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A HAHN-JORDAN DECOMPOSITION AND RIESZ-FRECHET REPRESENTATION THEOREM IN RIESZ SPACES¹

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ABSTRACT. We give a Hahn-Jordan decomposition in Riesz spaces which generalizes that of [B.A. WATSON, An Andô-Douglas type theorem in Riesz spaces with a conditional expectation, *Positivity*, **13** (2009), 543–558] and a Riesz-Frechet representation theorem for the T -strong dual, where T is a Riesz space conditional expectation operator. The result of Watson was formulated specifically to assist in the proof of the existence of Riesz space conditional expectation operators with given range space, i.e., a result of Andô-Douglas type. This was needed in the study of Markov processes and martingale theory in Riesz spaces. In the current work, our interest is a Riesz-Frechet representation theorem, for which another variant of the Hahn-Jordan decomposition is required.

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Key words: Riesz spaces, Andô-Douglas theorem, Radon-Nikodým theorem, Hahn-Jordan decomposition, Riesz-Frechet representation, strong dual, generalized inner products.

1. Introduction. We present a generalization of the Hahn-Jordan decomposition of measure theory to an abstract form in Riesz spaces with two objectives: first, that it include as an application the Hahn-Jordan decomposition given in [19];

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second, that it provide the generalization needed in duality theory for the proof of a Riesz-Frechet representation theorem in Riesz spaces, see later for details. In [19], the aim was to prove the existence of Riesz space conditional expectation operators with suitably specified range spaces. In particular a generalization of the results of Andô and Douglas [3, 6] was proved. This provided a critical result for the study of Markov processes in Riesz spaces and martingale theory in Riesz spaces, see [8, 17]. For other results in this area we refer the reader to [10, 13].

The setting used for our Hahn-Jordan decomposition is as follows. Let E be a Dedekind complete Riesz space with weak order unit, say e , and G be a Dedekind complete Riesz subspace of E which also contains e , so $\{e\} \subset G \subset E$. Denote the set of all components of e in E by $\mathcal{K}(e)$, and by $\mathcal{K}_G(e) = G \cap \mathcal{K}(e)$ the set of all components of e in E which are in G . It should be noted here that $\mathcal{K}(e)$ and $\mathcal{K}_G(e)$ are Boolean algebras with $\mathcal{K}_G(e)$ a subalgebra of $\mathcal{K}(e)$. The multiplication of the components of e is that of the f -algebra E_e which is taken to have algebraic unit e . Here E_e denotes the ideal in E generated by e , i.e., $E_e = \{f \in E \mid |f| \leq ke \text{ for some } k \in \mathbb{R}^+\}$. Moreover p, q are components of e in E if and only if there are band projections P, Q on E with $p = Pe$ and $q = Qe$. In this case, $pq = PQe = QPe = qp$ and $pq = p \wedge q$. It should be noted that E is an E_e -module, see [11, 12]. As e is a weak order unit for G , we have that G and E are G_e -modules. Here $\mathcal{K}_G(e) \subset G_e$.

Let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{K}(e)$ which contains $\mathcal{K}_G(e)$ and is order closed in E , i.e., if (p_α) is a net in $\mathcal{B}$ with $p_\alpha \downarrow p$ in E , then $p \in \mathcal{B}$. Take $\psi : \mathcal{B} \rightarrow G$ be a map such that:

- (i) If $p \in \mathcal{B}$ and $q \in \mathcal{K}_G(e)$ then $\psi(pq) = q\psi(p)$.
- (ii) If $p, q \in \mathcal{B}$ with $pq = 0$ then $\psi(p \vee q) = \psi(p + q) = \psi(p) + \psi(q)$ (additivity).
- (iii) If (p_α) is a net in $\mathcal{B}$ with $p_\alpha \downarrow p$ in E , then $\psi(p_\alpha) \rightarrow \psi(p)$ (order continuity of ψ).
- (iv) There is $g \in E^+$ so that $|\psi(p)| \leq g$ for all $p \in \mathcal{B}$ (ψ is order bounded).

We say that $q \in \mathcal{B}$ is strongly positive (resp. strongly negative) with respect to ψ if $\psi(p) \geq 0$ (resp. ≤ 0) for all $p \in \mathcal{B}$ with $p \leq q$. The Hahn-Jordan decomposition presented in Theorem 3.3 gives the existence of $q \in \mathcal{B}$ so that q is strongly positive with respect to ψ and $e - q$ is strongly negative with respect to ψ .

As noted earlier, we apply the Hahn-Jordan decomposition, Theorem 3.3, in the following setting to obtain a Riesz-Frechet representation of the T -strong dual of a Riesz space. Let E be a Dedekind complete Riesz space with weak order unit. By T being a conditional expectation operator on E we mean that T is a linear positive order continuous projection on E which maps weak order units to weak order units and has range $R(T)$ closed with respect to order limits in E . This gives that there is at least one weak order unit, say e , with $Te = e$, and that $R(T)$ is Dedekind complete when considered as a subspace of E . By T being strictly positive we mean that if $f \in E^+$, the positive cone of E , and $f \neq 0$ then $Tf \in E^+$ and $Tf \neq 0$.

