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Gauged permutation invariant matrix quantum mechanics: path integrals

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Abstract: We give a path integral construction of the quantum mechanical partition function for gauged finite groups. Our construction gives the quantization of a system of d, $N \times N$ matrices invariant under the adjoint action of the symmetric group S_N. The approach is general to any discrete group. For a system of harmonic oscillators, i.e. for the non-interacting case, the partition function is given by the Molien-Weyl formula times the zero-point energy contribution. We further generalise the result to a system of non-square and complex matrices transforming under arbitrary representations of the gauge group.

Keywords: Discrete Symmetries, Gauge Symmetry, Lattice Quantum Field Theory, Matrix Models

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1 Introduction

Permutation invariant matrix models, defined by matrix integration with a measure of the form $d^{N^2}Xe^{-V(X)}$, where X is an $N \times N$ matrix of variables, $d^{N^2}X$ is the standard Euclidean measure on N^2 real variables and $V(X)$ is a potential invariant under permutation transformations, were introduced in [1] with motivations from computational linguistics. The general Gaussian potential $V(X)$, of polynomial degree two in the matrix variables, invariant under permutation transformations $X \to UXU^T$ for $N \times N$ permutation matrices U, was solved using the representation theory of the symmetric group S_N [2]. Under this transformation by S_N matrices, X transforms as $V_N \otimes V_N$ where V_N is the natural representation of S_N. The potential $V(X)$ contains two linear invariants and eleven quadratic ones, which were each weighted by an independent parameter to give a solvable 13-parameter model. The 13-parameter models were used to find evidence of approximate Gaussianity in ensembles of word matrices constructed from natural language data [3, 4]. Motivated by holographic properties of analogous gauged U(N) invariant matrix theories [5–7], a large N-factorisation property for permutation invariant observables in an inner product defined using the simplest matrix model was established in [8].

Permutation invariant quantum mechanical matrix models were described in [9, 10]. An orthogonal basis of states for the permutation invariant sector of the Hilbert space was constructed for the simplest matrix harmonic oscillator, by employing the representation theory of partition algebras. The canonical quantization of the general 11-parameter matrix harmonic oscillator, defined using the 11 quadratic permutation invariant functions of $N \times N$ matrices, was given and the canonical partition function was computed.

In this paper we realise the permutation transformation of the quantum mechanical matrix degrees of freedom as a gauge symmetry within a path integral formulation. We

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do this by beginning with a lattice formulation of the canonical partition function. We take a lattice with Λ sites wrapped on a circle with period β so that the lattice spacing is $a = \beta/\Lambda$. Then a permutation invariant potential is placed at each site with the different sites connected in the standard lattice fashion by a discretized Laplacian but with the hopping terms connected by permutation matrices to parallel transport the permutation symmetry. This in effect is a lattice gauged matrix model with the discrete gauge group S_N. In the context of lattice gauge theory finite groups have in fact received significant attention since the inception of the subject [11–17]. See [18] for a recent discussion.

The invariance of the partition function is enforced by summing over the Λ linking gauge variables which are representations of the symmetric group in the natural representation acting by adjoint action. We first establish that all of the individual Λ group elements can be collected to the final connecting link as a single group element using the invariance of the measure. The sum over gauge fields then involves a single sum over the group elements. We then concentrate on the Gaussian case where the matrices can be integrated out. This gives an inverse square root of the determinant of the quadratic form in the lattice action which in turn depends on the single group element. The lattice partition function is then obtained by summing over all group elements. Finally we show that the resulting expression when the zero-point energy is removed (i.e. when the ground state energy is subtracted) is the classical Molien-Weyl expression (see [19] for a textbook discussion) for the generating function of the Hilbert series of the quantum matrix model. This allows us to easily discuss quantum models more generally, by e.g. adding more matrices transforming under the group, or extending to complex and general representations.

Our approach works quite generally for both continuous and discrete groups and we establish the rather general result (5.5) which is valid for a system of complex matrices with Gaussian action. Furthermore, one can easily incorporate an invariant local time-independent potential.

The principal results presented here are:

- We give a path integral derivation of the Molien-Weyl formula which in this context describes a system of non-interacting quantum matrix models with a U(N) gauge invariance in the adjoint representation and in a thermal bath.

- We then extend the derivation to include discrete groups such as the symmetric group S_N. Modifying the potential as in [1, 2, 9, 10] allows us to access the 11-parameter family of S_N invariant models.

- Finally we discuss the case when the gauge group acts in different representations on the left and right of the matrices which are now not necessarily square matrices. The analysis can easily be extended to gauged quiver matrix models.

The paper is organised as follows: to establish notation and introduce the lattice formulation that we will use throughout we begin in section 2 with a brief review of the path integral formulation of a simple harmonic oscillator. In section 3 we then describe a lattice version of the gauged system and show that on the lattice, the local gauge transformations can be accumulated into a holonomy around the thermal circle. This holonomy can be placed

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on any link but we choose to place it on the closing link of the periodic lattice. Section 4 discusses the case of discrete groups and recovers the classical Molien formula. Then in 5 we extend the discussion to arbitrary representations of the gauge groups. We end the paper with some conclusions and discussion. The paper also contains two short appendices where the relevant set of lattice determinants arising in the bulk of the paper are evaluated explicitly.

2 Path integral for a system of harmonic oscillator

A simple quantum oscillator is described by the Hamiltonian $H = m(a^\dagger a + \frac{1}{2})$. The partition function in this case is

$$Z(x) = \sum_{n=0}^{\infty} \mathbf{Tr}(\mathrm{e}^{-\beta H}) = \frac{x^{1/2}}{1-x} \quad \text{where} \quad x = \mathrm{e}^{-\beta m}\,. \tag{2.1}$$

Here we give a Euclidean lattice path integral derivation of this result which we will then generalise to the Molien-Weyl formula.

