Bornological aspects of asymmetric structures

Danny Mukonda

A Thesis Presented
to
The Faculty of Science, University of the Witwatersrand,
Johannesburg

in partial fulfilment of the requirements
for
the degree of Doctor of Philosophy
in Mathematics

March 10, 2021

Abstract

Over the last decades much progress has been made in the investigation of bornologies on metric spaces. In particular, Hu, Beer, Meroño, Garrido and others have published many papers on metric bornologies. The bornology of bounded sets in quasi-metric spaces was introduced by Piękosz and Wajch in 2015. They extended the Hu's metrization theorem to quasi-metric spaces and applied it to bornologies of bitopological spaces. In 2019, Olela Otafudu et al. used the asymmetric version of Hu's theorem to quasi-metrize the bornological universes on extended quasi-metric spaces. The principal aim of this thesis is to investigate the existence of bornologies of totally bounded sets and Bourbaki-bounded sets on asymmetric structures. In particular, we extend several results obtained by others on metric bornologies to quasi-metric settings. We show that a quasi-metric space can be bounded but not totally bounded and the bornology on a supseparable quasi-metric space agrees with the bornology of totally bounded sets. For Bourbaki-boundedness, it turns out that a set can be Bourbaki-bounded on a quasi-metric space but not on the metric space. In addition, we prove that every real-valued semi-Lipschitz in the small function is bounded if and only if the quasi-metric is Bourbaki-bounded. Consequently, we use semi-Lipschitz functions to characterize those bornologies on asymmetric normed spaces that can be realized as bornologies of Bourbaki-bounded sets. For example, we show that on quasi-metric spaces, the bornology of Bourbaki-bounded sets sits between the bornology of totally bounded sets and the bornology of bounded sets but on asymmetric normed spaces, the bornology of Bourbaki-bounded sets coincides with the bornology of bounded sets.

Declaration

This thesis titled 'Bornological aspects of asymmetric structures' was carried out by Danny Mukonda under the supervision of Professor Olivier Olela Otafudu of the Department of Mathematics and Applied Mathematics, University of the Western Cape and co-supervised by Dr Wilson Bombe Toko of the School of Mathematics, University of the Witwatersrand. It is being submitted for the degree of Doctor of Philosophy in Mathematics at the University of Witwatersrand, Johannesburg.

Signature of Candidate: ____________________ (Mr.D. Mukonda)

Date: March 10, 2021

Signature of Supervisor: ____________________ (Prof. O. Olela Otafudu)

Date: March 10, 2021

Signature of Co-supervisor: _______________ (Dr W. Bombe Toko)

Date: March 10, 2021

Publications

1. In 2018 during MSc studies, we extended the Beer's [4] bornological results from metric settings to quasi-metric framework. In 2019, as part of this PhD prelimary investigations, we revisited our previous studies. However, the improvement of my MSc results lead us to publish a paper titled **"On Bornology of extended quasi-metric spaces"** in Hacettepe Journal of Mathematics and Statistics.

2. Our first publication gave us insights and directions to look at other generalizations of metric bornologies. We have studied the paper of Beer et al. [6] on Total boundedness in metrizable spaces and generalized some results to quasi-metric settings. We have submitted a manuscript titled **"Bornologies of totally bounded sets in quasi-metric spaces"** in the Journal of Facta Universitatis, Series of Mathematics and Informatics.

3. We have also extended the results of Beer and Garrido [9] on metric bornologies with locally Lipschitz functions from metric to quasi-metric spaces. We managed to come up with new results and submitted them in a manuscript titled **"On Bourbaki-bounded sets on quasi-pseudometric spaces"** in the Journal of Commentationes Mathematicae Universitaticae Carolinae. Looking at his great contributions to the development of asymmetric structures, this manuscript was dedicated to the memory of late Professor Hans-Peter Künzi.

Dedication

I dedicate this PhD thesis to my lovely parents.

After my presentation during the 34th Summer Conference on Topology and its Applications in 2019, we had a fruitful discusions on asymmetric structures with the late Professor Hans-Peter Künzi (May his soul rest in internal peace). I therefore dedicate this work to his memory.

Acknowledgements

From the deepest of my heart, I would like to thank my supervisor, Professor Olivier Olela Otafudu for his fruitful suggestions, careful advice, wonderful directions, constant encouragements and comments in the completion of this thesis.
From the deepest of my heart, I would like to thank my supervisor, Professor Olivier Olela Otafudu for his fruitful suggestions, careful advice, wonderful directions, constant encouragements and comments in the completion of this thesis.

It is with great pleasure that I wish also to thank immensely my co-supervisor Dr Wilson Bombe Toko for his guidance and patience throughout my research.

Let me also acknowledge and thank Dr Hope Sabao for his suggestions and comments he made after going through this thesis.

Let me express my gratitude to the School of Mathematics for the teaching opportunities and a marvelous research environment offered to me during my studies.

I acknowledge financial support in the form of Post Graduate Merit Award (PGMA) given to me from the University of the Witwatersrand during my studies.

To my friends Dr Sebogodi Katlego, Mr Demco Mukongo, Mr Shashilan Singh and Mr Faith Zotto thank you for your support and encouragement.

Contents

Introduction

The systematic study of bornology in topological spaces started with the paper of Sze-tsen Hu [34] in 1949. Hu's results gave some necessary and sufficient conditions for which a bornological universe is metrizable. For instance, he showed that bornologies defined on metric spaces correspond to the usual bornologies on nonempty sets. After the study of Hu, there has been renewed interest in topology and bornologies stemming from the wide applications of Hausdorff metric convergences. For example, Lechicki et al. [35] initiated a comprehensive study of bornological convergence for nets in metric spaces. In addition, Meyer [29] provided the bornological and topological approaches to many problems. For instance, he proved that the bornological versions of Grothendiecks approximation property can be applied to convex analysis, noncommutative geometry and optimization theory.
On the other hand, Borwein et al. in [12] used bounded sets to develop the theory of differentions for functions on normed spaces. However, there have been generalizations of boundedness from nonempty sets to metric spaces, for instance, it is well known that a subset A of a metric space (X, d) is called bounded if it is contained in some ball of X, totally bounded if there exists a finite subset of X that covers it and finally Bourbaki-bounded if there is a finite subset $\{x_1, x_2, \ldots, x_k\}$ of X and some $n \in \mathbb{N}$ such that A is covered by $D_d^n(x_j, \epsilon)$ for $1 \leq j \leq k$. The word "Bourbaki-bounded" comes from the book of Bourbaki [13] where these subsets in uniform spaces were first studied.
In spite of the fact that there are several extensions of bounded sets, we realize that all of them are stable under finite unions, are hereditary and contain the singletons. A family with these three properties is called a *bornology* and denoted by $\mathscr{B}$ (see Definition 1.2.1). If we have a nonempty set X and a bornology $\mathscr{B}$ on X, then the pair $(X, \mathscr{B})$ is called a bornological universe. On a metric space (X, d), we denote the bornology of bounded sets by $\mathscr{B}_d(X)$, the bornology of totally bounded sets by $\mathscr{TB}_d(X)$ and the bornology of Bourbaki-bounded sets by $\mathscr{BB}_d(X)$.
Garrido and Meroño [17] studied and compared these bornologies in different contexts. Other authors like Beer et al. [6] studied bornologies of bounded

sets and totally bounded subsets on metric spaces and showed that a metric space is separable if the metric bornology agrees with the bornology of totally bounded sets. The study of metric bornologies and continuous functions has been the area of study for quite some time. For instance, Munkres [33] in the year 2000 proved that continuity is preserved under uniform convergence on the bornology of relatively compact subsets and Cauchy continuity is preserved under uniform convergence on the bornology of totally bounded subsets.

The notion of total boundedness has also been defined in quasi-metric spaces. In 2012, Olela Otafudu investigated total boundedness of the injective hull of a totally bounded ultra-quasi-metric space. He proved that total boundedness is preserved by the ultra-quasi-metrically injective hull of a quasi-metric space (see e.g [49, Proposition 5.4.1]). In addition, Kemajou et al. [37] showed that every quasi-metric space has a hyperconvex hull which is join-compact provided that it is totally bounded.

On the other hand, the theory of Bourbaki-boundedness in metric spaces was studied by Atsuji in [3] as the generalization of total boundedness. Atsuji proved that a metric space is Bourbaki-bounded if and only if every real-valued uniformly continuous function on it is bounded. However, the concept of Bourbaki-boundness got huge interest by many mathematicians (see [9,17–19]). For instance, in [19] the authors introduced a new tool for the completeness of metric spaces with respect to Bourbaki-boundedness ideas which they called Bourbaki-completeness and cofinal Bourbaki-completeness.

In the past years, some authors have made some attempts to develop the study of Bourbaki-boundedness in nonsymmetric spaces. For instance, Murdeshwar and Theckedath [32] studied Bourbaki-boundedness and observed that if the set $[0,1]$ is equipped with the T_0-quasi-metric

$$q(x,y) = \begin{cases} y-x & \text{if } x \leq y \\ 1 & x > y \end{cases}$$

and $\mathcal{U}$ is the quasi-uniformity generated by q on $[0,1]$ and $\mathcal{U}^{-1}$ is the conjugate quasi-uniformity of $\mathcal{U}$ on $[0,1]$, then $([0,1],\mathcal{U})$ and $([0,1],\mathcal{U}^{-1})$ are Bourbaki-bounded quasi-uniform spaces, but $([0,1],\mathcal{U}^s)$ is the discrete uniform space which is not Bourbaki-bounded. They also noted that when a quasi-uniformity is derived from a quasi-metric, it would be pertinent to compare Bourbaki-boundedness of the quasi-uniformity with the boundedness of the quasi-metric space.

In 1995, Künzi and Ryser [45] studied the properties of Bourbaki quasi-uniformity defined on the collection of all nonempty sets of quasi-uniform spaces. They actually generalized the well known Isbell-Burdick theorem from uniform spaces to quasi-uniform spaces. The theory of bornology can also be studied using asymmetric structures like quasi-metric spaces and asymmetric normed spaces. Recall that a structure is said to be asymmetric if it does not respect the symmetry condition. The lack of symmetry comes with difficulties in the classical concepts like completeness, compactness, boundedness, duality and others. For example, it is shown in [52] that on a quasi-metric space (X, q), being bounded on q or q^t does not mean bounded on q^s. The bornology of bounded sets in quasi-metric spaces was introduced by Piękosz and Wajch [53] in 2015. They first extended the Hu's metrization theorem of bornological universe to quasi-metric spaces and applied it to the bornologies of bitopological spaces.

In our work [52], we successfully extended the concept of bornology from metric settings to the framework of extended quasi-metrics. We used the asymmetric version of Hu's theorem and proved that for an extended quasi-metric q, the bornology $\mathscr{B}$ coincides with $\mathscr{B}_q(X)$. Though many classical ideas about bornology could be generalized from metric to quasi-metric spaces, these generalizations were not trivial and sometimes the nonsymmetric versions required interesting new variations from the old ones. Naturally, this has led to the speculation of how to extend the bornology of totally bounded sets and the bornology of Bourbaki-bounded sets from the metric point of view to quasi-metric settings. There is no doubt that uniformly continuous functions and asymmetric normed spaces are important tools in the characterization of Bourbaki-bounded sets on quasi-metric spaces.

The main purpose of this thesis is to investigate the existence of bornologies of Bourbaki-bounded sets and totally bounded sets on asymmetric structures. We intend to generalize the classical bornological results on totally bounded sets and Bourbaki-bounded sets from metric spaces to the category of quasi-metric spaces. For instance, given a compatible quasi-metric, we give some necessary and sufficient conditions for which the bornology of totally bounded sets and the bornology of Bourbaki-bounded sets agree with the quasi-metric bornology studied by Olela Otafudu et al. [52]. It actually turns out that if a quasi-metric space is supseparable then the quasi-metric bornology coincides with the bornology of totally bounded sets (see Theorem 4.2.4).

In order to have a better understanding of Bourbaki-boundedness in quasi-metric spaces, we present equivalent characterizations of this boundedness. It is interesting to note in Example 5.2.1 that if a set is Bourbaki-bounded on the symmetrized quasi-pseudometric space, then it is also Bourbaki-bounded on the quasi-pseudometric space, but the converse need not be true. Morever, we show how the studies of Beer and Garrido [9] on bornologies of Bourbaki-bounded sets with Lipschitz functions can be modified in order to obtain a theory that is appropriate for quasi-metric spaces.

Furthermore, we characterize Bourbaki-boundedness under uniformily continuous functions and semi-Lipschitz functions. For instance, we show that every real-valued semi-Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded (see Theorem 5.3.2). Moreover, we observe that for any two quasi-pseudometric spaces (X, q) and (Y, p), if the function $\varphi : (X, q) \to (Y, p)$ is uniformly continuous, then $\varphi : (X, q^s) \to (Y, p^s)$ is also uniformly continuous, but the converse is not true in general (see Example 5.1.1). All these are great motivations for generalizing the results on bornologies from symmetric to nonsymmetric settings.
In a metric space, it is proved by Garrido and Meroño [17] that the bornology of Bourbaki-bounded sets sits between the bornology of totally bounded sets and the bornology of bounded sets. In this thesis, we also study and compare the bornologies of bounded sets with the bornologies of totally bounded sets and the bornologies of Bourbaki-bounded sets. For example, we show in Remark 5.3.2 that on quasi-metric spaces, the bornology of Bourbaki-bounded sits between the bornology of totally bounded sets and the bornology of bounded sets but for asymmetric normed spaces, the bornology of Bourbaki-bounded sets coincides with the quasi-metric bornology. Finally, we use semi-Lipschitz functions to characterize those bornologies on asymmetric normed spaces that can be viewed as bornologies of Bourbaki-bounded sets.

Organization of the thesis

Chapter 1. In this first chapter, we recall some basic concepts and definitions to be used throughout this thesis. The first section defines quasi-pseudometric spaces with their topological structures and later give some interesting examples. In addition, we present some notions of bitopological spaces and thereafter recall some concepts of convergence and bicompleteness. The last section of this chapter discusses the bornologies on nonempty sets. We also present some examples of bornologies (Examples 1.2.1 and 1.2.2) and study some results on the closed and open bases of the bornological universe (see Corollary 1.2.1). Lastly, we review the concept of characteristic function which Meroño and Garrido [18] also used in order to obtain their metrization theorems for bornologies.

Chapter 2. In this chapter, we collect the fundamental results about aspects of metric bornologies. We mainly give an overview of the terminology and elementary results about different metric bornologies, most of which already appear in the literatures [4, 6, 9], but to be generalized to asymmetric spaces throughout the thesis. We begin by summarizing the results on metrizability of the bornological universe and the construction of the universal space of a metric space (see Theorems 2.1.7 and 2.1.8). In the second section, we present the concept of total boundedness on a metric space. It will be shown in Theorem 2.2.2 that if a metric space is separable, then the metric bornology becomes a bornology of totally bounded sets. Finally, in the last section we recall some results on bornologies of Bourbaki-bounded sets with Lipschitz functions on metric spaces. For example, Theorem 2.3.1 presents the results on bornologies of locally Lipschitz functions with relatively compact subsets.

Chapter 3. In this chapter, we start our own investigations by revisiting my MSc dissertation. In [52], we redefined and improved our own universal space constructed during the MSc studies (compare [31, Definition 3.2.1] and Section 3.1). The universal space was constructed on a set of (f, A) where f is a real-valued continuous function on q^s-closed set A. However, we show that the universal space of an extended quasi-metric space is a bicomplete space (Remark 3.1.2 and see also [52, Remark 3.1]). Furthermore, Propositions 3.1.1 and 3.1.2 provide the isometries between the extended quasi-metric spaces and the universal spaces. In the second section, we revisit the comparisons of quasi-metric bornologies determined by q, q^s and q^t (see Lemma 3.2.1). We prove in Theorem 3.2.2 that for an extended quasi-metric q, the bornological universe $(X, \mathscr{B})$ is quasi-metrizable by q and the bornology $\mathscr{B}$ coincides with the quasi-metric bornology on X. We end this chapter by proving in Theorem 3.2.3 that for a given bornology $\mathscr{B}$ on an extended quasi-metric space (X, q), there exists an extended bona fide quasi-metric q_1 on X, uniformly equivalent to q such that $\mathscr{B}_q(X) = \mathscr{B}_{q_1}(X)$. Note that we have published most of the results in this chapter (see [52]).

Chapter 4. This chapter continues our investigations into the bornology of totally bounded sets on a quasi-metric space (X, q). In the first section, we study and compare q-boundedness, q-precompactness, joincompactness, bicompleteness with q-total boundedness. We observe that for infinite dimension quasi-metric spaces, q-boundedness and q-total boundedness are two different notions. For example, it will be proved in Example 4.1.2 that every q-totally bounded quasi-pseudometric space (X, q) is q-bounded but the converse is not always true. In ending this section, Theorem 4.1.2 proves that a subset of the quasi-metric space (X, q) is joincompact if and only if it is both bicomplete and q-totally bounded. In the second section, we present our own results on bornologies of totally bounded sets. We prove in Theorem

4.2.1 that the bornology of q-totally bounded sets agrees with the bornology of $\tau(q)$-relatively compact sets if and only if the quasi-metric space is bicomplete. It also turns out that if a quasi-metric space (X, q) is supseparable, then the quasi-metric bornology studied by Olela Otafudu et al. [52] coincides with the bornology of q-totally bounded sets (see Theorems 4.2.4 and 4.2.5). We use asymmetric version of Hu's theorem (Theorem 3.2.1) to prove that the bornology of q-totally bounded sets with one point extension is quasi-metrizable and if the bornology has a countable base then it coincides with quasi-metric bornology (see Theorem 4.2.5). Finally, we prove in Theorem 4.2.7 that the family of nonempty sets of the quasi-metric space (X, q) becomes the bornology of q-totally bounded sets if and only if there exists a star-development on X.

Chapter 5 We continue with our investigations on quasi-metric bornologies of Bourbaki-bounded sets with semi-Lipschitz functions in this chapter. In the first section, we study the concepts of uniform continuity and semi-Lipschitz functions. In asymmetric structures, the concept of uniform continuity is not as direct as it is in symmetric ones. For instance, we show in Example 5.1.1 and Lemma 5.1.1 that on a quasi-metric space (X, q), the uniform continuity of q^s does not imply the uniform continuity of q. In the second section, we study the equivalent characterizations of Bourbaki boundedness on quasi-metric spaces. It is interesting to note that a set can be q-Bourbaki-bounded but not q^s-Bourbaki-bounded (see Example 5.2.1). After introducing the concept of δ-chain in Definition 5.2.4, we construct the relation $\asymp_\delta$ on (X, q) and prove that it is actually an equivalence relation (Lemma 5.2.3). In Section 5.3, we study and compare the bornology of q-totally bounded sets with other quasi-metric bornologies. For instance, Remark 5.3.2 shows that the bornology of q-Bourbaki-bounded contains the bornologies of q-totally bounded sets. Morever, Proposition 5.2.3 proves that given an asymmetric normed space, the bornology of q-Bourbaki-bounded sets coincides with quasi-metric bornology. In Theorem 5.3.1, we use the $\tau(q)$-compact subsets with locally semi-Lipschitz functions on asymmetric normed space to come up with a set on the quasi-metric bornology of real numbers. Theorem 5.3.2 concludes this chapter by proving that every real-valued semi-Lipschitz in the small function on asymmetric normed spaces is bounded if and only if it is Bourbaki-bounded.

Chapter 6 We conclude this work by reflecting on the main results of the thesis and highlighting some connections of this work with other studies. Furthermore, we mention some open problems to which one can constitute the topics for further research. The study of bornology in asymmetric structures can lead to many open problems. For instance, Sebogodi [22], one of the former PhD student of my supervisor, studied modular sets equipped with T_0-quasi-metric induced by modular quasi-pseudometrics. Indeed one can investigate the introduction of bornologies on modular quasi-pseudometric spaces. In additon, Murdeshwar and Theckedath studied boundedness and quasi-uniform continuous functions. They proved in [32, Theorem 2.2] that boundedness is preserved under quasi-uniform continuous functions. This could lead us to another open problem, for instance one could also investigate the bornology of Bourbaki-bounded sets using quasi-uniform continuous functions.

Chapter 1

Preliminaries

In this chapter, we begin by considering some basic definitions, properties, notations and some results that will be useful in the thesis. Firstly, we present the basic concepts about the theory of quasi-metric spaces. Secondly, we also summarize facts about boundedness which are often called bornologies. For further readings and recent results in the area of asymmetric topology and bornologies on sets, the reader is advised to consult [4, 14, 16, 32, 34, 43, 46].

1.1 Concepts of quasi-pseudometrics

In this section, we introduce the terminologies and notations for quasi-pseudometric spaces that we will use throughout this thesis.

Definition 1.1.1. *[48, Definition 2.1] Let X be a nonempty set and let $q : X \times X \to [0, \infty)$ be a function. Then q is called a* quasi-pseudometric *on X if:*

(i) $q(x, x) = 0$ for all $x \in X$;

(ii) $q(x, y) \leq q(x, z) + q(z, y)$ for all x, y, $z \in X$.

The pair (X, q) is called a quasi-pseudometric space*. If in addition, for any $x, y \in X$, $q(x, y) = 0 = q(y, x) \implies x = y$, then q is called a* T_0-quasi-metric *(or* quasi-metric*), and the pair (X, q) is called a T_0-quasi-metric space. If we replace the set $[0, \infty)$ by $[0, \infty]$ then we have the function q called T_0-extended quasi-metric and (X, q) is called T_0-extended quasi-metric space.*

If q is a quasi-pseudometric on a set X, then $q^t : X \times X \to [0, \infty)$ defined by $q^t(x, y) = q(y, x)$ for every x, $y \in X$, often called the conjugate quasi-pseudometric space of q, is also a quasi-pseudometric on X. The quasi-pseudometric on a set X such that $q = q^t$ is a pseudometric. Note that if

(X, q) is a quasi-metric space, then $q^s = \max\{q, q^t\} = q \vee q^t$ is also a metric on X.

Remark 1.1.1. *[14] Let (X, q) be a quasi-pseudometric space. The open ball of radius $\epsilon > 0$ centred at $x \in X$ is the set $D_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$. The collection of open balls yields a base for the topology $\tau(q)$ and it is called the topology induced by q on X. Similarly, the closed ball of radius $\epsilon \geq 0$ centred at $x \in X$ is the set $D_q[x, \epsilon] = \{y \in X : q(x, y) \leq \epsilon\}$. If (X, q) is a quasi-pseudometric space, then the pair $\{D_q[x, r]; D_{q^t}[x, s]\}$ where $x \in X$ and $r, s \in [0, \infty)$ is called a double ball. In general, $\{(D_q(x_i, r_i))_{i\in I}; (D_{q^t}(x_i, s_i))_{i\in I}\}$, with $x_i \in X$ and $r_i, s_i \in [0, \infty)$, is called the family of double balls. Note that the set $D_q(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\}$ is a $\tau(q^t)$-closed set, but not $\tau(q)$-closed in general. The following inclusions holds:*

$$D_{q^s}(x, \epsilon) \subset D_q(x, \epsilon) \text{ and } D_{q^s}(x, \epsilon) \subset D_{q^t}(x, \epsilon).$$

Example 1.1.1. *[22, Example 1.1.2] Let $\mathbb{R}$ be the set of real numbers and $x, y \in \mathbb{R}$. The function u defined by*

$$u(x, y) = (x - y)^+ = x \dot{-} y = \max\{x - y, 0\}$$

is a T_0-quasi-metric on $\mathbb{R}$. Morever the function $u^s(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ is the usual metric on $\mathbb{R}$.

