EXPLORING THE PREDICTIVE POWER OF EARLY ASSESSMENTS IN MATHEMATICS IN SOUTH AFRICA

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ABSTRACT

A significant body of evidence in mathematics education points to a set of early mathematical competencies being particularly powerful predictors of later mathematical learning. Many international studies highlight the importance of structural thinking in the early years for later performance.

However, such a study has not been done in South Africa at this scale. This study aims to find qualitatively and quantitatively the extent of the predictive power of structural thinking for later performance in mathematics in South African primary schools.

Therefore, the overall question my study addresses is, “Does my evidence support the international confirmation of the importance of structural thinking in the early years?” This longitudinal study will help demonstrate whether learners with sound mathematical structural thinking in Grade 2 are more likely to perform well in mathematics by the time they reach Grade 6.

An essential aspect of structural thinking in the early years of schooling is an awareness of number relationships underpinned by knowledge of the base ten build-up of our number system. The awareness of number relations leads later to learners using sophisticated strategies that include mental calculations and creatively finding answers to new problems.

My study is within the Wits Maths Connect Primary project (WMC-P), where I am a member. This research study is a mixed-method longitudinal study over just less than five years, with initial data collection taking place in February/March 2011, when the participants were in Grade 2, and later in November 2015 when they were in Grade 6. The sample is 64 participants from eight government schools in one province in South Africa. The participants were from four townships schools (n=38) and four schools in suburban areas (n=26).

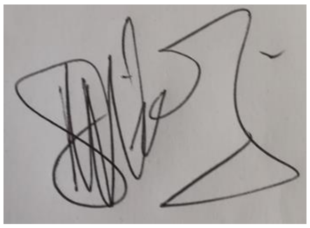
The study explored quantitatively and qualitatively whether mathematical structural thinking in Grade 2 predicts later mathematics performance in the South African context. I am exploring the predictive power of structural thinking by investigating the correlations between tests.

The results show that structural thinking is essential in the early years of primary school in order to perform well in higher grades. Teachers should ideally teach to make learners constantly aware of the relationship between numbers and mathematical structure.

Keywords:
Structural thinking, awareness of relationships, strategies, mixed-method, assessment, unit counting, predictive power, correlations, sophisticated strategies, relationship between quantities, unenumerated quantities.

DECLARATION

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.



Maria Susanna Weitz

16th day of March in the year 2022

To my dearest husband Chris Weitz and children Christian Weitz, Carina Weitz, Marisa, Byron, and Grace du Bois.

INTERNATIONAL AND NATIONAL CONFERENCES PRESENTATIONS EMANATING FROM THIS RESEARCH

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BOOK CHAPTER PUBLICATIONS EMANATING FROM THIS RESEARCH

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LIST OF ABBREVIATIONS

Base-ten thinking	Base-ten thinking
BNWS	Backward Number Word Sequence
CMMRD	Corroborative Mixed Method Research Design
CMMRD	Corroborative Mixed Method Research Design
DBE	Department of Basic Education
FNWS	Forward Number Word sequence.
G1 ANAdG2	Grade 1 Annual National Assessment delivered in Grade 2
G1 SANAdG2	Grade 1 Annual National Assessment delivered in Grade 2 for the suburban schools
G1 TANAdG2	Grade 1 Annual National Assessment delivered in Grade 2 for the township schools
G2 LFIN	Grade 2 Learning Framework I Number
G2 SLFIN	Grade 2 Learning Framework In Number for suburban schools
G2 TLFIN	Grade 2 Learning Framework In Number for the township schools
G6 CA	Grade 6 Common Assessment
G6 ESTA	Grade 6 Evidence of Structural Thinking Assessment
G6 SCA	Grade 6 Common Assessment for the suburban schools
G6 SESTA	Grade 6 Evidence of Structural Thinking Assessment for the suburban schools.
G6 TCA	Grade 6 Common Assessment for the township schools
G6 TESTA	Grade 6 Evidence of Structural Thinking Assessment for the township schools.
NI	Numeral Identification
NONRT	Not-only-number-related tasks
NRT	Number-related tasks
NWA	Number Word After
NWB	Number Word Before

PPMC	Pearson product-moment correlation coefficient (PPMCC)
S0: ICA	Incorrect answers
S0: ICA	Incorrect answers
S1: PGO	Perceptual sharing and grouping in ones with physical items available.
S1: UC	Unit counting of physical items
S2: FWOS	Formalising without structuring - Using the written vertical column method without structuring (counting on fingers to find answers to the partial sums and differences);
S2: PGM	Perceptual sharing and grouping in multiples where the physical items are available.
S3: FCC	The learner has developed counting strategies that do not rely on physical items to be visible or touchable.
S3: FWS	Formalising with structuring - Using the written vertical column method with structure (finding the answers to partial sums and differences mentally)
S4: IMA	Immediate mental answer - Answered with and immediate mental answer using separation, aggregation, wholistic and compensation strategies.
S4: RCWA	The strategies at this level depend on written algorithms.
S5: IMA	Answered with and immediate mental answer using multiplication known facts. The learner at this level can derive new multiplication facts from basic multiplication facts.
SBase-ten thinking	Base-ten thinking for the suburban schools
SBNWS	Backward Number Word Sequence for the suburban schools
SEAL	Stages of Early Arithmetical learning.
SFNWS	Forward Number Word sequence for the suburban schools.
SNI	Numeral Identification for the suburban schools
SSEAL	Stages of Early Arithmetical Learning for the suburban schools
TBase-ten thinking	Base-ten thinking for the township schools.

TBNWS	Backward Number Word Sequence for the township schools.
TFNWS	Forward Number Word Sequence for the township schools.
TNI	Numeral Identification for the township schools.
TSEAL	Stages of Early Arithmetical Learning for the township schools.
WMC-P	Wits Maths Connect Primary

CHAPTER 1: INTRODUCTION

1.1 Preface

Findings from international studies suggest that the extent of mathematical structural thinking in the early elementary years predicts later mathematical performance (Claessens & Engel, 2013; Papic, Mulligan, & Mitchelmore, 2011). This PhD study explores whether structural thinking in mathematics during the early years predicts later mathematics performance in the South African landscape.

The above paragraph points to two critical aspects: structural thinking and predictive power. I will deal first with structural thinking and then focus on the predictable power of structural thinking. I will also mention the types of relational knowledge essential to my study.

1.1.1 Structural thinking

For Warren (2003), structural thinking is knowledge about mathematical objects and their relationships. He argues that structural thinking is:

> "… concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality)" (2003, p. 125).

Warren also claims that with respect to algebra, young learners are expected to be familiar with the above concepts for understanding the subject later in their schooling days. Warren's (2003) assertion highlights the importance of structural thinking for learners in the early primary school years, and the importance of this study. The concepts Warren (2003) regards as part of structural thinking will be discussed in the literature review.

For me, an essential aspect of structural thinking in the early years of schooling involves the awareness of number relationships, which is underpinned by the knowledge of the base ten build-up of our number system. This awareness of number relations leads later to learners using sophisticated strategies that include mental calculations and creatively finding answers to new problems.

My starting point of structural thinking is based on the work of Venkat, Askew, Watson, & Mason (2019). They argue that it is vital that learners develop an awareness of mathematical relationships between elements to calculate (ibid, pp14). Learners with sound structural thinking in mathematics can then use their knowledge of relationships between numbers to calculate answers to mathematical problems.

For example, a learner with introductory-level of structural thinking will know there are three beads in a teacher's hand if the teacher shows only seven beads on a ten-bead string. Here, there is an awareness of number relationships in the context of concrete items. The learner knows the relationship between 7, 3, and 10, even though she can only see the seven beads.

The opposite of working with numbers with an awareness of relationships between numbers is when children cannot work without concrete items like fingers, counters or tallies. A child without structural thinking will need to see and feel the three beads in the teacher's hand before she can answer the question in the example above.

Another example of a Grade 2 learner working without structural thinking can be seen in the response on a Grade 1 Annual National Assessment delivered in Grade 2 task (G1 ANAdG2), one of the assessment tools of my study[1]. The Annual National Assessment (ANA) was introduced in South Africa in 2010 and was curriculum-orientated[2].

The task in the Grade 1 ANA was to find the value of $10 + 10 + 10 \ =$? The question was, "Write the correct answer in the box." The illustration in Figure 1.1 (Weitz & Venkat, 2013) reveals the response of a learner who works without structural thinking by working concretely with number to solve this problem.

[1] I discuss the G1 ANAdG2 in section 3.5.1
[2] The ANAs were conducted in all government schools in South Africa for Grades 1-6 and Grade 9 learners; however, the ANAs were discontinued in 2015 due to pressure from the teachers' unions.

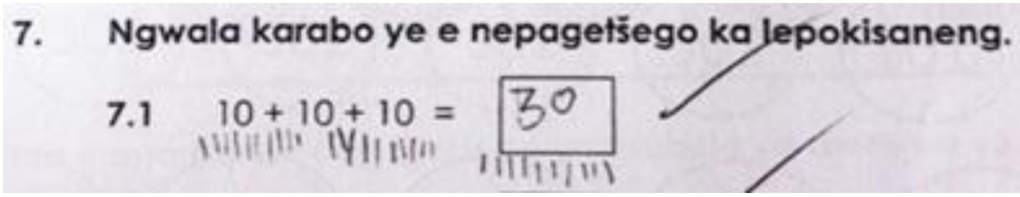


Figure 1.1 Illustration of a learner working without knowledge of structure

In the example in Figure 1.1, we can assume that the learner does not know the underlying base-ten structure of our Arabic number system (N. Thomas & Mulligan, 1999; N. D. Thomas, Mulligan, & Goldin, 2002). For this learner, it might be that the question is not about the relationship between numbers and knowing the base-ten number system; it is about drawing ten tallies three times concretely and counting thirty tallies to respond to the question. The answer is correct; however, the learner's thinking might be at a deficient level.

Gray, Pitta, and Tall (2000) argue that learners with structural thinking have a profound understanding of mathematics, leading to learners' ability to work with numbers creatively. An example of working with number in creative ways is to derive new facts from known facts. For example, a learner who knows that $5 + 5$ is 10 as a known fact can use the known facts and derive that $5 + 6 = 11$ because (s)he knows that six is one more than 5. Therefore, the answer will be one more than 10, which is 11.

Wright and colleagues (Wright, Ellemor-Collins, & Tabor, 2012) expressed these creative ways of working with number as 'sophisticated strategies' that can be observed in the methods learners use when doing mathematics. The evolution of sophisticated strategies involves learners' emancipation from using physical items (manipulatives) towards a conceptual understanding when calculating in mathematics (2012). The development of sophisticated strategies based on structural thinking and relational thinking leads learners to perform mental calculations, which develop when they begin to manipulate numbers in their heads to find answers to problems.

The latter brings us to mental calculations. Askew (1998) claims that mental calculations involve manipulating known facts to derive answers to new problems mentally. The ability to derive facts mentally shows knowledge of the relationship between numbers, between equations, and between mathematical elements.

Thus, the ability to work with the relational aspects between mathematical elements can be seen as structural thinking. To accentuate the importance of mental calculations for structural thinking, Gürbüz and Erdem (2016) point to the value of such calculations as critical to developing a structural understanding in mathematics which afford young children the opportunity to do mental mathematics.

1.1.2 Structural thinking as relational knowledge

Three different types of relational knowledge are essential to my study. Firstly, there is relational knowledge of unenumerated quantities (more/less/equal) that very small children possess (Sophian, 2008). An example of unenumerated knowledge of quantities is that if a very small child must choose between a set of 8 sweets or 3 sweets, he will select the 8 sweets if he likes sweets a lot without knowledge of 8 and 3 as numbers.

Secondly, they develop knowledge of relationships between numbers when they learn about number words and solving mathematical problems using operations (I Thompson, 2008). An example where learners start working with an awareness of number relations is when a teacher asks a learner to show 8 beads on an abacus. The learner reacts by taking away two counters on the abacus and indicates the counters that were not taken away as 8 counters. This learner knows the relationship between 10, 2, and 8 and knows that it is less time-consuming to take away 2 than to count out 8 counters. In this situation, the learner uses subtraction and addition and relational thinking, to answer.

Lastly, knowledge of the relationships between quantities involves an analysis of a mathematical situation in order to know how to solve a problem (Nunes, Bryant, Barros, & Sylva, 2012). An example of this type of relational knowledge is when a learner knows that if an addend's last digit is 9 he can add a multiple of 10 and compensate by doing the opposite.

Thus, structural thinking is about relationships between unenumerated quantities, relationships between numbers, and relationships between quantities.

1.1.3 Predictability of early mathematics

I now turn my discussion to the predictability of early mathematics. Early years' achievement in mathematics, in general, is a critical prerequisite for later academic achievement and is ultimately a precursor for life success (Jimerson, Egeland, & Teo, 1999, p. 116). Numerous studies have argued that particular early mathematical skills can predict later mathematical performance (Aubrey, Dahl, & Dogfrey, 2006; Aunola, Leskinin, Lerkkanen, & Nurmi, 2004; Claessens & Engel, 2013; Manfra, Dinehart, & Senbiante, 2012; Weitz & Venkat, 2017). These particular mathematical skills include an awareness of number relationships (Geary, Hoard, Nugent, & Bailey, 2013; Mulligan & Mitchelmore, 2009; Mulligan, Mitchelmore, & Prescott, 2005) and using these relational aspects in solving mathematical problems. Aubrey et al. (2006) agree and state that learners who bring relational number knowledge and awareness of number and quantities to their first year of schooling appear to be more advanced in their mathematics progression than learners who do not.

More specifically, an agreement based on studies in different parts of the world by the mathematics education community suggests that the learner's level of mathematical structural thinking in the early years is highly correlated with later mathematical performance (Aunola et al., 2004; Papic et al., 2011).

Aunola (2004) and Manfra et al. (2012) tested pre-schoolers' counting methods and claim that the way they count predicts later performance in mathematics. Further evidence emerged from other studies in the United Kingdom and the United States of America, suggesting that particular early mathematical skills are essential predictors of later mathematical performance (Claessens & Engel, 2013; Geary et al., 2013). These particular mathematical skills will be discussed in section 2.7.1.

To sum up, studies have investigated relationships between counting in kindergarten (Manfra et al., 2012), school-entry number system knowledge (Geary et al., 2013), structural thinking (Mulligan, Prescott, & Mitchelmore, 2004), and later mathematical performance over several years. Clements & Sarama (2009, p. 214) summarised that a common conclusion across these studies is that the "…effect of initial mathematics performance is unusually strong and notably persistent".

1.2 The importance of mathematics and structural thinking in the early years of schooling

The importance of primary school mathematics in the early years is emphasised by suggesting that the poor results in Grade 12 mathematics in South Africa are due to inadequate preparation of mathematics in the primary schools (Taylor, Fleisch, & Shindler, 2008). It is well known that

Grade 12 mathematics is essential for tertiary education, as many study fields at university require good performance in Grade 12 mathematics.

Concerning the value of structural thinking in the early years of primary school, I am highlighting two explanations as to why it is important.

Firstly, structural thinking is essential because concrete unit counting methods are insufficient and time-consuming (Ensor et al., 2009). Correct answers can be produced with concrete unit counting when the number range is small, but when the number range enlarges, learners make mistakes because they cannot keep track of their counting. Consequently, learners who do not possess structural awareness may underperform in the later years because some aspects of mathematics is of such nature, like algebra, where unit counting is not an option. Second, the predictive power of structural thinking emphasises the importance of fostering a structural awareness from a young age by focusing on the relational aspects of mathematics.

1.3 Background to the study

South Africa's brutal history of apartheid resulted in educational inequalities, with substantial differences between the education of the rich and the poor. The inequalities in South African education are well acknowledged by education specialists (Fleisch, 2007; Taylor et al., 2008). South African learners' poor performance in international numeracy assessments, low mathematics attainment in primary schools, the substandard teaching of the poor, and the crisis of primary school teachers' content knowledge are some of the inequalities that researchers highlight.

The Department of Basic Education (DBE) in South Africa has changed the mathematics curriculum 3 times since 1994, to redress the inequalities in the educational system. However, very little has been accomplished, and due to this lack of progress in mathematics, inequality has worsened in mathematics education (Graven, 2014).

Additionally, I add the lack of learners' structural thinking and the lack of teachers teaching mathematics structurally in South African primary schools to the schools in South Africa's problems.

International numeracy assessments suggest that South African learners' performances in numeracy were substantially lower than that of 11 other African countries (Ensor et al., 2009; Fleisch, 2008). In conjunction, the outcomes of the international TIMSS (Trends in International Mathematics and Science Study) reported that South Africa ranked second last out of 48 countries on both the Grade 4 and Grade 8 mathematics assessments. Both South African and international researchers in early mathematics education agree that there is a problem with mathematics performance in South African primary schools (Carnoy et al., 2008; Fleisch, 2007; Taylor et al., 2008). Fleisch (2008) argues that learners in primary schools in South Africa perform abysmally and that they are not on the grade level they should be.

One of the underlying causes of deprived performance in South Africa is the substandard teaching to the poor by, at times, underqualified teachers. Not all learners receive the necessary education they should (Fleisch, 2008; Spaull, 2013). Venkat and Spaull (2015) conducted a study whereby teachers of Grade 6 performed worse in the test than the learners they taught, who wrote the same assessment.

Ensor et al. (2009) claim that the problem with South African learners' mathematics is the lack of structural thinking, where learners prefer to use unit counting of concrete items to find answers to tasks.

The prevalence of unit counting shows that teachers in South African classrooms seemingly do not teach to make learners aware of number relationships in order to develop structural thinking (Graven & Venkat, 2019). Learners are not afforded the learning opportunity that would shift their thinking away from the unit counting of concrete items to structural ways where they do not need physical items to calculate. The teachers should focus on developing an awareness of relationships between numbers. They should also be aware of the value of understanding number relations for further learning (Ensor et al., 2009; M. Graven & Venkat, 2019; Schollar, 2008).

1.4 The aim of the study

I mentioned those international studies highlighting the importance of structural thinking in the early years for later performance. However, such a study has not been done in South Africa at this scale yet. This study aims to find qualitatively and quantitatively the extent of the predictive power of structural thinking for later performance in mathematics in South African primary schools.

Therefore, the overall question my study addresses is, "Does my evidence support the international confirmation of the importance of structural thinking in the early years?" This longitudinal study will help demonstrate whether learners with sound mathematical structural thinking in Grade 2 are more likely to perform well in mathematics by the time they reach Grade 6.

This study aimed to highlight the importance of structural thinking in the early years. It is important that teachers, teachers' trainers and policymakers recognise the importance of learners being less dependent on one-by-one concrete counting and shift their thinking to thinking that is more abstract and sophisticated when they calculate answers for mathematical problems.

1.5 Research question

The main research question this study addresses is:

<u>In what ways, if any, does structural thinking in the early years of primary school predict performance in the later years of primary school?</u>

Sub-questions arising from this include:

1. Is there evidence for a strong correlation between an assessment that measures structural thinking in the early years of school and mathematical performance in the later years of primary school?
2. How important is structural thinking in the early years of school?

These research questions are explored and made operational by the researcher's questions:

Researcher's questions

<u>In what ways do Grade 2 learners' performance on two tests - the Grade 1 Annual National Assessment (Grade 1 ANA) and Wright et al.'s (2006) Learning Framework in Number (LFIN) - assessments relate to their subsequent Grade 6 performance on two mathematics tests, the Grade 6 Common Assessment (Grade 6 CA) and a number focused interview test, the Grade 6 Evidence</u>

of Structural thinking Assessment (Grade 6 ESTA) aimed at exploring the extent of structural ways of working with number?

In order to answer the main research question, the following sub-questions will be answered:

1. In qualitative terms, in what ways, and to what extent do the 2011 ANA Grade 1 test, the 2011 LFIN, the Grade 6 Common Assessment, and the G6 ESTA assessments measure structural thinking with early number?
2. In quantitative and qualitative terms, what do the empirical findings regarding the extent of structural thinking tell us?
3. In quantitative terms, what is the correlation between learner performance in Grade 2 on the Grade 1 ANAdG2 and LFIN tests in 2011 and performance in Grade 6 on the Common Assessment and G6 ESTA tests in 2015?

1.6 Answering the researcher's questions

This section states how I am answering the three researcher's questions. I answer the first question by doing a product analysis and discussing the association between each assessment and mathematical structural thinking. I will do the product analysis qualitatively in terms of how well the tasks of the assessments measure mathematical structural thinking.

The second research question will provide the results of the empirical data in the form of graphs and discussions about the participants' responses.

Answering the third research question, I am providing quantitatively correlational statistics to show the relationship between the data of Grade 2 and Grade 6.

To summarise, I am answering the main research question of 'In what ways, if any, does structural thinking in the early years of primary school predict performance in the later years?' by discussing the three researcher's questions qualitative and quantitatively.

1.7 Research design

In this section, I discuss the methodology, the sample, and the assessment instruments of the study.

1.7.1 The methodology

This research study is a mixed-method longitudinal study over just less than five years, with initial data collection taking place in February/March 2011, when the participants were in Grade 2, and later in November 2015, when they were in Grade 6. The study explored quantitatively and qualitatively whether mathematical structural thinking in Grade 2 predicts later mathematics performance in the South African context. I am exploring the predictive power of structural thinking by asking to what extent early mathematical structural thinking is associated with later mathematical performance.

To sum up, the connections between two sets of data when the participants attended Grade 2 and two sets of data when the same participants attended Grade 6 form the study's foundation. Analysing the data helped me answer my research question of how important mathematical structural thinking is in the early years as a predictor of later performances in mathematics in the South African primary classroom.

1.7.2 The sample

My study is within the Wits Maths Connect Primary project (WMC-P), where I am a member.

In February/March 2011, members of the WMC-P collected data on two assessments from 10 government primary schools (five township schools and five suburban schools) of 230 Grade 2 learners in one district in Gauteng, South Africa. However, when I collected the second set of Grade 6 data in 2015, I could only find 64 learners with matched data.

Therefore, to test the assumption in South Africa that the level of mathematical structural thinking correlates well with later mathematical performance in South Africa, I took a sample of 64 learners in Gauteng, South Africa. The participants were from eight schools: four from townships and four from suburban areas.

In South Africa, suburban and township schools differ in many ways. For example, the types of schools do not have the same resources, and the learner-to-teacher ratios are different. The previous Apartheid education system segregated education departments according to race. Schools previously reserved for a particular race that wasn't white remain overwhelmingly populated by that group. Despite attempts by the post-Apartheid government to redress the historical imbalances, these township schools remain poorly resourced. The average learner-to-teacher ratio in South African schools is 33:1 (Meier & West, 2020); however, in some schools, the ratio goes up to 39:1 (White & Van Dyk, 2019). Personally, when visiting a township school, I saw more than 50 learners in one class.

1.7.3 The assessment instruments

A discussion follows of the two assessments in Grade 2, namely the Grade 1 Annual National Assessment that was delivered in Grade 2 (G1 ANAdG2) and the Learning Framework in Number (G2 LFIN) interview assessments and the two Grade 6 assessments, namely the Grade 6 Common Assessment (G6 CA) and the Grade 6 Evidence of Structural Thing Assessment (G6 ESTA).

Before saying more about each assessment, I want to foreground three types of assessments that are important to this study. Since the assessments have different purposes, they can be classified as diagnostic, formative and summative. The G1 ANAdG2 was a summative assessment, and the G6 CA was a formative assessment conducted twice a year, while the G2 LFIN and the G6 ESTA were diagnostic assessments. I elaborate on how and why I regard the data instruments as summative, formative and diagnostic in section 4.2.1.

A summative assessment evaluates the learners' understanding at the end of a teaching semester or after a year's learning. Summative assessments are mostly high-stakes assessments as they are used to progress learners to the next grade or level (Dixson & Worrell, 2016). Summative assessments are therefore set to capture what a student has learnt, to know the quality of the learning, and to judge performance against some standards. Thus, unlike formative assessments, which are generally used to give feedback to students and teachers regularly, summative assessments are usually used to get a final assessment of how much a learner knows (Gardner, 2010).

Formative assessment is used to inform teachers, parents, and students about learning continuously. Wiggins (1998, p. 7) argues: "The aim of formative assessment is primarily to educate and improve student performance, not merely to audit it." Thus, formative assessment comprises undertakings of teachers that make information available to adapt teaching and learning activities (Dixson & Worrell, 2016).

Grade 1 Annual National Assessment 2011 that was delivered in Grade 2 (G1 ANAdG2)

One assessment in Grade 2 was the Grade 1 Annual National Assessment (ANA). The ANA was a large-scale summative written assessment conducted by the DBE in South Africa. The ANAs for grades 1-9 were implemented in 2011. The Grade 1 ANA was delivered to the Grade 2 participants of my study in 2011 (G1 ANAdG2) due to difficulties in conducting the assessment in the previous year.

Learning Framework in Number test (LFIN) of Bob Wright and colleagues

The second assessment instrument of Grade 2 was the Learning Framework in Number (G2 LFIN) video-recorded interview of Bob Wright and colleagues (Wright, Martland, Stafford, & Stanger, 2006).

The LFIN involves assessing mathematical structural thinking, and many of the competencies described in the literature as critical. The analysis of how sophisticated the strategy is forming the basis of the G2 LFIN.

A framework to analyse the participants' responses accompanies the assessments. I used the framework of Wright et al. (2006) because its stages and levels deal with structural thinking development in a meaningful way. More about the connection between structural thinking and the levels and stages of the framework follows in Chapter 4 (section 4.6), where I answer my first researcher's question on the extent of measuring structural thinking of the four assessments.

Grade 6 Common Assessment (G6 CA)

One of the assessments in 2015, when the learners in my sample were in Grade 6, was the Common Assessment for the Grade 6 learners (G6 CA).

According to CAPS, the G6 CA is a summative assessment of the knowledge and skills that learners are expected to have developed by the end of Grade 6.

The G6 CA tasks include the following aspects of mathematics: number operations, whole numbers, fractions and decimal numbers, patterns, functions, and algebra, numeric patterns, geometric patterns, space and shape (2-D, 3-D, transformations and symmetry), measurement (length, volume, time, perimeter, area), data handling and probability.

Grade 6 Evidence of Structural thinking Assessment (G6 ESTA)

The other Grade 6 assessment, the G6 ESTA video-recorded interview, was developed within this study. I named the video-recorded interview in Grade 6 the Evidence of Structural thinking Assessment for Grade 6 (G6 ESTA).

Since there is a high association between structural thinking and mental calculations, I developed a research-informed mental assessment for Grade 6 to measure the extent of structural thinking drawing on research on assessment of efficient mental and written strategies to perceive evidence of structural thinking. However, if the learners could not answer mentally, they could use paper and pen to answer with a written response.

As said before, the G6 ESTA contains items appropriate for mental computation and each item was conducted to elicit mental computation; for example, what is 274+99? I allowed 8 seconds for scoring the highest mark on each item as mental answers; however, I allowed more than 8 seconds. I did allow more minutes because I was afraid that I would only have wrong answers considering the difficulties children have with mental calculations. I was also interested in other strategies like written algorithms that could be used with some structural thinking because a correct written response in the G6 ESTA scored a lower mark than a correct mental response. The G6 ESTA and G6 CA include items on all four operations.

Developing the assessment to examine structural thinking was not the purpose of the study; nevertheless, it contributes to primary mathematics teaching and learning.

To summarise, the Grade 6 ESTA was developed to examine structural thinking using tasks that elicit mental calculations; however, written responses were allowed to investigate other strategies learners use when confronted with the tasks. More about the Grade 6 ESTA follows in Chapter 4, where I discuss the assessments’ product analyses.

1.8 Organisation of the thesis

This study is structured into seven chapters. This first chapter has explained what structural thinking means and includes introducing the predictive power of mathematical structural thinking. It also considers the problem statement in South Africa's history, the aim of the study, research questions and an introduction of the research design.

The literature and theoretical framework that support my thesis are discussed in Chapter 2. Chapter 3 provides the research design and the rationale for using a mixed-method research design.

Answering the research questions is systematically discussed in Chapters 4, 5, and 6. The first research question of examining the extent of measuring structural thinking in all four assessments will be discussed qualitatively in Chapter 4. This discussion will be in the form of product analysis using different frameworks.

Chapter 5 highlights the empirical analysis and findings of the participants' responses to all the assessments.

Chapter 6 contains the quantitative analysis of the correlations in statistical terms between assessments and a discussion on the early assessments' predictive power.

Chapter 7 summarises the results, recommendations, and conclusions of the study. It includes the research limitations and offers suggestions for teaching and learning, and propositions for future research. It also encapsulates my PhD journey.

Chapter 2 : LITERATURE AND THEORETICAL FRAMEWORK

2.1 Introduction

A considerable body of evidence in early mathematics education literature points to a set of mathematical competencies being particularly powerful predictors of later mathematical learning (Aubrey et al., 2006; Aunola et al., 2004; Claessens & Engel, 2013). Examples of the mathematical competencies considered good predictors of later performance include recognising the next number in a sequence[3] and pattern recognition, number line accuracy, identifying the ordinal position of an object, solving simple word problems and 'abstract thinking' where the latter refers to participants' using knowledge of the relationship between numbers (Claessens & Engel, 2013; Geary et al., 2013).

From a more detailed investigation of all the competencies (and not only abstract thinking) indicated as particularly powerful predictors of later mathematical performance, I posit that learners need well-established knowledge of relationships between mathematical elements (for example, number) to develop these predictive competencies.

In early mathematics education, structural thinking can be observed when learners demonstrate an awareness of the relationships between mathematical elements (Mason, Stephens, & Watson, 2009; Venkat, Askew, Watson, & Mason, 2019). Therefore, my point of departure in this study is that learners' extent of structural thinking depends on their awareness of the relationships between mathematical elements (Venkat, Askew, Watson, & Mason, 2019).

Before young learners can identify relationships between mathematical elements (in this case number), they have to recognise the base-ten composition of our number system and know the preceding and succeeding numbers. The latter forms the foundation of our number system.

Because structural thinking deals with relationships between mathematical elements, I focus on what some researchers have discovered about mathematical relations. The literature distinguishes between number relations and quantitative relations (Nunes, Bryant, Barros, & Sylva, 2012). Some

[3] A sequence is a list of objects (numbers, for example) in a specific order with a specific rule.

research shows that relationships are first understood in the context of quantities and later as number relationships (Nunes et al., 2012).

Number relations is about arithmetical calculations and the knowledge of how to use different operations. Quantitative relations, on the other hand, refer to the analysis of mathematical situations and the knowledge required choosing the correct procedure (Thompson, 1993). Later in this chapter, I discuss the link between number and quantitative relations, and how an understanding of relations develops.

As previously stated, structural thinking is a broad and complex concept, with many characteristics and applications, and is difficult to delineate or convey in a single definition. Four applications of structural thinking are essential to my study, namely: base-ten structural thinking, additive structural thinking, multiplicative structural thinking, and the recognition of patterns in structural thinking. I discuss how knowledge of relationships relates to the applications of structural thinking

This study is a longitudinal study spanning nearly five years in order to observe the predictive power of the early Grade 2 assessments and the emergence of structural thinking. Young learners' structural thinking emerges as an embryonic awareness of relationships, which grows into an awareness of multiple relationships, which allows them to use known facts and derive new facts.

Along with the emergence of structural thinking, the empirical indicators of structural thinking are essential to my study. Two of the data collection instruments (the G2 LFIN and the G6 ESTA) in my study are video-recorded interviews, which specifically examine the strategies learners use as empirical indicators of structural thinking and include mental strategies. Thus, a discussion of strategies used as indicators of structural thinking is essential.

Since my investigation is a longitudinal study concerning the predictive power of structural thinking, I explore how other longitudinal studies address this topic. In the latter part of this chapter, I investigate a set of number knowledge competencies that other longitudinal studies consider important predictors of later mathematical performance.

To sum up, the chapter includes five themes around structural thinking:

1. The foundation of structural thinking

2. Structural thinking as number and quantitative relationships
3. The applications of structural thinking
   a. Base-ten structural thinking
   b. Additive structural thinking
   c. Multiplicative structural thinking
   d. Recognition of patterns in structural thinking
4. Strategies used as indicators of the emergence of structural thinking, and
5. Predictive power of structural thinking.

2.2. The foundation of structural thinking

Young learners are born with an initiation awareness of quantitative relationships of very small quantities (Spelke & Kinzler, 2007). This awareness of quantitative relationships includes making judgements about sameness and non-sameness of quantities (less/more). The latter forms the first foundation of structural thinking.

After realising relationships between small quantities at a very young age, they start to make sense of the relationships between more and bigger numbers and the stable ordering of numbers when they learn some number names through interaction with the environment before they go to school (Van den Heuvel-Panhuizen, 2008). When they go to school, they gain knowledge of even more and larger numbers and learn the cardinality and ordinality of some numbers.

Another foundation of structural thinking is the knowledge of two essential principles of the number system, namely: the sequel/prequel principle and the digit's place value principle.

The first principle of the ordered number list is the sequel/prequel principle that affirms a relationship between a number and its sequel or prequel. The sequel/prequel principle stipulates that each number n has a sequel $(n + 1, n > 0)$ and a prequel $(n - 1, n > 0)$ (Houdement & Tempier, 2019) and that no two numbers have the same sequel or prequel. For this reason, each number in the Hindu-Arabic system is unique and specifically ordered. This principle relates to 'what comes after/before' questions, which are often asked in primary school. Knowledge of the sequel and prequel is fundamental to the more/less/same quantitative relationship concept in structural thinking.

The second principle of the ordered number list is that digits' place values in a numeral are decimal based. Scholars assume that the choice of base-ten for our number system is related to our ten fingers (Houdement & Tempier, 2019). Our number system uses 10 digits (0,1, 2, 3, 4, 5, 6, 7, 8, 9) to organise and describe numbers' magnitudes. Knowledge of place value develops concurrently with an understanding of the base-ten structure. This is discussed shortly in the applications of structural thinking.

Thus, the initial knowledge of the sameness/less/more of quantities together with knowledge of the sequel/prequel principle and the digits' place value principle (decuple relationships) form the foundation of structural thinking.

2.3 Structural thinking as knowledge of relationships

The literature indicates that structural thinking in mathematics is made explicit when learners work with the relationships between mathematical objects (Mulligan & Mitchelmore, 2009; Venkat et al., 2019). These mathematical objects include numbers and quantities. Knowledge of number relations and knowledge of quantitative relations are not the same. However, they are connected. Drawing on Nunes (2009) and Thompson (1993), I describe the distinguishing aspects as follows:

Firstly, knowledge of number relationships involves knowing how numbers are composed in our Hindu-Arabic number system and knowing how to operate on numbers using addition, subtraction, multiplication, and division. Knowledge of number relationships requires learners to know how to perform procedures and operations correctly (P. W. Thompson, 1993). Knowledge of number relationships includes taught procedures.

Secondly, understanding quantitative relationships involves not only young learners' initial knowledge of unenumerated quantities, which is crudely comparative and includes equivalence (sameness) and non-equivalence (more or less) (Nunes et al., 2012; Spelke & Kinzler, 2007), but also includes the ability to analyse a mathematical situation and make useful decisions to calculate effectively (Nunes et al., 2012). Thompson points out that knowledge of quantitative relationships involves "the analysis of a situation into a quantitative structure" (P. W. Thompson, 1993, p. 1). Thus, quantitative relations highlight the importance of learners being able to analyse a mathematical situation within a network of multiple relationships in order to manipulate the numbers they are presented with (Nunes et al., 2012). The nature of quantitative relations differs with every mathematical situation. The major determination that the learner has to make is what the nature of the relations is. An example of this is deciding whether a relationship is additive or multiplicative.

Quantitative relations do not exclude numbers, but they are of lesser value regarding the reasoning about the situation. Thompson (1993) argues that:

> "Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values" (P. W. Thompson, 1993, p. 1).

Knowledge of number and quantitative relations is integrated. Integrated knowledge of number relationships and knowledge of quantitative relationships develop reciprocally: improved number knowledge can broaden a learner's knowledge of quantitative relationships, and improved knowledge of quantitative relationships can infuse number knowledge with meaning (P. W. Thompson, 1993). Integrated knowledge of quantitative and number relationships is essential for fostering structural thinking. I will refer to the integrated knowledge of number and quantitative relationships as quantitative reasoning.

2.4 Applications of structural thinking

As stated, structural thinking is a broad concept and difficult to define. In this section, I discuss those applications of structural thinking that are important to my study. These applications are base-ten structural thinking, additive structural thinking, multiplicative structural thinking, and pattern recognition.

2.4.1 Base-ten structural thinking

Base-ten thinking, and place value are inventions of mathematicians. The world is not naturally organised in groups of ten; instead, humans have arranged it in that way (Houdement & Tempier, 2019). Using base-ten thinking and place value, we can easily compare numbers to find relationships between them.

Together with base-ten thinking comes knowledge of place value. More aspects of base-ten thinking, when learners are doing mathematics, is base-ten compensation and bridging through 10.

Thus, in my discussion of base-ten structural thinking, I am focusing on the learners' knowledge of place value, compensation and bridging through 10.

Place value

Place value indicates the value of a digit, subject to its place or position in a number. In the French-speaking communities, place value is referred to as positional notation (Sun & Bussi, 2018). In my view, 'positional' notation is an excellent way to explain what is meant by place value because the position of the digit determines which group of the power of 10 the digit counts. For example, in 543 the 5 counts the 10^2 group and 4 counts the 10^1 group and the 3 counts the 10^0 group. Thus:

$$543 = 5 \times 10^2 + 4 \times 10^1 + 3 \times 10^0 = 5 \times 100 + 4 \times 10 + 3 \times 1 = 500 + 40 + 3.$$

Understanding place value involves the simultaneous understanding of two principles. Firstly, the positional principle, where the position of a digit tells us the value of the digit in the given number (for example, hundreds stand in the third place) and secondly, the decimal principle, where each position is equal to ten times the immediately lower order (for example, one thousand = ten hundred) (Houdement & Tempier, 2019; R. Howe, 2018). The latter leads logically to the numeration units in place value. Numeration units are ones, tens, hundreds, thousands, etc. (Houdement & Tempier, 2019). For example, the numeration unit of the digit 2 in 2364 is thousands.

Furthermore, place value is essential because of its effects on understanding number structures. It helps learners understand that numbers are interconnected and not just clusters of ones (Askew, 2013). For this reason, place value contributes to structural thinking and knowledge of relationships.

Compensation

The compensation strategy that learners use draws on their knowledge of base-ten thinking to solve addition and subtraction problems where the last digit is a number 8, 9, 1, or 2. Compensation in base-ten involves the rounding of an addend[4] or a subtrahend[5] (with a last digit of 1, 2, 8, 9) to the

---

[4] *Addends* are numbers used in an addition problem, 2 + 3 = 5. In this case, 2 and 3 are the *addends*, while 5 is the sum.

[5] *A subtrahend* is a number to be subtracted from another. For example: in $7 - 4 = 3$ the 4 is the subtrahend, the 7 is the minuend and 3 is the difference.

nearest multiple of 10 that is larger (in the case of 8 or 9) or smaller (if 1 or 2) than what was required. The compensation for this rounding happens by doing the opposite of the rounding to complete the problem. For example, $274 + 99$ would become $274 + 100 - 1$. In this way, a known fact, such as $274 + 100 = 374$, is used to derive an unknown fact such as $274 + 99$.

The compensation strategy that allows you to add a rounded multiple of 10 (which is easier to add) and then compensate by doing the opposite of the rounding requires quantitative reasoning. This is because the compensation strategy involves the analysis of the compensation situation in which the learner works out how to manipulate the numbers best to solve the problem (Nunes et al., 2012).

The knowledge that 99 is one less than 100 when adding 273 and 99 requires an understanding of number relationships. However, the knowledge that you can round up any variable addend that ends with a 9, no matter the size of the number, requires an understanding of quantitative relationships. Here the idea of a variable comes to mind, which is the start of algebra. The concept of adding more than (or less than) and then compensating is quite sophisticated and requires base-ten number knowledge and knowledge of quantitative relationships.

I add a compensation strategy task in my data collection instrument (G6 ESTA) that does not involve base-ten thinking but requires knowledge of number relationships. An example of these questions is: the teacher presents a card with $\boxed{65+18=83}$ as a 'referent equation' to the learner while asking, "If we know that $\boxed{65+18 = 83}$, can you find the answer to the following questions?"

a. If $\boxed{65 + 18 = 83}$
   What is? $\boxed{65 + 17 = \ ?}$
   And
b. If $\boxed{65 + 18 = 83}$
   What is? $\boxed{66 + 19 = \ ?}$

In (a), the learner has to recognise the number relationships between the second addends and realise that 17 is one less than 18. Therefore, the 83 in the referent equation should be one less, which is

82. In (b), the learner must recognise that both the first and the second addends increased by one. Therefore, the answer of 66+19 should be compensated with two more than 83, which is 85.

'Bridging through ten'

'Bridging through ten' is another sophisticated strategy that requires an understanding of base-ten. 'Bridging through ten' refers to the splitting of a second addend in addition or subtrahend in subtraction to find the answer (Tzur et al., 2017).

Let us refer to a sum $A + B = ?$. The splitting of the second addend B (let us say $B = c + d$) involves the addition of a number c that will result in an answer to the nearest multiple of 10 (which are sometimes referred to as 'friendly' numbers). Then the remaining part of B, which is d, should be added to the multiple of 10.

The 'bridging through ten' strategy can be illustrated in the following example:

Find the answer of $14 + 9 = ?$ $(A + B = ?)$

A learner who can use the 'bridging through ten' strategy will know that $14 + 6 = 20$ will give the next multiple of 10. Hence, the learner can split up the 9 into 6+3 and since $9 = 6 + 3$ $(B = c + d)$ the remaining 3 should be added to 20. Thus, the answer of

$14 + 9 = (14 + 6) + 3 = 20 + 3 = 23.$

The 'bridging through ten' strategy requires three foundational skills. Firstly, to understand the number bonds[6] of 10 in order to know which number to add to reach the nearest multiple of 10. Secondly, to know the number bonds of the digits $2, 3, 4, 5, 6, 7, 8, and\ 9$ to be able to know which number is remaining and needs to be added to 10. In the case of $9 = 6 + 3,$ the 3 should be added to 20. Lastly, the understanding that a multiple of 10 plus a one-digit-number, say k, results in a number that has k as a last digit in the number is also vital to find the final answer of $20 + 3 = 23$. In this case, 'k' is 3. I am explaining the three foundational skills for 'bridging through ten' in the following example.

[6] Number bonds, for example to 7, are pairs of numbers that when added together, give the number 7 (3 and 4 are number bonds to 7 because $3 + 4 = 7$).

In the example, $14 + 9 =?$, the learner's knowledge of the 'bonds of 10' may help him to make the decision to take 6 from the 9 and add 6 to 14 in order to make a 20. The knowledge of the bonds of 9 lets him know that there is still a leftover of 3 when he has removed the 6 from the 9. He finally puts the 3 with the 20 to give him the answer of 23.

An example of the 'bridging through ten' strategy in subtraction is as follows. In a question of what is $14 - 5$? a learner can say that 'if I take 4 away from 14 it will give me 10; however, I still need to subtract another one from 10, and I will get $10 - 1 = 9$. The learner used a 'bridging through 10' subtraction strategy in this example. The 14 has signalled to him the fact that he needs to take 4 away to change the 14 to a multiple of 10. This means that the number 5 has to be split into 4 plus 1, and the remaining 1 should be subtracted from the 10. Thus,

$$14 - 5 = (14 - 4) - 1 = 9$$

2.4.2 Additive structural thinking

In additive structural thinking, knowledge of part-part-whole relationships, knowledge of additive situations and operations, and knowledge of formal additive algorithms are important to this study because many of the tasks in the four assessment tools of the study include knowledge of all three above constructs.

Knowledge of part-part-whole relationships

Hunting (2003) argues that a primitive form of part-part-whole reasoning can be seen in early counting actions, where learners know the difference between the items already counted and those items yet to be counted. In this respect, the one part is the items that are already counted, and the other part is the items that still need to be counted. The whole is all the items together.

Learners must have the ability to reflect on a number as a whole, and then know how to isolate its parts (Hunting, 2003). The awareness of part-part-whole relationships helps learners discern how numbers are assembled as parts and wholes (Ekdahl et al., 2018).

Thus, learners who are aware of part-part-whole relationships can comprehend numbers as additive compositions of other numbers (Ekdahl et al., 2018; Hunting, 2003). An example is the bonds of 10, where the learner understands that 10 can be assembled in many ways. For instance: $1 + 9$, $2 + 8$, $3 + 7$ etc. Later, they know that more than two numbers can make up a number, for example, $10 = 2 + 2 + 6$ and $2 + 2 + 3 + 3 = 10$. More additive compositions include subtraction, where $10 = 12 - 2, 10 = 15 - 5$, etc.

Knowledge of additive situations and operations

The focus in this research is on four basic classes of additive situations, namely: join additive situations, separate additive situations, part-part-whole additive situations, and compare additive situations (Thomas P Carpenter, Fennema, Franke, Levi, & Empson, 1999)

i. Knowledge of join additive situations

Join additive situations happen over time where the problem starts with a number of items and then there is a change where the number of items increases. For example, an interviewer hides 4 red counters under a page of paper and later hides 3 blue counters under the page.

> She then asks, "Here are 4 red counters 4 under the page. I put another 3 blue counters under the page. How many counters there are now altogether?"

In this *join* additive situation, the words describe an action that can be taken by putting the 4 red counters and the 3 blue counters together with an 'action'.

Three different types of join additive situations can be generated by varying the unknown amount. They are 'result unknown,' 'change unknown,' and 'start unknown.' The different 'unknowns' vary in difficulty. The problem above is an example of the 'result unknown' type. For this type, the learner can join the numbers by adding them. Learners find the 'result unknown' of the join additive situation easier than the 'change' or 'start' unknown because in the result unknown, they can do an 'action' and just add the two numbers. However, in the change unknown or start unknown situations, the learners have to analyse the situation by knowing the relationship between the numbers, and then decide which operation to use. This analysis process involves knowledge of quantitative relations and structure based on join additive situations.

A summary of the join additive situations can be seen in Table 2.1, adapted from Carpenter et al. (1999).

Table 2.1. Examples of Join Additive situations

Examples of join additive situations		Example
Result unknown	An interviewer hides 4 counters under a piece of paper and later hides another 3 counters. She then asks, “How many counters are there now altogether?”	$4 + 3 = ?$
Change unknown	An interviewer says: “I have 4 counters screened under this piece of paper. I add some more counters, and now there are 7 counters. How many counters did I add to make 7 counters?”	$4 + ? = 7$
Start unknown	An interviewer says: “I screened some counters under this page. While you were not looking, I put 3 more counters under the page. There are now 7 counters under the page. How many counters were there before I put 3 more counters under the page?”	$? + 3 = 7$

(Thomas P Carpenter et al., 1999)

Wright et al. (2006) regard the ‘change unknown’ type of question as a ‘missing addend’ problem because in $4 + ? = 7$, the second addend is missing, or the ‘change amount’ is missing.

In the $? + 3 = 7$ problem, the learner can generate an answer by reasoning; for example, she can use her knowledge of $4 + 3 = 7$ as a number fact, or she can use the inverse property of addition and subtraction to say that $? + 3 = 7$ is the same as $7 - 3 = 4$, thus $\boxed{4} + 3 = 7$. The latter problem is also an example of a ‘missing addend’ problem; however, the first addend is missing in the question. What is important to note in *join* additive problems is that there is a change happening that takes place over time.

ii. <u>Knowledge of separate additive situations in additive reasoning</u>

Both separate additive scenarios and join additive scenarios happen over time; however, the amount that changes decreases and does not increase. For example,

> Thulela had 8 sweets; she gave 3 sweets to Corin. How many sweets does Thulela have left?

Here Thulela's 8 sweets is the 'start amount'. The 3 sweets she is giving away to Corin is the change amount and the 5 sweets that Thulela has left is the 'result amount'. This problem is also called a 'taking away' subtraction-type problem.

In *separate* additive situations, the result, the change, and the start can be unknown. The latter problem is an example of a 'result unknown' situation. In a 'result unknown' situation, the learner can do an 'action' of 'taking away' 3 from 8 because the words in the problem describe a 'giving away' scenario. Table 2.2 summarises the three different *separate* situations, and I adapted it from carpenter et al. (1999).

Table 2.2 Examples of separate additive situations

Examples of separate additive situations		Example
Result unknown	Thulela had 8 sweets; she gave 3 sweets to Corin. How many sweets does Thulela have left?	$8 - 3 =?$
Change unknown	Thulela has 8 sweets; she gave some to Corin, and now she has 5 sweets. How many sweets did she give to Corin?	$8 - ? = 5$
Start unknown	Thulela had some sweets; she gave 3 to Corin and had 5 left. How many did she have at the beginning?	$? - 3 = 5$

(Carpenter et al., 1999).

In the 'start unknown' problem, the learner has to know that the 5 that Thulela still had and the 3 that she gave away to Corin need to be added to know the start amount. Knowledge of part-part-whole relations is essential here, where Thulela and Corin each have a part, and the whole is what she had in the beginning. For the 'change unknown' situation, the learner has to know that she should subtract the 5 from the 8 to know the change amount.

As in join additive problems, separate additive problems also range in difficulty. In *separate* additive situations, the result unknown problem can be performed by an 'action' of 'taking away'. Learners find the result unknown in additive *separate* problems or 'taking away' problems easier than the other two separate additive situations. In the 'start unknown' and 'change unknown' in *separate* additive situations, learners have to know the number relationships and quantitative relationships to know which operation to use (P. W. Thompson, 1993).

iii. Knowledge of part-part-whole additive situations

Part-part-whole additive situations involve static situations where both sets of items are available simultaneously.

The literature identifies two types of part-part-whole relationship tasks. For example:

1) Parts are given, and the whole is unknown (called 'whole unknown' by Carpenter et al. 1999). Examples for 'whole unknown' are:
   - $(P + P = ?)$ *or* $(6 + 8 = ?)$
2) A whole and a part is given, and a part is unknown (called 'part unknown' by Carpenter et al. [1999]). Examples for 'part unknown' are:
   - $(P + ? = W)$ *or* $(8 + ? = 14)$

An example of a 'part unknown' can be

> There are 5 blue and 4 red counters screened. How many counters are there?

An example of 'whole unknown' can be

> There are 12 counters screened. 5 counters are red. How many are blue?

Learners with a developed awareness of structure can use knowledge of part-part-whole relationships to solve problems in additive situations.

iv. Knowledge of compare additive situations.

Compare additive situations, like part-part-whole additive situations, involve knowledge of the relationships between numbers rather than joining or unraveling them in an 'action'. Comparing additive situations involve a comparison between two sets, and it includes knowledge of relationships. According to Carpenter et al. (1999), the one set is compared to another set and they are expressed as the 'referent set' and the 'comparative set'. There is also a third amount, which is the 'difference' between the two sets. This amount is the number by which one set exceeds the other. This relational understanding requires a critical eye to identify and work structurally in comparative additive situations.

An example of a comparison problem is the following:

Mike has R68 and Sam has R50. How many more rand does Mike have than Sam does?

In this problem, the 'referent set' is Mike has R68, and the 'comparison set' is Sam has R50. The 'difference set' is Mike has R18 more than Sam. The latter example describes a 'difference set unknown' situation. A summary of the different compare additive situations is presented in table 2.3 and is adapted from Carpenter et al. (1999).

Table 2.3 Summary of the compare additive situations

Examples of compare additive situations		Example
Difference set unknown	Mike has R68 and Sam has R50. How many more rand does Mike have than Sam does?	$68 - 50 = ?$
Comparison set unknown	Mike has R68 and Sam has R18 more than Mike does. How many rand does Sam have?	$68 - 18 = ?$
Referent set unknown	Sam has R50. She has R18 more than Mike does. How many rand does Mike have?	$50 + 18 = ?$

(Carpenter et al., 1999).

In 'compare additive' situations, we can also have a scenario where the 'comparison set' is unknown or the 'referent set' is unknown. All the compare additive situations are challenging for students because the words in the problems do not describe an ‘action’ that can be carried out but need knowledge of number relationships.

In the case where the ‘comparison set’ is unknown, the problem of a referent unknown will appear as follows:

Mike has R68 and Sam has R18 more than Mike does. How many rand does Sam have?

A referent unknown problem will be phrased as follows:

Sam has R50. She has R18 more than Mike does. How many rand does Mike have?

Knowledge of formal additive written algorithms

Knowledge of formal addition and subtraction algorithms is essential when the number range gets bigger. In this section, I am focusing on the written vertical column algorithm for additive reasoning. Before the written column method can be taught to the learner, understanding of base-ten and place value should be established because then the written algorithm will be understood, and it would not be a mere ‘recipe’ to follow.

According to Thompson (2010) there are two strategies of written column calculations. They are the ‘expanded vertical column’ algorithm and the ‘vertical column’ algorithm, which is also called the ‘standard method’ of written calculations.

In the expanded column algorithm, the learner does not have knowledge of carrying to the next numeration unit and treats the units separately from the tens multiples. Both numbers have to be partitioned. For example: $68 + 56$.

$$
\begin{array}{rl}
68 & \\
\underline{+56} & \\
14 & (6+8=14) \\
\underline{110} & (60+50=110) \\
124 &
\end{array}
$$

Later learners start to use the ‘carrying’ of groups of multiples of 10 to the next numeration unit:

$$\begin{array}{r} {}^{1}3\,{}^{1}87 \\ +456 \\ \hline 843 \end{array}$$

The learner has to write the answer underneath the two numbers that must be added. In the example above, $6 + 7 = 13$. The learner has to write the 3 in the one’s column and ‘carry’ one group of 10 to the 8 tens, making the 8 tens 9 tens. Thus, 9 tens plus 5 tens is 14 tens. Now the learner writes the 4 in the tens column and carries one group of hundred to the hundreds column, making the 3 hundred 4 hundred. Then 4 hundred and 4 hundred are made 8 hundred. The resulting answer is then 843.

What is important to mention here is that a learner should know when it is better to use the column method and when an addition task can be calculated mentally. This will depend largely on the magnitude of the numbers and the learners’ fluency when working across the structural elements of the decimal number system, such as compensation and bridging through 10.

2.4.3 Multiplicative structural thinking

Learners find multiplicative reasoning more challenging than additive reasoning (Clark & Kamii, 1996; Dooren, Bock, & Verschaffel, 2010). Most textbooks introduce multiplication as a compact way of performing repeated addition. However, Kamii and Clark argue that multiplication requires higher-order thinking (Piaget, 1987), and therefore learners have difficulties understanding multiplicative reasoning (1996).

Among the difficulties that young learners have with multiplication is first that they add instead of multiplying. For example, I have 12 apples in each bag, and I have 25 bags. How many apples do I have? A learner who does not understand multiplication may answer $12 + 25 = 37$. The second problem learners may experience with multiplicative reasoning is that they struggle to derive a new fact from a known fact. For example, they find it challenging to use the knowledge of $6 \times 6 = 36$ to help them to figure out what 7×6 is. However, if a learner knows that $6 + 6$ equals 12 they can more easily figure out what 7 + 6 is. The third challenge that learners experience in multiplication is that they have difficulty understanding what the computation of multiplication means. For example, when O'Brien and Casey (1983) asked learners to write a story problem for the calculation $6 \times 3 = 18$, a mere 37% of the fourth graders and 44% of the fifth graders could produce an appropriate word problem for $6 \times 3 = 18$ (Clark & Kamii, 1996).

In this section, I discuss multiplicative structural thinking in four parts: Part 1: knowledge of multiplication as more than repeated addition. Part 2: knowledge of multiplicative situations and problem types. Part 3: knowledge of multiplicative reasoning bond of facts to derive new facts, and Part 4: knowledge of formal multiplicative operations.

Part 1: Knowledge of multiplication as more than repeated addition

Although multiplication is generally introduced as a faster way of doing repeated addition (Clark & Kamii, 1996), Piaget's work (1987) tells a different story. Piaget claimed that multiplication is an operation that requires more higher-order multiplicative thinking than addition and subtraction as operations. He went further and posited that addition is accomplished by the repeated addition of ones, while multiplication, on the other hand, is a more complex operation because it is constructed out of the addition of groups that are more than one.

Clark and Kamii's (1996) summary of Piaget's work (1983/1987) on multiplicative reasoning reveals that for Piaget, the difference between addition and multiplication is also that in multiplicative reasoning, the number of composite units[7] is more than in additive reasoning. Steffe (1992) argues that for a learner to be competent in multiplicative reasoning, she must have concurrent knowledge of the multiplicand[8] and the multiplier as composite units, and is thus on a higher level as the addition of repeated equal groups.

[7] To have knowledge of a number, say 3, as a composite unit, an understanding of 3 as 3 ones and 3 as 'one' composite unit must be established simultaneously.
[8] Multiplicand and multiplier are the two factors of a product. $Multiplicand \times multiplier = product$

The following example can illustrate repeated equal groups: 3 + 3 + 3 + 3 is equal to 12. Additive thinking in multiplication involves only one level of thinking. Here the learner adds 3 + 3 = 6, then 6 + 3 is 9 and finally 9 + 3 is 12. Only 3 is a composite unit.

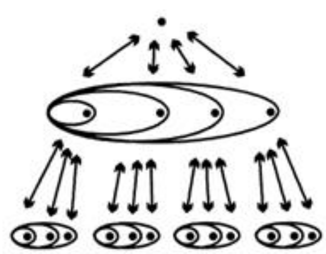
According to Piaget, this multiplication as repeated addition is on a horizontal form of reasoning because the knowledge of the relations between the numbers is on one level. Figure 2.1 illustrates this notion quite effectively.

(a) Additive

Figure 2.1 Multiplication as the addition of equal repeated groups on a horizontal form

(Clark & Kamii, 1996)

By contrast, multiplicative reasoning is more complex, and the learner must reason on two levels simultaneously. In the diagram below, the learner has to reason vertically and horizontally at the same time. The vertical perceptive of the 4 as a composite unit and the vertical perspective of 3 as a composite unit should be established. Furthermore, the horizontal reasoning includes the knowledge of the ratio of the 1:3. Then every 'one' in the 4 should include one composite unit of three. This leads, at the end, to 12 units. See figure 2.2.



(b) Multiplicative

Figure 2.2 Multiplicative reasoning on a horizontal and vertical form (Clark & Kamii, 1996, p. 42)

Drawing from work done by Vernguad, Askew (2018) argues that the difference between additive and multiplicative reasoning is that additive reasoning is ternary[9] and multiplicative reasoning quaternary[10]. All additive situations involve two parts and a whole (Hunting, 2003), for example: $\mathbf{3+4=7}$. Thus, additive reasoning is ternary as it involves three amounts.

In contrast, multiplicative reasoning is quaternary. Let us consider a simple multiplicative question to explain:

> Thulela bought 4 bags of apples. There are 6 apples in each bag. How many apples did she buy?

In the question above, there is a hidden 'one' in the words 'each bag' and a "subtle rate is invoked" (Askew, 2018, p. 3). The rate is that there are 'six' apples in 'one' bag. Table 2.4 shows the quaternary nature of multiplicative reasoning for the question above.

Table 2.4: The quaternary nature of multiplication

Bags	Apples
1	6
4	?

(Askew, 2018)

Another reason why multiplicative reasoning is more complicated than additive reasoning is because, for addition and subtraction problems, learners can use counting strategies to solve simple addition and subtraction tasks, but it is more challenging to use counting strategies in multiplicative reasoning (Thomas P Carpenter et al., 1999). The counting strategies in multiplication are skip counting for the multiplication and quotative division while keeping track on their fingers. For partitive division, learners use trial and error skip counting to find the amount of the group. More about the strategies learners use for multiplicative reasoning follows in section 2.5 of this chapter.

[9] Ternary: exists out of 3 parts
[10] Quaternary: exists out of 4 parts

Part 2: Knowledge of multiplicative situations and problem types

This section discusses the problem types and the solution strategies that young learners use. Even though I argued that multiplicative reasoning is quaternary (Table 2.4), there are always only 3 numbers that can be unknown in a multiplication or division task. The fourth number in multiplicative reasoning is always a constant and is a hidden 1. Multiplicative reasoning involves three quantities, as illustrated in the following equation $5 \times 3 = 15$:

Sam has 3 bags of apples. There are 5 apples in each bag. Altogether, she has 15 apples.

Potentially there are 3 different unknown numbers in the above equation. One of the numbers (5, 3, or 15) is unknown in multiplicative reasoning tasks. The three unknown amounts in this problem are the number of bags, apples in each bag, and the total number of apples. If the total number of apples is unknown, it is a multiplication problem; when the number of apples in each bag is unknown, the problem is a partitive division[11] problem; and if the number of bags is unknown, the problem is called a quotative division[12] problem. Table 2.5 shows the different types of situations in multiplicative reasoning, as adapted from Carpenter et al. (1999).

Table 2.5 The different types of situations in multiplicative reasoning

Type of situation	Example in words	Notation example
Multiplication (total number of apples is unknown)	Sam has 4 bags of apples. There are 5 apples in each bag. How many apples does Sam have altogether?	$4 \times 5 =?$
Quotative division (repeated grouping aspect of division): the number of bags is unknown.	Sam has 15 apples. She puts 5 apples into each bag. How many bags can she fill?	$15 \div 5 =?$
Partitive division (the sharing aspect of division): the number of apples in each bag is unknown.	Sam has 15 apples. She puts the apples into 3 bags with the same number of apples in each bag. How many apples are in each bag?	$15 \div 3 =?$

[11] Partitive division refers to the sharing aspect of division.
[12] Quotative division refers to the grouping aspect of division.

(Thomas P Carpenter et al., 1999, p. 34)

Young learners solve multiplication problems by using counters to make the required number of specified size of groups and counting the total number of objects. Here the learner counts out sets of 5 counters until she has 4 sets and then counts out the counters altogether to get an answer of 20 apples.

In quotative division situations, learners normally count out the given total number of objects. Then they use trial and error to make sets. For example, learners can count out 15 counters and make sets of 3, 4 or 5 counters in each set until there are no remainder counters. Then count how many sets there are to find the answer.

The third procedure in multiplicative reasoning is knowledge of partitive division (also called the sharing aspect of division). One way to deal with partitive division situations is to first count out the total number of items and partition or share all the items one by one into the number of groups until the total number of items is used up. The partitioning can be one-by-one or two-by-two or three-by-three, etc.

Learners also use some counting strategies, like skip counting, if they do not have counters to model the above situations. Multiplication and quotative division often involve skip counting in multiples of the multiplicand. Learners often use skip counting for partitive division, but use trial and error skip counting to find the number in the groups and the number of groups while keeping track on their fingers (Thomas P Carpenter et al., 1999).

Part 3: Knowledge of multiplicative bonds

Some learners have a set of known multiplicative facts or bonds in place to find answers to multiplicative reasoning tasks. Learners 'knowing' that $3 \times 5 = 15$ helps them derive an answer to 6×5. Another two examples: if they know that $8 \times 10 = 80$, they can derive a solution for 8×5 by saying the answer is 40, or a learner uses his knowledge of $7 \times 10 = 70$ to derive a solution of 63 to the question: 'what is $7 \times 9'$ by subtracting 7 from 70.

Part 4: Knowledge of multiplicative algorithms (formalisation)

Knowledge of multiplicative algorithms or formalisation involves written strategies. Van de Walle (1998, p. 238) distinguishes between three different written strategies that learners use for multiplication by a single-digit multiplier (63×5), namely, complete number strategies, partitioning strategies and multiplicative compensation strategies.

Complete number strategies include knowledge of repeated addition and doubling. See the example in Figure 2.3.

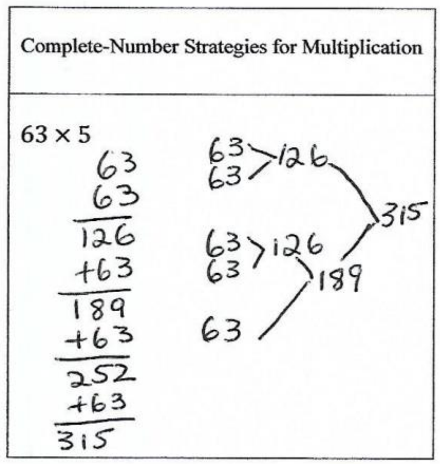


Figure 2.3 The complete-number written strategies for multiplication (Van de Walle, 1998, p. 238)

Often a learner will list long columns of numbers and add them all up. The doubling strategies follow the repeated addition strategy, where they start to realise that doubling makes the work less time-consuming.

In partitioning strategies, learners break numbers up in a variety of ways. Figure 2.4 provides some examples:

Partitioning strategies for multiplication		
By decades		By tens and ones
27 × 4 4×20=80 > 108 4×7=28	268 × 7 7×200=1400 > 1820 > 1876 7×60= 420 7×8 = 56	27 × 4 10×4=40 > 80 > 108 10×4=40 7×4=28
Partitioning the multiplier		Other partitions
46 × 3 Double 46 → 92 138		27 × 8 25×4=100 50 25×8=200 > 216 2×8 = 16

Figure 2.4 Partitioning written strategy of multiplication according to Van de Walle (1998)

These approaches in Figure 2.4 include partitioning according to decades[13], partitioning the multiplier by tens and ones, and other partitioning modes. Partitioning by decades and by tens and ones requires an understanding of base-ten structural thinking. Partitioning by the multiplier reflects their understanding of the number structures in doubling. For example, in $124 \times 5 = ?$ the learner may double 124 to get 248, then double 248 to get 496, and lastly, add another 124 to get the final answer. This learner knows that times 5 means to double twice (times 4) and then add another 124 to get to times 5.

The last example, in figure 2.4, is called 'other partitions' and reflects knowledge of known facts, place value and base-ten thinking.

The compensation strategies of Figure 2.5 reflect knowledge of base-ten thinking, where the learner changes the 27 into 30 and later compensates for the change by subtracting $3 \times 4 = 12$ from 120.

[13] Partitioning according to decades in 268×7 means that the number 268 can be broken up into 200, 60 and 8 and then multiplied by 7 separately.

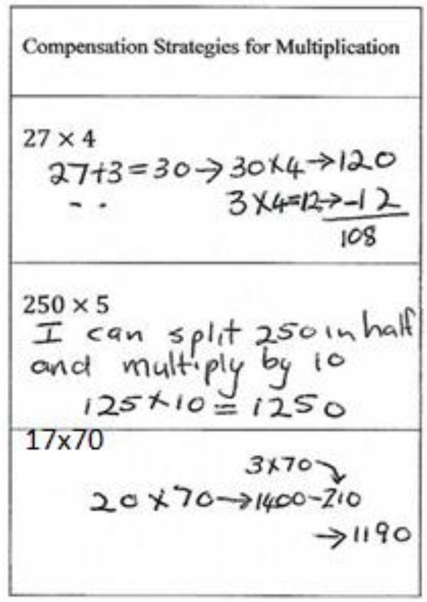


Figure 2.5 Compensation written strategies for multiplication according to Van de Walle (1998, p.238)

The above are examples where learners use different forms of invented strategies for multiplication; however, the formal multiplication and division algorithms need to be taught to learners because problems such as $'486 \times 372 = ?'$ can be challenging to find without an algorithm.

2.4.4 Recognition and utilisation of patterns and structural thinking

The last application of structural thinking that I regard as necessary to my study is recognising patterns[14] in numbers and later the utilisation of number patterns and pattern generalisation.

The reason why I want to explore pattern recognition in my study is that pattern recognition is widely mentioned in literature as an essential aspect of structural thinking (Mulligan & Mitchelmore, 2009) and is one of the predictors of later mathematical performance (Claessens & Engel, 2013; Geary et al., 2013; Mulligan et al., 2005).

[6] Here ‘pattern’ is taken to mean any predictable regularity involving numbers (Mulligan & Mitchelmore, 2009).

In an empirical study, Mulligan and Mitchelmore (2009) identified a salient underlying characteristic of structural thinking in early mathematics learning as the recognition of patterns in number sequences. They demonstrated the importance of recognising a pattern as an application of structural thinking.

Here ‘pattern’ is taken to mean any predictable regularity involving mathematical elements (Mulligan & Mitchelmore, 2009) such as +1 patterns (for example: 1, 2, 3, 4, 5…) and patterns in ever-increasing complexity. These elements can be numbers and shapes. Listing the elements (or terms) in a pattern is called a sequence with a specific ‘rule’ or ‘regularity’ to find the next term.

The ability to determine “regularity among an ordered set of units, termed patterning, is a crucial cognitive ability that precedes pre-algebraic mathematics skills” (Bock et al., 2015, p. 1).

As mentioned earlier, the +1 regularity of our number system is the base pattern learners must acquire (McIntosh, Reys, & Reys, 1992). Learners need to build a representational structure of the +1 number system in their minds to be able to recognise other more advanced patterns. Without the +1 pattern, there is no referent number sequence to which different patterns can be compared.

Theoretically, to recognise the next number in a sequence (where sequence means a list of numbers with a constant regularity between terms), learners need to find the number relationships between the terms by comparing them. The comparison includes recognising the number relationship between term one and term two and then recognising the number relationship between term two and term three.

This twofold recognition can make the learner aware of whether the sequence is increasing or decreasing. The next phase is that the learner has to find the predictable commonality between the terms.

The recognition of the predictable commonality between the terms involves knowledge of all four operations. Knowing how and when to use particular operations for discovering the commonality between the terms in a sequence and the concept of the commonality of a sequence are founded on a knowledge of number and quantitative relationships.

