



The Direct and Inverse Problem for a Class of Entire Functions Related to Sine Type Functions

Manfred Möller¹

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Abstract

For $a > 0$ and nonnegative integers m and n the classes $\mathcal{A}_{m,n}^a$ of entire functions are introduced. An entire function ω belongs to $\mathcal{A}_{m,n}^a$ if ω is real on the imaginary axis and has the form $(i\mu)^{n-m+1}\omega(\mu) = \chi_1(\mu)\cos\mu a + i\chi_2(\mu)\sin\mu a + \Psi(\mu)$, where χ_1 is a polynomial of degree $n+1$ with leading coefficient i^{n+1} , χ_2 is a polynomial of degree at most n , and Ψ is small with respect to the other terms. When $m = 0$, these functions are sine type functions. The functions in $\mathcal{A}_{m,n}^a$ have infinitely many zeros with explicitly given asymptotic behaviour, for which the first $n+1$ terms in the asymptotics are determined by the coefficients in the polynomials χ_1 and χ_2 . Conversely, any sequence of complex numbers with such an asymptotic representation determines a unique function from a class $\mathcal{A}_{m,n}^a$ whose zeros are this given sequence. Some explicit formulas for the direct and the inverse problem are provided.

Keywords Sine type functions · Asymptotic distribution of zeros · Inverse problem · Exponential type

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1 Introduction

The direct spectral problem for an ordinary differential operator is the task to find a representation of its spectrum. The inverse spectral problem consists of two questions.

- Uniqueness of the spectrum: Does the spectrum uniquely determine the spectral

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✉ Manfred Möller
manfred.moller@wits.ac.za

¹ School of Mathematics, University of the Witwatersrand, 1 Jan Smuts Ave, Johannesburg 2017, South Africa

problem? Often more than one problem with different boundary conditions for the same differential equation is needed.

- Existence of the differential operator: Given a sequence of complex numbers, is this sequence the spectrum of a differential operator?

The earliest comprehensive solution for direct and inverse regular Sturm-Liouville problems was given by Borg [4].

Often, the direct and inverse problem are split by the introduction of an intermediate function which contains the necessary information. For quite general operators or relations, singularities of the (analytic) Weyl function provide such information about the spectrum, see e. g. [2] and references therein. In special cases another approach is based on the spectral function as initiated by Gelfand and Levitan in [6].

However, this paper will focus on regular problems on a bounded interval. Then the spectrum is infinite and discrete, and in this case, the spectrum can be found as the zeros of the characteristic determinant of the differential operator, which is an entire function. Consequently, the direct spectral problem can be split into two tasks: firstly, to find the characteristic determinant, and secondly, to find the zeros of the characteristic determinant.

The characteristic determinant is defined via a fundamental system of the differential equation, which is explicitly known only in the simplest cases. Similarly, the zeros of entire functions are only explicitly found for the simplest entire functions. Given an asymptotic representation of a fundamental system results in an asymptotic representation of the characteristic determinant. Also, for some classes of entire functions, asymptotic representations of their zeros can be found, and this extends to the case that the entire function itself has an asymptotic representation.

Consequently, problems will be considered for classes $\mathfrak{D}$ of differential operators. For each such class $\mathfrak{D}$, the question arises how to describe the class $\mathfrak{A}$ of characteristic determinants of $\mathfrak{D}$. And then, for each such class $\mathfrak{A}$ the class $\mathfrak{E}$ of its zeros will be identified.

For the easiest Sturm-Liouville problems, the characteristic determinants are of the form $\sin(\rho)$, where ρ is linear in the square root of the eigenvalue parameter. For more general problems, the characteristic determinant behaves asymptotically like $\sin(\rho)$. Such a class, called sine type functions, was introduced by Levin [8]. He proved properties of the zeros λ_n of such functions and that the functions $e^{i\lambda_n x}$, $n \in \mathbb{N}$, form a basis in L_2 -spaces. This type of basis problems has been initiated by Payley and Wiener in [15]. Another particular basis property of exponential functions was developed by Beurling and Malliavin in [3]. Levin and Ostrovskii [9] gave a comprehensive account of properties of sine type functions, whereas in [1] perturbations of sine type bases were investigated.

It was realized that characteristic functions of Sturm-Liouville problems with various boundary conditions are sine type functions or closely related to sine type functions, see e. g. [5, 11, 12, 14, 16, 17]. Here the boundary conditions may contain the eigenvalue parameter.

For differential operators, only in the simplest cases the eigenfunctions are simple exponential functions, but rather perturbations of exponential functions in more general cases. Furthermore, when e. g. the differential equation contains a damping

term or the boundary conditions depend on the eigenvalue parameter, the operator may have to be considered with nonlinear dependence on the eigenvalue parameter. In such cases, only multiple completeness of the eigenfunctions can be expected, see [18]. Another feature when considering completeness or the basis property is that there may be multiple roots of the characteristic determinant. In this case, bases of the corresponding root spaces or even associated functions for multiple eigenvalues would have to be taken into account.

The spectral problem was solved for particular classes $\mathfrak{D}$ in [11]. In these classes, the boundary conditions depend on the eigenvalue parameter and therefore the eigenvalues are no longer real in general. In this case there is enough information to recover the differential operator from its spectrum for just one set of boundary conditions. In particular, there is a suitable one-to-one correspondence between the class of differential operators under consideration in [11] and the class of entire functions which are given by the characteristic determinants.

Since the completion of the thesis [11], the monograph [14] superseded some of the techniques used in [11] and those techniques and results from [14] can now be employed for larger classes $\mathfrak{D}$, which in turn leads to larger classes $\mathfrak{A}$ of entire functions. However, these classes $\mathfrak{A}$ and the classes $\mathfrak{E}$ of their zeros can easily be modified by introducing parameters in such a way that definitions, statements and proofs can be presented simultaneously for all parameter values. Indeed, the leading term of the characteristic determinant may contain a positive power of ρ , which motivates the introduction of the classes $\mathcal{A}_{m,n}^a$ of entire functions for nonnegative integers m and n , where m is the exponent of the leading power and n indicates the depth of the asymptotic representation. Although in the applications in mind only $m = 2$ will be used, it is anticipated that more general differential operators may lead to characteristic determinants which belong to different classes $\mathcal{A}_{m,n}^a$. Thus such spectral problems may already be partly solved in the current paper.

In this paper representations of the classes $\mathcal{Z}_{m,n}^a$ of the sequences of the zeros of functions in $\mathcal{A}_{m,n}^a$ are established, and it is shown that such a sequence uniquely determines the entire function. In a forthcoming publication the results of this paper will be applied to a string problem where a part of the string has zero density and where the remaining part of the string may be damped.

In Section 2 the classes $\mathcal{A}_{m,n}^a$ of entire functions are introduced, and some of their elementary properties are shown. It will also be briefly discussed how these classes relate to sine type functions. In Section 3, the asymptotic representation of the zeros of functions in $\mathcal{A}_{m,n}^a$ are stated and proved. Section 4 is devoted to the development of an algorithm to find the coefficients in the asymptotic representation of the sequences in $\mathcal{Z}_{m,n}^a$ from the coefficients of the functions in $\mathcal{A}_{m,n}^a$. The inverse problem of finding a function in $\mathcal{A}_{m,n}^a$ whose zeros have a prescribed form will be solved in Section 5. Section 6 is concerned with applications to Sturm-Liouville problems with possible damping. In this case, the characteristic function is a polynomial combination of fundamental solutions, which will be shown in the above mentioned forthcoming publication. For the case that the damping is proportional to the potential, it is shown in this paper that a transformation of the eigenvalue parameter results in a characteristic function which belongs to some class $\mathcal{A}_{m,n}^a$. Finally, in Section 7 some partial recovery of the coefficients of the characteristic functions considered in Section 6 from their

spectral asymptotics are obtained, which will be used for the inverse problem in the above mentioned forthcoming publication.

2 The Class $\mathcal{A}_{m,n}^a$

Throughout this paper, a is a positive real number and m and n are nonnegative integers. First, some well-known notation, definitions and results will be recalled.

As usual, $O(\lambda)$ and $o(\lambda)$ indicate boundedness or convergence to 0 as $|\lambda| \rightarrow \infty$ or $|\lambda| \rightarrow 0$. If not specified, it will be clear from the context if $|\lambda| \rightarrow \infty$ or $|\lambda| \rightarrow 0$ is meant.

An entire function which is real on the real axis is called a real entire function. An entire function of exponential type $\leq a$ is said to belong to the Paley-Wiener class $\mathcal{L}^a$ if its restriction to the real axis belongs to $L_2(\mathbb{R})$. Plancherel's theorem states that an entire function ω belongs to $\mathcal{L}^a$ if and only if it is of the form

$$\omega(\lambda) = \int_0^a \xi(t) \cos \lambda t \, dt + i \int_0^a \zeta(t) \sin \lambda t \, dt, \quad \lambda \in \mathbb{C}, \quad (2.1)$$

where $\xi, \zeta \in L_2(0, a)$, see e.g. [19, Theorems 48 and 50] and [20, Vol II, Chapter XVI, (7.2)]. Clearly, such functions ω are real on the imaginary axis if ξ and ζ are real-valued functions. It is well-known and easy to see that the integrability condition holds if and only if the function ω is in $L_2(iy + \mathbb{R})$ for any and hence for all $y \in \mathbb{R}$.

In view of (2.1) it is clear that $\omega \in \mathcal{L}^a$ if and only if ω is an entire function for which $\omega(\lambda) = O(e^{|\operatorname{Im} \lambda|a})$ as $|\lambda| \rightarrow \infty$ and ω is square integrable on $iy + \mathbb{R}$ for some $y \in \mathbb{R}$. Let $\tilde{\mathcal{L}}^a$ denote the class of functions f which are analytic in a region containing the set $\{\lambda \in \mathbb{C} : |\lambda| \geq R\}$ for some $R > 0$, which satisfy $O(e^{|\operatorname{Im} \lambda|a})$ as $|\lambda| \rightarrow \infty$ and which are square integrable on $iy + (\mathbb{R} \setminus (-R, R))$ for some $y \in \mathbb{R}$. The set of functions in $\tilde{\mathcal{L}}^a$ which are entire functions or which are analytic on $\mathbb{C} \setminus \{0\}$ with a pole at 0 is denoted by $\mathcal{L}_0^a$. Clearly, $\mathcal{L}^a$ is the set of all entire functions in $\mathcal{L}^a$.

Lemma 2.1 *For each $f \in \mathcal{L}_0^a$ there is a polynomial p_f with $p_f(0) = 0$ such that $f - p_f \in \mathcal{L}^a$, where P_f is defined by $P_f(\mu) = p_f(\mu^{-1}) \cos \mu a$. Furthermore, $f(\mu) e^{-|\operatorname{Im} \mu|a} = o(1)$ as $|\mu| \rightarrow \infty$, and P_f is even or odd, respectively, if f has this property.*

Proof If f is an entire function, let $p_f = 0$. Otherwise, also the function F defined by $F(\mu) = \frac{f(\mu)}{\cos \mu a}$ has a pole at 0, and the principal part of F at 0 equals $p_f(\mu^{-1})$ for some polynomial p_f with $p_f(0) = 0$. Therefore $F(\mu) - p_f(\mu^{-1})$ is an entire function. Then clearly $P_f(\mu) = p_f(\mu) \cos \mu a$ belongs to $\mathcal{L}_0^a$, and $f - P_f \in \mathcal{L}^a$ follows. Clearly, the principal part of F and hence P_f are even or odd if f has this property.

Since each function $g \in \mathcal{L}^a$ satisfies $e^{-|\operatorname{Im} \mu|a} g(\mu) = o(1)$ as $|\mu| \rightarrow \infty$, see [10, Lemma 1.3.1], and since obviously $e^{-|\operatorname{Im} \mu|a} P_f(\mu) = o(1)$ as $|\mu| \rightarrow \infty$, also the function $f = (f - P_f) + P_f$ satisfies $e^{-|\operatorname{Im} \mu|a} f(\mu) = o(1)$ as $|\mu| \rightarrow \infty$. $\square$

For an analytic function f , the conjugate function $\bar{f}$ is defined by $\bar{f}(\lambda) = \overline{f(\bar{\lambda})}$ when $\bar{\lambda}$ is in the domain of f . Then also $\bar{f}$ is analytic. When writing $f(\lambda)$ etc. for a function value with generic λ , then λ will be a complex number in the (natural) domain of the function f unless otherwise specified.

Definition 2.2 An entire function ω is said to belong to the class $\mathcal{A}_{m,n}^a$ if it is real on the imaginary axis and if it has the representation

$$\omega(\mu) = \left((i\mu)^m + \sum_{j=0}^n A_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^n B_j (i\mu)^{m-j-1} \right) \sin \mu a + \frac{\Psi(\mu)}{(i\mu)^{n-m+1}}, \quad (2.2)$$

where $A_j, B_j \in \mathbb{C}$ for $j = 0, \dots, n$, and where $\Psi \in \mathcal{L}_0^a$.

Remark 2.3 Multiplying (2.2) by $(i\mu)^{n-m+1}$ gives the identity

$$(i\mu)^{n-m+1} \omega(\mu) = \chi_1(\mu) \cos \mu a + i \chi_2(\mu) \sin \mu a + \Psi(\mu), \quad (2.3)$$

where

$$\chi_1(\mu) = (i\mu)^{n+1} + \sum_{j=0}^n A_j (i\mu)^{n-j}, \quad (2.4)$$

$$\chi_2(\mu) = \sum_{j=0}^n B_j (i\mu)^{n-j}. \quad (2.5)$$

In order to emphasize particular values of n , the notations ${}_n\Psi$, ${}_n\chi_1$, ${}_n\chi_2$, may be used.

Proposition 2.4 For $\omega \in \mathcal{A}_{m,n}^a$ the numbers A_j and B_j , $j = 0, \dots, n$, in (2.2) are real and uniquely determined by ω , and the function Ψ in (2.2) is real on the imaginary axis and uniquely determined by ω .

