip counting as the repetition of specific multiples and how skip counting is linked to multiplication (2/11 episodes).

5.4.3. Summary

Hazel's division instruction consisted predominantly of bald calculations worked with through sharing actions with one episode including a sharing situation. Focusing on only one situational divisional word problem appears to be a feature that was more evident in schools in a low socio-economic category. In only using sharing actions, the learners in Hazel's class were not exposed to any grouping actions, pointing to a one-sided encounter with division for these learners. However, there was evidence of the use of doubling, halving and remainder as specified within the FFL curriculum, with a strong emphasis on the inverse relationship between multiplication and division, set predominantly within low dividend number ranges.

5.4.4. Analysis of coherence in Hazel's lessons

Table 35 Analysis of episodes within signification pathways framework

Number of lessons	Category 1- Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	0	2	0	0	3	5
L2	0	0	1	0	1	2
L3	7	1	0	0	2	10
L4	1	0	0	0	3	4
L5	1	1	0		2	4
Total	**9**	**4**	**1**	**0**	**11**	**25**

Table 35 summarises the episodes as categorised within the signification pathway framework. Coherence was evident in n= 9/25 episodes, a little over a third of Hazel's instructional episodes. Recalling that there were 13 episodes overall involving bald calculations, it is interesting to note that 8/13 of these episodes were coherent. Episodes classified as coherent with limitations were evident in four episodes (n=4/25 episodes). As with Cathy, two linked notions underpinned the limitations category: "division always makes smaller" and the "biggest number goes first". The "biggest number goes first" is also seen in one other episode which was "uncoded" showing that the notion was more widely evident in her teaching than reported in only the coded episodes. The total avoidance of grouping is a key limitation in Hazel's instruction. One episode was classified as having "some ambiguity" with one number associated with two contradictory signifiers, leading to ambiguity and creating difficulties for learners as they solve tasks (Amit & Gilat, 2012).

5.5. Asha, G4: School A

In this fourth case study, I examine Asha's lessons. Asha taught a G4 cohort of advantaged learners in a previously advantaged school. I was following the same cohort of learners through from Grade 3 and there were between 23 and 30 learners in the class during the observed lessons. As noted earlier, the school fees were R 983.00 per month in this quintile 5 school. This case study had 25 episodes demarcated across three lessons (See Appendix D)

Table 36 Weighting of episode types across division foci in Asha's lessons

Sharing situations	Grouping situation	Bald calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
3	0	9	13

In Asha's lessons calculation situations were split between sharing situations and bald calculations (12/25 episodes) and other division-related foci in 13/25 episodes. The number ranges in sharing situations were higher than seen in Grade 3 lessons in School A, but low in relation to the FFL curriculum specifications with single-digit dividends in 5/25 episodes and two-digit dividends in 9/25 episodes. Dividends leading to remainders in relation to the divisor were included in 7 episodes. While three-digit dividends were included, the number range across sharing situations and bald calculations was below the FFL specifications which suggests that division tasks should have dividend values up to 2000.

5.5.1. Sharing situations

Table 37 Description of sharing situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (SHARING SITUATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
21 May 2012	L1	E5	T took out thirty marbles. She asked: "If you are dividing the marbles among your four friends and you, how many bowls should I have? Ls shouted: "Five". A L distributed two marbles at a time between the five bowls and then she distributed another two marbles at a time between the bowls and then finally another two marbles at time in each bowl. T then asked: "So how many marbles do you have in each bowl?" The L replied: "Six".	Sharing situation worked with through partial composite unit distribution sharing action. Coherent	1
	L1	E6	T then called another L to the front table and added another marble to the thirty marbles. She asked: "What are you going to do?" A L distributed the marbles one at a time between the five bowls. T asked: "What is left, you have six marbles in each bowl, what is in your hand, what are you going to do with it? A L stated: "I am going to share it?" T then asked: "What do you call this? Another L shouted: "Remainder". T accepted the answer. Another L stated: "Remainder one". T stated: "Six remainder one will be your answer".	Sharing situation with remainder (based on word problem offered and worked with through a one by one distribution sharing action). Coherent	1
22 May 2012	L2	E8	T showed the Ls a bowl of sweets and said that there were 71 sweets in the bowl. With three other bowls. T then stated: "What I want you to do is that I want you to divide these sweets for me into three containers (points to the three containers on the table). How are you going to do this?" Ls distributed first two sweets, then three sweets, then four sweets at a time between the three containers. T stated that after dividing the sweets equally amongst the three containers there was 1 remainder and showed the class one sweet. T asked: "How many do you think she has in each of the containers?" A L confirmed by counting that there are twenty-four sweets in one of the containers. Writing 71 ÷ 3 in a short division bracket and using the short division algorithm, the T produced the answer "23 r 2". She then noted that L must have counted incorrectly as she only had 1 remainder. As another L came to the front to distribute the sweets, T recounted the sweets in 2s and established that there were 71 sweets. The L distributed the sweets two at a time between the three containers, but again ended with one sweet left over. T reworked the short division algorithm and stated that the remainder should be 2. T called another L to the front who distributed the sweets two at a time between the three containers, and still ended up with one remainder. T returned to the board and again, went through the short division algorithm.	Sharing situation with remainder (based on word problem offered and worked with through partial composite unit distribution sharing action). The sharing action then produced an answer 23 remainder 1. To verify the answer, the T used a short division algorithm which produced the answer 23 remainder 2. Incoherent: incoherence of answers produced by algorithm and by sharing actions is not dealt with.	4

Table 38 Enactments of sharing situations in Asha's lessons

	Sharing situations (n= 3/25 episodes)
Initial signifier (dividend in square brackets)	Oral/written linguistic word problem situation (3/3 episodes): L1 – E5 [30], E6 [31] L2 – E8 [71]
Signifier sequence	Word problem > situation replayed with counters involving a sharing through partial composite units to produce the verbal answer with a remainder > solving a short division algorithm to verify the answer > a subsequent replay of the same situation with counters to produce incorrect answer (L2: E8 – Error) Word problem > situation replayed with counters involving a sharing either through 1 by 1-unit distribution or partial composite unit distribution to produce the verbal answer with/without a remainder (L1: E5, E6)
Calculation strategy	Sharing through 1 by 1 distribution (1/3 episodes: L1 E6) Sharing by partial composite units (2/3 episodes: L1: E5, L2:E8) Short division algorithm (1/3 episodes: L2: E8 – Repeated episode)

In relation to signifier sequences, the 3 episodes beginning with verbal/written sharing situations were solved through the use of counters or short division algorithms to produce either a verbal or written answer with and without a remainder. The distribution itself occurred in two ways, either through sharing by partial composite units with different values until the dividend value was reached (2/3 episodes) or sharing through one-by-one distribution (1/3 episodes).

Across the first two connected episodes (with the second involving a "1 more" dividend value), a sharing action was coherently enacted in both cases. Of interest was the lack of use of the answer produced in the first episode to help answer the question in the next episode; instead, there was a reversion to restarting the sharing. This kind of "separate" working with examples has been noted in the South African literature (Venkat & Askew, 2018). In the third example, the two different remainders from a concrete sharing action and a short division algorithm created dissonance, but this was not resolved. This episode points to extended time and potential for errors in keeping track of quantities when using concrete unit counting of objects to solve division tasks with larger dividend values. In contrast, the use of the short division algorithm with a large number range was effective in finding a precise answer (Van Putten et al., 2005). The failure to deal with two different answers may result in disconnections between algorithms and the distributing actions underlying them – a point that has been noted in the international literature (Stein, Engle, Smith, & Hughes, 2008). The inclusion of sharing through "partial composite units" reflects a second type of sharing action that was incorporated (1/3 episodes).

Taken together, what this analysis suggests is some shift towards more truncated symbolic signifiers in the use of a short division algorithm which can be used to more efficiently solve division problems with larger dividend values with increased number ranges. Together with this progression towards more truncated symbolic signifiers, a shift towards the use of more efficient calculations strategies was evident in the use of sharing through "partial composite units". Of the six teachers, Asha was the only one to use distribution by partial composite units when solving a sharing situation. While her teaching indicates progression, she does not move towards anything more than partial distribution in 2s for a dividend of 30 divided by 5.

5.5.2. Bald calculations

Table 39 Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATIONS EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
21 May 2012	L1	E4	T wrote 45÷5 on the board. She asked: "So how did you do it?" A L came to the board and drew a short division bracket and placed 45 inside the bracket and 5 on the outside. T then stated: "Does five go into four?" The L replied: "No". T asked: "Does five go into forty-five?" The L replied: "Yes" and the T asked: "How many?" L answered: "Seven". T said: "No". L then said: "Nine" and wrote "9" on top of the short division bracket.	Bald calculation is enacted through short division algorithm with learners input. Coherent	1
	L1	E7	Saying that the "breaking down" method would be used, T wrote "145÷5" and asked: "How do I break this up?" A L stated: "One hundred plus forty plus five". T wrote "100+40+5". T then said: "One hundred divided by five plus forty divided by five plus five divided by five" and wrote "(100÷5) + (40÷5) +(5÷5)". T divided each number by 5 resulting in "20 +8+1" and asked: "What do you think we can do?" A L stated: "We need to add them (T pointed to "20 +8+1" on the board) and proceeded to add and produced the answer "29".	Bald calculation is enacted through the use of expanded notation and sum of decomposed divisions with learner offers of immediate answers. Limitation: the method of using expanded place value notation to determine the answer works well for division by 5 but is less effective more generally for divisors other than 5/10.	2
	L1	E8	T then wrote "145÷5" in a short division bracket and asked: "Does five go into one (points to the "1" in "145"). The Ls replied: "No". T asked: "Does five go into fourteen (points to the "14" in "145")? The Ls replied: "Yes" and the T asked: "How many times?" The Ls shouted: "Twice" and the T wrote "2" on top of the short division bracket. T asked:" So what do we have left?" A L said: "One remainder" which the T did not accept. T then stated: "No" and wrote a small "4" before the "5" to make "45". T asked: "So five goes into forty-five how many times (pointed to the "45"). The Ls shouted: "Nine" and the T wrote "9" next to the "2" on top of the short division bracket producing the answer "29". T then made the point that they had produced the same answer when they used "expanded notation" and "short division algorithm".	Bald calculation is enacted through short division algorithm with learners input. Coherent	1
	L1	E9	Focusing on "145÷5". T wrote 145 - 5 in subtraction column algorithm format and asked: "What is the answer?" Together with the Ls, 5 was repeatedly subtracted from 145 until 105. The T asked: "Do you understand this method, so you continue until you come to?" The L shouted: "Zero" which the T agreed with. The T illustrated that repeated subtraction took longer by saying "it will go on and on and on and take the whole period, therefore short division works for me".	Bald calculation is enacted through repeated subtraction with learners input. However, the T does not enact repeated subtraction until the end and does not count the number of times 5 was subtracted to show the answer for the division calculation. Coherent	1
	L1	E11	T represented 43÷2 in a short division bracket. T asked: "Two goes into four how many times?" The Ls shouted:" Two" and the T wrote "2" on top of the short division bracket. T then asked: "Two goes into three?" the Ls replied: "Once". T wrote "1" next to the "2" on top of the short division bracket. T asked: "Where do you put your remainder?" The Ls stated: "Next to the one". T wrote in "r 1" producing the answer "21 r 1".	Bald calculation with remainder is enacted through short division algorithm with learners input. Coherent	1
22 May 2012	L2	E4	T drew a short division bracket and added a "635" inside the bracket and then "3" on the outside and produced the answer "211 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
	L2	E5	T drew a short division bracket and added a "815" inside the bracket and then "3" on the outside and produced the answer "271 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
	L2	E6	T drew a short division bracket and added a "965" inside the bracket and then "3" on the outside producing the answer "321 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
23 May 2012	L3	E6	T drew a short division bracket and added a "524" inside the bracket and then "7" on the outside producing the answer "74 r 6" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1

Table 40 Enactments of bald calculations in Asha's lessons

	Bald Calculations (n=9/25 episodes)
Initial signifier (dividend in brackets)	L1 – E4 [45], E7 [145], E8 [145], E9 [145], E11 [42] L2 – E4[635], E5 [815], E6[965] L3 – E6 [574]
Signifier sequence	Written symbolic signifier > dividend broken up into place values decomposition > each partial sum is divided by divisor with written answers produced through learner immediate offers > partial quotients are added to produce final written answer (L1:E7 – Limitation) Written symbolic signifier > the use of repeated subtraction to zero (suggested as alternative, longer method) (L1:E9) Witten symbolic signifier > the use of short division algorithm to produce written answers (L1:E4, E8, E11, L2: E4, E5, E6, L3: E6)
Calculation strategy	Repeated subtraction (1/9 episodes - L1:E9) Short division algorithm (7/9 episodes - L1:E4, E8, E11 & L2: E4, E5, E6 & L3: E6) Recalled number facts (1/9 episodes – L1: E7)

In relation to signifier sequences, in the 9 episodes beginning with bald calculations, the solving of the bald calculation occurred through breaking down of the number by place value decomposition (1 episode) or repeated subtraction (1 episode) or short division algorithms (7 episodes). All calculations were dealt with as "pure" number problems for calculation – there was no attempt to link these calculations with division situations or with the concrete distributing actions seen in her work with sharing and grouping situations.

However, the same division task was solved in three different ways: through the breaking down of the dividend into its place value and then dividing each place value by the divisor, through the use of repeated subtraction (noted as a longer method, and not followed through to produce the final answer), and through the use of a short division algorithm. The short division algorithm was described as the preferred strategy for solving division tasks. The use of repeated additive or subtractive strategies (Kouba, 1989), short division algorithms and acceptance of recalled number facts point to more sophisticated division calculation strategies being used with larger dividend values, and incorporates the inclusion of more abstract signifiers. There was a move to higher number ranges than seen in the Grade 3 lessons in this school.

Taken together, what this analysis reveals is Asha’s contrasting of what she sees as more and less efficient calculating strategies where the number range increases and her promotion of the use of more sophisticated calculation strategies – although the limitations of place value decomposition for division are not dealt with.

The 13 episodes which were not analysed focused on individual work in which the teacher did not lead (1/13 episodes), reciting of multiplication facts (2/13 episodes), mental maths based on general operations, homework and assessment activities (10/13 episodes). The extent of these "other" episodes in the lessons of this particular teacher limited the extent of direct working with division calculation approaches in the lessons where division was the named focus.

5.5.3. Summary

In Grade 4 lessons, the FFL focus stipulated is on division calculation tasks which emphasise four-digit dividend divided by one-digit divisor tasks. Asha included three 3-digit dividends in her bald calculations, as well as 2-digit dividends. Her number range in sharing situations was lower than required by the FFL, but did include one example that went beyond 10 times the divisor, and thus beyond the basic multiplication facts. She also paid some attention to different calculating strategies and the relative efficiency of these. Whilst there were no grouping episodes, the illustration of repeated subtraction and the use of short division language do tend to include attention to division as grouping in the "measuring" of the dividend in terms of the divisor. 13 of the episodes were coded as "other related division foci", and 5 of the episodes focused on mental maths activities and multiplication fluency. Asha’s focus on multiplication fluency appeared to assist learners in the tasks which required the use of recalled facts within a short division algorithm. The teacher also showed how the same bald calculation could be solved in three different ways. As in Hazel’s case, there was only one situational problem observed across the three lessons involving the use of sharing situations, where sharing by "partial composite units" was incorporated.

5.5.4. Analysis of coherence in Asha's lessons

Table 41 Analysis of episodes within signification pathways framework

Number of lessons	Category 1- Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	6	1			4	11
L2	3			1	4	8
L3	1				5	6
Total	**10**	**1**		**1**	**13**	**25**

Coherence was evident in n = 10/25 of the instructional episodes in Asha's lessons. 8/10 of the episodes labelled coherent focused on bald calculations. Almost all of Asha's work with bald calculations was coherent, with only 1/9 bald calculation episodes coded as coherent with limitation, due to the place value decomposition approach described earlier. In the other 2/10 coherent episodes, the teacher solved sharing situational problems.

In 5/8 coherent bald calculation episodes Asha made use of short division algorithms. The literature argues that when short division algorithms are effectively and efficiently used by teachers, learners are shifted from an informal approach to more formal approaches in solving division tasks. Work in the "uncoded" episodes on practice of multiplication facts to assist in solving a short division algorithm aligned well with the attention paid to formal calculation strategies. Bald calculations, in Asha's lessons had a higher number range than in those of the other two G4 teachers. The one episode classified as incoherent involved a situational sharing problem, where distributing counters did not provide the same answer for the sharing action as the short division algorithm. In this case, the teacher made no attempt to either explain what might have caused the difference, or to work towards a correct answer.

Overall, analysis of lesson episodes in Asha's class indicates that episodes of bald calculations dominated in her teaching and in her enactment of coherence. In contrast to Cathy, who taught the G3 class at school A, Asha's teaching shows a marked shift towards working almost solely with bald calculations, alongside widespread use of a short division algorithm. In this move to taking a "pure calculation" approach, there is much more coherence as a proportion of the total number of calculation focused episodes, but this comes at the cost of a more restricted focus on bald calculations and on calculation strategies, with the short division algorithm emphasized.

5.6. Joy, G4: School B

Joy taught a G4 cohort of previously disadvantaged learners in a previously advantaged school. There were between 30 and 35 learners in the class during the observed lessons. The school fees were R 375 per month, and the government identified the school as a quintile 5 school. Of the three schools in the study, this school was categorised as less privileged than the most privileged school and more privileged than the school with the lowest socio-economic background. In this case study, I delineate 32 episodes across three lessons (See Appendix F)

Table 42 Weighting of episode types across division foci in Joy's lessons

Sharing situations	Grouping situations	Bald calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
7	2	17	6

There was an uneven weighting of episodes across sharing situations, grouping situations, bald calculations and other division-related foci in the 32 episodes. The number ranges in sharing and grouping situation-based episodes was lower than specified in the FFL curriculum as outlined in Chapter 2, and dividends of less than 100 were used in 28/32 episodes. Dividends leading to remainders in relation to the divisor were included in 10 episodes.

