Lie Group Analysis of Equations Arising in non-Newtonian Fluids

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ABSTRACT

It is known now that the Navier-Stokes equations cannot describe the behaviour of fluids having high molecular weights. Due to the variety of such fluids it is very difficult to suggest a single constitutive equation which can describe the properties of all non-Newtonian fluids. Therefore many models of non-Newtonian fluids have been proposed.

The flow of non-Newtonian fluids offer special challenges to the engineers, modellers, mathematicians, numerical simulists, computer scientists and physicists alike. In general the equations of non-Newtonian fluids are of higher order and much more complicated than the Newtonian fluids. The adherence boundary conditions are insufficient and one requires additional conditions for a unique solution. Also the flow characteristics of non-Newtonian fluids are quite different from those of the Newtonian fluids. Therefore, in practical applications, one cannot replace the behaviour of non-Newtonian fluids with Newtonian fluids and it is necessary to examine the flow behaviour of non-Newtonian fluids in order to obtain a thorough understanding and improve the utilization in various manufactures.

Although the non-Newtonian behaviour of many fluids has been recognized for a long time, the science of rheology is, in many respects, still in its infancy, and new phenomena are constantly being discovered and new theories proposed. Analysis of fluid flow operations is typically performed by examining local conservation relations, conservation of mass, momentum and energy. This analysis gives rise to highly non-linear relationships given in terms of differential equations, which are solved using special non-linear techniques.

Advancements in computational techniques are making easier the derivation of solutions to linear problems. However, it is still difficult to solve non-linear problems analytically. Engineers, chemists, physicists, and mathematicians are actively developing non-linear analytical techniques, and one such method which is known for systematically searching for exact solutions of differential equations is the Lie symmetry approach for differential equations.

Lie theory of differential equations originated in the 1870s and was introduced by the Norwegian mathematician Marius Sophus Lie (1842 - 1899). However it was the Russian scientist Ovsyannikov by his work of 1958 who awakened interest in modern group analysis. Today, the Lie group approach to differential equations is widely applied in various fields of mathematics, mechanics, and theoretical physics and many results published in these area demonstrates that Lie's theory is an efficient tool for solving intricate problems formulated in terms of differential equations.

The conditional symmetry approach or what is called the non-classical symmetry approach is an extension of the Lie approach. It was proposed by Bluman and Cole 1969. Many equations arising in applications have a paucity of Lie symmetries but have conditional symmetries. Thus this method is powerful in obtaining exact solutions of such equations. Numerical methods for the solutions of non-linear differential equations are important and nowadays there several software packages to obtain such solutions. Some of the common ones are included in Maple, Mathematica and Matlab.

This thesis is divided into six chapters and an introduction and conclusion. The first chapter deals with basic concepts of fluids dynamics and an introduction to symmetry approaches to differential equations. In Chapter 2 we investigate the influence of a time-dependent magnetic field on the flow of an incompressible third grade fluid bounded by a rigid plate. Chapter 3 describes the modelling of a fourth grade flow caused by a rigid plate moving in its own plane. The resulting fifth order partial differential equation is reduced using symmetries and conditional symmetries. In Chapter 4 we present a Lie group analysis of the third oder PDE obtained by investigating the unsteady flow of third grade fluid using the modified Darcy's law. Chapter 5 looks at the magnetohydrodynamic (MHD) flow of a Sisko fluid over a moving plate. The flow of a fourth grade fluid in a porous medium is analyzed in Chapter 6. The flow is induced by a moving plate. Several graphs are included in the ensuing discussions. Chapters 2 to 6 have been published or submitted for publication. Details are given in the references at the end of the thesis.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

H. Mambili Mamboundou

DEDICATION

To my Family and my Friends

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Contents

Introduction

The study of non-Newtonian fluids involves the modelling of flow with dense molecular structure such as polymer solutions, slurries, pastes, blood and paints. These materials exhibit both viscous properties like liquids and elastic properties like solids and the understanding of their complex behaviour is crucial in many industrial applications. Due to increasing importance of non-Newtonian fluids in modern technology and industries, the investigation of such fluids is desirable. The flows of non-Newtonian fluids occur in a variety of applications, for example from oil and gas well drilling to well completion operations, from industrial processes involving waste fluids, synthetic fibers, foodstuffs, extrusion of molten plastic and as well as in some flows of polymer solutions. Some important studies dealing with the flows of non-Newtonian fluids are made by Abel-Malek et al. [1], Ariel et al. [3], Chen et al. [6], Fetecau and Fetecau [10], [11], [12], Hayat and Ali [18], Hayat and Kara [19], Hayat et al. [20], [22], [27], [34], Rajagopal and Gupta [46], Rajagopal and Na [48], [49] and Wafo-Soh [58].

Modelling visco-elastic flows is important for understanding and predicting the behaviour of processes and thus for designing optimal flow configurations and for selecting operating conditions. Because of the complex nature of these fluids there is not a single constitutive equation available in the literature which describes the flow properties of all non-Newtonian fluids. For this reason various models have been suggested and among those models, power-law and differential type fluids have acquired a great deal of attention. Some relevant contributions dealing with this type of fluids are given in references [13], [14], [15], [19] , [25], [35], [38], [39], [55] and [56].

Power-law fluids, also referred to as fluids of grade one are the simplest models of non-Newtonian fluids and it is well-known for accurately moodelling the shear stress and shear rate of non-Newtonian fluids, but it does not properly predict the normal stress differences that are observed in phenomena like die-swell and rod-climbing [53] which are manifestation of the stresses that develop orthogonal to the plane of shear which can be well modelled by extending the study to the fluid of grade two. In turn this does not fit shear thining and shear thickening fluids. The third grade fluid model represents a further, although inconclusive, attempt towards a more comprehensive description of the behavior of viscoelastic fluids. Accordingly certain effects may well be described by flow of fourth grade fluids [36].

On the other hand, the governing equations resulting from non-Newtonian fluid models are non-linear high order equations whose analysis present a particular challenge to researchers. Hence progress was limited until recent times and closed-form solutions are available to more problems of particular interest than before. Also the study of such flows in porous media are quite important in many engineering fields such as enhanced oil recovery, paper and textile, but little work seems to be available in the literature. Few recent studies [28], [37], [38], [55], [56] maybe mentioned in this direction. Similarly the study of hydrodynamic flows with application of magnetic field (MHD flows) is of particular interest in chemical engineering, electromagnetic propulsions and the study of the flow of blood and yet again the literature is scarce. Mention maybe made here to the recent study of the topic by Hayat et al.[21], [23], [24], [25] and [33].

Motivated by the above, in Chapter 2 we take a look at the Rayleigh problem for the flow of third grade fluid, when the initial profile is arbitrary. This chapter covers the mathematical modelling and solutions of the applied magnetic field on the viscoelastic flow of order three bounded by a rigid plate. The findings of this chapter has been written up as a paper and accepted in the Journal of Nonlinear Mathematical Physics [29].

In Chapter 3 the unsteady unidirectional MHD flow of an incompressible fourth order fluid bounded by a rigid plate is discussed. The flow is induced due to arbitrary velocity $V(t)$ of an insulated plate. The fluid is electrically conducting in the presence of a uniform magnetic field. The study of the motion of non-Newtonian fluids by the application of a magnetic field has many applications, such as the flow of liquid metals and alloys, flow of plasma, flow of mercury amalgams, flow of blood and lubrication with heavy oils and greases. We have submitted this work to the Journal of Nonlinear Analysis Series B for publication [30].

In the remaining Chapters, 4, 5 and 6, we have concentrated on the study of flows in a porous space. Recently Tan and Masuoka [55] analyzed the Stokes first problem for a second grade fluid in a porous medium. In another paper, Tan and Masuoka [56] studied the Stokes' first problem for an Oldroyd-B fluid in a porous medium. In these investigations, the authors have used the modified Darcy's law. Chapter 4's goal is to determine the analytical solution for an unsteady flow of a third grade fluid over a moving plate. The relevant problem is formulated using modified Darcy's law of third grade fluid. Two types of analytical solutions are presented and discussed, see [31]. Chapter 5 deals with the investigation of a MHD flow of a power law fluid filling the porous half-space. The formulation of the problem is given using modified Darcy's law for a Sisko fluid. The exact analytical and numerical solutions are constructed. The physical interpretation of the obtained results is made through graphs. This work was accepted in the Journal of Porous Media [40]. Chapter 6 concentrates on the exact analytical and numerical solutions for MHD flow of a non-Newtonian fluid filling the porous half-space. The formulation of the problem is given using modified Darcy's law for a Sisko fluid. The fluid is electrically conducting and a uniform magnetic field is applied normal to the flow by neglecting the induced magnetic field. The exact analytical solutions have been constructed using the similarity approach. We have reported this work in [32]. Finally Chapter 7 is concerned with the analysis of obtained results and the conclusion of the thesis.

Chapter 1

Fluid Dynamics and Computational Approaches

This chapter deals with some useful concepts in the field of fluid mechanics and the description of reduction methods of differential equations. We start by defining some notion of fluids. Then we present how the equations governing the motion of unidirectional flow of third and fourth grade fluids are derived and finally we review some techniques of solving nonlinear differential equations.

1.1 Definitions

This section contains some basic definitions necessary for the subsequent chapters. Some of this are taken from [17]. Others are referred to in the text.

1.1.1 Compressible and incompressible flows

An incompressible flow is a flow in which the variation of the mass per unit volume (density) within the flow is considered constant. In general, all liquids are treated as the incompressible fluids. On the contrary, flows which are characterised by a varying density are said to be compressible. Gases are normally used as the compressible fluids. However, all fluids in reality are compressible since any change in temperature or pressure result in changes in density. In many situations, though, the changes in temperature and pressure are so small that the resulting changes in density is negligible. The mathematical equation that describes the incompressibility property of the fluid is given by

$$\frac{D\rho}{Dt} = 0, \tag{1.1}$$

where D/Dt is the substantive derivative defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla, \tag{1.2}$$

in which $\mathbf{V}$ represents the velocity of the flow.

1.1.2 Unsteady and steady flows

A steady flow is one for which the velocity does not depend on time. When the velocity varies with respect to time then the flow is called unsteady.

1.1.3 Laminar and turbulent flows

Laminar flow is one in which each fluid particule has a definite path. In such flow, the paths of fluid particules do not intersect each other. In turbulent flow the paths of fluid particules may intersect each other (see figures below, taken from [8] and [9]).

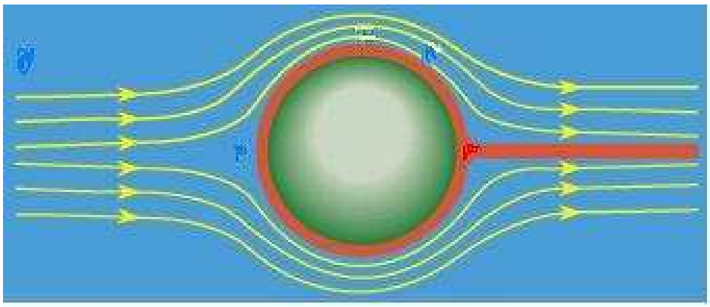

Figure 1.1: Laminar flow

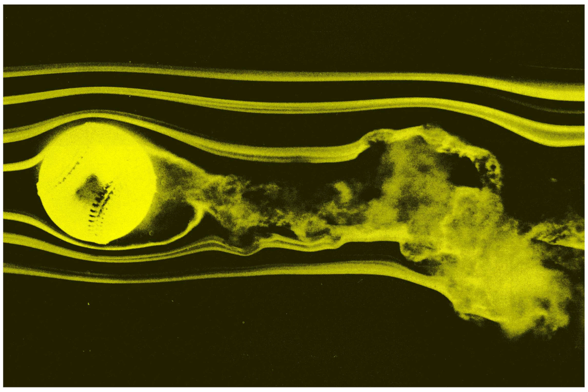

Figure 1.2: Turbulent flow

1.1.4 Newtonian and non-Newtonian fluids

Fluids such as water, air and honey are described as Newtonian fluids. These fluids are essentially modelled by the Navier-Stokes equations, which generally describe a linear relation between the stress and the strain rate. On the other hand, there is a large number of fluids that do not fall in the category of Newtonian fluids. Common examples like mayonnaise, peanut butter, toothpaste, egg whites, liquid soaps, and multi grade engine oils are non-Newtonian. A distinguishing feature of many non-Newtonian fluids is that they exhibit both viscous and elastic properties and the relationship between the stress and the strain rate is non-linear. Contrary to Newtonian fluids, there is not a single model that can describe the behaviour of all the non-Newtonian fluids. For this reason many models have been proposed and among those models, the fluid of differential type and rate type have acquired particular attention. In this thesis we discuss some models of differential type namely the third and fourth grade fluids.

1.2 Basic equations

In this section, we derive the equations that describe the motion of fluids in general. These equations are the basic laws of conservation of mass and momentum. The flow modeling of the subsequent chapters is based upon these two conservation laws.

1.2.1 Equation of continuity

Let Ω be a controlled volume and $\partial\Omega$ its bounded surface, so that the fluid can move in and out of the boundary. The conservation of mass states that the rate of change of matter into the controlled volume is constant. Therefore we have

$$\frac{d}{dt}\int_{\Omega}\rho d\Omega = 0, \tag{1.3}$$

in which ρ is the density of the fluid and d/dt is the ordinary time derivative. Using the Reynold's transport theorem, the left hand side of equation (1.3) gives

$$\frac{d}{dt}\int_{\Omega}\rho d\Omega = \int_{\Omega}\left(\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V})\right)d\Omega, \tag{1.4}$$

where **V** is the velocity at which the fluid is moving. Equations (1.3) and (1.4) yield

$$\int_{\Omega}\left(\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V})\right)d\Omega = 0. \tag{1.5}$$

For an arbitrary controlled volume Ω, we can drop the integral in equation (1.5) to obtain the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0. \tag{1.6}$$

In the case of an incompressible fluid, the density ρ does not vary and the continuity equation (1.6) is equivalent to the following equation

$$\nabla.\mathbf{V} = 0. \tag{1.7}$$

1.2.2 Equation of motion

The motion of a fluid is generally governed by the the equation of continuity derived above and the conservation of linear momentum. The conservation law of momentum states that the rate of change of linear momentum over a control volume Ω bounded by $\partial\Omega$ must equal what is created by external forces acting on the control volume, minus what is lost by the fluid moving out of the boundary. Thus

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{V}d\Omega = \int_{\Omega}\rho\mathbf{f}d\Omega - \int_{\partial\Omega}\mathbf{\Pi}.\mathbf{n}\; d\partial\Omega, \tag{1.8}$$

where $\mathbf{V}$ is the velocity of the fluid, ρ is the density, $\mathbf{f}$ is the body force per unit mass and $\mathbf{\Pi}$ is the linear momentum current density given by

$$\mathbf{\Pi} = (\rho\mathbf{V}) \otimes \mathbf{V} - \mathbf{T}, \tag{1.9}$$

in which $\mathbf{T}$ is the Cauchy stress tensor and $\otimes$ denotes the tensor product. Using the divergence theorem, the last part of the right hand side of equation (1.8) can be written as

$$\int_{\partial\Omega}\mathbf{\Pi}.\mathbf{n}\; d\partial\Omega = \int_{\Omega}(\nabla.\mathbf{\Pi})\, d\Omega. \tag{1.10}$$

Substituting equation (1.10) into equation (1.8) we have

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{V}d\Omega = \int_{\Omega}\rho\mathbf{f}d\Omega - \int_{\Omega}(\nabla.\mathbf{\Pi})\, d\Omega. \tag{1.11}$$

Since the control volume Ω is invariant in time, we can take the d/dt under the integral and equation (1.11) becomes

$$\int_{\Omega}\left(\frac{\partial}{\partial t}(\rho\mathbf{V}) + \nabla.\mathbf{\Pi}\right) d\Omega = \int_{\Omega}\rho\mathbf{f}d\Omega. \tag{1.12}$$

For an arbitrary volume Ω we can drop the integral and we have

$$\frac{\partial}{\partial t}(\rho\mathbf{V}) + \nabla.\mathbf{\Pi} = \rho\mathbf{f}. \tag{1.13}$$

The equation of motion in general is commonly represented by

$$\rho\left[\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\,\mathbf{V}\right] = \nabla.\mathbf{T} + \rho\mathbf{f}. \tag{1.14}$$

In the above equation the force $\mathbf{f}$ is the body force, for example, gravity and electromagnetic forces. In this thesis, flows of non-Newtonian fluids with applied magnetic field are considered.

1.3 Governing equations for unidirectional flow of a fourth grade fluid

In this section we derive the equation which governs the unsteady unidirectional flow of an incompressible fourth grade fluid. We make the derivation of such an equation by employing the following equations of continuity and momentum:

$$\mathrm{div}\mathbf{V} = 0, \tag{1.15}$$

$$\rho\left(\frac{\partial}{\partial t} + (\mathbf{V}.\nabla)\right)\mathbf{V} = -\nabla\mathbf{p} + \mathrm{div}\mathbf{S} + \mathbf{r}, \tag{1.16}$$

where $\mathbf{V}$ is the velocity, ρ is the fluid density, t is the time, p is the pressure, $\mathbf{S}$ is the extra stress tensor and $\mathbf{r}$ is the Darcy's resistance in a fourth grade fluid.

The constitutive equation in an incompressible fourth grade fluid is

$$\begin{aligned}
\mathbf{S} &= \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \mathbf{S}_1 + \mathbf{S}_2, && (1.17)\\
\mathbf{S}_1 &= \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2\right) + \beta_3\left(\mathrm{tr}\mathbf{A}_1^2\right), && (1.18)\\
\mathbf{S}_2 &= \gamma_1\mathbf{A}_4 + \gamma_2\left(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3\right) + \gamma_3\mathbf{A}_2^2 + \gamma_4\left(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2\right) && (1.19)\\
&+ \gamma_5\left(\mathrm{tr}\mathbf{A}_2\right)\mathbf{A}_2 + \gamma_6\left(\mathrm{tr}\mathbf{A}_2\right)\mathbf{A}_1^2 + \left(\gamma_7\mathrm{tr}\mathbf{A}_3 + \gamma_8\mathrm{tr}\left(\mathbf{A}_2\mathbf{A}_1\right)\right)\mathbf{A}_1, && (1.20)
\end{aligned}$$

in which μ is the dynamic viscosity and $\alpha_i\ (i = 1, 2)$, $\beta_j\ (j = 1 - 3)$ and $\gamma_k\ (k = 1 - 8)$ are the material constants. It should be noted that if $\alpha_i = \beta_j = \gamma_k = 0$, we recover the Navier-Stokes model. For $\gamma_k = 0$, we recover the third grade fluid and when $\gamma_k = \beta_j = 0$ then we have a second grade fluid. The Rivlin-Ericksen tensors $\mathbf{A}_i\ (i = 1 - 4)$ are defined through the following relations:

$$\begin{aligned}
\mathbf{A}_1 &= (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T, && (1.21)\\
\mathbf{A}_n &= \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1},\ n \geq 2. && (1.22)
\end{aligned}$$

For unidirectional flow, the velocity is given by

$$\mathbf{V} = (u(y, t), 0, 0), \tag{1.23}$$

where u is the velocity in x-direction. Note that the velocity defined in Eq. (1.23) satisfies the incompressibility condition automatically.

It is a well-known fact that Darcy's law holds for viscous flows having low speed [55], [56]. This law provides a relationship between the pressure drop and the velocity. Moreover, this law is invalid for a porous medium having boundaries. In such a situation, Brinkman law is valid. On the basis of the Oldroyd constitutive equation, the following expression of pressure drop has been suggested [56]:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\nabla p=-\frac{\mu\Phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{1.24}$$

in which Φ is the porosity, k is the permeability of the porous medium, λ is the relaxation time and λ_r is the retardation time. The pressure gradient in the above expression is also interpreted as the measure of the resistance to the flow. Therefore, the x-component of the Darcy's resistance through Eq. (1.24) is given by

$$r_x=-\frac{\mu\Phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)u. \tag{1.25}$$

Using Eq. (1.24) we have through Eqs. (1.21) and (1.22) the following expressions

$$\mathbf{A}_1=\begin{bmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.26}$$

$$\mathbf{A}_2=\begin{bmatrix} 0 & \frac{\partial^2 u}{\partial y\partial t} & 0 \\ \frac{\partial u}{\partial y\partial t} & 2\left(\frac{\partial u}{\partial y}\right)^2 & 0, \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.27}$$

$$\mathbf{A}_3=\begin{bmatrix} 0 & \frac{\partial^3 u}{\partial y^2\partial t} & 0 \\ \frac{\partial^3 u}{\partial y^2\partial t} & 6\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.28}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & \dfrac{\partial^3 u}{\partial y \partial t^2} & 0 \\ \dfrac{\partial^3 u}{\partial y \partial t^2} & 6\dfrac{\partial u}{\partial y}\dfrac{\partial^2 u}{\partial y \partial t} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.29}$$

$$\mathbf{A}_4 = \begin{bmatrix} 0 & \dfrac{\partial^4 u}{\partial y \partial t^3} & 0 \\ \dfrac{\partial^4 u}{\partial y \partial t^3} & 6\dfrac{\partial^2 u}{\partial y \partial t}^2\dfrac{\partial u}{\partial y}\dfrac{\partial^3 u}{\partial y \partial t^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.30}$$

with

$$\nabla \mathbf{V} = \begin{bmatrix} 0 & \dfrac{\partial u}{\partial y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.31}$$

$$(\nabla \mathbf{V})^T = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.32}$$

In view of Eqs. (1.16) and (1.23) one has

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} + r_x, \tag{1.33}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial S_{yy}}{\partial y} + r_y, \tag{1.34}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial S_{zy}}{\partial y} + r_z \tag{1.35}$$

where S_{xy}, S_{yy} and S_{zy} can be computed through relations (1.17)-(1.20) and r_x, r_y and r_z are the components of the Darcy's resistance in the x, y and z-directions.

Employing Eqs. (1.17)-(1.20) we have

$$
\begin{aligned}
S_{xy} &= \mu\frac{\partial u}{\partial y}+\alpha_1\frac{\partial^2 u}{\partial t\partial y}+\beta_1\frac{\partial^3 u}{\partial t^2\partial y}+2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3 \\
&\quad +\gamma_1\frac{\partial^4 u}{\partial t^3\partial y}+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial t\partial y}, \qquad (1.36)\\
S_{yy} &= -p+(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2+2(3\beta_1+\beta_2)\left(\frac{\partial^3 u}{\partial t\partial y}\right) \\
&\quad +6\gamma_1\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial t\partial y}\right)+2(\gamma_1+\gamma_2)\frac{\partial u}{\partial y}\left(\frac{\partial^3 u}{\partial t^2\partial y}\right) \\
&\quad +\gamma_3\left(\frac{\partial^2}{\partial t\partial y}\right)^2+2(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6)\left(\frac{\partial u}{\partial y}\right)^4, \qquad (1.37)\\
S_{zy} &= 0. \qquad (1.38)
\end{aligned}
$$

Utilizing the same idea as in Eq. (1.25) we have the following expression of the Darcy's resistance in the x-direction

$$
\begin{aligned}
r_x = -\Bigg(&\mu+\alpha_1\frac{\partial}{\partial t}+\beta_1\frac{\partial^2}{\partial t^2}+2\left(\beta_2+\beta_3\right)\left(\frac{\partial u}{\partial y}\right)^2+\gamma_1\frac{\partial^3}{\partial t^3} \\
&+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\Bigg)\frac{\Phi u}{k}, \qquad (1.39)
\end{aligned}
$$

and $r_y = r_z = 0$ (since $u = w = 0$). Here u and w are the y and z components of the velocity. Substituting Eqs. (1.36) and (1.39) into Eq. (1.33) and then neglecting the pressure gradient we get

$$
\begin{aligned}
&\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+\beta_1\frac{\partial^4 u}{\partial y^2\partial t^2}+6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} \\
&+\gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right] \\
&-\Bigg[\mu+\alpha_1\frac{\partial}{\partial t}+\beta_1\frac{\partial^2}{\partial t^2}2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2+\gamma_1\frac{\partial^3}{\partial t^3}+(6\gamma_2+2\gamma_3+2\gamma_4 \\
&+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\Bigg]\frac{\phi u}{k}. \qquad (1.40)
\end{aligned}
$$

1.4 Differential Equations: Algebraic and Computational Approaches

In this section we present some features, and useful methods for solving partial differential equations. We start by presenting the Lie classical approach for solving differential equations followed by the nonclassical method and later on we utilise computational methods based on finite differences and other associated methods. Here, we restrict the analysis for our use of symmetry method with respect to one dependent variable. The reader is referred to the books [5], [41] and [42].

1.4.1 Classical symmetry method for partial differential equations

We consider the p^{th} order partial differential equation in n independent variables $x = (x_1, ..., x_n)$ and one dependent variable u,

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0 \ , \tag{1.41}$$

where $u_{(k)}$, $1 \leq k \leq p$, represent the set of all k^{th} order derivatives of u, with respect to the independent variables defined by:

$$u_{(k)} = \{\frac{\partial^k u}{\partial x_{i_1}, ..., x_{i_k}}\}, \tag{1.42}$$

with

$$1 \leq i_1, i_2, ..., i_k \leq n. \tag{1.43}$$

To find the symmetries of equation (1.41), we fist construct the group of invertible transformations depending on a real parameter a, that leaves equation (1.41) invariant, namely

$$\bar{x}_1 = f^1(x, u, a), \ \bar{x}_2 = f^2(x, u, a), ..., \bar{x}_n = f^n(x, u, a), \ \bar{u} = g(x, u, a). \tag{1.44}$$

The transformations (1.44) have the closure property, are associative, admit inverses and identity transformation and are said to form a one-parameter group.

Since a is a small parameter the transformations (1.44) can be written by means of Taylor series expansions as:

$$\bar{x}_1 = x_1 + a\xi_1(x,u) + O(a^2), ..., \bar{x}_n = x_n + a\xi_n(x,u) + O(a^2),$$

$$\bar{u} = u + a\eta(x,u) + O(a^2). \tag{1.45}$$

The transformations (1.45) above are the infinitesimal transformations, and the finite transformations are found by solving the Lie equations

$$\xi_1(\bar{x},\bar{u}) = \frac{d\bar{x}_1}{da}, ..., \ \xi_n(\bar{x},\bar{u}) = \frac{d\bar{x}_n}{da}, \ \eta(\bar{x},\bar{u}) = \frac{d\bar{u}}{da}, \tag{1.46}$$

subject to the initial conditions

$$\bar{x}_1(\bar{x},\bar{u},a)_{|a=0} = x_1, ..., \bar{x}_n(\bar{x}, \ \bar{u},a)_{|a=0} = x_n, \ \bar{u}(\bar{x},\bar{u},a)_{|a=0} = u, \tag{1.47}$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$.

The transformations (1.44) can also be represented by the Lie symmetry generator

$$\chi = \xi^1(x,u)\frac{\partial}{\partial x_1} + ... + \xi^n(x,u)\frac{\partial}{\partial x_n} + \eta(x,u)\frac{\partial}{\partial u}. \tag{1.48}$$

The functions ξ^i and η are the coefficient functions of the operator χ which is also referred to as the *infinitesimal generator* or *operator*, and equation (1.45) can be represented using (1.48) by the following

$$\bar{x}_1 \approx (1 + a\chi)x_1, ..., \bar{x}_n \approx (1 + a\chi)x_n, \ \bar{u} \approx (1 + a\chi)u. \tag{1.49}$$

The operator (1.48) is a symmetry generator of Equation (1.41) if:

$$\chi^{[p]}E_{|E=0} = 0, \tag{1.50}$$

where $\chi^{[p]}$ denotes the p^{th} prolongation of the operator χ and is given by

$$\chi^{[1]} = \chi + \sum_{i=1}^{n} \eta^{x_i}\frac{\partial}{\partial u_{x_i}}, \tag{1.51}$$

$$\chi^{[2]} = \chi^{[1]} + \sum_{i=1}^{n}\sum_{j=1}^{n} \eta^{x_i x_j} \frac{\partial^2}{\partial u_{x_i x_j}}, \tag{1.52}$$

$$\vdots$$

$$\chi^{[p]} = \chi^{[1]} + \ldots + \chi^{[p-1]} + \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \eta^{x_{i_1} \ldots x_{i_p}} \frac{\partial^p}{\partial u_{x_{i_1} \ldots x_{i_p}}}, \tag{1.53}$$

where $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_{i_1} \ldots x_{i_k}} = \frac{\partial^k u}{\partial x_{i_1} \ldots x_{i_k}}$ and the additional coefficient functions satisfy

$$\eta^{x_i} = D_{x_i}(\eta) - \sum_{j=1}^{n} u_{x_j} D_{x_i}(\xi^j), \tag{1.54}$$

$$\eta^{x_i x_j} = D_{x_j}(\eta^{x_i}) - \sum_{k=1}^{n} u_{x_i x_k} D_{x_j}(\xi^k), \tag{1.55}$$

$$\vdots$$

$$\eta^{x_{i_1} \ldots x_{i_p}} = D_{x_{i_p}}(\eta^{x_{i_1} \ldots x_{i_{p-1}}}) - \sum_{j=1}^{n} u_{x_{i_1} \ldots x_{i_{p-1}} x_j} D_{x_{i_p}}(\xi^j), \tag{1.56}$$

where D_{x_i} is the total derivative operator given by

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_{x_i} \frac{\partial}{\partial u} + \sum_{j=1}^{n} u_{x_i x_j} \frac{\partial}{\partial x_j} + \ \ldots \tag{1.57}$$

The determining equation (1.50) gives a polynomial in the derivatives of the dependent variable u, and according to Lie's theory those derivatives are taken independent. After separation of equation (1.50) with respect to the partial derivatives of u and their powers, one gets an overdetermined system of linear homogeneous partial differential equations for the coefficient functions ξ^i's and η. Solving the overdetermined system leads to the following cases:

- There is no symmetry, which means that the Lie point symmetry ξ_i and η are all zero.
- The point symmetry has r arbitrary constants, in which case we obtain a r-dimensional Lie Algebra.
- The point symmetry admits some finite number of arbitrary constants and arbitrary functions; in this case we obtain a infinite-dimensional Lie Algebra.