Proof Two representations (2.3) with corresponding functions $\chi_1^{[p]}$, $\chi_2^{[p]}$ and $\Psi^{[p]}$ for $p = 1, 2$ lead to the identity

$$(\chi_1^{[1]}(\mu) - \chi_1^{[2]}(\mu)) \cos \mu a + i(\chi_2^{[1]}(\mu) - \chi_2^{[2]}(\mu)) \sin \mu a = \Psi^{[2]}(\mu) - \Psi^{[1]}(\mu). \quad (2.6)$$

Since the left hand side of (2.6) is an entire function, so is $\tilde{\Psi} = \Psi^{[2]} - \Psi^{[1]}$. From $\tilde{\Psi} \in \mathcal{L}_0^a$ it therefore follows that $\tilde{\Psi} \in \mathcal{L}^a$. Observing that $\tilde{\Psi}(\mu) = o(1)e^{|\operatorname{Im} \mu|a}$ as $|\mu| \rightarrow \infty$ by Lemma 2.1 and evaluating the above identity at $2k\pi a^{-1}$ and $(2k + \frac{1}{2})\pi a^{-1}$ for (nonzero) integers k , it follows that each of the two polynomials $\chi_l^{[1]} - \chi_l^{[2]}$, $l = 1, 2$, tends to zero on a sequence of points tending to infinity. Therefore both polynomials must be the zero polynomials, which shows that χ_1 and χ_2 are uniquely determined by

ω . Hence the complex numbers A_j and B_j for $j = 0, \dots, n$ are uniquely determined by ω . Then also Ψ is unique.

Since ω is real on the imaginary axis, it follows for $\mu \in i\mathbb{R}$ that $\omega(\mu) = \omega(\overline{-\mu}) = \overline{\omega(-\mu)} = \overline{\omega}(-\mu)$, which shows by the identity theorem that $\omega(\mu) = \overline{\omega}(-\mu)$ for all $\mu \in \mathbb{C}$. Hence the representation (2.2) leads to

$$\begin{aligned} \omega(\mu) = & \left((i\mu)^m + \sum_{j=0}^n \overline{A_j} (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^n \overline{B_j} (i\mu)^{m-j-1} \right) \sin \mu a \\ & + \frac{\overline{\Psi}(-\mu)}{(i\mu)^{n-m+1}}. \end{aligned}$$

Clearly, with $\Psi \in \mathcal{L}_0^a$ also $\overline{\Psi} \in \mathcal{L}_0^a$, and hence the uniqueness result implies that $A_j = \overline{A_j}$ and $B_j = \overline{B_j}$, which means that they are real. Finally $\Psi(\mu) = \overline{\Psi}(-\mu)$ shows that Ψ is real on the imaginary axis. $\square$

Remark 2.5 1. For $n > 0$, dividing (2.3) by $i\mu$ gives

$$\begin{aligned} (i\mu)^{n-m} \omega(\mu) = & {}_{n-1}\chi_1(\mu) \cos \mu a + i {}_{n-1}\chi_2(\mu) \sin \mu a \\ & + (i\mu)^{-1} A_n \cos \mu a + i (i\mu)^{-1} B_n \sin \mu a + (i\mu)^{-1} {}_n\Psi(\mu), \end{aligned}$$

which shows that this is indeed a representation (2.3) for $n-1$ with

$${}_{n-1}\Psi(\mu) = (i\mu)^{-1} A_n \cos \mu a + i (i\mu)^{-1} B_n \sin \mu a + (i\mu)^{-1} {}_n\Psi(\mu).$$

Therefore

$$\mathcal{A}_{m,n}^a \subset \mathcal{A}_{m,n-1}^a \quad (m \geq 0, n > 0). \quad (2.7)$$

2. When $n \geq m-1$, then the left hand side of (2.3) is an entire function. Thus the same holds for Ψ , and therefore $\Psi \in \mathcal{L}^a$ when $n \geq m-1$. However, this need not be the case when $n < m-1$, as the following simple example shows.

Example 2.6 Let

$$\tilde{\omega}(\mu) = (1 - (i\mu)^{-1}) \cos \mu a + \frac{\tilde{\Psi}(\mu)}{i\mu}$$

where $\tilde{\Psi}$ is of the form (2.1) with $\int_0^a \xi(t) dt = 1$ and $\zeta = 0$. Then $\tilde{\Psi} \in \mathcal{L}^a$ is an even function which is real on the imaginary axis and satisfies $\tilde{\Psi}(0) = 1$, showing that $\tilde{\omega} \in \mathcal{A}_{0,0}^a$. Now define

$$\omega(\mu) = ((i\mu)^2 + 1) \tilde{\omega}(\mu).$$

Then $\omega \in \mathcal{A}_{2,0}^a$ has the representation

$$\omega(\mu) = ((i\mu)^2 - (i\mu)) \cos \mu a + \frac{\Psi(\mu)}{(i\mu)^{-1}},$$

where

$$\Psi(\mu) = \left((i\mu)^{-1} - (i\mu)^{-2} \right) \cos \mu a + \left(1 + (i\mu)^{-2} \right) \tilde{\Psi}(\mu)$$

belongs to $\mathcal{L}_0^a$. But Ψ has a simple pole at 0, so that $\Psi \notin \mathcal{L}^a$.

Remark 2.7 It is sometimes advantageous to give the function Ψ some more structure. Indeed, writing

$$\Psi(\mu) = i B_{n+1} (i\mu)^{-1} \sin \mu a + \Psi_1(\mu) + (i\mu)^{-1} \Psi_0(\mu) \quad (2.8)$$

with $B_{n+1} \in \mathbb{C}$, $\Psi_0, \Psi_1 \in \mathcal{L}_0^a$ it follows that $\Psi \in \mathcal{L}_0^a$. Of course, $B_{n+1} = 0$, $\Psi_0 = 0$ and $\Psi_1 = \Psi$ is a trivial such representation. Hence, the representation (2.2) of $\omega \in \mathcal{A}_{m,n}^a$ can be supplemented by

$$\begin{aligned} \omega(\mu) = & \left((i\mu)^m + \sum_{j=0}^n A_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^{n+1} B_j (i\mu)^{m-j-1} \right) \sin \mu a \\ & + \frac{\Psi_1(\mu)}{(i\mu)^{n-m+1}} + \frac{\Psi_0(\mu)}{(i\mu)^{n-m+2}}. \end{aligned} \quad (2.9)$$

Proposition 2.8 For $\omega \in \mathcal{A}_{m,n}^a$ the number B_{n+1} in (2.9) is real and uniquely determined by ω if Ψ_1 is an odd function.

Proof With the obvious notation for two versions of (2.8), the right hand side of (2.6) multiplied by μ is

$$(B_{n+1}^{[2]} - B_{n+1}^{[1]}) \sin \mu a + \mu (\Psi_1^{[2]}(\mu) - \Psi_1^{[1]}(\mu)) - i (\Psi_0^{[2]}(\mu) - \Psi_0^{[1]}(\mu)).$$

Observing that each entire function can be uniquely written as a sum of an even and an odd entire function, it follows for the odd summand that $(B_{n+1}^{[2]} - B_{n+1}^{[1]}) \sin \mu a = o(1)$ for $\mu \in \mathbb{R}$. Therefore also B_{n+1} is uniquely determined by ω if Ψ_1 is odd. The fact that B_{n+1} is real follows as in the proof of Proposition 2.4. $\square$

Due to Propositions 2.4 and 2.8 it is convenient to make the following assumption whenever referring to (2.9).

Assumption 2.9 $B_{n+1} \in \mathbb{R}$.

Below is the definition of sine type functions as given in [9].

Definition 2.10 ([8, §1]) An entire function ω of positive exponential type is said to be a sine type function if

- (i) there is $h > 0$ such that all zeros of ω lie in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < h\}$,
- (ii) there are $h_1 \in \mathbb{R}$ and positive numbers $m < M$ such that $m \leq |\omega(\lambda)| \leq M$ holds for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda = h_1$,
- (iii) the exponential type of ω in the lower half-plane coincides with the exponential type of ω in the upper half-plane.

Proposition 2.11 *Each entire function of class $\mathcal{A}_{0,n}^a$ is a sine type function of exponential type a .*

Proof Let $\omega \in \mathcal{A}_{0,n}^a$. The representation (2.2) of ω shows that

$$\omega(\mu) = \cos \mu a + O(|\mu|^{-1}) e^{|\operatorname{Im} \mu| a}.$$

Observing

$$|\cos \mu a|^2 = \cos^2(\operatorname{Re} \mu a) + \sinh^2(\operatorname{Im} \mu a) \quad (2.10)$$

it is clear that the function ω inherits the properties (i), (ii) and (iii) in Definition 2.10 from the function $\cos \mu a$. $\square$

Proposition 2.12 *Let ω and ϖ be two entire functions and let p be a monic polynomial of degree m' such that $\omega = i^{m'} p \varpi$. Assume that two of the three functions ω , ϖ , $i^{m'} p$ are real on the imaginary axis.*

1. If $\varpi \in \mathcal{A}_{m,n}^a$, then $\omega \in \mathcal{A}_{m+m',n}^a$.
2. If $\omega \in \mathcal{A}_{m,n}^a$ and $m' \leq m$, then $\varpi \in \mathcal{A}_{m-m',n}^a$.

Proof Since two of the three functions ω , ϖ , $i^{m'} p$ are real on the imaginary axis, all three are real on the imaginary axis. In particular, $i^{m'} p$ can be written as

$$i^{m'} p(\mu) = (i\mu)^{m'} \left(1 + \sum_{j=1}^{m'} C_j (i\mu)^{-j} \right) \quad (2.11)$$

with $C_j \in \mathbb{R}$. Then there is $R > 0$ such that

$$\frac{1}{i^{m'} p(\mu)} = (i\mu)^{-m'} \left(1 + \sum_{j=1}^{\infty} D_j (i\mu)^{-j} \right) \quad (2.12)$$

with $D_j \in \mathbb{R}$ for $|\mu| > R$.

1. A representation (2.2) of ϖ with remainder function $\Psi^\dagger \in \mathcal{L}_0^a$ and (2.11) lead to a representation of ω of the form (2.2) with leading term $(i\mu)^{m+m'} \cos \mu a$ and with remainder term

$$\Psi(\mu) = O(\mu^{-1}) \cos \mu a + O(\mu^{-1}) \sin \mu a + O(1) \Psi^\dagger(\mu)$$

which is analytic on $\mathbb{C} \setminus \{0\}$. Clearly, $\Psi \in \mathcal{L}_0^a$, and therefore $\omega \in \mathcal{A}_{m+m',n}^a$.

2. This part of the proof is analogous to part 1 if one observes that the remainder function of ϖ is the difference of functions which are analytic on $\mathbb{C} \setminus \{0\}$. $\square$

For a nonzero entire function f with infinitely many zeros, let $\mathcal{Z}(f)$ denote the sequence of zeros of f , with multiplicities taken into account. The index set as well as the indexing are arbitrary, which means that $\mathcal{Z}(f)$ is an equivalence class, and whenever convenient, such a class is identified with any of its representatives. Finally, let

$$\mathfrak{Z}_{m,n}^a = \{\mathcal{Z}(\omega) : \omega \in \mathcal{A}_{m,n}^a\}.$$

3 Asymptotic Representations of Sequences in $\mathfrak{Z}_{m,n}^a$

In this section it will be shown that the zeros of functions in the class $\mathcal{A}_{m,n}^a$ have a very particular representation.

Theorem 3.1 *Let ω be an entire function of class $\mathcal{A}_{m,n}^a$ and put $\kappa = \lfloor \frac{m}{2} \rfloor + \frac{1}{2}$, $Z = \mathbb{Z}$ if m is odd and $Z = \mathbb{Z} \setminus \{0\}$ if m is even. Then there is a representation $(\mu_k)_{k \in Z}$ of $\mathcal{Z}(\omega)$ which has the following properties:*

- (i) *there is a nonnegative integer k_0 such that μ_k is pure imaginary for $|k| \leq k_0$;*
- (ii) *for $k > k_0$ it is true that $\operatorname{Re} \mu_k > 0$ and $\mu_{-k} = -\overline{\mu_k}$;*
- (iii) *as $k \rightarrow \infty$, the sequence has the asymptotic representation*

$$\mu_k = \frac{\pi}{a} (k - \kappa) + \sum_{j=1}^{n+2} i^{j+1} \frac{Q_j}{(k - \kappa)^j} + \frac{\beta_k}{(k - \kappa)^{n+1}}, \quad (3.1)$$

where $Q_j \in \mathbb{R}$ ($j = 1, \dots, n+2$) and where $(\beta_k)_{k=1}^\infty \in l_2$.

If $\ell \in \{0, 1\}$ and $i^\ell \Psi_1$ is real on the real axis for some representation (2.9) of ω , then $(k \operatorname{Re}(i^{n+\ell} \beta_k))_{k=1}^\infty \in l_2$.

Proof If $m = 0$, let $\widehat{\omega}(\mu) = \mu \omega(\mu)$. If $m = 1$, let $\widehat{\omega}(\mu) = -i\omega(\mu)$.

In case $m \geq 2$ let Γ_k be the boundary of the rectangle with vertices $\pm k\pi a^{-1} \pm ik\pi a^{-1}$, $k \in \mathbb{N}$. It easily follows from (2.10) that

$$|\cos \mu a| \geq \frac{1}{4} e^{|\operatorname{Im} \mu| a}, \quad \mu \in \Gamma_k, \quad k \in \mathbb{N}.$$

In view of

$$\omega(\mu) - (i\mu)^m \cos \mu a = O(\mu^{m-1} e^{|\operatorname{Im} \mu| a}) \text{ as } \mu \rightarrow \infty$$

there is $k \in \mathbb{N}$ such that

$$|\omega(\mu) - (i\mu)^m \cos \mu a| < |(i\mu)^m \cos \mu a| \text{ for all } \mu \in \Gamma_k.$$

By Rouché's theorem, the function ω has the same number of zeros inside Γ_k as $\mu \mapsto (i\mu)^m \cos \mu a$, counted with multiplicity, that is, exactly $m + 2k$ zeros. Let $\tilde{\mu}_1, \dots, \tilde{\mu}_{m-1}$ be any $m - 1$ such zeros. Then an entire function $\widehat{\omega}$ is defined by

$$\widehat{\omega}(\mu) = (-i)^m \omega(\mu) \prod_{j=1}^{m-1} (\mu - \tilde{\mu}_j)^{-1}.$$

By Proposition 2.12,2 and (2.7) it is true that $i\widehat{\omega} \in \mathcal{A}_{1,n}^a \subset \mathcal{A}_{1,0}^a$. Therefore, the function $\widehat{\omega}$ has the representation

$$\widehat{\omega}(\mu) = \mu \cos \mu a + \widehat{A}_0 \cos \mu a + \widehat{B}_0 \sin \mu a + \widehat{\psi}(\mu),$$

where $\widehat{A}_0, \widehat{B}_0 \in \mathbb{C}$ and $\widehat{\psi} \in \mathcal{L}^a$. By [14, Lemma 7.1.3.2] with $\alpha = 0$ and $\sigma = 1$, the sequence $(\widehat{\mu}_k)_{k \in \mathbb{Z}}$ of the zeros of $\widehat{\omega}$, counted with multiplicity, can be indexed in such a way that it has the asymptotic representation

$$\widehat{\mu}_k = \frac{\pi}{a} \operatorname{sgn} k \left(|k| - \frac{1}{2} \right) + \frac{\widehat{Q}_1}{k} + \frac{\widehat{\beta}_k}{k} \quad (3.2)$$

with $\widehat{Q}_1 \in \mathbb{C}$ and $(\widehat{\beta}_k)_{k \in \mathbb{Z} \setminus \{0\}} \in l_2$.

