BUNDLES IN THE CATEGORY OF FRÖLICHER SPACES AND SYMPLECTIC STRUCTURE

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 2006

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Bundles and morphisms between bundles are defined in the category of Frölicher spaces (earlier known as the category of smooth spaces, see [2], [5], [9], [6] and [7]). We show that the sections of Frölicher bundles are Frölicher smooth maps and the fibers of Frölicher bundles have a Frölicher structure. We prove in detail that the tangent and cotangent bundles of a n-dimensional pseudomanifold are locally diffeomorphic to the even-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^{2n}$. We define a bilinear form on a finite-dimensional pseudomanifold. We show that the symplectic structure on a cotangent bundle in the category of Frölicher spaces exists and is (locally) obtained by the pullback of the canonical symplectic structure of $\mathbb{R}^{2n}$. We define the notion of symplectomorphism between two symplectic pseudomanifolds. We prove that two cotangent bundles of two diffeomorphic finite-dimensional pseudomanifolds are symplectomorphic in the category of Frölicher spaces.

To my wife Gisèle MAYUBA TOKO and my parents Jean Bedel TOKO NKOBA and Elizabeth LUSINGAMA BOMBE.

Acknowledgements

I am grateful to my supervisors for their effective supports. Prof Kathy Driver has believed in my capacities and made my Masters' study possible. I specially want to acknowledge the numerous advices and suggestions Dr Augustin Tshidibi Batubenge has made to complete this degree. I also thank him for the fruitful discussions on various points of this dissertation during our multiple seminars on Symplectic Geometry and the category of Frölicher spaces.
I also thank my colleague Mr Mukinayi Hermé Tshilombo who later joint our discussions in the seminars on Symplectic Geometry and the category of Frölicher spaces and accepted to read our work.
I thank all of those beautiful persons, staff and administrative of the university of the Witwatersrand (particularly the School of Mathematics and the Geo-Maths Library), for their willing to assist me. Particularly, I shall always remember the following people for their kindness and their ubuntu spirit: Dr Margaret Archibold, Dr CAC Brinnan, Ms C Mennen, Ms NGZ Mkhize, Mr C NDipingwi, Mrs DM Whitman, Dr Tom Brown, Mrs WI Lawrence, Ms Yasmine Hill, Ms S Booley, Mrs IJ Eitzen (all from the School of Mathematics), Mrs MT Makoti and Ms T Dlhamini (both from the Geo-Maths Library).
To finish, I must thank my beautiful wife for her love and her words of encouragement. And also, I am deeply grateful my parents for their support and their patience during all years of my studies.
Above all, Alleluyah to the Almighty YaHWeH El Shadday who gives the wisdom and the intelligence to those who fear Him!

Contents

Introduction

This dissertation is aimed at extending the theory of bundles to the category of Frölicher spaces and investigating the symplectic structure on such bundles. Our work rests on the notions of Frölicher spaces as developed in [2],[3],[5],[6],[7],[9],[14] and [17] which the category is denoted by $\underline{\mathcal{F}RL}$. It was proved in [7] and [9] that the the category of differential spaces is non-cartesian closed and that of Frölicher spaces is cartesian closed (see also [14]). The advantages of the category $\underline{\mathcal{F}RL}$ were detailed in the introductory chapter of [2].

The first chapter of this dissertation presents the basic concepts of the category of Frölicher spaces as defined in [2], [5] and [6]. We compare the topology generated by the structure curves and that induced by the structure functions on a Frölicher space. We review some properties of Frölicher structure on a set and give some examples. We also review the characterization of diffeomorphisms (in the sense of the category of Frölicher spaces) and the category of pseudomanifolds. A pseudomanifold is a Frölicher space locally diffeomorphic to a closed subspace of the n-dimensional Euclidian canonical Frölicher space $\mathbb{R}^n$ [2]. We prove that the structure functions and the structure curves are smooth (in the sense of category of Frölicher spaces) and that the set of all the structure functions of a given Frölicher space is an algebra. A tangent vector will be defined on this algebra of structure functions. We define some additional concepts and properties of linear Frölicher structure on the set formed from the other Frölicher linear spaces because, the fibers of a vector Frölicher bundle all have a linear Frölicher structure (see the following paragraph).

The second chapter deals with the theory of bundles in the category of Frölicher spaces. The definition of a bundle in the category of Frölicher spaces, first given in [2], is recalled. We follow the common theory of bundles developed for other categories in [4],[10],[13] and [16]. We prove that a fiber of a bundle in the category of Frölicher spaces has a Frölicher structure. We also prove that a section on a Frölicher bundle is smooth (in the setting of Frölicher spaces) and we define a morphism between two Frölicher bundles. The Frölicher bundles and the morphisms between them form the category of Frölicher bundles. We then investigate some particular type of bundles in category of Frölicher spaces, such as the fiber bundles, the vector bundles and the principal bundles. We give particular attention to the vector Frölicher bundles. Some examples follow each type of Frölicher bundle.

The third chapter reviews in detail the tangent structures on a Frölicher space. The kinematic tangent structure is provided by the structure curves and the operational tangent structure is defined by the structure functions of the underlying Frölicher space, see [2], [6], [7] and [17]. In this work, we are interested in the operational tangent structure on a Frölicher space. We prove that a tangent space on a Frölicher space has a linear Frölicher structure. The linear tangent map between two operational tangent Frölicher spaces, as defined in [2], is reviewed and greater detail given. We prove that the linear tangent map between two tangent Frölicher spaces is linear and smooth (in the sense of the category of Frölicher spaces). We define, as in [2], [6] and [7], the notion of the tangent bundles on a Frölicher space. We prove that the tangent and the cotangent bundles on a Frölicher space have a Frölicher structure. We are interested in the tangent and the cotangent Frölicher bundles on a finite-dimensional pseudomanifold. We prove that the tangent and the cotangent Frölicher bundles on an n-dimensional pseudomanifold are locally diffeomorphic to the even-dimensional Euclidian canonical Frölicher space $\mathbb{R}^{2n}$. Then, the tangent and the cotangent Frölicher bundles on a finite-dimensional pseudomanifold are examples of vector Frölicher bundles. Detailed proofs of the tangent maps between two tangent Frölicher bundles, as defined in [2] are given as well as properties of vector fields and forms on a finite-dimensional pseudomanifold, viewed as the base space of a tangent Frölicher bundle or the base space of a wedge product Frölicher bundle. We investigate the correspondence between a form and exterior form on a finite-dimensional pseudomanifold, as in [10] for the category of differentiable manifolds. We conclude this chapter with a brief review of the Lie-bracket on the set of vector fields on a finite-dimensional pseudomanifold.

In the fourth chapter, we investigate the existence of a symplectic structure on a cotangent Frölicher bundle on a finite-dimensional pseudomanifold. We mainly refer to [2] for the symplectic structure on Frölicher spaces of constant dimension. The common theory on symplectic structure is found in [1], [10] and [15] for other categories. We review in detail the basic concepts of symplectic structures on linear (Frölicher) spaces. Some examples of symplectic linear Frölicher spaces follow. The concept of symplectic structures on a vector Frölicher bundle and some examples are given. We define a symplectic structure on a finite-dimensional pseudomanifold. The standard example of a symplectic pseudomanifold $\mathbb{R}^{2n}$ is given with respect to the canonical symplectic form and we show that there exists a symplectic structure on the cotangent Frölicher bundle of a finite-dimensional pseudomanifold by the pullback of the canonical symplectic form on $\mathbb{R}^{2n}$. The induced symplectic form is defined on a fiber of the tangent Frölicher bundle on a finite-dimensional pseudomanifold. Here, a finite-dimensional pseudomanifold is the state space of a cotangent Frölicher bundle on a finite-dimensional pseudomanifold. This is a second example of a symplectic pseudomanifold. We give a definition of a symplectomorphism between two symplectic pseudomanifolds and prove that the cotangent Frölicher bundles on two diffeomorphic finite-dimensional pseudomanifolds are symplectomorphic. We expect that the canonical form of the symplectic structure over a cotangent Frölicher bundle on an n-dimensional pseudomanifold might be induced from the canonical symplectic form of $\mathbb{R}^{2n}$. Since an n-dimensional pseudomanifold is locally diffeomorphic to a closed Frölicher subspace of $\mathbb{R}^n$ of constant maximal dimension and their cotangent Frölicher bundles are symplectic that is, these symplectic cotangent Frölicher bundles are symplectomorphic. But, we need some additional concept beyond this work to guaranty the induced canonical symplectic structure. In our opinion, this is an open question for a future work.

Chapter 1

Some concepts of Frölicher spaces

In this first chapter, we present the basic and useful concepts on Frölicher spaces.

1.1 Frölicher structures

In what follows M is a set, $\mathcal{F}_M$ a collection of real-valued functions f from M to $\mathbb{R}$ and $\mathcal{C}_M$ a collection of curves c of $\mathbb{R}$ to M. We have the following diagram:

$$\begin{tikzcd} & M \arrow[dr, "f"] & \\ \mathbb{R} \arrow[ur, "c"] \arrow[rr, "f \circ c"] & & \mathbb{R} \end{tikzcd}$$

Definition 1.1.1 *(See [2], [5],[6], [7], [14]) The triple* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *is a* **Frölicher space** *if the following properties, called compatibility condition, are satisfied:*

- $\Phi\mathcal{C}_M := \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_M\} = \mathcal{F}_M$;
- $\Gamma\mathcal{F}_M := \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_M\} = \mathcal{C}_M$.

The pair $(\mathcal{C}_M, \mathcal{F}_M)$ is called a **Frölicher structure** *on M. The elements of $\mathcal{C}_M$ are called* **structure curves** *in M and those of $\mathcal{F}_M$ are* **structure functions** *on M.*

In this work, when there is no confusion, we might mention $(M, \mathcal{C}, \mathcal{F})$ or simply M as a Frölicher space instead of $(M, \mathcal{C}_M, \mathcal{F}_M)$ with subscript M. We will write $\mathbb{F}$-space for means of Frölicher space and $\mathbb{F}$-structure for means of Frölicher structure.