As shown in [11], a strictly positive conditional expectation operator, T , on a Dedekind complete Riesz space with weak order unit, can be extended to a strictly positive conditional expectation operator, also denoted T , on its natural domain, denoted $L^1(T) := \text{dom}(T) - \text{dom}(T)$. We say that E is T -universally complete if $E = L^1(T)$. From the definition of $\text{dom}(T)$, see [11], E is T -universally complete if and only if for each upwards directed net $(f_\alpha)_{\alpha \in \Lambda}$ in E^+ such that $(Tf_\alpha)_{\alpha \in \Lambda}$ is order bounded in E_u , we have that $(f_\alpha)_{\alpha \in \Lambda}$ is order convergent in E . Here E_u denotes the universal completion of E , see [15, page 323]. E_u has an f -algebra structure which can be chosen so that e is the multiplicative identity.

Let E be a T -universally complete Riesz space, where T is a strictly positive conditional expectation operator on E , and let e be a weak order unit for E with $Te = e$. From [12] $R(T)$ is universally complete and an f -algebra, further $E = L^1(T)$ is an $R(T)$ -module. From [11, Theorem 5.3], T is an averaging operator, i.e., if $f \in R(T)$ and $g \in E$ then $T(fg) = fT(g)$. This prompts the definition of an $R(T)$ (vector valued) norm $\|\cdot\|_{T,1} := T|\cdot|$ on $L^1(T)$. The homogeneity is with respect to multiplication by elements of $R(T)^+$. The Hahn-Jordan decomposition of [19] is an application of Theorem 3.3 with $\psi(p) = T(pf)$ for $p \in \mathcal{B}$ and $f \in E$, see Section 3.

The definition of the Riesz space $L^2(T) := \{f \in L^1(T) \mid f^2 \in L^1(T)\}$ was given in [14]. By the averaging property, $L^2(T)$ is an $R(T)$ -module and the map

$$f \mapsto \|f\|_{T,2} := (T(f^2))^{1/2}, \quad f \in L^2(T),$$

is an $R(T)$ -valued norm on $L^2(T)$. Aspects for this development for $L^p(T)$ with general $1 < p < \infty$ can be found in [5, 7]. Here the multiplication is as defined in the f -algebra E_u , which when restricted to the f -algebra E_e is also the multiplication there. Proofs of various Hölder type inequalities in Riesz spaces with conditional expectation operators can be found in [12] and [5]. In particular,

$$T|fg| \leq \|f\|_{T,2}\|g\|_{T,2}, \quad \text{for all } f, g \in L^2(T). \quad (1.1)$$

Let $E = L^2(T)$. We say that a map $\mathfrak{f} : E \rightarrow R(T)$ is a T -linear functional on E if it is additive, $R(T)$ -homogeneous and order continuous. We recall, from [2, Theorem 4.10], that, since $R(T)$ is a Dedekind complete Riesz space and E is a Riesz space, a linear map from E to $R(T)$ is order bounded if and only if it is order continuous. We denote the space of T -linear functionals on E by E^* and call it the T -dual of E . We note that $E^* \subset \mathcal{L}_b(E, R(T))$, since $R(T)$ -homogeneity implies real linearity. Further as $R(T)$ is Dedekind complete, so is $\mathcal{L}_b(E, R(T))$, see [2, page 12].

If $\mathfrak{f} \in E^*$ and there is $k \in R(T)^+$ such that

$$|\mathfrak{f}(g)| \leq k\|g\|_{T,2}, \quad \text{for all } g \in E,$$

we say that $\mathfrak{f}$ is T -strongly bounded. We denote the space of T -strongly bounded T -linear functionals on E by

$$\hat{E} := \{\mathfrak{f} \in E^* \mid \mathfrak{f} \text{ } T\text{-strongly bounded}\}$$

and refer to it as the T -strong dual of E . Further,

$$\|\mathfrak{f}\| := \inf\{k \in R(T)^+ \mid |\mathfrak{f}(g)| \leq k\|g\|_{T,2} \text{ for all } g \in E\}$$

defines an $R(T)$ -valued norm on $\hat{E}$ with

$$|\mathfrak{f}(g)| \leq \|\mathfrak{f}\| \|g\|_{T,2} \quad (1.2)$$

for all $g \in L^2(T)$.

Applying Theorem 3.3 in this setting (with $\psi(p) = \mathfrak{f}(p)$), see Lemma 4.2 below, leads to a Riesz-Frechet representation of $\hat{E}$, see Theorem 4.3. This, when combined with Proposition 4.1, gives the following identification.

