CHARACTERIZATION OF OPERATOR SPACES

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Declaration

I declare that this research report is my own unaided work. It is being submitted in partial fulfilment of the requirements for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted for any other degree or examination in any other university.

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Abstract

This research report serves as an introduction to the concept of Operator Spaces which has gained considerable momentum in its acknowledgement and research interest in the last few decades. It will highlight a very important breakthrough on the characterization of Operator spaces which occurred in the last few years brought about by Z.J. Ruan. It investigates the relationship of this space in relation to Banach space theory by looking at an extension theorem for linear functionals.

To my father, mother and baby sister.

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Preface

Operator spaces is a very dynamic mathematical concept which is blossoming at an incredible rate. We discuss its nature set out in the following chapters as follows:

Chapter one: We introduce matricially ordered spaces and operator systems. The main aim of this chapter is to obtain Theorem 1.4 which relates these two ideas and is an important stepping stone in the development of chapter three.

Chapter two: This introduces matrix normed and operator spaces but more importantly it serves to show the abundance of such spaces. Important terminology is introduced to place this theory into sound footing such as completely bounded maps which could be related to as bounded maps in Banach space theory. Ruan's theorem is mentioned as a prelude to chapter three.

Chapter three: A very important chapter motivated towards the establishment of Theorem 3.7 which characterizes operator spaces.

Chapter four: The Haagerup norm is introduced and we discuss its application in injectivity and extension theory of maps. Ideas of 2–row and column summing is introduced which turns out to be an intrinsic property of operator spaces.

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Chapter 1 : Matrix Ordered Spaces

In this chapter we consider the concept of matrix ordered spaces and work towards a relationship between these spaces and operator systems.

Terminology

Let V, W be any two vector spaces. We let $L(V,W)$ be the vector space of linear functions from V to W and $V^d = L(V,\mathbb{R})$, $\big(L(V,\mathbf{C})\big)$ be the real (complex) dual space.

Let V be a complex vector space. We let $M_m(V)$ denote the linear space of $m \times m$ matrices $v = [v_{ij}]$, $v_{ij} \in V$, and let $M_m = M_m(\mathbf{C})$ where M_m has the canonical basis

$$\epsilon_{ij} = \begin{bmatrix} 0 & & 0 \\ & 1_{ij} & \\ 0 & & 0 \end{bmatrix}.$$

If $v \in M_m(V)$, $w \in M_n(V)$ we use the notation

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & m \end{bmatrix} \in M_{m+n}(V).$$

If $\varphi \colon V \to W$ is linear, we define the linear map $\varphi_m \colon M_m(V) \to M_m(W)$ by

$$\varphi_m[v_{ij}] = [\varphi(v_{ij})].$$

A $*$–vector space V is a complex vector space with a conjugate linear map $v \mapsto v^*$ such that $v^{**} = v$. We say v is hermitian if $v^* = v$. Denote V_h to be the real space of all such hermitian elements. In general

$$V = V_h + iV_h, \quad V_h \cap iV_h = \{0\}.$$

If V is a $*$–vector space, we define a $*$ operation on $M_m(V)$ by letting $[v_{ij}]^* = [v_{ji}^*]$. If V and W have $*$ operations and $\varphi \colon V \to W$ is linear, we define

$\varphi^*: V \to W$ by $\varphi^*(v) = \varphi(v^*)^*$. This defines a $*$ operation on $L(V,W)$. If $W = \mathbb{C}$, V^d becomes a $*$–vector space. If $\varphi: V \to W$ is hermitian, then it follows that $\varphi_m: M_m(V) \to M_m(V)$ is hermitian.

We say a complex vector space V is *partially ordered* (or ordered) if it is a $*$–vector space together with a cone (not necessarily proper) $V^+ \subseteq V_h$, i.e. $a, b \in V^+ \Rightarrow a + b \in V^+$, and $a \in V^+$, $\lambda \geq 0 \Rightarrow \lambda a \in V^+$. We write $v \geq w$ if $v - w \in V$. If V is an ordered complex space, an element $e \in V^+$ is an *order unit* for V if for each $v \in V_h$ there is a $t > 0$ such that $-te \leq v \leq te$. The cone V^+ is *Archimedian* if for each $v_0 \in V^+$, $(-t)v_0 \leq v$ for all $t > 0$ implies $v \in V^+$. We say a linear map $\varphi: V \to W$ is *positive* if $\varphi \in L(V,W)_h$ and $\varphi(V^+) \subseteq W^+$. If φ is a linear isomorphism from V onto W, we say φ is an *order isomorphism* if φ and φ^{-1} are positive.

The cone of positive maps $L(V,W)^+$ provides a partial ordering for $L(V,W)$. Letting $C^+ = [0,\infty)$, it follows that V^d is partially ordered by the cone $(V^d)^+$.

Real or complex spaces V and V_1 are said to be *in duality* if there exists a bilinear map $(v,f) \mapsto v.f$ from $V \times V_1$ to the scalars such that

$$(a) \quad v = 0 \text{ iff } v.f = 0 \quad \forall f \in V_1$$

$$(b) \quad f = 0 \text{ iff } v.f = 0 \quad \forall v \in V.$$

If V and V_1 are in duality we denote $B_\sigma(V,W)$ as the space of weakly continuous linear functions from V to W. Let $V^\delta = B_\sigma(V,\mathbb{R}) (= B_\sigma(V,\mathbb{C}))$ be the real (complex) dual.

If V and V^δ are complex spaces in duality then $M_m(V)$ and $M_m(V^\delta)$ are in duality under the bilinear function

$$[v_{ij}].[f_{ij}] = \sum f_{ij}(v_{ij}).$$

In particular if $V = V^\delta = \mathbf{C}$, this gives a duality of M_n with itself.

If V is a $*$–vector space with dual V^δ, we say V^δ is a $*$–vector dual if V^δ is a hermitian subspace of V^d i.e. $f \in V^\delta \Rightarrow f^* \in V^\delta$ then V^δ will be a $*$–vector space.

If V is an ordered complex space with $*$–vector dual V^δ we partially order V^δ by the dual cone $(V^\delta)^+ = V^\delta \cap (V^d)^+$, i.e. $f \in (V^d)^+$ iff $f(v) \geq 0$ for all $v \in V^+$.

Let H be a Hilbert space and $B(H) = B(H, H)$ denote the Von Neumann algebra of bounded linear operators on H. We may identify $M_m(B(H))$ with $B(H^m)$ where $H^m = H \oplus H \oplus \cdots \oplus H$ (m copies) by using the identification

$$[a_{ij}](x_1, \ldots, x_n) = \left(\sum a_{1j}x_j, \ldots, \sum a_{mj}x_j\right).$$

An *operator system* R on a Hilbert space H is a linear subspace of $B(H)$ containing the identity operator I such that if $r \in R$, then $r^* \in R$.

Let R, S be two operator systems. We say a mapping $\varphi\colon R \to S$ is *completely positive* if for all m, the mappings $\varphi_m : M_m(R) \to M_m(S)$ are positive. Also a linear isomorphism $\varphi\colon R \to S$ is a *complete order* isomorphism if both φ and φ^{-1} are completely positive.

A complex vector space V is *matrix ordered* if

(1) V is a $*$–vector space (and so is $M_m(V)$ for all $m \geq 1$).

(2) $M_m(V)$, for all $m \geq 1$ is partially ordered.

(3) If $\gamma = [\gamma_{jq}]$ is any $m \times n$ matrix of complex numbers,

$$\gamma^* M_m(V)^+ \gamma \subseteq M_n(V)^+.$$

Characterization of matrix ordered spaces

We now develop a discussion leading to the characterization of matrix ordered spaces. In particular Theorem 1.4 is of interest.

Given a matrix ordered space V, and an element $v \in M_m(V)$ we define $\Theta(v): M_m \to V$ by

$$\Theta(v)(\alpha) = \sum \alpha_{ij} v_{ij}. \tag{1}$$

Lemma 1.1 If V is a matrix ordered space, then the map

$$\Theta: M_m(v) \to L(M_m, V)$$

is an order isomorphism (with respect to $L(M_m, V)^{\mathscr{C}}$ i.e. the cone of completely positive maps $\varphi_n: M_n(M_m) \to M_n(V)$ for all n).