It is essential to note, at this point, that structural thinking does not only include the recognition of regularity in patterns but in the learner's application of it. The learner must be able to develop more terms of the sequence (Mulligan & Mitchelmore, 2009; Warren, 2005). After finding the regularity between the terms, learners have to use their knowledge of regularity to produce the next term by using addition, subtraction, multiplication and division, and the idea of repetition.

Mason et al. (2009) describes structural thinking as appreciating relationships between mathematical 'objects' (these objects can be numbers, triangles, functions, etc.); however, appreciating relationships between objects is not the only skill learners need to master. For Mason et al. structural thinking involves developing a 'deeper awareness' of quantitative relationships and knowing how the relationships are connected and explicated. This deeper awareness includes knowledge of the involved properties of the mathematical objects and the understanding of how to use the properties of the objects in a specific situation. Mason et al. state:

> "Recognizing a relationship amongst two or more objects is not in itself structural or relational thinking, which, for us, involves making use of the relationships as instantiations of properties. Awareness of the use of the properties lies in the core of structural thinking." (p. 10)

It now becomes essential to look at generalisation as the product/outcome of patterns and structural exploration of the relational aspects embedded in a sequenced set of numbers. It is so much more than just pattern noticing – it is the formation of an algebraic expression. The following quote from Radford and Peirce (2006) expounds what I consider generalisation to be in this context.

> "The process of generalizing includes two interrelated components. The first one is noticing a commonality in some given particular terms. The second one is to form a general concept … by generalizing the noticed commonality to all the terms of the sequence. In order for a generalization of patterns to be called algebraic, I have suggested a third component … a rule providing one with an expression of whatever term of the sequence." (Radford & Peirce, 2006, p. 28)

The above quote can be further elaborated on by an example from Mulligan and Mitchelmore (2009) which touches on the progression from recognising patterns to generalisation. In the sequence 2, 4, 6, 8…., the terms are organised in a particular way. To find the next local term of the sequence requires a recognition that it should be two more than the previous term. Generalisation of pattern recognition and exploration is the ability to know how a pattern is formed and not only recognise the next local term. To generalise the pattern is to understand that the 10^{th} term will be 20, and the 100^{th} term will be 200. In advancing to generalisation in algebra, using symbols, we can say that the sequence's function is $y = 2x$.

Even though my study is not about early algebra, it is important to mention how vital patterning and pattern generalisation in the early years of schooling are for future development in algebraic reasoning skills and the later learning of algebra as a concept in mathematics.

2.5 Strategies used as the indicators of the emergence of structural thinking

Many researchers consider it vital that mathematical development happens over time. This can be seen from the vast amount of literature that has been published that deals with the progressive dynamics of early mathematical skills (Askew, 1998; Ensor et al., 2009; Pirie & Kieren, 1994; Wright, Martland, Stafford, & Stanger, 2010).

The development of structural thinking entails learners proceeding along different levels of sense-making. The latter spans the acquisition of understanding from developing an awareness of unenumerated quantities, when they are very young, to learning number words, counting, and the introduction of number symbols. Later they determine informal, context-based answers in simple addition and subtraction tasks and create shortcuts when they start to count-on/back to solve simple addition and subtraction problems. When learners acquire insight into the underlying structure of numbers, they realise the relationships between numbers and quantities, and later they recognise relationships between operations (Van den Heuvel-Panhuizen, 2000).

Because structural thinking keeps emerging, it is essential to investigate the extent and nature of learners' structural thinking. Venkat et al. (2019) distinguishes between two structural thinking notions: 'emergent structure' and 'mathematical structure.' The emergent structure is about noticing and forming only local relationships that are internal to a specific case, and 'mathematical structure', which involves developing general relationships in many cases. For Venkat et al. (2019), 'emergent structure' is about noticing and forming local relationships that are internal to a specific case. For example, a learner can count on from 7 to 13 when asked to answer 7+6. Additionally, 'mathematical structure' involves forming general relationships. For example, a learner with mathematical structural thinking can add 6 to more numbers over bigger number ranges.

While the literature details applications of structural thinking and the emergence of structural thinking, this needs to be translated into indicators, or 'confirmations' of structural thinking that can be observed in the learners' responses in empirical data. The way in which learners use their knowledge of structural thinking is important.

Because of the above[15] viewpoint of Mason et al. (2009), about the utilisation of pattern recognition, I realised the importance of finding the next term in a pattern, and the prominence of using the relationships and properties in structural thinking. Thus, structural thinking indicators are located in the strategies that learners use. The video-recorded assessments called the Learning Framework in Number (LFIN) that Wright et al. (2006) developed, are an excellent example of how learners' strategies can give the educator some insight into the extent of structural thinking that is present in learners' reasoning.

The literature describes different levels of strategies learners use (Wright et al., 2012). Young learners' initial foundational strategies are dependent on actions on concrete items and develop beyond counting strategies to a mental calculation that references structured thinking.

In this section, I discuss the strategies learners use that serve as indicators of the emergence of structural thinking. I discuss the knowledge of unenumerated quantities when learners are very young, counting strategies through which learners develop shortcuts like counting-on/back, using knowledge of relationships between numbers and quantities, mental calculations, and algorithmic strategies.

[15] Section 2.4.4.

2.5.1 Knowledge of unenumerated quantities as a ‘starting point’ of structural thinking.

Literature in early-year education claims that research on infants and pre-schoolers shows that mathematical structural thinking begins with comparisons of unenumerated quantities when learners are very young (Catherine Sophian & Adams, 1987). Babies can make sense of number relationships (Davydov, 1995b; Dehaene, 1997; Sophian, 2008) in ways that involve a core knowledge of quantities without the use of number words. Thus, core knowledge is not the result of any mediation from adults or peers, but rather inherited (Sophian & Adams, 1987).

The core knowledge system represents the building blocks for later mathematical thinking and the development of an awareness of number relationships (Spelke & Kinzler, 2007) with larger number ranges.

Van den Heuvel-Panhuizen (2008) adds another knowledge of quantities that learners have which is termed the ‘preliminary ideas’ about certain quantities. She argues that learners come to school with an abundance of information, which may or may not contribute to mathematical structural knowledge development.

Preliminary ideas about quantities include three aspects of number knowledge and are essential for structural thinking. The first aspect of preliminary ideas about quantities is the ability to compare knowledge, where learners can make ‘bigger or smaller’ judgments on sets of items. This comparison idea may help learners, later on, to answer questions in school like ‘which number is bigger/smaller than 5?’ It can also help when learners have to compare two sets of numbers when they have to subtract two numbers (Clements & Sarama, 2009).

The second aspect of preliminary ideas about quantities is the understanding of increase/decrease, where the learner knows that if one cookie on the plate is removed, they have less to eat. Similarly, if nothing is taken away, it stays the same. The increase/decrease knowledge is the same idea as Piaget’s conservation notion. Piaget’s conservation of number (Kamii & DeClark, 1985) refers to the ability to determine, through reason, that the number of a set remains the same, even when the observed appearance is changed. If the increase/decrease idea is in place, it may help learners later with questions like, “What is 1 more or less than 5?” Or questions like: “If 26+17= 43, what is 27+17?”

The third aspect of preliminary ideas about quantities is part/whole knowledge, where learners demonstrate an inherent schema about how a set comes apart and goes together. The fourth aspect of learners' preliminary knowledge is an inherent capacity to subitize without serial enumeration (Mulligan et al., 2018). Learners can 'see' the number of items up to 5 without counting them one by one (Wright et al. 2010).

Both 'preliminary ideas' about quantities and core knowledge form the foundation for structural mathematical thinking (Resnick, 1990; Sophian, 2008).

Some researchers have investigated the point of departure of mathematical thinking as unit counting (Ensor et al., 2009; Wright, Martland, & Stafford, 2006). Others explored the origin of mathematical thinking as the making sense of the relationships between unenumerated quantities (core-knowledge) from a very early age.

In my study, I posit that learners make sense of unenumerated quantities (core-knowledge) from a very young age and then develop knowledge of numbers when they start to count. Learners use their core-knowledge when they begin to count, which enables them to grasp the concept of numbers better. Thus, core knowledge precedes counting. Since the participants in this study are in Grade 2 and Grade 6, we may think there is no relevance in mentioning unenumerated knowledge in this study; however, to have the complete picture of structural thinking and where it starts, it is essential to note because I want to what know structural thinking is to the fullest.

2.5.2 Counting

Counting is broadly seen as an essential skill that learners need to master (Clements & Sarama, 2009; Ensor et al., 2009; Gelman & Gallistel, 1986; Sfard, 2008; Wright, Martland, & Stafford, 2006). It constructs the first bridge from core knowledge of quantities to more advanced mathematical thinking (Butterworth, 2005).

Thus, when learners start to count, they do not lose the core knowledge. On the contrary, core knowledge forms the foundation of all mathematical relational sense-making and structural thinking (Wynn, 1995). When learners start to use words to count in the world of mathematics, the number words in a sequence reinforce the already existing core knowledge by coordinating the number words with the core knowledge of quantities. The result of number words in a sequence that is well established equips learners to be able to communicate quantities with others within a mathematics culture (Wynn, 1992).

Piaget (1971) argues that the growth of structural thinking in the learner stems not from the properties of the concrete items but from the counting actions on the tangible items (p. 21). In early structural thinking, learners make sense of their lived experience and their world of mathematics by actively creating their own understandings (Lunenburg, 2011). In the preschool years, learners make sense of the world around them by interacting with the 'world of life' (Van den Heuvel-Panhuizen, 2000, p. 4).

Therefore, when learners are introduced to number words and start to count, they do not do so as blank slates but start to count with core knowledge as knowledge of the world of their lives. When learners are introduced to number words and number symbols, they move from real-world situations to the world of mathematics. Freudenthal (in Van den Heuvel-Panhuizen, 2000) phrased the movement as horizontal mathematisation which "…involves going from the word of life into the world of symbols", while vertical mathematisation involves a movement within the world of symbols (Ibid, pp. 4).

Actions on concrete items have the potential to create "objects of the mind", and the actions underpin numerical concepts (Gray et al., 2000, p. 401). Thus, learners must start to count sets of physical items to understand numerical objects. Counting sets of physical items helps the learner understand the cardinality in number knowledge (Mason et al., 2009).

Some learners do not really understand the significance of the last word uttered when they count. Research shows that for some learners, the question 'How many?' means count. They keep on counting the objects without providing an answer (Hunting, 2003). Some learners know that the last word in the counting sequence is the correct answer but without understanding the number's cardinality. Others understand the cardinality of the number (Hunting, 2003) but do not see it related to other cardinalities (J Mason et al., 2009). For example, if a learner knows the cardinality of the number 4 and the number 6, and does not know that 4 comes before 6 and is two less than 6, he does not see the cardinalities in terms of each other.

Learners think structurally when they use a particular cardinality in relation to other cardinalities. Mason et al. (2009) claim that the cardinal principle is only well established when learners use cardinality in terms of other cardinalities. (Ensor et al., 2009). Therefore, structural thinking involves the ability to use the cardinal principle in relation to other cardinalities. When learners can see one cardinality of a number in relation to other cardinalities, they become aware of its ordinality. For example, they know the cardinalities of 4 and 6 and know that 4 comes before 6 and is two less than six. Sinclair and Coles (2015) say it is vital to develop an awareness of ordering numbers by challenging the more common emphasis on cardinal knowledge. They suggest that more emphasis should be placed on numbers' ordinality in the context of learners comparing numbers with numbers (ordinality of number) rather than comparing a number with sets of items (cardinality of number).

Furthermore, Gelman and Gallistel (1978) add to the importance of ordinality and claim that the list of the stable ordered number words holds a relationship between numbers, fosters ordinality, and should be established before the learner masters cardinality.

Marja van den Heuvel-Panhuizen (2008) suggests three different sets of counting levels: context-related counting or counting in class, concrete counting of physical items or object-based counting, and pure or formal counting.

Context-based counting refers to counting in classrooms, and it relates to the ability to say the number words in the correct order. Context-based counting does not include the ability to answer, 'how many?' and thus the learner with context-based counting does not know the cardinality of a number.

Object-based counting involves the ability of the learner to answer, 'how many in a set of objects?" and the learner with object-based counting does know the cardinality of numbers.

Pure-counting (counting mentally) refers to working with numbers without the physical items where the learner can reflect on knowledge of number relations. Pure-counting also includes the ability to use number relationships in answering mathematical questions and solving mathematical tasks.

Counting can happen on different levels. For example, a learner might count out 5 on his fingers recursively, which is at a process level, and eventually, he will stop to count out 5. He will show 5 fingers without counting by ones and employ the noun '5' to describe the counting process, which he has then reified. The moment the learner can say 5 without undertaking the counting process, the 5 becomes reified and becomes an object (Sfard, (2008). Mentioning the latter is essential to this study because when we see that the child does not count the 5 and counts on from 5, we can assume that the learner has reified 5 and knows 5 as a mathematical object. When the learner knows 5 as a mathematical object he can count on, and is on a higher level of structural thinking than before.

Reification is the modification from a concrete understanding, where calculations can be done only in the presence of physical items or tallies, to structural understanding where learners do not need to use tangible items. Structural understanding includes reified objects where the learner is aware of mathematical objects in relation to each other (Sfard, 2008, Venkat et.al., 2019) and can calculate without physical objects (Ensor et al., 2009).

If a learner counts all in a simple addition task, he counts both sets one by one, following with counting the total also one by one. If he counts on from the first set, he starts with the first number in his mind and counts on. The first number is a mathematics object for him, and he does not need to count it again. When a learner counts on in an addition problem, from the larger number, he is aware of the relationship between the two numbers and knows that if he counts from the larger number, it would be less labour-intensive, and the learner's thinking starts to become structural and progressively more sophisticated. In number work, learners with skills beyond unit-counting strategies will find answers more efficiently, which means that the strategy used was executed with the minimum effort (Askew, 1998).

2.5.3 Mental calculations as calculating with structure

Researchers claim that mental calculation improves learners' mathematical understanding and suggest that learners will have all kinds of advantages if exposed to mental calculations (Beishuizen & Anghileri, 1998; Cockroft, 1982; McIntosh, Reys, & Reys, 1997).

The literature refers to many different words for mental calculations. In this study, I will use the term mental calculations, where mental calculation refers to the application of nonstandard algorithms without using pen and paper (Trafton, 1978).

The advantages of mental calculations are as follows: mental calculations enable learners to realise how the properties of numbers and number sequence structures work (McIntosh et al., 1997) as the basis for developing estimation skills (Maclellan, 2001; Reys & Barger, 1994).

Mental calculation enables learners to create strategies (Heirdsfield & Cooper, 2002). It promotes independent thinking (Ian Thompson, 2010), and early mental work in mathematics enables flexible written calculations and algorithms later (Threlfall, 2002). For some researchers, mental calculations are so crucial that they suggest postponing the introduction of place value written column algorithms in the early years of teaching to later in primary school (Gravemeijer, 1994; Treffers, 1991).

Mental calculation involves knowledge of the relationships between numbers (Threlfall, 2002) to derive new facts (Askew, 1998), and requires learners to compare numbers and give quick, accurate answers by using structural thinking (Maclellan, 2001). When learners progress to numerical understanding, they accumulate several known facts and derive new facts with an awareness of number relationships, patterns and generalisations. Thus, mental calculations have two aspects that need to be developed: firstly, the learner has to establish a repertoire of known facts and recall them and secondly, know how to use the known facts to derive new facts (Askew, 1998).

The recalling and use of the repertoire of known facts to answer a mathematics question are the same as what Beishuizen (1997) and Anghileri (Anghileri, 1999) regarded as mental calculations ‘in the head’, while the development of new facts from the known facts is the same as mental calculation ‘with the head’. Doing mental calculations ‘with your head’ refers to derived facts that learners figure out using known number facts and number relationships. Later, the derived facts become part of the set of known facts, leading to more derived facts (Askew, 1998).

Another way of doing mental mathematics as set out in the literature is to construct a mental picture of a ‘paper and pencil’ column method (Thompson, 2010) and work through the procedure in their minds without using external devices. However, while the vertical column method's mental representation will help the learner get to the correct answer, it is not seen as a mental strategy that contributes to structural understanding (Threlfall, 2002) and is an inefficient mental strategy (Heirdsfield & Cooper, 2002).

Mental calculation is also about an awareness of the different mental strategies and choosing the most efficient one (Askew, 2000). An example of deciding the most efficient strategy is adding one number to another by separating the tens and the ones that are more situated in $55 + 24$ than in $33 + 29$. For example: in $55 + 24$ is $50 + 20 = 70$ and $5 + 4 = 9$, which gives us 79. To separate the tens and the ones in 33+29 is not as efficient as rounding off the 29 in $33 + 29$ to get to $33 + 30 - 1 = 62$. Anghileri (2006) claims that a learner who can match a mental method to the numbers in the problem will be the most successful at performing mental calculations and will be flexible in mental calculation.

Thus, flexible mental calculations involve choosing the best method for the mathematical situation at hand (Meindert Beishuizen, Van Putten, & Van Mulken, 1997). Furthermore, Torbeyns, Ghesquiere and Verschaffel (2009) argue that flexible mental calculations require an understanding of operations in terms of inverses, and commutative and associative properties.

The literature describes many different mental strategies. Heirdsfield and Cooper (2002) distinguish between separation, aggregation, and 'wholistic' mental strategies for addition and subtraction.

Separation[16] strategies refer to splitting up both numbers to be added or subtracted into tens and ones and then adding or subtracting the tens and ones separately. Using segregation strategies, the learner uses her base-ten and place value structural knowledge (discussed earlier in this chapter).

Aggregation[17] strategies refer to starting with one number and splitting up the other number into tens and ones to then add or subtract. Aggregation strategies require knowledge of base-ten and place value.

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[16] Separation strategies: the same as what Beishuizen, Van Putten and Van Mulken regard as 1010 strategies (1997)
[17] Aggregation strategies: the same as what Beishuizen, Van Putten and Van Mulken regard as N10 strategies (1997)

Ron Tzur and colleagues (2017) add to aggregation strategies and refer to break-apart-make-ten (BAMT), where learners split up a single-digit addend or subtrahend to add or subtract (Tzur et al., 2017). The BAMT strategy is the same as 'bridging through ten', described earlier in this chapter as base-ten structural knowledge.

'Wholistic' strategies do not split numbers but consider and change the given numbers to make calculations more 'comfortable' and then compensate appropriately (Heirdsfield & Cooper, 2002). It is more 'comfortable' to add 100 to 273 and subtract 1 afterward than to add $273 + 99$. Torbeyns et al. (2009) use the phrase 'shortcut strategies' for 'wholistic' strategies and include subtraction shortcut strategies.

For them, subtraction shortcut strategies can change the subtraction operation to addition when the learner realises the small difference between numbers in subtraction questions. They argue that shortcut strategies are based on a person's understanding of number relations and the properties of number operations (2009). Separation, aggregation, 'wholistic' or 'shortcut strategies', and BAMT reflect structural knowledge of numbers and relationships between operations.

The abovementioned mental strategies indicate that learners need to have base-ten structural knowledge, including compensation, bridging through ten and place value to perform mental calculations flexibly. To tackle problems effectively, they also need structural knowledge of all the additive (join, separate, and part-part-whole situations) and multiplicative (multiplication, quotative, and partitive division).

2.5.4 Algorithmic calculation strategies versus mental calculations

The place and value of mental computation versus standard written algorithms are heavily debated (Torbeyns & Verschaffel, 2016). A discussion about which calculation strategy should be used will follow.

Later in primary school, learners are introduced to written formal algorithmic calculation strategies, which include the vertical column algorithm for addition, subtraction, multiplication and division.

The introduction of the vertical column algorithm must be accompanied by knowledge of number and quantitative relationships. If learners do not learn standard algorithms together with number relationships and quantitative relationships, the standard algorithm can become a meaningless procedure. The learner will get to the correct answers but will not know what the numbers or the answers mean, and it does not contribute to the learner's structural understanding (Graven, Venkat, Westaway, & Tshesane, 2013; van den Heuvel-Panhuizen & Treffers, 2009). In such cases, the learner would always use a written algorithm and would not choose a more efficient strategy between a written algorithm and a mental calculation.

Researchers claim that early mathematics education should shift from a focus on the acquisition of memorised standard algorithms without understanding to the development of learners' ability to solve mathematical tasks efficiently, creatively, and flexibly with strategies where the learner understands number relationships (J Torbeyns, De Smedt, Ghesquiere, & Verschaffel, 2009).

Thus, the knowledge of the standard written algorithms is not about learning a step-by-step calculation procedure but needs to be underpinned by an awareness of number relations and an understanding of the properties of operations (van den Heuvel-Panhuizen & Treffers, 2009).

Learners should use mental calculations when it is appropriate. However, when they cannot do the calculation in their heads, they should use a written algorithm (Thompson & Thompson, 2010). To know when to select the correct strategy between a mental response and an algorithmic written response is part of structural thinking. For example: for $273 + 99 =?$, a compensation strategy would be the most appropriate strategy. However, for $275 + 928 =?$ the standard vertical algorithm would be the best strategy choice.

Thus, selecting the most appropriate and efficient strategy between written algorithmic and mental calculations depends on the extent of structural thinking that a learner has developed. Therefore, flexibility in mathematics calculation refers to the ability to make the best choice by taking the characteristics of the numbers at hand into consideration (Torbeyns et al., 2009) before choosing between a mental calculation or a written calculation.

2.5.5 Empirical indicators of structural thinking through the eyes of Wright et al. (2006)

Wright, Martland, Stafford and Stanger (2006) state that the model they developed has been adapted from research done by Steffe and colleagues (for example, Steffe, 1992; Steffe and Cobb, 1988; Steffe et al., 1983) and focuses mainly on the progression of learners' strategies for solving number problems.

In the work of Steffe and Cobb (1988), the word 'stage' is used in a theoretical sense that includes four characteristics: the consecutive stages form an invariant sequence, each stage stays constant over time, each stage includes all previous stages, and the changing to a next stage involves a cognitive reorganisation or shift.

Thus, Wright et al. (2006) distinguish between stages and levels in their framework. For them, a stage is the same as what it is for Steffe and Cobb (1988), and a subsequent stage involves a cognitive shift, while the next level requires a learner to satisfy specific performance criteria for that level.

Wright et al. (2006) describe the development and increasing sophistication of number in a framework called the Learning Framework in Number (LFIN), divided into four parts.

Part A entails early arithmetic strategies and base-ten understandings. Part B explores Forward Number Word Sequence (FNWS) and Backward Number Word Sequence (BNWS) counting skills and Numeral Identification (NI); Part C entails other aspects of Early Arithmetical learning like base-5, finger patterns, and subitizing; Part D entails early multiplication and division. The LFIN Framework can be seen in Table 2.6.

Table 2.6 Learning Framework I Number (LFIN)

PART A	**PART B**	**PART C**	**PART D**
Early arithmetical strategies and base-ten arithmetical strategies	**Forward (FNWS) and Backward (BNWS) Number Word sequence and Numeral Identification**	**Other aspects of Early Arithmetical Learning**	**Early Multiplication and Division**
a. SEAL Stages which include Addition and Subtraction	a. FNWS	Finger patterns	Initial grouping

b. Base-ten knowledge	b. BNWS	Spatial patterns and subitizing	Perceptual counting in groups
	c. Numeral Identification	Combining and Partitioning	Figurative composite grouping
		Base-five strategies	Repeated abstract composite grouping
			Multiplication and division as operations

(Wright, Martland, & Stafford, 2006, p. 20)

Wright et al. (ibid) view the skills tested in parts B, C, and D as feeding into the Stages of Early Arithmetical Learning (SEAL), which is the central component for identifying increasing sophistication with number in their model. A high SEAL stage indicates more sophisticated strategies than the lower stages of SEAL.

There are 11 dimensions to the LFIN. Wright et al. (2006) argue that these aspects should not be seen as various topics in early number, which are widely separated, but that they are tightly integrated. They are:

a) The Stages of Early Arithmetical Learning (SEAL)
b) Forward Number Word Sequence and Number Word After (FNWS)
c) Backward Number Word Sequence and Number Word Before (BNWS)
d) Numeral Identification (NI)
e) Base-ten Arithmetical Strategies (Base-ten)
f) Combining and Partitioning
g) Spatial Patterns and Subitizing
h) Temporal Sequences
i) Finger Patterns
j) Base Five Strategies
k) Multiplicative reasoning

(Wright, Martland, & Stafford, 2006, p. 17)

In this PhD study, I included only the first five aspects of the LFIN because of the curriculum of what a Grade 1 learner should know at the end of the year. Furthermore, tests 1.1, 2.1, and 1.2 only draw on the first 5 aspects. The aspects of Part A are: The Stages of Early Arithmetical Learning (SEAL) and Base-ten Arithmetical Strategies (Base-ten). Part B is: Forward Number Word Sequence and Number Word After (FNWS); Backward Number Word Sequence and Number Word Before (BNWS); and Numeral Identification (NI). A description of each aspect follows next.

2.6.5.1 Stages of Early Arithmetical Learning (SEAL)

Wright et al. (2006) regard the SEAL stages as hierarchical, where each successive stage is more sophisticated than the previous one. A learner progresses from one the next. Wright et al. name the different hierarchies of SEAL 'stages'; however, they name the various hierarchies of FNWS, BNWS, IN and base-ten, 'levels'. According to Wright et al., the reason for doing so is that the FNWS, BNWS, IN and Base-ten Strategies feed into the SEAL stages and provide information about the SEAL stage of a learner. Therefore, you can find the SEAL stage in which the learner is after coding FNWS, BNWS, IN, and Base-ten Strategies. Wright et al. (2006) emphasise that moving between the SEAL stages involves substantial growth, and hence these moves take significant time to achieve.

The focus is not only on whether the learner can find the answer to a task but also on how sophisticated the strategy deployed by the learner is. There are six SEAL stages: emergent; perceptual; figurative-counting; initial number sequence; intermediate number sequence; and facile number sequence. Table 2.6 shows the stages and descriptions for each stage.

Table 2.7 The Stages of Arithmetical Learning (SEAL) (Wright et al., 2006, p. 9)

Stages of Early Arithmetical Learning (SEAL)		
Stage number and name		Description
Stage 0	Emergent counting	The child cannot count visible items. She either does not know the number words or cannot coordinate the number words one-by-one when moving the items.
Stage 1	Perceptual counting	The child can count perceived items but not those that are screened (concealed).
Stage 2	Figurative counting	The child can count concealed items, but counts from one; does not count-on or count-down to solve addition and subtraction questions.

Stage 3	Initial number sequence	The child uses counting-on strategies rather than counting from one to solve addition questions. She can count-down-from to solve removed items tasks (take away strategies), but cannot count-down-to to solve missing subtrahend tasks (seeing the difference strategies).
Stage 4	Intermediate number sequence	The child can count down to solve removed items (taking away) and missing subtrahend tasks (seeing the difference). Child can choose the more efficient of count-down-to or count-down-from strategies. The child can distinguish between the 'take away' and 'difference' idea in subtraction and count-up or down accordingly.
Stage 5	Facile number sequence	Child uses a range of strategies that are referred to as non-count-by-ones strategies. In additive and subtractive situations, the child uses strategies like compensation, using a known fact, bridging through ten, knowing inverses, commutativity and knowing 10 as an abstract composite number.

(Wright, Martland, & Stafford, 2006, p. 22)

Wright et al. (2006) argue that when a learner is at Stage 0 of SEAL, she cannot say the number words in the correct order or link the 'moving' of an object with 'saying a number word.' Alternatively, she cannot determine which items/counters were already counted and which ones have not been counted.

Stage 1 demonstrates the ability to count correctly seen and felt objects. A learner on this stage is aware of the one-to-one correspondence between number words and the moving items and knows which items are already counted and which ones are not yet counted (Gelman & Gallistel, 1986).

Stage 2 refers to being able to count objects that are screened, usually by being able to associate the number heard with that number of fingers or tally marks. For example, a learner is told how many items there are in two collections of screened objects and asked how many there are altogether. A learner at stage 2 will give a correct answer but will count each collection and the sum from 1 on her fingers. Anghileri (2007) refers to this approach as the triple count strategy. For example: if two screened collections of 5 and 4 counters are presented to the learner, the learner would count out 1 to 5 on the one hand and 1 to 4 on the other hand and then count them all from 1 to 9.

A learner at stage 3 can count-on from the first number. Therefore, when a learner at this stage is presented with $7 + 4$ (screened objects), the learner would say $8, 9, 10, 11$ while keeping track of the 4 on her fingers. This stage also indicates the ability to count backward from the first number to solve removed item tasks. For example for: $17 - 3$, the learner would start with $17\ and\ say\ 16, 15, and\ 14$.

At stage 4, the learner demonstrates the ability to count-down to solve removed items (taking away $17 - 3$) and counting down to solve missing subtrahend tasks (seeing the difference in $17 - 14$). She can choose the most efficient of count-down-to ($17 - 3$) or count-down-from ($17 - 14$) strategies. The learner can distinguish between the 'take away' and 'difference' idea in subtraction. For example, in the case of $17 - 14$, the learner is aware of the relationship between 17 and 14 and would not count down from 17 to 3. The learner at this stage will know to count down from 17 to 14, or she could count up from 14 to 17 to find the answer of 3.

A learner who is functioning at stage 5 of the SEAL stages has a range of sophisticated strategies to solve arithmetical tasks, including bridging through ten, compensation and equivalence notions. Wright et al. (2006) emphasise that moving between the SEAL stages involves substantial growth, and hence these moves take significant time to achieve.

2.6.5.2 Base-ten arithmetical learning

The second aspect of Part A is base-ten arithmetical strategies. According to Wright et al. (2006), base-ten arithmetical strategies refer to the learner's ability to see 10 as a composite unit together with 10 as 10 units. Table 2.8 describes the levels relating to base-ten arithmetical strategies.

Table 2.8 Base-ten arithmetical strategies

Base-ten arithmetical strategies		
Name of the level		Description
Level 1	Initial concept of ten	The child does not see 10 as a unit but focuses on the 10 items. A child can already count forward and backward from a number on this level. That means that a child on this level needs to be on stage 3 of the SEAL stages.
Level 2	Intermediate concept of ten	This child sees 10 as a unit composed of 10 ones when concrete materials like 10 dot strips are presented to her.
Level 3	Facile concept of ten	The child can solve addition and subtraction tasks involving tens and ones without concrete materials like dot strips (Wright et al., 2010, p. 9).

(Wright, Martland, & Stafford, 2006, p. 22)

The SEAL stage descriptions link to the idea of structural understanding and will be discussed extensively in Chapter 5, where I will discuss the extent to which the assessments, including the LFIN, measure structural understanding.

Part B consists of Forward Number Word Sequence (FNWS), Backward Number Word Sequence (BNWS) and Numeral Identification.

2.6.5.3 Forward Number Word Sequence (FNWS)

FNWS measures the ability to say the number word sequence in the correct order and to be able to count-on from any number; for example, to count-on from 48. This aspect includes the ability to say the number word after (NWA) a given number. The table below illustrates the levels and how the levels are awarded.

Table 2.9 Forward Number Word Sequence and Number Word After

Forward number Word Sequence (FNWS) and Number Word After (NWA)		
Name of the level		Description
Level 0	Emergent FNWS	The child cannot produce the FNWS from 1 to 10.
Level 1	Initial FNWS up to 10	The child can produce the FNWS from 1 to 10 but cannot produce the number just after a given number.
Level 2	Intermediate FNWS up to 10	The child can produce the FNWS from 1 to 10 and can produce the number word just after a given number word, but drops back to 1 when doing so.
Level 3	Facile with FNWS up to 10	The child can produce the FNWS from 1 to 10 and can produce the number word just after a given number word without dropping back to 1.
Level 4	Facile with FNWS up to 30	The child can produce the FNWS from 1 to 30 and can produce the number word just after a given number word without dropping back to 1.
Level 5	Facile with FNWS up to 100	The child can produce the FNWS from 1 to 100 and can produce the number word just after a given number word without dropping back to 1.

(Wright, Martland, & Stafford, 2006, p. 23)

For Levels 1, 2, and 3, the number range is the same ($\mathbf{1-10}$). The learner in Levels 1, 2, and 3 can say the number words from 1-10 in the correct order. They differ according to whether hey can say the NWA, and in the strategy they use to find the NWA. In Level 1, the learner cannot say the NWA, but can say the NWA in Levels 2 and 3. The learner operating at level 3 can say the

NWA immediately; however, the learner performing at Level 2 has to drop back to 1 to say the NWA. The next aspect is backward-counting.

2.5.6.4 Backward Number Word Sequence (BNWS)

The different levels of BNWS demonstrate the learner's ability to say the number words in a backward order from any number. It also indicates if a learner can say the Number Word Before (NWB) for any number. Table 2.10 shows the way in which the levels are awarded:

Table 2.10 Backward Number Word Sequence and Number Word Before.

Backward Number Word Sequence (BNWS) and Number Word Before (NWB)		
Name of the Level		Description
0	Emergent BNWS	The child cannot produce the BNWS from 10 to 1.
1	Initial BNWS up to 10	The child can produce the BNWS from 10 to 1 but cannot produce the word just before a given number.
2	Intermediate BNWS up to 10	The child can produce the BNWS from 10 to 1 and produce the word just before a given number word but drops back to 1 when doing so.
3	Facile with BNWS up to 10	The child can produce the BNWS from 10 to 1 and produce the word just before a given number word in the range of 10 to 1 without dropping back to 1.
4	Facile with BNWS up to 30	The child can produce the BNWS from 30 to 1 and produce the word just before a given number word in the range of 30 to 1 without dropping back to 1.
5	Facile with BNWS up to 100	The child can produce the BNWS from 100 to 1 and produce the word just before a given number word in the range of 100 to 1 without dropping back to 1 (Wright et al., 2010, p. 11).

(Wright, Martland, & Stafford, 2006, p. 24)

5.6.5.5 Numeral Identification (NI)

The last aspect of Part B is Numeral Identification. This aspect shows the range in which a learner can identify numerals. Table 2.11 details the coding.

Table 2.11 Numeral Identification (IN)

Numeral Identification (NI)

Name of Level		Description.
Level 0	Emergent NI	The child cannot identify some or all numerals in the range of 1 to 10.
Level 1	NI up to 10	The child can identify numerals in the range of 1 to 10.
Level 2	NI up to 20	The child can identify numerals in the range of 1 to 20.
Level 3	NI up to 100	The child can identify one- and two-digit numerals.
Level 4	NI up to 1 000	The child can identify one-, two- and three-digit numerals.

(Wright, Martland, & Stafford, 2006, p. 24)

Wright et al. (2006) distinguish between numeral identification and numeral recognition. Numeral identification refers to saying the number word when a card with a number symbol is presented. Wright et al. (2006) describe number recognition as the ability to recognise the symbol when presented with a collection of cards and being asked: "Show me the 6." In this case, the learner does not say the word but has to point it out.

The results of the LFIN interviews are qualitative. To be able to compare the results quantitatively (in Chapter 6), I quantified the responses of the LFIN interviews. Quantification of qualitative data involves turning the qualitative data from responses to numbers (Ward, 2010). In a personal discussion with Bob Wright, he suggested that the importance of the SEAL aspect within the LFIN could carry a more substantial (double or more) weighting when quantifying these qualitative levels and stages. Given that the base-ten knowledge aspect depends on structural understanding, I doubled the base-ten aspect as well in my study. To quantify or to put a mark on the qualitative responses, I doubled the SEAL stages and base-ten knowledge mark and added all the other marks together to get a mark out of 30. See table 2.12 for the way in which the marks were allocated.

Table 2.12 Allocation of marks for the five aspects in the Learning Framework in Number (LFIN)

Learner	Marks
The five Aspects	
1. Stage of Early Arithmetical Learning (SEAL)	Out of 10 (5 stages x2)
2. Level of Forward Number Word Sequence (FNWS)	Out of 5 (5 levels)
3. Level of Backward Number Word Sequence (BNWS)	Out of 5 (5 levels)
4. Level of numeral identification	Out of 4 (4 levels)
5. Level of base-ten knowledge (Wright, Martland, & Stafford, 2006)	Out of 6 (3 levels x2)
	Total 30

(Wright, Martland, & Stafford, 2006)

Summary of the Learning Framework in Number (LFIN)

The LFIN test is a useful instrument for investigating predictive power. It covers a range of aspects in early number, and the particular mathematical skills that literature suggests are good predictors of later performance. The LFIN tests are structured to measure the extents of number range in counting and the strategic sophistication of strategy used.

When looking at the predictive power in literature and the four longitudinal studies, it is clear that a higher counting number range and the use of more sophisticated strategies, including advanced cognitive mathematical skills, are widely seen as related to better mathematics performance in later years. The four longitudinal studies include counting predictors like concrete counting one-by-one from (Claessens & Engel, 2013; Manfra et al., 2012), counting-on, counting-backward, count as far as possible, count-on from a number with a number (Aunola et al., 2004) and counting on from the bigger number in simple addition tasks (Geary et al., 2013). Worth mentioning is that the LFIN assessment tests that I used in my study did not include counting from a bigger number and counting-on from a number with a number. There was an aspect that the LFIN assessments included, which was not included in the other four longitudinal studies, which was the skill of saying the Number Word After (NWA) and the Number Word Before (NWB).

2.6 Predictive power of structural thinking

As argued before, mathematics is an emergent process where mathematical understanding in the early years provides a foundation for later mathematical knowledge (Geary, Hoard, Nugent, & Bailey, 2013). For example, high mathematical performance in the early primary school years predicts academic achievement in later years (Jimerson et al., 1999).

Some literature highlights the predictive power of general school-entry number knowledge skills but does not specify the specific skills (Claessens & Engel, 2013). All the studies I reviewed have aimed to address this gap, where the researchers investigated specific early mathematical skills.

I am using the studies of Claessens and Engel (2013), Geary and colleagues (2013), Manfra et al. (2012), and Aunola et al. (2004). The rationale for choosing these four studies is that all four of them focus on counting strategies as predictors, and two of them elaborate by including other advanced cognitive demanding skills as predictors (Claessens & Engel, 2013; Geary et al., 2013).

The specific mathematical skills the studies examined helped me design the Grade 6 Evidence of Structural thinking Assessment (G6 ESTA) (the assessment tool that I developed for this study, which I will discuss later[18]).

2.6.1 The differences and similarities between the four studies

I discuss the four studies' methodology in terms of who the participants were and where the research was undertaken, and the duration of the study.

As mentioned above, some literature highlights the predictive power of general school-entry number knowledge skills but does not explicitly identify the specific skills (Claessens & Engel, 2013). All the studies I reviewed aim to address this gap, in which the researchers investigated specific early mathematical skills.

Because my thesis investigates evidence of counting and structural thinking in participants at the beginning of Grade 2 and again at the end of Grade 6, it is necessary to investigate other longitudinal studies that report on research that includes different counting skills as predictors and more advanced number knowledge skills as counting alone. The LFIN assessment (R J Wright, J Martland, A K Stafford, et al., 2006), one of my assessment tools, includes counting strategies that learners use and other more advanced cognitive skills like base-ten thinking and Stages of Early Arithmetical Learning (SEAL).

Firstly, I discuss two short-term longitudinal studies interested in the associations between counting abilities and later performance in Grade 1 and 2. Secondly, I discuss the two long-term longitudinal studies. All four studies' starting point are the participants' kindergarten years.

The two short-term longitudinal studies I refer to are those of Manfra et al. (2012) and Aunola (2002). Both of them focus on counting abilities. Manfra et al. (2012) explored, in a short-term longitudinal study over just less than three years, the effects of counting objects and reciting number words at the beginning of kindergarten, and the mathematical performance at the end of Grade 2 of 3 247 of ethnically diverse, low-income learners in the USA.

[18] The discussion is in 3.5.4.

Aunola et al. (2004) examined the developmental dynamics of mathematics performance from preschool to Grade 2 of 194 Finnish learners. The study aimed to determine how learners' cognitive antecedents[19] were configured and how their mathematics performance[20] developed.

Both Manfra et al (2012) and Aunola et al (2004) found associations between counting abilities in kindergarten and later performance in Grade 1 and Grade 2. Manfra et al's (2012) results suggest that it is important to consider both reciting and counting-based items in early assessment. Counting can be seen as the bringing together of both an organised linguistic system and a system that is based on the one-to-one principle involving the tracking of concrete objects. Aunola et al.'s (2004) results demonstrate that the development of mathematical performance during pre-school and the first two years of primary school showed a cumulative pattern and that individual differences were stable. Learners with high marks early on tended to have high marks later. However, the results also suggested that individual differences in mathematical performance grow larger over time, backing other research findings.

Now I turn my attention to the two long-term longitudinal studies, namely the studies of Claessens and Engel (2013) and Geary et al. (2013)

Claessens and Engel (2013) questioned the importance of learners' mathematical understanding when they start school. The study participants were a demographically representative group of 7 655 kindergarten learners in the USA. They were tested in their kindergarten years and then every year up to Grade 8. The researchers added language, general knowledge and science tests to the mathematics tests. They aimed to explore how early mathematics skills related to later science, language, and mathematics achievement. I will only report on the mathematics results.

Claessens & Engels (ibid) noted that they were interested in "the examination of growth over time" (p. 7) in all areas. For successive years, they created tasks and nine mathematics proficiency levels

[19] For Aunola et al. (2004), cognitive antecedents include counting ability, visual attention, metacognitive knowledge and listening comprehension.
[20] For Aunola et al., mathematical performance includes knowledge of ordinal numbers, cardinal numbers, number identification, word problems and basic arithmetic, which includes addition, subtraction, multiplication, and division tasks.

to span the achievement from kindergarten to Grade 8. The mathematics tests were conducted in kindergarten, in the first year, the third year, the fifth year, and the eighth year of school.

Three proficiency levels were commonly seen in the kindergarten year dataset in their study. The three levels that were visible in the kindergarten years were described as follows: Proficiency level 1: reading all one-digit numerals, counting beyond 10, recognising a sequence of patterns, using non-standard units to compare lengths. Proficiency level 2: reading two-digit numerals, recognising the next number in a sequence, identifying the ordinal position of an object, solving a simple word problem. Proficiency Level 3: solving simple addition and subtraction problems.

Geary et al. studied Counting and Number System Knowledge as predictors. They tested learners in the USA during their kindergarten years and then again when adolescents in Grade 7. Their focus was on how counting and early number system knowledge influenced performance based on a ‘functional numeracy test’ that they used to measure young adults’ employability.

The kindergarten assessments of Geary et al. consisted of multiple assessments and included addition strategy choice, number sets and number line estimation.

They used addition strategy choice to measure the learners’ counting ability in kindergarten and classified learners’ responses based on their strategies. For example, Sum (or triple count) was used to describe responses where a learner, if asked to solve 4+5, ‘counts all’, meaning a count from 1 to 4 and then from 1 to 5 and then from 1 to 9 (Anghileri, 2006). Max (start with the smaller first number and count-on with the second larger number) was used to describe responses involving counting on from 4 to 9 for 4+5. Min (start with the larger number and count on with the smaller number) was used to describe responses involving counting-on from the larger number (five in the case of 4 + 5) and thus involves awareness of commutativity of addition and awareness of number relationships. Number Line Accuracy was used to measure accuracy in the placement of numerals on a number line. Number Sets Fluency indicated a learner’s ability to observe relations between cards with domino-like rectangles featuring different combinations of numerals and objects. Learners were required to perform quick addition by ‘seeing the sum,’ thus providing a check of the subitizing skills (perceiving small number quantities immediately without counting).

The Functional Numeracy test was administered in the learners' Grade 7 year and included arithmetical word problems, computational arithmetic (which included tasks like $12 + 42 + 53;\ 83 - 57;\ 85 \times 6\ and\ 728 \div 8$), fraction computational tasks (like $2\frac{1}{4} + \frac{1}{4};\ \frac{1}{4} \times \frac{1}{8};\ \frac{1}{3} \div \frac{1}{6}$) and fractions comparison tasks (for example: circle the largest number in $\frac{5}{6}\ or\ \frac{6}{5}$).

2.6.2 Findings across the four studies

The findings across the above studies are that concrete counting one-by-one (Claessens & Engel, 2013; Manfra et al., 2012) (Claessens & Engel, 2013; counting-on, counting-backward, counting as far as possible (Aunola et al., 2004), and counting on from the bigger number (Geary et al., 2013) were the best predictors of later mathematical performance in the short term. However, Claessens and Engel's (2013) and Geary et al's (2013) findings were that counting in terms of counting one-by-one and counting-on from the largest number in simple addition tasks were not good predictors of later mathematical performance in the long term.

The studies of Claessens and Engel (2013) and Geary et al. (2013) show that over the long term (between 7 and 8 years), other specific mathematical skills that do not include unit counting are better predictors of later mathematical success. These skills include reading two-digit numerals, recognising the next number in a sequence, identifying the ordinal position of an object, and solving simple word problems. It also includes understanding the magnitude of numbers, ordering of numbers, knowing number combinations, decomposition to solve problems, number line accuracy and observing links between numerals and pictures of sets.

As illustrated from this overview, a critical aspect of early number knowledge and skills is that counting is an essential starting point for developing number knowledge and is vital as a base for further development. However, a more abstract and structural understanding was a better predictor of later mathematical understanding. A common emphasis across these studies was on measuring the extent and sophistication of counting ability – both of which were found to be good predictors of later mathematical performance.

What is of interest is that Aunola et al. found that the development of mathematical performance during preschool and the first two years of primary school indicated a cumulative pattern. This supports the literature, which asserts that mathematical understanding is a developmental process.

They also found that individual differences were more or less stable, where learners with high marks early on tended to have high marks later. However, the results also suggested that individual differences in mathematical performance grow larger over time, confirming other research findings, which means that the gap between learners with a backlog and those with an advanced understanding of mathematics in kindergarten grows bigger over time.

2.7 Conclusion

The research literature informed me what structural thinking consists of. It made me attentive to the importance of the awareness of relationships in structural thinking. Knowledge of relationships entails knowledge of number relationships and knowledge of quantities and is interwoven and connected. This review shows the different applications of structural thinking and the importance of the indicators of structural thinking development as seen in the strategies that learners use.

The predictive power of structural thinking is an essential notion in research. The literature on the four longitudinal studies about the predictive power of a set of competencies helped me choose Bob Wright and colleagues' Grade 2 LFIN assessment (2006) as one of my data instruments.

The literature review helped me develop and conduct the Grade 6 Evidence of Structural thinking Assessment (G6 ESTA) as one of my data collection instruments.

CHAPTER 3 : RESEARCH DESIGN

3.1 Introduction

This research investigates the predictive power of structural thinking in the early years. It also confronts learners' difficulties with structural thinking in mathematics, specifically in primary South African schools.

According to Kirshenblatt-Gimblett (2006) the term 'research design' refers to a complete approach that a researcher chooses to integrate the study's various components in an understandable way that addresses and gives solutions to the research problem. Thus, a research design describes how the study plans for acceptable and replicable methods of data collection and sampling choices and discusses responsible and ethical analysis methods (Vogt, Gardner, Haeffele, & Vogt, 2014).

This chapter presents the research questions and discusses the methodology used to answer them. I state who the participants in the sample were and how and why they were selected. I discuss the data collection instruments and the reason I selected them. I then elaborate on the analysis process and the way the responses of the participants were analysed. In the last part of the chapter, I conclude with the reliability and ethics of the study.

3.2 The Research Questions

As mentioned, my study is about the predictive power of structural thinking. The predictive variables are considered in the two early assessments, namely G1 ANAdG2[21] and the G2 LFIN[22]. The predicted variables are G6 CA[23] and the G6 ESTA[24].

The overall research question this study addresses is:

<u>In what ways, if any, does structural thinking in the early years of primary school predict performance in the later years of primary school?</u>

Sub-questions arising from this include:

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[21] Grade 2 Annual National Assessment that was delivered in Grade 2.
[22] Grade 2 Learning Framework in Number assessment.
[23] Grade 6 Common Assessment
[24] Grade 6 Evidence of Structural thinking assessment

1. Is there evidence for a strong correlation between an assessment that measures structural thinking in the early years of school and mathematical performance in the later years of primary school?
2. How important is structural thinking in the early years of school?

These research questions are explored and operationalised by the researcher's questions:

Researcher's questions

<u>In what ways do Grade 2 learners' performance on two tests – the Grade 1 ANA and Wright et al.'s (2006) LFIN assessments – relate to their subsequent Grade 6 performance on two mathematics tests, the Grade 6 Common Assessment and a number-focused interview test (Grade 6 ESTA), aimed at exploring the extent of structural ways of working with number?</u>

In order to answer the main research question, the following sub-questions need to be answered:

1. In qualitative terms, in what ways, and to what extent do the 2011 ANA Grade 1 test, the 2011 LFIN, the Grade 6 Common Assessment and the G6 ESTA assessments measure structural thinking with early number?
2. In quantitative and qualitative terms, what do the empirical findings of the responses of the participants regarding the extent of structural thinking tell us?
3. In quantitative terms, what is the correlation, if any, between learner performance in Grade 2 on the Grade 1 ANAdG2 and the G2 LFIN tests in 2011 and performance in Grade 6 on the G6 CA and G6 ESTA tests in 2015?

3.3 Research Paradigm: Corroborative Mixed Method Research

The study used a Corroborative Mixed Method Research Design (CMMRD) to answer the research questions.

Johnson et al. (2007, p. 123) describe this as follows:

> Mixed methods research is the type of research in which a researcher combines elements of qualitative and quantitative research approaches (e. g., use of qualitative and quantitative

> viewpoints, data collection, analysis, inference techniques) for the broad purposes of breadth and depth of understanding and corroboration.

However, in CMMRD, quantitative and qualitative research methods serve each other for a broader purpose. It is not only the putting together of two research methods (Johnson & Onwuegbuzie, 2004). I will elaborate on the latter later in this chapter.

3.3.1 Justification for a Corroborative Mixed-Method Research approach

Purists of quantitative and qualitative research claim that qualitative and quantitative paradigms are incompatible (K. R. Howe, 2012). However, Barmby et al. (Barmbly, Bolden, & Raine, 2014) claim that a study can accommodate both paradigms. Johnson and Onwuegbuzie (2004) refer to this approach as a Mixed-Method Research Design.

The mixed-method design refers to a method where you do not choose between quantitative and qualitative but learn from each approach's strengths as they complement one another and minimise any flaws that quantitative and qualitative research may have on their own (Creswell, 2012; Johnson & Onwuegbuzie, 2004).

Johnson, Onwuegbuzie, and Turner (2007) asked a few academics to define the mixed-method approach. They listed the definitions provided by all the respondents in their paper. Hallie Preskill was one of the academics; her definition is quoted in Johnson et al. (2007) as:

> "Mixed-methods research acknowledges that all methods have inherent biases and weaknesses; and that using a mixed-method approach increases the likelihood that the sum of the data collected will be richer, more meaningful, and ultimately more useful in answering the research questions." (p.121)

A mixed-method research design gives us a fuller picture and different perspectives on the research. Using mixed methods can support the validity and reliability of the findings (Johnson & Onwuegbuzie, 2004).

3.3.2 The Corroborative Mixed-Method Research Design

Corroborative Mixed Method research is not only putting together quantitative and qualitative research in one study; quantitative and qualitative research corroborate and complement each other. As Lyn Shulha, in a paper by Johnson et al. (2007), argues, quantitative and qualitative research 'serve' each other in corroborative mixed-method research design. Neither research method can be done without the other.

In this study, qualitative and quantitative research methods were 'mixed' as follows: I undertook a qualitative analysis of the results of the interview data, quantified the results, and then did a statistical analysis of the quantified results. The statistical analysis was impossible without the initial qualitative analysis of the interview data.

In my study, I specifically undertook a qualitative analysis of the G2 LFIN 'interview schedules', using the framework of Wright et al (2006) to find the levels and stages[25] of each participant. Since the levels and stages are sequential, I used them to quantify the results.

I also analysed the G6 ESTA interview assessment qualitatively using the framework I developed[26]. I used the qualitative results of the G6 ESTA and quantified the data. (The quantification process of the data for both the G2 LFIN and the G6 ESTA follows later in the chapter.)

I then used the quantified data to complete the quantitative research in the form of statistical analysis (section 6.2).

3.4 The participants

In February/March 2011, I was part of the Wits Maths Connect Primary Project team (WMC-PP) that collected the first set of data in 10 schools in Gauteng. The participants were from five suburban schools and five township schools. Each school was asked to identify two top achievers in mathematics, two middle performers, and two weak performers in each Grade 2 class in the

[25] See the difference between stages and levels according to Wright et al (2006) in section 3.5.3.
[26] In Chapter 4 section 4.7, I discuss the framework for the G6 ESTA.

school. This added up to 86 participants from the suburban[27] schools and 156 participants from the township[28] schools. However on the day that we went to a specific school to record the videos, some learners were absent and we did not go back to the same school to catch up. Therefore at times we did not manage to video-record the full contingent of 6 learners per class.

The townships schools had more Grade 2 classes and more learners per class than the suburban schools, hence the larger number of participants from township schools. This situation is common in Gauteng (Ginsburg, Richter, Fleisch, & Norris, 2011).

At that time, I was busy with my master's degree and was part of the data collection team. We collected two sets of data in February/March 2011 of 242 Grade 2 participants. The one data set was based on a video-recorded interview assessment, developed by Bob Wright and colleagues (Wright, Martland, & Stafford, 2006), called the Learning Framework in Number (LFIN) assessment.

The other set of data was the South African Annual National Assessment (ANA) results for Grade 1 learners (G1 ANA). The participants wrote the Grade 1 ANA at the beginning of 2011 when they were in Grade 2. The latter was due to delayed administrative arrangements within the Department of Basic Education (DBE). I will refer to this assessment as the G1 ANA delivered in G2 (G1 ANAdG2)

In 2015, I conducted and collected another two follow-up mathematics assessments with the same sample of learners. By 2015, these learners were in Grade 6. Unfortunately, only 64 of the original sample of 242 learners were still in the same schools as they were in 2011 and could be selected as participants in the assessments.

This high dropout, school migration and or grade retention rate between Grades 2 and 6 was not a surprise as other South African studies also report high rates on this factor (Ginsburg, Richter, Fleisch, & Norris, 2011). Studies have shown that residential mobility patterns in South Africa's township areas are influenced by temporary labour needs that frequently change (Townsend,

[27] A location in the outer edges of a metropolitan city.
[28] A township in South Africa is a place or city of predominantly black occupation, designated for black occupation by the apartheid legislation (Oxford dictionary).

Madhavan, Tollman, Garenne, & Kahn, 2002). Learners change school as their parents migrate to find new jobs.

The 64 learners participating in the assessments attended eight Gauteng province schools; 26 of the learners were from four suburban schools, and 38 were from four township schools. In 2015 we collected data from 10 schools; however, when I went back to track the learners, I could only find learners in eight schools. One school withdrew from the study, and no learner could be found agter five years later in another school.

As stated earlier, there were two phases in the data analysis processes in the study. This is because of the two data collection timeframes. The data were collected in February 2011, when the participants were in Grade 2, and November 2015, when they were in Grade 6.

In February 2011, as WMC-P project members, we collected the G1 ANAdG2 and recorded all the learners' scores. We also conducted the LFIN interviews and recorded the quantitative results.

At the end of 2015, I conducted the Grade 6 ESTA with 64 learners. Only 64 learners were still at the same schools as in February 2011. I also collected the marked scripts of the G6 CA for the same cohort of 64 learners. After collecting the scripts of the G6 CA, I recorded the results for each learner on each task in an Excel sheet. I also recorded all the scores of the G2 LFIN, G1 ANAdG2, and the G6 ESTA on the same sheet. Table 3.1 summarises the sample at each point of data collection.

Table 3.1 Summary of the sample at each point of data collection

The year	**February 2011**		**November 2015**	
Assessment	G1 ANAdG2	G2 LFIN	G6 CA	G6 ESTA
Participants in townships schools	38	38	38	38
Participants in suburban schools	26	26	26	26

This study is extensive, using the data of 64 Grade 2 learners and data from the same cohort in Grade 6. Some of the 64 participants were from townships and some from suburban schools. I only disaggregated the data when I thought it would contribute to my investigation of the extent of structural thinking. I rationalised in each instance why I thought it would be necessary to disaggregate the data between the township and suburban schools when doing so.

3.5 Data collection instruments

Four data collection instruments were used in my study. Since my research is about the predictive power of early assessments, I have predictor and predicted assessments. They were the G1 ANAdG2 assessments and G2 LFIN, when the participants were in Grade 2 as the study's predictor assessments. The G6 CA assessments and G6 ESTA, served as as the study's predicted assessments.

3.5.1 The Grade 1 Annual National Assessment, delivered in Grade 2 (G1 ANAdG2) in 2011

A systemic Annual National Assessment (ANA) was introduced in South Africa in 2010. The assessments were curriculum-orientated, and they were conducted in all government schools in South Africa. Grades 1-6 and Grade 9 learners participated.

3.5.1.1 The foundation phase curriculum and the Grade 1 Annual National Assessment

The motivation for conducting the assessment was to monitor and improve educational outcomes in the schooling system. It aimed to achieve the following: 1) expose teachers to better assessment practices. 2) enable districts to prioritise schools requiring assistance. 3) encourage schools to celebrate outstanding performance, and 4) empower parents with important information about their children's performance (DBE, 2011b).

The ANAs was cancelled in 2015 due to pressure from South African teacher unions. Because of the cancellation of the ANAs, I used the Grade 6 Common Assessment, conducted by the DBE at schools in 2015, as one of my predicted assessments.

The G1 ANAdG2 was conducted orally. The teacher read the questions and the learners wrote answers in their scripts. The G1 ANAdG2 focused on correct answers, and no marks were allocated to the sophistication of strategy used.

According to the department directives (Government-Gazette, 2010), the teachers marked the assessments internally using a memo provided by the DBE. No observations were made of the learners' actions, processes, or methods. No attention was given to the learners' use of sophisticated strategies, their demonstration of structural understanding of numbers, or use of unit-counting. All correct answers were merely rewarded with the same marks.

Grades 1-3 make up the Foundation Phase of schooling in South Africa. The G1 ANAdG2 covered the content areas Numbers, Operations and Relationships, and Space and Shape. In the National Curriculum Statement (NCS), the Curriculum and Assessment Policy Statement (CAPS) gives the following weightings (illustrated in Table 3.1) for mathematics content areas in the Foundation Phase.

Table 3.2 Weighting of content in the Grade 1 curriculum

WEIGHTING OF CONTENT AREAS for Grade 1			
Content area	Grade 1	Grade 2	Grade 3
Numbers, Operations and Relationships	65%	60%	58%
Patterns, Functions and Algebra	10%	10%	10%
Space and Shape	11%	13%	13%
Measurement	9%	12%	14%
Data handling	5%	5%	5%
Total	100%	100%	100%

(DBE, 2011a, p. 10)

Table 3.2 shows that numbers, operations and relationships in Grade 1 weigh the most, with 65%, weighing less in Grades 2 and 3, with 60% and 58% respectively. The high weighting of numbers, operations and relationships in Grade 1 suggests that most of the time should be used for number, operations and relationships; however, 35% of the time should be used for patterns and algebra, space and shape, measurement and data handling.

Together with Table 3.2 comes Table 3.3, which stipulates the content knowledge together with number ranges for the specific content that learners should possess at the end of Grade 1.

Table 3.3: Number range for Grade 1

Content	Number range
1. Counting physical items	0-50
2. Count forward and backward in ones	0-100
3. Count forward in 10s, 5s, 2s	0-100
4. Describe, compare, and order numbers	0-20
5. Place value	11-19
6. Solve problems in context and context-free problems in + and -, repeated + leading to ×, grouping and sharing leading to ÷	0-20
7. Mental mathematics	0-20

(DBE, 2011a, p. 10)

The number range for reciting (Aunola et al., 2004) forward and backward in ones is 0-100 and also 0-100 for counting forward in tens. A Grade 1 learner should be able to count physical items from 0-50. Counting physical items is more challenging than reciting forward and backward and, therefore, the lower number range of 0-50. It is more challenging to count physical items than to just say the number words in the correct order. Counting physical items includes awareness of the one-to-one correspondence where the learner knows that moving the item and saying the number word should happen at the same time. It also includes the awareness of knowing which items were already counted and which items still need to be counted (Gelman & Gallistel, 1986).

A Grade 1 learner should be able to describe, compare and order numbers, solve context-free problems and do mental mathematics in a number range of 0-20. Knowledge of place value from 11-20 should also be established in Grade 1.

With the information above about the content and the number-ranges of the content that a Grade 1 learner should know at the end of the year, I now present the tasks of the G1 ANAdG2.

3.5.1.2 The tasks of the grade 1 Annual National Assessment Delivered in Grade 2

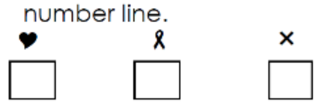
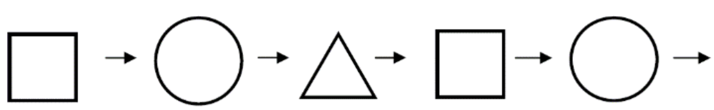
Although most of the content areas use numbers to solve mathematical problems, I distinguish between number-related and not-only-number-related content. The number-related content is numbers, operations and relationships, patterns and algebra, and measurement, and the not-only-number-related content is space and shape and data handing.

Table 3.4 shows the questions of the G1 ANAdG2. Nine questions were number-related tasks (NRT), and the last two questions focused on shapes and were not-only-number-related tasks

(NONRT). The range and the scope of the tasks are described in bold in brackets. The number range was 1-30 (excluding question 1.2, where knowledge counting in 10 to 100 was required), and most questions could, therefore, be answered by doing unit-counting. Table 3.4 shows the NRT and the NONRT.

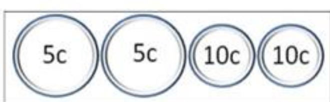
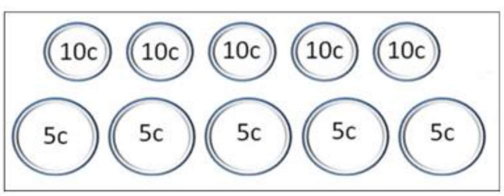
Table 3.4: The tasks of the G1 ANAdG2

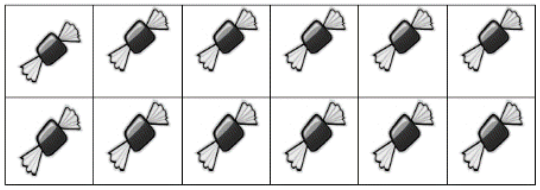
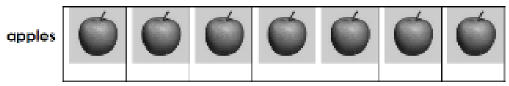
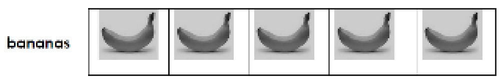
<table>
<tr><td></td><td>The tasks of the G1 ANAdG2 according to NRT and NONRT</td><td>Score</td></tr>
<tr><td>1</td><td>Fill in the missing numbers (NRT)
1.1 3; 4; __;6; 7; __;9; __;11 (Unit-counting task, range 1-10)
1.2 20; __;40; 50; 60; __; __; 90; __(Counting in 10 task, range 0-100)</td><td>2</td></tr>
<tr><td>2</td><td>Complete this table by filling in the blank spaces. (NRT) (Unit-counting, number symbols, number words, cardinality of number, range 1-10)
<table>
<tr><td>Shapes</td><td>Total number</td><td>Number in words</td></tr>
<tr><td>ΔΔΔΔΔΔΔΔΔ</td><td>9</td><td></td></tr>
<tr><td></td><td>7</td><td>seven</td></tr>
</table></td><td>2</td></tr>
<tr><td>3</td><td>What is the position of these pictures on the number line? The position of the first picture is given. (NRT) (Unit-counting, ordinality of number, range 1-10)

3.1 The ✂ is in position ________________.
3.2 Tick ☑ the box below the picture that is in position 6 on the number line.
</td><td>2</td></tr>
<tr><td>4</td><td>Extend this pattern by drawing the shape that comes next (repeated pattern using geometric shapes) (NONRT)
</td><td>1</td></tr>
<tr><td>5</td><td>Write the answer inside the box. (NRT) (Addition and subtraction of two numbers, number symbols, range 1-25)</td><td>2</td></tr>
</table>

<table>
<tr><td></td><td>5.1 20 + 3 = $\square$

5.2 18 – 4 = $\square$</td><td></td></tr>
<tr><td>6</td><td>Fill in missing numbers. An example has been done for you. (NRT) (Doubling and halving, range 1-20)
6.1 Double the given number
<table>
<tr><td>Number</td><td>Double</td></tr>
<tr><td>5</td><td>10</td></tr>
<tr><td>7</td><td></td></tr>
</table>
6.2 Halve the given number
<table>
<tr><td>Number</td><td>Halve</td></tr>
<tr><td>16</td><td>8</td></tr>
<tr><td>20</td><td>10</td></tr>
</table></td><td>2</td></tr>
<tr><td>7</td><td>Write the correct answer in the box. (NRT) (Addition and subtraction of more than two numbers)
7.1 10 + 10 + 10 = $\square$

7.2 10 – 2 – 2 = $\square$</td><td>2</td></tr>
<tr><td>8</td><td>Tick (√) the shape that is a square. (NONRT)
</td><td>1</td></tr>
<tr><td>9</td><td>Look at the 10c and 5c coins in each question. (NRT) (9.1 Additive reasoning and 9.2 multiplicative reasoning and repeated addition)
9.1 The total amount of money is_____. Write your answer in the box.

 $\square$
9.2 I have 4 coins in my pocket and they add up to 25c. Tick (√) the four coins that I have in my pocket.

</td><td>2</td></tr>
</table>

10	Busi and her 2 friends ate 12 sweets. They each ate an equal number of sweets. How many sweets did each one eat? (NRT) (Multiplicative reasoning, sharing task)  Each one ate ____________________ sweets.	2
11	Thabo bought apples and bananas at the shop. (NRT)   Write the correct number of each kind of fruit Thabo bought. He bought: 11.1 ____________ apples. 11.2 ____________ bananas.	2
	Total	20

Ninety percent of the tasks were NRT, and the rest were NONRT. A thorough theoretical analysis of the extent to which the G1 ANAdG2 measures structural thinking will follow in section 4.3.

3.5.2 Grade 6 Common Assessment

In 2015, when the participants were in Grade 6, I collected the Grade 6 Common Assessment (G6 CA) for the matched sample of 64 participants at the different schools.

When the ANAs were suspended in 2015, the Gauteng Department of Education introduced a new test, the G6 CA.

3.5.2.1 The Grade 6 Common Assessment and the Grade 6 curriculum

According to CAPS, the G6 CA is a summative assessment of the knowledge and skills that learners are expected to have developed by the end of Grade 6. Table 3.5 sets out the different content areas and the weighting of the content areas for Grade 6.

Table 3.5 Weighting of the Grade 6 content areas

WEIGHTING OF CONTENT AREAS	
Content area	Grade 6
Numbers, Operations and Relationships	50%
Patterns, Functions and Algebra	10%
Space and Shape	15%
Measurement	15%
Data handling	15%
Total	100%

(DBE, 2011b, p. 10)

Table 3.4 shows different content areas. Although all content in mathematics is number-related, I classify the content areas, (i) Numbers, operations and relationships and (ii) number patterns (number patterns are structural questions, which deals with patterns that use numbers as terms) and algebra and (iii) measurement as 'number-related' content.

I classify content of Data-handling, Space and Shape and Probability as 'not-only-number-related' content.

3.5.2.2 The tasks of the Grade 6 Common Assessment (G6 CA)

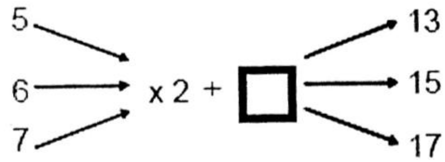
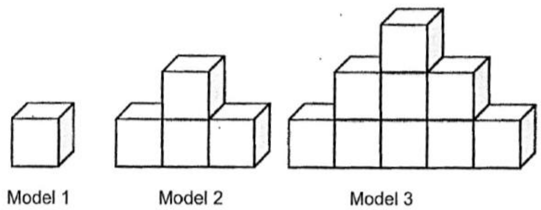
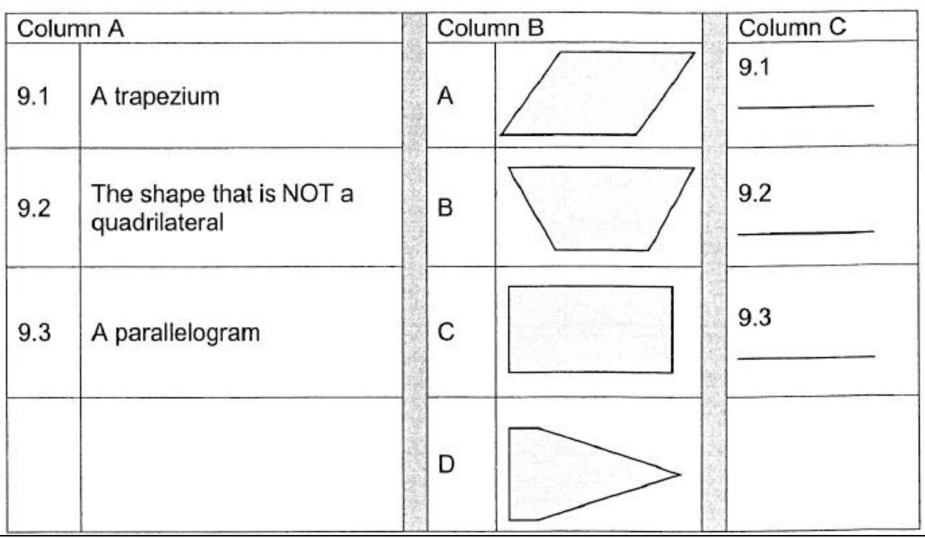
There are 22 tasks in the G6 CA. The total mark of the assessment is 75. Marks were awarded for correct answers only. Thus, no marks were awarded for the strategies used. The teachers marked the scripts using a memorandum that the DBE provided.