The Euclidean action is obtained by taking the Lagrangian of the oscillator, rotating to imaginary time on a circle of period β. So for the oscillator the Euclidean action is then

$$S[X] = \int_0^\beta d\tau \left(\frac{1}{2}(\partial_\tau X)^2 + \frac{1}{2}m^2 X^2\right) \tag{2.2}$$

The Euclidean lattice version of this consists of replacing the time interval with a periodic lattice of Λ points so that $\tau \to \tau_n = na$, with $a = \beta/\Lambda$ the lattice spacing, and n labeling the lattice points from 1 to Λ. Then we denote $X(\tau_n) = X_n$ with periodicity implying $X_{\Lambda+1} = X_1$. The Euclidean time derivative is then replaced by a finite difference which we take to be

$$\partial_\tau X \to \frac{X_{n+1} - X_n}{a} = \frac{\mathrm{e}^{a\partial_\tau} - 1}{a} X_n\,, \quad d\tau \to a \tag{2.3}$$

and where $\mathrm{e}^{a\partial_\tau}$ is the shift operator on the lattice, taking X_n to X_{n+1} The lattice action can then be expressed as

$$S_\Lambda[X] = \sum_{n,n'=1}^{\Lambda} \frac{a}{2} X_{n'} (\Delta_\Lambda + m^2)_{n',n} X_n\,, \tag{2.4}$$

where Δ_Λ is the lattice version of $-\partial_\tau^2$ and the only non-zero elements of $(\Delta_\Lambda + m^2)_{n',n}$ are

$$(\Delta_\Lambda + m^2)_{n,n} = \frac{2}{a^2} + m^2 \qquad (\Delta_\Lambda + m^2)_{n+1,n} = (\Delta_\Lambda + m^2)_{n,n+1} = -\frac{1}{a^2}\,. \tag{2.5}$$

The path integral is then given by the Λ dimensional Gaussian integration over X_n and yields

$$Z_\Lambda = \int_{\mathbb{R}^\Lambda} \frac{d^\Lambda X}{(2\pi a)^{\Lambda/2}} \mathrm{e}^{-S_\Lambda[X]} = \mathrm{Det}^{-1/2}(a^2(\Delta_\Lambda + m^2)) \tag{2.6}$$

where

$$\Delta_\Lambda = \frac{2 - \mathrm{e}^{a\partial_\tau} - \mathrm{e}^{-a\partial_\tau}}{a^2}\,. \tag{2.7}$$

The matrix

$$a^2(\Delta_\Lambda + m^2) = 2 + \frac{\beta^2 m^2}{\Lambda^2} - \mathrm{e}^{a\partial_\tau} - \mathrm{e}^{-a\partial_\tau} := M_{\Lambda,1} \tag{2.8}$$

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is a tri-diagonal matrix with $2+\frac{\beta^2 m^2}{\Lambda^2}$ on the diagonal, -1 in the leading off diagonals and the upper right and lower left corners (see (A.1) and the subsequent discussion).

The Λ eigenvectors of $a^2\Delta_\Lambda$, are obtained by noting the discrete lattice translation invariance implies that translation by Λ steps (i.e. corresponding to translations around one full period, $\beta=\Lambda a$) gives the identity i.e. $\mathrm{e}^{\Lambda a\partial_\tau}=1$ and the operator is diagonalized by a discrete Fourier transform. The eigenvalues are then given in terms of the roots of unity of $(w_k)^\Lambda=1$, i.e. $w_k=\mathrm{e}^{2\pi ik/\Lambda}$. The corresponding eigenvectors are $v_n^k=\frac{1}{\sqrt{\Lambda}}\mathrm{e}^{\frac{2\pi ink}{\Lambda}}$ with n the lattice site. Colectively v_n^k form the unitary matrix that diagonalizes Δ_Λ and $M_{\Lambda,1}$. The eigenvalues of $M_{\Lambda,1}$ are then

$$\lambda_k=2-2\cos\left(\frac{2\pi k}{\Lambda}\right)+\frac{\beta^2m^2}{\Lambda^2} \tag{2.9}$$

so that the partition function (2.6) is

$$Z_\Lambda=\mathrm{Det}^{-1/2}M_{\Lambda,1}=\prod_{k=1}^{\Lambda}\lambda_k^{-1/2}. \tag{2.10}$$

One can now compute Z_Λ by factoring the λ_n, i.e. writing $2\cos(\frac{2\pi k}{\Lambda})=w_k+1/w_k$, factoring

$$\lambda_k=2+\mu^2-w_k-1/w_k=\frac{1}{w_k}(z_+-w_k)(w_k-z_-) \tag{2.11}$$

and using

$$\prod_{k=1}^{\Lambda}(z-w_k)=z^\Lambda-1\quad\text{and}\quad\prod_{k=1}^{\Lambda}(-w_k)=-1 \tag{2.12}$$

to find

$$\mathrm{Det}M_{\Lambda,1}=\prod_{k=1}^{\Lambda}\lambda_k=(z_+^\Lambda-1)(1-z_-^\Lambda)=z_+^\Lambda+z_-^\Lambda-2 \tag{2.13}$$

so that the lattice partition function is given by

$$Z_\Lambda=(z_+^\Lambda-1)^{-\frac{1}{2}}(1-z_-^\Lambda)^{-\frac{1}{2}}=\frac{z_-^{\frac{\Lambda}{2}}}{1-z_-^\Lambda} \tag{2.14}$$

where $z_\pm=1+\frac{\mu^2}{2}\pm\sqrt{\mu^2(1+\frac{\mu^2}{4})}$, $\mu=\frac{m\beta}{\Lambda}$ and $z_+z_-=1$.