5.6.1. Sharing situations

Table 43 Description of sharing situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (SHARING SITUATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
23 April 2012	L1	E1	T stated that her mother bought 6 sweets for her and 2 Ls and she wanted to know how many did each one eat. A L replied: "2". Ls took out six counters and shared them equally between 3 people. The Ls distributed the counters two at a time making 3 groups of 2.	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L1	E2	The T stated that her mother had R50 and wanted to share it amongst 5 family members and asked: "How much will each one get?" A L responded: "10 Rand". T then drew five "R10" beneath each other in a column and as she drew them she named different members of the family. T then added the five "R10s" to confirm that R50 was shared equally.	Sharing situation: action interpretable as a complete composite unit sharing action through the use of symbols. Coherent	1
	L1	E15	T placed a flashcard on the board: "Mary bought 27 sweets for her 9 brothers. How many sweets did each one get?". She wrote 27 and 9 on the board. The teacher then drew 3 figurines showing the brothers. She pointed to the number "27" and asked Ls what they would do. A L replied: "Divide" and the T wrote in 27 ÷9 =. She then stated: "We going to share the sweets" and wrote the word "share" on the board. T said: "Lets share the sweets amongst the 9 boys (pointing to the 3 figurines)". She asked: "How many sweets does each boy get?" A L replied: "Three". T then mentioned another way to get to the answer was to go to the NL and she made 3 jumps of 9 on the NL from 0 to 27. Then she said: "Which means each one is going to get?" Ls shouted: "Three". T then indicated that 6 more boys needed to be added to the 3 figurines/brothers previously drawn to indicate nine boys. She paused and wrote "3" on top of the first figurine to indicate sweets and wrote the answer "3" for "27÷ 9 =".	Sharing situation: the answer is used to produce 3 jumps of 9 on a NL. The ambiguity was that the T initially drew three brothers; however, towards the end of the episode the T corrected this by gesturing that there needed to be nine brothers. Ambiguity: An incorrect signifier for a signified, but with this ambiguity subsequently corrected.	3
	L1	E16	On a flashcard the following was seen: "Share 45 oranges among 9 girls". T then wrote "45÷ 9 =". T asked for the answer and a L replied: "Five" and the T wrote "5" as the answer.	Sharing situation produces a number sentence with an immediate answer. Coherent	1
	L1	E17	On another flashcard the following was seen: "Divide 80 among 9 players". A L made use of the short division algorithm and produced an answer, "8 rem 8".	Sharing situation with remainder worked through a short division algorithm. Coherent	1
24 April 2012	L2	E1	T said that she had 7 sweets and drew seven small circles. She then stated: "But I have 3 friends" and she drew three big circles which she labelled with initials J – John, M- Mary and P – Peter respectively. T wrote "7 ÷ 3 =". T distributed the seven sweets one at a time between the 3 big circles resulting in two sweets in each circle. T added the answer "2". She then pointed to the seventh sweet and called it the remainder and proceeded to write in "r 1".	Sharing situation with remainder worked through a 1 by 1 sharing action. Coherent	1
25 April 2020	L3	E8	T stated: "I have 97 sweets and I want to divide the sweets among five children, what is the remainder". T then wrote "97 sweets ÷ 5 children" on the board and L then wrote 97 ÷ 5= and through jumping in 5's on the number line produced the answer 19 r 2.	Sharing situation with remainder. Teacher produced number sentence with matching NL that produces answer through grouping action. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation, although insertion of symbolic division sentence keeps signifiers aligned with previous signifier at all points	3

Table 44 Enactments of sharing situations in Joy's lessons

	Sharing situations (n= 7/32episode)
Initial signifier (dividend in square brackets)	Oral/written word problem situation (7/7 episodes): L1 – E1 [6], E2 [50], E15 [27], E16 [45], E17 [80] L2 – E1 [7] L3– E8 [97]
Signifier sequence	Word problem > initial incorrect drawing > written number sentence > correct immediate answer > answer used to produce 3 jumps of 9 on the NL > drawing corrected (L1: E15 – Some ambiguity) Word problem > written number sentence > jumping in multiples on a number line > producing a written answer with a remainder (L3: E8 – Some ambiguity) Word problem > drawing > written number sentence > sharing through 1 by 1 distribution of dots between circles to produce a written answer (L2: E1) Word problem > a short division algorithm to produce a written answer with a remainder (L1: E17) Word problem > correct immediate answer > Sharing by CCU of counters or distribution of written money to different people to verify the answer (L1: E1, E2) Word problem > written number sentence > written correct immediate answer (L1: E16)
Calculation strategy	Sharing through 1 by 1 distribution (1/7 episodes - L2:E1) Sharing CCU (counters or symbolic) (2/7 episodes - L1:E1, E2) Grouping by jumping in multiples (1/ 7 episodes - L3 E8) Short division algorithm (1/7 episodes – L1:E17) Recalled number facts (4/7 episodes – L1: E1 (repeated episode), E2 (repeated episode), E15, E16)

Table 44 summarises the enactments of 7 sharing situations in terms of signifiers and calculation strategies. In contrast to what was noted at Grade 3 level in School B, sharing situations with remainders were dealt with. In these episodes beginning with a sharing word problem, sharing through one-by-one distribution was accepted in one episode involving a single digit dividend. Sharing by "complete composite unit" (CCU) occurred in 2/7 episodes, through the use of symbols and counters. In one episode grouping by jumping on a NL was used to find the answer, with a NL used to represent the solution in another episode. The short division algorithm was used in one episode to produce a written answer with remainder. In 4/7 episodes, recalled number facts were in evidence.

Taken together, this analysis indicates inclusion of more abstract signifiers (NL and symbolic forms) and more efficient calculation strategies (e.g. CCU, jumping in multiples, short division algorithm). Of interest is that within these sharing situations some of the answers were produced through the distribution of symbols rather than the distribution of objects – marking a move towards more abstract signifiers (Fosnot & Dolk, 2001). Joy was the only teacher to use a number line to produce answers for sharing situations, via the inclusion of a symbolic division sentence, but without acknowledging the move from a sharing to a grouping interpretation of the situation in her working.

5.6.2. Grouping situations

Table 45 Description of grouping situation episodes across lessons

Date	Lesson	Episode	Task & its enactment (GROUPING SITUATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
23 April 2012	L1	E11	T instructed Ls to take out 27 counters. She then said: "Put them into groups of 9". T then said: "Let's say that the counters are sweets, can you share the 27 counters among 9 boys". Ls are seen making groups of 9 counters resulting in 3 rows of 9. T asked: "What is the answer?" The Ls shouted: "3".	Grouping situation inferred from the placing of counters into groups through 1 by 1 distribution she then offered a sharing situation. Ambiguity: unacknowledged move from grouping to sharing situation	3
25 April 2020	L3	E6	T takes 13 sticks. She took 5 sticks and gave it to one L and stated that it is a set. She then took another 5 sticks gave it to another L. T then counted the sticks left in her hand, which were 3, and said: "I have 2 sets and 3 units". T stated: "13 divided by 2?" and wrote "13÷ 2=" on the board. T then stated: "13 divided by 5" and showed Ls that there were 2 groups of 5 with a remainder of 3.	Grouping situation with remainder with played out through a grouping action. The T then writes a bald calculation which focuses on the number of groups as divisor (sharing). She then returns to a verbal number sentence which matches the grouping situation. Ambiguity: unacknowledged move from grouping to sharing situation	3

Table 46 The enactments of grouping situations

	Grouping situations (n=2/32 episodes)
Initial signifier (dividend in brackets)	Oral/written linguistic problem situation (2/2 episodes): L1 – E11 {27] L3 – E6 [13]
Signifier sequence	Word problem > situation replayed involving grouping through 1 by 1 distribution > incorrect verbal and written number sentence without answer > correct verbal number sentence with answer and remainder (L3: E6 – Some ambiguity) Word problem > situation replayed involving grouping the dividend value > verbal sharing situation > a verbal answer (L1: E11 – Some ambiguity)
Calculation strategy	Making of groups based on the divisor (1/2 episodes – L1:E11) Grouping distribution through 1 by 1 distribution (1/2 episodes L3:E6)

Table 46 summarises the enactments of grouping situations in terms of signifiers and calculation strategies. In relation to signifier sequences, in the 2 episodes with oral/written grouping situations, the initial playing through occurred via grouping counters to produce either verbal answers with or without a remainder or a written number sentence without an answer. However, in one episode, a linguistic sharing situation was incorporated before the answer was provided, reflecting a confusion between the language of "sharing between – people" and making "groups of – counters" that is evident in the work of the teachers already discussed. The distribution itself occurred, essentially, through one-by-one distribution into groups in both cases. Thus, only fundamental grouping strategies were evident, with no moves towards more efficient ways of producing an answer, as the literature advocates (Fosnot & Dolk, 2001).

5.6.3. Bald Calculations

Table 47 Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATION EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
23 April 2012	L1	E4	On board, a NL with multiples of 9 from 0 to 90 marked. T asked: "How many groups of 9 in 18?" and wrote "18÷ 9 = " on the board. A L answered: "Two" and pointed to "18" on the NL. T then added "2" to the number sentence as the answer and stated: "There are 2 sets of 9 in 18". Another L was asked for a different way of producing the answer. The L drew 18 sticks and circled nine sticks at a time and counted both circles to show that the answer was 2.	Bald calculation with correct answer offered immediately, answer linked to 18 on the NL and verified by circling groups of sticks. Coherent	1
	L1	E5	T wrote "27÷ 9 =" and asked a L to solve it by drawing sticks. The L drew 9 more sticks in a circle alongside the two circles of nine sticks and wrote "3" on top of the third circle. T asked the L to count the circles. The L counted the three circles wrote "3" as the answer for "27÷ 9 =".	Bald calculation with grouping actions through circling groups of sticks to produce the answer. Coherent	1
	L1	E6	Next the T wrote "9 ÷ 9 =". L drew nine sticks on the board, circled them and wrote 1 on top of the circle. T then wrote "1" as the answer and stated: "How many sets of 9 in 9, is just 1". Referring to the previous example. The T asked: "How many sets of 9 in 18?" as she pointed to "18÷9 =2". The T explained: "It is the same as saying I am sharing 18 oranges among 9 boys".	Bald calculation - The bald calculation was solved through grouping nine sticks at a time. To clarify the number of sets in the first division calculation the teacher used a previous example. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from a grouping to a sharing interpretation of the bald calculation.	3
	L1	E7	T wrote "45÷ 9 =" and stated: "I want to divide 45 by 9; I want to share 45 sweets between 9 girls". T then went to the number line and jumped in 9 from 0 to 45. As the T jumped on the number line the Ls counted in ones from one to five and the T wrote "5" on top of the last jump.	Bald calculation - the T instruction is to solve a sharing situational problem. Based on the sharing situation an answer was produced by jumping on the NL. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation.	3
	L1	E8	T then wrote "27÷ 9 =" and said: "I want to see if I get the same answer using the NL" as opposed to sticks. The T jumped 3 times in nines on the NL and pointed at the last jump asked for the answer, Ls shouted out: "3". T wrote "3" as the answer.	Bald calculation with grouping actions play out by jumping in multiples on a NL to produce the answer. Coherent	1
	L1	E9	T wrote: "45÷ 9 =; 63÷ 9 = ; 72÷ 9 =" and labelled them A, B and C respectively on the board. Three Ls then drew sticks based on the dividend value and circled groups of nine sticks producing answers such as 5, 7 and 8 respectively, next to each group of sticks. T said that they could use a quicker approach by using the NL. She then drew a NL and marked multiples of 9 from 0 to 45. T said: "If you at forty-five (points to 45 on the NL) just wait as you are sharing forty-five oranges" and pointed to "45" in the calculation "45÷ 9 =". She then asked: "So how many groups do you have?" before jumping 5 times on the NL and writing "5" as the answer to calculation A.	Bald calculation - The T accepts grouping actions through the grouping of drawn sticks and marking multiples on the NL. A sharing situation was offered and accepted. The teacher then asked how many groups and then jumped on the NL in multiples. Ambiguity: unacknowledged move from a grouping to a sharing interpretation of the bald calculation.	3
	L1	E10	Next the T pointed to a 10 by 9 array made up of dots. She reminded Ls that there were 9 oranges in a row by counting each orange in the first row. T then asked Ls to count in 9s from 0 to 90 as she pointed to each row. T used the array to produce the answers of "9 ÷ 9 = 1, 18÷ 9 = 2 and 27÷ 9 = 3" by counting sets of nine in the array.	Bald calculation was enacted through use of an array with learners input. Coherent	1
	L1	E13	T wrote "45÷ 9 =" next to the NL marked in multiples of 9. She instructed a L to circle "45" on the NL as the dividend value in the division calculation. The T made 5 jumps of 9 on the NL and wrote 5 as the answer. T then asked: "How many sets do we have in forty-five?" The Ls shouted: "Five". The T then	Bald calculation - the bald calculation was solved through grouping actions played out by jumping in nines on the NL to produce the answer followed by a sharing situation. Ambiguity: unacknowledged move	3

			said: "What if you shared forty-five oranges among nine boys, how many will each one eat?" A L stated: "Five".	from grouping to sharing situation. While a connection is implied between the grouping and sharing situation here, how the two differ and how they overlap is not discussed.	
	L1	E14	Next the T wrote "63÷ 9 =". T instructed a L to circle "63" on the previous NL. The L counted the number of jumps made previously on the NL (which were five) and then made another two jumps on the NL to end at 63. T wrote the answer 7 on the last jump.	Bald calculation with grouping actions by playing out by jumping in multiples on a NL to produce the answer. Coherent	1
	L2	E2	T wrote "28÷ 9 = _ r_ " on the board. Pointing at "28" she said she would go to 28 on the NL. She asked: "Is there 28 on the NL?" Ls replied: "No". Pointing to the NL the T asked: "Which number is nearer to 28?". A L replied: "27". T made an asterisk below the number "27" on the NL and asked: "How many sets are there in 27?". Thereafter the T jumped three times on the NL from 0 to 27 and wrote "3" above the last jump and over the first dash in the division calculation. She then asked: "How many steps can I count to get to 28?". A L responded: "1". T stated: "Which means that 1 is the remainder" and wrote "1" over the second dash.	Bald calculation with remainder - with grouping actions played out by jumping in nines on the NL to produce the answer. Limitation: Language of "nearer" number on multiples NL works correctly here, but not universally. Actions in all cases though indicate marking of nearest multiple below the dividend – so limitation at the language level, not in calculation	2
	L2	E3	Next the T wrote "66÷ 9 = _ r_ " and produced the answer 7 r 3 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2	2
	L2	E4	T wrote "87÷ 9 =" on the board and produced the answer 9 r 6 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2	2
25 April 2020	L3	E1	T wrote "37÷ 3 =" on the board. She drew a NL in multiples of 3 until "39". Pointing to "37" in the number sentence and she asked the class: "Where should we end?" L's replied: "36". The T corrected them and stated: "37". The T then asked:" Why must she stop at "36"? A L replied:" 36 is a multiple of 3 and 36 is smaller than 37". The T accepted offer and then noted that "37" lies between "36" and "39" and drew in a line after "36" to indicate "37" and added an asterisks above the "37". She then wrote "37÷ 3 =" on the board again and jumped in 3's, twelve times on the NL until "36" and added "12" as the answer. She then asked whether they should end at "36" or "37" and reinforced that they should end at "37". A L then stated: "Remainder one" and the T wrote in "rem 1" next to the answer "12" and pointed to "37" and made a small jumped from "36" to "37" to indicate "rem 1".	Bald calculation with grouping actions by playing out by jumping in multiples on a NL to produce the answer. Coherent	1
	L3	E2	T wrote "39÷ 3 =" on the board. A L immediately provided the answer: "Thirteen" and the T then wrote in "13" as the answer. To prove the answer a L jumped in 3s on the NL until "39".	Bald calculation with immediate answer offered and answer verified by jumps in multiples on a NL. Coherent	1
	L3	E3	T wrote "28 ÷ 3 =" on the board and produced the answer 9 r 1 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2.	2
	L3	E4	L drew a NL with multiples of 5 on it. T instructed the L to write "25÷ 5 =". T then went to the NL and jumped 5 times on it from 0 to 25. The L then wrote "5" as the answer.	Bald calculation - with grouping actions played out by jumping in fives on the NL. Coherent	1
	L3	E5	T called another L to the board. She instructed the L to write "42÷ 5 =" on the board and produced the answer 8 r 2 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2.	2

Table 48 Enactments of bald calculations in Joy's lessons

	Bald Calculations (n=17/32 episodes)
Initial signifier (dividend in brackets)	L1 – E4 [18], E5 [27], E6 [9], E7 [45], E8 [27], E9 [45], E10 [27], E13 [45], E14 [63] L2 – E2 [28], E3 [66], E4 [87] L3 – E1 [37], E2 [39], E3 [28], E4 [25], E5 [42]
Signifier sequence	Written symbolic signifier > drawing the dividend value > circling groups of sticks to produce written answers > in one episode jumping in multiples on a NL until dividend value to show a different approach to finding the answer > a sharing situation inserted (L1:E6, E9 – Some ambiguity) Written symbolic signifier > identifies the dividend value on the NL > jumps in multiples on a NL until dividend value > the written answer signifies the number of jumps > a sharing situation to describe the jumping in groups on the NL (L1 E13 – Some ambiguity) Written symbolic signifier > Sharing situation > jumps in multiples on a NL until dividend value > lines drawn in on the NL to signify the remainder > written answer (L1 E7 – Some ambiguity) Written symbolic signifier with remainder > identifies the dividend value on the NL > "nearer" multiple asked for > jumps to multiples below dividend on NL > lines drawn in on the NL to signify the remainder > written answer (L2: E2 - Limitation, E3 – Limitation, E4 – Limitation and L3: E1 – Limitation, E3 – Limitation, E5 - Limitation) Written symbolic signifier > adding through drawing a circle with sticks to the dividend value > circling groups of sticks > counting the number of circles to produce the written answer (L1 E5) Written symbolic signifier > either identified the dividend value on the NL first or jumped in multiples on a NL until dividend value to produce a written answer (L1:E8, E14, L3:E4) Written symbolic signifier > counting rows within an array to produce the written answer (L1: E10) Written symbolic signifier > an correct immediate answer > circling of groups of drawn sticks or jumped in multiples on a NL to verify the written answer (L1:E4, L3:E2)
Calculation strategy	No calculation strategies (1/17 episodes – L1:E10) Making of groups or counting groups based on the divisor (4/17 episodes - L1:E4, E5, E6, E9) Grouping action played out by jumping in multiples on the NL (11/17 episodes – L1:E7, E8, E13, E14 & L2: E2, E3, E4 & L3: E1, E3, E4, E5) Recalled facts (2/17 episodes – L1: E4 – repeated & L3:E2 – repeated)

Table 48 summarises 17 episodes of enactments of bald calculations in terms of signifiers and calculation strategies. In relation to signifier sequences, in episodes beginning with bald calculations (n=17/17 episodes), the solving of the bald calculation occurred via a translation into circling dots/sticks as equal groups or counting rows in array diagrams or jumping in multiples on the NL. In instances, when drawing sticks, the dividend value was first drawn then groups were made based on the divisor (4/17 episodes). The bald calculations could be separated into two groups: one group of bald calculations without remainders and another group of bald calculations with remainders. In most of the episodes the answers were produced through jumps on the NL (11/17 episodes). All bald calculations with a remainder were solved only through the use of a NL (6/17 episodes). In some instances, the answers were provided immediately followed by the use of the NL or drawing of sticks to verify the answer (2/17 episodes). The distribution occurred through counting groups of sticks or jumping in multiples on the NL, based on the quantity indicated by the divisor (Kouba, 1989). Grouping within arrays was also seen with immediate answers offered in 3/17 episodes.