Table 3.6 gives a more detailed summary of the tasks of the Grade 6 CA regarding number-related tasks (NRT) and 'not-only-number-related tasks' (NONRT). The bold text indicates if a task is an NRT or NONRT task.

Table 3.6: Tasks (NRT and NONRT) and scoring of the Grade 6 Common Assessment

A summary of the tasks and the scoring of the Grade 6 Common Assessment		
#	The tasks of the Grade 6 Common Assessment	Score
1.1	What is the value of the underlined digit in 6**4**3 979 568? (NRT)	1

<table>
<tr><td>1.2</td><td>Which is the largest number? 231 323 689, 231 230 689, 231 302 689, 231 320 689 (NRT).</td><td>1</td></tr>
<tr><td>1.3</td><td>Which number is the missing number in the number pattern 9,27; 9,25; 9,23; _, 9,19? (NRT).</td><td>1</td></tr>
<tr><td>1.4</td><td>Round 5697 off to nearest 10 (NRT).</td><td>1</td></tr>
<tr><td>1.5</td><td>Which improper fraction is equal to 1½? (NRT).</td><td>1</td></tr>
<tr><td>1.6</td><td>Which number (of 12; 36; 276; 112) is not a multiple of 12? (NRT).</td><td>1</td></tr>
<tr><td>1.7</td><td>How many small cubes fit exactly into the big cube? Small cube sides = 2cm and big cube sides = 6cm. (NRT).</td><td>1</td></tr>
<tr><td>1.8</td><td>Write $(8 \times 1\ 000\ 000\ +\ 5 \times 100\ 000\ +\ 1 \times 10\ +\ 8 \times 1)$ as a whole number (NRT).</td><td>1</td></tr>
<tr><td>1.9</td><td>A picture of a clock is presented. Question: What is the time on the clock? (NRT).</td><td>1</td></tr>
<tr><td>1.10</td><td>A picture of a scale is presented. Question: What is the mass on the scale? (NRT).</td><td>1</td></tr>
<tr><td>2.1</td><td>Calculate: $7\ 122\ 568 + 5\ 722\ 188$ (NRT).</td><td>2</td></tr>
<tr><td>2.2</td><td>Calculate: $4\ 071\ 274 - \ 2\ 128\ 863$ (NRT).</td><td>2</td></tr>
<tr><td>2.3</td><td>4748×36 (NRT).</td><td>3</td></tr>
<tr><td>2.4</td><td>$7695 \div 25$ (NRT).</td><td>3</td></tr>
<tr><td>2.5</td><td>$2\ \frac{6}{7}\ +\ 3\ \frac{1}{14}$ (NRT).</td><td>3</td></tr>
<tr><td>2.6</td><td>$5\frac{7}{8} - 1\ \frac{4}{8}$ (NRT).</td><td>2</td></tr>
<tr><td>2.7</td><td>$45{,}05 - 19{,}21$ (NRT).</td><td>2</td></tr>
<tr><td>2.8</td><td>$40\%\ of\ 150$ (NRT).</td><td>2</td></tr>
<tr><td>2.9</td><td>$10 \times (3 + 15) - 6$ (NRT).</td><td>2</td></tr>
<tr><td>2.10</td><td>$18{,}03 \times 100$ (NRT).</td><td>1</td></tr>
<tr><td>3</td><td>A cyclist travels at a speed of 30km/h. How far will she cycle in 4½ hours? (NRT).</td><td>3</td></tr>
<tr><td>4</td><td>Table with headings of fraction, decimal, and percentage. Fill in the open spaces (NRT).
<table><tr><th>Fraction</th><th>Decimal</th><th>Percentage</th></tr><tr><td>$\frac{4}{10}$</td><td>0,4</td><td>?</td></tr><tr><td>$\frac{75}{100}$</td><td>?</td><td>75%</td></tr></table></td><td>2</td></tr>
<tr><td>5.1</td><td>How many factors does 30 have? (NRT).</td><td>1</td></tr>
<tr><td>5.2</td><td>Which factors of 27 are prime? (NRT).</td><td>1</td></tr>
<tr><td>6</td><td>$9 \times 5 \div \_\_ = 1$ (NRT).</td><td>1</td></tr>
<tr><td>7</td><td>Complete the flow diagram by filling in the missing number (NRT).</td><td>1</td></tr>
</table>

<table>
<tr><td></td><td>
</td><td></td></tr>
<tr><td>8.1</td><td>How many blocks, for 4th model: (NRT).

</td><td>1</td></tr>
<tr><td>8.2</td><td>What is the rule of the pattern? Formula? (NRT).</td><td>1</td></tr>
<tr><td>8.3</td><td>How many rows for 36 blocks? (NRT)</td><td>1</td></tr>
<tr><td>9</td><td>Match column A and B (not only number-related tasks (NONRT))

<table>
<tr><td colspan="2">Column A</td><td colspan="2">Column B</td><td>Column C</td></tr>
<tr><td>9.1</td><td>A trapezium</td><td>A</td><td></td><td>9.1 ________</td></tr>
<tr><td>9.2</td><td>The shape that is NOT a quadrilateral</td><td>B</td><td></td><td>9.2 ________</td></tr>
<tr><td>9.3</td><td>A parallelogram</td><td>C</td><td></td><td>9.3 ________</td></tr>
<tr><td></td><td></td><td>D</td><td></td><td></td></tr>
</table></td><td>3</td></tr>
<tr><td>10.1</td><td>An angle smaller than 90° is called an ………….angle (NONRT).</td><td>1</td></tr>
<tr><td>10.2</td><td>If I add two obtuse angles, the resulting angle will always be________ (NONRT).</td><td>1</td></tr>
<tr><td>10.3</td><td>The angle that is equal to 360° is called ________________(NONRT).</td><td>1</td></tr>
<tr><td>11</td><td>The learners are provided with a diagram of a triangular pyramid and asked to name it and record the number of vertexes (NONRT).</td><td>2</td></tr>
<tr><td>12</td><td>Five nets are given. Circle the net that can be folded in a hexagonal pyramid (NONRT)</td><td>1</td></tr>
<tr><td>13</td><td>How many lines of symmetry does the letter M have? (NONRT).</td><td>1</td></tr>
<tr><td>14</td><td>Questions asked about the positions of figures on a Cartesian plane grid (NONRT).</td><td>2</td></tr>
<tr><td>15.1</td><td>Convert 2½ days into hours (NONRT).</td><td>1</td></tr>
<tr><td>15.2</td><td>Convert 4,15kg to g (NONRT).</td><td>1</td></tr>
</table>

16	Joe had 8½m ribbon. He sold 50cm for R9 and the rest was sold for R15 per m. How much money did he get for the entire roll? (NRT).	3
17.1	What is the area of a given triangle on a grid? (NONRT).	1
17.2	Draw the reflection of a given triangle. (NONRT).	1
18	What is the perimeter of an octagon if all sides are 6cm? (NONRT).	2
19	Answering questions of a graph based on learners' time that they watch TV and time spent doing homework. (NONRT).	4
20	Counting 50 tallies and drawing 32 tallies. (NONRT).	2
21	If a R5 coin is tossed, how many outcomes are there? (NONRT).	1
22	What is the mode of 5, 3, 3, 6, 4, 3, 4, 3, 7? (NONRT).	1
23	Seven teams take part in a soccer league. If every team plays every other team once, how many soccer matches will there be altogether? (NONRT).	2
	Total	75

Sixty-six percent of the tasks were number-related tasks, and 33,33% were not only-number-related tasks.

3.5.3 Grade 2 Learning Framework in Number video-recorded interviews (G2 LFIN)

In 2011 the Wits Maths Connect Primary project (WMC-P) members conducted tests 1.1 and 2.1 to all the participants and carried out test 1.2 only to the participants who showed base-ten knowledge in tests 1.1 and 2.1 of the LFIN test package. Furthermore, we did not include Part D, which consists of multiplicative reasoning in the G2 LFIN, because the Grade 1 curriculum does not include multiplicative reasoning since the tests were conducted early in Grade 2. The other interviews included questions about multiplication and division.

The reason for using only tests 1.1 and 2.1 on all participants was due to the length of the interviews and the specifications of the curriculum for a learner at the beginning of the Grade 2 year.

Video-recording of learners answering mathematical tasks is essential for finding out the extent of development of the Stages of Arithmetical Learning (Cobb, 2000; Wright, Ellemor-Collins, & Tabor, 2012) in learners. Using video-recording has considerable advantages for the interviewer as well as the researcher. Intensive investigations can be done after the interview. The learner's strategies can be assessed, and the responses can be recorded to evaluate their stage of development in arithmetical learning due to the sophistication of strategies used.

The LFIN evaluates sophistication of strategy used and has four parts. Part A entails Early Arithmetic Learning strategies and base-ten understanding. Part B explores Forward Number Word Sequence (FNWS) and Backward Number Word Sequence (BNWS) counting skills and Numeral Identification (NI); Part C entails other aspects of Early Arithmetical learning like base-5, finger patterns, and subitizing[29]; and Part D entails Early Multiplication and Division (even though we did not use the multiplicative reasoning part but it is part of the LFIN assessment). Parts B, C, and D contribute to the Stages of Early Mathematical Learning (SEAL).

SEAL is the central component for identifying increasing sophistication with number in the LFIN model. A higher SEAL stage indicates more sophisticated strategies than the lower stages of SEAL. Structural thinking and the way in which it relates to a high SEAL stage are discussed in section 4.6.

The LFIN Assessment comprises six sets of questions, called assessment interview schedules. Table 3.7 shows the different interview schedules of the LFIN assessment.

Table 3.7 Assessment Interview Schedules of the LFIN

Assessment Interview	Description
1.1	Early arithmetical strategies and numerical knowledge
1.2	Base-ten knowledge and advanced arithmetical strategies
2.1	Early grouping and structural thinking numbers 1-10
2.2	Advanced grouping – Structuring numbers 1-20
3.1	Early multiplication and division
3.2	Advanced multiplication and division

All the interviews were organised by asking an initial entry task, and if the participant answered correctly, a more advanced task was asked. Thus, the participants who answered the entry task correctly were not asked the less advanced task. If the participant answered incorrectly, the less advanced task was asked, and the advanced tasks would not be asked to such participants.

[29] Subitizing, according to Wright et al. (2006, p105), means “to apprehend directly the number of dots in an unstructured stimulus display without counting them”

Two schedules, 1.1 and 2.1, were used in all the participants' interviews. Schedule 1.1 tests the learners' competence in number sequences, counting forward and backward from a given number word and saying the number word after and the number word before a given number, numeral identification, and working with screened tasks in additive situations. Schedule 2.1 tests early grouping, and structural thinking base-five and base-ten. Tasks include subitizing and asking the learner to show a number of fingers without counting them first.

If a learner's responses to the questions in the first two interview schedules indicated awareness of ten as a structural composite unit, the learner was additionally asked the questions of schedule 1.2, which tests working with base-five and base-ten knowledge in additive reasoning. I am discussing the interview schedules in the following order: firstly interview schedule 1.1, then 2.1 and lastly 1.2.

Thorough discussions about the extent of structural thinking the G2 LFIN evaluates, will be discussed in section 4.6.

In all the interview schedules, I am showing the words of the interviewer in italics and what the task evaluates in bold.

Details of the Tasks of Interview Schedule 1.1

The tasks of interview schedule 1.1 focus on forward and backward counting in different number ranges. It also includes the saying of the number word after and the number word before a number. The last part of the interview schedule includes additive tasks like adding of two sets of counters and missing addend tasks. It also comprises subtraction tasks in the form of seeing the small difference between the minuend and subtrahend (Tzur et al., 2017), removed items tasks and missing subtrahend tasks.

The *** in the table is a kind of 'placeholder' for the different questions that should be asked. For example, in a) the interviewer would say, "Start to count from '1' and I will tell you when to stop". The interviewer will not say "to 32" but stop the participant when she/he gets to 32. Table 3.8 shows the tasks of interview schedule 1.1

Table 3.8: Tasks of interview schedule 1.1

<table>
<tr><td>Task 1
FNWS[30]</td><td>FNWS: Start counting from*** and I will tell you when to stop. (Forward counting from any number)
a. 1 (to 32) b. 48 (to 61) c. 76 (to 84) d. 93 (to 112)</td></tr>
<tr><td>Task 2
NWA[31]</td><td>Say the word that comes straight after *** (Shows the learner’s awareness of number relations and the ability to know the number just after a given number without dropping back to 1)
<table>
<tr><td colspan="2">Entry task: 14, 11, 19, 12, 23, 29, 20</td></tr>
<tr><td>Less advanced: 5, 9, 7, 3, 6</td><td>More advanced: 59, 63, 32, 70, 99</td></tr>
</table></td></tr>
<tr><td>Task 3
Numeral Identification</td><td>Show each card in turn, saying, what number is this? (Evaluates the ability to know how to read the formal symbol and say the number word)
<table>
<tr><td colspan="2">Entry task:10, 15, 47, 13, 21, 80, 12, 17, 99, 20, 66</td></tr>
<tr><td>Less advanced: 8, 3, 5, 7, 9, 6, 2, 4, 1</td><td>More advanced: 100, 123, 206, 341, 820</td></tr>
</table></td></tr>
<tr><td>Task 4
Numeral Recognition</td><td>Arrange cards 1-10 randomly and point to a card, asking, which number is this? (Evaluates the ability to recognise the formal symbol) 6, 4, 7, 9, 8</td></tr>
<tr><td>Task 5
BNWS[32]</td><td>Example, count backward from 3. Three, two, one. (Backward counting from a number and evaluates the ability to count-down for subtraction tasks)
Now count backwards from *** and I will tell you when to stop.
a. 10 (down to 1) b. 15 (down to 10) c. 23 (down to 16) e. 34 (down to 27) d. 72 (down to 67)</td></tr>
<tr><td>Task 6
NWB[33]</td><td>Say the number that comes just before ***. Example: say the number that comes just before 2. (Evaluates the learner’s awareness of number relations and the ability to know the number just before a given number without dropping back to 1)
<table>
<tr><td colspan="2">Entry task: 24, 27, 20, 11, 13, 21, 14, 30</td></tr>
<tr><td>Less advanced: 7, 10, 4, 8, 3</td><td>More advanced: 67, 50, 38, 100, 83, 41, 99</td></tr>
</table></td></tr>
<tr><td>Task 7
Ordering numerals</td><td>Show the 10 numbered cards face-up in random order, asking the learner to identify each number as you put it out. Then say, Can you place the cards in order? Start by putting the smallest number here. (Evaluates an awareness of place value of formal symbols)
<table>
<tr><td>Entry task: 46 to 55</td><td>Less advanced task: 1 to 10</td></tr>
</table></td></tr>
</table>

[30] Forward number word sequence is discussed in section 2.5.5.
[31] Number word after.
[32] Backward number word sequence is discussed in section 2.5.5
[33] Number word before is discussed in section 2.5.5.

<table>
<tr><td>Task 8
Additive tasks</td><td>Addition tasks:
Counters screened[34]. Use two different colours for the counters.
(Evaluates the ability to work structurally with addition)
There are three red counters under here and 1 yellow counter under here. How many counters are there all together?
Introductory task: $\boxed{3} + \boxed{1}$
a. Entry task: both are screened, $\boxed{5} + \boxed{4}$, $\boxed{9} + \boxed{6}$
b. Less advanced tasks.
i) Less advanced: $\boxed{5} + 2$, $\boxed{7} + 3$, $\boxed{9} + 4$ (First number screened)
ii) Unscreened: $5 + 2,\ 7 + 3,\ 9 + 4$
iii) Perceptual counting - Count to see how many counters there are altogether in this group.
Place out 13 counters. Place out 18 counters
c. Supplementary additive task. (Screened, use two colours).
$\boxed{8} + \boxed{5}$, $\boxed{9} + \boxed{3}$
d. Missing addend task.
There are 4 red counters under the paper. Now look away. While you were not looking, I put some more yellow counters under here. (Indicate to the screened counters. Now there are 6 counters altogether. How many yellow counters did I put under here?)
<table><tr><td colspan="2">Introductory task: $4 + [] = 6$</td></tr><tr><td>$Task{:}\ 7 + [] = 10$</td><td>$Task{:}\ 12 + [] = 15$</td></tr></table></td></tr>
<tr><td>Task 9
Subtractive tasks</td><td>a. Subtraction sentence. Present the tasks as a written number sentence on a card.
Say to the learner, ‘What does this say?’
Do you have a way to work out an answer?
(Note, using counters is not an option in this question.)
(Evaluates the ability to work structurally with subtraction tasks seeing the small difference between the minuend and the subtrahend)
<table><tr><td>Entry task</td><td>Supplementary task</td></tr><tr><td>$16 - 12$</td><td>$17 - 14$</td></tr></table>b. Missing subtrahend (Use counters of one colour only.)
Here are five counters (Ask the learner to look away.
Remove and screen two counters).
There were 5 counters and while you were not looking, I took some away. There are 3 counters now. How many did I take away?
<table><tr><td colspan="2">Introductory task: 5 to 3 (5 – [] = 3)</td></tr><tr><td>Entry tasks: 10 to 6 (10 – $[] = 6$),
12 to 9 (12 – $[] = 9$)</td><td>Supplementary ta
15 to 11 (15 –</td></tr></table>c. Removed items
Here are 3 counters. (Briefly display and then screen.)</td></tr>
</table>

[34] Screened counters are counters that are covered with a piece of paper.

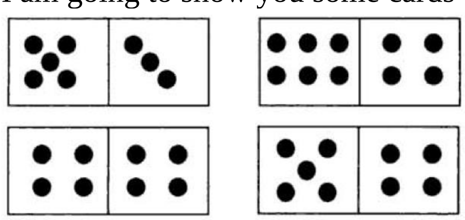
	If I take away one (remove one counter, display briefly, then re-screen), how many are left under here? (Indicate to the first screen.) Introductory task: $3-1$ Entry tasks: $6-2, 9-4, 15-3$ More advanced task: 27

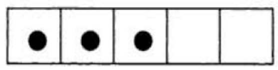
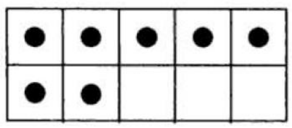
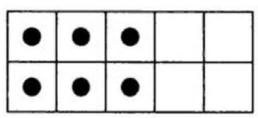
3.5.3.2 Details of the tasks of interview schedule 2.1

Interview schedule 2.1 (Table 3.8) evaluates early structural thinking of 10 and 5, and structural thinking numbers up to 10. The tasks in this interview schedule measure finger pattern 1-5 (task 2) and 6-10 (task 3).

Tasks 1, 4, 5, and 6 evaluates subitizing and spatial patterns in different forms. It comprises dots on a card that are structured (like one side of a domino) and unstructured (the dots are spread out in an irregular way), 5 and 10 framed cards, and cards with dots in a pair-wised pattern in a 10 frame. See the tasks of interview schedule 2.1 in Table 3.9.

Table 3.9: Tasks of interview schedule 2.1

Task 1 Subitizing and spatial patterns	Flash the cards one by one, saying, I am going to show you some cards very quickly. How may dots do you see? (Evaluates the ability to 'see' a number of dots at once without counting them) a. Flash regular or structured cards like domino cards (4, 3, 2, 5, 6) b. Flash irregular cards where the spacing of the dots is unstructured (6, 7, 4, 5, 8) c. Flash the domino cards one by one. Saying, I am going to show you some cards very quickly.  i. What do you see? ii. Tell me how many dots you see on one side. iii. Tell me how many dots there are on the other side. iv. How many are there altogether?
Task 2 Finger patterns: 1 to 5	Evaluates the ability to structure numbers up to 5 a. Show me 3, (2, 1, 5, 4) on your fingers. b. Show me 3, (2, 5, 4) fingers on two hands
Task 3	Evaluates the ability to structure numbers up to 10

Finger patterns: 6 to 10	a. Show me 6 on your fingers. b. Show me 6 in a different way. c. Show me 9 on your fingers. d. Show me 10 on your fingers. e. Show me 8 on your fingers. f. Show me 8 in a different way.
Task 4 Five frame patterns: 1-5	Flash the five-frame card. (Evaluates the ability to structure numbers up to 5)  a. What do you see? b. How many dots did you see? Cards with 3, 2, 5, 1, 4. dots on each.
Task 5 Five-wise patterns on a ten frame: 1-10	Five-wise patterns on a ten-frame (Evaluates the ability to structure numbers on a structured artefact)  a. What do you see? b. How many dots did you see? Cards with 7, 10, 8, 6, 9 dots on each..
Task 6 Pair-wise patterns on a ten-frame	Pair-wise patterns on a ten-frame (Evaluates the ability to structure numbers on a structured artefact)  a. What do you see? b. How many dots did you see? Cards with 4, 2, 5, 1, 3, 7, 10, 8, 6, 9 dots on each.
Task 7 Combining to make 5	I will say a number and you will say a number that goes with it to make 5. (Evaluates base-five knowledge) 4, 2, 1, 3, 5
Task 8 Combining to make 10	Base-ten knowledge a. Tell me two numbers that add up to 10. b. Tell me two other numbers that add up to 10. c. Can you tell me another two? d. I have 8 apples, how many more do I need to make 10 apples? e. I have 4, how many more to make 10? f. I have 7, how many more to make 10?

Five-wise and ten-wise finger patterns are an important part of interview schedule 2.1. An interesting question in this interview is task 2 b) where the participants was asked to show 3, 2, 5, and 4 with two hands.

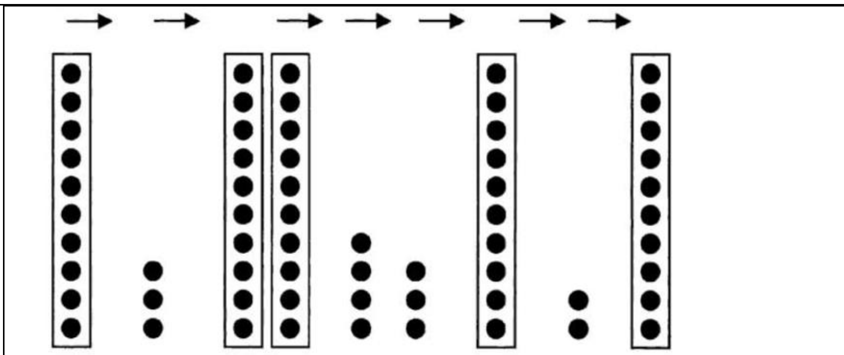
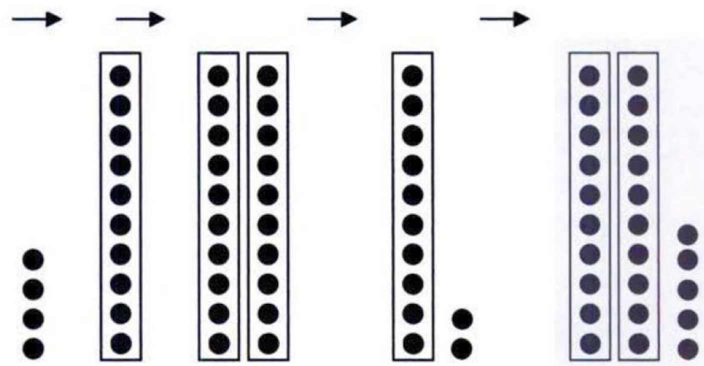
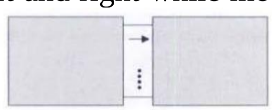
3.5.3.3 Details of the Tasks in Interview Schedule 1.2

Interview schedule 1.2 (Table 3.10) was only used for learners who could structure numbers up to 10 and whose knowledge was Stage 3 and beyond of SEAL. A stage 3 learner can count-on and down when he/she is confronted with addition and subtraction tasks.

The first task in interview schedule 1.2 tests base-ten knowledge, and the second task evaluates the ability to work structurally and formally with numbers and symbols. The last task tests strategies that involve approaches other than count-by-one, like compensation and knowledge of addition and subtraction as inverses. Table 3.10 shows a summary of the tasks in interview schedule 1.2.

Table 3.10: Tasks for interview schedule 1.2

Task 1 Tens and ones	Counting in tens with ‘ten-dot strips’ (Evaluates base-ten knowledge)
	Counting in tens with ‘ten-dot strips’ i. Put down a ten-dot strip and ask: How many do we have? ii. Put down a one ten-dot strip at a time and ask: How many altogether? iii. Pick up all the strips. How many dots do we have? How many strips do we have?
	Incrementing by 10 (Evaluates base-ten knowledge) i. Place out a four-dot strip. How many dots are there? ii. Place out a ten-dot strip to the right of the four-dot strip. How many are there altogether? iii. Continue to place ten-dot strips to the right of the 4-dot strip. How many are there altogether? 24, 34, 44, 54, 64, 74 iv. If necessary, repeat the whole task with either the 3- dot strips or 7-dots strip.
	Board 1 Uncovering tasks: cover the whole board with a piece of paper and then start to uncover board 1 by moving the cover paper to the right. Upon each uncovering, ask: How many dots are there now?

	 Board 2 Uncovering tasks: cover board 2 with two pieces of paper from the left and the right and uncover by showing only the dots that you are busy with. Upon each uncovering, ask: How many dots are there now?  For example, cover from left and right while moving both papers to the right simultaneously. 
Task 2 Mathematical sentences	Horizontal Sentences. (Evaluates the ability to work structurally and formally with numbers and symbols) Do you have a way to figure out what is 16 + 10 =? So what is 16+9? Do you have a way to figure out what is 42 + 23 =? If correct, do you have another way to work it out or to check? Do you have a way to figure out what is 38 + 24 =? If correct, do you have another way to work it out or to check? Repeat the above questions for 39 + 53 =? 56 − 23 =? 43 − 15 =? 73 − 48 =?
Task 3 Relational thinking	Tasks to elicit non-count-by-one strategies and relational thinking (Evaluates the ability to work structurally and formally with numbers and symbols) a Given $9 + 3 = 12$,

		Can you use $9+3=12$ to find the answer for $9+4, 9+5, 9+6$?
	b	Given $6+6=12$ Can you use $6+6=12$ to find the answer for $7+5, 8+5, 9+3$?
	c	Given $7-5=2$ Can you use $7-5=2$ to find the answer for $27-5, \quad 47-5$ What do you think the next question will be?
	d	Given $15+3=18$, Can you use $15+3=18$ to help you to do $18-3$?
	e	If I tell you that $21-16=5$ Can you use $21-16=5$ to help you to find $21-5$?
	f	Given $14+4=18$ Can you use $14+4=18$ to make a takeaway sum? Can you make another one?

Interview schedule 1.2 is different from interview schedule 1.1 and 2.1 because the tasks require knowledge that goes beyond unit counting. To be successful in interview schedule 1.2, learners should be able to work structurally with numbers and knows about the relationships in base-ten.

3.5.3.4 Allocation of scores

Administering the LFIN furnished me with 'data' on learner responses. I analysed the data qualitatively by using the LFIN Framework of Wright et al. (2006) discussed in section 2.5.5. To be able to compare the results quantitatively (in section 6.2), I quantified the responses of the LFIN interviews. Quantification of qualitative data involves turning the qualitative data from responses to numbers (Ward, 2010). The method I used to turn the qualitative data into quantitative data follows next.

The levels and stages of the learning framework in number are sequential, therefore I used the stages and levels as scores. However, I doubled the SEAL score because, in a personal discussion with Bob Wright, he suggested that, due to the importance of the 'SEAL' aspect within the LFIN, SEAL could carry a more substantial (double or more) weighting when quantifying these qualitative levels and stages.

Furthermore, given that base-ten knowledge depends more on structural understanding than forward and backward counting or numeral identification, I doubled the score for base-ten reasoning in my quantification process. Thus, to quantify or assign a score for the qualitative responses, I doubled SEAL and base-ten knowledge score and added all the other scores together to get a score out of 30. Table 3.11 shows how the scores were allocated.

Table 3.11: Allocation of scores

Learner		Scoring
	The five Aspects	
2	Stage of Early Arithmetical Learning (SEAL)	Out of 10 (5 stages x2)
3	Level of Forward Number Word Sequence (FNWS)	Out of 5 (5 levels)
4	Level of Backward Number Word Sequence (BNWS)	Out of 5 (5 levels)
5	Level of Numeral Identification	Out of 4 (4 levels)
6	Level of Base-ten knowledge (Wright, Martland, & Stafford, 2006)	Out of 6 (3 levels x2)
		Total 30

I converted all scores of the G2 LFIN to percentages.

3.5.4 Grade 6 Evidence of Structural Thinking Assessment (G6 ESTA)

I developed G6 ESTA exclusively for this study. Data from administering the G6 ESTA was collected through video-recording the assessment interviews. G6 ESTA measures mathematical structural thinking (Mulligan, Prescott, & Mitchelmore, 2004) and sophistication of strategy used by the sample of learners in Grade 6. As discussed in section 2.5.5, the better the learners' structural understanding, the more sophisticated their strategies will be.

The participants in the assessment could use mental or written methods. The tasks of the G6 ESTA were based on literature about structural thinking, evidence of structural thinking, and mental calculation. All the questions could be answered mentally, but I made counters, pens, and paper available and told learners they could use whichever method or strategy they preferred because I wanted to capture all evidence of structural thinking, even in written work. Furthermore, all strategies, mental and written, were relevant for my study as they indicate the learners' levels of mathematical structural understanding. The assessment entails two parts: additive reasoning and multiplicative reasoning.

The interview consists of 50 questions. The additive and multiplicative reasoning questions were mixed. Questions 1-19 are additive reasoning questions and question 20 – 45, 47 and 49 are multiplicative reasoning questions. Questions 46 and 47 were questions that include additive and multiplicative reasoning.

The scoring of the responses of the G6 ESTA was done by coding each question on a stage. While the coding of the G2 LFIN happened after analysing several questions' answers on the same concept, I decided not to code the G6 ESTA in the same way. Because in the G2 LFIN some tasks that deals with the same concept were answered inconsistently. For example, in the G 2 LFIN, participants could count-on to find the answer to one addition task, but for another similar task they could not. This difference in responses resulted in looking at the average level or stage of the responses on all the addition tasks, which was sometimes challenging to decide which stage/level the learner was in. Consequently, I coded each question following the frameworks I developed.

I developed a framework for the additive reasoning tasks (which include addition and subtraction tasks), multiplication, and division, but I coded each task separately according to the framework. Thus, the G6 ESTA comprises three frameworks: one for addition and subtraction tasks, one for multiplication tasks, and one for division tasks.

I developed the additive reasoning framework. However, I used the multiplicative reasoning framework of Wright et al. (2006) to guide me to develop my farmeworks, which are two separate frameworks for both multiplication and division, to develop two different frameworks, one for multiplication and one for division. Furthermore, The G6 ESTA only includes two or more-digit numbers, but I did not explicitly state it..

3.5.4.1 Additive reasoning tasks and evaluation framework

In the early years, children use counting strategies to add and subtract for varying periods of time. A learner's ability to change from counting individual items to identifying the structure of mathematical objects is fundamental to developing their number knowledge (Hunting, 2003). I categorised the responses of the additive tasks into four hierarchical stages that evaluate the development of structural thinking. (The evaluation framework follows shortly.) The strategies ranged from working with numbers by employing unit-counting and concrete items or tallies to approaches that demonstrate mathematical structural understanding using not-count-by-one-strategies.

I present the tasks first (Table 3.12) and then the evaluating framework (Table 3.13).

Specific additive reasoning tasks were selected to evaluate and show evidence of structural thinking in the participants' strategies. A summary of the tasks follows in Table 3.12.

Table 3.12: Summary of additive reasoning tasks

	Addition
Example	20+30- example question. We agreed that the answer was 50.
1.	Tasks with openings for the use of separation (1010) and aggregation (N10) strategies (Heirdsfield & Cooper, 2002) (number range 1-100). 12+85
2.	17+56
3.	Tasks with openings for the use of wholistic strategies (Heirdsfield & Cooper, 2002) with no given root equation (number range 200-300): Addition Compensation Find the answer to 275+99. (Askew & Brown, 2004)
4.	Tasks with openings for the use of wholistic strategies (Heirdsfield & Cooper, 2002): Subtraction compensation or changing the subtraction operation to addition on realising the small difference between numbers in subtraction questions (Torbeyns, Ghesquiere, & Verschaffel, 2009) Find the answer to 435-399. (Askew & Brown, 2004)
5.	Tasks with openings for the use of subtraction as 'seeing a small difference' versus 'taking away', changing the subtraction operation to addition on realising the small difference between numbers in subtraction questions (Torbeyns, Pol Ghesquiere, et al., 2009; Wright et al., 2012) Find the answer to 403-396.
6.	Tasks with openings for the use subtraction as 'taking away' (Wright et al., 2012) Find the answer to 504-7
7.	Tasks with openings for the use of subtraction as comparison (Askew, 1998; T.P Carpenter, Fennema, France, Levi, & Empson, 1999) I want to buy a pair of socks that costs R45. I have R39 in my purse. My mother says she will give me the rest. How much does she need to give me?
8.	Word problem (Askew, 1998): More than Mike has R14 more than me. He has R62. How much money do I have?

9.	Tasks with openings for the use of structural thinking: Bonds of 10 and 100 (Bridging through 100) Find the sum of the numbers in the box: 479 32 21 Which two numbers in the box will you add first to find the sum of the numbers quickly?
11-13	Tasks with openings for the use of wholistic strategies (Heirdsfield & Cooper, 2002) with a given root equation (number range 1-100): Addition compensation Present a card with 65+18=83 on it and ask, "Can you find the answer to the following using the information on the card?" c. 65+17= d. 66+18= e. 67+17= (Wright, J Martland, & Stafford, 2006)
14-17	Tasks with openings for the use of wholistic strategies (Heirdsfield & Cooper, 2002) with a given root equation (number range 0-500): Addition compensation Present a card with 164+57=221 on it and ask, "Can you find the answer to the following, using the information on the card?" 163+57= 164+58= 165+56= 165+58= (Wright, J Martland, & Stafford, 2006)
18	Tasks with openings for the use of subtraction as the inverse of addition (number range 1-230): Present a card with 164+57=221 on it and ask, "Can you find the answer to the following, using the information on the card?" 221-57= (Askew, 1998; Wright, Martland, & Stafford, 2006)
19	Tasks with openings for the use of base-ten knowledge (number range 1-230) Present a card with 164+57=221 on it and ask, "Can you find the answer to the following, using the information on the card?" 1640+570= (Wright, J Martland, & Stafford, 2006)

The additive reasoning tasks of the G6 ESTA were selected very carefully. It includes tasks that require knowledge of aggregation (N10) and separation (1010), mental strategies (Tzur et al., 2017) and wholistic strategies (Heirdsfield & Cooper, 2002) like compensation. Wholistic strategies include ‘seeing the small difference’ in a subtraction task and then count-on from the subtrahend.

I developed a framework consisting of four sequential stages to evaluate the strategy the learner used. Table 3.13 below shows the five sequential stages of this framework.

Table 3.13: Evaluation framework for additive reasoning

Additive reasoning: Levels for strategies used in Addition and Subtraction		
Level and Name		Description
Level 0	Incorrect answers	The learner cannot find the correct answer
Level 1	Unit counting	Strategies that involve unit counting and physical objects
Level 2	Formalising without structure	Strategies that involve some use of place value structure. Learners at level 2 use an algorithmic rule or form with unit counting to find the partial sums and differences. The Level 2 learner uses formalisation with limited structural understanding.
Level 3	Formalising with Structural understanding for numbers less than 20	Strategies that involve some use of place value structure use an algorithmic rule or form with working out the partial sums and differences mentally. Learners at this level used formalisation with structural understanding of numbers below 20. Delayed mental response (mental column algorithm) is included in Level 3
Level 4	Mental calculation using structural thinking	Strategies that involve the use of place value, structural understanding of quantity, as well as an awareness of relative size of number.

3.5.4.2 Multiplication tasks and evaluation framework in the Grade 6 Evidence of Structural Thinking Assessment

Part two of the G6 ESTA deals with multiplication and division. I distinguish between multiplication and division tasks with different frameworks for each. It was important for me to have two different frameworks for each multiplication and division because I wanted to know how they respond to the tasks separately.

The multiplication tasks consist of questions using cards with 9 dots on them asking how many dots 8 cards will have. Some tasks comprise an array with 7 dots in each row with different number of rows. The rest of the tasks are about multiplication facts, multiplication combined with addition, and division and multiplication as inverses of each other.

The G6 ESTA multiplication assessment tasks are shown in Table 3.14 below.

Table 3.14: G6 ESTA multiplication assessment tasks

23-45	<u>Multiplication</u>
	<u>A. Multiplication with repeated equal groups (Wright et al., 2012)</u> 23 We have 8 cards, each containing 9 dots. Show one card and screen the rest, asking how many are there altogether?
	<u>B. Multiplication with an array (Wright et al., 2012)</u> 24 On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 10 in each column). Screen the inside and show only the first row and first column. How many dots are there altogether? 25 What is 7x10? 26 On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 12 in each column: 7x12). Screen the inside and show only the first row and first column. How many dots are there altogether? 27 On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 9 in each column - 7x9). Screen the inside and show only the first row and first column. How many dots are there altogether?
	<u>C. Multiplication Basic Facts (Wright et al., 2012)</u> 28-39 Ask: what is 1x 8, 2x8, 3x8, 4x8, 5x8, 6x8, 7x8…….12x8?
	<u>D. Multiplication and addition combined tasks:</u> 46 There are 5 cupcakes in a box. Zandile bought 3 boxes and 2 loose cupcakes. How many cupcakes did Zandile buy? 47 Thulela bought 33 cupcakes. How many boxes and how many loose ones did she buy?
	<u>Division and multiplication as inverses of each other</u> 48 If we know that 8x7=56, what is 56 divided by 7? 49 If we know that 48 divided by 6 is 8, what is 6x8?

In the G6 ESTA, two evaluation frameworks are used to analyse the participants' responses to the multiplicative reasoning tasks; one framework is used for multiplication and the other one for division. I used the levels of Wright et al. (2006) to analyse the multiplication and division tasks. Table 3.15 shows the levels for multiplication.

Table 3.15: Evaluation framework for multiplication

Multiplication		
Name of Level		Description
Level 0	Incorrect answers	This level is for participants that cannot do or understand multiplication
Level 1	Perceptual grouping in ones	The learner at this level can establish equal groups, but she uses counters and unit counting to find the numerosity of both the multiplicand and multiplier. She can make a specified number of groups of a specified size and can find the total number of items.
Level 2	Perceptual grouping in multiples	The learner at this level can group a number up to 5 items in a group without counting them first; however, the items must still be visible. The learner at Level 2 will 'shift' four items as a group without counting the items of the group first. The learner can skip-count with counters by moving groups of counters with one movement. The groups are not bigger than 5.
Level 3	Figurative composite counting	This learner has developed counting strategies that do not rely on all items or counters to be visible or touchable, and learners do not need to count the equal groups one by one. For example, a learner uses skip-counting in multiples while one hand keeps track of how many multiples she already counted. A learner at Level 3 has a structural understanding of the multiplicand but not yet the multiplier because he still needs his fingers to know the numerosity of the multiplier.
Level 4	Formalised repeated composite grouping	The strategies at this level depend on a written algorithm. For example, learners use repeated addition by keeping on doubling. I included the formal vertical algorithm for multiplication in Level 4. Knowing multiplication table facts and deriving from them are included in this level.
Level 5	Immediate mental answer	The learner at this level can give an immediate mental answer by knowing multiplicative table facts. She regards both the multiplicand and the multiplier as composite units. The learner at this level knows and understands that division is

		the inverse of multiplication. An indication that a learner in my sample is on level 5 will be an immediate correct response. A learner at this level may use functional relations and intuition to answer multiplication tasks.

The sequential levels show the development of multiplication from working with counters that can be seen or touched, to working with multiplication mentally. Learners who can work mentally give an answer on a multiplication question immediately.

3.5.4.3 Division tasks and evaluation framework in the Grade 6 Evidence of Structural Thinking Assessment (G6 ESTA)

The G6 ESTA division assessment tasks are set out in Table 3.16 below.

Table 3.16: The G6 ESTA division assessment tasks

	Division tasks
20-22	Quotitive sharing – (grouping aspect of division): 20 I have 20 apples. I want to put them in bags of 4. How many bags do I need? (Wright, Martland, & Stafford, 2006)
	Partitive sharing (sharing aspect of division): 21 Here are 24 sweets. If we share them equally among 3 learners, how many will each one get? 22 If we now share them equally among 4 children, how many will each one get? (Wright, J Martland, & Stafford, 2006).
	Division and multiplication as inverses of each other 48 If we know that $8 \times 7 = 56$, what is 56 divided by 7? 49 If we know that 48 divided by 6 is 8, what is 6×8?
	50 Division as grouping Show a card with an array of 54 dots and 9 dots in a row. Ask: How many dots are there in one row? (while showing the first row) The learner may count and answers 9. Then ask: How many rows are there if there are 54 dots altogether? (while screening all the dots except the first row)

The evaluation table for division is set out in Table 3.17 below. The levels are sequential and show a development from learners working with counters to find answers to partitive and quotative division questions to learners that can give an immediate mental answer to division questions.

Table 3.17: An evaluation framework for the division tasks

Division		
Name of Level		Description
Level 1	Perceptual sharing in ones	She can share a set of items into a specified number of groups, for example: share 20 items in 5 groups (partitive sharing). She can also make groups of 4 when given 20 counters, and asked how many groups of 4 can be made (quotitive sharing). However, she needs the concrete items to do so. She also uses on-by-one counting when she performs partitive and quotitive sharing. Strategies at this level show that the learner cannot see the dividend or the divisor as a composite unit.
Level 2	Perceptual sharing in compound groups	The learner at this level can group a number up to 5 items in a group without counting them first; however, the items must still be visible and available.
Level 3	Figurative composite counting	The learner has developed counting strategies that do not rely on items or counters to be visible or touchable for the dividend or the divisor; however, she still uses her fingers to keep track of the quotient. For example, I have 20 apples. I put them in bags of 4, how many bags do I need? Learners at this level count in 4s, saying 4, 8, 12, 16, 20 while keeping track on their fingers saying 1, 2, 3, 4, 5, and answering “5 bags”.
Level 4	Formalised repeated composite subtraction	The learner at this level shows strategies that include any form of written algorithms in division, like repeated subtraction and long-division strategies. I included a delayed mental response (more than 8 seconds) in Level 4.
Level 5	Immediate mental answer	Learners understand division as an operation and know that division is the inverse of multiplication. They also know the division table facts and that multiplication and division are inverses. The learner uses strategies that show an understanding of division as an operation where he knows the meaning of the dividend, divisor, and the quotient. He can answer questions using amended, known facts or intuition. Strategies

		included in this level are an immediate mental answer as a recalled known fact.

3.5.5 The rationale for selecting /designing the data instruments

I am using the G1 ANAdG2, G2 LFIN, G6 CA and the G6 ESTA as the data instruments for the study because the analysis of each instrument and the analysis of the participants' responses on the tasks in the assessments helped me to answer the research questions. My main research question relates to the predictive power of structural thinking in the early years. Therefore, the selection of data instruments relates to how the instruments emphasise structural thinking.

I used the G2 LFIN (Wright, J Martland, & Stafford, 2006) as one of the early data collection instruments for this study because many of the questions in the interviews evaluate an ability to work structurally with numbers in increasing number ranges. Thus, as a predictor of structural thinking, it was useful to include the G2 LFIN in this study. The LFIN interviews cover a range of aspects in early number, and the particular mathematical skills that literature identifies are, therefore, good predictors of later performance. When looking at the predictive power in the literature, it is clear that a higher counting number range and more sophisticated strategies are widely seen as related to better mathematics performance in later years. The LFIN assessments are structured to measure the extent of number range in counting and the sophisticated strategies learners use to find answers to mathematical problems.

The G1 ANAdG2 rewarded only correct answers, regardless of the sophistication of strategy used, and thus did not focus on learners' structural thinking. Therefore, since the G1 ANAdG2 focus was not on structural thinking while the G2 LFIN was, the G1 ANAdG2 instrument contrasted with the G2 LFIN, where structural thinking was emphasised. Hence, the two predictor variables are the G1 ANAdG2 and the G2 LFIN, and since their foci are different, it was meaningful to use them because they could shed light on the predictive power of structural thinking.

Since my research question considers the means by which a later assessment can be predicted, I used the G6 CA as one of the predicted assessment instruments, because the G6 CA was a high-stake assessment for all Grade 6 learners in Gauteng in 2015. The marks were essential to parents and teachers. The Grade 6 CA evaluates the knowledge and skills that learners are expected to have acquired by the end of Grade 6. I used the G6 CA because I wanted to know how structural thinking in the early years predicts later mathematical performance in a high-stakes assessment and all aspects of mathematics.

The DBE conducted the Common assessments for all Grade 3 and 6 learners in Gauteng in 2015, and the results were collected from all the schools in the WMCP-project. The DBE administered Grade 6 CA; however, the teachers marked the scripts.

The only assessment in which I had control over the tasks was the G6 ESTA. I selected each item as a research-based item with the intention that the tasks evaluate structural thinking.

3.6 Data analysis process

As stated earlier, there were two time phases in the data analysis processes in the study. This is because of the two data collection timeframes. The data were collected in February 2011, when the participants were in Grade 2, and November 2015, when the participants were in Grade 6.

In February 2011 we, as WMC-P project members, collected the G1 ANAdG2 and recorded all the learners' marks. We also conducted the LFIN interviews and recorded the quantitative results.

At the end of 2015, I conducted the Grade 6 ESTA with 64 of the learners. Only 64 learners were still at the same schools as in February 2011. I also collected the marked scripts of the G6 CA for the same cohort of 64 learners. After collecting the scripts of the G6 CA, I recorded the results for each learner on each task in an Excel sheet. I also recorded all the scores of the G2 LFIN, G1 ANAdG2 and the G6 ESTA on the same sheet.

3.6.1 Qualitative analysis of the participants' responses in the G2 LFIN and G6 CA

I developed the G6 ESTA with a framework to analyse qualitatively the strategies the participants used for each question. I analysed the data of the G6 ESTA interviews quantitatively, using the analytical framework that I developed for the G6 ESTA. I then recorded the results of the 64 video-recorded interviews on an Excel sheet.

In 2015, I also decided to reanalyse the data gathered in the interviews of the G2 LFIN quantitatively, focusing on the 64 learners, because I made a small change to base-ten thinking in the analytical framework of Wright et al. (2006). I discuss the change in section 4.6.2, where I investigate the ways in which the instruments measure structural thinking.

The results of the qualitative analysis of the data of the two video-recorded interviews were quantified. Since the levels of the G2 LFIN and G6 ESTA are sequential, I used the levels for each learner as a score for the video-recorded assessments.

3.6.2 Quantitative analysis

The results of the responses on all the assessments were compared quantitatively, using statistical correlational analysis, to find the relationships between the results on instruments to answer the third sub-research question regarding the predictive power of the assessments in 2011 in quantitative terms. I discuss the correlational relationships between the responses of the participants on the assessments in section 6.2.

To be able to find the relationships between the assessments, I used the Pearson Correlation Coefficient (r) to examine the strength of the relationship between the early assessments and the later assessments. In this study, the predictor variables are the results of the Grade 2 LFIN assessment and Grade 2 ANA assessment, and the predicted variables are the scores on the G6 CA and the Grade 6 ESTA.

To understand the nature of the relationship between Grade 2 and Grade 6 assessments, I employed bivariate analysis using the Pearson correlation coefficient. However, Watkins et al. (2004) argue that, for a more in-depth interrogation of the data, it is helpful to do a univariate analysis first, which will help one to understand the measures of central tendency, variances, and the spread of the distributions of the assessments.

To be able to use Pearson's correlation coefficient, the distributions should be tested for normality since the non-normality of distributions in bivariate analysis violates the robustness of Pearson's r (Kowalski, 1972).

Therefore, I discuss the normality of the distributions using Brase and Brase's test (Brase & Brase, 2011) and report on the findings in section 6.1.

The bivariate analysis will consist of different collections of correlations. Firstly, I discuss the correlations between the two Grade 2 assessments and the Grade 6 assessments as 'whole' tests. Then I will look at the correlations between the early Grade 2 assessments and the G6 CA's NRT and NONRT.

I will then discuss the relationships between the early assessments against the G6 ESTA's additive reasoning and multiplicative reasoning tasks.

Lastly, I will also report on the associations between the Grade 6 assessments and the five early number aspects of the G2 LFIN, namely SEAL, FNWS, BNWS, NI, and base-ten thinking.

3.6.2.1 Correlations between the two Grade 2 assessments and the Grade 6 assessments as 'whole' tests

In this section, I will look at the correlation coefficient between the G1 ANAdG2, the whole G6 CA and the whole G6 ESTA to find how strong a predictor the G1 ANAdG2 was. I will do the same with the G2 LFIN to see how strong its predictive power is. Figure 3.1 shows the first set of associations that will be tested.

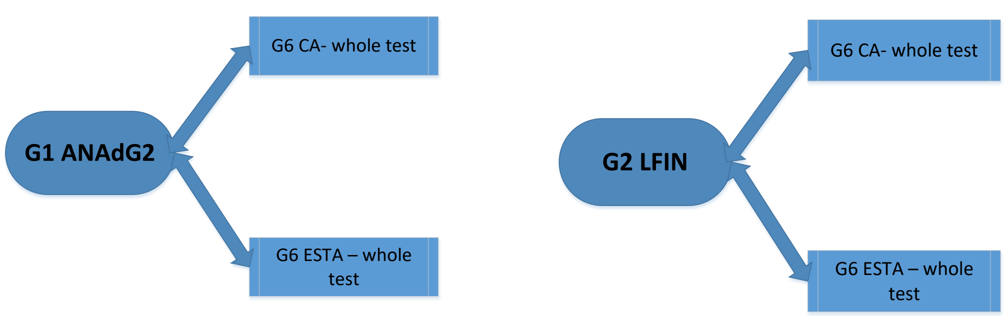


Figure 3.1: G6 CA and G6 ESTA compared with Grade 2 assessments

3.6.2.2 Correlations between the Grade 2 assessments and the G6 CA's NRT and NONRT.

The strength or weakness of the associations between the Grade 2 assessments and the number-related and not-only-number-related tasks in the G6 CA will be interesting to know. Figure 3.2 shows another set of associations that will be tested in section 6.2.

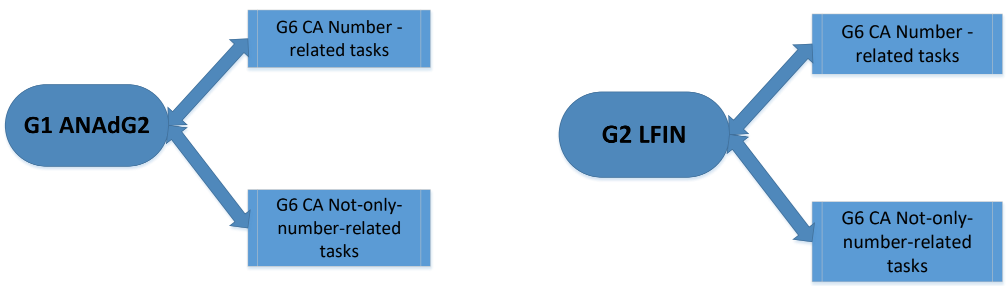


Figure 3.2 Associations between the Grade 2 assessments and the number-related and not-only-number-related tasks

Analysing the strength or weakness of the associations between the Grade 2 assessments and the number-related and not-only-number-related tasks in the G6 CA will tell me if there is a difference in predictability for NRT and NONRT.

3.6.2.3 Correlations between the Grade 6 assessments and the five early number aspects of the G2 LFIN, namely SEAL, FNWS, BNWS, NI, and base-ten thinking

These correlations will help me understand how strong the predictive power is of the different aspects of the G2 LFIN. I will compare the predictive power of stages of early arithmetical learning, forward and backward counting, numeral identification and base-ten thinking. The associations between the aspects of the Grade 2 LFIN and the subsets of the Grade 6 assessments will also be discussed. Figure 3.3 shows the last set of the associations that will be investigated.

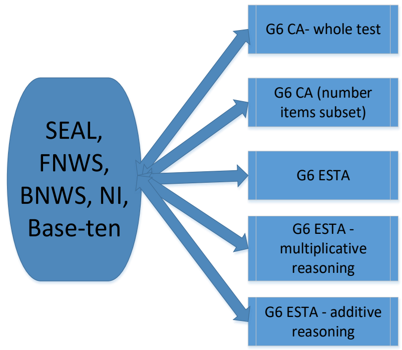


Figure 3.3: The Grade 6 assessments compared with the five aspects of the LFIN

3.7 Validity and reliability

The data instruments had some limitations. The limitations of the G1 ANAdG2 were that the learners could use unit-counting to answer all number-related questions. There were no observations made to record the methods learners used.

Because the G2 LFIN was an orally conducted interview, language could play a role in the performance of the participants. Although we, as WMCP-Project team members, were assisted by translators to interpret words for the participants, it could still be that they did not understand the wording and the demands of the tasks.

The participants could have been tense and under pressure when they wrote the G6 CA because the G6 CA was a high-stake assessment since the scores were essential to parents and teachers. The G6 CA was a written assessment in which the participants had to read the questions and answer on a script; hence, reading difficulties could have affected the results.

A limitation of the G6 ESTA could be that the participants could be stressed because it was an interview assessment. However, I did not use a camera operator or a translator. It was only me. All interviews were in English. I selected the tasks so that learners could use mental methods.

The collaborative mixed-method approach used for this study is intended to eliminate bias. This approach increases the possibility that the sum of the data collected will be richer and more meaningful in answering the research questions (Johnson et al., 2007). Furthermore, the collaborative mixed method approach taken in the study contributes to the reliability of the study which as a result can be replicated under the same conditions. Statistical methods were used for the analysis and contributed to the rigor with which results were interpreted and reported.

3.8 Ethics

I received ethical permission from the Gauteng Department of Education and the University of the Witwatersrand through the usual official channels. I am part of the Wits Maths Connect Primary Project, and my data was collected in the project. I obtained the permission of principals, teachers, parents, and learners, who provided signed consent. I also received signed consent letters from learners and parents for the video recordings of interviews. Learner anonymity was always respected during this research. The names of the schools were not used in writing up of the thesis, and schools and learners were also informed that they may withdraw from the research project at any time, without such a withdrawal affecting their relationship in any way with the WMCPP.

The interviews could be intimidating for the participants because, at times, three adults were observing their reactions - an interviewer, a camera operator and a translator. I attempted to ensure that the learner felt calm and comfortable during interviews.

3.9 Conclusion

In this chapter, I discourse on the study’s research design. I voice the research paradigm of Corroborative Mixed Method Research Design and the reason it is important to this study. I discuss the study sample and how the participants were selected, following with a discourse about the four assessment instruments. I then turn my attention to the qualitative and quantitative analysis process and end with a discussion about the study’s validity, reliability, and ethics.

CHAPTER 4 THEORETICAL EXTENT OF EVALUATING STRUCTURAL THINKING IN THE ASSESSMENTS' ITEMS/FRAMEWORKS

4.1 Introduction

The research question to be answered in this chapter is, in theoretical instances, to what extent and in what ways do the G2 LFIN, G2 ANAdG2, G6 ESTA, and G6 CA assessments evaluate mathematical structural thinking. I am evaluating each assessment through product analysis by looking at the relationships between the assessments and structural thinking.

As previously stated, mathematical structural thinking refers to the learner's awareness of relationships between unenumerated quantities, relationships between numbers and relationships between quantities[35]. Thus, in this chapter, I am analysing each assessment's capacity to evaluate the extent of structural thinking by referring to knowledge of relationships between mathematical objects.

Many researchers have written about structural thinking, and many do not agree with what structural thinking is. So, structural thinking is not easy to pinpoint.

Therefore, in this thesis I present my view of what structural thinking is. Firstly, I argue that structural thinking emerges in learners from young to older. What is considered an advanced strategy of structural thinking in Grade 2 does not mean that the same strategy is deemed to be advanced in Grade 6. For example, to count-on when a learner is in Grade 2 is an appropriate strategy to use and is regarded as a sophisticated strategy to answer, "what is $8 + 7 = ?$". However,

[35] I discuss the different relational knowledge in section 2.5.1.

to answer, “what is $17 + 56 =?$” and use a counting-on strategy in Grade 6 is not an appropriate strategy to use.

The foundation of structural thinking is knowledge of two principles: the first principle of the ordered number list is the sequel/prequel principle that affirms a relationship between a number and its sequel or its prequel. The second principle of the ordered number list is that digits’ place values in our number system are decimal-based.

Furthermore, the literature indicates that structural thinking in mathematics can be observed when learners use their knowledge of mathematical structure when they work in mathematics with an awareness of the relationships between mathematical objects (Mulligan & Mitchelmore, 2009; Venkat et al., 2019). So, for me, the first sign of structural thinking is that a learner works with knowledge of the relationships between the numbers in our number system, and then later on in their lives, works with the awareness of place values. Thus, structural thinking is about knowing relationships and working with an awareness of how numbers and even mathematical objects relate to each other.

Thus, the participants’ observed strategies are essential in the video-recorded assessments (G2 LFIN and G6 ESTA) to be able to see whether the participants worked with an awareness of number relationships.

There are different relationships to consider — knowledge of unenumerated quantities, number relationships, and quantitative relationships. Knowledge of unenumerated relationships is the awareness of more/same/less that we find in babies (Spelke & Kinzler, 2007).

Knowledge of number relations and knowledge of quantitative relations are not the same. However, they are linked. Nunes (2009) and Thompson (1993) describe the differences and similarities as follows: knowledge of number relationships involves knowing how numbers are composed in our Hindu-Arabic number system and knowing how to operate on numbers using addition, subtraction, multiplication and division (Thompson, 1993). Knowing the operations and/or learned procedure in additive and multiplicative reasoning shows knowledge of number relationships. Understanding quantitative relationships on the other hand involves young learners’ initial knowledge of unenumerated quantities (Nunes et al., 2012; Spelke & Kinzler, 2007) and includes knowledge of quantitative relationships. Quantitative relations highlight the importance of learners' ability to analyse a mathematical situation within a network of multiple connections to manipulate the numbers they are presented with to solve mathematics problems. The utilisation of pattern recognition and place value proves knowledge of quantitative relationships.

It is essential to notice that the three elements of structural thinking are sequential. For example, a learner starts with unenumerated quantities they are born with; then they develop knowledge of number relationships, followed by knowledge of quantities. Furthermore, a learner with knowledge of quantitative relationships knows all three pieces of knowledge of structural thinking. And a learner with knowledge of number relationships also knows unenumerated quantities.

Given that we cannot look into learners' heads, we must investigate their observable strategies to assess the extent of structural thinking. In this study, I consider an immediate mental answer as an indication of being aware of mathematical structure. An immediate mental answer is the highest level of structural thinking in my framework and includes rapid recall, place value, compensation, bridging through 10, separation and aggregation strategies. The only way to know which strategy the learner used is to ask, "how did you work out the answer?" on an immediate mental answer. While rapid recall answers do not always prove the learner's structural thinking, I regarded a rapid recall answer as structural thinking because all rapid recall answers were checked. I checked all correct answers with a question: "How did you get the answer?" and I could observe if the understanding was structural or not.

Moreover, the participants' observed strategies are essential in the video-recorded assessments (G2 LFIN and G6 ESTA) to be able to see whether the participants worked with an awareness of number relationships.

Now I will discuss how I evaluate the extent of measuring structural thinking of the assessments. The method I used to evaluate each assessment is distinct because they serve different purposes. The two video-recorded assessments, namely, the G2 LFIN and the G6 ESTA, are diagnostic video-recorded tests and are not summative like the G1ANAdG2 and the G6 ESTA, which were written tests. For the diagnostic assessments, I evaluate the frameworks to see whether each following stage or level requires a higher level of structural thinking. But for the written assessments, I developed a framework to evaluate the extent of structural thinking by finding the minimal required knowledge of each task or question to be answered successfully.

My focus in this study is how well number-related understanding at the beginning of Grade 2 predicts later performance in all aspects of the Grade 6 curriculum. For this reason, and since the G2 LFIN has only number-related tasks, I decided to take out the two geometrical tasks (tasks 4 and 8) in the G1 ANAdG2.

However, because I was interested in how well the Grade 2 assessments predict all the mathematics aspects in the Grade 6 curriculum, I used all the tasks in the G6 CA, which includes number and not-only-number-related questions.

Not all frameworks have the same number of levels. For example, the framework of the G2 LFIN of Wright et al. (2006) has a different number of stages or levels for the different aspects (SEAL-6 stages, FNWS – 6 levels, BNWS - 6 levels, NI – 5 levels, and Base-ten – 4 levels[36]) of the LFIN.

For the ESTA, I found it appropriate to have five stages for the additive reasoning part and six stages for the multiplicative reasoning part. I did the latter because I found it suitable to fit all the additive reasoning's observable strategies into five stages. However, I could not fit the observable strategies for multiplicative reasoning into five stages and thus developed six stages for multiplicative reasoning. Wright et al. (2006) developed 5 stages for their multiplicative reasoning part in their framework, however my framework for multiplicative reasoning includes 6 stages.

Before I discuss the assessments concerning measuring structural thinking, I want to foreground three types of assessments that are important to this study. They are diagnostic, formative, and summative assessments.

A summative assessment evaluates the learners' understanding at the end of a teaching semester or after a year's learning. Summative assessments are mostly high-stakes assessments as they are used to progress learners to the next grade or level (Dixson & Worrell, 2016). Summative assessments are therefore set to capture what a student has learned, to know the quality of the learning, and to judge performance against some standards. Thus, unlike formative assessments, which are generally used to give feedback to students and teachers regularly, summative

[36] I discussed the levels and stages in section 2.5.5.

assessments are usually used to get a final assessment of how much a learner knows (Gardner, 2010).

Formative assessment is used to continuously inform teachers, parents, and students about learning. Wiggins (1998, p. 7) argues: "The aim of formative assessment is primarily to educate and improve student performance, not merely to audit it." Thus, formative assessment comprises undertakings of teachers that make information available to adapt teaching and learning activities (Dixson & Worrell, 2016).

Both formative and diagnostic mathematical assessments can reveal deficits in a teacher's instructional practices and help them teach more successfully, but they are different in the following ways: a formative assessment aims to assess how students are assimilating the work they have been taught. However, a diagnostic mathematical assessment takes place prior to a formative assessment. A diagnostic assessment aims to analyse the knowledge that students acquired in the past from different teachers at different times. Another aim of a diagnostic assessment is to provide a profile of a learner's specific areas of weaknesses and strengths. These results of a diagnostic assessment can be used to offer valuable edifying information needed to plan remedial instructional programmes and/or interventions (Ketterlin-Geller & Yovanoff, 2009).

4.2 Differences and similarities between the four assessments

Now I turn my attention to the differences and similarities between the four assessments, according to the following: how the assessments were administered, the aim of the assessment (what kind of assessment it is), the content, and the mark allocation of each assessment.

4.2.1 The purpose and the method of assessment

As previously stated, the mathematics Annual National Assessments (ANAs) are standardised summative national assessments for Grades 1-6 and 9 in South Africa. The ANAs were introduced to inform schools and the DBE about possible learning gains in these grades, as well as to signal learners' performances in these summative assessments. The ANAs are standardised because the questions, conditions for conducting, scoring procedures and administrations are standard or consistent. The Grade 1 ANA that was carried out in Grade 2 (G1 ANAdG2) was orally administered, and the learners wrote their answers in an answer book.

In 2015, when the participants were in Grade 6, and I intended to collect the Grade 6 ANA scripts, the ANAs were discontinued, and I decided to use the Grade 6 Common Assessments (G6 CA). The Common Assessments were administered by the Gauteng Department of Education and were implemented for all Grade 3 and Grade 6 learners in Gauteng. All the schools that participated in this study are in Gauteng. The G6 CA was a written test in which the learners read the questions themselves and wrote down the answers in the exam book. A June and a November Common Assessment was introduced to inform the students, parents, teachers, and schools about the learning gains on a semi-annual basis.

The G2 LFIN and the G6 ESTA were interview schedule-based assessments. An interview schedule-based assessment takes the form of a one-to-one conversation, where the interviewer asks questions, and the interviewee responds. The interviewer and the interviewee take turns talking. These interview schedule assessments were video-recorded so that other researchers and I could analyse the interviewee's responses later and in more depth.

Both the G2 LFIN and the G6 ESTA were diagnostic assessments. The G2 LFIN assessment is diagnostic since it aims to provide a profile of each learner's knowledge about SEAL, FNWS, BNWS, NI, and base-ten knowledge (Wright, Martland, & Stafford, 2006).

The G6 ESTA provides a profile of how the learner responds to additive and multiplicative reasoning questions. It affords the researcher an in-depth understanding of how the learner reacted to questions regarding structural thinking. Learners' observable strategy serves as an indicator of knowledge of relationships between mathematical elements and structural thinking. It also provides information about the learners' understanding of multiplication and division separately.

Thus, the G1 ANAdG2 and the G6 CA were similar in that they were both written tests, and the strategies the learners used to find answers were not captured. In addition, the G2 LFIN and G6 ESTA were the same as they were both diagnostic tests that were video-recorded.

Table 4.1 summarises the differences and similarities between the four assessment instruments regarding the type and how the assessments were taken.

Table 4.1: Differences and similarities between the four assessments regarding the type and the way the assessments were taken

		Type of assessment		
G2 LFIN		*Summative*	*Formative*	*Diagnostic*
Written or interview assessment.	Video-taped interview schedule			X
	Written test	X		
G1 ANAdG2				
Written or interview assessment.	Video-taped interview schedule			
	Written test	X		
G6 CA				
Written or interview assessment.	Video-taped interview schedule			
	Written test		X	
G6 ESTA				
Written or interview assessment.	Video-taped interview schedule			X
	Written test			

The G1 ANAdG2 was a high-stakes summative assessment. However, the G6 CA was a formative assessment conducted twice a year. The G1 ANAdG2 and the G6 CA assessments were developed to inform the schools, departments and parents about learning gains.

4.2.2 The mark allocation and the content of the assessments

The mark allocations of the assessments were different. For scoring the two written assessments, the G1 ANdG2 and the G6 CA, I collected the marks from the eight schools. The teachers marked the scripts according to a memorandum provided by the Department of Basic Education and the Gauteng Department of Education. Only correct responses were rewarded with marks.

For the G2 LFIN and G6 ESTA, marks were allocated according to the level and stages of mathematical structural understanding.

I used Bob Wright and colleagues' (2006) LFIN levels and stages to allocate marks to each learner[37] to score the G2 LFIN. Thus, the mark allocation of the G2 LFIN was theory-based.

For the G6 ESTA, I developed a framework using literature and theory on mathematical structural thinking to compose the G6 ESTA assessment and a framework to analyse the learners' responses. I used the stages of the framework to allocate marks. Thus, the mark allocation was also theory-based.

Furthermore, the G6 ESTA assessment is comprised of two parts: additive reasoning and multiplicative reasoning. I developed two frameworks for the G6 ESTA, one for additive reasoning and one for multiplicative reasoning. The multiplicative reasoning is separated into another two frameworks, namely a framework for multiplication and a framework for division.

Even though the two Grade 2 assessments, namely the G2 LFIN and G1 ANAdG2, feature only additive reasoning number-related questions, I included multiplicative reasoning in the G6 ESTA because the Grade 6 curriculum specifies that multiplicative reasoning is part of the number-related content.

Since the primary focus was on to what extent structural thinking in Grade 2 predicts additive and multiplicative reasoning separately in Grade 6, I did not include the other content areas of the Grade 6 curriculum in the G6 ESTA.

The G1 ANAdG2 was comprised mostly of number-related content, and all the questions were additive. The G6 CA included all the content areas that the Grade 6 curriculum stipulates. Thus, the G6 CA included number and number operations, functions and algebra and numeric patterns, geometry, measurement data handling and probability.

There were similarities and differences in the administration of the assessments. The administration of the G1 ANAdG2 and the G6 CA was the same. In both assessments, the teachers collected the tests and marked them. The assessments were also similar in that the education department designed them.

[37] Section 2.5.5.

Bob Wright and colleagues (2006) developed the G2 LFIN, and the WMC-P Project members and I administered and coded the participants' responses.

I designed the G6 ESTA, conducted the assessments and coded the participants' responses according to an evaluation framework that I also developed. Table 4.2 summarises the differences and similarities regarding the assessments' mark allocation, administration and content.