Proof: It can be clearly seen that Θ is a surjective linear isomorphism. Choose $r \in M_m(V)$ such that $\Theta(r)$ is completely positive. It follows then that

$$\Theta(r)_m: M_m(M_m) \to M_m(V)$$

is positive. Now choose $\epsilon = \delta^* \delta$ where

$$\delta = \begin{bmatrix} \epsilon_{11} & \cdots & \epsilon_{1m} \\ & 0 & \end{bmatrix}.$$

This implies that $\epsilon = [\epsilon_{ij}] \in M_m(M_m)^+$. Hence

$$r = [r_{ij}] = [\Theta(r)(\epsilon_{ij})] = \Theta(r)_m(\epsilon) \geq 0.$$

Conversely, suppose that $r \in M_m(V)^+$. Given $n \geq 1$, we must show that if $\sigma \in M_n(M_m)^+$, then

$$\Theta(r)_n(\sigma) \geq 0.$$

The elements of M_{nm}^{+} are of the form

$$[\bar{\beta}_\mu \beta_\nu]_{\mu,\gamma=1}^{nm}.$$

([1], sect 69). Hence using the map Θ, every element of $M_n(M_m)^+$ is a sum of the elements of the form

$$\sigma = \left[[\bar{\gamma}_{ip}\gamma_{jq}]_{i,j=1}^{m} \right]_{p,q=1}^{n}$$

where we have $\gamma_{jq} = \beta_m(q-1) + j$.

Thus we have

$$\Theta(r)_n(\sigma) = \left[\Theta(r)\left[\bar{\gamma}_{ip}\gamma_{jq}\right]\right] = \left[\sum_{ij} \gamma_{ip} r_{ij} \gamma_{jq}\right] = \gamma^* r \gamma \in M_n(R)^+.$$

__Lemma 1.2__ If V is a matrix ordered space with a *–vector dual space V^δ, then V^δ is a matrix ordered space.

__Proof:__ We prove that property (3) follows: Let $f \in M_m(V^\delta)^+$ and γ be an $m \times n$ matrix and let $v \in M_n(V)^+$ arbitrary. Then

$$\begin{aligned} v.\gamma^* f \gamma &= [v_{pq}].\left[\sum_{ij} \bar{\gamma}_{ip}\gamma_{jq} f_{ij}\right] = \sum \bar{\gamma}_{ip}\gamma_{jq}(v_{pq}.f_{ij}) \\ &= \left[\sum_{pq} \bar{\gamma}_{ip}\gamma_{jq} v_{pq}\right].[f_{ij}] = (\gamma^{t^*} v \gamma^t).f \end{aligned}$$

where γ^{t^*} is the transpose of γ^*. Since V is an ordered matrix space and property (3) follows we have that $\gamma^* M_m(V^\delta)\gamma \subseteq M_n(V^\delta)^+$. ■

We will say that V^δ is the matrix ordered dual of V. Let V^δ and W^δ be the vector space duals of V and W. If $\varphi: V \to W$ is a weakly continuous map we define the *adjoint* map $\varphi^\delta: W^\delta \to V^\delta$ by $v.\varphi^\delta g = \varphi v.g$.

It turns out that the map $\varphi \mapsto \varphi^\delta$ determines a surjective linear isomorphism

$$\delta \colon B_\sigma(V,W) \to B_\sigma(W^\delta, V^\delta)$$

(see [2], Section 21.1). Also it is easy to note that $(\varphi_m)^\delta = (\varphi_m^\delta)$ and that if V and W are *–vector spaces with *–vector duals V^δ, W^δ, then $(\varphi^*)^\delta = (\varphi^\delta)^*$. It follows that $\varphi \colon V \to W$ is completely positive iff $\varphi^\delta \colon W^\delta \to V^\delta$ is completely positive.

Now assume that V and V^δ are dual ordered spaces. Let $v \in M_m(V)$ and define $\Lambda(v) \colon V^\delta \to M_m$ by $\Lambda(v)(f) = [f(v_{ij})]$. Then note that $\Lambda(v) = \Theta(v)^\delta$ since if $f \in V^\delta$ and $\alpha \in M_m$,

$$\Theta(v)(\alpha).f = \sum \alpha_{ij} f(v_{ij}) = \alpha.[f(v_{ij})] = \alpha\Lambda(v)(f).$$

And thus we have the following commutative diagram:

$$\begin{array}{ccc} & \overset{\Theta}{\nearrow} & B_\sigma(M_m,V) = L(M_m,V) \\ M_m(V) & & \downarrow \delta \\ & \underset{\Lambda}{\searrow} & B_\sigma(V^\delta, M_m) \end{array}$$

and since Θ is an order isomorphism and δ is an order isomorphism with relative complete order cones, we can conclude the following lemma:

<u>**Lemma:**</u> Let V and V^δ be dual matrix ordered spaces. Then the map

$$\Lambda \colon M_m(V) \to B_\sigma(V^\delta, M_m)$$

is an order isomorphism (with respect to $B_\sigma(V^\delta, M_m)^\oplus$).

Now we come to a very useful lemma pertaining to identity preserving maps:

Lemma 1.3 Suppose that $A \subseteq B(H)$ is an operator system, R a von-Newmann algebra, and $\varphi\colon A \to R$ is completely positive. Letting $\varphi(1) = b$, there exists a completely positive map $\psi\colon A \to R$ such that $\psi(1) = 1$ and $\psi(a) = b^{1/2}\psi(a)b^{1/2}$.

Proof: Let x be a unit vector in H and e be the support projection for b. Define a completely positive map $\psi^n\colon A \to R$ by

$$\psi^n(a) = \big(b + (1/n)\big)^{-1/2}\varphi(a)\big(b + (1/n)\big)^{-1/2} + (ax.x)(1 - e).$$

The sequence $b^{1/2}\big(b + (1/n)\big)^{-1/2}$ increases with the least upperbound e and thus converges strongly to e. If $a \in A$ such that $0 \leq a \leq 1$, we have $0 \leq \varphi(a) \leq b$ implies that $\varphi(a)^{1/2} = sb^{1/2}$ for some operator $s \in B(H)$ [3]. Thus $\psi(a)^{1/2}\big(b + (1/n)\big)^{-1/2}$ is a bounded sequence that converges strongly. By taking adjoints we have $\varphi(a)^{1/2} = b^{1/2}s^*$ and it is also true for $\big(b + (1/n)\big)^{-1/2}\varphi(a)^{1/2}$. Since A^+ is a bounded set and multiplication is strongly continuous on a bounded set, $\psi^n(a)$ converges strongly for each $a \in A^+$. Since $1 \in A$, each element $a \in A$ is a linear combination of four positive elements and thus $\psi^n(a)$ converges to some $\psi(a) \in R$. It is easily checked that $\psi(a)$ has the desired properties. ■

Theorem 1.4 If V is a matrix ordered space, then V is completely order isomorphic to an operator system iff V^+ is a proper cone with an order unit e, and each of the cones $M_m(V)^+$ is Archimedian. If this is the case, the isomorphism may be assumed to map e to 1.

Proof: The necessity comes from considering the case $m = 1$.

We now show that $M_m(V)^+$ is Archimedian $(m \geq 1)$. Let $X \in M_m(V)^+$.

Then

$$(-t)X = Y \Rightarrow (-t)[x_{ij}] = [y_{ij}] \Rightarrow (-t)x_{ij} = y_{ij} \Rightarrow y_{ij} \in V^+$$

since V^+ is Archimedian and thus $Y \in M_m(V)^+$. Therefore $M_m(V)^+$ is Archimedian.

We show that $M_m(V)^+$ is proper $(m > 1)$. If $v \in M_m(V)^+ \cap (-M_m(V)^+)$, then from Lemma 1.1 $\Theta(v) \geq 0$ and $\Theta(v) \leq 0$. If $\alpha \in M_m^+$, then $\Theta(v)(\alpha) \in V^+ \cap (-V)^+ = \{0\}$ since V^+ is proper. Therefore $\Theta(v) = 0$ and $v = 0$.

Now we show that

$$e_m = \begin{bmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{bmatrix}$$

is an order unit for $M_m(V)$. Firstly note that if $v \in V_h$ and $-te \leq v \leq te$ then

$$\begin{bmatrix} te & v \\ v & te \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[te+v][1\ 1] + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}[te-v][1\ -1] \geq 0.$$

For a general x, choose t with $-te \leq Re\, v,\ Im\, v \leq te$, and note that

$$\begin{bmatrix} 2te & v \\ v^* & 2te \end{bmatrix} = \begin{bmatrix} te & Re\, v \\ Re\, v & te \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} te & Im\, v \\ Im\, v & te \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \geq 0.$$

Thus if $v = [v_{ij}] \in M_m(V)_h$, choose t such that $-te \leq Re\, v_{ij},\ Im\, v_{ij} \leq te$ for all i, j, and then we have

$$2mte_m + v = \frac{1}{2}\sum \gamma_{ij}^* \begin{bmatrix} 2te & v_{ij} \\ v_{ji} & 2te \end{bmatrix} \gamma_{ij} \geq 0$$

where γ_{ij} is the $2 \times m$ matrix with 1 at the $(1,i)$th and $(2,j)$th positions and zero elsewhere. Similarly obtainable is that $2mte_m - v \geq 0$. Hence e_m is an order unit for $M_m(V)$.