Example 1.1.2. *[14, Example 1.1.6] Let x and y be in $\mathbb{R}$, and let us define a quasi-metric q by $q(x, y) = y - x$, if $x \leq y$ and $q(x, y) = 1$, if $x > y$. A basis of $\tau(q)$- open neighbourhood of a point $x \in \mathbb{R}$ is formed by the family $[x, x + \epsilon)$, $0 < \epsilon < 1$. The family of intervals $(x - \epsilon, x]$, $0 < \epsilon < 1$ forms the basis of $\tau(q^t)$- open neighbourhood of a point x. If $q^s(x, y) = 1$ for $x \neq y$ then $\tau(q^s)$ is called a discrete topology on $\mathbb{R}$.*

Example 1.1.3. *[42, Example 1] Let $X = \{-\frac{1}{n+1}, \frac{1}{n+1} : n \in \mathbb{N}\}$. If for all x and y in X we define $q(x, y) = 1$ for $y < 0 < x$ and $q(x, y) = |x - y|$ otherwise, then q is a quasi-metric on X.*

Suppose (X, q) is a quasi-pseudometric space, then for $\delta > 0$, $x \in X$ and $F \subseteq X$, we define $\text{dist}_q(x, F)$ by $\text{dist}_q(x, F) = \inf_{f\in F} q(x, f)$. Moreover,

$$\text{dist}^t_q(x, F) = \inf_{f\in F} q^t(x, f) \text{ and } \text{dist}^s_q(F, x) = \max\{\text{dist}_q(F, x), \text{dist}^t_q(F, x)\}.$$

Definition 1.1.2. *Let (X, q) be a quasi-pseudometric space. A subset B of X is q-bounded if there exists $\epsilon > 0$ such that $q(x, y) < \epsilon$, whenever $x, y \in B$.*

Let us recall some concepts about quasi-uniform spaces from [32, 43, 52].

Definition 1.1.3. *Let X be a nonempty set. A quasi-uniformity on a set X is a* filter *$\mathcal{U}$ on $X \times X$ such that:*

(i) for each $U \in \mathcal{U}$, $\{(x,x) : x \in X\} \subseteq U$;

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Here we have that $V^2 = V \circ V = \{(x,z) \in X \times X :$ there is $y \in X$ such that $(x,y) \in V$ and $(y,z) \in V\}$.

A quasi-uniform space is a pair $(X,\mathcal{U})$ such that X is a set and $\mathcal{U}$ is a quasi-uniformity on X.

Definition 1.1.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $A \subseteq X$. The set A is said to be* bounded or Bourbaki-bounded *if for any $B \in \mathcal{U}$, there exists a positive integer n and a finite set $F \subseteq X$ such that $A \subseteq B^n[F]$, where*

$$B[F] = \bigcup_{f \in F} B(f) = \{x \in X : \text{ there exists } s \in F \text{ such that } (s,x) \text{ is in } B\}.$$

It has been observed in [32] that if a set is bounded in the sense of the above definition, then it is also bounded in the metric sense.

Definition 1.1.5. *[2, Definition 3] A quasi-pseudometric space (X,q) is q-precompact if for each $\epsilon > 0$ there exists a finite set $\{f_1, f_2, f_3, \ldots f_k\} \subset X$ such that $X \subseteq \bigcup_{j=1}^{k} D_q(f_j, \epsilon)$.*

A quasi-pseudometric space (X,q) is said to be *q-totally bounded (or totally bounded)* provided the pseudometric (X,q^s) is totally bounded.

Definition 1.1.6. *An extended quasi-metric space (X,q) is said to be* locally compact *with respect to $\tau(q)$ if every point of X has a $\tau(q)$-compact neighbourhood.*

Remark 1.1.2. *As a space with two topologies τ_1 and τ_2, a quasi-metric space can be viewed as a bitopological space in the sense of Kelly [36] and so, all the results valid for bitopological spaces apply to a quasi-metric space.* A bitopological space *is simply a set X endowed with two topologies τ_1 and τ_2.*

Definition 1.1.7. *A topological space (X,τ) is* quasi-metrizable *if there exists a quasi metric q on X such that $\tau = \tau(q)$.*

A bitopological space (X, τ_1, τ_2) is *quasi-metrizable* if there exists a quasi-metric q on X such that $\tau_1 = \tau(q)$ and $\tau_2 = \tau(q^t)$.

Here are some concepts about the convergence and bicompleteness in quasi-metric spaces from Cobzas [14] and Salbany [23].

Definition 1.1.8. *If (X, q) is a quasi-pseudometric space, then a sequence $(x_n) \in X$ is said to be:*

(i) $\tau(q)$-convergent *to x if for each $\epsilon > 0$, there exists a positive $N = N(\epsilon)$ such that $q(x_n, x) < \epsilon$ for all $n \geq N$;*

(ii) $\tau(q^t)$-convergent *to x if for each $\epsilon > 0$, there exists a positive $N = N(\epsilon)$ such that $q(x, x_n) < \epsilon$ for all $n \geq N$.*

Definition 1.1.9. *Let (X, q) be a quasi-pseudometric space, a sequence (x_n) is called:*

(i) q-Cauchy *if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \epsilon$ for all $m \geq n \geq n_\epsilon$;*

(ii) q^t-Cauchy *if for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $q(x_m, x_n) < \epsilon$ for all $m \geq n \geq n_\epsilon$.*

(iii) q^s-Cauchy *if it is both q-Cauchy and q^t-Cauchy.*

Note that (x_n) is a q^s-Cauchy sequence in (X, q) if and only if (x_n) is a Cauchy sequence in the pseudometric space (X, q^s).

Proposition 1.1.1. *[14, Proposition 1.1.2] Let (x_n) be a sequence in the quasi-metric space (X, q) :*

(a) if (x_n) is $\tau(q)$-convergent to x and $\tau(q^t)$-convergent to y, then $q(x, y) =$ 0;

(b) if (x_n) is $\tau(q)$-convergent to x, and $q(y, x) = 0$; then (x_n) is also $\tau(q)$-convergent to y.

Proposition 1.1.2. *[49, Proposition 0.1.1] A quasi-pseudometric space (X, q) is bicomplete if and only if the metric space (X, q^s) is complete.*

A *bicompletion* of the quasi-metric space (X, q) is a bicomplete quasi-metric space $(\widetilde{X}, \widetilde{q})$ in which (X, q) can be quasi-isometrically embedded as a $\tau(\widetilde{q}^s)$-dense subspace.

Definition 1.1.10. *[15, p. 146] Let (X, q) and (Y, p) be quasi-pseudometric spaces. A function $\varphi : (X, q) \to (Y, p)$ is called* quasi-uniformly continuous *(or* uniformly continuous*) if for any $\epsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leq \delta$, then $p(\varphi(x), \varphi(y)) < \epsilon$ for all $x, y \in X$.*

A subset B of a quasi-metric space (X, q) is said to be $\tau(q)$*-relatively compact* if B is $\tau(q^s)$-relatively compact.

Definition 1.1.11. *[1, Definition 2.1] Let (X, q) be a T_0-quasi-metric space and A be a subset of X. We say that A is* supseparable *if A is $\tau(q^s)$-separable.*

Let (X, q) be a quasi-pseudometric space. Then, for each set A in X, we write $\mathrm{cl}_{\tau(q)}(A)$ for its *closure* with respect to $\tau(q)$ and $\mathrm{int}_{\tau(q)}(A)$ for its *interior* with respect to $\tau(q)$.
Let (X, q) and (Y, p) be quasi-pseudometric spaces, then the mapping $\varphi : (X, q) \longrightarrow (Y, p)$ is an *isometric mapping* if for each $x, y \in X$ we have $p(\varphi(x), \varphi(y)) = q(x, y)$. Two quasi metric spaces (X, q) and (Y, p) are called *isometric* if there exists a bijective isometry between them.

We recall the concepts of asymmetric norms and semi-Lipschitz functions in quasi-metric spaces.

Definition 1.1.12. *[14] An* asymmetric norm *on a real vector space X is a function $\|.|: X \to [0, \infty)$ satisfying the conditions:*

(i) $\|x|= \|-x|= 0$ *then* $x = 0$;

(ii) $\|ax|= a\|x|$;

(iii) $\|x + y|\leq \|x|+\|y|$,

for all $x, y \in X$ and $a \geq 0$. Then the pair $(X, \|.|)$ is called an asymmetric normed space.

The conjugate asymmetric norm $|.\|$ *of* $\|.|$ *and the* symmetrized norm $\|.\|$ *of* $\|.|$ *are defined respectively by*

$$|x\|:= \|-x| \quad \textit{and} \quad \|x\|:= \max\{|x\|, \|x|\}\,\textit{for any}\, x \in X.$$

An asymmetric norm $\|.|$ on X induces a quasi-metric $q_{\|.|}$ on X defined by

$$q_{\|.|}(x, y) = \|x - y| \text{ for any}\, x, y \in X.$$

If $(X, \|.\|)$ is a normed lattice space, then the function $\|x|:= \|x^+\|$ with $x^+ = \max\{x, 0\}$ is an asymmetric norm on X.

Definition 1.1.13. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then a function $\varphi : (X, q) \to (Y, \|.|)$ is called* k-semi-Lipschitz (or semi-Lipschitz) *if there exists $k \geq 0$ such that*

$$\|\varphi(x) - \varphi(y)| \leq kq(x, y) \quad \textit{for all } x, y \in X. \tag{1.1.1}$$

A number k satisfying inquality (1.1.1) is called *semi-Lipschitz constant* for φ.

1.2 Concept of bornology on a set

The study of bornologies started with Sze-tsen Hu [34] in 1949. Hu studied the bornologies on a nonempty set which he called boundedness (see [34]). This section is devoted to give some basic concepts of bornologies on a nonempty set X. There is substantial number of publications about bornologies in the major reviews of Hu [34] and Beer et al. [4, 6, 7, 9, 10].

Definition 1.2.1. *[4, Definition 1.1] Let X be a nonempty set. A family $\mathscr{B}$ of subsets of X is called a* bornology *on X provided the following conditions are satisfied:*

(i) $\mathscr{B}$ forms a cover of X, i.e., $X = \bigcup_{B \in \mathscr{B}} B$;

(ii) $\mathscr{B}$ is hereditary under inclusion, i.e., whenever $B \in \mathscr{B}$ and A is a subset of X contained in B, then $A \in \mathscr{B}$;

(iii) $\mathscr{B}$ is stable under finite union, i.e., if $B_1, B_2, ..., B_n \in \mathscr{B}$ then we get $\bigcup_{i=1}^{n} B_i \in \mathscr{B}$.

Given a bornology $\mathscr{B}$ and the set X, a pair $(X, \mathscr{B})$ is called a bornological universe*.*

Let us look at the following examples of bornologies.

Example 1.2.1. *[9, p. 257] For a nonempty set X the collection $\mathscr{B}$ of all subsets in X, is a bornology on X called the trivial bornology. Obviously if we have a bornology $\mathscr{B}$ on X and $X \in \mathscr{B}$, i.e., X is bounded, then $\mathscr{B}$ is a trivial Bornology.*

Example 1.2.2. *[6, p. 2] Let (X, τ) be a topological space and if $\mathscr{B}$ is a collection of all compact subsets of a topological space X, then $\mathscr{B}$ is a bornology called the compact bornology.*

Example 1.2.3. *[7, Example 2] Let* $\mathscr{A}$ *be a family of subsets of a set* X *such that it forms a cover of* X *and let* $\mathscr{B}$ *be a family of subsets of* X *which consists of the totality of the subsets of the finite unions of the family* $\mathscr{A}$*. The family* $\mathscr{B}$ *is the bornology generated by the cover* $\mathscr{A}$*.*

We give the definition of the base for a bornology in the following remark.

Remark 1.2.1. *[18, p. 286] Given a nonempty set* X *and bornology* $\mathscr{B}$ *on* X*, a family* $\mathcal{C} \subseteq \mathscr{B}$ *is said to be a* base *for* $\mathscr{B}$ *if for every* $B \in \mathscr{B}$ *there is* $D \in \mathcal{C}$ *such that* $B \in D$*, i.e., every set in* $\mathscr{B}$ *is contained in some set of* $\mathcal{C}$*. For instance, the (countable) family* $\mathcal{C} = \{D(x_0, n) : n \in \mathbb{N}\}$ *of the open balls of center* $x_0 \in X$ *and radius* $n \in \mathbb{N}$*, is a base for the bornology* $\mathscr{B}_d(X)$ *of the bounded sets in the metric space* (X, d)*.*

Let $(X, \mathscr{B})$ be a bornological universe. The bornology $\mathscr{B}$ is said to have a *countable base* if it possesses a base consisting of a sequence of bounded sets and it is said to be second countable if its base is countable.

We recall the following definition and results about bornologies from Hu's paper [34].

Definition 1.2.2. *Let* (X, τ_1, τ_2) *be a bitopological space. Then, a bornology* $\mathscr{B}$ *in* X *is called* proper *if and only if for every* $A \in \mathscr{B}$*, there exists a* $B \in \mathscr{B}$ *such that* $cl_{\tau_2}(A) \subseteq int_{\tau_1}(B)$*.*

Theorem 1.2.1. *[34, Theorem 3.5] Let* (X, τ) *be a topological space. For any given bornology* $\mathscr{B}$ *in a topological space* X*, the following conditions are equivalent:*

(i) $\mathscr{B}$ *is closed;*

(ii) $\mathscr{B}$ *is generated by its subfamily of the bounded closed sets (it has a closed base);*

(iii) The closure of every bounded set is a bounded set.

Proof. (see the proof of [34, Theorem 3.5]). □

The proof of the following result is in [34] and gives us the base of a bornology.

Theorem 1.2.2. *[34, Theorem 3.6] Let* (X, τ) *be a topological space. For any given bornology* $\mathscr{B}$ *in a topological space* X*, the following conditions are equivalent:*

(i) $\mathscr{B}$ *is open;*

(ii) $\mathscr{B}$ *is generated by its subfamily of the bounded open sets (it has an open base);*

(iii) every bounded set is contained in the interior of some bounded set.

Corollary 1.2.1. *Let* (X, τ) *be a topological space. For any given bornology* $\mathscr{B}$ *in a topological space* X*, the following conditions are equivalent:*

(i) $\mathscr{B}$ *is proper;*

(ii) $\mathscr{B}$ *is generated by its subfamily of the bounded open sets and its family of bounded closed sets (it has both open and closed base);*

(iii) The closure of every bounded set is contained in the interior of some bounded set.

Proof. It is a consequence of Theorem 1.2.1 and Theorem 1.2.2. □

We end this chapter by recalling the notion of the characteristic function that Meroño and Garrido [18] introduced in order to obtain their metrization theorem for bornological universes.

Definition 1.2.3. *[18, Definition 2.1] Let* $(X, \mathscr{B})$ *be a bornological universe. A* characteristic function *of* $(X, \mathscr{B})$ *is a real-valued non negative continuous function* $\chi : X \longrightarrow [0, \infty)$ *such that a subset* $E \subset X$ *is bounded if and only if* $\chi(E)$ *is bounded in* $[0, \infty)$*, that is,*

$$\mathscr{B} = \{E \subset X : \exists k > 0, \chi(E) \leq k\}.$$

Proposition 1.2.1. *[18] Let* (X, d) *be a metric space and* $\mathscr{B}$ *be a bornology on* X*. Then the bornological universe* $(X, \mathscr{B})$ *admits a continuous characteristic function if and only if* $\mathscr{B}$ *has a countable base* $\{B_n : n \in \mathbb{N}\}$ *such that for some* $\delta > 0$*,*

$$B_n^{\delta} \subset B_{n+1}, \textit{ for every } n \in \mathbb{N}.$$

Proof. The proof can be found in [18, Proposition 2.2]. □

Chapter 2

Some bornologies in metric spaces

It is hard to believe that bornology on metric spaces has already been investigated for more than fifty years. The concept of bornology on a set was introduced by Hu [34] in 1949 and was investigated later by many authors (see e.g [4, 9, 11]). In this chapter, we mainly give an overview of the terminology and elementary results about different metric bornologies, most of which already appear in the literature, but to be generalized to quasi-metric spaces throughout the thesis. Except otherwise stated, most of the material in this chapter can be found in [4, 6, 9, 18, 56].

2.1 Metric boundedness in metric spaces

In this section, we summarize Beer's results in [4] on the bornology of extended metrics spaces. We first present the construction of the universal space, thereafter prove some results on metrizability of metric bornologies.

Definition 2.1.1. *Let X be an extended metric space. For any $x, y, \in X$, we define a relation $\mathcal{E}_d$ on X by $x\, \mathcal{E}_d\, y$ provided $d(x, y) < \infty$.*

The proof of the following lemma will be provided in asymmetric settings in the next chapter.

Lemma 2.1.1. *If (X, d) is an extended metric space, then the relation $\mathcal{E}_d$ is an equivalence relation on X.*

If (X, d) be an extended metric space and $x \in X$ then the equivalence class of x on X denoted by $mc_d(x)$ is called *metric component* of X.

Definition 2.1.2. *[4, p. 6] Let* $\Delta_X = \{(f, A) : f \in C(A)\}$ *and* $A \in \mathscr{P}_0(X)$*. We equip the set* Δ_X *with an extended metric* ρ_X *defined by*

$$\rho_X((f,A),(g,B)) = \begin{cases} \sup_{a\in A}|f(a)-g(a)|) & \text{if } A{=}B \\ \\ \infty, & \text{if } A \neq B. \end{cases}$$

The pair (Δ_X, ρ_X) is called the *universal space* for extended metric space (X, d).
The following theorem shows that the universe space of a metric space is complete. We have also presented its asymmetric version in Remark 3.1.2.

Theorem 2.1.1. *[4, Proposition 3.1] Let* X *be a Hausdorff space. Then* (Δ_X, ρ_X) *is a complete extended metric space.*

Proof. We show that every Cauchy sequence in (Δ_X, ρ_X) converges. Let (f_n, A_n) be a Cauchy sequence in (Δ_X, ρ_X). There exists an $\epsilon > 0$ and $N \in \mathbb{N}$ such that for $n, j \geq N$, $\rho_X((f_n, A_n),(f_j, A_j)) < \epsilon$. Assuming $A_n = A_j = A_i$, then for $n = j \neq i \geq N$, we have,

$$\rho_X((f_n, A_i),(f_j, A_i)) = \sup_{a\in A_i}\{|f_n(a) - f_j(a)|\} < \epsilon.$$

Thus, a sequence is Cauchy and it converges to some $f \in C(A)$. □

Theorem 2.1.2. *[4, Proposition 3.3] Let* (X, d) *be an extended metric space. Then for each compatible metric* d *on* X*,* $(\mathscr{P}_0(X), H_d)$ *can be isometrically embedded in* (Δ_X, ρ_X)*.*

Proof. The proof of this result can be found in [4, Proposition 3.3]. □

The next result illustrates an embedding from extended metric space to the universal space.

Theorem 2.1.3. *[4, Theorem 3.2] Let* (X, d) *be a metric space. Then, the function* $\pi : (X, d) \longrightarrow (\Delta_X, \rho_X)$ *defined by* $\pi(x) = (d(x, \cdot)|_{mc_d(x)}, mc_d(x))$*, for* $x \in X$ *is an isometry.*

Proof. [4, proof of Proposition 3.2]. This proof is also given in nonsymmetric case (see Proposition 3.1.1). □

The following theorem proves the existence of an isometry from one universal space to another. Theorem 3.1.1 is the generalization of this theorem to quasi-metric settings.

Theorem 2.1.4. *[4, Theorem 3.4] Let (X, d) and (Y, d_1) be extended metric spaces and suppose there exists a surjection map σ from X to Y. Then there exists an isometry from (Δ_X, ρ_X) into (Δ_Y, ρ_Y).*

Proof. For us to show that there exists an isometry from (Δ_X, ρ_X) into (Δ_Y, ρ_Y). Let the mapping $\psi : (\Delta_X, \rho_X) \longrightarrow (\Delta_Y, \rho_Y)$ be defined by:

$$\psi(f, A) = (fo\sigma|_{\sigma^{-1}(A)}, \sigma^{-1}(A)).$$

We need to show that ψ is an isometry. Let $A_1 \neq A_2$ be closed subsets of Y, and since σ is a continuous surjective mapping, we have $\sigma^{-1}(A_1) \neq \sigma^{-1}(A_2)$ as nonempty closed subsets of X as well. Now, if $f \in C(A_1)$ and $g \in C(A_2)$ then by the definition of ρ_X, we have

$$\begin{aligned} \rho_X(\psi(f, A_1), \psi(g, A_2)) &= \rho_X(fo\sigma|_{\sigma^{-1}(A_1)}, \sigma^{-1}(A_1)), (go\sigma|_{\sigma^{-1}(A_2)}, \sigma^{-1}(A_2))) \\ &= \infty \\ &= \rho_Y((f, A_1), (g, A_2)). \end{aligned}$$

Again, let $A_1 = A_2 = A$ and $f, g \in C(A)$, since σ is a continuous surjective mapping, we compute the suprimum as follows:

$$\rho_X(\psi(f, A_1), \psi(g, A_2)) = \sup_{x \in \sigma^{-1}(A)} |f(\sigma(x)) - g(\sigma(x))| = \rho_Y((f, A_1), (g, A_2)).$$

□

In what follows, we study and present Beer [4]'s results on metrizability of the bornological universes.

Definition 2.1.3. *[4] Let (X, d) be an extended metric space. If X can be partitioned into disjoint unions of sets $\{X_i : i \in I\}$ and each X_i is metrizable, then choosing an extended metric d_i for X_i, the function d on X defined by*

$$d(x, w) = \begin{cases} d_i(x, w) \;\; \text{if } \exists i \;\; \text{with} \;\; \{x, w\} \subseteq X_i \\ \\ \infty \;\; \text{otherwise}, \end{cases}$$

is an extended metric *of X and has $\{X_i : i \in I\}$ as metric components.*

In the next theorem, we prove that the metric bornology $\mathscr{B}_d(X)$ is proper, locally bounded and it has a countable base.

Theorem 2.1.5. *[34, Theorem 10.1] (X, d) be an extended metric space. Then, the metric bornology $\mathscr{B}_d(X)$ is proper, locally bounded and with countable base.*

Proof. Let $\mathscr{B} = \mathscr{B}_d(X)$ and $A \in \mathscr{B}$. For $\epsilon, \delta > 0$ and since diameter of cl(A) is equal to diameter of diam$_d(A)$, we have cl$(A) \in \mathscr{B}$. However, the ϵ-neighbourhood of A is an open set that is bounded with diameter less than or equal to diameter of $A + 2\epsilon$, Hence, $\mathscr{B}$ is proper.
Let $x \in \mathscr{B}$ be an arbitrary point, then the ϵ-neigbourhood $U_x \in \mathscr{B}$. Therefore, $\mathscr{B}$ is locally bounded.
To show that $\mathscr{B}$ has a countable base, let us choose a fixed point $x_0 \in X$ and for $n \in \mathbb{N}$ let G_n denote the n-neighbourhoods of x_0. We need to prove that $\mathscr{G} = \{G_1, G_2, \dots\}$ form a basis for bornology $\mathscr{B}$. Let M be an arbitrary bounded set in $\mathscr{B}$ and we choose a point $x \in M$ with $\rho = d(x_0, x_1)$ and $\delta =$diam$_d(M)$. Thus, we have $d(x_0, x) \leq \rho + \delta$ for each $x \in M$. Hence M is contained in G_n if $n > \rho + \delta$. □

The following theorem is the well known Hu's theorem.

Theorem 2.1.6. *[34, Thorem 2] Let $((X, \tau), \mathscr{B})$ be a bornological universe. Then $\mathscr{B}$ is metrizable if and only if the following conditions are satisfied:*

(i) topological space X is metrizable;

(ii) the bornology $\mathscr{B}$ has a countable base, a closed base and an open base;

(iii) for each $B_1 \in \mathscr{B}$ there is a $B_2 \in \mathscr{B}$ with cl$(B_1) \subseteq$ int(B_2).

Proof. [34, Proof of Theorem 2]. See also Theorem 3.2.1 where this proof has been generalized to the category of quasi-metric spaces. □

We state the following lemma and its asymmetric proof is provided in Lemma 3.2.3

Lemma 2.1.2. *[4, Lemma 2.1] Let (X, d) be an extended metric space. Then*

(i) the family of all finite unions of open balls forms a base for $\mathscr{B}_d(X)$;

(ii) the metric bornology $\mathscr{B}_d(X)$ contains the bornology of relatively compact subsets of X;

(iii) whenever (x_n) is a Cauchy sequence in X, then $\{x_n : n \in \mathbb{N}\} \in \mathscr{B}_d(X)$.