Table 4.2 Differences and similarities regarding the assessment's mark allocation, administration and content

The mark allocation, administration and the content of the four assessments				
Assessment and design of the assessment	**Allocation of marks was theoretically/not theoretically underpinned**	**Marks allocated for:**	**Administered and marked/coded by:**	**Content of the assessments**
G2 LFIN Wright et al. (2006)	Learning Framework in Number of Bob Wright et al. – LFIN is supported by mathematical structural thinking	Marks were allocated for working in mathematics with an awareness of structural thinking	The WMC-P project members, including the researcher of the study	Additive Number-related questions
G6 ESTA The researcher of the study	Evidence of Structural thinking – Mathematical structure	Marks were allocated for working in mathematics with an awareness of structural thinking	The researcher of the study	Additive and multiplicative number-related questions
G2 ANAdG2 DBE	Not theoretically underpinned – marking memo for right answers	Marks were allocated for correct answers according to a memorandum	Teachers of the schools	Additive number-related questions and geometry
G6 CA DBE	Not theoretically underpinned – marking memo for right answers	Marks were allocated for correct answers according to a memorandum	Teachers of the schools	Additive and multiplicative number-related questions and geometry, data,

				probability, algebra

4.3 The evaluation framework to analyse the G1 ANAdG2, and G6 CA in terms of the extent of structural thinking they measure

Since both the G1 ANAdG2 and the G6 CA were written assessments, and there was no recording of how the participants answered the questions, I analysed the extent to which they measured structural thinking by considering each task theoretically and not empirically. Thus, in this framework, I am not interested in the child's strategy but only the minimal required knowledge the task entails.

I developed an analytical tool to help me investigate the extent of measuring structural thinking in each task in the G1 ANAdG2 and the G6 CA.

Tasks, in and of themselves, can be completed in various ways. For example, $20 + 3 \ = ?$ could be done by counting (lower-level thinking) or by recall/known-fact (higher-level structural thinking). I categorised the G1 ANAdG2, and the G6 CA tasks according to the minimal required knowledge the task entails. For example, in the case of $10 + 10 + 10$, a learner could answer with knowledge of structure by knowing the base-ten structure of our number system; however, a learner could draw 10 tallies three times to get 30 tallies counted one-by-one. Since I did not observe the learners' strategy, I categorised 10+10+10=? as knowledge of counting because structural thinking was not essential to correctly answer this question. In the G6 CA, some tasks entail insight into mathematical algorithms, formal mathematical knowledge and mathematical structure. For these questions, structural thinking was essential, and knowledge of counting was not sufficient to answer correctly.

My objective was to discover to what extent mathematical structural knowledge was essential to answering the questions correctly. Therefore, I categorised the tasks into three categories: knowledge of counting, knowledge of algorithms accompanied by knowledge of formal mathematical knowledge where structural thinking was not necessary to answer correctly, and insight into mathematical structure. Table 4.3 shows the minimum expected knowledge for a task to be completed successfully.

Table 4.3 Summary of the minimum expected knowledge or insight for the tasks in the assessment instrument to be completed successfully

The minimal required knowledge or insight for a task to be completed successfully	
1. Knowledge of counting, number symbols and number words	KC
2. Knowledge of algorithms accompanied by knowledge of formal mathematical knowledge where structural thinking was not necessary to answer correctly	KA
3. Insight into mathematical structure.	IMS

It is vital to notice that each successive category includes the previous category. For example, a task that falls into category 3 includes knowledge of counting, algorithms and formal mathematical knowledge together with an insight into mathematical structure.

4.3.1 Category 1: Tasks with counting as minimal means of answering correctly

This category includes tasks with counting as the minimal means of answering, where learners recite and count physical items or tallies in ones. Thus, knowledge of actions and processes on concrete items or drawn tallies is the minimal required knowledge to solving mathematical problems in this category. Knowledge of the number words in the correct sequence, number symbols and the counting process, which includes the one-to-one correspondence, is vital to execute the tasks in this category. This category also includes counting, addition and subtraction tasks in the number range 1 to 35 and counting in tens in the number range from 10 to 100 as minimal means to answer correctly. Tasks in category 1's minimal means of knowledge are knowledge of the relationship between unenumerated quantities and relationships between numbers in the number range 1 to 35.

For example, the addition and subtraction tasks in the G1 ANAdG2 need children to know what $20 + 3$ means. Furthermore, a taught procedure of drawing 20 tallies, then 3 tallies, and then count-all the tallies is the minimal required knowledge the learner needs to have to answer successfully. This category is also for tasks that require understanding, doubling and halving. Thus, I categorise tasks that required an understanding of the addition and subtraction operations and the terms 'double' and 'half' in this category.

4.3.2 Category 2: Tasks with knowledge of algorithms or taught procedures accompanied by formal mathematical knowledge where structural thinking was not necessary to answer correctly, as minimal means of answering correctly

The second-category tasks' minimal required knowledge is knowing algorithms or taught procedures and formal mathematical knowledge. In terms of knowledge of relationships, this category entails knowledge of the relationship between numbers in a number range 1 to 10^8.

The G6 CA tasks' minimal required knowledge to answer correctly is knowledge algorithms and formal mathematical knowledge; for example, the written vertical column algorithm. Knowledge of algorithms is necessary when operating with relatively large numbers that are difficult to calculate mentally (Van den Heuvel-Panhuizen, 2008). Thus, I include tasks requiring learners to apply written column algorithms for addition, subtraction, multiplication and division in this category. In successfully applying a written column algorithm, knowledge of mathematical structure is not essential. A learner can use counting by ones to find the partial sums and differences and complete the algorithm successfully without knowing what the numbers represent or understanding the relationships between them. For some learners, an algorithm is only a 'recipe' to calculate with unrelated digits. In these tasks, a correct answer can be given without knowing the relationships between the numbers and thus without structural thinking.

In the South African educational setting, the preference for unit counting is a significant concern and has prompted much apprehension among academics (Ensor et al., 2009; Weitz & Venkat, 2016). Since the preference for unit-counting is so predominant in the South African landscape, we can expect that some learners may use it to find the partial sums and differences when doing the written column algorithm, or even use unit-counting for all the numbers.

For example, consider the question, "What is $4\ 071\ 274 + 2\ 128\ 863$?":

A learner may use her fingers to add the partial sums when doing the vertical column method. Thus, to find $4 + 3$, she will count 4 fingers on one hand and then another 3 fingers on another hand, then count both hands' fingers to get the answer.

I also included tasks requiring learners to know formal mathematical knowledge in this category. The tasks of the G1 ANAdG2 and the G6 CA required learners to know some formal mathematical knowledge, which may include mathematical vocabulary like names of 3-D objects, knowing words like multiple, factors, prime numbers, names of angles, symmetry, knowing statistical graphs, and know the Cartesian plane.

Thus, this category is for tasks that can be done by an algorithm or having formal mathematical knowledge, which is just an application of a remembered procedure, where structural thinking is not essential to answer successfully.

4.3.3 Category 3: Insight into mathematical structure as a means of answering correctly

The third category of tasks is for questions that cannot be answered by counting or knowing an algorithm but call for an understanding of mathematical structure. Category 3 entails knowledge of quantitative relationships and thus quantitative reasoning, where a learner can analyse a mathematical situation and make useful decisions of how to solve the problem effectively (Nunes et al., 2012). I was specifically interested in the number of tasks of the assessments where mathematical structural thinking was essential to answering the question.

This category includes tasks that entail an awareness and knowledge of relationships between numbers. Our number system is formed by specific relationships that place particular elements in an exact arrangement (Venkat et al., 2019). Thus, an awareness of structure is the same as seeing numbers and mathematical objects in relationship to each other. To complete tasks in this category effectively requires that the learners:

- Have an awareness of inverses of operations, for example: In $52 - 48 =?$ The learner recognises that the difference between the numbers is small and gives an answer of 4 immediately by changing the subtraction into addition and counting-on from 48 to 52. Alternatively, the learner may also count back from 52 to 48.
- Understands compensation, for example:
$$55 + 19 =? \;\rightarrow\; 55 + 20 = 75 \;\rightarrow\; 75 - 1 = 74.$$
- Can give the value of the 4 in the number 347 203 311

- Knows the number facts $1 - 20$ and can apply them when dealing with larger numbers, for example, if:

$$6 + 7 = 13 \text{ then } 86 + 7 = 93$$

Thus, I include tasks in this category where mathematical structural thinking is essential to answer correctly.

4.4 Analysis of the tasks of the G1 ANAdG2 in terms of the measurement of the structural thinking framework

Now I turn my attention to the question: In theoretical terms, how and to what extent do the tasks in the G1 ANAdG2 assessment evaluate structural thinking competencies with early number?

I will start with arguments of why I classify each task in the G1 ANAdG2 in a particular category by stating the task's minimal required knowledge to answer successfully.

4.4.1 The rationale for categorising a G1 ANAdG2 task according to its minimal required knowledge or insight it entails

I am now explaining how I categorised each task according to its minimal required knowledge or insight to answer correctly.

1. Fill in the missing numbers.

1.1

3	4		6	7		9		11

Since the child can count from 3 to 11 and fill in the missing number, the minimal required knowledge is counting and knowing the number symbols. So, I classified this task as a category 1 task.

1.2

20		40	50	60			90	

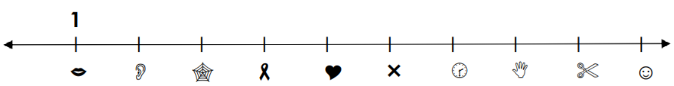
This task required the knowledge of counting in tens and knowing the number symbols. This task is also a category 1 task.

2.1 and 2.2 Complete this table by filling in the blank spaces:

Shapes	Total number	Number in words
Δ Δ Δ Δ Δ Δ Δ Δ Δ	9	
	7	seven

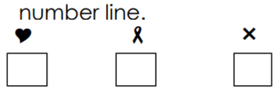
Tasks 2.1 and 2.2 are classified as category 1 tasks because the minimal required knowledge for these tasks is knowledge of counting, knowing the number words and number symbols.

3.1 and 3.3 What is the position of these pictures on the number line? The position of the first picture is given.



3.1 The ✂ is in position ________________.

3.2 Tick ☑ the box below the picture that is in position **6** on the number line.



The minimal required knowledge of these tasks is knowledge of the order of the number symbols and counting. So, tasks 3.1 and 3.2 are category 1 tasks.

5 Write the answer inside the box

5.1 $20 + 3 = \square$

5.2 $18 - 4 = \square$

Knowledge of counting in the number range 1 to 35 is the minimal means of answering successfully. The learner can draw 20 tallies and then another 3. To get to a correct answer, she counts all the tallies to get to 23. I categorise these tasks as category 1 tasks.

6 Fill in missing numbers. An example has been done for you.

6.1 Double the given number

Number	Double
5	10
7	

6.2 Halve the given number

Number	Half
16	8
20	

I regard the minimal required knowledge for tasks 6.1 and 6.2 as knowing counting to answer successfully. The child can draw 7 tallies twice and count-all to find 14 for the doubling task. The taught procedure for the halving task is that the child can share 20 tallies into two groups and count the tallies in one of the groups to know what half or 20 is. Tasks 6.1 and 6.2 are classified as category 1 tasks.

7 Write the correct answer in the box.

7.1 $10 + 10 + 10 = \square$

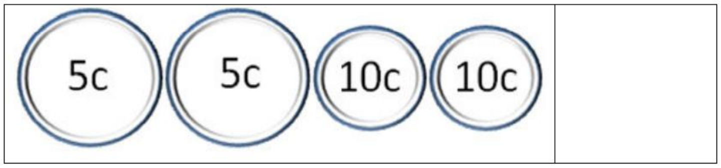
7.2 $10 - 2 - 2 = \square$

In task 7.1, $10 + 10 + 10 = 30$ can be done by mathematical structure; however, the minimal means of answering successfully is counting in ones because the learner can draw 10 tallies 3 times and then count all the tallies to get to 30.

For 7.2, the learner can draw 10 tallies and scratch out 2 and then another two tallies to get to the answer of 6. Thus, counting is the minimum required knowledge for these tasks, so these tasks are category 1 tasks.

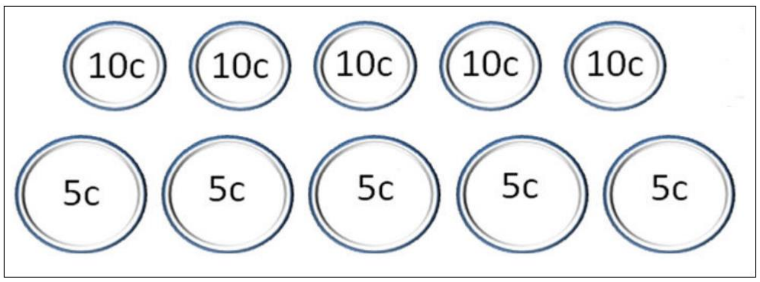
9 Look at the 10c and 5c coins in each question

9.1 The total number of money is______ (write the answer in the box)



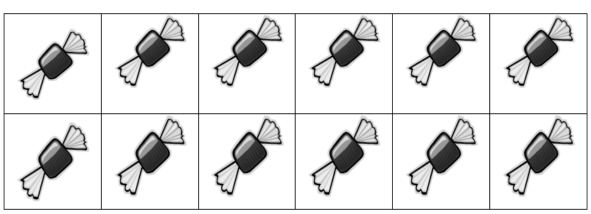
The minimal required knowledge is knowledge of counting because the correct answer can be obtained by drawing 5 tallies and 10 tallies twice to get to the right answer of 30c. Thus, task 9.1 is considered to be a category 1 task.

9.2 I have 4 coins in my pocket, and they add up to 25c. Tick (√) the 4 coins that I have in my pocket.



In this task, the learner must determine which 4 given coins add up to 25c. Task 9.2 is a category 3 task because the minimal required knowledge is mathematical structure. The task cannot be done by unit-counting or a learned algorithm. The learner has to use structural thinking to find the answer because she must know the relationship between the 10c, the 5c and the 25c. Furthermore, she must consider the number of coins to make the 25c while she is bearing in mind the relationship between the 5c, the 10c and the 25c coins.

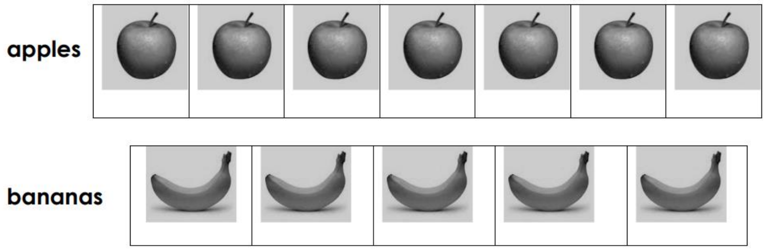
10 Busi and her 2 friends ate 12 sweets. They each ate an equal number of sweets. How many sweets did each one eat?



Each one ate ____________________ sweets.

The minimal required knowledge to answer successfully is knowledge of a learned procedure. A learner can use the sharing taught procedure for division by sharing the 12 sweets into 3 groups using tallies. So, this task is a category 2 task.

11 Thabo bought apples and bananas at the shop.



Write the correct number of each kind of fruit Thabo bought.

He bought:

11.1 ___________ apples.

11.2 ___________ bananas.

For both the tasks of question 11, the minimal required knowledge is counting. Thus I considered these tasks as category 1 tasks.

A summary of the classification of each G1 ANAdG2 task in terms of the minimal required knowledge a task entails to answer successfully follows in the next section.

4.4.2 Summary of categorising tasks in the G1 ANAdG2 according to the minimal required knowledge of insight to answer correctly

This section summarises the minimal required knowledge each task in the G1 ANdG2 demands, and I give the weighting each category carries in percentages. Table 4.4 summarises how I categorised the tasks in the G1 ANAdG2.

Table 4.4 Categorisation of tasks in the G1 ANAdG2 according to the minimal required knowledge or insight to answer successfully

Categorisation of tasks in the G1 ANAdG2 according to the lowest expected knowledge or insight to answer successfully				
Task	Score	Knowledge of counting	Knowledge of algorithms without insight into mathematical structure accompanied by knowledge of formal	Insight into mathematical structure

			mathematical knowledge	
1.1 $3;\ 4;\ \_\_;6;\ 7;\ \_\_;9;\ \_\_;11$	1	X(1)		
1.2 $20;\ \_\_;40;\ 50;\ 60;\ \_\_;\ \_\_;\ 90;\ \_$	1	X(1)		
2.1 Complete this table by filling in the blank spaces.	1	X(1)		
2.2 Complete this table by filling in the blank spaces.	1	X(1)		
3.1 What is the position of these pictures on the number line? The position of the first picture is given.	1	X(1)		
3.2 What is the position of these pictures on the number line? The position of the first picture is given.	1	X(1)		
5.1 20+3	1	X(1)		
5.2 18-4	1	X(1)		
6.1 Double 7	1	X(1)		
6.2 Halve 20	1	X(1)		
7.1 $10 + 10 + 10$	1	X(1)		
7.2 $10 - 2 - 2$	1	X(1)		
9.1 Add up coins value	1	X(1)		
9.2 Determine which 4 coins add up to 25c (5x10c and 5x5c)	1			X(1)
10 Sharing of 12 sweets between Busi and 2 friends	2		X(2)	
11.1 Count out how many apples	1	X(1)		
11.2 Count out how many bananas	1	X(1)		
Total score	18 marks	15 marks	2 marks	1 mark

Most tasks in the G1 ANAdG2 minimal means of answering correctly were unit-counting. These tasks could be carried out without an awareness of taught procedures or the relationship between the numbers. Task 10 required knowledge of the addition and subtraction learned procedures.

The only task that requires an awareness of structure is question 9.2. Task 9.2 provides pictures of five 5c coins and five 10c coins. The question was: tick the coins in my pocket that will add up to 25c if I have 4 coins in my pocket.

Table 4.5 summarises the weighting in percentage each category carries.

Table 4.5 Summary of each category's weighting in percentage in the G1 ANAdG2

The minimal required knowledge or insight for a task to be done successfully	**Number of tasks**	**Weighting in percentage**
Knowledge of counting and number symbols	9	$\frac{15}{18} \times 100 = 83\%$
Knowledge of taught procedures that is accompanied by formal mathematical knowledge where insight into mathematical structure is not essential	8	$\frac{2}{18} \times 100 = 11\ \%$
Insight into mathematical structure	1	$\frac{1}{18} \times 100 = 6\%$

Since only one question in the G1 ANAdG2 requires knowledge of mathematical structure, high marks could be scored in the G1 ANAdG2 without an awareness of the relationship between numbers or mathematical structures. The number range of all the tasks in the number range of the G1 ANAdG2 is 1 – 30, except for a question that entails counting in tens up to 100. The low number range of the G1 ANAdG2 made it possible for learners to use unit-counting in many tasks. To draw up to 30 tallies is not too time-consuming. Therefore, knowledge of counting would be sufficient to score high marks in the G1 ANAdG2.

To answer the question related to the extent to which the G1 ANAdG2 measures structural thinking, I argue that since many tasks merely require unit-counting knowledge, knowledge of number words, symbols, and taught procedures, I conclude that the G1 ANAdG2 does little to assess mathematical structural thinking where all three elements of structural thinking are essential.

4.5 Analysis of the tasks of the G6 CA in terms of the measurement of the structural thinking framework

Now I turn my attention to the Grade 6 CA. The G6 CA is different from the G1 ANAdG2 in that it is more focused on mathematical structural thinking than the G1 ANAdG2. The assessment took the form of a test in which knowledge of formal mathematical knowledge, algorithms, and mathematical structure played a significant role. A learner with only knowledge of counting, number words and number symbols would struggle to achieve reasonable results in the G6 CA.

4.5.1 The rationale for categorising a G6 CA task according to its minimal required knowledge or insight it entails to answer correctly

I am now giving arguments of why I classify each task in the G6 CA in a particular category by stating the task's minimal required knowledge to answer successfully.

1 Multiple choice questions 1.1 – 1.10

1.1 What is the value of the underlined digit in 6<u>4</u>3 979 568?

I classify task 1.1 as a category 2 task because the minimal required knowledge this task entails is knowledge of a learned procedure. The taught procedure is: you write down the underlined digit and replace the digits on the right side of the underlined number with zeros.

1.2 Which number is the largest? 231 323 689; 231 230 689; 231 302 689; or 231 320 689?

In this task, the minimal required knowledge is knowing place values, where the child has to consider the millions, thousands and the hundreds. So, this is a category 3 task.

1.3 What is the missing number in the sequence: 9,27; 9,25; 9,23; __?___; 9,19?

Task 1.3's minimal required knowledge is counting backward in two-hundredths from 9,27 and therefore it is a category 1 task.

1.4 Round 5697 to the nearest 10

This task's minimal required knowledge is knowing what rounding off to the nearest 10 means and how to manipulate the digits in the base-ten number system. So, this task I classified as a category 3 task because knowledge of base-ten and place values are the minimal required knowledge to answer correctly. If the question were to round off 5 637 to the nearest 10, I would classify the task in category 2; however, task 1.4 required more structural thinking than rounding off 5 637 to the nearest 10.

1.5 Which improper fraction is equal to 2½?

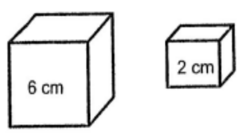
This task's minimal required knowledge is knowing the algorithm for changing a mixed fraction into an improper fraction. The algorithm is, 'write down the denominator of the fraction part of the mixed fraction (2) as the denominator of the improper fraction. To find the numerator of the improper fraction, multiply the whole number (2) with the denominator of the fraction part of the mixed fraction and add the numerator ($2 \times 2 + 1 = 5$). So the improper fraction is $\frac{5}{2}$. This task is a category 2 task.

1.6 Which number is not a multiple of 12? 12, 36, 276,112

This task is a category 2 task because the algorithm the learner can use is to find the divisibility of 2 and 3 in the given numbers. All the numbers are even numbers because they all have an even last digit. Furthermore, 12, 36, and 276 are divisible by 3 because the sum of the digits is divisible by 3, but not 112 ($1 + 1 + 2 = 4$ is not divisible by 3). Another algorithm to use is long division to determine if 12 can divide into the given number. If there is a remainder, the number is not a multiple of 12.

1.7 How many small cubes can fit exactly into the big cube?

A 9
B 12
C 20
D 27

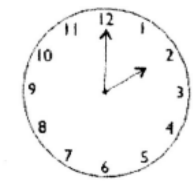


I classified this task as a category 3 task because the minimal required knowledge to answer correctly is structural thinking. The learner has to use structural thinking to imagine how many 2cm cubed blocks will fit into a 6cm cubed block. To know how many blocks will fit in, the learner must change their thinking from 3 dimensions to 2 dimensions by focusing on the faces of the small and the big blocks. Seeing that three 2cm squares will fit onto one side of the 6cm squared face and therefore nine 2cm squares onto a 6cm square is knowing the structure of the 3-D object. The learner has to change their thinking again, this time from 2-D to 3-D, and say $3 \times 3 \times 3 = 27$ to get the correct answer.

1.8 Write $(6 \times 1\,000\,000) + (5 \times 100\,000) + (1 \times 1\,000) + (8 \times 1)$ as a whole number

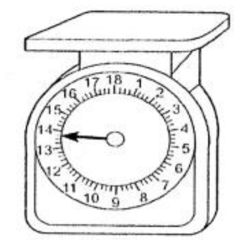
The learner's minimal required knowledge to answer correctly is knowledge of a learned procedure. For example, the learner can write down $6\,000\,000 + 500\,000 + 1\,000 + 8$ and use the written column method to answer correctly with 6 501 008. So this is a category 2 task.

1.9 Look at the clock. What is the time in 24 hours' time if it is now in the afternoon?



This task entails knowledge of a learned procedure because the learner can read off 2 o'clock; however, if it is in the afternoon, the algorithm is to add 12 hours to the read-off time to give an answer of 14:00. So this is another category 2 task.

1.10 What is the mass being shown on the scale?



In this task, a learner has to use structural thinking and recognise the 4 units between 13 and 14, so the learner must divide 1 by 4 and get 0,25. Thus, each small unit represents 0,25. The arrow is on the third small unit after the 13, so the answer is $13{,}75\ (13 + 0{,}25 + 0.25 + 0{,}25)$. So, I classify task 1.10 as a category 3 task.

Calculate the answers for questions 2.1 – 2.10.

All the tasks of question 2 are classified as category 2 tasks because for all of them, the minimal required knowledge is an algorithm.

2.1. $7\,211\,568\ +\ 5\,722\,188$

The child can use the written column algorithm, where the numbers' corresponding place values are written down beneath each other. The partial sums can be obtained by using fingers.

2.2 $4\,071274\ -\ 2\,128\,863$

In this task, the learner can use the column subtraction algorithm by writing down the numbers beneath each other and find the partial differences using fingers.

2.3 4748×36

The multiplicative column algorithm can be used to find the answer.

2.3 $7695 \div 25$

The long-division algorithm can be used to find the correct answer.

2.4 $2\ \frac{6}{7}\ +\ 3\ \frac{1}{14}$

The minimal required knowledge to answer this question is to change the proper fractions into improper fractions, change the denominators for both fractions to 14, and add the two numerators.

2.5 $5\ \frac{7}{8} - 1\ \frac{4}{8}$

The minimal required knowledge for this task is knowing how to count and know the whole number part and the fraction part of a mixed fraction. The learner can subtract the whole numbers, saying $5 - 1$, and then subtract the fractions $\frac{7}{8} - \frac{4}{8}$. Knowledge of converting a mixed fraction into an improper fraction and knowledge of changing the fractions so that they have a common denominator was not essential to answer correctly.

2.6 $45{,}05 - 19{,}21$

The algorithm is to write the decimal numbers beneath each other so that the place values of the numbers correspond and subtract.

2.7 40% of 150

The algorithm is to write 40% as $\frac{40}{100}$ because % means to multiply by $\frac{1}{100}$. Then multiply $\frac{40}{100}$ by $\frac{150}{1}$. The multiplication results in $\frac{40}{100} \times \frac{150}{1} = \frac{60}{1}$. Thus, this is a category 2 task.

2.8 $10 \times (3 + 15) - 6$

The order of operations is the algorithm for task 2.9. So this is a category 2 task.

2.9 $18{,}03 \times 100$

The algorithm is the shifting of the comma by two places. Hence this is a category 2 task.

3 A cyclist travels at a constant speed of 30km/h. How far will he travel in 4½ hours?

This task's minimal required knowledge is knowledge of mathematical structure. The learner must understand the relationships between speed, time and distance. The learner must also know that the cyclist moves at 30km/h. Thus, for 4,5 hours, the cyclist moves $4{,}5 \times 30km = 135km$. Therefore, this is a category 3 task.

4. Complete the table: Fraction/decimal/percentage

FRACTION	DECIMAL	PERCENTAGE
$\frac{4}{10}$	0,4	
$\frac{75}{100}$		75%

Task 4's minimal required knowledge is knowledge of algorithms. The learner should know the algorithms of changing a rational number from a fraction to a decimal fraction and a percentage. It also required the learner to know the algorithm to change a rational number from a percentage to a decimal fraction. So this is a category 2 task.

5.1 How many different factors does the number 30 have?

This task is classified as a category 3 task because the continuous division of the prime numbers learned procedure can lead to the prime factors. For example: first, the child divides 2 into 30 and gets 15, then divides 3 into 15 and gets 5. Then he divides 5 into 5 and gets 1. So the prime numbers are 2, 3, and 5. To find all the factors, the learner should think structurally and use the prime factors by saying $2 \times 3 = 6, and\ 2 \times 5 = 10, and\ 3 \times 5 = 15.$

Furthermore, the child must remember that 1 is a factor of any number, and 30 is also a factor of 30. So, this is a category 3 task.

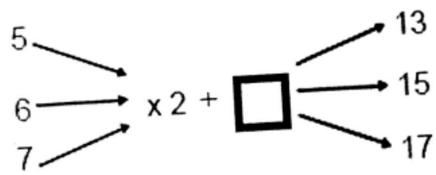
5.2 Which factors of 27 are also prime numbers?

To find the factors of 27 (continuous division of the prime numbers can give you that $27 = 3 \times 3 \times 3$) and knowing the prime numbers requires structural thinking. So, I classified this task as a category 3 task.

6. Complete: $9 \times 5 \div \_\_ = 1$

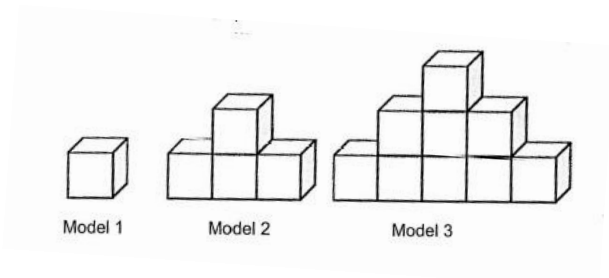
Task 6's minimal required knowledge is knowing that any number (variable) divided by itself will give you an answer of 1. So, this task is a category 3 task.

7 Complete the flow diagram: missing part of a rule.



I classify task 7 as a category 3 task because the minimal required knowledge is knowing that if I take any number (5, 6, or 7) and multiply it by 2, what should I add to the product (10, 12, and 14) to get 13, 15, and 17?

8. Look at the pattern made from blocks.



8.1 How many blocks do you need to form model 4?

I classify task 8.2 as a category 1 task because the minimal required knowledge is knowledge of counting and addition. The number of blocks of the previous model and the number of blocks in the current model's bottom row will give the current model. The first model had 1 block at the bottom, the second row had 3 blocks, and model 3 had 5 blocks at the bottom. Thus, model 4 will have 7 blocks at the bottom. Since model 3 has 9 blocks altogether, model 4 will have $7 + 9 = 16$ blocks. Or the learner can see the pattern and draw a few models to answer this question. I consider the knowledge to find the answer of 16 in task 8.1 as knowledge of a learned procedure.

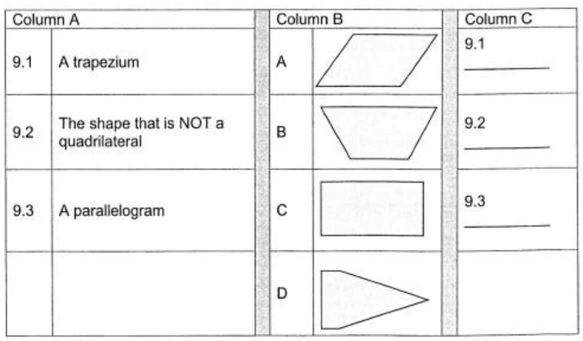
8.2 What is the rule for calculating the number of blocks if the model number is given?

In task 8.2, the minimal required knowledge is knowing mathematical structure and variables. The learner has to know that 1, 4, 9, and 16 are square numbers and say the rule is $T_n = n^2$ where T stands for the term and n stands for the position of the term. So task 8.2 is a category 3 task.

8.3. How many rows tall will a model be if it is made with 36 blocks?

The minimal required knowledge for task 8.3 is a learnt procedure of trial and error by drawing many models until the blocks in a model add up to 36 blocks. So this is a category 2 task.

9 Match column A with column B. Write the correct letter in Column C



Column A		Column B		Column C
9.1	A trapezium	A		9.1 ______
9.2	The shape that is NOT a quadrilateral	B		9.2 ______
9.3	A parallelogram	C		9.3 ______
		D		

Tasks 9.1, 9.2, and 9.3's minimal required knowledge is the formal mathematical knowledge of polygons. The learner should know the properties of quadrilaterals and identify them. Thus these tasks are classified as category 2 tasks.

10.1 An angle smaller than 90° is called____?

Task 10.1 is a category 2 task because the minimal required knowledge to answer correctly is knowledge of angles' names.

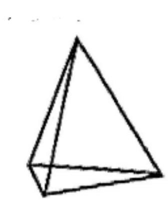
10.2 If I add two obtuse angles, the resulting angle will always be______.

This task required children to know angles' names as well as knowing addition of angles and knowledge of the 360º revolution. The learner should also be able to reason why it cannot be another obtuse angle. So, I classify this task as a category 3 task.

10.3 An angle that is equal to 360° is called______.

Task 10.3 is a category 2 task because the minimal required task is knowing the names of angles.

11 Look at the object and answer the questions that follow.



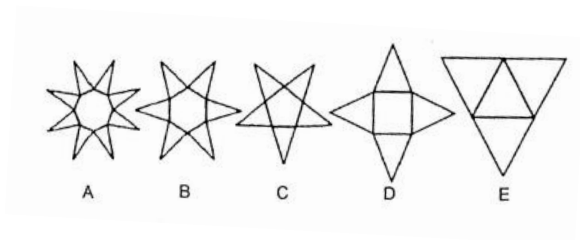
11.1 What is the name of the above 3-D object?

In this question, the minimal required knowledge is knowing the names of polyhedra. So, the task is a category 2 task.

11.2 How many vertices does the 3-D object have?

Knowledge of polyhedra properties and counting are the minimal required knowledge task 11.2 requires. Thus, this task is a category 2 task.

12 Circle the net that can be folded into a hexagonal pyramid.

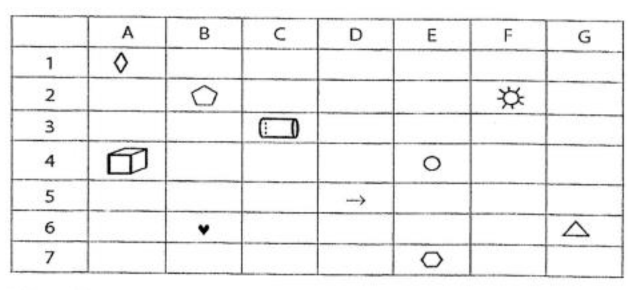


Knowledge of polyhedra names and properties is the minimal required knowledge this task entails. Therefore task 12 is a category 2 task.

13 How many lines of symmetry does the following image have?

Task 13 is a category 2 task because the minimal required knowledge is knowledge of symmetry.

14 Answer the questions based on the grid.



14.1 What is the position of the block where the cylinder can be found?

14.2 What shape can be found in block G6?

The minimal required knowledge a learner should have for tasks 14.1 and 14.2 is knowing the names of polygons and cylinders and how to indicate their position in this Cartesian plane. Therefore, I classify them as category 2 tasks.

15 Convert the following:

15a 2½ days = _______ hours.

The minimal required knowledge the learner can use is knowing that there are 24 hours in one day and 12 hours in half a day. Thus, this is a category 2 task.

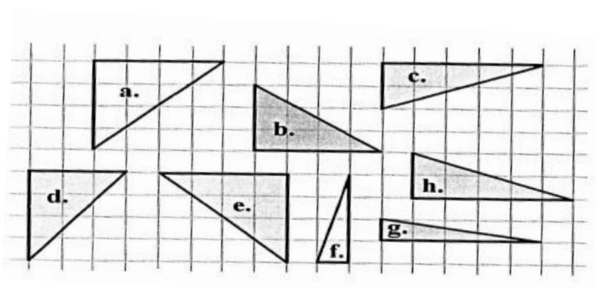
15b 4,15 kg = ________g.

This task's minimal required knowledge is a learned procedure of shifting the comma 3 places to the right to change kilograms into grams. Hence, task 15b is a category 2 task.

16 Joe had a roll of ribbon that was 8.5m long. He cut off a piece that was 50cm long and sold it for R9. The rest of the roll was sold for R15 per meter. How much money did he get for the entire roll?

Task 16 is a category 3 task because the task's minimal required knowledge to answer successfully is structural thinking. This problem-solving task required the learner to know the relationships between length and cost. The learner needs to see the relationship between 50cm and 8.5m.

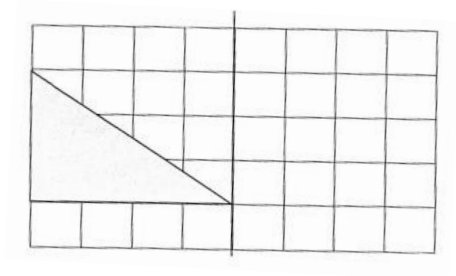
17.1 Look at the triangles in the grid, where each block represents a square that is 1cm by 1cm.



What is the rea of triangle **a** in square units?

Task 17.1 is a category 1 task as the minimal required knowledge is counting the square blocks of triangle **a.**

17.2 Draw a reflection of triangle b in the grid provided.

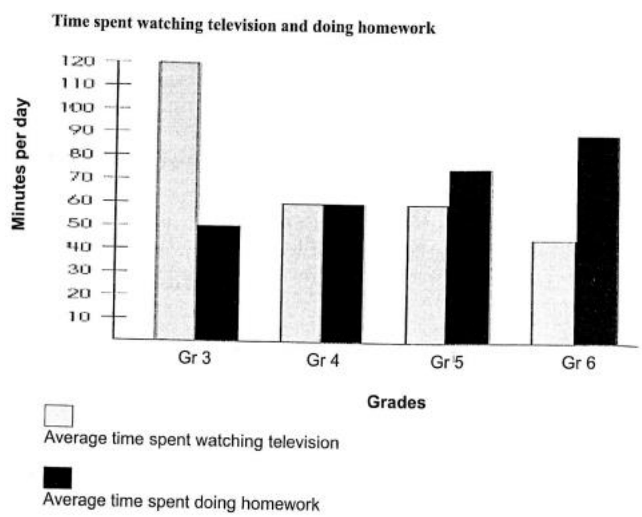


Task 17.2 is a category 2 task because the minimal required knowledge to answer successfully is to know translations.

18 What would the perimeter of an octagon be if all the sides measured 6cm?

The minimal required knowledge is knowledge of polygons and addition. Thus, task 18 is a category 2 task.

19 Answer the questions based on the double-bar graph showing the number of minutes that students spend on homework and watching television.



19.1 Which grade spends the most time watching TV?

19.2 How many more minutes do the Grade 6s spend on homework than the Grade 3s?

19.3 Which grade spends an equal amount of time doing homework and watching TV?

The minimal required knowledge for tasks 19.1, 19.2, and 19.3 is knowing how to read a double-bar graph, including how to use the graph's key. For task 19.1, the learner needs to find the grade that spends the most time watching TV. For tasks 19.2 and 19.3, the minimal required knowledge is to compare the minutes. So, all the questions in task 19 are category 2 tasks.

20 complete the tally table below by filling in the answers to **A** and **B**

Favourite cool-drinks of the Grade 6 class		
Cooldrinks	**Tally**	**Total**
Coke	卌 卌 卌 卌 卌 卌 卌卌 卌 卌	A
Sprite	B	32

I classified this task as a category 2 task because the minimal required knowledge is knowledge of how to draw the grouping-in-fives-tallies in a frequency table, which is formal mathematical knowledge.

21 A R5 coin is tossed; how many possible outcomes are there?

Task 21 is a category 2 task because the minimal required knowledge is knowing what possible outcomes are in probability.

22 What is the mode of the following set of data?

5, 3, 3, 6, 4, 3, 4, 3, 7

The minimal required knowledge is knowing how to find the mode in a data set, so the task is a category 2 task.

23 Seven teams take part in a soccer league. Each team plays every other team once. How many soccer games will there be altogether?

Task 23 minimal required knowledge is knowledge of mathematical structure. The learner must know the relationship between each team in terms of every other team. So, task 23 is a category 3 task.

Table 4.6 shows a summary of each task of the Grade 6 CA, according to the three categories of the analysis tool.

4.5.2 Summary of categorising tasks in the G1 ANAdG2 according to the minimal required knowledge of insight to answer correctly

The next part shows a summary of each task in the G6 CA and its category. I also include a table where I summarise the weighting in percentage each category carries.

Table 4.6 Categorised tasks in the G6 CA according to the minimal required knowledge or insight to answer successfully

Task	Knowledge of counting and number symbols	Knowledge of algorithms that is accompanied by formal mathematical knowledge where insight into mathematical structure is not essential	Insight into mathematical structure
Multiple choice questions 1.1 – 1.10			
1.1 What is the value of the underlined digit in 643 979 568?		X(1)	
1.2 Which number is the largest? 231 323 689; 231 230 689; 231 302 689; or 231 320 689?			X(1)
1.3 What is the missing number in the sequence: 9,27; 9,25; 9,23; __?___; 9,21?	X(1)		
1.4 Round 5 697 to the nearest 10			X(1)
1.5 Which improper fraction is equal to 1½?		X(1)	
1.6 Which number is not a multiple of 12? 12, 36, 276,112		X(1)	
1.7 How many small cubes can fit exactly into the big cube?			X(1)
1.8 Write $(6 \times 1\ 000\ 000) + (5 \times 100\ 00) + (1 \times 1\ 00) + (8 \times 1)$ as a whole number		X(1)	

1.9 Look at the clock. What is the time in 24 hours' time if it is now in the afternoon?		X(1)	
1.10 What is the mass being shown on the scale?			X(1)
Calculate the answers for questions 2.1 – 2.10			
2.1. $7\,211\,568 + 5\,722\,188$		X(2)	
2.2 $4\,071274 - 2\,128\,863$		X(2)	
2.3 4748×36		X(3)	
2.4 $7695 \div 25$		X(3)	
2.5 $2\,\frac{6}{7} + 3\,\frac{1}{14}$		X(3)	
2.6 $5\,\frac{7}{8} - 1\,\frac{4}{8}$		X(2)	
2.7 $45{,}05 - 19{,}21$		X(2)	
2.8 40% of 150		X(2)	
2.9 $10 \times (3 + 15) - 6$		X(2)	
2.10 $18{,}03 \times 100$		X(1)	
3. A cyclist travels at a constant speed of 30km/h. How far will he travel in 4½ hours?			X(3)
4. Complete the table: Fraction/decimal/percentage		X(2)	
5.1 How many different factors does the number 30 have?			X(1)
5.2 Which factors of 27 are also prime numbers?			X(1)
6 Complete: $9 \times 5 \div \_\_ = 1$			X(1)
7 Complete the flow diagram - missing part of rule.			X(1)
9. Look at the pattern made from blocks.			
8.1 How many blocks do you need to form model 4?	X(1)		
8.2 What is the rule of calculating the number of blocks if the model number is given?			X(1)
8.3. How many rows tall will a model be if it is made with 36 blocks?		X(1)	
9 Match column A with column B. Write the correct letter in column C		X(3)	
10.1 An angle smaller than 90° is called ____?		X(1)	

10.2 If I add two obtuse angles, the resulting angle will always be______			X(1)
10.3 An angle that is equal to 360° is called____		X(1)	
11 Look at the object and answer the questions that follow.			
11.1 What is the name of the above 3-D object?		X(1)	
11.2 How many vertices does the 3-D object have?		X(1)	
12 Circle the net that can be folded into a hexagonal pyramid.		X(1)	
13 How many lines of symmetry does the following image have?		X(1)	
14 Answer the questions based on the grid.			
14.1 What is the position of the block where the cylinder can be found?		X(1)	
14.2 What shape can be found in block G6?		X(1)	
15 Convert the following:			
15a 2½ days = _____ hours.		X(1)	
15b 4,15 kg = ______g.		X(1)	
16 Joe had a roll of ribbon that was 8.5m long. He cut off a piece that was 50cm long and sold it for R9. The rest of the roll was sold for R15 per meter. How much money did he get for the entire roll?			X(3)
17.1 Look at the triangles in the grid where each block represents a square that is 1cm by 1cm. What is the rea of triangle **a** in square units?	X(1)		
17.2 Draw a reflection of triangle b in the grid provided.		X(1)	
18 What would the perimeter of an octagon be if all the sides measured 6cm?		X(2)	
19 Answer the questions based on the double bar graph showing the number of minutes students spend			

on homework and watching television.			
19.1 Which grade spend the most time watching TV?		X(1)	
19.2 How many more minutes do the Grade 6s spend on homework than the Grade 3s?		X(2)	
19.3 Which grade spends an equal amount of time doing homework and watching TV?		X(1)	
20 Complete the tally table below by filling in the answers to A and B		X(2)	
21 A R5 coin is tossed. How many possible outcomes are there?		X(1)	
22 What is the mode of the following set of data? 5, 3, 3, 6, 4, 3, 4, 3, 7		X(1)	
23 Seven teams take part in a soccer league. Each team plays every other team once. How many soccer games will there be altogether?			X(2)
Total score	3 marks	54 marks	18 marks

Table 4.7 provides a summary of the number of tasks in the G6 CA, classified according to the minimum required knowledge or insight for a task to be answered successfully and the weighting each category carries.

Table 4.7 A summary of the weighting each category carries in the G6 CA, according to the minimal required knowledge or insight for a task to be successfully completed

Minimum required knowledge or insight to answer successfully	Number of tasks	Weighting in percentage the category carries
Knowledge of counting	3	$\frac{3}{75} \times 100 = 4\%$
Knowledge of algorithms accompanied by formal mathematical knowledge where structural thinking is not required	10	$\frac{54}{75} \times 100 = 72\%$

Insight into mathematical structure	33	$\frac{18}{75} \times 100 = 24\%$

The summary above illustrates that the G6 CA's tasks were more demanding because they required learners to demonstrate mathematical insight and formal mathematical knowledge, including mathematical vocabulary.

To answer the question related to the extent to which the G6 CA evaluates structural thinking, I argue that a significant number of tasks (72% of the marks) required knowledge of algorithms and formal mathematical knowledge beyond unit-counting, which is knowledge of relationships of number. Nearly a quarter of the questions' marks need knowledge of structural thinking where all three elements of structural thinking are essential. It follows that the G6 CA evaluates algorithms and mathematical structural thinking to a far greater extent than the G1 ANAdG2.

4.6 Analysis of the G2 LFIN framework in terms of measuring the extent of structural thinking

To answer this question related to the extent to which and in what ways the G2 LFIN assessment evaluates mathematical structural competencies in early number, I will now discuss the G2 LFIN framework in terms of structural thinking by focusing on number relationships, quantitative relationships (Nunes, 2009), and composite units (Steffe & Cobb, 2012) or reified objects (Sfard, 1991).

When a learner knows, say, 8 as a composite unit, she simultaneously knows that 8 is one unit and that 8 exists within 8 units. Sfard uses the term 'reified object', where she referred to a reified object as procedural and structural understanding at the same time. Thus, a learner has reified the number 8 when she simultaneously knows that if she counts 8 items in a set, one-by-one (procedural), that 8 is one unit within 8 items (structural). Sfard phrased this duality as "two sides of the same coin" (1991, p. 1).

The Grade 2 LFIN interview assessments are grounded in structural thinking (Wright, Martland, & Stafford, 2006). To analyse the extent of the measurement of structural thinking, I discuss each level and stage[38] in terms of the knowledge of relationships between numbers and quantities. The

[38] The differences between levels and stages according to Wright et al. (2006) will become clearer as you continue reading.

latter will allow one to observe the extent to which the Grade 2 LFIN measures structural understanding and observe how structural thinking's hierarchical nature becomes clear through the stages and levels.

Wright, Martland, Stafford and Stanger (2006) describe the development and increasing sophistication of number in a framework they call the Learning Framework in Number (LFIN). The latter is comprised of four parts. As stated earlier, this study is concerned with Parts A and B only because interview schedules 1.1, 2.1, and 1.2[39] only draw on the first five aspects.

Part A deals with early arithmetic strategies (SEAL) and base-ten knowledge. Part B explores forward number word sequence (FNWS) and backward number word sequence (BNWS) counting skills and numeral identification (NI).

Wright et al. distinguish between stages and levels. For them, a 'stage' is used in a "theoretical sense" (2006, p. 52). Wright et al. share the same understanding of stages as von Glasersfeld and Kelly (1982). According to this model, stages possess four characteristics:

- Stages stay constant over time,
- Different stages form an invariant sequence,
- Each stage builds on and includes the preceding stages, and
- Each new stage consists of a substantial conceptual reform.

Therefore, SEAL is structured in terms of stages because each new stage announces a conceptual change to a higher level. Thus, it is predicated on the principle that learners undergo a substantial conceptual shift at each new stage. Wright et al. (2006) emphasise that moving between the SEAL stages involves substantial growth, and hence these moves take significant time to achieve.

Even though Wright et al. (2006) call the different hierarchies of SEAL stages, they referred to the various hierarchies of FNWS, BNWS, IN and base-ten as levels. The learners must perform to a specific condition of a level to move to the next level (Wright et al., 2006). The reason for doing so, according to Wright et al., is that the FNWS, BNWS, IN and base-ten strategies feed into the SEAL stages and supply us with information about the SEAL stage of a learner. Therefore, you can find the SEAL stage in which the learner is operating after coding FNWS, BNWS, IN, and

[39] Discussed in 2.5.5.

base-ten strategies. The stages and levels are empirical indicators of the mathematical comprehension a learner possesses.

4.6.1 Stages of Early Arithmetical Learning (SEAL)

The first aspect of Part A[40] of the LFIN is the Stages of Early Arithmetical Learning (SEAL). This section highlights the correspondence between the observable strategy or empirical indicator of the SEAL and mathematical structural thinking.

Number knowledge emerges from innate knowledge (Clements & Sarama, 2009) and reality (for example, actions on concrete counters) and expands continuously to higher levels of comprehension (Van den Heuvel-Panhuizen, 2008).

The SEAL stages focus on a learner's engagement with reality and offer a detailed description and indicators of the progression from using concrete items to a mathematical structural understanding of number where the learner has moved to higher levels of structural thinking. In other words, from counting by seeing and feeling to a point where the learner has an abstract view of addition, subtraction, multiplication and division, where the learner uses different sophisticated strategies to calculate. These sophisticated strategies include compensation, knowing and using number bonds, knowing number facts and bridging through 10, knowing that if $9 + 3 = 12$ then $12 - 3 = 9$, and multiplication facts, to name a few.

The focus of the SEAL stages is not only on whether the learner can find the answer to a task but is also on how sophisticated the strategy is. There are six SEAL stages: emergent, perceptual, figurative-counting, initial number sequence, intermediate number sequence, and facile number sequence.

SEAL shows the progression of structural thinking in the following way:

<u>Stage 0 – Emergent counting</u>: The stage 0 learner counts in an idiosyncratic manner, with no order or structure. Wright et al. (2006) argue that a learner on stage 0 of SEAL has no understanding of counting physical objects because she either lacks the ability to recite[41] the number words in the

[40] I discussed the parts and aspects of the LFIN in section 2.5.5.
[41] The word 'recite' comes from Manfra, Dinehart, & Senbiante, (2012) where they refer to 'recite' as saying the word in the correct order without the ability to count physical objects.

correct order or is unable to coordinate every spoken word with each item to be counted (Gelman & Gallistel, 1986). Thus, the empirical indicator of a learner functioning at stage 0 is that she cannot say the words in the correct order and/or cannot count physical items.

At this stage, the learner does not have an awareness of relationships between the spoken words or the relationship between the movement of items and the spoken words. No knowledge of the number words or cardinality of number is comprehended yet. However, one cannot assert that there is no knowledge of unenumerated quantitative relationships.

Therefore, I would suggest that tasks that measure unenumerated quantitative relationships be included in the LFIN. An example of a task like that is to show 5 red counters and 8 blue counters and ask, “Which one is more?”

<u>Stage 1 – Perceptual counting</u>: Stage 1 refers to the learner who can count one set of objects, but only when they are seen and felt. Thus, the empirical indicator of a learner on stage 1 is that she needs physical/concrete items to count. The learner at stage 1 has achieved two skills, namely reciting (Manfra et al., 2012) the words in the correct order and knowing the one-to-one correspondence principle (Gelman & Gallistel, 1986), which she could not master in the previous stage. Thus, the learner at this stage has established an awareness of the relationship between the spoken word and the movement of physical objects.

Not only did she master the counting process, but she also comprehended that a specified set of items encompasses a fixed number of items (Mitchelmore & White, 2007). She realises that the cardinality of a set of items stays the same if no item was taken away or added (Piaget, 1964). Thus, the learner at stage 1 can conserve number for one set of items; however, she has difficulties establishing the numerosity of two sets of unscreened counters (Wright, Martland, & Stafford, 2006). In terms of screened objects, the learner at this stage cannot count items that are concealed.

This stage shows progression in structural thinking from the previous stage because the learner now knows the relationship between the spoken word and the physical item’s movement. For example, a learner in this stage knows the number words in the correct order and knows the one-to-one relationship, which the level 0 learner could not do. There is a growth of number knowledge from the stage 0 learner to the stage 1 learner because the stage 1 learner can now count physical items correctly. Another proof that the stage 1 learner is functioning on a higher level is that she knows the relationship between the counted items and the items that still need to be counted.

The learner in the perceptual counting stage can conserve one set of unscreened items but has difficulties conserving two sets of items.

Stage 2 – Figurative counting: The stage 2 learner can conserve two sets of items and count them when they are concealed, while the learner in the perceptual counting stage (previous stage) cannot.

The ability to conserve two sets of items and to be able to count the screened sets is a substantial conceptual shift from stage 1. Consequently, the stage 2 learner masters two skills that the stage 1 learner could not. Firstly, making sense of two sets of items demonstrates that the learner knows the relationship between the items of the first set and the second set simultaneously. Secondly, the stage 2 learner can 'imagine' two sets of counters when counting the concealed counters. Learners usually transfer the 'imagined' sets of counters to their fingers and then start to do the 'triple count'[42]. Thus, the empirical indicator for a learner functioning at stage 2 is that she can count two sets of items with physical objects and count them when they are concealed.

For a learner, making sense of two sets of items indicates that she is ready to do simple addition tasks. The learner on stage 2 knows the relationships between two addends and the sum in addition. The stage 2 learner does not yet make sense of any subtractive tasks.

To count screened sets of counters requires the ability to transfer the number of screened items to the number of fingers or tallies. The learner knows the relationship between the number of screened counters and the number of fingers. Thus, she can 'imagine' the number of screened objects and transfer the screened number of items to her fingers and do the calculation by counting her fingers one by one; however, she has to count every time from 1 when dealing with 2 sets of screened counters (Anghileri, 2006).

For example, a learner is asked to find the total number of counters if there are 5 red counters and 4 blue counters screened underneath 2 pieces of paper. From a cognitive perspective, the learner had to count from 1 to 5 to make sense of 5 in '5+4=?' and counts from 1 to 4 to make sense of

[42] Triple counting in additive reasoning refers to a child who counts the first set, then the second set and lastly counts the total number of counters in both sets (Anghileri, 2006).

the 4. To add these two numbers, she has to count from 1 every time. Since the learner has to count from 1 every time to make sense of the numbers at hand, she does not yet see the 5 and the 3 as composite units, or does not reify 5 or 3 yet.

Learners operating at stage 2 progressed in structural thinking, because learners in stage 2 know the relationship between two sets of items and can do addition calculations with concealed items by transferring the concealed items to their fingers.

<u>Stage 3 – the Initial Number Sequence</u>: The stage 3 learner can count-on because she can 'anticipate' the numerosity of the counts in the first set (in $5 + 4$) and keeps track of the counts of the first set in her mind. Thus, the learner does not recount the 5 in $5 + 4$ but rather counts-on and says "6, 7, 8, 9… the answer is 9" (Wright, Martland, & Stafford, 2006).

According to Wright et al., this 'anticipation' and keeping track is the hallmark of stage 3. Thus, the empirical indicator or observed strategy of a learner operating at stage 3 is to count-on from a number.

Therefore, the stage 3 learner has accomplished 3 skills that the stage 2 learner could not do. Firtsly, the stage 3 learner can count-on to find the sum of two sets. Secondly, she can count-up-from to find the missing addend. And lastly, she recognises subtraction for the first time.

This stage refers to learners who understand one of the sets as a composite unit and can count-on. Since the learner starts to count on from 6, she seems to be aware that the 5 has already been counted, and she has formed a reified understanding of 5. When the learner counts-on, she knows the relationship between the number that is a composite unit and its successor number. Thus, the 5 is a composite unit for a learner at this stage and does not need to be counted again.

This stage also includes the ability to count-up-from to solve missing addend tasks like $6 + ? = 9$. The learner would say "7, 8, 9" (while keeping track on 3 fingers) and answer "3".

Stage 3 is the first stage in which the learner recognises subtraction of two sets for the first time and understands subtraction as 'taking away'. She can count backward to solve tasks that involve taking-away. This stage does not include the learner's ability to count-down-to to solve the missing subtrahend like $17 - 14$ (saying $16, 15, 14$: the answer is 3) because she cannot see the 'small difference' relationship between 17 and 14 yet. The learner in this stage will try to take away 14 by counting 14 backward from 17.

This stage indicates that the learner understands the differences between addition and subtraction and that addition is a process of 'adding more' and subtraction is a process of 'taking away'.

Stage 3 refers to the learner with a reified understanding of the first part of the addition and subtraction operations, but not yet the second part. Counting by ones is still an essential strategy at this stage.

The anticipation of the numerosity of the items in the first set shows a higher level of structural understanding. The learner can count-on because she knows the numerosity of the first number and sees the first number in relation to the adjacent numbers. The latter means she reified the first number. Since she knows the first number concerning the adjacent numbers, she is capable of counting-on or counting backward. Thus, stage 3 indicates a higher level of structural understanding because she knows the relationships between a number and its sequel/prequel.

<u>Stage 4 – Intermediate Number Sequence</u>: The learner in stage 3 (previous stage) can count-on from 6 in $(6 + 3)$ for addition tasks and count-backward from 9 in $(9 - 3)$ for 'taking away' tasks in subtraction. The empirical indicator of a learner operating at stage 4 is that she has knowledge of number relationships in subtraction tasks where there is a small difference between the minuend and subtrahend, and counting-on from the subtrahend is a more appropriate strategy than 'to take the subtrahend away'. Another empirical indicator is the stage 4 learner can find the missing subtrahend by counting 'down-to'.

For example, the learner recognising the small difference between the minuend and the subtrahend in $17 - 14$: the learner on level 3 will try to take-away 14 from 17. However, the learner in stage 4 realises the relationship between 17 and 14. She realises the small difference between 17 and 14, and knows that the shortest way to find the answer is to count-up from 14 to 17 or count-down from 17 to 14 to find the answer.

The stage 4 learner changes the subtraction task $(17 - 14)$ into an addition task, indicating a higher structural understanding than the learner in the previous stage. Changing the subtraction task into an addition situation requires knowledge of quantitative relationships because she knows how to manipulate the numbers by changing the subtraction task into an addition strategy.

The learner in stage 4 can also find the missing subtrahend by counting down. If we set the following problem: $15 - ? = 13$ (missing subtrahend) in a problem context, a learner on stage 4 knows that she can count down from 15 and say 14, 13, and state the answer of 2. Alternatively, a learner may count on from 13 to 15 and say the answer is 2.

Thus, the observable strategy for a learner to be on stage 4 is to count on his fingers from 14 to 17 to find the answer of 3 for $17 - 14$, or to count on his fingers from 13 to 15 to find the missing subtrahend in $15 - ? = 13$.

To sum up, stage 4 refers to the learner who can count-down-to to solve tasks like $17 - 14 = ?$ and missing subtrahend tasks like $15 - ? = 13$. The learner in stage 4 can choose the more efficient way to do addition or subtraction problems efficiently because she knows number relationships and quantitative relationships. The learner reified both the numbers and saw the numbers in relation to each other, which refers to mathematical structural thinking of a higher level than before.

Stage 5 – Facile Number Sequence: The learner in stage 5 uses a variety of procedures, of which count-by-ones is not among them. Thus, the observable strategy is an immediate mental answer. The learner functioning at stage 5 may use strategies like compensation, known facts, knowing subtraction as the inverse of addition, base-ten knowledge (which includes bridging through 10), and commutativity in additive and subtractive tasks. The learner also realises, and understands, the inverse relationship between addition and subtraction and can use it interchangeably to solve problems.

Strategies may include using a known fact, $3 + 3 = 6$, and deriving that $53 + 3 = 56$. All the strategies mentioned above require knowledge of number relationships and quantitative relationships because, to use these strategies, the learner has to understand the mathematical situation and know how to manipulate the numbers effectively.

Thus, the learner at stage 5 has progressed to the highest level of mathematical structure in number knowledge because she can fluently use mathematical structural knowledge in effective ways to solve mathematical problems immediately.

4.6.2 Base-ten arithmetical learning

The second aspect of Part A is base-ten arithmetical strategies. According to Wright et al. (2006), base-ten arithmetical strategies refer to the learner’s ability to see 10 as a composite unit together with 10 as 10 units.

The SEAL is founded on base-ten thinking, an essential aspect of the LFIN framework, and indicates the extent of a learner’s structural thinking of the number 10. Knowledge of the base-ten structure is necessary when learners use strategies like bridging through 10 and compensation to solve mathematical problems.

<u>Level 0 No Concept of 10</u>: Level 0 refers to a learner who cannot count-on or count down and does not understand 10 is a reified object that exits out of 10 units but is also a composite unit. Additionally, the empirical indicator for a learner to be on level 0 is that the learner is on stage 2 of SEAL or lower. The latter means that she cannot yet count-on; furthermore, she cannot see the number relationships of 4 and 6, or 3 and 7, and regarding 10. So she does not understand the number-bonds that add up to 10.

Hence, the observable strategy for a learner to be on level 0 is that she cannot count-on from 5 to find the answer of $5 + 4 =?$ When asked, she also cannot answer, “If I give you 6, what would you give me to make 10?”

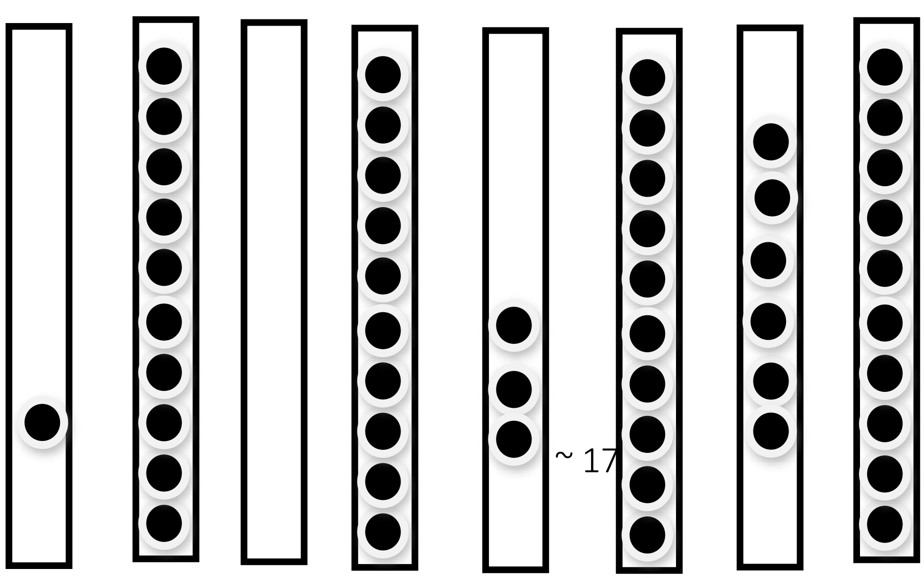
<u>Level 1 Initial Concept of 10</u>: The learner on level 1 can count-on or down from a number; however, the learner cannot see 10 ones as a composite unit and focuses on the 10 individual items that make up the 10 (Wright, Martland, Stafford, & Stanger, 2010, p. 9).

However, when I remarked on the levels of base-ten in my study, I used knowledge of bonds of 10 to be an indicator of level 1 because it can be argued that the initial indicator of a learner knowing base-ten is an understanding of the bonds of 10.

The difference between level 0 and level 1 of base-ten thinking is that the level 1 learner knows the bonds of 10, and the level 0 learner does not.

Thus, the observable strategy for a learner to be on level 1 of base-ten is that the learner would answer “4” when asked: “I give you 6, what should you give me to make 10?” Here the learner knows the number relationships between 4, 6 and 10.

<u>Level 2: Intermediate Concept of 10</u>: The learner sees 10 as a unit and sees 10 as 10 ones simultaneously but cannot solve problems involving 10s and 1s without using physical items, like screened 10-strips and 1s. At level 2, the learner has constructed 10 as a composite and can see 10 as a mathematical object concerning other numbers; however, the learner needs a physical representation of the mathematical object 10 (for example, a screened 10-strip). An example of an addition problem in a physical representation of the composite unit 10 concerning other numbers is shown in Figure 4.1 below.



3+ 10+ 2+ 10+ 5+ 10+ 8+ 10 = __?___

(= 13) (= 15) (= 25) (= 30) (= 40) (= 48) (= 58)

Figure 4.1 An example of the screened ten-strip tasks

The interviewer screened all the strips except the one that had to be added. For example, a question like this will start with the interviewer asking how many dots are there while screening all the strips except the first strip of 3 dots. The learner must keep the 3 in her head. The interviewer will screen all the strips from both sides of the first 10-strip and ask, “How many if you add these?” The learner has to keep 13 in mind because the 3-strip and the 10-strip will be screened next, and the learner will only see the next strip (the 2-strip). The process will continue until the last strip, with an answer of 58.

A learner on level 2 can solve a problem like the one featured above, where the physical representation of the 10-strip is available when adding another addend. Furthermore, the level 2 learner cannot answer written number sentences involving 10s and 1s.

Thus, the observable strategy for a learner to be on level 2 of base-ten thinking is to calculate with physical items like 10-strips when confronted with addition and subtraction problems involving 10s and 1s.

<u>Level 3: Facile Concept of 10:</u> The learner on level 3 of base-ten thinking sees 10 as one unit and as 10 ones simultaneously and can solve addition and subtraction problems appropriately using 10s and 1s without using physical items. The level 3 learner has reified the number 10 and can act upon it. For example, a learner that knows $34 + 28 = 62$ will know $340 + 280 = 620$.

Furthermore, the level 3 learner can find answers to problems involving 10s and 1s without the help of physical items like 10-strips. The observable strategy that a learner on level 3 would give is an immediate answer to questions involving 10s and 1s in the form of written number sentences or word problems.

Thus, the base-ten knowledge levels specify the extent of structural thinking as each level indicates a greater awareness of the relationship between numbers.

4.6.3 Forward Number Word Sequence (FNWS)

FNWS evaluates the ability to say the number words in the correct sequence and give the number word after a given number (NWA). The development of FNWS is designed in the following way:

<u>Level 0 – Emergent FNWS:</u> The learner on level 0 cannot recite the FNWS in the correct order. Thus, the observable strategy of a learner functioning on level 1 is that the learner cannot say the number sequence from 1 to 10 in the correct order, and therefore, no mathematical structural knowledge is evident.

The number ranges of the levels are different. The number range for levels 1, 2 and 3 is 1 to 10, and learners in levels 1, 2 and 3 can say the number words of 1 to 10 in the correct order. However, they are different in the following ways:

<u>Level 1 – Initial FNWS:</u> At level 1, the learner can recite the sequence from 1 to 10 but cannot say the NWA a number in the number range 1 to 10 because she does not understand the concept of NWA or cannot see the number relationship between the number and its successor. No mathematical structural thinking is visible in the level 1 learner.

<u>Level 2 – Intermediate FNWS ‘up to 10’ and Level 3 – Facile FNWS ‘up to 10’</u> refer to the learner who understands the NWA and can say the NWA but uses different strategies to answer. To find the NWA, the level 2 learner has to ‘drop back’ to 1. For example, in a question: “What is the number after 6?” the learner has to count from 1 to find what the number word after 6 is by saying 1, 2, 3, 4, 5, 6,7… and answering, “7 comes after 6”.

Thus, the level 2 learner is busy developing a sense of structure but uses actions of unit-counting to find the NWA. He has not yet acquired an understanding of the number relationships between the number and its sequel to be able to give an immediate answer of the sequel.

The level 3 learner has established knowledge of a relationship between the number at hand and its sequel. Thus, he can produce the NWA without returning to one. When the learner can give the

NWA without returning to one, an awareness of the relationship between the relevant number and its sequel has formed, and the learner has attained structural knowledge of numbers 1 to 10.

The descriptions of levels 4 and 5 are the same as level 3; however, they differ in the number range.

For Level 4 – Facile 'up to 30', the learner knows the relationships between numbers and their sequels in the number range 1 to 30, and for Level 5 – Facile 'up to 100' the learner knows the relationships between numbers and their sequels in the number range of 1 to 100.

To sum up: levels 0 to 3 demonstrate a progression in structural thinking. In level 1, no structural knowledge is visible. The level 1 learner knows the structure of the number words in the correct order but does not know the number and its sequel and thus does not know numbers in relation to each other. The level 2 learner knows a number and its sequel but has to count from 1 to find the sequel, which indicates a dependency on unit-counting. This learner does not show knowledge of number relationships and therefore acts on a deficient level of structural thinking. The level 2 learner's thinking is still processual[43]. The level 3, 4, and 5 learner knows the numbers concerning each other and can use structural thinking to find the NWA, but it requires knowledge in different number ranges.

Level 3 requires structural knowledge in the number range is 1 to 10, level 4 requires knowledge of structure in the number range 1 to 30, and level 5 requires structural thinking in the number range 1 to 100. Thus, when the number range increases, the learner's mathematical structural thinking expands.

4.6.4 Backward Number Word Sequence (BNWS)

The BNWS and FNWS are similar in that they correspond in terms of the number ranges, but for BWNS, the levels require a learner to count backward.

[43] Sfard describes processual as relating to processes and actions rather than relating to reified objects. In the context of early number, the action of counting objects to solve addition and subtraction problems can relate to processuality. Structurality, in contrast, is described as follows: "Seeing a mathematical entity (like a number) as an object means being capable of referring to it as if it was a real thing" - i.e. it having a static structure, existing somewhere in space and time. It also means being able to recognise the idea at a glance "and to manipulate it as a whole, without going into details" (1991, p. 4).

Thus, the different levels of BNWS indicate the learner's ability to say the number words in backward order from any number. It also requires a learner to tell the number word before (NWB) of any number.

Level 0 – Emergent BNWS: A learner on level 0 cannot count backward from 10 to 1 and, therefore, does not understand the NWB a number.

Level 1 – Initial BNWS: The learner on level 1 can repeat the BNWS from 10 to 1 but cannot produce the NWB, whereas the learner on levels 2 and 3 can produce the NWB using different strategies.

Level 2 – Intermediate BNWS: The level 2 learner counts from one to find the NWB. This learner's understanding is still processual, and the learner is not yet aware of the relationships between numbers.