When $m = 0$, then $\widehat{\omega}(0) = 0$ and it can be assumed that the sequence $(\widehat{\mu}_k)_{k \in \mathbb{Z}}$ is indexed in such a way that $\widehat{\mu}_0 = 0$. For $m = 0$ or $m = 1$ and $k \in \mathbb{Z}$ let $\tilde{\mu}_k = \widehat{\mu}_k$. Then $(\tilde{\mu}_k)_{k \in \mathbb{Z}}$ is the sequence of the zeros of ω .

Now let $m \geq 2$ and put $\kappa' = \lfloor \frac{m}{2} \rfloor$. For integers $k \geq m$ let $\tilde{\mu}_k = \widehat{\mu}_{k-\kappa'}$. For these k ,

$$\begin{aligned} \tilde{\mu}_k &= \frac{\pi}{a} \left(k - \kappa' - \frac{1}{2} \right) + \frac{\widehat{Q}_1}{k - \kappa'} + \frac{\widehat{\beta}_{k-\kappa'}}{k - \kappa'} \\ &= \frac{\pi}{a} (k - \kappa) + \frac{\widetilde{Q}_1}{k - \kappa} + \frac{\widetilde{\beta}_k}{k - \kappa} \end{aligned} \quad (3.3)$$

with $\widetilde{Q}_1 = \widehat{Q}_1$ and $(\widetilde{\beta}_k)_{k=m}^\infty \in l_2$. Clearly, also $\tilde{\mu}_1, \dots, \tilde{\mu}_{m-1}$ can be written in this way. For integers $k < 0$ if m is even or integers $k \leq 0$ if m is odd let $\tilde{\mu}_k = \widehat{\mu}_{k+\kappa'}$. When m is even, then $\tilde{\mu}_{-1} = \widehat{\mu}_{\kappa'-1}$ and $\tilde{\mu}_m = \widehat{\mu}_{m-\kappa'} = \widehat{\mu}_{\kappa'}$, and when m is odd, then $\tilde{\mu}_0 = \widehat{\mu}_{\kappa'}$ and $\tilde{\mu}_m = \widehat{\mu}_{m-\kappa'} = \widehat{\mu}_{\kappa'+1}$, so that in any case the sequence $(\tilde{\mu}_k)_{k \in \mathbb{Z}}$ is the concatenation of the sequences $(\widehat{\mu}_k)_{k=1}^{m-1}$ and $(\widehat{\mu}_k)_{k \in \mathbb{Z}}$, that is, it is the sequence of the zeros of ω , counted with multiplicity. For $k < -\kappa'$ it follows in view of (3.2) that

$$\begin{aligned} \tilde{\mu}_k &= \widehat{\mu}_{k+\kappa'} = -\frac{\pi}{a} \left(-(k + \kappa') - \frac{1}{2} \right) + \frac{\widehat{Q}_1}{k + \kappa'} + \frac{\widehat{\beta}_{k+\kappa'}}{k + \kappa'} \\ &= -\frac{\pi}{a} (-k - \kappa) + \frac{\widetilde{Q}_1}{k + \kappa} + \frac{\widetilde{\beta}_k}{k + \kappa}, \end{aligned} \quad (3.4)$$

where $(\widetilde{\beta}_k)_{k=-\kappa'-1}^{-\infty} \in l_2$. By (3.2), the representations (3.3) and (3.4) are also valid for $m = 0$ and $m = 1$.

In view of $\omega(\mu) = \overline{\omega}(-\mu)$ the zeros of ω are symmetric with respect to the imaginary axis. Hence (3.3) and (3.4) show that $\tilde{\mu}_{-k} = -\overline{\tilde{\mu}_k}$ for sufficiently large positive integers k . Possibly reindexing finitely many terms of the sequence $(\tilde{\mu}_k)_{k \in \mathbb{Z}}$ gives a sequence $(\mu_k)_{k \in \mathbb{Z}}$ which is the sequence of the zeros of ω and which satisfies properties (i) and (ii).

The proof of (iii) starts with the introduction of the auxiliary function f defined by

$$f(z, w) = \left(1 + \sum_{j=0}^n i^{-j-1} A_j w^{j+1} (1 + wz)^{-j-1} \right) \sin za - \left(\sum_{j=0}^{n+1} i^{-j} B_j w^{j+1} (1 + wz)^{-j-1} \right) \cos za. \quad (3.5)$$

Here any representation (2.9) of ω is taken. The function f is analytic in both variables near $(0, 0)$ with $f(z, 0) = \sin za$. Hence $f(\cdot, 0)$ has a simple zero at 0. Therefore, there are positive numbers ε and δ such that for all $|w| < \varepsilon$, the function $f(\cdot, w)$ has exactly one (simple) zero $h(w)$ with $|h(w)| < \delta$, and the power series expansion

$$h(w) = \sum_{j=1}^{\infty} i^{j+1} a_j w^j \quad (3.6)$$

holds, where the a_j are complex numbers, see e. g. [14, Theorem 9.1.1]. Since the numbers A_j and B_j are real by Proposition 2.4 and Assumption 2.9, it follows for $z, w \in i\mathbb{R}$ sufficiently close to 0 that also $f(z, w) \in i\mathbb{R}$, which shows that the zeros of such $f(\cdot, w)$ must be symmetric with respect to the imaginary axis. But since the zero $h(w)$ of $f(\cdot, w)$ is unique, it must therefore lie on the imaginary axis. It follows that $a_j \in \mathbb{R}$ for all $j \in \mathbb{N}$.

Since

$$\frac{\partial}{\partial z} f(0, 0) = a,$$

there is $\eta \in (0, \min\{\varepsilon, \delta\})$ such that

$$\frac{\partial}{\partial z} f(z, w) - a = O(|z| + |w|) \quad \text{if } |z| < \eta \text{ and } |w| < \eta.$$

For integers $k \geq 1$ let $w_k = \frac{a}{\pi} (k - \kappa)^{-1}$ and let $v_k = \mu_k - \frac{1}{w_k}$. By definition of w_k , $w_k = O(k^{-1})$ as $k \rightarrow \infty$. Then by (3.3) and (3.6), also $v_k = O(k^{-1})$ and $h(w_k) = O(k^{-1})$ as $k \rightarrow \infty$. Observing that $\cos \mu_k a = (-1)^{k-\kappa'} \sin v_k a$ and $\sin \mu_k a = (-1)^{k-\kappa'+1} \cos v_k a$, it follows from (2.2) that

$$(-1)^{k-\kappa'-1} \left(\frac{\Psi_1(\mu_k)}{(i\mu_k)^{n+1}} + \frac{\Psi_0(\mu_k)}{(i\mu_k)^{n+2}} \right) = f(v_k, w_k) = f(v_k, w_k) - f(h(w_k), w_k)$$

$$\begin{aligned}
&= \int_{h(w_k)}^{v_k} \frac{\partial}{\partial z} f(\zeta, w_k) d\zeta \\
&= a(v_k - h(w_k)) + \int_{h(w_k)}^{v_k} \left(\frac{\partial}{\partial z} f(\zeta, w_k) - a \right) d\zeta \\
&= (v_k - h(w_k))(a + O(k^{-1}))
\end{aligned}$$

as $k \rightarrow \infty$. Therefore, there are $k_1 \in \mathbb{N}$ and a bounded sequence $(\delta_k)_{k=k_1}^\infty$ such that

$$v_k = h(w_k) + (-1)^{k-\kappa'-1} \left(\frac{\Psi_1(\mu_k)}{(i\mu_k)^{n+1}} + \frac{\Psi_0(\mu_k)}{(i\mu_k)^{n+2}} \right) \left(\frac{1}{a} + \frac{\delta_k}{k} \right), \quad k \geq k_1.$$

This proves (iii) with

$$Q_j = a_j \left(\frac{a}{\pi} \right)^j \in \mathbb{R}, \quad j = 1, \dots, n+2, \quad (3.7)$$

and

$$\begin{aligned}
\beta_k &= (k - \kappa)^{n+1} w_k^{n+3} \sum_{j=n+3}^{\infty} i^{j+1} a_j w_k^{j-n-3} \\
&\quad + (-1)^{k-\kappa'-1} (k - \kappa)^{n+1} \left(\frac{\Psi_1(\mu_k)}{(i\mu_k)^{n+1}} + \frac{\Psi_0(\mu_k)}{(i\mu_k)^{n+2}} \right) \left(\frac{1}{a} + \frac{\delta_k}{k} \right), \quad k \geq k_1.
\end{aligned}$$

Here it is clear that the infinite sum including the factor in front of it satisfies $O(k^{-2})$. Since $\tilde{\Psi}_j := \Psi_j - P_{\Psi_j} \in \mathcal{L}^a$ ($j = 1, 2$) by Lemma 2.1, their integral representations (2.1) lead to $(\tilde{\Psi}_j(\mu_k))_{k=k_1}^\infty \in l_2$, e. g. by applying [14, Lemma 12.2.1] separately for even and odd k . Furthermore,

$$P_{\Psi_j}(\mu_k) = O(\mu_k^{-1}) \cos(\mu_k a) = O(\mu_k^{-1}) \sin(v_k a) = O(k^{-2}). \quad (3.8)$$

Altogether, $(\beta_k)_{k=1}^\infty \in l_2$.

Finally, assume that $i^\ell \Psi_1$ is real on the real axis and put

$$\beta_{k,1} = (-1)^{k-\kappa'-1} \frac{k^{n+1}}{a(i\mu_k)^{n+1}} \Psi_1(\mu_k), \quad k \geq k_1.$$

Then the previous reasoning shows that $(k(\beta_k - \beta_{k,1}))_{k=k_1}^\infty \in l_2$. Since

$$\frac{k}{\mu_k} = k w_k (1 + w_k v_k)^{-1} = \frac{\pi}{a} k (k - \kappa)^{-1} \left(1 + O(k^{-2}) \right),$$

it remains to show that $(k \operatorname{Im} i^\ell \Psi_1(\mu_k))_{k=k_1}^\infty \in l_2$. Recall that $\mu_k = (k - \kappa) \frac{\pi}{a} + v_k$, where $v_k = O(k^{-1})$. Applying [14, Lemma 12.2.1] separately for even and odd k it

follows that

$$\tilde{\Psi}_1(\mu_k) = \tilde{\Psi}_1\left((k - \kappa)\frac{\pi}{a}\right) + k^{-1}g(k),$$

where $(g(k))_{k=1}^{\infty} \in l_2$. With (3.8) it follows that $k \operatorname{Im} i^\ell \Psi_1(\mu_k) = \operatorname{Im} i^\ell g(k) + O(k^{-1})$, and the proof is complete. $\square$

Remark 3.2 The asymptotic expansion (3.1) has an ambiguity when $Q_{n+2} \neq 0$ since the term $i^{n+3} Q_{n+2} (k - \kappa)^{-n-2}$ can be incorporated into $\beta_k (k - \kappa)^{-n-1}$. However, this is not possible if additionally $(k \operatorname{Re}(i^{n+1} \beta_k))_{k=1}^{\infty} \in l_2$ is required. In this case, not only $Q_0, \dots, Q_{n+1}$ but also Q_{n+2} is uniquely determined by the asymptotic expansions (3.1). Hence the additional statement in Theorem 3.1 is most useful in case $\ell = 1$.

Remark 3.3 In the applications in mind, with $n \geq m - 1$, a result on continuous dependence of the eigenvalues on a parameter will be used. First it will be shown here that a simple bound of the imaginary parts of the eigenvalues can be found in terms of the coefficients and the remainder terms in (2.2). Recall that $\Psi \in \mathcal{L}^a$ by Remark 2.5,2. In view of (2.1),

$$\Psi(\mu) = \int_0^a \xi(t) \cos \mu t \, dt + i \int_0^a \zeta(t) \sin \mu t \, dt, \quad \mu \in \mathbb{C},$$

with $\xi, \zeta \in L_2(0, a)$. For $\omega \in \mathcal{A}_{m,n}^a$ with the representation (2.2) define

$$C_\omega := \sum_{j=0}^n (|A_j| + |B_j|) + a^{\frac{1}{2}} (\|\xi\|_{L_2} + \|\zeta\|_{L_2}). \quad (3.9)$$

Proposition 3.4 Let $n \geq m - 1$, $\omega \in \mathcal{A}_{m,n}^a$ and $\mu \in \mathbb{C}$ such that $|\mu| \geq 1$. Then the estimate $|\omega(\mu) - (i\mu)^m \cos \mu a| \leq |\mu|^{m-1} C_\omega \cosh(\operatorname{Im} \mu a)$ holds.

Proof The identity (2.10), the corresponding identity for $\sin \mu a$ and Hölder's inequality for the representation of Ψ show that

$$\begin{aligned} |\omega(\mu) - (i\mu)^m \cos \mu a| &\leq |\mu|^{m-1} \sum_{j=0}^n (|A_j| + |B_j|) \cosh(\operatorname{Im} \mu a) \\ &\quad + |\mu|^{m-1} \cosh(\operatorname{Im} \mu a) a^{\frac{1}{2}} (\|\xi\|_{L_2} + \|\zeta\|_{L_2}) \\ &= C_\omega |\mu|^{m-1} \cosh(\operatorname{Im} \mu a). \end{aligned}$$

$\square$

Corollary 3.5 Let $n \geq m - 1$, $\omega \in \mathcal{A}_{m,n}^a$ and $\mu \in \mathbb{C}$ such that $|\mu| \geq 1$. If $\operatorname{Re} \mu a = k\pi$ with $k \in \mathbb{Z}$ and $|\operatorname{Re} \mu| > C_\omega$ or if $|\operatorname{Im} \mu| > \max\{a^{-1} \coth^{-1} 2, 2C_\omega\}$, then the estimate $|\omega(\mu) - (i\mu)^m \cos \mu a| < |(i\mu)^m \cos \mu a|$ holds.

Proof If the condition on $\operatorname{Re} \mu$ is satisfied, then Proposition 3.4 and (2.10) give

$$|\omega(\mu) - (i\mu)^m \cos \mu a| < |\mu|^m \cosh(\operatorname{Im} \mu a) = |(i\mu)^m \cos(\mu a)|.$$

If the condition on $\operatorname{Im} \mu$ is satisfied, then similarly

$$\begin{aligned} & |\omega(\mu) - (i\mu)^m \cos \mu a| \\ & < \frac{1}{2} |\mu|^m |\coth(\operatorname{Im} \mu a)| |\sinh(\operatorname{Im} \mu a)| < |(i\mu)^m| |\sinh(\operatorname{Im} \mu a)| \\ & \leq |(i\mu)^m \cos(\mu a)|. \end{aligned} \quad \square$$

An immediate consequence of the above corollary is the following rather explicit simple estimate of the imaginary parts of the zeros of functions in $\mathcal{A}_{m,n}^a$ which may be of independent interest. If desired, it could be improved.