If we let $V^\delta = V^d$ we have $M_m(V)^+$ is weakly closed and thus V and V^d are dual matrix ordered spaces.

Let Φ_m be the set of completely positive maps $(m \geq 1)$ such that $\forall \varphi \in \Phi_m$ we have $\varphi: V \to M_m$ and $\|\varphi(e)\| \leq 1$. Then $\oplus M_m^\varphi$ with all $M_m^\varphi = M_m$ is a C^* algebra. We identify $M_n(\oplus_\varphi M_m^\varphi)$ with $\oplus_\varphi M_n(M_m^\varphi)$ by letting

$$\left[(\alpha_{pq}^\varphi)_{\varphi\in\Phi_m}\right] = \left([\alpha_{pq}^\varphi]\right)_{\varphi\in\Phi_m}.$$

If $v \in V$, define a linear map $J^m: V \to \oplus M_m^\varphi$ by $J^m(v) = \left(\varphi(v)\right)_{\varphi\in\Phi_m}$. It is so defined because whenever $-te \leq v \leq te$ we have

$$-t.1 \leq -t\varphi(e) \leq \varphi(v) \leq t\varphi(e) \leq t.1$$

and thus $\|\varphi(v)\| \leq t$ for all $\varphi \in \Phi_m$.

We claim J^m is completely positive because if $v \in M_n(V)$, then

$$(J^m)_n(v) = \left[J^m(v_{pq})\right] = \left[(\varphi(v_{pq}))\right] = \left(\varphi_n(v)\right) \geq 0.$$

We claim that

$$(J^m)_m: M_m(V)^+ \to \oplus_\varphi M_m(M_m)^\varphi$$

is an order isomorphism onto its image.

In order to prove this we must prove that $(J^m)_m$ is one–to–one and that for $v \in M_m(v)$, $(J^m)_m(v) = \left(\varphi_m(v)\right)_{\varphi\in\Phi_m} \geq 0$ implies that $v \geq 0$.

Note that the second statement implies the first since if $(J^m)_m(v) = 0$, then $v \geq 0$, and with $v \leq 0$ implies $v = 0$ because $M_m(V)^+$ is a proper cone.

We know that $M_m(V)^+$ is weakly closed and hence it will be sufficient to prove that if $f = [f_{pq}] \in M_m(V^\delta)^+$, then $f(v) \geq 0$.

Let $\epsilon = [\epsilon_{ij}] \in M_m(M_m)^+$. Then we have that

$$\begin{aligned} f(v) &= \sum f_{ij}(v_{ij}) = \sum_{ij} \left[f_{pq}(v_{ij})\right]_{pq} \epsilon_{ij} = \sum_{ij} \Lambda(f)(v_{ij})\epsilon_{ij} \\ &= \left[\Lambda(f)(v_{ij})\right]\epsilon = \Lambda(f)_m(v)\epsilon. \end{aligned}$$

But we know that $\Lambda(f)$ is completely positive from lemma and may assume that $\varphi_0 = \Lambda(f) \in \Phi_m$. Then we have that

$$f(v) = (\varphi_0)_m(v)\epsilon \geq 0$$

and thus we have that $v \geq 0$.

Set $A = \oplus_m(\oplus_{\varphi \in \Phi_m} M_m^{\varphi})$. Define a linear map $J: V \to A$ by $J(v) = \oplus J^m(v)$. We have that J is completely positive and J is also a complete order isomorphism since if $v \in M_m(V)$ such that $J_m(v) \geq 0$, then $(J^m)_m(v) \geq 0$ and from above $v \geq 0$.

Assume that A is faithfully represented on a Hilbert space H. Let $b = J(e)$ and using Lemma 1.3 there exists a complete order isomorphism J' of V into $B(H)$ such that $J'(e) = 1$. Therefore $J'(V)$ is an operator system.

Chapter 2 : Matrix Normed and Operator Spaces

This chapter will serve as an introduction to the abovementioned spaces and give a hint to the abundant existence of them.

Let X be a vector space over $\mathbb{C}$ and let $M_{n,m}(X)$ denote the vector space of n by m matrices consisting of entries from X and we endow it with the operator norm by considering it as a linear transformation from the Hilbert space $\mathbb{C}^m$ to the Hilbert space $\mathbb{C}^n$.

__Definition 2.1__ X is called a *matrix normed space* if each $M_{n,m}(X)$ is given a norm $\|\cdot\|_{n,m}$ such that for each $A \in M_{n,p}$, $B \in M_{p,q}(X)$, $C \in M_{q,m}$ we have

$$\|ABC\|_{n,m} \leq \|A\| \, \|B\|_{p,q} \, \|C\|.$$

Set $M_n(X) = M_{n,n}(X)$. We assume that the norms $\|\cdot\|$ on each $M_n(X)$ satisfy the following:

[i] For each $B \in M_n(X)$, $0 \in M_m(X)$, $\|B + 0\|_{n+m} = \|B\|_n$.

[ii] For each $B \in M_n(X)$, A, $C \in M_n$ we have $\|ABC\|_n \leq \|A\| \, \|B\|_n \, \|C\|$.

We assign norms to $M_{n,m}(X)$ by embedding it in $M_k(X)$ such that $k = \max\{n, m\}$ by adjoining rows or columns of zeros and take the norms on M_k. For this reason we assign norms to $M_n(X)$.

If X is matrix normed, we set $M_{n,1}(X) = R_n(X)$, $M_{1,n}(X) = C_n(X)$, $R_n(\mathbb{C}) = R_n$ and $C_n(\mathbb{C}) = C_n$. Notice that, for example, by the identification $M_{n,s}(R_n(X)) = M_{n,s}(M_{n,1}(X)) = M_{n^2,s}(X)$, we have R_n to be a matrix normed space. Similarly, $C_n(X)$, R_n, C_n are matrix normed spaces.

Let X and Y be matrix normed spaces and $\varphi: X \to Y$ linear. Define

$\varphi^{(n)}: M_n(X) \to M_n(Y)$ via $\varphi^{(n)}((x_{ij})) = (\varphi(x_{ij}))$. We set $\|\varphi\|_{cb} = \sup\|\varphi^n\|$ and say that φ is ***completely bounded*** when $\|\varphi\|_{cb} < \infty$. We say that φ is ***completely contractive*** if $\|\varphi\|_{cb} \leq 1$ and φ a ***complete isometry*** if each φ^n is an isometry. Two matrix normed spaces are ***completely isometrically isomorphic*** if there is a complete isometry of the first space onto the second.

By the method of adjoining zeros we can see that R_n, and C_n are not complete isometries even though R_n and C_n are isometrically isomorphic by the natural identification.

We now mention an important matrix normed space.

Let H be a Hilbert space and set $H^n = H \oplus \cdots \oplus H$ (n times). Let $B(H) = B(H,H)$ denote the space of bounded linear operators on H. $B(H)$ becomes a matrix normed space if we identify $M_n(B(H))$ with $B(H^n, H^n)$ through the identification $(L_{ij})(h_1 \oplus \cdots \oplus h_n) = (\sum_j L_{1j}(h_j)) \oplus \cdots \oplus (\sum_j L_{nj}(L_j))$ and give this space the operator norm. [Using the fact that if A is any C^* algebra, then A is isomorphic to a subalgebra of $B(H)$, it is interesting to note that every C^*–algebra is a matrix normed space.]

We now make a very important definition.

<u>Definition 2.2</u> An *operator space* X is a matrix normed space which has a completely isometric representation as a subspace of a C^*–algebra.

In Chapter 3, we discuss a very important characterization of operator spaces given by Ruan [4] as L^∞–matrix normed spaces. He defines a L^∞–matrix normed space as a matrix normed space X such that if $A \in M_n(X)$ and $B \in M_m(X)$ then $\|A \oplus B\|_{n+m} = \max\{\|A\|_n, \|B\|_m\}$. He proves that if X is an L^∞-normed space then there exists a completely isometric representation of X as a subspace of a C^*-algebra and conversely that every operator space is a L^∞-matrix normed space.

Let us now discover more matrix normed spaces.

Let X be a matrix normed space. Then X^{op}, the opposite space, where given $x.y \in X$ then $y.x \in X^{op}$, is a matrix normed space through the norm $\|(x_{ij})\|_n^{op} = \|(x_{ij})^t\|_n$ where t represents the transpose map. A nice example is R_n^{op} which is completely isometrically isomorphic to C_n.