Level 3 – Facile BNWS 'up to 10': The level 3 learner gives the NWB immediately. An immediate answer to the number just before the mentioned number is an indication that the learner understands the number and its prequel. At this level, the learner has a mathematical structural understanding of the numbers between 1 and 10.

Level 4 – Facile BNWS 'up to 30' and Level 5 – Facile BNWS 'up to 100': The learner on levels 4 and 5 can count backward and understand the relevant number and its prequel in the number range 30 to 1 (level 4) and 100 to 1 (level 5), respectively. (Wright, Martland, Stafford, & Stanger, 2010, p. 11). The realisation of the relationships between the numbers and their prequels expands as the number range increases. Each successive level indicates a more advanced level of structural thinking because, as the number range becomes larger, the knowledge of mathematical structure expands.

4.6.5 Numeral Identification (NI)

The last aspect of Part B is numeral identification and shows the range in which a learner can identify numerals.

The different levels of numeral identification depend on the number range only. Level 0 – Emergent Numeral Identification: Cannot identify any numbers; Level 1 – Numerals up to 10: can identify one-digit numbers; Level 2 – Numerals up to 20: can identify numbers in the number range 1 to 30; Level 3 – Numerals up to 100: can identify 2-digit numbers; and Level 4 – Numerals up to 1 000: can identify 3-digit numbers and 1 000 (Wright et al., 2010, p. 12). Identifying numerals in higher number ranges shows a higher level of structural thinking.

4.7 Analysis of the Grade 6 Evidence of Structural Thinking Assessment's (G6 ESTA) evaluation framework in terms of the extent of structural thinking

The Grade 6 ESTA interview assessment was, in essence, developed to find evidence of structural thinking in observable strategies learners use. I video-recorded the interviews because, as in the G2 LFIN assessment, I wanted to go back and look at the learners' responses intensively to see if I could find evidence of structural thinking in their actions when they answered mathematical problems.

The assessment was developed as a mental mathematical calculation test. However, we know that in the South African landscape, mental calculations are not the preferred way of doing mathematics (Fleisch, 2008). Therefore, I gave the learners more options by providing pen and paper and counters if they preferred to use written or counting strategies. In this way, I was also able to observe possible empirical evidence of mathematical structural thinking in written calculations.

I established an assessment framework and used the term 'stages' to describe the development of strategies used for the G6 ESTA. Wright et al. (2006) borrowed the term 'stages' from von Glasersfeld and Kelly (1982) and applied it to the development of early arithmetical learning. I will also use the word 'stages' to explain the hierarchical development evidently when assessing the observable strategies to the G6 ESTA. The rationale for using the term 'stages' is that the G6 ESTA framework has four characteristics, namely: the stages stay constant over time, the different stages form an invariant sequence, each stage builds on and includes the preceding stages, and each new stage includes a substantial conceptual reform (von Glasersfeld & Kelley, 1982).

Different empirical indicators in the form of 'strategies used' determine the stage at which the learner is functioning. The more sophisticated a strategy is, the higher the stage. A more sophisticated strategy indicates a higher level of structural thinking.

The assessment entails two parts: additive reasoning and multiplicative reasoning. The learners' strategies ranged from working with concrete items to using sophisticated strategies, like giving an immediate mental answer. I developed an assessment framework for both additive and multiplicative reasoning.

4.7.1 Stages for Additive Reasoning (AR)

The stages for additive reasoning were formulated to show the progression of structural thinking in additive reasoning.

AR Stage 0 – incorrect or no answer: The observable strategy for a learner to be placed at stage 0 is that he cannot answer additive reasoning questions like $12 + 85 =?$, $435 - 399 = ?$, and $273 + 99 =?$

AR Stage 1 – Perceptual counting: The observable strategy for a learner to be placed at this stage is to use strategies involving physical items and unit-counting to find answers. For example, in, $12 + 85$, the learner may start with 12 and count-on to 97 or may begin with the larger number, 85, and count-on to 97. Both these strategies are included in stage 1. The stage 1 learner does not show knowledge of mathematical structure and is dependent on physical items and unit-counting to find the answers.

AR Stage 2 – Formalising without structural understanding: Learners in this category use a written calculation or algorithm combined with physical items to support the written algorithm. This learner knows the written algorithm for addition and subtraction but uses perceptual counting to find the partial sums and differences. For example, a learner uses the written vertical column algorithm and their fingers to find the partial sums and differences. Learners at this level can execute the written column algorithm; however, they cannot use structural thinking to find the partial sums and differences. For example, when they have to add $353 + 876$, they count on their fingers to get the answers $3 + 6$, $5 + 7$, and $3 + 8$. Thus, they do not have the bonds of 20 in their repertoire of known facts.

AR Stage 3 – Formalising with structure: A learner at this stage calculates through the written-column method but can find the answers to the partial sums and differences using structural thinking. I also include a delayed mental answer, with no visible unit counting in stage 3. The strategies in stage 3 are responses where the learner answered the question with a written algorithm, as in stage 2; however, the learner did not use his fingers to calculate the partial sums and differences but found the partial sums and differences mentally. This strategy is more sophisticated than the previous level because the learner did not rely on unit-counting but could determine the answers of the partial sums mentally, which showed a mathematical structural understanding of the bonds between all the numbers in a range from 1 to 20. Level 3 also includes strategies where the learners gave delayed mental answers. When I asked them how they obtained the answer, they explained that they used the vertical column algorithm in their minds. Learners capable of performing a column method in their heads can do the partial sums and differences mentally and keep track of the calculations in their minds.

AR Stage 4 – Structural Additive Reasoning: The last category is for responses that took the form of an immediate mental answer (less than 8 seconds). Thus, the observable strategy is an immediate mental answer. An immediate answer implies knowing quantitative relations and, therefore, structural thinking. The learner at this level could work out the solution by using structural thinking in the form of either compensation, bridging through 10 or 100, inverses, or derived facts, to name a few.

4.7.2 Stages for Multiplicative Reasoning (MR) for the Grade 6 Evidence of Structural Thinking (G6 ESTA)

Multiplicative reasoning is comprised of multiplication (Mul) and division (Div). I developed two different evaluation frameworks for multiplication and division. The first evaluation framework that I discuss is the evaluation framework for multiplication. I will then turn my attention to the evaluation framework for division. I borrowed the names of the stages for multiplicative reasoning from Wright et al. (2006).

Stages for Multiplication

Mul. Stage 0 – Incorrect answers: Learners were classified as level 0 when they provided an incorrect answer or did not give an answer.

Mul. Stage 1 – Perceptual sharing and grouping in ones: The learner at this level can establish equal groups, but she uses counters and unit-counting. The learner at this level uses concrete items like fingers or tallies to find the numerosity of both the multiplicand and multiplier.

She can make a specified number of groups of a specified size and find the total number of items. For example, the learner can make four groups with five items in each and say there are 20 items; however, she needs the items to be visible, and she uses unit-counting to find the number of items in each group. The task involved showing the learner 8 cards containing 9 dots on each card and asking how many dots there are there altogether. A stage 1 learner would draw 9 tallies 8 times and then count from 1 up to 72 to execute the answer. Strategies at this level demonstrate that the learner has not yet acquired the capacity to reify the multiplicand or the multiplier.

The observable strategy for a learner on level 1 is that she is still reliant on physical items and does not use structural thinking to answer multiplication problems because she does not know the relationships between the multiplicand and the multiplier.

<u>Mul. Stage 2 – Perceptual sharing and grouping in multiples</u>: The learner at this level can group a number up to 5 items in a group without counting them first; however, the items must still be visible. The observable strategy for a learner at Level 2 is that she will 'shift' 5 items as a group without counting them first. In response to a question as to how many items are in 4 groups of 5 items, she will 'shift' 5 items in 4 groups and then count in multiples of 4; 5, 10, 15, 20, or use skip-counting to arrive at an answer of 20. Although the learner does not need to count the 5 items, she still needs to see and feel them.

The learner has not yet reified 4 (multiplicand) or 5 (multiplier) as mathematical objects and thus cannot imagine 5 without seeing the physical items when working with this problem. Still, she has an idea of what 5 physical items are without counting them. Learners have a preliminary knowledge of an inherent capacity to subitize without serial enumeration (Mulligan et al., 2018). Learners in this stage can subitize the number of items up to 5 without counting them one by one (Wright et al., 2010).

This stage is on a higher level of structural thinking because the stage 1 learner cannot subitize five items without counting them first, whereas the stage 2 learner can 'see' the 5 items without counting. When 'seeing' 5 items without counting, the child knows the cardinality of 5. When counting in 5s, the learner understands the relationship between 5, 10, 15, and 20 and thus progresses in structural thinking.

Mul. Stage 3 – Figurative composite counting: The observable strategy is that the learner in stage 3 can calculate answers for multiplicative problems without using physical items anymore. He has developed counting strategies that do not rely on items or counters to be visible or touchable. For example, learners use skip-counting in multiples while one hand keeps track of how many multiples she has already counted. The skip-counting in stage 2 is different from the skip-counting in stage 3, as in stage 3 the learner can skip-count without using counters. In contrast, for the stage 2 learner, the counters must be visible and available.

The stage 3 learner would skip-count without seeing or feeling the physical items. An example of such a problem is: "Cupcakes are sold as boxes of 5. Zandile bought 3 boxes of cupcakes. How many cupcakes did she buy?" A strategy at this stage would be for the learner to skip-count in 5s up to 15 while keeping track of the 3 boxes on the one hand and answering, "15 cupcakes". A learner at stage 3 has reified the multiplicand but not yet the multiplier, because he still needs his fingers to know the numerosity of the multiplier.

The stage 3 learner has a more developed structural understanding than the stage 2 learner because the former can imagine the set of 5 items in his head, compared to the stage 2 child, who has to work concretely. Stage 3 is on a higher stage.

Mul. Stage 4 – Repeated composite grouping using a written algorithm: The strategies at this stage depend on a written algorithm. For example, they use repeated addition by continuing to double the value if the multiplier is even. Or keep on doubling and add a single value if the multiplier is uneven. Alternatively, they use the vertical column method for multiplication. For example, in response to the question "What is 7×8?" the learner in this stage would add $7 + 7 = 14$, four times, and $14 + 14 = 28$ twice, and then give the answer $28 + 28 = 56$.

At this stage, the learner has reified the multiplicand but not yet the multiplier; however, the learner uses formal written algorithms.

Mul. Stage 5 – Immediate mental answer – Multiplication as an operation of known fact: This stage is the highest level for multiplication in my framework because the learner functions using structural thinking as she can give an immediate mental answer.

She can coordinate two reified numbers in the context of multiplication by quickly deriving new multiplication facts from basic multiplication facts. The learner at this stage knows the commutativity aspect of multiplication and knows that division is the inverse of multiplication. The observable strategy for a learner functioning at stage 5 is an immediate correct response.

Stages for division (Div)

<u>Div. Stage 0 – Incorrect or no answer:</u> The learner in stage 0 answers incorrectly or cannot answer the division question.

<u>Div. Stage 1 – Perceptual sharing in ones</u>: The observable strategy that a learner uses at this stage is that he cannot calculate division tasks without physical items and unit-counting. He will make groups of a specified size from a set of items but needs to count the items one by one. For example: "Given 20 counters, how many groups of 4 can you make?" The learner can make up 5 groups of 4; however, she needs to count the counters in each group of 4 one by one. Creating groups of 4 is called quotative sharing and is the same as the grouping aspect of division.

The learner can also share a set of items into a specified number of groups. For example, the learner in stage 1 can share 20 items into 5 groups, but she needs to share the counters one by one in each group when doing so. An example of such a problem is: "I have 20 apples, and I want to put them in 5 bags. How many apples will go into each bag?" Learners that share counters in 5 groups one by one until all the counters are shared will fall into stage 1. The sharing of the items one by one is referred to as partitive sharing and is the same as the 'sharing' aspect of division.

A learner in stage 1 cannot subitize the number 4 yet and cannot see a group of 4 as a unit in quotative sharing. Learners at stage 1 have not yet reified the divisor or the dividend, and use unit-counting to find both dividend and divisor numerosity. Observable strategies at this stage show that the learner cannot see the dividend or the divisor as mathematical objects and thus is deficient in structural thinking.

<u>Div. Stage 2 – Perceptual sharing in groups</u>: The observable strategy is that the learner at stage 2 is still dependent on physical items but can subitize the divisor, whereas the stage 1 learner cannot.

A learner functioning at this stage can group a number of up to 5 items in a group without counting them first; however, the items must still be available and visible.

In response to a question, “Share 24 sweets among 4 friends”, the learners would count out 24 counters and share counters in groups of 2s, 3s, and 4s. For example, a learner at this stage would share groups of 3 counters by ‘shifting’ them without counting. He would then realise that there are still counters left. He will keep sharing another 3 counters to each friend until all the counters are shared. The sharing then resulted in an answer of 6 sweets for each friend. There are more options, for example: to share 2 counters 3 times with each friend or share 4 counters and then another 2 counters to each friend. All the later observable strategies show that the learner is on stage 2.

The stage 2 learner is on a higher level of structural thinking because he can subitize the groups he is sharing with each friend. He is no longer dependent on unit-counting to find the group of items he is sharing; however, the physical items still need to be available.

Div. Stage 3 – Figurative grouping: The observable strategy to be in stage 3 is that the learner is less dependent on physical items. The learner has developed counting strategies that do not rely on items or counters to be visible or touchable for the dividend or the divisor; however, she still uses her fingers to keep track of the quotient. For example, “I have 20 apples. I put them in bags of 4. How many bags do I need?” Learners at this stage will count in 4s, saying 4, 8, 12, 16, 20 while keeping track on their fingers and replying “5 bags”.

Trial and error strategies, where the learner would try out different quotients, were also evident for a learner to be in this stage. For example, in a task: “Share 24 sweets among 4 learners”, learners would try out $4 + 4 + 4 + 4$ and see that the values were insufficient, then try out $5 + 5 + 5 + 5$ and still discover that the division was incomplete. They would then try out $6 + 6 + 6 + 6$ and realise that each learner would get 6 sweets.

This stage is on a higher level of structural thinking because the child is less dependent on physical items but can imagine the groups without using counters.

Div. Stage 4 – Repeated composite sharing or grouping using a written algorithm: The learner uses structural thinking and formal (written algorithm) understanding of division where written algorithms are visible as observable strategies. I included a delayed mental response in stage 4 and written algorithms like repeated subtraction on paper as well as long-division algorithms.

The learner at this stage is on a higher level of structural understanding because she is using various written algorithms and formal mathematical knowledge to calculate division problems.

Div. Stage 5 – Immediate mental answer: The learner uses strategies indicating he has a reified understanding of the multiplier and the multiplicand. To be on stage 5 the child uses structural thinking. For example, learners understand division as an operation and know that division is the inverse of multiplication.

The learner used strategies that show the understanding of division as an operation where he knows the meaning of the dividend, divisor and quotient. He can find answers to questions using amended or known facts. For example, the learner can find the answer to 9×7 because he knows $10 \times 7 = 70$ and subtracts 7 from 70. The observable strategies in stage 5 are an immediate mental answer as a recalled known fact.

4.8 Conclusion

From the analysis of the assessments, I concluded that the G1 ANAdG2 did not measure structural thinking to any significant extent because only 6% of the tasks required insight into mathematical structure accompanied by formal mathematical knowledge. However, the G6 CA measured structural thinking to a greater extent than the G1 ANAdG2 because 63% of the tasks required insight into mathematical structure accompanied by formal mathematical knowledge. Therefore, a high mark in the G1 ANAdG2 does not necessarily show an advanced level of structural thinking, but a high mark in the G6 CA does.

The G2 LFIN is in essence grounded in structural thinking and the G6 ESTA was specifically developed to measure structural thinking. Thus, a high mark in both suggests a high level of structural understanding.

To conclude, both the Grade 6 assessments and the G2 LFIN measured structural thinking. However, the G1 ANAdG2 did not measure structural thinking significantly.

CHAPTER 5 THE EXTENT OF STRUCTURAL THINKING EMPIRICALLY IN QUANTITATIVE TERMS

5.1 Introduction

This chapter discusses the evidence of the extent of structural thinking the 64 participants showed through their empirical responses on all four of the assessment instruments. With graphs, tables, and discussions, I represent how the participants responded to the questions/tasks in all the assessments/tests that reveal structural thinking.

While in Chapter 4 I established whether the assessments addressed structural thinking, the purpose of Chapter 5 is to investigate whether structural thinking was revealed by the participants' responses empirically. In doing so, I will examine the participants' responses qualitatively by contemplating the three elements of structural thinking, namely knowledge of relationships between unenumerated quantities, numbers, and quantities

Therefore, in this chapter, my focus will be on those items where structural thinking was prevalent; however, I will also mention how well they performed in some cases where structural thinking was not essential. I am starting my qualitative dialogue by showing the participants' responses on the two Grade 2 assessments (G1 ANAdG2 and the G2 LFIN), followed by the two Grade 6 assessments (G6 CA and the G6 ESTA).

Before starting the discussion, it is imperative to repeat that the assessments are different in purpose and mark allocation. The G2 LFIN and the G6 ESTA are diagnostic tests and the G1 ANAdG2 and G6 CA are not. The data of the diagnostic tests were qualitative, and a correct answer could be coded in different levels or stages depending on how sophisticated the strategy was when the participant answered the question.

Since the G6 ESTA was developed for this study, I emphasise its empirical findings because the tasks were specifically constructed to measure structural thinking. My discussion shows how well or insufficient the participants perform on each assessment question. I am also showing how differently, from what I expected, the participants respond to tasks in the G6 ESTA that were amendable to measure structural thinking.

5.2 G1 ANAdG2 - Empirical evidence of the extent of structural thinking

The analysis in the previous chapter regarding the extent of structural thinking, the four assessment instruments evaluated, showed that the G1 ANAdG2 did not measure structural thinking to a great extent.

Furthermore, the G1 ANAdG2 assessment was of such a nature that most of the questions could be answered by unit-counting strategies, and structural thinking was not essential to answer most of the questions. The inability to measure structural thinking is also because of the low number range of 1 to 35 (except for one question where the learners had to count in 10 to 100). The low number range allowed learners to draw tallies up to 35 (which is not many) or use their fingers to find answers to tasks. Using tallies and fingers does not show structural thinking. Thus, we expect the learners to do very well in the G1 ANAdG2.

However, some questions were challenging. The graph below shows the percentage of correct responses for every question in the G1 ANADG2 in ascending order. Thus, the first question in the graph was the most difficult question, with the least correct answers.

The bar graph below (Figure 5.1) shows the frequency of learners with correct responses in percentages on the number-related questions in the G1 ANAdG2.

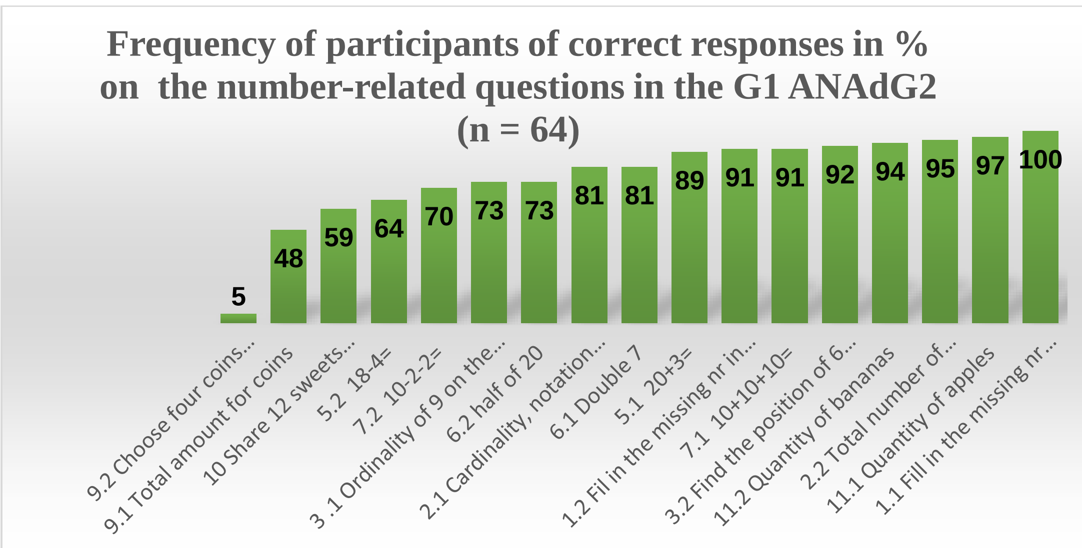


Figure 5.1 Percentage of correct responses on the number-related questions in the G1 ANAdG2

The highest number of correct answers was for question 1, with 100% correct answers. In question 1, the participants had to fill in missing numbers. The number sequence 3, 4, ?, 6, 7, ?, 9, ?, 11 was given, and the question required learners to fill in the missing numbers. This task requires counting from 0 to 11 and knowing the number symbols from 0 to 11 only. No knowledge of relationships or structural thinking was needed to answer this question and hence the high number of correct responses. The high number of correct responses could also be because the learners repeatedly count in ones in Grade 1. Besides, most of the counting sequences in the early primary mathematic textbooks are counting in ones.

Another two tasks with very high correct answers reported were questions 11.1 and 11.2, where the participants had to count apples/bananas in a picture. (See task 11 in Figure 5.2.).

Thabo bought apples and bananas at the shop.

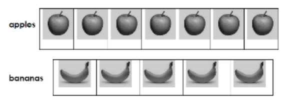


Figure 5.2 Task 11 in the G2 ANAdG2

For both questions, knowledge of the counting process and the notations '7' and '5' were required. Structural thinking or knowledge of relationships was not a prerequisite for this question. These questions did not need the learners to know the relationships between numbers, but only how to count and write the number symbols.

The bar graph shows that the percentage of correct responses to the subtraction questions was relatively lower than the addition operation questions. What is also necessary to mention is that the participants found the instruction "half of 20" (73% correct responses) much harder than "double seven" (81% correct responses). The 'double 7' can be found by saying $7 + 7$ and unit-counting, but to find 'half of 20' requires knowledge of relationships and thus number structure. The 'double' and 'halve' questions need learners to know number relationships.

Questions 3.1 and 3.2 are about the position or ordinality of numbers; however, the correct responses differ broadly. For question 3.1, there were 73% correct answers, while question 3.2 had 92% correct responses.

Figure 5.3 shows questions 3.1 and 3.2

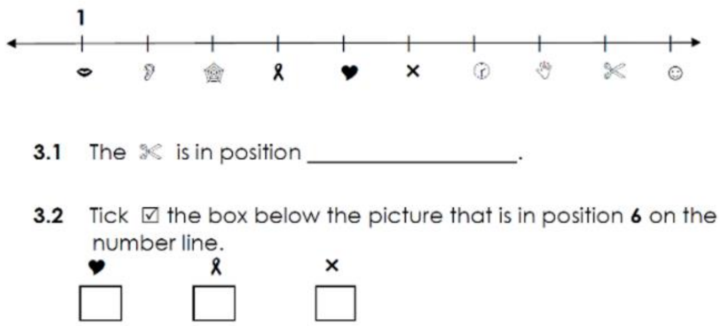


Figure 5.3 Question 3.1 and 3.2 of the G1 ANAdG2 - Ordinality

Question 3.2 had so many more correct answers because, in question 3.2, the participants had to choose between three given options. They only had to tick the right box, while in question 3.1, they had to count to 9 and find the answer by themselves (there were no options to choose from).

The question with the least correct answers was question 9.2, with only three correct responses. (See question 9.2 in Figure 5.4.)

9.2 I have 4 coins in my pocket, and they add up to 25c. Tick (√) the four coins that I have in my pocket

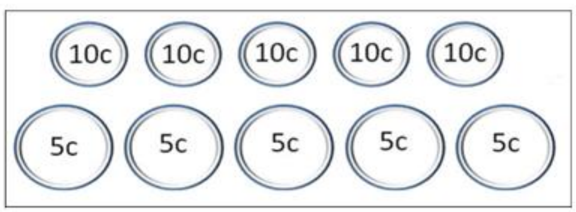


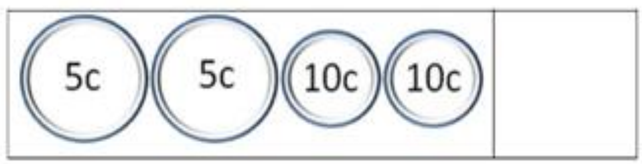
Figure 5.4 Question 9.2 in the G1 ANAdG2

Question 9.2 is about 5 given 5c coins and 5 given 10c coins drawn. The question asked the participants to tick (√) off *four* coins that would add up to 25c. The learner should use structural thinking to answer because the question requires knowledge of a 10c coin and a 5c coin as composite units.

Furthermore, knowledge of relationships between a 5c coin, a 10c coin, and 25c is needed. The participants had to realise that only 4 coins should be ticked off. Thus, they have to recognise that five 5c coins are not the answer, and two 10c coins with one 5c coin also, because it involves three and five coins and not four coins. The latter reasoning necessitates structural thinking, because to find the answer to question 9.2, the learner had to reason quantitatively and figure out which way they could work out the solution.

The second most challenging question for the participants, with 48% correct responses, was also about coins where they had to find the amount of money for two 5c coins and two 10c coins. (See Figure 5.5 for question 9.1.)

9.1 Look at the 10c and 5c coins in the question.



The total amount of money is_____. Write your answer in the box.

Figure 5.5 Question 9.1 in the G1 ANAdG2

The third most challenging question for the participants, with 59% correct responses, was question 10, where the participants had to share 12 sweets among three children. Question 10 requires knowledge of partitive sharing of division and entails structural thinking.

To sum up, the tasks in the G1 ANAdG2 that could be answered with unit-counting strategies showed many correct answers, while the tasks that required structural thinking did not. Most of the tasks required counting and knowledge of number relationships, and one task required knowledge of quantities.

5.3 G2 LFIN - Empirical evidence of the extent of structural thinking

The G2 LFIN is one of the data instruments exists out of five aspects which are the Stages of Early Arithmetical Learning (SEAL), forward-counting (FNWS), backward-counting (BNWS), numeral identification (NI), and base-ten thinking. In this section, I discuss the results of the participants' responses on each aspect.

5.3.1 Stages of Early Arithmetical Learning (SEAL)

Wright et al. (2006) regard the SEAL as the essential aspect of the LFIN because all the other aspects contribute to the SEAL a learner would be in. The stages in SEAL are S0: emergent counting, S1: perceptual counting, S2: figurative counting, S3: initial number sequence, S4: intermediate number sequence, and S5: facile number sequence. Figure 5.6 shows the percentage of participants in each stage.

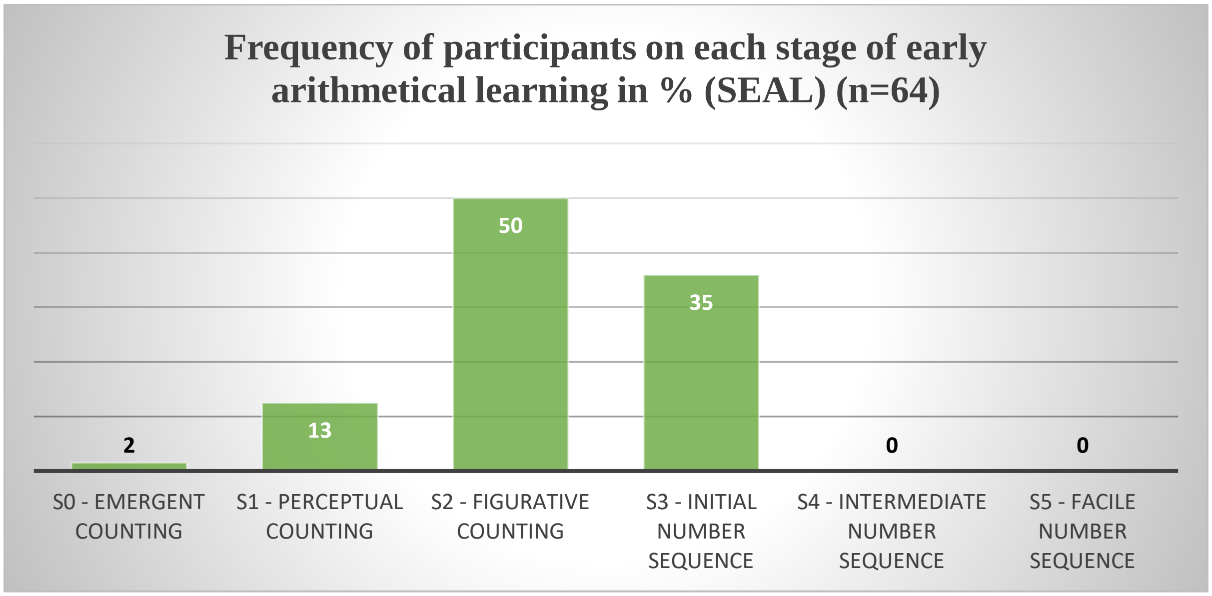


Figure 5.6 Percentage of participants on each stage of early arithmetical learning

One of the participants (2% rounded off) could not count to find answers to addition and subtraction tasks or could not say the number words in the correct order.

A learner in stage 1 can do addition and subtraction tasks only when physical items are available. The 8 participants in stage 1 (perceptual counting) could count-all to find answers to addition and subtraction tasks; however, they could only do so if they used counters to feel and touch.

On the other hand, a stage 2 learner can do addition and subtraction tasks with screened objects where the learners transfer the counts to their fingers and then 'count from one' every time to calculate (triple count of Anghileri (2007)). Half of all the participants were in stage 2 (figurative counting). This stage is also referred to as the count-all stage.

Moreover, the graph shows that 35% were in stage 3 and could count-on or count-down to solve addition and subtraction tasks. No participants were in stage 4 or 5, which tells us that the sample participants could not count down from 17 to14 to solve tasks like 17-14, nor could they count up from 14 to 17 to solve the same task (stage 4). No participant in the sample used strategies referred to as not-count-by-one-strategies, including compensation, using a known fact, commutativity, knowing inverses (stage 5).

Since stages 1 and 2 do not necessitate structural thinking, I can say that 65% of the participants did not show structural thinking significantly, while 35% of them could imagine a 5 in their minds and count-on to solve $5 + 4 =?$ Stage 3 learners can count-on from 5 and say 6, 7, 8, 9 while keeping track on 4 fingers. Stage 3 is where learners stop counting-all and start counting-on or down to solve additive reasoning tasks. Stage 3 is a hallmark stage for Wright et al. (2006) because the learner in stage 3 starts to think structurally as he has a concept of what 5 is and does not need to count it again. He also knows how to count-on and knows the relationship between 5 and 6 from where he starts to count-on.

I also considered the differences and similarities in performance of the participants from township schools and suburban schools. The graph in Figure 5.7 shows the differences.

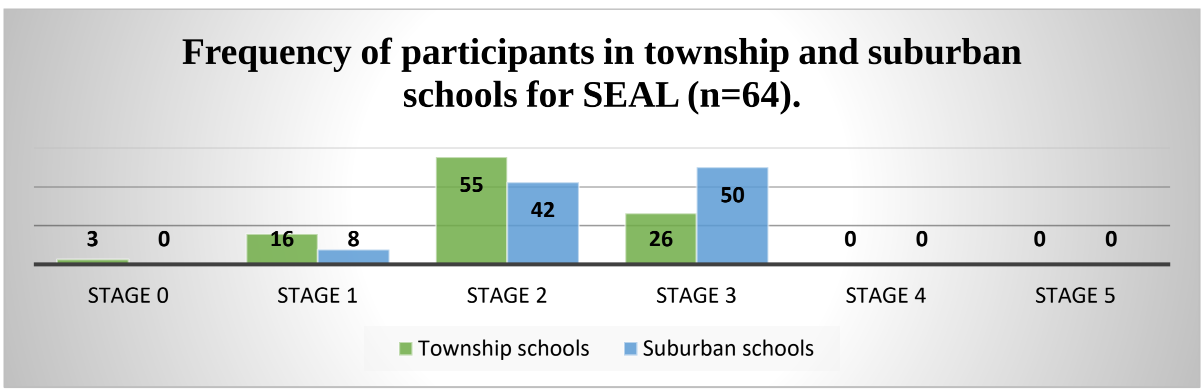


Figure 5.7 Percentage of participants in each SEAL stage in terms of the township and suburban schools

Sixteen percent of the township school participants were on level 1 (where the learner cannot calculate without physical objects), and 8% of the suburban schools' participants were on this level. The 16% of township schools and the 8% of suburban schools participants' reliance on physical items to calculate is worrying if we consider that they were in Grade 2.

Fifty percent of the suburban schools' participants could count-on, while only 26% of the township schools' participants could. The latter suggests that the participants in township schools were more dependent on count-all strategies than those in suburban schools. The findings indicate that the suburban school participants performed better in SEAL than the township schools' participants.

To sum up, if we consider that the assessment was done at the beginning of the participants' Grade 2 year and that some of the questions were in a number range of 1 to 10, the findings suggest that the participants did not perform the way they should because CAPS stipulates that a Grade 1 learner should know addition and subtraction facts to 10.

5.3.2 Forward Number Word Sequence

The Forward Number Word Sequence[44] (FNWS) aspect of the G2 LFIN shows how well a learner can count forward and say the number word after (NWA) without 'dropping back' to one. The difference between level 4 and level 5 is that for level 5, the learner can count and say the number word after in the range 1 to 100, while the level 3 learner can count and say the NWA in the number range of 1 to 30.

The graph below in figure 5.8 shows the percentage of learners in each level of FNWS.

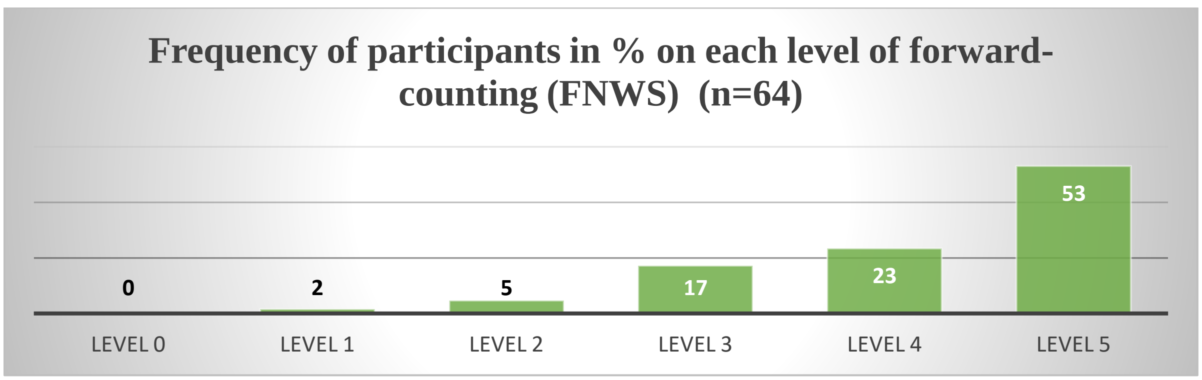


Figure 5.8 Percentage of participants on each level of forward counting (FNWS)

The number of learners in level 5 was just more than 50%. That means that half of the participants could count up to 100 and say the NWA without dropping back to one. About a quarter of the participants could count and say the NWA up to 30. Approximately one-sixth of the learners could count and say the NWA up to 10.

The differences between the township and suburban schools are shown in the bar graph in Figure 5.9. The results show that the participants in the suburban schools performed better in the FNWS tasks than the participants in the township schools.

[44] I discuss the levels of FNWS in section 2.5.5.

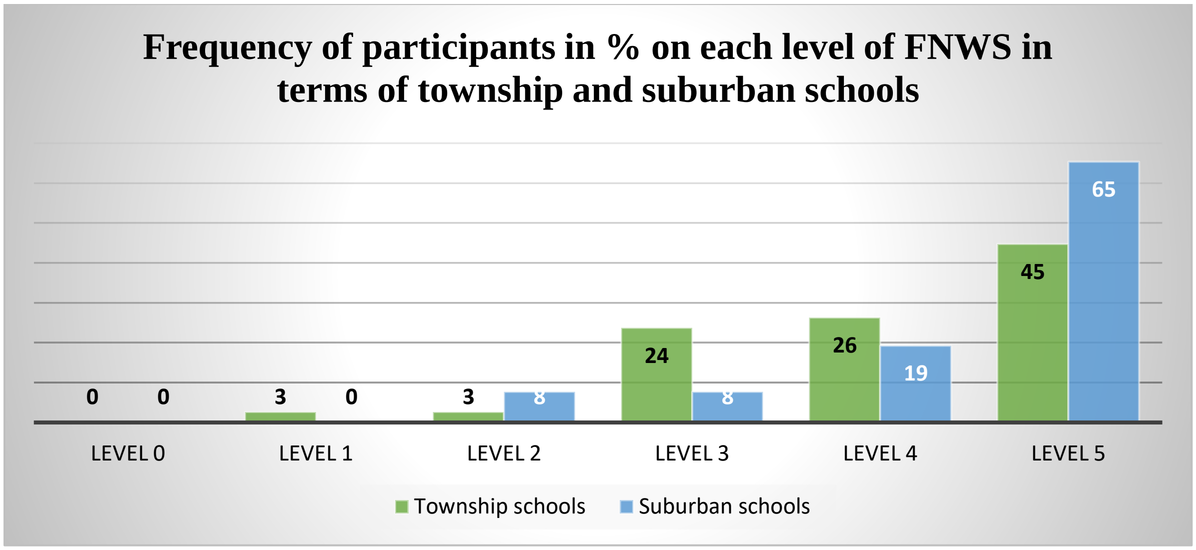


Figure 5.9 Percentage of participants in each level of FNWS in terms of the township and suburban schools

Of the suburban schools' participants, 65% produced the FNWS from any number and could say the NWA in the range 1 to 100, with 45% of the township schools' participants being able to do the same (level 5). The big difference of 20 percentage points between the two groups of participants is significant: it suggests that the townships schools' participants show counting difficulties in the higher number ranges in terms of forward-counting.

A level 4 learner can say the FNWS from any number and the NWB in the number range of 1 to 30 and produce the NWA immediately without dropping down to 1, and a level 3 learner can do the same in the number range from 1 to 10. A substantial number (nearly a quarter) of township schools' participants could only say the FNWS and the NWA immediately in the range from 1 to 10 (level 3).

According to CAPS, a Grade 1 learner (the assessment was taken at the beginning of the participants' Grade 2 year) should be able to say the FNWS in the range of 1 to 100. The findings suggest that only 53% of the participants were in the level they should be regarding forward-counting.

CAPS also stipulates (under mental mathematics) that a learner should be able to name the number before and after a given number in the number range 1 to 20. Thus, regarding the NWA, 24% of all the participants could not name the number after in the number range 1 to 20 (Figure 5.9).

5.3.3 Backward Number Word Sequence (BNWS)

The participants' responses on backward-counting tasks were not as good as on the forward-counting tasks. See Figure 5.10 for the results for backward-counting[45].

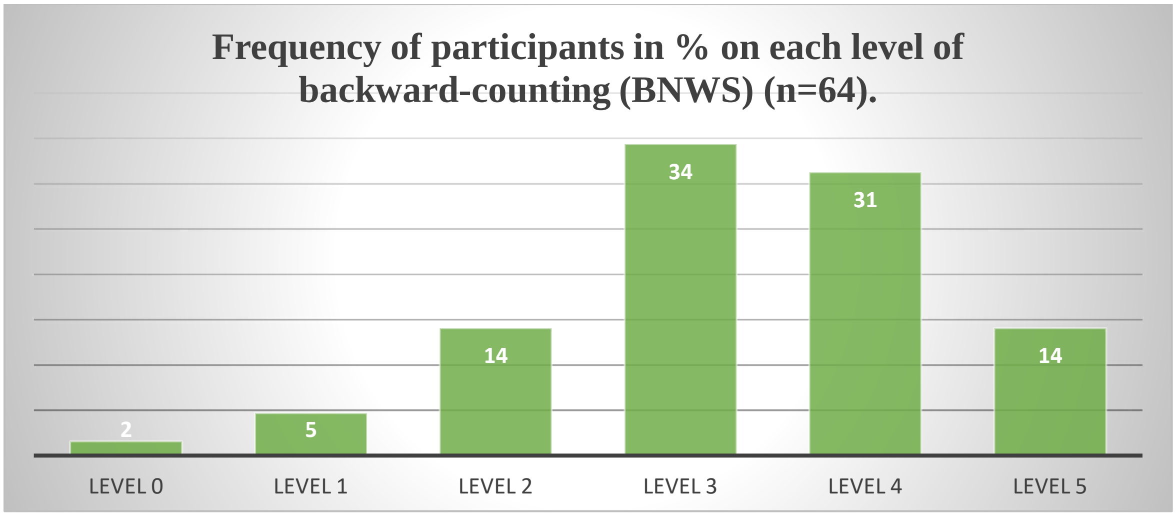


Figure 5.10 Percentage of participants on each level of backward-counting (BNWS)

Figure 5.8 shows that 31% of participants could say the numbers in the correct order and produce the NWB (number word before) in the range from 1 to 30 (level 4). Fourteen percent of the participants could do the same in the number range from 1 to 100 (level 5). A substantial number of participants (34%) could say the BNWS and the NWB in a deficient number range of 1 to 10 (level 3).

The difference between the responses of the suburban schools and township schools can be seen in Figure 5.11.

[45] I discuss the levels of BNWS in section 2.5.5.

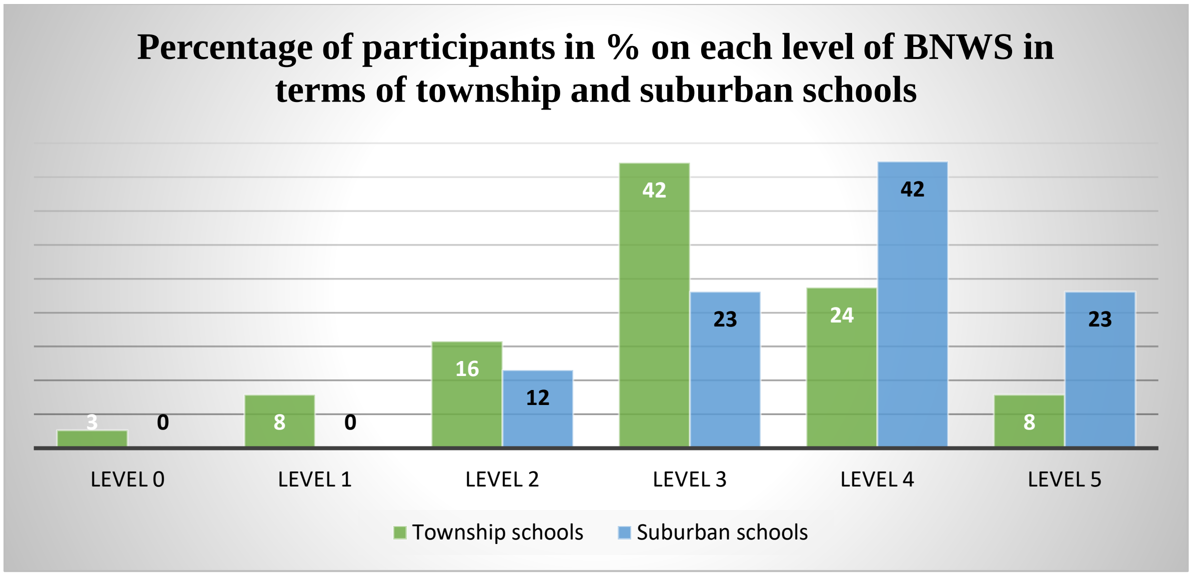


Figure 5.11 Percentage of participants in each level of BNWS in terms of the township and suburban schools

Twenty-three percent of the participants in the suburban schools could count backward from 100 to 0 and say the 'number word before' immediately (without having to count from 1) in the same range (level 5). By contrast, 8% of the participants in the township schools could do the same. The difference of 15 percentage points is significant. Forty-two percent of the participants in the suburban schools could say the backward sequence and produce the NWB without dropping' back to 1 in the range of 1 to 30, with 24% of the participants in the township schools being able to do the same (level 4).

There were no participants in the suburban schools on level 0 or 1, which suggests that all of them could count backward from 10 to 1 and produce the NWB without falling back to 1 in the range of 1 to 10. At the same time, 11% of the participants in the township schools were categorised in level 0 and level 1. The latter suggests that 11% of the participants in the township schools could not count backward from 10 to 1 (level 0), or they could count from 10 to 1 but could not say the NWB in the range 10 to 0 (level 1).

According to CAPS, a Grade 1 learner (the assessment was taken at the beginning of the participants' Grade 2 year) should be able to say the BNWS in the range of 1 to 100. The findings suggest that only 14% of all the participants were on the level they should be (Figure 5.10).

5.3.4 Numeral Identification (NI)

The different levels of NI[46] depend on number range only. Level 0: cannot identify any numbers, Level 1: can identify one-digit numbers, Level 2: can identify numbers in the range 1 to 20, Level 3: can identify all two-digit numbers, and level 4: can identify three-digit numbers (Wright, Martland, Stafford, & Stanger, 2010, p. 12).

Figure 5.12 shows how well the participants could identify numbers.

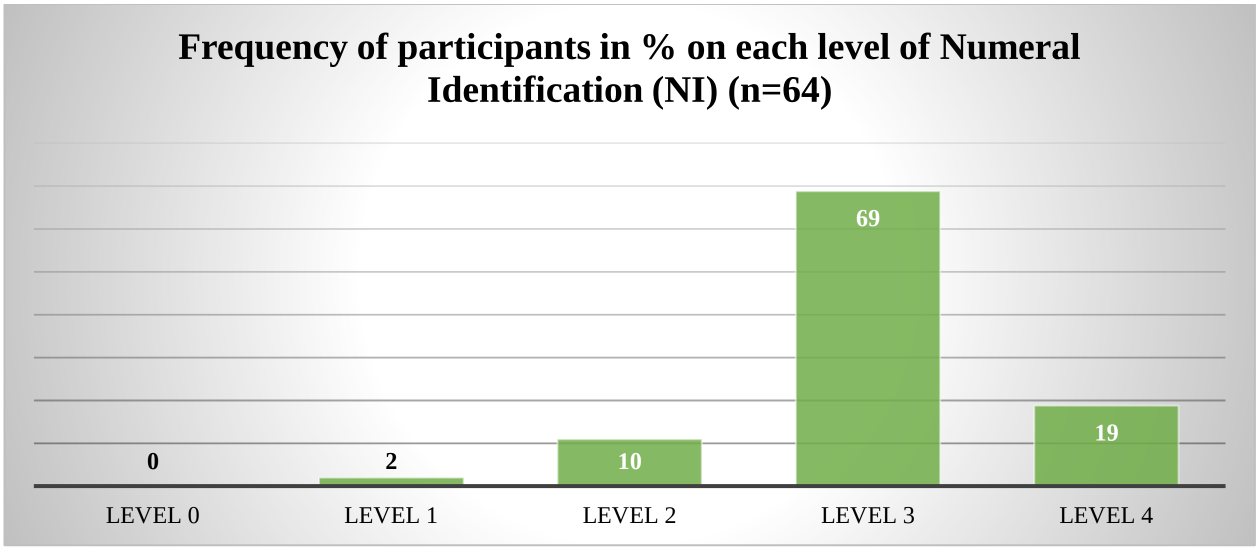


Figure 5.12 Percentage of participants on each level of numeral identification (NI)

The Grade 1 Foundation Phase CAPS Document stipulates that learners recognise, identify and read number symbols 1 to 100. A vast majority of the participants (69%) could identify numbers up to 100, and 19% could even identify three-digit numbers. The latter suggests that most of the participants were on the level they should be according to CAPS regarding numeral identification.

The graph below (Figure 5.13) demonstrates the differences and similarities of how well and in which number range they could identify numbers for the suburban and township schools separately.

[46] I discuss the levels of NI in section 2.5.5.

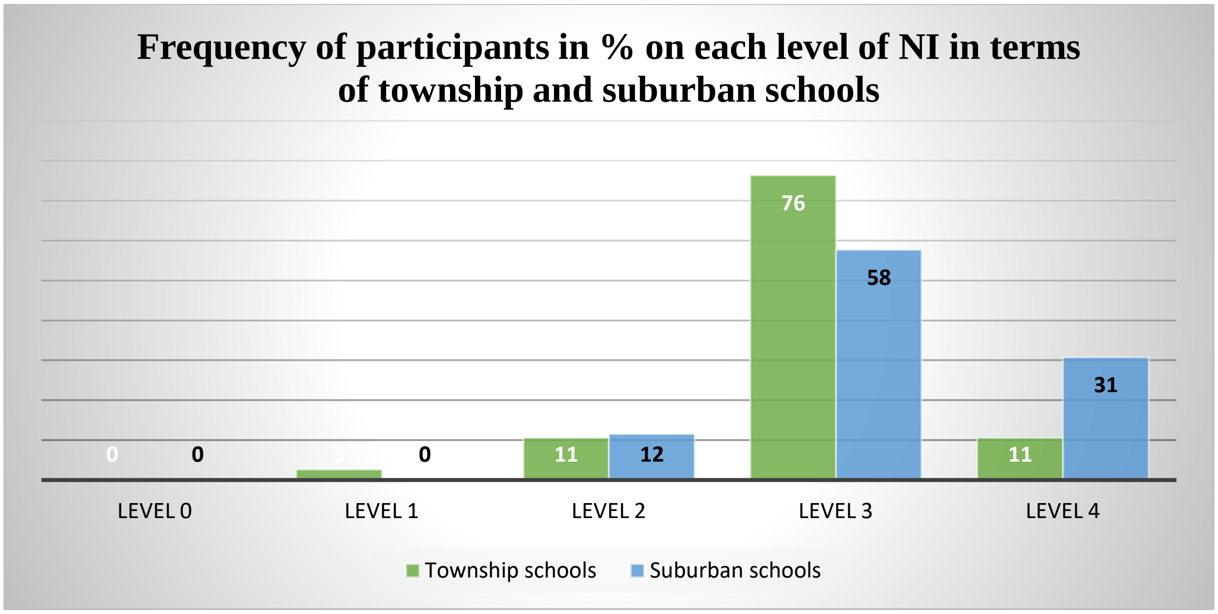


Figure 5.13 Percentage of participants in each level of NI in terms of the township and suburban schools

As shown in Figure 5.13, 31% of the suburban schools' participants could identify three-digit numbers (level 4); in comparison, only 11% of the township schools' participants could identify any number less than 1 000. The difference of 20 percentage points suggests that the suburban schools' participants were more familiar identifying numbers between 100 and 1 000 than those in suburban schools.

Since none of the participants are located in level 0, all participants could identify numbers ranging from 1 to 10. Furthermore, the suburban school group could identify numbers from 1 to 20 while 3% of the township school group could not.

It is essential to mention that 88% of the township school group and 89% of the suburban school group could identify two- and three-digit numbers.

5.3.5 Base-ten arithmetical strategies

The levels of base-ten thinking are[47] level 0: the learner cannot see 10 as a unit of any kind; level 1: the learner knows the bonds of 10; level 2: the learner can compute with hidden 10-strip artefacts; level 3: the learner can solve additive reasoning tasks with units of 10s and 1s.

[47] I discuss the levels of base-ten in section 2.5.5.

Figure 5.14 shows that many of the participants (88%) could not see 10 as a unit of any kind. Nine percent of the participants knew 10 bonds, and 3% could calculate with hidden 10-strips.

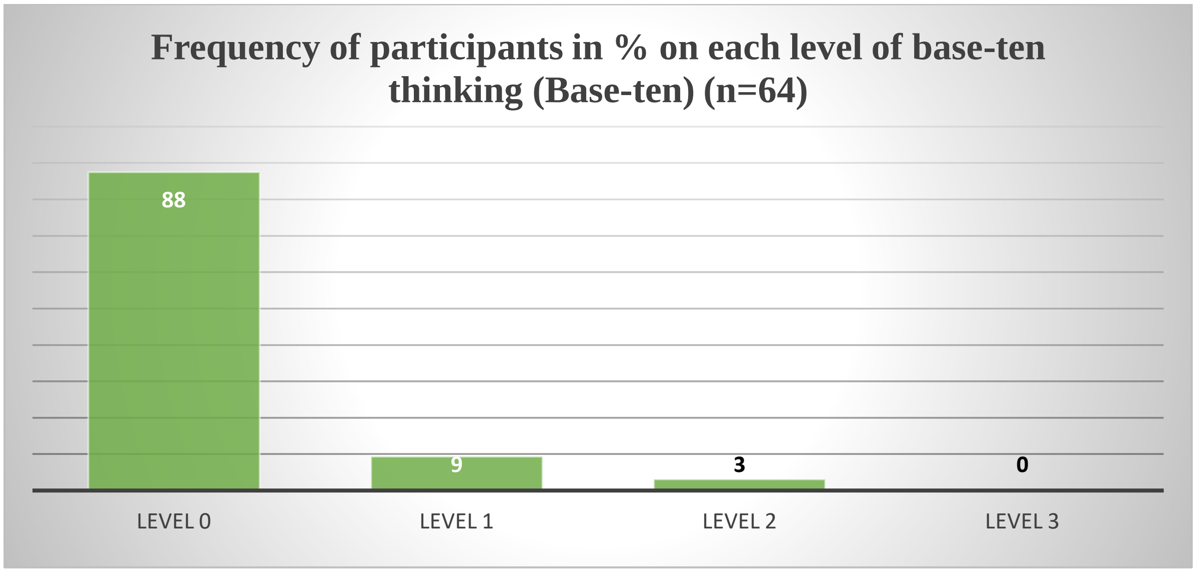


Figure 5.14 Percentage of participants on each level of base-ten thinking

The graph in Figure 5.14 illustrates the levels for each group of participants related to base-ten thinking. It is evident from the graph that 15% of the suburban schools' participants were familiar with the bonds of 10 (level 1) compared to 5% of the participants in the township schools for the same conceptual content.

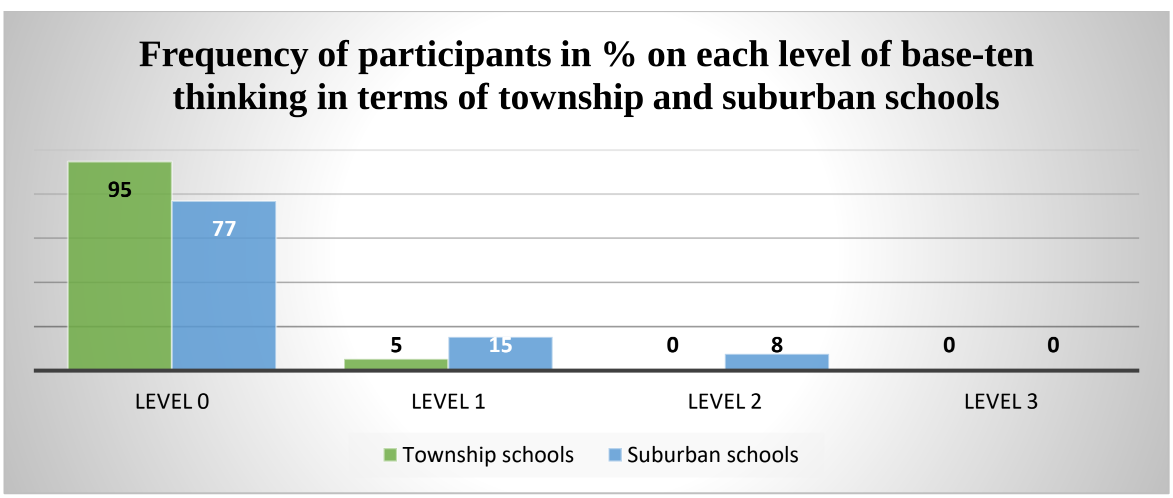


Figure 5.15 Percentage of participants in each level of base-ten thinking in terms of the township and suburban schools

Further to this, 8% of the participants in the suburban schools could work with hidden 10-strip[48] artefacts, which indicates that they could work with 10 as a unit when a physical 10-strip was available (or using open hands of 10 fingers), while none of the suburban school participants could.

None of the participants could solve addition and subtraction tasks involving 10s and 1s without using a representation of physical materials, nor could they solve written number sentences involving 10s and 1s (level 3).

A concerning matter is that a vast majority of both school groups were on level 0, which suggests that they cannot work with 10 as a unit or produce 10 bonds.

CAPS stipulates that a Grade 1 learner should know the bonds to 10 as rapid recall facts. Considering that the learners were in Grade 2, the findings suggest that 88% of the participants who could not find the bonds to 10 were not on the level of base-ten thinking they should be.

5.4 Grade 6 Common Assessment (G6 CA) - Empirical evidence of the extent of structural thinking

In this section, I summarise the participants' responses to all the questions in the G6 CA in bar graphs and pi-charts. Since the G6 CA was meant to measure how well the participants scored on each question, I will illustrate their reactions in terms of the total score for each question.

The G6 CA consists of both number-related tasks (NRT) and not-only-number-related tasks (NONRT). Therefore, I am discussing the NRT and NONRT separately. I separate them because I compare the correlations between the G2 LFIN and G1 ANAdG2 with the results of the G6 CA in terms of NRT and NONRT separately in Chapter 6 pp. I separate the tasks because I want to know how well number-related assessments in the early years predict performance in number-related assessments and not-only-number-related assessments later in primary school.

5.4.1 Number-related questions

This section focused on the participants' responses to the number-related tasks first. I will compare the questions with the same weightings. For example, I will group and compare the empirical results of the questions that counted one mark, two marks, and three marks separately.

[48] The 10-strip artefacts were discussed in section 4.6.

Furthermore, the responses will be illustrated using bar graphs with the percentage of level 4 responses in ascending order. Thus, questions with the most 'full marks' or 'correct responses' (illustrated in blue) will be at the top of the stacked bar graph.

5.4.1.1 Number-related questions out of one mark

I put together questions of the G6 CA with a score of one mark, including the ten multiple-choice questions and other questions with a score mark of one together. Figure 5.16 shows the findings of the questions with a score of one mark.

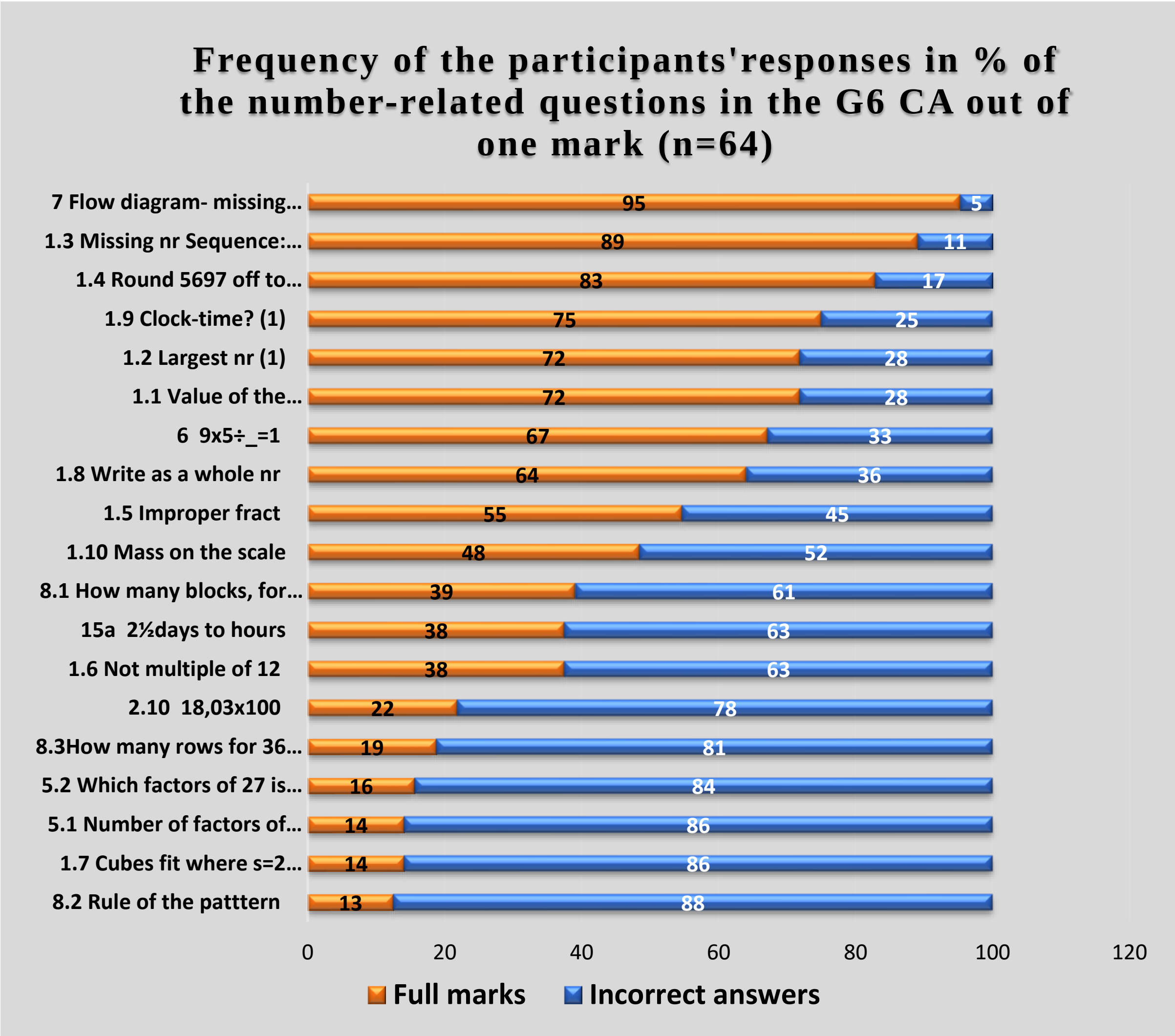


Figure 5.16 Results of the responses in %, of the number-related questions in the G6 CA out of one mark

The task where 95% of the participants could answer correctly was task 7, where the participants had to complete a flow diagram by filling in the missing constant of the linear equation. The input numbers were 5, 6, and 7, and the output was 13, 15, and 17. So, the relationship were a linear relationship $y = mx + c$ where m was given as 2, and c was the unknown. Firstly the participants had to take the input numbers (x values) and multiply them by two. Then they had to consider the output numbers and (y values) to figure out what to add to the products to give the output numbers. The 95% correct answers were a surprise because the task requires knowledge of number and quantitative relationships and, thus, structural thinking.

The question with the second-highest correct responses (89%) was question 1.3. The participants were asked to complete the number pattern $9{,}27;\ 9{,}25;\ 9{,}23;\ \_\_\_;\ 9{,}21$. The high percentage of correct responses was understandable because the participants could count backward in twos to find the missing number in the number pattern. Question 1.3 required knowledge of number relationships only and knowledge of relationships between quantities was not required.

The questions with the least percentage (13%,14%, and 16%) of correct responses were tasks that require knowledge of number and quantitative relationships and thus structural thinking. The question with 13% correct responses required a rule for the square numbers 1, 4, and 9. With 14% correct responses, the task: “What is the number of different factors for 30?” required structural thinking, and the results were not a surprise. Lastly, the question with 16% correct responses asked which factors of 27 are also prime numbers. The latter task also requires knowledge of structure.

5.4.1.2 Number-related questions out of 2

The number-related questions out of two can be seen in the graph below (Figure 5.17).

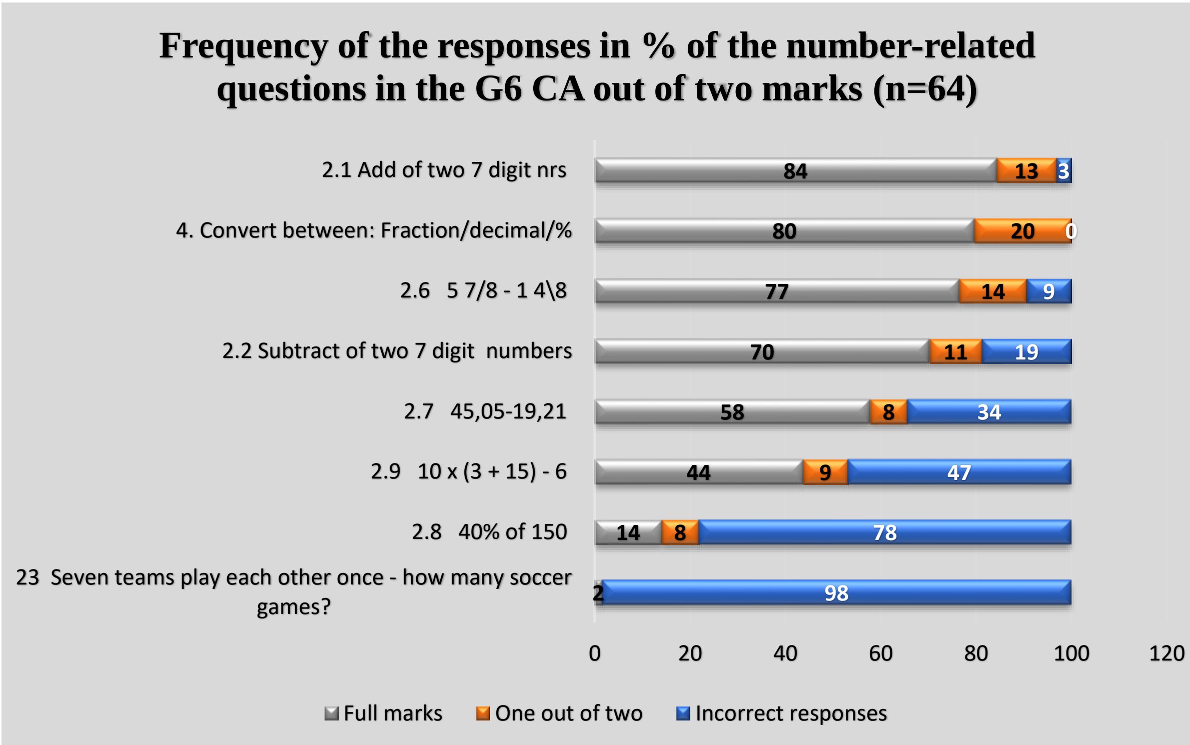


Figure 5.17 Results of the responses in %, of the number-related questions in the G6 CA out of two marks

All the tasks in the bar graph require knowledge of an algorithm where structural thinking is not essential except for question 23, with 2% of correct responses. Task 23 required knowledge of structure and was very challenging for the participants. Question 23 focused on probability calculations, a problematic concept in mathematics. Furthermore, question 2.8 was answered correctly by 14% of the participants and asked to find 40% of 150.

The bar graph above (Figure 5.9) shows that question 2.1 had the most correct responses (84% correct). Question 2.1 asks to add two 7-digit numbers ($7\ 211\ 586 + 5\ 722\ 188 = ?$) and requires knowledge of the written vertical column algorithm for addition. For question 2.1, structural thinking was not essential to answer correctly. The percentage of correct responses for task 2.2, which asked to subtract two 7-digit numbers ($4\ 071\ 274 - 2\ 128\ 863 = ?$) was only 70%. The subtraction question also requires knowledge of the written vertical column algorithm. The big difference of the percentage points (14 percentage points) of correct responses of questions 2.1 and 2.2 is significant. The big difference suggests that the subtraction question was more challenging than the addition question for the participants in my sample. The latter confirms the literature that says that the subtraction written column method is more difficult for learners than the addition written column method (Ian Thompson, 2010).

The participants did not perform well on the questions above, where structural thinking was not essential except for task 23. The findings suggest that the participants performed well in the written column addition task. They could convert between fraction/decimal/% but could not find 40% of 150. They also did not perform well in the order of operations task 2.9.

5.4.1.3 Number-related tasks out of three marks

The number-related questions out of three marks were tasks that can be regarded as challenging. Two of the questions were word problems (question 3 and 16), two were multiplicative reasoning questions (question 2.3 and 2.4), and question 2.5 focused on the addition of two mixed fractions. The graph below (Figure 5.18) shows the responses of the participants.

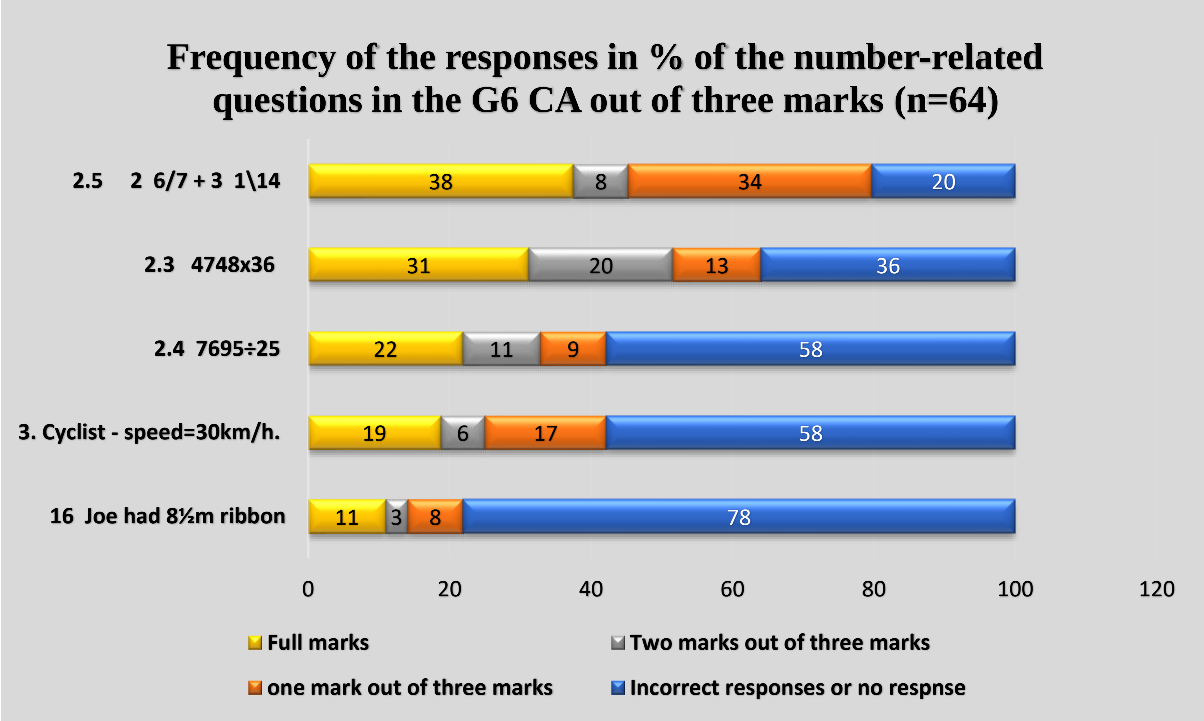


Figure 5.18 Results of the responses in percentage of the number-related questions in the G6 CA out of three marks

The percentage of full marks responses varies between 11% and 38%, which is relatively low comparing to the percentage of full marks for the question out of the two and one mark in the previous two graphs (Figures 5.8 and 5.9).