Corollary 3.6 *Let $n \geq m - 1$, $\omega \in \mathcal{A}_{m,n}^a$ and $\mu \in \mathbb{C}$ such that $|\mu| \geq 1$ and $|\operatorname{Im} \mu| > \max\{a^{-1} \coth^{-1} 2, 2C_\omega\}$. Then $\omega(\mu) \neq 0$.*

Theorem 3.7 *Let $n \geq m - 1$. Let X be a connected topological space and let $F : X \times \mathbb{C} \rightarrow \mathbb{C}$ be continuous with $F(x, \cdot) \in \mathcal{A}_{m,n}^a$ for all $x \in X$ and such that $\{C_{F(x, \cdot)} : x \in X\}$ is a bounded set. Furthermore, assume that there are two disjoint open subsets Ω_1 and Ω_2 of $\mathbb{C}$ such that for all $x \in X$ all zeros of $F(x, \cdot)$ belong to $\Omega_1 \cup \Omega_2$. For $j = 1, 2$ and $x \in X$ let $N_j(x)$ be the number of zeros of $F(x, \cdot)$ in Ω_j , counted with multiplicity. Then N_1 and N_2 are constant functions.*

Proof Choose real numbers $c_1 > \sup\{C_{F(x, \cdot)} : x \in X\}$, $c_2 > \max\{a^{-1} \coth^{-1} 2, 2c_1\}$, and a positive integer $k_1 > a\pi^{-1}c_1$. Let R_0 be the rectangle with vertices $\pm k_1 a^{-1} \pm i c_2$, for integers $k \geq k_1$ let R_k^+ be the rectangle with vertices $ka^{-1} \pm i c_2$ and $(k+1)a^{-1} \pm i c_2$, and R_k^- the rectangle with vertices $-ka^{-1} \pm i c_2$ and $-(k+1)a^{-1} \pm i c_2$. By Corollary 3.5, the estimate $|F(x, z)(\mu) - (i\mu)^m \cos \mu a| < |(i\mu)^m \cos \mu a|$ holds for all $x \in X$ and all z on the boundaries of R_0 and R_k^+ , R_k^- ($k \geq k_1$). This estimate and Corollary 3.6 imply that all zeros of all $F(x, \cdot)$ ($x \in X$) lie in the interiors of R_0 and the R_k^+ , R_k^- , and by Rouché's theorem, the numbers of the zeros inside each of these rectangles, counted with multiplicity, is independent of x . Applying [13, Corollary 5] to the restriction of F to the interior of each of these rectangles completes the proof. $\square$

Remark 3.8 The above theorem is of course only useful if one of the functions N_1 or N_2 has a finite value. But it may be worthwhile to mention explicitly that the numbers of zeros of $F(x, \cdot)$ inside the intersections of Ω_j with any of the rectangles occurring in the proof of Theorem 3.7 is finite and independent of x .

4 The Coefficients in the Asymptotics of the Sequences in $\mathfrak{J}_{m,n}^a$

Recall the following well-known formula. Let q be a positive integer and let $\sum_{j=0}^{\infty} c_{p,j}$, $p = 1, \dots, q$, be q absolutely convergent series of complex numbers. Then

$$\prod_{p=1}^q \left(\sum_{j=0}^{\infty} c_{p,j} \right) = \sum_{j=0}^{\infty} \sum_{\substack{l \in \mathbb{N}_0^q \\ |l|=j}} \prod_{p=1}^q c_{p,l_p}, \quad (4.1)$$

where $|l| = l_1 + \dots + l_p$ for $l = (l_1, \dots, l_q) \in \mathbb{N}_0^q$.

Writing (3.5) as

$$f(z, w) = (1 + \psi_s(z, w)) \sin za - \psi_c(z, w) \cos za \quad (4.2)$$

it follows that

$$\begin{aligned} \psi_s(z, w) &= \sum_{j=0}^n i^{-j-1} A_j w^{j+1} \sum_{k=0}^{\infty} \binom{-j-1}{k} w^k z^k \\ &= \sum_{r=1}^{\infty} \sum_{k=\max\{r-n-1, 0\}}^{r-1} \binom{k-r}{k} i^{k-r} A_{r-k-1} w^r z^k, \end{aligned}$$

and therefore

$$\psi_s(h(w), w) = \sum_{r=1}^{n+1} \sum_{k=0}^{r-1} \binom{k-r}{k} i^{k-r} A_{r-k-1} w^r (h(w))^k + O(w^{n+3}).$$

An application of (4.1) gives

$$(h(w))^k = \left(\sum_{j=1}^{\infty} i^{j+1} a_j w^j \right)^k = \sum_{j=k}^{\infty} i^{j+k} w^j g_{k,j},$$

where

$$g_{k,j} = \sum_{\substack{l \in \mathbb{N}^k \\ |l|=j}} \prod_{p=1}^k a_{l_p} \text{ when } k > 0, \quad g_{0,0} = 1, \quad g_{0,j} = 0 \text{ when } j > 0. \quad (4.3)$$

Also observe that

$$g_{1,j} = a_j \text{ for } j \geq 1. \quad (4.4)$$

Hence

$$\begin{aligned} \psi_s(h(w), w) &= \sum_{r=1}^{n+1} i^{-r} w^r \sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} A_{r-j-k-1} g_{k,j} \\ &\quad + O(w^{n+2}). \end{aligned}$$

In the same way one obtains

$$\begin{aligned} \psi_c(h(w), w) &= \sum_{r=1}^{n+1} i^{-r+1} w^r \sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} B_{r-j-k-1} g_{k,j} \\ &\quad + O(w^{n+2}). \end{aligned}$$

Furthermore,

$$\begin{aligned} \sin(h(w)a) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} (h(w)a)^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{a^{2k+1}}{(2k+1)!} \sum_{j=2k+1}^{\infty} i^{j+2k+1} w^j g_{2k+1,j} \\ &= \sum_{j=1}^{n+1} i^{j+1} w^j \sum_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{a^{2k+1}}{(2k+1)!} g_{2k+1,j} + O(w^{n+2}), \end{aligned}$$

and similarly

$$\cos(h(w)a) = \sum_{j=0}^{n+1} i^j w^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{a^{2k}}{(2k)!} g_{2k,j} + O(w^{n+2}).$$

Substituting these representations into (4.2) and observing that $f(h(w), w) = 0$ it is seen that the coefficient of w^s for $s = 1, \dots, n+1$ satisfies the identity

$$\begin{aligned} 0 &= i^{s+1} \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \frac{a^{2k+1}}{(2k+1)!} g_{2k+1,s} \\ &\quad + \sum_{r=1}^{s-1} \left[i^{-r} \sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} A_{r-j-k-1} g_{k,j} \right] \times \\ &\quad \times \left[i^{s-r+1} \sum_{k=0}^{\lfloor \frac{s-r-1}{2} \rfloor} \frac{a^{2k+1}}{(2k+1)!} g_{2k+1,s-r} \right] \end{aligned}$$

$$\begin{aligned}
& - \sum_{r=1}^s \left[i^{-r+1} \sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} B_{r-j-k-1} g_{k,j} \right] \times \\
& \times \left[i^{s-r} \sum_{k=0}^{\lfloor \frac{s-r}{2} \rfloor} \frac{a^{2k}}{(2k)!} g_{2k, s-r} \right]. \quad (4.5)
\end{aligned}$$

Moving the term for $k = 0$ in the first sum on the right hand side to the left hand side, dividing both sides by $i^{s-1}a$ and observing (4.4) gives

$$\begin{aligned}
a_s &= - \sum_{k=1}^{\lfloor \frac{s-1}{2} \rfloor} \frac{a^{2k}}{(2k+1)!} g_{2k+1, s} \\
& - \sum_{r=1}^{s-1} (-1)^r \left[\sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} A_{r-j-k-1} g_{k,j} \right] \times \\
& \times \left[\sum_{k=0}^{\lfloor \frac{s-r-1}{2} \rfloor} \frac{a^{2k}}{(2k+1)!} g_{2k+1, s-r} \right] \\
& + \sum_{r=1}^s (-1)^r \left[\sum_{j=0}^{r-1} \sum_{k=0}^{\min\{j, r-j-1\}} \binom{k-r+j}{k} (-1)^{k+j} B_{r-j-k-1} g_{k,j} \right] \times \\
& \times \left[\sum_{k=0}^{\lfloor \frac{s-r}{2} \rfloor} \frac{a^{2k-1}}{(2k)!} g_{2k, s-r} \right]. \quad (4.6)
\end{aligned}$$

Note that by (4.3) the terms $g_{k,j}$ depend on a_p for exactly $p \leq j - k + 1$ when $k > 0$. Hence the right hand side of (4.6) depends at most on the a_p with $p < s$. Consequently, the numbers Q_s can be found as follows.

Proposition 4.1 *The numbers Q_s , $s = 1, \dots, n+2$ in (3.1) can be evaluated recursively by (3.7) and (4.6).*

Remark 4.2 It is easy to see that $a_1 = -B_0 a^{-1}$ and that, for $s = 2, \dots, n+2$, the A_l and B_l with the highest indices on the right hand side of (4.6) occur when r has the largest possible value and $j = k = 0$ in the corresponding sums. This shows for $s = 2, \dots, n+2$ that

$$aa_s + (-1)^{s-1} aa_1 A_{s-2} + (-1)^{s-1} B_{s-1}$$

only depends on a_j with $j \leq s-1$, A_j with $j \leq s-3$ and B_j with $j \leq s-2$. In particular, $a_2 = (B_1 - A_0 B_0) a^{-1}$. Also recall from Remark 3.2 that Q_{n+2} is not unique without an additional assumption on the sequence $(\beta_k)_{k=1}^\infty$.

A tedious but straightforward calculation gives some more terms under the assumption that $Q_2 = 0$. We summarize these results below.

Proposition 4.3 1. The numbers Q_j for $j = 1, 2$ in Theorem 3.1 are

$$Q_1 = -\frac{B_0}{\pi}, \quad Q_2 = (B_1 - A_0 B_0) \frac{a}{\pi^2}.$$

2. If $Q_2 = 0$, then the numbers Q_j for $j = 3, 4, 5$ are

$$\begin{aligned} Q_3 &= \frac{a^2}{\pi^3} \left(D_1 - \frac{1}{3} B_0^3 \right) - \frac{a}{\pi^3} B_0^2, \quad Q_4 = \frac{a^3}{\pi^4} (D_2 + A_0 D_1), \\ Q_5 &= \frac{a^4}{\pi^5} q_{5,4} + \frac{a^3}{\pi^5} q_{5,3} + \frac{a^2}{\pi^5} q_{5,2}, \end{aligned}$$

where

$$\begin{aligned} D_1 &= A_1 B_0 - B_2, \quad D_2 = B_3 - A_2 B_0, \\ q_{5,4} &= -B_4 + A_0 D_2 + \left(A_0^2 + B_0^2 - A_1 \right) D_1 + A_3 B_0 - \frac{1}{5} B_0^5, \\ q_{5,3} &= 4 B_0 D_1 - \frac{4}{3} B_0^4, \quad q_{5,2} = -2 B_0^3. \end{aligned}$$

5 The Inverse Problem

Theorem 5.1 Let $Z = \mathbb{Z}$ or $Z = \mathbb{Z} \setminus \{0\}$, let κ' be a nonnegative integer and put $\kappa = \kappa' + \frac{1}{2}$. Let $m = 2\kappa'$ if $Z = \mathbb{Z} \setminus \{0\}$ and let $m = 2\kappa' + 1$ if $Z = \mathbb{Z}$. Let $n \geq m - 1$ and let $(\mu_k)_{k \in \mathbb{Z}}$ be a sequence of complex numbers satisfying the following conditions:

- (i) there is a nonnegative integer k_0 such that μ_k is pure imaginary for $|k| \leq k_0$;
- (ii) for $k > k_0$ it is true that $\operatorname{Re} \mu_k > 0$ and $\mu_{-k} = -\overline{\mu_k}$;
- (iii) as $k \rightarrow \infty$, the sequence admits the asymptotic representation

$$\mu_k = \frac{\pi}{a} (k - \kappa) + \sum_{j=1}^{n+2} i^{j+1} \frac{Q_j}{(k - \kappa)^j} + \frac{\beta_k}{(k - \kappa)^{n+1}}, \quad (5.1)$$

where $a > 0$, $Q_j \in \mathbb{R}$ for $j = 1, \dots, n+2$, and where $(\beta_k)_{k=1}^{\infty} \in l_2$.

Then there is a unique entire function $\omega \in \mathcal{A}_{m,n}^a$ such that $(\mu_k)_{k \in \mathbb{Z}}$ is a representation of $\mathcal{Z}(\omega)$. Furthermore, there is a nonzero real number c such that

$$\omega(\mu) = c(i\mu)^d \lim_{j \rightarrow \infty} \prod_{\substack{k \in \mathbb{Z}, |k| \leq j \\ \mu_k \neq 0}} \left(1 - \frac{\mu}{\mu_k} \right), \quad (5.2)$$

where d is the number of occurrences of the value 0 in the sequence $(\mu_k)_{k \in \mathbb{Z}}$.

If $(k \operatorname{Im}(i^n \beta_k))_{k=1}^{\infty} \in l_2$, then the function Ψ_1 in the representation (2.9) of ω can

be chosen in such a way that Ψ_1 is an odd function which is real on the imaginary axis.

Proof Let $\tilde{k}_0 = \max\{0, k_0 - \kappa'\}$. For $k \in \mathbb{Z} \setminus \{0\}$ set

$$\tilde{\mu}_k = \begin{cases} \mu_{k+\kappa'} & \text{if } k > 0, \\ \mu_{k-\kappa'} & \text{if } k < 0. \end{cases}$$

Then $\tilde{\mu}_k$ is pure imaginary when $0 < |k| \leq \tilde{k}_0$. When $k > \tilde{k}_0$ then $\operatorname{Re} \tilde{\mu}_k = \operatorname{Re} \mu_{k+\kappa'} > 0$ and

$$\tilde{\mu}_{-k} = \mu_{-k-\kappa'} = -\overline{\mu_{k+\kappa'}} = -\overline{\tilde{\mu}_k}.$$

In the case $k > 0$,

$$\begin{aligned} \tilde{\mu}_k &= \frac{\pi}{a} \left(k - \frac{1}{2} \right) + \sum_{j=1}^{n+2} \frac{i^{j+1} Q_j}{(k - \frac{1}{2})^j} + \frac{\beta_{k+\kappa'}}{(k - \frac{1}{2})^{n+1}} \\ &= \frac{\pi}{a} \left(k - \frac{1}{2} \right) + \sum_{j=1}^{n+2} \frac{i^{j+1} Q_j}{(k - \frac{1}{2})^j} + (-1)^n \frac{\tilde{\beta}_k}{k^{n+1}}, \end{aligned}$$

where clearly $(\tilde{\beta}_k)_{k=1}^\infty \in l_2$.