Remark 1.1.1 *There are two natural topologies underlying an $\mathbb{F}$-space $(M, \mathcal{C}, \mathcal{F})$: the topology generated by the structure curves in M, denoted by $\tau_{\mathcal{C}}$, and the topology induced by the structure functions on M, denoted by $\tau_{\mathcal{F}}$.*

Proposition 1.1.1 *The topology $\tau_{\mathcal{C}}$ is finer than the topology $\tau_{\mathcal{F}}$, that is $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$.*

Proof. Let $(M, \mathcal{C}, \mathcal{F})$ be a $\mathbb{F}$-space, $\tau_{\mathcal{C}}$ the final topology induced by structure curves in M and $\tau_{\mathcal{F}}$ the initial topology generated by structure functions on M. The topology $\tau_{\mathcal{C}}$ is generated by subsets U of M such that $c^{-1}(U)$ are open sets in $\mathbb{R}$, where $c \in \mathcal{C}$ (see [9] and [14]). The topology $\tau_{\mathcal{F}}$ is generated by the basis $\{f^{-1}(0, +\infty)\}_{f \in \mathcal{F}}$ (see [2]). We need to show that for any subset W of M, if $W \in \tau_{\mathcal{F}}$ then $W \in \tau_{\mathcal{C}}$. That is, $c^{-1}(W)$ is an open set in $\mathbb{R}$. Given $W \subseteq M$ such that $W \in \tau_{\mathcal{F}}$, the set can be written as

$$\begin{aligned} W &= \bigcup \{f^{-1}(0, +\infty)\}_{f \in \mathcal{F}} \\ &= \bigcup_{f \in \mathcal{F}} f^{-1}(0, +\infty). \end{aligned}$$

For some $c \in \mathcal{C}$, using the properties of reciprocal images, we have

$$\begin{aligned} c^{-1}(W) &= c^{-1}\left(\bigcup_{f \in \mathcal{F}} f^{-1}(0, +\infty)\right) \\ &= \bigcup_{f \in \mathcal{F}} c^{-1}\left(f^{-1}(0, +\infty)\right) \\ &= \bigcup_{f \in \mathcal{F}} \left(c^{-1} \circ f^{-1}\right)(0, +\infty) \\ &= \bigcup_{f \in \mathcal{F}} (f \circ c)^{-1}(0, +\infty). \end{aligned}$$

Since $c \in \mathcal{C}$ and $f \in \mathcal{F}$, the composite $f \circ c$ is a C^{∞}-map from $\mathbb{R}$ into $\mathbb{R}$. Therefore, $f \circ c$ is a continuous map. And, since the set $(0, +\infty)$

is an open set in $\mathbb{R}$, then $(f \circ c)^{-1}(0, +\infty)$ is an open set in $\mathbb{R}$. Hence $c^{-1}(W) = \bigcup_{f \in \mathcal{F}} (f \circ c)^{-1}(0, +\infty)$ is an open set in the usual topology of $\mathbb{R}$. Thus, $W \in \tau_{\mathcal{C}}$. That is, $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$. $\square$

If the two topologies $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{F}}$ coincide then the space M is said *balanced* [5]. Moreover, if the space M is balanced and compact Hausdorff then M is called *base space* [2]. In our work, we will restrict ourselves to the base spaces and it will be assumed that a set and its subsets are all endowed with the same topological structure.

Definition 1.1.2 *(See [2]) Let $\mathcal{F}_o$ be any subset of $\mathbb{R}^M$. The set $\mathcal{F}_o$ is called the* **generating set** *of the $\mathbb{F}$-structure $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ on M (generated by functions), if*

- $\Gamma\mathcal{F}_o := \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_o\}$;
- $\Phi\Gamma\mathcal{F}_o := \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \Gamma\mathcal{F}_o\}$.

Similarly, any subset $\mathcal{C}_o$ of $M^{\mathbb{R}}$ is called the **generating set** *of the $\mathbb{F}$- structure $(\Gamma\Phi\mathcal{C}_o, \Phi\mathcal{C}_o)$ on M (generated by curves), if*

- $\Phi\mathcal{C}_o := \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_o\}$;
- $\Gamma\Phi\mathcal{F}_o := \{c : \mathbb{R} \to M \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \Phi\mathcal{C}_o\}$.

Now consider an $\mathbb{F}$-structure $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ (or $(\Gamma\Phi\mathcal{C}_o, \Phi\mathcal{C}_o)$) on a set M. In what follows, we will show that $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$ (or $\mathcal{C}_o \subseteq \Gamma\Phi\mathcal{F}_o$) and the uniqueness of the $\mathbb{F}$-structure.

Lemma 1.1.1 *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be subsets of $M^{\mathbb{R}}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ subsets of $\mathbb{R}^M$. We have:*

1. *If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ then $\Phi\mathcal{C}_2 \subseteq \Phi\mathcal{C}_1$*
2. *If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\Gamma\mathcal{F}_2 \subseteq \Gamma\mathcal{F}_1$.*

Proof.

1. Let $f \in \Phi\mathcal{C}_2$. We have, by Definition 1.1.1, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_2$. In particular, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_1$ since $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Thus, $f \in \Phi\mathcal{C}_1$.

2. Let $c \in \Gamma\mathcal{F}_2$. Then, by Definition 1.1.1, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_2$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have particulary $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_1$. That is $c \in \Gamma\mathcal{F}_1$. $\square$

Remark 1.1.2 *One can observe from Lemma 1.1.1 that the functors Γ and Φ are order reversing. As a consequence, a small set generates a richer $\mathbb{F}$-structure.*

Proposition 1.1.2 *Consider an $\mathbb{F}$-space M, let $\mathcal{F}_o$ be a subset of $\mathbb{R}^M$ and $\mathcal{C}_o$ a subset of $M^{\mathbb{R}}$. Then*

1. $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$
2. $\mathcal{C}_o \subseteq \Gamma\Phi\mathcal{C}_o$

Proof.

1. Let $\mathcal{F}_o$ be a subset of $\mathbb{R}^M$ and assume that $f \in \mathcal{F}_o$ but $f \notin \Phi\Gamma\mathcal{F}_o$. It follows, from Definition 1.1.1 and $f \in \mathcal{F}_o$, that there exists $c \in \Gamma\mathcal{F}_o$ such that $f \circ c \notin C^\infty(\mathbb{R}, \mathbb{R})$. But $c \in \Gamma\mathcal{F}_o$ means that $g \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $g \in \mathcal{F}_o$. In particular, $g = f$ yields a contradiction. Thus, if $f \in \mathcal{F}_o$ then $f \in \Phi\Gamma\mathcal{F}_o$. That is, $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$.
2. As above, let $\mathcal{C}_o$ be a subset of $M^{\mathbb{R}}$ and assume that $c \in \mathcal{C}_o$ but $c \notin \Gamma\Phi\mathcal{C}_o$. That is, according to Definition 1.1.1 and the fact that $c \in \mathcal{C}_o$, there exists $f \in \Phi\mathcal{C}_o$ such that $f \circ c \notin C^\infty(\mathbb{R}, \mathbb{R})$. From Definition 1.1.2, $f \in \Phi\mathcal{C}_o$ implies $f \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\tilde{c} \in \mathcal{C}_o$. In particular, $f \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ for $\tilde{c} = c \ \in \mathcal{C}_o$. This is a contradiction. Thus, if $c \in \mathcal{C}_o$ then $c \in \Gamma\Phi\mathcal{C}_o$. That is, $\mathcal{C}_o \subseteq \Gamma\Phi\mathcal{C}_o$. $\square$

Proposition 1.1.3 *The following identities hold for the functors Φ and Γ:*

1. $\Gamma\Phi\Gamma = \Gamma$
2. $\Phi\Gamma\Phi = \Phi$

Proof.

1. Let $\mathcal{F}_o$ be a subset of $\mathbb{R}^M$. From Proposition 1.1.2(1), it yields $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$. Applying Lemma 1.1.1(2) to $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$, we obtain $\Gamma\Phi\Gamma\mathcal{F}_o \subseteq \Gamma\mathcal{F}_o$. Since $\Gamma\mathcal{F}_o$ is a subset of $M^{\mathbb{R}}$, Proposition 1.1.2(2) gives $\Gamma\mathcal{F}_o \subseteq \Gamma\Phi\Gamma\mathcal{F}_o$. From the two inclusions $\Gamma\Phi\Gamma\mathcal{F}_o \subseteq \Gamma\mathcal{F}_o$ and $\Gamma\mathcal{F}_o \subseteq \Gamma\Phi\Gamma\mathcal{F}_o$, it follows that $\Gamma\Phi\Gamma\mathcal{F}_o = \Gamma\mathcal{F}_o$ for any subset $\mathcal{F}_o$ of $\mathbb{R}^M$. Thus, $\Gamma\Phi\Gamma = \Gamma$.

2. In a similar way, let $\mathcal{C}_o$ a subset of $M^{\mathbb{R}}$. From Proposition 1.1.2(2), it yields $\mathcal{C}_o \subseteq \Gamma\Phi\mathcal{C}_o$. By Lemma 1.1.1(1), it follows from $\mathcal{C}_o \subseteq \Gamma\Phi\mathcal{C}_o$ that $\Phi\Gamma\Phi\mathcal{C}_o \subseteq \Phi\mathcal{C}_o$. By Definition 1.1.1, $\Phi\mathcal{C}_o$ is a subset of $\mathbb{R}^M$. Then Proposition 1.1.2(1) gives $\Phi\mathcal{C}_o \subseteq \Phi\Gamma\Phi\mathcal{C}_o$. The two inclusions $\Phi\Gamma\Phi\mathcal{C}_o \subseteq \Phi\mathcal{C}_o$ and $\Phi\mathcal{C}_o \subseteq \Phi\Gamma\Phi\mathcal{C}_o$ imply that $\Phi\Gamma\Phi\mathcal{C}_o = \Phi\mathcal{C}_o$ for any subset $\mathcal{C}_o$ of $M^{\mathbb{R}}$. Thus, $\Phi\Gamma\Phi = \Phi$. □

Remark 1.1.3 *As an interpretation of Proposition 1.1.3, we conclude that one can obtain more than one $\mathbb{F}$-structure on a set. But the $\mathbb{F}$-structure generated by a fixed set $\mathcal{F}_o$ or $\mathcal{C}_o$ is unique.*

Example 1

1. Let $M = \mathbb{R}$ be the real line and the set $\mathcal{C}_o = \{id_{\mathbb{R}}\}$, where $id_{\mathbb{R}}$ is the identity map on $\mathbb{R}$. The structure functions are given by the set

$$\Phi\mathcal{C}_o := \{f : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_o\}.$$

We have $f \circ c = f \circ id_{\mathbb{R}} = f$. Hence

$$\Phi\mathcal{C}_o = \{f : M \to \mathbb{R} \mid f \in C^\infty(\mathbb{R}, \mathbb{R})\} = C^\infty(\mathbb{R}, \mathbb{R}).$$

The structure curves are given by

$$\Gamma\Phi\mathcal{C}_o := \{c : M \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \Phi\mathcal{C}_o\}.$$

For any $f \in \Phi\mathcal{C}_o = C^\infty(\mathbb{R}, \mathbb{R})$, we want to show that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ implies $c \in C^\infty(\mathbb{R}, \mathbb{R})$. If we assume that $c \notin C^\infty(\mathbb{R}, \mathbb{R})$ then, on taking $f = id_{\mathbb{R}} \in C^\infty(\mathbb{R}, \mathbb{R})$, it yields a contradiction

$$f \circ c = id_{\mathbb{R}} \circ c = c \notin C^\infty(\mathbb{R}, \mathbb{R}).$$

Thus,

$$\begin{aligned} \Gamma\Phi\mathcal{C}_o &= \{c : \mathbb{R} \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in C^\infty(\mathbb{R}, \mathbb{R})\} \\ &= C^\infty(\mathbb{R}, \mathbb{R}). \end{aligned}$$

The $\mathbb{F}$-structure $(C^\infty(\mathbb{R}, \mathbb{R}), C^\infty(\mathbb{R}, \mathbb{R}))$ obtained is called the *canonical $\mathbb{F}$-structure* on $\mathbb{R}$, see [2].