The participants struggled the most with the problem-solving word problems (tasks 16 and 3), with 78% and 58% incorrect answers where structural thinking was essential to answer correctly.

5.4.2 Not-only-number-related questions

Since I also do correlational research on how well the Grade 2 assessments predicted the not-only-number-related tasks, I present the graphs for these asks in Appendix C.

5.5 Grade 6 Evidence of Structural thinking Assessment (G6 ESTA) - Empirical evidence of the extent of structural thinking.

As mentioned before, the G6 ESTA measures the participants' structural thinking ability at the end of their Grade 6 year. I developed the G6 ESTA for this study, and all questions were amenable to be answered mentally. Since the G6 ESTA was developed to measure structural thinking for Grade 6 learners, I include tasks that consist of only 2 and 3-digit numbers. There were no tasks with single-digit numbers.

Research has suggested that mental computation should play a more prominent role in number calculation (McIntosh, Reys, & Reys, 1992; Torbeyns et al., 2009). A primary reason mental computation should be promoted is that it permits learners to discern how numbers work to make conclusions on actions and help them create their own strategies (Heirdsfield & Cooper, 2002). Mental calculation involves flexibility and efficiency based on awareness and the use of number structure, relations and properties.

Even though the G6 ESTA was designed to be answered mentally, I made paper and pencil available to see whether there were other strategies, such as written procedures, used by the participants.

Learning procedures (like column methods for addition and subtraction) are essential, but it is of little value if it is only a procedure without appreciating mathematical structure. Furthermore, a load of procedures without understanding becomes burdensome and difficult to remember. Moreover, learners who know mathematical structures, and use more sophisticated and effective strategies, do not need to rely on procedures to find answers to more advanced problems like algebra, where unit-counting is not an option. (Schmittau, 2005).

The G6 ESTA consists of tasks on additive and multiplicative reasoning. This section discusses the participants' responses on all tasks in the G6 ESTA.

5.5.1 Additive reasoning

Firstly, I discuss the results of the additive reasoning tasks of the G6 ESTA, starting with a summary (Table 5.1) of the strategies of the responses in the different stages of additive reasoning[49].

Table 5.1 Summary of the stages for the additive reasoning tasks in the Grade 6 Evidence of Structural thinking Assessment (G6 ESTA)

Stage	Abbreviation	Observed strategies	Possible indiscernible strategies
Stage 0 – Incorrect answers	(S0: ICA)	Incorrect answers	Do not understand
Stage1 – Unit counting	(S1: UC)	Unit-counting of physical items	Concrete understanding of number
Stage 2 – Formalisation without structural thinking	(S2: FWOS)	Using the written vertical column method and counting on fingers to find answers to the partial sums and differences	No structural understanding of addition and subtraction of single digits, however knowledge of an algorithm.
Stage 3 – Formalisation with structural thinking	(S3: FWS)	Using the written vertical column method with no counting on fingers	Some mathematical structural understanding to find the answers to partial sums and differences mentally and knowledge of an algorithm
Stage 4 – Immediate mental answer	(S4: IMA)	Answered with and immediate mental answer	Structural understanding where the unobserved strategy or mental strategy involves separation, aggregation, wholistic and compensation strategies

This section discusses the participants' strategies to answer the addition and subtraction tasks. Thus, I am reporting on participants' responses on tasks that were amenable to be solved using separation and aggregation, compensation and wholistic mental strategies (Heirdsfield & Cooper, 2002).

5.5.1.1. Learners' responses to tasks amenable to be solved using separation (1010) and aggregation (N10) strategies (Tasks 1 and 2)

Task 1 ($12 + 85$) and task 2 ($17 + 56$) were meant for using mental separation and aggregation strategies. Mental aggregation strategies (N10) are when a learner starts with the biggest number

[49] The stages of additive reasoning are discussed in section 4.7.1.

and splits the second number into tens and ones (See Chapter 2 p). On the other hand, is the mental separation strategy (1010), when a learner splits up both numbers and adds the tens and ones separately (Beishuizen & Anghileri, 1998; Heirdsfield & Cooper, 2002).

When I developed the assessment, I decided to ask task 2 only to the participants who could immediately answer task 1 (indicated below as the purple sector in the bar of the pie chart). Thus task 1 was asked to 64 participants, while task 2 was only asked to the 30 participants who could answer task 1 with an immediate mental answer.

I aimed to understand how the participants responded with mental answers and not how well they could perform another written vertical column strategy. In the first task, no 'carrying[50]' was necessary, but in task 2, carrying was essential. Thus, task 2 was more challenging than task 1. It is necessary to mention that task 2 was the only task I did not put to all the participants. Figure 5.19 shows the results of the strategies used by the participants.

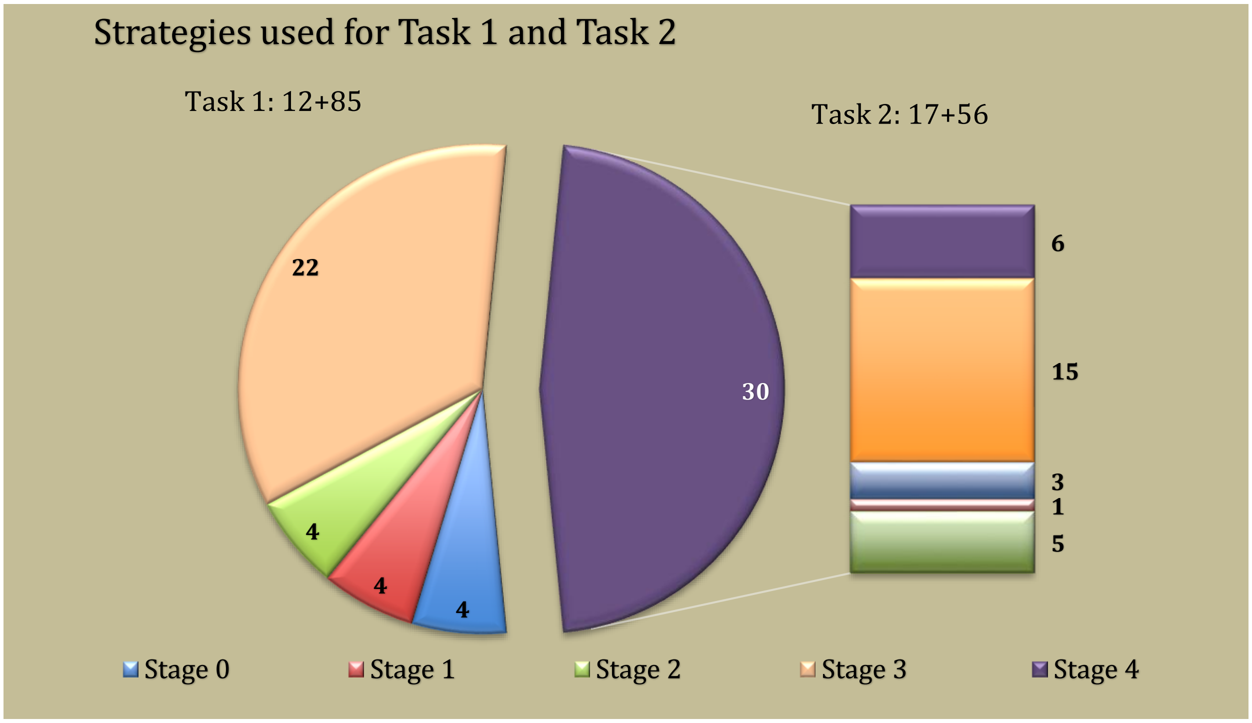


Figure 5.19 Strategies used for task 1 (n=64) and task 2 (n=30) of the G6 ESTA

[50] In elementary mathematics, 'carrying' is when a digit is transferred from one column of digits to another more appropriate column of digits.

Thirty (47%) of the participants answered the question "What is $12 + 85$?" correctly and immediately, which means they either used aggregation or separation mental strategies. If the learner used the mental separation strategy, she could calculate that $10 + 80 = 90$ and then add $2 + 5 = 7$ and get 97, which is a mental separation strategy. For aggregation, the learner starts with 85, adds 10 to give 95, and then adds another 2, giving 97. I chose "What is $12 + 85$?" because no 'carrying' or 'bridging through ten' was necessary to answer the question.

Twenty-six (41%) of the participants used the vertical column method, of which 4 of the participants used unit-counting to get the answers of the partial sums. Thus, 22 of the 26 participants who used the vertical column method could mentally find the answers for the partial sums.

Four of the learners counted on 12 from 85 to get 97 using fingers or counters. None of them counted on from 12. The participants who counted on from 85 knew the commutative law for addition. Lastly, 4 of the participants gave an incorrect answer for task 1.

Regarding task 2, I asked task 2 ("What is $17 + 56?$") only to the 30 (47%) participants who could answer with a mental answer in task 1. I was not interested in how well they could do another written column strategy. Still, I wanted to know how well the participants who could answer task 1, where no bridging through ten or carrying was necessary, would respond to a task where carrying and/or bridging through ten was essential.

As in task 1, task 2 was amenable to answer using aggregation or separation strategies. If the learner used the mental separation strategy, she could add $10 + 50 = 60$ first and then add $7 + 6 = 13$. While keeping 60 and 13 in mind, the next step should have been to add 60 and 13 to get to 73, which is much more complicated than the previous task.

If the learner used the mental aggregation strategy, he could have started with 56 and added 10, which is 66. To add 7 to 66 mentally, the learner had to break up 7 into 4 and 3, add 4 to 66 to get 70 and add the 3 to get 73. This aggregation strategy required knowledge of bridging through 70.

Using either the separation or the mental aggregation strategy in task 2 required knowledge of relationships at a higher level than task 1. Only 6 (20%) of the 30 learners who could answer task 1 mentally could answer task 2 correctly with a mental answer.

Eighteen (60%) participants of the 30 participants answered task 2 correctly using the vertical written column method. It is important to mention that 15 of the 18 participants who used the written vertical column method could find the partial sums mentally.

To sum up, the evidence shows that approximately 50% of the participants can give a mental answer to a task that does not include bridging through ten and carrying. Nevertheless, when the task involves knowledge of structure, like bridging through ten and carrying, the same participants who could answer mentally with a correct response had difficulties responding mentally to a task where bridging through ten and carrying were involved and had to regress to vertical column methods.

5.5.1.2 Learners' responses to tasks amenable to be solved using the compensation strategy

This section illustrates how students respond to tasks that are amenable to solving using mental compensation strategies. I am splitting the compensation tasks into two parts: tasks where a referent equation was given and another part where no referent equation was given.

Learners' responses to tasks amenable to solve using a referent equation to do compensation

The compensation tasks where a referent equation was given had one or two changes from the referent equation. Table 5.2 shows how the mathematical expression asked differed from the given referent equation.

Table 5.2 Tasks amenable to solve using a referent equation to do compensation

Task number	Task	Nature of the task with referent equation given
11	If $\mathbf{66} + 17 = 83$, What is $\mathbf{65} + 17$?	One change: one less than **66**
12	If $66 + \mathbf{17} = 83$, What is $66 + \mathbf{18}$?	One change: one more than **17**
13	If $\mathbf{66} + 17 = 83$, What is $\mathbf{67} + 17$?	One change: one more than **66**
14	If $\mathbf{164} + 57 = 221$ What is $\mathbf{163} + 57$?	One change: one less than **164**
15	If $164 + \mathbf{57} = 221$ What is $164 + \mathbf{58}$?	One change: one more than **57**
16	If $\mathbf{164} + \mathbf{57} = 221$ What is $\mathbf{165} + \mathbf{56}$?	Two changes: one more than **164** and one less than 57

17	If $\mathbf{164} + \mathbf{57} = 221$ What is $\mathbf{165} + \mathbf{58}$?	Two changes: one more than **164** and one more than **57**

The participants' performance on tasks 11-17 is shown in the bar graph below (Figure 5.20). The graph shows the frequency of participants in Stage 4 (immediate mental answer) in ascending order, which means that the task at the top of the graph has the lowest frequency of immediate mental responses (Stage 4) and the task at the bottom of the graph shows the highest frequency of participants in Stage 4.

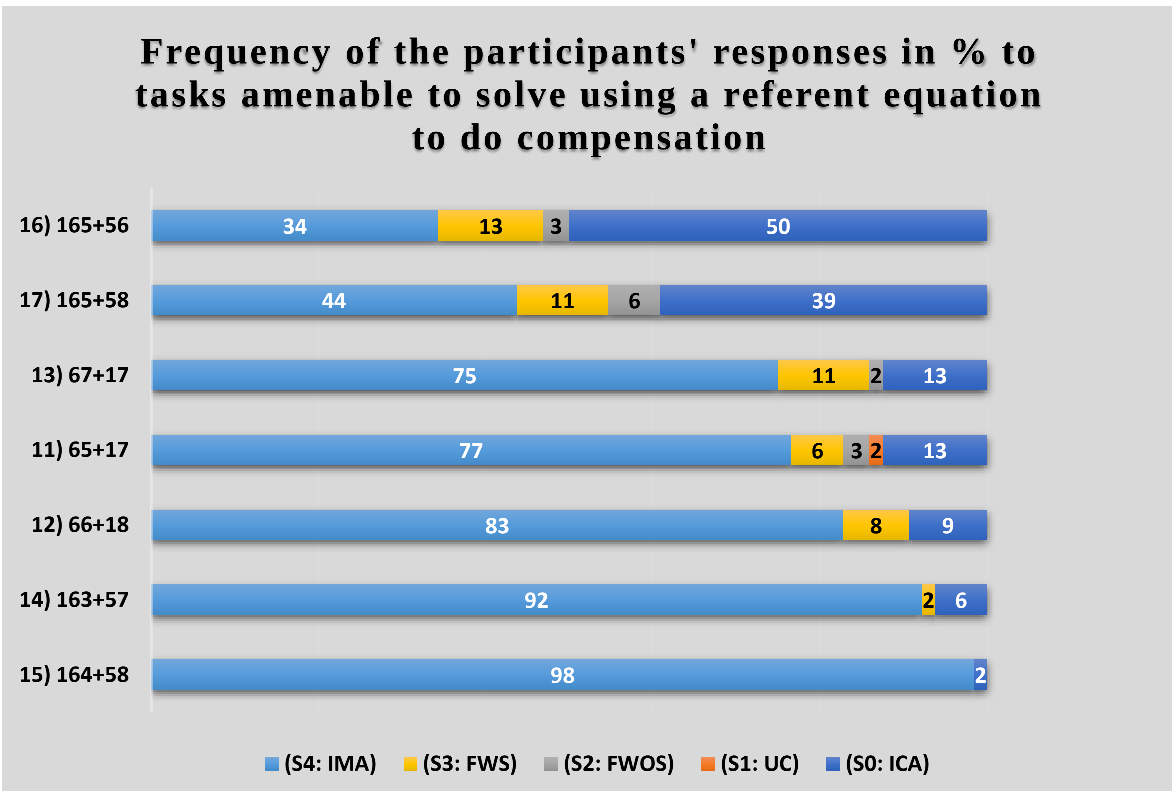


Figure 5.20 Students' responses to tasks amenable to solve using a referent equation to do compensation

Firstly, evidence from the graph in Figure 5.18 shows that only one participant used unit-counting strategies successfully in task 11 ($what\ is\ 65 + 17?$). Thus, no other participant used unit-counting strategies successfully to answer any of the above compensation tasks. The absence of unit counting strategies in all the other tasks suggests that the participants did not regard unit-counting as a possible strategy to find the answers to these tasks. The reason why participants did not use unit-counting strategies successfully could be the high number range of the tasks.

The graph (Figure 5.20) also shows that tasks 16 and 17 had the most incorrect answers, with 32 (50%) and 25 (39%) participants respectively. The nature of both these tasks was that they had two changes. In task 17, both numbers, namely 164 and 57, changed with one more each to 165 and 58, while in task 16, one number (164) changed with one more to 165 and the other (57) changed with one less to 56. Twenty-eight (44%) of the 64 participants could give an immediate mental answer to task 17, while 22 (34%) participants gave a Stage 4 response to task 16.

The tasks with the highest number of correct responses were the tasks with only one change in the equation. As you can see, the tasks with one change were tasks that included the addition of two-digit numbers (like $65 + 17$: two/two-digit addition) and tasks that had one three-digit number and one two-digit number (like $164 + 85$: three/two-digit addition). Interestingly, the participants performed better on the three/two-digit addition tasks than the two/two-digit addition tasks.

The better performance in the three/two-digit addition task could be since the three/two-digit tasks were asked later than the two/two-digit tasks, and the participants could realise the pattern of the compensation with a referent equation strategy.

Learners' responses to tasks amenable to solve using compensation without a referent equation

Table 5.3 shows the tasks without a referent equation.

Table 5.3 Compensation tasks in the G6 ESTA without a referent equation

Task number	Task	Nature of the question
4	Addition Compensation: What is $375 + 99$?	Learner can use $375 + 100 - 1$
5	Subtraction Compensation: What is $435 - 399$?	Learner can use $435 - 400 + 1$

Task 5 could also be answered using the subtraction strategy of counting on from 399 to 400 to 435 and saying the answer is $1 + 35 = 36$. However, only one student gave an immediate mental answer for task 5. I asked him *how did you find the answer?* and he responded with: "I subtract 400 and then I add one."

The graph (Figure 5.21) below shows the participants' strategies for tasks 4 and 5.

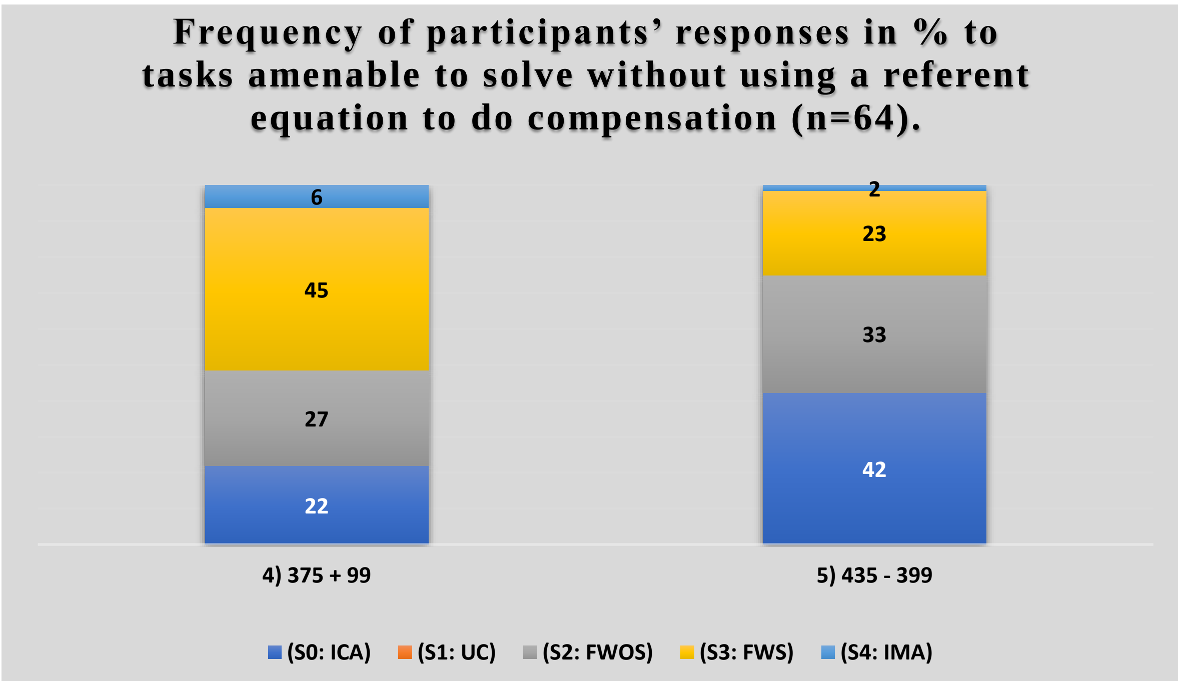


Figure 5.21 Students' responses to tasks amenable to solve without using a referent equation to do compensation

None of the participants in the sample used unit-counting strategies for these two tasks, which is understandable because of the high number range of the tasks. Thus, no S1:UC results for both tasks.

Seventy-eight percent of the participants correctly answered the addition compensation tasks, while only 58% answered the subtraction compensation task correctly. The big difference between 78% and 58% is significant and highlights learners' difficulties with subtraction.

Interesting to mention is the difference in Stage 2 and 3 responses on the two tasks. More participants could do the vertical column algorithm without using their fingers in $375 + 99$ (29 participants – 45%) than they could in $435 - 399$ (15 participants – 23%). Thus, the participants were more reliant on their fingers in the subtraction compensation tasks than on the addition subtraction task.

To sum up, the participants scored very high on the compensation tasks with a referent equation, which is gratifying. The high score on the tasks with a referent equation suggests that the participants understood the idea of the compensation strategy; however, they find it challenging to produce a referent equation.

5.5.1.3 Subtraction tasks

The subtraction tasks in the G6 ESTA consist of two number sentences given to the participants on a card and two problem-solving tasks that were asked verbally. See the tasks in Table 5.4.

Table 5.4 Subtraction tasks in the G6 ESTA

Task number	**Item**	**Nature of the question**
6	What is $403 - 396$?	Seeing the small difference: counts-on from 396 to 403 by bridging through 400 or counts-down from 403 to 396
7	What is $504 - 7$?	Take away by bridging through 500
8	Thulela wants to buy socks that cost R45. She has R39 in her purse. Her mother said she would give her the rest. How much should her mother give her to buy the socks?	Socks $39 + ? = 45$ Count-on from 39 to 45 by bridging through 40 (how much more?)
9	Mike has R64. I have R48. How much more does Mike have than me?	Mike $48 + ? = 64$ Compare- count-on in tens from 48, 58 then bridge through 60 to arrive at 64.

Task 5 (What is $403 - 396 = ?$) was meant to be answered by 'seeing the small difference' between 403 and 397 (Tzur et al., 2017). Knowing the relationship between these two numbers and how they are related to each other is essential to see the small difference between them concerning subtraction. In this task, the participants could count forward from 397 to 400 and then to 403 to arrive at 6. Task 7 (What is $504 - 7 = ?$) was developed to measure subtraction's 'taking away' strategy. Tasks 8 and 9 were meant to measure 'compare additive situations'. The strategies used by the participants on these tasks can be seen in the bar graph in Figure 5.22.

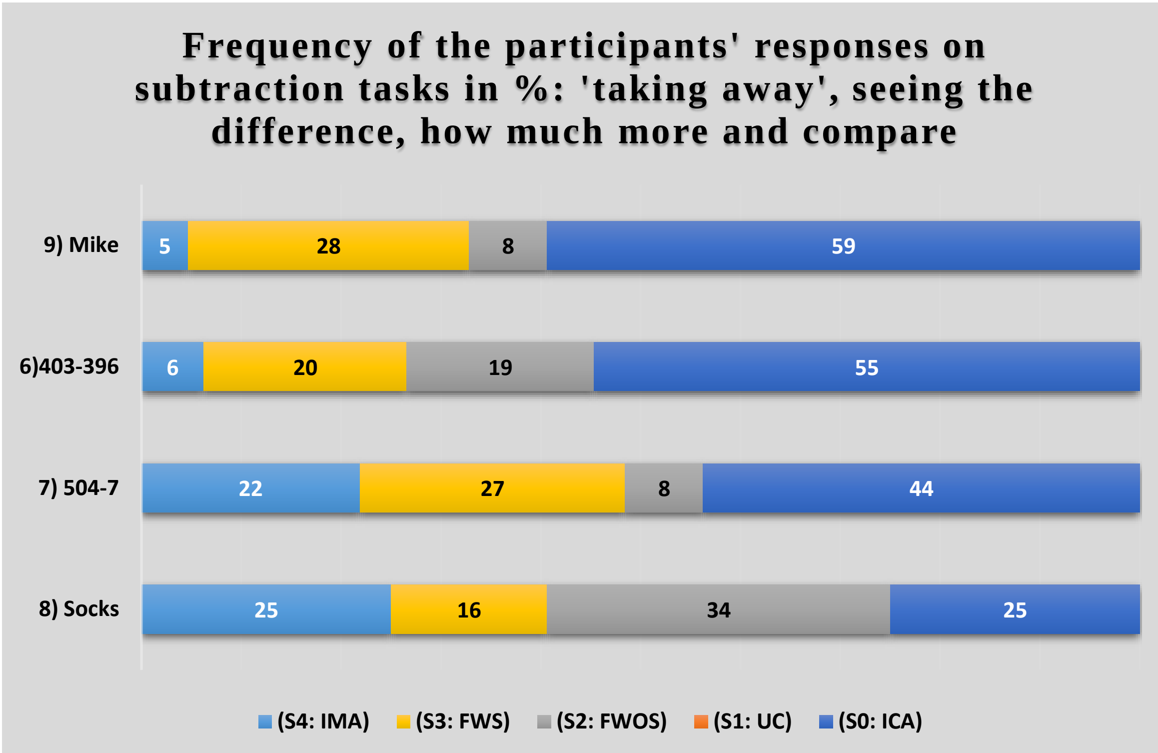


Figure 5.22 Students' responses to subtraction task in the G6 ESTA: 'taking away', seeing a small difference, 'how much more' and compare

The task with the least 'immediate mental' responses (stage 4) was task 9, with 5% mental responses, and the task with the most mental responses was task 8(25%). Even though the participants had to find the missing addend for tasks 8 and 9, they responded very differently in these two tasks.

Task 9 asked: "Mike has R14 more than me. He has R62. How much money do I have?"

Task 8 asked: "Thulela wants to buy a pair of socks that cost R45. She has R39 in her purse. Her mother says she will give her the rest. How much does her mother need to give her?"

The difference in the performance on the two problem-solving tasks could be that they associated themselves more with the context of buying shoes than with the understanding of the 'more than' matter of task 9 (Mike). Task 8 requires knowledge of counting-on from 39 to 45 only. Task 9 requires knowledge of quantitative relationships where the learner had to figure out how to manipulate the numbers to get to an answer. Another possible reason for the difference in the responses to tasks 8 and 9 could be the higher number range of task 9 (Mike). Furthermore, more than half of the participants (59%) could not answer task 9 correctly. In comparison, 25% of the participants could not answer task 8 correctly, suggesting that task 9 was significantly challenging.

The Stage 4 responses of the two subtraction number sentences show that 14 (22%) participants could do the ‘taking away’ strategy in task 6. By contrast, only four (6%) could see the ‘small difference’ between 396 and 403 of task 6. The latter confirms literature saying that learners have difficulties to see subtraction as the ‘difference’ between numbers but rather see it as ‘taking away’.

5.5.1.4 Base-ten knowledge

Task 19 was developed with the compensation tasks with a given referent equation. The task read: “If $164 + 57 = 221$, what is $1640 + 570?$” The idea was to see whether the participants could use their base-ten knowledge to do task 19 by multiplying both sides of the equation with 10 and arriving at an answer of 2 210.

The responses to task 19 can be seen in the graph below (Figure 5.23).

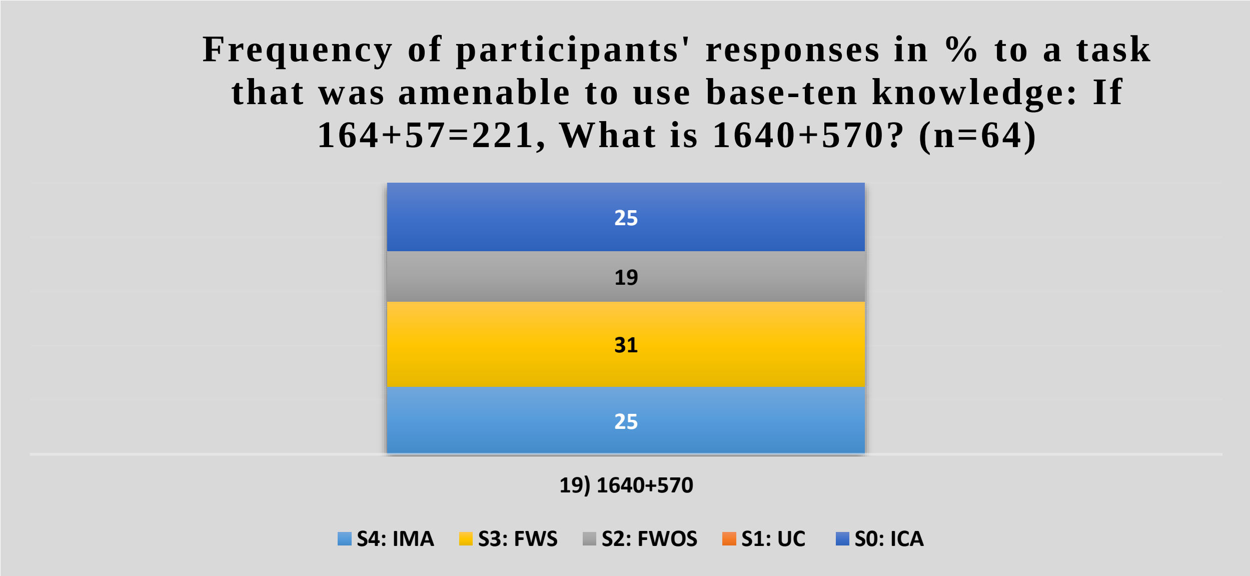


Figure 5.23 Students’ responses to a task that was amenable to use base-ten knowledge

Only 16 (25%) participants could answer with an immediate mental answer, showing that 25% know the relationships and structure between 164 and 1 640, 57 and 570, and 221 and 2 210. Thirty-two (50%) participants used the vertical column algorithm, of which 20 of the 64 knew the answers to the partial sums mentally, and 12 of the 64 had to count on their fingers to find the answers for the partial sums.

5.5.1.5 Addition and subtraction as inverses of each other

An important skill that learners should apply is reasoning with additive inverses. Task 18 was about additive reasoning inverses. The task asked: "$If\ 164 + 57 = 221$, what is $221 - 57$?" This task was also asked together with the compensation tasks where a referent equation was given. The graph in Figure 5.24 shows how the participants responded.

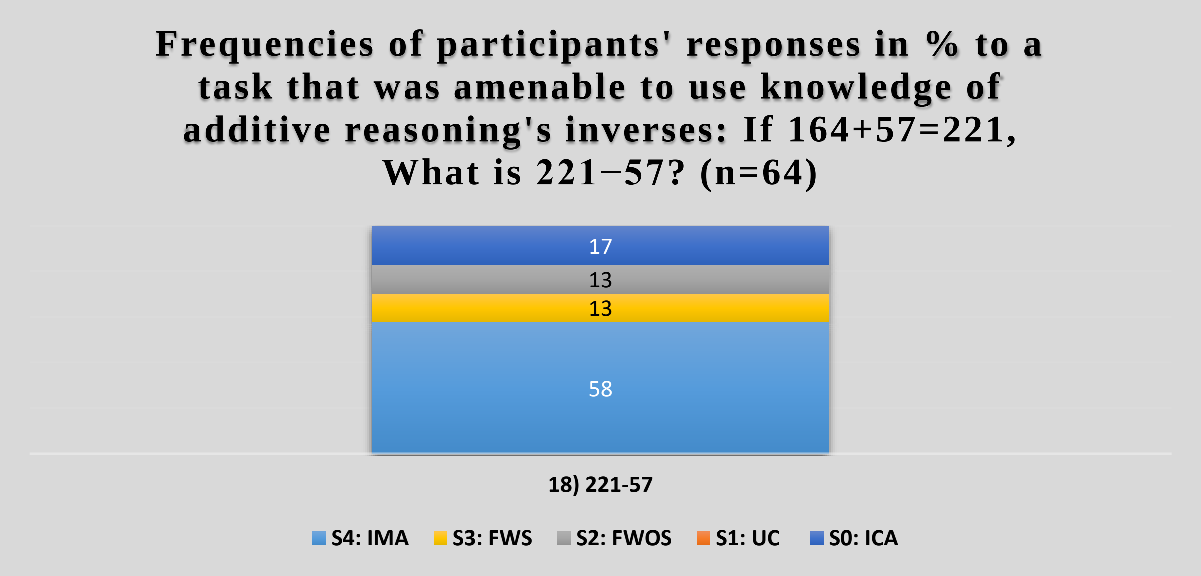


Figure 5.24 Students' responses to a task that was amenable to use knowledge of additive reasoning inverses

More than half of the participants (37) knew the relationships between addition and subtraction as each other's inverse operations. Sixteen participants used the written vertical column algorithm, of which eight could find the answers of the partial differences mentally while the other eight participants could not. It is important to note that the participants used no unit-counting strategies, which is understandable because of the high number range.

5.5.1.6 Additive reasoning in terms of structural thinking

Task 10 was amenable to testing structural thinking in additive reasoning. The task read: "If I ask you to add 479, 32, and 21, which two numbers will you add first to make it easier to calculate?" Task 10 did not ask to do an operation or find the sum. However, it asked the participants to reason about the relationship between the numbers and bonds to ten and base-ten. The graph below (Figure 5.25) shows the responses for task 10.

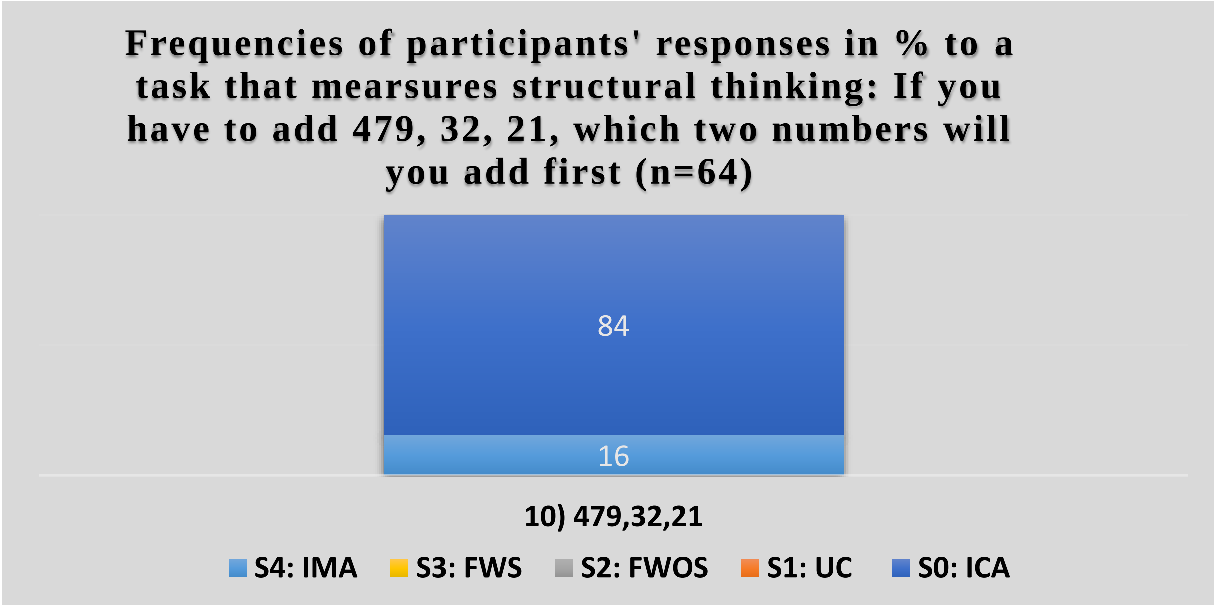


Figure 5.25 Structural thinking: If you must add 479, 32 and 21, which two numbers will you add first to make it easier to calculate?

Ten (16%) participants could answer task 10 with an immediate mental answer, and 54 (84%) could not give a correct solution. I asked these participants why they said so, and they responded by saying they would add 479 and 21 first to arrive at 500 and then add 32 to get an answer of 532. I regarded a correct response without a correct explanation as a wrong answer because if the learner could not explain the answer, it meant it was a guess.

5.5.2 Multiplicative reasoning

The multiplicative reasoning tasks consist of two parts, multiplication and division. The strategies learners use to answer the multiplication and division tasks follow next.

5.5.2.1 Multiplication tasks results

The multiplication tasks are about multiplication with repeated equal groups, multiplication with an array, basic multiplication facts, multiplication/division and addition combined tasks, and division and multiplication as inverses of each other.

A summary of the framework to code the multiplication tasks can be seen in Table 5.5.

Table 5.5 A summary of the framework to code the multiplication tasks with abbreviations

Stage	Abbreviation	Observed strategy	Possible indiscernible strategies
Stage 0 – Incorrect answers	S0: ICA	Incorrect or no answers	The learner has not yet gained an understanding of multiplication.
Stage1 – Perceptual grouping in ones	S1: PGO	Perceptual sharing and grouping **in ones** where the physical items are available.	The learner cannot subitize a group of 5 items or less. She has to count them first.
Stage 2 – Perceptual sharing and grouping in multiples	S2: PGM	Perceptual sharing and grouping **in multiples** where the physical items are available. Strategies include skip counting in multiples; however, the physical items must be present.	The learner can subitize a group of 5 physical items or less without counting them first.
Stage 3 – Figurative composite counting without using physical items.	S3: FCC	Sharing and grouping **in multiples without physical items.** Strategies include skip-counting in multiples of the multiplicand while keeping track of the multiplier on their fingers.	The learner has developed skip-counting strategies that **do not rely on physical items** to be visible or touchable. She has reified the multiplicand but not yet reified the multiplier.
Stage 4 - Repeated composite grouping using a written algorithm	S4: RCWA	The strategies at this stage depend on a written algorithm. For example, they use repeated addition by continuing to double the value. The strategy at this level depends on **written algorithms.**	The learner has developed written algorithms like repeated column addition. She has reified the multiplicand but not yet the multiplier and knows an algorithm.
Stage 5 – Immediate mental answer	S5: IMA	Answered with an immediate mental answer.	The learner at this level does have a reified understanding of the multiplier and the multiplicand and can derive new multiplication facts from basic multiplication facts mentally.

Multiplication with repeated equal groups

For task 23, I showed the participants one card containing 9 dots. I then presented another 7 cards also containing 9 dots. Without revealing the dots on all the cards, I said: “I now have 8 cards with 9 dots on each card. How many dots are there altogether?”

The strategies the participants used can be seen in the graph below (Figure 5.26).

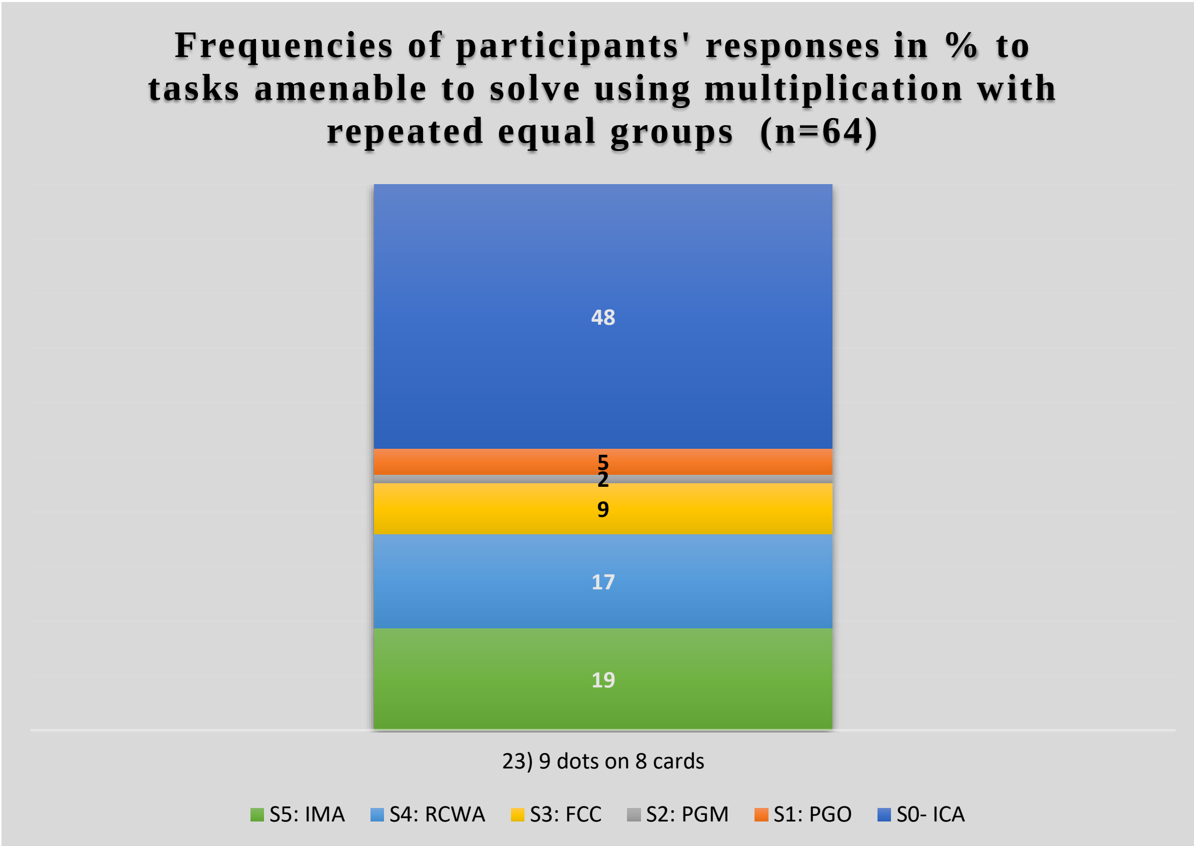


Figure 5.26 Students' responses to tasks amenable to solve using multiplication as repeated equal groups

Thirty-one (48%) participants could not answer task 23 successfully; however, 12 (19%) of participants could give an immediate mental answer. The participants used repeated addition on fingers and counting-on from a known multiple of 9 or counting-on from a previous multiple.

The participants used a counting strategy by drawing 8 groups of 9 tallies and then counting them all. The tallies were structured in the following way (Figure 5.25):

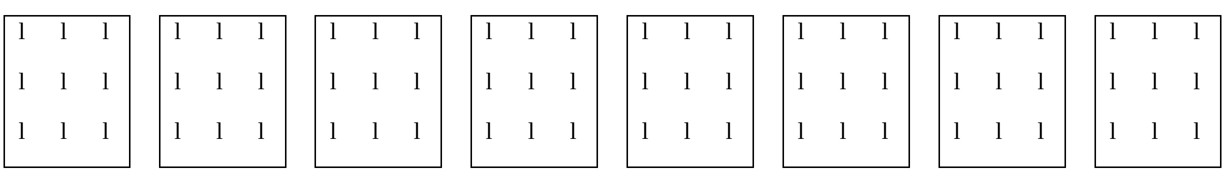


Figure 5.27 A counting strategy that entails unit counting by drawing 8 groups of 9 tallies

Another unit-counting strategy that some of the participants used was to draw only one group of 9 structured tallies and count-on repeatedly while keeping track on their fingers until they reached 8 fingers.

Written strategies included doubling 9 to get to 18 and then doubling 18 using the written vertical column algorithm and arriving at 36. In the end, they doubled 36 and arrived at the correct answer of 72 (Van de Walle, 1998).

Multiplication with an array

I am now focusing on the multiplication tasks that involve arrays. The tasks are presented in table 5.6.

Table 5.6 Multiplication tasks that involve an array (tasks 24-27)

Task number	Task
24	On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 10 in each column). Screen the inside and show only the first row and first column. How many dots are there altogether? (7×10)
25	Then ask: What is 7×10?
26	On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 12 in each column: 7×12). Screen the inside and show only the first row and first column. How many dots are there altogether?
27	On this page, we have an array of dots. How many are there in each row? How many are there in each column? (7 in each row and 9 in each column: 7×9). Screen the inside and show only the first row and first column. How many dots are there altogether?

The 'pages' of tasks 24, 26, and 27 can be seen in Figure 5.26

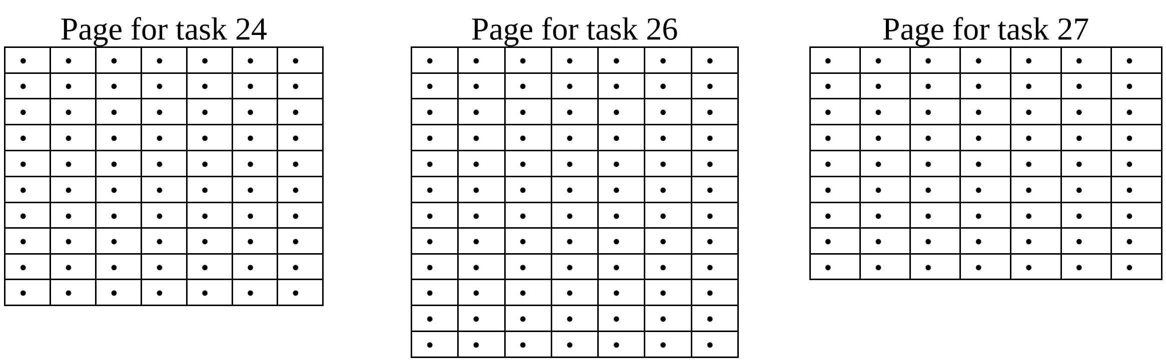


Figure 5.28 the 'pages' of tasks 24, 26 and 27 of the G6 ESTA

The tasks were presented as follows. Firstly, a page was introduced, and question 24 was asked. Then a card with "What is $7 \times 10 =$?" on it was shown to the participant for task 25, following with tasks 26 and 27.

See the responses of the participants of tasks 24-27 in Figure 5.29.

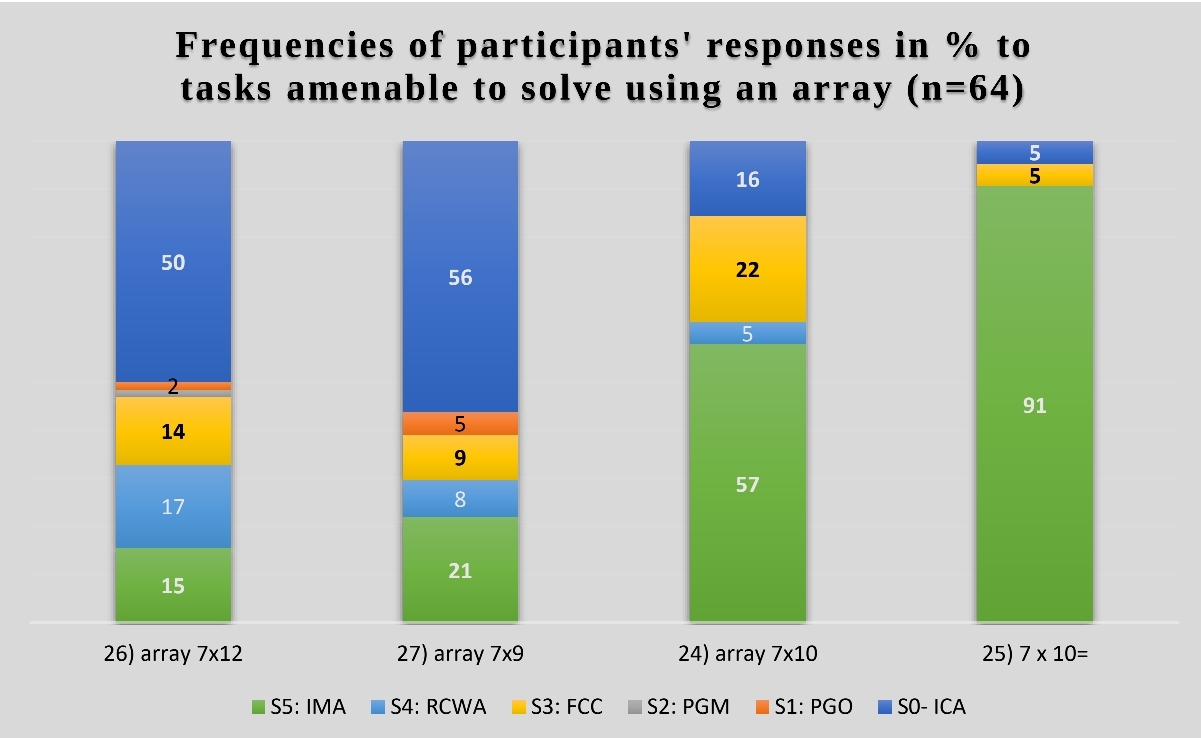


Figure 5.29 Students' responses to tasks amenable to solving using an array

Task 25 (7×10) had the most (58%) immediate mental (stage 5) answers, which is understandable because the array of 7 by 10 dots was done in the previous task and because the 10-times table is one of the first multiplication tables that learners find easy to remember (known fact). That there were only 10 (16%) participants who could not give a correct answer to 7×10 was a concern because a Grade 6 learner should know the 10-times table.

The task with the least mental answers was task 26 (7×12); furthermore, the high number (50%) of incorrect answers for task 26 is concerning if we consider that these participants were in Grade 6 at the time of data collection.

The idea behind tasks 24 to 27 was to see whether the participants could use a known fact like 7×10 to find answers to 7×9 (subtracting 7 from 70) and (adding 14 to 70) using an array. I wanted to see whether they could become aware of the relationships structurally between $7 \times 10 \quad and\ 7 \times 9$ by subtracting 7 from 70 and between 7×10 and 7×12 by adding 14 to 70. Only 16% for task 26 and 22% of participants for task 27 could see the relationships structurally or knew the expressions as known facts.

Nevertheless, the results showed me that most participants (84% for task 26 and 78% for task 27) could not give an immediate mental answer. Furthermore, many participants could not answer correctly (50% of task 26 and 56% for task 27).

Multiplication Basic Facts – 8 times table (Wright et al., 2012)

Tasks 28 to 39 asked participants to give answers to $1 \times\ 8,\ 2 \times 8,\ 3 \times 8,\ 4 \times 8,\ 5 \times 8,\ 6 \times 8,$ $7 \times 8 \ldots \ldots .12 \times 8$? These tasks were asked in the order above. I put the responses of the participants in 5 levels. The table below describes the different stages in table 5.7. The highest stage is where learners know the multiplication tables as known facts.

Table 5.7 The levels for coding the responses for the 8-times table questions

Stage	Response
0	Incorrect answers
1	The participant could answer 1×8 and 2×8 as known facts and then started to count-on in ones on their fingers to find the next multiple.
2	Delayed correct mental answer with no visible unit counting for $1 \times\ 8,\ 2 \times 8,\ 3 \times 8,$ $and\ \ 4 \times 8$
3	Delayed correct mental answer with no visible unit counting for $1 \times\ 8,\ 2 \times 8,\ 3 \times 8,\ 4 \times 8,$ $5 \times 8,\ and\ 6 \times 8$
4	Delayed correct mental answer with no visible unit counting for $1 \times\ 8,\ 2 \times 8,\ 3 \times 8,\ 4 \times 8,$ $6 \times 8,\ and\ 7 \times 8$ with immediate mental responses for $5 \times 8\ , 10 \times 8\ and\ 11 \times 8$.
5	Immediate mental answer for all the 8 times table questions.

The graph below (Figure 5.30) shows the results of the participants' responses to the 8-times table questions.

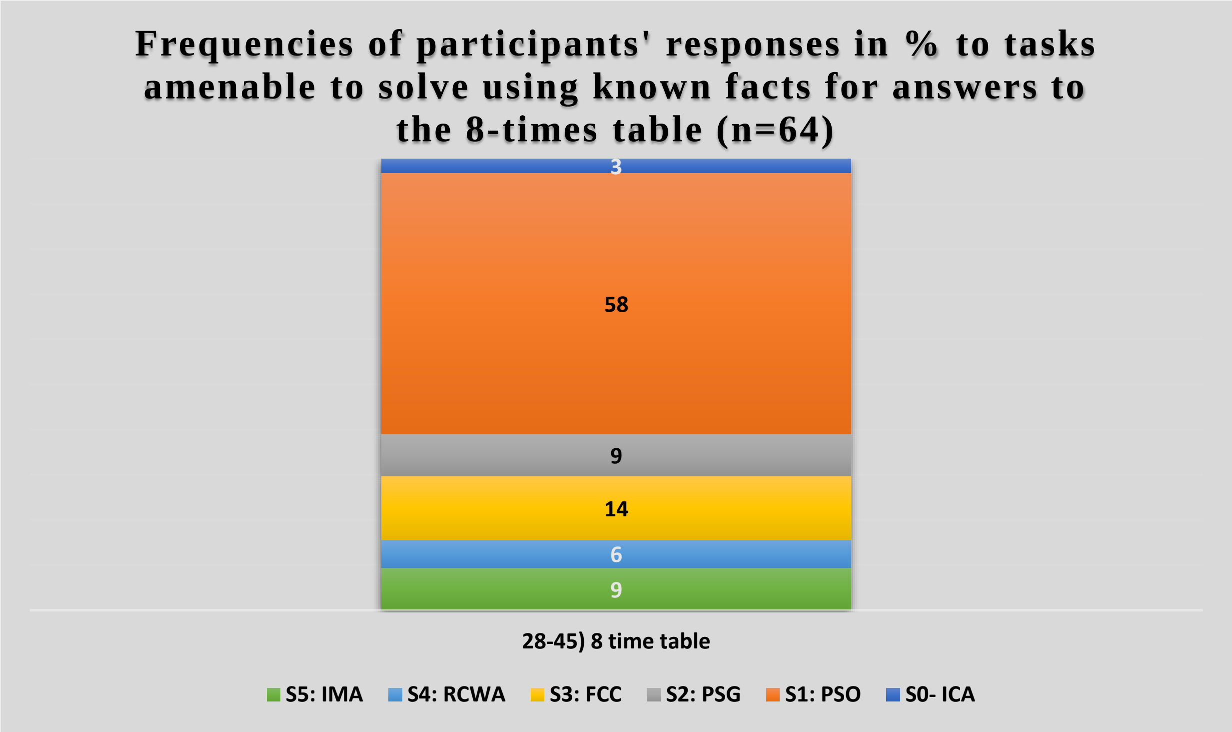


Figure 5.30 Students' responses to tasks amenable to solve using known facts in multiplication

Only 37 (58%) participants could give an immediate mental response on 1×8 and 2×8 and then count on from 16 to 24 using their fingers. Only 6 participants could provide an immediate mental answer to all the questions of the 8-times table.

The high percentage (58%) of stage 1 responses shows a prevalence for using unit-counting, which is not what we expect from Grade 6 learners.

Multiplication/division and addition combined tasks:

The tasks where multiplication/division operations are combined with addition are shown in Table 5.8. Both tasks are about a school fair where they sell 5 cupcakes in a box, and when you want to buy fewer cupcakes than 5, you can buy them as loose cupcakes, which means they are not in a box.

Table 5.8 Multiplication/division and addition combined tasks

Task number	Task
46	There are 5 cupcakes in a box. Zandile bought 3 boxes and 2 loose cupcakes. How many cupcakes did Zandile buy?
47	Thulela bought 33 cupcakes. How many boxes and how many loose ones did she buy?

The responses of how the participants answered these two tasks can be seen in the graph below (Figure 5.31).

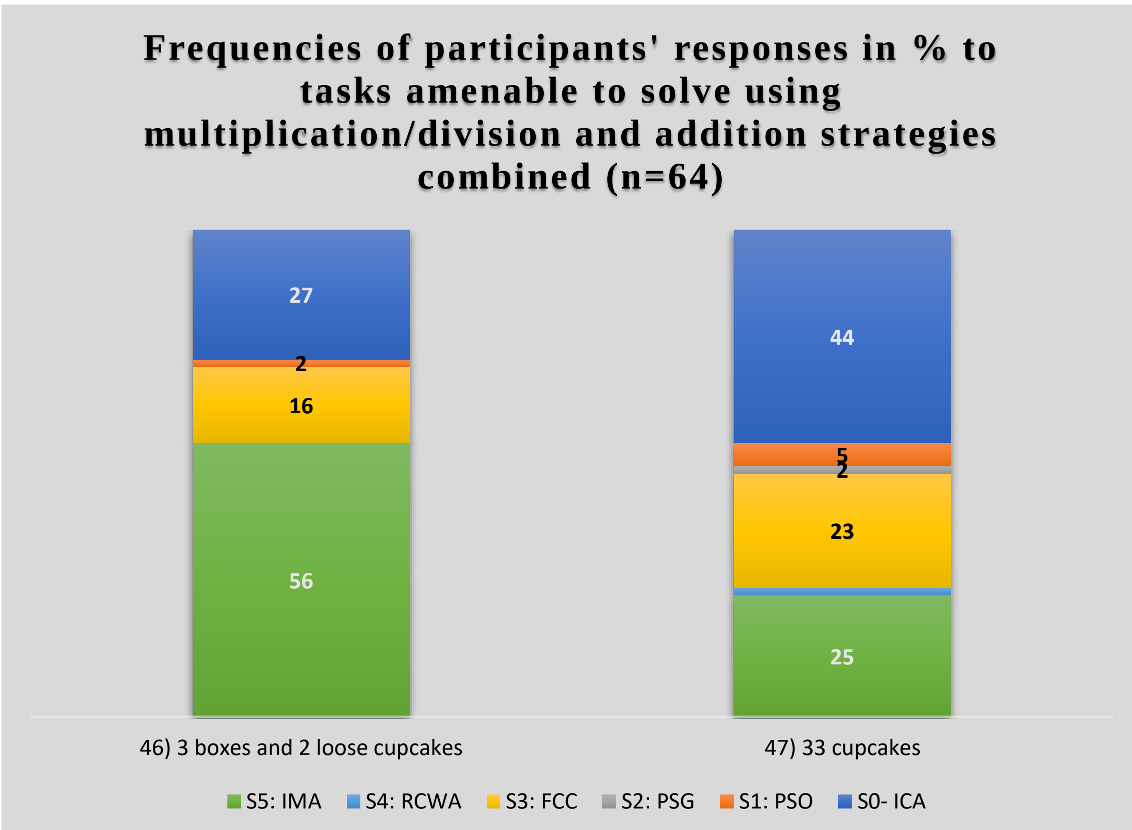


Figure 5.31 Students' responses to tasks amenable to solve using multiplication/division and addition strategies combined

Figure 5.31 shows that more participants (36 participants – 56%) could immediately answer with a mental answer in task 46 than they could answer task 47, with only 16 (25%) of participants answering with an immediate mental response.

The difference between the tasks is that for task 46, the participants could use repeated addition to find the number of 3 boxes of cupcakes by counting in fives and then adding another two cupcakes to arrive at an answer of 17 cupcakes. Still, the same strategy would not work for task 47. For task 47, the participants first knew that 30 is a multiple of 5 and $5 \times \mathbf{6} = 30$. Thus, 33 cupcakes will be sold in 6 boxes with 3 loose cupcakes. Task 47 was concerned with division and addition, while task 45 required multiplication and addition knowledge to complete.

Division and multiplication as inverses

To have a structural understanding of multiplication and division, a learner should clearly understand how the two operations relate to each other. Thus, the importance of measuring whether learners know multiplicative inverses is foregrounded here.

The tasks dealing with the inverses of multiplication and division are shown in Table 5.9.

Table 5.9 Division and multiplication as inverses

Task number	Task
48	If we know that $8 \times 7 = 56$, what is 56 divided by 7?
49	If we know that 48 divided by 6 is 8, what is 6×8?

The responses can be seen in the graph below (Figure 5.32).

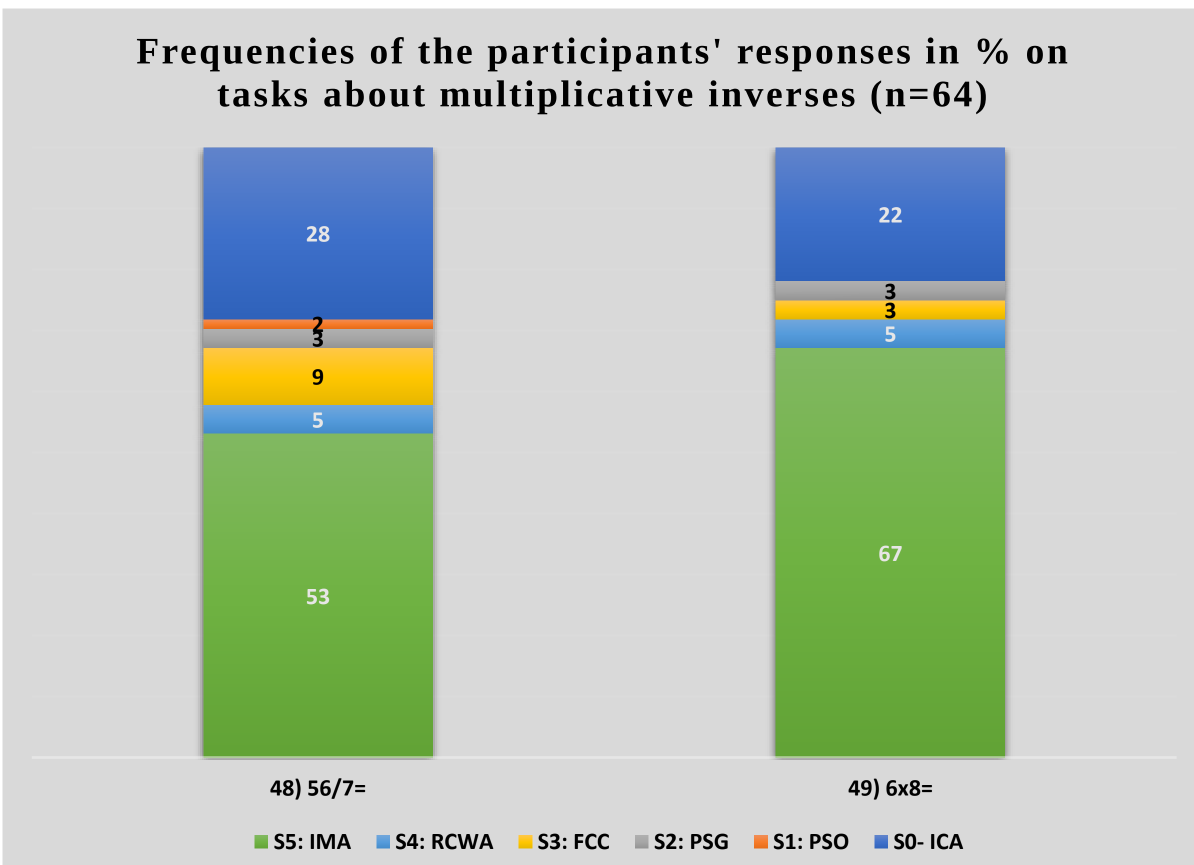


Figure 5.32 Students' responses on tasks about multiplicative inverses

The results of the two tasks about the inverses of multiplicative reasoning show that a vast majority of the participants could answer with an immediate mental answer, suggesting that they either know the multiplicative reasoning inverses or know the answers to $56 \div 7 = 8 \; and \; 6 \times 8 = 56$ as known facts.

However, the high number of students (28% for task 48 and 22% for task 49) who could not respond with a correct answer suggests that the participants who did answer incorrectly do not understand multiplication and division in relationship with each other.

The absence and the low number of stage 1 responses in the multiplicative inverses are understandable because unit-counting will not be a strategy to use as inverses require an understanding of the structure and relationship between multiplication and division.

5.5.2.2 Division tasks results

A summary of the stages in which I analysed the division tasks is presented in Table 5.10.

Table 5.10 A summary of the framework to code the division tasks with abbreviations

Level	Abbreviation	Observed strategy	Possible indiscernible strategies
Stage 0 – Incorrect answers	S0: ICA	Incorrect answers	Do not understand division
Stage 1 – Perceptual sharing or grouping in ones.	S1: PSO	The learner works with physical items and counts in ones when making groups of specified sizes	The learner understands division as sharing or grouping but cannot subitize a group of items without counting them first
Stage 2 – Perceptual sharing and grouping in groups.	S2: PSG	The learner is still dependent on physical items, but can make groups without counting them first.	The learner understands division and can make a group of items (5 or less items) without counting them first.
Stage 3 – Figurative composite counting without using physical items.	S3: FCC	The learner does not use physical items to make the groups but instead develops counting strategies like skip-counting backward (in multiples of the dividend) and keeping track of the divisor on their fingers.	The learner has developed counting strategies that do not rely on physical items to be visible or touchable. In this stage, the learner has a composite understanding of the dividend but not the divisor.
Stage 4 - Repeated composite sharing or grouping using a written	S4: RCWA	The learner uses written strategies to calculate the division answers.	The strategies at this level depend on written algorithms. The learner can use algorithms like drawing representations of

algorithm without physical items			the sharing and grouping aspects of division.
Stage 5 – Immediate mental answer.	S5: IMA	Answered with and immediate mental answer.	The learner used multiplication known facts. Learners understand division as an operation and know that division is the inverse of multiplication.

The division tasks involve partitive and quotative sharing as problem-solving tasks and a quotative sharing task that was presented in an array of dots.

Quotative and partitive sharing – (grouping and sharing aspect of division):

Tasks about quotative and partitive sharing are tasks 20, 21 and 22. The task are presented in Table 5.10.

Table 5.11 Quotitive and partitive sharing tasks in the G6 ESTA

Task number	Task
20	I have 20 apples. I want to put them in bags of 4. How many bags do I need? –Quotative sharing.
21	Here are 24 sweets. If we share them equally among 3 children, how many will each one get? – Partitive sharing.
22	If we now share them equally among 4 children, how many will each one get? – Partitive sharing.

(Wright, Martland, & Stafford, 2006)

The students' responses on the quotative and partitive tasks are presented in the graph below (Figure 5.33).

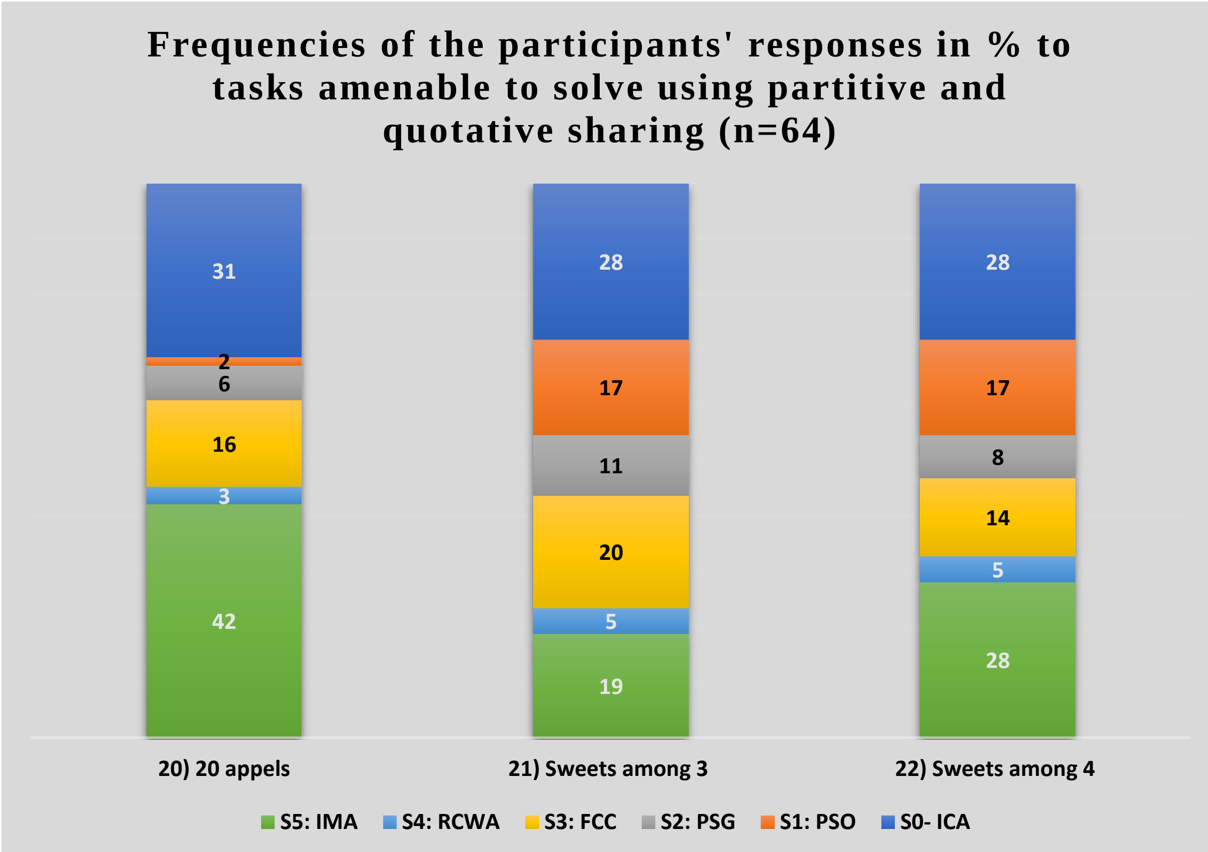


Figure 5.33 Students' responses to tasks amenable to solve using partitive and quotative sharing

Responses to Stage 1 understanding included a sharing strategy where the learners drew 24 tallies and shared them in 3 (task 21) or 4 (task 22) 'drawn circles'. The sharing was done by crossing out the tallies and then putting tallies into the circles one by one while scratching out the first drawn tallies. For the question "I have 20 apples; I want to pack them in bags of 4. How many bags do I need?" they counted 4 fingers repeatedly until they got to 20 and answered, "5 bags".