In [5] it is shown that if X is a matricially normed space, then the canonical inclusion of X into the continuous functions on the unit ball of its dual aids it to identify X with a subspace of a C^*-algebra and thus endow X to be an operator space. Bletcher and Paulsen in [6] denote this operator space by $MIN(X)$ and denote the norm by $\|\cdot\|_n^{\min}$. Since

$$\begin{aligned}\|(x_{ij})\|_n^{\min} &= \sup\left\{\|f(x_{ij})\| : \|f\| \le 1\right\} \\ &\le \sup\left\{\|(x_{ij})\|\, \|f\| : \|f\| \le 1\right\} \\ &\le \|(x_{ij})\|\end{aligned}$$

we see that $MIN(X)$ is the minimum of all possible matrix norms on X.

Let X be a matrix normed space with the norm $\|(x_{ij})\|_n^{\max} = \sup\left\{\|\varphi(x_{ij})\|\right\}$ where the supremum is taken over all Hilbert space H and over all contractive linear maps φ from X to $B(H)$. Bletcher denotes this space by $MAX(X)$.

If X is a normed space, then define a norm on $M_n(X) = M_n \otimes X$ and give it the projective tensor norm. Bletcher and Paulsen denote this matrix normed space as $PROJ(X)$. This norm turns out to be the largest of all matrix norms on X that are crossnorms on $M_n \otimes X$.

Let X be a matrix normed space and let X^* be the dual space. X^* becomes a matrix normed space by the identification of $M_n(X^*)$ and $M_n(X)^*$ through the duality

$$\langle (f_{ij}),(x_{ij})\rangle = \sum_{ij} f_{ij}(x_{ij}),\ f_{ij}\in X^*,\ x_{ij}\in X.$$

This identification induces a norm on X^*. Bletcher and Paulsen denote it as the *Choi–Effros* dual of X. Similarly, the *left dual* matrix normed space of X, X_ℓ is obtained by the identification of $M_n(X^*)$ with $B\big(C_n(X),C_n\big)$ with

$$(f_{ij})(x_1,\dots,x_n)^t = \Big(\sum_j f_{1j}(x_j),\dots,\sum_j f_{nj}(x_j)\Big)^t,\ f_{ij}\in X^*,\ x_j\in X.$$

Similarly, the *right dual* matrix normed space of X, X_r is obtained by the identification $M_n(X)^*$ with $B\big(R_n(X),R_n\big)$ through

$$(x_1,\dots,x_n)(f_{ij}) = \Big(\sum_i f_{i1}(x_i),\dots,\sum_i f_{in}(x_i)\Big).$$

Let X and Y be matrix normed spaces and $B(X,Y)$ denote the space of bounded linear maps from X to Y. The *left matrix normed* space $B_\ell(X,Y)$ is obtained by identifying $M_n\big(B(X,Y)\big)$ with $B\big(M_n(X),M_n(Y)\big)$ through

$$(L_{ij})(x_{ij}) = \Big(\sum_k L_{ik}(x_{kj})\Big),\ L_{ij}\in M_n\big(B(X,Y)\big).$$

Similarly the *right matrix* normed space $B_r(X,Y)$ is obtained by the identification of $M_n\big(B(X,Y)\big)$ with $B\big(M_n(X),M_n(Y)\big)$ through

$$(x_{ij})(L_{ij}) = \Big(\sum_k L_{kj}(x_{ik})\Big).$$

We now have an interesting proposition:

Proposition 2.3: Let X be a matrix normed space. Then X_ℓ^* and $B_\ell(X,\mathbb{C})$ is completely isometric isomorphic through the identification of X^* with $B(X,\mathbb{C})$.

Proof: $M_n(X_\ell^*) = B(C_n(X), C_n)$ and $M_n(B_\ell(X,C)) = B(M_n(X), M_n)$. We know that $M_n(X_\ell^*)$ can be identified as $M_n(X_\ell)^*$ by $\langle (f_{ij}), (x_{ij}) \rangle = \sum_{ij} f_{ij}(x_{ij})$, $f_{ij} \in X^*$. Consider $(f_{ij})(x_1, \ldots, x_n)^t$ as $(f_{ij})(x_{ij})$ and consider the following norms

(1) $\sup\left\{\sum_i \left|\sum_j f_{ij}(x_j)\right|^2 : \left\|(x_1,\ldots,x_n)^t\right\|_{n,1} \le 1\right\}$.

(2) $\sup\left\{\left\|\left(\sum_k f_{ik}(x_{kj})\right)\right\|^2 : \left\|(x_{ij})\right\|_n \le 1\right\}$.

We see that (1) is less than (2) since (2) has only the first column non-zero. Notice that

$$\left\|\sum_k f_{ik}(x_{kj})\right\|^2 = \sup\left\{\sum_i \left|\sum_{k,j} f_{ik}(x_{kj})\lambda_j\right|^2 : |\lambda_1|^2 + \cdots + |\lambda_n|^2 \le 1\right\}$$

and setting $x'_k = \sum_j x_{kj}\lambda_j$ and realising that $\left\|(x'_1, \ldots, x'_n)\right\| \le 1$ we have that (2) is less than (1). ■

Let X and Y be matrix normed spaces and $CB(X,Y)$ be the space of completely bounded maps from X to Y. Let the norm be

$$\begin{aligned}\left\|(L_{ij})_n\right\|_n = \sup\Big\{\left\|\left(L_{ij}(x_{k\ell})\right)\right\|_{mn} : \left\|(x_{k\ell})\right\|_m \\ \le 1,\ (L_{ij}) \in M_n(CB(X,Y)),\ (x_{k\ell}) \in M_m(X)\Big\}.\end{aligned}$$

This becomes a matrix normed space.

By letting $L: X \to M_n(Y)$ given by $L(x) = (L_{ij}(x))$, we can identify $M_n(CB(X,Y))$ with $CB(X, M_n(Y))$ and $\|(L_{ij})\|_n = \|L\|_{cb}$. With these identifications, we can study $CB(X,Y)$ more:

Proposition 2.4: Let Y be an operator space. Then $CB(X,Y)$ is an operator space.

Proof: Let Y be an operator space. Then by Ruan's characterization, mentioned on page 12, Y is a L^∞–matrix normed space i.e. if $A \in M_n(Y)$, $B \in M_m(Y)$ then $\|(A \oplus B)\|_{n+m} = \max\{\|A\|_n, \|B\|_m\}$.
Now let $A \in M_n(CB(X,Y)) = CB(X, M_n(Y))$ and $B \in M_m(CB(X,Y)) = CB(X, M_m(Y))$. Then $\|A \oplus B\|_{m+n} = \|M_n(Y) \oplus M_m(Y)\| = \max\{\|A\|, \|B\|\}$ since Y is an operator space. Thus $CB(X,Y)$ is an L^∞–matrix normed space and thus an operator space. ∎

Remark:

When $Y = \mathbb{C}$ we can identify X^* with $CB(X, \mathbb{C})$ isometrically and isomorphically. This allows X^* to be an operator space. Bletcher and Paulsen [6], refer to it as the ***standard operator space dual*** of X denoted X^*. They point out the importance of this dual by pointing out a few facts: Let X and Y be operator spaces.

(i) Let $T: X \to Y$ be completely bounded, then the adjoint map $T^*: Y^* \to X^*$ is completely bounded and $\|T^*\|_{cb} = \|T\|_{cb}$.

(ii) If $i: X \to Y$ is a completely isometric inclusion, then $i^{**}: X^{**} \to Y^{**}$ is a completely isometric inclusion.

(iii) $MIN(X)^* = MAX(X^*)$ and $MAX(X)^* = MIN(X^*)$.

An important point which shows a breakoff from Banach space theory is that if X is a matrix normed space then the canonical embedding of X into X^{**} is not a complete isometry since X^{**} is an operator space. However, we have

Theorem 2.5 Let X be an operator space. Then the canoncial embedding of X into X^{**} is a complete isometry.

Proof: Since we assume X is an operator space we may assume that X is subspace in $B(H)$ for some H. Let $\mathcal{F}$ be the family of finite dimensional subspace F of H. For each $F \in \mathcal{F}$, let φ_F be the compression map from $B(H)$ to $B(F)$. If $\tilde{}: X \to X^{**}$ is the canonical embedding we see that for $[x_{ij}] \in M_n(X)$ we have

$$\begin{aligned}
\left\|[\widetilde{x_{ij}}]\right\|_n &= \sup\left\{\left\|\left[[\widetilde{x_{ij}}](f_{k\ell})\right]\right\| : [f_{k\ell}] \in Ball\big(M_m(X^*)\big)\right\} \\
&= \sup\left\{\left\|\left[f_{k\ell}(x_{ij})\right]\right\| : [f_{k\ell}] \in Ball\big(M_n(X^*)\big)\right\} \\
&= \sup\left\{\left\|T_n\big([x_{ij}]\big)\right\| : T \in Ball\big(CB(X, M_m)\big)\right] \\
&\leq \left\|[x_{ij}]\right\|_n .
\end{aligned}$$

Since φ_F is a contractive map, then $\left\|[x_{ij}]\right\| = \sup\left\{(\varphi_F)^n\big([x_{ij}]\big) : F \in \mathcal{F}\right\} \leq \left\|[x_{ij}]\right\|_n$. ■

Chapter 3 : Characterization of Operator Spaces

We now deal with an important characterization of operator spaces as L^∞–spaces which was done by Ruan [5]. We first go through some preliminaries.