The continuum limit is obtained by noting that $\lim_{\Lambda\to\infty}z_-^\Lambda\to\mathrm{e}^{-\beta m}=x$ so that

$$Z_\Lambda=\frac{z_-^{\frac{\Lambda}{2}}}{1-z_-^\Lambda}\longrightarrow\lim_{\Lambda\to\infty}Z_\Lambda=\frac{\mathrm{e}^{-\frac{m\beta}{2}}}{1-\mathrm{e}^{-m\beta}}=\frac{x^{1/2}}{1-x}, \tag{2.15}$$

as required. Hence, by simply taking the limit of an infinite number of lattice points with fixed βm one recovers exactly the continuum result.

3 Gauged Gaussian matrix models

We now consider X to be an $N\times N$ hermitian matrix and the action involves a trace over matrices. The resulting action has a global U(N) invariance[1] under the adjoint action of

[1]This is effectively a global SU(N) invariance since the U(1) cancels in the adjoint action. One could equally consider traceless matrices, but it convenient to leave the U(1) and the trace in our expressions.

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U(N) so that $X \to gXg^{-1}$. This invariance can be promoted to a local gauge symmetry. The Euclidean action is then

$$S[X] = \int_0^\beta d\tau \mathbf{tr}\left(\frac{1}{2}(\mathcal{D}_\tau X)^2 + \frac{1}{2}m^2X^2\right) \tag{3.1}$$

where $\mathcal{D}_\tau = \partial_\tau - i[A, \cdot\,]$ is the covariant derivative with the gauge field $A(\tau)$ being an $N \times N$ hermitian matrix.

When discretising the covariant derivative $\mathcal{D}_\tau$ we need the lattice analogue of (2.3). It is naturally defined to be

$$\mathcal{D}_\tau X \to \frac{g_{n,n+1}X_{n+1}g_{n+1,n} - X_n}{a} = \frac{\mathrm{e}^{a\mathcal{D}_\tau} - 1}{a^2}X_n \tag{3.2}$$

where $g_{n,n+1}$ are the link gauge variables that parallel transport the group element at $n+1$ back to that at n and $g_{n+1,n} = g_{n,n+1}^{-1}$. The corresponding lattice Laplacian is given by

$$\Delta_{\Lambda,g} = \frac{2 - e^{a\mathcal{D}_\tau} - e^{-a\mathcal{D}_\tau}}{a^2} \tag{3.3}$$

and the action is a quadratic form as in (2.4) with Δ_Λ replaced by $\Delta_{\Lambda,g}$. By substituting a path ordered product, denoted $\mathcal{P}$, one sees

$$g_{n,n+1} = \mathcal{P}\exp\left[i\int_{na}^{na+a} d\tau\, A(\tau)\right], \tag{3.4}$$

and taking a to zero in (3.3) one recovers the continuum Laplacian. The corresponding partition function is then

$$Z_{N,\Lambda} = \int_{\mathrm{U}(N)^\Lambda}\int_{\mathbb{R}^{N^2\Lambda}} \prod_{k=1}^{\Lambda} \mu(g_{k,k+1})\frac{d^{N^2}X_k}{(2\pi a)^{N^2}}\mathrm{e}^{-S_{\Lambda,g}} \tag{3.5}$$

where

$$S_{\Lambda,g} = \sum_{n,n'=1}^{\Lambda}\frac{a}{2}\mathbf{tr}(X'_n(\Delta_{\Lambda,g} + m^2)_{n',n}X_n)\,. \tag{3.6}$$

Eq. (3.5) is a matrix integral over the matrix variables, X_n, with Dyson measure $d^{N^2}X_n$, for each lattice site, n, together with Λ integrals over the individual link group elements $g_{n,n+1}$. These latter integrals have Haar measure, $\mu(g)$, normalised to unity.

As shown in [20] one can further simplify the dependence of the action to a dependence on a single group element. First observe that reorganising the lattice action (3.6) gives

$$S_{\Lambda,g} = \sum_{n=1}^{\Lambda}\frac{1}{a}\mathbf{tr}\left\{-X_ng_{n,n+1}X_{n+1}g_{n+1,n} + \left(1 + \frac{\beta^2m^2}{2\Lambda^2}\right)X_n^2\right\}, \tag{3.7}$$

and we have used that periodicity of the lattice implies $X_{\Lambda+1} = X_1$. Since the X_n will be integrated over we can make a change of variables $X'_1 = X'_1$, $X'_2 = g_{1,2}X_2g_{2,1}$ so that $X_1g_{1,2}X_2g_{2,1}$ becomes $X'_1X'_2$, and $X_2g_{2,3}X_3g_{3,2} = g_{2,1}X'_2g_{1,2}g_{2,3}X_3g_{3,2}$ which, with cyclicity

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under the trace, allows us to redefine $X'_3 = g_{1,2}g_{2,3}X_3 g_{3,2}g_{2,1}$, etc. We can continue, using the transformation

$$X'_r = \prod_{k=1}^{r-1} g_{k,k+1}\, X_r \prod_{k=1}^{r-1} g_{k+1,k}\,, \tag{3.8}$$

until we arrive at $X_\Lambda g_{\Lambda,1}X_1 g_{1,\Lambda}$ which becomes $X'_\Lambda g_{1,2}\dots g_{\Lambda,1}X'_1 g_{1,\Lambda}\dots g_{2,1}$. Due to periocicity of the lattice the accumulated group element, $g = g_{1,2}\dots g_{\Lambda,1} = \prod_{k=1}^{\Lambda} g_{k,k+1}$, can no longer be absorbed.[2] The group elements in g form a closed loop, and the product is a holonomy element. This holonomy element, g, can be placed anywhere on the circle but here it is placed on the link between $n = \Lambda$ and $n = 1$. The lattice action then becomes

$$S_{\Lambda,g} = -\frac{1}{a}\mathbf{tr}\left\{\sum_{n=1}^{\Lambda-1} X'_n X'_{n+1} + X'_\Lambda g\, X'_1 g^{-1}\right\} + \frac{1}{a}\sum_{n=1}^{\Lambda-1}\mathbf{tr}\left\{\left(1+\frac{a^2\beta^2m^2}{2}\right)X'^2_n\right\}. \tag{3.9}$$