2. Let $M = \mathbb{R}^2$ be the 2-dimensional Euclidian space and $\mathcal{C}_o$ the set of all smooth curves in $\mathbb{R}^2$. If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all smooth curves c then, from Boman's Theorem

in [14] or in [6], all iterated partial derivatives of f exist and are continuous i.e. $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Thus,

$$\Phi\mathcal{C}_o := \{f : \mathbb{R}^2 \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_o\} = C^\infty(\mathbb{R}^2, \mathbb{R}).$$

Therefore,

$$\Gamma\Phi\mathcal{C}_o := \{c : \mathbb{R} \to \mathbb{R}^2 \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \Phi\mathcal{C}_o\}.$$

Sinc $\Phi\mathcal{C}_o = C^\infty(\mathbb{R}^2, \mathbb{R})$, it follows that

$$\Gamma\Phi\mathcal{C}_o := \{c : \mathbb{R} \to \mathbb{R}^2 \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}.$$

Since the set of smooth curves is a subset $\Gamma\Phi\mathcal{C}_o$, see Proposition 1.1.2(2), it follows that some structure curves (if not all) in $\mathbb{R}^2$ are smooth. Hence, $(\Gamma\Phi\mathcal{C}_o, \Phi\mathcal{C}_o)$ defines an $\mathbb{F}$-structure on $\mathbb{R}^2$, where $\Gamma\Phi\mathcal{C}_o \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ and $\Phi\mathcal{C}_o \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

3. Let $M = \mathbb{R}^n$ be the n-dimensional Euclidian space and $\mathcal{F}_o$ the set of all smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. According to Boman's result given in [7], $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for each smooth curve $c : \mathbb{R} \to \mathbb{R}^n$. Hence,

$$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \to \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_o\} = C^\infty(\mathbb{R}, \mathbb{R}^n)$$

and

$$\Phi\Gamma\mathcal{F}_o := \{f : \mathbb{R}^n \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \Gamma\mathcal{F}_o\} = C^\infty(\mathbb{R}^n, \mathbb{R}).$$

Thus, $(C^\infty(\mathbb{R}, \mathbb{R}^n), C^\infty(\mathbb{R}^n, \mathbb{R}))$ is the canonical $\mathbb{F}$-structure on $\mathbb{R}^n$ and $\mathbb{R}^n$ is called the n-dimensional canonical Euclidian $\mathbb{F}$-space [2].

4. Let $M = \mathbb{R}$ be the real line and the set $\mathcal{F}_o = \{id_{\mathbb{R}}, |x| \mid x \in \mathbb{R}\}$. The structure curves are elements of the set

$$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \to \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_o\}.$$

Since $f(x) = |x|\,(x \in \mathbb{R})$, the compatibility condition requires that, for $t \in \mathbb{R}$,

$$(f \circ c)(t) = f(c(t)) = c(t) \in C^\infty(\mathbb{R}, \mathbb{R})$$

and

$$(f \circ c)(t) = f(c(t)) = |c(t)| \in C^\infty(\mathbb{R}, \mathbb{R}).$$

We have then

$$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \to \mathbb{R} \mid c(t) \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ and } |c(t)| \in C^\infty(\mathbb{R}, \mathbb{R})\}.$$

Therefore $\Phi\Gamma\mathcal{F}_o :=$

$\{f : \mathbb{R} \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C^\infty(\mathbb{R}, \mathbb{R})$ s.t $|c| \in C^\infty(\mathbb{R}, \mathbb{R})\}$.

Obviously, $id_{\mathbb{R}}$ and $|x|$ for $x \in \mathbb{R}$ are structure functions since we have always $\mathcal{F}_o \subseteq \Phi\Gamma\mathcal{F}_o$ as proved in Proposition 1.1.2(1). Thus, $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ is an $\mathbb{F}$-structure on $\mathbb{R}$.

Remark 1.1.4 *It follows that:*

1. *Example 1 (1), (2), (3) above are particular examples of finite dimensional smooth manifolds which are* $\mathbb{F}$*-spaces. That means that if* $(M, \mathcal{F})$ *is a differential space then* $(M, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ *is the associated* $\mathbb{F}$*-space, see [2], [6], [7].*
2. *Example 1 (4) above does not hold at* $x = 0$ *in the category of differential spaces.*

1.2 Frölicher smooth maps

Definition 1.2.1 *(See [2] and [14]) A map* $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \to (N, \mathcal{C}_N, \mathcal{F}_N)$ *between two* $\mathbb{F}$*-spaces is* **Frölicher smooth map** *if one of the following conditions holds :*

1. *For each* $c \in \mathcal{C}_M$ *the composite* $\varphi \circ c$ *lies in* $\mathcal{C}_N$*,*
2. *For each* $f \in \mathcal{F}_N$ *the composite* $f \circ \varphi$ *lies in* $\mathcal{F}_M$*,*
3. *For each* $c \in \mathcal{C}_M$ *and for each* $f \in \mathcal{F}_N$*,* $f \circ \varphi \circ c$ *lies in* $C^\infty(\mathbb{R}, \mathbb{R})$*.*

$$\begin{tikzcd}
 & \mathbb{R} & \\
M \arrow[ur] \arrow[rr, "\varphi"] & & N \arrow[ul, "f"'] \\
 & \mathbb{R} \arrow[ul, "c"] \arrow[ur] &
\end{tikzcd}$$

We shall use $\mathbb{F}$-smooth in short or smooth if there is no risk of confusion means Frölicher smooth or smooth in the setting of Frölicher spaces. The Frölicher spaces and $\mathbb{F}$-smooth maps between them form a category called the **category of Frölicher spaces** and denoted by $\underline{\mathcal{FRL}}$ [2], [6], [7].

Proposition 1.2.1 *Let M be an $\mathbb{F}$-space. The identity map id_M on M is an $\mathbb{F}$-smooth map.*

Proof. One can consider the fact that $\varphi = id_M$ and $N = M$ in Definition 1.2.1 . Hence, the proposition holds. $\square$

Proposition 1.2.2 *The composite of two $\mathbb{F}$-smooth maps is an $\mathbb{F}$-smooth map.*

Proof. If M, N, and P are $\mathbb{F}$-spaces and $\varphi_1 : M \to N$ and $\varphi_2 : N \to P$ are $\mathbb{F}$-smooth maps then for some $f \in \mathcal{F}_P$ and for some $c \in \mathcal{C}_M$, the composite $f \circ (\varphi_2 \circ \varphi_1) \circ c = (f \circ \varphi_2) \circ (\varphi_1 \circ c)$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ since $(f \circ \varphi_2) \in \mathcal{F}_N$ and $(\varphi_1 \circ c) \in \mathcal{C}_N$. That is, $(\varphi_2 \circ \varphi_1) : M \to P$ is an $\mathbb{F}$-smooth map. $\square$

Some examples of $\mathbb{F}$-smooth maps will be given in the next section after introducing the notion of Frölicher subspaces.

Proposition 1.2.3 *Let M be an $\mathbb{F}$-space. The structure curves in M and the structure functions on M are $\mathbb{F}$-smooth.*

Proof. Let us consider the canonical $\mathbb{F}$-structure on $\mathbb{R}$. Each structure curve in M is a map $c : \mathbb{R} \to M$ such that for all $f \in \mathcal{F}_M$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) = \mathcal{F}_\mathbb{R}$. And each structure function on M is a map $f : M \to \mathbb{R}$ such that for all $c \in \mathcal{C}_M$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) = \mathcal{C}_\mathbb{R}$. Hence, according to Definition 1.2.1(1),(2), the structure curve c and the structure function f are $\mathbb{F}$-smooth. $\square$

Remark 1.2.1 *Given an $\mathbb{F}$-space M, by Proposition 1.2.3, we may set $\mathcal{C}_M = C^\infty(\mathbb{R}, M)$ and $\mathcal{F}_M = C^\infty(M, \mathbb{R})$. The notation*

$$(M, C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$$

is used, see [14].

Definition 1.2.2 *(See [2]) A* **Frölicher diffeomorphism** *between two $\mathbb{F}$-spaces M and N is a bijective $\mathbb{F}$-smooth map $\varphi : M \to N$ such that φ^{-1} is $\mathbb{F}$-smooth.*

For short, we shall say $\mathbb{F}$-diffeomorphism or simply diffeomorphism if there is no risk of confusion. We will also say M and N are $\mathbb{F}$-diffeomorphic or diffeomorphic.

Proposition 1.2.4 *Let M be an $\mathbb{F}$-space. The identity map id_M on M is an $\mathbb{F}$-diffeomorphism.*

Proof. According to Proposition 1.2.1, id_M is $\mathbb{F}$-smooth. But, id_M is a bijective map. We have also $id_M^{-1} = id_M$ is $\mathbb{F}$-smooth. Thus, by Definition 1.2.2, id_M is an $\mathbb{F}$-diffeomorphism. □

Proposition 1.2.5 *If φ is an $\mathbb{F}$-diffeomorphism. Then φ^{-1} is also an $\mathbb{F}$-diffeomorphism.*

Proof. By assumption, φ is an $\mathbb{F}$-diffeomorphism. That is, by Definition 1.2.2, φ is a bijective $\mathbb{F}$-smooth map and φ^{-1} is $\mathbb{F}$-smooth. But, the inverse φ^{-1} of a bijective map φ is also a bijective map. Therefore, φ^{-1} is a bijective $\mathbb{F}$-smooth map. We have that $(\varphi^{-1})^{-1} \equiv \varphi$ is $\mathbb{F}$-smooth. That is, by Definition 1.2.2, φ^{-1} is an $\mathbb{F}$-diffeomorphism. □

Proposition 1.2.6 *The composite of two $\mathbb{F}$-diffeomorphisms is an $\mathbb{F}$-diffeomorphism.*

Proof. Let f and g be two $\mathbb{F}$-diffeomorphisms. That is, by Definition 1.2.1, f and g are bijective $\mathbb{F}$-smooth maps, so are f^{-1} and g^{-1}. According to Proposition 1.2.2, $f \circ g$ and $g^{-1} \circ f^{-1}$ are then $\mathbb{F}$-smooth . Since f and g are bijective maps, the composite $f \circ g$ is also a bijective map. We have obviously $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, which is $\mathbb{F}$-smooth. Thus, by Definition 1.2.2, $f \circ g$ is $\mathbb{F}$-diffeomorphism. □

Some examples of $\mathbb{F}$-diffeomorphisms will be given at Section 1.6.

1.3 Frölicher subspaces

Proposition 1.3.1 *Let M be an $\mathbb{F}$-space and A be a subset of M. The set A can be endowed with $\mathbb{F}$-structure induced from that of M making the canonical injection $\mathbb{F}$-smooth.*

Proof. Assume that $(M, \mathcal{C}_M, \mathcal{F}_M)$ is an $\mathbb{F}$-space and A a subset of M. Consider the canonical injection $\iota : A \hookrightarrow M$.

$$
\begin{tikzcd}
& \mathbb{R} & \\
A \arrow[ur, "\tilde{f}"] \arrow[rr, "\iota"] & & M \arrow[ul, "f"'] \\
& \mathbb{R} \arrow[ul, "\tilde{c}"] \arrow[ur, "c"'] &
\end{tikzcd}
$$

Let $\mathcal{F}_{oA} := \{f \circ \iota | f \in \mathcal{F}_M\}$. We need to prove that $\mathcal{F}_{oA}$ generates an $\mathbb{F}$-structure on A. Let $c \in \mathcal{C}_M$ and $\tilde{f} \in \mathcal{F}_{oA}$, that is, $\tilde{f} = f \circ \iota$. Now, consider the maps $\tilde{c} : \mathbb{R} \to A$. We have $c = \iota \circ \tilde{c}$. Hence

$$
\begin{aligned}
\tilde{f} \circ \tilde{c} &= (f \circ \iota) \circ \tilde{c} \\
&= f \circ (\iota \circ \tilde{c}) \\
&= f \circ c.
\end{aligned}
$$

Since $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, such is $\tilde{f} \circ \tilde{c}$. So, the structure curves in A are given by

$$\Gamma\mathcal{F}_{oA} := \{\tilde{c} : \mathbb{R} \to A | \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \tilde{f} \in \mathcal{F}_{oA}\}.$$

It follows that the structure functions are given by

$$\Phi\Gamma\mathcal{F}_{oA} := \{\tilde{f} : A \to \mathbb{R} | \tilde{f} \circ \tilde{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \tilde{c} \in \Gamma\mathcal{F}_{oA}\}.$$

Thus, $(\Gamma\mathcal{F}_{oA}, \Phi\Gamma\mathcal{F}_{oA})$ defines the induced $\mathbb{F}$-structure on $A \subseteq M$. By construction made above in the proof of Proposition 1.3.1, one has to take $f \circ \iota \in \mathcal{F}_A$ for $f \in \mathcal{F}_M$. According to Definition 1.2.1(2), ι is $\mathbb{F}$-smooth. $\square$

We mean that a subset of an $\mathbb{F}$-space is also an $\mathbb{F}$-space, which is not always true in the category of differentiable manifolds. Sometimes, a subspace of an $\mathbb{F}$-space will be called $\mathbb{F}$-*subspace*.