1.4.2 Example on classical symmetry method

In this section, we illustrate the use of the classical symmetry method on the well-known Korteweg-de Vries equation given by

$$u_t + u_{xxx} + uu_x = 0. \tag{1.58}$$

We look for the operator of the form

$$\chi = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{1.59}$$

Equation (1.59) is a symmetry operator of equation (1.58) if

$$\chi^{[3]}\left(u_t + u_{xxx} + uu_x\right)_{|u_t=-u_{xx}-uu_x} = 0, \tag{1.60}$$

where the third prolongation in this case is:

$$\chi^{[3]} = \eta\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}}. \tag{1.61}$$

Thus the determining equation (1.60) becomes

$$\left(\eta^t + \eta^{xxx} + u_x\eta + u\eta^x\right)_{|u_{xxx}=-u_t-uu_x} = 0, \tag{1.62}$$

and using equations (1.54) to (1.56) we get

$$\begin{aligned} \eta^t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \\ &= \eta_t + u_t\eta_u - u_t(\xi^1_t + u_t\xi^1_u) - u_x(\xi^2_t + u_t\xi^2_u), \end{aligned} \tag{1.63}$$

$$\begin{aligned} \eta^x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) \\ &= \eta_x + u_x\eta_u - u_t(\xi^1_x + u_x\xi^1_u) - u_x(\xi^2_x + u_x\xi^2_u), \end{aligned} \tag{1.64}$$

$$\begin{aligned} \eta^{xx} &= D_x(\eta^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) \\ &= \eta_{xx} + u_x(2\eta_{xu} - \xi^2_{xx}) - u_t\xi^1_{xx} + u_x^2(\eta_{uu} - 2\xi^2_{xu}) \\ &\quad -2u_t u_x\xi^1_{xu} - u_x^3\xi^2_{uu} - u_x^2 u_t\xi^1_{uu} + (\eta_u - 2\xi^2_x)u_{xx} \\ &\quad -2u_{tx}\xi^1_x - 3u_x u_{xx}\xi^2_u - u_t u_{xx}\xi^1_u - 2\xi^1_u u_x u_{tx}, \end{aligned} \tag{1.65}$$

$$
\begin{aligned}
\eta^{xxx} &= D_x\left(\eta^{xx}\right) - u_{xxt}D_x\left(\xi^1\right) - u_{xxx}D_x\left(\xi^2\right) \\
&= \eta_{xxx} + u_x\left(2\eta_{xxu} - \xi^2_{xxx}\right) - u_t\xi^1_{xxx} + u_x^2\left(\eta_{xuu} - \xi_{xxu}\right) - 2u_tu_x\xi^1_{xxu} \\
&\quad -u_x^3\xi^2_{xuu} - u_x^2u_t\xi^1_{xuu} + u_{xx}\left(\eta_{xu} - 2\xi^2_{xx}\right) - 2u_{tx}\xi^1_{xx} - 3u_xu_{xx}\xi^2_{xu} \\
&\quad -u_tu_{xx}\xi^1_{xu} - 2u_xu_{tx}\xi^1_{xu} + u_x\left(\eta_{xxu} + u_x(2\eta_{xuu} - \xi^2_{xxu}) - u_t\xi^1_{xxu}\right. \\
&\quad +u_x^2(\eta_{uuu} - 2\xi^2_{xuu}) - 2u_tu_x\xi^1_{xuu} - u_x^3\xi^2_{uuu} - u_x^2u_t\xi^1_{uuu} \\
&\quad \left.+u_{xx}(\eta_{uu} - 2\xi^2_{xu}) - 2u_{tx}\xi^1_{xu} - 3u_xu_{xx}\xi^2_{uu} - u_tu_{xx}\xi^1_{uu} - u_xu_{tx}\xi^1_{uu}\right) \\
&\quad +u_{tx}\left(-\xi^1_{xx} - 2u_x\xi^1_{xu} - u_x^2\xi^1_{uu} - u_{xx}\xi^1_u\right) + u_{xx}\left(2\eta_{xu} - \xi^2_{xx}\right. \\
&\quad \left.+2u_x(\eta_{uu} - 2\xi^2_{xu}) - 2u_t\xi^1_{xu} - 3u_x^2\xi^2_{uu} - 2u_xu_t\xi^1_{uu} - 3u_{xx}\xi^2_u - 2u_{tx}\xi^1_u\right) \\
&\quad +u_{txx}\left(-3\xi^1_x - 3u_x\xi^1_u\right) + u_{xxx}\left(\eta_u - 3\xi^2_x - 4u_x\xi^2_u - u_t\xi^1_u\right). \qquad (1.66)
\end{aligned}
$$

We substitute the coefficient functions into the determining equation (1.62) and the resulting is a polynomial in the derivatives of u. Since the unknown functions ξ^1, ξ^2 and η are independent of the derivatives of u, we can then separate with respect to the derivatives and their powers. One gets after simplifications the following linear system of partial differential equations

$$
\begin{aligned}
u_{txx}u_x &: \quad \xi^1_u = 0 && (1.67) \\
u_{txx} &: \quad \xi^1_x = 0 && (1.68) \\
u_{xx}^2 &: \quad \xi^2_u = 0 && (1.69) \\
u_{xx}u_x &: \quad \eta_{uu} = 0 && (1.70) \\
u_{xx} &: \quad 3\eta_{xu} - 3\xi^2_{xx} = 0 && (1.71) \\
u_t &: \quad -3\xi^2_x + \xi^1_t = 0 && (1.72) \\
u_x &: \quad \eta - u\xi^2_x + 3\eta_{xxu} - \xi^2_{xxu} + u\xi^1_t - \xi^2_t = 0 && (1.73) \\
1 &: \quad u\eta_x + \eta_{xxx} + \eta_t = 0 && (1.74)
\end{aligned}
$$

We solve the system of equations (1.67) to (1.74) and obtain

$$\xi^1 = c_1 - \frac{3}{2}c_4 t, \tag{1.75}$$

$$\xi^2 = c_2 + c_3 t - \frac{1}{2}c_4 x, \tag{1.76}$$

$$\eta = c_3 + c_4 u, \tag{1.77}$$

where c_1 to c_4 are arbitrary constants of integration. We obtain a 4-dimensional Lie symmetry algebra generated by the operators

$$\chi_1 = \frac{\partial}{\partial t} \tag{1.78}$$

$$\chi_2 = \frac{\partial}{\partial x} \tag{1.79}$$

$$\chi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{1.80}$$

$$\chi_4 = -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \tag{1.81}$$

where each χ_i is obtained by substituting (1.75) to (1.77) into (1.59) and letting c_i to be some non zero constant, while $c_j = 0$ with $j \neq i$. For instance χ_1 is obtained by taking $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$.

1.4.3 Non-classical symmetry method

There exist equations whose exact solutions cannot be obtained by using the classical Lie symmetry method. The non-classical method was first introduced by Bluman and Cole [4]. As compared to the classical symmetries, non-classical symmetries are symmetries of the equation along side an auxiliary condition, also called the invariant surface condition. Considering the p^{th} order equation (1.41) augmented by a side condition as given below

$$\begin{aligned} E(x, u, u_{(1)}, ..., u_{(p)}) &= 0 \\ \xi^1(x,u)u_{x_1} + \xi^2(x,u)u_{x_2} + ... + \xi^n(x,u)u_{x_n} - \eta(x,u) &= 0, \end{aligned} \tag{1.82}$$

one only requires the subset of the solutions of (1.41) to be invariant. Moreover, this provides a more general approach than the classical method. The general method for finding non-classical symmetries consists in computing the p^{th} prolongation of (1.41)

$$\chi^{[p]}\left(E(x, u, u_{(1)}, ..., u_{(p)})\right) = 0, \tag{1.83}$$

and then reduce the resulting prolongation using the invariant surface condition. In general, many difficulties arise and the algorithm proposed by Peter Clarkson and Elizabeth Mansfield [7], avoid some of the difficulties encountered.

The algorithm of calculating non-classical symmetry is not as straightforward as for the classical method. The computations are not easy to perform by hand and the resulting determining equation is an overdetermined system of non-linear equations.
There are usually n cases to be considered

$$\begin{aligned} \xi^1 u_{x_1} + \xi^2 u_{x_2} + \cdots + \xi^{n-1} u_{x_{n-1}} + u_{x_n} &= \eta, && (1.84)\\ \xi^1 u_{x_1} + \xi^2 u_{x_2} + \cdots + u_{x_{n-1}} &= \eta, && (1.85)\\ &\vdots && (1.86)\\ u_{x_1} &= \eta. && (1.87) \end{aligned}$$

1.4.4 Group invariant solution

In this section, we state the use of symmetry for finding special exact solutions to differential equations. Given a symmetry operator of the form

$$\chi = \xi^1(x,u)\frac{\partial}{\partial x_1} + ... + \xi^n(x,u)\frac{\partial}{\partial x_n} + \eta(x,u)\frac{\partial}{\partial u}, \tag{1.88}$$

the group invariant solution $u = u(x)$ is obtained from $X(u - u(x))|_{u=u(x)} = 0$ and is deduced by solving the corresponding characteristic system defined as

$$\frac{dx_1}{\xi^1(x,u)} = \frac{dx_2}{\xi^2(x,u)} = ... = \frac{dx_n}{\xi^n(x,u)} = \frac{du}{\eta(x,u)}. \tag{1.89}$$

Chapter 2

The Rayleigh problem for a third grade electrically conducting fluid with variable magnetic field

In this chapter the influence of time-dependent magnetic field on the flow of an incompressible third grade fluid bounded by a rigid plate is investigated. The flow is induced due to motion of a plate in its own plane with an arbitrary velocity. The solution of the equations of mass and momentum is obtained analytically. We also present a numerical solution with particular choices of the magnetic field and boundary conditions. Emphasis has been given to study the effects of various parameters on the flow characteristics. In the present analysis we extend Wafo Soh's [58] analysis for the flow of a third grade fluid, when the initial profile is $g(y)$. Section 2.1 contains the basic equations. Section 2.2 deals with the formulation of the problem. Analytical solutions are presented in sections 2.3 and 2.4. Numerical solutions are presented in section 2.5. Section 2.6 contains the concluding remarks.

2.1 Basic equations

Employing Eqs. (1.7) and (1.14), the fundamental equations governing the MHD flow of an incompressible electrically conducting fluid are

$$\mathrm{div}\,\mathbf{V} = 0, \quad \rho\frac{D\mathbf{V}}{Dt} = \mathrm{div}\,\mathbf{T} + \mathbf{J}\times\mathbf{B}, \quad \mathbf{J} = \sigma\,(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{2.1}$$

where $\mathbf{V}$ is the fluid velocity, ρ is the density of the fluid, σ is the fluid electrical conductivity, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the magnetic induction so that $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$ ($\mathbf{B_0}$ and $\mathbf{b}$ are the applied and induced magnetic fields respectively), and D/Dt is the material time derivative. Employing Eqs. (1.17)-(1.22), the Cauchy stress tensor for a third grade fluid satisfies the following constitutive equation

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2[\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1] + \beta_3(\mathrm{tr}\,\mathbf{A}_1^2)\mathbf{A}_1,$$

$$\mathbf{A}_1 = \mathrm{grad}\,\mathbf{V} + (\mathrm{grad}\,\mathbf{V})^{\mathrm{T}}, \tag{2.2}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\,\mathrm{grad}\,\mathbf{V} + (\mathrm{grad}\,\mathbf{V})^{\mathrm{T}}\mathbf{A}_{\mathrm{n}-1},$$

where the isotropic stress $p\,\mathbf{I}$ is due to the constraint of incompressibility, μ denotes the dynamic viscosity, $\alpha_i\ (i = 1, 2)$ and $\beta_i\ (i = 1 - 3)$ are the material constants, T indicates the matrix transpose and $\mathbf{A}_i\ (i = 1 - 3)$ are the first three Rivlin-Ericksen tensors. Moreover, the Clausius-Duhem inequality and the result that the Helmholtz free energy is minimum in equilibrium provide the following restrictions [16]:

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0, \beta_3 \geq 0, \mid \alpha_1 + \alpha_2 \mid \leq \sqrt{24\mu\beta_3}. \tag{2.3}$$

2.2 Mathematical formulation

Consider the unidirectional flow of a third grade fluid, obeying equations (2.2) and (2.3), maintained above a non-conducting plate by its motion in its own plane with arbitrary velocity $V(t)$. The fluid is magnetohydrodynamic with small magnetic Reynold's number so that the induced magnetic field is negligible. By taking the velocity field $\Big(u(y,t), 0, 0\Big)$, the conservation of mass equation is identically satisfied and in the absence of modified pressure gradient the momentum balance equation (2.1) along with equations (2.2) and (2.3) yields

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial y^2} - \alpha\frac{\partial^3 u}{\partial t\partial y^2} - \epsilon\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + MH^2(t)u = 0, \tag{2.4}$$

where

$$\nu = \frac{\mu}{\rho}, \alpha = \frac{\alpha_1}{\rho}, \epsilon = \frac{6\beta_3}{\rho}, M = \frac{\sigma\bar{\mu}^2}{\rho}, \mathbf{H}_0 = \frac{\mathbf{B}_0}{\bar{\mu}}. \tag{2.5}$$

In equations (2.4) and (2.5) ν is the dynamic viscosity and $\bar{\mu}$ is the magnetic permeability of the fluid. By neglecting the modified pressure gradient the dependence of (2.4) on α_2 has been removed.

The relevant boundary and initial conditions are

$$\begin{aligned} &u(0,t) = V(t), \quad t > 0, \\ &u(y,t) \to 0 \ \text{ as } \ y \to \infty, \quad t > 0, \\ &u(y,0) = g(y), \quad 0 < y < \infty, \end{aligned} \tag{2.6}$$

where $V(t)$ and $g(y)$ are as yet arbitrary functions. These functions are constrained in the next section when we seek exact solutions using the Lie point symmetries method. Also in section 2.5, for the numerical solution, we choose specific functions for $V(t)$ and $g(y)$. Section 2.6 contains the concluding remarks.

2.3 Symmetry analysis

We present a complete Lie point symmetry analysis of the nonlinear partial differential equation (2.4). We find two cases for which equation (2.4) admits a Lie point symmetry algebra. These algebras are used to reduce the initial and boundary value problem (2.4)-(2.6) to solvable form.

An operator (see Chapter 1, section 1.4)

$$\chi = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}, \tag{2.7}$$

where $\tau,\ \xi$ and η are functions of $t,\ y$ and u, is a Lie point symmetry generator of the partial differential equation (2.4) if (see Chapter 1, section 1.4)

$$\chi^{[3]}\left\{u_t - \nu u_{yy} - \alpha u_{tyy} - \epsilon u_y^2 u_{yy} + MH^2(t)u\right\}\Big|_{(2.4)} = 0, \tag{2.8}$$

where

$$\chi^{[3]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^y\frac{\partial}{\partial u_y} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \eta^{tyy}\frac{\partial}{\partial u_{tyy}} \tag{2.9}$$

is the third prolongation of the operator (2.7) in which the additional coefficient functions satisfy

$$\eta^t = D_t(\eta) - u_t D_t(\tau) - u_y D_t(\xi),$$

$$\eta^y = D_y(\eta) - u_t D_y(\tau) - u_y D_y(\xi),$$

$$\eta^{yy} = D_y(\eta^y) - u_{ty} D_y(\tau) - u_{yy} D_y(\xi),$$

$$\eta^{tyy} = D_t(\eta^{yy}) - u_{tyy} D_t(\tau) - u_{yyy} D_t(\xi)$$

with total derivative operators given by

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{ty}\frac{\partial}{\partial u_t} + \cdots$$
$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + \cdots .$$

The determining equation (2.8), after separation with respect to the partial derivatives of u and their products and their powers, gives rise to an overdetermined system of linear homogeneous partial differential equations for the coefficient functions τ, ξ and η. Solution of this system gives $\xi = a_1$, where a_1 is a constant, and that H^2 is constrained by the ordinary differential equation

$$\frac{dH^2}{dt} + \frac{2\beta a_3 e^{2\beta t}}{a_2 + a_3 e^{2\beta t}} H^2 = \frac{2\beta^2 a_3 e^{2\beta t}}{M(a_2 + a_3 e^{2\beta t})}\ , \tag{2.10}$$

where a_2, a_3 are further constants and $\beta = \nu/\alpha$.

The solution for H^2 gives two sets of Lie point symmetries depending on the values of the constants in the ordinary differential equation (2.10).

2.3.1 Case 1: $a_3 = 0, a_2 \neq 0$

Equation (2.10) yields $H^2 = C_1$, where C_1 is a constant and the solutions for the coefficient functions are

$$\xi = a_1, \quad \tau = a_2, \quad \eta = a_4 e^{-MH^2 t},$$

where a_4 is a constant.

2.3.2 Case 2: $a_2 = 0, a_3 \neq 0$

When $a_2 = 0, a_3 \neq 0$, $H^2 = \beta/M + C_2 e^{-2\beta t}$ where C_2 is a constant and the Lie point symmetry coefficients are

$$\xi = a_1, \quad \tau = a_3 e^{2\beta t}, \quad \eta = -a_3 \beta u e^{2\beta t} + a_5 L(t),$$

where a_5 is a constant with

$$L(t) = \exp\left(-M \int_0^t H^2(s) ds\right).$$

Thus we obtain two sets of three-dimensional Lie algebras, generated in each case by

Case 1:

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = L(t)\frac{\partial}{\partial u}, \tag{2.11}$$

Case 2:

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = e^{2\beta t}\frac{\partial}{\partial t} - \beta u e^{2\beta t}\frac{\partial}{\partial u}, \quad X_3 = L'(t)\frac{\partial}{\partial u}. \tag{2.12}$$

2.4 Physical invariant solutions

Given the generator (2.7) the invariant solutions corresponding to χ are obtained by solving the characteristic system (cf. Chapter 1, section 1.4)

$$\frac{dy}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}.$$

We use only the operators which give meaningful physical solutions of the problem consisting of Eqs. (2.4) and (2.6). This means that only X_2 in each case is considered.

2.4.1 Invariant solution corresponding to case 1

The form of the invariant solution in this case corresponding to X_2 is the steady state

$$u(y,t) = F(y). \tag{2.13}$$

The substitution of (2.13) into the partial differential equation (2.4) results in the following reduced second-order ordinary differential equation

$$\left(\nu + \epsilon\left(F'(y)\right)^2\right)F''(y) - C_1MF(y) = 0. \tag{2.14}$$

The boundary conditions imply

$$F(0) = u_0, \qquad F(\infty) = 0, \tag{2.15}$$

and

$$V(t) = u_0, \tag{2.16}$$

where u_0 is a constant. Note that no steady state solution exists when the magnetic field is zero as equation (2.14) reduces to $F'' = 0$. Applying the boundary condition (2.15) on this equation results in the trivial solution $F = 0$.

The double integration of (2.14) leads to

$$\int \frac{\pm\sqrt{\epsilon}dF}{\left[-\nu \pm \left[\nu^2 + 2\epsilon(C_1MF^2 + \lambda_2)\right]^{1/2}\right]^{1/2}} = y + \lambda_1, \tag{2.17}$$

where λ_1 and λ_2 are constants that need to be fixed by (2.15). We observe that we cannot impose the boundary condition $u(y,0) = g(y)$ for arbitrary $g(y)$. The expression of the vorticity is

$$\begin{aligned} w_z &= -\frac{dF}{dy} \\ &= \pm\sqrt{\frac{-\nu \pm \sqrt{\nu^2 + 2\epsilon(C_1 M F^2 + \lambda_2)}}{\epsilon}}. \end{aligned} \tag{2.18}$$

We focus our attention on the solution (2.17). For a real solution we require the positive sign in the integrand. From (2.15) the boundary condition $F(0) = u_0$ implies

$$\pm\sqrt{\epsilon}\int^{u_0} \frac{dF}{\left[(\nu^2 + 2\epsilon C_1 M F^2 + 2\epsilon\lambda_2)^{1/2} - \nu\right]^{1/2}} = \lambda_1. \tag{2.19}$$

Thus equation (2.17) becomes

$$\pm\sqrt{\epsilon}\int_{u_0}^{F} \frac{dF}{\left[(\nu^2 + 2\epsilon C_1 M F^2 + 2\epsilon\lambda_2)^{1/2} - \nu\right]^{1/2}} = y. \tag{2.20}$$

We now investigate when the boundary condition $F(\infty) = 0$ of (2.14) is satisfied for the solution (2.20). Clearly the integral of (2.20) must be divergent as $F \to 0$ so as to ensure that $y \to \infty$. It is seen that if $\lambda_2 \neq 0$, then the integrand behaves like

$$\left[\sqrt{\nu^2 + 2\epsilon\lambda_2} - \nu\right]^{-1/2} \text{ as } F \to 0 \tag{2.21}$$

which means that the integral in (2.20) is convergent as $F \to 0$. For $\lambda_2 = 0$, the integral in (2.20) is divergent since the left hand side of (2.20) is

$$\begin{aligned} & \pm\left(\frac{\epsilon}{\nu}\right)^{1/2}\int_{u_0}^{F} \frac{dF}{\left[(1 + 2\epsilon\nu^{-2} C_1 M F^2)^{1/2} - 1\right]^{1/2}} \\ = \;& \pm\left(\frac{\nu}{C_1 M}\right)^{1/2}\int_{u_0}^{F}\left(\frac{1}{F} + O(F)\right) dF \quad \text{as} \quad F \to 0 \qquad (2.22) \\ = \;& \pm\left(\frac{\nu}{C_1 M}\right)^{1/2}\ln\left(\frac{F}{u_0}\right) + O(F^2) \quad \text{as} \quad F \to 0. \qquad (2.23) \end{aligned}$$

This tends to infinity if the negative sign is taken. Therefore the solution (2.20) subject to (2.15) is

$$\left(\frac{\epsilon}{\nu}\right)^{1/2}\int_{F}^{u_0} \frac{dF}{\left[(1 + 2\epsilon\nu^{-2} C_1 M F^2)^{1/2} - 1\right]^{1/2}} = y. \tag{2.24}$$

The vorticity (2.18) upon using (2.24) is

$$w_z = \left(\frac{\nu}{\epsilon}\right)^{1/2}\left[\left(1+2\epsilon\nu^{-2}C_1MF^2\right)^{1/2}-1\right]^{1/2} \tag{2.25}$$

We present the numerical solution of (2.14) for various values of parameters in Figures 2.1 to 2.4.

We have plotted Figures 2.1 to 2.4 to observe the behaviour of the flow when varying different emerging parameters. Figure 2.1 just leads to the numerical solution of the problem consisting of Eqs. (2.14) and (2.15). This indicates that u decreases as y increases. It can be noted from Figure 2.2 that the velocity increases when the kinematic viscosity ν increases. Figure 2.3 depicts that the velocity is an increasing function of the parameter ϵ. In Figure 2.4 it is observed that the velocity decreases as the Hartmann number M increases.

2.4.2 Invariant solution corresponding to case 2

The invariant solution for this case for X_2 is given by

$$u(y,t) = F(y)e^{-\beta t}. \tag{2.26}$$

The insertion of (2.26) into (2.4) gives the reduced equation

$$\gamma\bigg(F'(y)\bigg)^2F''(y) - F(y) = 0, \tag{2.27}$$

where $\gamma = \epsilon/(C_2M)$. Through Eqs. (2.6) and (2.26) and selecting $V(t) = C_2e^{-\beta t}$, the corresponding boundary conditions are

$$F(0) = C_2, \qquad F(\infty) = 0. \tag{2.28}$$

To integrate the boundary value problem (2.27) and (2.28), we let

$$F'(y) = K(F) \tag{2.29}$$

and substitute (2.29) into (2.27) to obtain

$$F - \gamma K^3\frac{dK}{dF} = 0. \tag{2.30}$$

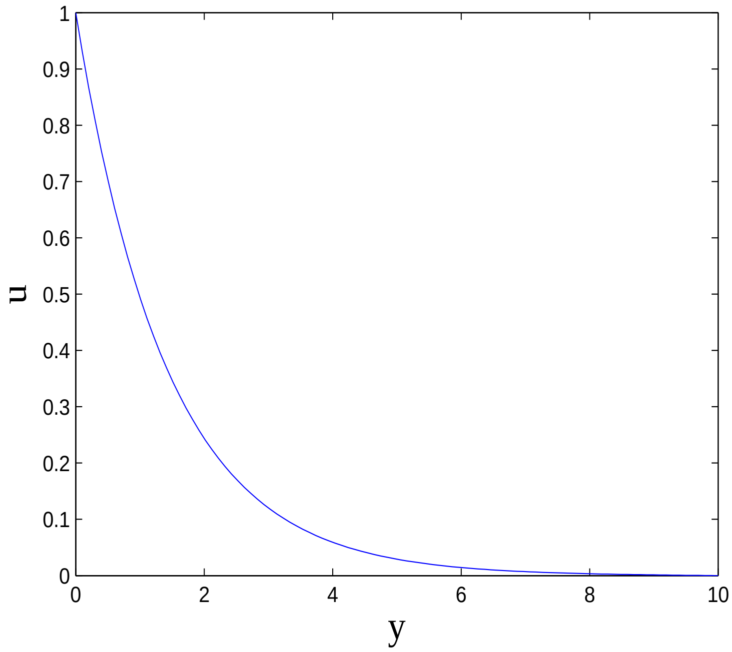

Figure 2.1: Numerical solution of equations (2.14) and (2.15) when $M = 1$, $C_1 = 1$, $\epsilon = 0.1$, $\nu = 0.5$

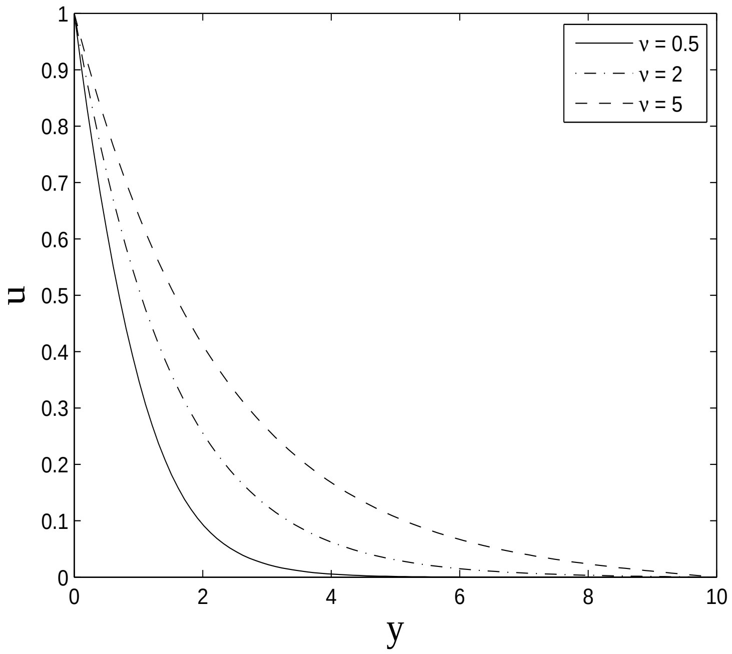

Figure 2.2: Numerical solution of equations (2.14) and (2.15) varying the kinematic viscosity ν when $M = 1$, $C_1 = 1$ and $\epsilon = 2$

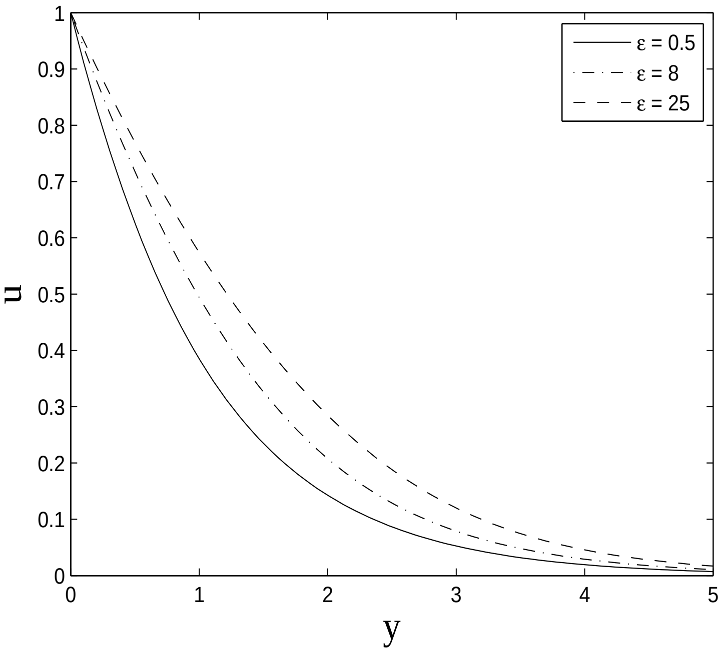

Figure 2.3: Numerical solution of equations (2.14) and (2.15) varying the parameter ϵ when $M = 1$, $C_1 = 1$ and $\nu = 1$

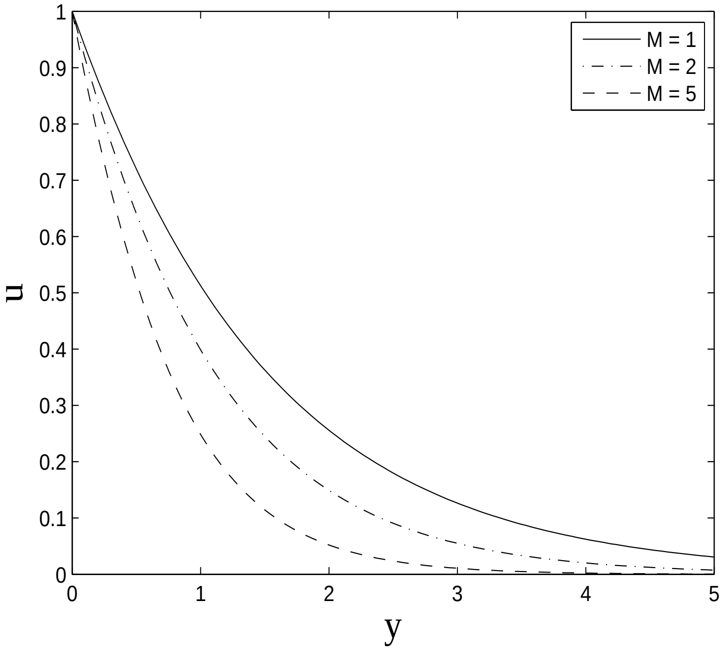

Figure 2.4: Numerical solution of equations (2.14) and (2.15) varying the Hartmann number M with $\nu = 2$, $C_1 = 1$ and $\epsilon = 2$

The integration of (2.30) gives

$$K(F) = \left(\frac{2}{\gamma}F(y)^2 + B_1\right)^{1/4}, \tag{2.31}$$

where B_1 is a constant. Equation (2.29) together with (2.31) yields

$$\frac{dF}{dy} = \left(\frac{2}{\gamma}F(y)^2 + B_1\right)^{1/4}. \tag{2.32}$$

Thus

$$\int \left(\frac{2}{\gamma}F^2 + B_1\right)^{-1/4} dF = y + B_2, \tag{2.33}$$

where B_2 is another constant.