The structure of (3.9) is now of the form (2.6)

$$S_{\Lambda,g} = \sum_{n,n'=1}^{\Lambda}\frac{1}{2}\mathbf{tr}X_{n'}a(\Delta_{\Lambda,g}+m^2\mathbf{1})_{n',n}X_n \tag{3.10}$$

where $\mathbf{1}$ is the unit matrix and group identity element. The matrix $(a^2\Delta_{\Lambda,g} + \frac{\beta^2m^2}{\Lambda^2}\mathbf{1})_{n',n}$ is a ΛN^2 dimensional tri-diagonal matrix with $g\otimes g^{-1}$ in the right upper corner and its inverse in the lower corner. The diagonal elements are all $(2+\frac{\beta^2m^2}{\Lambda^2})\mathbf{1}$ and the off diagonals are $-\mathbf{1}$.

The group invariance and normalization to unity of the $\mu(g)$ measure ensures that the resulting path integral has a single integration over this final gauge transformation, g. The partition function (3.5) is then

$$Z_{N,\Lambda} = \int \mu(g)\mathbf{Det}^{-1/2}\left(a^2\Delta_{\Lambda,g}+\frac{\beta^2m^2}{\Lambda^2}\mathbf{1}\right) = \int \mu(g)\mathbf{Det}^{-1/2}M_{\Lambda,g} \tag{3.11}$$

where

$$M_{\Lambda,g} := a^2\Delta_{\Lambda,g} + \frac{\beta^2m^2}{\Lambda^2} = 2+\mu^2 - e^{a\mathcal{D}_\tau} - e^{-a\mathcal{D}_\tau} \tag{3.12}$$

and our notation is **Det** for the $\Lambda N^2\times\Lambda N^2$ determinant, Det for the $\Lambda\times\Lambda$ lattice determinant and **det** for the $N^2\times N^2$ or group determinant.

The group element entering the expressions creates no additional complication as it acts as a commuting variable with respect to the $\Lambda\times\Lambda$ matrix entries and as shown in the appendix A one arrives at the determinant of the lattice determinant as

$$\mathrm{Det}M_{\Lambda,g} = z_+^\Lambda + z_-^\Lambda - g\otimes g^{-1} - g^{-1}\otimes g = z_+^\Lambda(\mathbf{1}-z_-^\Lambda g\otimes g^{-1})(\mathbf{1}-z_-^\Lambda g^{-1}\otimes g)\,. \tag{3.13}$$

The partition function of the gauged system (3.11) is then given by taking the remaining matrix determinant (which we denote **det**) and integration over the gauge group gives the lattice partition function

$$Z_{N,\Lambda} = \int \mu(g)\mathbf{det}[(z_+^\Lambda\mathbf{1}-g^{-1}\otimes g)(\mathbf{1}-z_-^\Lambda g\otimes g^{-1})]^{-\frac{1}{2}} = \int \mu(g)\frac{z_-^{\frac{N^2\Lambda}{2}}}{\mathbf{det}[\mathbf{1}-z_-^\Lambda g\otimes g^{-1}]}. \tag{3.14}$$

[2] Note $g_{1,\Lambda}\dots g_{2,1} = g^{-1}$.

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To obtain this we have used that $z_- = z_+^{-1}$ and that the adjoint representation is Hermitian so that $\mathbf{det}[\mathbf{1} - z_-^\Lambda g \otimes g^{-1}] = \mathbf{det}[\mathbf{1} - z_-^\Lambda g^{-1} \otimes g]$.

We can now take the limit of Λ to infinity keeping βm fixed to obtain the continuum result

$$Z_{N,\infty} = \int \mu(g) \frac{\mathrm{e}^{-\frac{N^2 \beta m}{2}}}{\mathbf{det}[\mathbf{1} - \mathrm{e}^{-\beta m} g \otimes g^{-1}]} \,. \tag{3.15}$$

The numerator $\mathrm{e}^{-\frac{N^2\beta m}{2}}$ comes from the zero point energy of the quantum system of oscillators. If we remove this zero-point energy to consider the normal ordered Hamiltonian we obtain the generating function for the Hilbert-Poincaré series as the Molien-Weyl formula

$$Z_N(x) = \int \mu(g) \frac{1}{\mathbf{det}[\mathbf{1} - x g \otimes g^{-1}]} \tag{3.16}$$

For the gauge group U(N) we can simplify this expression further by diagonalizing the holonomy matrix g. Diagonalization reduces it to the Cartan torus with eigenvalues $z_i = \mathrm{e}^{i\theta_i}$. The measure becomes a product of Vandermonde determinants

$$\Delta(z) = \prod_{1 \le i < j \le N} (z_i - z_j) \tag{3.17}$$

and we have the explicit expression

$$Z_N(x) = \frac{1}{N!} \oint \prod_{i=1}^N \frac{dz_i}{2\pi i} \Delta(z) \Delta(z^{-1}) \prod_{i=1}^N \prod_{j=1}^N \frac{1}{1 - x z_i z_j^{-1}} \tag{3.18}$$

The N contour integrals are around the unit circle and can be evaluated in sequence to give

$$Z_N(x) = \prod_{n=1}^N \frac{1}{1 - x^n} = P_N(x) = \sum_{n=0}^\infty p_N(n) x^n \tag{3.19}$$

i.e. $Z_n(x) = P_N(x)$ is the generating function of partitions of n with at most N parts.