Preliminaries

We have already defined the concept of matricially normed spaces in Chapter 2. In addition we have:

Definition 3.1: A matricially normed space E with norm $\| \quad \|_n$ (noted as $(E, \| \quad \|_n)$) is a L^∞–space if it satisfies:

$$\|x \oplus y\|_{n+m} = \sup\{\|x\|_n, \|y\|_m\} \text{ for all } x \in M_n(E),\, y \in M_m(E).$$

We then have the following proposition:

Proposition 3.1: Let $(E, \| \quad \|_n)$ be a matricially normed space. Then we have that:

(1) $\|\alpha x \beta\|_m \leq \|\alpha\| \, \|\beta\| \, \|x\|_n$, $\alpha \in M_{mn}$, $\beta \in M_{nm}$, $x \in M_n(E)$.

(2) $\|x_{ij}\|_1 \leq \left\|[x_{ij}]\right\|_n \leq \sum_{i,j=1}^n \|x_{ij}\|_1$ for all $[x_{ij}] \in M_n(E)$.

Proof:

(1) Let $\alpha \in M_{m,n}, \beta \in M_{n,m}$ and $x \in M_n(E)$. *Case 1: $m > n$:* Consider

$[\alpha, 0], \begin{bmatrix} \beta \\ 0 \end{bmatrix} \in M_m$ then we have that

$$\|\alpha x\beta\|_m = \left\| [\alpha, 0] \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ 0 \end{bmatrix} \right\|_m \le \|[\alpha, 0]\| \left\| \begin{bmatrix} x & \\ 0 & 0 \end{bmatrix} \right\|_m \left\| \begin{bmatrix} \beta \\ 0 \end{bmatrix} \right\|$$
$$= \|\alpha\| \|\beta\| \|x\|_n.$$

Case 2: $m < n$: Consider $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}, [\beta, 0] \in M_n$. Then we have that

$$\|\alpha x\beta\|_m = \left\| \begin{bmatrix} \alpha x\beta & 0 \\ 0 & 0 \end{bmatrix} \right\|_n \le \left\| \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \right\| \|x\|_n \|[\beta, 0]\|$$
$$= \|\alpha\| \|\beta\| \|x\|_n.$$

Case 3: $m = n$: The result follows immediately from Definition 2.1 (ii):

(2) Let $[x_{ij}] \in M_n(E)$ and $\{E_{ij}\}_{i,j=1}^n$ be the system of matrix units of M_n. Then

$$\|x_{ij}\|_1 = \|E_{1i}[x_{ij}]E_{j1}\|_n \le \|[x_{ij}]\|_n.$$

Let $A^* = \{A^i : n \times n$ matrices such that 1st and ith rows is the unit element 1_n in $M_n.\}$

Then A^* is the set of unitary matrices in M_n and we also have that $x_{ij} \otimes E_{ij} = A^i(x_{ij} \otimes E_{ij})A^j$. Then

$$\|x_{ij} \otimes E_{ij}\|_n = \|x_{ij} \otimes E_{11}\|_n = \|x_{ij}\|_1.$$

Thus

$$\|[x_{ij}]\|_n \le \sum_{i,j=1}^n \|x_{ij} \otimes E_{ij}\|_n = \sum_{i,j=1}^n \|x_{ij}\|_1$$

and so

$$\|x_{ij}\|_1 \le \|[x_{ij}]\|_n \le \sum_{i,j=1}^n \|x_{ij}\|_1. \blacksquare$$

Remark: From (2) we see that E is complete under $\| \ \|_1$ iff $M_n(E)$ is complete under $\| \ \|_n$ for every $n \in \mathbb{N}$.

Relationship between L^∞–spaces and Operator Spaces

We will now work our way to prove a g about L^∞-spaces which ill state that it is isometric to an operator space.

Assume E to be an L^∞-space. Let $\bar{E} = \{\bar{x} : x \in E\}$. $\bar{E}$ is a vector space under the operations $\bar{x}+\bar{y} = \overline{x+y}$ and $\lambda.\bar{x} = \overline{\bar{\lambda}x}$. Define the norm on $M_n(\overline{E})$ by

$$\|[\bar{x}_{ij}]\|_n = \|[x_{ji}]\|_n$$

and thus $\overline{E}$ is also an L^∞ matricially normed space.

Now consider the vector space $L = E \oplus \overline{E} \oplus \mathbb{C}$. We identify an element of L say $x \oplus \bar{y} \oplus \lambda$ with a 2×2 matrix

$$\begin{bmatrix} \lambda & x \\ \bar{y} & \lambda \end{bmatrix}$$

for every $x, y \in E$ and $\lambda \in \mathbb{C}$. L is a *–vector space under the involution J defined by

$$\begin{bmatrix} \lambda & x \\ \bar{y} & \lambda \end{bmatrix}^* = \begin{bmatrix} \bar{\lambda} & y \\ \bar{x} & \bar{\lambda} \end{bmatrix}.$$

We can identify

$$\left[\begin{bmatrix} \lambda_{ij} & x_{ij} \\ \bar{y}_{ij} & \lambda_{ij} \end{bmatrix}\right] \text{ in } M_n(L) = M_n(E) \oplus M_n(\overline{E}) \oplus M_n$$

with the matrix

$$\begin{bmatrix} \alpha & x_n \\ \overline{y_n} & \alpha \end{bmatrix}$$

where $\alpha = [\lambda_{ij}] \in M_n$, $x_n = [x_{ij}] \in M_n(\overline{E})$ and $\overline{y_n} = [\overline{y_{ij}}] \in M_n(\overline{E})$. For each $n \in \mathbb{N}$, $M_n(L)$ is a *–vector space with the involution J_n defined by

$$\begin{bmatrix} \alpha & x_n \\ \overline{y_n} & \alpha \end{bmatrix}^* = \begin{bmatrix} \alpha^* & \overline{y_n}^* \\ x_n^* & \alpha^* \end{bmatrix}$$

where

$$x_n^* = [\overline{x_{ij}}] \text{ and } \bar{y}_n^* = [y_{ji}].$$

The self–adjoint part of $M_n(L)$ is written as

$$M_n(L)_h = \left\{ \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} : \alpha \text{ is self–ajoint in } M_n, x_n = [x_{ij}] \in M_n(E) \right\}.$$

We also define

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)_h$$

to be positive if $\alpha \geq 0$ in M_n and for any $\epsilon > 0$,

$$\left\| \alpha_\epsilon^{-1/2} x_n \alpha_\epsilon^{-1/2} \right\|_n \leq 1$$

where $\alpha_\epsilon = \alpha + \epsilon 1_n$ and $\alpha_\epsilon^{-1/2} = (\alpha_\epsilon^{+1/2})^{-1}$ in M_n. We denote $M_n(L)^+$ as the set of all positive elements in $M_n(L)_h$. Firstly, a lemma:

<u>**Lemma 3.3:**</u> For every $n \in \mathbb{N}$, $M_n(L)^*$ is a positive cone in $M_n(L)_h$. With this matrix order, $(L, J, M_n(L)^+)$ becomes a matrix ordered space.

<u>**Proof:**</u> The zero element of $M_n(L)$ is in $M_n(L)^+$.