Now we impose the boundary (2.28) on the solution (2.33). The boundary condition $F(0) = C_2$ of (2.28) imposed on the solution (2.33) yields

$$\int^{C_2} \frac{dF}{\left(B_1 + \frac{2}{\gamma}F^2\right)^{1/4}} = B_2. \tag{2.34}$$

Therefore using the condition (2.34), solution (2.33) becomes

$$\int_{C_2}^{F} \frac{dF}{\left(B_1 + \frac{2}{\gamma}F^2\right)^{1/4}} = y. \tag{2.35}$$

Now we invoke $F(\infty) = 0$ of (2.28) on the solution (2.35). We first consider the case $B_1 \neq 0$. The integral (2.35) needs to be divergent as $F \to 0$ as $y \to \infty$. For small F, the left hand side of (2.35) is convergent. Hence the boundary condition $F(\infty) = 0$ is not satisfied. For the case $B_1 = 0$, the solution (2.35) gives

$$\left(\frac{\gamma}{2}\right)^{1/4}\left[2F^{1/2} - C_2^{1/2}\right] = y \tag{2.36}$$

which means that the integral is convergent. Thus the solution (2.35) does not satisfy the boundary condition $F(\infty) = 0$. Therefore for this case we do not obtain a time-dependant solution using symmetry.

2.5 Numerical solution of the PDE (2.4)

In this section we present a numerical solution of the partial differential equation (2.4) subject to (2.6) at various times for different values of the magnetic field and for some emerging parameters.

In Figures 2.5 to 2.7, we have plotted numerically the velocity profile for different values of time, when the magnetic field is first taken to be zero (Figure 2.5), then constant (Figure 2.6) and finally dependent on time (Figure 2.7). In the case of $H = 0$, the velocity decreases and then increases when the value of t is increased. The velocities in Figures 2.5 to 2.7 are similar in a qualitative sense. However the shape of velocity differs slightly depending on the form of the magnetic field considered. It can be said that the shape of the velocity is more parabolic in the case when the strength of the magnetic field is non zero. Finally the variation of ϵ on the velocity is seen in Figure 2.8. It shows that velocity first decreases slightly and then increases when ϵ is increased.

2.6 Concluding remarks

In this chapter we have considered the Rayleigh problem of an incompressible non-Newtonian fluid of third grade with variable magnetic field. After having formulated the problem, we obtained a non-linear third-order partial differential equation. The Lie symmetry approach was employed for the reduction of this equation. We obtained a first-order differential equation that the magnetic field H had to satisfy in order for the existence of point symmetries. Two cases arose and one symmetry was invoked for each case. Invoking the relevant symmetry, we were able to reduce the partial differential equation to a second-order non-linear ordinary differential equation for each of the cases. In the first case, we presented the steady state for various values of the emerging parameters and in the second case, the analytical solution that was obtained did not satisfy the boundary conditions. Thus we have shown that static solutions only exist for $H \neq 0$ by using symmetry methods. Finally, we presented numerical

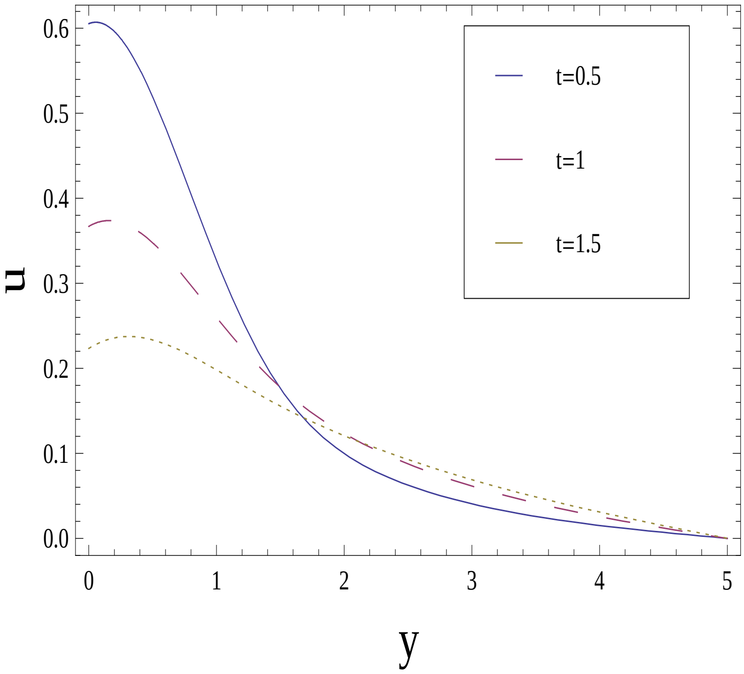

Figure 2.5: Numerical solution of (2.4) when there is no magnetic field ($H = 0$), with $\epsilon = 1.1$, $\nu = 1.85$, $\alpha = 1$, $M = 1$, $V(t) = e^{-t}$ and $g(y) = e^{-y^2}$

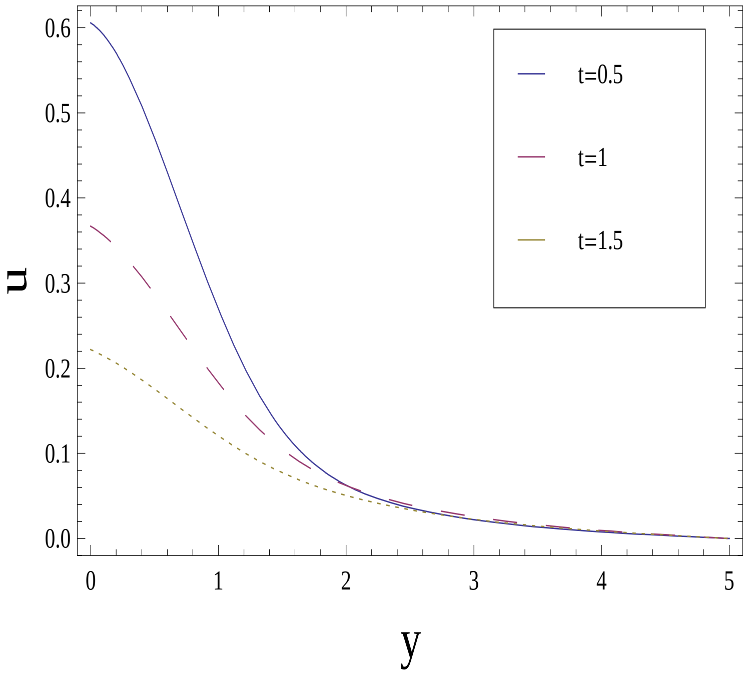

Figure 2.6: Numerical solution of (2.4) when the magnetic field is taken constant ($H = 1$), with $\epsilon = 1.1$, $\nu = 1.85$, $\alpha = 1$, $M = 1$, $V(t) = e^{-t}$ and $g(y) = e^{-y^2}$

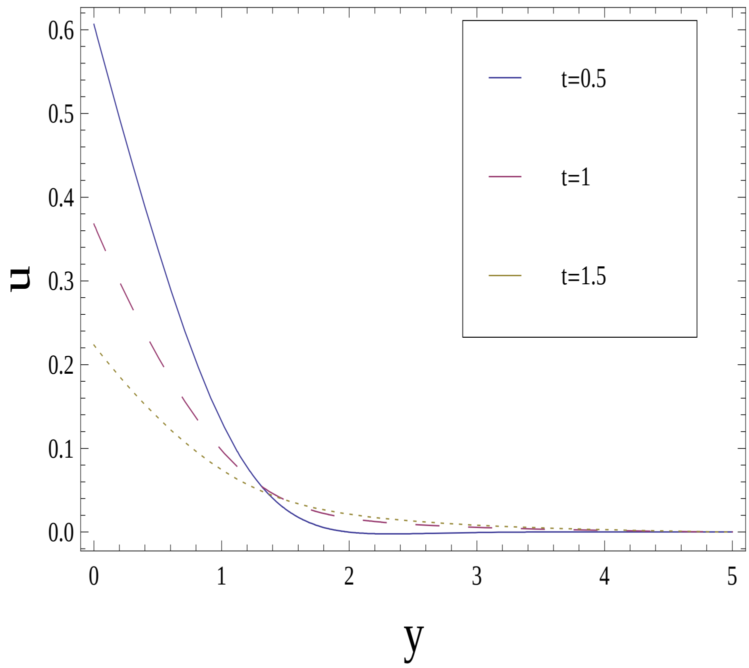

Figure 2.7: Numerical solution of (2.4) for a time dependent magnetic field ($H = 1 + 2e^{-2t}$), with $\epsilon = 1.1$, $\nu = 1.85$, $\alpha = 1$, $M = 1$, $V(t) = e^{-t}$ and $g(y) = e^{-y^2}$

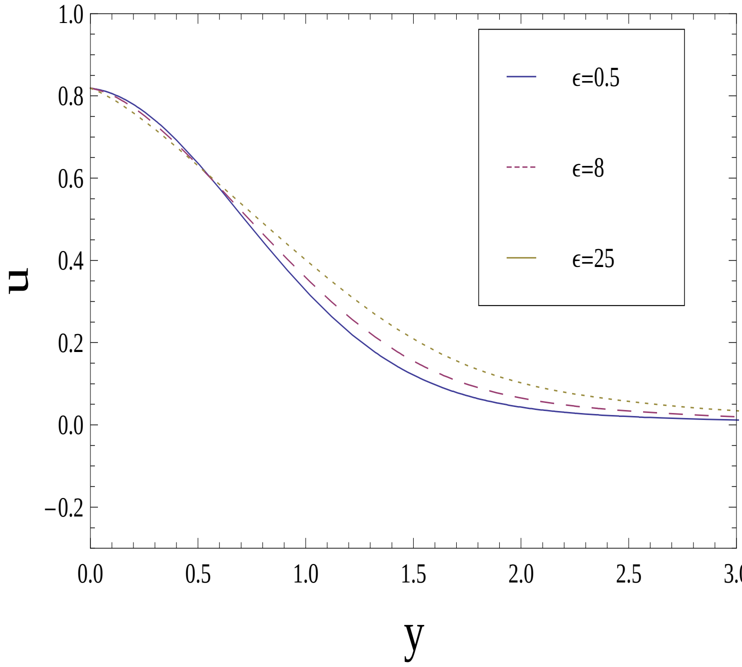

Figure 2.8: Numerical solution of (2.4) varying ϵ, with $\nu = 1.85$, $H = 1 + 2e^{-2t}$, $\alpha = 1$, $M = 1$, $t = 0.2$, $V(t) = e^{-t}$ and $g(y) = e^{-y^2}$

solutions of the PDE with a choice of variable magnetic field as well as suitable boundary conditions. These solutions were presented graphically in Figures 2.5 to 2.8. We briefly comment on the characteristic diffusion distance. The velocity diffuses a characteristic distance $(\nu t)^{1/2}$ in time t. This is clearly illustrated in figures 2.5 to 2.8. This may explain why u decays significantly after $t \approx 1$.

Chapter 3

Effect of magnetic field on the flow of a fourth order fluid

This chapter deals with an unsteady flow engendered in a semi-infinite expanse of an incompressible fluid by an infinite rigid plate moving with an arbitrary velocity in its own plane. The fluid is considered to be fourth order and electrically conducting. The magnetic field is applied in the transverse direction to the flow. The study of the motion of non-Newtonian fluids with magnetic field has importance in the flow of liquid metals and alloys, flow of plasma, flow of mercury amalgams, flow of blood and lubrication with heavy oils and greases. Both analytical and numerical solutions are developed. Lie symmetry analysis is performed for an analytical solution. Limiting cases of interest are deduced as the special cases of the presented analysis by choosing suitable parameters values. This chapter is organized in the following fashion. Section 3.1 deals with the position of the problem. In sections 3.2 and 3.3 the analytical and numerical solutions of the problem are given. Section 3.4 deals with the concluding remarks.

3.1 Mathematical formulation

Let us consider an infinite plate located at $y = 0$ and incompressible fourth grade fluid which is in contact with the plate and occupies the region $y > 0$. The fluid is electrically conducting and plate is insulating. The magnetic field's strength is taken as constant. The plate is moving in its own plane with time-dependent velocity for $t > 0$. Initially, both the fluid and the plate are at rest. The unsteady motion of the conducting fluid in the Cartesian coordinate system is governed by the conservation laws of momentum and mass which are presented in Eq. (2.1) i.e.

$$\rho \frac{D\mathbf{V}}{Dt} = \mathrm{div}\mathbf{T} + \mathbf{J} \times \mathbf{B}, \tag{3.1}$$

$$\mathrm{div}\mathbf{V} = 0, \tag{3.2}$$

where all the quantities appearing in the above equations have been already defined in the previous chapter. Through Eqs. (1.7)-(1.20), the Cauchy stress $\mathbf{T}$ for a fourth order fluid [36] is related to the fluid motion in the form

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S}, \tag{3.3}$$

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \tag{3.4}$$

$$\mathbf{S}_1 = \mu \mathbf{A}_1, \tag{3.5}$$

$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{3.6}$$

$$\mathbf{S}_3 = \beta_1 \mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{3.7}$$

$$\begin{aligned} \mathbf{S}_4 = {} & \gamma_1 \mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\ & + \gamma_5(\mathrm{tr}\,\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\mathrm{tr}\,\mathbf{A}_2)\mathbf{A}_1^2 + [\gamma_7\,\mathrm{tr}\,\mathbf{A}_3 + \gamma_8\, tr\,(\mathbf{A}_2\mathbf{A}_1)]\mathbf{A}_1. \end{aligned} \tag{3.8}$$

Here μ is the dynamic viscosity, α_i $(i = 1, 2)$, β_i $(i = 1, 2, 3)$ and γ_i $(i = 1, 2, \cdots, 8)$ are material constants. The kinematical tensors $(\mathbf{A}_n)$ are defined via Eqs. (1.21) and (1.22) in the following form

$$\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T, \tag{3.9}$$

$$\mathbf{A}_n = \left(\frac{\partial}{\partial t} + \mathbf{V}.\nabla\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T\mathbf{A}_{n-1}, \quad n > 1, \tag{3.10}$$

in which grad is the gradient operator. In our analysis the fluid is electrically conducting and there is an imposed magnetic field in the $y-$direction normal to the plate. In the low-magnetic-Reynolds-number consideration (see, e.g., Shercliff [52]), in which the induced magnetic field can be ignored, the magnetic body force $\mathbf{J}\times\mathbf{B}$ becomes $\sigma(\mathbf{V}\times\mathbf{B}_0)\times\mathbf{B}_0$ when imposed and induced electric fields are negligible and only the magnetic field $\mathbf{B}_0$ contributes to the current $\mathbf{J}=\sigma(\mathbf{V}\times\mathbf{B}_0)$. Here σ is the electrical conductivity of the fluid. The Lorentz force on the right hand side of Eq. (3.1) becomes

$$\mathbf{J}\times\mathbf{B}=-\sigma\mu_1^2H^2\mathbf{V}, \tag{3.11}$$

in which $\mathbf{B}_0=\mu_1 H$ and μ_1 is the magnetic permeability of the medium. For unidirectional flow, we seek solutions to the equation of motion in the form

$$\mathbf{V}=[u(y,t),0,0]. \tag{3.12}$$

It is clear that with the above equation, the continuity equation (3.2) is satisfied identically and Eq. (3.3) after using Eqs. (3.4) to (3.10) takes the component form (cf. Chapter 1)

$$T_{xx}=-p+\alpha_2\left(\frac{\partial u}{\partial y}\right)^2+2\beta_2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+2\gamma_2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2}+\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2+2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4, \tag{3.13}$$

$$\begin{aligned}T_{yy}=&-p+(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2+2(3\beta_1+\beta_2)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+6\gamma_1\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\right)\\&+2(\gamma_1+\gamma_2)\frac{\partial^3 u}{\partial y\partial t^2}\frac{\partial u}{\partial y}+\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2+2(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6)\left(\frac{\partial u}{\partial y}\right)^4,\end{aligned} \tag{3.14}$$

$$T_{zz}=-p, \tag{3.15}$$

$$\begin{aligned}T_{xy}=&\mu\frac{\partial u}{\partial y}+\alpha_1\frac{\partial^2 u}{\partial y\partial t}+\beta_1\frac{\partial^3 u}{\partial y\partial t^2}+2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3+\gamma_1\frac{\partial^4 u}{\partial y\partial t^3}\\&+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t},\end{aligned} \tag{3.16}$$

$$T_{xz}=0,\ \ T_{zy}=0, \tag{3.17}$$

while the equation of motion (3.1) takes the component form

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial\hat{p}}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}-\sigma\mu_1^2H^2u+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+\beta_1\frac{\partial^4 u}{\partial y^2\partial t^2}+6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$

$$+\gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right], \tag{3.18}$$

$$0=-\frac{\partial \hat{p}}{\partial y}=-\frac{\partial \hat{p}}{\partial z}, \tag{3.19}$$

in which the modified pressure $\hat{p}$ is

$$\hat{p}=p-(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2-2(3\beta_1+\beta_2)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}-6\gamma_1\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\right)$$

$$-2(\gamma_1+\gamma_2)\frac{\partial^3 u}{\partial y\partial t^2}\frac{\partial u}{\partial y}-\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2-2(2\gamma_3+2\gamma_4+2\gamma_5+\gamma_6)\left(\frac{\partial u}{\partial y}\right)^4 \tag{3.20}$$

and Eq. (3.19) implies that $\hat{p}$ is not a function of y and z. Hence $\hat{p}$ is at most function of x and t. In the absence of modified pressure gradient, Eq. (3.18) becomes

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial y^2\partial t}+\frac{\beta_1}{\rho}\frac{\partial^4 u}{\partial y^2\partial t^2}+\frac{6(\beta_2+\beta_3)}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\frac{\gamma_1}{\rho}\frac{\partial^5 u}{\partial y^2\partial t^3}$$

$$+\frac{6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)}{\rho}\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]-MH^2u, \tag{3.21}$$

where ν is the kinematic viscosity and

$$M=\frac{\sigma\mu_1^2}{\rho}. \tag{3.22}$$

The appropriate boundary conditions are

$$u(0,t)=u_0V(t),\ \ t>0, \tag{3.23}$$

$$u(\infty,t)=0,\ \ t>0, \tag{3.24}$$

$$u(y,0)=g(y),\ \ y>0, \tag{3.25}$$

where u_0 is the reference velocity.

It should be pointed out that the mathematical problem consisting of Eqs. (3.21)-(3.25) reduces to that of the second grade and third grade fluids when $\alpha_i\ (i=1-2),\ \beta_i\ (i=1-3)$ and $\gamma_i\ (i=1-8)$ are all zero respectively.

3.2 Solutions of the problem

We rewrite Eq. (3.21) as

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + \beta_1\frac{\partial^4 u}{\partial y^2\partial t^2} + \beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}$$
$$+\gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right] - \rho M H^2 u, \tag{3.26}$$

where

$$\beta = 6(\beta_2+\beta_3) \quad \text{and} \quad \gamma = 6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8.$$

Note that Eq. (3.26) has two translation symmetry generators, viz.,

$$X_1 = \frac{\partial}{\partial y} \quad \text{and} \quad X_2 = \frac{\partial}{\partial t}.$$

We now look for invariant solutions under the operator $X_1 + cX_2$, which represents wave-front type travelling wave solutions with constant wave speed c. Symmetry invariant solutions deal with applications of Lie groups to differential equations. For more details on this subject we refer the interested reader to Chapter 1 and the books [5], [41] and [42]. Travelling wave solutions are the simplest kind of invariant solutions. These are solutions of the form

$$u(y,t) = U(x_1) \quad \text{where} \quad x_1 = y - ct \tag{3.27}$$

and can be obtained easily without recourse to Lie group methods. Substituting Eq. (3.27) into Eq. (3.26) yields a fifth-order ordinary differential equation for $U(x_1)$:

$$-c\rho\frac{dU}{dx_1} = \mu\frac{d^2U}{dx_1^2} - c\alpha_1\frac{d^3U}{dx_1^3} + c^2\beta_1\frac{d^4U}{dx_1^4} + \beta\left(\frac{dU}{dx_1}\right)^2\frac{d^2U}{dx_1^2} - c^3\gamma_1\frac{d^5U}{dx_1^5}$$
$$-c\gamma\frac{d}{dx_1}\left[\left(\frac{dU}{dx_1}\right)^2\frac{d^2U}{dx_1^2}\right] - \rho M H^2 U, \tag{3.28}$$

Looking at the solution of the form

$$U(x_1) = u_0\exp(\Omega x_1), \tag{3.29}$$

where Ω is a constant to be determined, it can be seen that equation (3.28) admits the solution

$$U(x_1) = u_0 \exp\left(\frac{\beta x_1}{3c\gamma}\right) \tag{3.30}$$

provided

$$-c\rho\frac{\beta}{3c\gamma} = \mu_1\left(\frac{\beta}{3c\gamma}\right)^2 - c\alpha_1\left(\frac{\beta}{3c\gamma}\right)^3 + \beta_1 c^2\left(\frac{\beta}{3c\gamma}\right)^4 - c^3\gamma_1\left(\frac{\beta}{3c\gamma}\right)^5 - \rho M H^2 \tag{3.31}$$

and hence Eq. (3.26) subject to Eq. (3.31) admits the solution

$$u(y,t) = u_0 \exp\left(\frac{(y-ct)\beta}{3c\gamma}\right). \tag{3.32}$$

Equation (3.31) can be solved in order to find the wave speed c and doing so leads to the equation

$$c^2 = \frac{\left[\mu_1\left(\frac{\beta}{3\gamma}\right)^2 - \alpha_1\left(\frac{\beta}{3\gamma}\right)^3 + \beta_1\left(\frac{\beta}{3\gamma}\right)^4 - \gamma_1\left(\frac{\beta}{3\gamma}\right)^5\right]}{\rho M H^2 - \rho\left(\frac{\beta}{3\gamma}\right)}. \tag{3.33}$$

Thus

$$c = +\frac{\left[\mu_1\left(\frac{\beta}{3\gamma}\right)^2 - \alpha_1\left(\frac{\beta}{3\gamma}\right)^3 + \beta_1\left(\frac{\beta}{3\gamma}\right)^4 - \gamma_1\left(\frac{\beta}{3\gamma}\right)^5\right]^{1/2}}{\left[\rho M H^2 - \rho\left(\frac{\beta}{3\gamma}\right)\right]^{1/2}}, \tag{3.34}$$

or

$$c = -\frac{\left[\mu_1\left(\frac{\beta}{3\gamma}\right)^2 - \alpha_1\left(\frac{\beta}{3\gamma}\right)^3 + \beta_1\left(\frac{\beta}{3\gamma}\right)^4 - \gamma_1\left(\frac{\beta}{3\gamma}\right)^5\right]}{\rho M H^2 - \rho\left(\frac{\beta}{3\gamma}\right)}. \tag{3.35}$$

Equation (3.34) represents the speed of waves propagating toward the plate in the $-y$ direction and Equation (3.35) the speed of waves propagating away from the plate.

We take c positive. We note that from equation (3.33), the value of c cannot be infinity as $MH^2 \neq \beta/3\gamma$ since β and γ have opposite signs in order for the solution (3.32) to satisfy the boundary conditions (3.24). Also there will be no waves ($c = 0$) if the condition

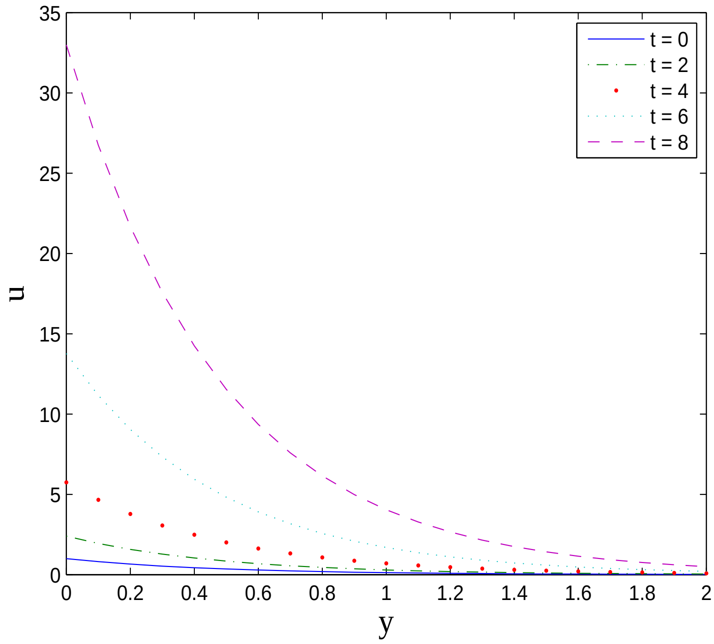

Figure 3.1: Travelling wave solution varying time t when $\beta = 0.6, \mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1, M = 2$, $u_0 = 1$, and $H = 1$.

$\mu_1 - \alpha_1(\beta/3\gamma) + \beta_1(\beta/3\gamma)^2 - \gamma_1(\beta/3\gamma)^3 = 0$ is satisfied.

The solution (3.32) is plotted in Figures 3.1 to 3.5 for various values of the emerging parameters.

The effect of the magnetic field on the unsteady solution (3.32), is shown in figures 3.4 and 3.5.

3.3 Numerical solutions

We present the numerical solution of (3.26) subject to the initial and boundary conditions:

$$u(0,t) = u_0 V(t), \quad u(\infty, t) = 0 \;\; t > 0, \tag{3.36}$$

$$u(y,0) = g(y), \quad y > 0, \tag{3.37}$$

where $g(y)$ is a given function of y. This solution (see figure 3.6) is plotted using Mathematica's solver **NDSolve**. To study the effect of the magnetic field on the flow, we plot the figures

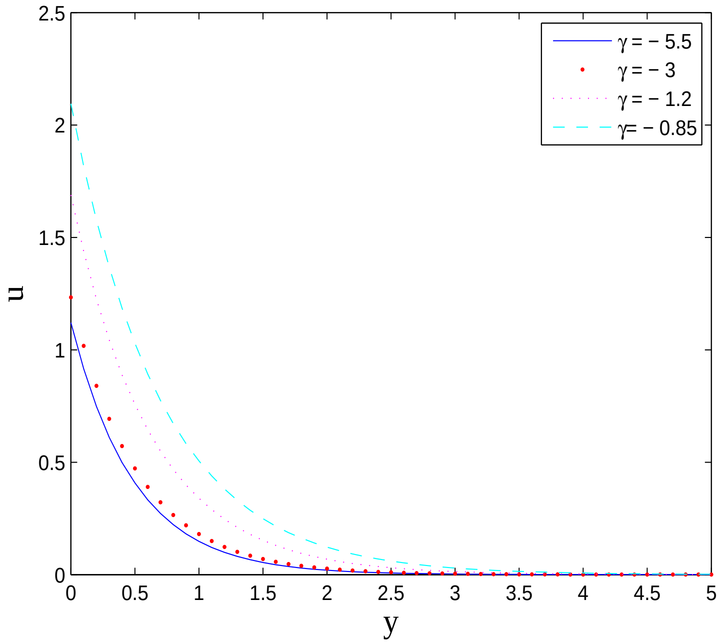

Figure 3.2: Travelling wave solution varying the fluid parameter γ when $\beta = 0.6, \mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \rho = 1, M = 2$, $t = \pi/2$, $u_0 = 1$, and $H = 1$.

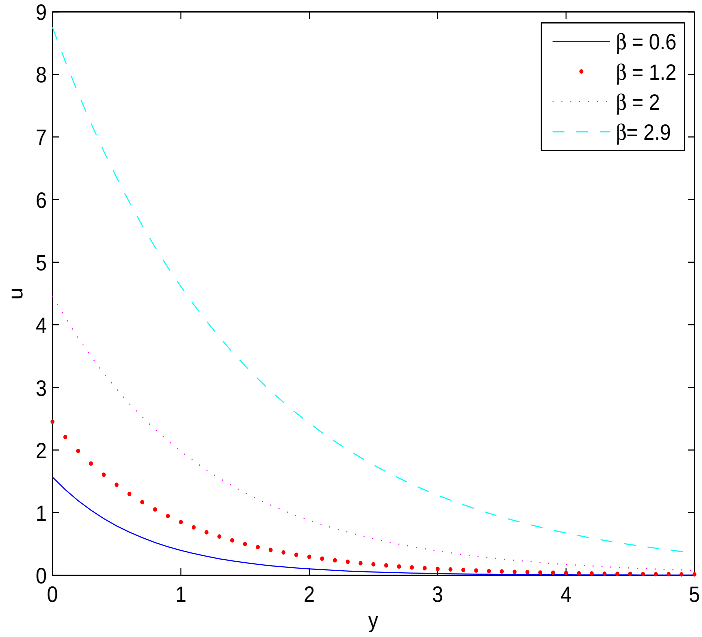

Figure 3.3: Travelling wave solution varying the fluid parameter β when $\mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1, M = 2, t = \pi/2, u_0 = 1$, and $H = 1$.

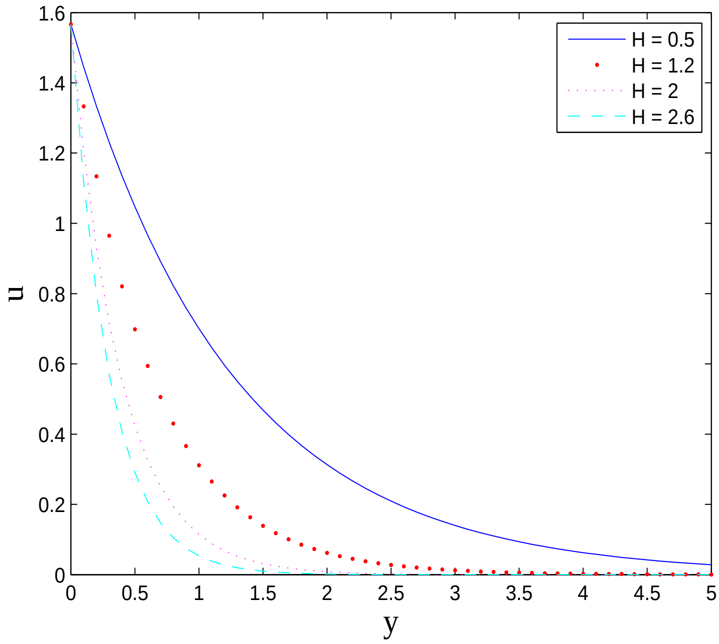

Figure 3.4: Travelling wave solution varying the magnetic field H when $\beta = 0.6, \mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1$, $t = \pi/2$, $u_0 = 1$, and $M = 2$.