Overall, what this analysis suggests is that most of the bald calculations were solved through the use of grouping actions on the NL (Fosnot & Dolk, 2001; Orrill & Brown, 2012). Joy attempted to connect grouping and sharing via pointing to the same calculation, but her explanation tended not to draw attention to how signifiers overlap and how they are different.

Three episodes which were not analysed focused on individual work in which the teacher did not lead. In a further episode there was a focus on forward and backward counting in multiples of 9 on the NL. Two of the episodes not analysed focused on group work (2/6 episodes).

5.6.4. Summary

More sophisticated calculation strategies with more truncated signifiers were evident in the solving of sharing situations and bald calculations. In sharing situations, a shift to more sophisticated signifiers in sharing through CCU by using symbols and the use of recalled number facts instead of using counters were evident. The shift to more truncated use of signifiers was seen in NL's and short division algorithms to produce the answer. Only in solving grouping situations were fundamental grouping actions (grouping distribution through one-by-one) used. In the solving of bald calculations, the frequent use of grouping language is evident. Furthermore, bald calculations are evident in a shift away from concrete objects to a reliance on diagrammatic signifiers such as the use of circles and sticks but predominantly the use of NL's and, in some instances, the use of arrays. Truncated signifiers and sophisticated division calculations strategies exposed learners to a range of ways of solving sharing situations and bald calculations in more efficient and abstract ways.

5.6.5. Analysis of coherence in Joy's lessons

Table 49 Analysis of episodes within signification pathways framework

Number of lessons	Category 1- Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	9		6		4	19
L2	1	4				4
L3	3	1	2		2	9
Total	**13**	**5**	**8**		**6**	**32**

Table 49 summarises the episodes as categorised within the signification pathway framework. In these episodes coherence was evident in n= 13/32. Episodes in which Joy worked with bald calculations made up just more than half of the coherent category (8/13 episodes). In 5/13 coherent episodes she solved sharing situational problems. One possible reason why bald calculations were more coherent is that there were repeated episodes relating to the same divisor and the same NL signifier. Episodes classified as coherent with limitations were evident in five episodes (n=5/32 episodes). In these 5 episodes, the limitation code was used because in the teacher's instructions to learners, there were limitations in her use of mathematical language. However, the use of the number line is correct. 8/32 episodes were classified as having "some ambiguity". Of these, seven episodes showed a shift between signifiers concerning division models or actions with no acknowledgement of the shift. In one episode, an incorrect signifier for a signified was provided, with no acknowledgement of the incorrect signifier, but with this ambiguity subsequently corrected (L1:E15). The episodes categorized as having "some ambiguity" contained half divisional situational word problems and half bald calculations.

Overall, what this analysis suggests is that there is an almost even spread between episodes that are coherent and those that are less than coherent in Joy's explanations.

Of interest, as with the previous teachers (Hazel and Mary), the lack of precise language plays an essential part in producing episodes that can be coded either as coherent or coherent with limitations. The lack of precision in mathematical language is one of the characteristics that to a certain extent can effect schools of a lower status.

5.7. Sam, G4: School C

Sam taught a G4 cohort of previously disadvantaged learners in a previously disadvantaged school. Between 34 and 40 learners were observed in the class. The school fees were R 78 per month, and the government identified the school as a quintile 5 school. In this case study, I delimited 14 episodes across the three lessons for study (See Appendix F).

Table 50 Weighting of episode types across division foci in Sam's lessons

Sharing situation	Grouping situation	Bald calculations	Other division-related focus (naming/identifying, vocabulary, matching representations, multiplication fluency, multiplication concepts)
0	0	7	7

In the lessons observed in Sam's classroom the 14 episodes were divided solely between bald calculations and other division – related calculation foci. In these episodes there was no reference to any situational word problems (sharing or grouping). Dividends leading to remainders in relation to the divisor were included in 2 episodes. In the "other" category, the seven episodes not analysed focused on individual and group work in which the teacher did not lead (2/7 episodes), naming parts of division sentences within a short division algorithm (2/7 episodes), homework and assessment activities (2/7 episodes), and in one episode the teacher redoing a previous long division example (L3:E2).

5.7.1. Bald calculations

Table 51 Description of bald calculation episodes across lessons

Date	Lesson	Episode	Task & its enactment (BALD CALCULATIONS EPISODES)	Division focus in episode/ coherence-related comment	Coherence category
25 July 2012	L1	E3	Stating a focus on long division, T placed a flashcard with the steps: "multiply, divide, subtract, bring down" on the board. She wrote 84÷ 4 in a division bracket. Ls identified 84 as the dividend and 4 as the divisor. T asked: "If I do division what is the opposite?" Ls replied: "Multiplication". T pointed to the 4 and asked: "What times table will I use?" A L answered: "The 4 times table". T placed a flashcard on the board containing the 4 times table from 1x4=4 to 10x4=40 and asked: "What are we going to do first?" and proceeded to say: "4 goes into 8 how many times?" A L replied: "Twice". T wrote the number 2 above the division bracket. She then said: "Now in long division, you say 2 times 4" as she pointed to the 2 above the bracket, the word "multiply" and the 4 outside of the bracket. A L answered: "Eight". T wrote 8 under the 8 in the dividend. Then the T pointed at the flashcard with words "multiply, divide, subtract, bring down" and asked: "Which sign am I going to use?" One L suggested "divide" while another learner said "subtract". T accepted "subtract" and wrote the subtraction symbol to the left of the 8 and drew a line beneath both 8s. She then said: "8 minus 8" and Ls shouted: "Nought" and the T placed a dot beneath the line. She then asked: "What is the next step?" A L suggested that they "bring down" the 4 and the T wrote the number 4 next to the dot. T then established that the divisor 4 needed to be divided into the dividend 4 and said: "4 into 4?" A L replied: "One" and the T wrote 1 above the bracket, making the answer 21. She then said: "What will I say? 4 times 1 (pointing to the 1 and the divisor 4)". A L replied: "Is 4" and the T wrote 4 under the dividend 4. She then asked: "What sign will I use again?" Ls shouted: "Subtraction" and the T wrote the subtraction symbol and drew a line under both 4s. She then subtracted 4 from 4 and got 0 as an answer.	Bald calculation is enacted through long division algorithm with learners input. Coherent	1
	L1	E4	T wrote "65÷5" in a division bracket, and produced the answer 13 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
	L1	E5	T wrote "96÷ 6" in a division bracket, and produced the answer 16 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
	L2	E2	T wrote "99÷ 9" in a division bracket, and produced the answer 11 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
	L2	E4	T wrote "97÷5" in a division bracket, and produced the answer 19 rem 2 using the same steps linked with long division as in L1 E3.	Bald calculation with remainder- same reason as L1:E3	1
	L2	E6	Then the T called 2 Ls to the front to solve "96÷4". L 1 set the problem up within a division bracket on the board. He wrote 2 above the bracket and -8 beneath 9 in the dividend. The T reminded the L to draw in a line beneath the -8. L 2 pointed to the 9 and the 8 and L 1 wrote 1 under the line. Then L 2 wrote 6 next to the 1 making the new sub-dividend 16. Next L 2 wrote 1 above the bracket and the T said: "4 goes into 16, 4 times". L 2 erased the 1 and wrote 4 above the bracket instead. Then L 2 wrote - 16 below the dividend 16, drew a line under the 16s and wrote 0 as the difference.	Bald calculation is enacted through long division algorithm with learner input. Coherent	1
	L3	E3	T wrote "77÷5" in a division bracket, and produced the answer 15 rem 2 using the same steps linked with long division as in L1 E3.	Bald calculation with remainder - same reason as L1:E3	1

Table 52 Enactments of bald calculations in Sam's lessons

	Bald Calculations (n=7/14 episodes)
Initial signifier (dividend in brackets)	L1 – E3 [84], E4 [65], E5 [96] L2 – E2 [99], E4 [97], E6[96] L3 – E3 [77]
Signifier sequence	Written symbolic signifier > a long division algorithm to produce a written answer with or without a remainder (L1:E3, E4, E5, L2:E2, E4 E6 & L3 E3)
Calculation strategy	Long division algorithm, with interim quotients inserted as recalled, or stated, facts (7/ 7 episodes -L1: E3, E4, E5 & L2: E2, E4, E6 & L3:E3)

In relation to signifier sequences, all 7 episodes beginning with bald calculations involved the solving of the bald calculation via long division algorithms to produce the result with or without a remainder. The FFL requirement to include situational problems in G4 and higher grades was not evident in Sam's division teaching, and the bald calculation episodes were not linked to sharing or grouping situations.

Bald calculations rely on truncated signifiers to produce the answer with/without a remainder (Carruthers & Worthington, 2006). While grouping language was evident within Sam's work in phrases like: "4 goes into 8 how many times?" – this is used in "digit-based" ways that fit with the traditional algorithm with no reference to the overall dividend quantity (Takker & Subramaniam, 2018). Sam was the only teacher in the sample to use long division algorithms – a marker of traditional instruction, in tasks where the dividend was always more than 10 times the divisor, but never more than 100 (Threlfall, 1996; Van Putten et al., 2005).

5.7.2. Analysis of coherence in Sam's lessons

Table 53 Analysis of episodes within signification pathways framework

Number of lessons	Category 1-Coherent	Category 2 – Coherent with limitation	Category 3 – Some ambiguity	Category 4 – Incoherence	Not analysed episodes	Total
L1	3				2	5
L2	3				3	6
L3	1				2	3
Total	**7**				**7**	**14**

For Sam, coherence was evident in all of the instructional episodes, with all involving the long division algorithm. Whether the sidelining of attention to division situations contributed to this coherence is not known, but this lack of attention meant that many aspects of the language of division were ignored. Furthermore, the uncoded episodes also show that Sam focused on "vocabulary" (naming of division parts of a short division algorithm) – a feature that was more evident in schools in lower socio-economic categories, which formed part of my school sample.

5.8. Synthesis across the six case studies

In Tables 54 and 55, I summarise, firstly, the categorisation of coherence within episodes in relation to *coherent, coherent with limitations, some ambiguities and incoherence,* and secondly, the categorisation of division tasks within episodes concerning *division related foci, initial signifier, division calculation strategies and number range* as they appear in each episode for Grade 3. Commentary on what these summaries indicate across the Grade 3 teachers then follows. The same procedure is then followed in summarising the findings in relation to the Grade 4 teachers

5.8.1. Comparing G3 teaching across schools differentiated in terms of socio-economic status

Table 54 Summary categorization of episodes in relation to coherence: Grade 3 teachers

Teacher	Coherent			Coherent with limitations			Some ambiguities			Incoherence			Not analysed
	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	
School A													
Cathy G3	n = 36 episodes			n = 4 episodes			n= 9 episodes						15
	11	10	15	3		1	3	6					
School B													
Mary G3	n = 5 episodes						n = 5 episodes			n = 1 episodes			9
		2	3				1		4			1	
School C													
Hazel G3	n = 9 episodes			n= 4 episodes			n= 1 episode						11
	1		8			4			1				

Table 55 Summary categorization of division related foci, initial signifier, division calculation strategies and number range: Grade 3 teachers

Teacher	Division focus			Initial signifier			Division calculation strategies									Number Range
							Sharing division calculations				Grouping division calculation strategies					
	Sharing	Grouping	Bald Calculations	Initial linguistic signifier	Initial diagrammatic signifier	Initial symbolic signifier	One to one distribution	Partial composite units	Group based distribution	Complete composite units CCU	Grouping Approaches (1 by 1 grouping distributions or making groups based on the divisor)	Skip counting	Repeated subtractive	Algorithm (Short & Long)	Recalled Number Facts & Multiplication facts	
School A																
Cathy G3	17	16	16	16	17	16	5		3	7	21	2		3	4	93
School B																
Mary G3	1	2	8	3		8				5	7	3			7	36
School C																
Hazel G3	1		13	1		13	9			1		2			11	2000
Total	**19**	**18**	**37**	**20**	**17**	**37**	**14**		**3**	**13**	**28**	**7**		3	**22**	

In looking across the three Grade 3 teachers, some key contrasts and key overlaps are apparent. Firstly, Cathy in School A, the most advantaged of the three schools, teaches more division calculation-related tasks than the other two teachers (49 calculation-related episodes in comparison to 11 for Mary and 14 for Hazel. This finding accords with findings from South African research that have highlighted problems with coverage and pacing in schools serving more disadvantaged populations (Ensor et al., 2009).

Looking at the range of division calculation tasks, it is also clear that there is a broader range of coverage across sharing and grouping situations and bald calculations in Cathy's class compared to the other two classes. In Hazel's class, the imbalance is extreme, with 13/14 episodes beginning with bald calculation tasks.

Cathy incorporated more a greater range of strategies oriented towards sharing and grouping. Mary used CCU distribution – where the answer has to be known ahead of problem-solving in sharing situations. In grouping situations, she used grouping through one-by-one distribution or making of groups based on the divisor. Hazel used one-by-one sharing distributions most commonly and relied on recalled facts in grouping-oriented calculations. G3 teachers basically used two digits by one-digit divisors to do division calculations in most of their episodes. School A was the only school that more routinely included three-digit dividend values in repeated episodes.

Issues related to the extent of coherence correlated with the range of division tasks in different ways across the three teachers. For Cathy, issues related to limitation and some ambiguity were more common when dealing with sharing or grouping situations (12/33 situational signifier episodes involved limitation/some ambiguity) than when she was dealing with bald calculations (1/16 episodes involved limitation). In Mary's case, 1/3 situational signifier tasks involved ambiguity, but 5/8 bald calculation tasks also included some ambiguity/error. In Hazel's teaching, her one situational signifier task was coherent, and 5/13 bald calculation tasks involved either limitation or some ambiguity.

Across all the bald calculation teaching, when limitation and ambiguity featured, this tended to happen when the teachers attempted to link the bald calculation to a division situation. Thus, in looking across the Grade 3 teaching, there is evidence of both more division teaching tasks, a wider division calculation task range and a wider range of calculation strategies, and more coherent handling of these tasks in the most privileged school of the three.

5.8.2. Comparing G4 teaching across school differentiated in terms of socio-economic status

Table 56 Summary categorization of episodes in relation to coherence: Grade 4 teachers

Teacher	Coherent			Coherent with limitations			Some ambiguities			Incoherence			Not analysed
	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	Sharing	Grouping	Bald calculation	
School A													
Asha G4	n = 10 episodes			n = 1 episode						n = 1 episode			13
	2		8			1				1			
School B													
Joy G4	n = 13 episodes			n = 5 episodes			n = 8 episodes						6
	5		8			5	2	2	4				
School C													
Sam G4	n = 7 episodes												7
			7										

Table 57 Summary categorization of division related foci, initial signifier, division calculation strategies and number range: Grade 4 teachers

Teacher	Division related focus			Initial signifier			Division calculation strategies									Number Range
							Sharing division calculations				Grouping division calculation strategies					
	Sharing	Grouping	Bald Calculations	Initial linguistic signifier	Initial diagrammatic	Initial symbolic signifier	One by one distribution	Partial composite units	Group based distribution	Complete composite units CCU	Grouping Approaches (1 by 1 grouping distributions or making groups based on the	Skip counting	Repeated subtractive	Algorithm (short & Long)	Recalled Number Facts Multiplication facts	
School A																
Asha G4	3		9	3		9	1	3					1	8	1	965
School B																
Joy G4	7	2	17	9		17	1			2	20	2		1	6	97
School C																
Sam G4			7			7								7	7	88
Total	**10**	**2**	**33**	**12**		**33**	**2**	**3**		**2**	**20**	**2**	**1**	16	**14**	

In looking across the three Grades 4 teachers, these findings emerge. Firstly, Joy in School B, the school with a previously disadvantaged cohort of learners (second privileged school), teaches more division calculation – related tasks then the other two teachers (26 calculations – related episodes compared to 12 for Asha and 7 for Sam). In Grade 4, in contrast to Grade 3, bald calculations were the predominant division focus across all three teachers, but Joy also included more attention to sharing and grouping situations than the other two teachers. Sam only dealt with bald calculations with no reference to either sharing or grouping calculations. Joy incorporated more of a range of strategies oriented towards sharing and grouping. Asha and Sam relied mainly on short and long division algorithms to solve division tasks. School A used the largest number range across the three schools: repeated three-digit numbers divided by a one-digit number.

Looking at the grade 4 division teaching with a socio-economic lens, two interconnected points are of interest. Firstly, in Asha's teaching in the most privileged school, as in Grade 3 in School A, division tasks across sharing and bald calculations were included. In Joy's lower socio-economic status school, tasks across sharing, grouping and bald calculations were included. In contrast, in Sam's teaching in the poorest school in my sample, only bald calculation tasks were worked with. The differences in range of division tasks however, was also linked with findings related to coherence. In Asha's case, 10/12 episodes were coded as coherent, with these ten episodes involving sharing and bald calculations. In Joy's teaching, a much smaller proportion of episodes were coded as coherent (13/26). All 4 episodes where she worked with grouping tasks contained some ambiguity, and 4 episodes involving work with bald calculations also contained ambiguity. Five further episodes included the kinds of limitations described earlier. In Sam's case, all episodes were coherent, but – as noted already – her instruction was restricted solely to working with bald calculations using the long division algorithm. This points to double challenges for children in the two poorer schools – relating to both task range and breakdowns in coherence, with these issues compounded by the lower dividend number ranges seen in Joy and Sam's teaching.

5.8.3. Comparing G3 teaching with G4 teaching

Across the three Grade 3 and three Grade 4 teachers, some key differences and key similarities are evident. Firstly, school A, the most advantaged of the three schools, teaches more division calculation-related tasks than the other two schools (61 calculations – related episodes compared to 37 for School B and 21 for School C).

Looking at the range of division calculation tasks, it is also clear that there is a broader range of coverage across sharing and grouping situations in School A than in the other two schools. In School C, the imbalance is extreme, with no episodes involving grouping situational word problems. Across two schools, grouping approaches were dominant.

In this study, the number range stipulated by the FFL was not adhered to by the six teachers. The number range suggested by the FFL is between 1000 and 2000. A low number range was used in lessons across the three schools with only Asha using three-digit dividend values that extended from 200 to 1000, and Hazel including one straight forward example of a four-digit dividend. Only School A ensured that most of the tasks in G4 involved three digits by one-digit numbers, although these were still far below the curriculum specifications. Thus, patterns of enactment suggest that privileges of broader access (even at the more advantaged end of the South African quintile system) follow the schools' socio-economic profiles, in which the most privileged school had more coherence with more variety of divisional tasks (sharing/grouping situations/ bald calculations).