Defining the numbers a_j , $j = 1, \dots, n+2$, by (3.7), Remark 4.2 shows that there are real numbers A_j , $j = 0, \dots, n$ and B_j , $j = 0, \dots, n+1$ such that (4.6) holds for $s = 1, \dots, n+2$. Observe that these numbers are not unique. But if $B_0, A_0, B_1, \dots, A_{s-3}, B_{s-2}$ are given and A_{s-2} is arbitrarily chosen, then (4.6) for s is satisfied for a unique B_{s-1} . Let

$$\widehat{\omega}_0(\mu) = \left((i\mu)^{n+1} + \sum_{j=0}^n A_j (i\mu)^{n-j} \right) \cos \mu a + i \left(\sum_{j=0}^{n+1} B_j (i\mu)^{n-j} \right) \sin \mu a.$$

Clearly, $\widehat{\omega}_0$ is an entire function which is real on the imaginary axis. There are real polynomials ξ and ζ such that, for $k = 0, \dots, n+1$,

$$\begin{aligned} i^k \int_0^a t^k \xi(t) dt &= -\widehat{\omega}_0^{(k)}(0) \text{ if } k \text{ is even,} \\ i^k \int_0^a t^k \zeta(t) dt &= -\widehat{\omega}_0^{(k)}(0) \text{ if } k \text{ is odd.} \end{aligned}$$

Then

$$\widehat{\psi}(\mu) := \int_0^a \xi(t) \cos \mu t dt + i \int_0^a \zeta(t) \sin \mu t dt$$

defines an entire function $\widehat{\psi} \in \mathcal{L}^a$ which is real on the imaginary axis. Since

$$\widehat{\psi}^{(k)}(\mu) = \int_0^a t^k \xi(t) \cos^{(k)} \mu t dt + i \int_0^a t^k \zeta(t) \sin^{(k)} \mu t dt$$

for $k \in \mathbb{N}$, it follows that

$$\widehat{\omega}_0^{(k)}(0) + \widehat{\psi}^{(k)}(0) = 0, \quad k = 0, \dots, n+1.$$

Let

$$\widehat{\omega}(\mu) := (i\mu)^{-(n+1)} (\widehat{\omega}_0(\mu) + \widehat{\psi}(\mu)). \quad (5.3)$$

Then $\widehat{\omega}$ is an entire function which belongs to $\mathcal{A}_{0,n}^a$. By Theorem 3.1 with $m = 0$ the zeros $\widehat{\mu}_k$, $k \in \mathbb{Z} \setminus \{0\}$, of $\widehat{\omega}$ have a representation like $\widetilde{\mu}_k$ with the same Q_j , $j = 1, \dots, n+2$, since (4.6) is satisfied. Therefore

$$\widehat{\mu}_k = \frac{\pi}{a} \left(k - \frac{1}{2} \right) + \sum_{j=1}^{n+2} \frac{i^{j+1} Q_j}{(k - \frac{1}{2})^j} + \frac{\widehat{\beta}_k}{k^{n+1}} \text{ for } k > 0,$$

where $(\widehat{\beta}_k)_{k=1}^\infty \in l_2$.

Choosing $k_1 > 0$ such that $\operatorname{Re} \widetilde{\mu}_k > 0$ and $\operatorname{Re} \widehat{\mu}_k > 0$ for $k \geq k_1$ it follows that

$$\widetilde{\mu}_k = \widehat{\mu}_k + \frac{\gamma_k}{(\widehat{\mu}_k)^{n+1}} \text{ for } k \geq k_1 \quad (5.4)$$

with $(\gamma_k)_{k=k_1}^\infty \in l_2$. Due to $\widetilde{\mu}_{-k} = -\overline{\widetilde{\mu}_k}$ and $\widehat{\mu}_{-k} = -\overline{\widehat{\mu}_k}$ for $k \geq k_1$, (5.4) extends to $k \leq -k_1$, with $(\gamma_k)_{k=-\infty}^{-k_1} \in l_2$. In view of Proposition 2.11 the function $\widehat{\omega}$ is a sine type function of exponential type a . Hence [14, Lemma 11.3.15 and Remark 11.3.16] shows that there is a sine type function $\widetilde{\omega}$ of exponential type a such that $(\widetilde{\mu}_k)_{k \in \mathbb{Z} \setminus \{0\}}$ is a sequence of the zeros of $\widetilde{\omega}$ and that there are complex numbers $\widetilde{C}_1, \dots, \widetilde{C}_{n+1}$ and a function $\widetilde{\psi} \in \mathcal{L}^a$ such that

$$\widetilde{\omega}(\mu) = (1 + \widetilde{C}_1 \mu^{-1} + \dots + \widetilde{C}_{n+1} \mu^{-n-1}) \widehat{\omega}(\mu) + \widetilde{\psi}(\mu) \mu^{-n-1}, \quad \mu \in \mathbb{C} \setminus \{0\} \quad (5.5)$$

It is clear that with $\widehat{\omega}$ also $\widetilde{\omega}$ has a representation of the form (2.2) with $m = 0$. The function $\mu \mapsto \widetilde{\omega}(-\overline{\mu})$ is a sine type function of exponential type a with the same zeros as $\widetilde{\omega}$. But up to a constant factor, sine type functions are uniquely determined by their zeros, see [14, Lemma 11.2.29]. From $\widehat{\omega} \in \mathcal{A}_{0,n}^a$ and from (5.5) one finds

$$\lim_{k \rightarrow \pm\infty} \widetilde{\omega}(2\pi k a^{-1}) = 1 = \lim_{k \rightarrow \pm\infty} \widetilde{\omega}(2\pi k a^{-1}),$$

which implies that this constant factor is 1, and hence $\widetilde{\omega}(\mu) = \widetilde{\omega}(-\mu)$. Therefore $\widetilde{\omega}$ is real on the imaginary axis, and it follows that $\widetilde{\omega} \in \mathcal{A}_{0,n}^a$.

Let

$$P(\mu) = \prod_{\substack{k \in Z \\ |k| \leq \kappa'}} (i(\mu - \mu_k)).$$

For $k \in Z$ with $|k| < k_0$ the number μ_k is pure imaginary, and therefore each of the factors $i(\mu - \mu_k)$ in P with $|k| < k_0$, if any, is real for pure imaginary μ . The remaining factors, if any, in P occur in pairs with the indices k and $-k$ for $k_0 \leq k \leq \kappa'$. But for these k and $\mu \in i\mathbb{R}$ we have

$$i\mu - i\mu_{-k} = i\mu + i\overline{\mu_k} = \overline{i\mu - i\mu_k}.$$

Altogether it follows that P is real on the imaginary axis.

The function P is a polynomial of degree m , and

$$\omega := P\tilde{\omega} \quad (5.6)$$

is an entire function whose zeros coincide with the sequence $(\mu_k)_{k \in Z}$. From $\tilde{\omega} \in \mathcal{A}_{0,n}^a$, (5.5) and (5.6) it follows that ω has the representation (2.2), and ω is real on the imaginary axis since P and $\tilde{\omega}$ have this property. Thus it has been shown that $\omega \in \mathcal{A}_{m,n}^a$.

Now let $\omega_0 \in \mathcal{A}_{m,n}^a$ be such that $(\mu_k)_{k \in Z}$ is a representation of $\mathcal{Z}(\omega_0)$. By Proposition 2.12, $\tilde{\omega}_0 = \frac{\omega_0}{P}$ belongs to $\mathcal{A}_{0,n}^a$, which is a sine type function of exponential type a by Proposition 2.11, and $(\tilde{\mu}_k)_{k \in Z \setminus \{0\}}$ is the sequence of its zeros. Again, because of the normalization, this function is uniquely determined by its zeros, and therefore $\tilde{\omega}_0 = \tilde{\omega}$, which leads to $\omega_0 = \omega$. Thus it has been shown that ω is unique.

Since $\tilde{\omega}$ is a sine type function, it follows from [14, Lemma 11.2.29] that there is a complex number $\tilde{c}$ such that

$$\tilde{\omega}(\mu) = \tilde{c}\mu^{\tilde{d}} \lim_{j \rightarrow \infty} \prod_{\kappa' < |k| \leq j, \mu_k \neq 0} \left(1 - \frac{\mu}{\mu_k}\right), \quad (5.7)$$

where $\tilde{d}$ is the number of occurrences of the value 0 in the sequence $(\mu_k)_{|k| > \kappa'}$. Then (5.2) with a nonzero complex number c follows from (5.6), (5.7), and the definitions of P and $\tilde{\mu}_k$. From the fact that $(i\mu)^{\tilde{d}}$, $1 - \mu\mu_k^{-1}$ for $|k| \leq k_0$ with $\mu_k \neq 0$, $(1 - \mu\mu_k^{-1})(1 - \mu\mu_{-k}^{-1})$ for $k > k_0$, and $\omega(\mu)$ are real for pure imaginary μ it follows that c is real.

Now assume that $(k \operatorname{Im}(i^n \beta_k))_{k=1}^\infty \in l_2$. There is a number $k_1 \geq \max\{k_0, \kappa'\} + 1$ such that

$$v_k := \frac{\pi}{a}(k - \kappa) + \sum_{j=1}^{n+2} \frac{i^{j+1} Q_j}{(k - \kappa)^j} + \frac{i^{-n+1} \operatorname{Im}(i^n \beta_k)}{(k - \kappa)^{n+1}}, \quad k \geq k_1,$$

satisfies $\operatorname{Re} v_k > 0$ for $k \geq k_1$. For $k \leq -k_1$ put $v_k = -\overline{v_{-k}}$, and for $k \in Z$ with $|k| < k_1$ put $v_k = \mu_k$. The sequence $(v_k)_{k \in Z}$ satisfies the assumptions (i)–(iii), with

possibly larger k_0 and with n replaced by $n + 1$. But by what has already been shown, it follows that there is $\chi \in \mathcal{A}_{m,n+1}^a$ whose sequence of zeros is $(v_k)_{k \in \mathbb{Z}}$. Putting

$$\beta_{k,R} := \operatorname{Re}(i^n \beta_k), \quad k \geq k_1,$$

it follows for $k \geq k_1$ that

$$\begin{aligned} \mu_k - v_k &= \frac{i^{-n}}{(k - \kappa)^{n+1}} \beta_{k,R} \\ &= \varepsilon v_k^{-n-1} \delta_k, \end{aligned}$$

where

$$\varepsilon = i^{-n} \quad \text{and} \quad \delta_k = \left(\frac{v_k}{k - \kappa} \right)^{n+1} \beta_{k,R}.$$

Since the polynomial P , defined above, is the same for ω and χ , $\frac{\omega}{\chi}$ can be written as a quotient of two sine type functions. Clearly, $(\varepsilon \delta_k)_{|k| > \kappa'} \in l_2$, and in the proof of [14, Lemma 11.3.15, (11.3.32)] it was shown that, for some $H > 0$ and all $\mu \in \mathbb{C}$ with $\operatorname{Im} \mu \geq H$,

$$\log \frac{\omega(\mu)}{\chi(\mu)} = - \sum_{j=1}^{n+1} \widehat{C}_j (i\mu)^{-j} + q(\mu) (i\mu)^{-n-1}$$

for suitable branches of the logarithm, where

$$\begin{aligned} \widehat{C}_{j+1} &= i^{j+1} \lim_{l \rightarrow \infty} \sum_{|k|=k_1}^l \int_{v_k}^{\mu_k} \tau^j d\tau, \quad j = 0, \dots, n, \\ q(\mu) &= i^{n+1} \lim_{l \rightarrow \infty} \sum_{|k|=k_1}^l \int_{v_k}^{\mu_k} \frac{\tau^{n+1}}{\tau - \mu} d\tau. \end{aligned}$$

Here it has to be observed that $\omega(iy)(\chi(iy))^{-1} \rightarrow 1$ as $|y| \rightarrow \infty$, $y \in \mathbb{R}$, due to the representations (2.2) for ω and χ and that n in [14, Lemma 11.3.15)] has to be replaced by $n + 1$. Therefore

$$\omega(\mu) = \widetilde{\chi}(\mu) \exp \left(q(\mu) (i\mu)^{-n-1} \right),$$

where

$$\widetilde{\chi}(\mu) = \chi(\mu) \exp \left(- \sum_{j=1}^{n+1} \widehat{C}_j (i\mu)^{-j} \right).$$

Since $\chi \in \mathcal{A}_{m,n+1}^a$, an expansion of the exponential function into a power series in $(i\mu)^{-1}$ gives

$$\begin{aligned}\tilde{\chi}(\mu) &= \left((i\mu)^m + \sum_{j=0}^{n+1} \tilde{A}_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^{n+1} \tilde{B}_j (i\mu)^{m-j-1} \right) \sin \mu a \\ &\quad + \frac{\tilde{\Psi}(\mu)}{(i\mu)^{n-m+2}}\end{aligned}$$

with $\tilde{A}_j, \tilde{B}_j \in \mathbb{C}$ and $\tilde{\Psi} \in \mathcal{L}_0^a$.

In the proof of [14, Lemma 11.3.15, pp.342–343] the function q was written as $q = q_1 + q_2$, and the estimates there show that q_2 is analytic and bounded on each set $\mathbb{C} \setminus \Lambda_\delta$ with $\Lambda_\delta = \{\mu \in \mathbb{C} : \exists k \in \mathbb{Z} \, |\mu - \nu_k| < \delta\}$. Also, the estimate [14, (11.3.34)] holds for all $\mu \in \mathbb{C}$ which have a uniformly positive distance from all line segments from ν_k to μ_k , $|k| \geq k_1$. Together with [14, (11.2.28)] one can conclude that for each $\eta > 0$ there is $R_\eta > 0$ such that q_1 is bounded on

$$\Omega_\eta = \{\mu \in \mathbb{C} : |\mu| > R_\eta \text{ and } |\mu - \nu_k| > \eta, \, |k| \geq k_1\}.$$

In [14, p. 343] it was also shown that $q \in L_2(iy + \mathbb{R})$ for sufficiently large $y > 0$, say $y > R_\eta$. The proof there easily extends to the case $y < -R_\eta$.