For a system of d matrices transforming simultaneously as $X_a \to g X_a g^{-1}$ with all of the masses m_a distinct, so that $x_a = \mathrm{e}^{-\beta m_a}$, the resulting normal ordered partition function is

$$Z_N(x_1, \cdots, x_d) = \frac{1}{N!} \oint \prod_{i=1}^N \frac{dz_i}{2\pi i} \Delta(z) \Delta(z^{-1}) \prod_{a=1}^d \prod_{i=1}^N \prod_{j=1}^N \frac{1}{1 - x_a z_i z_j^{-1}} \tag{3.20}$$

The case of $d = 2$ has received much attention due to its relevance to the counting of $\frac{1}{4}$-BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills [22, 23].

The specialisation to SU(N) from U(N) is rather trivial since in the path integral formulation the additional degree of freedom associated under the adjoint action is a U(1) factor which cancels. Similarly the trace of each matrix is itself a singlet and removing the trace is equivalent to multiplying $Z_n(x_1, \cdots, x_d)$ by $\prod_{a=1}^d (1 - x_a)$.

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4 Discrete groups and more general representations

Taking the gauge group to be discrete causes little complication. Though it is not clear how to think of the continuum expression (3.1) with $X(\tau) \to g(\tau)X(\tau)g(\tau)^{-1}$ for a $g(\tau) \in G$ with G a discrete group, such as the symmetric group S_N, the lattice expressions (3.2) and (3.3) and lattice actions (3.6) and (3.7) and the steps leading to (3.9) (where we collect the gauge transformations on the final link) make perfect sense. We can therefore proceed as in the continuous group case but with the Haar measure replaced by $1/|G|$ and the integration by the sum over the group elements.

The sum of lattice determinants (3.11) in the finite group case then becomes

$$Z_{N,\Lambda} = \frac{1}{|G|}\sum_{g\in G} \mathbf{Det}^{-1/2}(a^2(\Delta_{\Lambda,g} + \beta^2 m^2 \mathbf{1})) = \frac{1}{|G|}\sum_{g\in G} \mathbf{Det}^{-1/2} M_{\Lambda,g} \tag{4.1}$$

We proceed (see the appendix A) by expanding the determinant in minors and establish

$$\mathrm{Det} M_{\Lambda,g} = z_+^\Lambda + z_-^\Lambda + g\otimes g^{-1} - g^{-1}\otimes g\,, \tag{4.2}$$

directly.

From (4.2) by taking Λ to infinity in (4.1) we obtain the classical Molien form of the expression as

$$Z_N(x) = \frac{1}{|G|}\sum_{g\in G} \frac{1}{\mathbf{det}[\mathbf{1} - xg\otimes g^{-1}]} \tag{4.3}$$

In the case of $G = S_N$, the symmetric group on N elements, explicitly for $N = 2, 3$ and 4 we have

$$\begin{aligned}
Z_{S_2}(x) &= \frac{1}{2!}\frac{1}{(1-x)^4} + \frac{1}{2}\frac{1}{(1-x^2)^2}\\
Z_{S_3}(x) &= \frac{1}{3!}\frac{1}{(1-x)^9} + \frac{1}{2}\frac{1}{(1-x)(1-x^2)^4} + \frac{1}{3}\frac{1}{(1-x^3)^3}\\
Z_{S_4}(x) &= \frac{1}{4!}\frac{1}{(1-x)^{16}} + \frac{1}{4}\frac{1}{(1-x)^4(1-x^2)^6} + \frac{1}{2!2^2}\frac{1}{(1-x^2)^8} + \frac{1}{3}\frac{1}{(1-x)(1-x^3)^5} + \frac{1}{4}\frac{1}{(1-x^4)^4}\,.
\end{aligned} \tag{4.4}$$

We have written the expression with the individual symmetry factors and used that for $N = 2$ the two elements of S_2 in the natural representation can be taken to be the identity and the Pauli matrix σ^1. For $N = 3$ the sum (4.3) is over the 6 possible permutations on three elements

$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\right\} \tag{4.5}$$

with the identity matrix giving the first term, the next 3 matrices in (4.5) giving the second and the last two giving the final term.

The sum of group elements breaks into conjugacy classes of permutations with the permutations labeled by partitions of N. So

$$Z_{S_N}(x) = \frac{1}{N!}\sum_{p\vdash N}\sum_{g\in C(p)} \frac{1}{\mathbf{det}[\mathbf{1} - xg\otimes g^{-1}]} = \sum_{p\vdash N}\frac{1}{\mathrm{Sym}_p}\frac{1}{\mathbf{det}[\mathbf{1} - xg_p\otimes g_p^{-1}]}, \tag{4.6}$$

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where $C(p)$ is the conjugacy class of p, Sym_p is the number of elements of the stabiliser group of the conjugacy class (so that $\frac{N!}{\mathrm{Sym}_p}$ is the number of elements in $C(p)$) and g_p is a single representative of the conjugacy class. The symmetry factor Sym_p is given by

$$\mathrm{Sym}_p = \prod_{i=1}^{s} a_i^{p_i} p_i! \tag{4.7}$$

where we have used the form $p = [a_1^{p_1}, a_2^{p_2}, \cdots, a_s^{p_s}]$, with a_i distinct non-zero parts (cycle lengths) such that $1 \leq a_i \leq N$ and p_i are positive integers ordered as $a_1 < a_2 < \cdots < a_s$ and $N = \sum_{i=1}^{s} a_i p_i$,

By choosing the representative one can evaluate the determinant explicitly [21] and show that in we show that

$$Z_{S_N}(p,x) = \frac{1}{\mathbf{det}[\mathbf{1} - x g_p \otimes g_p^{-1}]} = \prod_i \frac{1}{(1-x^{a_i})^{a_i p_i^2}} \prod_{i<j} \frac{1}{(1-x^{L(a_i,a_j)})^{2G(a_i,a_j)p_i p_j}} \tag{4.8}$$

where $L(a_i, a_j)$ is the least common multiple of a_i and a_j and $G(a_i, a_j)$ is the greatest common denominator of a_i, a_j.