Levels of coherence were related to the range of division tasks in different ways across the three schools. For School A, more coherence was evident when the teacher focused on bald calculations (23/61 episodes involved coherence) and situational signifier tasks (23/61 episodes involved coherence), the remaining 15/61 episodes involved limitations/some ambiguity/error across situational signifier and bald calculation tasks. In School B, 5/37 situational signifier tasks involved some ambiguity, but 14/37 bald calculation tasks included limitations /some ambiguity/ error. At the same time, bald calculations made up more than half of the coherent episodes. In School C, 5/21 bald calculation tasks included limitations/errors with only one situational signifier task being coherent. Across the six teachers, 39 episodes were categorized either as including some limitations or some ambiguity or errors. The greatest number of episodes (11/39 episodes) identified as having limitations were bald calculations, usually when teachers attempted to provide "rules" when solving division tasks such as "the answers are always smaller" (Tirosh & Graeber, 1991). In looking across the grade 3 and 4 teaching, there is evidence of more coherent handling of these tasks in the most privileged school (School A), thus compounding the advantage for learners noted above in terms of task and calculation strategy range.

In summary, the focus in my analysis has been on key types of disruptions to coherence as seen across the six teachers in this study. In overviewing these disruptions, three main categories emerged. The first key disruptor to coherence was seen in the use of language that erroneously portrayed a generality that could not be applied to the mathematical concept. Tirosh & Graeber's (1991) noting of the statement that "division makes smaller" was widely seen in this study, alongside division involving "start with the bigger number" and "multiplication makes bigger". The disruption to coherence is that although the terminology is applicable to the example, it is not generalisable to all division tasks. The lack of generalizable terminology creates a limitation or localized explanation (Venkat & Askew, 2018)

Ambiguity as a disruption to coherence was seen in two forms. One form related to signifiers that were linked to more than one signified in contradictory ways. Most commonly in this study, this ambiguity was related to the divisor. In this form of ambiguity, the divisor was linked to the dividend unit (e.g. 5 dog biscuits) at some points in an explanation and to the divisor unit (e.g. 5 children) at other points. This reflects Lesh & Doerr (2003) note in their work with learners' mathematical activity that one form of ambiguity resides in multiple meanings that cannot be reconciled.

A second form of ambiguity was seen in unacknowledged shifts between division models and situations in the course of teachers' instructional explanations. Rowland (2012) has pointed to the phenomena of a shift from, in his student teacher's case, the initial grouping explanation to a sharing interpretation of the word problem. This was the more predominant ambiguity type of disruption to coherence in this study.

The final key disruption to coherence was a more general lack of attention to precise mathematical language when solving division tasks. Beckmann (2011) and Anghileri (2007) note that the use of precise mathematical language when solving division tasks are critical. The lack of precise mathematical language was seen – in this study - within the solving of bald calculations, the interpretation of word problems, the use of diagrams and the enactment of sharing and grouping actions, and featured within instances of both limitations and ambiguity.

5.9 Conclusion

In this chapter I have presented findings from an analysis of lesson episodes in terms of coherence in six teachers' solving of division tasks across lessons observed in 2011 and 2012 using the categories *coherent, coherent with limitations, some ambiguities and incoherence.* The findings from this analysis indicate that coherence is located mainly in the teachers' use of bald calculations. Findings from analysis of the episodes show both forward and backward shifts in teachers' use of truncated signifiers. These findings also show limited progression from fundamental to sophisticated division calculation strategies in the lesson episodes analysed. The six case studies also suggest that there may be a link between the socio-economic status of a school and the opportunities for quality learning of division. In the final chapter, I will be looking at the empirical and theoretical findings, as well as the limitations of this study, concluding with a reflection on this journey.

6.1 Introduction

Between 2011 and 2012, I conducted research in three schools in the South African province of Gauteng. I selected three schools of differing socio-economic status within the upper quintiles to follow up on an earlier study which had found a relationship between the socio-economic profile of the school, G3 and G4 teachers' views on mathematical tasks and their perceptions of learners' results. School A was categorized as a high socio-economic status school with most of the learners from higher economic status homes. School B was in a previously historically advantaged area; however, most learners came from lower socio-economic status households. School C was in a historically disadvantaged area with learners from lower socio-economic status homes. In each of the schools, two teachers were selected, a teacher from G3 and a teacher from G4, which allowed me to investigate the division teaching offered to the same cohorts of learners.

Initially, I had intended to look at teachers' classroom mathematical tasks, focusing on differences in the teaching and in the nature and range of mathematical tasks provided to learners in G3 and G4. Division was selected as the focal topic as it was an area in which learners performed particularly poorly.

A shift from this initial focus occurred after my early observations of the enactment of division tasks in classrooms. I had assumed that the teaching of division tasks would be mathematically coherent, but as the teachers taught and solved the division tasks, I began to observe differences in the extent of coherence in parts of what they said, did and wrote. These differences were located in ambiguities and breakdowns in connections between parts of the instructional explanations offered or accepted by teachers as they worked with various division tasks. Following similar observations in several lessons, I shifted the study's focus from the teaching, nature and range of division tasks to investigating coherence in division teaching, while also focusing on range and progression in signifier use, division calculation strategies and number.

In this final chapter I review the empirical and theoretical findings that emerged from the data analysis. The chapter concludes with a consideration of the study's limitations and my reflections on this research journey.

6.2 Empirical findings

This study's first significant finding is that the teachers in the study had what I consider to be a "*fluid*" engagement with sharing and grouping division models. In the mathematics education literature sharing and grouping situations have "separate" actions associated with them: making sense of sharing situations involves distribution with sharing actions, while making sense of grouping situations involves grouping actions. However, findings from this study indicate that teachers frequently linked sharing and grouping situations and included actions drawn from the other "situation" without acknowledgment of this move. This indicated that connections in the mathematical discourse linked to coherent early working with sharing and grouping models were, relatively often, not evident in the teaching of division. Some ambiguous episodes were present across 23/6 of the teachers and in – 1/5 of the episodes analysed in this study, with the most frequent involving a shift between models of division without the teacher acknowledging the shift. While Rowland (2012) has mentioned this kind of shift, in this study the extent of unacknowledged shifts was extensive. This shifting often involved a move, within a signification pathway, between the divisor being the number of circles or the number in a circle, without acknowledgement that the division model had changed. While shifts between division situations are mathematically possible and often useful and sensible, such shifts need to be acknowledged in order for explanations to be coherent for learners being introduced to the meanings of division through the signifiers linked to division situations, actions and calculation strategies.

The second significant finding is that bald calculations (symbolic signifiers) predominated over work with situational problems. In all six classes, the use of bald calculations as initial signifiers was evident. In Grade 4, these bald calculations were frequently presented and worked with through short or long division algorithms. Literature suggests that reliance on bald calculations limits the use of language. Given that most of the learners and the teachers in schools B and C in this study used English as an additional language, the predominance of work with bald calculations may have been linked to difficulties with language, when working with word problems. Lubienski (2000) has noted more generally that a more limited discourse is evident in classes serving learners from lower socio-economic areas. In this study, an implication of the emphasis on division work with bald calculations is that learners, and particularly in Schools B and C, are likely to understand division as involving purely numerical calculation rather than as linked to situations and actions.

Importantly, regardless of the initial situation/bald calculation, tasks were more frequently enacted with group-based distributions. Gravemeijer (1997) argues that teachers tend to opt for sharing division models with sharing actions. However, in this study, across grouping and certain sharing situations, the teachers' predominant actions were different grouping approaches. In both G3 and G4 classes, the data pointed to bald calculation mostly being solved through the use of a range of grouping-based distribution actions/language/procedures rather than sharing distributions, in contrast with Gravemeijer's (1997) findings.

The finding highlighted in the research literature noting the misconception around "division makes smaller" (e.g. Tirosh & Graeber, 1991) was found in this study as well, but in the signification pathway framework, this was labelled as "limitation" rather than as misconception or error. This choice was due to the relatively common occurrence of episodes labelled as containing "some ambiguity", with these problems containing more serious challenges to coherence and learner sense-making than what I saw in the limitation category.

The fifth significant finding relates to the extent of coherence in the teaching episodes. 80/119 coded episodes were coherent. The majority of the episodes in the coherent category began with bald calculations (49/80 episodes). Further, in Grade 4, of the 23 episodes with coherent bald calculations, 16/23 were worked with using short/long division algorithms. This combination of findings related to the coherence category and the predominance of bald calculations in this category suggests that teachers were more likely to find bald calculations easier to enact coherently than division situation tasks.

Finally, there is evidence of a lack of progression from less sophisticated division calculation strategies to more advanced division calculation strategies. Grouping using one-by-one distribution at the one extreme and short and long division algorithms at the other were frequently used by the teachers, particularly in G4 and there was little evidence of movement between these two division calculation strategies.

6.3 Theoretical contribution

When using semiotics to study the teaching of mathematics, Walkerdine (1988) and Presmeg (1997) suggest the use of signification chains. The formation of a signification chain is based on an initial signified entity that is carried forward in a linear development of the initial sign so that a chain of clear signifiers contributes to the explanation. However, in order to analyse the data collected in this research I found it necessary to introduce the concept of *signification pathways*, because I identified several episodes in which a teacher would begin talking about one signified entity, then move to talking about a different signified entity, and in some cases, then return to the first sign produced, thus developing an explanation that could not be traced with a single chain. The analysis therefore examined the formation of sequences of signifiers within these signification pathways, where, in some cases, later signifiers could not be traced back to the initial signified situation, or to a signifier earlier in the sequence. The use of semiotics in the form of signification pathways highlighted key limitations in teachers' mathematical discourse as they solved various division tasks and was particularly useful for analysing ambiguity.

The signification framework investigated the idea of coherence within episodes. Coherence came to be regarded as located along a continuum in which the first category showed how a division task produced an answer with a pathway involving links between signifiers that are mathematically justifiable at all points within the episode. The second category of coherence highlighted signifiers, which, although coherent, had limitations in the localization of these signifiers. The broader literature has tended to view instances of what I call limitation as error – for example, statements like "division makes smaller" (Tirosh & Graeber, 1991). For my purposes, and given the more serious problems of ambiguity and error, I found it useful to see these instances as locally correct, but not generally so. The third category is evident in episodes where the signification pathway has some ambiguities, which are corrected by the end of the signification pathway. Last, there were pathways in which incorrect signifiers were produced and overlooked, or the answer provided was incorrect.

The second theoretical contribution relates to progression in sharing situations, emphasising Group Based Distributions (GBD). Mathematics education literature states that sharing involves distribution by "partial composite" units, involving more than one item distributed at a time. In addition to sharing through "partial composite" units, in this study many episodes showed sharing through *"complete composite units*" (CCU). Distribution by CCU units refers to distributing items by the quotient value immediately (Zweng, 1964). I describe this more efficient strategy as distribution of sets based on the answer either being known prior to problem–solving or the answer being already available from an offer earlier in the episode. A different phenomenon that was sometimes evident in the data in this study is sharing situations that to led to what I refer to as Group Based Distribution (GBD) in which the divisor is used to form groups up to the dividend value. In other words, the teacher would ask: "Share

10 apples between 5 boys. How many apples will each boy get?" The teacher would write 10 ÷ 5 = and then draw five apples and another five apples and then count the groups of apples and insert the answer "2".

6.4 Limitations

The study has provided me with several insights into how division is taught in the early grades in selected South African classrooms, but it also has limitations. The first limitation is that the study did not consider learners' conceptions of division. Although learners' responses were acknowledged and included within the signification pathways as part of teachers' instructional discourse, the study did not focus directly on learners' discourse. In this sense, the study offers a rather "one-sided" view of teachers' work. While the focus in this study was on teaching, teachers, like learners, were observed subjects, whose work was described from my interpretations of the literature on what division means and how it should be taught. I did not engage with the teachers' reasons for their selection of tasks and reasons as to why they solved them the way they did.

Another limitation is that the study focused on examining coherence only within episodes and not across lessons. It is possible that looking across teacher's lessons could result in more in-depth insights into how coherence is enacted and what this means mathematically.

6.5 Implications for curriculum development and professional development

The implication for curriculum development and professional development are based on the findings, which suggest the need for extensive work with teachers on the key language and actions related to division (and multiplication). Both the literature and the evidence gathered in this study point to the need for attention to situations involving "equal groups" and to identifying the "givens" and "unknowns" in division word problems involving total quantities, the number of groups and numbers within groups.

Second, attention needs to be given to progression in modes of signifiers and calculation strategies in relation to the increase in number range, through discussing scenarios of different approaches to solving division problems. Discussing different approaches shows their relative efficiency, and detailing the trajectory that learners need to be encouraged to work through to get to efficient ways of solving division problems.

6.6 Self Reflection

My first reflection relates to making sense of the teaching of division. What was crucial to understanding how division is taught was identifying the two-division situations, namely sharing and grouping. These two situations inform the selection of signifiers and division calculation strategies, together with the number range. Furthermore, the teacher needs to explicitly show how progression occurs when using truncated signifiers, sophisticated division calculation strategies and an increased number range. The teaching of division is both nuanced and complex, and the teacher needs to be cognizant of these realities when teaching division.

Another reflection concerns how different teachers understand and teach the same concept in the same curriculum. Even though all six teachers were using the same curriculum document none of them taught the concept division as prescribed in the curriculum document. This study has demonstrated that very few of the approaches, signifiers and solving processes used by the six teachers were uniform despite the presence of a standard curriculum. What this reflection highlights is that many of the assumptions made by the curriculum designers about teachers' understanding of division and how division must be taught in G3 and G4 may be incorrect. Furthermore, a lack of a common approach to the teaching of division, resulted in teachers teaching division from a limited perspective. This includes sometimes working from their own prior knowledge of the answer from the outset. Teachers knowing the answer without providing coherent explanations is not helpful for learners, who do not have access to the answer at the start of the problem–solving process.

The critical role of mathematical discourse in teaching and learning is another reflection. Teachers do not always realize the importance of appropriate discourse in determining the selection of examples, signifiers and division calculation strategies. Mathematical discourse is critical in articulating exactly the division situation that will be the focus within division tasks. Such discourse is critical in making explicit the difference between different situations, signifiers, calculation strategies and number ranges. It needs to be carefully thought through before the teaching of any division task to circumvent limitation, ambiguity and incoherence.

The sequence of signifiers that the teacher uses in the solving of division tasks gave rise to a further reflection. Within their mathematical discourse, teachers need to produce a sequence of signifiers as they teach division in which the sequence of signifiers is aligned with the signified situation and is made explicit to the learner. Teachers need to ensure that the sequence of signifiers that is produced is mathematically justifiable throughout the solving of the division task. This alignment is important for developing coherence when teaching division. Coherence in the lower grades is more nuanced in that teachers can produce the correct answer despite creating some disruptions to coherence. In G3 and G4, episodes with the correct answer were categorized as demonstrating limitations and some ambiguity. Making teachers aware of the types of limitations and ambiguities which contribute to the disruption of coherence could be a useful forward move from this study, and this reflection leads me to the implications of this study for teacher professional development and pre-service teaching. My research has highlighted the importance of coherence and the critical issues that create disruptions to coherence when teaching division. Both pre- and in-service teachers should be exposed to the need for an alignment between the division situation, the selection of signifiers at the start of the task and as the sequence of signifiers is produced, and the enactment of division calculation strategies when teaching division.

I also reflect here on my development as an academic. I have extended my knowledge about division, and the process of academic study has also helped me as a researcher in terms of the dynamics of research within mathematics education and of the importance of mathematical discourse. This study has highlighted the importance of the sequence of signifiers with reference to how signifiers, division calculation strategies and number range are selected and unpacked and has shown me that this sequencing has a critical influence on what is required to develop coherence in the teaching of division. Over the past few years, the research process has highlighted the importance of being clear in my explanations and being consistent in my analysis of each episode, lesson and teacher.

Lastly, this research has taken me beyond what I thought was possible reflectively, psychologically, spiritually and academically. One of the lessons that I have learnt is that doctoral studies are a collective production that includes working with teachers, writing articles and listening to other academics. I have learnt that this "process of production" develops how one articulates one’s thinking, difficulty and apprehensions, enables trust in the process and acceptance that each step is valuable. Having developed insights that are illuminating, one realizes the responsibility one has to ensure that the resulting knowledge is used to assist both teachers and learners to be less afraid of division as a concept. Mathematics still makes me afraid, but I have learnt that being afraid is "ok". Not knowing has become the motivation for development as I have striven through the use of mathematical and semiotic discourses to make sense of, in this case, the formidable concept that is division.