Division as quotative sharing presented in an array of dots

For task 50, I showed a card with an array of 54 dots and 9 dots in a row. Only the first row was visible to the participants. I then asked: *how many dots are there in one row*? I allowed the participants to count the dots in the first row. They answered 9 dots. Then I asked: *how many rows are there if there are 54 dots altogether*?

The page[51] that I showed them was as follows (Figure 5.32):

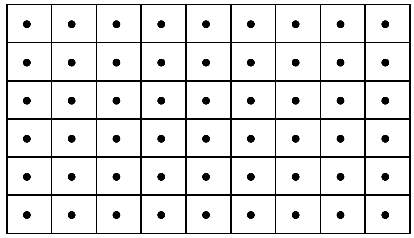


Figure 5.34 The array of 54 dots for task 50

The results of the responses are presented in Figure 5.35.

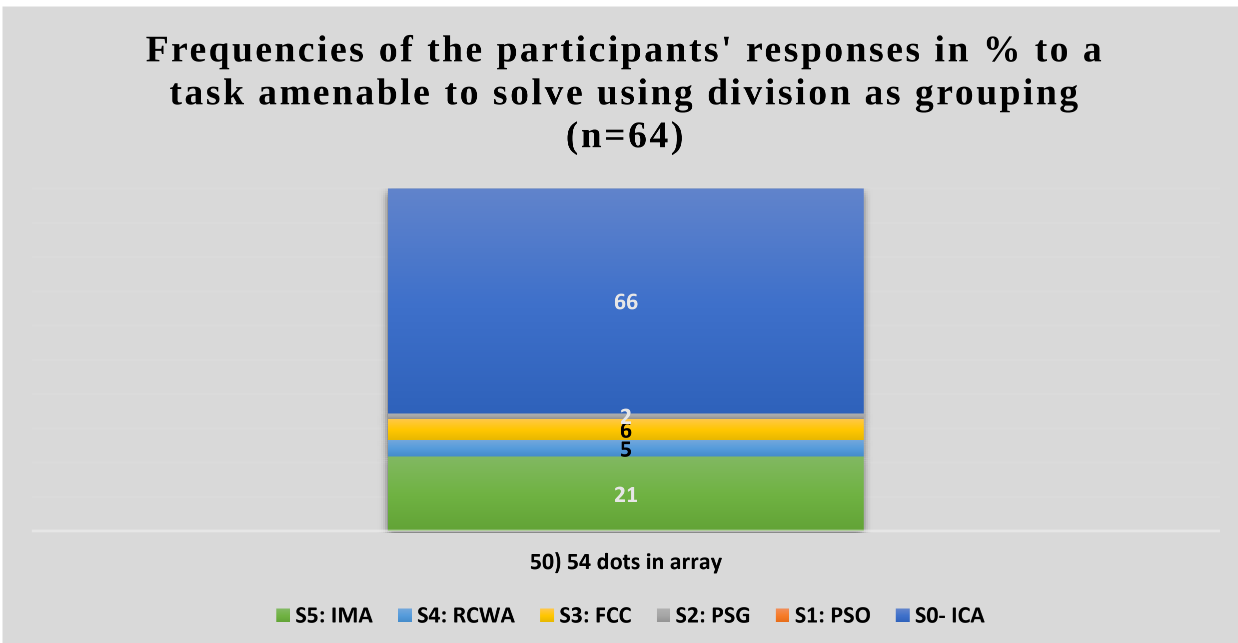


Figure 5.35 Students' responses to tasks amenable to solve using quotative sharing in division

A vast majority could not give a correct answer for task 50, while 14 (21%) participants gave an immediate mental answer, but 66% could not answer successfully.

The findings suggest that the participants do not know the 9-times table, nor do they understand how to link the groups of 9 dots in one row with the 54 dots. The knowledge of division as quotative

[51] See Chapter 3 p for the tasks of the G6 ESTA

sharing requires an understanding of relationships between numbers, which the participants could not manage. Thus, the participants could not use structural thinking to answer this question.

5.6 Conclusion

Chapter 5 gave empirical evidence of the participants' extent of structural thinking when answering the questions in all the data instruments by looking at the strategies the participants used and how well/badly they scored in the assessments.

Firstly, I focused on the G1 ANAdG2 that the participants wrote when they were in Grade 2. The participants scored very well on this assessment. We also mentioned that the excellent performance could be the low number range of the G1ANAdG2. Ninety percent of the questions could be answered using unit-counting.

Then we analysed the performance and strategies used by the participants on the G2 LFIN. The participants could count forward in a higher number range than for backward counting. A vast majority of participants could identify numbers from 1-20, but only a few could identify two-digit numbers higher than 20. No participants in the sample could identify any three-digit numbers. Most of the participants did not see 'ten units' and 'one ten' as a unit at the same time. Thus, the score for base-ten thinking was deficient. The stage at which most of the participants could be classified for SEAL was stage 2 (50%). This suggests that 50% of the participants still needed to count-all when doing addition tasks. Thirty-six percent of the participants counted-on to solve addition tasks and counted-down to solve taking-away questions in subtraction.

The results on the G6 CA showed that the participants find the G6 CA much more challenging than the G1 ANAdG2. They did not score as well on the G6 CA as they scored on the G2 ANAdG2.

The strategies the participants used in the G6 ESTA showed that they are still very dependent on unit-counting. The findings indicate that the vast majority of the participants do not use mental calculations but mainly use written column algorithms for questions that were amenable to mental strategies. To add to the lack of mental strategies used, a substantial portion of the written-column algorithm responses included unit-counting. If the students had not been video-taped, the written-column algorithm would have hidden the low stage of structural knowledge evident from the accompanying unit-counting.

Implications arising from the findings include attention to learners' reluctance to use mental calculations and low-level number knowledge that is obscured in written-column algorithms. Given that learners can appreciate number relations and mathematical structure in the early years, teachers and policymakers should emphasise teaching numbers structurally in the early years.

CHAPTER 6 QUANTITATIVE ANALYSIS – FITNESS FOR PURPOSE AND PREDICTIVE POWER

Chapter 6 answers the third researcher's question: In quantitative terms, what is the nature of the relationship, if any, between learner performance in Grade 2 on both the Grade 1 ANA and the G2 LFIN assessment and their performance in Grade 6 on each of the G6 CA and the G6 ESTA assessments? This chapter is mainly about the predictive power of the early assessments; however, to interrogate the data in bivariate terms more rigorously, it is crucial to determine if the datasets are fit for the purpose of doing bivariate analysis.

Thus, I do two things in this chapter. First, I conducted a univariate analysis, which helped me understand the different assessments' measures of central tendency, variance, spread, normality and fitness of purpose. Secondly, I used bivariate analysis to understand the nature of the correlations between the Grade 2 and Grade 6 assessments to answer the third research question.

The correlations between the four assessments are at the heart of this study (G1 ANAdG2 and G2 LFIN versus G6 CA and G6 ESTA). I report the correlations between the data sets of the assessments as intact[52] assessments. However, I will also note the correlations between the data as parts of the assessments. For example, I will examine the correlations between the SEAL, FNWS, BNWS, NI, and base-ten knowledge[53] of the Grade 2 LFIN framework against both the Grade 6 assessments. Moreover, in some cases, I am discussing the correlations of the earlier and later tests by disaggregating the data according to the two groups in my data context, namely township and suburban schools.

Furthermore, I include relational analyses of how accurately the Grade 2 assessments predicted the additive and multiplicative reasoning of the G6 ESTA and the number and not-number-related parts of the G6 CA.

[52] As 'whole' assessments.
[53] The SEAL, FNWS, BNWS, NI and base-ten knowledge are known as the 'aspects' of the G2 LFIN (Wright, Martland, & Stafford, 2006)

Because the study dataset includes participants from township[54] and suburban[55] schools, I will include correlational analysis to distinguish between data from townships and suburban schools where I think it is valuable to do so.

I used SPSS, Excel and Fathom to do the statistical analysis and graphs.

6.1 The nature and distributions of the scores of the G1 ANAdG2, G2 LFIN, G6 CA and G6 ESTA as single variables

Univariate analysis summarises data on a single variable and is essential for descriptive analysis. I will also look at the normality of the data sets of the assessment instruments by using Brase and Brase's guidelines for normality (2013). Furthermore, according to McMillan and Schumacher (2010), a thorough univariate analysis of group data can be reported in the following way:

- Explaining the origin and nature of the data.
- Presenting graphic portrayals of the data.
- Describing the measures of spread.
- Discussing measures of central tendency and debating measures of variability.

In the following paragraphs, I will discuss the data sets of the assessments according to McMillan and Schumacher's (ibid) guidelines for doing univariate analysis. Since it is essential that the datasets are normal, I include the guidelines for normality in the discussion.

6.1.1 Origin and nature of the data

As mentioned before, the sample consists of 64 learners from 8 schools in Gauteng, which were assessed on two assessments in February 2011, when the learners were in Grade 2, and two assessments in November 2015, when the participants were in Grade 6 with the same cohort of learners. The one Grade 2 assessment was an oral video-recorded diagnostic test known as Bob Wright and colleagues' Learning Framework in Number assessment (2006) (G2 LFIN). The second Grade 2 assessment was the Annual National Assessment (G1 ANAdG2), a national assessment conducted by the South African Department of Basic Education (DBE).

[54] A location in the outer edges of a metropolitan city.
[55] A township in South Africa is a place or city of predominantly black occupation, designated for black occupation by the apartheid legislation (Oxford dictionary).

When the learners were in Grade 6, they were assessed on the G6 ESTA test, an oral video-recorded diagnostic test developed during this study. The second Grade 6 test was a national assessment conducted by the Gauteng education department, known as the Common Assessment (G6 CA).

The G1 ANAdG2 and the G6 CA were only concerned with correct answers, while the additional two diagnostic assessments were concerned with the strategies learners used.

6.1.2 Graphic portrayals of the data sets

This section presents graphical representations of the participants' performance on all assessment tools. All scores are recorded as percentages and will be presented in graphs. Since I am examining the correlations between the assessments in terms of 'all participants', participants in township schools and participants in suburban schools, I also present graphical illustrations of the participants' performances as township schools' participants and suburban schools' participants.

This section shows the distributions of the G1 ANAdG2, G2 LFIN, G5 CA, and the G6 ESTA as intact tests. When displaying the graphical portrayals of the assessments, I will first show the dispersion in histograms to see where the data points were situated in bins and then as combined boxplots to see the relationships between the assessments in quartiles. I used Excel to draw the histograms and presented all sets' data in seven bins and SPSS to illustrate the combined boxplots.

6.1.2.1 The G1 ANAdG2 distribution – all schools

The marks of the G1 ANAdG2 (all schools) test scores appear to be slightly skewed to the left, with very high scores (Figure 6.1).

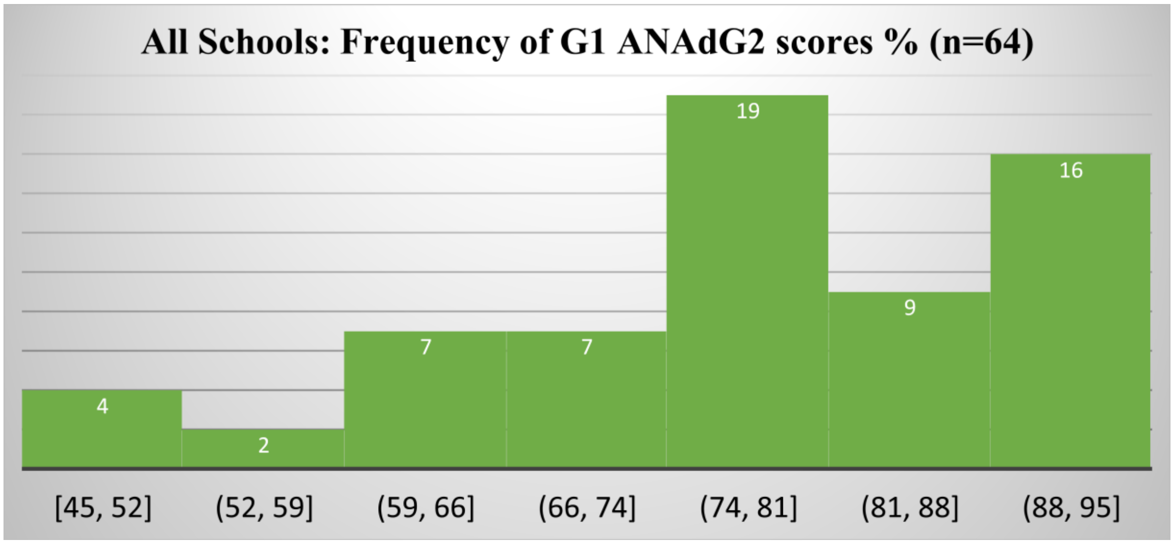


Figure 6.1 All schools: Frequency of G1 ANAdG2

Figure 6.1 reveals that more than two-thirds (68,75% ((19+9+16) out of 64) of the participants scored higher than 74%. A mode bin with 74% and 81% limits shows that 18 participants (28%) had scores between these two limits. The lowest four scores for the G1 ANAdG2 were between 45% and 52%, and the highest 16 scores (25%) were between 88% and 95%. The performance in the G1 ANAdG2 reveals that the assessment was quite facile, and many participants scored very high marks.

Figures 6.2 and 6.3 show the differences between the participants' scores from the townships and the suburbs.

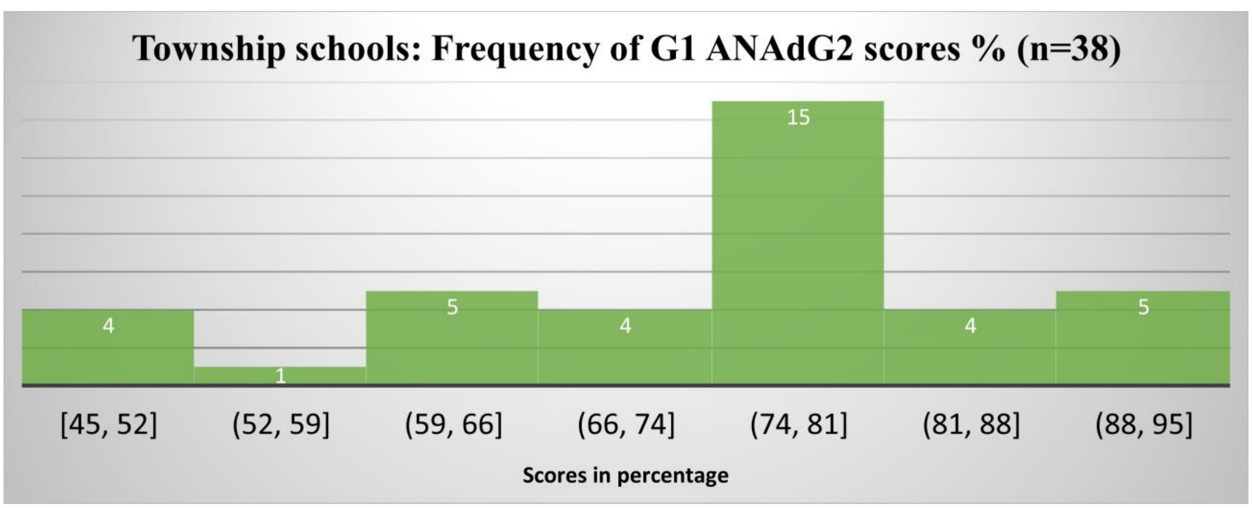


Figure 6.2 Township schools: Frequency of G1 ANAdG2 scores

From Figures 6.2 and 6.3, we observe that 63% of the participants from the township schools scored higher than 74%, while 77% of the suburban schools' participants scored higher than 72%.

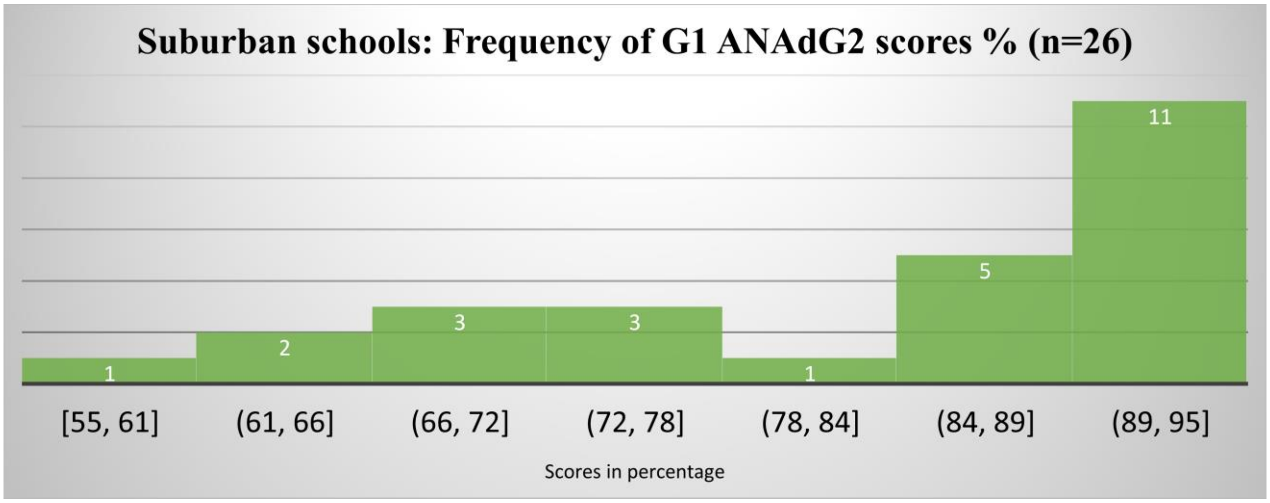


Figure 6.3 Suburban schools: Frequency of G1 ANAdG2 scores

Furthermore, 62% of the suburban schools' participants scored higher than 84%, while only 24% of the participants in the township schools scored higher than 82%. Of the participants in the township schools, 38% scored lower than 66% in this assessment, while only 11% of the participants in the suburban schools scored lower than 66%. Figures 6.2 and 6.3 signal that the participants in the suburban schools scored better in the G1 ANAdG2 than the participants from the township schools.

In summary, regarding the data of the G1 ANAdG2, the boxplots in Figure 6.4 illustrate the dispersion of the school groups' data in relationship with each other regarding the G1 ANAdG2.

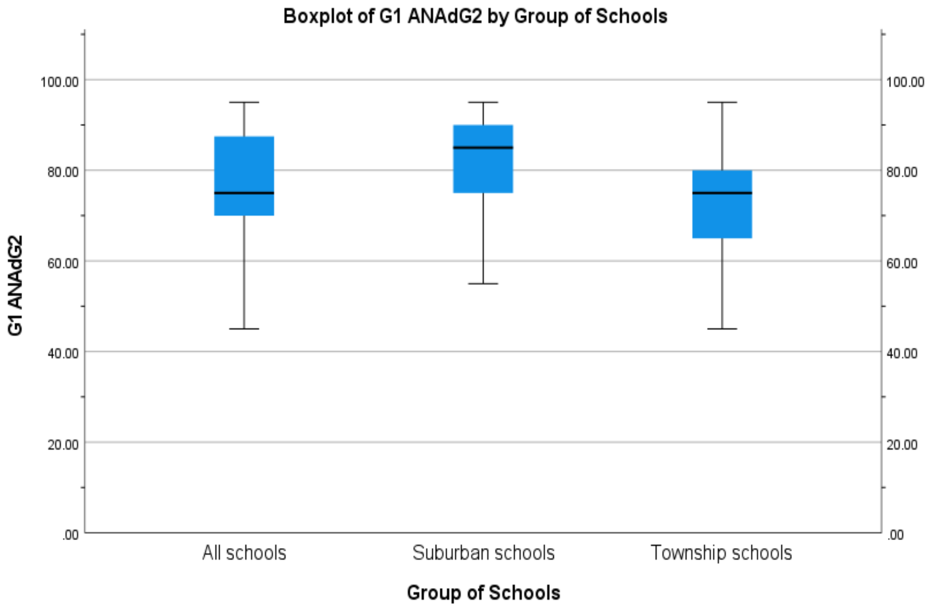


Figure 6.4 Combined boxplot of the scores of the G1 ANAdG2 by schools

Figure 6.4 approves the suggestion that the suburban schools scored better than the township schools in the G1 ANAdG2 assessment, also because the middle 50% (Inter Quartile Range - IQR) of scores for the township schools was more or less between 65% and 80%, while the central 50% of the scores for the suburban schools were between 75% and 90%.

6.1.2.2 The G2 LFIN distribution

The scores of the G2 LFIN show (Figure 6.5) that the majority of the participants' scores were between 40% and 64% (70%), with three high scores between 72% and 80% and six deficient scores with scores between 23% and 31%.

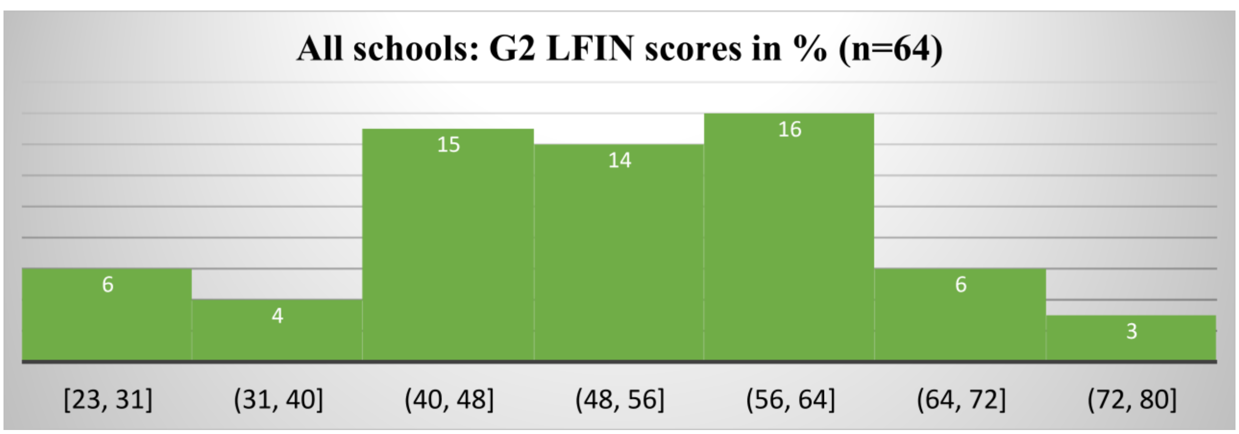


Figure 6.5 All schools: Frequency of G2 LFIN scores

From Figures 6.6 and 6.7, we notice that there are dissimilarities between the scores of the two school groups. The maximum score bin for the township schools' participants is between 60% and 67%, with 13% of the township schools' participants' scores between these limits. The minimum bin is between 23% and 30%.

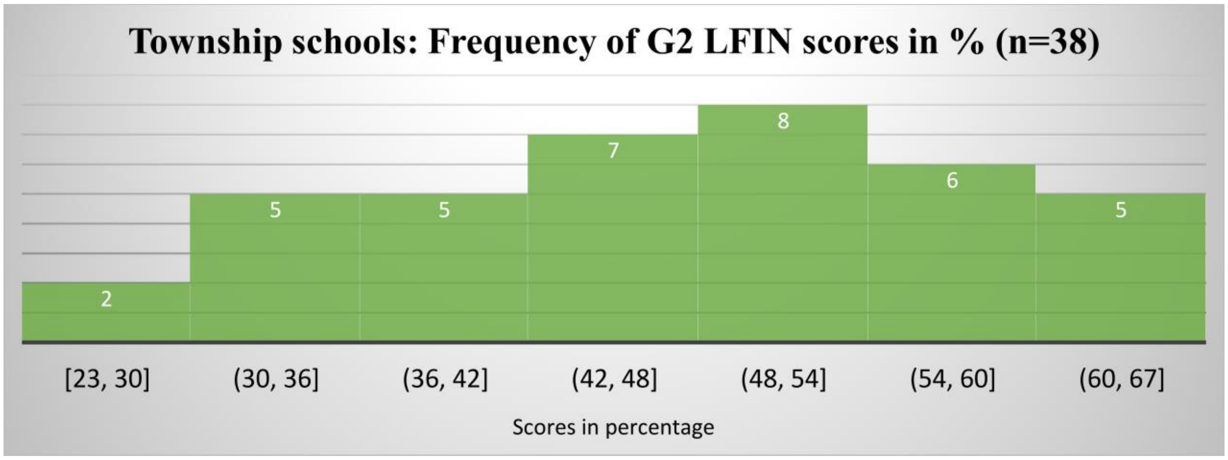


Figure 6.6 Township schools: Frequency of G2 LFIN scores

In comparison, we perceive from Figure 6.7 that the maximum score bin for the suburban school group is between 72% and 80%, with a frequency of 3 participants. The minimum score bin is between 27% and 34%.

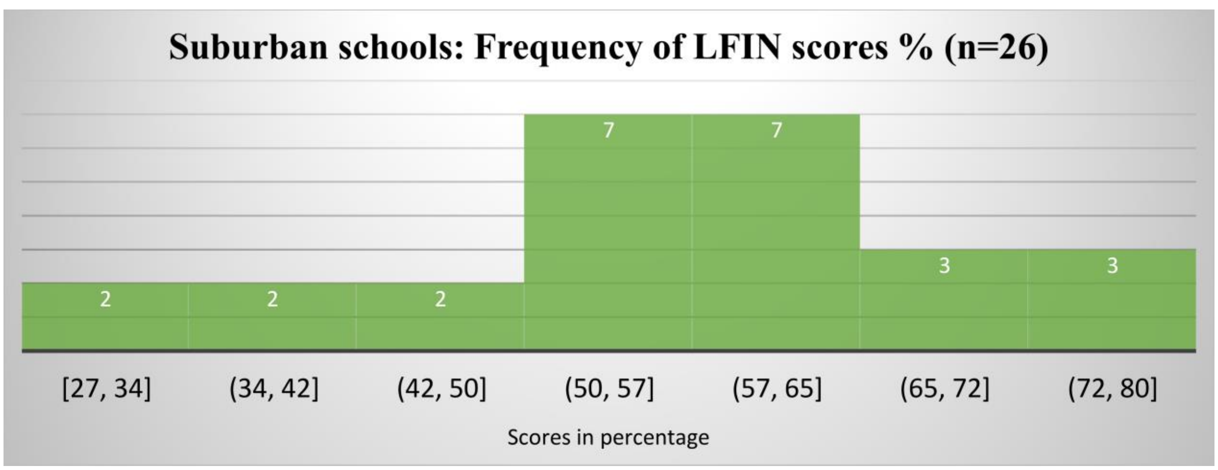


Figure 6.7 Suburban schools: Frequency of LFIN scores

To sum up the discussion about the distributions of the Gr2 LFIN as all schools, township schools, suburban school groups, I conclude that Figures 6.5, 6.6, and 6.7 demonstrate that the distributions of the data set of the G2 LFIN show relatively normal distributions. Furthermore, the suburban schools' participants scored better in the G2 LFIN than the participants from the township schools.

To see the distributions in relationship with each other, I drew a combined boxplot to see the distributions in relation to each other (Figure 6.8).

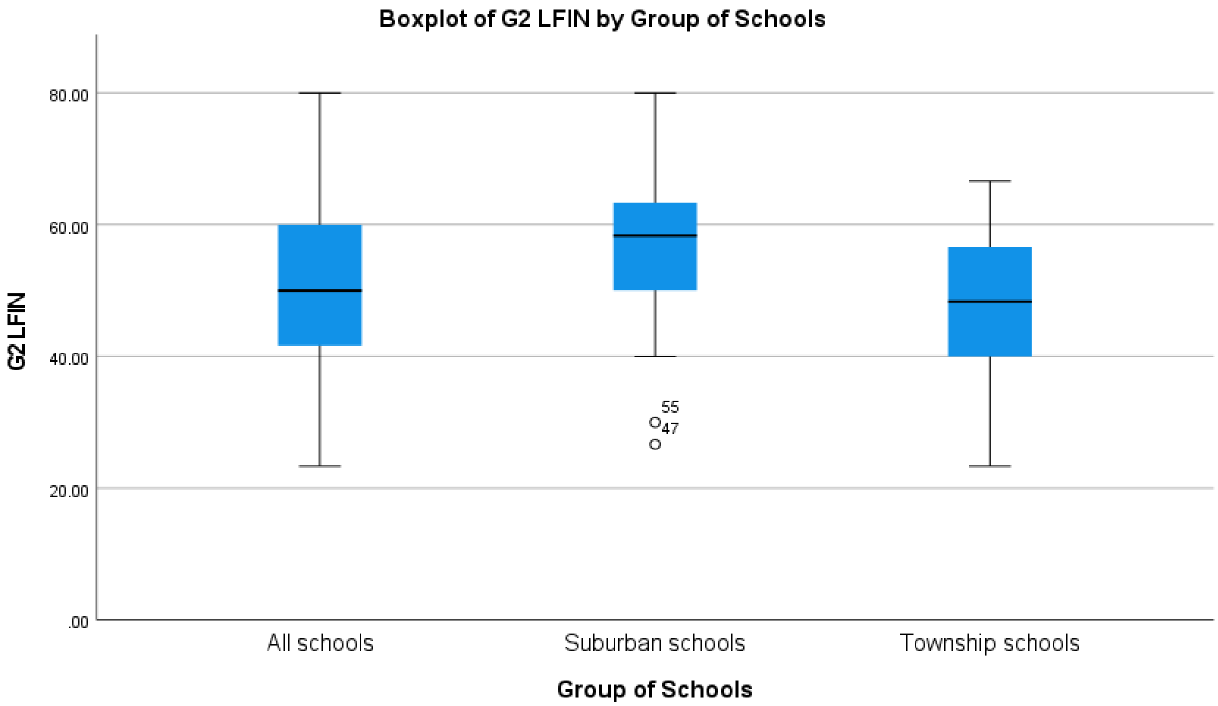


Figure 6.8 Combined boxplot of the scores of the G2 LFIN by schools

From Figure 6.8 it is evident that the suburban schools' participants showed a better understanding of numbers in terms of structural thinking than the township schools' participants when they were in Grade 2.

6.1.2.3 G6 ESTA distribution

The G6 ESTA's distribution can be seen in Figure 6.9, where a clear modal bin is located between 41% and 51% showing that 39% of all participants scored between these limits. The lowest bin reflects 4% of all the participants, between 20% and 31%. Eleven percent of all the participants scored in the maximum bin, between 81% and 92%.

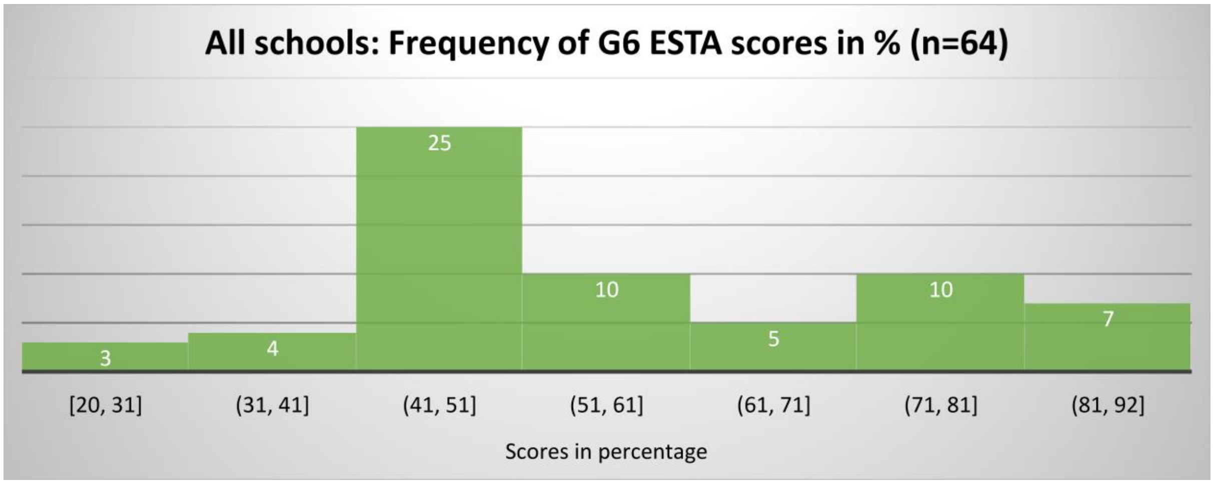


Figure 6.9 All schools: Frequency of G6 ESTA scores

The differences between the township and suburban schools' data sets are evident in Figures 6.10 and 6.11.

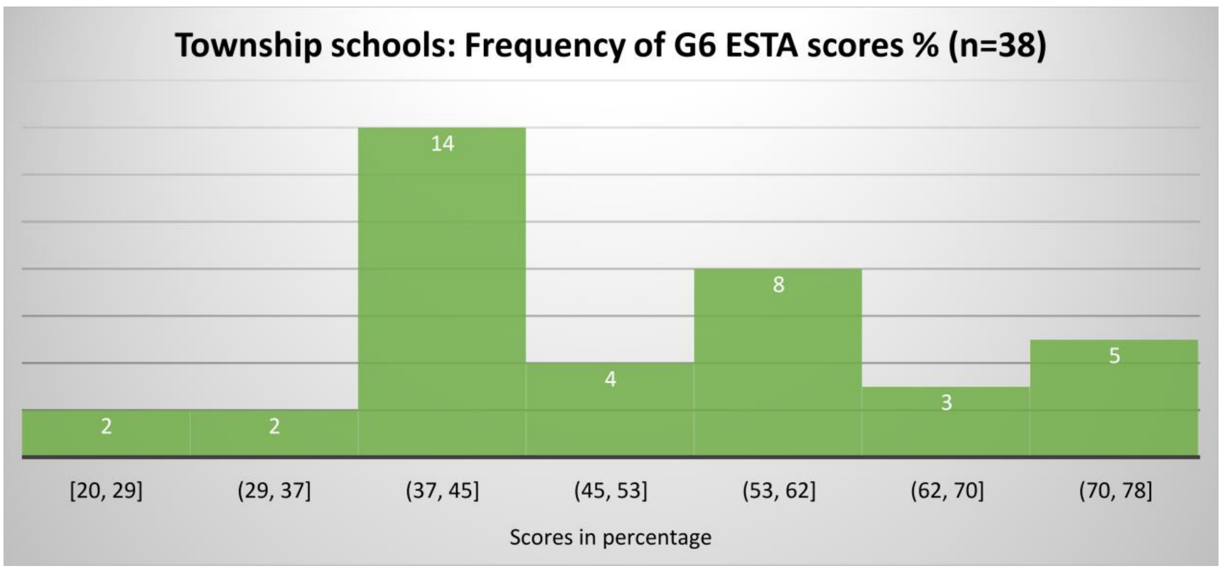


Figure 6.10 Township schools: Frequency of G6 ESTA scores

The apparent difference between the data sets of the township schools and the suburban schools is that the suburban schools' participants had higher marks than the township schools' participants.

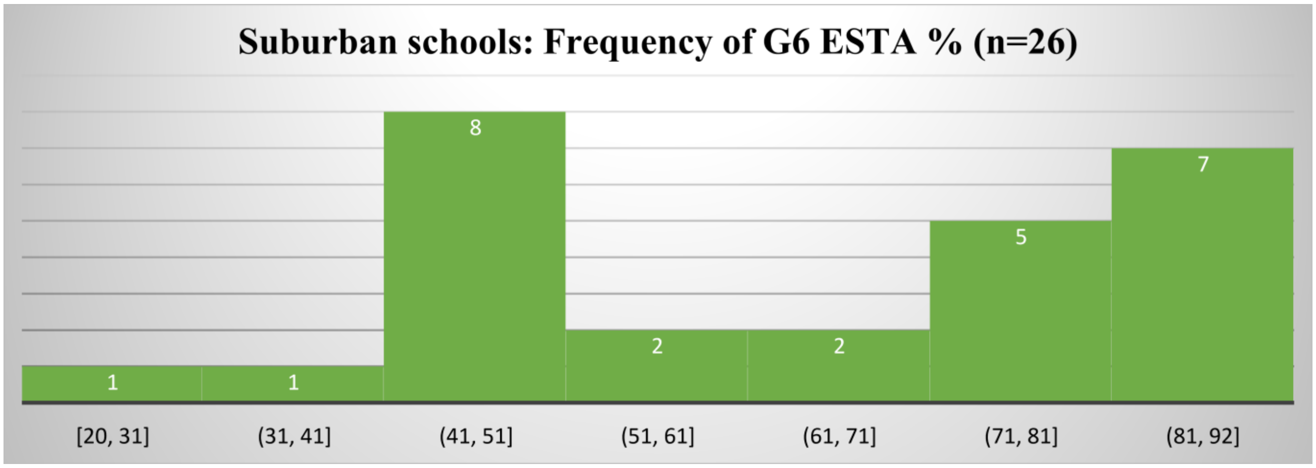


Figure 6.11 Suburban schools: Frequency of G6 ESTA

As illustrated in Figures 6.10 and 6.11, the highest bin scores for the suburban schools' participants were between 81% and 92%, with 26% of participants in this category recording scores between these limits. The scores of the township schools' participants, which ranged from 70% to 78%, represented 13% of the participants in this category.

To sum up the dispersions of the school groups and all schools scores, I present the boxplots in Figure 6.12.

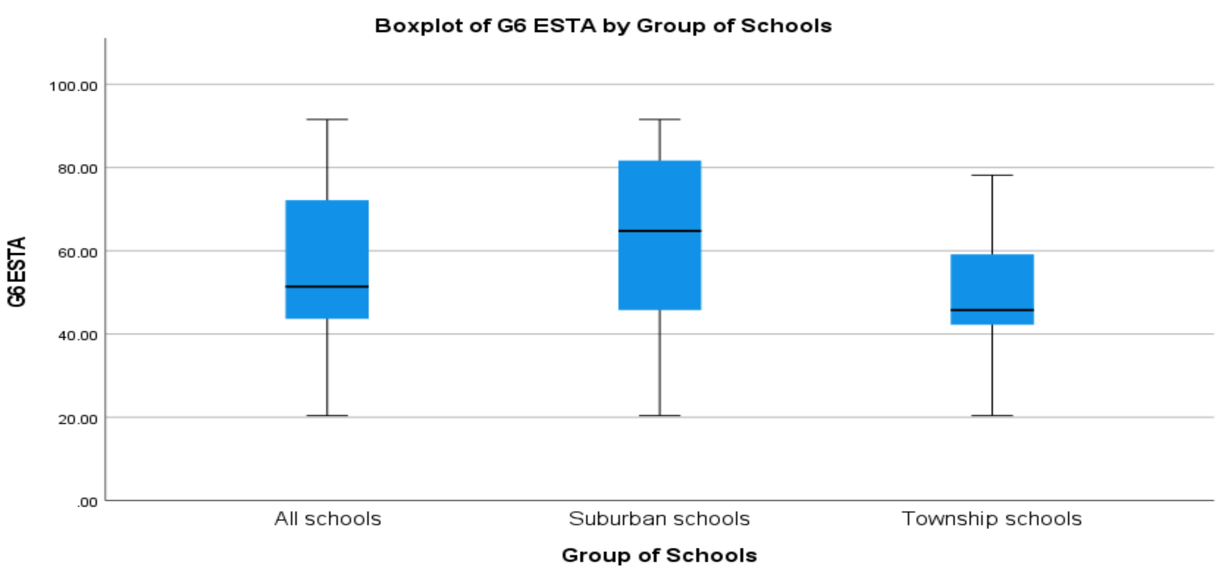


Figure 6.12 Clustered boxplot of the scores of the G6 ESTA by schools

From Figure 6.12, it is evident that the suburban schools' participants scored higher marks in the G6 ESTA than the participants in the township schools, as we can see from the histograms. Since the G6 ESTA measured structural thinking, we can say the suburban school participants' understanding of structural thinking was higher than the participants in the township schools.

6.1.2.4 Grade 6 CA distribution

The scores of the G6 CA confirm that this assessment was possibly challenging for the participants as they did not score as high as for the DBE's assessment at the beginning of Grade 2. The distribution of the data set of all schools' participants is slightly skewed to the right. The scores for the G6 CA range from 21% to 88%, as illustrated in Figure 6.13.

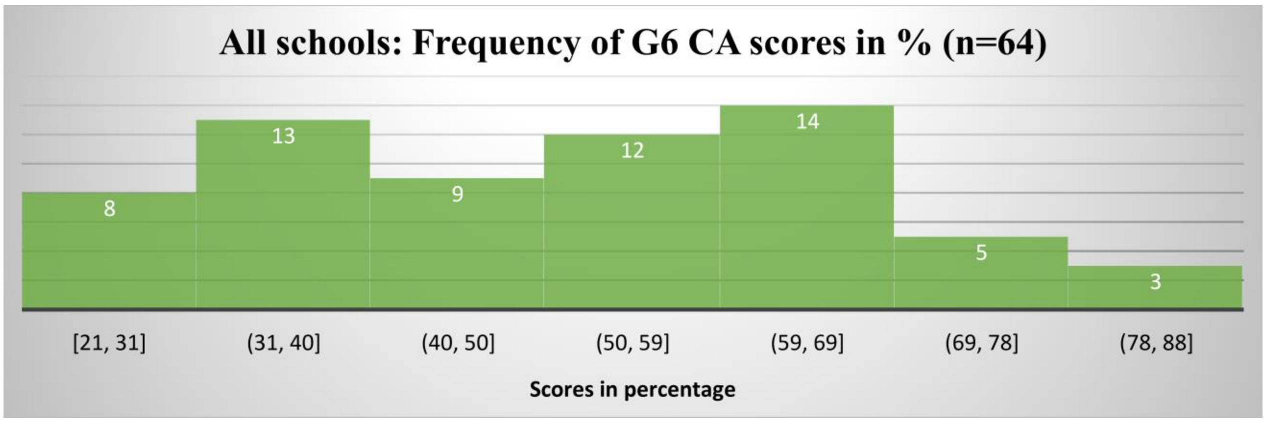


Figure 6.13 All schools: Frequency of G6 CA scores

The distribution of the data set of the township schools is slightly skewed to the right, with 61% of participants in suburban schools scoring lower than 47% in the G6 CA (Figure 6.14).

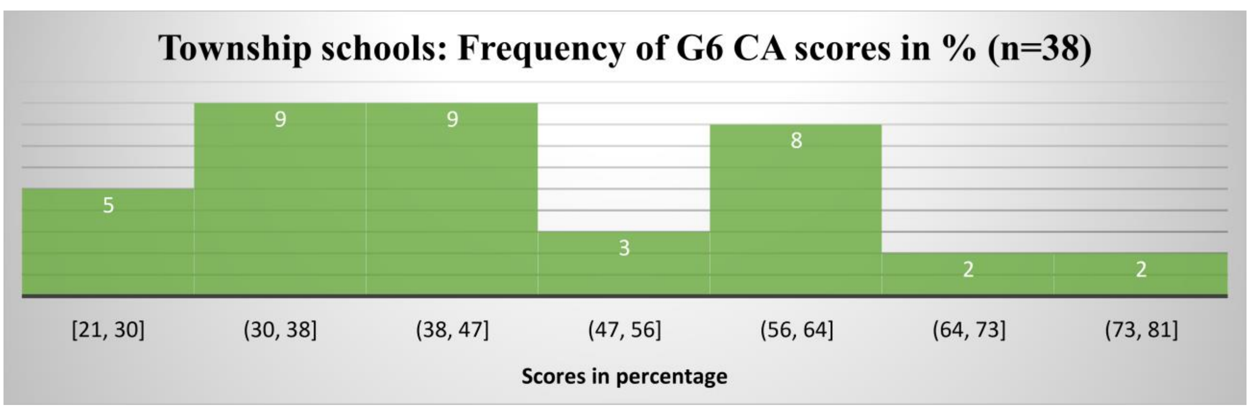


Figure 6.14 Township schools: Frequency of G6 CA scores

The scores of the suburban schools (Figure 6.15) are relatively normal, with scores between 24% and 88%. The modal bin of the scores is between 61% and 70%, with 27% of participants scoring between the two limits.

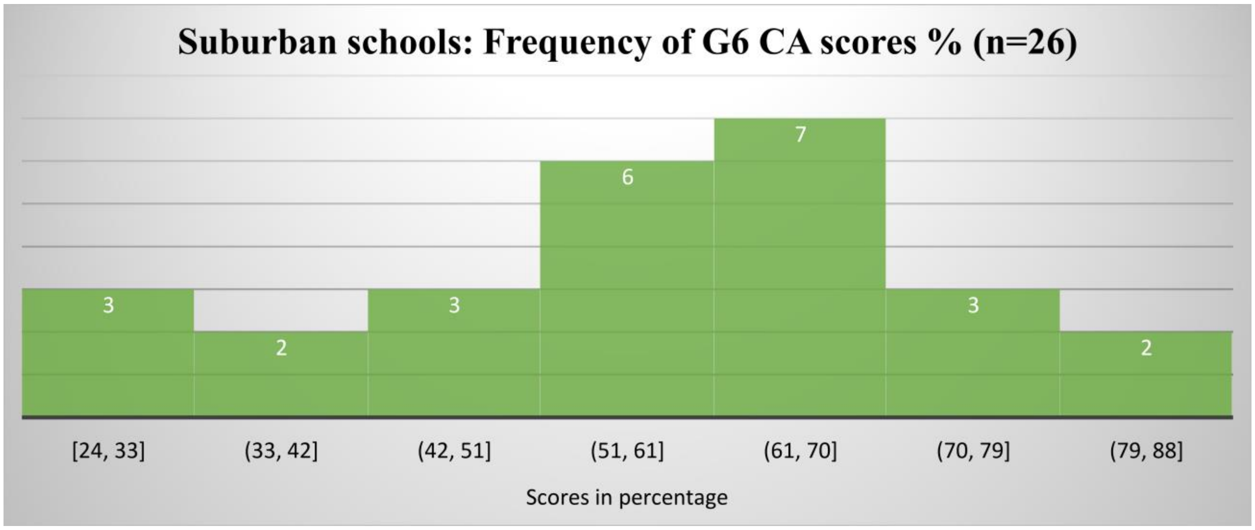


Figure 6.15 Suburban schools: Frequency of G6 CA scores

As illustrated in Figure 6.14 and Figure 6.15, the participants' scores in suburban schools were higher than those in township schools.

In summary, Figure 6.16 illustrates how these data sets' distributions relate to each other in terms of the G6 CA scores.

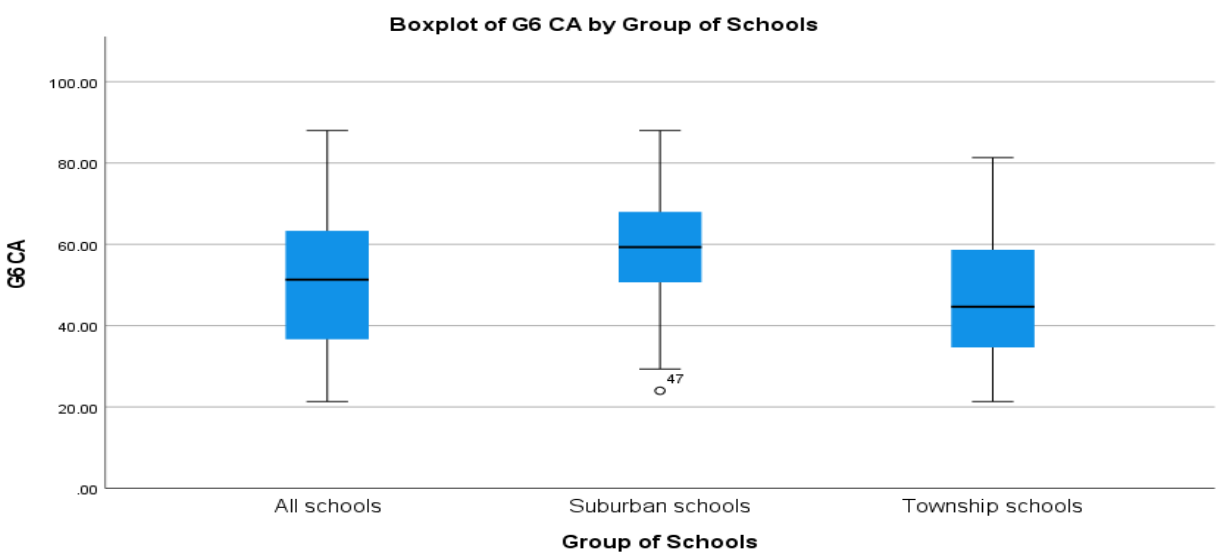


Figure 6.16 Clustered boxplot of the scores of the G6 CA by schools

Figure 6.16 illustrates that the IQR for the suburban schools' distribution is higher than the IQR of the township schools' data set. Thus, the township schools' performance was lower than the suburban schools in all the assessment instruments.

6.1.3 Measures of spread in terms of range and interquartile range

This section will deal with the measures of the spread of the tests for both school groups combined. Table 6.2 summarises the ranges and IQRs for all the assessment instruments.

Table 6.1 The distributions in terms of the range and IQR for all schools

	G2 LFIN	G6 CA %	G1 ANAdG2	G6 ESTA %
Maximum value	80,0	88,0	95,0	92,0
Minimum value	23,0	21,0	45,0	20,0
Range	57,0	67,0	50,0	72,0
Quartile 3	60,0	63,5	87,5	72,5
Quartile 1	41,5	36,5	70,0	44
IQR	18,5	27,0	17,5	28,5

The ranges of the assessments differ in the following way: the G6 ESTA assessment data points have the most extensive range of 72% with values between 92% and 20%. The G1 ANdG2 with a number range of 50 and values between 95% and 45% has the smallest number range. The number ranges of the other assessments have values between 80% and 23% for the G2 LFIN, and 88% and 21% for the G6 CA. The latter suggests that the data points of the G6 ESTA have the lowest variation and the G1 ANAdG2 the most significant variation.

The IQR of the assessments is important because this measure tells us where the middle 50% of the data items are located. The IQRs of the two Grade 2 assessments are 17,5 and 16,3, which means that 50% of the results lie within about 17 percentage points. However, the IQRs are situated in different places (between 60% and 42,5% for the G2 LFIN and between 86,3% and 70% for the G1 ANAdG2). The IQR of the Grade 6 assessments differ by 2,3 percentage points, with an IQR of 28,3 and 26.

The location of the IQR of the G1 ANAdG2 indicates that 50% of the scores were situated in high scores. Interestingly, 25% of the scores for the G1 ANAdG2 lie between 87,6% and 95%, which is also extraordinarily high and could be interpreted as an assessment that did not provide a significant challenge to the participants.

To sum up the measure of spread for the scores of the assessment instruments, I present multiple boxplots for the G1 ANAdG2, G2 LFIN, G6 CA, and the G6 ESTA in Figure 6.21.

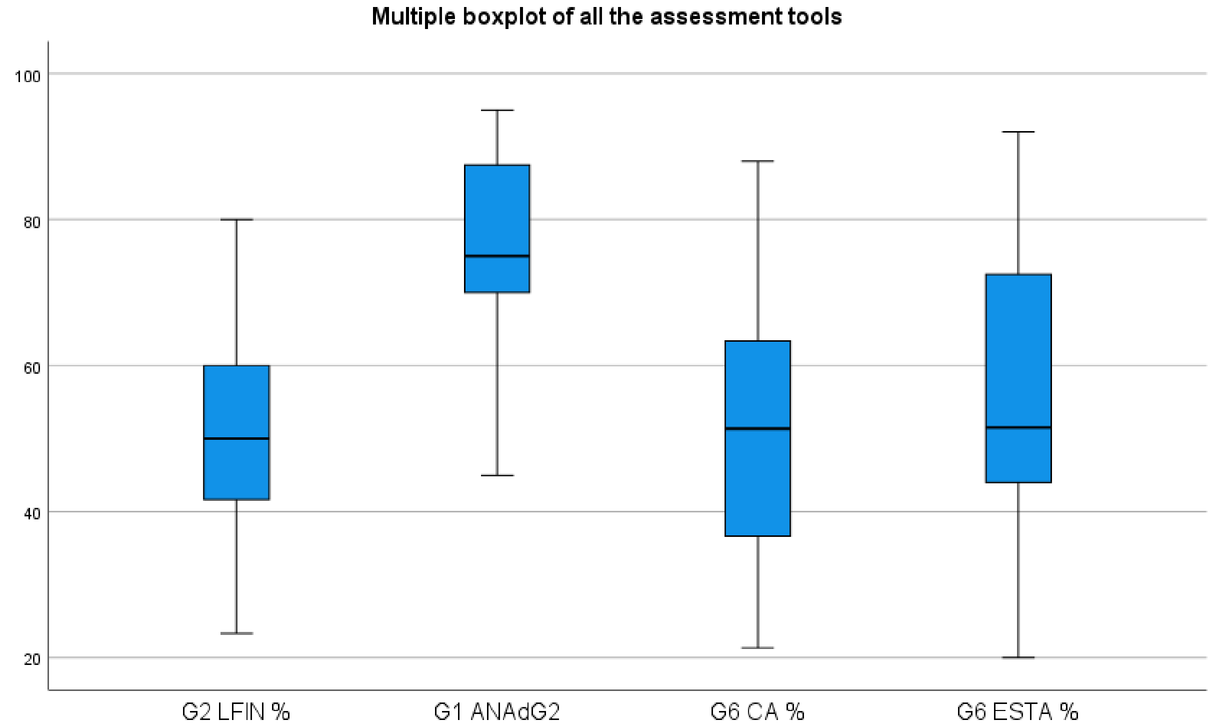


Figure 6.17 A multiple box-and-whisker-plot for the G1 ANAdG2, G2 LFIN, G6 CA, and the G6 ESTA

In the box-and-whisker plots of Figure 6.21, we see how the data sets are distributed concerning each other. The G2 LFIN, the G6 CA and the G6 ESTA medians are approximately the same, with an unusually high median for G1 ANAdG2.

6.1.4 Measures of central tendency

The positions of the mean and the median are an indication of the skewness of the distribution. If the mean and median lie far from each other, the graph will likely be skewed. The table below summarises the means and medians of the different assessments.

Table 6.2 Measures of central tendency of the assessment instruments

	G2 LFIN	G6 CA %	G1 ANAdG2	G6 ESTA %
Mean	51.0	50.8	77.0	55.8
Median	50.0	51.5	75.0	51.5
Difference between the mean and median	1.0	-0.7	2.0	4.3

As shown in Table 6.2, the mean and the median differences are not significant enough to claim that the distributions are skewed. In a Gaussian graph, the mean and the median are equal. The minor differences between the means and the medians can be a first indication that the distributions of the assessments are relatively normal. In the next section (6.1.5), I discuss the normality of the data sets in more detail.

The means of three of the assessments are lower than 60; however, the 77,1 mean of the G1 ANAdG2 is relatively high. The high median of the G1 ANAdG2 tells us that 50% of the scores were above 75.

6.1.5 Measures of variability

The standard deviation of a data set indicates the dispersion of the data about the mean.

Table 6.3 Measures of the variability of the datasets of the assessment instruments

	G2 LFIN	G1 ANAdG2	G6 CA %	G6 ESTA %
Mean	51,0	77,0	50,8	55,8
Standard deviation	13,1	13,145	16,25	18,035

A higher standard deviation indicates a more spread-out or flat dispersion. Table 6.3 shows that the G1 ANAdG2 and the G2 LFIN have the smallest standard deviation and are the least spread-out about the mean than the two Grade 6 assessments.

The scores of the G6 ESTA assessment vary the most, with a standard deviation of 18,035. This higher standard deviation tells us that the data set of the G6 ESTA are more spread-out about the mean than the others. Thus Furthermore, Table 6.3 shows that 68% of the G1 ANAdG2 scores lie much higher than the scores of all the other assessment instruments.

6.1.6 Normality of the distributions of the data instruments

It is essential to know whether the data sets of the assessment distributions are Gaussian, as some bivariate assessments require the data to be normal. According to Brase and Brase (2011), a normality investigation of the assessments can be done to assess whether the distribution is normal or not.

A Gaussian distribution (normal distribution) is symmetrical about the mean. The data items in this study show minimal differences between the mean and medians of all the distributions of the data instruments. Watkins, Scheaffer and Cobb (2004) claim that a small difference between the mean and median indicates that a distribution is not skewed. However, Brase and Brase (2011) give guidelines to determine normality by arguing that the distributions must comply with the following conditions:

1. The distribution should be roughly bell-shaped.
2. There should not be more than one outlier, if any.
3. Gaussian distributions are symmetrical[56].
4. The normal quantile plot should be relatively straight (Brase & Brase, 2011, p. 323).

In the following paragraphs, I will follow the four guidelines of Brase and Brase (2011) to discuss the normality of the data sets of the assessment instruments.

Histograms are roughly bell-shaped: All the distributions for the four assessment tools are roughly bell-shaped. I used Fathom to draw a density standard normal curve over the histograms. Figure 6.18 shows relative normality for the G2 LFIN and the G6 CA. However, the G1 ANAdG2 and the G6 ESTA data sets need more investigation to ensure normality.

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[56] Pearson's index gives a measure of skewness for a sample of data for skewness. The Pearson's index must have a value between minus one and one.

$$Pearson's \;\; index = \frac{3(\bar{x} - median)}{standard\ deviation}$$

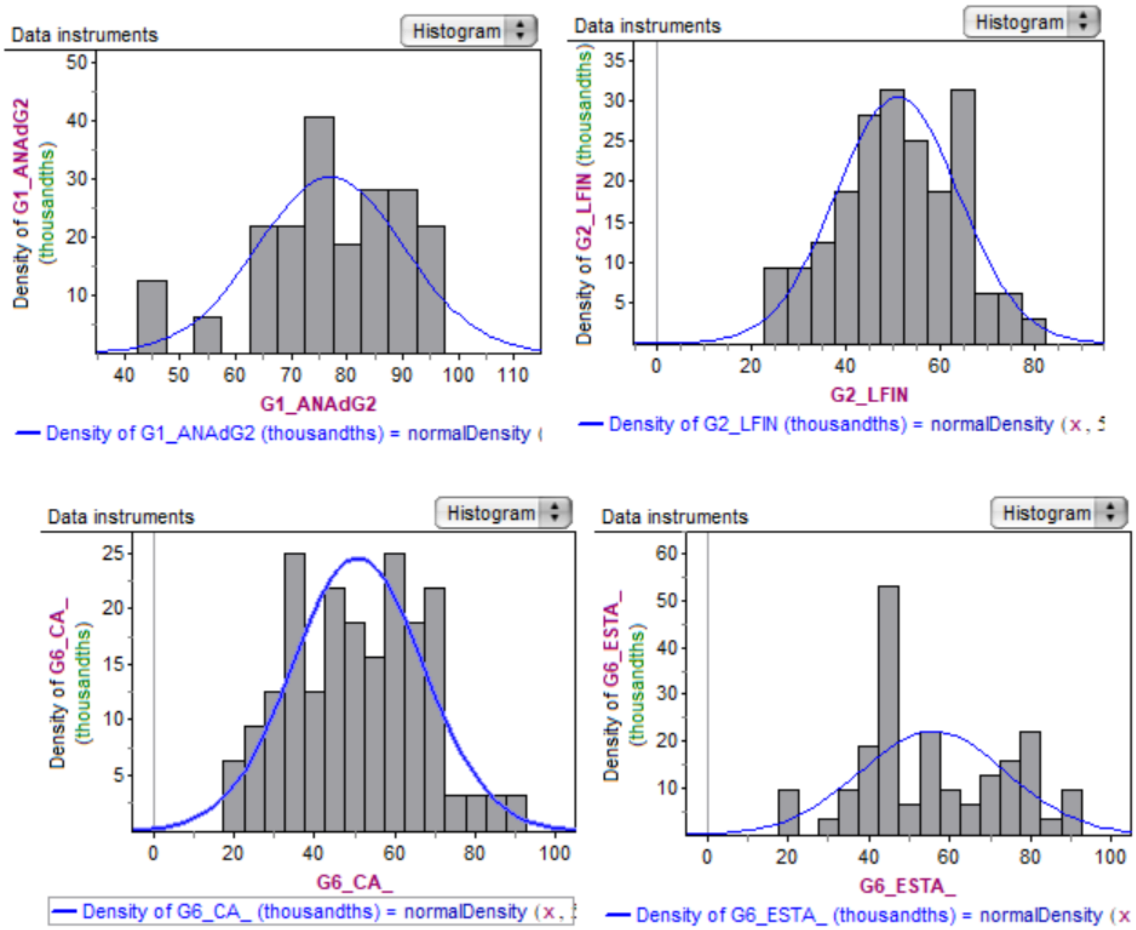

Figure 6.18 Histograms of all the data instruments with density standard normal curves

Not more than one outlier: The boxplots in section 6.1.3 show the distributions of all the data sets of the assessment tools. The boxplots indicate that none of the data sets of the assessments contains an outlier.

Pearson's index for normality: For the assessment outcomes, the Pearson index for skewness must take on a value between -1 and 1 for the distribution to be classified as normal. In Table 6.4, the values of the Pearson indexes of the assessment instruments are between -1 and 1. The contents in Table 6.4 confirm the previous finding that the distributions are relatively normal.

Table 6.4 Pearson's indexes for the assessment instruments

Assessment	G2 LFIN	G1 ANAdG2	G6 ESTA	G6 CA
Pearson's index	0.22705	0.46358	0.72155	-0.09987

The normal quantile plot should be relatively straight: A quantile plot takes the data scores, sorts them in ascending order, and then plots them versus quantiles calculated from a standard normal curve in a scatter plot. Figures 6.26 to 6.30 show the normal quantile plots for all the assessment instruments.

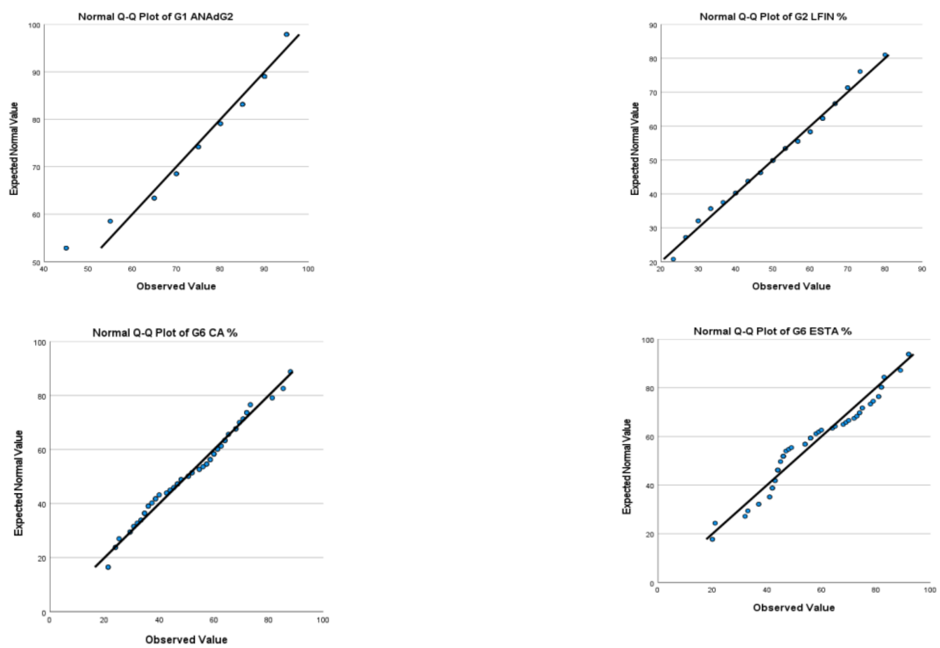

Figure 6.19 Normal quantile plots for all assessments

As we can see in Figure 6.19, the quantile plots of all the data of the assessment instruments are relatively straight, which indicates that all data sets of the assessment instruments meet this requirement of the quantile plots to be normal. A summary of the guidelines of testing normality according to Brase and Brase is in Table 6.5.

Table 6.5 Summary of the guidelines of testing normality according to Brase and Brase

Assessment	Histogram relatively bell-shape	Not more than one outlier	Pearson's index -1 and 1	Normal quantile plot relatively straight	Gaussian or not
G2 LFIN	Yes	No outlier	0.52623481	Yes	Gaussian
G1 ANAdG2	Yes	No outlier	0.47644586	Yes	Gaussian
G6 ESTA	Yes	No outlier	0.483917127	Yes	Gaussian
G6 CA	Yes	No outlier	0.007867	Yes	Gaussian

According to Brase and Brase's (2011) guidelines for normality, the G1 ANAdG2, G2 LFIN, G6 CA, and the G6 ESTA are relatively normal and fit for bivariate analysis.

6.2 The predictive power of the early assessments

The predictive power of the early assessments is the focus of the research. The question to be answered is:

"In what ways, if any, does structural thinking in the early years of primary school predict performance in the later years of primary?"

The sub-questions arising from this include:

1) How important is structural thinking in the early years of school?
2) Is there evidence for a strong correlation between an assessment that measures structural thinking in the early years of school and mathematical performance in the later years of primary school?

I used Pearson's product-moment correlation coefficient (PPMCC) to analyse the predictive power of the early assessments. This analysis was conducted following the earlier theory-based analysis (chapter 4) of the assessment marking criteria, indicating that both the G2 LFIN and the G6 ESTA assessments score for strategic efficiency are based on mathematical structure. Both national and provincial assessments (the G1 ANAdG2 and the G6 CA) undervalued attention to structure by grading the assessment based on the correct answers given regardless of the extent to which the use of structure was foregrounded in the approach taken by the participants.

Furthermore, as the assessment instruments included different facets/aspects, I illustrate what I mean by the different facets or aspects of all the assessment instruments in Table 6.6.

Table 6.6 Different facets/aspects of the assessment instruments

Assessments	Different aspects in the assessment instruments
G1 ANAdG2	Number and not-only-number-related questions. (Since only two questions are not-only-number-related, I will focus on the number-related questions only)
G2 LFIN	SEAL, FNWS, BNWS, NI, and base-ten (known as aspects of the G2 LFIN)
G6 CA	Number-related questions and not-only-number-related questions
G6 ESTA	Additive and multiplicative reasoning

I analysed the following correlations:

- Grade 2 assessments against Grade 6 assessments - as intact assessments with the township and suburban schools combined/separately.
- The Grade 2 LFIN aspects (SEAL, FNWS, BNWS, NI, and base-ten) against the Grade 6 assessments with the township and suburban schools combined and separately.
- The Grade 2 assessments against the multiplicative and additive reasoning components of the Grade 6 ESTA assessment and number- and not-only-number-related parts of the G6 CA - with the township and suburban schools combined and separately.

I present the Pearson correlations in tables with P-values[57].

6.2.1 Grade 2 versus Grade 6 assessments as intact[58] tests for all schools

The Pearson product-moment correlation coefficient (PPMCC) was considered to understand the relationships between these assessments. I used SPSS to obtain the PPMCC. The PPMCCs between the various assessments are summarised in Table 6.7.

[57] The P-value is the probability that the Pearson-product-correlation-coefficient happened by chance. If the P-value is lower than the conventional 5% ($P<0.05$), the correlation coefficient is called statistically significant. Thus, a P-value of 0,05 means a 5% chance that the mentioned correlations happened by chance, and a P-value of 0.01 indicates a 1% chance that the mentioned correlations happened by chance.

[58] Whole tests.

Table 6.7 Correlations between the data of the Grade 2 and Grade 6 assessments as intact tests for all schools

		G2 LFIN	G1 ANAdG2	G6 CA	G6 ESTA
G2 LFIN	PPMCC	1			
G2 ANA	PPMCC	.493**	1		
G6 CA	PPMCC	.738**	.534**	1	
G6 ESTA	PPMCC	.750**	.446**	.724**	1

**. Correlation is significant at the 0.01 level (2-tailed).

The results of the correlations in Table 6.7 indicate that the G2 LFIN shows a much better correlation with the two G6 assessments (G6 CA r = 0.738, n = 64, p = 0.01; G6 ESTA r = 0.750, n = 64, p = 0.01) than does the G1 ANAdG2 (G6 CA r = 0.534, n = 64, p = 0.01; G6 ESTA r = 0.446, n = 64, p = 0.01). These correlations implicate that the LFIN was a better predictor of the Grade 6 assessments than the G1 ANAdG2. It is important to mention that the G2 LFIN measured structural thinking. At the same time, the G1 ANAdG2 scoring was only concerned about the right answers and no attention was given to the sophistication of the strategy used.

The stronger correlation between the G2 LFIN and the two Grade 6 assessments is of interest because it backs up the literature and theory-driven hypothesis that it is the kind of structural thinking that underlies descriptions of strong number sense that is particularly important for later mathematical performance, rather than an ability to simply produce correct answers using any approach.