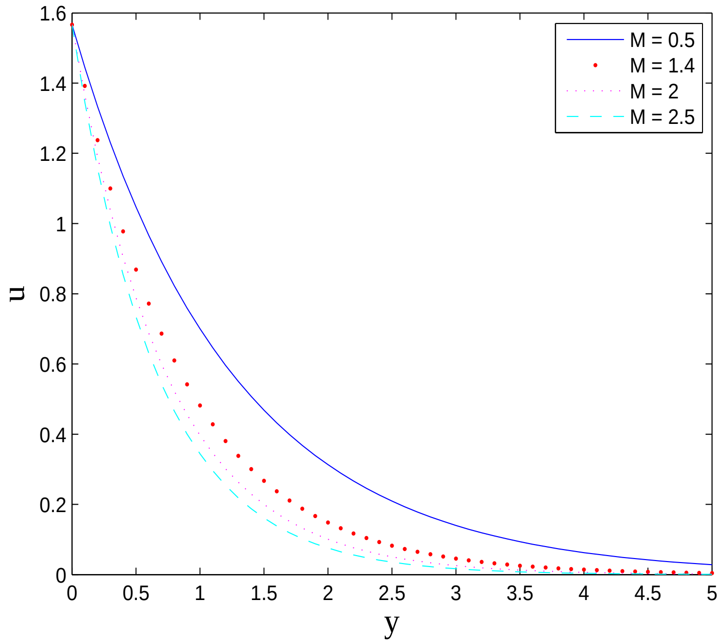

Figure 3.5: Travelling wave solution varying M when $\beta = 0.6, \mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1$, $t = \pi/2$, $u_0 = 1$, and $H = 1$.

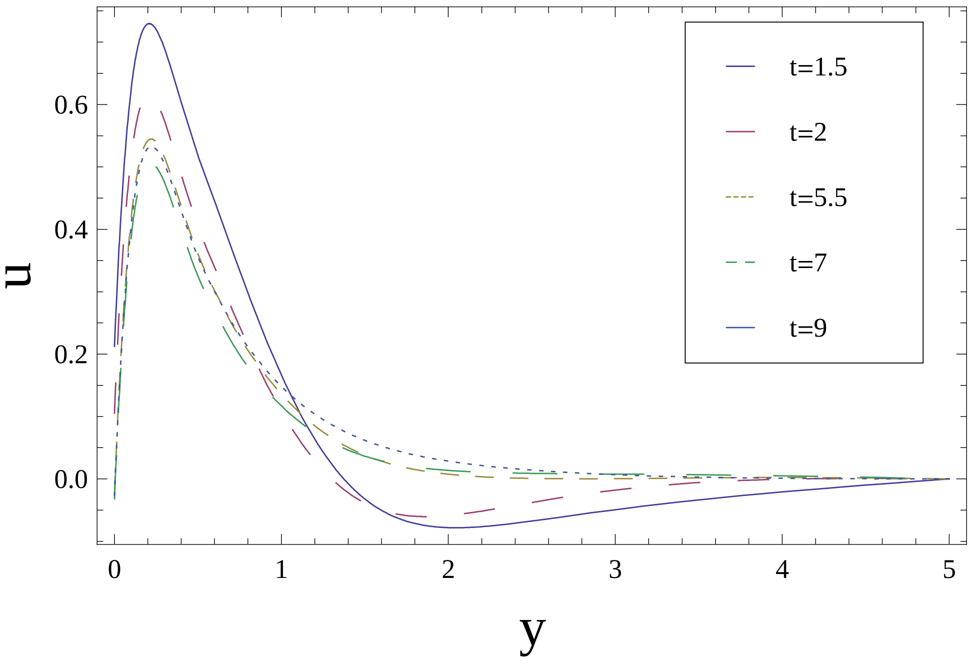

Figure 3.6: Numerical solution of (3.26) varying time t, with $\beta = 0.6, \mu = 0.5, \alpha_1 = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1, M = 2$, $V(t) = e^{-t}$, $u_0 = 1$, $g(y) = e^{-y^2}$ and $H = 1$.

3.7 and 3.8 below for various values of H and M.

3.4 Results and discussion

In this work, the flow of a fourth order fluid over a suddenly moving plate is analyzed. Equation (3.26) under the conditions (3.23) – (3.24) have been solved using symmetry and reductions. In order to see the variation of various physical parameters on the velocity, the Figures 3.1- 3.8 have been plotted. The effect of time t on the velocity profile is shown in Figure 1. This figure depicts that velocity increases for large values of time. The velocity profile decays for large values of y.

The effects of the fluid parameters γ and β are given in figures 3.2 and 3.3, respectively. These figures depict that γ and β have the same effect on the velocity. Moreover the effect of the magnetic field on the travelling wave solution is sown in figures3.4 and 3.5. It is seen that

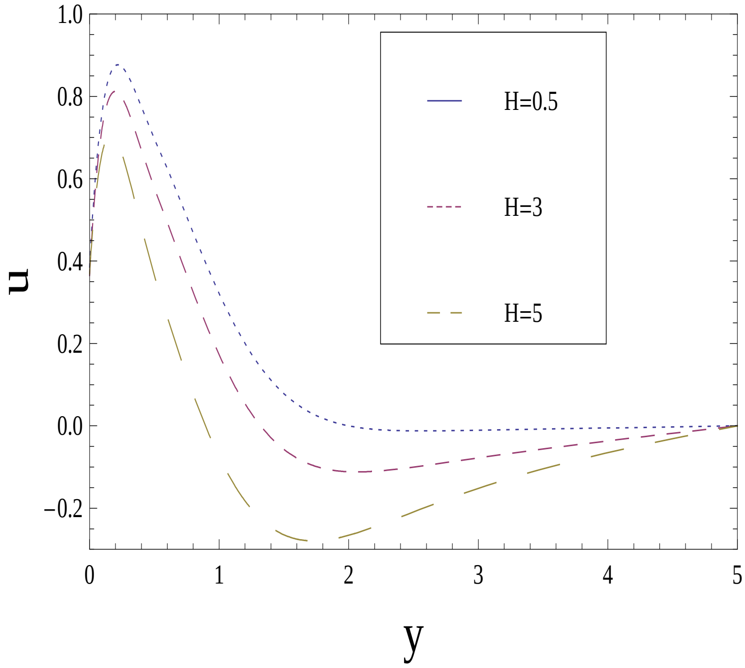

Figure 3.7: Numerical solution of (3.26) varying H , with $\beta = 0.6, \mu = 0.5, \alpha = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1, M = 1$, $V(t) = e^{-t}$, $u_0 = 1$, $g(y) = e^{-y^2}$ and $t = 1$.

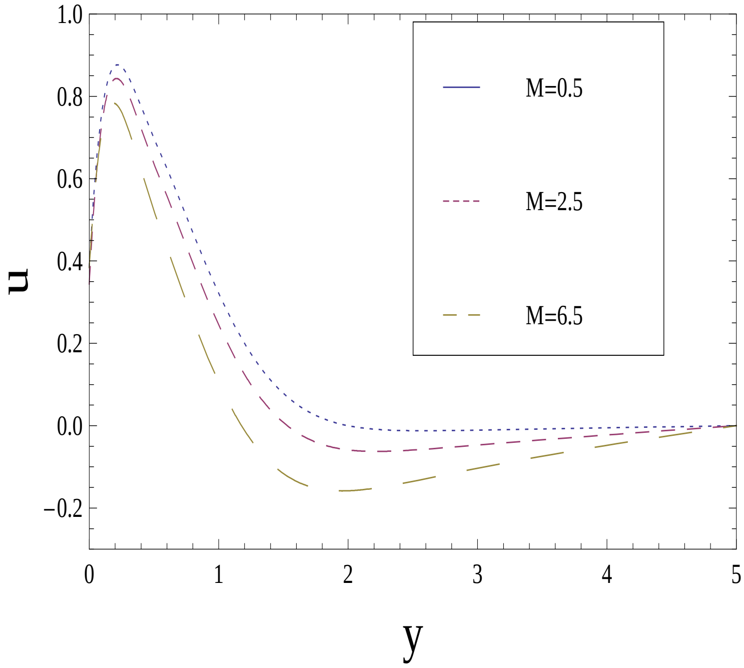

Figure 3.8: Numerical solution of (3.26) varying M , with $\beta = 0.6, \mu = 0.5, \alpha = 2, \beta_1 = 1.5, \gamma_1 = 0.8, \gamma = -0.7, \rho = 1, H = 1$, $V(t) = e^{-t}$, $u_0 = 1$, $g(y) = e^{-y^2}$ and $t = 1$.

the velocity decreases when the magnetic field is increased.

In Figure 3.6, we plotted numerically the velocity profile for various time and it is observed that the velocity oscillates with time.

Finally, in figures 3.7 and 3.8, we investigate numerically the effects of the magnetic field on the flow. It shows like in the case of the travelling wave solution that the velocity decreases when the magnitude of the magnetic field is increased.

Chapter 4

Solutions in a third grade fluid filling the porous space

The unsteady flow of a third grade fluid in a porous semi-infinite space is discussed. The flow is caused by a plate moving in its own plane. Modified Darcy's law is employed in the flow modelling. Two type of analytical solutions are developed. The results are analyzed through graphical representations.

4.1 Problem formulation

Let us introduce a Cartesian coordinate system OXYZ with y-axis in the upward direction. The third grade fluid fills the porous space $y > 0$ and is in contact with an infinite moved plate at $y = 0$. For unidirectional flow the velocity field is

$$\mathbf{V} = (u(y,t), 0, 0)\,, \tag{4.1}$$

where the above definition of velocity automatically satisfies the incompressibility condition. The equation of motion in a porous medium is similar to Eq. (1.16) and can be rewritten here

$$\rho(\frac{\partial}{\partial t} + (\mathbf{V}.\nabla))\mathbf{V} = -\nabla\mathbf{p} + \mathrm{div}\mathbf{S} + \mathbf{r}, \tag{4.2}$$

where ρ is the fluid density, **T** is the Cauchy stress tensor and **r** is the Darcy's resistance in a porous space. The Cauchy stress tensor of an incompressible third grade fluid has the form

Eq. (2.2) [57]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\text{tr}\mathbf{A}_1^2)\mathbf{A}_1, \quad (4.3)$$

in which p is the pressure, $\mathbf{I}$ is the identity tensor, α_i $(i = 1, 2)$ and β_i $(i = 1 - 3)$ are the material constants and $\mathbf{A}_i$ $(i = 1 - 3)$ are the first Rivlin-Ericksen tensors [51] which may be defined through equation (2.2) as

$$\mathbf{A}_1 = (\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^T \quad (4.4)$$

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^T\mathbf{A}_{n-1}; \quad n > 1. \quad (4.5)$$

In studying fluid dynamics, it is assumed that the flow meets the Clausius-Duhem inequality and that the specific Helmholtz free energy of the fluid is a minimum at equilibrium. These conditions are satisfied when [16] (see equation (1.24))

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0,$$

$$\mid \alpha_1 + \alpha_2 \mid \leq \sqrt{24\mu\beta_3}. \quad (4.6)$$

On the basis of constitutive equation in an Oldroyd-B fluid, the following expression in a porous medium has been proposed [2]:

$$(1 + \lambda\frac{\partial}{\partial t})\nabla p = -\frac{\mu\phi}{k}(1 + \lambda_r\frac{\partial}{\partial t})\mathbf{V}, \quad (4.7)$$

where λ and λ_r are the relaxation and retardation times and ϕ and k are the porosity and permeability of the porous medium, respectively. It should be pointed out that for $\lambda_r = 0$, Eq. (4.7) reduces to the expression which holds for a Maxwell fluid [2] and when $\lambda = 0$, it reduces to that of second grade fluid [56].

Keeping the analogy of Eq. (4.7) with the constitutive equation of an extra stress tensor in a Oldroyd-B fluid, the following expression in the present problem has been suggested:

$$\frac{\partial p}{\partial x} = -\frac{\phi}{k}[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2]u\ . \quad (4.8)$$

Since the pressure gradient in Eq. (4.8) can also be interpreted as a measure of the flow resistance in the bulk of the porous medium and r_x is the measure of the flow resistance offered by the solid matrix in x-direction then (cf. Eq. (1.39))

$$r_x = -\frac{\phi}{k}[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2]u. \tag{4.9}$$

From Eqs. (4.1)-(4.6) and (4.9) we have (cf. Eq. (1.40))

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\phi u}{k}. \tag{4.10}$$

The relevant boundary and initial conditions are

$$u(0,t) = U_0 V(t), \;\; t > 0, \tag{4.11}$$

$$u(\infty,t) = 0, \;\; t > 0, \tag{4.12}$$

$$u(y,0) = g(y), \; y > 0, \tag{4.13}$$

in which U_0 is a reference velocity.

4.2 Solutions of the problem

We rewrite Eq. (4.10) as

$$\frac{\partial u}{\partial t} = \mu_*\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \gamma_1\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \gamma_2 u\left(\frac{\partial u}{\partial y}\right)^2 - \phi_1 u \tag{4.14}$$

where

$$\mu_* = \frac{\mu}{\rho + \alpha_1\frac{\phi}{k}}, \;\; \alpha = \frac{\alpha_1}{\rho + \alpha_1\frac{\phi}{k}}, \;\; \gamma_1 = \frac{6\beta_3}{\rho + \alpha_1\frac{\phi}{k}}, \;\; \gamma_2 = \frac{2\beta_3\frac{\phi}{k}}{\rho + \alpha_1\frac{\phi}{k}}, \;\; \phi_1 = \frac{\mu\frac{\phi}{k}}{\rho + \alpha_1\frac{\phi}{k}}.$$

4.2.1 Lie symmetry analysis

We look for an operator (see section 1.4)

$$\chi = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u}, \tag{4.15}$$

such that

$$\chi^{[3]}\left(-\frac{\partial u}{\partial t}+\mu_*\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial^3 u}{\partial y^2\partial t}+\gamma_1\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\gamma_2 u\left(\frac{\partial u}{\partial y}\right)^2-\phi_1 u\right)|_{(4.14)}=0, \quad (4.16)$$

where

$$\chi^{[3]}=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}+\eta^t\frac{\partial}{\partial u_t}+\eta^y\frac{\partial}{\partial u_y}+\eta^{yy}\frac{\partial}{\partial u_{yy}}+\eta^{tyy}\frac{\partial}{\partial u_{tyy}}, \quad (4.17)$$

and η^t, η^y, η^{yy}, η^{tyy} defined in Chapter 1, are the relations (1.54) - (1.56). In equation (4.16) the unknown functions τ, ξ and η are independent of the derivatives of u. Thus separating with respect to the derivatives of u and their powers leads to the overdetermined system of partial differential equations given by

$$u_{ty} \;:\; \xi_{yy}=0, \quad (4.18)$$

$$u_y^2u_{yy} \;:\; \tau_t+2\eta_u-2\xi_y=0, \quad (4.19)$$

$$u_yu_{yy} \;:\; 0=0, \quad (4.20)$$

$$u_{yy} \;:\; \alpha\eta_{tu}+\mu_*\tau_t=0, \quad (4.21)$$

$$u_y^3 \;:\; \xi_{yy}=0, \quad (4.22)$$

$$u_y^2 \;:\; -\eta_u-u\tau_t-\eta=0, \quad (4.23)$$

$$u_y \;:\; \xi_{yy}=0, \quad (4.24)$$

$$u_t \;:\; \xi_y=0, \quad (4.25)$$

$$1 \;:\; \phi_1 u\left(\eta_u-\tau\right)-\phi_1-\eta_t=0. \quad (4.26)$$

We solve the system (4.18)-(4.26) and we obtain two sets of Lie point symmetry generators depending on the value of ϕ_1.

Case 1 $\phi_1\neq\frac{\mu_*}{\alpha}$,

$$\xi = a_1, \quad (4.27)$$

$$\tau = a_2, \quad (4.28)$$

$$\eta = 0. \quad (4.29)$$

In this case the symmetry Lie algbra is two dimensional and is generated by

$$X_1 = \frac{\partial}{\partial y}, \tag{4.30}$$

$$X_2 = \frac{\partial}{\partial t}. \tag{4.31}$$

Case 2 $\phi_1 = \frac{\mu_*}{\alpha}$**,**

$$\xi = a_1, \tag{4.32}$$

$$\tau = a_2\alpha/(2\mu_*)e^{(2\mu_*/\alpha)t} + a_3, \tag{4.33}$$

$$\eta = -(a_2/2)e^{(2\mu_*/\alpha)t}u. \tag{4.34}$$

The corresponding Lie algebra is three dimensional spanned by

$$X_1 = \frac{\partial}{\partial y}, \tag{4.35}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{4.36}$$

$$X_3 = e^{(2\mu_*/\alpha)t}\frac{\partial}{\partial t} - \frac{\mu_*}{\alpha}e^{(2\mu_*/\alpha)t}u\frac{\partial}{\partial u}\ , \tag{4.37}$$

where a_i's are constants of integration.

4.2.2 Travelling wave solutions

We now look for invariant solutions under the operator $X_1 + cX_2$, which represents wave-front type travelling wave solutions with constant wave speed c. The invariant is given by (cf. section 1.4.4)

$$u(y,t) = U(x_1) \quad \text{where} \quad x_1 = y - ct. \tag{4.38}$$

Substituting Eq. (4.38) into Eq. (4.14) yields a third-order ordinary differential equation for $U(x_1)$:

$$\phi_1 U(x_1) = c\frac{dU}{dx_1} - \gamma_2 U(x_1)\left(\frac{dU}{dx_1}\right)^2 + \mu_*\frac{d^2U}{dx_1^2} + \gamma_1\left(\frac{dU}{dx_1}\right)^2\frac{d^2U}{dx_1^2} - c\alpha\frac{d^3U}{dx_1^3}. \tag{4.39}$$

Using the same method as in section 3.2, it can be seen that this equation admits the solution

$$U(x_1) = U_0 \exp\left(\frac{\sqrt{\gamma_2}x_1}{-\sqrt{\gamma_1}}\right) \tag{4.40}$$

provided the wave speed is

$$c = \frac{\phi_1 - \mu_* \gamma_2/\gamma_1}{\sqrt{\gamma_2/\gamma_1}\,(\alpha\gamma_2/\gamma_1 - 1)}. \tag{4.41}$$

We note that when $c < 0$ the waves are propagating toward the plate and when $c > 0$ the waves are propagating away from the plate. and hence Eq. (4.14) subject to Eq. (4.11)-(4.13) admits the solution

$$u(y,t) = U_0 \exp\left(-\frac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}} y\right) \exp\left(\frac{\phi_1 - \mu_* \gamma_2/\gamma_1}{\alpha\gamma_2/\gamma_1 - 1} t\right). \tag{4.42}$$

This solution is plotted in Figures 4.1 and 4.4 for various values of the emerging parameters. These solutions satisfy the initial and boundary conditions (4.11)-(4.13) for

$$V(t) = \exp\left(\frac{\phi_1 - \mu_* \gamma_2/\gamma_1}{\alpha\gamma_2/\gamma_1 - 1} t\right). \tag{4.43}$$

On the other hand, we find group invariant solutions corresponding to operators which give meaningful physical solutions of the initial and boundary value problem (4.10) to (4.13). This here means X_2 and X_3.

4.2.3 Group invariant solutions corresponding to X_2

The invariant solution admitted by X_2, is the steady state solution

$$u(y,t) = F(y). \tag{4.44}$$

The substitution of (4.44) into (4.14) yields the third order ordinary differential equation for $F(y)$

$$\gamma_1 \left(F'(y)\right)^2 F''(y) + \mu_* F''(y) - \gamma_2 F(y) \left(F'(y)\right)^2 - \phi_1 F(y) = 0, \tag{4.45}$$

subject to the boundary conditions $(V(t) = v_0/U_0, U_0 \neq 0)$

$$F(0) = v_0\ , \tag{4.46}$$

$$F(l) = 0, \quad l > 0, \tag{4.47}$$

where l is sufficiently large.

Let

$$K(F) = \frac{dF}{dy},$$

Eq. (4.45) transforms to

$$\gamma_1 K(F)^3 K'(F) + \mu_* K(F) K'(F) - \gamma_2 F K(F)^2 - \phi_1 F = 0. \tag{4.48}$$

The integration of (4.48) gives

$$\frac{\gamma_1}{\gamma_2} K(F) + \frac{\mu_* \gamma_2 - \gamma_1 \phi_1}{\gamma_2 \sqrt{\gamma_2 \phi_1}} \, \mathbf{Arctan}[\sqrt{\frac{\gamma_2}{\gamma_1}} \, K(F)] = \frac{1}{2} y^2 + C, \tag{4.49}$$

where C is a constant. Eq. (4.49) is equivalent to the following first order ODE in F

$$\frac{\gamma_1}{\gamma_2} F'(y) + \frac{\mu_* \gamma_2 - \gamma_1 \phi_1}{\gamma_2 \sqrt{\gamma_2 \phi_1}} \, \mathbf{Arctan}[\sqrt{\frac{\gamma_2}{\gamma_1}} \, F'(x)] = \frac{1}{2} y^2 + C \tag{4.50}$$

This solution is plotted in figure 4.5.

4.2.4 Group invariant solutions corresponding to X_3

The invariant solution admitted by X_3 is

$$u = U_0 \exp(-\frac{\mu_*}{\alpha} t) B(y), \tag{4.51}$$

where $B(y)$ as yet undetermined function of y. Substituting Eq. (4.51) into Eq. (4.14) yields the linear second-order ordinary differential equation

$$B'' - \frac{\gamma_2}{\gamma_1} B = 0. \tag{4.52}$$

From Eqs. (4.11) - (4.12), the appropriate boundary conditions for Eq. (4.52) are

$$B(0) = 1, \quad B(l) = 0, \; l \to \infty, \tag{4.53}$$

where

$$V(t) = \exp(\frac{-\mu_*}{\alpha} t).$$

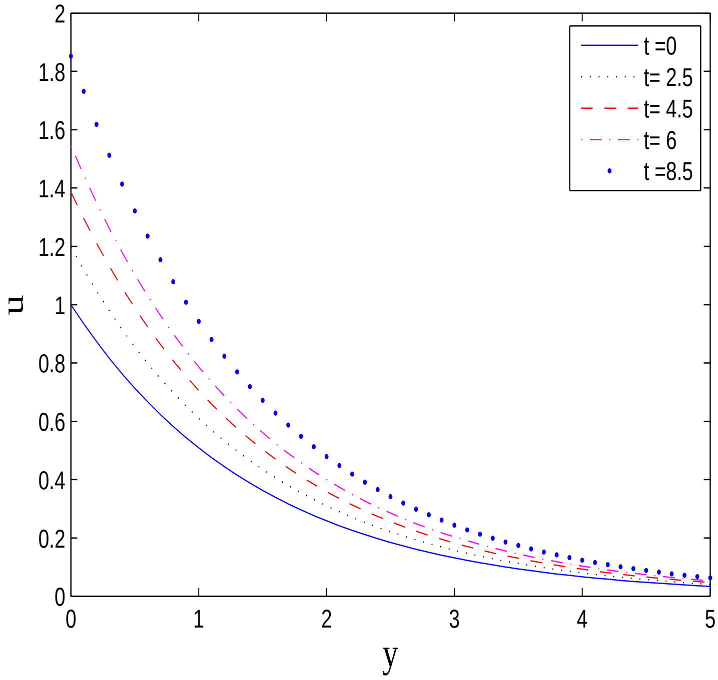

Figure 4.1: Travelling wave solution varying time t, when $\gamma_1 = 1.5,\ U_0 = 1,\ \gamma_2 = 1$ and $c = 1$

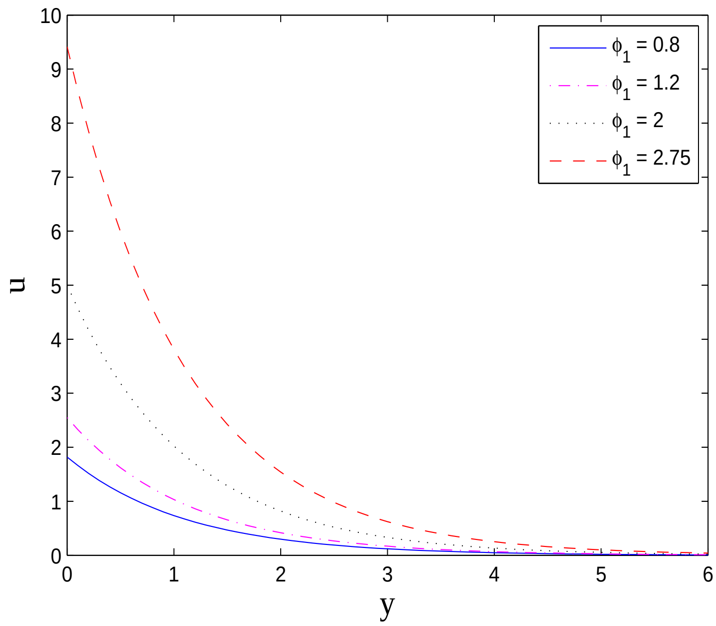

Figure 4.2: Travelling wave solution varying ϕ_1, when $\gamma_1 = 2.2,\ \gamma_2 = 1.8,\ \alpha = 3.5,\ \mu_* = 0.11,\ t = \pi/2$ and $U_0 = 1$.

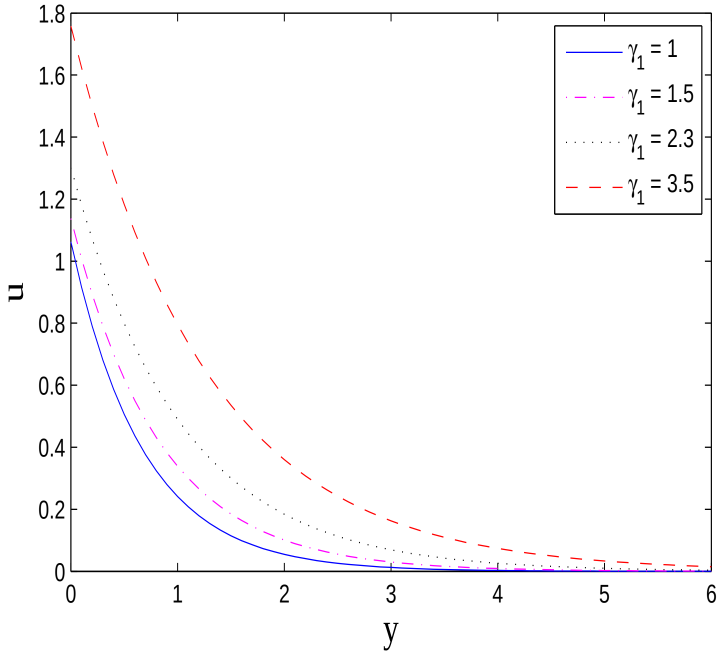

Figure 4.3: Travelling wave solution varying γ_1, when $\gamma_2 = 2.2,\ \phi_1 = 0.5,\ \alpha = 3.5,\ \mu_* = 0.11,\ t = \pi/2$ and $U_0 = 1$

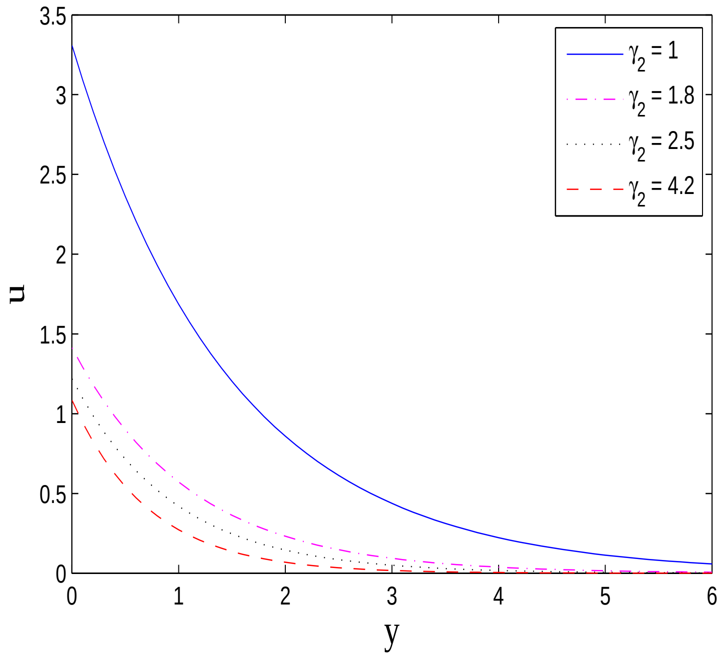

Figure 4.4: Travelling wave solution varying the fluid parameter γ_2, wwhen $\gamma_1 = 2.2,\ \phi_1 = 0.5,\ \alpha = 3.5,\ \mu_* = 0.11,\ t = \pi/2$ and $U_0 = 1$

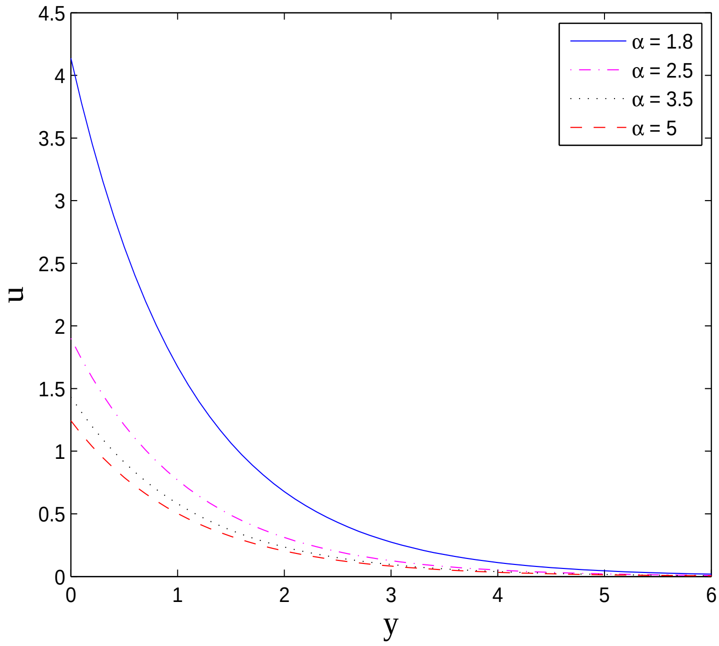

Figure 4.5: Travelling wave solution varying the fluid parameter α, when $\gamma_1 = 1.8,\ \gamma_2 = 2.2,\ \mu_* = 0.7,\ \phi_1 = 1,\ t = \pi/2$ and $U_0 = 1$

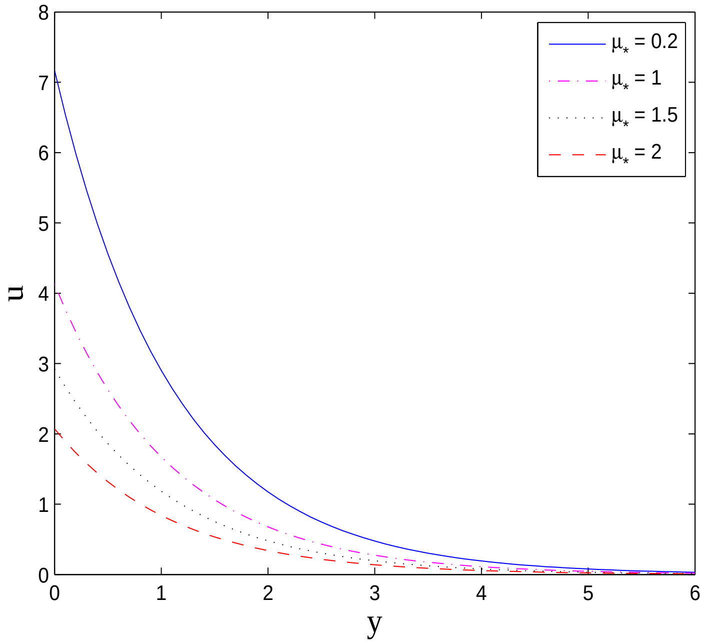

Figure 4.6: Travelling wave solution varying the fluid parameter μ_*, when $\gamma_1 = 1.8,\ \gamma_2 = 2.2,\ \alpha = 3.5,\ \phi_1 = 1,\ t = \pi/2$ and $U_0 = 1$

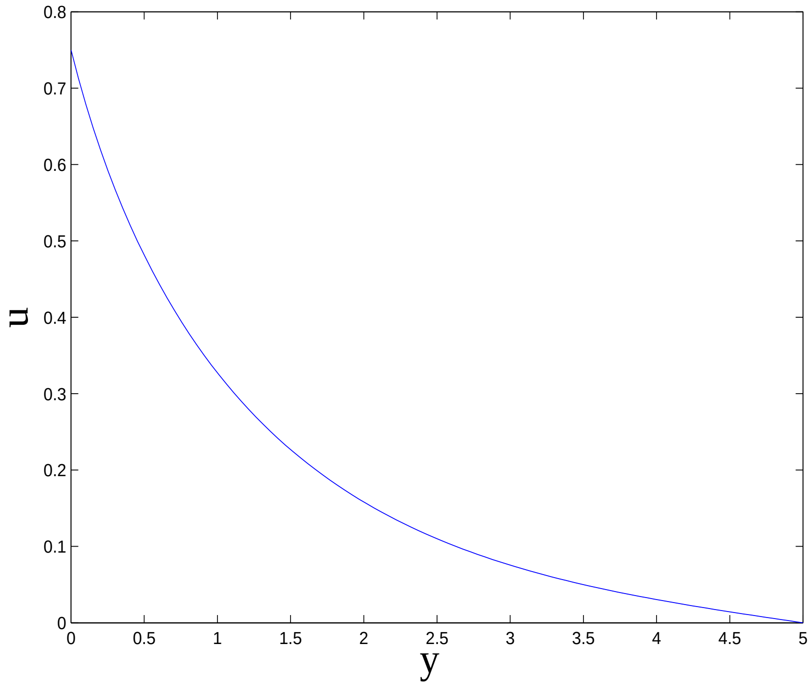

Figure 4.7: Numerical solution of Eq. (4.48), when $\mu_* = 1.5,\ \gamma_1 = 0.75,\ \gamma_2 = 1.5$ and $\phi_1 = 1$

We solve Eq. (4.52) subject to conditions (4.53) for positive γ_2/γ_1 (we can select the parameters in such a way that we get γ_2/γ_1 positive). We obtain

$$B(y) = \exp(-\sqrt{\gamma_2/\gamma_1}y) \tag{4.54}$$

Hence we have

$$u(y,t) = U_0 \exp(-\frac{\mu_*}{\alpha}t) \exp(-\sqrt{\gamma_2/\gamma_1}y) \tag{4.55}$$

The solutions (4.55) are plotted for positive γ_2/γ_1 in figure 4.6.