By including the full generality of the quadratic invariant potential $V(x) = G_{ijkl} X_{ij} X_{kl}$, with G in general involving 11-parameters and discussed in [2, 9], we can access the full 11-parameter (7 independent variables when diagonalized) S_N invariant partition functions. The resulting expression

$$Z_N(G) = \sum_{p \vdash N} \frac{1}{\mathrm{Sym}_p} \frac{1}{\mathbf{det}[\mathbf{1} - G g_p \otimes g_p^{-1}]} \tag{4.9}$$

will factor into a product form as described in [21].

5 Generalizations to more general representations

One may further generalize these arguments to cover the case when the matrix X is neither Hermitian nor square and has a different representation acting on the left and right so that the group action is $X \to g_L(g) X g_R^{-1}(g)$. The Euclidean action for an $N_L \times N_R$ matrix then becomes

$$S[X] = \int_0^\beta d\tau \left(\mathbf{tr}(\mathcal{D}_\tau X^\dagger)(\mathcal{D}_\tau X) + m^2 X^\dagger X \right) \tag{5.1}$$

The arguments for accumulating the gauge transformations on the final link still go through and the lattice version becomes

$$S_{\Lambda,R} = \sum_{n,n'=1}^{\Lambda} a \mathbf{tr} X_n^\dagger (\Delta_{\Lambda,R} + m^2 \mathbf{1})_{n,n'} X_{n'} \tag{5.2}$$

where R is the accumulated group action on the left and right, i.e. $R(g) = g_L(g) \otimes g_R^{-1}(g)$. The integration over X and $X^\dagger$ now yields[3]

$$\begin{aligned} Z_{\Lambda,R} &= \mathbf{Det}^{-1}[a^2 \Delta_{\Lambda,R} + \frac{\beta^2 m^2}{\Lambda^2} \mathbf{1}] \\ &= z_+^{-N_L N_R \Lambda} \mathbf{det}^{-1} \left[(\mathbf{1} - z_-^\Lambda R(g))(\mathbf{1} - z_-^\Lambda R(g)^{-1}) \right] \end{aligned} \tag{5.3}$$

[3]Here the comlex Gaussian integration

$$\int \frac{dz d\bar{z}}{\pi a} \mathrm{e}^{-Mz\bar{z}} = \int \frac{dxdy}{2\pi a} \mathrm{e}^{-\frac{1}{2}M(x^2+y^2)} = \frac{1}{aM}\,.$$

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The factor, $z_+^{-N_L N_R \Lambda}$ in the $\Lambda \to \infty$ limit gives $z_+^{-N_L N_R \Lambda} \to \mathrm{e}^{-N_L N_R \beta m}$ which is the contribution from the zero-point energy. In this limit the normal-ordered expression corresponding the Molien-Weyl formula is therefore given by

$$Z_{N_L,N_R}(x) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\mathbf{det}[(1 - xR(g))(1 - xR^{-1}(g))]} \tag{5.4}$$

or correspondingly an integral over the Haar measure in the case of a continuous group.

One can equally take a family of non-interacting matrices that collectively transform under the group i.e. $X^i \to g_L(g) X^i g_R(g)^{-1}$, $i = 1, \cdots, d$ the resulting integrations over the matrices X^A lead to determinants and the evaluations proceed as before, so that we obtain

$$Z_{N_L,N_R}(x,d) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\mathbf{det}^d[(1 - xR(g))(1 - xR^{-1}(g))]} \tag{5.5}$$

of the equivalent expression with Haar measure for a continuous group.

The case of fermions is treated in the companion paper [21] where the partition function is derived.[4] For a system of d matrices of fermionic oscillators transforming in the adjoint representation it is given by

$$Z_N^{\text{Fermions}}(x,d) = \frac{1}{|G|} \sum_{g \in G} \mathbf{det}^d[1 + xg \otimes g^{-1}] \tag{5.6}$$

of the equivalent expression with Haar measure for a continuous group.

A further possible generalization is to restrict the group elements, $g_{n,n+1}$ in (3.7), to a given conjugacy class $C(p)$. The resulting path integral will again give the Molien-Weyl expression

$$Z_N(C(g)) = \sum_{g \in C(g)} \frac{1}{\mathbf{det}[\mathbf{1} - Gg \otimes g_p^{-1}]} = \frac{N!}{\mathrm{Sym}_p} Z_{S_N}(p,x). \tag{5.7}$$

The normalization $\frac{N!}{\mathrm{Sym}_p}$ accounts for the number of elements in the conjugacy class. Eq. (5.7) then provides a path integral interpretation for the individual p dependent partition functions $Z_{S_N}(p,x)$. Equivalently, they correspond to path integrals with holonomy around the thermal circle constrained to a fixed conjugacy class.

6 Conclusions

In the above we provided a path integral derivation for the partition function for a gauged matrix quantum mechanical system. We concentrated on the Gaussian case corresponding to a system of oscillators subject to the constraint that all states are invariant under the group action. We began with a review of the case of a continuous U(N) symmetry and the evaluation of the partition function led to the Molien-Weyl formula for counting U(N) invariant states of a system of oscillators (3.20) when the zero point energy was removed.[5]

[4] The lattice derivation is not given here as it involves the additional complication of adding a Wilson term to deal with Fermion doubling and it would take us outside the scope of this paper.