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Appendix A – Cathy's episodes across lessons

Date	Lesson	Episode	Task & its enactment	Division focus in episode/ coherence-related comment	Coherence category
16-May-11	L1	E1	Lesson aims described in terms of learning to solve and explain division problems.		
	L1	E2	Skip counts in 2's and 3's.	Skip counting fluency	
	L1	E3	12÷4=3 written on the board. Learners asked to identify parts of the symbolic sentence. In explaining these parts, teacher stated that in division: "The answer will always be smaller".	Naming parts of symbolic division sentence..	
	L1	E4	On transparency: 7 column by 2 row array of mice. T asked: "If I give 14 mice to 2 cats. How many mice will each cat get?" T accepted answer "7" pointing to each row of 7 mice.	Sharing situation. Limitation: Answer tied to presence of array.	2
	L1	E5	Transparency: 4-2=2 and 2-2=0 in 1st column, with 2 mice per row alongside each subtraction in the next column. Columns 3 and 4 showed repeated subtraction of 2 starting from 6, and columns 5 & 6, starting from 8. T talked through each repeated subtraction as: "a cat eats 2 mice at a time". Asking for the division number sentence matching each column of repeated subtractions, T initially says 6 ÷ 2 = 3 and then 6 ÷ 3 = 2 for column 4. She then states 8 ÷ 4 = 2 for column 6.	Grouping situation: inferred from repeated subtraction & initial teacher talk of "two mice" eaten "at a time". Final statement of number sentences as 6 ÷ 3 = 2 and 8 ÷ 4 = 2 instead of 6 ÷ 2 = 3 and 8 ÷ 2 = 4, breaks with initial situation. Ambiguity: unacknowledged move from grouping to sharing situation.	3
	L1	E6	Verbal task: "We have 12 flowers and three girls. How many flowers can we give to each of the girls?". Ls: "Four". Asked to prove this, Ls distributed 4 flowers at a time to each of 3 girls. T wrote 12÷ 3 = 4 on board.	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L1	E7	T: "Can we do equal sharing with stars?". She requested a number sentence with the dividend being smaller than 9 which needed to be divided by 2. L offers: "6 divided by 2" and places a set of 3 stars into 1 column and another set of 3 stars into a 2nd column. L then wrote: 6÷ 2 =3 below on the board.	Sharing situation, based on request for equal sharing worked with through complete composite unit sharing action. Coherent.	1
	L1	E8	On a transparency: "How many sets of 2 in 18?" with a 6 column by 3 row array. T asked for a number sentence. "18 divided by 2" was offered and accepted. Ls proceeded to write "18÷2=9" on their whiteboards. T then stated: "Remember how you share 1 by 1". L then drew 2 circles and distributed counters 1 at a time between the 2 circles. T asked: "What is 18 divided by 2?" L responded: "9".	Grouping situation inferred from task presentation. In final section, a sharing action is offered and accepted. Ambiguity: unacknowledged move from grouping to sharing situation. No use of array.	3
	L1	E9	On transparency: 5 column by 3 row array of chocolate bars. T offered story situation: "Share these bars of chocolates amongst 3 Ls equally with nothing left". T asked for the total number of chocolates and only to provide a number sentence. T accepted 15 and "15 divided by 3" as the answers.	Array linked to sharing situation, matched to verbal number sentence. Coherent.	1
	L1	E10	On transparency: 5 branches with 6 blue gum leaves on each branch and 5 koala bears. T accepted 6 as answer for number of blue gum leaves each koala bear eats. She then accepted: "Thirty divided by five" as number sentence for the situation. As T distributed 6 leaves to each koala bear, Ls asked to check answer by counting in 6s. T stated: "We said that your answer is definitely going to be smaller". The word "smaller" pointed to on the board. 30 ÷ 5 = 6 is written.	Sharing situation, with answer checked through counting in multiples of 6. Limitation: division answer reiterated as "definitely ... smaller".	2
	L1	E11	T told a story about John having 8 sweets to share among himself and 3 friends. T asked: "How many did each one get?" A L, given 8 sweets to distribute between himself and 3 Ls, deals them out one-by-one. T asked for number sentence: "8 divided by 4" is offered. T: "Why 4?" Ls: "Because it's him and his 3 friends". T: "How many did each one get?" T accepted the answer 2.	Sharing situation, worked with through a one by one distribution sharing action. Coherent.	1

	L1	E12	Individual worksheet activity: 1. 24 Pencils are out in 4 containers so that there is an equal amount in each container. Pack your beads in groups. Now write a number sentence for the problem. 2. Taylor wants to share 16 beads with her 3 friends. Pack your beads in groups. Now write a number sentence for the problem.		
17-May-11	L2	E1	T asks class to identify the division bracket and symbol.	Identifying division notation.	
	L2	E2	T stated focus on "equal sharing" between 3 people. She took 6 sweets and distributed 2 sweets at a time to each of 3 Ls and asked for number sentence. T stated number sentence as 6 ÷ 3 = 2.	Sharing situation: action interpretable as complete composite unit sharing action. Coherent.	1
	L2	E3	Based on a 3 column by 3 row array of dots (three dots seen in vertical circles). T asked "Can you give me the division sum for these groups?" "9 divided by 3 equals 3" is offered and accepted.	Grouping situation inferred through circles within array. Coherent.	1
	L2	E4	Based on a 3 column by 2 row array of dots (three dots seen in horizontal circles). T asked: "I want a division sum for this grouping?" "6 divided by 2 equals 3" is offered and accepted as the answer.	Grouping situation inferred through circles within array. Coherent.	1
	L2	E5	Based on 6 column by 3 row array of dots (three dots seen in vertical circles). T asked: "Who can give me the number sentence?" "18 divided by 6" is offered and accepted as the answer. T pointed to the 6 vertical groups of 3 dots.	Grouping situation inferred through circling of groups within an array. Coherent.	1
	L2	E6	Poster with 3 columns. Each column contains 3 rabbits holding groups of Easter eggs: column 1, 3 rabbits each holding 4 Easter eggs; column 2, 3 rabbits each holding 3 Easter eggs; column 3: 3 rabbits holding 2 Easter eggs each. T showed number sentences: 12÷3=, 9÷3= and 6÷3= on flashcards. Ls asked to match number sentences with correct column on poster.	Sharing situation, matching diagrams with written number sentences. Coherent.	1
	L2	E7	On a flashcard 7 groups of 3 Easter eggs shown. Ls counted the 21 Easter eggs and instructed to identify the groups of 3 resulting in 7 groups. T asked for number sentence and accepted: "21 divided by 7". She then stated: "We have 7 groups of 3 because if you count in 3s 7 times you will get 21". She then asked for a story sum. A L provided the following story sum: "There are 21 Easter eggs and 7 children how many will each one get?"	Grouping situation inferred from instruction to make groups of three. Situation matched to a number sentence. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from grouping action to sharing oriented number sentence.	3
	L2	E8	On a transparency: "How many sets of 2 in 18?" with a 6 column by 3 row array. T asked for a number sentence. "18 divided by 2" was offered and accepted. Ls proceeded to write "18÷2=9" on their whiteboards. T then stated: "Remember how you share 1 by 1". L then drew 2 circles and distributed counters 1 at a time between the 2 circles. T asked: "What is 18 divided by 2?" L responded: "9".	Grouping situation inferred from task presentation. In final section, a sharing action is offered and accepted. Ambiguity: unacknowledged move from grouping to sharing situation. No use of array.	3
	L2	E9	On transparency: 7 column by 3 row array. T: "Three girls eat 21 smarties. If shared equally, how many smarties did each one get?" T instructed Ls to count out 21 counters and to provide a number sentence. A L provided the number sentence "21÷ 3 =", then drew 3 circles, distributed the counters one at a time between the three circles, counted the counters in one circle and wrote "7" as the answer for "21÷ 3 =".	Sharing situation, empty array present but not used. Situation worked with through one by one distribution sharing action. Coherent.	1
	L2	E10	On a transparency: "How many 5's in 15? Write a number sentence as quick as you can?" L stated: "15 divided by 5 equals 3".	Grouping situation matched to verbal number sentence. Coherent.	1
	L2	E11	On transparency: "Jared shared 21 beads with him and 2 friends. How many beads will each 1 get, if shared equally?" L wrote "21÷3=" and drew 3 circles on the WB (White board). L distributed counters 1 at a time between the 3 circles and counted the counters in one circle, stated: "7" and wrote in "7" as the answer.	Sharing situation, worked with through one by one distribution sharing action. Coherent	1

	L2	E12	On board: "9÷3=", "16÷4=", "6÷3=", "8÷2 =" and "4÷2=". T accepted correct answers that were offered immediately with no further elaboration.	Bald calculations answered correctly and immediately. Coherent.	1
	L2	E13	On board: "93 ÷3 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent.	1
	L2	E14	On board: "46 ÷2 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent	1
	L2	E15	On board: "63 ÷3 =" within division bracket. Asking Ls to solve it, T worked through the short division algorithm focusing on individual digits at a time.	Bald calculation enacted through short division algorithm. Coherent	1
	L2	E16	Individual work done without the T. Ls were asked to make their own division tasks using a two-digit number by one-digit number that needed to be solved through a short division algorithm.		
	L2	E17	Group work not T led		
01-Aug-11	L3	E1	T recapped through a discussion the following terms written on flashcards - share, left, remainder, smaller and less and then stated: "That the answer is always smaller".	Vocabulary and 'rules/results'. Uncoded as no calculation in focus, but "the answer is always smaller" rule is reiterated here.	
	L3	E2	T asked Ls to place 9 counters in groups of 3. Ls made three groups with 3 counters in each. T asked: "Are there remainders?" Ls answered: "No". T asked for, accepted and wrote up "9 ÷ 3 = 3" as the number sentence. She then asked for a story sum for the number sentence. A L offers: "I share 9 marbles with 2 friends" (gesturing herself as well as part of the group) is accepted.	Grouping situation: 1 by 1 distribution into groups of 3. Towards end of episode T introduces a sharing situation. Ambiguity: unacknowledged move from grouping to sharing situation.	3
	L3	E3	Diagram showing 11 dolls. T stated: "So we are going to share. 11 divided by 3". T then wrote "11 ÷ 3 =" on the board and counted in 3s. As they counted in 3s the T made circles around groups of 3 dolls. She then asked: "Do we have any left?" Ls replied: "2 remainders". T then reiterated that they made 3 groups of 3 with 2 remainders. She then added "3 rem 2" as the answer.	Grouping situation with remainder played through by grouping through 1 by 1 distribution. Number sentence with answer produced. While the word "share" is used in opening part of this episode, the enactment within the episode is coherent. Coherent	1
	L3	E4	With 13 counters, T asked Ls to make groups of 4. A L made 3 groups of 4 counters and the T circled each group and instructed the L to indicate that the 1 counter is the remainder. T asked for a number sentence and accepted: "13 divided by 4" and wrote "13 ÷ 4=" on the board and wrote on top of the 4 "in a group". She then asked: "How many groups did you make?" A L stated: "3". T wrote "3" in the answer. She then asked: "Is that it?" L stated: "And 1 remainder" T wrote in "rem 1".	Grouping situation with a remainder; enacting a concrete experience and a number sentence played through by grouping through 1 by 1 distribution. Coherent	1
	L3	E5	With 14 counters, T instructed the Ls to make groups of 3. A L made 4 groups of 3 counters with 2 as the remainder. She then asked for a number sentence. A L stated: "14 divided by 3 equals 4 and remainder 2" and then wrote "14 ÷ 3 = 4 2 rem". T asked for a story sum and accepted: "I have 14 apples and 3 friends, how many will each one get?"	Grouping situation with remainder played through by grouping through 1 by 1 distribution. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from grouping to sharing situation.	3
	L3	E6	T wrote "19÷5=" on the board. T stated: "We going to group them into 5s". The Ls made groups of 5 counters resulting in 3 groups of 5 with 4 remainders. T then wrote in the answer "3 rem 4".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E7	Stating focus on sharing, T showed 2 large circles with 4 sweets in each, and 3 sweets outside of these circles. T: "Between how many people were the sweets divided? How many sweets did each person get? How many sweets are there in total?" T stated that the sweets should be shared between 2 people, each would receive 4 sweets with a remainder of 3. She then counted the 11 sweets pointing to each sweet in turn. Then, pointing to 1st and 2nd group of 4 sweets and then the 3 loose sweets, T stated that because of the remainder "3", "11 divided by 2" would not be appropriate. Circling each	Sharing situation with remainder initially indicated. The four objects in each large circle come to be associated with two contradictory signifiers: as a "group of four sweets" and as "four people". Ambiguity: one object associated with two contradictory signifiers. Limitation also flagged in instruction to interpret divisors as "numbers in a group".	3

			group of sweets, she accepted 4 as answer to: "How many did each one get?" Stating that she wanted to prove that 11 sweets could not have been divided between 2 people she wrote "11÷ 2=" on board. Accepting "5" as answer, T stated: "Each one will get 5" with "one remainder". "5 rem 1" was recorded as answer. T then stated that the correct number sentence for the diagram was: "11 divided by 4 equals 2 and 3 remainder" and wrote "11÷ 4= 2 rem 3". Telling Ls to work with groups, T "over-circled" the 2 groups of 4 sweets and asked: "11 divided by 4, each one will get?" "2" accepted. T stated 3 as remainder (pointing to the 3 loose sweets). She asked: "Between how many people were the sweets divided (pointing to the 4 sweets in each circle)?". "4" accepted. T stated: "11 divided by 4, each one will get a group of 2 (the teacher pointed at 2 circles), what will the remainder be?" Ls: "3". T then drew 5 groups with 2 circles in each with 1 single dot outside. Listing 11÷ 5=, 11÷ 2=,11÷ 3= , T indicated divisors as the number of items in a group.		
	L3	E8	Stating focus on problems about sharing, T showed 4 squares with 2 dog biscuits in each and 1 dog biscuit on the outside with questions: Between how many dogs were the biscuits divided? How many biscuits did each dog get? How many biscuits are there in total? T: "Between how many dogs were the biscuits divided?" Ls offered: "2" and "4". T accepted: "2" (pointing to the 2 biscuits in each square). She then stated: "Into how many groups did we put it?" (pointing to 4 squares). Ls: "4" which T accepted. T: "Who can give me a number sentence?" (pointing across all 4 squares). T accepted: "9 divided by 2 equals 4 with 1 remainder".	Sharing situation with remainder with matching diagram. Verbal number sentence requested and accepted. Ambiguity in T producing two signifiers for the objects inside the square as "dogs" and as "biscuits". Ambiguity: one object associated with two contradictory signifiers.	3
	L3	E9	Diagram: 3 squares each containing 3 slices of bread and 1 slice on the outside with questions: "Between how many boys was the sliced bread divided? How many slices of bread did each boy get? How many slices of bread are there in total?" For the number of boys involved, T accepted the answer "3". For the number sentence, she accepted: "9 divided by 3 equals 3 and 1 remainder". T went on to distribute 10 counters (slices) one at a time between 3 boys. She pointed at each boy and noted that each boy received 3 slices. She then stated: "9 divided by 3 equals 3 and this is my remainder (pointing to the 1 slice). But you can also do this 9 divided by 3, I take my 9 slices, I will put them into groups of 3 and I know 3 times 3 is 9". A L corrected the T’s number sentence with T’s agreement to "10 divided by 3 equals 3 and 1 remainder".	Sharing situation with remainder based on word problem. Situation worked with through one-by-one unit sharing action, with acknowledgement of the "slip" in the dividend value. Shift to group based distribution is also acknowledged explicitly as an alternative option here for the orally stated number sentence. Coherent	1
	L3	E10	On transparency: "Share 24 biscuits among 5 children?" T: "Share 24 among 5. You can count in 5". T wrote "24÷5=" on board and added: "in a group" on top of the 5. She listed more division calculations (16÷3=,14÷4=,13÷2=) and pointed to the divisors as the number of objects in a group. Returning to "24÷5=" she drew 5 dots and repeated the groups of 5, 4 times. Then circled the groups in five and counted in fives to make 20 and drew a group of 4 dots and inserted that the answer was 4 remainder 4.	Sharing situation with remainder. Grouping model acknowledged with "you can count in five" and solved by group based distribution. Limitation also flagged in instruction to interpret divisors as "numbers in a group".	2
	L3	E11	T wrote "16÷3=" on the board and instructed the Ls to draw groups of 3. Working with 2 Ls and their WB, she demonstrated how to draw groups of 3 and ended with 5 groups of 3 with 1 remainder.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E12	T wrote "14÷4=". Based on the task, T instructed the Ls to solve the calculation by drawing the diagrams. Ls are seen drawing 3 groups with 4 dots in each with 2 dots. T wrote in the answer "3 rem 2".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L3	E13	T wrote "13÷2=". Ls then wrote "13÷2=" on the WB and then drew 6 groups of 2 with 1 remainder and added the answer "6 rem 1"	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1

	L3	E14	Individual work without the T – Division with remainder. 1. Read and complete the following. A picture of 19 stars was provided. Group the stars in 4's (circle)_____. How many groups? _____ How many stars left over? _______ Write a number sentence for the problem? ___ 2. Problem solving use your WB to do the following, you can also use counters. Three birds share fish equally. How many fish will each get? Remember to write a number sentence. Picture of 11 fish was provided. Number sentence____. 3. Share 19 books among five people. Are there any left? Picture of 17 books were provided. Number sentence___.		
03-Aug-11	L4	E1	T recapped through discussion the following terms: "division", "share", "smaller", "less", "remainder", "less" and mentions that "the answer for division is always smaller".	Vocabulary and "rules/results" Uncoded as no calculation in focus, but "the answer is always smaller" rule is reiterated here.	
	L4	E2	Holding 9 sweets, T stated: "Let's share 9 sweets. How many groups can she make?" Ls responded, "3". T counted 3 groups of 3 sweets. Calling 3 Ls to the front, T gave each one 3 sweets and stated: "9 divided by 3 and each child will get 3". T took out 2 more sweets in her hand and asked: "If I have 11 sweets what do I call the 2 sweets". Ls responded: "11 divided by 3 equals 3 with 2 remainder". T wrote "9÷3=3", and then corrected this to "11÷3=3 2 rem".	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L4	E3	T: "If you have 13 sweets and there are 3 children can you group it?" Ls told to write a number sentence for "13 divided by 3" and calculate answer by making groups of 3 with counters. T wrote "13÷3=" on board and proceeded to draw a group of 3 dots at a time and circle them while counting in multiples of 3 resulting in 4 groups of 3. She then drew in one dot to show the remainder and added "4 rem 1" as the answer.	Sharing situation with remainder, linked to a number sentence, with diagram produced through group based distribution. T produces two signifiers for "3": as "3 children" and as "3 sweets". Ambiguity: one object associated with two contradictory signifiers.	3
	L4	E4	T wrote "9÷2=". Ls drew answer on WBs. To check the answer T took a WB with the symbolic number sentence "9÷2= 4 1 rem" with 4 groups with 2 dots in each with 1 dot on the outside. She then asked the Ls to count the dots in 2s and added the 1 loose dot as she pointed to the 9.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E5	T wrote "17÷5=" and asked: "You are going to do your grouping in?" Ls responded: "5". A L wrote "17÷5=" on their WB and then drew 3 groups of 5 with 2 dots left outside and provided the answer "3 rem 1".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E6	T wrote "14÷4=". She then accepted 3 circles with 4 dots in each with 2 dots on the outside and "14÷4= 3 2 rem". One L did not answer correctly. To assist the T offered 7 divided by 3 which produced 2 groups of 3 with 1 remainder. She then stated to the Ls that they needed to remember that: "When they do division the answer is always smaller".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Limitation: division answer always smaller.	2
	L4	E7	T wrote "17÷4=". She then stated: "How many groups of 4?" A L responded: "4". A L responded, "4 and 1 remainder". The T proceeded to write "4 1 rem" as the answer.	Bald calculation with remainder produced through recalled number facts. Coherent	1
	L4	E8	Skip counts in 5's,10's and 3's.	Skip counting fluency	
	L4	E9	T asked class to draw 5 groups of 3 and to write down a division written number sentence with answer. 15 ÷ 3 = 5 written down by Ls. T then instructed Ls to add 2 counters and stated that the 2 counters were called the remainder. She said: "When we divide our answer is always smaller". Finally, she told the Ls to change the initial division calculation of 15 ÷ 3= 5 to 17 ÷3= 5 rem 2.	Grouping situation with remainder – limitation: answer to division "always smaller". Ambiguity in what is in focus: "answer is always smaller" could refer to either quotient integer or remainder or both together. Ambiguity: Set of words associated with two or more contradictory signifiers.	3