THEOREM 1.1. *The map Ψ defined by $\Psi(f)(g) := T_f(g) = T(fg)$ for $f, g \in L^2(T)$ is a bijection between $E = L^2(T)$ and, its $R(T)$ -homogeneous T -strong dual, $\hat{E}$. This map is additive, $R(T)$ -homogeneous and $R(T)$ -valued norm preserving in the sense that $\|T_f\| = \|f\|_{T,2}$ for all $f \in L^2(T)$.*

We thank the referees for their useful suggestions and their interest in our work.

2. Preliminaries. For essential background in Riesz space theory we refer the reader to [1, 2, 15, 20, 21]. First we discuss properties of the Boolean subalgebra $\mathcal{B}$ of $\mathcal{K}(e)$ and the map $\psi : \mathcal{B} \rightarrow G$ with the conditions (i) to (iv) given in the introduction.

Concerning condition (i), $q \in \mathcal{K}_G(e)$ gives that $q \in \mathcal{B}$, and $\mathcal{B}$ is a Boolean algebra, so $pq \in \mathcal{B}$. Thus $\psi(pq)$ is defined and is in G . Further, $\psi(p) \in G$ and G is a G_e module, but $q \in \mathcal{K}_G(e) \subset G_e$ so ensuring that $q\psi(p)$ is defined in G . By (i) with $q = 0$ we have $\psi(0) = 0$. With regards to condition (iii), we note that it also holds for upward directed nets.

For $q \in \mathcal{B}$ we set

$$C(q) := \{\psi(p) \mid p \in \mathcal{B}, p \leq q\}.$$

Since $0 \in \mathcal{B}$, $0 \leq q$ and $\psi(0) = 0$ we have that $0 \in C(q)$. Further, as ψ is order bounded, so is $C(q)$. Thus, as G is Dedekind complete, we can define

$$\alpha(q) := \sup C(q),$$

and here $\alpha(q) \in G^+$, since $0 \in C(q)$.

For $q \in \mathcal{B}$ let

$$\mathcal{M}(q) := \{p \in \mathcal{B} \mid 2\psi(p) \geq p\alpha(q), p \leq q\}.$$

It should be note that the 2 in the above inequality (and the resultant work below) could be replaced by any fixed real number greater than 1. For all $p \leq q$ we have $p\psi(p) \leq p\alpha(q)$, but $\mathcal{M}(q)$ gives those $p \leq q$ for which $p\psi(p)$ is close to $p\alpha(q)$ (to within a given multiplicative factor of $p\alpha(q)$, which we have chosen as $1/2$).

LEMMA 2.1. *For each $q \in \mathcal{B}$, $0 \in \mathcal{M}(q)$, $\mathcal{M}(q)$ has a maximal element, $\hat{q} \in \mathcal{B}$, and if $r, s \in \mathcal{M}(q)$ with $rs = 0$, then $r + s \in \mathcal{M}(q)$.*

Proof. As $0 \in G \subset \mathcal{B}$, $0 \leq q$ and $2\varphi(0) = 0 = 0\alpha(q)$ it follows that $0 \in \mathcal{M}(q)$.

Let (p_γ) be a totally ordered increasing net in $\mathcal{M}(q)$ and $p_\gamma \leq e$, so, from the Dedekind completeness of E , $p_\gamma \rightarrow \bar{p} := \sup_\gamma p_\gamma \in E$. Since $\mathcal{B}$ is order closed in E , $\bar{p} \in \mathcal{B}$. Further, from the order continuity of ψ , we have that

$$\bar{p}\alpha(q) \leftarrow p_\gamma\alpha(q) \leq 2\psi(p_\gamma) \rightarrow 2\psi(\bar{p})$$

in order. Thus $\bar{p}\alpha(q) \leq 2\psi(\bar{p})$. Also $\bar{p} \leftarrow p_\gamma \leq q$ in order, so $\bar{p} \leq q$. Hence $\bar{p} \in \mathcal{M}(q)$. Thus $\bar{p}$ is an upper bound for (p_γ) from $\mathcal{M}(q)$ and Zorn's lemma gives that $\mathcal{M}(q)$ has a maximal element, say $\hat{q}$.

Finally, if $r, s \in \mathcal{M}(q)$ with $rs = 0$, then, property (ii) of ψ , $\psi(r + s) = \psi(r) + \psi(s)$ and from the definition of $\mathcal{M}(q)$ it follows that $r + s \in \mathcal{M}(q)$. $\square$

LEMMA 2.2. For $q \in \mathcal{B}$ and each $p \in \mathcal{M}(q)$,

$$(2\psi(p) - \alpha(q))p \geq 0,$$

i.e., $(\alpha(q) - 2\psi(p))^+$ and p are disjoint in E .