The properties of q stated above show that

$$\exp\left(q(\mu)(i\mu)^{-n-1}\right) = 1 + q(\mu)(i\mu)^{-n-1} + q(\mu)O(\mu^{-2n-2}), \quad \mu \in \Omega_\eta.$$

Therefore

$$\begin{aligned}\omega(\mu) &= \left((i\mu)^m + \sum_{j=0}^n \tilde{A}_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^n \tilde{B}_j (i\mu)^{m-j-1} \right) \sin \mu a \\ &\quad + \frac{\Psi^\dagger(\mu)}{(i\mu)^{n-m+1}},\end{aligned}$$

where

$$\begin{aligned}\Psi^\dagger(\mu) &= \tilde{A}_{n+1}(i\mu)^{-1} \cos \mu a + i \tilde{B}_{n+1}(i\mu)^{-1} \sin \mu a \\ &\quad + (i\mu)^{-m} \tilde{\chi}(\mu) \left[q(\mu) + q(\mu)O(\mu^{-n-1}) \right] + (i\mu)^{-1} \tilde{\Psi}(\mu), \quad \mu \in \Omega_\eta.\end{aligned}$$

For any $0 < \eta < \frac{\pi}{2a}$ the number R_η can be chosen in such a way that the disks $\{\mu \in \mathbb{C} : |\mu| \leq R_\eta\}$ and $\{\mu \in \mathbb{C} : |\mu - \nu_k| \leq \eta\}$, $|\nu_k| \geq R_\eta$, are mutually disjoint and that $|\nu_k| + \eta \leq R_\eta$ when $|\nu_k| < R_\eta$. Since $\Psi^\dagger$ and $\tilde{\Psi}$ are analytic on $\mathbb{C} \setminus \{0\}$, the third summand of $\Psi^\dagger$ has an analytic extension to $\mathbb{C} \setminus \{0\}$. Due to $\tilde{\chi} \in \mathcal{L}_0^a$ and the boundedness of q on Ω_ε , it follows from the maximum principle that this third summand satisfies the estimate $O(e^{|\operatorname{Im} \lambda|a})$ as $|\lambda| \rightarrow \infty$. Furthermore, Lemma 2.1

then shows that this third summand is also square integrable along any line $iy + \mathbb{R}$ with $y > R_\eta$. Therefore $\Psi^\dagger \in \tilde{\mathcal{L}}^a$. Since 0 is a pole or a removable singularity of $\Psi^\dagger$, it follows that $\Psi^\dagger \in \mathcal{L}_0^a$.

Clearly, $2k\frac{\pi}{a} \in \Omega_\varepsilon$ for sufficiently large $k \in \mathbb{N}$, and therefore $\Psi^\dagger(2k\frac{\pi}{a}) \rightarrow 0$ as $k \rightarrow \infty$. Using the corresponding notations $\chi_j^\dagger$ as in (2.4) and (2.5) and with the representation (2.2) for ω it follows as in the proof of Proposition 2.4 that $\chi_1(2k\frac{\pi}{a}) - \chi_1^\dagger(2k\frac{\pi}{a}) \rightarrow 0$ as $k \rightarrow \infty$ and therefore $\chi_1^\dagger = \chi_1$. Repeating this procedure now with a sufficiently small $\delta > 0$ for the points $2k\frac{\pi}{a} + \delta$ shows that $\chi_1^\dagger \sin \delta a = \chi_2 \sin \delta a$. As in the proof of Proposition 2.4 one can now conclude that $A_j^\dagger = A_j \in \mathbb{R}$ and $B_j^\dagger = A_j \in \mathbb{R}$ for $j = 0, \dots, n$, and therefore $\Psi^\dagger = \Psi$.

Defining

$$\begin{aligned}\Psi_1(\mu) &:= \frac{1}{2}(\Psi(\mu) - \Psi(-\mu)), \\ \Psi_0(\mu) &:= \frac{i\mu}{2}(\Psi(\mu) + \Psi(-\mu)) - i\tilde{B}_{n+1} \sin \mu a,\end{aligned}$$

gives a formal representation (2.9) with $\Psi_1 \in \mathcal{L}_0^a$. Clearly, Ψ_1 is an odd function which is real on the imaginary axis.

To obtain the representation (2.9) for ω it remains to be shown that $\Psi_0 \in \mathcal{L}_0^a$. Note that Proposition 2.8 then would give $B_{n+1} \in \mathbb{R}$ and that B_{n+1} is uniquely determined. Write

$$\Psi_0(\mu) = \Psi_{0,1}(\mu) + \Psi_{0,2}(\mu) + \Psi_{0,3}(\mu) + \Psi_{0,4}(\mu), \quad \mu \in \Omega_\eta,$$

where

$$\begin{aligned}\Psi_{0,1}(\mu) &= \frac{1}{2}i\mu \cos \mu a [q(\mu) + q(-\mu)], \\ \Psi_{0,2}(\mu) &= \frac{1}{2}[(i\mu)^{-m+1} \tilde{\chi}(\mu) - i\mu \cos \mu a] q(\mu) \\ &\quad - \frac{1}{2}[(-i\mu)^{-m+1} \tilde{\chi}(-\mu) + i\mu \cos \mu a] q(-\mu), \\ \Psi_{0,3}(\mu) &= (i\mu)^{-m} \tilde{\chi}(\mu) q(\mu) O(\mu^{-n}) + (i\mu)^{-m} \tilde{\chi}(-\mu) q(-\mu) O(\mu^{-n}), \\ \Psi_{0,4}(\mu) &= \frac{1}{2}[\tilde{\Psi}(\mu) - \tilde{\Psi}(-\mu)].\end{aligned}$$

Clearly, $\Psi_{0,4} \in \mathcal{L}_0^a$ and $\Psi_{0,j}(\mu) = \max\{|q(\mu)|, |q(-\mu)|\} O(e^{|\operatorname{Im} \mu a|})$ for $j = 2, 3$ and $\mu \in \Omega_\eta$.

It remains to investigate $\Psi_{0,1}$. For $|k| \geq k_1$ and $\mu \in \Omega_\eta$ consider

$$\bar{\varepsilon} \int_{v_k}^{\mu_k} \frac{\mu \tau^{n+1}}{\tau - \mu} d\tau = q_{k,1} + q_{k,2}(\mu),$$

where

$$q_{k,1} := -\bar{\varepsilon} \int_{v_k}^{\mu_k} \tau^{n+1} d\tau,$$

$$q_{k,2}(\mu) := \bar{\varepsilon} \int_{v_k}^{\mu_k} \frac{\tau^{n+2}}{\tau - \mu} d\tau.$$

Then

$$\begin{aligned} \widehat{\Psi}_{0,1}(\mu) &:= \frac{1}{2} i [\mu q(\mu) - (-\mu) q(-\mu)] \\ &= -\frac{1}{2} \lim_{l \rightarrow \infty} \sum_{|k|=k_1}^l [(q_{k,1} + q_{k,2}(\mu)) - (q_{k,1} + q_{k,2}(-\mu))] \\ &= -\frac{1}{2} \sum_{k=k_1}^{\infty} [q_{k,2}(\mu) + q_{-k,2}(\mu) - q_{k,2}(-\mu) - q_{-k,2}(-\mu)]. \end{aligned}$$

It is clear that η and R_η can be chosen in such a way that $|v_k - \operatorname{Re} v_k| < \frac{\eta}{2}$ when $|v_k| \geq R_\eta$. For $k \geq k_1$ and $\mu \in \Omega_\eta$ one calculates

$$\begin{aligned} q_{k,2}(\mu) &= \bar{\varepsilon} \int_0^1 \frac{(v_k + t(\mu_k - v_k))^{n+2}}{v_k + t(\mu_k - v_k) - \mu} (\mu_k - v_k) dt \\ &= v_k \delta_k \int_0^1 \frac{(g_k(t))^{n+2}}{v_k g_k(t) - \mu} dt, \end{aligned}$$

where

$$g_k(t) := 1 + t \varepsilon v_k^{-n-2} \delta_k.$$

Observe that

$$g_k(t) = 1 + O(k^{-2}) \beta_{k,R}$$

for $k \geq k_1$ and uniformly in $t \in [0, 1]$. Furthermore,

$$\operatorname{Im} v_k = O(k^{-2}), \quad k \geq k_1.$$

The straightforward estimates

$$\begin{aligned} \frac{v_k \delta_k (g_k(t))^{n+2}}{v_k g_k(t) - \mu} - \frac{\operatorname{Re} v_k \delta_k (g_k(t))^{n+2}}{v_k g_k(t) - \mu} &= \frac{O(k^{-2}) \beta_{k,R}}{\operatorname{Re} v_k - \mu}, \\ \frac{\operatorname{Re} v_k \delta_k (g_k(t))^{n+2}}{v_k g_k(t) - \mu} - \frac{\operatorname{Re} v_k \delta_k}{v_k g_k(t) - \mu} &= \frac{O(k^{-1}) \beta_{k,R}}{\operatorname{Re} v_k - \mu}, \end{aligned}$$

$$\begin{aligned} \frac{\operatorname{Re} v_k \delta_k}{v_k g_k(t) - \mu} - \frac{\operatorname{Re} v_k \delta_k}{\operatorname{Re} v_k - \mu} &= \frac{\operatorname{Re} v_k \delta_k (\operatorname{Re} v_k - v_k g_k(t))}{(v_k g_k(t) - \mu)(\operatorname{Re} v_k - \mu)} \\ &= \frac{O(k^{-2}) \beta_{k,R}}{\operatorname{Re} v_k - \mu} \end{aligned}$$

show that

$$\frac{v_k \delta_k (g_k(t))^{n+2}}{v_k g_k(t) - \mu} - \frac{\operatorname{Re} v_k \delta_k}{\operatorname{Re} v_k - \mu} = \frac{O(k^{-1}) \beta_{k,R}}{\operatorname{Re} v_k - \mu}.$$

With

$$\begin{aligned} \delta_k - \bar{\delta}_k &= \frac{\beta_{k,R}}{(k - \kappa)^{n+1}} \left(v_k^{n+1} - (\bar{v}_k)^{n+1} \right) \\ &= \left(\frac{\operatorname{Re} v_k}{k - \kappa} \right)^{n+1} \beta_{k,R} \left(\left(1 + i \frac{\operatorname{Im} v_k}{\operatorname{Re} v_k} \right)^{n+1} - \left(1 - i \frac{\operatorname{Im} v_k}{\operatorname{Re} v_k} \right)^{n+1} \right) \\ &= \left(\frac{\operatorname{Re} v_k}{k - \kappa} \right)^{n+1} \beta_{k,R} \left(\left(1 + O(k^{-3}) \right)^{n+1} - \left(1 + O(k^{-3}) \right)^{n+1} \right) \\ &= O(k^{-3}) \beta_{k,R} \end{aligned}$$

it finally follows that

$$q_{k,2}(\mu) = \frac{\operatorname{Re} v_k \operatorname{Re} \delta_k}{\operatorname{Re} v_k - \mu} + \frac{O(k^{-1})}{\operatorname{Re} v_k - \mu} \beta_{k,R}, \quad k \geq k_1, \quad \mu \in \Omega_\eta.$$

For $k \geq k_1$ and $\mu \in \Omega_\eta$ it follows due to $v_{-k} = -\bar{v}_k$ and $\mu_{-k} = -\bar{\mu}_k$ that

$$q_{-k,2}(\mu) = \bar{\varepsilon} \int_0^1 (-1)^{n+2} \frac{(\bar{v}_k + t(\bar{\mu}_k - \bar{v}_k))^{n+2}}{\bar{v}_k + t(\bar{\mu}_k - \bar{v}_k) + \mu} (\bar{\mu}_k - \bar{v}_k) dt.$$

Therefore

$$q_{-k,2}(-\mu) = \overline{q_{k,2}(\bar{\mu})},$$

which gives

$$q_{k,2}(\mu) - q_{-k,2}(-\mu) = \frac{O(k^{-1})}{\operatorname{Re} v_k - \mu} \beta_{k,R}, \quad k \geq k_1.$$

Then Hölder's inequality shows that

$$|\widehat{\Psi}_{0,1}(\mu)|^2 = O(1) \sum_{|k| \geq k_1} \frac{k^{-2}}{|\operatorname{Re} v_k - \mu|^2} \sum_{k \geq k_1} \beta_{k,R}^2,$$

where $\operatorname{Re} v_k + \mu = -(\operatorname{Re} v_{-k} - \mu)$ has been used. In particular, $\widehat{\Psi}_{0,1}$ is bounded on Ω_η . But

$$\int_{-\infty}^{\infty} \frac{dx}{|\operatorname{Re} v_k - (x + iy)|^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2 + y^2} = \frac{\pi}{|y|}$$

shows that

$$\int_{-\infty}^{\infty} |\widehat{\Psi}_{0,1}(x + iy)|^2 dx < \infty, \quad y \geq R_\eta.$$

Since $\Psi_{0,1}(\mu) = \cos(\mu a) \widehat{\Psi}_{0,1}(\mu)$, it follows that all four functions $\Psi_{0,j}$, $j = 1, \dots, 4$, satisfy the estimate $O(e^{|\operatorname{Im} \mu a|})$ on Ω_η and that they are square integrable on $iy + \mathbb{R}$ for $y > R_\eta$. Then also their sum Ψ_0 has this property. Furthermore, Ψ_0 is either an entire function or is meromorphic with exactly one pole at 0. The maximum principle then shows as above that $\Psi_0 \in \mathcal{L}_0^a$. $\square$

The general statements of Theorems 3.1 and 5.1 can be considered as the main result of this paper. By Theorem 3.1, $\mathcal{Z}(\omega)$ for $\omega \in \mathcal{A}_{m,n}^a$ has the properties (i)–(iii) in Theorem 3.1. Conversely, Theorem 5.1 shows that any sequence $(\mu_k)_{k \in \mathbb{Z}}$ with these properties gives rise to a unique $\omega \in \mathcal{A}_{m,n}^a$ such that $\mathcal{Z}(\omega)$ has a representation $(\mu_k)_{k \in \mathbb{Z}}$. Hence the following result has been shown.

Theorem 5.2 *For each $a > 0$ and all nonnegative integers m and n the sequences in $\mathcal{Z}_{m,n}^a$ are described by the properties (i)–(iii) in Theorem 3.1 or 5.1, respectively. The map $\mathcal{Z}$ from $\mathcal{A}_{m,n}^a$ to $\mathcal{Z}_{m,n}^a$ is bijective.*

6 The Functions s_0 and s_1

Consider even real entire functions s_0 and s_1 which have representations

$$s_0(\lambda) = \frac{\sin \lambda a}{\lambda} + \sum_{j=1}^{n+1} \eta_j \frac{\sin^{(j)} \lambda a}{\lambda^{j+1}} + \frac{\psi_0(\lambda)}{\lambda^{n+2}}, \quad (6.1)$$

$$s_1(\lambda) = \cos \lambda a + \sum_{j=0}^n \kappa_j \frac{\sin^{(j)} \lambda a}{\lambda^{j+1}} + \frac{\psi_1(\lambda)}{\lambda^{n+1}}, \quad (6.2)$$

where the η_j and κ_j are real numbers and ψ_0 and ψ_1 belong to $\mathcal{L}^a$.