Example 2 Let us consider $\mathbb{Q}$ the set of all rational numbers. Since $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ is a canonical $\mathbb{F}$-space, one can conclude by Proposition 1.3.1 that $\mathbb{Q}$ has an $\mathbb{F}$-structure induced from $\mathbb{R}$ via canonical injection. $\mathbb{Q}$ is then called *discrete* $\mathbb{F}$-*space*, see [2], [6] and [7].

1.4 Product Frölicher spaces

Proposition 1.4.1 *Let M_1 and M_2 be two $\mathbb{F}$-spaces. There exists an $\mathbb{F}$-structure on the product set $M := M_1 \times M_2$ making the two projections $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ $\mathbb{F}$-smooth.*

Proof. Consider two $\mathbb{F}$-spaces $(M_1, \mathcal{C}_1, \mathcal{F}_1)$ and $(M_2, \mathcal{C}_2, \mathcal{F}_2)$ and the projections $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$.

$$\begin{tikzcd}
 & \mathbb{R} & \\
M_1 \arrow[ur, "f_1"] & M \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "f"] & M_2 \arrow[ul, "f_2"'] \\
 & \mathbb{R} \arrow[ul, "c_1"] \arrow[u, "c"] \arrow[ur, "c_2"'] &
\end{tikzcd}$$

By construction, the set

$$\mathcal{F}_o := \{f_1 \circ \pi_1 | f_1 \in \mathcal{F}_{M_1}\} \cup \{f_2 \circ \pi_2 | f_2 \in \mathcal{F}_{M_2}\}$$

generates the $\mathbb{F}$-structure on M. Note that if $f \in \mathcal{F}_o$ then $f = f_1 \circ \pi_1$ or $f = f_2 \circ \pi_2$, where $f_i \in \mathcal{F}_{M_i}$ $(i = 1, 2)$. Therefore, $c : \mathbb{R} \to M$ is given by $c(t) = (c_1(t), c_2(t))$ with $c_1 \in \mathcal{C}_{M_1}$ and $c_2 \in \mathcal{C}_{M_2}$. Hence

$$\begin{aligned} f \circ c &= (f_i \circ \pi_i) \circ c \\ &= f_i \circ (\pi_i \circ c) \\ &= f_i \circ c_i. \end{aligned}$$

Since $f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R})$ for $i = 1, 2$, one concludes that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. Therefore,

$$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \to M | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_o\},$$

where $c(t) = (c_1(t), c_2(t))$ with $c_i \in \mathcal{C}_{M_i}$ $(i = 1, 2)$, is the set of structure curves in M and

$$\Phi\Gamma\mathcal{F}_o := \{f : M \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \Gamma\mathcal{F}_o\}$$

is the set of structure functions on M.
Let show that π_1 and π_2 are $\mathbb{F}$-smooth. Consider $c \in \mathcal{C}_M$ and $f_i \in \mathcal{F}_{M_i}$ $(i = 1, 2)$. Hence $\pi_i \circ c = c_i \in \mathcal{C}_{M_i}$ for $i = 1, 2$ and

$$\begin{aligned} f_i \circ \pi_i \circ c &= f_i \circ (\pi_i \circ c) \\ &= f_i \circ c_i. \end{aligned}$$

Since $f_i \circ c_i \in C^\infty(\mathbb{R},\mathbb{R})$, we have $f_i \circ \pi_i \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for $i = 1, 2$. Then π_1 and π_2 are $\mathbb{F}$-smooth by Definition 1.2.1(3).
Thus $(\Gamma\mathcal{F}_o, \Phi\Gamma\mathcal{F}_o)$ is the $\mathbb{F}$-structure generated by $\mathcal{F}_o$ on the product set M, making the natural projections of M on M_i $(i = 1, 2)$ $\mathbb{F}$-smooth. □

Proposition 1.4.1 above can be generalized as follows (see also [7]):

Proposition 1.4.2 *Let $\{M_i, \mathcal{C}_{M_i}, \mathcal{F}_{M_i}\}_{i\in I}$ be a countable collection of $\mathbb{F}$-spaces. Let $M = \prod_{i\in I} M_i$ be the product of $\{M_i\}_{i\in I}$ and $\pi_i : M \to M_i$ for $i \in I$ denote the i-th projection map. There exists an $\mathbb{F}$-structure on M making the projections π_i $\mathbb{F}$-smooth.*

Proof. Alike in the proof of Proposition 1.4.1, the $\mathbb{F}$-structure on $M = \prod_{i\in I} M_i$ is generated by the set

$$\mathcal{F}_o := \bigcup_{i\in I} \{f_i \circ \pi_i \mid f_i \in \mathcal{F}_{M_i}\}$$

making the projections π_i $\mathbb{F}$-smooth. □

Example 3 The n-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^n$ is a product $\mathbb{F}$-space since $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$, where $\mathbb{R}$ is the one-dimensional Euclidian canonical $\mathbb{F}$-space.

1.5 Cartesian closedness

Definition 1.5.1 *(See [9]) Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Let ρ and λ be two functors: $\rho : \mathcal{A} \to \mathcal{B}$ and $\lambda : \mathcal{B} \to \mathcal{A}$. Then ρ is called* **right adjoint** *to λ (and λ* **left adjoint** *to ρ) if there are bijections between the following sets:*

$$\mathcal{A}(\lambda A, B) \cong \mathcal{B}(B, \rho A) \text{ for } A \in \mathcal{A}, B \in \mathcal{B}$$

in such a way that they form a natural transformation between the functors

$$\mathcal{A}(\lambda\ \ ,\ \ ), \mathcal{B}(\ \ , \rho\ \ ) : \mathcal{B}^{op} \times \mathcal{A} \to \underline{Set},$$

where $\mathcal{B}^{op}$ is the category opposite to $\mathcal{B}$, i.e. $\mathcal{B}^{op}(A, B) := \mathcal{B}(B, A)$ for all objects A, B and $\underline{Set}$ is the category of sets. This natural transformation is called an **adjunction** *between ρ and λ.*

Definition 1.5.2 *(See [9]) A category $\mathcal{A}$ is called* **cartesian closed** *if and only if*

1. *$\mathcal{A}$ has a terminal object T, i.e. an object such that for any object A there exists exactly one morphism $A \to T$;*

2. *For any pair of objects a product exists, i.e. the functor*

$$\mathcal{A} \to \mathcal{A} \times \mathcal{A}, A \mapsto (A, A), f \mapsto (f, f)$$

   has a right adjoint $\Pi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$;

3. *For every object B the functor ΠB has a right adjoint.*

Theorem 1.5.1 *(See [9] and [14]) The category $\underline{\mathcal{FRL}}$ is Cartesian closed i.e. for any $\mathbb{F}$-spaces M, N and P there is a natural isomorphism*

$$C^\infty(M, C^\infty(N, P)) \cong C^\infty(M \times N, P),$$

which can be written as **exponential law**

$$P^{M \times N} \cong (P^N)^M$$

if one uses the notation P^N for $C^\infty(N, P)$.

Proof. The proof of this theorem requires some notions which are beyond this work. The complete materials and proof are found in [7], [9] and [14]. Take C^∞, $\times$ and the adjunction in such way that the underlying set of $C^\infty(M, N)$ is the set of ($\mathbb{F}$-smooth) morphisms $\mathcal{A}(M, N)$ and that of the underlying sets of M and N. The adjunction

$$C^\infty(M, C^\infty(N, P)) \cong C^\infty(M \times N, P)$$

is given by the map

$$f \mapsto \check{f}, \text{ where } \check{f}(x)(y) := f(x, y);$$

the inverse map is

$$g \mapsto \tilde{g}, \text{ where } \tilde{g}(x, y) := g(x)(y).$$

□

Corollary 1.5.1 *(See [17]) Let M, N and P be $\mathbb{F}$-spaces. Then the following maps are smooth:*

1. *$ev : C^\infty(M, N) \times M \to N$, $ev(f, x) = f(x)$.*

2. *$ins : M \to C^\infty(N, M \times N)$, $ins(x)(y) = (x, y)$.*
3. *$comp : C^\infty(N, P) \times C^\infty(M, N) \to C^\infty(M, P)$, $comp(g, f) = g \circ f$.*

Proof.

1. Let $c_1 : \mathbb{R} \to C^\infty(M, N)$ and $c_2 : \mathbb{R} \to M$ be curves into $C^\infty(M, N)$ and M respectively. Then one has

$$ev(c_1, c_2)(t) = \quad \hat{c_1} \circ (id_{\mathbb{R}}, c_2)(t),$$

where $\quad \hat{c_1} : \mathbb{R} \times M \to N$ is the smooth map associated by

$$C^\infty(\mathbb{R}, C^\infty(N, P)) \cong C^\infty(\mathbb{R} \times N, P)$$

to the map c_1, see Theorem 1.5.1. Since $\quad \hat{c_1} \circ (id_{\mathbb{R}}, c_2)$ is smooth, it follows that ev is smooth.

2. Let $c : \mathbb{R} \to M$ be a structure curve in M and let $y \in N$. For $t \in \mathbb{R}$, one has

$$ins(c(t))(y) = (c \times id_N)(t, y).$$

But $c \times id_N$ is smooth since c and id_N are smooth. Then $ins \circ c$ is smooth. Hence ins is smooth by

$$C^\infty(M, C^\infty(N, M \times N)) \cong C^\infty(M \times N, M \times N),$$

see Theorem 1.5.1.

3. Let us consider two curves $c_1 : \mathbb{R} \to C^\infty(N, P)$ and $c_2 : \mathbb{R} \to C^\infty(M, N)$, and $x \in M$. It follows that

$$(comp(c_1, c_2)(t))(x) = ev(id \times ev)(c_1 \circ pr_1, c_2 \times id_M)(t, x),$$

where $pr_1 : \mathbb{R} \times M \to \mathbb{R}$ is the natural projection on the first component and id is the identity map of the set of functions $\mathbb{R} \times M \to C^\infty(M, P)$. The right hand side of the equality above is smooth since ev, id, c_1, pr_1, c_2, and id_M are smooth. Hence, the left hand side is also smooth. Therefore , since $c_1(t) \in C^\infty(N, P)$ and $c_2(t) \in C^\infty(M, N)$, $comp$ is smooth by Cartesian closedness, according to Theorem 1.5.1.