The next two theorems prove the metrizabiliy of the bornological universe of the extended metrics.

Theorem 2.1.7. *[4, Theorem 4.1] Let (X, d) be an extended space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_d(X)$ if and only if there exists $\mathcal{A} \subset \mathscr{B}$ such that $\downarrow (\Sigma(\mathcal{A})) = \mathscr{B}$ and a partition $\{\mathcal{A}_i : i \in I\}$ of $\mathcal{A}$ with the following properties:*

(i) each partition $\mathcal{A}_i$ of $\mathcal{A}$ contains a nonempty subset of X;

(ii) for all $A_1 \in \mathcal{A}_i$, there exists $A_2 \in \mathcal{A}_i$ with $cl(A_1) \subseteq int(A_2)$ for each $i \in I$;

(iii) whenever $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ for $i \neq j$, then $A_i \cap A_j = \emptyset$;

(iv) each $\mathcal{A}_i$ has a countable subfamily which is cofinal in $\mathcal{A}_i$ with respect to inclusion.

Proof. See [4, Theorem 4.1]. This proof has also been generalized to the quasi-metric case in Theorem 3.2.2. □

We now turn our attention to the uniform metrization of a bornology.

Theorem 2.1.8. *[4, Theorem 4.3] Let (X, d) be an extended metric space. The following conditions are equivalent:*

(a) the set of metric components induced by d is countable;

(b) there exists an extended metric d_1 such that $\mathscr{B}_{d_1}(X) = \mathscr{B}_d(X)$.

Proof. [4, Proof of Theorem 4.3]. See the asymmetric version of this result in Theorem 3.2.3. □

We next define the notion of the weakly bounded sets in metric spaces.

Definition 2.1.4. *Let (X, d) be a Hausdorff space. We define a subset A of X to be weakly bounded if its intersection with each metric component is bounded.*

Theorem 2.1.9. *[4, Theorem 4.4] Let (X, d) be a Hausdorff space that is a free union of $\{X_i : i \in I\}$. Then, the following conditions are equivalent:*

(i) the set I is finite and X is locally compact and second countable;

(ii) there exists an extended metric d with metric components $\{X_i : i \in I\}$ such that each closed and weakly bounded subset is compact.

2.2 Total boundedness in metric spaces

In this section, we summarize results of total boundedness and bornologies of totally bounded sets on metric spaces studied by Beer and others in [6]. We will generalize these concepts to the framework of quasi-metrics later in this thesis.

Definition 2.2.1. *[56, Definition 3] A pseudometric space (X, d) is* totally bounded *if for each $\epsilon > 0$ there exists a finite subset $\{x_1, x_2, x_3, \ldots x_k\}$ of X such that $X \subseteq \cup_{j=1}^{k} D_d(x_j, \epsilon)$.*

The following lemma compares totally boundedness and the usual metric boundedness. Its proof is given in asymmetric version (see Lemma 4.1.1).

Lemma 2.2.1. *Every totally bounded pseudometric space (X, d) is bounded.*

The following example illustrates a converse of the above lemma and it is generalized to quasi-metric spaces in Example 4.1.2.

Example 2.2.1. *Let us equip the set of natural numbers $\mathbb{N}$ with discrete metric*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & x = y. \end{cases}$$

The metric space $(\mathbb{N}, d)$ is bounded but not totally bounded.

Proof. For all $x, y \in \mathbb{N}$, we can find an $\epsilon > 0$ such that $d(x, y) \leq \epsilon$. But any finite set $\{x_1, x_2, x_3, \ldots x_n\}$ of $\mathbb{N}$ with the d, the set $\mathbb{N} \not\subseteq \bigcup_{i=1}^{n} D_d(x_i, \epsilon)$. □

The next result proves the metrizability of the bornology of totally bounded sets. We will present it in quasi-metric settings later in this thesis.

Theorem 2.2.1. *[6, Theorem 3.1] Let (X, d) be a metric space and let $x_0 \in X$. The following conditions are equivalent:*

(1) there exists an equivalent metric d' such that $\mathscr{B}_d(X) = \mathscr{T}\mathscr{B}_{d'}(X)$;

(2) the metric space (X, d) is separable;

(3) there is an embedding Φ from X into some metrizable space (Y, d') such that the family $\{cl_Y(\Phi(D_d[x_0, n])) : n, s \in \mathbb{N}\}$ is cofinal in $\mathscr{K}_0(Y)$;

(4) There exists an equivalent metric d' with $\mathscr{B}_d(X) = \mathscr{T}\mathscr{B}_{d'}(X) = \mathscr{B}_{d'}(X)$.

Proof. [6, Proof of Theorem 3.1]. Theorem 4.2.4 is the asymmetric version of this result. □

Definition 2.2.2. *[6] Let (X, d) be metric space. Given the point $p \notin X$ and a metric bornology $\mathscr{B}_d(X)$ on X, we define the* one-point extension *of X associated with $\mathscr{B}_d(X)$ by $X' = X \cup \{p\}$.*

If τ is the topology X, then the corresponding topology on X' is defined by

$$\tau \cup \left\{ \{p\} \cup X \backslash B : B = \mathrm{cl}(B) \in \mathscr{B}_d(X) \right\}.$$

The metric bornology associated with X' is denoted by $\mathscr{B}_d(X')$.

Remark 2.2.1. *[6] If $p \notin X$ and $\mathcal{B}_0$ is a closed base of the bornology then $\{\{p\} \cup X \backslash B : B \in \mathcal{B}_0\}$ forms a neighbourhood base at the point p.*

The next result illustrates the concept of one point extension on a bornology of totally bounded sets.

Theorem 2.2.2. *[6, Theorem 3.4] Let (X, d) be a metric space The following conditions are equivalent:*

(a) the bornology $\mathscr{TB}_d(X)$ has a base which is countable;

(b) there is an equivalent metric d' such that $\mathscr{TB}_d(X) = \mathscr{B}_{d'}(X)$;

(c) the one-point extension of X associated with $\mathscr{TB}_d(X)$ is metrizable;

(d) the one-point extension of X associated with $\mathscr{TB}_d(X)$ has a neighbourhood base at the ideal point.

Proof. $(a) \Rightarrow (b)$: Since $\mathscr{TB}_d(X)$ has a countable base by Hu's theorem, there exists an equivalent metric d' such that $\mathscr{TB}_d(X) = \mathscr{B}_{d'}(X)$.
$(b) \Rightarrow (c)$: By (2), $\mathscr{TB}_d(X) = \mathscr{B}_{d'}(X)$. Also from [5, Theorem 4.3] the metric bornology $\mathscr{B}_d(X')$ is metrizable. Thus, $\mathscr{TB}_d(X')$ is metrizable.
$(c) \Rightarrow (d)$ Since $\mathscr{TB}_d(X')$ is metrizable by Hu's theorem it has a closed base. Thus, by the Remark 2.2.1, $\mathscr{TB}_d(X')$ has a neighbourhood base at the ideal point.
$(d) \Rightarrow (a)$: If the bornology $\mathscr{TB}_d(X')$ has a neighbourhood base at each point, then $\mathscr{TB}_d(X)$ has a countable base. □

The next result indicates when the metric bornology becomes a bornology of totally bounded sets. We have generalized this result to quasi-metric spaces in Theorem 4.2.6.

Theorem 2.2.3. *[6, Theorem 4.1] Let $\mathscr{B}$ be a family of nonempty subsets of a metric space (X,d). Then $\mathscr{B} = \mathscr{TB}_d(X)$ if and only if there exists an embedding $\Psi : (X,d) \longrightarrow (\widetilde{X}, \widetilde{d})$ with the following property:*

$$\mathscr{B} = \left\{ A \in \mathscr{P}_0(X) : \Psi(A) \text{ is relatively compact in } \widetilde{X} \right\}.$$

Proof. Let $\mathscr{B} = \mathscr{TB}_d(X)$ and $\Psi : (X,d) \longrightarrow (\widetilde{X}, \widetilde{d})$. If $A \in \mathscr{TB}_d(X)$ then $\Psi(A) \in \mathscr{TB}_d(\Psi(A)) \subseteq \mathscr{TB}_d(\widetilde{X})$. The set $\mathrm{cl}[\psi(E)] \subseteq \mathscr{TB}_d(\widetilde{X})$ too, thus it is totally bounded and complete implying that it is compact.
On the other hand, let $\Psi : (X,d) \longrightarrow (\widetilde{X}, \widetilde{d})$ and let $E = \mathrm{cl}[\psi(X)] \subseteq \widetilde{X}$. Since E is a closed subspace of $\widetilde{X}$, it is completely metrizable. Let $\widetilde{d}$ be a compatible complete metric for E and d be its trace on $\Psi(X) \times \Psi(X)$. If $A \in \mathscr{B}$, then $\mathrm{cl}_{\widetilde{X}}(\Psi(A)) = \mathrm{cl}_E(\Psi(A))$ is compact and $\Psi(A)$ is $\widetilde{d}$-totally bounded but since $\Psi(A) \subseteq \Psi(X)$, we get $\Psi(A) \in \mathscr{TB}_d(\Psi(X))$. Conversely if $\Psi(A) \in \mathscr{TB}_d(\Psi(X))$ then by completeness $\mathrm{cl}_{\widetilde{X}}(\Psi(A)) = \mathrm{cl}_E(\Psi(A))$ is compact. So $A \in \mathscr{B}$. □

The next proposition proves that the bornology of nonempty separable subsets of a metrizable space becomes a bornology of totally bounded sets.

Proposition 2.2.1. *[6, Proposition 4.4]. Let $\mathcal{S}(X)$ be the bornology of nonempty separable subsets of a metric space (X,d). Then $\mathcal{S}(X) = \mathscr{TB}_d(X)$ if and only if X is separable.*

Proof. See the proof of [6, Proposition 4.4]. This proof in quasi-metric framework is given in Proposition 4.2.1. □

Definition 2.2.3. *[56, Definition 23.6] Let (X,d) be a metric space. A star-development for a space X is a sequence $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \ldots$ of open covers of X such that $\mathcal{D}_n$ refines $\mathcal{D}_{n-1}$ and each $x \in X$, $\{\mathrm{st}(x, \mathcal{D}_n) : n \in \mathbb{N}\}$ is a neighbourhood base.*

If S is a subset of X then the star of S with respect to $\mathcal{D}$ is the union of all the sets $D \in \mathcal{D}$ that intersects S, i.e., $\mathrm{st}(S, \mathcal{D}) = \cup\{D : D \in \mathcal{D}, S \cap D \neq \emptyset\}$.

The proof of the following theorem can be found in [6, p. 12].

Theorem 2.2.4. *Let $\mathscr{B}$ be a family of nonempty subsets of a metric space (X,d). Then $\mathscr{B} = \mathscr{TB}_d(X)$ if and only if there exists a star-development $\{\mathcal{D}_n : n \in \mathbb{N}\}$ for X such that*

$$\mathscr{B} = \left\{ A \in \mathscr{P}_0(X) : \ \forall n \in \mathbb{N}, \ \mathcal{D}_n \text{ has a finite subcover of A} \right\}.$$

2.3 Bourbaki-boundedness in metric spaces

In this section, we present the Beer and Garrido's results on metric bornologies of Bourbaki-bounded sets with locally Lipschtz functions (see [9]). We will generalize these results to quasi-metric spaces in chapter 5.

In what follows, we define the concept of Lipschitz functions on metric spaces and thereafter give some examples.

Definition 2.3.1. *[20, p. 283] The function $f : (X, d) \longrightarrow (Y, \rho)$ is said to be* Lipschitz *if there exists $k \geq 0$ such that $\rho(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$.*

Definition 2.3.2. *The function $f : (X, d) \longrightarrow (Y, \rho)$ is said to be locally Lipschitz if for every $x \in X$, there exists $k_x \geq 0$ and $r_x > 0$ such that $\rho(f(x), f(y)) \leq k_x d(x, y)$, whenever $d(x, y) < r_x$.*

Definition 2.3.3. *A locally Lipschitz function is said to be:*

(i) uniformly locally Lipschitz *if r in Definition 2.3.2 is chosen independent of x;*

(ii) Lipschitz in the small *if k and r in Definition 2.3.2 are chosen independent of x.*

We present some examples of Lipschitz functions on metric spaces.

Example 2.3.1. *[20, Example 1] Let $X \subset \mathbb{R}$ and $X = \cup_{n\in\mathbb{N}}[2n, 2n + 1]$ and $f(x) = g_n(x - 2n)$ for $x \in [2n, 2n + 1]$ where (g_n) is any sequence of functions on $[0, 1]$ uniformly converging to $g(x) = \sqrt{x}$. The function f is uniformly locally Lipschitz.*

Example 2.3.2. *[20, Example 2] Let $X = ([0, \infty) \times \{0, 1\}) \cup (\{0\} \times [0, 1])$ with the usual metric on $\mathbb{R}^2$. The function defined by $f(x, 0) = x$, $x \geq 0$ and $f(y, 0) = 0$, $y \in [0, 1]$ is Lipschitz in the small but not Lipschitz.*

We recall the definition of an enlargement on a metric space.

Definition 2.3.4. *[9] Let (X, d) be metric space, $x \in X$, $\epsilon > 0$ and $D_d(x, \epsilon)$ be an open ball of radius ϵ. For $A \subset X$, we define the ϵ- enlargement of A as: $A^\epsilon = D_d(A, \epsilon) = \{x \in X : \inf\{d(x, a) : a \in A\} < \epsilon\} = \bigcup_{a\in A} D_d(a, \epsilon)$.*

Let (X, d) be a metric space and $x \in X$. We define the sets $D_d^n(x, \delta)$ for $n = 0, 1, 2, \ldots$ recursively so that $D_d^0(x, \delta) = \{x\}$ and $D_d^{n+1}(x, \epsilon) = D_d(D_d^n(x, \epsilon), \epsilon)$.

The proof of the next lemma is given in nonsymmetric version in Lemmas 5.2.1 and 5.2.2.

Lemma 2.3.1. *[9, p. 258] Let (X, d) be a metric space and A be a subset of X. If $\epsilon, \delta > 0$ then $D_d(D_d(A, \epsilon), \delta) \subset D_d(A, \epsilon + \delta)$ and $D_d^n(x, \epsilon) \subseteq D_d^{n+1}(x, \epsilon)$.*

Definition 2.3.5. *[3] Let (X, d) be a metric space. An ϵ-chain of length n from x to y in X is a finite sequence $\{x_0, x_1, x_2, x_3, \ldots x_n\} \subset X$ such that $x = x_0$; $y = x_n$ and for each $i \in \{1, 2, 3, \ldots\}$, $d(x_{i-1}, x_i) < \delta$.*

Definition 2.3.6. *[19, Definition 1] Let (X, d) be a metric space. We call a subset A of X Bourbaki-bounded if for each $\epsilon > 0$ there is a finite subset $F = \{x_1, x_2, \ldots, x_k\}$ of X and $n \in \mathbb{N}$ such that $A \subseteq \cup_{j=1}^k D_d^n(x_j, \epsilon)$.*

Note that if we put $n = 1$ in Definition 2.3.6 above, then we have the notion of total boundedness presented in the previous section.

We give the following examples of Bourbaki-boundedness in metric spaces.

Example 2.3.3. *[18, Example 5.1] Let $(X, m_{||.||})$ be an infinite dimensional normed space. For $\epsilon > 0$, the ball $D_{m_{||.||}}(0, \epsilon)$ is Bourbaki-bounded.*

Example 2.3.4. *[18, Example 5.2] Let $X = \mathbb{R}$ and $m^* = \min\{1, m\}$ be a metric where m is the usual metric. Then every subset in (X, m^*) is bounded but not Bourbaki-bounded.*

Lemma 2.3.2. *Let (X, d) and (Y, ρ) be two metric spaces and the function f from (X, d) to(Y, ρ). If $B \in \mathscr{B}_d(X)$ and f is Lipschitz, then $f(B) \in \mathscr{B}_\rho(Y)$.*

Proof. Let $B \in \mathscr{B}_d(X)$. Then there exists $k \geq 0$ such that for all $x, y \in B$, $d(x, y) < k$. But since f is Lipschitz, $\rho(f(x), f(y)) \leq kd(x, y) < k$. Thus, $f(B) \in \mathscr{B}_\rho(Y)$. □

The next result is about bornologies of locally Lipschitz functions with relatively compact subsets. Theorem 5.3.1 is an asymmetric version of this result.

Theorem 2.3.1. *[9, Theorem 3.1] Let (X, d) be a metric space and let A be a subset of X. The following conditions are equivalent:*

(i) the subset A is relatively compact;

(ii) *if* (Y, ρ) *is a metric space and* $f : (X, d) \longrightarrow (Y, \rho)$ *is continuous, then* $f(A) \in \mathscr{B}_\rho(Y)$*;*

(iii) *if* (Y, ρ) *is another metric space and* $f : (X, d) \longrightarrow (Y, \rho)$ *is locally Lipschitz, then* $f(A) \in \mathscr{B}_\rho(Y)$*;*

(iv) *if* $f : (X, d) \longrightarrow (\mathbb{R}, \rho)$ *is locally Lipschitz, then* $f(A) \in \mathscr{B}_\rho(\mathbb{R})$.

Proof. $(i) \Rightarrow (ii)$: Since A is relatively compact and f is continuous, $f(B)$ is a relatively compact. Thus, $f(A)$ is bounded.
(ii) $\Rightarrow (iii)$: Let (Y, ρ) be a metric space and f be a locally Lipschitz function. Since f is locally Lipschitz and continuous by (ii) $f(A) \in \mathscr{B}_\rho(Y)$.

$(iii) \Rightarrow (iv)$: Letting $Y = \mathbb{R}$ in (iii), we get $f(A) \in \mathscr{B}_\rho(\mathbb{R})$.
$(iv) \Rightarrow (i)$: See the last implication of [9, Proof of Theorem 3.1]. □

Definition 2.3.7. *Given a metric space* (X, d)*, a subset* A *of* X *is said to be uniformly discrete if there exists* $\epsilon > 0$ *such that whenever* $x, y \in X$ *and* $x \neq y$ *we have* $d(x, y) \geq \epsilon$.

The following theorem is about bornologies of the uniformly locally Lipschitz functions with totally bounded subsets of metric spaces.

Theorem 2.3.2. *[9, Theorem 3.2] Let* (X, d) *be a metric space and let* A *be a nonempty subset. The following conditions are equivalent:*

(i) *the subset* A *is totally bounded;*

(ii) *whenever* (Y, ρ) *is a metric space and* $f : (X, d) \longrightarrow (Y, \rho)$ *maps Cauchy sequences to Cauchy sequences, then* $f(A) \in \mathscr{B}_\rho(Y)$*;*

(iii) *whenever* (Y, ρ) *is a metric space and* $f : (X, d) \longrightarrow (Y, \rho)$ *is uniformly locally Lipschitz, then* $f(A) \in \mathscr{B}_\rho(Y)$*;*

(iv) *whenever* $f : (X, d) \longrightarrow (\mathbb{R}, \rho)$ *is uniformly locally Lipschitz, then* $f(A) \in \mathscr{B}_\rho(\mathbb{R})$.

Proof. $(i) \Rightarrow (ii)$: Let $f : (X, d) \longrightarrow (Y, \rho)$ be the function that maps Cauchy sequences to Cauchy sequences. If (x_n) is a Cauchy sequence in X then $f(x_n)$ is Cauchy sequence in Y. By [21, p. 155], $f(A)$ is a totally bounded subset of Y.
(ii) $\Rightarrow$ (iii) : Let (Y, ρ) be a metric space and $f : (X, d) \longrightarrow (Y, \rho)$ be a locally Lipschitz function. Since f is uniformly locally Lipschitz function it

is uniformly continuous. Thus by (ii), $f(A) \in \mathscr{B}_\rho(Y)$.

(iii) $\Rightarrow$ *(iv)* : Letting $Y = \mathbb{R}$ in (iii), we get $f(A) \in \mathscr{B}_\rho(\mathbb{R})$.

(iv) $\Rightarrow$ *(i)*: See (4) $\Rightarrow$ (1) of [9, Proof of Theorem 3.2]. □

We next turn to bornologies of Lipschitz in the small functions with Bourbaki-bounded subsets. The last part of this theorem is the modification of [3, Theorem 2]. In Chapter 5, we have generalized this theorem to quasi-metric spaces.

Theorem 2.3.3. *[9, Theorem 3.3] Let (X, d) be a metric space and let A be a nonempty subset. The following conditions are equivalent:*

(i) the set A is Bourbaki-bounded;

(ii) let (Y, ρ) be a metric space and $f : (X, d) \longrightarrow (Y, \rho)$ be uniformly continuous, then, $f(A) \in \mathscr{B}_\rho(Y)$;

(iii) whenever (Y, ρ) is a metric space and $f : (X, d) \longrightarrow (Y, \rho)$ is Lipschitz in the small, then $f(A) \in \mathscr{B}_\rho(Y)$;

(iv) if $f : (X, d) \longrightarrow (\mathbb{R}, \rho)$ is Lipschitz in the small, then $f(A) \in \mathscr{B}_\rho(\mathbb{R})$.

Proof. The proof can be found in [9, Theorem 3.3]. Also see Theorem 5.3.2 where we have provided its proof in the category of quasi-metric spaces. □

Chapter 3

Bornology of bounded sets on quasi-pseudometric spaces

In this chapter, we start our own investigations by revisiting our previous studies (see [31]). In [52], we redefined and improved our own universal space constructed on an extended quasi-metric space (X, q) during my MSc dissertation (compare [31, Definition 3.2.1], [52, Section 3] and Definition 3.1.2). In the first section, we present the universal space which was constructed on a set of (f, A) where f is a real-valued continuous function on $\tau(q^s)$-closed set A. In addition, we also prove that the universal space is a bicomplete space (see also [52, Remark 3.1]). However, Propositions 3.1.1 and 3.1.2 provide the isometries between the extended quasi-metric spaces and the universal spaces. Morever, in [52] we improved the quasi-metrizability theorems of the bornological universes. For instance, we present the improved versions of quasi-metrizability theorems and show that the bornology of nonempty sets coincides with the quasi-metric bornology (compare [31, Theorems 4.2.1 and 4.2.2] and Theorems 3.2.2 and 3.2.3). Note that most of the results we present in this chapter are published in our paper [52].

3.1 Universal space of extended quasi-metric spaces

This section presents the universal space of an extended T_0-quasi-metric space constructed in [31] and improved by us in [52].

Definition 3.1.1. *Let (X, q) be an extended T_0-quasi-metric space. Then for any $x, y, \in X$, we define a relation $\mathcal{R}_q$ on X by $x \, \mathcal{R}_q \, y$ provided $q(x, y) < \infty$ and $q(y, x) < \infty$.*

The next lemma proves that the relation in Definition 3.1.1 is actually an equivalence relation.

Lemma 3.1.1. *If (X, q) is an extended T_0-quasi-metric space, then the relation $\mathcal{R}_q$ is an equivalence relation on X.*

Proof. For $x, y \in X$, let $q(x, y) < \infty$ and $q(y, x) < \infty$ then $q(x, x) = 0 < \infty$ and $x\mathcal{R}_q x$.
We know that $x\mathcal{R}_q y$ if and only if $q(x, y) < \infty$ and $y\mathcal{R}_q x$ if and only if $q(y, x) < \infty$.
For $x, y, z \in X$, if $x\mathcal{R}_q y$ and $y\mathcal{R}_q z$ then $q(x, y) < \infty$ and $q(y, z) < \infty$ respectively. By the triangle inequality, $x\mathcal{R}_q z$. Therefore, $\mathcal{R}_q$ is an equivalence relation on X. □

Note that if $q = q^t$ then the relation $\mathcal{R}_q$ is exactly the equivalence relation in Lemma 2.1.1.
Let $x \in X$. Then the equivalence class of x denoted by $\mathrm{qmc}_q(x)$ will be called the *quasi-metric component* of x.