Firstly, let $\lambda > 0$ and

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)^+.$$

Then

$$\lambda \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} = \begin{bmatrix} \lambda\alpha & \lambda x_n \\ (\lambda x_n)^* & \lambda\alpha \end{bmatrix}$$

such that $\lambda\alpha \geq 0$ in M_n. Let $\epsilon > 0$. Then

$$\begin{aligned} \left\| (\lambda\alpha)_\epsilon^{-1/2} (\lambda x_n) (\lambda\alpha)_\epsilon^{-1/2} \right\|_n &= \left\| \left(\frac{\alpha + \epsilon 1_n}{\lambda^{-1/2}} \right)^{-1/2} \lambda x_n \left(\frac{\alpha + \epsilon 1_n}{\lambda^{-1/2}} \right)^{-1/2} \right\| \\ &= \left\| \alpha_\epsilon^{-1/2} x_n \alpha_\epsilon^{-1/2} \right\| \leq 1 \end{aligned}$$

and thus

$$\lambda \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)^+.$$

Secondly, let $X \in M_{nm}$ and

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)^+.$$

Then

$$X^* \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} X = \begin{bmatrix} X^*\alpha X & X^* x_n X \\ (X^* x_n X)^* & X^*\alpha X \end{bmatrix} \in M_m(L)_h$$

such that $X^*\alpha X \geq 0$ in M_m. Assume that $X \neq 0$ and let $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{\|X^*X\|} > 0$. Then

$$\begin{aligned}
&\left\|(X^*\alpha X)_\epsilon^{-1/2}(X^* x_n X)(X^*\alpha X)_\epsilon^{-1/2}\right\|_n \\
&= \left\|(X^*\alpha X)_\epsilon^{-1/2} X^* \alpha_{\epsilon_1}^{1/2}(\alpha_{\epsilon_1}^{-1/2} x_n \alpha_{\epsilon_1}^{-1/2})\alpha_{\epsilon_1}^{1/2} X (X^*\alpha X)_\epsilon^{-1/2}\right\|_n \\
&= \left\|Y^*(\alpha_{\epsilon_1}^{-1/2} x_n \alpha_{\epsilon_1}^{-1/2}) Y\right\|_n \text{ such that } Y = \alpha_{\epsilon_1}^{1/2} X (X^*\alpha X)_\epsilon^{-1/2} \in M_{nm} \\
&\leq \|Y^*\| \left\|\alpha_{\epsilon_1}^{-1/2} x_n \alpha_{\epsilon_1}^{-1/2}\right\|_n \|Y\| \text{ by Proposition 3.1} \\
&\leq \|Y^*\| \, \|Y\| \text{ since } \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)^\dagger \text{ and } \left\|\alpha_{\epsilon_1}^{-1/2} x_n \alpha_{\epsilon_1}^{-1/2}\right\| \leq 1 \\
&= \left\|(X^*\alpha x)_\epsilon^{-1/2} X^* \alpha_{\epsilon_1} X (X^*\alpha X)_\epsilon^{-1/2}\right\| \\
&\leq \left\|(X^*\alpha X)_\epsilon^{-1/2}(X^*\alpha X)_\epsilon (X^*\alpha X)_\epsilon^{-1/2}\right\| = \|1_n\| = 1.
\end{aligned}$$

Since

$$X^*\alpha_{\epsilon_1} X = X^*\left(\alpha + \frac{\epsilon}{\|X^*X\|}\right) X \leq X^*\alpha X + \epsilon 1_n.$$

Therefore

$$X^* \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} X \in M_m(L)^+.$$

Thirdly, suppose that

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix}, \begin{bmatrix} \beta & y_n \\ y_n^* & \beta \end{bmatrix} \in M_n(L)^+.$$

Let

$$Q = \begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \oplus \begin{bmatrix} \beta & y_n \\ y_n^* & \beta \end{bmatrix} = \begin{bmatrix} \alpha \oplus \beta & (x_n \oplus y_n) \\ (x_n \oplus y_n)^* & \alpha \oplus \beta \end{bmatrix} \in M_{2n}(L)_h.$$

Then $(\alpha \oplus \beta) \geq 0$ in M_{2n}. Let $\epsilon > 0$. Then

$$\begin{aligned}
&\left\|(\alpha \oplus \beta)_\epsilon^{-1/2}(x_n \oplus y_n)(\alpha \oplus \beta)_\epsilon^{-1/2}\right\|_{2n} \\
&= \left\|(\alpha_\epsilon^{-1/2} \oplus \beta_\epsilon^{-1/2})(x_n \oplus y_n)(\alpha_\epsilon^{-1/2} \oplus \beta_\epsilon^{-1/2})\right\|_{2n} \\
&= \left\|(\alpha_\epsilon^{-1/2} x_n \alpha_\epsilon^{-1/2}) \oplus (\beta_\epsilon^{-1/2} y_n \beta_\epsilon^{-1/2}\right\|_{2n} \\
&= \max\left\{\left\|\alpha_\epsilon^{-1/2} x_n \alpha_\epsilon^{-1/2}\right\|_n, \left\|\beta_\epsilon^{-1/2} y_n \beta_\epsilon^{-1/2}\right\|_n\right\} \leq 1
\end{aligned}$$

by the L^∞-condition. Therefore $Q \in M_{2n}(L)^+$.

Then

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} + \begin{bmatrix} \beta & y_n \\ y_n^* & \beta \end{bmatrix} = [1_n, 1_n] Q \begin{bmatrix} 1_n \\ 1_n \end{bmatrix} \in M_{2n}(L)^+.$$

Thus $M_n(L)^+$ is a positive cone in $M_n(L)_h \forall n \in \mathbb{N}$ and $(L, J, M_n(E)^+)$ becomes a matrix ordered space.

<u>Lemma 3.4:</u> Let

$$e = \begin{bmatrix} 1 & 0 \\ \bar{0} & 1 \end{bmatrix} \in L^+$$

and $e_n = e \oplus \cdots \oplus e_n \in M_n(L)^+$. Then L^+ is a proper cone in L_h with order unit e for every $n \in \mathbb{N}$ and $M_n(L)^+$ is Archimedian with order unit e_n.

<u>Proof:</u> Assume

$$\begin{bmatrix} \lambda & x \\ \bar{x} & \lambda \end{bmatrix} \in L^+ \cap (-L^+).$$

Since $\lambda \geq 0$ and $\lambda \leq 0$ this implies that $\lambda = 0$.

Let $\epsilon > 0$, then

$$\left\|(\lambda + \epsilon)^{-1/2} x (\lambda + \epsilon)^{-1/2}\right\| = \left\|\frac{x}{\lambda + \epsilon}\right\| = \frac{\|x\|}{\epsilon} \leq 1$$

which implies that $\|x\| = 0$.

Therefore $L^+ \cap (-L^+) = \{0\}$ and thus L^+ is a proper cone.

Let

$$\begin{bmatrix} \lambda & x \\ \bar{x} & x \end{bmatrix} \in L_h.$$

Choose $t > 0$ sufficiently large enough such that $t \pm \lambda \geq \|x\| \geq 0$. Then

$$te \pm \begin{bmatrix} \lambda & x \\ \bar{x} & \lambda \end{bmatrix} = \begin{bmatrix} t \pm \lambda & x \\ \bar{x} & t \pm \lambda \end{bmatrix} \geq 0$$

in L_h and thus e is an order unit of L^+.

Similarly e_n is an order unit of $M_n(L)^+$ for every $n \in \mathbb{N}$. Suppose

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)_h$$

such that

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} + te_n \geq 0 \; \forall t > 0.$$

Then $\alpha + t1_n \geq 0$ for all $t \geq 0$ and this implies that $\alpha \in M_n^+$. Let $\epsilon > 0$. Then

$$\|\alpha_\epsilon^{-1/2} x_n \alpha_\epsilon^{-1/2}\|_n = \left\|(\alpha + \frac{\epsilon}{2})\frac{\epsilon^{-1/2}}{2} x_n (x + \frac{\epsilon}{2})\frac{\epsilon^{-1/2}}{2}\right\|_n \leq 1.$$

Thus

$$\begin{bmatrix} \alpha & x_n \\ x_n^* & \alpha \end{bmatrix} \in M_n(L)^+$$

and $M_n(L)^+$ is Archimedian for every $n \in \mathbb{N}$. ■

<u>Corollary 3.5:</u> There exists a complete order isomorphic injection $\tau: L \to B(H)$ for some Hilbert space H such that $\tau(e) = 1_H$.

<u>Proof:</u> From Theorem 1.4, we find that L is completely order isomorphic to an operator system, i.e. there exists a map τ such that $\tau(e) = 1$ where $\tau: L \to B(H)$ for some Hilbert space and being a complete order isomorphic injection. ■

We have basically shown that $\tau(L) = L_H \subseteq B(H)$ and thus L_H is an operator system. The other implication is that its matricial norm can be expressed in terms of the matrix order of $\mathcal{B}(\mathcal{H})$. Effros describes the norm as

$$\left\|[x_{ij}]\right\|_n = \inf\left\{\lambda > 0;\ \begin{bmatrix} \lambda 1_n & [x_{ij}] \\ [x_{ij}] & \lambda 1_n \end{bmatrix} \geq O_{2n} \text{ in } M_{2n}(\mathcal{B}(\mathcal{H}))\right\}$$

for every $[x_{ij}] \in M_n(E)$.