	L4	E10	Diagram: 2 circles, 4 sweets in each and 3 sweets on the outside with the following questions: Between how many people were the sweets divided? How many sweets did each person get? How many sweets are there in total? T stated: "You see all these sweets" (points to all 11 sweets). Without pointing to the diagram again T asked: "How many sweets are there altogether?" Ls: "11". T: "How many in a group?" Ls: "4". T: "How many groups of 4?" Ls: "2". T: "How many left over?". Ls: "3". T: "Between how many people were the sweets divided?" Ls: "2". T: "How many sweets did each one get?" Ls responded: "4". T asked: "What is your remainder?" Ls replied: "3". T then asked: "How many sweets are there in total?" Ls replied: "11".	Situation repeats sharing situation with remainders in L3 E7, but sweets and children remain unambiguously pointed to throughout the signification pathway here. Remaining in the situation leads to coding as: Coherent.	1
	L4	E11	Same example and explanations done with a previous group.		
	L4	E12	Same example and explanations done with a previous group.		
	L4	E13	Same example and explanations done with a previous group.		
	L4	E14	T placed a chart with 3 rows of teddy bears in groups. In the first row there are 11 teddy bears in a row made up of four groups of teddy bears. In the first 3 groups there were 3 teddy bears in the last group there were 2 teddy bears. She asked: "How many teddy bears in a group?" She then established with a re-created diagram of circles and dots that the first 3 groups had "3" teddy bears, the last group had 2 teddy bears.	Grouping situation with a remainder matched to diagram of circles and dots. Coherent.	1
	L4	E15	T wrote "24÷5=". Ls told to draw groups of 5. T assisted Ls by drawing circles 4 groups of 5 dots and then drawing in 4 separate dots. She ends by "4 4 rem" as the answer.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E16	T wrote "11÷2 =" , Ls told to make groups of 2 using counters. Ls made groups of 2 and added the answer "5 1 rem" to "11÷2 =" on the WB.	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E17	T wrote "14÷3=". She told Ls: "Remember to count, 14 sweets in groups of 3". Ls packed 4 groups of 3 with 2 counters. Two offers for remainder: "1" and "2". T affirmed that remainder was "2".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent	1
	L4	E18	T wrote "18÷5=". She told the Ls to pack out 18 counters. asked: "How many in a group?" Ls answered:"5". She then asked: "And what is the 3?" Ls responded: "Remainder". T stated: "My answer is 3 rem 3".	Bald calculation with remainder translated into groups through 1 by 1 distribution. Coherent.	1
	L4	E19	On transparency: "Divide 19 sweets in groups of 3 __ left __". T instructed the Ls to make groups of 3 using 19 counters, Ls ended with 6 groups of 3 with one remainder, some Ls wrote "6 1 rem" as the answer.	Grouping situation with a remainder played through by grouping through 1 by 1 distribution. Coherent.	1
	L4	E 20	Individual work without the T – Division with remainder. Picture of 20 stars was provided. Group the stars in 3's. How many groups are there? How many stars are left? Write a number sentence for the problem. Picture of 13 circles was provided. Count the dolls and fill in the correct answer. In a table with 3 columns by 4 rows. How many groups of 4, 2,3,5 and 6 could be made, and what is your remainder? 3		
	L4	E21	T placed a flashcard with 3 rows of teddy bears - 1st row, 11 bears in 4 groups, 3 teddy bears in the first 3 groups & 2 teddy bears in the last group; 2nd row, 7 bears in 3 groups, 2 groups of 2 bears and 1 more bear; 3rd row: 7 bears in 2 groups, one with 4 teddy bears and another group of 3 teddy bears. T asked for number sentences for each row. As number sentences were produced, she circled groups and wrote the number sentences: 11÷3= 3 rem 2 for 1st row, 7÷2= 3 rem 1 for 2nd row, and 7÷4= 1 rem 3 for last row.	Grouping situation with a remainder played through by grouping through 1 by 1 distribution. Coherent.	1

Appendix B – Mary 's episodes across lessons

Date	Lesson	Episode	Task and its enactment	Division focus in episode and coherence – related comment	Signification Pathway – coherence category
1 June 2011	L1	E1	Count in 2's from 2 to 100, 3's from 3 to 120, 5's from 5 to 150	Skip counting fluency	
	L1	E2	T wrote ÷, x, +, - on the CB and asked the Ls to identify each one.	Identifying mathematical notations	
	L1	E3	T mentioned that when we talk about divide and division we are talking about sharing.	Vocabulary and 'rules/results'	
	L1	E4	10 ÷ 2 on flashcard and a two row by five column array drawn on board. T: "I have ten sweets and then I have to share them to two people", pointing to array. T circles two dots in the array at a time and gestured that five learners would each receive two dots. T: "How many big circles must I give these children? L: "Five". T: "Five big circles" T: "Let's share the sweets to five people." Teacher drew five circles, put two dots in each and Ls said: "Two sweets" in each circle. T pointed to circles: "We have ten sweets into five big circles which means ten divided by two is?". Ls counted in twos as T pointed to each circle. T wrote the answer "5" on the board and on the flashcard.	Bald calculation interpreted as a sharing situation but divisor is then worked with as the number in a group from the array. Situation involving sharing between 5 people is then redrawn in equal groups diagram, with skip counting in 2s to 10, after which T writes in 5 as answer. Ambiguity in that 2 refers at one point to 2 Ls and later to 2 sweets: one number is associated with two contradictory signifiers.	3
	L1	E5	21 ÷ 3 = on flashcard. T: "We shared what? How many? Who can give me the answer?" L: "Seven". T wrote "7" T: "How many big circles can I make for getting the answer?" L: "Seven". T drew seven big circles. T: "Into the big circles how many sweets must I put?" L: "Three sweets". T added three dots to each big circle while Ls counted 1-3. T: "21 sweets were distributed between the seven children". Ls counted in threes from 3 to 21 as the T pointed to each big circle. T: "Which means your answer is?" Ls: "Seven".	Bald calculation interpreted as a sharing situation with immediate answer provided. Problem then worked with through grouping through 1 by 1 distribution based on this quotient. Skip counting used to count up to dividend. The quotient is used as the basis for the drawing of diagrams pointing to the number of groups and the divisor is then interpreted as the number in each group. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation.	3
	L1	E6	15 ÷ 5= on flashcard and T wrote the answer 3; T produced answer in the same way as in L1:E5	Bald calculation – Same reason as L1:E5	3
	L1	E7	20 ÷ 2 = on flashcard and T wrote in the answer 10; T produced answer in the same way as in L1:E5	Bald calculation – Same reason as L1:E5	3
	L1	E8	15 ÷ 3 = on flashcard. T told Ls to make groups of three with counters. T: "What is the answer?" Ls: "Five!" T: "How many big circles?" Ls: "Five". T drew four circles with three dots in each.	Bald calculation interpreted as a grouping problem with immediate answer which produced diagrams through grouping through 1 by 1 distribution - Instead of drawing five circles the teacher only drew four circles and did not correct the error. Incoherent: incorrect diagram with no answer provided.	4

02 June 2011	L2	E1	T wrote: "There are 36 fish in the boat. All 4 fishermen caught the same number of fish. How many did each one get?" Ls asked to raise four fingers to show that fish will be shared amongst four people. T wrote "36 ÷ 4 = _". T: "How many big circles should we draw to share the thirty-six fish to those men?" pointing at "4". "Four" & "nine" offered. "Nine" accepted. "9" added to calculation. T drew nine circles. T: "And how many fish do we put into each of the circles?" L: "Four!" T accepted the answer. T drew four dots in each circle with Ls counting 1- 4. T: "If we count all the fish in the circles it will give us?" Ls: "Thirty-six fish". T pointed to the calculation: "Thirty-six fish are divided between four men; how many do these four men get?" L: "Nine" T: "Each man gets nine" as she pointed to the nine circles.	Sharing situation produces a number sentence. "9" accepted as answer # of circles & inserted as answer to number sentence. T's diagram uses 9 as # of groups & 4 as # in each group: a grouping interpretation of symbolic calculation. Group of dots drawn into circles. Ambiguity: unacknowledged move from sharing to a grouping interpretation of the bald calculation.	3
	L2	E2	The T placed three different types of charts on the WB for each group to solve; not T led.		
13 Sept 2011	L3	E1	Using a division sheet the Ls recited the division calculation.	Division facts fluency	
	L3	E2	The T wrote the word "division" on the CB. She said, "By division we mean sharing".	Vocabulary and 'rules/results'	
	L3	E3	T wrote: "A soccer team scores 15 goals. Each scorer scores 3 goals. How many players score goals?" T: "Something is being grouped into something" L: "You make groups of three until you get to fifteen" T: "What are we grouping?" L: "Each soccer team must be grouped into three". T circled the "15" in the word problem: "How many groups (pointing to the three) can give me the fifteen?" L: "Five groups of three" T: "She is going to divide or group or share the fifteen goals into groups of three". T drew three dots and circled the three dots, she continued to draw groups of three until she had five groups of three. T: "How many groups of three am I having here?" while pointing to the five groups. L: "Five groups of three". Ls counted the five circles as told to do. T reiterated that five groups of three makes fifteen.	Grouping situation played out through 1 by 1 distribution, linked to a word problem, with diagram produced. While the word "share" is used in the middle of this episode and there is a slip -up between "soccer team" and "goals", the enactment within the episode is coherent. Coherent	1
	L3	E4	T wrote 15 ÷ 3 =. T stated: "These types of number sentences have nothing to do with grouping, they can be answered as a mental exercise using multiplication". T read the number sentence and added "5" as the answer.	Bald calculations answered correctly and immediately. Coherent	1
	L3	E5	T wrote 24 ÷ 3 = L: "Twenty-four divided by three is eight" T: "How did you get to the answer?" L: "We group twenty-four into threes". T added "8" as the answer to the symbolic calculation.	Bald calculations answered correctly and immediately. Coherent	1
	L3	E6	T wrote 30 ÷ 5 = T: "How did you get to the answer?" L: "Maybe a person could draw 30 circles and make groups into five" which T accepted. T then added the number "6" to the division symbolic calculation.	Bald calculations answered through a verbal description of grouping through 1 by 1 distribution based on the divisor to produce the answer. Coherent	1
14 Sept 2011	L4	E1	With the aid of a division sheet, the Ls recited the division calculation	Division facts fluency	
	L4	E2	T recapped homework.		
	L4	E3	T wrote: "A teacher has a bag of 32 sweets. He gives each child 4 sweets. How many children are there?" L: "I drew thirty-two sticks and then I circled four sticks and counted the circles and got to the answer eight". T accepted the answer. T drew thirty-two sticks as Ls counted from 1 to 32. T counted four sticks and then circled them and Ls counted the circles. T drew eight circles altogether. T wrote "32 ÷ 4 = 8".	Grouping situation translated into a diagram and number sentence, and played out through a grouping 1 by 1 distribution. Coherent	1
	L4	E4	Ls work individually - Solve 45÷5=, 21÷3= and 12÷6=		

Appendix C – Hazel's episodes across lessons

Date	Lesson	Episode	Task and its enactment	Division focus in episode and coherence – related comment	Signification Pathway – coherence category
10-May-11	L1	E1	Discussion to define what the word "sharing" means	Vocabulary and "rules/results"	
	L1	E2	Ls asked to count in 2's from 2 to 24, then in 3's from 3 to 36. T wrote "12x 3=36" and acknowledged "3x12=36" as the same. Ls then asked to count in 4's from 4 to 48 and T wrote "12x4= 48" on the board. T then stated: "Times is the opposite of division".	Linking skip counting and multiplication calculations and stating that multiplication is the opposite of division.	
	L1	E3	T stated: "Times is the opposite of division". On a 200 number chart on the board, T covered the multiples of 2 from 2 to 24 with magnets. Writing 12x2=24 on the board, T asked for a division symbolic calculation. Ls offers the division calculation 24÷2=12. T stated: "Whenever you do a divided sum the biggest of the numbers always goes first".	Bald calculation with correct answer offered immediately – limitation: in division the biggest number always goes first.	2
	L1	E4	T wrote 24÷2=12 and added a printed three column sheet on the board. She mentioned that because the divisor was 2 she would use two columns. She asked: "How many counters do I need, 24, 2 or 12?". T rejected an offer of 12 stating 24 as the answer. She reiterated: "The biggest number ..?" Ls responded: "always goes first". T picked out 24 magnets and a L distributed these between the two columns one by one. T then counted the number of magnets in one column and stated 12 as the answer.	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution – limitation: in division the biggest number always goes first.	2
	L1	E5	Group work, not T led		
11-May 11	L2	E1	Division described as "sharing or grouping" and the need to equally share was stated. T reminded Ls that if something was left over after dividing it was the remainder. Asking for reasons why a L had provided "24" as an answer for 12 ÷ 2 =, a L stated that they had treated the division calculation as a multiplication calculation. T reiterated multiplication as the opposite of division, and reminded the class: "What was the rule yesterday, the biggest number goes first".	Vocabulary and "rules/results". Uncoded as no calculation in focus, but 'biggest number goes first' rule is reiterated here.	
	L2	E2	T wrote 1x2=, 2x2= and 3x2=. Ls completed these calculations correctly. Next to each multiplication calculation the T drew one/ two/three circles respectively and placed two magnets in each. She then created division calculations as follows 2÷2=1, 4÷2=2, 6÷2=3, drawing two circles next to each division calculation. Pointing to the divisors, she stated: "We are sharing equally, only share between two" pointing to the two columns on the printed three columns sheet. T then distributed one magnet at a time between the two columns or drew in one dot at time between the two circles for 2÷2=1, 4÷2=2. To solve 6÷2=3, a L placed three counters in the first column and then three counters in the second column and then proceeded to draw three dots in the first circle and then three dots in the next circle. T then wrote 8÷2= 4 on the board. A L came to the board, drew two circles and proceeded to draw a dot one at a time between the two circles until four dots appeared in each circle.	Bald division calculations with correct answer offered immediately, answer verified through 1 by 1 sharing distribution and sharing through complete composite units - Ambiguity in teacher producing two contradictory signifiers for "2": "2 magnets inside each circle" for multiplication calculations and as "2 circles" for division calculations without acknowledging this shift. Ambiguity is seen in that one number is associated with two contradictory signifiers.	3
12-May-11	L3	E1	T recapped that dividing can be understood as both sharing and grouping.	Vocabulary and "rules/results"	

	L3	E2	T wrote 6x2=12, 7x2=14,8x2=16 on board, and rewrote them as 12÷2=6, 14÷2=7, 16÷2=8. Saying that more division calculations could be made from the same numbers, T wrote 12÷6=2, 14÷7=2 and 16÷8=2 on the board. She stated again that the biggest number always goes first and added: "When we divide the answer is always smaller and when you multiply your answer is always bigger".	Bald calculation with correct answer offered immediately - Limitation: the biggest number always goes first in division and the answer is always smaller. [Also, in multiplication the answer is always bigger].	2
	L3	E3	T wrote x2 and ÷2 on board and explained that when multiplying by two, doubling occurs which is repeated addition and when you divide by 2, halving happens. She wrote 2x2=4, 3x2=6 and 4x2=8 to illustrate doubling and then stated that the opposite of doubling is halving. Writing 2÷2=1, 4÷2=2,6÷2=3, she linked these calculations to halving saying: "half of two is one, half of four is two and half of eight is four". She mentioned that halving meant: "Share equally".	Bald calculations with correct answer offered immediately. Doubling highlighted as any number multiplied by two and halving is any number divided by two. Coherent	1
	L3	E4	T said that not all division calculations can be shared equally and wrote 7÷3=_. T asked for seven magnets and used a three column printed sheet. She distributed the magnets one at a time between the three columns producing two magnets in each column with one magnet outside the printed sheet. She then wrote "r" next to the outside magnet. Pointing to the two magnets in a column and the one magnet outside she wrote 2 r 1 as the answer.	Bald calculation with remainder with correct answer offered immediately, answer verified solved through sharing 1 by 1 distribution. Coherent	1
	L3	E5	T wrote 9 ÷2= and took out nine magnets to distribute between two columns stating the divisor in the division task was 2. T distributed the magnets one at a time between the two columns (labelled Tom and Dick) resulting in 5 magnets (Tom) in one column and 4 magnets (Dick) in another column. She asked: "Did I share them equally?" Ls said: "No". T asked: "Who got more than the other?" Ls responded: "Tom". T asked: "How can I make it even?" A L responded that the T should take away one from Tom which she did. Stating that there were 4 magnets in each column with one as the remainder T wrote "2 r 1" as the answer.	Bald calculation with remainder solved through 1 by 1 sharing distribution. Coherent	1
	L3	E6	T wrote 36÷3=. Ls answered: "12". T stated: "36 needs to be divided by 3" and drew three circles as she pointed to 3 in the division calculation. T asked: "How much are you going divide?" Ls responded: "36". T drew in crosses one at a time between the three circles. T pointed at one circle and counted crosses in threes, then asked: "How many will each one get?' Ls responded: "Twelve". T wrote "12" as the answer.	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution and skip counting. Coherent	1
	L3	E7	T wrote 15÷5= on the board and solved the task as in L3:E6	Bald calculation – worked with as L3:E6	1
	L3	E8	Pointing to 12÷6= T asked: "How many groups would she need to draw?" Ls responded: "6". T drew six circles on the board and asked: "How much will she have to share?" Ls answered: "Twelve". Ls drew crosses one at a time between the six circles counting up to 12. T asked: "How much did each one get?" A L stated: "2". T wrote in the answer "2".	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution. Coherent	1
	L3	E9	Group work not T led		
	L3	E10	T wrote 37÷3=. She said: "Thirty-seven beans divided by three". On printed three column mats Ls distributed 37 counters one at a time between the three columns resulting in 12 counters in each column with one counter on the outside. T asked: "How many did each	Bald calculation with remainder solved through 1 by 1 sharing distribution. Coherent	1

			one get?" Ls responded: "12" as T wrote in 12 as the answer, then L stated: "and 1 remainder" and the T added "rem 1"		
13-May-11	L4	E1	The T recapped on concepts such as remainders, division tasks with different divisors, fractions and working with whole numbers.	Vocabulary	
	L4	E2	Not analysed - preparation for an assessment		
	L4	E3	T stated: "Dad has got two thousand sunflower seeds. He plants them into two rows. How many seeds will he plant in each row?" Ls responded: "One thousand". T then asked: "What kind of number sentence will I write for that sum?" A L stated: "Two thousand divided into two equals one thousand". The T wrote 2000÷2=1000 on the board. T reiterated that dividing is "to share equally".	Sharing situation with correct answer offered immediately. Coherent	1
	L4	E4	Not analysed - Ls wrote assessment		
19-Sep-11	L5	E1	Skip counts in 25's and 3's	Skip counting fluency	
	L5	E2	T pointed to 6 x 5= and noted that 5 x 6 is the same and wrote 30 as the answer for both. She remarked that the opposite of multiplication is division and asked: "What happens when we divide, does our answer get bigger or smaller?" Ls replied: "Smaller". T then asked Ls for a division calculation for 6 x 5=30. A L answered: "Thirty divided by five equals six". T wrote 30÷5 = 6 and asked: "How do I draw that sum? How many groups?" Ls replied: "Five groups". T drew five circles and reiterated: "I must share between five groups". T then asked: "How much must I divide?" Ls responded: "Thirty". T then asked: "Each one will get?" A L answered: "Six". T then drew in dots one at a time between the five circles resulting in six dots in each circle. "Six groups" was accepted again as the "number of groups".	Bald calculation with correct answer offered immediately, answer verified through 1 by 1 sharing distribution - Limitation: in division, the answer is always smaller.	2
	L5	E3	T pointed to 9 x 3= and said: "What is the answer to nine times three?" 27 was offered and written in as the answer. T asked for a linked division calculation and accepted and wrote "27÷3=9" on the board. As in a previous episode, T accepted "3" as number of groups and "9" as number in each group. T then asked for a different division calculation. A L suggested: "Twenty-seven divided by nine equals three". T then wrote "27÷9=3". T stated: "If I had to draw that sum, it would be nine groups and I needed to share twenty-seven between the nine groups. There would be three in each group".	Bald calculation with correct answer offered immediately. Coherent	1
	L5	E4	Group work not T led		