$\square$

Proposition 1.5.1 *Let M and N be $\mathbb{F}$-spaces. The set $C^\infty(M, N)$ has an $\mathbb{F}$-structure.*

Proof. Let $\mathcal{C}_{M,N}$ be the set of all curves $c : \mathbb{R} \to C^{\infty}(M,N)$ such that $\hat{c} : \mathbb{R} \times M \to N$ given by $\hat{c}(t,x) := c(t)(x)$ is smooth. Since $\underline{\mathcal{FRL}}$ is Cartesian closed, it turns out that curves into $C^{\infty}(M,N)$ are smooth and $C^{\infty}(M,N)$ has then the $\mathbb{F}$-structure generated by $\mathcal{C}_{M,N}$ $\square$

Corollary 1.5.2 *Let M be an $\mathbb{F}$-space. The sets $\mathcal{C}_M$ and $\mathcal{F}_M$ have an $\mathbb{F}$-structure.*

Proof. Let M be an $\mathbb{F}$-space. Let $N := M$ and $M := \mathbb{R}$ into the set $C^{\infty}(M,N)$ in Proposition 1.5.1. Thus, $\mathcal{C}_M = C^{\infty}(\mathbb{R}, M)$ has an $\mathbb{F}$-structure since $\mathbb{R}$ and M are $\mathbb{F}$-spaces. Similarly, one can substitute N by $\mathbb{R}$ into the set $C^{\infty}(M,N)$ in Proposition 1.5.1. Thus, $\mathcal{F}_M = C^{\infty}(M,\mathbb{R})$ has an $\mathbb{F}$-structure since M and $\mathbb{R}$ are $\mathbb{F}$-spaces. $\square$

Corollary 1.5.3 *Let E and F be two linear $\mathbb{F}$-spaces. Let $LIN(E,F)$ denotes the space of all $\mathbb{F}$-smooth linear maps from E to F. $LIN(E,F)$ has an $\mathbb{F}$-structure.*

Proof. One can see that $LIN(E,F) \subset C^{\infty}(M,N)$. Thus, according to Proposition 1.3.1, $LIN(E,F)$ can be equiped with an induced $\mathbb{F}$-structure. $\square$

Remark 1.5.1 *In general, if E_i with $i = 1, 2, \ldots, n$ and F are linear $\mathbb{F}$-spaces,*

$$\begin{aligned} &LIN(E_1, E_2, \ldots, E_k, E_{k+1}, \ldots, E_n; F) \cong \\ &\qquad\qquad LIN(E_1, E_2, \ldots, E_k, LIN(E_{k+1}, \ldots, E_n; F)), [14]. \end{aligned}$$

Thus, $LIN(E_1, E_2, \ldots, E_n; F)$ has an $\mathbb{F}$-structure.

Proposition 1.5.2 *Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an $\mathbb{F}$-space. The set $\mathcal{F}_M$ is an algebra.*

Proof. According to Definition 1.1.1 and the fact that, for all $x \in M$, the map $\theta : M \to \mathbb{R}$ given by $\theta(x) = 0$ is in $\mathcal{F}_M$, we have $\mathcal{F}_M \neq \emptyset$ and $\mathcal{F}_M \subseteq \mathbb{R}^M$. Since $\mathbb{R}^M$ is an algebra, we have to show that $\mathcal{F}_M$ is a closed space under pointwise multiplication of functions. In effect, for any $f \in \mathcal{F}_M$ and $g \in \mathcal{F}_M$, the map

$$\varphi : M \to \mathbb{R}^2 \text{ defined by } \varphi(x) = (f(x), g(x))$$

is smooth since $\varphi(x) := ins(ev(g,x))(ev(f,x))$, where ins and ev are smooth functions (see Corollary 1.5.1). Then, the function

$$\omega : \mathbb{R}^2 \to \mathbb{R} \text{ defined by } \omega(f(x), g(x)) = f(x).g(x)$$

is a smooth real-valued function. That is, ω is infinitely differentiable. Thus, since f and g are elements of $\mathcal{F}_M$ and $\mathcal{F}_M$ has the differential structure associated with the $\mathbb{F}$-structure $(\mathcal{C}_M, \mathcal{F}_M)$, we have that $\omega \circ (f, g)$ is an element of $\mathcal{F}_M$ (see also [2] and [6]). We conclude that if $f \in \mathcal{F}_M$ and $g \in \mathcal{F}_M$ then $f.g \in \mathcal{F}_M$. That is, $\mathcal{F}_M$ is closed under pointwise multiplication. Hence, $\mathcal{F}_M$ is an algebra. $\square$

1.6 More about Frölicher diffeomorphisms and product Frölicher spaces

In the following, we review some results from [17] in detail.

Proposition 1.6.1 *Let M be an $\mathbb{F}$-space, and let N be a set, and let $S = \{f_i : M \to N,\ i \in I\}$ be a family of set maps. Define the set map*

$$\varphi : M \to N^I \text{ by setting } \varphi(x) = (f_i(x))_{i\in I} \text{for all } x \in M,$$

where $N^I = \prod_I N$. If φ is bijective then M is $\mathbb{F}$-diffeomorphic to the subspace $\varphi(M)$ of the $\mathbb{F}$-space N^I.

Proof. The family

$$\mathcal{F}_o = \{f : N \to \mathbb{R} \,|\, f \circ f_i \in \mathcal{F}_M \text{ for all } i \in I\}$$

generates an $\mathbb{F}$-structure on N making the maps f_i $\mathbb{F}$-smooth. Hence, there exists a product $\mathbb{F}$-structure on N^I induced from that on N according to Proposition 1.4.2. Then, $\varphi := (ins(ev(f_i, x)))_{i\in I}$ is $\mathbb{F}$-smooth since ins and ev are smooth (see Corollary 1.4.1). Now consider $\varphi^{-1} : \varphi(M) \to M$. According to Proposition 1.3.1, the subspace $\varphi(M)$ of the $\mathbb{F}$-space N^I is an $\mathbb{F}$-space. Curves in $\varphi(M)$ have the form

$$c(t) = ((f_i \circ \tilde{c})(t))_{i\in I},$$

where $\tilde{c} : \mathbb{R} \to M$ is a structure curve in M. It follows from the definition of φ above that

$$\varphi^{-1}((f_i \circ \tilde{c})(t))_{i\in I} = \varphi^{-1}(f_i(\tilde{c}(t)))_{i\in I} = \varphi^{-1}(\varphi(\tilde{c}(t))) = \tilde{c}(t).$$

That is, by Definition 1.2.1(1), φ^{-1} is $\mathbb{F}$-smooth since $(f_i \circ \tilde{c})_{i\in I} \in \mathcal{C}_{\varphi(M)}$ and $\tilde{c} \in \mathcal{C}_M$. $\square$

Corollary 1.6.1 *Let M and P be $\mathbb{F}$-spaces, and let N be a set, and let $S = \{f_1, \ldots, f_n : M \to N\}$ and $S' = \{g_1, \ldots, g_m : P \to N\}$ be families of set maps. Suppose that $\varphi := (f_1, \ldots, f_n)$ and $\psi := (g_1, \ldots, g_m)$ are bijective maps $M \to N^n$ and $P \to N^m$ respectively. Then, the map $\alpha : M \times P \to N^{n+m}$ defined by*

$$\alpha = (f_1 \circ \pi_1, \ldots, f_n \circ \pi_1, g_1 \circ \pi_2, \ldots, g_n \circ \pi_2)$$

is one-to-one, and the product $\mathbb{F}$-space $M \times P$ is $\mathbb{F}$-diffeomorphic to the subspace $\alpha(M \times P)$ of the $\mathbb{F}$-space N^{n+m}.

Proof. Since M and P are $\mathbb{F}$-spaces, according to Proposition 1.4.1, the product $M \times P$ is an $\mathbb{F}$-space. The map $\alpha : M \times P \to N^{n+m}$, where $N^{n+m} = \prod_{i=1}^{n+m} N$ defined by

$$\alpha(x) = (h_1(x), \ldots, h_n(x), h_{n+1}(x), \ldots, h_{n+m}(x)),$$

where $h_i = f_i \circ \pi_1$ if $1 \leq i \leq n$ and $h_i = g_i \circ \pi_2$ if $n+1 \leq i \leq n+m$, is bijective since h_i are bijective. The $\mathbb{F}$-structure on N is induced by the family $\{h_1, \ldots, h_n, h_{n+1}, \ldots, h_{n+m}\}$ (see the proof of Proposition 1.6.1). Thus, according to Proposition 1.6.1, $M \times P$ is diffeomorphic to $\alpha(M \times P)$. $\square$

Proposition 1.6.2 *Let M be an $\mathbb{F}$-space with a generating set F_o of structure functions containing a function that separates points of M. Then the inclusion*

$$\iota : M \to \mathbb{R}^{F_o},$$

where $\mathbb{R}$ is a one-dimensional canonical Euclidian $\mathbb{F}$-space, sending x to $(f(x))_{f \in F_o}$ is $\mathbb{F}$-smooth and identifies M with an $\mathbb{F}$-subspace of $\mathbb{R}^{F_o}$.

Proof. Let us show that $\iota : M \to \mathbb{R}^{F_o}$ is $\mathbb{F}$-smooth. In effect, let $c \in \mathcal{C}_M$. Then, by Cartesian closedness of $\underline{\mathcal{FRL}}$ (see Theorem 1.5.1),

$$\iota \circ c : \mathbb{R} \to \mathbb{R}^{F_o}$$

is $\mathbb{F}$-smooth if only if the map

$$\iota \,\tilde{\circ}\, c : \mathbb{R} \times F_o \to \mathbb{R}, \text{ where } \iota \,\tilde{\circ}\, c(t, g) = \iota(c(t))(g) = g(c(t)),$$

is $\mathbb{F}$-smooth. But, $\iota \,\tilde{\circ}\, c$ is smooth since $g(c(t)) = ev((id_{F_o} \times c)(g, t))$, where ev is the evaluation map (see Corollary 1.5.1(2)) and id_{F_o} the identity map of F_o. Thus, $\iota \circ c$ is $\mathbb{F}$-smooth. Then, by Definition 1.2.1(1), ι is $\mathbb{F}$-smooth since c is an $\mathbb{F}$-smooth curve. On the other hand, we need to show that if

$c : \mathbb{R} \to \iota(M)$ is a structure curve, where $\iota(M)$ is an $\mathbb{F}$-subspace of $\mathbb{R}^{F_o}$, then $\iota^{-1} \circ c$ is a structure curve in M. Let us assume that c is a structure curve in $\iota(M)$. By Cartesian closedness of $\underline{\mathcal{FRL}}$, $c : \mathbb{R} \to \iota(M)$ is $\mathbb{F}$-smooth if and only if

$$\tilde{c} : \mathbb{R} \times F_o \to \mathbb{R} \text{ defined by } \tilde{c}(t, g) = c(t)(g)$$

is $\mathbb{F}$-smooth. But

$$c(t)(g) = ((\iota \circ \iota^{-1})(c(t)))(g) = \iota(\iota^{-1}(c(t)))(g) = g(\iota^{-1}(c(t))) = (g \circ \iota^{-1} \circ c)(t)$$

(see above). Since, for all $g \in F_o$, $g \circ \iota^{-1} \circ c$ is $\mathbb{F}$-smooth then $\iota^{-1} \circ c$ is also $\mathbb{F}$-smooth. Thus, according to Definition 1.2.1(1), ι^{-1} is $\mathbb{F}$-smooth since c is a structure curve in $\iota(M)$. □

Proposition 1.6.3 *Let N be an $\mathbb{F}$-space, and the pair $(\mathcal{C}_M, \mathcal{F}_M)$ the $\mathbb{F}$-structure induced on the set M via maps $f_i : M \to N$, $i \in I$. Assume that the map*

$$\varphi : M \to N^I, \text{ given by } \varphi(x) = (f_i(x))_{i \in I},$$

is bijective. Then φ is an $\mathbb{F}$-diffeomorphism of M onto the $\mathbb{F}$-subspace $\varphi(M)$ of N^I.