Since even real entire functions are also real on the imaginary axis, the functions s_0 and s_1 are real on the imaginary axis. Then (6.1) and (6.2) show that

$$\frac{\psi_0(\lambda)}{\lambda^{n+2}} \in \mathbb{R} \text{ and } \frac{\psi_1(\lambda)}{\lambda^{n+1}} \in \mathbb{R} \text{ for } \lambda \in i\mathbb{R} \setminus \{0\}. \quad (6.3)$$

Let $\varrho \in \mathbb{R}$. Putting $\tilde{\tau}(\lambda) = \sqrt{\lambda^2 - 2i\varrho\lambda}$ and observing that s_0 and s_1 are even functions, it follows that $s_j \circ \tilde{\tau}$ is a well-defined entire function for $j = 0, 1$. Letting

$\mu = \lambda - i\rho$ gives

$$\tau(\mu) := \tilde{\tau}(\lambda) = \sqrt{\mu^2 + \rho^2}. \quad (6.4)$$

With some abuse of notation, the number $\tau(\mu)$ may be written as τ when convenient. For definiteness, take a branch of the square root for which $\tau(\mu) > 0$ when $\mu > 0$ and such that τ is analytic on $\mathbb{C} \setminus i[-|\rho|, |\rho|]$. Consider the entire function

$$\varphi(\mu) = \phi(\lambda) = \left(\sum_{j=0}^m \alpha_j (i\lambda)^j \right) s_1(\tau) + \left(\sum_{j=0}^m \beta_j (i\lambda)^j \right) s_0(\tau), \quad (6.5)$$

where the numbers α_j and β_j are real numbers with $\alpha_m = 1$.

This function is similar to the function considered in [14, (7.3.10)]. However, whereas the leading term of the function considered in [14, (7.3.10)] is a multiple of $\mu \sin \mu a$, the leading term of φ is a multiple of $\mu^m \cos \mu a$. Therefore, nontrivial modifications are needed.

Lemma 6.1 *The function φ belongs to the class $\mathcal{A}_{m,n}^a$. If n is even, φ has a representation of the form (2.9) with an odd function Ψ_1 which is real on the imaginary axis.*

Proof The function φ is real on the imaginary axis because s_1 and s_2 have this property. Since asymptotic representations will be shown, the following representations of τ and $\frac{1}{\tau}$ in terms of μ when $|\mu| > |\rho|$ will be used:

$$\tau = \mu \left(1 + \frac{\rho^2}{\mu^2} \right)^{\frac{1}{2}} = \mu + \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} \frac{\rho^{2j}}{\mu^{2j-1}}, \quad (6.6)$$

$$\frac{1}{\tau} = \mu^{-1} \left(1 + \frac{\rho^2}{\mu^2} \right)^{-\frac{1}{2}} = \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \frac{\rho^{2j}}{\mu^{2j+1}}. \quad (6.7)$$

For nonnegative integers j and for $t \in \mathbb{R}$ one has

$$\sin^{(j)} \tau t = \sin^{(j)} \mu t \cos(\tau - \mu)t + \cos^{(j)} \mu t \sin(\tau - \mu)t. \quad (6.8)$$

The expansion (6.6) of τ shows that $\cos(\tau - \mu)a$ and $\sin(\tau - \mu)a$ have power series expansions in terms of μ^{-1} for $|\mu| > |\rho|$ with the constant terms in these power series being 1 for $\cos(\tau - \mu)a$ and 0 for $\sin(\tau - \mu)a$. From (6.1), (6.2), (6.5), (6.6), (6.7) and (6.8) it follows for $|\mu| > |\rho|$ that

$$\varphi(\mu) = \left((i\mu)^m + \sum_{j=0}^{\infty} A_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^{\infty} B_j (i\mu)^{m-j-1} \right) \sin \mu a$$

$$\begin{aligned}
& + \frac{\tilde{\Psi}(\mu)}{(i\mu)^{n-m+2}} \\
& = \left((i\mu)^m + \sum_{j=0}^n A_j (i\mu)^{m-j-1} \right) \cos \mu a + i \left(\sum_{j=0}^{n+1} B_j (i\mu)^{m-j-1} \right) \sin \mu a \\
& + \frac{\Psi(\mu)}{(i\mu)^{n-m+2}}, \tag{6.9}
\end{aligned}$$

where $A_j, B_j \in \mathbb{R}$ for $j = 0, 1, \dots$,

$$\tilde{\Psi}(\mu) = (i\mu)^{n-m+2} \left(\sum_{j=0}^m \alpha_j (i\lambda)^j \right) \frac{\psi_1(\tau)}{\tau^{n+1}} + (i\mu)^{n-m+2} \left(\sum_{j=0}^m \beta_j (i\lambda)^j \right) \frac{\psi_0(\tau)}{\tau^{n+2}},$$

and

$$\Psi(\mu) = \tilde{\Psi}(\mu) + \left(\sum_{j=n+1}^{\infty} A_j (i\mu)^{n-j+1} \right) \cos \mu a + i \left(\sum_{j=n+2}^{\infty} B_j (i\mu)^{n-j+1} \right) \sin \mu a.$$

For $r = 0, 1$ let

$$\tilde{\psi}_r(\mu) = \mu(\psi_r(\tau) - \psi_r(\mu)).$$

Since s_j is an even function for $j = 0, 1$, ψ_r is an even function when $n+r$ is even and an odd function when $n+r$ is odd. Therefore, because of $\psi_r \in \mathcal{L}^a$, the representation (2.1) shows that there is $\xi_r \in L_2(0, a)$ such that

$$\psi_r(\tau) = \int_0^a \xi_r(t) \cos^{(n+r)} \tau t \, dt.$$

In view of (6.8) with $j = n+r+1$,

$$\begin{aligned}
\tilde{\psi}_r(\mu) &= \mu \int_0^a \xi_r(t) \cos^{(n+r)} \mu t (\cos(\tau - \mu)t - 1) \, dt \\
&\quad - (\tau - \mu)\mu \int_0^a t \xi_r(t) \sin^{(n+r)} \mu t \, dt \\
&\quad - \mu \int_0^a \xi_r(t) \sin^{(n+r)} \mu t (\sin(\tau - \mu)t - (\tau - \mu)t) \, dt. \tag{6.10}
\end{aligned}$$

Since $\cos(\tau - \mu)t - 1 = O([\!(\tau - \mu)t\!]^2) = O((\tau - \mu)^2) = O(\mu^{-2})$, $\sin(\tau - \mu)t - (\tau - \mu)t = O([\!(\tau - \mu)t\!]^3) = O(\mu^{-3})$ and $(\tau - \mu)\mu = O(1)$, it follows that the first and the third term on the right hand side of (6.10) satisfy $O(|\mu|^{-1}e^{|\operatorname{Im} \mu|a})$ as $|\mu| \rightarrow \infty$, whereas the second term is the product of a bounded function and a function in $\mathcal{L}^a$. Hence $\tilde{\psi}_r \in \tilde{\mathcal{L}}^a$.

When $n + r$ is even, the function ψ_r is an even function, so that $\mu \mapsto \psi_r(\tau(\mu))$ is an entire function. Therefore also $\tilde{\psi}_r$ is an entire function, which shows that $\tilde{\psi}_r \in \mathcal{L}^a$ when $n + r$ is even.

If $n + r$ is odd, let

$$\widehat{\psi}_r(\mu) = \tau \psi_r(\tau) - \mu \psi_r(\mu).$$

In this case, $\mu \mapsto \tau(\mu)\psi_r(\tau(\mu))$ is an entire function, and therefore also $\widehat{\psi}_r$ is an entire function. Furthermore,

$$\begin{aligned} \widehat{\psi}_r(\mu) &= \tau(\psi_r(\tau) - \psi_r(\mu)) + (\tau - \mu)\psi_r(\mu) \\ &= \tau\mu^{-1}\tilde{\psi}_r(\mu) + (\tau - \mu)\psi_r(\mu). \end{aligned}$$

Since $\tilde{\psi}_r \in \tilde{\mathcal{L}}^a$ and $\psi_r \in \mathcal{L}^a$ and since $\tau\mu^{-1}$ and $\tau - \mu$ are bounded as $|\mu| \rightarrow \infty$, it follows that $\widehat{\psi}_r \in \mathcal{L}^a$ when $n + r$ is odd.

Clearly,

$$\tilde{\Psi}(\mu) = (i\mu)^{n+2}\tau^{-n-1}\psi_1(\tau) + O(1)\psi_1(\tau) + O(1)\psi_0(\tau).$$

If n is even, then

$$\begin{aligned} \tau^{-n-1}\psi_1(\tau) &= \tau^{-n-2}\tau\psi_1(\tau) \\ &= \mu^{-n-2}\left[1 + O\left(\mu^{-2}\right)\right](\mu\psi_1(\mu) + \widehat{\psi}_1(\mu)). \end{aligned}$$

If n is odd, then

$$\tau^{-n-1}\psi_1(\tau) = \mu^{-n-2}\left[1 + O\left(\mu^{-2}\right)\right](\mu\psi_1(\mu) + \tilde{\psi}_1(\mu)).$$

Putting

$$\Psi_{1,1}(\mu) = i^{n+1}\psi_1(\mu), \quad \Psi_{0,1}(\mu) = \tilde{\Psi}(\mu) - i\mu\Psi_{1,1}(\mu),$$

it follows that $\Psi_{1,1} \in \mathcal{L}^a$ and $\Psi_{0,1} \in \tilde{\mathcal{L}}^a$. Furthermore,

$$\Psi(\mu) - \tilde{\Psi}(\mu) = A_{n+1} \cos \mu a + O\left(\mu^{-1}\right) \cos \mu a + O\left(\mu^{-1}\right) \sin \mu a.$$

Letting

$$\begin{aligned} \Psi_{1,2}(\mu) &= -iA_{n+1}\left(\frac{1}{\mu} \cos \mu a - \frac{1}{\mu^2 a} \sin \mu a\right), \\ \Psi_{0,2}(\mu) &= \Psi(\mu) - \tilde{\Psi}(\mu) - i\mu\Psi_{1,2}(\mu), \end{aligned}$$

it is true that $\Psi_{1,2} \in \mathcal{L}^a$ and $\Psi_{0,2} \in \tilde{\mathcal{L}}^a$. With

$$\Psi_1 := \Psi_{1,1} + \Psi_{1,2} \in \mathcal{L}^a, \quad \Psi_0 := \Psi_{0,1} + \Psi_{0,2} \in \tilde{\mathcal{L}}^a,$$

the identity

$$\frac{\Psi(\mu)}{(i\mu)^{n-m+2}} = \frac{\Psi_1(\mu)}{(i\mu)^{n-m+1}} + \frac{\Psi_0(\mu)}{(i\mu)^{n-m+2}}$$

holds. Hence φ belongs to the class $\mathcal{A}_{m,n}^a$.

Clearly, $\Psi_{1,2}$ is an odd function which is real on the imaginary axis. If n is even, then ψ_1 is an odd function and pure imaginary on the imaginary axis by (6.3). Hence $\Psi_{1,1}$ is an odd function which is real on the imaginary axis. Therefore the function Ψ_1 is an odd function which is real on the imaginary axis if n is even. $\square$

Corollary 6.2 Put $\kappa = \lfloor \frac{m}{2} \rfloor + \frac{1}{2}$, $Z = \mathbb{Z}$ if m is odd and $Z = \mathbb{Z} \setminus \{0\}$ if m is even. Then the zeros of ϕ defined in (6.5), counted with multiplicity, can be written as a sequence $(\lambda_k)_{k \in Z}$ with the following properties:

- (i) there is a nonnegative integer k_0 such that λ_k is pure imaginary for $|k| \leq k_0$;
- (ii) for $k > k_0$ it is true that $\operatorname{Re} \lambda_k > 0$ and $\lambda_{-k} = -\bar{\lambda}_k$;
- (iii) as $k \rightarrow \infty$, the sequence $(\lambda_k)_{k \in Z}$ has the asymptotic representation

$$\lambda_k = \frac{\pi}{a} (k - \kappa) + \sum_{j=0}^{n+2} i^{j+1} \frac{Q_j}{(k - \kappa)^j} + \frac{\beta_k}{(k - \kappa)^{n+1}}, \quad (6.11)$$

where $Q_j \in \mathbb{R}$ ($j = 0, \dots, n+2$) with $Q_0 = \varrho$ and where $(\beta_k)_{k=1}^\infty \in l_2$.

If $\ell \in \{0, 1\}$ and $i^\ell \Psi_1$ is real on the real axis for some representation (2.9) of φ , then $(k \operatorname{Re}(i^{n+\ell} \beta_k))_{k=1}^\infty \in l_2$.

Proof In view of Lemma 6.1, the function φ satisfies the assumptions of Theorem 3.1. With μ_k from Theorem 3.1 it follows that $\lambda_k = \mu_k + i\varrho$ has the stated properties. $\square$

Remark 6.3 Replacing (3.1) with (6.11), Remark 3.2 also applies to Corollary 6.2 accordingly.

Corollary 6.4 Let $Z = \mathbb{Z}$ or $Z = \mathbb{Z} \setminus \{0\}$, let κ' be a nonnegative integer and put $\kappa = \kappa' + \frac{1}{2}$. Let $m = 2\kappa'$ if $Z = \mathbb{Z} \setminus \{0\}$ and let $m = 2\kappa' + 1$ if $Z = \mathbb{Z}$. Let $n \geq m - 1$ and let $(\lambda_k)_{k \in Z}$ be a sequence of complex numbers satisfying the following conditions:

- (i) there is a nonnegative integer k_0 such that λ_k is pure imaginary for $|k| \leq k_0$;
- (ii) for $k \geq k_0$ it is true that $\operatorname{Re} \lambda_k > 0$ and $\lambda_{-k} = -\bar{\lambda}_k$;
- (iii) as $k \rightarrow \infty$, the sequence admits the asymptotic representation

$$\lambda_k = \frac{\pi}{a} (k - \kappa) + \sum_{j=0}^{n+2} \frac{i^{j+1} Q_j}{(k - \kappa)^j} + \frac{\beta_k}{(k - \kappa)^{n+1}}, \quad (6.12)$$

where $(\beta_k)_{k=1}^\infty \in l_2$, $a > 0$, and $Q_j \in \mathbb{R}$ for $j = 0, \dots, n+2$.