Appendix D – Asha's episodes across lessons

Date	Lesson	Episode	Task & its enactment	Division focus in episode and coherence-related comment	Signification Pathway Coherence category
21 May 2012	L1	E1	Not analysed - T gives instruction on what needs to be done for homework.		
	L1	E2	Six times table is recited	Multiplication facts fluency	
	L1	E3	Not analysed - Mental maths activity		
	L1	E4	T wrote 45÷5 on the board. She asked: "So how did you do it?" A L came to the board and drew a short division bracket and placed 45 inside the bracket and 5 on the outside. T then stated: "Does five go into four?" The L replied: "No". T asked: "Does five go into forty-five?" The L replied: "Yes" and the T asked: "How many?" L answered: "Seven". T said: "No". L then said: "Nine" and wrote "9" on top of the short division bracket.	Bald calculation is enacted through short division algorithm with learners input. Coherent	1
	L1	E5	T took out thirty marbles. She asked: "If you are dividing the marbles among your four friends and you, how many bowls should I have? Ls shouted: "Five". A L distributed two marbles at a time between the five bowls and then she distributed another two marbles at a time between the bowls and then finally another two marbles at time in each bowl. T then asked: "So how many marbles do you have in each bowl?" The L replied: "Six".	Sharing situation worked with through partial composite unit distribution sharing action. Coherent	1
	L1	E6	T then called another L to the front table and added another marble to the thirty marbles. She asked: "What are you going to do?" A L distributed the marbles one at a time between the five bowls. T asked: "What is left, you have six marbles in each bowl, what is in your hand, what are you going to do with it? A L stated: "I am going to share it?" T then asked: "What do you call this? Another L shouted: "Remainder". T accepted the answer. Another L stated: "Remainder one". T stated: "Six remainder one will be your answer".	Sharing situation with remainder (based on word problem offered and worked with through a one by one distribution sharing action). Coherent	1
	L1	E7	Saying that the "breaking down" method would be used, T wrote "145÷5" and asked: "How do I break this up?" A L stated: "One hundred plus forty plus five". T wrote "100+40+5". T then said: "One hundred divided by five plus forty divided by five plus five divided by five" and wrote "(100÷5) + (40÷5) +(5÷5)". T divided each number by 5 resulting in "20 +8+1" and asked: "What do you think we can do?" A L stated: "We need to add them (T pointed to "20 +8+1" on the board) and proceeded to add and produced the answer "29".	Bald calculation is enacted through the use of expanded notation and sum of decomposed divisions with learner offers of immediate answers. Limitation: the method of using expanded place value notation to determine the answer works well for division by 5 but is less effective more generally for divisors other than 5/10.	2

	L1	E8	T then wrote "145÷5" in a short division bracket and asked: "Does five go into one (points to the "1" in "145"). The Ls replied: "No". T asked: "Does five go into fourteen (points to the "14" in "145")? The Ls replied: "Yes" and the T asked: "How many times?" The Ls shouted: "Twice" and the T wrote "2" on top of the short division bracket. T asked:" So what do we have left?" A L said: "One remainder" which the T did not accept. T then stated: "No" and wrote a small "4" before the "5" to make "45". T asked: "So five goes into forty-five how many times (pointed to the "45"). The Ls shouted: "Nine" and the T wrote "9" next to the "2" on top of the short division bracket producing the answer "29". T then made the point that they had produced the same answer when they used "expanded notation" and "short division algorithm".	Bald calculation is enacted through short division algorithm with learners input. Coherent	1
	L1	E9	Focusing on "145÷5". T wrote 145 - 5 in subtraction column algorithm format and asked: "What is the answer?" Together with the Ls, 5 was repeatedly subtracted from 145 until 105. The T asked: "Do you understand this method, so you continue until you come to?" The L shouted: "Zero" which the T agreed with. The T illustrated that repeated subtraction took longer by saying "it will go on and on and on and take the whole period, therefore short division works for me".	Bald calculation is enacted through repeated subtraction with learners input. However, the T does not enact repeated subtraction until the end and does not count the number of times 5 was subtracted to show the answer for the division calculation. Coherent	1
	L1	E10	Individual work - Solve forty-three divided by two using a short division algorithm.		
	L1	E11	T represented 43÷2 in a short division bracket. T asked: "Two goes into four how many times?" The Ls shouted:" Two" and the T wrote "2" on top of the short division bracket. T then asked: "Two goes into three?" the Ls replied: "Once". T wrote "1" next to the "2" on top of the short division bracket. T asked: "Where do you put your remainder?" The Ls stated: "Next to the one". T wrote in "r 1" producing the answer "21 r 1".	Bald calculation with remainder is enacted through short division algorithm with learners input. Coherent	1
22-May-12	L2	E1	Not analysed - Ls wrote down the homework		
	L2	E2	Six times table is recited	Multiplication facts fluency	
	L2	E3	Not analysed - Mental maths activity		
	L2	E4	T drew a short division bracket and added a "635" inside the bracket and then "3" on the outside and produced the answer "211 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
	L2	E5	T drew a short division bracket and added a "815" inside the bracket and then "3" on the outside and produced the answer "271 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
	L2	E6	T drew a short division bracket and added a "965" inside the bracket and then "3" on the outside producing the answer "321 r 2" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1
	L2	E7	Not analysed - Teacher checks the homework from the previous day.		

	L2	E8	T showed the Ls a bowl of sweets and said that there were 71 sweets in the bowl. With three other bowls. T then stated: "What I want you to do is that I want you to divide these sweets for me into three containers (points to the three containers on the table). How are you going to do this?" Ls distributed first two sweets, then three sweets, then four sweets at a time between the three containers. T stated that after dividing the sweets equally amongst the three containers there was 1 remainder and showed the class one sweet. T asked: "How many do you think she has in each of the containers?" A L confirmed by counting that there are twenty-four sweets in one of the containers. Writing 71 ÷ 3 in a short division bracket and using the short division algorithm, the T produced the answer "23 r 2". She then noted that L must have counted incorrectly as she only had 1 remainder. As another L came to the front to distribute the sweets, T recounted the sweets in 2s and established that there were 71 sweets. The L distributed the sweets two at a time between the three containers, but again ended with one sweet left over. T reworked the short division algorithm and stated that the remainder should be 2. T called another L to the front who distributed the sweets two at a time between the three containers, and still ended up with one remainder. T returned to the board and again, went through the short division algorithm.	Sharing situation with remainder (based on word problem offered and worked with through partial composite unit distribution sharing action). The sharing action then produced an answer 23 remainder 1. To verify the answer, the T used a short division algorithm which produced the answer 23 remainder 2. Incoherent: incoherence of answers produced by algorithm and by sharing actions is not dealt with.	4
23 May 2012	L3	E1	Not analysed - Ls wrote down the homework		
	L3	E2	Not analysed - Circulation of division assessment		
	L3	E3	Not analysed - T checks the homework from the previous day.		
	L3	E4	Not analysed - Circulation of assignment		
	L3	E5	Not analysed - Assessment		
	L3	E6	T drew a short division bracket and added a "524" inside the bracket and then "7" on the outside producing the answer "74 r 6" using the same steps as in L1 E11.	Bald calculation with remainder – Same as L1:E11	1

Appendix E – Joy's episodes across lessons

§Date	Lesson	Episode	Task & its enactment	Division focus in episode and coherence-related comment	Signification Pathway Coherence category
23 April 2012	L1	E1	T stated that her mother bought 6 sweets for her and 2 Ls and she wanted to know how many did each one eat. A L replied: "2". Ls took out six counters and shared them equally between 3 people. The Ls distributed the counters two at a time making 3 groups of 2.	Sharing situation: action interpretable as a complete composite unit sharing action. Coherent.	1
	L1	E2	The T stated that her mother had R50 and wanted to share it amongst 5 family members and asked: "How much will each one get?" A L responded: "10 Rand". T then drew five "R10" beneath each other in a column and as she drew them she named different members of the family. T then added the five "R10s" to confirm that R50 was shared equally.	Sharing situation: action interpretable as a complete composite unit sharing action through the use of symbols. Coherent	1
	L1	E3	T demonstrated through role play how nine oranges were bought and returned to enact the forward and backward counting in multiples of 9. T then pointed to an empty NL on the board and placed flashcards with multiples of 9 on the NL from 0 to 99.	Counting forward and backwards in multiples of nine and linking to the number line.	
	L1	E4	On board, a NL with multiples of 9 from 0 to 90 marked. T asked: "How many groups of 9 in 18?" and wrote "18÷ 9 = " on the board. A L answered: "Two" and pointed to "18" on the NL. T then added "2" to the number sentence as the answer and stated: "There are 2 sets of 9 in 18". Another L was asked for a different way of producing the answer. The L drew 18 sticks and circled nine sticks at a time and counted both circles to show that the answer was 2.	Bald calculation with correct answer offered immediately, answer linked to 18 on the NL and verified by circling groups of sticks. Coherent	1
	L1	E5	T wrote "27÷ 9 =" and asked a L to solve it by drawing sticks. The L drew 9 more sticks in a circle alongside the two circles of nine sticks and wrote "3" on top of the third circle. T asked the L to count the circles. The L counted the three circles wrote "3" as the answer for "27÷ 9 =".	Bald calculation with grouping actions through circling groups of sticks to produce the answer. Coherent	1
	L1	E6	Next the T wrote "9 ÷ 9 =". L drew nine sticks on the board, circled them and wrote 1 on top of the circle. T then wrote "1" as the answer and stated: "How many sets of 9 in 9, is just 1". Referring to the previous example. The T asked: "How many sets of 9 in 18?" as she pointed to "18÷9 =2". The T explained: "It is the same as saying I am sharing 18 oranges among 9 boys".	Bald calculation - The bald calculation was solved through grouping nine sticks at a time. To clarify the number of sets in the first division calculation the teacher used a previous example. In the final section, a sharing story sum is offered and accepted. Ambiguity: unacknowledged move from a grouping to a sharing interpretation of the bald calculation.	3
	L1	E7	T wrote "45÷ 9 =" and stated: "I want to divide 45 by 9; I want to share 45 sweets between 9 girls". T then went to the number line and jumped in 9 from 0 to 45. As the T jumped on the number line the Ls counted in ones from one to five and the T wrote "5" on top of the last jump.	Bald calculation - the T instruction is to solve a sharing situational problem. Based on the sharing situation an answer was produced by jumping on the NL. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation.	3

	L1	E8	T then wrote "27÷ 9 =" and said: "I want to see if I get the same answer using the NL" as opposed to sticks. The T jumped 3 times in nines on the NL and pointed at the last jump asked for the answer, Ls shouted out: "3". T wrote "3" as the answer.	Bald calculation with grouping actions play out by jumping in multiples on a NL to produce the answer. Coherent	1
	L1	E9	T wrote: "45÷ 9 =; 63÷ 9 =; 72÷ 9 =" and labelled them A, B and C respectively on the board. Three Ls then drew sticks based on the dividend value and circled groups of nine sticks producing answers such as 5, 7 and 8 respectively, next to each group of sticks. T said that they could use a quicker approach by using the NL. She then drew a NL and marked multiples of 9 from 0 to 45. T said: "If you at forty-five (points to 45 on the NL) just wait as you are sharing forty-five oranges" and pointed to "45" in the calculation "45÷ 9 =". She then asked: "So how many groups do you have?" before jumping 5 times on the NL and writing "5" as the answer to calculation A.	Bald calculation - The T accepts grouping actions through the grouping of drawn sticks and marking multiples on the NL. A sharing situation was offered and accepted. The teacher then asked how many groups and then jumped on the NL in multiples. Ambiguity: unacknowledged move from a grouping to a sharing interpretation of the bald calculation.	3
	L1	E10	Next the T pointed to a 10 by 9 array made up of dots. She reminded Ls that there were 9 oranges in a row by counting each orange in the first row. T then asked Ls to count in 9s from 0 to 90 as she pointed to each row. T used the array to produce the answers of "9 ÷ 9 = 1, 18÷ 9 = 2 and 27÷ 9 = 3" by counting sets of nine in the array.	Bald calculation was enacted through use of an array with learners input. Coherent	1
	L1	E11	T instructed Ls to take out 27 counters. She then said: "Put them into groups of 9". T then said: "Let's say that the counters are sweets, can you share the 27 counters among 9 boys". Ls are seen making groups of 9 counters resulting in 3 rows of 9. T asked: "What is the answer?" The Ls shouted: "3".	Grouping situation inferred from the placing of counters into groups through 1 by 1 distribution she then offered a sharing situation. Ambiguity: unacknowledged move from grouping to sharing situation	3
	L1	E12	The T distributed various tasks to different groups to solve the following: "45÷ 9 =", "54÷ 9 =", "63÷ 9 =", "72÷ 9 =", "9÷ 9 =", "18÷ 9 =", "27÷ 9 =", "36÷ 9 =", "18÷ 9 =", "27÷ 9 =", "36÷ 9 =","45÷ 9 =", "81÷ 9 =", "90÷ 9 =", "99÷ 9 =" and "72÷ 9 =" through either counters, array recalled number facts. All the activities were solved under the guidance of the T.		
	L1	E13	T wrote "45÷ 9 =" next to the NL marked in multiples of 9. She instructed a L to circle "45" on the NL as the dividend value in the division calculation. The T made 5 jumps of 9 on the NL and wrote 5 as the answer. T then asked: "How many sets do we have in forty-five?" The Ls shouted: "Five". The T then said: "What if you shared forty-five oranges among nine boys, how many will each one eat?" A L stated: "Five".	Bald calculation - the bald calculation was solved through grouping actions played out by jumping in nines on the NL to produce the answer followed by a sharing situation. Ambiguity: unacknowledged move from grouping to sharing situation. While a connection is implied between the grouping and sharing situation here, how the two differ and how they overlap is not discussed.	3
	L1	E14	Next the T wrote "63÷ 9 =". T instructed a L to circle "63" on the previous NL. The L counted the number of jumps made previously on the NL (which were five) and then made another two jumps on the NL to end at 63. T wrote the answer 7 on the last jump.	Bald calculation with grouping actions by playing out by jumping in multiples on a NL to produce the answer. Coherent	1
	L1	E15	T placed a flashcard on the board: "Mary bought 27 sweets for her 9 brothers. How many sweets did each one get?". She wrote 27 and 9 on the board. The teacher then drew 3 figurines showing the brothers. She pointed to the number "27" and asked Ls what they would do. A L replied: "Divide" and the T wrote in 27 ÷9 =. She then stated: "We going to share the sweets" and wrote the word "share" on the board. T said: "Lets share the	Sharing situation: the answer is used to produce 3 jumps of 9 on a NL. The ambiguity was that the T initially drew three brothers; however, towards the end of the episode the T corrected this by gesturing that there needed to be nine brothers.	3

			sweets amongst the 9 boys (pointing to the 3 figurines)". She asked: "How many sweets does each boy get?" A L replied: "Three". T then mentioned another way to get to the answer was to go to the NL and she made 3 jumps of 9 on the NL from 0 to 27. Then she said: "Which means each one is going to get?" Ls shouted: "Three". T then indicated that 6 more boys needed to be added to the 3 figurines/brothers previously drawn to indicate nine boys. She paused and wrote "3" on top of the first figurine to indicate sweets and wrote the answer "3" for "27÷ 9 =".	Ambiguity: An incorrect signifier for a signified, but with this ambiguity subsequently corrected.	
	L1	E16	On a flashcard the following was seen: "Share 45 oranges among 9 girls". T then wrote "45÷ 9 =". T asked for the answer and a L replied: "Five" and the T wrote "5" as the answer.	Sharing situation produces a number sentence with an immediate answer. Coherent	1
	L1	E17	On another flashcard the following was seen: "Divide 80 among 9 players". A L made use of the short division algorithm and produced an answer, "8 rem 8".	Sharing situation with remainder worked through a short division algorithm. Coherent	1
	L1	E18	Individual work - Solving bald calculations and sharing word problem.		
	L1	E19	Individual work - Mental Maths activity		
24 April 2012	L2	E1	T said that she had 7 sweets and drew seven small circles. She then stated: "But I have 3 friends" and she drew three big circles which she labelled with initials J – John, M- Mary and P – Peter respectively. T wrote "7 ÷ 3 =". T distributed the seven sweets one at a time between the 3 big circles resulting in two sweets in each circle. T added the answer "2". She then pointed to the seventh sweet and called it the remainder and proceeded to write in "r 1".	Sharing situation with remainder worked through a 1 by 1 sharing action. Coherent	1
	L2	E2	T wrote "28÷ 9 = _ r_ " on the board. Pointing at "28" she said she would go to 28 on the NL. She asked: "Is there 28 on the NL?" Ls replied: "No". Pointing to the NL the T asked: "Which number is nearer to 28?". A L replied: "27". T made an asterisk below the number "27" on the NL and asked: "How many sets are there in 27?". Thereafter the T jumped three times on the NL from 0 to 27 and wrote "3" above the last jump and over the first dash in the division calculation. She then asked: "How many steps can I count to get to 28?". A L responded: "1". T stated: "Which means that 1 is the remainder" and wrote "1" over the second dash.	Bald calculation with remainder - with grouping actions played out by jumping in nines on the NL to produce the answer. Limitation: Language of "nearer" number on multiples NL works correctly here, but not universally. Actions in all cases though indicate marking of nearest multiple below the dividend – so limitation at the language level, not in calculation	2
	L2	E3	Next the T wrote "66÷ 9 = _ r_ " and produced the answer 7 r 3 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2	2
	L2	E4	T wrote "87÷ 9 =" on the board and produced the answer 9 r 6 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2	2
25 April 2020	L3	E1	T wrote "37÷ 3 =" on the board. She drew a NL in multiples of 3 until "39". Pointing to "37" in the number sentence and she asked the class: "Where should we end?" L's replied: "36". The T corrected them and stated: "37". The T then asked:" Why must she stop at "36"? A L replied:" 36 is a multiple of 3 and 36 is smaller than 37". The T accepted offer and then noted that "37" lies between "36" and "39" and drew in a line after "36" to indicate "37" and added an asterisks above the "37". She then wrote "37÷ 3 =" on the board again and jumped in 3's, twelve times on the NL until "36" and added "12" as the answer. She then asked whether they should end at "36" or "37" and reinforced that they	Bald calculation with grouping actions by playing out by jumping in multiples on a NL to produce the answer. Coherent	1