Remark 3.1.1. *For any extended T_0-quasi-metric space (X, q), it is easy to see that $\mathrm{qmc}_{q^s}(x) \subseteq \mathrm{qmc}_q(x)$ whenever $x \in X$. Here $\mathrm{qmc}_{q^s}(x)$ is a metric component presented in Section 2.1.*

Let (X, q) be an extended T_0-quasi-pseudometric space and let $x \in X$. Obviously $\mathrm{qmc}_q(x) \neq \emptyset$, since $x \in \mathrm{qmc}_q(x)$. We equip $\mathrm{qmc}_q(x)$ with the quasi-metric q_x defined by $q_x := q|_{\mathrm{qmc}_q(x)}$. It follows that

$$\tau(q) = \bigcup_{x\in X} \tau(q_x) \text{ and } X = \bigoplus_{x\in X} \mathrm{qmc}_q(x).$$

Moreover, if X can be partitioned into nonempty $\tau(q)$-clopen sets $\{\mathrm{qmc}_q(x) : x \in X\}$ which are quasi-metrizable, then choosing a compatible quasi-metric q_x for $\mathrm{qmc}_q(x)$ whenever $x \in X$, the extended T_0-quasi-metric q can be defined by

$$q(y, z) = \begin{cases} q_x(y, z) & \text{if } \{y, z\} \in \mathrm{qmc}_q(x) \text{ for some } x \in X, \\ \infty & \text{otherwise.} \end{cases}$$

Let (X, q) be a quasi-pseudometric space. In the sequel, we set

$$\mathcal{C}(X, q) = \{(f_1, f_2) : f_1 \text{ and } f_2 \text{ are continuous real-valued functions on } X\}$$

and

$$\mathcal{C}^b(X, q) = \{(f_1, f_2) \in \mathcal{C}(X, q) : f_1 \text{ and } f_2 \text{ are bounded on } (X, q^s)\}.$$

We equip $\mathcal{C}(X,q)$ with the extended T_0-quasi-metric space Q_q defined by

$$Q_q(f,g) = \sup_{x\in X}(f_1(x)\dot{-}g_1(x)) \vee \sup_{x\in X}(g_2(x)\dot{-}f_2(x))$$

whenever $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{C}(X,q)$ (see [37]).

Obviously, $(\mathcal{C}^b(X,q), Q)$ is a T_0-quasi-metric space, when $Q = Q_q|_{\mathcal{C}^b(X,q)}$. Note that for a fixed $a \in X$ and for any $b \in X$, we define the function pair $e_X(b) = ((e_X)_1(b), (e_X)_2(b))$, where

$$(e_X)_1(b) = q(b,x) - q(a,x) \text{ and } (e_X)_2(b) = q(x,b) - q(x,a),$$

whenever $x \in X$. It is clear that $(e_X)_1$ and $(e_X)_2$ are bounded by $q^s(a,b)$. The map $e_X : (X,q) \longrightarrow (\mathcal{C}^b(X,q), Q)$ defined by $b \mapsto e_X(b)$ yields an isometric embedding. Indeed, for any $b, b' \in X$, it follows that

$$\sup_{x\in X}[(q(b,x) - q(a,x))\dot{-}(q(b',x) - q(a,x))] = q(b,b')$$

and

$$\sup_{x\in X}[(q(x,b') - q(x,a))\dot{-}(q(x,b) - q(x.a))] = q(b,b').$$

Therefore $Q(e_X(b), e_X(b')) = q(b,b')$.

We observe that $Q^s(f,g) = \sup_{x\in X}|f_1(x) - g_1(x)| \vee \sup_{x\in X}|g_2(x) - f_2(x)|$ whenever $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{C}^b(X,q)$.

Let $\triangle_q(X,q) = \{(f;A) : f = (f_1,f_2) \in \mathcal{C}(A,q)$ and $A \in \mathscr{C}_0(X,q^s)\}$. Here, $\mathscr{C}_0(X,q^s)$ is the set of nonempty subsets of X which are closed with respect to $\tau(q^s)$.

Definition 3.1.2. *Let (X,q) be an extended T_0-quasi-metric. We define an extended T_0-quasi-metric E_X on $\triangle_q(X,q)$ by*

$$E_X((f;A),(g;B)) = \begin{cases} \infty & \text{if } A \neq B \\ \sup_{x\in A}(f_1(x)\dot{-}g_1(x)) \vee \sup_{x\in A}(g_2(x)\dot{-}f_2(x)) & \text{if } A = B. \end{cases}$$

The following remark can be compared to Theorem 2.1.1, it illustrates the bicompleteness of the universal space on a quasi-metric space.

Remark 3.1.2. *We observe that*

$$(E_X^s)((f;A),(g;B)) = \begin{cases} \infty & \text{if } A \neq B \\ \sup_{x\in A}|(f_1(x) - g_1(x))| \vee \sup_{x\in A}|(g_2(x) - f_2(x))| & \text{if } A = B. \end{cases}$$

Furthermore, $(\triangle_q(X,q), E^s_X)$ is a complete extended metric space by Theorem 2.1.1. Therefore, by Preposition 1.1.2, the extended T_0-quasi-metric space $(\triangle_q(X,q), E_X)$ is bicomplete.

The next results are about isometries between the extended quasi-metric spaces and the universal spaces.

Proposition 3.1.1. *(compare Theorem 2.1.3) Let (X,q) be a T_0-quasi-metric space. Then the map $\theta : (X,q) \longrightarrow (\triangle_q(X,q), E_X)$ given by $\theta(x) = ((\theta_1(x), \theta_2(x)); qmc_q(x))$, where $\theta_1(x) = q(x,\cdot)|_{qmc_q(x)}$ and $\theta_2(x) = q(\cdot,x)|_{qmc_q(x)}$ whenever $x \in X$ is an isometry. Moreover, θ is injective.*

Proof. Let $x, y \in X$ such that $x \neq y$.
Case 1: If $q(x,y) = \infty$ and $q(y,x) = \infty$, then $\mathrm{qmc}_q(x) \neq \mathrm{qmc}_q(y)$. We have $E_X(\theta(x), \theta(y)) = \infty = q(x,y)$.

Case 2: If $q(x,y) = \infty$ and $q(y,x) < \infty$, then $\mathrm{qmc}_q(x) \neq \mathrm{qmc}_q(y)$. So $E_X(\theta(x), \theta(y)) = \infty = q(x,y)$.

Case 3: If $q(x,y) < \infty$ and $q(y,x) = \infty$, then case is similar to Case 2.

Case 4: If $q(x,y) < \infty$ and $q(y,x) < \infty$, then $\mathrm{qmc}_q(x) = \mathrm{qmc}_q(y)$. One sees that $\sup_{a \in \mathrm{qmc}_q(x)}(q(x,a) - q(y,a)) = q(x,y)$ by letting $y = a$ and by the triangle inequality. Similarly, $\sup_{a \in \mathrm{qmc}_q(x)}(q(a,y) - q(a,x)) = q(x,y)$. Thus, $E_X(\theta(x), \theta(y)) = q(x,y)$. Hence, θ is an isometry map. Since (X,q) is T_0-quasi-metric space, it follows by [47, Lemma 4] that the map θ is an injective.

□

Example 3.1.1. *(compare [37, Remark 4]) Let (X,q) be a T_0-quasi-metric space and let $\mathcal{P}_0(X)$ be the set of nonempty subsets of X. Then for any $A \in \mathcal{P}_0(X)$, set $(f_A)_1(x) := dist(A,x)$ and $(f_A)_2(x) := dist(x,A)$ whenever $x \in X$. Then for the function pair $f_A = ((f_A)_1, (f_A)_2)$, we have $(f_A)_1 : (X,q^t) \longrightarrow (\mathbb{R},u)$ is a nonexpansive map and $(f_A)_2 : (X,q) \longrightarrow (\mathbb{R},u)$ is a nonexpansive map. Furthermore, $q_H(A,B) = Q_q(f_A, f_B)$ whenever $A, B \in \mathcal{P}_0(X)$, where $q_H(A,B)$ is the extended Hausdorff quasi-pseudometric on $\mathcal{P}_0(X)$.*

Proposition 3.1.2. *If (X,m) is a quasi-metric space, then the map $\rho(f) = (f,f)$ defines an isometric embedding of $(\triangle_m(X,m), \rho_X)$ into $(\triangle_q(X,m), E_X)$.*

Proof. Suppose that $(f;A) \in \triangle_m(X,m)$. Then $((f,f);A) \in \triangle_q(X,m)$ as (f,f) is a pair of continuous functions and A is a nonempty $\tau(m)$-closed subset of X.

If $A \neq B$, then obviously we have

$$\rho_X((f;A),(g;B)) = \infty = E_X[((f,f);A),((g,g);B)].$$

$\square$

Theorem 3.1.1. *(compare Theorem 2.1.4) Let (X, q_X) and (Y, q_Y) be quasi-metric spaces. If $\rho : (X, q_X) \to (Y, q_Y)$ is a continuous surjective map, then there exists an isometry $\psi : (\triangle_q(Y, q_Y), E_Y) \to (\triangle_q(X, q_X), E_X)$.*

Proof. Suppose that $A \subseteq Y$ and $f = (f_1, f_2) \in \mathcal{C}(A, q_Y)$. Then we define ψ by

$$\psi((f;A)) = [(((f_1 \circ \rho)|_{\rho^{-1}(A)}, (f_2 \circ \rho)|_{\rho^{-1}(A)}); \rho^{-1}(A)].$$

We prove that ψ is an isometry.
Case 1: If A is $\tau(q_Y^s)$-closed subset of Y and we have functions given as $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{C}(A, q_Y)$, then $\rho^{-1}(A)$ is $\tau(q_X^s)$-closed subset of X. Since ρ is surjective, we have

$$\begin{aligned} & E_X(\psi((f;A)), \psi((g;A))) \\ =& \sup_{x\in\rho^{-1}(A)} [(f_1 \circ h)(x) \dot{-} (g_1 \circ h)(x)] \vee \sup_{x\in\rho^{-1}(A)} [(g_2 \circ h)(x) \dot{-} (f_2 \circ h)(x)] \\ =& \sup_{y\in A}(f_1(y) \dot{-} g_1(y)) \vee \sup_{y\in A}(g_2(y) \dot{-} f_2(y)) \\ =& E_Y((f;A),(g;A)). \end{aligned}$$

Case 2: Let A_1 and A_2 be two different $\tau(q_Y^s)$-closed subsets of Y. Then by continuity and surjectiveness of ρ, we have the sets $\rho^{-1}(A_1)$ and $\rho^{-1}(A_2)$ as the two different $\tau(q_X^s)$-closed subsets of X. Therefore,

$$E_X(\psi((f;A_1)), \psi((g;A_2))) = \infty = E_Y((f;A_1),(g;A_2))$$

whenever $f = (f_1, f_2) \in \mathcal{C}(A_1, q_Y)$ and $g = (g_1, g_2) \in \mathcal{C}(A_2, q_Y)$. $\square$

3.2 Bornology of extended quasi-pseudometric spaces

This section is as a result of the inspiration that we gave in [52] about q-boundedness and q^s-boundedness. It was proved that a nonempty subset A of a quasi-metric space can be q-bounded but not q^s-bounded (see Remark 3.2.1). It is important to emphasise that the results about bornological quasi-metrizability theorems in this section are the improvements of my MSc studies [31].

We begin this section with the following definition of boundedness using the double ball in quasi-metric spaces.

Definition 3.2.1. *[48, p. 4] Let (X, q) be a quasi-pseudometric space. An arbitrary subset A of X is called* q-bounded *if and only if there exists $x \in X$, $r > 0$ and $s > 0$ such that $A \subseteq D_q(x, r) \cap D_{q^t}(x, s)$.*

We point out that Definition 3.2.1 is equivalent to Definition 1.1.2 but slightly different with [53, Definition 1.5]. In [53], a subset A of X can be q-bounded but not q^t-bounded. In our context a subset A is q-bounded if and only if it is q^t-bounded.

Remark 3.2.1. *It is proved in [52, Remark 4.2]) that if a set is q^s-bounded, then it is q-bounded but the converse is not true.*

Moreover, if q is an extended quasi-pseudometric on X, then a subset B of X can be included in $D_q(x, \epsilon)$ for some $x \in X$ and its diameter is given by $\operatorname{diam}(B) = \{q(y, z) : y, z \in B\} = \infty$ (see [55, p. 2022]).

Definition 3.2.2. *Let (X, q) be a quasi-pseudometric space. Then $\mathscr{B}_q(X)$ is the collection of all q-bounded subsets of X in the sense of Definition 3.2.1.*

It has been observed in [52,53] that the collection $\mathscr{B}_q(X)$ forms a bornology on X called *quasi-metric bornology*. Moreover, we observe that $\mathscr{B}_{q^s}(X)$ is a metric bornology presented in Section 2.1.

We recall that two bornologies on set are *equivalent* if they determine the same collection of bounded sets (see [4, p. 3]).

From Remark 3.2.1, we have the following observation.

Lemma 3.2.1. *Let (X, q) is a quasi-metric space. Then the following statement is true:*

$$\mathscr{B}_{q^s}(X) \subseteq \mathscr{B}_q(X) \tag{3.2.1}$$

and the quasi-metric bornologies $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^t}(X)$ are equivalent.

Proof. Let $A \in \mathscr{B}_{q^s}(X)$, then A is q^s-bounded. By Remark 1.1.1, A is q-bounded too. Thus $A \in \mathscr{B}_q(X)$. The equivalence of $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^t}(X)$ comes from the fact that any subset A of X is q-bounded if and only if it is q^t-bounded. □

Let us recall that any quasi-pseudometric space (X, q) can be seen as a bitopological space $(X, \tau(q), \tau(q^t))$. This motivates the following definition that we translate from [53] to our context.

Definition 3.2.3. *Let (X, q) be a quasi-pseudometric space and $\mathscr{B}$ be a bornology on X. Then $(X, q, \mathscr{B})$ is a* bornological bi-universe*.*

In the above definition if $q = q^t$, then $(X, q, \mathscr{B})$ is bornological universe studied in Section 2.1.

Definition 3.2.4. *[52] Let (X, q) be a quasi-pseudometric space. A bornological bi-universe $(X, q, \mathscr{B})$ is called* quasi-metrizable *if $\mathscr{B} = \mathscr{B}_q(X)$.*

Let (X, q) be a quasi-metric space. Then q is said to induce a bornological bi-universe $(X, q, \mathscr{B})$ if $\mathscr{B} = \mathscr{B}_q(X)$.

Remark 3.2.2. *Let (X, q) be a quasi-metric space. Note that the bornological bi-universe $(X, q, \mathscr{B})$ is quasi-metrizable if and only if the bornological bi-universe $(X, q^t, \mathscr{B})$ is quasi-metrizable.*

Example 3.2.1. *[53] Consider the real line $\mathbb{R}$. We have the following bornologies in $\mathbb{R}$ defined by:*

$$UB(\mathbb{R}) = \{A \subseteq \mathbb{R} : \text{ there exists } r \in \mathbb{R} \text{ such that } A \subseteq (-\infty, r)\}$$

and

$$LB(\mathbb{R}) = \{A \subseteq \mathbb{R} : \text{ there exists } r \in \mathbb{R} \text{ such that } A \subseteq (r, +\infty)\}.$$

The topology $u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$ in $\mathbb{R}$ is called the *upper topology* and the topology $l = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$ in $\mathbb{R}$ is called the *lower topology.*

Lemma 3.2.2. *[52, Lemma 4.15] Let (X, q) be a quasi-metric space. If the bornological bi-universe $(X, q, \mathscr{B})$ is quasi-metrizable, then there exists q-characteristic function for the bornology $\mathscr{B}$.*

Definition 3.2.5. *Let (X, q) be a quasi-metric space. A bornology $\mathscr{B}$ on X will be called* q-proper *if for each $A \in \mathscr{B}$, there exists $B \in \mathscr{B}$ such that $cl_{\tau(q^t)}(A) \subseteq int_{\tau(q)}(B)$.*

Remark 3.2.3. *From [53, Proposition 4.6], it follows that if $(X, q, \mathscr{B})$ is the bornological bi-universe such that $\mathscr{B}$ has a q-characteristic function, then $\mathscr{B}$ is q-proper and second countable.*

The following result extends the well known Hu's theorem (Theorem 2.1.6) from metric point of view to quasi-metric settings.

Theorem 3.2.1. *[52, Theorem 4.18] Let (X, q) be a quasi-metric space. If $\mathscr{B}$ is a bornology on X, then the following conditions are equivalent:*
(a) the bornological bi-universe with respect to $\mathscr{B}$ is quasi-metrizable;
(b) there exists a q-characteristic function for $\mathscr{B}$;
(c) $\mathscr{B}$ is q-proper and it has countable base.

Let X be a set. If $\mathscr{A} \subseteq \mathcal{P}(X)$, then $\downarrow \mathscr{A}$ and $\sum \mathscr{A}$ are defined respectively by

$$\downarrow \mathscr{A} := \{B \in \mathcal{P}(X) : B \subseteq A \text{ for some } A \in \mathscr{A}\}$$

and

$$\sum \mathscr{A} := \left\{ \bigcup_{i=1}^{n} A_i : A_i \in \mathscr{A} \text{ whenever } i \in \{1, \cdots, n\} \text{ with } n \in \mathbb{N} \right\}.$$

If $\mathscr{B}$ is a bornology on X, then we say that a family $\mathscr{B}_0$ of subsets of X is a *base* of $\mathscr{B}$ if $\downarrow (\mathscr{B}_0) = \mathscr{B}$.
Note that the family of $\tau(q)$-relatively compact subsets of X is a natural bornology on (X, q).

Before we present the results about metrizability of the bornological universe, we state and prove the following important lemma.

Lemma 3.2.3. *(comapre Lemma 2.1.2) Let (X, q) be a T_0-extended quasi-metric space. Then*

(a) the family of all finite unions of double $\tau(q)$-open balls forms a base for quasi-metric bornology $\mathscr{B}_q(X)$;

(b) the quasi-metric bornology $\mathscr{B}_q(X)$ contains the bornology of $\tau(q)$-relatively compact subsets of X;

(c) if the sequence (x_n) is $\tau(q)$-convergent and $\tau(q^t)$-convergent on X, then $\{x_n : n \in \mathbb{N}\}$ is contained in a quasi-metric bornology $\mathscr{B}_q(X)$.

Proof. (a) Consider

$$\mathscr{F}_B = \left\{ \bigcup_{k=1}^{n} D_q(x_k, r_k) \cap D_{q^t}(x_k, s_k) : x_k \in X \text{ and } r_k, s_k \in (0, \infty) \text{ for } n \in \mathbb{N} \right\}.$$

We show that $\downarrow(\mathscr{F}_B) = \mathscr{B}_q(X)$.
Let $A \in \mathscr{F}_B$. Then $A = \bigcup_{k=1}^{n} D_q(x_k, r_k) \cap D_{q^t}(x_k, s_k)$. Therefore, A is in $\mathscr{B}_q(X)$ as A is q-bounded. So $\downarrow(\mathscr{F}_B) \subseteq \mathscr{B}_q(X)$.

Conversely, if $B \in \mathscr{B}_q(X)$, then there exists $x \in X$ and $r, s > 0$ such that $B \subseteq D_q(x, r) \cap D_{q^t}(x, s)$. Since $D_q(x, r) \cap D_{q^t}(x, s) \in \mathscr{F}_B$, it follows that $B \in \downarrow(\mathscr{F}_B)$.

(b) Let A be a $\tau(q)$-relatively compact subset of X. Then A is $\tau(q^s)$-bounded. Hence $A \in \mathscr{B}_q(X)$ by Lemma 3.2.1.

(c) Suppose that the sequence (x_n) is $\tau(q)$-convergent to $x \in X$. Then, for $\epsilon = 1$, there exists $N' \in \mathbb{N}$ such that $q(x_n, x) < \epsilon$ whenever $n \geq N'$.
If $r = \max\{1, q(x_0, x), q(x_1, x), \cdots, q(x_{N'}, x)\}$, then $q(x_n, x) < r$ whenever $n \in \mathbb{N}$.
By similar arguments, if (x_n) is $\tau(q^t)$-convergent to $x \in X$, then $q(x, x_n) < s$ whenever $n \in \mathbb{N}$. Therefore, $(x_n) \in D_q(x, r) \cap D_{q^t}(x, s)$ whenever $n \in \mathbb{N}$.
□

The next theorem is about metrizability of the bornological universe of a quasi-metric space.

Theorem 3.2.2. *(compare Theorem 2.1.7) Let (X, q) be a T_0-extended quasi-metric space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_q(X)$ if and only if there exists $\mathscr{A} \subseteq \mathscr{B}$ such that $\downarrow(\sum(\mathscr{A})) = \mathscr{B}$ and a partition $\left\{\mathscr{A}_i : i \in I\right\}$ of $\mathscr{A}$ with the following properties:*

(1) $\mathscr{A}_i$ contains a nonempty subset of X whenever $i \in I$;

(2) for all $A_1 \in \mathscr{A}_i$, there exists $A_2 \in \mathscr{A}_i$ with $cl_{\tau(q^t)}(A_1) \subseteq int_{\tau(q)}(A_2)$ whenever $i \in I$;

(3) whenever $A_i \in \mathscr{A}_i$ and $A_j \in \mathscr{A}_j$ for $i \neq j$, then $A_i \cap A_j = \emptyset$;

(4) each $\mathscr{A}_i$ has a countable subfamily which is cofinal in $\mathscr{A}_i$ with respect to inclusion.

Proof. Suppose q is a T_0-extended quasi-metric on X and $\mathscr{B} = \mathscr{B}_q(X)$.
If $\{X_i : i \in I\}$ is the quasi-metric components of X then,

$$\mathscr{A}_i = \{D_q(y, r) \cap D_{q^t}(y, s) : y \in X_i \text{ and } s, r > 0\}$$

and $\mathscr{A}$ is the collection of double open balls in X. By Lemma 3.2.3 (a)

$$\sum\left(\bigcup_{i\in I}\mathscr{A}_i\right)=\sum\left(\bigcup_{i\in I}\{D_q(y,r)\cap D_{q^t}(y,s):y\in X_i \text{ and } s,r>0\}\right)$$

and $\left\{\mathscr{A}_i : i\in I\right\}$ is a partition of $\mathscr{A}$. Thus, we have

$$\mathscr{B}_q(X)=\downarrow\left(\sum\left(\bigcup_{i\in I}\mathscr{A}_i\right)\right)=\downarrow\left(\sum\left(\mathscr{A}\right)\right).$$

If $A\in\mathscr{A}_i$ whenever $i\in I$, then A is double open ball. Therefore, $A\neq\emptyset$ and the property (1) holds.
Moreover, since $\mathscr{A}_i$ is a bornology on X_i whenever $i\in I$, then by Theorem 3.2.1 $\mathscr{A}_i$ is q_i-proper and has a countable base. Therefore, properties (2), (3) and (4) hold.
Conversely, suppose that there exists $\mathscr{A}$ with $\mathscr{A}\subseteq\mathscr{B}$ such that $\downarrow(\sum(\mathscr{A}))=\mathscr{B}$ for some $\mathscr{A}$ with $\left\{\mathscr{A}_i : i\in I\right\}$ is a partition of $\mathscr{A}$, where $\mathscr{A}$ satisfies properties (1), (2), (3) and (4).
Let $X_i=\bigcup_{i\in I}\mathscr{A}_i$. Then $X_i\neq\emptyset$ since the family $\mathscr{A}_i$ contains a nonempty subset of X whenever $i\in I$.
Furthermore, by property (2) we have that X_i is $\tau(q)$-open whenever $i\in I$. Moreover, from property (3) $\{X_i : i\in I\}$ is a pairwise disjoint family. Then we have that X_i is $\tau(q^t)$-closed and $\tau(q)$-open whenever $i\in I$.
For each $i\in I$, we set $\mathscr{B}_i:=\downarrow\left(\sum(\mathscr{A}_i)\right)$. It follows that for each $i\in I$, $\mathscr{B}_i$ is bornology on X_i since $\mathscr{A}_i$ is a cover of X_i. Any member of $\mathscr{B}_i$ is a finite union of elements of $\mathscr{A}_i$ and since the $\tau(q^t)$-closure of a finite union is the union of $\tau(q^t)$-closures, then family $\mathscr{B}_i$ satisfies the property (2) whenever $i\in I$. Similarly, the family $\mathscr{B}_i$ satisfies the property (3) whenever $i\in I$.
Finally, if $\mathscr{C}_i$ is a countable and cofinal family in $\mathscr{A}_i$ whenever $i\in I$, then $\sum(\mathscr{C}_i)$ is countable and cofinal in $\mathscr{B}_i$. That is whenever $i\in I$, the family $\mathscr{B}_i$ satisfies the property (4). Then by Theorem 3.2.1, there exists a quasi-metric q_i on X_i such that $\mathscr{B}_i=\mathscr{B}_{q_i}(X_i)$.
Since $\downarrow\left(\sum(\mathscr{A})\right)$ is assumed to be a cover of X, we conclude that $\mathscr{A}$ is a cover of X and so is $\{X_i : i\in I\}$ a cover of X. So we can define

$q : X \times X \to [0, \infty]$ by

$$q(x,y) = \begin{cases} q_i(x,y), & \text{there exists}\, i \in I \text{ with } \{x,y\} \subseteq X_i \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\downarrow \left(\sum (\bigcup_{i \in I} \mathscr{A}_i) \right) = \downarrow \left(\sum (\mathscr{A}) \right) = \mathscr{B}. \tag{3.2.2}$$

Since any member of $\mathscr{B}_i$ is a finite union of elements of $\mathscr{A}_i$, with q defined above, we have

$$\mathscr{B}_q(X) = \sum (\bigcup_{i \in I} \mathscr{B}_i) = \downarrow \left(\sum (\bigcup_{i \in I} \mathscr{A}_i) \right). \tag{3.2.3}$$

Combining (3.2.2) and (3.2.3), we have $\mathscr{B}_q(X) = \mathscr{B}$.