Then there is a unique entire function $\varphi \in \mathcal{A}_{m,n}^a$ such that $(\lambda_k)_{k \in \mathbb{Z}}$ is a sequence of the zeros of the function ϕ defined by $\phi(\lambda) = \varphi(\lambda - iQ_0)$. Furthermore, there is a nonzero real number c such that

$$\phi(\lambda) = c(i\lambda)^d \lim_{j \rightarrow \infty} \prod_{\substack{k \in \mathbb{Z}, |k| \leq j \\ \lambda_k \neq 0}} \left(1 - \frac{\lambda}{\lambda_k}\right), \quad (6.13)$$

where d is the number of occurrences of the value 0 in the sequence $(\lambda_k)_{k \in \mathbb{Z}}$. If $(k \operatorname{Im}(i^n \beta_k))_{k=1}^\infty \in l_2$, then the function Ψ_1 in the representation (2.9) of φ can be chosen in such a way that Ψ_1 is an odd function which is real on the imaginary axis.

Proof Putting $\mu_k = \lambda_k - iQ_0$ for $k \in \mathbb{Z}$, it follows that the assumptions of Theorem 5.1 are satisfied for the sequence $(\mu_k)_{k \in \mathbb{Z}}$. Taking $\varphi = \omega$ from Theorem 5.1 completes the proof. For the representation (6.13) just observe that

$$1 - \frac{\mu}{\mu_k} = \frac{\mu_k - (\lambda - iQ_0)}{\mu_k} = \frac{\lambda_k}{\mu_k} \frac{\lambda_k - \lambda}{\lambda_k} = \frac{\lambda_k}{\mu_k} \left(1 - \frac{\lambda}{\lambda_k}\right)$$

when $\lambda_k \neq 0$ and $\mu_k \neq 0$, that, for $|k| \geq k_0$,

$$\frac{\lambda_k}{\mu_k} \frac{\lambda_{-k}}{\mu_{-k}} = \left(1 + \frac{iQ_0}{\mu_k}\right) \left(1 - \frac{iQ_0}{\overline{\mu_k}}\right) = 1 + \frac{iQ_0(\overline{\mu_k} - \mu_k) + Q_0^2}{\mu_k \overline{\mu_k}} = 1 + O(k^{-2})$$

is real, and that for $Q_0 \neq 0$, $i\lambda = -Q_0 \left(1 + \frac{\mu}{iQ_0}\right)$ and $i\mu = Q_0 \left(1 - \frac{\lambda}{iQ_0}\right)$. $\square$

7 Determining Coefficients for Functions of the Form (6.5)

In order to find functions of the form (6.5) whose zeros are as given in Corollary 6.4, numbers α_j and β_j as well as functions s_0 and s_1 have to be found. Determining s_0 and s_1 is beyond the scope of this paper. Hence it will be assumed that s_0 and s_1 are given, and the aim is to explore relations between the coefficients $\alpha_0, \dots, \alpha_{m-1}, \beta_0, \dots, \beta_m$ in (6.5) and the coefficients $Q_0, \dots, Q_{n+2}$ in the asymptotic representation (6.12) of the zeros of (6.5). From Corollary 6.2 it is known that $\varrho = Q_0$, that is, τ is determined by Q_0 .

Remark 7.1 A comparison of the right hand sides of (2.2) and (5.2) for $\mu_{r,\iota} = (2j + \frac{\iota}{2})\frac{\pi}{a}$ with $r \in \mathbb{N}$, $\iota \in \{0, 1\}$, allows to find the values $c, A_0, \dots, A_n, B_0, \dots, B_n$. A corresponding procedure holds when comparing (6.5) and (6.13).

For example, with $\varrho = Q_0$,

$$c = \phi(\lambda)(i\lambda)^{-d} \lim_{j \rightarrow \infty} \prod_{\substack{k \in \mathbb{Z}, |k| \leq j \\ \lambda_k \neq 0}} \left(1 - \frac{\lambda}{\lambda_k}\right)^{-1}$$

$$= \varphi(\mu)(i\mu - \varrho)^{-d} \lim_{j \rightarrow \infty} \prod_{\substack{k \in \mathbb{Z}, |k| \leq j \\ \lambda_k \neq 0}} \left(1 - \frac{\mu + i\varrho}{\lambda_k}\right)^{-1}.$$

Assuming that the λ_k and $\mu_{r,0} + i\varrho$ are mutually distinct it is true that

$$c = \lim_{r \rightarrow \infty} (i\mu_{r,0} - \varrho)^{m-d} \lim_{j \rightarrow \infty} \prod_{\substack{k \in \mathbb{Z}, |k| \leq j \\ \lambda_k \neq 0}} \left(1 - \frac{\mu_{r,0} + i\varrho}{\lambda_k}\right)^{-1}.$$

However, this approach using (6.13) would require the knowledge of all eigenvalues. Hence, alternatively the coefficients of the eigenvalue asymptotics will be used to find the values of the A_j 's and the B_j 's in cases which are particularly interesting for applications.

7.1 The Case $n = 2m$

Here $2m + 1$ real numbers $Q_1, \dots, Q_{2m+1}$ in (6.12) relate to $2m + 1$ real numbers $\alpha_0, \dots, \alpha_{m-1}, \beta_0, \dots, \beta_m$ in (6.5).

It is best to express (6.5) in terms of μ . To this end write

$$\left(\sum_{j=0}^m \alpha_j (i\lambda)^j\right) = \left(\sum_{j=0}^m \tilde{\alpha}_j (i\mu)^j\right), \quad \left(\sum_{j=0}^m \beta_j (i\lambda)^j\right) = \left(\sum_{j=0}^m \tilde{\beta}_j (i\mu)^j\right). \quad (7.1)$$

The functions s_0 and s_1 are even functions. Defining $\tilde{s}_j(\mu) = s_j(\tau)$, $j = 0, 1$, and observing (6.6) and (6.8) it follows that $\tilde{s}_0$ and $\tilde{s}_1$ are even functions with the representations

$$\begin{aligned} \tilde{s}_0(\mu) &= \frac{\sin \mu a}{\mu} + \sum_{j=1}^{m+1} \tilde{\eta}_{c,j} \frac{\cos \mu a}{(i\mu)^{2j}} + i \sum_{j=1}^m \tilde{\eta}_{s,j} \frac{\sin \mu a}{(i\mu)^{2j+1}} + \frac{\tilde{\psi}_0(\mu)}{\mu^{n+2}}, \\ \tilde{s}_1(\mu) &= \cos \mu a + \sum_{j=1}^m \tilde{\kappa}_{c,j} \frac{\cos \mu a}{(i\mu)^{2j}} + i \sum_{j=0}^m \tilde{\kappa}_{s,j} \frac{\sin \mu a}{(i\mu)^{2j+1}} + \frac{\tilde{\psi}_1(\mu)}{\mu^{n+1}}, \end{aligned}$$

with real numbers $\tilde{\eta}_{c,j}, \tilde{\eta}_{s,j}, \tilde{\kappa}_{c,j}, \tilde{\kappa}_{s,j}$. Observing that by Propositions 2.4 and 2.8 and Lemma 6.1 the coefficients A_j for $j = 0, \dots, 2m$ and B_j for $j = 0, \dots, 2m + 1$ in (6.9) are uniquely determined by φ , comparing (6.9) and (6.5) and observing (7.1) gives for $0 \leq p \leq m - 1$ that

$$A_p = \tilde{\alpha}_{m-p-1} + \sum_{j=1}^{\lfloor \frac{p+1}{2} \rfloor} \tilde{\alpha}_{m-p-1+2j} \tilde{\kappa}_{c,j} + \sum_{j=1}^{\lfloor \frac{p+1}{2} \rfloor} \tilde{\beta}_{m-p-1+2j} \tilde{\eta}_{c,j} \quad (7.2)$$

and for $0 \leq p \leq m$ that

$$B_p = \tilde{\beta}_{m-p} + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \tilde{\beta}_{m-p+2j} \tilde{\eta}_{s,j} + \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \tilde{\alpha}_{m-p+2j} \tilde{\kappa}_{s,j}, \quad (7.3)$$

where $\tilde{\alpha}_m = 1$. Hence, if A_p for $p = 0, \dots, m-1$ and B_p for $p = 0, \dots, m$ are known, then the coefficients $\tilde{\beta}_m, \tilde{\alpha}_{m-1}, \tilde{\beta}_{m-1}, \dots, \tilde{\alpha}_0, \tilde{\beta}_0$ can be recovered recursively.

However, Remark 4.2 shows that these numbers A_p and B_p are not uniquely determined by the Q_j . Hence more identities are needed. Indeed, for $m \leq p \leq 2m$ it is true that

$$A_p = \sum_{j=\lceil \frac{p-m+1}{2} \rceil}^{\lfloor \frac{p+1}{2} \rfloor} \tilde{\alpha}_{m-p-1+2j} \tilde{\kappa}_{c,j} + \sum_{j=\lceil \frac{p-m+1}{2} \rceil}^{\lfloor \frac{p+1}{2} \rfloor} \tilde{\beta}_{m-p-1+2j} \tilde{\eta}_{c,j} \quad (7.4)$$

and for $m+1 \leq p \leq 2m+1$ that

$$B_p = \sum_{j=\lceil \frac{p-m}{2} \rceil}^{\lfloor \frac{p}{2} \rfloor} \tilde{\beta}_{m-p+2j} \tilde{\eta}_{s,j} + \sum_{j=\lceil \frac{p-m}{2} \rceil}^{\lfloor \frac{p}{2} \rfloor} \tilde{\alpha}_{m-p+2j} \tilde{\kappa}_{s,j}. \quad (7.5)$$

7.2 The Case $m = 2, n = 4$

In this subsection $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\beta}_0, \tilde{\beta}_1$ and $\tilde{\beta}_2$ will be recovered without the knowledge of $\tilde{s}_0$ and $\tilde{s}_1$ in case $m = 2$ and $n = 4$. First write down (7.2)–(7.5) explicitly:

$$A_0 = \tilde{\alpha}_1, \quad A_1 = \tilde{\beta}_2 \tilde{\eta}_{c,1} + \tilde{\alpha}_0 + \tilde{\kappa}_{c,1}, \quad A_2 = \tilde{\beta}_1 \tilde{\eta}_{c,1} + \tilde{\alpha}_1 \tilde{\kappa}_{c,1}, \quad (7.6)$$

$$A_3 = \tilde{\beta}_0 \tilde{\eta}_{c,1} + \tilde{\beta}_2 \tilde{\eta}_{c,2} + \tilde{\alpha}_0 \tilde{\kappa}_{c,1} + \tilde{\kappa}_{c,2}, \quad A_4 = \tilde{\beta}_1 \tilde{\eta}_{c,2} + \tilde{\alpha}_1 \tilde{\kappa}_{c,2}, \quad (7.7)$$

$$B_0 = \tilde{\beta}_2 + \tilde{\kappa}_{s,0}, \quad B_1 = \tilde{\alpha}_1 \tilde{\kappa}_{s,0} + \tilde{\beta}_1, \quad B_2 = \tilde{\beta}_2 \tilde{\eta}_{s,1} + \tilde{\alpha}_0 \tilde{\kappa}_{s,0} + \tilde{\beta}_0 + \tilde{\kappa}_{s,1}, \quad (7.8)$$

$$B_3 = \tilde{\beta}_1 \tilde{\eta}_{s,1} + \tilde{\alpha}_1 \tilde{\kappa}_{s,1}, \quad B_4 = \tilde{\beta}_0 \tilde{\eta}_{s,1} + \tilde{\beta}_2 \tilde{\eta}_{s,2} + \tilde{\alpha}_0 \tilde{\kappa}_{s,1} + \tilde{\kappa}_{s,2}, \quad (7.9)$$

$$B_5 = \tilde{\beta}_1 \tilde{\eta}_{s,2} + \tilde{\alpha}_1 \tilde{\kappa}_{s,2}. \quad (7.10)$$

Next find relations between the numbers $\tilde{\alpha}_j, \tilde{\beta}_j, A_j$ and B_j which do not depend on the coefficients in $\tilde{s}_0$ and $\tilde{s}_1$. One of these is clearly

$$A_0 = \tilde{\alpha}_1, \quad (7.11)$$

which gives $\tilde{\alpha}_1$. Another one is

$$B_1 - A_0 B_0 = \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_2. \quad (7.12)$$

The next identities depend on the coefficients of $\tilde{s}_0$ and $\tilde{s}_1$, but in a very particular way:

$$A_0 A_1 - A_2 = \tilde{\alpha}_0 \tilde{\alpha}_1 - (\tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_2) \tilde{\eta}_{c,1}, \quad (7.13)$$

$$B_1 + B_3 - A_0 B_2 = (1 - \tilde{\alpha}_0) \tilde{\alpha}_1 (B_0 - \tilde{\beta}_2) + \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0 + (\tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_2) \tilde{\eta}_{s,1}. \quad (7.14)$$

If $\tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_2 = 0$, then also (7.13) and (7.14) are independent of $\tilde{s}_0$ and $\tilde{s}_1$, and (7.14) can be written as

$$\begin{aligned} B_1 + B_3 - A_0 B_2 &= \tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0 + (1 - \tilde{\alpha}_0) \tilde{\alpha}_1 B_0 \\ &= \tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0 + (1 - \tilde{\alpha}_0) B_1. \end{aligned}$$

Hence

$$\tilde{\alpha}_0 B_1 + B_3 - A_0 B_2 = \tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0. \quad (7.15)$$

Under the additional assumption that $A_0 \neq 0$ one finds $\tilde{\alpha}_0$ from (7.13).

In the application in mind one has $\tilde{\eta}_{c,1} = \tilde{\kappa}_{s,0}$, and this together with $\tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_2 = 0$ will now be assumed. Then

$$A_2 + A_4 - A_0 A_3 = (\tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0) (B_0 - \tilde{\beta}_2) + (1 - \tilde{\alpha}_0) A_2. \quad (7.16)$$

Substituting $\tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0$ from (7.15) into (7.16) and assuming $\tilde{\alpha}_0 \tilde{\beta}_1 - \tilde{\alpha}_1 \tilde{\beta}_0 \neq 0$, one gets a unique solution for $\tilde{\beta}_2$. Then also $\tilde{\beta}_1 = \tilde{\alpha}_1 \tilde{\beta}_2$ is known, and finally one finds $\tilde{\beta}_0$ from (7.15). The details will be provided in the forthcoming paper mentioned above.

Supplementary information

The results in Proposition 4.3 have been verified with the sagemath program Proposition4.3.ipynb.

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1007/s11785-025-01728-0>.

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