			should end at "37". A L then stated: "Remainder one" and the T wrote in "rem 1" next to the answer "12" and pointed to "37" and made a small jumped from "36" to "37" to indicate "rem 1".		
	L3	E2	T wrote "39÷ 3 =" on the board. A L immediately provided the answer: "Thirteen" and the T then wrote in "13" as the answer. To prove the answer a L jumped in 3s on the NL until "39".	Bald calculation with immediate answer offered and answer verified by jumps in multiples on a NL. Coherent	1
	L3	E3	T wrote "28 ÷ 3 =" on the board and produced the answer 9 r 1 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2.	2
	L3	E4	L drew a NL with multiples of 5 on it. T instructed the L to write "25÷ 5 =". T then went to the NL and jumped 5 times on it from 0 to 25. The L then wrote "5" as the answer.	Bald calculation - with grouping actions played out by jumping in fives on the NL. Coherent	1
	L3	E5	T called another L to the board. She instructed the L to write "42÷ 5 =" on the board and produced the answer 8 r 2 through the use of the number line as in L2 E2	Bald calculation with remainder - same reason as L2: E2.	2
	L3	E6	T takes 13 sticks. She took 5 sticks and gave it to one L and stated that it is a set. She then took another 5 sticks gave it to another L. T then counted the sticks left in her hand, which were 3, and said: "I have 2 sets and 3 units". T stated: "13 divided by 2?" and wrote "13÷ 2=" on the board. T then stated: "13 divided by 5" and showed Ls that there were 2 groups of 5 with a remainder of 3.	Grouping situation with remainder with played out through a grouping action. The T then writes a bald calculation which focuses on the number of groups as divisor (sharing). She then returns to a verbal number sentence which matches the grouping situation. Ambiguity: unacknowledged move from grouping to sharing situation	3
	L3	E7	T then distributed different worksheets to different groups and solved: "88÷ 4=", "79÷ 6 =", "39÷ 3 =", "78÷ 6 =","37÷ 3 =", "65÷ 5 =", "38÷ 3 =","79÷ 6 =","4 ÷ 3 =", "6 ÷ 2 =", "8 ÷ 2 =","10÷ 2 =","12÷ 2=", "18÷ 2 =", "9 ÷ 3 =", "14÷ 2 =", "32÷ 3 =", "40÷5 =", "54÷ 5 =" and "21÷ 3 =". T reminded Ls that the NL needed to be used based on the divisor within the division calculation.		
	L3	E8	T stated: "I have 97 sweets and I want to divide the sweets among five children, what is the remainder". T then wrote "97 sweets ÷ 5 children" on the board and L then wrote 97 ÷ 5= and through jumping in 5's on the number line produced the answer 19 r 2.	Sharing situation with remainder. Teacher produced number sentence with matching NL that produces answer through grouping action. Ambiguity: unacknowledged move from a sharing to a grouping interpretation of the bald calculation, although insertion of symbolic division sentence keeps signifiers aligned with previous signifier at all points	3
	L3	E9	Individual work - Mental Maths activity		

Appendix F – Sam's episodes across lessons

Date	Lesson	Episode	Task & its enactment	Division focus in episode and coherence-related comment	Signification Pathway Coherence category
25 July 2012	L1	E1	T placed a flashcard on the board with a short division bracket with 123 written inside the bracket and 4 on the outside with 31 above the bracket with the following words quotient, dividend and divisor written on flashcards. T proceeded to ask Ls to identify the quotient, dividend and divisor. The Ls identified 31 as the quotient, 123 as the dividend and 4 as the divisor and placed the respective flashcards next to the identified number.	Naming parts of a short division algorithm - the answer was incorrect it should be 30 rem 3.	
	L1	E2	The following division tasks were written on the board and Ls needed to identify different parts of the division calculation. For 10÷5=2, Ls identified 5 as the divisor, in 24÷8 =3, Ls identified 8 as the divisor, in 32÷4=8, 32 was identified as the dividend, in 42÷6 =7, Ls identified 6 as the divisor, in 50÷10=5, Ls identified 5 as the quotient and 42÷6 =7, Ls identified 7 as the quotient.	Naming parts of a division number sentence.	
	L1	E3	Stating a focus on long division, T placed a flashcard with the steps: "multiply, divide, subtract, bring down" on the board. She wrote 84÷ 4 in a division bracket. Ls identified 84 as the dividend and 4 as the divisor. T asked: "If I do division what is the opposite?" Ls replied: "Multiplication". T pointed to the 4 and asked: "What times table will I use?" A L answered: "The 4 times table". T placed a flashcard on the board containing the 4 times table from 1x4=4 to 10x4=40 and asked: "What are we going to do first?" and proceeded to say: "4 goes into 8 how many times?" A L replied: "Twice". T wrote the number 2 above the division bracket. She then said: "Now in long division, you say 2 times 4" as she pointed to the 2 above the bracket, the word "multiply" and the 4 outside of the bracket. A L answered: "Eight". T wrote 8 under the 8 in the dividend. Then the T pointed at the flashcard with words "multiply, divide, subtract, bring down" and asked: "Which sign am I going to use?" One L suggested "divide" while another learner said "subtract". T accepted "subtract" and wrote the subtraction symbol to the left of the 8 and drew a line beneath both 8s. She then said: "8 minus 8" and Ls shouted: "Nought" and the T placed a dot beneath the line. She then asked: "What is the next step?" A L suggested that they "bring down" the 4 and the T wrote the number 4 next to the dot. T then established that the divisor 4 needed to be divided into the dividend 4 and said: "4 into 4?" A L replied: "One" and the T wrote 1 above the bracket, making the answer 21. She then said: "What will I say? 4 times 1 (pointing to the 1 and the divisor 4)". A L replied: "Is 4" and the T wrote 4 under the dividend 4. She then asked: "What sign will I use again?" Ls shouted: "Subtraction" and the T wrote the subtraction symbol	Bald calculation is enacted through long division algorithm with learners input. Coherent	1

			and drew a line under both 4s. She then subtracted 4 from 4 and got 0 as an answer.		
	L1	E4	T wrote "65÷5" in a division bracket, and produced the answer 13 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
	L1	E5	T wrote "96÷ 6" in a division bracket, and produced the answer 16 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
26 July 2012	L2	E1	Not analysed - Recapped homework		
	L2	E2	T wrote "99÷ 9" in a division bracket, and produced the answer 11 using the same steps linked with long division as in L1 E3.	Bald calculation – same reason as L1 E3	1
	L2	E3	Not analysed - Individual work the Ls needed to solve the following through the use of long division 96÷4 =, 75÷5 =, 97÷5 = and 88÷5 =		
	L2	E4	T wrote "97÷5" in a division bracket, and produced the answer 19 rem 2 using the same steps linked with long division as in L1 E3.	Bald calculation with remainder- same reason as L1:E3	1
	L2	E5	Not analysed. - Group work not T led		
	L2	E6	Then the T called 2 Ls to the front to solve "96÷4". L 1 set the problem up within a division bracket on the board. He wrote 2 above the bracket and -8 beneath 9 in the dividend. The T reminded the L to draw in a line beneath the -8. L 2 pointed to the 9 and the 8 and L 1 wrote 1 under the line. Then L 2 wrote 6 next to the 1 making the new sub-dividend 16. Next L 2 wrote 1 above the bracket and the T said: "4 goes into 16, 4 times". L 2 erased the 1 and wrote 4 above the bracket instead. Then L 2 wrote - 16 below the dividend 16, drew a line under the 16s and wrote 0 as the difference.	Bald calculation is enacted through long division algorithm with learner input. Coherent	1
27 July 2020	L3	E1	Not analysed - T gives learners instructions to do assessment.		
	L3	E2	Not analysed - The T redoes 84÷4 = using a long division algorithm as in a previous episode.		
	L3	E3	T wrote "77÷5" in a division bracket, and produced the answer 15 rem 2 using the same steps linked with long division as in L1 E3.	Bald calculation with remainder - same reason as L1:E3	1

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MATHEMATICS EDUCATION

DOCTORAL RESEARCH PROJECT

Corin Mathews
8 Mortlake Street
Johannesburg
2092
Tel.: 011 717 3051
073 296 8507
Email: corin.mathews@wits .ac.za

The Principal

Dear Sir/Madam

REQUEST FOR PERMISSION TO USE YOUR SCHOOL AS A RESEARCH SITE

I am a student at the University of the Witwatersrand. I am currently studying for a PhD degree in mathematics education. As part of the requirements of the programme, I am proposing a research study of teachers' selection and design of tasks/tests. I am requesting permission to use your school as a research site.

My research focuses on teachers' selection of tasks/tests related to division within Grade 3 and Grade 4 numeracy/ mathematics teachers in three schools. The specific activities will be: observations of teachers teaching division and interviews related to the process of selecting and designing tasks/tests. In these activities I will be observing and interviewing the teacher of these grades. In the research, I will be focusing on how participation in these activities may result in shifts in their practices with regard to the selection of tasks/tests. The research is longitudinal and extends over 2 years when learners are taught division. The teachers will be working on their own in their particular grade. I propose to meet the teachers after they have taught their lessons to clarify and reflect on their lessons. The research procedures will include video-recorded lesson observations and individual teacher interviews which will be audio-recorded.

I am aware of the ethics in human participation research and I have put in place a number of provisions to safeguard both teachers and learners in this project. For example, I don't intend to disrupt any school activity. Lessons will be observed at the time and in the way they are conducted the normal school schedule. Teacher interviews will be conducted after school on the day of the observation of the lesson. Other ethical considerations include:

- Data collected will be used strictly for the purposes of the research, and will be reported in the thesis to be submitted to Wits and in papers for conference presentations and publications. The name of participating schools/principals/teachers and learners will be anonymised.

- Written consent will be obtained from teachers and learners.
- I will make the final report and summary available to your school after its examination by the University of Witwatersrand.

Please feel free to contact me at any time if you have questions or concerns about the research.

Yours sincerely

___________________ ______________________

Corin Mathews Date

Researcher

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MATHEMATICS EDUCATION DOCTORAL RESEARCH PROJECT

Corin Mathews
8Mortlake Street
Johannesburg
2092
Tel.: 011 717 3051
073 296 8507
Email: corin.mathews@wits .ac.za

The Teacher

Dear Sir/Madam

REQUEST FOR PERMISSION TO USE YOUR CLASS AS A RESEARCH SITE

I am a student at the University of the Witwatersrand. I am currently studying for a PhD degree in mathematics education. As part of the requirements of the programme, I am proposing a research study of teachers' selection and design of tasks/tests. I am requesting permission to use your class as a research site.

My research focuses on teachers' selection of tasks/tests related to division within Grade 3 and Grade 4 numeracy/ mathematics teachers in three schools. The specific activities will be: observations of teachers teaching division and interviews related to the process of selecting and designing tasks/tests. In these activities I will be observing and interviewing the teacher of these grades. In the research, I will be focusing on how participation in these activities may result in shifts in their practices with regard to the selection of tasks/tests. The research is longitudinal and extends over 2 years when learners are taught division. The teachers will be working on their own in their particular grade. I propose to meet the teachers after they have taught their lessons to clarify and reflect on their lessons. The research procedures will include video-recorded lesson observations and individual teacher interviews which will be audio-recorded.

I am aware of the ethics in human participation research and I have put in place a number of provisions to safeguard both teachers and learners in this project. For example, I don't intend to disrupt any school activity. Lessons will be observed at the time and in the way they are conducted the normal school schedule. Teacher interviews will be conducted after school on the day of the observation of the lesson. Other ethical considerations include:

- Data collected will be used strictly for the purposes of the research, and will be reported in the thesis to be submitted to Wits and in papers for conference presentations and publications. The name of participating schools/principals/teachers and learners will be anonymised.
- Written consent will be obtained from teachers and learners.
- I will make the final report and summary available to your school after its examination by the University of Witwatersrand.

Please feel free to contact me at any time if you have questions or concerns about the research.

Yours sincerely

____________________ ______________________

Corin Mathews Date

Researcher

Consent form for teachers' participation in the study – audio-taping.

I, …………………………………………… of………………………………………… (school) have read and understood the procedures involved in the study and what is expected of me as a participant. I willingly give the following consent:

Please put a tick in the appropriate box

I give consent for being interviewed, with audio-taping.

☐ Yes ☐ No

Thank you.

-------------------------   ----------------------

Signature of teacher   ***Date***

Please print your name

Consent form for teachers' participation in the study – video-taping.

I, ………………………………………… of……………………………………… (school) have read and understood the procedures involved in the study and what is expected of me as a participant. I willingly give the following consent:

Please put a tick in the appropriate box

I give consent for my lessons to be observed, with video-taping.

☐ Yes ☐ No

Thank you.

------------------------- ----------------------

Signature of teacher ***Date***

Please print your name

Consent form for teachers' participation in the study – photocopy of lesson plans, notes, tasks and tests.

I, …………………………………………… of………………………………………… (school) have read and understood the procedures involved in the study and what is expected of me as a participant. I willingly give the following consent:

Please put a tick in the appropriate box

I give consent for my lessons plans and/or notes, tasks/tests to be photocopied.

☐ Yes ☐ No

Thank you.

------------------------ ---------------------

Signature of teacher ***Date***

Please print your name

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MATHEMATICS EDUCATION DOCTORAL RESEARCH PROJECT

Corin Mathews
8 Mortlake Street
Johannesburg
2092
Tel.: 011 717 3051, 073 296 8507
Email: corin.mathews@wits .ac.za

Dear Parent/Learner,

INFORMATION AND REQUEST FOR PARTICIPATION CONSENT

I am currently doing some research in Mathematics Education at the University of the Witwatersrand. The study's focus is on the ways in which teachers select tasks/tests related to division in Grade 3 (G3) and Grade 4 (G4). I am interested in looking at how and what influences teachers' choices of these tasks/tests related to division. I will be observing teachers in a number of classrooms over the course of the next two years while the teachers teach division lessons. I believe that through my research, I can make a meaningful contribution to current debates around the purpose and nature of mathematical tasks/tests in primary schools and look at ways in which the selection of tasks and tests has impacted upon mathematical teaching and learners' performance within primary schools.

To this end I would like to observe your child's numeracy G3 and mathematics G4 class whenever division is being taught over the next two years. The focus of my observation will be on the kinds of tasks/tests and classroom interaction that characterise division lessons, and I will observe and take notes within these classes. I would also like to photocopy the class's workbooks and tests relating to division in order to understand learners' interaction with the tasks the teacher selected. Only my supervisor, Prof H. Venkatakrishnan, and I will have access to the field notes and workbooks/tests. The school will be anonymous and all names in the transcript will be pseudonyms. Later this year, I will carry out some interviews with teachers. When reporting my findings, it is my intention to illuminate the process and decisions involved in how teachers select and design tasks/tests related to division. In this regard I undertake to ensure that no untoward references are made about the pupils or the teachers. There are no negative consequences to learners from participating in the study or influenced on their participation from data gathered.

Lessons will continue as normal and as scheduled, with my presence within the classroom. I must stress that participation is voluntary. Your child is under no obligation to participate and there are no consequences should you or he/she choose not to. All participants also have the right to withdraw from the study at any point. I would be very grateful if you would grant me this opportunity. If you would like to grant me permission, please read and complete the attached consent form and return it to school.

If you have any questions or concerns or would like to discuss anything regarding my research, please do not hesitate to contact me on 073 296 8507. Should you wish to, you can also contact my supervisor, Prof H Venkatakrishnan on (011) 717-3742.

Yours sincerely

____

Corin Mathews

CONSENT FORM (Parent/Guardians) for Video-taping

I, ……………………………………..…of ……………………………………………………………… (address) have read and understood the procedures involved in the study and what is required of my child as a participant. I willingly give the following consents:

Please put a tick in the appropriate box

I am willing to allow my child to participate in the research study. This will involve video-taping of division lessons within mathematics with the focus on the teacher.

☐ Yes ☐ No

Thank you

Name of school: ____________________________________-

Name of learner: ______________________

Name of Parent/guardian *(Please print)*

____

Signature of parent/guardian *(Please print)*

CONSENT FORM (Parent/Guardians) for photocopying workbooks and tests

I, ……………………………………..…of ……………………………………………………………. (address) have read and understood the procedures involved in the study and what is required of my child as a participant. I willingly give the following consents:

Please put a tick in the appropriate box

I am willing to allow my child to participate in the research study. This will involve photocopying of workbooks and tests relating to division.

☐ Yes ☐ No

Thank you

Name of school: ____________________________________-

Name of learner: ______________________

Name of Parent/guardian *(Please print)*

____

Signature of parent/guardian *(Please print)*

Consent form for learners' participation in the study - video-taping.

I……………………………………………of………………………………………… (school) have read and understood the procedures involved in the study and what is expected of me as a participant. I willingly give the following consents:

Please put a tick in the appropriate box

I am willing to participate in the study. I understand that this will involve my division lessons being video-taped. I give consent for being video-taped during mathematics lessons.

☐ Yes ☐ No

Thank you.

------------------------ ---------------------

Signature of learner ***Date***

Please print your name and surname

Consent form for learners' participation in the study- photocopying tasks and tests.

I……………………………………………of……………………………………… (school) have read and understood the procedures involved in the study and what is expected of me as a participant. I willingly give the following consents:

Please put a tick in the appropriate box

I am willing to participate in the study. I understand that this will involve my division tasks and tests being photocopied. I give consent for my tasks and tests being photocopied.

☐ Yes ☐ No

Thank you.

------------------------- ----------------------

Signature of learner ***Date***

Please print your name and surname