Ruan defines a norm on $M_n(L)$ by

$$\|u\|_n = \inf\left\{\lambda > 0;\ \begin{bmatrix} \lambda e_n & u \\ u^* & \lambda e_n \end{bmatrix} \geq O_{2n} \text{ in } M_{2n}(L)\right\}$$

for all

$$u = \begin{bmatrix} \alpha & x \\ \bar{y}_n & \alpha \end{bmatrix} \in M_n(L)$$

where $x_n = [x_{ij}], \bar{y}_n = [\bar{y}_{ij}] \in M_n(\overline{E})$ and $\alpha = [\lambda_{ij}] \in M_n$.

τ is an order isomorphism and with this norm τ is a complete isometry.

<u>**Lemma 3.6:**</u> There exists a complete isometry $\eta: E \to L$.

<u>**Proof:**</u> Let $\eta: E \to L$ be a map defined such that

$$\eta(x) = \begin{bmatrix} 0 & x \\ \bar{0} & 0 \end{bmatrix}.$$

Let $x_n \in M_n(E)$ such that $\|x_n\|_n = 1$. Then we have that

$$\eta_n(x_n) = \begin{bmatrix} 0 & x_n \\ \bar{0}_n & 0 \end{bmatrix}$$

with 0 and 0_n as zero elements in M_n and $M_n(E)$ respectively.

By the definition of matricial normed spaces we have that

$$\left\|\begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix}\right\|_{2n} = \left\|\begin{bmatrix} x_n & 0_n \\ 0_n & 0_n \end{bmatrix}\right\|_{2n} = \|x_n\|_n = 1.$$

Let $\epsilon > 0$. Then

$$\left\|(1_{2n})_\epsilon^{-1/2} x_n (1_{2n})_\epsilon^{-1/2}\right\|_{2n} = \left\|(1_{2n})_\epsilon^{-1/2} \begin{bmatrix} 0 & x_n \\ 0 & 0 \end{bmatrix} (1_{2n})_\epsilon^{-1/2}\right\|_{2n}$$

$$= \frac{\left\|\begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix}\right\|_{2n}}{1+\epsilon} = \frac{\|x_n\|_n}{1+\epsilon} \leq 1.$$

Also using the matricial norm mentioned before we see that

$$\begin{bmatrix} e_n & \psi(x_n) \\ \psi(x_n)^* & e_n \end{bmatrix} = \begin{bmatrix} 1_{2n} & x_d \\ x_d^* & 1_{2n} \end{bmatrix} \geq 0_{2n} \text{ in } M_{2n}(L)$$

where

$$x_d = \begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix} \in M_{2n}(E).$$

We also see that $\left\|\psi_n(x_n)\right\|_{2n} \leq 1$.

Now let $r = \left\|\psi_n(x_n)\right\|_{2n}$. Then note that

$$\begin{bmatrix} r1_{2n} & x_d \\ x_d^* & r1_{2n} \end{bmatrix} = \begin{bmatrix} re_n & \psi_n(e_n) \\ \psi_n(x_n)^* & re_n \end{bmatrix} \geq 0_{2n} \text{ in } M_{2n}(L).$$

Let $\epsilon > 0$. Then

$$\left\|(r1_{2n})_\epsilon^{-1/2} \begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix} (r1_{2n})_\epsilon^{-1/2}\right\|_{2n} \leq 1$$

and we also have

$$\left\|(r1_{2n})_\epsilon^{-1/2} \begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix} (r1_{2n})_\epsilon^{-1/2}\right\|_{2n} = \frac{\left\|\begin{bmatrix} 0_n & x_n \\ 0_n & 0_n \end{bmatrix}\right\|_{2n}}{r+\epsilon}$$

$$= \frac{\|x_n\|_n}{r+\epsilon}.$$

Therefore,

$$1 = \|x_n\|_n \leq r = \left\|\psi_n(x_n)\right\|_{2n} \leq 1.$$

Thus $\left\|\psi_n(x_n)\right\|_{2n} = \|x_n\|_n$ which implies ψ_n is a complete isometry. ■

We have finally come to the main theorem:

Theorem 3.7: Let $(E, \| \ \|_n\}$ be a matricially normed space. Then E is an L^∞ matricially normed space iff E is completely isometric to an operator space.

Proof: $\Leftarrow$ If E is completely isometric to an operator space then E can be considered as a linear subspace of $\mathcal{B}(H)$ where H is a Hilbert space. Using the operator norm, all conditions are satisfied to make E a L^∞ matricially normed space.

$\Rightarrow$ Assume E is an L^∞ matricially normed space and consider the map $\xi = \tau \circ \eta : E \to B(H)$ which gives a complete isometry. ∎

Chapter 4 : The Haagerup Norm

In this chapter, we involve in a discussion to motivate the existence of Hahn–Banach analogoues of completely bounded maps in operator space theory. To do this we introduce the Haagerup norm and also show that it is completely injective.

Definition 4.1: Let $A = [x_{ij}] \in M_{n,k}(X)$ and $B = [y_{ij}] \in M_{k,m}(Y)$. The operation $A \odot B$ is an element of $M_{n,m}(X \otimes Y)$ given by the definition

$$A \odot B = \left(\sum_{r=1}^{k} x_{ir} \otimes y_{rj} \right).$$

Remark: This operation was firstly introduced by Effros.

Definition 4.2: Let X and Y be matrix normed spaces. Then the Haagerup matrix norm on $X \otimes Y$ is defined by

$$\|U\|_h = \inf \left\{ \sum_{k=1}^{m} \|A_k\| \, \|B_k\| : U = \sum_{k=1}^{m} A_k \odot B_k \right\}$$

where the infimum is taken over all such representations of U.

Definition 4.3: A matrix normed space X is called *2-row summing* if $x = (x_1, \ldots, x_k) \in R_k(X)$, $y = (x_{k+1}, \ldots, x_{k+j}) \in R_j(X)$ implies that $z = (x_1, \ldots, x_{k+j}) \in R_{k+j}(X)$ satisfies

$$\|z\|^2 \leq \|x\|^2 + \|y\|^2.$$

Similarly we can define *2-column summing* analogously. X is said to be 2-summing if it is both 2-row and 2-column summing.

The above definition is very useful since it gives a different perception on the calculation of the Haagerup norm as seen in the following norm:

<u>**Lemma 4.4:**</u> If X is 2–row summing and Y is 2–column summing, then for u in $X \otimes Y$,

$$\|u\|_h = \inf\left\{\|A\|\,\|B\| : u = A \odot B\right\}.$$

<u>**Proof:**</u> If $u = \sum_{i=1}^{k} A \odot B$ such that $A_i \in M_{1,n_i}(X)$, $B_i \in M_{n_i,1}(Y)$ then $u = A \odot B$ where

$$A = \begin{bmatrix} r_1 A_1 \\ r_2 A_2 \\ \vdots \\ r_k A_k \end{bmatrix} \in M_{1,n}(X) \text{ and } B = \left[r_1^{-1} B_1, \ldots, r_k^{-1} B_k\right] \in M_{n,1}(Y).$$

with the condition that $n = n_1 + \cdots + n_k$ and $r_i > 0$. Now, using the hypothesis of 2 row and 2 column summing of X and Y respectively we have that

$$\|A\|^2 \|B\|^2 \leq \left(\sum_{i=1}^{k} r_i^2 \|A_i\|^2\right)\left(\sum_{i=1}^{k} r_i^{-2} \|B_i\|^2\right).$$

Let $r_i = \|B_i\|/\|A_i\|$. Then

$$\|A\|\,\|B\| \leq \sum_{i=1}^{k} \|A_i\|\,\|B_i\| \text{ and we obtain } \|u\|_h. \blacksquare$$

To discuss the injectivity, we firstly do some preliminaries on tensors:
If $u \in X \otimes Y$ and $u = \sum_{i=1}^{n} x_i \otimes y_i$, let us set $E = span\{x_1, \ldots, x_n\}$, $F = span\{y_1, \ldots, y_n\}$. We choose a basis in F say $\{y_1, \ldots, y_k\}$. Then we can write $u = \sum_{i=1}^{k} x_i' \otimes y_i$ after reexpressing $y_{k+1}, \ldots, y_n$ in terms of the basic elements. Similarly, by choosing basis elements $x_1', \ldots, x_m'$ for the span of $\{x_1', \ldots, x_k'\}$ we can write $u = \sum_{i=1}^{m} x_i' \otimes y_i'$. We can continue this process until we express $u = \sum_{i=1}^{t} \tilde{x}_i \otimes \tilde{y}_i$ for some $t \in \mathbb{N}$.