[5] The zero-point energy is unaffected by gauging the system as the ground state is a gauge singlet.

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The two matrix version of the U(N) invariant case arises in $\mathcal{N} = 4$ super Yang-Mills as the counting problem for $\frac{1}{4}$-BPS states [23].

We then proceeded along similar lines for the finite group case beginning again with a lattice formulation where the gauge field gives parallel transport on the links, replacing the continuous group elements with finite group elements. Our main interest was in the permutation group S_N, however, the analysis is equally applicable to any finite group. Taking the continuum limit led us to the Molien formula (4.3) for the counting of states in the finite group case. Using this for S_N gives a sum over partitions (4.6) which can be analysed directly for small N and leads to our general result (4.8) which is derived and described in detail in [21].

We were able to extend the formulation to arbitrary representations and obtain a form for the general complex case which involves the product of two Molien-Weyl factors (see eq. (5.4)).

The extension to Fermions is straightforward but a lattice formulation requires the elimination of fermion doublers. This is largely a technical complication and we have not gone into these details here. In the companion paper [21] it is shown that the final result is as expected. One gets a determinant instead of an inverse determinant, and the variable x is replaced by $-x$ due to the fermions requiring anti-periodic thermal boundary conditions.

It would also be interesting to study S_N invariant systems in higher dimensions. These can help elucidate features of higher dimensional non-abelian theories along the lines of [11] where the finite subgroups of SU(2) were studied on the lattice. A significant difference between the U(N) and S_N invariant quantum matrix models is the absence of the Vandermonde determinant in the case of S_N. For U(N) gauge systems the Vandermonde is responsible for repulsion between eigenvalues of the U(N). This sets up a competition between the energetic and entropic aspects of the model and as the temperature is reduced leads to a Hagedorn transition. The absence of this repulsion in the S_N and more generally for finite groups leads one to expect the corresponding Hagedorn transition to be driven to zero temperature. This is presumably why the critical temperature was driven to zero in the old study of Rebbi [11]. We will return to the thermodynamics of the permutation invariant matrix quantum mechanical systems in a future paper [24].

Including interactions such as a general potential $V(X)$ for the matrices causes no additional difficulties other than the usual need to resort to perturbation theory. The determinants that arise in the current analysis would be the 1-loop approximations to the partition functions of the interacting systems and one could proceed perturbatively to evaluate approximate partition functions. It is clear, however, from our discussion that there are no additional difficulties introduced by including interactions for the discrete gauged system and our lattice formulation would provide the starting point for non-perturbative Hybrid Monte Carlo studies. It would be especially interesting to find novel S_N invariant interacting systems such as supersymmetric interacting models or potential generalisations of the BFSS [25] and BMN [26] matrix models of M-theory. These latter models have received intense interest and study by hybrid Monte Carlo methods in recent years see for example [20, 27] for the BFSS model and [28–30] for the BMN model.

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A Evaluating lattice determinants

Here we evaluate $\mathbf{Det}(a^2\Delta_{\Lambda,g}+\mu^2)$ where all the gauge field is accumulated in the final link can be evaluated directly by noting that it is a tri-diagonal matrix with the gauge field in the right upper corner and its inverse in the lower corner. The diagonal elements are all $2+\mu^2$ and the off diagonals are -1 and perform a determinant expansion by minors.

Let us evaluate the determinant of the more general matrix

$$M_\Lambda = \begin{pmatrix} d_1 & -\alpha_1 & 0 & 0 & 0 & \cdots & 0 & -g_1 \\ -\beta_1 & d & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & d & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & d & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & d & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & -1 \\ -g_\Lambda & 0 & 0 & 0 & 0\cdots & \cdots & -1 & d \end{pmatrix} \tag{A.1}$$

which is our desired matrix if we set $d_1 = d = 2+\mu^2$, $\alpha_1 = 1$, $\beta_1 = 1$, $g_1 = g \otimes g^{-1}$ and $g_\Lambda = g^{-1} \otimes g$. We will evaluate to determinant for arbitrary commuting parameters d_i,α_1,β_1, g_1 and g_Λ. Expanding the determinant in the first row

$$\mathrm{Det}M_\Lambda = d_1\mathrm{Det}M_{\Lambda-1,0} - \alpha_1\beta_1\mathrm{Det}M_{\Lambda-2,0} - g_1\beta_1 - g_\Lambda\alpha_1 - g_1.g_\Lambda\mathrm{Det}M_{\Lambda-2,0} \tag{A.2}$$

where we have used the notation $M_{\Lambda,0}$ for the tri-diagonal $\Lambda\times\Lambda$ matrix with -1 on the off diagonals and d on the diagonal, i.e. M_Λ with g_1 and g_Λ mapped to zero, which physically $M_{\Lambda,0}$ corresponds to imposing Dirichlet boundary conditions on the lattice Laplacian. Also we denote by $M_{\Lambda,1}$ the periodic tri-diagonal matrix g_1 and g_Λ mapped to 1 and with -1 on the off diagonals and d on the diagonal.