4.2.5 Numerical solution

We present the numerical solution of equation (4.14) subject to the initial and boundary conditions:

$$u(0,t) = U_0 V(t),\ \ u(\infty,t) = 0\ \ t > 0, \tag{4.56}$$

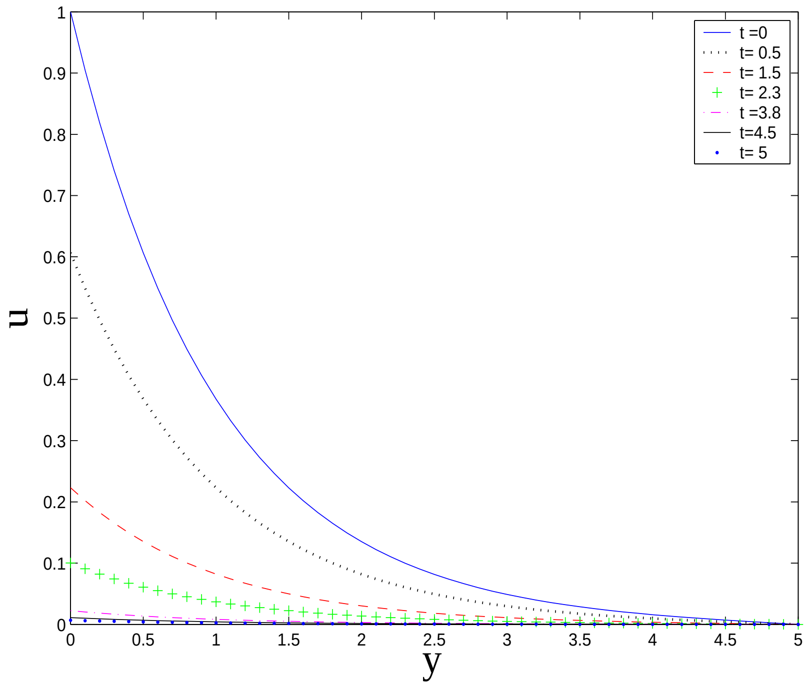

Figure 4.8: Analytical Solution (4.55) varying time t, when $\mu_* = 1,\ \gamma_1 = 1.5,\ \gamma_2 = 1,\ U_0 = 1$ and $\alpha = 1.5$

$$u(y, 0) = g(y) \ \ y > 0, \tag{4.57}$$

where $g(y)$ is yet an arbitrary function of y. We choose $U_0 = 1$. This solution is plotted using Mathematica's solver NDSolve.

4.3 Results and discussion

In order to see the variation of various physical parameters on the velocity, the Figures 4.1 – 4.7 have been plotted.

The effect of varying time t on the velocity u is shown in figure 4.1. This figure depicts that velocity increases for large values of time. The influence of the fluid parameter ϕ_1 on the velocity profile has been presented in figure 4.2. It is revealed that velocity increases by increasing ϕ_1. Moreover, the effects of the fluid parameters γ_1, γ_2, α and μ_* are given in figures 4.3, 4.4, 4.5 and 4.6 respectively. Figures 4.3, 4.4, depict that γ_1 and γ_2 play opposite roles

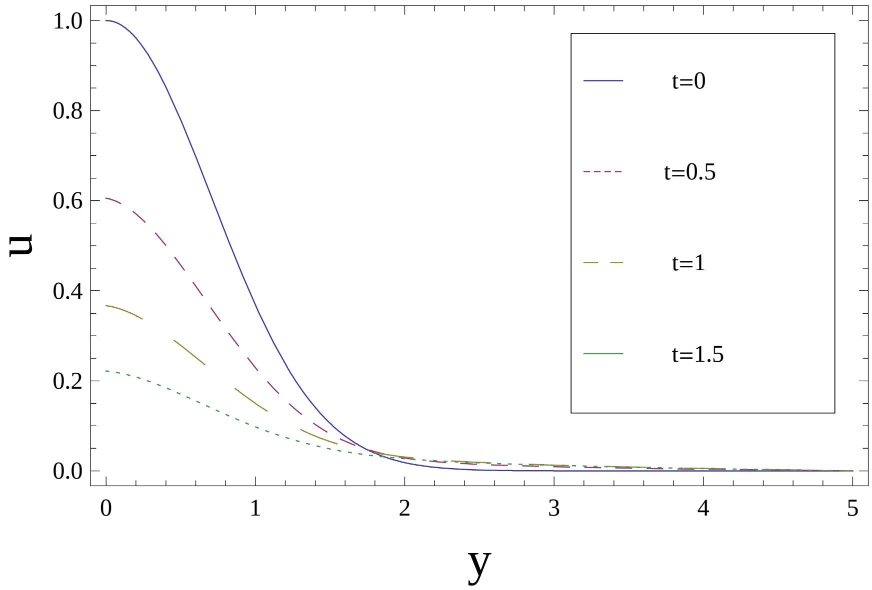

Figure 4.9: Numerical Solution of Eq. (4.14), with $V(t) = e^{-t}$, $g(y) = e^{-y^2}$, $\mu_* = 2.5$, $\alpha = 2$, $\gamma_1 = 1.5$, $\gamma_2 = 2.6$, and $\phi_1 = 0.8$

on the velocity. These figures show that velocity increases for large values of γ_1 whereas it decreases for increasing γ_2. Figures 4.5, 4.6, show tha α and μ_* have the same effect on the velocity. It is observed that the velocity is a decreasing function of α and μ_*. In figure 4.7, the study state solution is ploted, and the velocity profile is the same as observed in the case of travelling wave solution.

Further, the analytical solutions (4.52) for $\gamma_2/\gamma_1 > 0$ is plotted in figures 4.8. Here it indicates that the velocity profile decreases for large values of t. Ultimately when $t \geq 3.8$, there is almost no variation in velocity. The study state is achieved.

Finally, in figure 4.9, we plotted numerically the velocity profile for small variations of time. It is observed that the velocity first decreases as time increases but up to a certain point, like observed in Figure 4.8 and then increases as in Figure 4.1.

Chapter 5

Reduction and solutions for MHD flow of a Sisko fluid in a porous medium

This chapter concentrates on the exact analytical and numerical solutions for MHD flow of a non-Newtonian fluid filling the porous half-space. The formulation of the problem is given using modified Darcy's law for a Sisko fluid. The fluid is electrically conducting and a uniform magnetic field is applied normal to the flow by neglecting the induced magnetic field. The results are analyzed through graphs. The governing non-linear differential equation is obtained by employing reduction and solutions have been developed using the similarity approach. We also present numerical solutions for the reduced equations. The influence of various parameters of interest has been shown and discussed through several graphs. A comparison of the present analysis shows excellent agreement between analytic and numerical solutions. Besides the importance of exact solutions the non-Newtonian flows in a porous medium are important in biorheology and engineering fields such as enhanced oil recovery, paper and textile coating and composite manufacturing processes.

5.1 Governing equations

Here, the continuity and momentum equations (1.15) and (1.16) for an incompressible MHD fluid reduce to

$$\mathrm{div}\mathbf{V} = 0, \tag{5.1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \mathrm{div}\mathbf{S} - \sigma B_0^2 \mathbf{V} + \mathbf{R}, \tag{5.2}$$

where $\mathbf{V}$ is the velocity field, ρ the density, p the pressure, $\mathbf{S}$ the extra stress tensor, σ the electrical conductivity, B_0 the applied magnetic field and $\mathbf{R}$ represents the Darcy's resistance in the porous medium. In this chapter we consider a Sisko fluid and its extra stress tensor is [54]

$$\mathbf{S} = \left[a + b \left| \sqrt{\frac{1}{2} tr\left(\mathbf{A}_1^2\right)} \right|^{n-1} \right] \mathbf{A}_1, \tag{5.3}$$

where $a,\ b$ and n are constants defined differently for different fluids and $\mathbf{A}_1$ the rate of deformation tensor defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{L} = \mathrm{grad}\mathbf{V}. \tag{5.4}$$

For a two-dimensional flow the velocity and the stress fields have the form

$$\mathbf{V} = [u(x,y,t), v(x,y,t), 0] \quad \text{and} \quad \mathbf{S} = \mathbf{S}(x,y,t). \tag{5.5}$$

where u and v are the x and y-components of the velocity vector. Now substituing equation (5.5) into equation (5.3) along with equation (5.4) we get

$$S_{xx} = 2\left(\frac{\partial u}{\partial x}\right) \left[a + b \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right|^{\frac{n-1}{2}} \right], \tag{5.6}$$

$$S_{xy} = S_{yx} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \left[a + b \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right|^{\frac{n-1}{2}} \right], \tag{5.7}$$

$$S_{yy} = 2\left(\frac{\partial v}{\partial y}\right) \left[a + b \left| 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right|^{\frac{n-1}{2}} \right], \tag{5.8}$$

$$S_{xz} = S_{zx} = S_{yz} = S_{zy} = S_{zz} = 0. \tag{5.9}$$

Accordingly, the momentum Eq. (5.2) gives

$$\begin{aligned}
&\rho\left[\tfrac{\partial u}{\partial t} + u\tfrac{\partial u}{\partial x} + v\tfrac{\partial u}{\partial y}\right] \\
&= -\tfrac{\partial p}{\partial x} + \tfrac{\partial}{\partial x}\left\{2\left(\tfrac{\partial u}{\partial x}\right)\left[a + b\left|2\left(\tfrac{\partial u}{\partial x}\right)^2 + 2\left(\tfrac{\partial v}{\partial y}\right)^2 + \left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]\right\} \\
&+\tfrac{\partial}{\partial y}\left\{\left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)\left[a + b\left|2\left(\tfrac{\partial u}{\partial x}\right)^2 + 2\left(\tfrac{\partial v}{\partial y}\right)^2 + \left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]\right\} - \sigma B_0^2 u + R_x,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
&\rho\left[\tfrac{\partial u}{\partial t} + u\tfrac{\partial v}{\partial x} + v\tfrac{\partial v}{\partial y}\right] \\
&= -\tfrac{\partial p}{\partial y} + \tfrac{\partial}{\partial x}\left\{\left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)\left[a + b\left|2\left(\tfrac{\partial u}{\partial x}\right)^2 + 2\left(\tfrac{\partial v}{\partial y}\right)^2 + \left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]\right\} \\
&+\tfrac{\partial}{\partial z}\left\{2\left(\tfrac{\partial v}{\partial y}\right)\left[a + b\left|2\left(\tfrac{\partial u}{\partial x}\right)^2 + 2\left(\tfrac{\partial v}{\partial y}\right)^2 + \left(\tfrac{\partial u}{\partial y} + \tfrac{\partial v}{\partial x}\right)^2\right|^{\frac{n-1}{2}}\right]\right\} - \sigma B_0^2 v + R_y,
\end{aligned} \tag{5.11}$$

where R_x and R_y are the components of R in the x and $y-$ directions, respectively.

For the problem considered here we define the velocity and the stress fields of the following form

$$\mathbf{V} = [u(y,t), 0, 0], \qquad \mathbf{S} = \mathbf{S}(y,t), \tag{5.12}$$

Following [55] and [56], the constitutive relationship between the pressure drop and the velocity for the unidirectional flow of a Sisko fluid is

$$\frac{\partial p}{\partial x} = -\frac{\phi}{k}\left[a + b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u, \tag{5.13}$$

where $k\ (>0)$ and $\phi\ (0 < \phi < 1)$ are the permeability and porosity, respectively.

The pressure gradient in Eq. (5.13) is regarded as a measure of the flow resistance in the bulk of the porous medium. If R_x is a measure of the flow resistance offered by the solid matrix in the $x-$direction then R_x through Eq. (5.13) is inferred by

$$R_x = -\frac{\phi}{k}\left[a + b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u. \tag{5.14}$$

Under (5.12), the continuity equation (5.1) is satisfied identically and from Eqs. (5.2), (5.3), (5.12) and (5.14) we have

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\left(a+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right]-\frac{\phi}{k}\left[a+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u-\sigma B_0^2u. \quad (5.15)$$

5.2 Problem formulation

We consider a cartesian coordinate system with y-axis in the vertical upward direction and x-axis parallel to the rigid plate at $y = 0$. The flow of an incompressible and electrically conducting Sisko fluid is bounded by an infinite rigid plate. The Sisko fluid occupies the porous half-space $y > 0$. The flow is produced by the motion of the plate with the time-dependent velocity $U_0V(t)$. For zero pressure gradient, the resulting problem is

$$\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial y}\left[\left(a+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right]-\frac{\phi}{k}\left[a+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u-\sigma B_0^2u, \quad (5.16)$$

$$u(0,t)=U_0V(t), \qquad t>0,$$

$$u(\infty,t)=0, \qquad t>0, \quad (5.17)$$

$$u(y,0)=g(y), \qquad y>0,$$

in which U_0 is the characteristic velocity. The above equations can be non-dimensionalized using the following variables

$$u^* = \frac{u}{U_0}, \quad y^*=\frac{yU_0}{\nu}, \quad t^*=\frac{tU_0^2}{\nu}, \quad (5.18)$$

$$b^* = \frac{b}{a}\left|\frac{U_0^2}{\nu}\right|^{n-1}, \quad \frac{1}{K}=\frac{\phi\nu^2}{kU_0^2}, \quad M^2=\frac{\sigma B_0^2\nu}{\rho U_0^2}. \quad (5.19)$$

Accordingly the above boundary value problem after dropping the asterisks becomes

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial y}\left[\left(1+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right]-\frac{1}{K}\left[1+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u-M^2u, \quad (5.20)$$

$$u(0,t)=V(t), \qquad t>0,$$

$$u(\infty,t)=0, \qquad t>0, \quad (5.21)$$

$$u(y,0)=h(y), \qquad y>0,$$

where $h(y) = g(y)/U_0$.

5.3 Exact solutions

In this section, we are interested in obtaining the reductions and solutions of Eqs. (5.20) and (5.21). We firstly investigate Eqs. (5.20) and (5.21) when the velocity gradient is positive. Then we can write Eq. (5.20) as

$$u_t = u_{yy} + bnu_y^{n-1}u_{yy} - \frac{1}{K}u - M^2u - \frac{b}{K}uu_y^{n-1}, \tag{5.22}$$

where the suffices refer to partial derivatives.

In the special case of a Newtonian fluid, we have $b = 0$ or $a = 0$ and $n = 1$ in equation (5.16). In the Newtonian case Eq. (5.22) for $b = 0$ reduces to

$$u_t = u_{yy} - \left(\frac{1}{K} + M^2\right)u. \tag{5.23}$$

By means of the transformation

$$U(y,t) = u(y,t)\exp\left[\left(\frac{1}{K} + M^2\right)t\right], \tag{5.24}$$

one can reduce equation (5.23) to the classical heat equation

$$U_t = U_{yy}, \tag{5.25}$$

with boundary conditions becoming

$$\begin{aligned} &U(0,t) = V(t)\exp\left[\left(\frac{1}{K} + M^2\right)t\right], \\ &U(\infty,t) = 0, \\ &U(y,0) = h(y). \end{aligned} \tag{5.26}$$

A solution of Eq. (5.25) can be found by the reduction

$$U = f(yt^{-1/2}). \tag{5.27}$$

The substitution of Eq. (5.27) into Eq. (5.25) gives the following ordinary differential equation

$$f'' + \frac{1}{2}\lambda f' = 0, \tag{5.28}$$

subject to the boundary conditions

$$f(0) = l, \quad f(\infty) = 0, \tag{5.29}$$

where l is a constant and prime denotes the derivative with respect to $\lambda = yt^{-1/2}$. The solution of Eqs. (5.28) and (5.29) is given by

$$f(\lambda) = l\left[1 - \text{erf}(\lambda/2)\right], \tag{5.30}$$

in which erf is the error function. Thus in the case of a Newtonian fluid the solution is given by

$$u(y,t) = l\left[1 - \text{erf}\left(\frac{1}{2}yt^{-1/2}\right)\right]\exp\left[-(\frac{1}{K} + M^2)t\right], \tag{5.31}$$

with

$$V(t) = l\exp\left[-(\frac{1}{K} + M^2)t\right], \quad h(y) = 0. \tag{5.32}$$

This solution is graphically represented in figures 5.1 to 5.3, for various parameters.

We now look at the non-Newtonian case when there is no magnetic field and there is no porous space. Then our problem reduces to

$$u_t = u_{yy} + bnu_y^{n-1}u_{yy}, \tag{5.33}$$

subject to the boundary conditions (5.21). Travelling wave solutions of constant wave speed c of Eq. (5.33) are represented by

$$u\left(y,t\right) = f\left(x_1\right), \quad x_1 = y - ct, \tag{5.34}$$

which describes waves travelling away from the boundary. The substitution of Eq. (5.34) into Eq. (5.33) yields the equation for f

$$-cf' = f'' + bn\left(f'\right)^{n-1}f'', \tag{5.35}$$

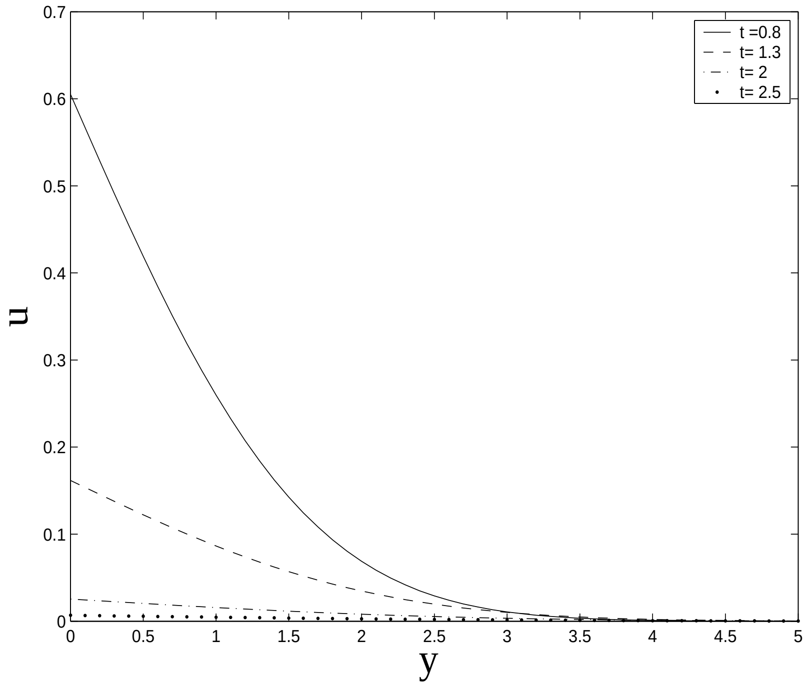

Figure 5.1: Velocity profile in the case of Newtonian fluid, for variuos values of time t, when l = 5, $M = 0.8$ and $K = 0.5$.

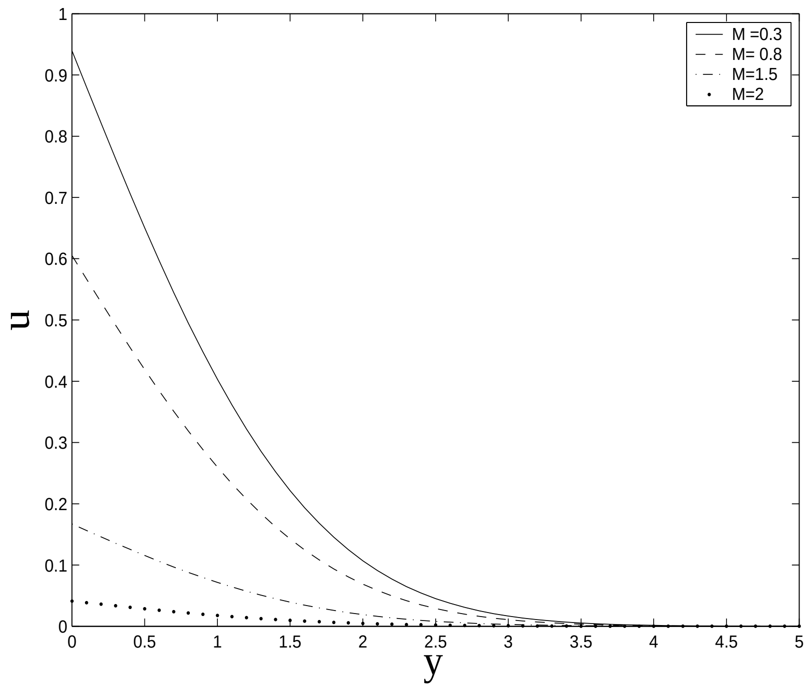

Figure 5.2: Velocity profile in the case of Newtonian fluid, when varying the Hartmann M, with l = 5, $t = 0.8$ and $K = 0.5$

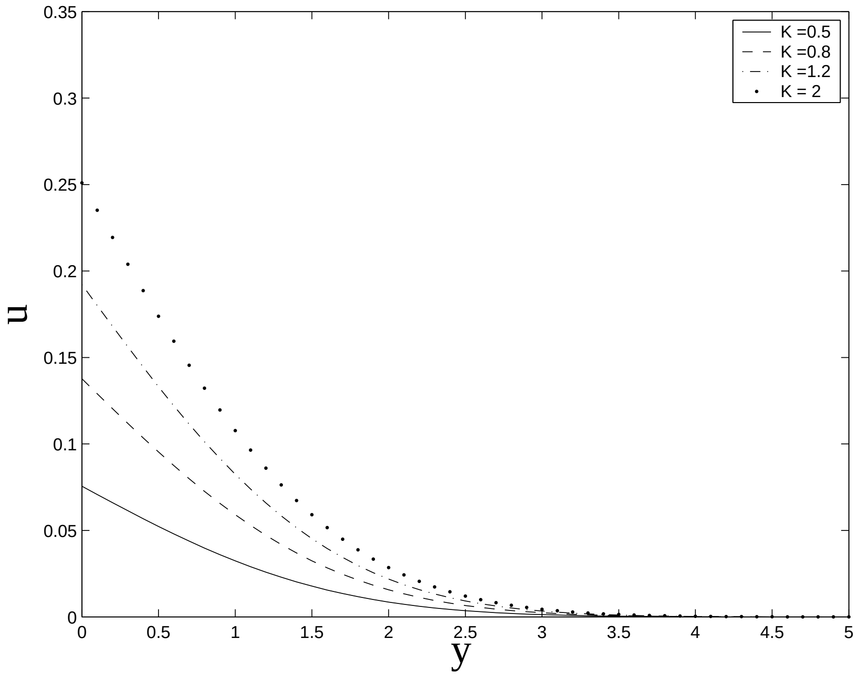

Figure 5.3: Velocity profile in the case of Newtonian fluid, for various values of the parameter K, when l = 5, $t = 0.8$ and $M = 1.8$

where prime refers to the total derivative with respect to x_1. The equation (5.35) can be reduced to a first-order equation which is

$$f' + b\left(f'\right)^n + cf + A = 0, \tag{5.36}$$

where A is a constant of integration. The solutions of Eq. (5.36) subject to the boundary conditions

$$f\left(0\right) = l, \qquad f(\infty) = 0, \tag{5.37}$$

are sought. For $n = 2$, Eq. (5.36) yields

$$f'(x_1) = \frac{\sqrt{-4b(A + cf(x_1)) + 1} - 1}{2b}, \tag{5.38}$$

with

$$f(0) = l, \tag{5.39}$$

$$f(M_1) = 0, \quad M_1 > 0. \tag{5.40}$$

For $n = 3$, Eq. (5.36) reduces to a cubic equation in f' which has as one of its roots

$$f'(x_1) = \frac{\sqrt[3]{2}\left(-9\left(A + cf\left(x_1\right)\right)b^2 + \sqrt{3}\sqrt{b^3\left(27b\left(A + cf\left(x_1\right)\right)^2 + 4\right)}\right)^{2/3} - 2\sqrt[3]{3}b}{6^{2/3}b\sqrt[3]{-9\left(A + cf\left(x_1\right)\right)b^2 + \sqrt{3}\sqrt{b^3\left(27b\left(A + cf\left(x_1\right)\right)^2 + 4\right)}}}, \tag{5.41}$$

with

$$f(0) = l, \tag{5.42}$$

$$f(M_1) = 0, \quad M_1 > 0, \tag{5.43}$$

where M_1 is sufficiently large. Eq. (5.41) can be obtained using Mathematica.

The numerical solutions of Eqs. (5.35) are plotted in figures 5.4 and 5.5 for various values of the emerging parameters in the non-Newtonian case. (The numerical solution for the newtonian case $n = 2$ presented singularity and could not be solved numerically.)