Lemma 4.5 Let $u \in X \otimes Y$ such that $u = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{k} v_j \otimes w_j$ and $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, $\{v_1, \ldots, v_k\}$, $\{w_1, \ldots, w_k\}$ are linearly independent. Then $span\{x_1, \ldots, x_n\} = span\{v_1, \ldots, v_m\}$ and $span\{y_1, \ldots, y_n\} = span\{w_1, \ldots, w_m\}$.

Proof: Let $f_i \colon X \to \mathbf{C}$ be linear functionals such that $f_i(x_j) = \delta_{ij}$. Then for $f_i \otimes 1 \colon X \otimes Y \to Y$, it follows that $y_i = (f_i \otimes 1)(u) = \sum_{k=1}^{m} f_i(v_k) w_k = \sum_{k=1}^{m} f_i(v_k) w_k$ which implies that $y_i \in span\{w_1, \ldots, w_m\}$. The rest is proved similarly.

Theorem 4.6: Let X and Y be matrix normed spaces such that X is 2–row summing and Y column 2–summing. Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be subspaces. Then the inclusion of $X_1 \otimes_h Y_1$ into $X \otimes_h Y$ is an isometry.

Proof: Let $\|u\|_1$ denote the norm of u in $X_1 \otimes_h Y_1$ and let $\|u\| = \inf\{\|A\| \, \|B\| : u = A \odot B\}$, $A \in M_{1,n}(X)$, $B \in M_{n,1}(Y)$ be the norm of u in $X \otimes_h Y$. Then, $\|u\| \leq \|u\|_1$.

Set $u = A \odot B$, $A = (x_1, \ldots, x_n) \in M_{1,n}(X)$, $B = (y_1, \ldots, y_n)^t \in M_{n,1}(Y)$ and assume that $y_1, \ldots, y_k$ are linearly independent. We write $B = L(b_1, \ldots, b_k)^t$ where L is an $n \times k$ matrix of scalars. If we let $L = UP$ be the polar form of L, then U is an $n \times k$ isometry and P is a $k \times k$ invertible matrix. Let $B' = (b'_1, \ldots, b'_k)^t = P(b_1, \ldots, b_k)^t$ and so $u = A \odot (UB') = (AU) \odot B'$. Note that

$$\|A\| \, \|B\| = \|A\| \, \|UB'\| = \|A\| \, \|B'\| \geq \|AU\| \, \|B'\|.$$

Thus, we have written $u = A' \odot B'$ such that $\|A'\| \, \|B'\| \leq \|A\| \, \|B\|$ and with the entries of B' linearly independent.

Since $u \in X \otimes Y$, by Lemma 4.5 we see that the entries of A'' and of B''

belong to X_1 and Y_1, respectively. Hence $\|u_1\| \leq \|A''\| \, \|B''\| \leq \|A\| \, \|B\|$ and since A and B are arbitrary, $\|u_1\| \leq \|u\|$.

Proposition 4.7: Let X and Y be matrix normed spaces. Then the map $\Phi \colon C_n(X) \otimes_h R_n(Y) \to M_n(X \otimes_h Y)$ given by $\Phi(A \otimes B) = A \odot B$ is a completely isometric isomorphism.

Proof: Let $U = \sum_{i=1}^{k} A_i \otimes B_i = (A_1, \ldots, A_k) \odot (B_1, \ldots, B_k)^t$, then $\Phi(U) = \sum_{i=1}^{k} A_i \odot B_i$. If we perceive $V = (A_1, \ldots, A_k)$ as an element of $M_{n,k}(X)$ with columns A_i, and $W = (B_1, \ldots, B_k)^t$ as an element of $M_{k,n}(Y)$ with rows B_i then $\Phi(U) = V \odot W$.

Since $\|(A_1, \ldots, A_k)\|$ in $M_{1,k}(C_n(X))$ is the same as $\|V\|$ in $M_{n,k}(X)$ and $\|(B_1, \ldots, B_k)^t\|$ in $M_{k,1}(R_n(Y))$ is the same as $\|W\|$ in $M_{k,n}(Y)$, it follows that $\|U\| = \|\Phi(U)\|_h$, and so Φ is an isometry. The proof for complete isometry is identical.

Before we proceed we must realise a very important point.

Lemma 4.8: All operator spaces are 2–summing.

Proof: By Ruan's theorem we know that operator space X is a L^∞–space. Let $(x_1, \ldots, x_n) = V \in R_n(X)$ and $(y_{n+1}, \ldots, y_j) = W \in R_j(X)$. Let $Z = (x_1, \ldots, x_n, y_{n+1}, \ldots, y_j)$. Then $\|z\| = \|V \oplus W\| = \max\{\|V\|, \|W\|\} = \|W\|$ say, without loss of generality. Therefore $\|z\|^2 = \|W\|^2 \leq \|W\|^2 + \|V\|^2$ and thus X is row–2–summing. Similarly, it can be proved that X is column–2–summing. ■

Theorem 4.9: Let X and Y be operator spaces, $X_1 \subseteq X$, $Y_1 \subseteq Y$ subspaces. Then the inclusion of $X_1 \otimes_h Y$ into $X \otimes_h Y$ is a complete isometry.

Proof: $C_n(X)$ and $R_n(Y)$ are operator spaces and thus 2-summing. Therefore $C_n(X_1) \otimes_h R_n(Y_1)$ is isometrically embedded into $C_n(X) \otimes_h R_n(Y)$ by Theorem 4.6 and by Proposition 4.7, $X_1 \otimes_h Y$ into $X \otimes_h Y$ is completely isometric. ∎

We note the following with the defined maps. Let X, Y and Z be matrix normed spaces and let $\varphi: X \otimes_h Y \to Z$ be completely bounded and define $L_\varphi: X \to B_\ell(Y, Z)$ and $R_\varphi: Y \to B_r(X, Z)$ by $L_\varphi(x)(y) = \varphi(x \otimes y)$ and $R_\varphi(y)(x) = \varphi(x \otimes y)$.

If $A \in M_{n,k}(X)$ and $B \in M_{k,n}(Y)$ we have that $\varphi^n(A \odot B) = \left(L_\varphi^n(A)(B)\right) = R_\varphi^n(B)(A)$ and thus we have that

$$\|\varphi\|_{cb} = \|L_\varphi\|_{cb} = \|R_\varphi\|_{cb}.$$

With these identifications we have that:

Proposition 4.10: Let X, Y, Z be matrix normed spaces. Then using the maps $\varphi \to L_\varphi$ and $\varphi \to R_\varphi$ we have the isometries from the space $CB(X \otimes_h Y, Z)$ onto $CB(X, B_\ell(Y, Z))$ and onto $CB(Y, B_r(X, Z))$ respectively.

We now have the Hahn–Banach extension theorem analogue:

Theorem 4.11: Let X be an operator space, and let $Y \subseteq X$ be a subspace. Then every completely bounded map $\varphi: Y \to M_n$ can be extended to a completely bounded map $\tilde{\varphi}: X \to M_n$ with

$$\|\varphi\|_{cb} = \|\tilde{\varphi}\|_{cb}.$$

Proof: R_n and X are operator spaces and by Theorem 4.9 we have that $R_n \otimes_h Y$ is completely isometrically embedded in $R_n \otimes_h X$. Therefore, every bounded linear functional on $R_n \otimes_h Y$ extends to a bounded linear functional on $R_n \otimes_h X$. Using the isometries $CB(R_n \otimes_h Y, \mathbb{C}) \cong CB(Y, B_r(R_n, \mathbb{C}))$ and $CB(R_n \otimes_h X, \mathbb{C}) \cong CB(X, B_r(R_n, \mathbb{C}))$ we see that every completely bounded linear map from Y to $B_r(R_n, \mathbb{C})$ extends to a map from X to $B_r(R_n, \mathbb{C})$ preserving the cb-norm. Since $B_r(R_n, \mathbb{C})$ is isomorphic to C_n, we have an extension theorem for maps into C_n. In [7], Paulsen and Smith have shown that the Haagerup tensor product of operator spaces is again an operator space. Now consider the operator spaces $Y \otimes_h C_n \subseteq X \otimes_h C_n$. By a similar argument as above every map in $CB(Y \otimes_h C_n, C_n)$ extends to a map in $CB(X \otimes_h C_n, C_n)$. However, we know that $CB(Y \otimes_h C_n, C_n) \cong CB(Y, B_\ell(C_n, C_n))$ and $CB(X \otimes_h C_n, C_n) \cong CB(X, B_\ell(C_n, C_n))$. Therefore we have an extension theorem for maps from Y into $B_\ell(C_n, C_n)$. To complete the proof, we must realise that $B_\ell(C_n, C_n) = M_n$ through the identification of a matrix with a linear transformation. ■

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