□

We end this chapter by proving in Theorem 3.2.3 that for any bornology on an extended quasi-metric space (X, q), there exists a uniformly equivalent quasi-metric on X such that their bornologies coincide.

Theorem 3.2.3. *(compare Theorem 2.1.8) Let (X, q) be a T_0-extended quasi-metric space. Then the set of quasi-metric components induced by q is countable if and only if there exists a compatible quasi-metric q' such that $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$.*

Proof. Let I be a countable set and $(x_i)_{i \in I}$ be a family of points in X. Suppose $\{\mathrm{qmc}_q(x_i) : i \in I\}$ is the set of distinct quasi-metric components induced by q. Then $\sum(\{D_q(x_i, n) \cap D_{q^t}(x_i, m) : i \in I, n, m \in \mathbb{N}\})$ is a countable base for $\mathscr{B}_q(X)$ and for each $A \in \mathscr{B}_q(X)$, there exists $B \in \mathscr{B}_q(X)$ with $\mathrm{cl}_{\tau(q^t)}(A) \subseteq \mathrm{int}_{\tau(q)}(B)$. Then by Theorem 3.2.1, there exists a compatible quasi-metric $q' = \min\{1, q\}$ such that $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$.

Conversely, suppose that there exists a compatible quasi-metric q' such that $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$. Then by Theorem 3.2.1, $\mathscr{B}_q(X)$ has a countable base and hence $\sum(\{D_q(x_i, n) \cap D_{q^t}(x_i, m) : i \in I, n, m \in \mathbb{N}\})$ contains a countable cofinal family within $\mathscr{B}_q(X)$. It follows that I is countable as $\{\{x_i\} : i \in I\}$ is a family of q-bounded sets. □

Chapter 4

Bornology of totally bounded sets on quasi-pseudometric spaces

In this chapter, we study totally bounded sets and generalize some well known results from metric spaces to the framework of quasi-metric spaces. In the first section, we present some comparisons of classic notions like boundedness, completeness and compactness with total boundedness. For example, it is proved in Theorem 4.2.1 that a subset of a quasi-metric space (X, q) is joincompact if and only if it is both bicomplete and q-totally bounded. In addition, we also show in Example 4.1.2 that a set can be q-bounded but not q-totally bounded in general. In the second section, we study bornologies of q-totally bounded sets. It turns out that if a quasi-metric space (X, q) is supseparable then the quasi-metric bornology studied by Olela Otafudu et al. [52] coincides with the bornology of totally bounded sets (see Theorems 4.2.4 and 4.2.5). We also use asymmetric version of Hu's theorem (Theorem 3.2.1) to study the bornology of totally bounded sets with one point extension.

4.1 Total boundedness in quasi-metric spaces

Many authors have defined total boundedness in quasi-pseudometric spaces (e.g [1,2,14,49]) but little or nothing is done in comparing total boundedness with other classic notions like boundedness, completeness, precompactness and compactness. It is for this reason that we revisit total boundedness in this section.
We first recall the following definitions from Section 1.1.

Definition 4.1.1. *A quasi-pseudometric space* (X, q) *is said to be q-precompact* *if for each* $\epsilon > 0$ *there is a set* $\{f_1, f_2, f_3, \dots f_k\} \subset X$ *such that* $X \subseteq \bigcup_{j=1}^{k} D_q(f_j, \epsilon)$.

Definition 4.1.2. *[26, Definition 5] A quasi-pseudometric space* (X, q) *is called* q-totally bounded *if for each* $\epsilon > 0$ *there is a set* $\{x_1, x_2, x_3, \dots x_k\} \subset X$ *such that* $X \subseteq \bigcup_{j=1}^{k} D_{q^s}(x_j, \epsilon)$.

Example 4.1.1. *If we equip the finite dimension space* $X = [0, 1]$ *with* T_0*-quasi-metric* q *defined in Example 1.1.2, then the pair* (X, q) *is* q*-totally bounded space.*

Proof. If we pick $A = \{0, 1\} \subset [0, 1]$ and $\epsilon = 1/2$, then $X = D_{q^s}(0, 1/2) \cup D_{q^s}(1/2, 1)$. Thus $X = [0, 1]$ is covered by $D_{q^s}(0, 1/2)$ and $D_{q^s}(1/2, 1)$. □

Lemma 4.1.1. *Every* q*-totally bounded quasi-pseudometric space* (X, q) *is* q*-bounded.*

Proof. Let (X, q) be a q-totally bounded subset of X. Then for $\epsilon > 0$ there exists a finite set $F = \{x_0, x_2, x_3, \dots, x_k\} \subset X$ such that $X \subseteq \cup_{j=1}^{k} D_{q^s}(x_j, \epsilon)$. Now fix x_1 and put $\delta = \max\{q^s(x_1, x_j) : j = 1, 2, 3, 4, \dots\} + \epsilon$ and we have $A \subset D_{q^s}(x_1, \delta)$. By Lemma 3.2.1, X is q-bounded. □

The following example illustrates the converse of Lemma 4.1.1. It shows that for infinite dimension spaces, q-boundedness does not imply q-total boundedness.

Example 4.1.2. *(compare Example 2.2.1) Let us equip the set of natural numbers* $\mathbb{N}$ *with the* T_0*-quasi-metric*

$$q(x, y) = \begin{cases} 0 & \textit{if } x \leq y \\ 1 & \textit{otherwise.} \end{cases}$$

The T_0*-quasi-metric space* $(\mathbb{N}, q)$ *is* q*-bounded but not* q*-totally bounded.*

Proof. For all $x, y \in \mathbb{N}$ we can find $k \geq 0$ such that $q(x, y) \leq k$. But any finite set $\{x_1, x_2, x_3, \dots x_n\} \subset \mathbb{N}$ with the discrete metric q^s, the set $\mathbb{N}$ can not be covered by $D_{q^s}(x_i, \epsilon)$ for $1 \leq i \leq n$. Hence, $(\mathbb{N}, q)$ is not q-totally bounded. □

Theorem 4.1.1. *(compare [2, Proposition 9]) Let* (X, q) *be a* T_0*-quasi-metric space. A subset* B *of* X *is* q*-precompact if and only if the* $cl_{\tau(q^t)}(B)$ *is* q*-precompact.*

Proof. If B is a q-precompact subset of X, then for $\epsilon > 0$ there exists points $\{x_1, x_2, x_3, \ldots, x_n\}$ in B with $B \subseteq \bigcup_{i=1}^{n} D_q(x_i, \epsilon/2) \subseteq \bigcup_{i=1}^{n} (D_q[x_i, \epsilon/2])$.

$\mathrm{cl}_{\tau(q^t)}(B) \subseteq \mathrm{cl}_{\tau(q^t)} \cup_{i=1}^{n} (D_q[x_i, \epsilon/2]) \subseteq \cup_{i=1}^{n} \mathrm{cl}_{\tau(q^t)}(D_q[x_i, \epsilon/2]) \subseteq \cup_{i=1}^{n} D_q(x_i, \epsilon)$.

Conversely, if $\mathrm{cl}_{\tau(q^t)}(B)$ is q-precompact then there exists $\epsilon > 0$ and finite points $\{x_1, x_2, x_3, \ldots, x_n\}$ in $\mathrm{cl}_{\tau(q^t)}(B)$ such that $\mathrm{cl}_{\tau(q^t)}(B) \subseteq \bigcup_{i=1}^{n} D_q(x_i, \epsilon/2)$. Since $B \subseteq \mathrm{cl}_{\tau(q^t)}(B)$ then for a fixed i, there is some $a_i \in B$ such that $q^{-1}(x_i, a_i) = q(a_i, x_i) < \epsilon/2$. Finally, we show that $D_q(x_i, \epsilon/2) \subseteq D_q(a_i, \epsilon/2)$. Let $y \in D_q(x_i, \epsilon/2)$ then $q(x_i, y) < \epsilon/2$, using triangle inequality, $q(a_i, y) < \epsilon$. Thus, $y \in D_q(a_i, \epsilon/2)$ and $D_q(x_i, \epsilon/2) \subseteq D_q(a_i, \epsilon/2)$. $\square$

We state the following corollary from Theorem 4.1.1 above.

Corollary 4.1.1. *Let (X, q) be a T_0-quasi-metric space. A subset B of X is q-totally bounded if and only if $cl_{\tau(q)}(B)$ is q-totally bounded.*

Proposition 4.1.1. *(compare [40, Lemma 3.73.]) A quasi-pseudometric space (X, q) is q-totally bounded if and only if every sequence in (X, q^s) has a Cauchy subsequence.*

Proof. Suppose (x_n) is a q^s-sequence in a q-totally bounded X. Then there exists a finite set $F \subseteq X$ such that for $n \in \mathbb{N}$, $X = \bigcup_{x \in F} D_{q^s}(x, \frac{1}{n})$.
Let $N_0 = \mathbb{N}$ and construct inductively infinite sets as follow $N_0 \supset N_1 \supset N_2 \supset N_3 \ldots$. For every $n \in N_{k-1}$ there exists $p \in F$ such that $q^s(x_n, p) < 1/k$. Since N_{k-1} is infinite while F is finite, there exists $p \in F$ such that the set $N_k := \{n \in \mathbb{N} : d(x_n, p) < 1/k\}$ is infinite.
Now define a q^s-subsequence a_{n_k} by letting n_k be some element of N_k such that $n_k > n_{k-1}$. For $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $2/M < \epsilon$. For $j, k \geq M$ we have $n_j, n_k \in N_M$, hence there is $p \in F$ such that $q^s(a_{n_k}, p) < 1/M$ and $q^s(a_{n_j}, p) < 1/M$ and by triangle inequality, we get $q^s(a_{n_k}, a_{n_j}) < 2/M$.
We prove the other implication contrapositively. With fixed $\epsilon > 0$ let us choose $x_1 \in X$ and if $X \neq D_{q^s}(x_1, \epsilon)$ then we can choose $x_2 \in X \backslash D_{q^s}(x_1, \epsilon)$ with $q^s(x_1, x_2) \geq \epsilon$. Again if $X \neq D_{q^s}(x_1, \epsilon) \bigcup D_{q^s}(x_2, \epsilon)$ then we can choose $x_3 \in X \backslash (D_{q^s}(x_1, \epsilon) \bigcup D_{q^s}(x_2, \epsilon))$ with both $q^s(x_1, x_2) \geq \epsilon$ and $q^s(x_1, x_3) \geq \epsilon$. Continuing this process, we get a q^s-sequence $x_1, x_2, \ldots x_n$ such that $q^s(x_i, x_j) \geq \epsilon$ for all i, j. This q^s-sequence has no q^s-Cauchy subsequence. $\square$

Definition 4.1.3. *[1, p. 85] Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called joincompact provided that the metric space (X, q^s) is compact.*

A T_0-quasi-metric space is said to be boundedly joincompact if every $\tau(q^s)$-closed bounded subset is joincompact. We denote by $\mathscr{K}_0(X)$ the family of nonempty joincompact subsets of X.

Theorem 4.1.2. *(compare [40, Theorem 3.78]) Let (X, q) be a T_0-quasi-metric space. A set $B \subseteq X$ is joincompact if and only if B is both bicomplete and q-totally bounded.*

Proof. Let B be joincompact and (x_n) be a q^s-Cauchy sequence in B, by the joincompactness of B, the sequence (x_n) has a subsequence that $\tau(q^s)$-converges to x. Thus B is bicomplete. Since (x_n) has a q^s-Cauchy subsequence which $\tau(q^s)$-converges, by Proposition 4.1.1, B is q-totally bounded set.
On the other hand, if B is both bicomplete and q-totally bounded then by Proposition 4.1.1, there exists a q^s-sequence (x_n) in B that has a q^s-Cauchy subsequence that converges in B. Hence, the set $B \subseteq X$ is joincompact. □

We rephrase the above theorem in the following Corollary as proved by Fletcher and Lindgreen in quasi-uniform spaces.

Corollary 4.1.2. *(compare[16, Proposition 3.36]) Let (X, q) be a T_0-quasi-metric space. Then (X, q) is q-totally bounded if and only if $(\widetilde{X}, \widetilde{q})$ is joincompact.*

Definition 4.1.4. *[14, Definition 1.44] Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called* supseparable *provided that the metric space (X, q^s) is separable.*

Let us recall that a set A is said to be subdense in a T_0-quasi-metric space (X, q) if it is dense in X with respect to $\tau(q^s)$.

Just like in metric spaces, we also show in the next proposition that a q-totally quasi-metric space is supseparable.

Proposition 4.1.2. *(compare[40, Proposition 3.72]) A q-totally bounded quasi-pseudometric space (X, q) is supseparable.*

Proof. Suppose X is q-totally bounded, for any positive integer n, we can find a finite set $A_n \subseteq X$ such that for all $x \in X$, $q^s(x, A_n) < \frac{1}{n}$. Now let $B = \cup_{n \in \mathbb{N}} A_n$. The set B is either finite or infinitely countable , thus countable. To show the $\tau(q^s)$-density of B, let us pick $x \in X$. Then we have $q^s(x, B) \leq q^s(x, A_n) < \frac{1}{n}$ implying that $q^s(x, B) = 0$ and $x \in \mathrm{cl}_{\tau(q^s)}(B)$. This proves that x is a q^s-limit point of B and hence B is a supdense subset of X. Thus, (X, q) is supseparable. □

4.2 Bornologies of totally bounded sets

In this section, we study and compare different bornologies with the bornologies of totally bounded sets on quasi-metric spaces. For instance, it is studied in Theorem 4.2.1 that the bornology of q-totally bounded sets agrees with the bornology of $\tau(q)$-relatively compact sets if and only if the quasi-metric space is bicomplete. In particular, we study those quasi-metric bornologies that are bornologies of q-totally bounded sets.

Remark 4.2.1. *Let (X, q) be a quasi-metric space. In the sequel, we denote by $\mathscr{TB}_q(X)$ the collection of all q-totally bounded subsets in (X, q). We notice that $\mathscr{TB}_q(X)$ satisfies the following conditions:*

(i) whenever $x \in X$, then $\{x\} \in \mathscr{TB}_q(X)$;

(ii) whenever $F \in \mathscr{TB}_q(X)$ and $G \subseteq F$, then $G \in \mathscr{TB}_q(X)$;

(iii) whenever $F, G \in \mathscr{TB}_q(X)$, then $F \cup G \in \mathscr{TB}_q(X)$.

Therefore, the collection $\mathscr{TB}_q(X)$ forms a bornology on X that we call the bornology of q-totally bounded sets *in (X, q).*

It is important to note that $\mathscr{TB}_{q^s}(X)$ is a bornology of q^s-totally bounded sets presented in Section 2.2.

The following note is a consequence of Definition 4.1.2.

Note 4.2.1. *For a quasi-pseudometric space (X, q), the following hold:*

$$\mathscr{TB}_q(X) = \mathscr{TB}_{q^s}(X) = \mathscr{TB}_{q^t}(X).$$

The converse of the following lemma is given in Example 4.1.2.

Lemma 4.2.1. *Let (X, q) be a quasi-metric space. Then $\mathscr{TB}_q(X) \subseteq \mathscr{B}_q(X)$.*

Proof. If $B \in \mathscr{TB}_q(X)$, then for $\epsilon > 0$ there exists a set $\{f_1, f_2, f_3, \dots f_k\}$ in B such that $B \subseteq \cup_{j=1}^{k} D_{q^s}(f_j, \epsilon)$. The set B is a finite family of q^s-bounded subsets thus it is q^s-bounded. Hence, $B \in \mathscr{B}_q(X)$ by Lemma 3.2.1. □

Recall that a subset A of a quasi-metric space (X, q) is said to be $\tau(q)$-*relatively compact* if A is $\tau(q^s)$-relatively compact. It was observed in [52] that the family of $\tau(q)$-relatively compact subsets of X forms a bornology denoted by $\mathscr{RK}_q(X)$.

Theorem 4.2.1. *(compare [40, Corollary 3.84]) Let (X,q) be a T_0-quasi-metric space. Then X is bicomplete if and only if $\mathscr{RK}_q(X) = \mathscr{TB}_q(X)$.*

Proof. Let (X,q) be a bicomplete quasi-metric space and B be a q-totally bounded subset of X. In order to prove that B is relatively compact, we need to show that $\mathrm{cl}_{\tau(q)}(B)$ is $\tau(q^s)$-compact. Let (x_n) be a q^s-sequence in $\mathrm{cl}_{\tau(q)}(B)$. Since $\mathrm{cl}_{\tau(q)}(B)$ is q-totally bounded too (see Corollary 4.1.1) by Proposition 4.1.1, (x_n) in $\mathrm{cl}_{\tau(q)}(B)$ has a q^s-Cauchy subsequence. Again $\mathrm{cl}_{\tau(q)}(B)$ is a $\tau(q^s)$-closed subset of a bicomplete space X. It therefore means any q^s-subsequence of (x_n) $\tau(q^s)$-converges in $\mathrm{cl}_{\tau(q)}(B)$. Thus, $\tau(q^s)$-compact. Conversely, let $\mathscr{RK}_q(X) = \mathscr{TB}_q(X)$ and $B \in \mathscr{RK}_q(X)$. Since $\mathrm{cl}_{\tau(q)}(B)$ is $\tau(q^s)$-compact and q-totally bounded then by Proposition 4.1.1, every q^s-sequence in $\mathrm{cl}_{\tau(q)}[B]$ has a q^s-Cauchy subsequence that $\tau(q^s)$-converges in $\mathrm{cl}_{\tau(q)}[B] \subseteq X$. Hence, (X,q) is bicomplete. □

Definition 4.2.1. *[30]Given a Hilbert cube $H = [0,1]^{\mathbb{N}}$, the product topology is defined in a usual way by a quasi-pseudometric*

$$\rho_q(x,y) = \sum_{n=1}^{\infty} \frac{u(x_n,y_n)}{2^n}$$

where $u(x_n,y_n) = \max\{x_n - y_n, 0\}$.

Theorem 4.2.2. *[30, Theorem 3.10] Every supseparable quasi-metric space is embeddable as subspace of the Hilbert cube $H = [0,1]^{\mathbb{N}}$.*

Theorem 4.2.3 (Tychonoff's Theorem). *The topological product of a family of compact spaces is compact*

Definition 4.2.2. *[52] Let (X,q) be a T_0-quasi-metric space. A T_0-quasi-metric $q_1 : X^2 \to [0,\infty)$ defined by $q_1(x,y) = \min\{1, q(x,y)\}$ is called a* bona-fide quasi-metric *of q.*

Note that the quasi-metric $q_1(x,y) = \min\{1, q(x,y)\}$ is always q_1-bounded even if $q(x,y)$ is not bounded.

The following theorem shows that if a quasi-metric space (X,q) is supseparable then the quasi-metric bornology studied by Otafudu et al. [52] coincides with the bornology of q-totally bounded sets.

Theorem 4.2.4. *(compare Theorem 2.2.1) Let (X,q) be a quasi-metric space and let $x_0 \in X$. The following conditions are equivalent:*

(1) there exists an equivalent quasi-metric ρ such that $\mathscr{B}_q(X) = \mathscr{TB}_\rho(X)$;

(2) the quasi-metric space (X, q) is supseparable;

(3) there is an embedding Φ of (X, q) into some quasi-metrizable space (Y, p) such that the family $\{cl_{\tau(p^s)}(\Phi(D_q[x_0, n] \cap D_{q^t}[x_0, s]) : n, s \in \mathbb{N})\}$ is cofinal in $\mathscr{K}_0(Y)$;

(4) there exists an equivalent quasi-metric ρ with $\mathscr{B}_q(X) = \mathscr{T}\mathscr{B}_\rho(X) = \mathscr{B}_\rho(X)$.

Proof. $1 \Rightarrow 2$: If there exists an equivalent quasi-metric space ρ such that $\mathscr{B}_q(X) = \mathscr{T}\mathscr{B}_\rho(X)$, then $X = \bigcup_{i=1}^{n} B_i$ where B_i are ρ-totally bounded subsets. This means that X is a countable union of ρ-totally bounded sets, thus it is ρ-totally bounded and by Proposition 4.1.2, the quasi-metric space (X, q) is supseparable.
$2 \Rightarrow 3$: **First case:** If (X, q) is q-bounded, then by Theorem 4.2.2, we can find an embedding $\Phi : (X, q) \longrightarrow ([0, 1]^{\mathbb{N}}, \rho_q)$. Let $Y = \mathrm{cl}_{\tau(\rho_q^s)}(\Phi(X))$ and choose $n, s \in \mathbb{N}$ so that $Y = \mathrm{cl}_{\tau(\rho_q^s)}(\Phi(D_q[x_0, n] \cap D_{q^t}[x_0, s]))$. Since $[0, 1]^{\mathbb{N}}$ is joincompact with respect to product topology, its subset Y is joincompact and confinal in $\mathscr{K}_0(Y)$.
Second case: If (X, q) is q-unbounded, then we can define the bounded quasi-metric $q_1(x, y) = \min\{1, q(x, y)\}$ and use the same embedding $\Phi : (X, q_1) \longrightarrow ([0, 1]^{\mathbb{N}}, \rho_q)$ as in the first case.
$3 \Rightarrow 4$: If ρ is a quasi-metric equivalent to q, then $\mathscr{B}_q(X) = \mathscr{B}_\rho(X)$ by Hu's theorem (see Theorem 3.2.1). To prove that $\mathscr{B}_q(X) = \mathscr{T}\mathscr{B}_\rho(X)$, let $B \in \mathscr{T}\mathscr{B}_\rho(X)$ and $(Y, \widetilde{\rho})$ be a bicompletion of ρ. Since $\widetilde{\rho}$ is bicomplete by Theorem 4.2.1, the set $\mathrm{cl}_{\tau(\widetilde{\rho}^s)}(B)$ is $\tau(\widetilde{\rho}^s)$-compact. Given the cofinality of $\mathscr{K}_0(Y)$, let us choose $n \in \mathbb{N}$ with $\mathrm{cl}_{\tau(\widetilde{\rho}^s)}(B) \subseteq \mathrm{cl}_{\tau(\widetilde{\rho}^s)}(D_q[x_0, n] \cap D_{q^t}[x_0, s])$. But this means that

$$B \subseteq \mathrm{cl}_{\tau(q^s)}(B) \subseteq \mathrm{cl}_{\tau(q^s)}(D_q[x_0, n] \cap D_{q^t}[x_0, s]) = D_q[x_0, n] \cap D_{q^t}[x_0, s].$$

Thus $B \in \mathscr{B}_q(X)$ and it follows that $\mathscr{T}\mathscr{B}_\rho(X) \subseteq \mathscr{B}_q(X)$. For the reverse inclusion, if $B \in \mathscr{B}_q(X)$, we can choose $n \in \mathbb{N}$ with $B \subseteq D_q[x_0, n] \cap D_{q^t}[x_0, s]$. By Theorem 4.2.1, $B \subseteq \mathrm{cl}_{\tau(\widetilde{\rho}^s)}(D_q[x_0, n] \cap D_{q^t}[x_0, s])$ is compact and $\widetilde{\rho}$-totally bounded. Therefore, $B \in \mathscr{T}\mathscr{B}_\rho(X)$. The equivalence $4 \Rightarrow 1$ follows from Hu's theorem (Theorem 3.2.1). □

Definition 4.2.3. *[44, Definition 2] Let (X, q) be a T_0-quasi-metric space and $p \notin X$. We say that X has one point-extension property if for any $A \subset X$, the set $A' = A \cup \{p\}$ is a one point extension quasi-metric space of A.*

Definition 4.2.4. *Let (X, q) be a T_0-quasi-metric space. Given the point $p \notin X$ and a quasi-metric bornology $\mathscr{B}_q(X)$ on X. We define the one-point extension of X associated with $\mathscr{B}_q(X)$ by $X' = X \cup \{p\}$.*

If $\tau(q)$ is the topology on X, then the corresponding topology on X' is defined by

$$\tau(q) \cup \left\{ \{p\} \cup X \backslash B : B = \mathrm{cl}_{\tau(q)}(B) \in \mathscr{B}_q(X) \right\}.$$

The quasi-metric bornology associated with X' is denoted by $\mathscr{B}_q(X')$.