We next construct similarity solutions of the form

$$u = t^{\alpha} f(yt^{-\alpha}) \tag{5.44}$$

for Eq. (5.33). Insertion of Eq. (5.44) into Eq. (5.33) suggests that $\alpha = \frac{1}{2}$. The reduced equation is then

$$f'' + bn\left(f'\right)^{n-1} f'' + \frac{1}{2}\gamma f' - \frac{1}{2}f = 0, \tag{5.45}$$

where $\gamma = yt^{-1/2}$ and the prime denotes differentiation with respect to γ. The boundary conditions (5.21) become

$$f(0) = l_1, \quad f(\infty) = 0, \tag{5.46}$$

where l_1 is a constant and $V(t)$ of the boundary conditions (5.21) takes the form

$$V(t) = l_1\sqrt{t}, \quad h(y) = 0. \tag{5.47}$$

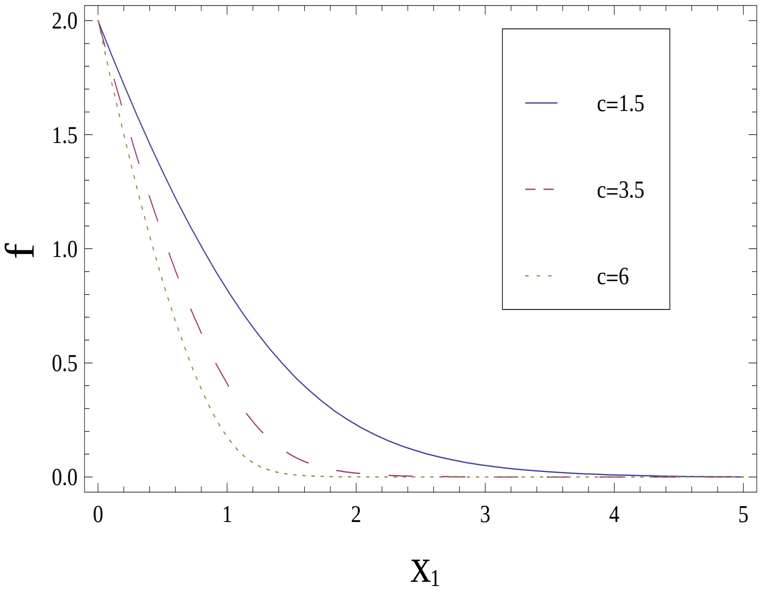

Figure 5.4: Travelling wave solution varying the fluid parameter c in the non-Newtonian case when there is no magnetic field and no porous space and the flow behaviour index $n = 3$, with $b = 1.5$

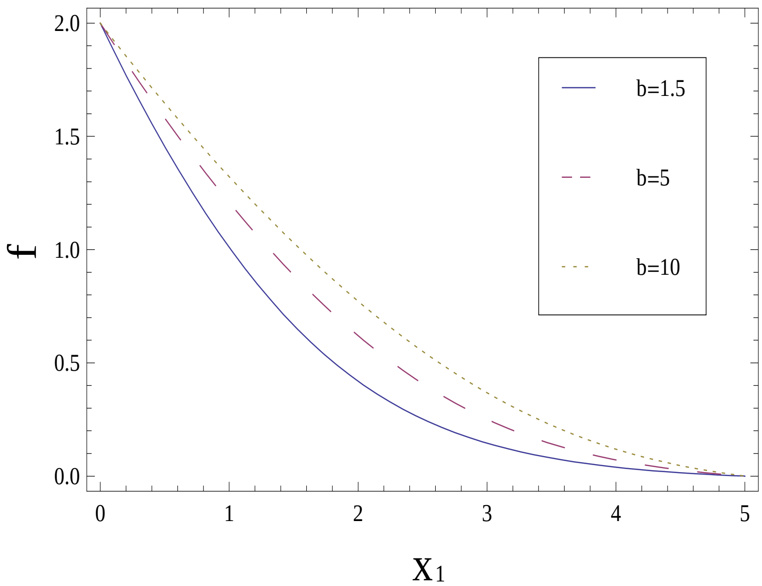

Figure 5.5: Travelling wave solution varying the fluid parameter b in the non-Newtonian case when there is no magnetic field and no porous space and the flow behaviour index $n = 3$, with $c = 1$

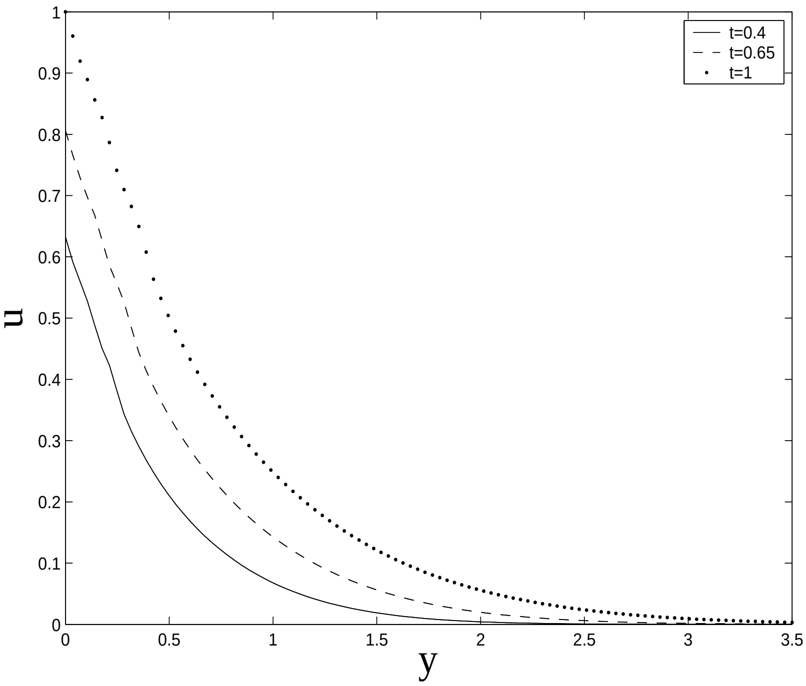

Figure 5.6: Similarity solution $u = t^{1/2}f(yt^{-1/2})$ varying time t in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 2$, with $b = 0.5$

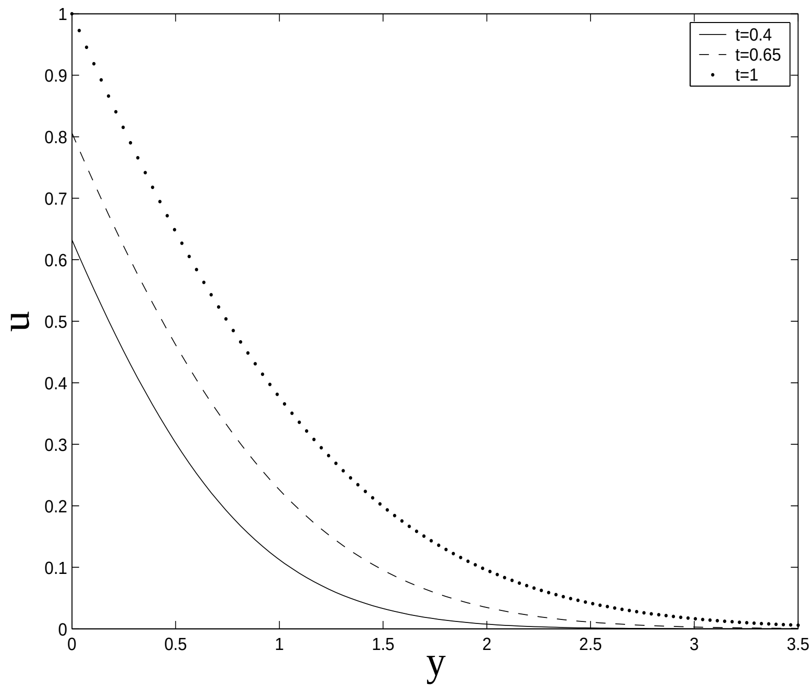

Figure 5.7: Similarity solution $u = t^{1/2}f(yt^{-1/2})$ varying time t in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n \neq 2 (n = 5)$, with $b = 0.5$

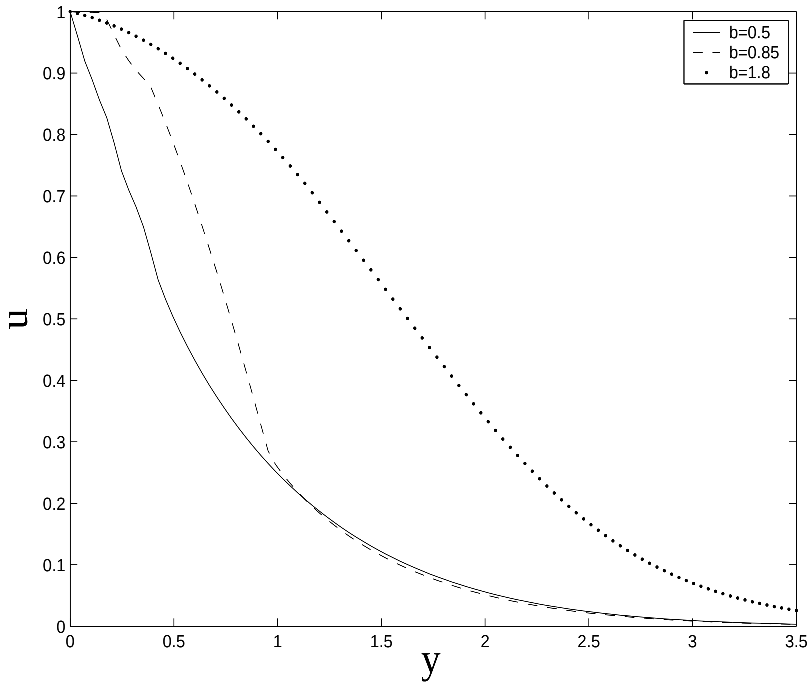

Figure 5.8: Similarity solution $u = t^{1/2} f(yt^{-1/2})$ varying the parameter b in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 2$, with $t = 1$

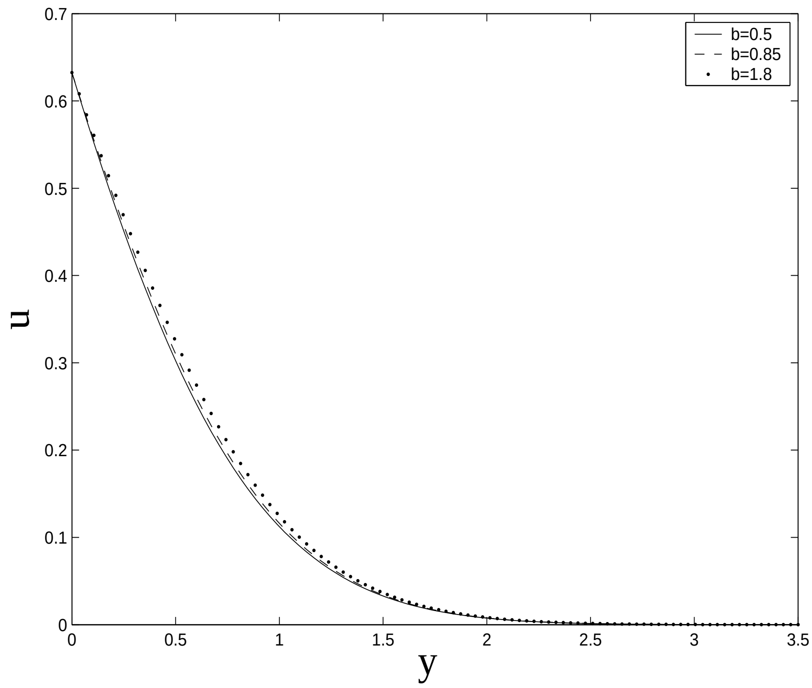

Figure 5.9: Similarity solution $u = t^{1/2} f(yt^{-1/2})$ varying the parameter b in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 5$, with $t = 1$

We plot the solution (5.44), $\alpha = \frac{1}{2}$, for f satisfying (5.46) for $(n = 2)$ and $(n \neq 2)$ varying the constant parameters in figures 5.6 to 5.9.

The Lie symmetry analysis of equation (5.33) reveals that it admits four symmetries given by

$$\chi_1 = \frac{\partial}{\partial t} \tag{5.48}$$

$$\chi_2 = \frac{\partial}{\partial y} \tag{5.49}$$

$$\chi_3 = \frac{\partial}{\partial u} \tag{5.50}$$

$$\chi_4 = 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} \tag{5.51}$$

By looking for the invariant solutions under the operator $\chi_1 + c\chi_4$, another similarity solution can be constructed by the following

$$u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right). \tag{5.52}$$

Note that the similarity solution (5.44) is obtained by considering the invariant solution under the operator χ_4.

The substitution of Eq. (5.52) into Eq. (5.33) yields the following non-linear ordinary differential equation

$$f'' + bn\left(f'\right)^{n-1} f'' + c\xi f' - cf = 0, \tag{5.53}$$

subject to

$$f(0) = l, \quad f(\infty) = 0, \quad h(y) = f(y), \tag{5.54}$$

where $\xi = y(1+2tc)^{-1/2}$ and prime denotes the derivative with respect to ξ. This solution is plotted in figures 5.10 to 5.15 for various values of the parameters.

We also present the numerical solutions of Eqs. (5.23) and (5.33) using the Matlab solver PDEPE in figures 5.16 and 5.17 for $V(t) = \exp(-t)$, $h(y) = \exp(-y^2)$ and $U_0 = 1$.

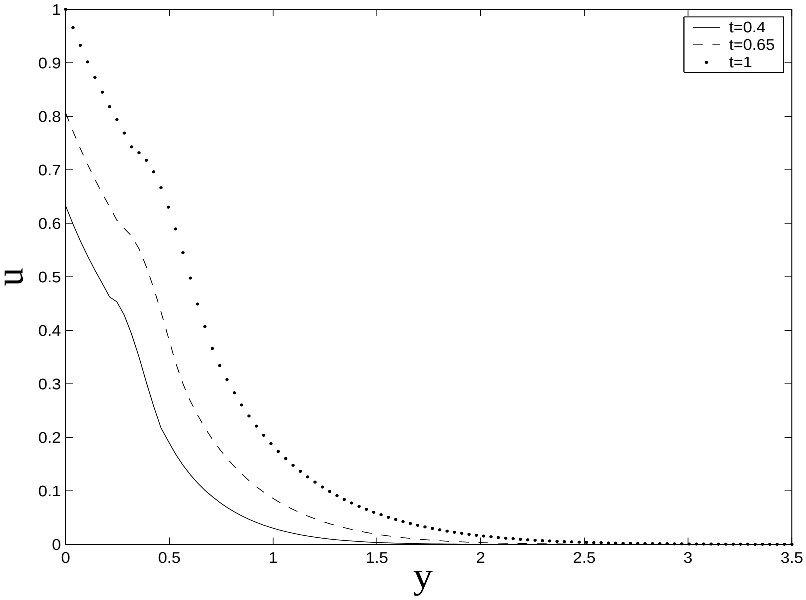

Figure 5.10: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying time t in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 2$, with $b = 0.5, c = 1$

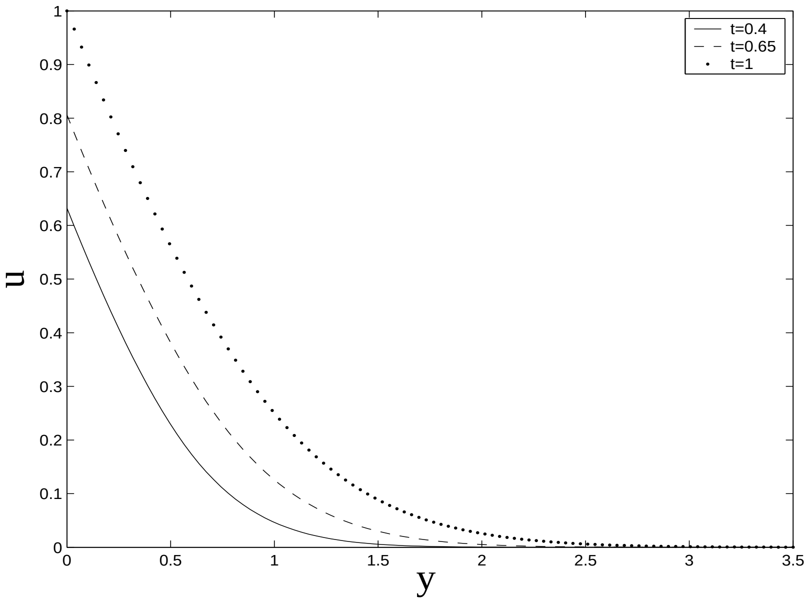

Figure 5.11: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying time t in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 5$, with $b = 0.5, c = 1$

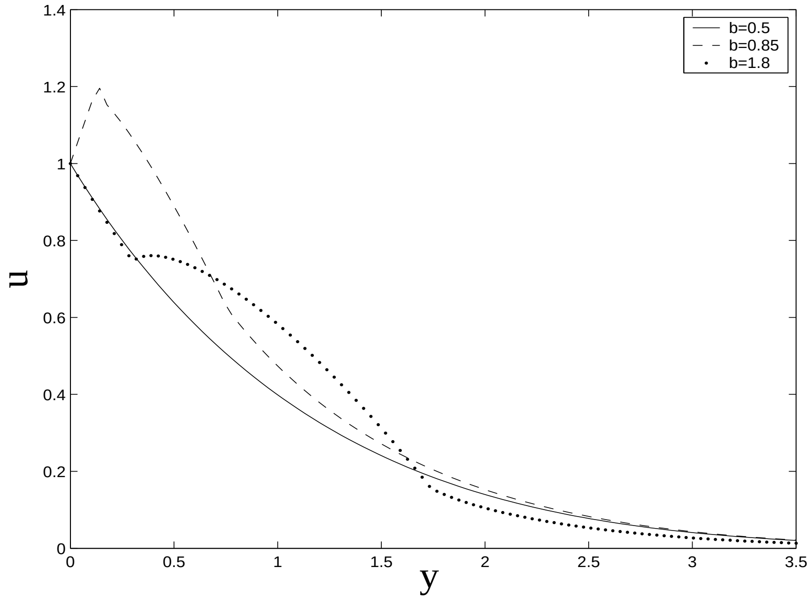

Figure 5.12: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying the parameter b in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 2$, with $t = 1, c = 0.3$

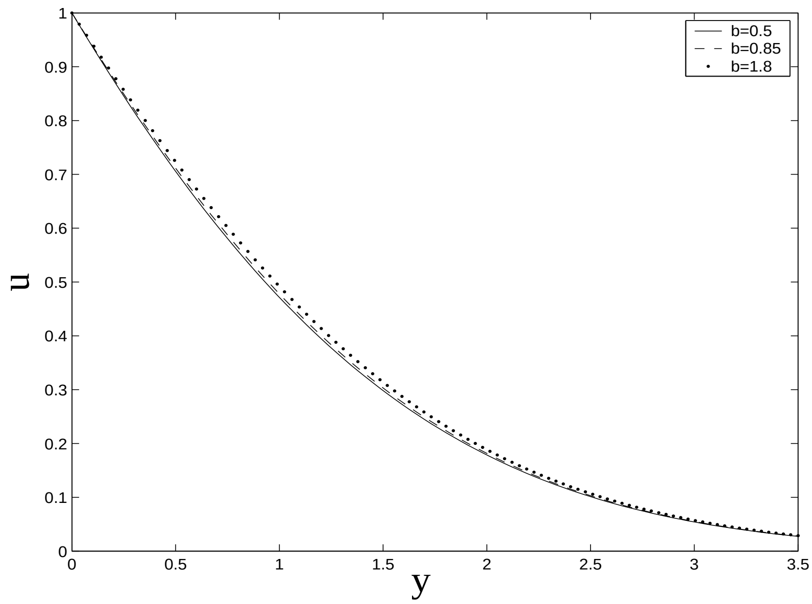

Figure 5.13: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying the parameter b in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 5$, with $t = 1, c = 0.3$

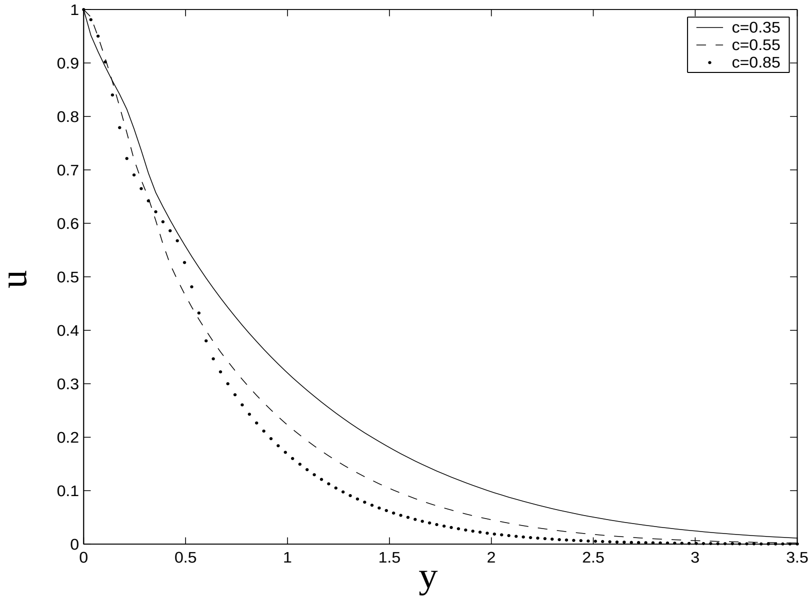

Figure 5.14: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying the parameter c in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 2$, with $t = 1, b = 0.5$

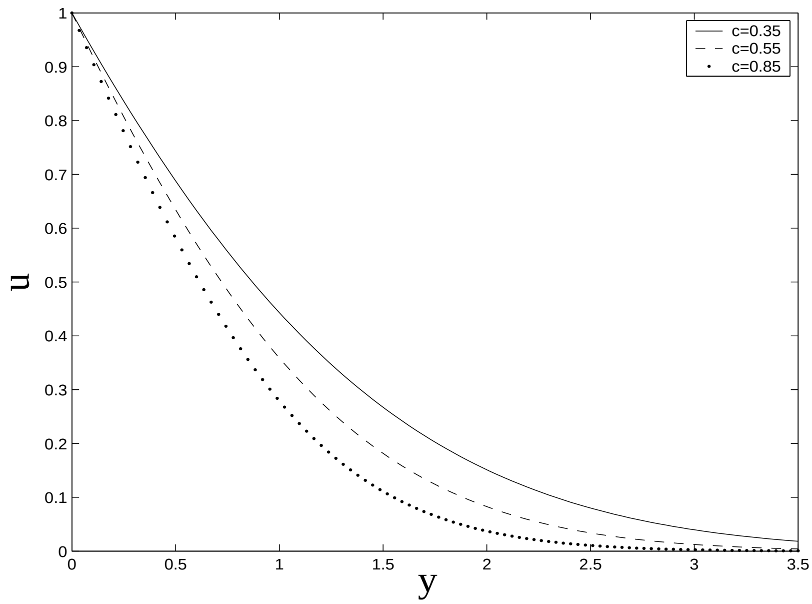

Figure 5.15: Similarity solution $u(y,t) = (1+2tc)^{1/2} f\left(y(1+2tc)^{-1/2}\right)$ varying the parameter c in the non-Newtonian case when there is no magnetic field and no porous space and the flow index is $n = 5$, with $t = 1, b = 0.5$

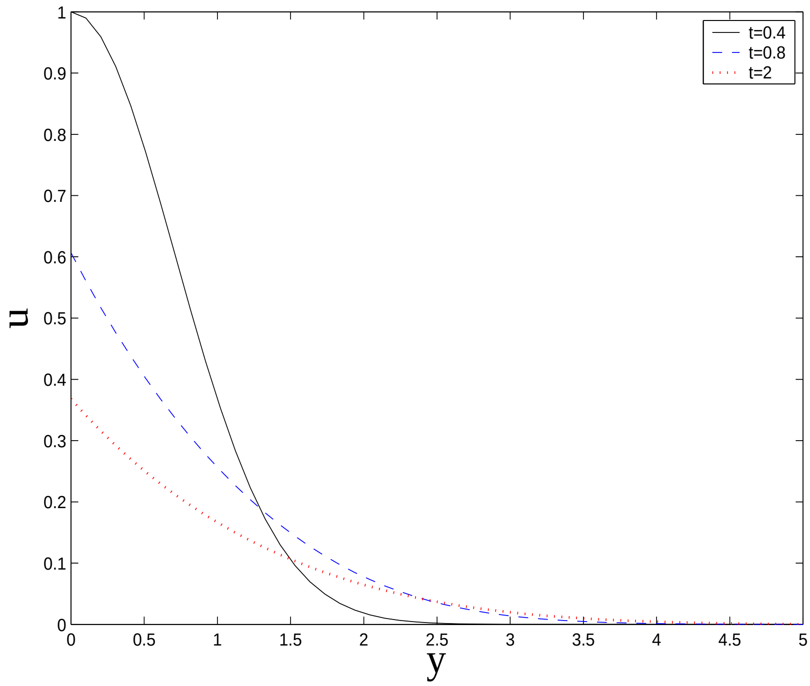

Figure 5.16: Numerical solution of the PDE (5.23) corresponding to the Newtonian case for various values of time t, with $K = 1.2$ and $M = 0.8$

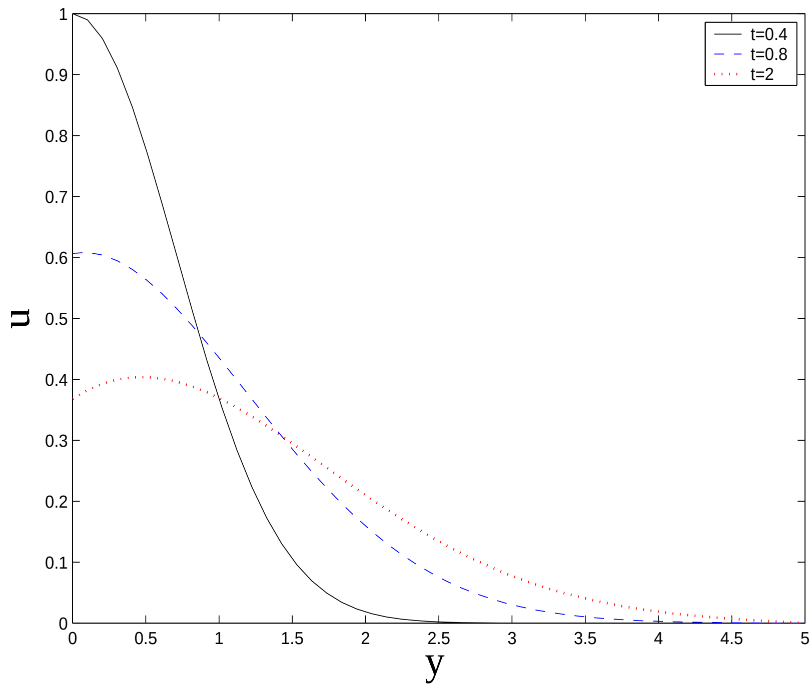

Figure 5.17: Numerical solution of the PDE (5.33) corresponding to the non-Newtonian case for various values of time t, with $n = 5, b = 0.5$

5.4 Exact solutions with magnetic field and porosity

We now look at the case when both the magnetic field and porosity effects are taken into account, that is we consider the general equation (5.22). Time-independent or the steady state solution of the form

$$u(y,t) = F(y) \tag{5.55}$$

are searched The substitution of Eq. (5.55) into Eq. (5.22) gives rise to the differential equation for $F(y)$, viz.

$$\frac{d}{dy}\left[\left(1+b\left(\frac{dF}{dy}\right)^{n-1}\right)\frac{dF}{dy}\right]-\frac{1}{K}\left[1+b\left(\frac{dF}{dy}\right)^{n-1}\right]F-M^2F=0, \tag{5.56}$$

where the relevant conditions are

$$F(0)=F_0, \quad F(\infty)=0, \quad h(y)=F(y), \tag{5.57}$$

in which F_0 is a constant. Here $V(t)$ is taken as constant and the Eq. (5.56) can be reduced to the form

$$\frac{F'+bn(F')^n}{\frac{b}{K}(F')^{n-1}+\frac{1}{K}+M^2}dF' = FdF, \tag{5.58}$$

where the prime indicates the differentiation with respect to y. For example, for $n=2$, we can deduce the first order equation

$$\left(-3K-4M^2K^2\right)\left(\frac{b}{K}F'+\frac{1}{K}+M^2\right)+\left(1+3M^2K+2M^4K\right)\ln\left(\frac{b}{K}F'+\frac{1}{K}+M^2\right)$$
$$+K^2\left(\frac{b}{K}F'+\frac{1}{K}+M^2\right)^2=\frac{b^2}{2K}F^2+A, \tag{5.59}$$

where A is a constant. We plot the solutions of Eq. (5.56) subject to (5.57) in figures 5.18 to 5.23 varying $b,\ K$ and M.

The travelling wave solutions of Eq. (5.22) are obtained by substituting Eq. (5.34) into Eq.(5.22). We obtain the second order equation

$$-cf'=f''+bn\left(f'\right)^{n-1}f''-\frac{1}{K}f-M^2f-\frac{b}{K}f\left(f'\right)^{n-1} \tag{5.60}$$

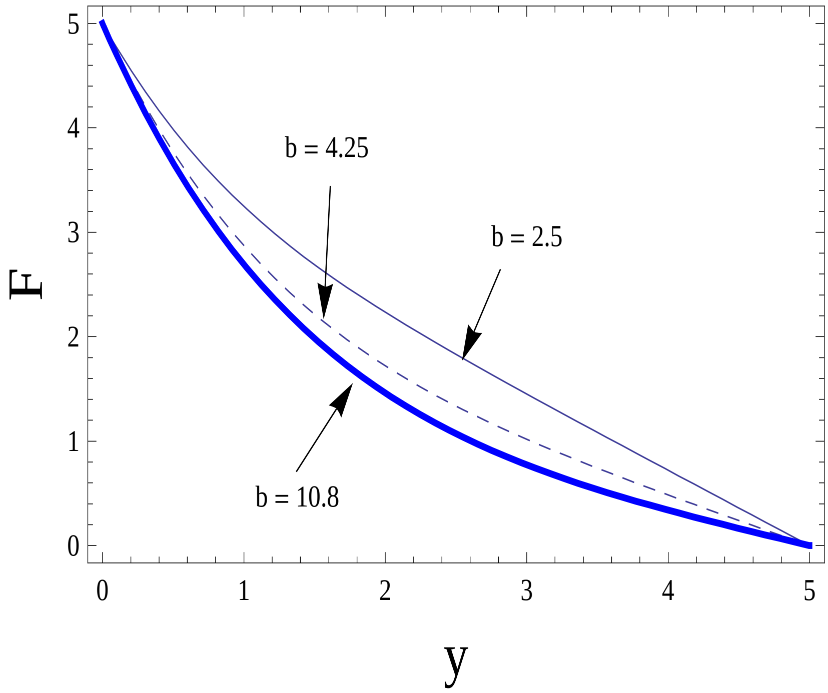

Figure 5.18: Steady state solution varying the parameter b taking into account the magnetic field and porosity when $n = 2$, with $M = 0.8, K = 1.2$

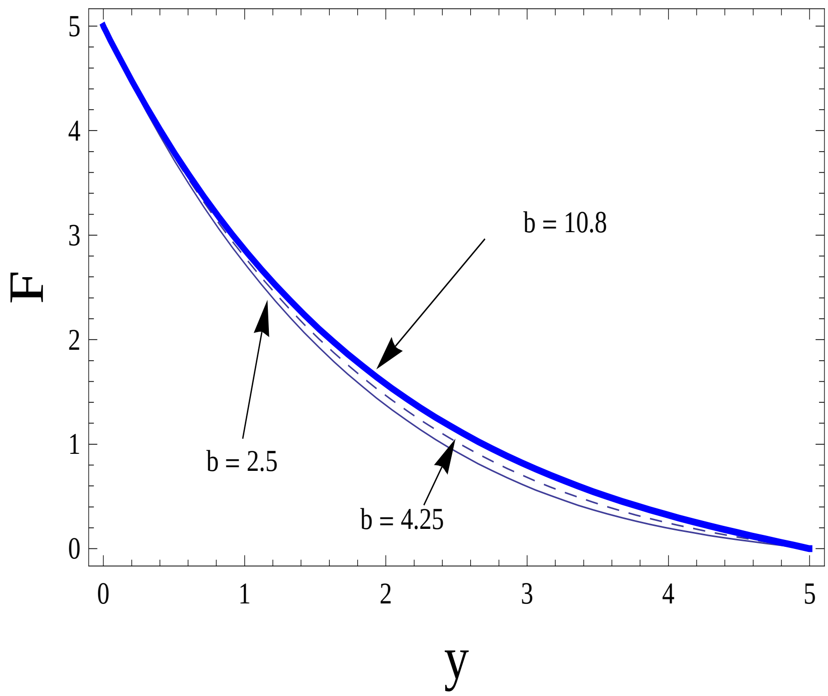

Figure 5.19: Steady state solution varying the parameter b taking into account the magnetic field and porosity when $n = 3$, with $M = 0.8, K = 1.2$