Proof. Let $c \in \mathcal{C}_M$. Then, by the definition of φ,

$$(\varphi \circ c)(t) = ((f_i \circ c)(t))_{i \in I}$$

for all $t \in \mathbb{R}$. Since the $\mathbb{F}$-structure on N^I is generated by the family

$$\{g \circ \pi_i \,|\, g \in \mathcal{F}_N \text{ and } \pi_i \text{ the } i^{th} \text{ projection of } N^I \text{ on } N,\ i \in I\},$$

so the structure curves in N^I have the form $(f_i(c))_{i \in I}$. Let $f \in \mathcal{F}_{\varphi(M)}$. It follows by setting $f := g \circ \pi_i \circ \iota$, that the map $\varphi \circ c : \mathbb{R} \to \varphi(M)$ is a structure curve in the $\mathbb{F}$-subspace $\varphi(M)$ of N^I. Hence, according to Definition 1.2.1(1), φ is $\mathbb{F}$-smooth since c is a structure curve in M. Now, let $(x_i)_{i \in I} \in \varphi(M)$ and since $f_i = \pi_i \circ \varphi$, we have by the definition of the bijective map φ and the associativity of the composition of maps,

$$\begin{aligned}
(g \circ f_i \circ \varphi^{-1})((x_i)_{i \in I}) &= (g \circ (\pi_i \circ \varphi) \circ \varphi^{-1})((x_i)_{i \in I}) \\
&= (g \circ \pi_i \circ (\varphi \circ \varphi^{-1}))((x_i)_{i \in I}) \\
&= (g \circ \pi_i)((x_i)_{i \in I}).
\end{aligned}$$

It follows, according to Definition 1.2.1(2), that φ^{-1} is $\mathbb{F}$-smooth since $g \circ f_i$ is a structure function on M and $g \circ \pi_i$ is a structure function on $\varphi(M)$. Therefore, by Definition 1.2.2, φ is $\mathbb{F}$-diffeomorphism. □

Corollary 1.6.2 *Let M be a set, and let $f_1, \ldots, f_n : M \to \mathbb{R}$ be real-valued functions on M such that the map*

$$\varphi : M \to \mathbb{R}^n, \text{ given by } \varphi(x) = (f_1(x), \ldots, f_n(x)),$$

is bijective. If $(\mathcal{C}_M, \mathcal{F}_M)$ is an $\mathbb{F}$-structure generated by the family $\{f_1, \ldots, f_n\}$, then φ is an $\mathbb{F}$-diffeomorphism of M to the $\mathbb{F}$-subspace $\varphi(M)$ of $\mathbb{R}^n$.

Proof. Let $N = \mathbb{R}$, the one dimensional canonical Euclidian $\mathbb{F}$-space. According to Proposition 1.4.2 and Example 3, if $I = \{1, \ldots, n\}$ then $N^I = \mathbb{R}^n$ is a product space of n copies of $\mathbb{R}$ and hence an $\mathbb{F}$-space. Thus, by Proposition 1.6.3, φ is an $\mathbb{F}$-diffeomorphism of M to the $\mathbb{F}$-subspace $\varphi(M)$ of $\mathbb{R}^n$. □

Definition 1.6.1 *An n-dimensional* **pseudomanifold** *is an $\mathbb{F}$-space which is locally $\mathbb{F}$-diffeomorphic to a closed subset of the n-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^n$ of constant dimension and nonempty interior.*

More details about pseudomanifolds can be found in [2]. A pseudomanifold is a constant dimentional $\mathbb{F}$-space. That is, the pointwise dimension is the same everywhere on the $\mathbb{F}$-space. We will sometimes restricts ouselves to this category of pseudomanifolds, a subcategory of $\underline{\mathcal{F}RL}$, mostly when the notion of finite dimensional $\mathbb{F}$-space will be needed.

Example 4

1. Let $M = (0, 2\pi)$ be a subspace of the one dimensional canonical Euclidian $\mathbb{F}$-space $\mathbb{R}$, and let $f_1, f_2 : M \to \mathbb{R}$ be maps given by $f_1(x) = cosx$, $f_2(x) = sinx$, and let $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ denote the identity map of $\mathbb{R}$. Obviously, according to Corollary 1.6.2, $M = (0, 2\pi)$ is an $\mathbb{F}$-space and if $\varphi := (f_1, f_2)$, we have that $\varphi : M \to \mathbb{R}^2$ is a bijective map of M onto $S^1 \setminus \{(1, 0)\}$. Hence, by Corollary 1.6.1 and Proposition 1.6.2, the map $\alpha : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, defined by $\alpha = (f_1 \circ \pi_1, f_2 \circ \pi_1, id_{\mathbb{R}} \circ \pi_2)$, is a bijective map and the strip $(0, 2\pi) \times \mathbb{R}$ is $\mathbb{F}$-diffeomorphismic to the $\mathbb{F}$-subspace
$$\{(cosx, sinx, z) \in \mathbb{R}^3 \,|\, 0 \leq x < 2\pi, \; -\infty < z < +\infty\}$$
of $\mathbb{R}$
2. Let M be the n-closed unit ball $\mathcal{B}^n$, and let $\varphi : \mathcal{B}^n \to \mathbb{R}^{n+1}$, given by $\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - \|x\|^2})$ for all $x = (x_1, \ldots, x_n) \in \mathcal{B}^n$. The functions $\pi_i, z : \mathcal{B}^n \to \mathbb{R}$, defined by $\pi_i(x) = x_i$ for $1 \leq i \leq n$ and $z(x) = \sqrt{1 - \|x\|^2}$, generate an $\mathbb{F}$-structure on $\mathcal{B}^n$ and is bijective. Thus, by Corollary 1.6.2, φ is an $\mathbb{F}$-diffeomorphism of $\mathcal{B}^n$ onto the top hemisphere $z(x) = \sqrt{1 - \|x\|^2}$ viewed as an $\mathbb{F}$-subspace of $\mathbb{R}^{n+1}$.

1.7 Frölicher Lie groups

Definition 1.7.1 *(See [17]) Assume that G is a group. A triple $(G, \mathcal{C}_G, \mathcal{F}_G)$ is called a* **Frölicher Lie group** *if:*

- *$(G, \mathcal{C}_G, \mathcal{F}_G)$ is an $\mathbb{F}$-space*
- *the mapping $\sigma : G \times G \to G$ given by $\sigma(x, y) = xy^{-1}$ is $\mathbb{F}$-smooth.*

We will say $\mathbb{F}$-Lie Group for short and it will be denoted by G. So G should mean the underlying set of elements of the structured space. And here the structured space can be the group G or the Group G together with the $\mathbb{F}$-structure if there is no risk of confusion. According to Proposition 1.4.1, the product set $G \times G$ has an $\mathbb{F}$-structure since G is an $\mathbb{F}$-space.

Proposition 1.7.1 *If H is an arbitrary subgroup of the underlying group G of an $\mathbb{F}$-Lie group $(G, \mathcal{C}_G, \mathcal{F}_G)$ and ι is the natural imbedding of H into G, then $(H, \Gamma\Phi(\iota(\mathcal{C}_G)), \Phi(\iota(\mathcal{C}_G)))$ is an $\mathbb{F}$-Lie group.*

Proof. Since H is a subspace of the $\mathbb{F}$-space G, the $\mathbb{F}$-structure on H can be generated by the set

$$\iota(\mathcal{C}_G) := \{c : \mathbb{R} \to H \mid \iota \circ c \in \mathcal{C}_G\},$$

making ι $\mathbb{F}$-smooth. Hence, by Proposition 1.3.1, $(H, \Gamma\Phi(\iota(\mathcal{C}_G)), \Phi(\iota(\mathcal{C}_G)))$ is an $\mathbb{F}$-space. According to Proposition 1.4.1, the product set $H \times H$ has an $\mathbb{F}$-structure since H has an $\mathbb{F}$-structure. Now, let us show that the map

$$\sigma_H : H \times H \to H, \text{ given by }, \sigma_H(x, y) = xy^{-1}$$

is $\mathbb{F}$-smooth. Let c be a structure curve in $H \times H$. Therefore, there exists structure curves c_1 and c_2 in H such that $c(t) = (c_1(t), c_2(t))$ for all $t \in \mathbb{R}$. Since ι is $\mathbb{F}$-smooth, the map

$$d : \mathbb{R} \to G \times G, \text{ given by } d(t) = ((\iota \circ c_1)(t), (\iota \circ c_2)(t))$$

is a structure curve in $G \times G$. Since $\iota : H \to G$ is a canonical inclusion of groups, one has

$$\iota \circ \sigma_H \circ c = \sigma \circ d,$$

where σ is given in Definition 1.7.1. But, $\sigma \circ d : \mathbb{R} \to G$ is $\mathbb{F}$-smooth, since σ and d are $\mathbb{F}$-smooth. Hence $\iota \circ \sigma_H \circ c$ is $\mathbb{F}$-smooth. Since ι is $\mathbb{F}$-smooth, $\sigma_H \circ c$ is also $\mathbb{F}$-smooth. It follows, by Definition 1.2.1(1), that σ_H is $\mathbb{F}$-smooth because c is $\mathbb{F}$-smooth as a structure curve in $H \times H$. $\square$

Example 5 Let M be an $\mathbb{F}$-space, and let $\mathrm{Diff}(M)$ be the set of all $\mathbb{F}$-diffeomorphisms of M. Then, $\mathrm{Diff}(M)$ has an $\mathbb{F}$-structure since $\mathrm{Diff}(M)$ is a subspace of the $\mathbb{F}$-space $C^\infty(M, M)$. We have that $\mathrm{Diff}(M) \neq \emptyset$ since $id_M \in \mathrm{Diff(M)}$ and $\mathrm{Diff}(M)$ is a subspace of the Lie-group $GL(M)$. Thus, according to Definition 1.2.2, Proposition 1.2.5 and Proposition 1.2.6, the map

$$\sigma : \mathrm{Diff(M)} \times \mathrm{Diff(M)} \longrightarrow \mathrm{Diff(M)} \text{ defined by } \sigma(f, g) = f \circ g^{-1}$$

is $\mathbb{F}$-smooth. More details on the notion of $\mathbb{F}$-Lie group can also be found in [17].

1.8 Quotient Frölicher spaces

Proposition 1.8.1 *Let M be an $\mathbb{F}$-space and ρ an equivalence relation in M. There exists an $\mathbb{F}$-structure on the quotient set M/ρ making the projection map*

$$q : M \longrightarrow M/\rho,\ a \longrightarrow q(a) = [a],$$

$\mathbb{F}$-smooth.