Proof. For $q \in \mathcal{B}$ and $p \in \mathcal{M}(q)$ we have from the definition of $\mathcal{M}(q)$ that $p \leq q$ and

$$2\psi(p) - p\alpha(q) \geq 0.$$

Multiplying $2\psi(p) - p\alpha(q)$ by p and noting that $0 \leq p = p^2$, the result follows. $\square$

LEMMA 2.3. Let Q denote the band projection onto the band, in E , generated by $(\alpha(q) - 2\psi(\hat{q}))^+$ and let $k := Qe$, then $k \in \mathcal{K}_G(e) \subset \mathcal{B}$. Further, if $k > 0$ then $k\alpha(q) > 0$ and there exists $p_0 \in \mathcal{B}$ with $p_0 \leq q$ such that

$$k(2\psi(p_0) - \alpha(q)) \not\leq 0.$$

Proof. We note that k is in the band generated by $\alpha(q)$, since

$$2\psi(\hat{q}) \geq \hat{q}\alpha(\hat{q}) \geq 0$$

which gives

$$\alpha(q) \geq (\alpha(q) - 2\psi(\hat{q}))^+.$$

Hence $k > 0$ gives $k\alpha(q) > 0$. Since $\alpha(q), \psi(\hat{q}) \in G$, we have $k \in G$ and thus $k \in \mathcal{B}$.

If there does not exist $p_0 \in \mathcal{B}$ with $p_0 \leq q$ such that

$$k(2\psi(p_0) - \alpha(q)) \not\leq 0,$$

then, for all $p \in \mathcal{B}$ with $p \leq q$, we have $2k\psi(p) \leq k\alpha(q)$. Taking the supremum over $p \in \mathcal{B}$ with $p \leq q$ in the above gives $2k\alpha(q) \leq k\alpha(q)$. Thus $k\alpha(q) \leq 0$ which contradicts $k\alpha(q) > 0$. $\square$

THEOREM 2.4. For each $q \in \mathcal{B}$, if $\hat{q}$ is a maximal element of $\mathcal{M}(q)$, then

$$\alpha(q) \leq 2\psi(\hat{q}) \leq 2\alpha(q).$$

Proof. From the definition of $\alpha(q)$, as $\hat{q} \in \mathcal{M}(q)$, it follows that $2\psi(\hat{q}) \leq 2\alpha(q)$.

We now show that $2\psi(\hat{q}) \geq \alpha(q)$. If this were not the case, then $(\alpha(q) - 2\psi(\hat{q}))^+ > 0$. Let Q denote the band projection onto the band, in E , generated by $(\alpha(q) - 2\psi(\hat{q}))^+$ and $k := Qe > 0$, then $k \in \mathcal{K}_G(e)$. By Lemma 2.3, there exists $p_0 \in \mathcal{B}$ with $p_0 \leq q$ such that

$$k(2\psi(p_0) - \alpha(q)) \not\leq 0,$$

and thus

$$k(2\psi(p_0) - \alpha(q))^+ = (k(2\psi(p_0) - \alpha(q)))^+ > 0, \quad (2.1)$$

as $k \geq 0$.

Let P_1 be the band projection onto the band generated by $(2\psi(p_0) - \alpha(q))^+$ and $p_1 = P_1e \in \mathcal{K}_G(e)$. Let $\bar{q} := kp_1 \in \mathcal{K}_G(e)$, then from (2.1), $\bar{q} > 0$ and

$$\bar{q}(2\psi(p_0) - \alpha(q)) = kp_1(2\psi(p_0) - \alpha(q)) = k(2\psi(p_0) - \alpha(q))^+ > 0. \quad (2.2)$$

Further, as $\alpha(q) \geq 0$ and $0 \leq p_0 \leq e$, we have $p_0\alpha(q) \leq \alpha(q)$. Thus, from equation (2.2) and the definition of $\bar{q}$,

$$2\bar{q}\psi(p_0) > \bar{q}\alpha(q) \geq \bar{q}p_0\alpha(q). \quad (2.3)$$

Here $k, p_1, \bar{q} \in \mathcal{K}_G(e)$ and $p_0 \in \mathcal{B}$ so $\bar{q}\psi(p_0) = \psi(\bar{q}p_0)$ and $\bar{q}p_0 \in \mathcal{B}$, thus from (2.3),

$$2\psi(\bar{q}p_0) > \bar{q}p_0\alpha(q), \quad (2.4)$$

giving $\bar{q}p_0 \in \mathcal{M}(q)$. Here $\bar{q}p_0 > 0$, as otherwise (2.4) would give $0 > 0$. Further

$$\bar{q}p_0 = kp_1p_0 \leq kp_0 \leq kq$$

and, by Lemma 2.2, $k\hat{q} = 0$. Thus $\bar{p}p_0 \wedge \hat{q} = 0$, so, by Lemma 2.1,

$$\hat{q} < \bar{p}p_0 + \hat{q} \in \mathcal{M}(q)$$

which contradicts the definition of $\hat{q}$. □

If F is a universally complete Riesz space with weak order unit, say e , then F is an f -algebra and e can be taken as the algebraic unit, see [18, Theorem 3.6].