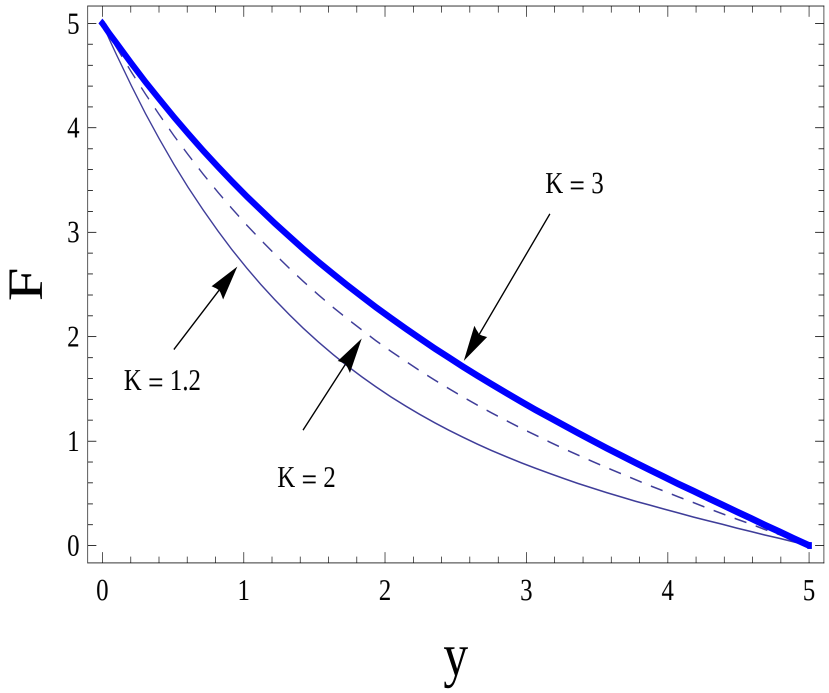

Figure 5.20: Steady state solution varying the parameter K taking into account the magnetic field and porosity when $n = 2$, with $M = 0.8, b = 10.8$

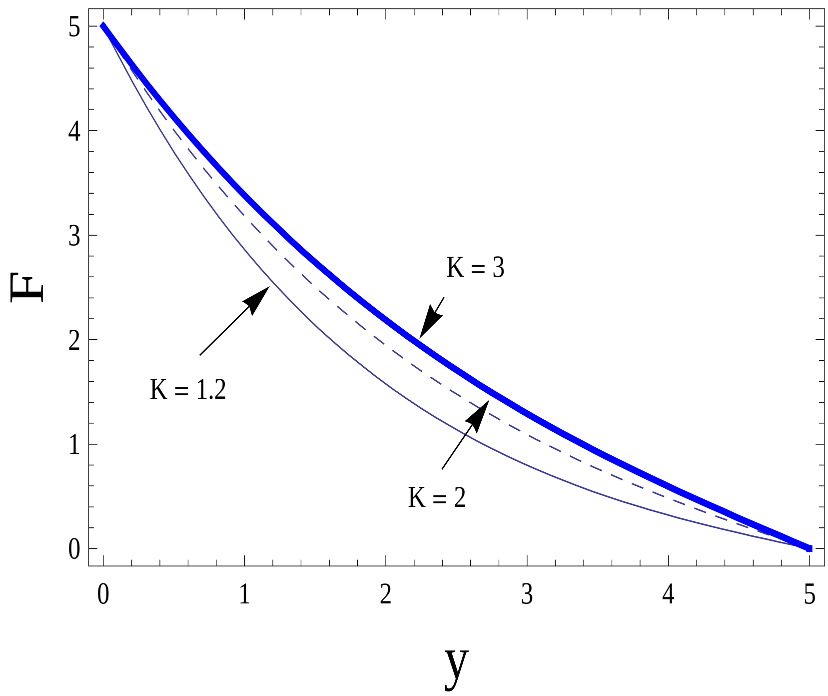

Figure 5.21: Steady state solution varying the parameter K taking into account the magnetic field and porosity when $n = 3$, with $M = 0.8, b = 10.8$

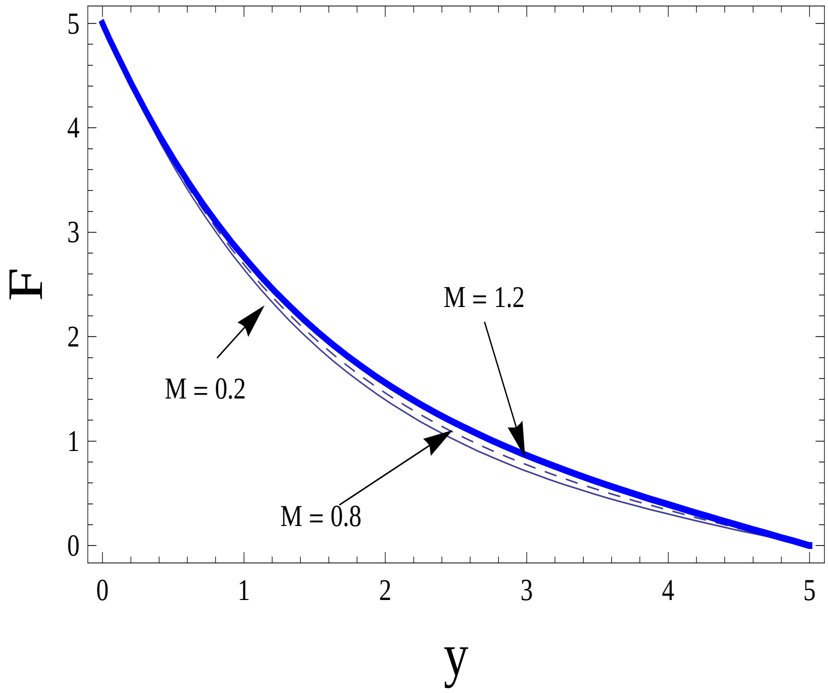

Figure 5.22: Steady state solution varying the parameter M taking into account the magnetic field and porosity when $n = 2$, with $K = 1.2, b = 10.8$

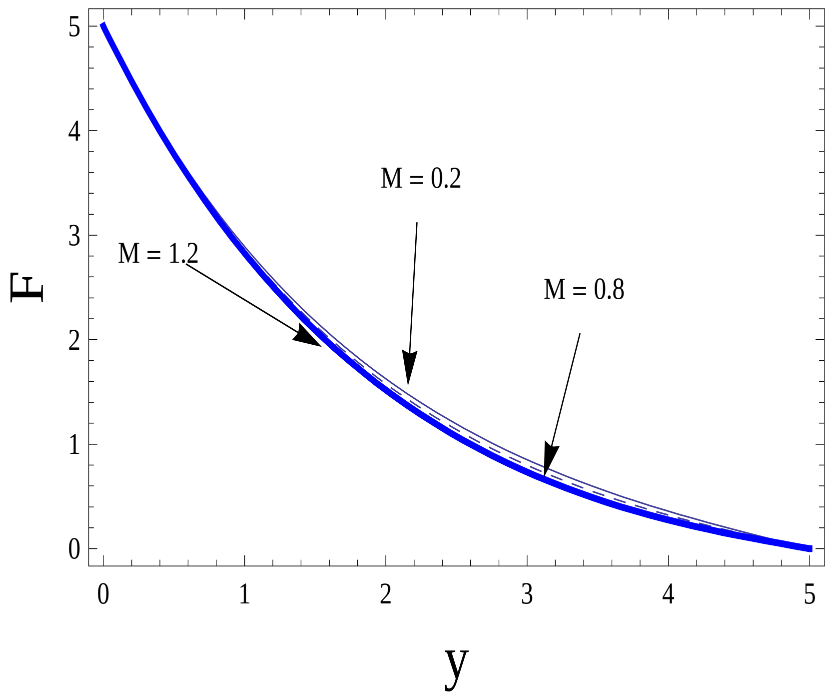

Figure 5.23: Steady state solution varying the parameter M taking into account the magnetic field and porosity when $n = 3$, with $K = 1.2, b = 10.8$

for $f(x_1)$, $x_1 = y - ct$. A solution of equation (5.60) can be obtained by looking at the form

$$f\left(x_1\right) = f_0 \exp\left(\Omega x_1\right). \tag{5.61}$$

This leads to the solution

$$f = f_0 \exp\left[\frac{1}{nK}\left(1 - n - nKM^2\right)t - \frac{y}{\sqrt{nK}}\right], \tag{5.62}$$

where f_0 is a constant. Thus, a solution of Eq. (5.22) which satisfies the boundary condition (5.21) with

$$V(t) = f_0 \exp\left[\frac{1}{nK}\left(1 - n - nKM^2\right)t\right], \quad h(y) = f_0 \exp\left(-\frac{y}{\sqrt{nK}}\right) \tag{5.63}$$

is given by

$$u = f_0 \exp\left[\frac{1}{nK}\left(1 - n - nKM^2\right)t - \frac{y}{\sqrt{nK}}\right]. \tag{5.64}$$

The solution (5.64) for various values of the emerging parameters are plotted in figures 5.24 to 5.29.

We present the numerical solution of Eq.(5.60) for different values of b, K and M in figures 5.30 to 5.32. The numerical solutions are plotted in the non-Newtonian case $n = 3$. In the Newtonian case $n = 2$, singularity occurs and the numerical solution could not be obtained.

Remark 1: The solutions (5.64) are valid when the magnetic field is zero. However, we cannot take the porosity to be zero. In this case (5.64) is not valid.

Remark 2: We now make a comment on solutions for which the velocity gradient is negative. For $n - 1$ even in equation (5.20), one still has the solutions as before. However for $n - 1$ odd, one needs to replace the parameter b by $-b$ in the aforementioned.

Remark 3: The numerical integration of the special case $n = 2$ resulted in some solutions having unusual behaviour. This is du to singularity presented by the equations at some point during the integration.

Finally, we obtain a numerical solution of Eqs. (5.21) and (5.22) for specific chosen values of $V(t)$ and $h(y)$. This solution is plotted in figures 5.33 and 5.34 using the Matlab solver

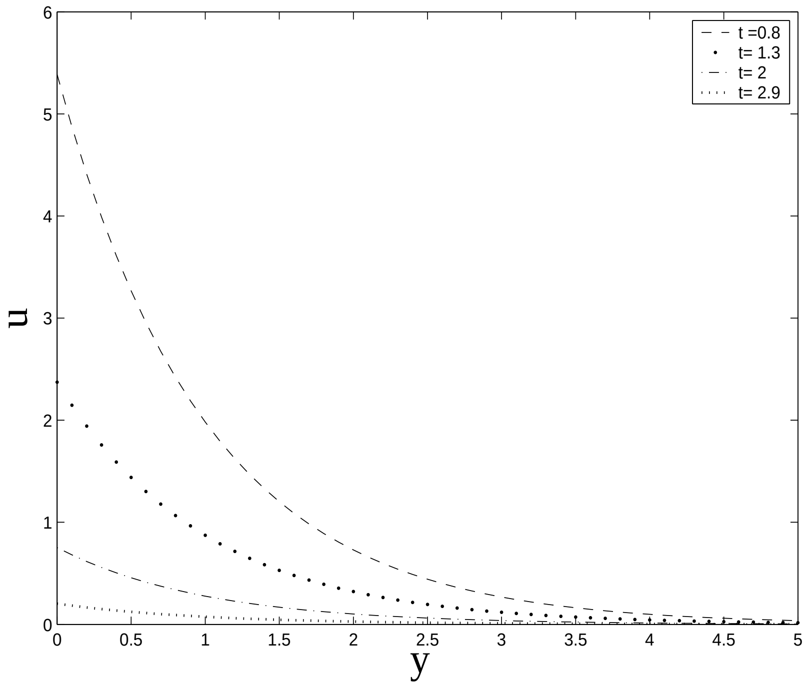

Figure 5.24: Travelling wave solution varying time t, taking into account the magnetic field and porosity when the flow exponent is $n = 2$, with $f_0 = 20, M = 0.8$ and $K = 0.5$

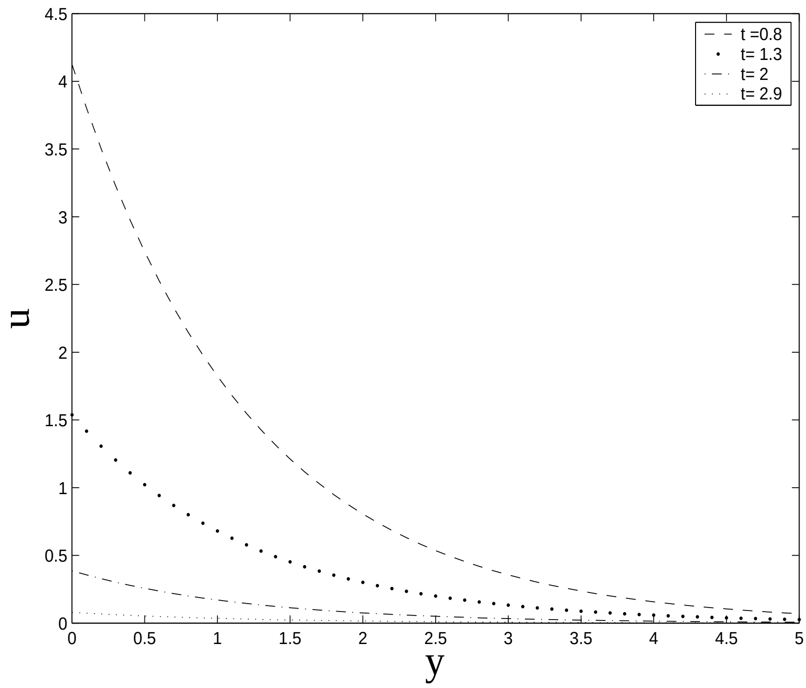

Figure 5.25: Travelling wave solution varying time t, taking into account the magnetic field and porosity when the flow exponent is $n = 3$, with $f_0 = 20, M = 0.8$ and $K = 0.5$

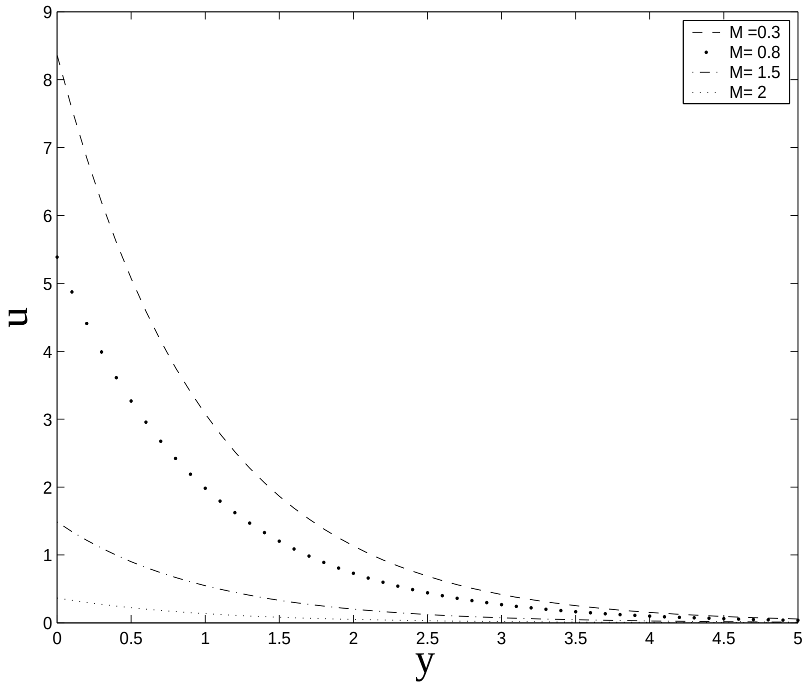

Figure 5.26: Travelling wave solution varying the parameter M, taking into account the magnetic field and porosity when the flow exponent is $n = 2$, with $f_0 = 20, t = 0.8$ and $K = 0.5$

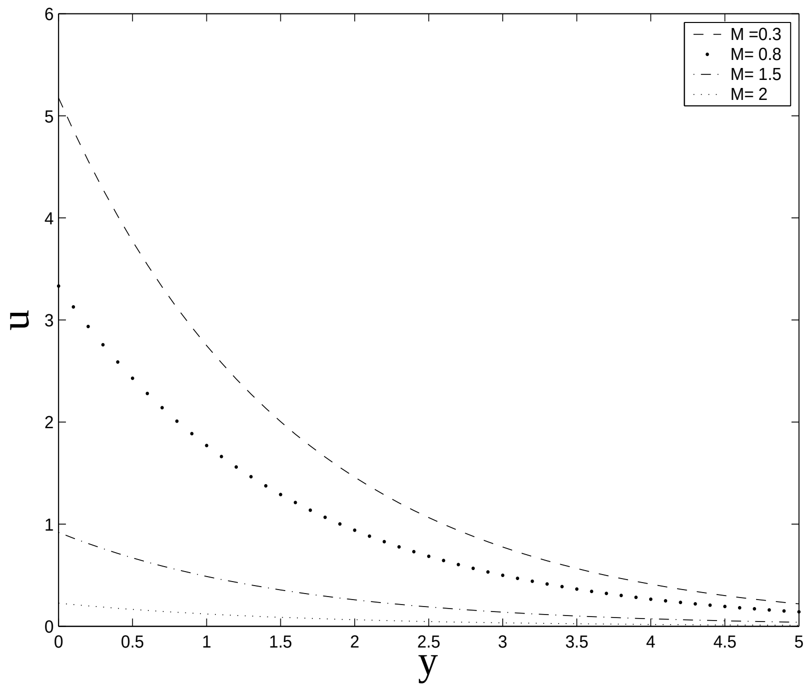

Figure 5.27: Travelling wave solution varying the parameter M, taking into account the magnetic field and porosity when the flow exponent is $n = 5$, with $f_0 = 20, t = 0.8$ and $K = 0.5$

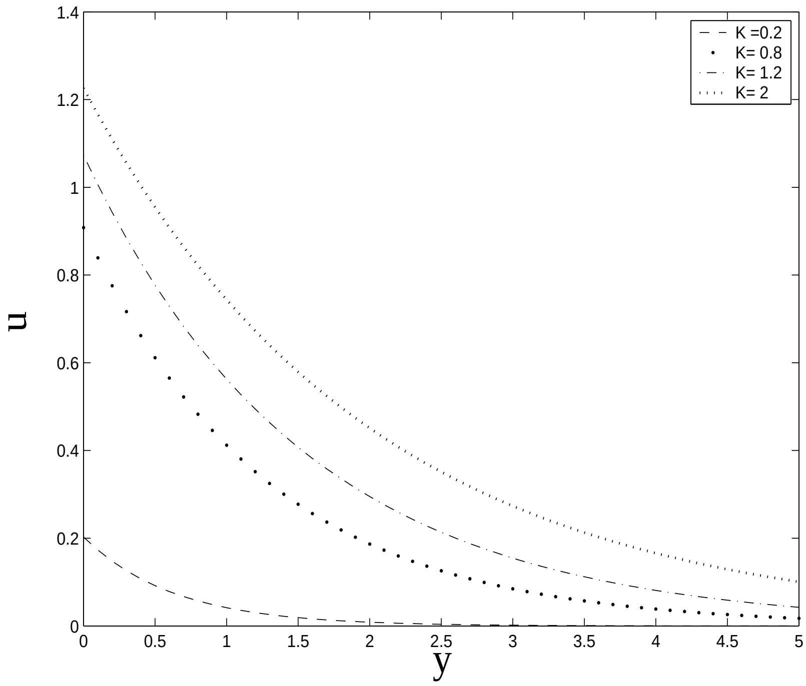

Figure 5.28: Travelling wave solution varying the parameter K, taking into account the magnetic field and porosity when the flow exponent is $n = 2$, with $f_0 = 20, t = 0.8$ and $M = 1.8$

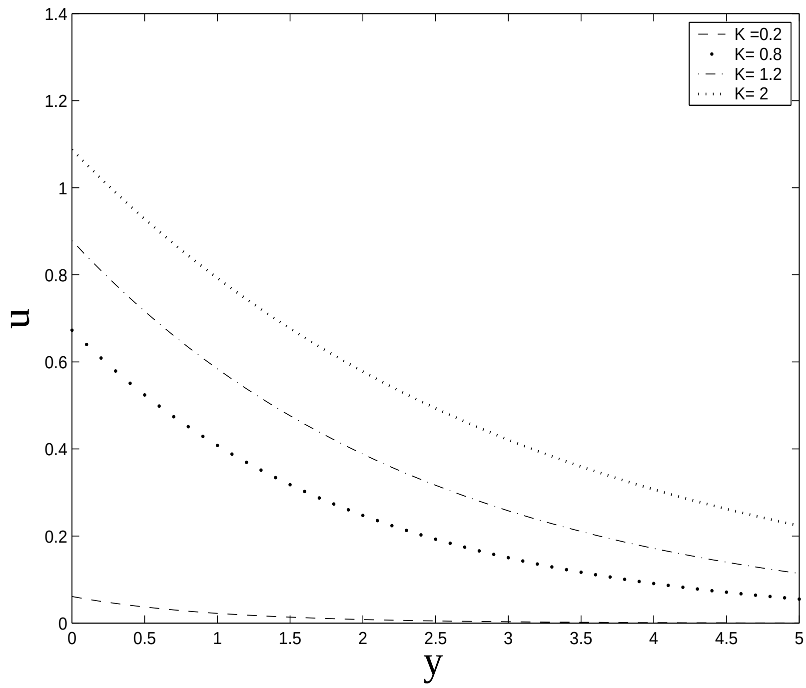

Figure 5.29: Travelling wave solution varying the parameter K, taking into account the magnetic field and porosity when the flow exponent is $n = 5$, with $f_0 = 20, t = 0.8$ and $M = 1.8$

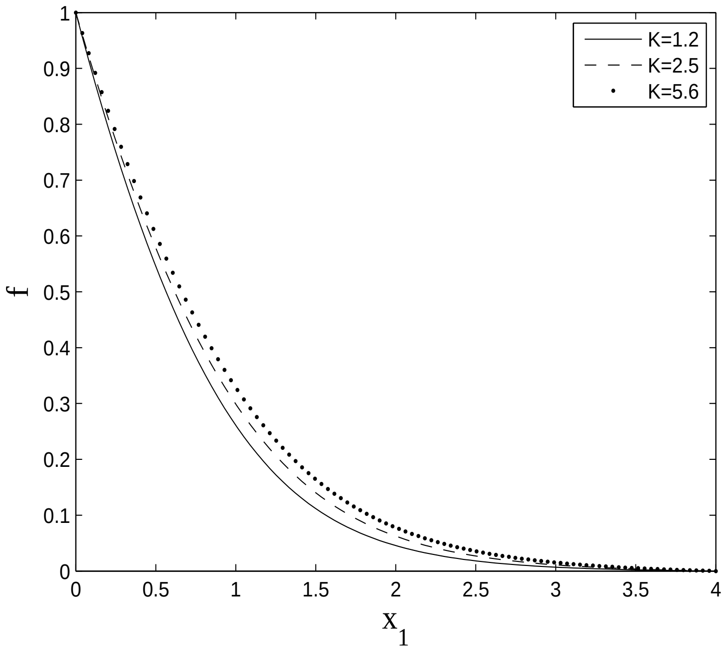

Figure 5.30: Numerical solution of the ODE (5.60) varying the parameter K when the flow exponent $n = 3$, with $c = 1, b = 1$ and $M = 0.8$

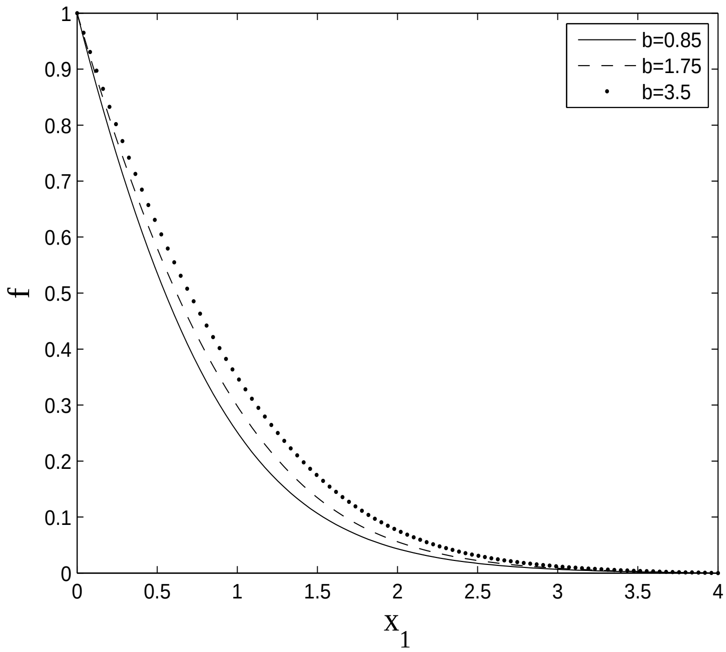

Figure 5.31: Numerical solution of the ODE (5.60) varying the the fluid parameter b when the flow exponent $n = 3$, with $c = 1, M = 0.8$ and $K = 1.2$

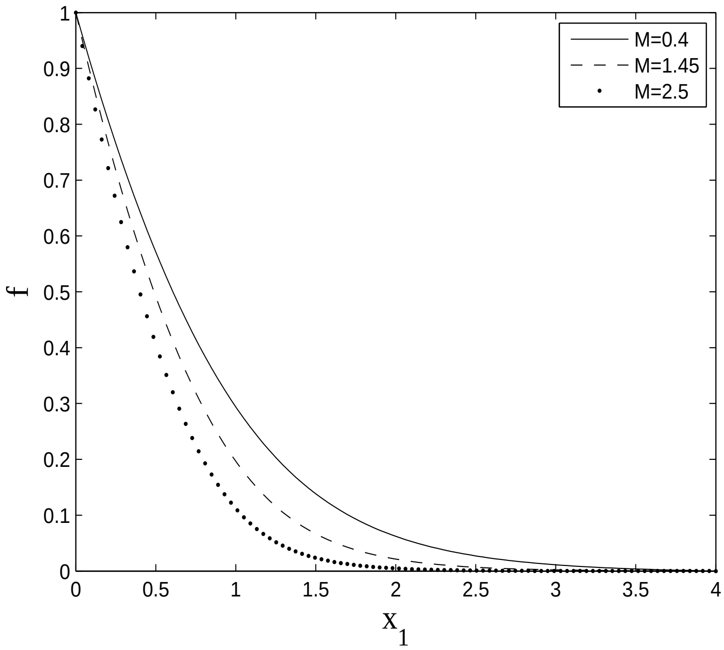

Figure 5.32: Numerical solution of the ODE (5.60) varying the parameter M when the flow exponent $n = 3$, with $c = 1, b = 1$ and $K = 1.2$

PDEPE, with $V(t) = \exp(-t)$, $h(y) = \exp(-y)$ and $U_0 = 1$.

It is seen form the figures 5.33 and 5.34 that the solution profile for the velocity has the same behaviour as the ones found analytically before in Eq. (5.64).

5.5 Results and discussion

In this chapter, we have investigated the unidirectional flow of a Sisko fluid filling the porous half-space. Equation (5.22) has been solved for similarity and numerical solutions. The solutions were first found for the case of a Newtonian fluid, then for the case of a non-Newtonian fluid where there is no magnetic field and porosity and finally when both magnetic field and porosity are taken into account.

In figures 5.1 to 5.3, we represented the solution (5.31) for the Newtonian case for different

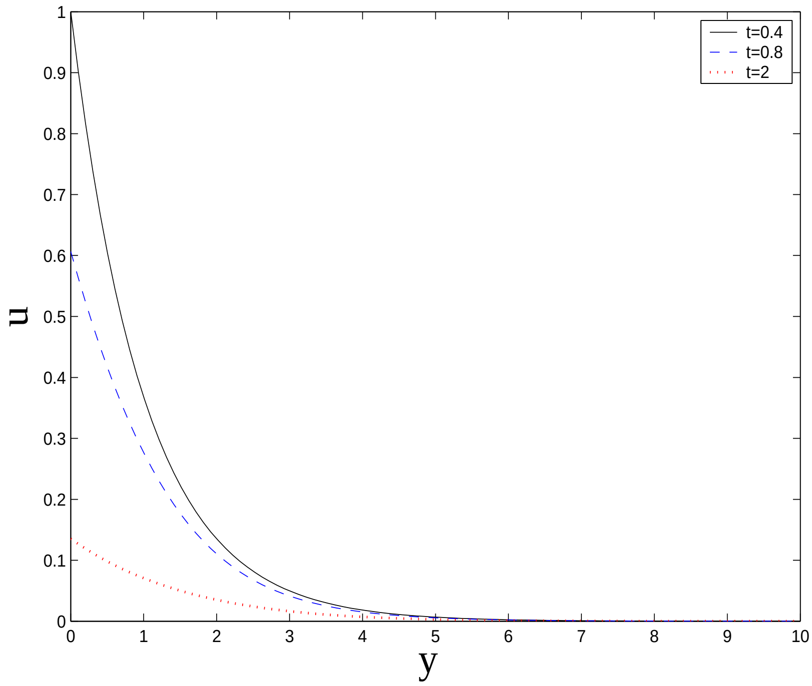

Figure 5.33: Numerical solution of the PDE corresponding to the Newtonian case when the magnetic field and porosity are taken into account by varying t, with n=1, b=0, M=0.8 and K=1.2

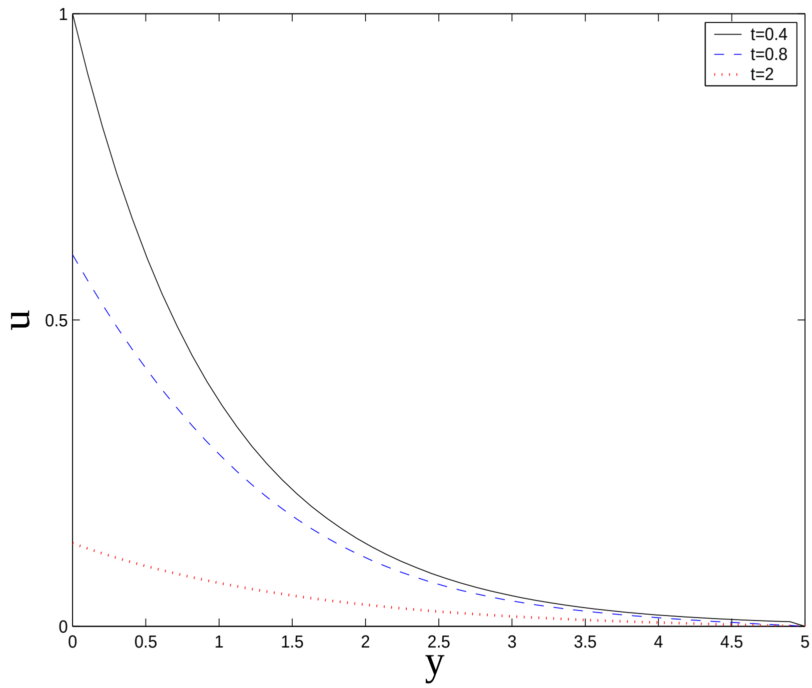

Figure 5.34: Numerical solution of the PDE corresponding to the non-Newtonian case when the magnetic field and porosity are taken into account by varying t, with n=5, b=0.5, M=0.8 and K=1.2

values of t, M and K. It is observed that the velocity is a decreasing function of time and of the parameter M, while it increases by increasing values of K. Since M is the ratio between the magnetic force and the viscous force, an increase in M is interpreted as the magnetic force taking over the viscous force, which results in the decrease of the velocity. To depict the influence of the parameters on the travelling wave solution (5.34), figures 5.4 and 5.5 were plotted for the case $n = 3$. It can be seen in both cases that the velocity decreases by increasing the wave speed c, while on the other hand the velocity increases for large values of b. Further we present similarity solutions in figures 5.6 to 5.15. The study was made on the effect of time t and of the constant parameters b and c. For each varying parameter, we looked at both cases $n = 2$ and $n \neq 2$. It is noted that the flow decreases for increasing t. Figures 5.8, 5.9, 5.12 and 5.13 show that the velocity increases for large values of b, although the corresponding variation of the flow is not very significant in the case when n is not equal to two. The effect of the wave speed c is presented in figures 5.14 and 5.15. Here the velocity decreases as c increases. Moreover, in figures 5.16 and 5.17 we plotted numerically the velocity varying the time t and it is observed that the velocity decreases with time.