Remark 4.2.2. *If $p \notin X$ and $\mathcal{B}_0$ is a $\tau(q)$-closed base of the bornology then $\{\{p\} \cup X \backslash B : B \in \mathcal{B}_0\}$ forms a $\tau(q)$-neighbourhood base at the point p.*

Lemma 4.2.2. *Let (X, q) be a T_0-quasi-metric space. If the bornology $\mathscr{B}_q(X)$ is quasi-metrizable then the associated bornolgy $\mathscr{B}_q(X')$ on X' is quasi-metrizable.*

In what follows, we prove that the bornology of q-totally bounded sets with one point extension and a countable base coincides with quasi-metric bornology.

Theorem 4.2.5. *(compare Theorem 2.2.2) Let (X, q) be a quasi-metric space. The following conditions are equivalent:*

(1) the bornology $\mathscr{TB}_q(X)$ has a countable base;

(2) there exists an equivalent quasi-metric q' such that $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$;

(3) the one-point extension X' of X associated with $\mathscr{TB}_q(X)$ is quasi-metrizable;

(4) the one-point extension X' of X associated with $\mathscr{TB}_q(X)$ has a $\tau(q)$-neighbourhood base at the ideal point.

Proof. (1) $\Rightarrow$ (2): Since $\mathscr{TB}_q(X)$ has a countable base by Hu's theorem (Theorem 3.2.1) there exists an equivalent quasi-metric q' such that $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$;
(2) $\Rightarrow$ (3): By (2), $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$. From Lemma 4.2.2 $\mathscr{B}_q(X')$ on X' is quasi-metrizable, thus $\mathscr{TB}_q(X')$ is quasi-metrizable;
(3) $\Rightarrow$ (4) By Corollary 4.1.1 $\mathscr{TB}_q(X')$ has a $\tau(q^s)$-closed base, thus by the Remark 4.2.2 $\mathscr{TB}_q(X')$ has a $\tau(q^s)$-neighbourhood base at the ideal point;
(4) $\Rightarrow$ (1): If the bornology $\mathscr{TB}_q(X')$ has a $\tau(q^s)$-neighbourhood base at each point, then $\mathscr{TB}_q(X)$ has a countable base. □

The next result proves that the bornology on a quasi-metrizable space agrees with a bornology of q-totally bounded sets with respect to some compatible quasi-metric.

Theorem 4.2.6. *(compare Theorem 2.2.3) Let $\mathscr{B}$ be a family of nonempty subsets of a quasi-metric space (X,q). Then $\mathscr{B}= \mathscr{TB}_q(X)$ if and only if there exists an embedding $\psi : (X,q) \longrightarrow (\widetilde{X},p)$ with the following property:*

$$\mathscr{B}= \left\{ B \in \mathscr{P}_0(X) : \varphi(B) \text{ is relatively compact in } \widetilde{X} \right\}.$$

Proof. Let $\mathscr{B}= \mathscr{TB}_q(X)$ and $\varphi ; (X,q) \longrightarrow (\widetilde{X},\widetilde{q})$. If $B \in \mathscr{TB}_q(X)$ then $\varphi(B) \in \mathscr{TB}_q(\varphi(B)) \subseteq \mathscr{TB}_q(\widetilde{X})$. The set $\mathrm{cl}_{\tau(q^s)}[\varphi(B)]$ is q-totally bounded and bicomplete thus compact by Theorem 4.1.2.
On the other hand, let $\varphi : (X,q) \longrightarrow (\widetilde{X},p)$ and $A = \mathrm{cl}_{\tau(p^s)}[\psi(X)]$. Since A is a $\tau(p^s)$-closed subspace of $\widetilde{X}$ it is also a bicomplete space. Let $\widetilde{q}$ be a compatible bicomplete quasi-metric for A and q be its trace on $\psi(X)\times\varphi(X)$. If $B \in \mathscr{B}$, then $\varphi(B)$ is relatively compact. By Theorem 4.2.1 $\varphi(B)$ is $\widetilde{q}$-totally bounded and since $\varphi(B) \subseteq \psi(X)$, $\psi(B) \in \mathscr{TB}_q(\varphi(X))$. Also if $\psi(B) \in \mathscr{TB}_q(\varphi(X))$ then by bicompleteness of $\widetilde{X}$ and Theorem 4.2.1 again $\varphi(B)$ is relatively compact. Thus, $\mathscr{B}= \mathscr{TB}_q(X)$. □

Recall that a quasi-metric space (X,q) is supseparable if the metric space (X,q^s) is separable.

Proposition 4.2.1. *(compare Proposition 2.2.1) Let $\mathcal{S}(X)$ be the bornology of nonempty supseparable subsets of a quasi-metric space (X,q). Then $\mathcal{S}(X) = \mathscr{TB}_q(X)$ if and only if X supseparable.*

Proof. If (X,q) is supseparable, then $\mathcal{S}(X) = \mathscr{P}_0(X)$, so that if q is a compatible q-totally bounded quasi-metric, then we have $\mathscr{TB}_q(X) = \mathcal{S}(X)$.
Conversely, suppose $\mathscr{TB}_q(X) = \mathcal{S}(X)$ and (X,q) is not supseparable. Let Y and φ be as guaranteed by Theorem 4.2.6 with respect to the family $\mathcal{S}(X)$. Since X is not supseparable, $\mathrm{cl}_{\tau(\widetilde{q}^s)}[\psi(X)]$ is not $\tau(q^s)$-compact. Thus, there exists a q^s-sequence (y_n) in $\mathrm{cl}_{\tau(\widetilde{q}^s)}[\psi(X)]$ without a q^s-limt point. Let $(y_n) \in \psi(X)$, choosing for each $n \in \mathbb{N}$, $(x_n) \in X$ with $\varphi((x_n)) = (y_n)$ we obtain a supseparable subset $\{x_n : n \in \mathbb{N}\}$ mapped by φ to a relatively compact set. Hence, we have a contradiction. Thus, X is supseparable. □

Definition 4.2.5. *(compare Definition 2.2.3) A* star-development *for a quasi-metirc space (X,q) is a sequence $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \dots$ of $\tau(q)$-open covers of X such that $\mathcal{D}_n$ refines $\mathcal{D}_{n-1}$ and for each $x \in X$, $\{\mathrm{st}(x,\mathcal{D}_n) : n \in \mathbb{N}\}$ is a $\tau(q)$-neighbourhood base of X.*

Definition 4.2.6. *(compare [56, Definition 21.6]) Let (X, q) be a quasi-metric space. The $\tau(q)$-covering $\mathcal{D}$ in X is said to be a* refinement *of $\mathcal{D}'$ if every $D \in \mathcal{D}$ is contained in $D' \in \mathcal{D}'$.*

If S is a subset of X, then the star of S with respect to $\mathcal{D}$ is the union of all sets $D \in \mathcal{D}$ that intersect S.
The covering $\mathcal{D}$ is said to be a *star-refinement* of $\mathcal{D}'$ if for every $D \in \mathcal{D}$, $\mathrm{st}(D, \mathcal{D})$ is contained in some $D' \in \mathcal{D}'$.

In Theorem 4.2.7, we show that the family of nonempty sets of the quasi-metric space (X, q) becomes the bornology of q-totally bounded sets if and only if there exists a star-development on X.

Theorem 4.2.7. *(compare Theorem 2.2.4) Let $\mathscr{B}$ be a family of nonempty subsets of a quasi-metric space (X, q). Then $\mathscr{B} = \mathscr{TB}_q(X)$ if and only if there exists a star-development $\{\mathcal{D}_n : n \in \mathbb{Z}^+\}$ for X such that*

$$\mathscr{B} = \left\{ C \in \mathscr{P}_0(X) : \forall n \in \mathbb{Z}^+,\ \mathcal{D}_n \text{ admits a finite subcover of C} \right\}. \quad (4.2.1)$$

Proof. Suppose $\mathscr{B} = \mathscr{TB}_q(X)$ and let the star-development of X be $\mathcal{D}_n = \{D_{q^s}(x, \frac{1}{3^n}) : x \in X\}$ for all $n \in \mathbb{Z}^+$. If $C \in \mathscr{B}$ then for each n, the set C can be covered by finitely many $\tau(q^s)$-open balls of radius $\frac{1}{3^n}$, i.e., for each $n \in \mathbb{Z}^+$, C can be covered by a subfamily of $\mathcal{D}_n$. Thus, we get Equation 4.2.1. Conversely, suppose there exists a star development $\mathcal{D}_n$, then we can find a quasi-metric q such that for all $n \geq 2$:

(i) $\mathcal{D}_n$ refines $\{D_{q^s}(x, \frac{1}{2^{n-1}}) : x \in X\}$;

(ii) $\{D_{q^s}(x, \frac{1}{2^n}) : x \in X\}$ refines $\mathcal{D}_{n-1}$.

Now suppose $\mathscr{B}$ satisfies Equation 4.2.1. Let us fix $C \in \mathscr{B}$ and for an abitrary $\epsilon > 0$ choose $2^{1-n} < \epsilon$ so that $\epsilon > \frac{1}{2^{n-1}}$. By Equation 4.2.1, we can choose $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots, \mathcal{D}_{n_k}\} \subseteq \mathcal{D}_n$ with $C \subseteq \cup_{j=1}^{n_k} \mathcal{D}_j$. Using (i) we can choose for all $j \leq k_n$, x_j with $\mathcal{D}_j \subseteq D_{q^s}(x_j, \frac{1}{2^{n-1}})$. This yields $C \subseteq \cup_{j=1}^{k_n} D_{q^s}(x_j, \epsilon)$ so that $C \in \mathscr{TB}_q(X)$. Now If $C \in \mathscr{TB}_q(X)$ then for $n \in \mathbb{Z}^+$ there exists a finite subset $F = \{x_1, x_2, x_3, \ldots, x_k\}$ of X with $C \subseteq \cup_{j=1}^{k} D_{q^s}(x_j, \frac{1}{2^{n-1}})$. Using (ii), we yield $C \in \mathscr{B}$. □

Chapter 5

Bourbaki-boundedness on quasi-pseudometric spaces

In the previous chapter, we proved for instance that for a supseparable quasi-pseudometric space, the quasi-metric bornology becomes the bornology of q-totally bounded sets. In this chapter, we continue with our investigations by showing how the studies of Beer and Garrido [9] on bornologies of Bourbaki-bounded sets with Lipschitz functions can be modified in order to obtain a theory for quasi-metric spaces. For us to have a better understanding of Bourbaki-boundedness in quasi-metric spaces, equivalent characterizations are studied in section 5.2. It is interesting to note that a set on a quasi-metric space (X, q) can be q-Bourbaki-bounded but not q^s-bourbaki bounded (see Example 5.2.1). In addition, we prove that given an asymmetric normed space, the q-Bourbaki-boundedness coincides with q-boundedness (see Proposition 5.2.3). In Section 5.3, we compare the bornology of q-Bourbaki-bounded sets with other quasi-metric bornologies (see Remark 5.3.2). We also prove that every real-valued semi-Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded (Theorem 5.3.2).

5.1 Uniformly continuous and semi-Lipschitz functions

In asymmetric spaces, the concepts of uniform continuity and Lipschitz functions are not as direct as they are in symmetric spaces. This section is devoted to generalize the notions of uniformly continuous and semi-Lipschitz functions from metric spaces to quasi-metric settings.

We start this section with the following lemma about uniformly continuous.

Lemma 5.1.1. *Let (X, q) and (Y, p) be quasi-pseudometric spaces. If the function $\varphi : (X, q) \to (Y, p)$ is uniformly continuous, then the function $\varphi : (X, q^s) \to (Y, p^s)$ is uniformly continuous.*

Proof. Let $\epsilon > 0$. If the function $\varphi : (X, q) \to (Y, p)$ is uniformly continuous, then for all $x, y \in X$, there exists $\delta > 0$ such that if $q(x, y) < \delta$ then $p(\varphi(x), \varphi(y)) \leq \epsilon$. Furthermore, $q^s(x, y) < \delta$ implies that

$$q(x, y) \leq q^s(x, y) < \delta \quad \text{for} \quad p(\varphi(x), \varphi(y)) \leq \epsilon \tag{5.1.1}$$

and

$$q(y, x) \leq q^s(x, y) < \delta \quad \text{for} \quad p(\varphi(y), \varphi(x)) \leq \epsilon. \tag{5.1.2}$$

For all $x, y \in X$ we have $p^s(\varphi(x), \varphi(y)) < \epsilon$ from (5.1.1) and (5.1.2). Therefore, the function $\varphi : (X, q^s) \to (Y, p^s)$ is uniformly continuous. □

We give the following example to show that the converse of the Lemma 5.1.1 does not hold in general, i.e., the uniform continuity of the function $\varphi : (X, q^s) \to (Y, p^s)$ does not imply the uniform continuity of the function $\varphi : (X, q) \to (Y, p)$.

Example 5.1.1. *We equip $X = \mathbb{R}_+ = [0, \infty)$ with the quasi-metric q defined by $q(x, y) = (y - x)^+$ for any $x, y \in [0, \infty)$ and $Y = \mathbb{R}$ is equipped with the T_0-quasi-metric p defined by $p(x, y) = (y - x)^+$ for any $x, y \in \mathbb{R}$. Then*

(i) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is uniformly continuous from $(\mathbb{R}_+, |.|)$ into $(\mathbb{R}, |.|)$.

(ii) the function $f(x) = -\sqrt{x}$ whenever $x \in \mathbb{R}_+$ is not uniformly continuous from $(\mathbb{R}_+, q)$ into $(\mathbb{R}, p)$.

Proof. (i) Let $\epsilon > 0$ be given. The function f is continuous and hence uniformly continuous on $[0, 1]$. Thus, there exists $\delta' > 0$ such that

$$|y - x| < \delta' \Rightarrow |f(y) - f(z)| < \epsilon, \tag{5.1.3}$$

for all $x, y \in [0, 1]$. Put

$$\delta := \min\{\delta', \epsilon\}, \tag{5.1.4}$$

and let $x, y \in \mathbb{R}_+$ with

$$0 < |y - x| < \delta. \tag{5.1.5}$$

If $x, y \in [0,1]$, then, by (5.1.3) and (5.1.4), $|f(y) - f(x)|< \epsilon$. If $x > 1$ and $y > 1$, then

$$|f(x) - f(y)|= \frac{|y-x|}{\sqrt{x}+\sqrt{y}} < |y-x|< \epsilon \quad \text{(by (5.1.4) and (5.1.5))}.$$

This proves the uniform continuity of f from $(\mathbb{R}_+, |.|)$ into $(\mathbb{R}, |.|)$.

(ii) To show that f is not uniformly continuous, let $\epsilon = 1$ and $\delta > 0$ be chosen arbitrary. Then, for any $x > 0$, $q(x, 0) = (0 - x)^+ = 0 < \delta$, but for all $x > 1$.

$$p(f(x), f(0)) = (f(0) - f(x))^+ = (\sqrt{x})^+ = \sqrt{x} > 1 = \epsilon.$$

□

Given an asymmetric normed space $(Y, \|.|)$, we equip X with its specialization order $\leq_q$ given by $x \leq_q y \Leftrightarrow q(x,y) = 0 \Leftrightarrow x \in \overline{\{y\}}$. The function $\varphi : (X, q) \to (\mathbb{R}, u)$ satisfies

$$\varphi(x) - \varphi(y) \leq q(x,y) \quad \text{for all } x, y \in X. \tag{5.1.6}$$

If $u(t) = \max\{t, 0\}$, $t \in \mathbb{R}$, then (5.1.6) is the same as $u(\varphi(x) - \varphi(y)) \leq q(x,y)$ for all $x, y \in X$, that is, φ is semi-Lipschitz from (X, q) and $(\mathbb{R}, u)$.
Any semi-Lipschitz function is monotone with respect to the specialization order, i.e., $x \leq_q y \Rightarrow \varphi(x) \leq \varphi(y)$. In addition, φ is semi-Lipschitz if and only if:

(i) φ is $\leq_q$-monotone;

(ii) $\|\varphi|:= \sup\left\{\frac{\varphi(x)-\varphi(y)}{q(x,y)} : q(x,y) > 0\right\}$ for any function φ mapping from (X, q) to $(\mathbb{R}, u)$.

Definition 5.1.1. *(compare Definitions 2.3.2 and 2.3.3) Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then:*

(a) A function $\varphi : (X, q) \to (Y, \|.|)$ is called locally semi-Lipschitz *provided that for all $x \in X$, there exists $\delta(x) > 0$ such that $\varphi|_{D_q(x,\delta(x))}$ is semi-Lipschitz;*

(b) A function $\varphi : (X, q) \to (Y, \|.|)$ is called uniformly locally semi-Lipschitz *provided that for all $x \in X$, there exists $\delta > 0$ (δ does not depend to x) such that $\varphi|_{D_q(x,\delta)}$ is semi-Lipschitz.*

Lemma 5.1.2. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. If function $\varphi : (X, q) \to (Y, \|.|)$ is locally semi-Lipschitz, then $\varphi : (X, q^s) \to (Y, \|.\|)$ is locally semi-Lipschitz.*

Proof. Let $\varphi : (X, q) \to (Y, \|.|)$ be locally semi-Lipschitz. If $x \in X$, then there exists $\delta(x) > 0$ and $k \geq 0$ such that for any $y, z \in D_q(x, \delta(x))$, we have

$$\|\varphi(y) - \varphi(z)| \leq kq(y, z) \leq kq^s(y, z) \tag{5.1.7}$$

and

$$\|\varphi(z) - \varphi(y)| \leq kq(z, y) \leq kq^s(y, z). \tag{5.1.8}$$

Combining (5.1.7) and (5.1.8) we get $\|\varphi(y) - \varphi(z)\| \leq kq(y, z) \leq kq^s(y, z)$ whenever $y, z \in D_q(x, \delta(x))$. Thus, the function $\varphi : (X, q^s) \to (Y, \|.\|)$ is locally semi-Lipschitz. □

The following lemma follows directly from the definitions of semi-Lipschitz in the small function and uniformly continuous.

Lemma 5.1.3. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. If a function $\varphi : (X, q) \to (Y, \|.|)$ is semi-Lipschitz in the small, then $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous*

Remark 5.1.1. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space:*

(1) If a function $\varphi : (X, q) \to (Y, \|.|)$ is locally semi-Lipschitz, then $\varphi|_{D_q(x,\delta_x)}$ is continuous whenever $x \in X$ and for some $\delta_x > 0$;

(2) Let $F \subseteq X$. If $\varphi_i : (X, q) \to (\mathbb{R}, u)$ is semi-Lipchitz restricted to F for $i = \{1, 2, \cdots, n\}$, then $\max\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ is semi- Lipschitz restricted to F.

5.2 Bourbaki-boundedness in quasi-metric spaces

In this section, we study equivalent characterizations of Bourbaki-boundedness on quasi-pseudometrics spaces.

In order to proceed smoothly with this section, there is need to recall the following definition.

Definition 5.2.1. *(compare Definition 2.3.4) Let (X, q) be a quasi-pseudometric space. For any $\delta > 0$ and $\emptyset \neq F \subset X$, we define the δ-* enlargement $D_q(F, \delta)$ *of F by*

$$D_q(F, \delta) := \{x \in X : dist(F, x) < \delta\} = \bigcup_{f \in F} D_q(f, x)$$

and

$$D_{q^t}(F, \delta) := \{x \in X : dist^t(F, x) < \delta\} = \bigcup_{f \in F} D_{q^t}(f, x).$$

Furthermore,

$$D_{q^s}(F, \delta) := \max\left\{D_q(F, \delta), D_{q^t}(F, \delta)\right\} = \bigcup_{f \in F} D_{q^s}(f, x).$$

The following lemma is the generalization of Lemma 2.3.1 the first part.

Lemma 5.2.1. *Let (X, q) be a quasi-pseudometric space. For $\epsilon, \delta > 0$, we have $D_q(D_q(F, \epsilon), \delta) \subset D_q(F, \epsilon + \delta)$.*

Proof. Let $y \in D_q(D_q(F, \epsilon), \delta) = \bigcup_{v \in D_q(F, \epsilon)} D_q(v, \delta)$. Then there exists a point $v \in D_q(F, \epsilon)$ such that $y \in D_q(v, \delta)$. It follows that there exists $f \in F$ such that $q(f, v) < \epsilon$ and $q(v, y) < \delta$. Moreover, $q(f, y) \leq q(f, v) + d(v, y) < \epsilon + \delta$. Hence, $y \in D_q(f, \epsilon + \delta)$. □

Definition 5.2.2. *Let (X, q) be a quasi-pseudometric space and $\delta > 0$. If $x \in X$ and $n = 0, 1, 2, \cdots$, we define the sets $D_q^n(x, \delta)$ by $D_q^0(x, \delta) = \{x\}$ and $D_q^{n+1}(x, \delta) = D_q(D_q^n(x, \delta))$.*

The next lemma follows by induction.

Lemma 5.2.2. *(compare Lemma 2.3.1) Let (X, q) be a quasi-pseudometric space and $\delta > 0$. For any $x \in X$ and $n = 0, 1, 2, \cdots$, we have $D_q^n(x, \delta) \subseteq D_q^{n+1}(x, \delta)$.*

Definition 5.2.3. *(compare Definition 2.3.5) Suppose that we have a quasi-pseudometric space (X, q). For any given $x, y \in X$ and $\delta > 0$. A δ-*chain *of length n from x to y in (X, q) is a finite sequence of points $x_0, x_1, \cdots, x_n$ such that $x = x_0$, $x_n = y$ and $q(x_{i-1}, x_i) < \delta$ for any i with $1 \leq i \leq n$.*

Proposition 5.2.1. *Let (X, q) be a quasi-pseudometric space and $\delta > 0$. If there exists a δ-chain of length n from x to y in (X, q), then there exists a δ-chain of length n from y to x in (X, q^t) whenever $x, y \in X$.*

Proof. If $(x_i)_{i=0}^n$ is a δ-chain of length n from x to y on (X, q). Then $x_0 = x$ and $x_n = y$ and for any i with $1 \leq i \leq n$, we have

$$q(x_{i-1}, x_i) < \delta. \tag{5.2.1}$$

We set $y_j := x_{n-j}$ for $j = \{0, 1, 2, \cdots\}$. We claim that $(y_j)_{j=0}^n$ is a δ-chain from y to x on (X, q^t).
Indeed, we have $y_0 = x_{n-0} = y$ and $y_n = x_{n-n} = x_0 = x$. Moreover, for any j with $1 \leq j \leq n$ we have

$$q^t(y_{j-1}, y_j) = q^t(x_{n-j+1}, x_{n-j}) = q(x_{n-j}, x_{n-j+1}) < \delta \quad \text{from (5.2.1)}.$$

Hence $(y_j)_{j=0}^n$ is a δ-chain from y to x in (X, q^t). □

Corollary 5.2.1. *Let (X, q) be a quasi-pseudometric space and $\delta > 0$. If there exists a δ-chain of length n from x to y in (X, q^t), then there exists a δ-chain of length n from y to x in (X, q) whenever $x, y \in X$.*

The following remark is a consequence of Proposition 5.2.1, Corollary 5.2.1 and the definition of $D_{q^t}^n(y, \delta)$.