DEFINITION 2.5. Let F be a universally complete Riesz space with weak order unit, say e , and take e as the algebraic unit of the associated f -algebra structure. We say that $g \in F$ has a partial inverse if there exists $h \in F$ such that $gh = hg = P_{|g|}e$ where $P_{|g|}$ denotes the band projection onto the band generated by $|g|$. We refer to h as the canonical partial inverse of g if, in addition, to being a partial inverse to g , we have that $(I - P_{|g|})h = 0$, i.e., $h \in \mathcal{B}_{|g|}$, where $\mathcal{B}_{|g|}$ is the band generated by $|g|$.

The following result gives existence, uniqueness and positivity results concerning partial inverses and canonical partial inverses. We denote by $\mathcal{B}_f$ the band generated by f and by P_f the band projection onto $\mathcal{B}_f$.

The following theorem follows directly from [9, Theorem 5] and [16, Remark 3.3].

THEOREM 2.6. *Let F be a universally complete Riesz space with weak order unit, say e , which also take as the algebraic unit of the associated f -algebra structure. Each $g \in F$ has a partial inverse $h \in F$. The canonical partial inverse of g exists, is unique and in this case g is also the canonical partial inverse of h . If $g \in F^+$ then so is its canonical partial inverse.*

3. Hahn-Jordan decomposition. We are now in a position to develop the generalized abstract Hahn-Jordan decomposition, which generalizes [19, Theorem 3.5] (details of this are given as an application at the end of this section) and [21, page 184].

THEOREM 3.1. *If $p \in \mathcal{B}$ with $\psi(p) < 0$ then there exists $v \in \mathcal{B}$ with $v \leq p$ and v strongly negative with respect to ψ and $\psi(v) \leq \psi(p)$.*

Proof. We begin by noting from Theorem 2.4 that for each $q \in \mathcal{B}$, if $\hat{q}$ is a maximal element of $\mathcal{M}(q)$, then

$$0 \leq \alpha(\hat{q}) \leq \alpha(q) \leq 2\psi(\hat{q}) \leq 2\alpha(\hat{q}). \quad (3.1)$$

Let $p \in \mathcal{B}$ with $\psi(p) < 0$. Set $\alpha_1 := \alpha(p) \geq 0$ and $p_1 := \hat{p} \in \mathcal{M}(p)$. From (3.1) applied to p we have $\alpha_1 \leq 2\psi(p_1)$. Inductively we define

$$\alpha_{n+1} := \alpha\left(p - \bigvee_{i=1}^n p_i\right)$$

and

$$p_{n+1} := \left(\widehat{p - \bigvee_{i=1}^n p_i}\right) \in \mathcal{M}\left(p - \bigvee_{i=1}^n p_i\right).$$

Here

$$\alpha_{n+1} \leq 2\psi(p_{n+1})$$

and $p_{n+1} \leq p - \bigvee_{i=1}^n p_i$, so $p_i p_j = 0$ for all $i \neq j$. Since $\mathcal{B}$ is order closed and E is Dedekind complete, we can define

$$\bar{p} := \sum_{i=1}^{\infty} p_i = \bigvee_{i=1}^{\infty} p_i \in \mathcal{B}$$

and we have $\bar{p} \leq p$. The additivity and order continuity of ψ give that

$$\psi(\bar{p}) = \psi\left(\sum_{i=1}^{\infty} p_i\right) = \sum_{i=1}^{\infty} \psi(p_i),$$

which, along with the definition of p_n , gives

$$0 \leq \sum_{i=1}^{\infty} \alpha_i \leq \sum_{i=1}^{\infty} 2\psi(p_i) = 2\psi(\bar{p}). \quad (3.2)$$

Let $v := p - \bar{p}$ then $v \leq p$ and $\psi(v) + \psi(\bar{p}) = \psi(p)$, so, by (3.2), $\psi(v) \leq \psi(p)$.

If $q \leq v$ then $q \leq p - \sum_{i=1}^n p_i$ making $\psi(q) \leq \alpha_{n+1}$, for all $n \in \mathbb{N}$. This together with (3.2) gives that

$$n\psi(q) \leq \sum_{i=2}^{n+1} \alpha_i \leq 2\psi(\bar{p}),$$

for all $n \in \mathbb{N}$. Since E is Archimedean, this gives that $\psi(q) \leq 0$. Hence v is strongly negative with respect to ψ and v is as required by the theorem. $\square$

COROLLARY 3.2. *If $\psi(q) \not\geq 0$ for some $q \in \mathcal{B}$, then there exists $v \in \mathcal{B}$ with $v \leq q$ such that v is strongly negative with respect to ψ and $\psi(v) \leq -\psi(q)^-$.*

Proof. Let $g := P_{\psi(q)} - e$, then $g \in \mathcal{K}_G(e)$. Let $p := gq \in \mathcal{B}$ then $\psi(p) = g\psi(q) = -\psi(q)^- < 0$. Applying the previous theorem we have that there is $v \in \mathcal{B}$ with $v \leq p = gq \leq q$ so that v is strongly negative with respect to ψ and $\psi(v) \leq \psi(p) = -\psi(q)^-$. $\square$

We are now in a position to conclude the proof of the Hahn-Jordan decomposition of ψ with respect to $\mathcal{B}$.