Furthermore, we made a similar analysis for the case when the magnetic field and the porosity are not neglected. Figures 5.18 to 5.34 representing the numerical and analytical solutions are plotted. The variation of the constant parameters are studied on each of the solutions. Figures 5.18 to 5.21 represent the numerical solutions of Eq. (5.56) by varying b, K and M. In the special case $n = 2$, the velocity decreases with increasing b whereas the opposite effect is observed when $n \neq 2$. In figures 5.20 and 5.21, it can be seen that an increase in the value of K causes the velocity to increase. The effect of M is shown in figures 5.22 and 5.23. These figures depict that the velocity increases for large values of M when $n = 2$ and decreases for large values of M when $n \neq 2$. The solution (5.64) is presented in figures 5.24 to 5.29 for various parameters. The parameters t and M have the same effect on the variation of the velocity as they both reduce the velocity when increased. On the other end when K increases, the velocity increases as well. The numerical solution of Eq. (5.60) is plotted in figures 5.30 to 5.32. The variation of b is shown in figures 5.31. As observed earlier, the velocity varies slightly for different values of b. The parameters K and M in figures 5.30 to 5.32 depict opposite role on the velocity profile; while large values of K causes the velocity to increase and the increase of M reduces the velocity profile.

Finally, in figures 5.33 and 5.34 we plotted the numerical solutions of the PDE (5.22) for Newtonian and non-Newtonian cases by varying the time t. A similar observation is made on the velocity profile, as u decreases by increasing t.

Chapter 6

A note on some solutions for the flow of a fourth grade fluid in a porous space

The main goal of the present chapter is to discuss the time-dependent unidirectional flow of a fourth grade fluid. The hydrodynamic fluid occupies the porous half-space. The modified Darcy's law proposed by Hayat et al. [35] in a fourth grade fluid is used in the problem formulation. Analytical solutions of the arising non-linear problem is obtained. Travelling wave and conditional symmetry solutions are deduced. The physical interpretation of the obtained results is made through graphs.

6.1 Flow development

Consider a cartesian coordinate system with the x-axis taken parallel to the plate and the y-axis in the vertically upward. The rigid plate coincides with the plane $y = 0$ and the incompressible fourth grade fluid fills the porous half-space $y > 0$. For unidirectional flow, the velocity field **V** is defined by

$$\mathbf{V} = (u(y,t), 0, 0)\,, \tag{6.1}$$

where u is the velocity in the x-direction and t is the time. It should be noted that Eq. (6.1) satisfies the incompressibility condition. The equation of motion is ([56], [25], [39], [38] and [35])

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \text{div}\mathbf{S} + \mathbf{R}, \tag{6.2}$$

in which ρ is the fluid density, p is the pressure, $\frac{D}{Dt}$ is the material derivative and **R** is the Darcy resistance.

The expression of extra stress tensor **S** is given by equations (1.17)-(1.20) as

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \mathbf{S}_1 + \mathbf{S}_2, \tag{6.3}$$

$$\mathbf{S}_1 = \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2\right) + \beta_3\left(\text{tr}\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{6.4}$$

$$\mathbf{S}_2 = \gamma_1\mathbf{A}_4 + \gamma_2\left(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3\right) + \gamma_3\mathbf{A}_2^2 + \gamma_4\left(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2\right)$$

$$+\gamma_5\left(\text{tr}\mathbf{A}_2\right)\mathbf{A}_2 + \gamma_6\left(\text{tr}\mathbf{A}_2\right)\mathbf{A}_1^2 + \left(\gamma_7\text{tr}\mathbf{A}_3 + \gamma_8\text{tr}(\mathbf{A}_2\mathbf{A}_1)\right)\mathbf{A}_1. \tag{6.5}$$

In the above equations, μ is the dynamic viscosity, $\alpha_i\ (i = 1, 2)$, $\beta_i\ (i = 1 - 3)$ and $\gamma_i\ (i = 1 - 8)$ are the material constants. Furthermore, the kinematical tensors $\mathbf{A}_n$ are given through the following relations [51] (cf. Chapter 1)

$$\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T, \tag{6.6}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_{n-1}, \quad n > 1, \tag{6.7}$$

in which T is the transpose.

Adopting the same procedure as in ref. ([55] and [56] for Oldroyd-B, Maxwell and second grade fluids, the x-component of the Darcy resistance r_x in the present case of fourth grade fluid is (cf. Chapter 1):

$$r_x = -\Bigg[\mu + \alpha_1\frac{\partial}{\partial t} + \beta_1\frac{\partial^2}{\partial t^2} + 2(\beta_2 + \beta_3)\left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1\frac{\partial^3}{\partial t^3}$$

$$+(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\Bigg]\frac{\phi u}{k}, \tag{6.8}$$

where Φ and k are the porosity and permeability of the porous space, respectively.

Upon making use of Eqs. (6.1) and (6.3)-(6.8), Eq. (6.2) gives

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+\beta_1\frac{\partial^4 u}{\partial y^2\partial t^2}+6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}$$
$$+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]$$
$$-\left[\mu+\alpha_1\frac{\partial}{\partial t}+\beta_1\frac{\partial^2}{\partial t^2}+2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\right.$$
$$\left.+\gamma_1\frac{\partial^3}{\partial t^3}+(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\right]\frac{\phi u}{k}. \tag{6.9}$$

The boundary and initial conditions are

$$u(0,t)=u_0V(t),\ \ t>0, \tag{6.10}$$

$$u(\infty,t)=0,\ \ t>0, \tag{6.11}$$

$$u(y,0)=f(y),\ y>0, \tag{6.12}$$

$$\frac{\partial u}{\partial t}(y,0)=g(y),\frac{\partial^2 u}{\partial t^2}(y,0)=h(y),\ \ y>0, \tag{6.13}$$

where u_0 is a reference velocity and f, g and h are functions yet to be determined.

6.2 Solutions of the problem

Equation (6.9) can be rewritten as

$$\frac{\partial u}{\partial t}=\mu_*\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial^3 u}{\partial y^2\partial t}+\beta\frac{\partial^4 u}{\partial y^2\partial t^2}+\beta_*\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\gamma\frac{\partial^5 u}{\partial y^2\partial t^3}+\gamma_*\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]$$
$$-\phi_1 u-\phi_2\frac{\partial^2 u}{\partial t^2}-\phi_3 u\left(\frac{\partial u}{\partial y}\right)^2-\phi_4\frac{\partial^3 u}{\partial t^3}-\phi_5 u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}\ , \tag{6.14}$$

whence

$$\mu_*=\frac{\mu}{\rho+\alpha_1\dfrac{\phi}{k}},\ \ \alpha=\frac{\alpha_1}{\rho+\alpha_1\dfrac{\phi}{k}},\ \ \beta=\frac{\beta_1}{\rho+\alpha_1\dfrac{\phi}{k}},\beta_*=\frac{6(\beta_2+\beta_3)}{\rho+\alpha_1\dfrac{\phi}{k}},\ \ \gamma=\frac{\gamma_1\dfrac{\phi}{k}}{\rho+\alpha_1\dfrac{\phi}{k}},$$

$$\gamma_* = \frac{(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8)}{\rho + \alpha_1 \dfrac{\phi}{k}},$$

$$\phi_1 = \frac{\mu \dfrac{\phi}{k}}{\rho + \alpha_1 \dfrac{\phi}{k}}, \quad \phi_2 = \frac{\beta_1 \dfrac{\phi}{k}}{\rho + \alpha_1 \dfrac{\phi}{k}}, \quad \phi_3 = \frac{2(\beta_2 + \beta_3) \dfrac{\phi}{k}}{\rho + \alpha_1 \dfrac{\phi}{k}}, \quad \phi_4 = \frac{\gamma_1 \dfrac{\phi}{k}}{\rho + \alpha_1 \dfrac{\phi}{k}},$$

$$\phi_5 = \frac{(6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8) \dfrac{\phi}{k}}{\rho + \alpha_1 \dfrac{\phi}{k}}.$$

6.2.1 Conditional symmetry solutions

We consider the PDE (6.14) with the invariant surface condition

$$u_t = \lambda u\,, \tag{6.15}$$

where λ is a constant to be found. This corresponds to the operator

$$X = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}\,. \tag{6.16}$$

The invariant solution corresponding to (6.16) is

$$u = u_0 \exp(\lambda t) F(y), \tag{6.17}$$

where u_0 is a constant and $F(y)$ is yet an undetermined function of y. Substituting Eq. (6.17) into Eq. (6.14) leads to the following quadratic equation in λ

$$(\beta\phi_4 - \gamma\phi_2)\,\lambda^2 + [\phi_4\,(\alpha - 3\gamma_*) + \gamma\,(\phi_5 - 1)]\,\lambda + [\phi_4\,(\mu_* - \beta_*) + \gamma\,(\phi_3 - \phi_1)] = 0 \tag{6.18}$$

and the linear second-order ordinary differential equation in F given below

$$(\beta_* + 3\gamma_*\lambda)\,F'' - (\phi_3 + \phi_5\lambda)\,F = 0. \tag{6.19}$$

We can rewrite Eq. (6.19) in the following form

$$F'' - \Theta F = 0, \tag{6.20}$$

where

$$\Theta = \frac{\phi_3 + \phi_5\lambda}{\beta_* + 3\gamma_*\lambda} \tag{6.21}$$

Through conditions (6.10) - (6.11) one can write the boundary conditions of (6.19) as

$$F(0) = 1, \quad F(l) = 0,\ l \to \infty, \tag{6.22}$$

where

$$V(t) = \exp(\lambda t). \tag{6.23}$$

We solve equation (6.18) for λ and consider two cases.

Case 1 : $\beta\phi_4 - \gamma\phi_2 = 0$

In the case $\beta\phi_4 - \gamma\phi_2 = 0$, we get

$$\lambda = \frac{\phi_2\left(\mu_* - \beta_*\right) + \beta\left(\phi_3 - \phi_1\right)}{\phi_2\left(\alpha - 3\gamma_*\right) + \beta\left(\phi_5 - 1\right)}. \tag{6.24}$$

We then solve (6.20) subject to (6.22) for positive Θ and get

$$u(y,t) = u_0 \exp\left(\lambda t\right) \exp\left(-y\sqrt{\frac{\phi_3 + \phi_5\lambda}{\beta_* + 3\gamma_*\lambda}}\right), \tag{6.25}$$

where λ is given by (6.24). This solution is plotted in figures 6.1 - 6.4, varying time t and some parameters.

Case 2 : $\beta\phi_4 - \gamma\phi_2 \neq 0$

In this case, we solve the quadratic equation (6.18) and obtain

$$\lambda = \frac{\phi_4\left(\alpha - 3\gamma_*\right) + \gamma\left(\phi_5 - 1\right)}{2(\beta\phi_4 - \gamma\phi_2)}\left(-1 + \sqrt{1 - \frac{4(\beta\phi_4 - \gamma\phi_2)[\phi_4(\mu_* - \beta_*) + \gamma(\phi_3 - \phi_1)]}{[\phi_4(\alpha - 3\gamma_*) + \gamma(\phi_5 - 1)]^2}}\right), \tag{6.26}$$

or

$$\lambda = \frac{\phi_4\left(\alpha - 3\gamma_*\right) + \gamma\left(\phi_5 - 1\right)}{2(\beta\phi_4 - \gamma\phi_2)}\left(-1 - \sqrt{1 - \frac{4(\beta\phi_4 - \gamma\phi_2)[\phi_4(\mu_* - \beta_*) + \gamma(\phi_3 - \phi_1)]}{[\phi_4(\alpha - 3\gamma_*) + \gamma(\phi_5 - 1)]^2}}\right).$$

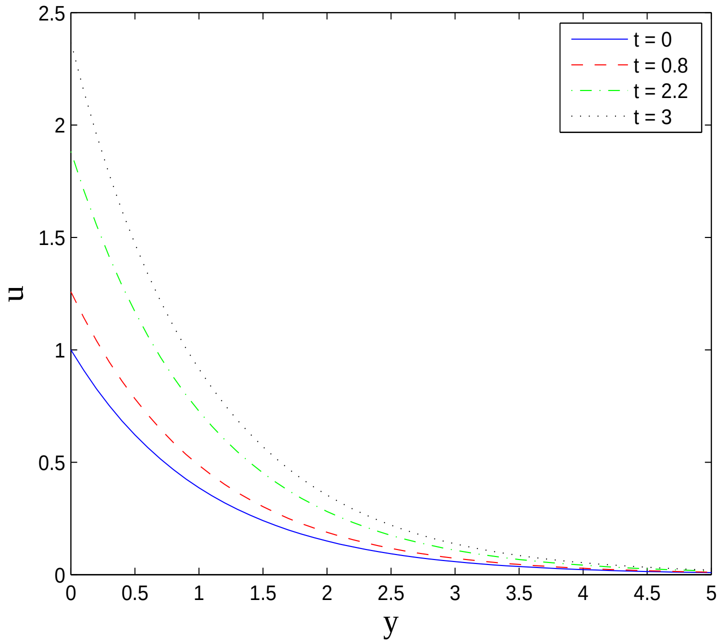

Figure 6.1: Analytical solution (6.25) varying time t, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ \mu_* = 1,\ \alpha = 3.8,\ \beta = 1.5,\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

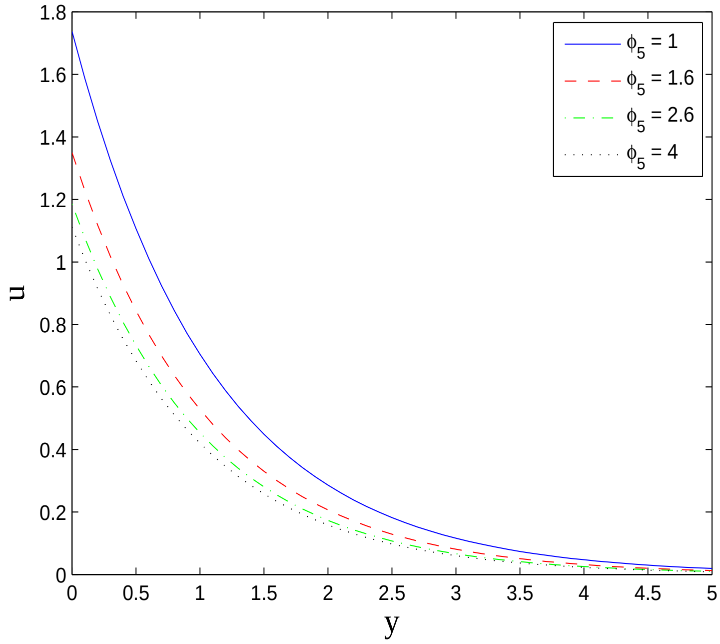

Figure 6.2: Analytical solution (6.25) varying ϕ_5, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ t = 0.8,\ \mu_* = 1,\ \alpha = 3.8,\ \beta = 1.5,\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

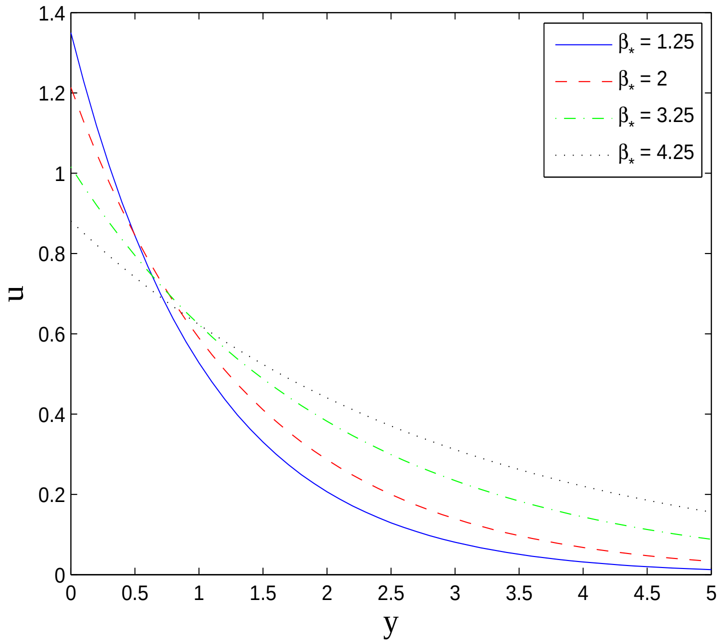

Figure 6.3: Analytical solution (6.25) varying β, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ \mu_* = 1,\ \alpha = 3.8,\ t = 0.8\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

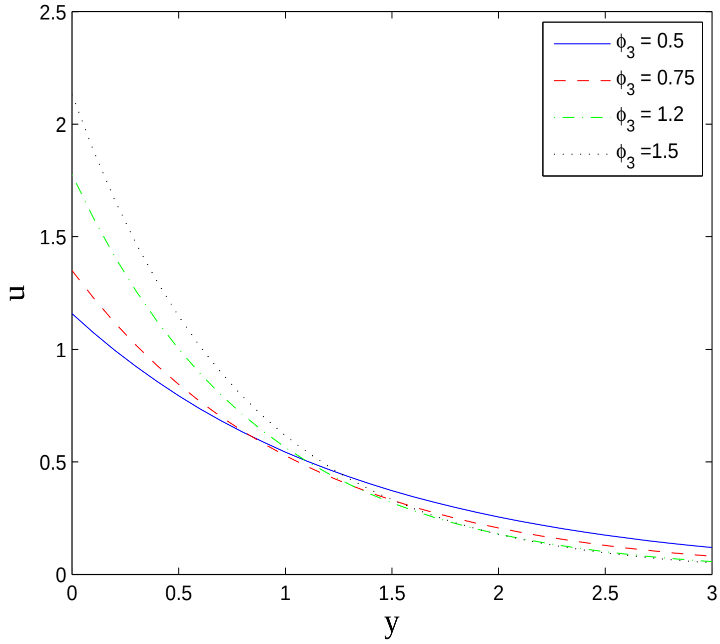

Figure 6.4: Analytical solution (6.25) varying ϕ_3, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ t = 0.8,\ \mu_* = 1,\ \alpha = 3.8,\ \beta = 1.5,\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

$$(6.27)$$

It follows that Eq. (6.14) subject to (6.10) - (6.11) admits the solution

$$u(y,t) = u_0 \exp(\lambda t) \exp\left(-y\sqrt{\frac{\phi_3 + \phi_5\lambda}{\beta_* + 3\gamma_*\lambda}}\right), \tag{6.28}$$

where λ is given by (6.26) or (6.27).

Note that we can choose the fluid parameters in such a way that Θ is positive. The solutions (6.28) are plotted for positive Θ in figures 6.5 and 6.6.

6.3 Numerical solution of the PDE (6.14)

In this section we present a numerical solution of the partial differential equation (6.14) subject to (6.10)-(6.13), when $V(t) = e^t,\ f(y) = e^{-y},\ g(y) = e^{-5y}$ and $h(y) = e^{-10y}$, using Mathematica solver NDSolve.

6.4 Summary and Conclusions

In order to see the influence of emerging parameters in the solution (6.14), we display the figures 6.1 - 6.7. Here main emphasis is given to the variations of t, β_*, ϕ_3 and ϕ_5.

Figure 6.1 describes the variation of t on the velocity u. It can be seen that u is a increasing function of t. Further, the variation of ϕ_5 is plotted in Figure 6.2. It elucidates that the behavior of ϕ_5 is quite opposite to that of t i-e u is an decreasing function of ϕ_5. Moreover the effect of the parameters β_* and ϕ_3 are displayed in Figures 6.3 and 6.4. It can be observed that β_* and ϕ_3 have an opposite effect on the velocity. The velocity first decreases up to a certain point and then increases for large values of ϕ_4 whereas it decreases by increasing and then decreases while increasing ϕ_3.

Further the Numerical solution (6.14) is plotted in Figures 6.5. In Figure 6.5, the velocity increases for large values of time. Furthermore it is depicted that the solution oscillates.

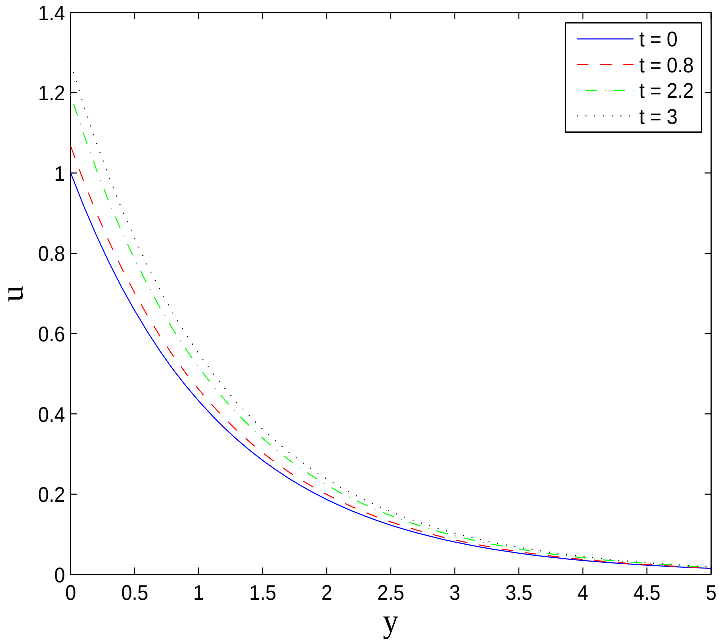

Figure 6.5: Analytical solution (6.25) with λ given by (6.26), varying time t, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ \mu_* = 1,\ \alpha = 3.8,\ \beta = 1.5\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

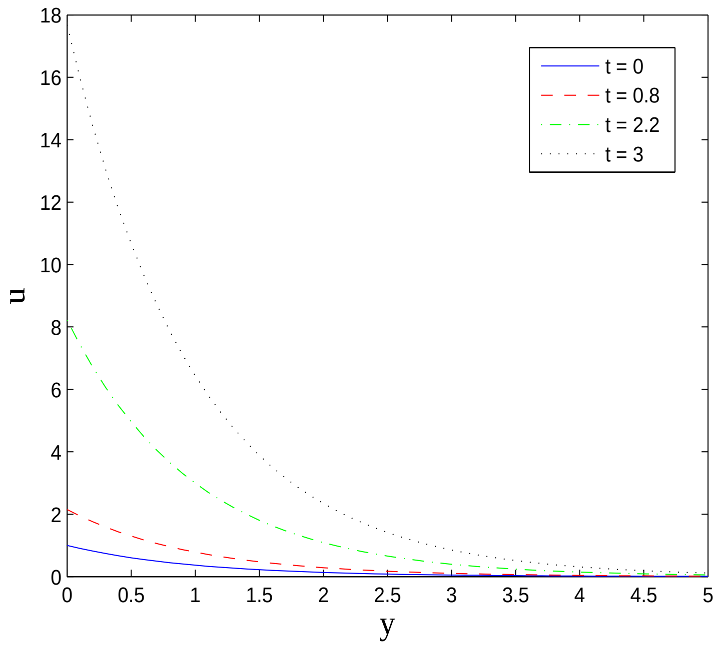

Figure 6.6: Analytical solution (6.25) with λ given by (6.27), varying time t, when $\phi_1 = 0.2,\ \phi_2 = 0.35,\ \phi_3 = 0.75,\ \phi_3 = 0.75,\ \mu_* = 1,\ \alpha = 3.8,\ \beta = 1.5,\ \beta_* = 1.25,\ \gamma = 2$ and $\gamma_* = 0.25$

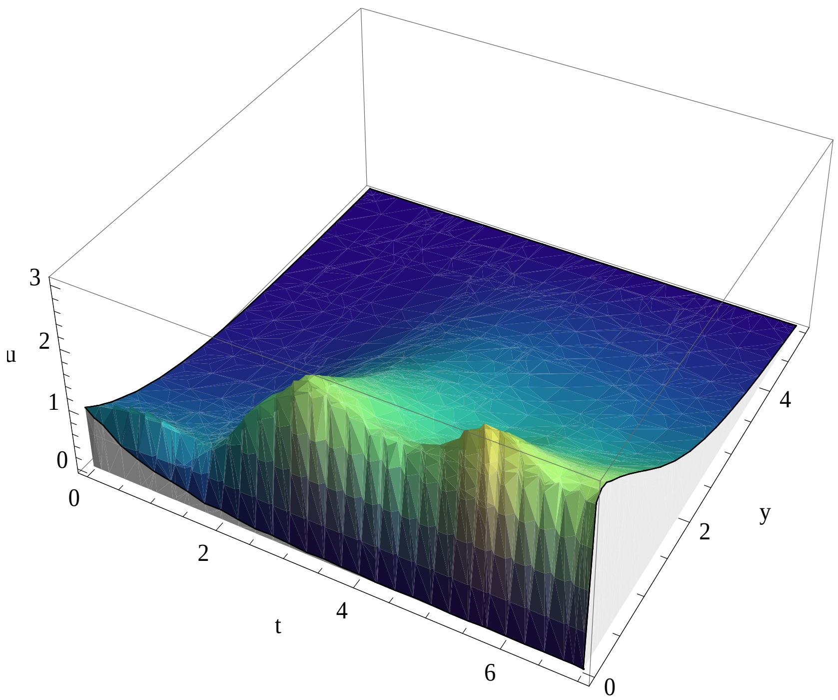

Figure 6.7: Numerical solution of Eq. (6.14), when $\alpha = 0.1,\ \beta = 0.1,\ \beta_* = 0.1,\ \gamma = 0.1,\ \gamma_* = 0.1,\ \phi_1 = 0.1,\ \phi_2 = 0.1,\ \phi_3 = 0.1,\ \phi_4 = 0.1,\ \phi 5 = 0.1,$ and $\mu_* = 0.1$

Conclusions

The goal of this thesis was to construct mathematical models and solutions to some problems arising in the study of non-Newtonian fluid with applied magnetic field. In our analysis, we considered the differential type model to study the Rayleigh problem in Chapter 2, the unsteady flow of fourth grade fluid in Chapter 3, while Chapter 4 dealt with some flow of third grade fluids on a porous media. In Chapter 5 we looked at the Sisko fluid on a porous plate and we ended in Chapter 6 with the flow of fourth grade fluid in a porous medium.

The resulting high order non-linear partial differential equations were analysed and solved using reduction methods, namely symmetry approaches as well as numerically. Symmetry methods are very powerful for solving nonlinear differential equations and some of the most important properties of symmetries is that they can be used to derive new solutions from known ones. For ordinary differential equations, they can be used to reduce the order and for partial differential equations symmetry can be used to reduce the number of independent variables. Throughout the thesis the governing PDEs were reduced to ODEs and in most of the cases the resulting ODEs could not be solved analytically. We then had to resort to numerical methods to solve the resultant problems. The resulting PDEs were also solved numerically using the powerful solvers NDSOLVE in Mathematica and PDEPE in Matlab. In both analytical and numerical techniques a study of the effect of non-Newtonian parameters and of the magnetic field was conducted and it was observed that the solutions depended strongly on the rheological parameters. A comparison of both techniques was conducted and the results are summerised as follows.

In Chapter 2, the analysis on third grade fluid with applied magnetic field was presented. The mathematical modelling led to a nonlinear third order PDE, which was reduced using symmetry. A form of the magnetic field was obtained and then used to solve the resulting ODEs both analytically and numerically. Furthermore, the original PDE was solved numerically and compared to the solutions obtained by the use of symmetries. The observation revealed that the velocity profile was similar in both cases. Moreover, the effects of emerging fluid parameters were studied and it was noted that the increase of the Hartmann number reduces the magnitude of the velocity whereas an increase of the dynamic viscosity forces the diffusion of the velocity and vorticity to increase.

Secondly in Chapter 3, we looked at the effect of a magnetic field on an electrically conducting fourth grade flow. The governing equation was a fifth order nonlinear PDE and travelling wave solutions were derived and analysed in terms of the non-Newtonian parameters. Other solutions were found using conditional symmetry. We also noted that the magnetic field strength H depended strongly on the rheological parameters $\beta(= 6(\beta_2 + \beta_3))$ and $\gamma(= 6\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_7 + 2\gamma_8)$. The variation of the material parameters β and γ revealed that the increase of β (γ) causes the velocity to decrease (increase). The effect of the magnetic field and Hartmann number were then studied numerically and it was depicted that they both force the flow velocity to decrease.

Further, in Chapter 4, a study was made on the solution in a third grade fluid filling a porous space. We obtained a third order nonlinear PDE. Group invariant solutions were found and what was noticeable was the influence of the parameters γ_1 and γ_2 on the velocity. It could be seen that an increase of γ_1 causes the flow velocity to increase, while γ_2 creates the opposite effect on the velocity.

Furthermore, a MHD flow of a Sisko fluid in a porous medium was analysed in Chapter 5. The resulting PDE was second order nonlinear and it was reduced by the means of the similarity approach. The solutions of this problem was first found for the Newtonian case, then in the non-Newtonian case with no porous half-space and no magnetic field and finally in the case of non-Newtonian fluid in a porous space with applied magnetic field. The solutions were obtained analytically (similarity) as well as numerically and we observed that in the cases of no porous half-space and no magnetic field, the velocity tended to a almost linear profile in comparison to the case of both porosity and magnetic field taken into account, where the flow is more parabolic. An increase in the permeability of the medium increases the flow velocity.

Finally, in Chapter 6, we investigated the time independent unidirectional flow of a fourth grade fluid filling a porous half space. Conditional symmetry and travelling wave solutions were obtained. Graphs of these solutions were plotted and discussed.

Most of the work presented in this thesis is new (it has been published or submitted) and the presented analysis gives us insights in understanding the behaviour of such non-Newtonian fluids. In our analysis we have only considered unidirectional flow which makes the non-linear part of the inertia $(v.\nabla).v$ to vanish. What we would like to do in future work is to consider multidimensional flows. Also for simplicity reasons the induced magnetic field was not considered in our models. What would be interesting is to see how this affects the model once taken into consideration.

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