Remark 5.2.1. *Let $\delta > 0$ and x and y be any two points in a quasi-pseudometric space (X, q). It is easy to check that there exists a δ-chain of length n from x to y if and only if $y \in D_q^n(x, \delta)$ if and only if $x \in D_{q^t}^n(y, \delta)$.*

Definition 5.2.4. *Let (X, q) be a quasi-pseudometric space and $\delta > 0$. For $x, y \in X$, we define the relation $\asymp_\delta$ on X by $x \asymp_\delta y$ if there exists a δ-chain of some length from x to y.*

The next lemma proves that the relation in Definition 5.2.4 is an equivalence relation.

Lemma 5.2.3. *For any quasi-pseudometric space (x, q) and $\delta > 0$, the relation $\asymp_\delta$ is an equivalence relation.*

Proof. Let x, y and $z \in X$. From Remark 5.2.1, we have $x \asymp_\delta x$ since $x \in D_q^0(x, \delta)$.
Furthermore, if $x \asymp_\delta y$ then $y \asymp_\delta x$ from Proposition 5.2.1.
Suppose that $x \asymp_\delta y$ and $y \asymp_\delta z$. Since $x \asymp_\delta y$, there exists a δ-chain $(x_i)_{i=0}^n$ of length n from x to y such that $x_0 = x$, $x_n = y$ and

$$q(x_{i-1}, x_i) < \epsilon \quad \text{for any } i \text{ with } 1 \leq i \leq n. \tag{5.2.2}$$

Moreover, from $y \asymp_\delta z$, there exists a δ-chain $(y_j)_{j=1}^m$ of length m from y to z such that $y_0 = y$, $y_m = z$ and

$$q(y_{j-1}, y_j) < \epsilon \quad \text{for any } j \text{ with } 1 \leq j \leq m. \tag{5.2.3}$$

By a concatenation of the δ-chains $(x_i)_{i=0}^n$ and $(y_j)_{j=1}^m$, we obtain a δ-chain $(z_k)_{k=0}^{n+m}$ with $z_0 = x_0 = x$, $z_n = x_n = y_0 = y$, and $z_{n+m} = y_m = z$. Furthermore, from (5.2.2) and (5.2.3) we have

$$q(z_{k-1}, z_k) < \delta \text{ for any } k \text{ with } 1 \leq k \leq n+m.$$

Hence $(z_k)_{k=0}^{n+m}$ is a δ-chain of length $n+m$ from x to z. Therefore, $x \asymp_\delta z$. $\square$

Let $x_{\asymp_\delta}$ be the equivalence class of x on (X, q), then

$$x_{\asymp_\delta} := \{y \in X : y \asymp_\delta x\} = \{y \in X : y \in D_q^n(x, \delta)\} = \bigcup_{n=0}^{\infty} D_q^n(x, \delta).$$

Remark 5.2.2. *Let (X, q) be a quasi-pseudometric space. For any $\delta > 0$ and $x, y \in X$, it is easy to see that if $(x_i)_{i=0}^n$ is a δ-chain in (X, q^s) of length n from x to y, then $(x_i)_{i=0}^n$ is also a δ-chain in (X, q) and in (X, q^t) of length n from x to y. However, with regard to Remarks 1.1.1 and 5.2.1, one has:*

$$D_{q^s}^n(x, \delta) \subseteq D_q^n(x, \delta) \quad \textit{and} \quad D_{q^s}^n(x, \delta) \subseteq D_{q^t}^n(x, \delta). \tag{5.2.4}$$

The following example shows that the inclusions (5.2.4) in Remark 5.2.2 cannot be reversed.

Example 5.2.1. *Consider the four points set $X = \{1, 2, 3, 4\}$. Let us equip X with T_0-quasi-metric q defined by the distance matrix*

$$Q = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i, j) = q_{i,j}$ whenever $i, j \in X$. Note that the symmetrized metric q^s of q is induced by the matrix

$$Q^s = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

Let $\delta = 1,5 > 0$. If we consider the sequecnce $(f_i)_{i=0}^2 := (4, 2, 1)$. Then we have

$$q(f_0, f_1) = q(4, 2) = 1 = q(f_1, f_2) = q(2, 1) < \delta.$$

Hence the sequence $(f_i)_{i=0}^2 := (4, 2, 1)$ is a δ-chain in (X, q) of length 2 from 4 to 1. But the same sequence $(f_i)_{i=0}^2 := (4, 2, 1)$ is not a δ-chain in (X, q^s) of length 2 from 4 to 1 because $q^s(f_0, f_1) = q^s(4, 2) = 2 > \delta$.

Let us present the next definiton about Bourbaki-boundedness on quasi-pseudometric spaces.

Definition 5.2.5. *(compare Definition 2.3.6) Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. We say that F is* q-Bourbaki-bounded, *if for any $\delta > 0$ there exists a finite subset $\{f_1, f_2, \cdots, f_k\}$ of X and for some $n \in \mathbb{N}$ such that $F \subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$.*

If $n = 1$ and $q = q^s$ in Definition 5.2.5, then we have the concept of q-total boundedness defined in the previous chapter.

The converse of the next lemma does not hold (see Example 5.2.1).

Lemma 5.2.4. *Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. If F is q^s-Bourbaki-bounded, then F is q-Bourbaki-bounded.*

Proof. Let $\delta > 0$. If F is q^s-Bourbaki-bounded, then there exists a finite set $\{f_1, f_2, \cdots, f_k\} \subset X$ such that $F \subseteq \bigcup_{i=1}^{k} D_{q^s}^n(f_i, \delta)$ for some positive integer n. By the first inclusion of (5.2.4), we have $F \subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$ for some positive integer n. Hence, F is q-Bourbaki-bounded. □

Analogously, one shows that q^s-Bourbaki-boundedness implies q^t-Bourbaki-boundedness by using the second inequality of (5.2.4).

We state the following lemma that we will use in our next proposition.

Lemma 5.2.5. *Let (X, q) be a quasi-pseudometric space. For some positive integer n, $\delta > 0$ and $x \in X$, we have*

$$\bigcup_{i=1}^{k} D_q^n(x_i, \delta) \subseteq \bigcup_{i=1}^{k} D_q(x_i, n\delta).$$

Proof. Let $y \in \bigcup_{i=1}^{k} D_q^n(x_i, \delta)$, then for some j with $1 \leq j \leq k$, $y \in D_q^n(x_j, \delta)$. Moreover, for some j with $1 \leq j \leq k$, there exists $\{f_0, f_1, \cdots, f_n\}$ a δ-chain of length n from x_j to y such that $f_0 = x_j$, $f_n = y$ and $q(f_{i-1}, f_i) < \delta$ for all

i with $1 \leq i \leq n$. Furthermore, we have

$$\begin{aligned} q(x_j, y) = q(f_0, f_n) &\leq q(f_0, f_1) + q(f_1, f_2) + \cdots + q(f_{n-1}, f_n) \\ &< \delta + \delta + \cdots + \delta \\ &< n\delta. \end{aligned}$$

Thus, for some j with $1 \leq j \leq k$, $y \in D_q(x_j, n\delta)$. Hence, $y \in \bigcup_{i=i}^{k} D_q(x_i, n\delta)$.

□

Proposition 5.2.2. *Given a quasi-pseudometric space (X, q). If F is a subset of X and $\delta > 0$, then we have the following conditions:*

(a) if F is q-totally bounded, then F is q-Bourbaki-bounded;

(b) if F is q-Bourbaki bounded, then F is q-bounded;

(c) if F is q-Bourbaki bounded, then $cl_{\tau(q)}[F]$ is also q-Bourbaki-bounded.

Proof. (a) Let $\delta > 0$. Suppose F is q-totally bounded, then there exists a set $\{f_1, f_2, \cdots, f_k\} \subseteq X$ such that

$$F \subseteq \bigcup_{i=1}^{k} D_{q^s}(f_i, \delta) = \bigcup_{i=1}^{k} D^1_{q^s}(f_i, \delta) \subseteq \bigcup_{i=1}^{k} D^1_q(f_i, \delta)$$

for some positive integer $n = 1$. Therefore, F is q-Bourbaki-bounded.
(b) Since F is q-Bourbaki-bounded, there exists a set $\{x_1, x_2, \cdots, x_k\} \subseteq X$ and some positive integer n such that for $\delta > 0$ we have $F \subseteq \bigcup_{i=1}^{k} D^n_q(x_i, \delta)$.
By Lemma 5.2.5, $F \subseteq \bigcup_{i=1}^{k} D^n_q(x_i, \delta) \subseteq \bigcup_{i=1}^{k} D_q(x_i, n\delta)$. Hence, F is q-bounded.
(c) Follows, since F is included in the union of $\tau(q)$-open balls.

□

Remark 5.2.3. *Note that the statement (a) of Proposition 5.2.2 can be obtained by the fact that if F is q-totally bounded, then F is q^s-totally bounded. It also follows from [9, 41] that if F is q^s-Bourbaki-bounded, then F is q-Bourbaki-bounded (see also Lemma 5.2.4).*

Corollary 5.2.2. *For a given quasi-pseudometric (X, q) let F be a subset of X and $\delta > 0$. The following conditions hold:*

(a) if F is q^t-totally bounded on (X, q), then F is q^t-Bourbaki-bounded;

(b) if F is q^t-Bourbaki-bounded, then F is q^t-bounded;

(c) if F is q^t-Bourbaki-bounded, then $cl_{\tau(q^t)}F$ is also q^t-Bourbaki-bounded.

Proposition 5.2.3. *Let $(X, \|.|)$ be an asymmetric normed space and F be a subset of X. Then F is $q_{\|.|}$-bounded if and only if F is $q_{\|.|}$-Bourbaki-bounded.*

Proof. The sufficiant condition ($\Leftarrow$) follows from Proposition 5.2.2 (b).
The other condition ($\Rightarrow$) suppose that F is $q_{\|.|}$-bounded then $F \subseteq D_{q_{\|.|}}(x_0, \epsilon)$ for some $x_0 \in X$ and $\epsilon > 0$. For any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{\epsilon}{n} < \delta$.
Let $f \in F$. We define $z_i := x_0 + \frac{i}{n}(f - x_0)$ whenever i with $1 \leq i \leq n$ and $z_0 = x_0$. Then

$$\begin{aligned} \|q_{\|.|}(z_{i-1}, z_i)| &= \|z_{i-1} - z_i| \\ &= \left\| \left[x_0 + \frac{i-1}{n}(f - x_0)\right] - \left[x_0 + \frac{i}{n}(f - x_0)\right] \right| \\ &= \left\| \frac{x_0}{n} - \frac{f}{n} \right| = \left\| \frac{1}{n}(x_0 - f) \right| \\ &< \frac{\epsilon}{n} < \delta. \end{aligned}$$

Thus, for any $f \in F$ we have obtained a δ-chain of length n on $(X, q_{\|.|})$ from z_0 to f. Therefore, $f \in \bigcup_{k=0}^{n} D^1_{q_{\|.|}}(z_k, \delta)$. $\square$

5.3 Bornologies and semi-Lipschitz functions

We end this chapter with the study of bornology of Bourbaki-bounded sets with semi-Lipschitz functions on quasi-metric spaces and asymmetric normed spaces. We compare the bornology of Bourbaki-bounded sets with other quasi-metric bornologies. In addition, we prove for instance in Theorem 5.3.2 that every real-valued semi-Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded.

Remark 5.3.1. *Let (X, q) be a quasi-metric space. In the sequel, we denote by $\mathscr{BB}_q(X)$ the collection of all q-Bourbaki-bounded subsets in (X, q). We observe that $\mathscr{BB}_q(X)$ satisfies the following conditions:*

(i) whenever $x \in X$, then $\{x\} \in \mathscr{B}\mathscr{B}_q(X)$;

(ii) whenever $F \in \mathscr{B}\mathscr{B}_q(X)$ and $G \subseteq F$, then $G \in \mathscr{B}\mathscr{B}_q(X)$;

(iii) whenever $F, G \in \mathscr{B}\mathscr{B}_q(X)$, then $F \cup G \in \mathscr{B}\mathscr{B}_q(X)$.

Therefore, the collection $\mathscr{B}\mathscr{B}_q(X)$ forms a bornology on X that we call the bornology of q-Bourbaki-bounded sets *in (X, q).*

Note 5.3.1. *From Proposition 5.2.1, Corollary 5.2.1 and Remark 5.2.1, it can be observed that the bornologies $\mathscr{B}\mathscr{B}_q(X)$ and $\mathscr{B}\mathscr{B}_{q^t}(X)$ are equivalent.*

In light of the inclusion 3.2.1 on bornologies $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^s}(X)$. One would ask for the connections between $\mathscr{B}\mathscr{B}_q(X)$, $\mathscr{T}\mathscr{B}_q(X)$ and $\mathscr{B}_q(X)$.

Let us provide the connections between these bornologies in the following remark.

Remark 5.3.2. *If (X, q) is a quasi-pseudometric space, then we have the following inlusions:*

$$\mathscr{T}\mathscr{B}_q(X) \subseteq \mathscr{B}\mathscr{B}_{q^s}(X) \subseteq \mathscr{B}\mathscr{B}_q(X) \subseteq \mathscr{B}_q(X) \tag{5.3.1}$$

$$\mathscr{T}\mathscr{B}_q(X) \subseteq \mathscr{B}\mathscr{B}_{q^s}(X) \subseteq \mathscr{B}\mathscr{B}_{q^t}(X) \subseteq \mathscr{B}_{q^t}(X). \tag{5.3.2}$$

But if $(X, \|.|)$ is an asymmetric normed space, then in the light of Proposition 5.2.3 we have

$$\mathscr{B}\mathscr{B}_{q_{\|.|}}(X) = \mathscr{B}_{q_{\|.|}}(X). \tag{5.3.3}$$

The first two inclusions of (5.3.1) and (5.3.2) can be found in [4, 18] and the last inclusions of (5.3.1) and (5.3.2) are consequences of Proposition 5.2.2 and Corollary 5.2.2 respectively.

The next definition is important in proving our next result. Hence, there is need to recall it.

Definition 5.3.1. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. A function $\varphi : (X, q) \to (Y, \|.|)$ is called* locally semi-Lipschitz *provided that for all $x \in X$, there exists $\delta(x) > 0$ such that $\varphi|_{D_q(x,\delta(x))}$ is semi-Lipschitz.*

The next result is about bornologies of locally semi-Lipschitz functions with $\tau(q)$-compact sets on an asymmetric normed space.

Theorem 5.3.1. *(compare Theorem 2.3.1) Let (X,q) be a quasi-metric space and $\emptyset \neq F \subseteq X$. Then the following conditions are equivalent:*

(1) $cl_{\tau(q)}(F)$ is $\tau(q)$-compact;

(2) if $(Y,\|.|)$ is an asymmetric normed space and $\varphi : (X,q) \to (Y,\|.|)$ is continuous, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;

(3) if $(Y,\|.|)$ is an asymmetric normed space and $\varphi : (X,q) \to (Y,\|.|)$ is locally semi-Lipschitz, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;

(4) if $\varphi : (X,q) \to (\mathbb{R},u)$ is locally semi-Lipschitz, then $\varphi(F)$ is a u-bounded set of real numbers.

Proof. (1) $\Rightarrow$ (2) If $\mathrm{cl}_{\tau(q)}(F)$ is $\tau(q)$-compact and $\varphi : (X,q) \to (Y,\|.|)$ is continuous. Then $\mathrm{cl}_{\tau(q)}(F)$ is $\tau(q^s)$-compact and $\varphi : (X,q^s) \to (Y,\|.\|)$ is continuous too. It follows that $\varphi(\mathrm{cl}_{\tau(q)}(F))$ is $\tau(q^s)$-compact and $\varphi(F)$ is $q_{\|.\|}$-bounded. Hence, $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$.
(2) $\Rightarrow$ (3) Follows from Remark 5.1.1, i.e., the locally semi-Lipschitz function is always continuous.
(3) $\Rightarrow$ (4) Follows directly.
(4) $\Rightarrow$ (1) Suppose that $\mathrm{cl}_{\tau(q)}(F)$ is not $\tau(q)$-compact. Then we can find a sequence (f_n) in F with $f_j \neq f_i$ for $i \neq j$ and the sequence (f_n) in F does not converge with respect to $\tau(q)$. For any $n \in \mathbb{N}$, let $\mu_n := q(f_n, \{f_j : j \neq n\}) > 0$ and $\epsilon_n := \left\{\frac{1}{n}, \frac{\mu_n}{3}\right\}$.
It follows that the family $\{D_q(f_n,\epsilon_n) : n \in \mathbb{N}\}$ is such that whenever $i \neq k$ we have $D_q(f_i,\epsilon_i) \neq D_q(f_k,\epsilon_k)$ and $\epsilon_i + \epsilon_k < \max\{\mu_i,\mu_k\}$. For any $n \in \mathbb{N}$, let $\phi_n : (X,q) \to (\mathbb{R},u)$ be a function defined by $\phi_n(x) := n - \frac{n}{\epsilon_n} q(f_n,x)$ for any $x \in X$. Then for any $x,y \in X$, we have

$$\begin{aligned} u(\phi_n(x),\phi_n(y)) &= \phi_n(x) - \phi_n(y) = \left[n - \frac{n}{\epsilon_n} q(f_n,x)\right] - \left[n - \frac{n}{\epsilon_n} q(f_n,y)\right] \\ &= \frac{n}{\epsilon_n}\left[q(f_n,y) - q(f_n,x)\right] \\ &\leq \frac{n}{\epsilon_n}\left[q(f_n,x) + q(x,y) - q(f_n,x)\right] \\ &= \frac{n}{\epsilon_n} q(x,y). \end{aligned}$$

Hence $\phi_n : (X,q) \to (\mathbb{R},u)$ is k-semi-Lipschitz function with $k = \dfrac{n}{\epsilon_n}$. Observe that $\phi_n(x) > 0$ if and only if $q(f_n,x) < \epsilon_n$ for $n \in \mathbb{N}$. Let $\varphi : (X,q) \to (\mathbb{R},u)$

be the function defined by

$$\varphi(x) = \begin{cases} \phi_n(x) & \text{if} \quad x \in D_q(f_n, \epsilon_n) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varphi(f_n) = \phi_n(f_n) = n - \dfrac{n}{\epsilon_n} q(f_n, f_n) = n$, it follows that $\varphi(F)$ is u-unbounded. To complete the proof, we need to show that φ is locally semi-Lipschitz. Let us consider an arbitrary point $x_0 \in X$. Since $\epsilon_n < \frac{1}{n}$ for any $n \in \mathbb{N}$ and the sequence (f_n) does not $\tau(q)$- converge to x_0. There exists $\delta > 0$ such that $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) = \emptyset$ or $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$ atmost finitely n. Let us say $n_1, n_2, \cdots, n_k$ and consider the following cases:

Case 1: If $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) = \emptyset$. Then $\varphi|_{D_q(x_0,\delta)} = 0$.

Case 2: Since $\phi_n : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz. From the second part of Remark 5.1.1 and if $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$ then whenever $q(x_0, x) < \delta$,

$$\varphi(x) = \max\{0, \phi_{n_1}(x), \phi_{n_2}(x), \cdots, \phi_{n_k}(x)\} \quad \text{is semi-Lipschitz.}$$

Either way, $\varphi|_{D_q(x_0,\delta)}$ is semi-Lipschitz. □

In the previous theorem, we have proved that every real-valued locally semi-Lipschitz on a quasi-metric space is $\tau(q)$-compact if and only if the quasi-metric is Bourbaki-bounded. Now a similar question arises for the semi-Lipschitz in the small functions. Before answering this question in Theorem 5.3.2, we first state the following definition.

Definition 5.3.2. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. A function $\varphi : (X, q) \to (Y, \|.|)$ is called* semi-Lipschitz in the small *if there exists $\delta > 0$ and $k \geq 0$ such that if $q(x, y) < \delta$ then $\|\varphi(x) - \varphi(y)| \leq kq(x, y)$.*

Theorem 5.3.2. *(compare Theorem 2.3.3) Let (X, q) be a quasi-metric space and $\emptyset \neq F \subseteq X$. Then the following conditions are equivalent:*

(1) the subset F is q-Bourbaki-bounded;

(2) if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;

(3) if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|.|)$ is semi-Lipschitz in the small function, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;

(4) if $\varphi : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz in the small function, then $\varphi(F)$ is a u-bounded set of real numbers.

Proof. (1) $\Rightarrow$ (2) If $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous then there exists $\delta > 0$ such that whenever $x, y \in X$ with $q(x, y) < \delta$, we have

$$q_{\|.|}(\varphi(x), \varphi(y)) = \|\varphi(x) - \varphi(y)| < 1. \tag{5.3.4}$$

By the q-Bourbaki-boundedness of F, there exists $A := \{a_1, a_2, \cdots, a_m\} \subseteq X$ such that

$$F \subseteq \bigcup_{i=1}^{m} D_q^n(a_i, \delta)$$

for some positive integer n. If we take f artbitrary in F, then there exists k with $1 \le k \le m$ such that $f \in D_q^n(a_k, \delta)$. Then for some k with $1 \le k \le m$, there exists a δ-chain $\{f_0, f_1, \cdots, f_n\}$ with $f_0 = a_k$, $f_n = f$ and

$$q(f_{i-1}, f_i) < \delta \text{ whenever } i \text{ with } 1 \le i \le m. \tag{5.3.5}$$

It follows from the uniform continuity of φ and inequality (5.3.4) that

$$q_{\|.|}(\varphi(f_{i-1}), \varphi(f_i)) < 1 \text{ whenever } i \text{ with } 1 \le i \le m. \tag{5.3.6}$$

Hence, for some k with $1 \le k \le m$, we have

$$\begin{aligned} q_{\|.|}(\varphi(a_k), \varphi(f)) = q_{\|.|}(f_0, f_n) &\le q_{\|.|}(f_0, f_1) + q_{\|.|}(f_1, f_2) + \cdots + q_{\|.|}(f_{n-1}, f_n) \\ &< n. \end{aligned}$$

Thus, $\varphi(f) \in \bigcup_{i=1}^{m} D_{q_{\|.|}}(\varphi(a_i), n)$ for any $f \in F$ and $\varphi(F) \subseteq D_q(\varphi(A), n)$. Therefore, $\varphi(F)$ is $q_{\|.|}$-bounded.

(2) $\Rightarrow$ (3) Follows from Lemma 5.1.3.

(3) $\Rightarrow$ (4) Follows directly by replacing $(Y, \|.|)$ with $(\mathbb{R}, u)$ in (3).

(4) $\Rightarrow$ (1). Suppose that F is not q-Bourbaki-bounded. Then there exists a $\delta > 0$ such that if $\{f_1, f_2, \cdots, f_k\} \subseteq X$ and a positive integer n, we have $F \not\subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$. We have two cases on the structure of F.