THEOREM 3.3. (Abstract Hahn-Jordan Decomposition)

There exists $q \in \mathcal{B}$ which is strongly positive with respect to ψ and which has $e - q$ strongly negative with respect to ψ .

Proof. Let

$$\mathcal{H} := \{p \in \mathcal{B} \mid p \text{ strongly negative w.r.t. } \psi\},$$

then $0 \in \mathcal{H}$, so $\mathcal{H}$ is non-empty. By the order continuity of ψ and the order closedness of $\mathcal{B}$, $\mathcal{H}$ is order closed with respect to limits of upwards directed nets. Hence, by Zorn's lemma, $\mathcal{H}$ has maximal elements. If $p_1, p_2 \in \mathcal{H}$ and $q \in \mathcal{B}$, then $p_1 p_2 q, (e - p_1) p_2 q, (e - p_2) p_1 q \in \mathcal{H}$ and thus $\psi(p_1 p_2 q) \leq 0, \psi((e - p_1) p_2 q) \leq 0, \psi((e - p_2) p_1 q) \leq 0$, making

$$\psi((p_1 \vee p_2)q) = \psi(p_1 p_2)q + \psi((e - p_1) p_2 q) + \psi((e - p_2) p_1 q) \leq 0.$$

Hence $p_1 \vee p_2 \in \mathcal{H}$ and $\mathcal{H}$ is closed with respect to pairwise suprema. Thus the maximal element of $\mathcal{H}$ is unique. Denote this maximal element by q .

It remains only to show that $e - q$ is strongly positive with respect to ψ . Suppose that this is not the case. Then there exists $p \in \mathcal{B}$ with $p \leq e - q$ and $\psi(p) \not\geq 0$. Corollary 3.2 now gives that there exists $m \in \mathcal{B}$ with $m \leq p$ strongly negative

with respect to ψ . Thus $mq = 0$ and $q \vee m = q + m > q$ is strongly negative, contradicting the maximality of q . Hence $e - q$ is positive. $\square$

The Hahn-Jordan decomposition of [19, Theorem 3.5] follows from Theorem 3.3 by taking: E a Dedekind complete Riesz space with weak order unit; T a Riesz space conditional expectation operator on E with $Te = e$, where e is some chosen weak order unit; E to be T -universally complete; $G = R(T)$; F an order closed Riesz subspace of E with $R(T) \subset F$; $\mathcal{B}$ the set of all components of e ; and $\psi_f(p) := T(pf)$ for $p \in \mathcal{B}$ and $f \in E$.

4. A Riesz-Frechet representation theorem. The Hahn-Jordan decomposition of Theorem 3.3 is used in Lemma 4.2 which provides a crucial step for the development of the Riesz-Frechet representation theorem, Theorem 4.3.

Let $\mathcal{B}$ denote the lattice of band projections on $L^2(T)$. For brevity of notation, if $f \in E^+$ then P_f will denote the band projection onto the band generated by f in E and $p_f := P_f e$ where e is the chosen weak order unit of E . Here p_f is a component of e in E . Further, due to E being an E_e -module, and the definition of the multiplicative structure, $P_f g = p_f g$ where on the left is the action of the band projection P_f on g and on the right is the product of p_f and g , for $g \in E$.

PROPOSITION 4.1. *For $y \in E := L^2(T)$ let $T_y(x) := T(xy)$ for all $x \in E$, then $T_y \in \hat{E}$ and*

$$\|T_y\| = \|y\|_{T,2}. \quad (4.1)$$

The map $\Psi : y \mapsto T_y$ is $R(T)$ -homogeneous, additive and injective.

Proof. T is linear and from the averaging property of T in $L^2(T)$, see [11], we have that T_y is $R(T)$ -homogeneous. The order continuity of T and of multiplication in $L^2(T)$ give that T_y is in E^* . That T_y is in $\hat{E}$ and that $\|T_y\| \leq \|y\|_{T,2}$ follow directly from (1.1). However,

$$\|T_y\| \|y\|_{T,2} \geq |T_y(y)| = \|y\|_{T,2} \|y\|_{T,2}. \quad (4.2)$$

Let β be the canonical partial inverse of $\|y\|_{T,2}$ in $R(T)$. Multiplying (4.2) by β we obtain

$$\begin{aligned} \|T_y\| &\geq P_{\|y\|_{T,2}} \|T_y\| \\ &= \beta \|y\|_{T,2} \|T_y\| \\ &\geq \beta |T_y(y)| \\ &= \beta \|y\|_{T,2}^2 \\ &= P_{\|y\|_{T,2}} \|y\|_{T,2} \\ &= \|y\|_{T,2}. \end{aligned}$$

Thus (4.1) holds, also see [4, Theorem 8].