The matrix $M_{\Lambda,g} = a^2\Delta_{\Lambda,g}+\mu^2$ is obtained from (A.1) by setting $d_1 = d = 2+\mu^2$, $\alpha = \beta = 1$, $g_1 = g\otimes g^{-1}$ and $g_\Lambda = g\otimes g^{-1}$ in which case (A.2) becomes

$$\mathrm{Det}M_{\Lambda,g} = (2+\mu^2)\mathrm{Det}M_{\Lambda-1,0} - 2\mathrm{Det}M_{\Lambda-2,0} - g\otimes g^{-1} - g^{-1}\otimes g \tag{A.3}$$

We have already met $\mathrm{Det}M_{\Lambda,1}$ in (2.8) and evaluated it in (2.13) finding that it is given by

$$\mathrm{Det}M_{\Lambda,1} = (2+\mu^2)\mathrm{Det}M_{\Lambda-1,0} - 2\mathrm{Det}M_{\Lambda-2,0} - 2 = z_+^\Lambda + z_-^\Lambda - 2 \tag{A.4}$$

we can therefore conclude that[6]

$$\mathrm{Det}M_{\Lambda,g} = z_+^\Lambda + z_-^\Lambda - g\otimes g^{-1} - g^{-1}\otimes g = z_+^\Lambda(\mathbf{1} - z_-^\Lambda g\otimes g^{-1})(\mathbf{1} - z_-^\Lambda g^{-1}\otimes g) \tag{A.5}$$

[6]The determinant of $M_{\Lambda,g}$ could equally be evaluated by the eigenvalue method, however that would involve taking the Λth root of g. We avoid this complication by a direct evaluation expanding in minors. This allows us to easily extend the analysis to discrete groups.

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and therefore taking the remaining determinant over the group variables we have

$$\mathbf{Det}M_{\Lambda,g} = z_+^{N^2\Lambda}\mathbf{det}[\mathbf{1} - z_-^{\Lambda} g \otimes g^{-1}]\mathbf{det}[\mathbf{1} - z_-^{\Lambda} g^{-1} \otimes g] \tag{A.6}$$

where we have used that the dimension of $\mathbf{1}$ is N^2 — this corresponds to the number of oscillators. As discussed in the text above, taking the $N \to \infty$ of the inverse of the square root of the determinant and summing over g and yields the Molien-Weyl formula

$$Z_N(x) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\mathbf{det}[\mathbf{1} - xg \otimes g^{-1}]}\,. \tag{A.7}$$

More generally if we take $g_1 = R(g)$ and $g_\Lambda = R(g)^{-1}$ with R any representation of the transformation group we obtain

$$\mathbf{Det}M_{\Lambda,R} = z_+^{\nu\Lambda}\mathbf{det}[\mathbf{1} - z_-^{\Lambda} R(g)]\mathbf{det}[\mathbf{1} - z_-^{\Lambda} R(g)^{-1}] \tag{A.8}$$

where we denote the dimension of $\mathbf{1}$ by ν.

Taking the limit of infinite Λ and noting that limit $\lim_{\Lambda\to\infty} z_\pm^{\Lambda} \to = x^{\mp 1}$ yields the general result:

$$\mathbf{Det}M_{\nu,R} = x^{\nu}\mathbf{det}[\mathbf{1} - xR(g)]\mathbf{det}[\mathbf{1} - xR(g)^{-1}] \tag{A.9}$$

Note: this derivation has made minimal assumptions as to the nature or the representation, R, or the group. The Molien-Weyl result is then obtained by summing over the group of the relevant power of the determinant. For generic non-hermitian matrices transforming as $X \to R \circ X$ one obtains

$$Z_\nu(x) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\mathbf{det}[\mathbf{1} - xR(g)]\mathbf{det}[\mathbf{1} - xR(g)^{-1}]} \tag{A.10}$$

while in the real hermitian case we recover the familiar Molien-Weyl formula

$$Z_\nu(x) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\mathbf{det}[\mathbf{1} - xR(g)]} \tag{A.11}$$

with R a real representation.

B Matrices with Dirichlet boundary conditions

As an aside one can proceed to evaluate the Dirichlet determinants (i.e. $\mathrm{Det}M_{\Lambda,0}$) directly as in the periodic case by the eigenvalue method, but it is slightly more involved. One can show that for even Λ ($\Lambda = 2n$)

$$\mathrm{Det}M_{2n,0} = 1 + \sum_{k=1}^{2n}(z_-^{2k} + z_+^{2k}) \tag{B.1}$$

and for odd Λ ($\Lambda = 2n+1$)

$$\mathrm{Det}M_{2n+1,0} = \sum_{k=1}^{2n+1}(z_-^{2k} + z_+^{2k}) \tag{B.2}$$

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These results can be proven by induction. One notes $z_\pm = \frac{1}{2}(d \pm \sqrt{d^2-4})$ so that $d = z_- + z_+$ and $z_- = z_+^{-1}$ and therefore

$$\begin{aligned} d\mathrm{DetM}_{2n-1} &= (z_- + z_+) \sum_{k=1}^{2n+1} (z_-^{2k} + z_+^{2k}) = \mathrm{Det}M_{2n,0} + \mathrm{Det}M_{2n-2,0} \\ &\text{and} \\ d\mathrm{DetM}_{2n} &= (z_- + z_+) \left(1 + \sum_{k=1}^{2n} (z_-^{2k} + z_+^{2k})\right) = \mathrm{Det}M_{2n+1,0} + \mathrm{Det}M_{2n-1,0} \end{aligned} \tag{B.3}$$

which are the recursion relations to be solved.

Performing the sums and simplifying (using $z_+ = z_-^{-1}$) we find

$$\mathrm{DetM}_{\Lambda,0} = f(z_+, \Lambda) = f(z_-, \Lambda) \qquad \text{where} \qquad f(x,\Lambda) = \frac{x^{\Lambda+1} - \frac{1}{x^{\Lambda+1}}}{x - \frac{1}{x}} \tag{B.4}$$

Observing that

$$\left(x + \frac{1}{x}\right) f(x, \Lambda - 1) - 2f(x, \Lambda - 2) = x^\Lambda + \frac{1}{x^\Lambda} \tag{B.5}$$

directly establishes (A.4). Also (B.3) is equivalent to

$$\left(x + \frac{1}{x}\right) f(x, \Lambda - 1) = f(x, \Lambda) + f(x, \Lambda - 2)\,. \tag{B.6}$$

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