Case 1: If $f \in F$, then there exists a positive integer n such that for all $j \in \mathbb{N}$

$$F \cap D_q^n(f, \delta) = F \cap f_{\precsim_\delta}.$$

Let f_1 be an arbitrary point of F. We choose a positive integer n_1 such that

$$F \cap D_q^{n_1}(f_1, \delta) = F \cap f_{1_{\asymp_\delta}}.$$

Since F is not q-Bourbaki-bounded, there exists $f_2 \in F$ such that $f_2 \notin D_q^{n_1}(f_1, \delta)$. It follows that $f_{1_{\asymp_\delta}} \neq f_{2_{\asymp_\delta}}$ by the choice of n_1.
One chooses another $n_2 \in \mathbb{Z}^+$ such that $n_2 > n_1$ and $F \cap D_q^{n_2}(f_2, \delta) = F \cap f_{2_{\asymp_\delta}}$.
Moreover, since $F \not\subseteq \bigcup_{j=1}^{2} D_q^{n_2}(f_j, \delta)$, we can find $f_3 \in F \setminus (f_{3_{\asymp_\delta}} \cup f_{2_{\asymp_\delta}})$.
Continuing this procedure by induction, we can find a sequence (f_j) with distinct terms in F such that for any $i \neq j$ we have $f_{i_{\asymp_\delta}} \neq f_{j_{\asymp_\delta}}$. Therefore, we define a function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$$\varphi(x) = \begin{cases} j & \text{if } \quad x \asymp_\delta f_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the function φ is constant on $D_q(x, \delta)$ and it is unbounded on F since $\varphi(f_j) = j$. Therefore, the function φ is semi-Lipschitz in the small function.
Case 2: If there exists $f \in F$ and for all positive integer n, there exists $j \in \mathbb{N}$ such that

$$F \cap D_q^n(f, \delta) \subset F \cap D_q^{n+j}(f, \delta).$$

For $x \asymp_\delta f$, let $n(x)$ be the smallest positive integer n such that

$$x \in F \cap D_q^n(f, \delta). \tag{5.3.7}$$

We then define the function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$$\varphi(x) = \begin{cases} (n(x) - 1)\delta + \operatorname{dist}_q(x, D_q^{n(x)-1}(f, \delta)) & \text{if } \quad x \neq f \text{ and } x \asymp_\delta f \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the function φ is unbounded on F. We now have to show that if $x \neq y$ and $q(x, y) < \delta$, then for $k = 2$,

$$u(\varphi(x), \varphi(y)) \leq kq(x, y).$$

If either x or y is not related to f with respect to $\asymp_\delta$, then since $x \neq y$, both x and y are not related to f with respect to $\asymp_\delta$ and

$$u(\varphi(x), \varphi(y)) = 0 < 2q(x, y).$$

If $x \asymp_\delta f$ and $y \asymp_\delta f$, then we have some cases on $n(x)$ and $n(y)$:
If $n(x) > n(y)$. Suppose that $n(y) = 0$ then $y = f$ and $0 < q(x, y) < \delta$ which implies that $y \in D_q(x, \delta)$ hence $n(x) = 1$.

Furthermore,

$$\begin{aligned} u(\varphi(x),\varphi(y)) &= u[(1-1)\delta + \text{dist}_q(x, D_q^0(f,\delta)), 0] \\ &= \text{dist}_q(x, \{y\}) \\ &= q(x,y) < 2q(x,y). \end{aligned}$$

If $n(y) \geq 1$ and $n(x) = n(y)$, then

$$\begin{aligned} u(\varphi(x),\varphi(y)) &= \max\{[\text{dist}_q(x, D_q^{n(x)-1}(f,\delta)) - \text{dist}_q(y, D_q^{n(x)-1}(f,\delta))], 0\} \\ &\leq q(x,y) < 2q(x,y). \end{aligned}$$

If $n(y) \geq 1$ and $n(x) > n(y)$ (i.e., $n(x) = n(y)+1$) with $\varphi(x) \leq \varphi(y)$, then there is nothing to prove since $u(\varphi(x),\varphi(y)) = 0 < 2q(x,y)$.

If $\varphi(x) > \varphi(y)$, then
$u(\varphi(x),\varphi(y)) = \varphi(x) - \varphi(y)$
$= [(n(x)-1)\delta + \text{dist}_q(x, D_q^{n(x)-1}(f,\delta))] - [(n(y)-1)\delta + \text{dist}_q(y, D_q^{n(y)-1}(f,\delta))]$
$= [(n(y)+1-1)\delta - [(n(y)-1)\delta] - [\text{dist}_q(x, D_q^{n(y)+1-1}(f,\delta))] - [(n(y)-1)\delta - \text{dist}_q(y, D_q^{n(y)-1}(f,\delta))]$.
Furthermore,

$$\begin{aligned} u(\varphi(x),\varphi(y)) &= \delta + [\text{dist}_q(x, D_q^{n(y)}(f,\delta))] - [(n(y)-1)\delta - \text{dist}_q(y, D_q^{n(y)-1}(f,\delta))] \\ &\leq \delta + q(x,y) + \text{dist}_q(y, D_q^{n(y)}(f,\delta)) - \text{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \end{aligned}$$

Since $n(w)$ is the smallest n such that $y \in F \cap D_q^n(f,\delta)$, it therefore means

$$\text{dist}_q(y, D_q^{n(y)}(f,\delta)) = 0.$$

Thus, we have

$$u(\varphi(x),\varphi(y)) \leq \delta + q(x,y) - \text{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \tag{5.3.8}$$

We claim that,

$$\delta - q(x,y) \leq \text{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \tag{5.3.9}$$

Suppose otherwise, i.e., $\text{dist}_q(y, D_q^{n(y)-1}(f,\delta)) < \delta - q(x,y)$, then

$$\begin{aligned} \text{dist}_q(x, D_q^{n(y)-1}(f,\delta)) &\leq q(x,y) + \text{dist}_q(y, D_q^{n(y)-1}(f,\delta)) \\ &< q(x,y) + \delta - q(x,w) \\ &< \delta. \end{aligned}$$

So $x \in D_q^{n(y)-1}(f,\delta)$ which implies that $n(x) \leq n(y) - 1 + 1$ but this is a contradiction since $n(x) > n(y)$.

Combining (5.3.8) and (5.3.9) we have

$$u(\varphi(x), \varphi(y)) \leq \delta + q(x, y) - \delta + q(x, y) \leq 2q(x, y).$$

Therefore, the proof is complete.

□

Chapter 6

Conclusion

6.1 Summary of the achieved work

In this thesis, many aspects of metric bornologies that do not satisfy the symmetry condition have been investigated. In this chapter, we discuss the conclusion of our investigations and point out some open problems encountered throughout this present work. These open problems may comprise the topics of future research.
In the first part of our investigations, we started by revisiting my MSc dissertation [31] in which we introduced the concept of bornology on an extended quasi-metric space (X, q). We redefined and improved our own universal space constructed during our previous studies (compare [31, Definition 3.2.1] and [52, Section 3], see also Definition 3.1.2). The universal space was constructed on the set (f, A) where f is a real-valued continuous function on $\tau(q^s)$-closed set A.
In addition, we showed that the universal space is a bicomplete extended quasi-metric space and Propositions 3.1.1 and 3.1.2 proved that there are always isometries between the extended quasi-metric spaces and the universal spaces. In the same chapter, it is shown (see also [52]) that on a quasi-metric space (X, q), being bounded on q or q^t does not mean bounded on q^s. We proved also that given an extended quasi-metric space (X, q), the bornological universe $(X, \mathscr{B})$ is quasi-metrizable by q and the bornology $\mathscr{B}$ coincides with the quasi-metric bornology $\mathscr{B}_q(X)$. We concluded this chapter by proving in Theorem 3.2.3 that for the quasi-metric bornology $\mathscr{B}_q(X)$ on an extended quasi-metric space (X, q), there exists an extended bone fide quasi-metric q_1 on X, uniformly equivalent to q such that $\mathscr{B}_q(X) = \mathscr{B}_{q_1}(X)$.

In the second part of this work, we continued with our own investigations

into the bornology of q-totally bounded sets on quasi-metric spaces. Before proving the results on quasi-metric bornologies, we presented some comparisons of q-total boundedness with other notions. For example, we showed in Lemma 4.1.1 that every q-totally bounded quasi-pseudometric space (X, q) is q-bounded but the converse is not always true (see Example 4.1.2). In addition, we proved that a subset of a quasi-metric space (X, q) is joincompact if and only if it is both bicomplete and q-totally bounded (Theorem 4.1.2).
Morever, we studied the bornologies of q-totally bounded sets on a quasi-metric space (X, q). It was interesting to note in Theorem 4.2.1 that the bornology of q-totally bounded set agrees with the bornology of $\tau(q)$- relatively compact sets if and only if the quasi-metric space (X, q) is bicomplete. It also turned out that if a quasi-metric space (X, q) is supseparable then the quasi-metric bornology studied by Otafudu et al. [52] coincides with the bornology of q-totally bounded sets (see Theorems 4.2.4 and 4.2.5). In addition, we used asymmetric version of Hu's theorem (Theorem 3.2.1) to prove that the bornology of q-totally bounded sets with one point extension and a countable base coincides with quasi-metric bornology in [52] (see Theorem 4.2.5). Lastly, we proved in Theorem 4.2.7 that the family of nonempty sets of the quasi-metric space (X, q) becomes the bornology of q-totally bounded sets if and only if there exists a star-development on X.

In the last part of this work, we investigated the bornologies of Bourbaki-bounded sets with semi-Lipschitz functions on asymmetric normed spaces. In the first section, we studied the concepts of uniform-continuity and semi-Lipschitz functions. For instance, we illustrated in Example 5.1.1 and Lemma 5.1.1 that on the quasi-metric space (X, q) the uniform continuity of q^s does not imply the uniform continuity of q. However, we also studied the equivalent characterizations of Bourbaki-boundedness on quasi-metric spaces. It was noted that on a quasi-metric space (X, q), a set can be q-Bourbaki-bounded and q^t-Bourbaki-bounded but not q^s-Bourbaki-bounded (see Example 5.2.1). After introducing the concept of δ-chain in Definition 5.2.4, we constructed the relation $\asymp_\delta$ and proved in Lemma 5.2.3 that it is actually an equivalence relation.
In Section 5.3, we characterized those bornologies on asymmetric normed spaces that can be realized as bornologies of Bourbaki-bounded sets. We studied and compared the bornology of q-Bourbaki-bounded sets with other bornologies. For instance, Remark 5.3.2 proved that on quasi-metric spaces, the bornology of q-Bourbaki-bounded sits between the bornology of q-totally bounded sets and the bornology of bounded sets but for asymmetric normed spaces, the bornology of q-Bourbaki-bounded sets coincides with the quasi-metric bornology in [52]. Finally, we proved that every real-valued semi-

Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded (see Theorem 5.3.2).

6.2 Research areas for future work

Our conclusion leads us to list some open problems that we came across throughout our investigations. We would like to study these problems in future work.

6.2.1 Bornology of bourbaki-bounded sets in quasi-uniform spaces.

In 2009, Vroegrijk [55] studied the bornology of Bourbaki-bounded sets and the bornology of totally bounded on uniform spaces. He extended the Hu's results to uniform spaces by giving a characterization of uniformizable bornological universes. He also proved that for a locally convex topological vector space, the classical notion of boundedness coincides with Bourbaki-boundedness with respect to the canonical uniformity. In 2012, Garrido and Meroño [18] have characterized bornologies on a nonempty set X and obtained necessary and sufficient conditions for a bornology to be uniformly metrizable. More precisely, they considered the compact bornology, the totally bounded bornology and the bornology of the Bourbaki-bounded sets.

Definition 6.2.1. *[55, Definition 2.2] A subset A of uniform space $(X,\mathcal{U})$ is called* Bourbaki-bounded *if for any entourage $B \in \mathcal{U}$, there exists a nutural number n and a finite set $F \subseteq X$ such that $A \subseteq B^n[F]$, where*

$$B[F] = \bigcup_{f\in F} B(f) = \{y \in X : \exists\, f \in F \text{ such that } (f,y) \in B\}.$$

A subset B of a uniform space $(X,\mathcal{U})$ is totally bounded if for each entourage $\mathcal{U}$ there is a finite partition $(B_i)_{i=1}^n$ of B such that $B_i \times B_i \subseteq \mathcal{U}$ for each $i \in I$.

Lemma 6.2.1. *[55, Lemma 2.6] For a set $\mathcal{C}$ of real valued mappings on a nonempty set X, the following are equivalent:*

(i) $f(B)$ *is bounded for all* $f \in \mathcal{C}$;

(ii) B *is totally bounded in* $(X,\mathcal{C})$;

(iii) B *is Bourbaki-bounded in* $(X,\mathcal{C})$.

One can still ask about how to extend the bornological results of Vroegrijk and Garrido-Meroño in [18, 45] to bornological universe of quasi-uniform spaces. In short, how does one approach the following problems?

Problem 6.2.1. *What is the relationship between the bornology of Bourbaki-bounded sets on the quasi-uniform spaces* $(X,\mathcal{U})$ *and* $(X,\mathcal{U}^t)$?

Problem 6.2.2. *Is it posible to use asymmetric version of Hu's metrization theorem to quasi-metrize the Vroegrijk's bornological universe on quasi-uniform spaces?*

6.2.2 Bornology of modular quasi-pseudometric spaces

Chistyakov in 2008 introduced the concept of a modular metric space induced by F-modulars. In his PhD thesis, Sabogodi [22] introduced the concept of modular quasi-pseudometric on a nonempty set, a concept that generalizes modular pseudometrics to the framework of quasi-metric spaces. He also investigated the topological aspects of a set equipped with a modular quasi-pseudometric.

Definition 6.2.2. *[22, Definition 3.1.1] Consider a nonempty set* X*. A function* $w : (0, \infty) \times X \times X \to [0, \infty]$ *is said to be* modular quasi-metric *on* X *if it satisfies the following conditions:*

(i) $w(\lambda, x, x) = 0$ *whenever* $x \in X$ *and* $\lambda \in (0, \infty)$;

(ii) $w(\lambda + \mu, x, y) = w(\lambda, x, z) + w(\mu, z, y)$ *whenever* $x, y, z \in X$ *and* $\lambda, \mu \in (0, \infty)$;

(iii) $w(\lambda, x, y) = 0$ *and* $w(\lambda, y, x) = 0$ *implies that* $x = y$.

Theorem 6.2.1. *[22, Theorem 3.1.2] Consider a nonempty set* X *and a modular quasi-metric on* X*. Then the function* q_w *on* X *defined by*

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$$

whenever $x, y \in X$ *is an extended* T_0*-quasi-metric on* X.

In our future work, we want to answer these questions in the framework of modular quasi-pseudometric spaces.

Problem 6.2.3. *How can one define the bornology of bounded sets of modular quasi-pseudometric spaces?*

Problem 6.2.4. *What are the connections between the bornologies induced by modular quasi-pseudometrics* w *and* q_w?

Problem 6.2.5. *How do we study the bornology of* q_w*-totally bounded sets induced by a modular quasi-pseudometric* q_w?

6.2.3 Bourbaki-completeness and cofinal Bourbaki completeness in quasi-metric spaces

In the year 2004, Lechicki and others [35] initiated a study of bornological convergence for nets in metric spaces. They proved that convergences are pretopologies and their families are essentially bornologies on X. In addition, Beer and Levi [8] found another necessary and sufficient conditions through the novel notion of strong uniform convergence on bornologies. Munkres [33] added by proving that Cauchy continuity is preserved under uniform convergence on the bornology of totally bounded subsets but uniform continuity is not preserved under uniform convergence on bounded subsets. In [19], the authors introduced a new tool for the completeness of metric spaces with respect to Bourbaki-boundedness ideas, that they called Bourbaki-completeness and cofinal Bourbaki completeness. They also mentioned that the concept of Bourbaki-completeness is important to the development of mathematical tools for bornologies. The above explaination lead us into the following questions:

Problem 6.2.6. *How do we introduce completeness of quasi-metric spaces with respect to q-Bourbaki-boundedness ideas?*

Problem 6.2.7. *How do we study quasi-uniform convergence on the bornology of Bourbaki-bounded sets?*

References

[1] C. Agyingi, P. Haihambo and H A.Künzi, *Endpoints in T_0-quasi-metric spaces*, Topology Appl. 168 (2014), 82-93.

[2] C. Alegre, L. Ferrando and S. Garca-Raffi , *Compactness in asymmetric normed spaces*, Topology Appl. 155 (2008), 527-539.

[3] M. Atsuji, *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. 8 (1958), 11-16.

[4] G. Beer, *The structure of extended Real-valued metric spaces*, Set-Valued Var. Anal. 4 (2013), 591-602.

[5] G. Beer, *On convergence to infinity*, Monat. Math. 129 (2000), 267-280.

[6] G. Beer, C. Costantini and S. Levi, *Total boundedness in metrizable spaces*, Houst. J. Math. 37 (2011), 1347-1362.

[7] G. Beer and S. Levi, *Total boundedness and bornologies* Topology Appl. 156 (2009), 1271-1288.

[8] G. Beer and S. Levi, *Strong uniform continuity*, J. Math. Anal. Appl. 350 (2009), 568-589.

[9] G. Beer and M.I. Garrido, *Bornologies and locally lipschitz*, Bull. Aust. Math. Soc. 90 (2014), 257-263.

[10] G. Beer and J. Vanderwerff, *Structural Properties of Extended Normed Spaces*. Set-Valued Var. Anal. 23 (2015), 613-630.

[11] A. Berarducci, D. Dikranjan and J. Pelant , *Uniform quasi components, thin spaces and compact separation*, Topology Appl. 122 (2002), 51-64.

[12] J. Borwein, M. Fabian and J. Vanderwerff, *Locally Lipschitz functions and bornological derivatives*, CECM Report No. 93:012.

[13] N. Bourbaki, *Elements of mathematics, general topology*, part 1, Hermann, Paris, 1966.

[14] S. Cobzaş, *Functional analysis in asymmetric normed spaces*, Frontiers in Mathematics, Springer, Basel, (2013).

[15] D. Doitchinov, *On completeness of quasi-metric spaces*, Topology Appl. 30 (1988), 127-148.

[16] P. Fletcher and W.F. Lindgren, *Quasi-uniform Spaces, Lecture Notes in Pure and Applied Mathematics* 77 Marcel Dekker, New York, 1982.

[17] M. I. Garrido and A. S. Meroño, *Some classes of bounded sets in metric spaces. - In: Contribuciones matemticas en homenaje a Juan Tarrés, eds. M. Castrillón et al. Universidad Complutense de Madrid*, Madrid, (2012), 179-186.

[18] M. I. Garrido and A. S. Meroño, *Uniformly metrizable bornologies*, J. Convex Anal. 20 (2013), 285-299.

[19] M. I. Garrido and A. S. Meroño, *New types of completeness in metric spaces*, Ann. Acad. Sci. Fenn. Math. 39 (2014), 733-758.

[20] M. I. Garrido and J. Jaramillo, *Lipschitz-type functions on metric spaces*, J. Math. Anal. Appl. 340 (2008), 282-290.

[21] R. Goldberg, *Methods of Real Analysis*, 2nd edn Wiley, New York, (1976).

[22] K. Sabogodi, *Some topological aspects of modular quasi-metric spaces*, Ph.D. thesis, University of the Witswaterand, (2019).

[23] S. Salbany, *Bitopological spaces, compactifications and completions*, Math. Monographs, no. 1, University of Cape Town, Depart. Math. 1974

[24] S. Romaguera and M. Sanchis, *Applications of utility functions defined on quasi-metric spaces*. J. Math. Anal. Appl. 283 (2003), 219-235.

[25] S. Romaguera and M. Sanchis, *Semi-Lipschitz functions and best approximation in quasi-metric spaces*, J. Approx. Theory. 103 (2000), 292-301.

[26] S. Romaguera and M. Schellekens, *Quasi-metric properties of complexity spaces*, Topology Appl. 98 (1999), 311-322.

[27] A. Stojmirović *Quasi-metrics, Similarities and Searches, aspects of geometry of protein datasets*, Ph.D. thesis, Victoria University of Wellington, (2015).

[28] J. Marín, *Weak bases and quasi-pseudo-metrization of bi-spaces*. Topology Appl. 156 (2009), 3070-3076.

[29] R. Meyer, *Analytic cyclic cohomology*, Ph.D. thesis, Cornell Unversity, (1999).

[30] E. Minguzzi, *Quasi-pseudo-metrization of topological preordered spaces*, Topology Appl. 12 (2012), 288-298.

[31] D. Mukonda, *Bornology of extended quasi-metric spaces*, MSc. disertation, North-West University, (2018).

[32] M. G. Murdeshwar and K.K. Theckedath, *Boundedness in a quasi-uniform space*, Canada. Math. Bull. 13 (1970), 367-370.

[33] J. Munkres, *General topology*, 2nd. edition, Prentice-Hall, Upper Saddle River, New Jersey, (2000).

[34] S.T. Hu, *Boundedness in a topological space*, J. Math. Pures Appl. 228 (1949), 287-320.

[35] A. Lechicki, S. Levi and A. Spakowski, *Bornological convergences*, J. Math. Anal. Appl. 297 (2004) 751-770.

[36] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. 13 (1963), 71-89.

[37] E. Kemajou, P. H. Künzi and O. Olela Otafudu, *The Isbell-hull of a di-space*, Topology Appl. 159 (2012), 2463-2475.

[38] R. D. Kopperman, *Which topologies are quasi-metrizable?* Topology Appl. 52 (1993), 99-107.

[39] R. D. Kopperman, *Asymmetry and duality in topology*, Topology Appl. 66 (1995), 1-39.

[40] C. S. Kubrusly, *The elements of operator theory*, Birkhuser Basel 2011.

[41] S. Kundua, M. Aggarwala and S. Hazrab, *Finitely chainable and totally bounded metric spaces*, Topology Appl. 216 (2017), 59-73.

[42] H.P. Künzi, *A construction of the B-completion of a quasi-metric space*, Topology Appl. 170 (2014), 25-39.

[43] H.P. Künzi, *Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology. In Handbook of the history of general topology*, Vol. 3, Topol. Kluwer Acad. Publ. Dordrecht, 2001.

[44] H. P. Künzi and M. Sanchis, *The Katětov construction modified for a T_0-quasi-metric space*, Topology Appl. 159 (2012), 711-720.

[45] H. P. Künzi and C. Ryser, *The Bourbaki quasi-uniformity*, Topology Proc. 20 (1995), 161-183.

[46] H. P. Künzi, *An introduction to quasi-uniform spaces*, Contemp. Math. 486 (2009), 239-304.

[47] H. P. Künzi and C. Makitu Kivuvu, *A double completion for an arbitrary T_0-quasi-metric space*, J. log. Algebr. Program. 76 (2008), 251-269.

[48] H. P. Künzi and O. Olela Otafudu, *q-Hyperconvexity in quasi-pseudometric spaces and fixed point theorems*, J. Func. Spaces Appl. 10 (2012) 1-18.

[49] O. Olela Otafudu, *Convexity in quasi-metric spaces*, Ph.D. thesis, University of Cape Town, (2012).

[50] O. Olela Otafudu and D. Mukonda, *Bornologies of totally bounded sets in quasi-metric spaces*, Facta. Univer. Ser. Math. (under review).

[51] O. Olela Otafudu and D. Mukonda, *On Bourbaki-bounded sets on quasi-pseudometric spaces*, Comment. Math. Univer. Carolinae. (under review).

[52] O. Olela Otafudu, W. Toko and D. Mukonda, *On Bornology of extended quasi-metric spaces*, Hacet. J. Maths. Stat. 48 (2019), 1767-1777.

[53] A. Piękosz and E. Wajch, *Quasi-Metrizability of Bornological Biuniverses in ZF*, J. Convex Anal. 22 (2015), 1041-1060.

[54] H. Vaughan, *On locally compact metrizable spaces*. Bull. Am. Math. Soc. 43 (1937), 532-535.

[55] T. Vroegrijk, *Uniformizable and realcompact bornological universes*, Appl. Gen. Topol. 2 (2009), 277-287.

[56] S . Willard, *General topology*. Dover Publications Inc., Mineola, New York, (1970).