Proof. Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ be an $\mathbb{F}$-space.

$$\begin{tikzcd}
& \mathbb{R} & \\
M \arrow[ur, "f"] \arrow[rr, "q"] & & M/\rho \arrow[ul, "\bar{f}"'] \\
& \mathbb{R} \arrow[ul, "c"] \arrow[ur, "\bar{c}"'] &
\end{tikzcd}$$

Let $b \in M/\rho$ be an equivalence class, then there exists $a \in M$ such that $q(a) = b$. Now, consider $f \in \mathcal{F}_M$ and $\bar{f} : M/\rho \to \mathbb{R}$ such that $f = \bar{f} \circ q$. We have

$$\bar{f}(b) = \bar{f}([a]) = \bar{f}(q(a)) = (\bar{f} \circ q)(a) = f(a).$$

Let

$$\bar{\mathcal{C}_o} := \{\bar{c} : \mathbb{R} \to M/\rho \,|\, \bar{c} = q \circ c \text{ for all } c \in \mathcal{C}_M\}.$$

We have to prove that $\bar{\mathcal{C}}_o$ generates an $\mathbb{F}$-structure on M/ρ. Assume that $\bar{c} \in \bar{\mathcal{C}}_o$ and $f \in \mathcal{F}_M$. Hence,

$$\begin{aligned} \bar{f} \circ \bar{c} &= \bar{f} \circ (q \circ c) \\ &= (\bar{f} \circ q) \circ c \\ &= f \circ c. \end{aligned}$$

Since $f \circ c \ \in C^\infty(\mathbb{R}, \mathbb{R})$, then it is clear that $\bar{f} \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R})$. So, the set of structure functions is

$$\Phi\bar{\mathcal{C}}_o := \{\bar{f} : M/\rho \to \mathbb{R} \,|\, \bar{f} \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \bar{c} \in \bar{\mathcal{C}}_o\}.$$

It follows that structure curves are given by

$$\Gamma\Phi\bar{\mathcal{C}}_o := \{\bar{c} : \mathbb{R} \to M/\rho \,|\, \bar{f} \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \bar{f} \in \Phi\bar{\mathcal{C}}_o\}.$$

Now, we may prove the $\mathbb{F}$-smoothness of q as follows. For any structure function $\bar{f}$ on M/ρ and any structure curve c in M, we have

$$\begin{aligned} \bar{f} \circ q \circ c &= (\bar{f} \circ q) \circ c \\ &= f \circ c. \end{aligned}$$

Since $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, we have $\bar{f} \circ q \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. By Definition 1.2.1(3), q is $\mathbb{F}$-smooth. Thus, $(\Phi\bar{C}_o, \Gamma\Phi\bar{C}_o)$ is an $\mathbb{F}$- structure on M/ρ making the map q $\mathbb{F}$-smooth. □

Example 6 Let G be an $\mathbb{F}$-Lie group and H an $\mathbb{F}$-Lie subgroup of G (see Proposition 1.7.1). Let $g \in G$ and $g' \in G$. Define the relation $\sim$ by $g \sim g'$ if there exists $h \in H$ such that $g' = gh$. One can show that $\sim$ is an equivalence relation in G. If we denote by G/H the quotient set then, according to Proposition 1.8.1, there exists an $\mathbb{F}$-structure on G/H making the quotient map $\pi : G \to G/H$ $\mathbb{F}$-smooth.

1.9 Coproduct Frölicher space

Proposition 1.9.1 *Let $(M_i, \mathcal{C}_i, \mathcal{F}_i)$, $i \in J$, be a family of $\mathbb{F}$-spaces and $\bar{M} := \bigsqcup_{i \in J} M_i$ the coproduct. There exists an $\mathbb{F}$-structure on $\bar{M}$ making the injection maps $j_i : M_i \to \bar{M}$ $\mathbb{F}$-smooth.*

Proof. Given the injections maps j_i of M_i into $\bar{M}$, and an $\mathbb{F}$-structure $(\mathcal{C}_i, \mathcal{F}_i)$ on M_i for each $i \in J$. Let $\bar{\mathcal{C}}_o := \cup_{i \in J}\{j_i \circ c_i \,|\, c_i \in \mathcal{C}_i\}$. In what follows, we show that $\bar{\mathcal{C}}_o$ generates an $\mathbb{F}$-structure on M.

$$\begin{tikzcd}
& \mathbb{R} & \\
M_i \arrow[ur, "f_i"] \arrow[rr, "j_i"] & & \bar{M} \arrow[ul, "\bar{f}"'] \\
& \mathbb{R} \arrow[ul, "c_i"] \arrow[ur, "\bar{c}"'] &
\end{tikzcd}$$

Let $\bar{c} \in \bar{\mathcal{C}}_o$. There exists $i \in J$ and $c_i \in \mathcal{C}_i$ such that $\bar{c} = j_i \circ c_i$. Let us consider $\bar{f} : \bar{M} \to \mathbb{R}$ such that $f_i = \bar{f} \circ j_i$ with $i \in J$ and $f_i \in \mathcal{F}_i$. Then, we have

$$\begin{aligned}
\bar{f} \circ \bar{c} &= \bar{f} \circ (j_i \circ c_i) \\
&= (\bar{f} \circ j_i) \circ c_i \\
&= f_i \circ c_i.
\end{aligned}$$

Since $f_i \circ c_i \in C^\infty(\mathbb{R}, \mathbb{R})$, so is $\bar{f} \circ \bar{c}$. Then, the structure functions are given by

$$\Phi\bar{\mathcal{C}}_o := \{\bar{f} : M \to \mathbb{R} \mid \bar{f} \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \bar{c} \in \bar{\mathcal{C}}_o\}$$

and structure curves from the set

$$\Gamma\Phi\bar{\mathcal{C}}_o := \{\bar{c} : \mathbb{R} \to M \mid \bar{f} \circ \bar{c} \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \bar{f} \in \Phi\bar{\mathcal{C}}_o\}.$$

Now, we show that the injections j_k, for $k \in J$, are $\mathbb{F}$-smooth. For any $c_k \in \mathcal{C}_k$ with $k \in J$ and for any $\bar{f} \in \mathcal{F}_{\bar{M}}$, we have

$$\begin{aligned}
\bar{f} \circ j_k \circ c_k &= (\bar{f} \circ j_k) \circ c_k \\
&= f_k \circ c_k.
\end{aligned}$$

By the compatibility condition given in Definition 1.1.1, we have $f_k \circ c_k \in C^\infty(\mathbb{R}, \mathbb{R})$. That is, $\bar{f} \circ j_k \circ c_k \in C^\infty(\mathbb{R}, \mathbb{R})$. Then, j_k are $\mathbb{F}$-smooth according to Definition 1.2.1(3). Thus, $\bar{\mathcal{C}}_o$ generates an $\mathbb{F}$-structure $(\Gamma\Phi\bar{\mathcal{C}}_o, \Phi\bar{\mathcal{C}}_o)$ on $\bar{M}$ making the injection maps j_i of M_i into $\bar{M}$ $(i \in J)$ $\mathbb{F}$-smooth. □

Example 7 The discret $\mathbb{F}$-space $\mathbb{Q}$ can be identified with a countable union of points.

1.10 Construction of linear Frölicher spaces

Proposition 1.10.1 *Given two linear $\mathbb{F}$-spaces E and F. The direct sum $E \oplus F$ has an $\mathbb{F}$-structure.*

Proof. Recall that the direct sum of two linear spaces E and F is denoted by $E \oplus F$ and is the cartesian product $E \times F$ obtained by defining the operations componentwise:

- $(e_1, f_1) + (e_2, f_2) = (e_1 + e_2, f_1 + f_2)$
- $r(e, f) = (re, rf)$ for $e, e_1, e_2 \in E$; $f, f_1, f_2 \in F$, and $r \in \mathbb{R}$ (in this work).

The subspace $E \times 0$ of $E \oplus F$ is isomorphic to E and is often identified with E; similarly for $0 \times F$ and F. With this identification, every element of $E \oplus F$ can be written in one and only one way as the sum of an element of E and an element of F. So, by Proposition 1.4.1, $E \oplus F$ has an $\mathbb{F}$-structure. □

Proposition 1.10.2 *Let E be a linear $\mathbb{F}$-space. The algebraic dual E^* of E has an $\mathbb{F}$-structure.*

Proof. According to Corollary 1.5.3, $E^* := LIN(E, \mathbb{R})$ has an $\mathbb{F}$-structure. □

Proposition 1.10.3 *Let E and F be two linear $\mathbb{F}$-spaces. The tensor product $E \otimes F$ has an $\mathbb{F}$-structure.*

Proof. We have that $E \otimes F \cong LIN(E, F; \mathbb{R})$ (see [14]). So, according to Remark 1.5.1, $E \otimes F$ has an $\mathbb{F}$-structure. □

Remark 1.10.1 *In general, if $E_1, E_2, \ldots, E_n$ are linear $\mathbb{F}$-spaces then $E_1 \otimes E_2 \otimes \cdots \otimes E_n$ has also an $\mathbb{F}$-structure. If $E_1 = E_2 = \cdots = E_n = E$ then $E_1 \otimes E_2 \otimes \cdots \otimes E_n$ is denoted by $\bigotimes^n E$ [14].*

Proposition 1.10.4 *Let E be a linear $\mathbb{F}$-space. The wedge product $\bigwedge^n E$ of E has an $\mathbb{F}$-structure.*

Proof. Recall, see [14], that the wedge product is given as the image of the alternator map

$$alt : \bigotimes^n E \longrightarrow \bigotimes^n E, \text{ given by } x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_n$$
$$:= \frac{1}{n!} \sum_{\sigma \in S_n} sign(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)},$$

where S_n is the set of permutations on $\{1, 2, \ldots, n\}$. So, $\bigwedge^{n} E$ is the linear subspace of all alternating tensors in $\bigotimes^{n} E$. According to Remark 1.10.1, $\bigotimes^{n} E$ has an $\mathbb{F}$-structure. Thus, according to Proposition 1.3.1, $\bigwedge^{n} E$ has an induced $\mathbb{F}$-structure. □

Definition 1.10.1 *Let E be a finite-dimensional real linear $\mathbb{F}$-space. $\bigwedge^{0}(E) = \mathbb{R}$, $\bigwedge^{1}(E) = E^*$, and in general, $\bigwedge^{k}(E) = LIN^k_{alt}(E, \mathbb{R})$ is the linear $\mathbb{F}$-space of* **alternating tensors** *in $\bigotimes^{k} E$ or* **skew (anti) symmetric** *k* **multilinear forms** *or* **exterior** *k***-forms** *on E.*

We presented the fundamental concepts of $\mathbb{F}$-spaces. We proved that the topology generated on an $\mathbb{F}$-space by the structure curves is finer than the topology induced by the structure functions. We assumed that the two topologies coincide. We also assumed that the $\mathbb{F}$-spaces are compact Hausdorff. We obtained an $\mathbb{F}$-structure on the following sets : the subset of an $\mathbb{F}$-space, the product set of $\mathbb{F}$-spaces, the set of the smooth maps between two $\mathbb{F}$-spaces, the quotient set of an $\mathbb{F}$-space and the coproduct set of $\mathbb{F}$-spaces. Finally, we proved that the linear space formed from the other linear $\mathbb{F}$-spaces has an $\mathbb{F}$-structure.

Chapter 2

Bundles in the category of Frölicher spaces

In this second chapter, we define and investigate the common notions of bundles and the maps between them (as in [10], [13], [16] and [19]) restricted to the category $\underline{\mathcal{FRL}}$. Such bundles are called *Frölicher bundles* or $\mathbb{F}$-*bundles* for short. We study some typical bundles such as the fiber Frölicher bundles, the vector Frölicher bundles, the principal Frölicher bundles and the associated Frölicher bundles.

2.1 Notion of Frölicher bundles

Definition 2.1.1 *An* $\mathbb{F}$**-bundle** *is a triple* (E, π, M)*, where* E *and* M *are* $\mathbb{F}$*-spaces respectively called the* **total space** *and the* **base space** *of the bundle and* $\pi : E \to M$ *is a surjective* $\mathbb{F}$*-smooth map called the* **projection** *of the bundle. For any* $x \in M$*,* $F_x := \pi^{-1}(x)$ *is called the* **fiber** *at (over)* x *(see also [2]).*

We will use sometimes $\lambda, \xi, \zeta, ...,$ like in [13], to denote an $\mathbb{F}$-bundle. We may denote the total space of λ by $E(\lambda)$ and the base space of λ by $M(\lambda)$. Sometimes, we shall denote the bundle over M with total space E by the map $\pi : E \to M$.