It is also of interest that, while the correlation between the G1 ANAdG2 and G2 LFIN (the two Grade 2 assessments) was relatively low (r = 0. 493, p = 0.01, n = 64), the correlation between the G6 CA and G6 ESTA (the two Grade 6 assessments) was much stronger (r = 0.724, p = 0.01, n = 64). The latter result emphasises that the two G2 assessments measured different skills. The G1 ANAdG2 was about answer-production using any method, in contrast with the G2 LFIN, which focuses on the sophistication of strategy used. The higher correlation between the Grade 6 assessments is understandable since the G6 CA tasks included formal mathematical knowledge that could not be answered correctly by counting strategies alone. Furthermore, in the much higher number ranges that are part of the G6 CA, the stronger correlation suggests that the ability to work with number structure forms a common underpinning skill across the two assessments.

Correlations do not always imply cause and effect. A careful regression analyst always looks at the residual plots to see if no lurking variable could influence the correlation coefficients (Watkins et al., 2004). For this reason, I drew the residual plots for the four regressions G1 ANAdG2 versus G6 CA, G1 ANAdG2 versus G6 ESTA, G2 LFIN versus G6 CA, and the G2 LFIN versus G6 ESTA.

Figure 6.20 displays the scatterplot between the G1 ANAdG2 and the G6 CA and the residual plot.

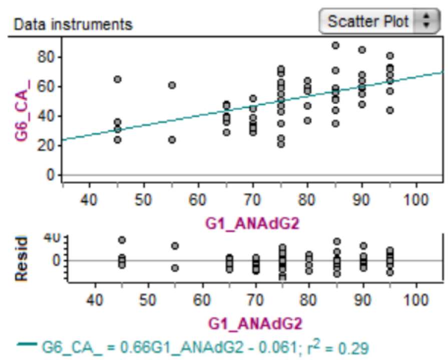

Figure 6.20 Scatter and residual plot of the relationship between the G1 ANAdG2 and the G6 CA

The residual plot shows no points separated from the rest of the points in the residual plot and no pattern can be observed, which indicates that there is no lurking variable that could course the correlations. Thus, we can assume that correlations between the data sets are of value.

Figures 6.21 to 6.23 show the residual plots for the rest of the regressions. No pattern or clustered-together points can be seen in the residual plots. The plots suggest that there are no lurking variables that influence the correlations.

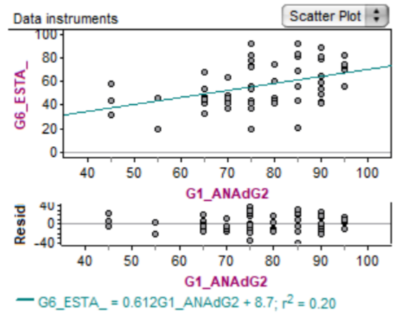

Figure 6.21 Scatter and residual plot of the relationship between the G1 ANAdG2 and the G6 ESTA

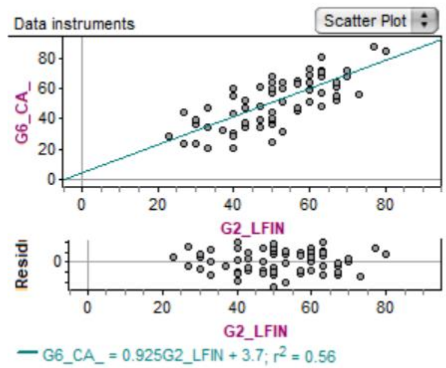

Figure 6.22 Scatter and residual plot of the relationship between the G2 LFIN and the G6 CA

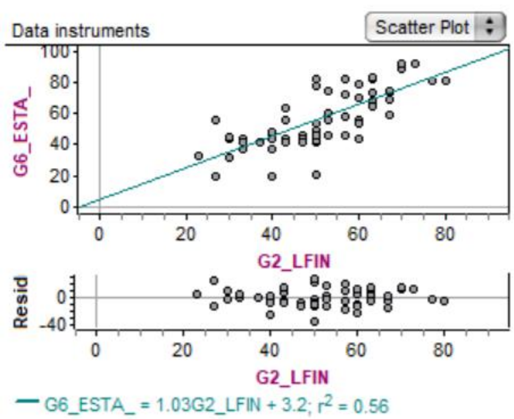

Figure 6.23 Scatter and residual plot of the relationship between the G2 LFIN and the G6 ESTA

6.2.2 Grade 2 and Grade 6 assessments as related to the suburban and township schools

I was interested in how the correlations would differ for suburban and township schools separately. In the following tables, G2 SLFIN refers to Grade 2 LFIN for the suburban school, and G2 TLFIN refers to the Grade 2 LFIN for the township schools. The same notation is used to signal the other assessment instruments. The Pearson-product-moment-correlation-coefficients (PPMCC) for all assessments for suburban and township schools are summarised in Table 6.8.

Table 6.8 Pearson's correlations for all assessments instruments – Suburban and township schools separately

Suburban schools		Gr 2 SLFIN	Gr 2 SANA	Gr 6 SCA	G6 SESTA
G2 SLFIN	PPMCC	1			
G1 SANAdG2	PPMCC	.627**	1		
G6 SCA	PPMCC	.768**	.708**	1	
G6 SESTA	PPMCC	.794**	.533**	.720**	1
Township schools		**Gr 2 TLFIN**	**Gr 2 TANA**	**Gr 6 TCA**	**G6 TESTA**
G2 TLFIN	PPMCC	1			
G1 TANAdG2	PPMCC	.326*	1		
G6 TCA	PPMCC	.655**	.347*	1	
G6 TESTA	PPMCC	.647**	.280	.664**	1

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (2-tailed).

Table 6.8 indicates that the suburban schools' results signify stronger correlations between the Grade 2 LFIN and the Grade 6 assessments (G6 SCA r = 0.768, n = 26, p = 0,01; G6 SESTA r = 0.794, n = 26, p = 0,01) compared to the township schools (G6 TCA r = 0.655, n = 38, p = 0,01; G6 TESTA r = 0.647, n = 38, p = 0,01). However, the township schools' correlations between the G2 LFIN and the Grade 6 assessments were still relatively strong, with correlations stronger than 0,6.

Nevertheless, the suburban schools' correlational results between the G1 ANAdG2 and the Grade 6 assessments indicated a much stronger relationship with the Grade 6 CA than with the Grade 6 G6 ESTA (G6 SCA r = 0.708, n = 26, p = 0,01; G6 SESTA r = 0.533, n = 26, p = 0,01). This stronger relationship can be ascribed to the fact that both the Grade 2 ANA and the Grade 6 CA are national assessments. A consequence is that the suburban schools are more concerned with the results of the national assessments and therefore focus more on achieving well in the national assessments than the township schools do. The finding that the G1 ANAdG2 in the suburban schools correlates strongly with the G6 CA was a surprise because there was a very weak correlation between the G1 ANAdG2 and the G6 CA for the schools combined (Table 6.7). For the suburban schools, an assessment that evaluates structural thinking (G2 LFIN) and an assessment that does not measure structural thinking per se (G1 ANAdG2) as good predictors contradicts what the literature reveals about structural thinking being a good predictor of later performance.

The very poor correlations between the G1 ANAdG2 and the Grade 6 assessments (G6 TCA $r = 0.347$, $n = 26$, $p = 0,01$; G6 TESTA $r = 0.280$, $n = 26$, $p = 0,01$) for the township schools illustrates that the Grade ANAdG2 (which does not focus on structural thinking but instead on answers only) was not as good a predictor as the G2 LFIN for the township schools.

There is a good relationship between the two Grade 6 assessments for both suburban (G6 SCA $r = 0.720$, $n = 26$, $p = 0,01$) and township (G6 TCA $r = 0.664$, $n = 38$, $p = 0,01$) schools. However, the Grade 2 assessments correlate well with each other for the suburban schools (G6 SCA $r = 0.627$, $n = 26$, $p = 0,01$) but not for the township schools (G6 TCA $r = 0.326$, $n = 38$, $p = 0,01$).

The results of the correlations for the township schools point to weaker correlations between the Grade 2 assessments and the Grade 6 assessments than the correlations in the suburban schools. Nevertheless, the correlation coefficients were stronger between the Grade 2 LFIN and the Grade 6 assessments than the correlations between the G1 ANAdG2 and the Grade 6 assessments for both suburban and township schools. These findings indicate that structural thinking (which the G2 LFIN evaluates) in Grade 2 predicted performance in Grade 6.

6.2.3 Aspects of the Grade 2 LFIN (SEAL, FNWS, BNWS, NI and base-ten knowledge) versus the Grade 6 assessments as intact tests[59] - with suburban and township schools combined.

The SEAL, FNWS, BNWS, NI, and base-ten thinking, which are the aspects of the Grade 2 LFIN, are critical concepts in early mathematics, and I wanted to know how these aspects correlated with the two Grade 6 assessments. Table 6.9 records the correlations between the aspects of the Grade 2 LFIN (namely, the SEAL, FNWS, BNWS, NI, and base-ten thinking) and the two Grade 6 assessments, with all schools included.

[59] Whole tests.

Table 6.9 Pearson's Correlations – Aspects of G2 LFIN versus the Grade 6 assessments – All the schools combined

		G6 CA	G6 ESTA
SEAL	PPMCC	.644**	.673**
FNWS	PPMCC	.598**	.571**
BNWS	PPMCC	.523**	.553**
NI	PPMCC	.645**	.563**
Base-ten	PPMCC	.483**	.505**

**. Correlation is significant at the 0.01 level (2-tailed).

The strong correlation between the SEAL of the G2 LFIN for the Grade 6 assessments (G6 CA r = 0.644, n = 64, p = 0.01; G6 ESTA r = 0.673, n = 64, p = 0.01) tells us that the Stages of Early Arithmetical Learning predicted the Grade 6 assessments the best. The SEAL measures how sophisticated (the learner using structural thinking) the strategies are when the learner does addition and subtraction tasks. The more sophisticated the strategy is, the higher the SEAL. Thus, since the correlations between the SEAL and the Grade 6 assessments are strong, it implies that SEAL's structural thinking predicted the Grade 6 assessments the best.

The NI correlation of the G2 LFIN against the G6 CA also shows a strong correlation (G6 CA r = 0.645, n = 64). The correlation indicates that a good knowledge of the numerals predicted how well the participants would perform in the G6 CA.

Stronger correlations can be seen between FNWS and the Grade 6 assessments (G6 CA r = 0.598, n = 64, p = 0.01; G6 ESTA r = 0.571, n = 64, p = 0.01), which implies that forward-counting predicted the Grade 6 performances better than backward (BNWS) counting (G6 CA r = 0.523, n = 64, p = 0.01; G6 ESTA r = 0.553, n = 64, p = 0.01).

For the whole group, where all schools are included, base-ten understanding correlates better with the G6 ESTA than with the G6 CA (G6 ESTA r = 0,505, n = 64, p = 0,01; G6 CA r = 0.483, n = 64, p = 0,01). Both correlations are weak and indicate that base-ten knowledge was not as strong a predictor as I thought.

To sum up, SEAL was the best predictor of the G6 CA and the G6 ESTA. Furthermore, NI is a good predictor of the G6 CA, but not for the G6 ESTA.

6.2.4 Aspects of the G2 LFIN (SEAL, FNWS, BNWS, NI, and base-ten thinking) versus the Grade 6 assessments with suburban and township schools separately

I was interested in how the aspects of the G2 LFIN would predict the Grade 6 assessments in the two different school groups. Table 6.10 records the correlations.

Table 6.10 Pearson's Correlations – Aspects of the LFIN versus the Grade 6 assessments – suburban and township schools separately

Suburban schools		Gr 6 SCA	G6 SESTA
SSEAL	PPMCC	.716**	.817**
SFNWS	PPMCC	.630**	.610**
SBNWS	PPMCC	.527**	.577*
SNI	PPMCC	.596**	.583**
SBase-ten	PPMCC	.629*	.523**
Township schools		Gr 6 TCA	G6 TESTA
TSEAL	PPMCC	.551**	.532**
TFNWS	PPMCC	.555**	.527**
TBNWS	PPMCC	.422**	.455**
TNI	PPMCC	.651**	.497**
TBase-ten	PPMCC	.168	.340*

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (2-tailed).

Substantial strong correlations exist between the SEAL and the Grade 6 assessments for the suburban schools' scores (G6 SCA r = 0.716, n = 26, p = 0,01; G6 SESTA r = 0.817, n = 26, p = 0,01). However, this is not the case for the township schools' scores (G6 TCA r = 0.551, n = 38, p = 0,01; G6 TESTA r = 0.532, n = 38, p = 0,01).

There is also a strong positive correlation between the FNWS and the G6 assessments for the suburban schools' scores (G6 SCA r = 0.630, n = 26, p = 0,01; G6 SESTA r = 0.610, n = 26, p = 0,01). This is not the case for the township schools' scores (G6 TCA r = 0.555, n = 38, p = 0,01; G6 TESTA r = 0.527, n = 38, p = 0,01).

Another positive correlation worth noticing is the correlation between base-ten thinking and the G6 CA for the suburban schools' scores (G6 SCA r = 0.629, n = 26, p = 0,01), which is noticeably stronger than the correlation between base-ten thinking and the G2 ESTA (G6 SESTA r = 0.523, n = 26, p = 0,01) for the township schools' scores.

6.2.5 Grade 2 assessments versus multiplicative and additive reasoning of the G6 ESTA and number- and not-only-number-related tasks of the G6 CA with suburban and township schools combined

The Grade 6 G6 ESTA assessment was organised in two parts: multiplicative reasoning and additive reasoning, and the G6 ESTA consists of number-related tasks and not-only-number-related tasks.

Table 6.11 records the correlations between the Grade 2 assessments and the G6 ESTA assessments regarding additive and multiplicative reasoning and the Grade 6 CA regarding the number and not-only-number-related tasks.

Table 6.11 Grade 2 assessments versus multiplicative and additive reasoning of the G6 ESTA and number- and not-only-number-related tasks of the G6 CA – township and suburban schools combined

All schools		Gr 2 LFIN	G1 ANAdG2
G6 ESTA Additive reasoning (G6 ESTAadd)	PPMCC	.663**	.379**
G6 ESTA Multiplicative reasoning (G6 ESTAmul)	PPMCC	.701**	.428**
G6 CA Number-related tasks (G6 CAnumr)	PPMCC	.728**	.535**
G6 CA not-only-number-related tasks (G6 CAnotnumr)	PPMCC	.583**	.401**

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (2-tailed).

There is a stronger positive correlation between the Grade 2 LFIN and both aspects of the Grade 6 G6 ESTA (G6 ESTAadd r = 0.663, n = 64, p = 0,01; G6 ESTAmul r = 0.701, n = 64, p = 0,01) compared to the correlation between the G1 ANAdG2 and both aspects of the G6 ESTA (G6 ESTAadd r = 0.379, n = 64, p = 0,01; G6 ESTAmul r = 0.428, n = 64, p = 0,01). This suggests that the G2 LFIN predicted multiplicative and additive reasoning better than the G1 ANAdG2.

Furthermore, the G2 LFIN correlated better with multiplicative reasoning (G6 ESTAmul r = 0.701, n = 64, p = 0,01) than with additive reasoning (G6 ESTAadd r = 0.663, n = 64, p = 0,01) of the G6 ESTA. The latter suggests that structural thinking in the early years predicts multiplicative reasoning better than additive reasoning for the participants of the study.

What is of interest is that the G2 LFIN did not correlate as strongly with the G6 CA not-number-related tasks as it did with the number-related tasks (G6 CAnotnumr r = 0.583, n = 64, p = 0,01; G6 CAnumr r = 0.728, n = 64, p = 0,01).

The findings suggest that the G2 LFIN predicted the number-related tasks of the G6 CA but not the not-number-only-related tasks. Furthermore, a difference in the predictability of the G2 LFIN for the multiplicative and additive reasoning parts of the G6 ESTA can be noticed.

6.2.6 Grade 2 assessments versus multiplicative and additive reasoning of the G6 ESTA and number- and not-only-number-related tasks of the G6 CA with suburban and township schools separately

My subsequent interest was to consider the correlations revealed when I separated the township schools' scores from the suburban schools' scores.

Table 6.12 shows the variation in the correlations recorded for the suburban schools and the township schools. Correlations are moderate to strongly positive, ranging from 0.426 to 0.775 in the suburban school group, and weaker, yet positive, in the township schools' group, ranging from 0.185 to 0.645.

Table 6.12 Grade 2 assessments versus multiplicative and additive reasoning of the G6 ESTA and number- and not-only-number-related tasks of the G6 CA – township and suburban schools

Suburban schools		G2 SLFIN	G1 SANAdG2
G6 SESTA Additive reasoning (G6 SESTAadd)	PPMCC	.748**	.426*
G6 SESTA Multiplicative reasoning (G6 SESTAmul)	PPMCC	.723**	.541**
G6 SCA Number-related tasks (G6 SCAnumr)	PPMCC	.775**	.719**
G6 SCA not-only-number-related tasks (G6 SCAnotnumr)	PPMCC	.609**	.552**
Township schools		G2 TLFIN	Gr 2 TANA
G6 TESTA Additive reasoning (G6 TESTAadd)	PPMCC	.522**	.254
G6 TESTA Multiplicative reasoning (G6 TESTAmul)	PPMCC	.610**	.235.
G6 TCA Number related tasks (G6 TCAnumr)	PPMCC	.645**	.364*
G6 TCA not-only-number-related tasks (G6 TCAnotnumr)	PPMCC	.456**	.185

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (2-tailed).

For the suburban schools' data set, strong positive correlations can be seen between the G2 LFIN and both the G6 ESTA's multiplicative and additive reasoning parts (G6 SESTAadd $r = 0.748$, $n = 26$, $p = 0{,}01$; G6 SESTAmul $r = 0.723$, $n = 26$, $p = 0{,}01$). The stronger positive correlation between the G2 LFIN and the NRT part of the G6 CA (G6 SCAnumr $r = 0{,}775$, $n = 26$, $p = 0{,}01$) is an important result.

The G2 LFIN does not correlate with the not-only-number-related tasks component of the G6 CA as strongly as it correlates with the number-related tasks part of the G6 CA for the suburban schools, however, it is still strong (G6 SESTAnumr $r = 0.775$, $n = 26$, $p = 0{,}01$; G6 SESTAnotnumr $r = 0.609$, $n = 26$, $p = 0{,}01$).

Surprisingly, there is a strong positive correlation between the G1 ANAdG2 and the number-related tasks of the G6 CA for the suburban schools' data set (G6 SCAnumr $r = 0{,}719$, $n = 26$, $p = 0{,}01$). The latter suggests that for the suburban school's data set, the G1 ANAdG2 has strong predictive power for the number-related tasks of the G6 CA, but not for the not-only-number-related part of the G6 CA. NONRT requires formal mathematical knowledge and the knowledge of the relationship between numbers and cannot be answered by counting strategies. Therefore, the strong correlation between the scores of the G1 ANAdG2 (which was primarily number-related and could be done without the knowledge of number relationships) and the NRT of the G6 CA was understandable.

The township schools' data set shows lower correlations between the G2 LFIN and the different parts/aspects of the G6 assessments than the suburban schools' data sets. This finding suggests that neither the G1 ANAdG2 nor the G2 LFIN predicted the Grade 6 assessments for the township schools.

6.3 In summary

At the beginning of the chapter, I stated that I performed univariate and bivariate analysis to investigate the predictive power of the assessment instruments. The univariate analysis of this chapter shows that all the assessment instruments are relatively normal. The G2 LFIN, G6 CA and G6 ESTA's means, and medians were between 50% and 56%, where the mean and median of the G1 ANAdG2 were 75% and 77%, respectively. The findings suggest that the scores of the G1 ANAdG2 were very high.

I followed the normality guidelines to determine the normality of Brase and Brase (2011). I found that all assessment tools' data sets were relatively normal, with standard deviations between 12,893 and 18,035, illustrating that all the distributions are dispersed more or less the same.

This study is about the predictive power of the extent of structural thinking in the early years of schooling. Thus, a bivariate correlational analysis is at the heart of it. Since the G2 LFIN evaluates the degree of structural thinking and the G1 ANAdG2 focused on answering correctly without marks allocated for sophisticated strategies, stronger correlations between the G2 LFIN and the G6 assessments than correlations between the G1 ANAdG2 and the Grade 6 assessments will indicate that the extent of structural thinking predicts later performance.

The bivariate analysis shows that the Grade 2 LFIN predicted the Grade 6 assessments better than the G1 ANAdG2 for all schools combined; however, when I separated the township and suburban schools' data sets, I found that the G1 ANAdG2 was also a good predictor of the G6 CA, but not for the G6 ESTA. The only reason I can think of is that the suburban schools are more concerned about the national assessments than township schools and thus the strong correlation between the G1 ANAdG2 and the G6 CA.

The SEAL of the LFIN had the strongest predictive power.

The analysis above indicates that the Grade 2 LFIN was a better predictor of the Grade 6 assessments. Since the G2 LFIN measures structural thinking, I conclude that structural thinking predicts later performance.

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CHAPTER 7 DISCUSSION AND CONCLUSION – IMPORTANCE OF THE EXTENT OF STRUCTURAL THINKING IN THE EARLY YEARS OF SCHOOLING

7.1 Introduction

This study's centre of attention is how important the extent of structural thinking is in the early years of schooling. To examine the importance of the extent of structural thinking, I investigated the predictive power of the degree of structural thinking in the early years of education for later mathematical performance by looking at the strength of correlations between the Grade 2 and the Grade 6 assessments.

Earlier in the thesis I discussed international studies that highlight the predictive power and the value of structural thinking in the early elementary years for later performance in mathematics (Claessens& Engel, 2013; Papic, Mulligan, & Mitchelmore, 2011).

However, such a study has not been done in South Africa at this scale yet. My objective was to determine the degree of the predictive power of structural thinking for later performance in mathematics in South African primary schools and to see if my findings are in line with the international findings.

This study includes two types of questions: research questions and researcher's questions. My research questions are more general questions where I do not highlight the background in which the study was done. Nevertheless, the researcher's question focuses on the South African context of the study.

Thus, my research questions are broader and have wider applicability outside the South African context. As a reminder, the questions were:

"In what ways, if any, does structural thinking in the early years of primary school predict performance in the later years of primary school?"

Two sub-research questions underpinned this question:

1. Is there evidence for a strong correlation between an assessment that measures structural thinking in the early years of school and mathematical performance in the later years of primary school?

2. How important is structural thinking in the early years of school?

These research questions were examined and made operational by the researcher's questions, including the study's setting, which were:

"In what ways do Grade 2 learners' performance in two tests – the G1 ANAdG2 and Wright et al.'s (2006) G2 LFIN assessments, pertain to their subsequent Grade 6 performance in two mathematics tests, the Grade 6 CA and the Grade 6 ESTA?"

To answer the main research question, I answered the sub-questions in Chapters 4, 5, and 6:

1. In qualitative terms, in what ways, and to what extent do G1 ANAdG2, the G2 LFIN, the Grade 6 CA, and the G6 ESTA assessments measure structural thinking with early number?

2. In quantitative and qualitative terms, what do the empirical findings regarding the extent of structural thinking tell us?

3. In quantitative terms, what is the correlation between learner performance in Grade 2 on the Grade 1 ANAdG2 and LFIN tests in 2011 and performance in Grade 6 on the Common Assessment and G6 ESTA tests in 2015?

This chapter summarises the study's findings and discusses its limitations and implications, formulating recommendations for teaching and learning and providing suggestions for future research. It includes reflections on my research journey, and I conclude by addressing the question: *How important is structural thinking in the early years*?

7.2 Discussion of the findings and results

I commence by discussing the findings where I address the ability of the assessment instruments to measure structural thinking. Then I will focus on the empirical findings where I discuss the strategies the participants used to answer the questions on the assessment instruments. Lastly, I discourse on the correlations between the early assessments and the G2 LFIN's aspects and the Grade 6 assessments.

Before I discuss the findings, I want to say more about the research paradigm of the study. The study used a Corroborative Mixed Method Research Design (CMMRD) to answer the research questions.

Johnson et al. (2007, p. 123) describe this as follows:

> "Mixed methods research is the type of research in which a researcher combines elements of qualitative and quantitative research approaches (e. g., use of qualitative and quantitative viewpoints, data collection, analysis, inference techniques) for the broad purposes of breadth and depth of understanding and corroboration."

In CMMRD, quantitative and qualitative research methods serve each other for a broader purpose. CMMRD is not only the putting together of two research methods (Johnson & Onwuegbuzie, 2004).

In this study, I first collected the G1 ANAdG2 and the videos data of the G2 LFIN in 2011. Then I qualitatively analysed the learners' responses by looking at the sophistication of the strategy used in the videos and recorded the different stages and levels for each participant. The analysis of the videos was qualitative.

In 2015 I collected the data of the G6 CA, conducted the video-recorded interviews of the G6 ESTA, and analysed the videos qualitatively using the frameworks I developed for the G6 ESTA. In addition, in chapter 4, I analysed the assessments qualitatively to see how well/not well the assessments evaluated structural thinking.

In chapter 5 I presented the participants' empirical results but before I could analyse the responses, I quantified the qualitative results of the video-recorded assessments. Finally, in chapter 6, I quantitatively analysed the predictive power of the assessments. Thus, this study is a collaborative mixed-method research design because the quantitative and qualitative analysis serves each other, and the quantitative analysis could not be done without the quantitative analysis, and vice versa.

7.2.1 The assessment instruments' ability to measure mathematical structure

The ability of the assessment instruments to measure mathematical structure reveals exciting findings. From the theoretical analysis of the assessments' ability to measure structural thinking, I concluded that the G1 ANAdG2 did not to any great extent measure structural thinking because only 6% of the tasks required learners to engage in structural thinking in responding to the assessment items. In fact, 83% of the items on the G1 ANAdG2 assessment could be completed successfully using unit-counting strategies.

On the G6 CA assessment tool, 24% of the marks required learners to engage in mathematical structural thinking, with only 4% of the marks of the assessment items relying on unit-counting strategies. Since 24% of the marks in the G6 CA required structural thinking and the G1 ANAdG2 only required 6%, the G6 CA assessment tool measured structural thinking to a greater extent than the G1 ANAdG2. Interestingly, 72% of the marks in the G6 CA task's required algorithms and formal mathematical knowledge as minimal knowledge to answer successfully.

Therefore, a high score in the G1 ANAdG2 assessment did not necessarily transfer to an advanced level of structural thinking, but a high score in the G6 CA assessment did, to some extent.

The two assessments that were dependent on the sophistication of strategy used, the G2 LFIN and the G6 ESTA, were developed to measure structural thinking. The strategy that the participants exhibited was coded into sequential levels, where the highest level was achieved by the participants showing the highest level of structural thinking. Thus, a high mark in both the G2 LFIN and G6 CA suggested a high level of engagement with structural thinking.

In summary, the G2 LFIN, the G6 ESTA and the G6 CA showed an ability to measure structural thinking, whereas the G1 ANAdG2 did not.

7.2.2 Empirical findings and strategies used

To detail the empirical findings regarding the extent of structural thinking, I discuss the empirical results of the two government written assessments first (G1 ANAdG2 and the G6 CA), followed by a discussion of the empirical results of the G2 LFIN and G6 ESTA.

Even though the study's context included two different locations of participants, namely those who attend schools in the townships and those attending schools in the suburbs, I decided to aggregate the empirical results of the suburban schools and township schools for the G6 CA, the G1 ANAdG2 and the G6 ESTA and the G1 ANAdG2. I keep the participant data combined since my study is not chiefly about the empirical results for the participants but rather about the predictive power of the early assessments. However, I disaggregated the empirical results of the G2 LFIN because my thesis explores the predictive power of early assessments in different ways. Since the G2 LFIN consists of various aspects like SEAL, FNWS, BNWS, NI, and base-ten thinking, I want to study how these aspects of the LFIN predict later performance in both school groups. Furthermore, I also wanted to examine how the two groups performed in the G2 LFIN separately on SEAL, FNWS, BNWS, NI and base-ten thinking. Thus, when I discussed the predictive power of the early assessment, I separated the two groups of participants.

The empirical findings of the two government assessments show different results. The participants scored very high marks on the G1 ANAdG2, where most of the questions had 73% or more correct responses. Empirical findings of the G6 CA show that the participants did not score as high as they did in the G1 ANAdG2, which suggests that the G6 CA was much more challenging than the G1 ANAdG2.

The findings of how well each question was answered in the G6 CA showed that the participants were challenged by questions where the skill of structural thinking was required. The participants scored very low on the tasks where they had to work structurally with numbers. For example, tasks that include questions like: which factors of 27 are also prime numbers, give the number of factors for 30, give a rule to a pattern, answer questions on probability, answer questions on percentage, do word problems, do tasks about fractions, and do long division.

The G2 LFIN's empirical findings of the strategy used showed that only 36% of the participants could count-on or count-down to answer addition and subtraction tasks (Stage 3), which tells us that 64% of the participants were still reliant on count-all strategies. It is essential to mention that none of the participants could count-down to answer subtraction tasks like $17 - 14$ (16, 15, 14 answer is 3 – Stage 4) or could respond with a non-count-by-one[60] strategy (Stage 5). Stages 4 and 5 required a learner to be more aware of number relationships than stages 1 to 3.

[60] These strategies involve compensation, using a known result, commutativity, inverses, and base-ten thinking (Wright, Martland, & Stafford, 2006).

The findings of the G2 LFIN revealed that 53% of the participants could count forward up to 100 and give the NWA without dropping back to one. However, only 14% could count backward in the number range 1 to 100 and give the NWB without dropping back to one. These findings signal that the participants found forward-counting easier than backward-counting and are in sync with other research findings (Wright et al., 2012).

The G2 LFIN's findings were that only 19% of the participants could identify 3-digit numbers, and further to this, 69% could successfully identify two-digit numbers.

The vast majority of participants (88%) did not show understanding of base-ten thinking and did not know how to work with ten as 'one unit' and 10 as ten separate units simultaneously. The 88% of participants not showing an understanding of base-ten thinking and not knowing number bonds of ten is a serious finding and needs to be addressed. It also points to the participants' dependency on unit-counting.

Another finding was, when I disaggregated the empirical responses of the G2 LFIN, I found that the suburban schools group performed better than the township schools' group. The implications of the findings are that more attention should be given to teaching in township schools.

The findings of the G6 ESTA indicate clearly that most of the participants did not make use of mental calculations. They mainly used written-column algorithms for questions that were amenable to mental strategies. The latter shows a reliance on procedural algorithms to complete calculations that could be ideally achieved by engaging in cognitive processes. Knowing a procedure and applying it is evidence of knowledge of number relations. The persistent dependence on written algorithms for tasks that could be done quickly through mental calculations shows that the participants are not yet reason qualitatively.

Further highlighting the inability to do calculations mentally in G6 ESTA is that the participants were still very reliant on unit-counting strategies. This was evident in the substantial portion of participants who used their fingers to determine the answers for the partial sums and differences when engaging the vertical column method for addition, subtraction, and multiplication in the G6 ESTA. This suggests there is a lack of structural thinking among the participants.

Furthermore, if the students had not been videotaped when doing the G6 ESTA, the use of the written-column algorithm would have hidden the low stage of structural knowledge evident from the accompanying unit-counting. Implications arising from the findings is that attention should be given to learners' inability to use mental calculations and to the low-level number knowledge that is obscured in written-column algorithms.

Another important empirical finding of the G6 ESTA was the results of tasks 1 and 2. Task 1 was "What is $12 + 85$?" and task 2 was "What is $17 + 56$?". In task 1 no bridging through 10 was necessary. The learner could just add the 1 and the 8 following with the additions of 2 and 5 resulting in an answer of 97. In task 2 the child had to consider bridging through 10. Just less than a half of the participants answered task 1 with an immediate mental answer. However, only one fifth of the participants who could answer task 1 with an immediate mental answer could answer task 2 immediately. This finding suggests that tasks that required knowledge of structural thinking, like bridging through 10, were challenging for the participants.

7.2.3 The predictive power of early assessments in quantitative terms

The predictive power of the early assessments is at the heart of my study. The context where I collected my data is eight schools in Gauteng, South Africa, of which four schools were from township schools and four schools from suburban areas. Because of the importance of the predictive power of the early assessments, I found it necessary to disaggregate the data in all schools, township schools and suburban schools, to elaborate on the difference between the correlations regarding all schools, township schools and suburban schools.

All the schools that participated in my study serve learners from previously disadvantaged families. Some parents sent their children from the townships to suburban schools using taxis or other transport.

Findings from the correlations of all schools combined showed that the G2 LFIN's predictive power was much more robust than the predictive power of the G2 ANA. The implications of the latter finding suggest that an assessment that evaluates structural thinking in Grade 2 does predict later performance in Grade 6.

However, when I separate the township schools' data from the suburban schools' data, it reveals higher correlations between the G2 LFIN and the Grade 6 assessments for the suburban schools but not as high for the township schools.

A strong correlation between the G1 ANAdG2 and the G6 CA for the suburban schools but not township schools or all schools surprised me. This finding suggests that both the G2 LFIN and the G1 ANAdG2 predict later performance for the suburban schools but not for the township schools. A reason for the high correlation between the G1 ANAdG2 and the G6 CA can be that both these assessments are national tests and that the suburban schools are more concerned about the performance on the national assessments than the township schools. The impact of this finding is that there can be a possibility that the suburban schools' teaching focuses on how to answer the national mathematics tests. The teachers might teach a learned procedure without finding whether the learners understand the procedure. It can be that the 'teaching to the test' hinders teachers from teaching structurally, where they make learners aware of the relationships between numbers.

7.2.5 Bringing together the quantitative/qualitative results

The qualitative analysis of the extent each assessment measure in Chapter 5, shows that the G2 LFIN evaluates structural thinking the best. The G2 LFIN measures the sophistication of the strategy used by the learner. The sophistication of a strategy provides evidence of structural thinking (Wright et al., 2006). The G1 ANAdG2 focuses only on obtaining the correct answer, with no focus on the strategy that is used to obtain the answer. Furthermore, the correlations of the quantitative analysis show that the G2 LFIN is the best predictor of the Grade 6 assessments. Thus, the G2 LFIN was more successful in predicting later performance than the G1 ANAdG2.

Both the quantitative together with the qualitative results of this study, suggest that structural thinking in the early years is essential, which is in line with the findings of other longitudinal studies (Aubrey, Dahl, & Dogfrey, 2006; Aunola, Leskinin, Lerkkanen, & Nurmi, 2004; Claessens & Engel, 2013; Manfra, Dinehart, & Senbiante, 2012; Weitz & Venkat, 2017).

In the early years of primary school, mathematical structural thinking can be observed when learners demonstrate an awareness of the relationships between mathematical elements (Mason, Stephens, & Watson, 2009; Venkat, Askew, Watson, & Mason, 2019). Since structural thinking is challenging to define in one single definition, I presented the applications of structural thinking in chapter 2. This discussion shows how extensive the range of concepts the child should develop in Primary school is. Structural thinking includes knowledge of relationships between unenumerated quantities, relationships between numbers, and relationships between quantities (Nunes et al., 2012) and is starting to develop soon after birth. Since this development process starts so early in the child's life, teachers, lecturers, academics, and policymakers should bear in mind that a learner comes to school with an existing structure of number, either unenumerated quantities, numbers, or quantities. Thus, teachers should ideally be open to the child's existing structural thinking and build on that.

7.2.4 Limitations of the study

A limitation of the study could be due to the data collection process of the video-recording of the G2 LFIN assessment. Conducting the G2 LFIN included three adults, namely an interviewer, a camera operator, and a translator (in cases where the language of learning and teaching (LOLt) was not English). Three adults constantly viewing and managing the data collection process could occasion a fair amount of discomfort or anxiety. Thus, some of the data of the G2 LFIN may not accurately reflect the actual ability of the participants.

Another limitation was that the result for the G2 LFIN assessment could be that the translator conveys the interviewers' questions and the participants' responses at times, not accurately presenting the meaning to both participant and interviewer. Thus, the element of a translator included in the interviews could result in certain information not always coming across precisely, with aspects of meaning getting 'lost in translation', so to speak.

7.4 Suggestions for future research

The data that I collected lends itself to further investigations. For example, I can investigate the empirical findings regarding the differences and similarities between the township and suburban schools' responses and find a rationale for the differences and similarities.

Another study I can conduct with the data I already have is to investigate structural thinking's development from Grade 2 to Grade 6 using the G2 LFIN and the G6 ESTA videos. Such a study can be done as case studies of the participants of one school.

Since this study only includes learners from previously disadvantaged families in Gauteng, it would be interesting to find out the predictive power of structural thinking in other provinces and other South African schools, including schools that do not fall into the government school classification.

7.5 Implications and contribution to knowledge

This study aimed to highlight the importance of structural thinking in the early years. The strong correlation between an assessment that was specifically developed to measure structural thinking through sophisticated strategies and the performance in the Grade 6 assessments, suggests the prominence of structural thinking and its development at an early age for future success in mathematics.

Educators should ideally be trained to create opportunities where pedagogy enhances structural thinking and assesses learners in the early years as far as structural thinking is concerned. This study has shown that a focus on structural thinking in assessments in the early years strengthens the later learning of more sophisticated mathematics. Therefore, government-controlled assessments should include opportunities to assess learners' structural thinking at an early age (M. Weitz, 2014).

The curriculum for early-age learners should be redesigned so that they not only count but also are confronted with number relationships so that structural thinking can be enhanced and ultimately mastered as a skill. Teachers must be sensitised to recognise opportunities that they can use to teach the understanding of number relationships and thus improve structural thinking.

Freudenthal (1991) spoke of 'guided reinvention', with an emphasis on the character of the learning process. The idea of 'guided reinvention' is to permit learners to come to notice the knowledge they already have as their own, which they are responsible for by themselves, and then give them the opportunity to build their own mathematical structural knowledge from there. Small children already possess relational knowledge of unenumerated quantities (more/less/equal) from a very early age (Sophian & Adams, 1987). Learners come to school with preliminary mathematical knowledge and teachers should be aware of the knowledge learners already have so that they can lead them in building their own knowledge networks as well as knowledge of number relationships.

In other words, each learner has a repertoire of conceptions and knowledge of number relationships, numerated and/or unenumerated, with which to construct new knowledge. The role of the teacher is to offer the environment for learning to take place (Chaille, 2008).

Not only does this study contribute to early mathematical education regarding structural thinking, but it also contributes to assessing structural thinking. The G6 ESTA and the three frameworks that accompany it were specifically designed to measure structural thinking in Grade 6. The G6 ESTA can be used to measure structural thinking of intermediate learners in future.

Furthermore, as my findings suggest, structural thinking in the early years is important for future performance in mathematics. Thus all stakeholders should be able to measure structural thinking in the early years of a child's schooling. For this purpose, early mathematics assessments need to be changed to measure structural thinking. As the G2 LFIN was a one-on-one video-recorded assessment, it is impractical to use the G2 LFIN to measure structural thinking in the early years. To measure learners' structural thinking, I suggest strongly that written examinations be developed with the explicit goal to measure learners' structural thinking. However, creating such assessments is both time consuming and labour intensive. Tasks in an exam like this would include questions where the solution is and not only focused on the outcome of an operation but rather the strategy the learner would follow to obtain the answer. An example of such a question is:

"*Which two numbers would you add first to make it easier to find the sum of the following three numbers: 3, 15, 17. Give a reason for your answer*".

A learner with sound structural thinking would add the 3 and 17, giving 20, and then add the 15 to the 20 to obtain the answer 35. Thus the possible answer to the question above is not 35 but rather:

"*I will add 3 and 17 because it is 20, and it is easier to add 15 to a multiple of 10*".

Developing assessments in the early years that includes tasks like the above, will be my next focus beyond this study.

7.6 My journey doing the PhD

Even though my PhD journey was challenging, I would not trade it in for anything. I'm saying this because I learned so much about learners' understanding of mathematics and structural thinking during the last few years, and the research took my mind to places I would otherwise never have gone. I have learned the importance of structural thinking in the early years. As a high school teacher, I always knew that children with structural thinking perform better in mathematics than others. Before starting my thesis, I had come to the conclusion that perhaps structural thinking would be a good predictor of later performance. However, I didn't know that the correlation between structural thinking and later performance would be as strong as the findings have shown.

Furthermore, the high correlation between structural thinking in the early years and later performance indicates that structural thinking is critical when learners are in Grades 1, 2, and 3. I want to raise the importance of structural thinking to teachers, policymakers and even parents. I would also like to supervise other postgraduate students in assessing and investigating learners' structural thinking.

Since my PhD uses four different assessments as data instruments with different purposes, I learned a great deal about assessments. Through analysing them I realised that most of the assessments that we use to assess learners in schools are about answering a question correctly and testing if a taught procedure can be done correctly. Most written assessments do not test the learners' conceptual understanding development. I also realise that structural thinking is not easy to assess, but it is possible. It is easier to evaluate structural thinking in video-recorded interviews, but it isn't easy to assess it in written tests. I think it is possible to have questions in written tests where we do not focus on the answer but instead on the reasoning of how to get to the solution. Moreover, I want to see whether I can develop written assessments that measure structural thinking in the early years of schooling.

If somebody asks me which researchers you want to discuss your study with, I have a few to consider. Firstly, I would like to talk to Robert Wright and colleagues (2006) to show them how I used their framework to measure structural thinking. I used their framework and linked each concurrent level or stage to a higher level of structural thinking than the previous stage or level. I would also want to ask them why there is such I big gap between stage 4 and stage 5 in the SEAL of the LFIN. A level 4 learner can realise the 'small difference' between 17 and 14 and count on from 14 to 17 when confronted by "What is $17 - 14$?". Stage 5 shows knowledge of compensation bridging through 10 known facts and the stage 3 learner would have taken 14 away from 17 by counting backward from 17 to 3. This is compared to the stage 5 learner who can do compensation, bridging through ten, and knows inverses of all the operations. In my opinion, this is a significant jump from stage 4 to stage 5.

Secondly, I would like to discuss my findings with Mulligan et al. (2005) to discuss their two-year-long longitudinal study of how structural thinking develops.

Thirdly, I would like to talk to Mason et al. (2009) about the Appreciation of Mathematical Structure in their paper by the same name. For them, using structural thinking when doing mathematics is essential. In other words, structural thinking can be assessed when learners demonstrate their structural thinking when calculating (Mason et al., 2009), and I agree strongly with them. Lastly, I would like to talk to Nunes et al. (2012) to discuss the difference between structural thinking and what they call Quantitative Reasoning.

When I look back on my research journey, I thought about how I would change the research if I had a chance to start all over again. There are many things that I would do differently, but there's only one thing that I will mention. I would change the G6 ESTA in the following way: I would insert more questions that do not require an answer or could be done by a taught procedure but involve discussing the strategy the learner would use to answer correctly. An example of such a question was asked in the G6 ESTA. It was: "Which two numbers will you add first if you have to add 179, 21, and 32 to make it easier to answer?" In this question, it's not about the answer of the sum of the numbers, but the reasoning of which two numbers you would add first to make it easier for yourself. The learner with knowledge of structural thinking will say, I will add 179 + 21 first because it gives 200, which is a multiple of 10. I would also change the framework for these questions because the framework that I developed for all the questions did not fit well with this question.

7.7 In conclusion

This study investigated whether structural thinking in the early years of primary school predicts mathematical performance in the later years of primary school and found that structural thinking does predict later mathematical performance.

The predictive power of structural thinking was shown by the evidence of the strong correlations between the G2 LFIN, which measures structural thinking, and the performance of the Grade 6 assessments, namely the G6 CA and G6 ESTA.

In the end, the question is, “How important is structural thinking in the early years of schooling?” According to my study, I can answer that structural thinking was important in the early years of the participants of this study because the better they scored in the G2 LFIN, the better they knew the structure of our number system and had an awareness of numbers in relationship with each other, and the better they performed in the later years. Thus, the evidence provided in this study supports the international confirmation of the importance of structural thinking in the early years of schooling.

Given that learners can appreciate number relationships and mathematical structure in the early years, teachers and policymakers should ideally emphasise teaching numbers structurally in the beginning of primary school.

APPENDIX A – THE GRADE 1 ANNUAL NATIONAL ASSESSMENT

<u>Grade 1 ANA 2011</u>

Note to the teacher: Read each question to the learners reasonably slowly and audibly, at least twice while the learners follow in their booklets, and then give them time to write answers independently in the spaces provided. Once they have finished answering a question, please continue to read the following question in the same way and give them time to write answers. Follow the same procedure as the last question. The duration of the test is 60 minutes.

1. Fill in the missing numbers:

1.1 3; 4; __;6; 7; __;9; __;11

1.2 20; __;40; 50; 60; __; __; 90; __

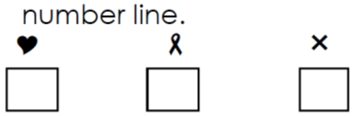
2. Complete this table by filling in the blank spaces:

Shapes	Total number	Number in words
∆ ∆ ∆ ∆ ∆ ∆ ∆ ∆ ∆	9	
	7	seven

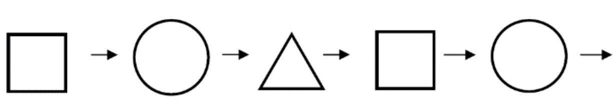
3. What is the position of these pictures on the number line? The position of the first picture is given.

3.1 The ✄ is in position ___________________.

3.2 Tick ☑ the box below the picture that is in position **6** on the number line.



4. Extend this pattern by drawing the shape that comes next:



5. Write the answer inside the box.

5.1 20 + 3 =

5.2 18 – 4 =

6. Fill in missing numbers. An example has been done for you.

6.1 Double the given number:

Number	Double
5	10
7	

6.2 Halve the given number:

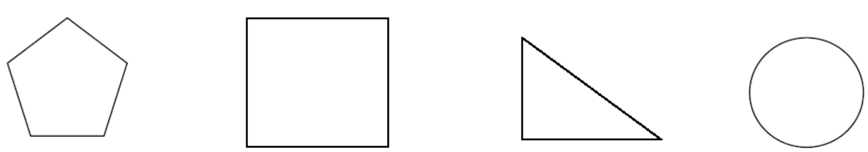
Number	Half
16	8
20	

7. Write the correct answer in the box:

7.1 10 + 10 + 10 =

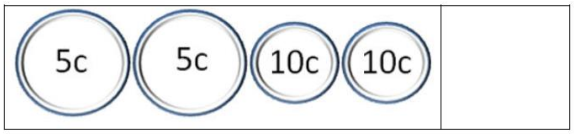
7.2 10 – 2 – 2 =

8. Tick (√) the shape that is a square:

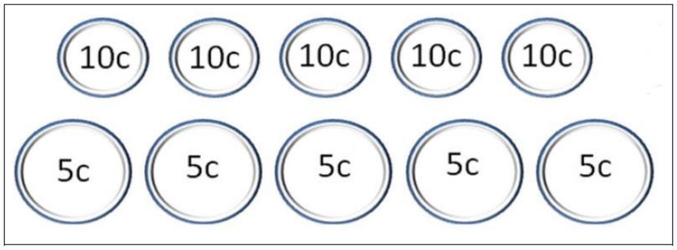


9. Look at the 10c and 5c coins in each question.

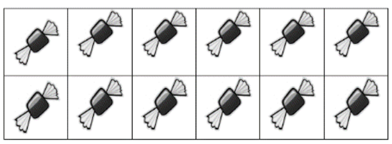
9.1 The total amount of money is ... (Write the answer in the box.)



9.2 I have 4 coins in my pocket and they add up to 25c.
Tick (✓) the 4 coins that I have in my pocket.



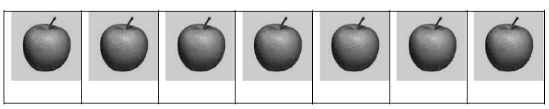
10. Busi and her 2 friends ate 12 sweets. They each ate an equal number of sweets. How many sweets did each one eat?



Each one ate ____________________ sweets.

11. Thabo bought apples and bananas at the shop.

apples



bananas

Write the correct number of each kind of fruit Thabo bought.

He bought:

11.1 ____________ apples.

11.2 ____________ bananas.

End of test!

Thank you

Appendix B – Grade 6 Common assessment

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GAUTENG DEPARTMENT OF EDUCATION

PROVINCIAL EXAMINATION

2015

GRADE 6

MATHEMATICS

DISTRICT____

SCHOOL NAME ____

EMIS NUMBER (9 digits)

CLASS (e.g. 6A) ____

SURNAME ____

NAME____

GENDER (✓)

BOY		GIRL	

16 pages

P.T.O.

GAUTENG DEPARTMENT OF EDUCATION
PROVINCIAL EXAMINATION

MATHEMATICS

MARKS: 75

TIME: 1½ hours

INSTRUCTIONS

1. Read all the instructions carefully.
2. Question 1 consists of 10 multiple-choice questions. You must circle the letter of the correct answer.
3. Answer questions 2 to 23 in the spaces or frames provided.
4. All calculations must be shown on the question paper and must not be done on rough paper.
5. The test duration is 90 minutes.
6. The test is out of 75 marks.
7. The use of a calculator is not allowed.

SECTION A

MULTIPLE-CHOICE QUESTIONS

Circle the letter of the correct answer.

1 1.1 What is the value of the underlined digit in 6<u>4</u> 379 568?

A 4 x 10 000 000
B 4 x 100 000
C 4 x 1 000 000
D 40 000 000 (1)

1.2 Which is the largest number?

A 231 323 689
B 231 230 689
C 231 302 689
D 231 320 689 (1)

1.3 Which number is missing in this number pattern?

9,27 ; 9,25 ; 9,23 ; _____ ; 9,19

A 0,21
B 9,20
C 9,12
D 9,21 (1)

1.4 Round off 5 697 to the nearest 10.

A 5 700
B 5 695
C 5 690
D 5 600 (1)

1.5 Which improper fraction is equal to $2\frac{1}{2}$?

A $\frac{5}{4}$

B $\frac{5}{2}$

C $\frac{5}{1}$

D $\frac{7}{2}$ (1)

1.6 Which number is NOT a multiple of 12?

A 12

B 36

C 276

D 112 (1)

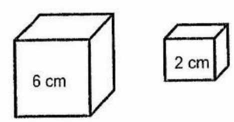
1.7 How many small cubes fit exactly into the big cube?

A 9

B 12

C 20

D 27



(1)

1.8 Write as a whole number:

$(6 \times 1\,000\,000) + (5 \times 100\,000) + (1 \times 1\,000) + (8 \times 1) =$

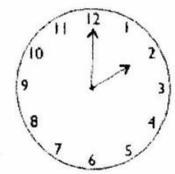
A 6 501 800

B 6 510 008

C 6 501 008

D 6 501 080 (1)

1.9 Look at the clock. What is the time in 24 hours' time if it is now in the afternoon?



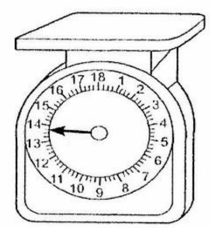
A 12:00

B 20:00

C 14:00

D 02:00 (1)

1.10 What is the mass being shown on the scale?



A 14,75 kg

B 13,75 kg

C 13,5 kg

D 13,8 kg (1)

[10]

SECTION B

Calculate the answers for questions 2.1 to 2.10. You may use any method. Show all your calculations.

2 2.1 7 211 568 + 5 722 188

(2)

2.2 4 071 274 – 2 128 863

(2)

2.3 4 748 × 36

(3)

2.4 7 675 ÷ 25

(3)

2.5 $2\frac{6}{7} + 3\frac{1}{14}$

(3)

2.6 $5\frac{7}{8} - 1\frac{4}{8}$

(2)

2.7 $45{,}05 - 19{,}21$

(2)

2.8 What is 40% of 150?

(2)

2.9 $10 \times (3 + 15) - 6$

(2)

2.10 $18{,}03 \times 100 =$ ______________________ (1)

3. A cyclist travels at a constant speed of 30 km per hour. How far will he/she travel in $4\frac{1}{2}$ hours?

(3)

4 Complete the table:

FRACTION	DECIMAL	PERCENTAGE
$\frac{4}{10}$	0,4	
$\frac{75}{100}$		75%

(2)

5 5.1 How many different factors does the number 30 have?

______________________ (1)

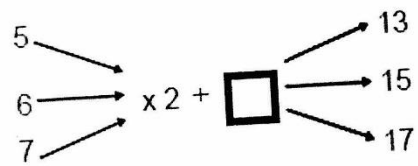
5.2 Which factors of 27 are also prime numbers?

______________________ (1)

6 Complete:

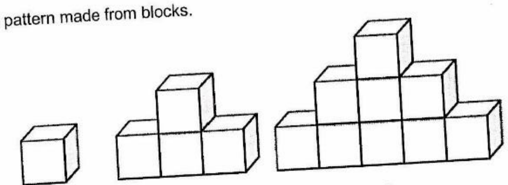
$9 \times 4 \div$ ________ $= 1$ (1)

7 Complete the flow diagram by filling in the missing number.



(1)

8 Look at the pattern made from blocks.



Model 1 Model 2 Model 3

8.1 How many blocks will you need to build model 4?

______________________________ (1)

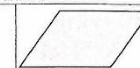
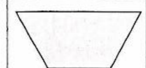
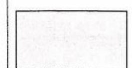
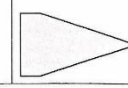
8.2 What is the rule for calculating the number of blocks, if the model number is given?

______________________________ (1)

8.3 How many rows tall will a model be, if it is made with 36 blocks?

______________________________ (1)

9 Match column A with column B. Write the correct letter in column C.

Column A		Column B		Column C
9.1	A trapezium	A		9.1 ______
9.2	The shape that is NOT a quadrilateral	B		9.2 ______
9.3	A parallelogram	C		9.3 ______
		D		

(3)

10 Use the types of angles listed in the box to complete the statements below.

an acute angle	an obtuse angle	a revolution
a straight line	a reflex angle	a right angle

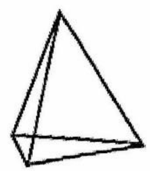
10.1 An angle smaller than 90 degrees is called ______________ (1)

10.2 If I add two obtuse angles, the resulting angle will always be

______________________________ (1)

10.3 An angle that is equal to 360 degrees is called ______________ (1)

11 Look at the object and answer the questions that follow.



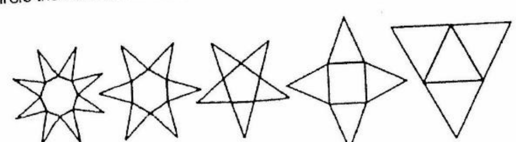
11.1 What is the name of the above 3D object?

_______________________________________ (1)

11.2 How many vertices does the object have?

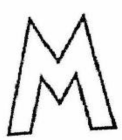
_______________________________________ (1)

12 Circle the net that can be folded into a hexagonal pyramid.



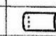
(1)

13 How many lines of symmetry does the following image have?



_______________________ (1)

14 Answer the questions based on the grid.

	A	B	C	D	E	F	G
1	◊						
2		⬠				☼	
3							
4					○		
5				→			
6		♥					△
7					⬡		

14.1 What is the position of the block where the cylinder can be found?

_______________________________________ (1)

14.2 What shape can be found in block G6?

_______________________________________ (1)

15 Convert the following:

a. $2\frac{1}{2}$ days = ________ hours

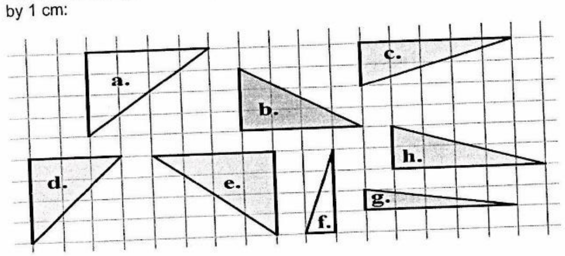
b. 41,5 kg = __________ g

(2)

16 Joe had a roll of ribbon that was $8\frac{1}{2}$ m long. He cut off a piece that was 50 cm long and sold it for R9. The rest of the roll was sold for R15 per meter. How much money did he get for the entire roll?

(3)

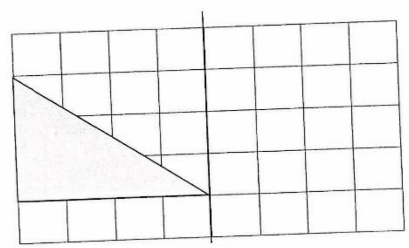
17 Look at the triangles on the grid where each block represents a square that is 1 cm by 1 cm:



17.1 What is the area of triangle ***a*** in square units?

(1)

17.2 Draw a reflection of triangle **b** on the grid provided:

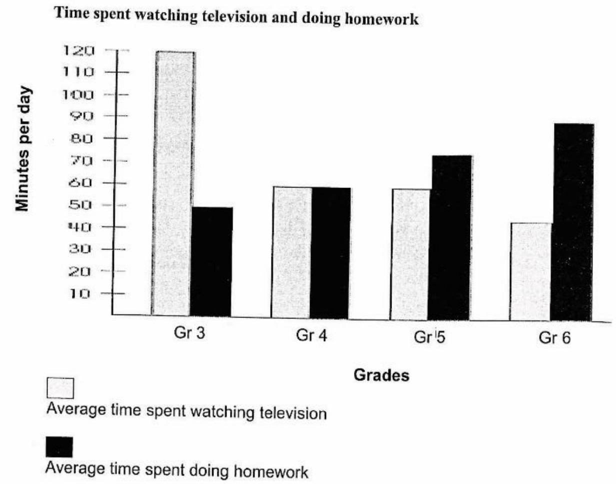


(1)

18 What would the perimeter of an octagon be with all sides measuring 6 cm?

(2)

19 Answer the questions based on the double bar graph showing the number of minutes that students spend on homework and watching television.



19.1 Which grade spends the most time watching TV?

(1)

19.2 How many more minutes do the grade sixes spend on homework, than the grade threes?

(2)

19.3 Which grade spends an equal amount of time watching TV and doing homework?

(1)

20 Complete the tally table below by filling in the answers to A and B.

Favourite cool-drinks of the Grade 6 class		
Cooldrinks	**Tally**	**Total**
Coke	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸𝍸 𝍸 𝍸	A
Sprite	B	32

(2)

21 A five-rand coin is tossed. How many possible outcomes are there?

____ (1)

22 What is the mode for the following set of data?

5 ; 3 ; 3 ; 6 ; 4 ; 3 ; 4 ;3 ;7

____ (1)

23 Seven teams take part in a soccer league. If each of the teams plays every other team in the league once, how many soccer matches will there be altogether?

(2)

[65]

TOTAL: 75

END

APPENDIX C – NOT ONLY NUMBER

Not-only-number-related questions

The not-only-number-related tasks are presented as questions out of one and two. There is one question out of three and another out of four marks. I will report on these questions separately.

Not-only-number-related questions out of one mark

Tasks in this category involve geometry, data handling and probability. See the results of the participants' responses in the bar graph below (Figure 5.14):

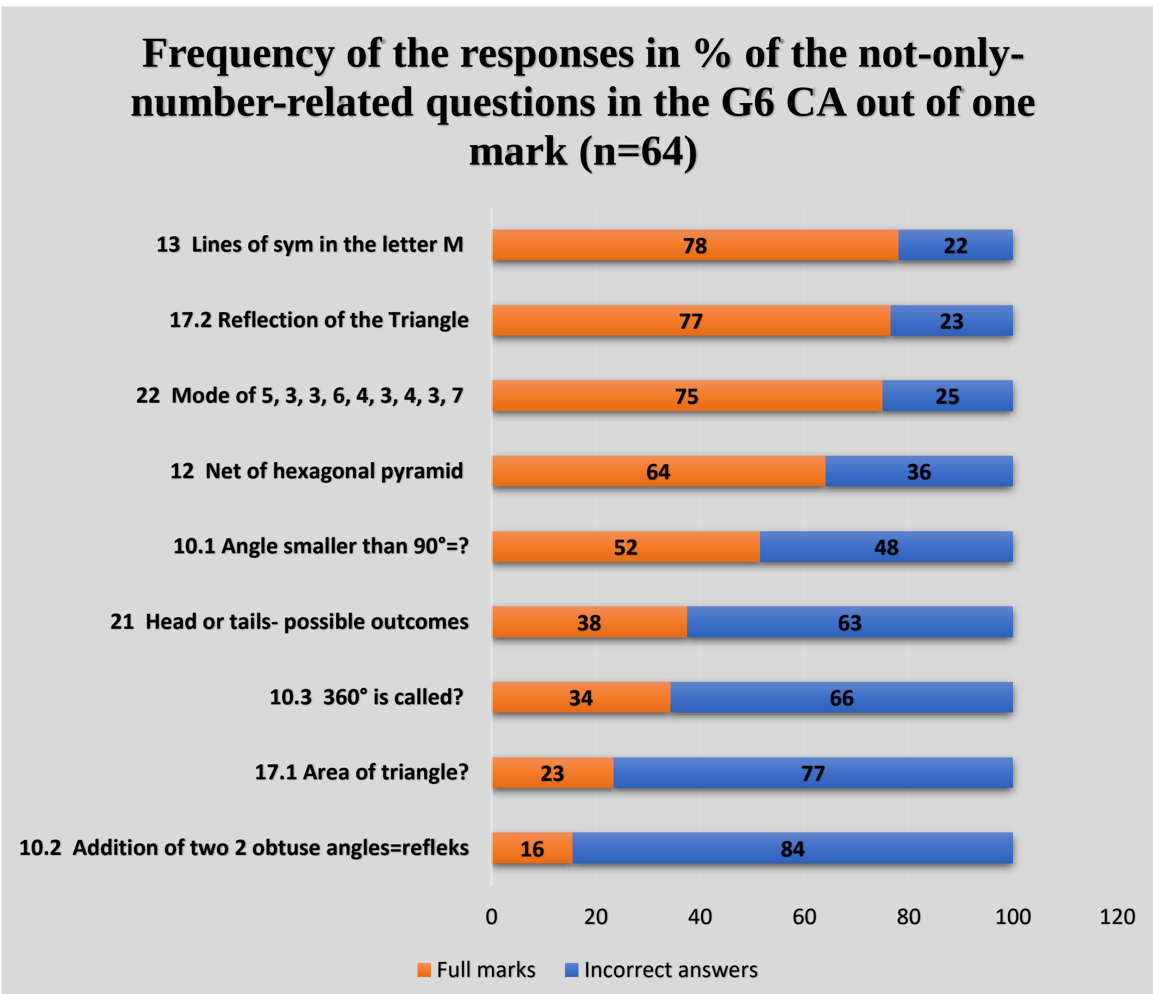


Figure 0.1 Results of the responses in %, of the not-only-number-related questions in the G6 CA out of one mark

The graph above shows that the task with the most significant percentage of correct responses was task 13. In this task, the participants had to tell how many lines of symmetry the letter 'M' has. Seventy-eight participants answered correctly. The task with the least correct responses was task 10.2, which asked: "If I add two obtuse angles, the resulting angle will always be ________."

Not-only-number-related questions out of two marks

The not-only-number-related tasks out of two marks tested geometry and data handling. The graph below (Figure 5.15) shows the results of the responses in percentages.

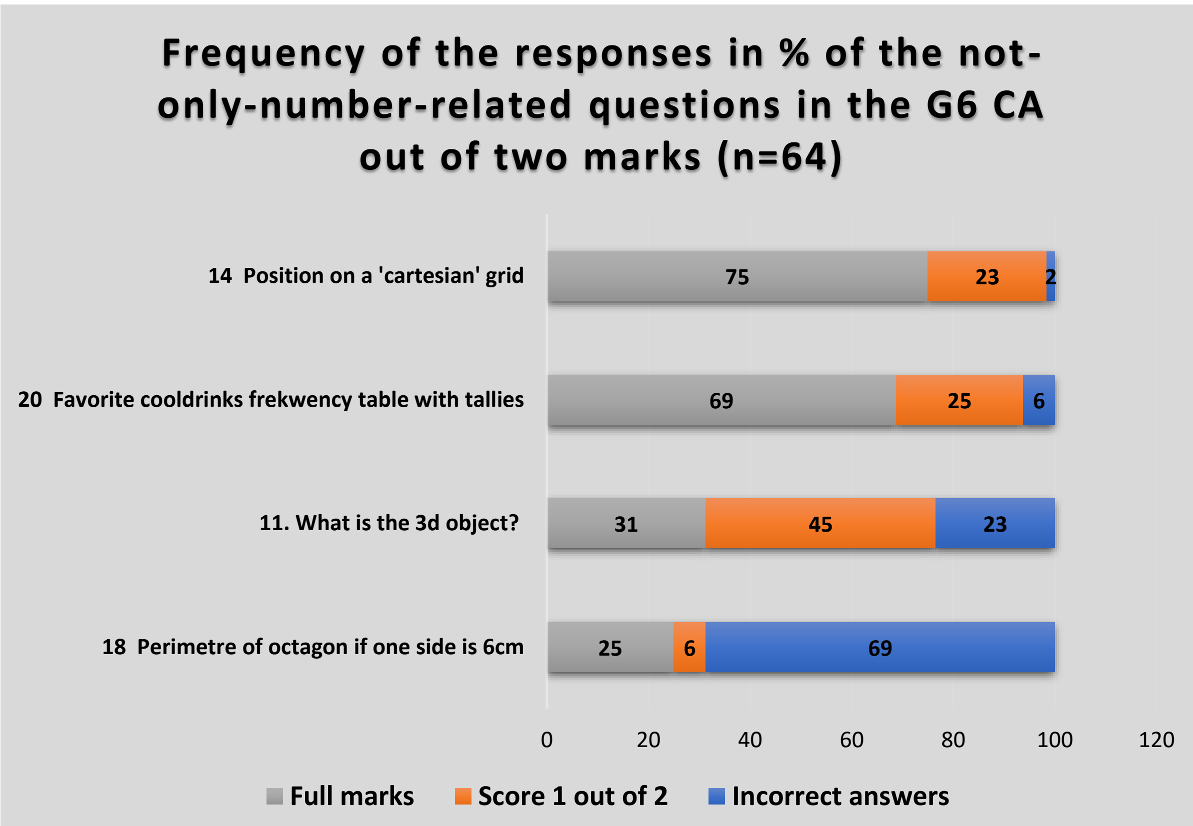


Figure 0.2 Results of the responses in % of the not-only-number-related questions in the G6 CA out of two marks

Question 18, with the least correct responses, asked what the perimeter is of an octagon where one side is 6 cm. Formal mathematical knowledge (perimeter and octagon) was necessary to answer question 18 correctly. The participants scored better on question 14, which required a position on the Cartesian plane. They also scored well on the question where they had to draw and count tally marks for frequencies of the number of favourite cool drinks.

Not-only-number-related questions out of 3 and 4 marks

The two not-only-number-related questions out of 3 and 4 were questions 9 and 19. In question 9 (scored out of 3 marks), the participants had to match the names of polygons with the correct diagram of the polygons; it was scored out of 4 marks. The graph below (Figure 5.16) shows the results of the responses for questions 9 and 19.

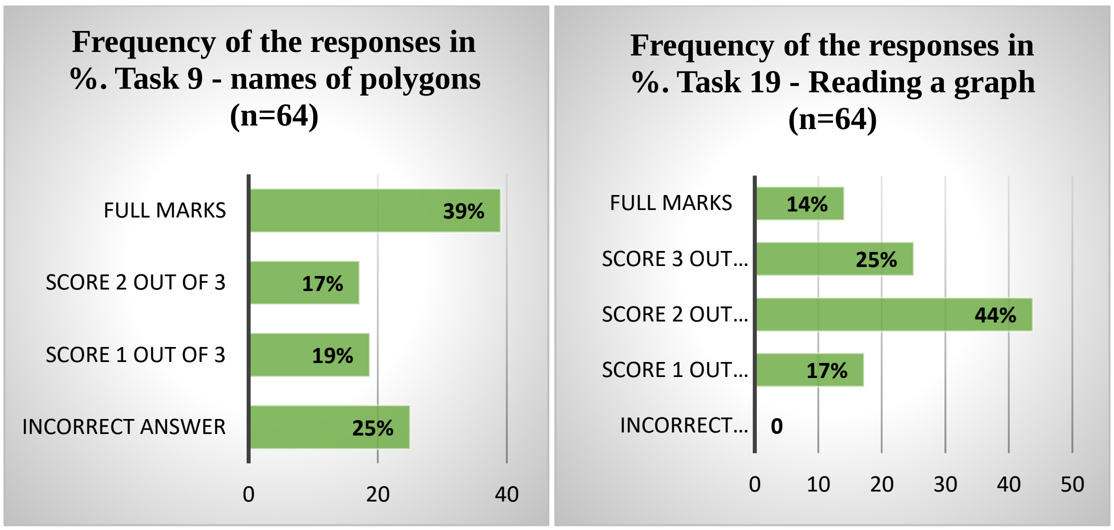


Figure 0.3 Results of the responses (in percentage) of tasks 9 and 19

Although the tasks did not weigh the same, I am comparing them. The two graphs above suggest that the participants performed better in naming polygons than on reading a graph because the number of participants scoring full marks for question 9 was 39, and the number of participants scoring full marks for question 19 was only 14%.

To sum up, the evidence shows that approximately 50% of the participants can give a mental answer to tasks that do not include bridging through ten and carrying. Nevertheless, when the tasks include knowledge of structure, like bridging through ten and carrying, the same participants who could answer mentally with a correct response to a task that does not include these strategies, had difficulties responding mentally to a task where bridging through ten and carrying were involved, and had to regress to vertical-column methods.

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