That the map $y \mapsto T_y$ is linear is straight forward, $R(T)$ -homogeneity follows from T being an averaging operator, while injectivity follows from the strict positivity of T and (4.1). $\square$

As a direct application of Proposition 4.1, $T = T_e \in E^*$.

The following lemma, critical to the proof of the Riesz-Frechet representation theorem, relies on the Abstract Hahn-Jordan Decomposition, Theorem 3.3.

LEMMA 4.2. For each $\mathfrak{g} \in \hat{E}$, there is a component $q_{\mathfrak{g}}^+$ of e in E so that $\mathfrak{g}(pq_{\mathfrak{g}}^+) \geq 0$ and $\mathfrak{g}((e - q_{\mathfrak{g}}^+)p) \leq 0$ for all components p of e in E .

Proof. We take $G = R(T)$, $\psi(p) = \mathfrak{g}(p)$ and $\mathcal{B}$ to be the set of all components of e in E . Here $|\psi(p)| \leq \|\mathfrak{g}\|$ for each $p \in \mathcal{B}$. Theorem 3.3 is applicable and gives that there is $q \in \mathcal{B}$ which is strongly positive with respect to ψ and has $e - q$ strongly negative with respect to ψ . Take $q_{\mathfrak{g}}^+ := q$. $\square$

THEOREM 4.3. (Riesz-Frechet representation theorem in Riesz space) For each $\mathfrak{f} \in \hat{E}$ there exists $y(\mathfrak{f}) \in E := L^2(T)$ such that $\mathfrak{f} = T_{y(\mathfrak{f})}$.

Proof. For each $\mathfrak{g} \in \hat{E}$ take $q_{\mathfrak{g}}^+$ as in Lemma 4.2. Let $\mathfrak{f} \in \hat{E}$.

We build our monotonic approximating sequence (s_n) to $y(\mathfrak{f})^+$ diadically. As $\mathfrak{f}, T \in \hat{E}$ we have $\mathfrak{f} - \frac{k}{2^n}T \in \hat{E}$ for $n \in \mathbb{N}$ and $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For all $p \in \mathcal{B}$ with $p \leq q_{\mathfrak{f}-2^{-n}kT}^+$ we have

$$\left(\mathfrak{f} - \frac{k}{2^n}T\right)(p) \geq 0$$

and

$$\mathfrak{f}(p) \geq 2^{-n}kT(p).$$

For all $p \in \mathcal{B}$ with $p \leq e - q_{\mathfrak{f}-2^{-n}(k+1)T}^+$

$$\left(\mathfrak{f} - \frac{k+1}{2^n}T\right)(p) \leq 0$$

and

$$\mathfrak{f}(p) \leq 2^{-n}(k+1)T(p).$$

Let

$$h_k^1 = q_{\mathfrak{f}-2^{-1}kT}^+(e - q_{\mathfrak{f}-2^{-1}(k+1)T}^+)$$

then $h_k^1 h_j^1 = 0$ for all $k \neq j$ so

$$\sum_{k=0}^{\infty} h_k^1 =: q^+$$

is a component of e . This limit exist in E_u , due to the disjointness of the summands, and as the sum is bounded by e it is also in $E = L^2(T)$. Here $\mathfrak{f}(p) \geq 0$ for all $p \in \mathcal{B}$ with $p \leq q^+$ and $\mathfrak{f}(p) \leq 0$ for all $p \in \mathcal{B}$ with $p \leq e - q^+$.

For $n \in \mathbb{N}$ define

$$h_{2k}^{n+1} = h_k^n q_{\mathfrak{f}-2^{-n-1}(2k+1)T}^+$$

and

$$h_{2k+1}^{n+1} = h_k^n (e - q_{\mathfrak{f}-2^{-n-1}(2k+1)T}^+).$$

Here $h_k^n h_j^n = 0$ for all $k \neq j$ and

$$\sum_{k=0}^{\infty} h_k^n = q^+.$$

The summation

$$s_n := \sum_{k=0}^{\infty} \frac{k}{2^n} h_k^n$$

gives a diadic approximation to $y(f)^+$. Since $(h_k^n)^2 = h_k^n$,

$$s_n^2 = \sum_{k=0}^{\infty} \frac{k^2}{2^{2n}} h_k^n.$$

Here these sums exist in E_u due to disjointness of the summands.

For all $p \in \mathcal{B}$,

$$\frac{k}{2^n} T(ph_k^n) \leq f(ph_k^n) \leq \frac{k+1}{2^n} Tph_k^n. \quad (4.3)$$

Here s_n exists in $L^1(T)$ as $L^1(T)$ is T -universally complete and the sequence of partial sums $\left(\sum_{k=0}^N \frac{k}{2^n} h_k^n\right)_{N \in \mathbb{N}}$ is increasing with

$$T\left(\sum_{k=0}^N \frac{k}{2^n} h_k^n\right) = \sum_{k=0}^N \frac{k}{2^n} Th_k^n \leq \sum_{k=0}^N f(h_k^n) \leq f(q^+).$$

Now taking the order limit as $N \rightarrow \infty$, in the above, gives $Ts_n \leq f(q^+)$ for all $n \in \mathbb{N}$.

By construction, the sequence (s_n) is increasing in $L^1(T)$ and as noted above $Ts_n \leq f(q^+)$ for all $n \in \mathbb{N}$ so, by the T -universal completeness of $L^1(T)$, (s_n) converges in order to some s in $L^1(T)$.

It remains to show that $s \in L^2(T)$. Now $f(q^+p) \leq \|f\| \in R(T)$ for all $p \in \mathcal{B}$ and as $R(T)$ is an f -algebra contained in $E = L^2(T)$ we have $(f(q^+p))^2 \leq \|f\|^2 \in R(T)$. Working in E_u^+ , from (1.2) and (4.3), we have that

$$\begin{aligned} \left(\sum_{k=0}^N \frac{k^2}{2^{2n}} Th_k^n\right)^2 &\leq \left(\sum_{k=0}^N \frac{k}{2^n} f(h_k^n)\right)^2 \\ &= \left(f\left(\sum_{k=0}^N \frac{k}{2^n} h_k^n\right)\right)^2 \\ &\leq \|f\|^2 T\left(\left(\sum_{k=0}^N \frac{k}{2^n} h_k^n\right)^2\right) \\ &= \|f\|^2 T\left(\sum_{k=0}^N \frac{k^2}{2^{2n}} h_k^n\right) \\ &= \|f\|^2 \left(\sum_{k=0}^N \frac{k^2}{2^{2n}} Th_k^n\right). \end{aligned}$$

Multiplying the above by the partial inverse of $\sum_{k=0}^N \frac{k^2}{2^{2n}} T h_k^n$, we obtain

$$T \left(\sum_{k=0}^N \frac{k^2}{2^{2n}} h_k^n \right) \leq \|f\|^2 \in R(T)^+$$

for all $N \in \mathbb{N}$. Thus, by the T -universal completeness of $L^1(T)$,

$$s_n^2 = \sum_{k=0}^{\infty} \frac{k^2}{2^{2n}} h_k^n \in L^1(T)$$

which gives that $s_n \in L^2(T)$. From the above $T(s_n^2) \leq \|f\|^2$ for all $n \in \mathbb{N}$ and (s_n^2) increases in order to s^2 . Thus, from the T -universal completeness of $L^1(T)$, $s^2 \in L^1(T)$, making $s \in L^2(T)$. Working from (4.3), we have

$$T(ps_n) = \sum_{k=0}^{\infty} \frac{k}{2^n} T(ph_k^n) \leq f(pq^+) \leq \sum_{k=0}^{\infty} \frac{k+1}{2^n} Tph_k^n \leq T(ps_n) + \frac{1}{2^n} T(p), \quad (4.4)$$

Taking the order limit as $n \rightarrow \infty$ in (4.4) gives

$$T(ps) = f(pq^+). \quad (4.5)$$

Applying Freudenthal's theorem along with the order continuity and linearity of T and f to (4.5) we have that

$$T(gs) = f(gq^+), \quad (4.6)$$

for all $g \in E^+$. This extends by linearity to all $g \in E$. We note here that s is in the band generated by q^+ .

Applying the above to $-f$ we have that there is $\sigma \in E^+$ and $q^- \in \mathcal{B}$ so that: $-f(q^-p) \geq 0$ and $-f((e - q^-)p) \leq 0$ for all $p \in \mathcal{B}$; σ is in the band generated by q^- ; and

$$T(g\sigma) = -f(gq^-), \quad (4.7)$$

for all $g \in E$.

Let $y(f) := s - \sigma$, then $y(f) \in L^2(T) = E$ and

$$T_{y(f)}(g) = T(gy(f)) = T(gs) - T(g\sigma) = f(g(q^+ + q^-)),$$

for all $g \in E$. Now

$$f(g) = f(g(q^+ + q^-)) - f(gq^+q^-) + f(g(e - q^+)(e - q^-)).$$

Here

$$f(pq^+q^-) = 0 = f(p(e - q^+)(e - q^-))$$

for all $p \in \mathcal{B}$, and thus by Freudenthal's theorem

$$f(gq^+q^-) = 0 = f(g(e - q^+)(e - q^-))$$

for all $g \in E$. Hence $f(g) = T_{y(f)}(g)$ for all $g \in E$. □

Combining Proposition 4.1 and Theorem 4.3 gives Theorem 1.1.

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