Proposition 2.1.1 *Let* (E, π, M) *be an* $\mathbb{F}$*-bundle. The fiber* F_x *over* $x \in M$ *is also an* $\mathbb{F}$*-space.*

Proof. Let (E, π, M) be an $\mathbb{F}$-bundle. Since π is a surjective map, the set $F_x := \pi^{-1}(x)$ is not empty and $F_x \subseteq E$. By Proposition 1.3.1, the fiber F_x

over $x \in M$ is an $\mathbb{F}$-space since E is an $\mathbb{F}$-space. □

Let us mention that, in general, the proposition above is not true in the category of differentiable manifolds.

Example 8

1. Let B and F be two $\mathbb{F}$-spaces. According to Proposition 1.4.1, there exists an $\mathbb{F}$-structure on the Cartesian product $B \times F$ making the projection $\pi_1 : B \times F \to B$ a surjective $\mathbb{F}$-smooth map. Therefore, the triple $(B \times F, \pi_1, B)$ is an $\mathbb{F}$-bundle and will be called the *trivial* $\mathbb{F}$-*bundle* over B.

2. The triple (S^2, π, M) is an $\mathbb{F}$-bundle, where the sphere
$$S^2 := \{u = (x, y, z) \in \mathbb{R}^3 \,|\, \|u\| = 1\} \subset \mathbb{R}^3$$
is the total space and the segment
$$M := \{(0, 0, z) \,|\, z \in [-1, +1]\} \subset \mathbb{R}$$
is the base space. According to Proposition 1.3.1, S^2 and M are $\mathbb{F}$-spaces since $\mathbb{R}^3$ and $\mathbb{R}$ are the canonical $\mathbb{F}$-spaces. The surjective $\mathbb{F}$-smooth map
$$\pi : (x, y, z) \in S^2 \mapsto (0, 0, z) \in M$$
is the projection of the $\mathbb{F}$-bundle. The fiber
$$F_u := \pi^{-1}(u) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1 - z^2 \text{ and } -1 \leq z \leq +1\}$$
over $u = (0, 0, z) \in M$ is a circle on S^2, in the plane through u and parallel to the xy plane.

3. The triple $(G, \pi, G/H)$ is an $\mathbb{F}$-bundle, where G is an $\mathbb{F}$-Lie group and H a $\mathbb{F}$-Lie subgroup of G. Therefore, the base space G/H is an $\mathbb{F}$-space in light of Example 6 of Section 1.8, making the (natural) projection $\pi : G \to G/H$ $\mathbb{F}$-smooth.

Definition 2.1.2 *An* $\mathbb{F}$**-subbundle** *of the* $\mathbb{F}$*-bundle* (E, π, M) *is an* $\mathbb{F}$*-bundle* (E', π', M'), *where* E' *is a subspace of* E *and* M' *a subspace of* M, *and* $\pi' = \pi_{|_{E'}}$ *the restriction of the surjective* $\mathbb{F}$*-smooth map* π *to* E'.

Remark 2.1.1 *In Definition 2.1.2, we need not to mention that* E' *and* M' *are* $\mathbb{F}$*-spaces since a subset of an* $\mathbb{F}$*-space is also an* $\mathbb{F}$*-space (see Proposition 1.3.1).*

Proposition 2.1.2 *Let (E, π, M) be an $\mathbb{F}$-bundle and E_o a subspace of E. The triple $(E_o, \pi_{|_{E_o}}, \pi(E_o))$ is an $\mathbb{F}$-subbundle of (E, π, M).*

Proof. Let $E_o \subseteq E$. The direct image $\pi(E_o)$ by a surjective map π is such that $\pi(E_o) \subseteq \pi(E) \subseteq M$. The restriction $\pi_{|_{E_o}}$ of the surjective $\mathbb{F}$-smooth map π of E_o is also a surjective $\mathbb{F}$-smooth map from E_o onto $\pi(E_o)$. Then, E_o and $\pi(E_o)$ are $\mathbb{F}$-spaces and $\pi_{|_{E_o}}$ is a surjective $\mathbb{F}$-smooth map. Thus, the triple $(E_o, \pi_{|_{E_o}}, \pi(E_o))$ is an $\mathbb{F}$-bundle. By assumption, $E_o \subseteq E$ and $\pi(E_o) \subseteq M$. Then, $(E_o, \pi_{|_{E_o}}, \pi(E_o))$ is an $\mathbb{F}$-subbundle of (E, π, M). □

Proposition 2.1.3 *Let (E, π, M) be an $\mathbb{F}$-bundle and M_o a subspace of M. The triple $(\pi^{-1}(M_o), \pi_{|_{\pi^{-1}(M_o)}}, M_o)$ is an $\mathbb{F}$-subbundle of (E, π, M).*

Proof. Let $M_o \subseteq M$. The inverse image $\pi^{-1}(M_o)$ by the surjective map π is such that $\pi^{-1}(M_o) \subseteq \pi^{-1}(M) \subseteq E$. Hence, $\pi^{-1}(M_o) \subset E$ and the map $\pi_{|_{\pi^{-1}(M_o)}}$ is the restriction of the surjective $\mathbb{F}$-smooth map π to $\pi^{-1}(M_o)$. Thus, $\pi_{|_{\pi^{-1}(M_o)}}$ is a surjective $\mathbb{F}$-smooth map from $\pi^{-1}(M_o)$ onto M_o. The triple $(\pi^{-1}(M_o), \pi_{|_{\pi^{-1}(M_o)}}, E_o)$ is indeed an $\mathbb{F}$-subbundle of (E, π, M). □

Note that, in Proposition 2.1.3, if ξ denotes the $\mathbb{F}$-bundle (E, π, M) then $\xi_{|_{M_o}}$ denotes his $\mathbb{F}$-subbundle $(\pi^{-1}(M_o), \pi_{|_{\pi^{-1}(M_o)}}, M_o)$ [13].

Definition 2.1.3 *An* **$\mathbb{F}$-smooth cross-section** *or simply a* **section** *of an $\mathbb{F}$-bundle (E, π, M) is an $\mathbb{F}$-smooth map $s : M \to E$ such that $\pi \circ s = id_M$.*

One can see that $s(x)$ is in the fiber at $x \in M$. For this definition to make sense, we prove that such a map $s : M \to E$ exists and is $\mathbb{F}$-smooth. Let us consider the following proposition.

Proposition 2.1.4 *Let (E, π, M) be an $\mathbb{F}$-bundle. There exists an $\mathbb{F}$-smooth map $s : M \to E$ such that $\pi \circ s = id_M$.*

Proof. Let (E, π, M) be an $\mathbb{F}$-bundle. The map $s : M \to E$ such that $\pi \circ s = id_M$ exists in the category of topological spaces, differential spaces and manifolds (see [19]). We need to show that s is an $\mathbb{F}$-smooth map. Let $f \in \mathcal{F}_E$ and $g \in \mathcal{F}_M$ such that $f = g \circ \pi$. Let $c \in \mathcal{C}_M$. We have, by the associativity of the composition of functions,

$$\begin{aligned} f \circ s \circ c &= (g \circ \pi) \circ s \circ c \\ &= g \circ (\pi \circ s) \circ c \\ &= g \circ id_M \circ c \\ &= g \circ c. \end{aligned}$$

But $g \in \mathcal{F}_M$ and $c \in \mathcal{C}_M$, then $g \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. It follows that $f \circ s \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$. That is, s is an $\mathbb{F}$-smooth map by Definition 1.2.1(3).□

The set of all $\mathbb{F}$-smooth cross-sections of an $\mathbb{F}$-bundle (E, π, M) will be denoted by $\Gamma(M, E)$ [16].

Example 9 (see [9]) Let $(B \times F, p_1, B)$ be a trivial $\mathbb{F}$-bundle. An $\mathbb{F}$-smooth cross-section of such a bundle has the form $s(b) = (b, f(b))$, where $f : B \to F$ is an $\mathbb{F}$-smooth map uniquely defined by s and $b \in B$. That is, by assumption, $s(b) = (b, f(b)) \in B \times F$ since $f(b) \in F$. Then,

$$(p_1 \circ s)(b) = p_1(s(b)) = p_1(b, f(b)) = b$$

since the map p_1 is the projection of $B \times F$ on the first factor. Thus, $(p_1 \circ s)(b) = b$ for every $b \in B$. That is, $p_1 \circ s = id_B$. The map s is $\mathbb{F}$-smooth since $s(b) := ev_b(id_B, f)$, where ev_b is the point evaluation at $b \in B$ which is smooth, by Corollary 1.5.1.

2.2 Frölicher bundle morphisms

Definition 2.2.1 *Let $\xi := (E, \pi, M)$ and $\xi' := (E', \pi', M')$ be $\mathbb{F}$-bundles. An* **$\mathbb{F}$-bundle morphism** *of ξ into ξ' is a pair (H, h) of $\mathbb{F}$-smooth maps $H : E \to E'$ and $h : M \to M'$ such that $\pi' \circ H = h \circ \pi$, i.e. making the following diagram commutative.*

$$\begin{tikzcd} E \arrow[r, "H"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "h"] & M' \end{tikzcd}$$

It means that H carries the fiber at $x \in M$ into the fiber at $h(x) \in M'$ determining completely the map h. Note that H will be also called the $\mathbb{F}$-*bundle morphism over* h [10], [13].

Example 10: Properties of $\mathbb{F}$-bundle morphisms
In this example we refer the reader to the terminology used in [10].

1. The pair (id_E, id_M) is an $\mathbb{F}$-bundle morphism called the *identical morphism* of the $\mathbb{F}$-bundle (E, π, M), where id_E and id_M are respectively identity maps of E and M.

2. Let (E', π', M') be an $\mathbb{F}$-subbundle of (E, π, M). Let $\iota : M' \hookrightarrow M$ and $I : E' \hookrightarrow E$ be inclusion maps, then (I, ι) is an $\mathbb{F}$-bundle morphism called the *canonical morphism*.

3. Let (H, h) be an $\mathbb{F}$-bundle morphism of ξ into ξ' and (K, k) be an $\mathbb{F}$-bundle morphism of ξ' into ξ''. The pair $(K \circ H, k \circ h)$ is an $\mathbb{F}$-bundle morphism of ξ into ξ'' called the *composite morphism*.

4. Let (H, h) be an $\mathbb{F}$-bundle morphism of ξ into ξ', where H and h are $\mathbb{F}$-diffeomorphisms. Then the pair (H^{-1}, h^{-1}) is an $\mathbb{F}$-bundle morphism of ξ' into ξ.

Definition 2.2.2 *(see [13]) Let $\xi := (E, \pi, M)$ and $\xi' := (E', \pi', M)$ be two $\mathbb{F}$-bundles over M. An $\mathbb{F}$-bundle morphism of ξ into ξ' over M or simply M-* **morphism** *is an $\mathbb{F}$-smooth map $H : E \to E'$ such that $\pi = \pi' \circ H$ making the following diagram commutative.*

$$\begin{tikzcd} E \arrow[rr, "H"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

The $\mathbb{F}$-bundle morphism H over M is just the $\mathbb{F}$-bundle morphism (H, id_M), where id_M is the identity map of M, and we can simply write $H : \xi \to \xi'$.

Definition 2.2.3 *Let (H, h) be an $\mathbb{F}$-bundle morphism of ξ into ξ'. The two $\mathbb{F}$-bundles ξ and ξ' are said to be* **$\mathbb{F}$-diffeomorphic** *if H and h are $\mathbb{F}$-diffeomorphisms. Two $\mathbb{F}$-bundles ξ and ξ' over M are said* **equivalent** *if H is an $\mathbb{F}$-diffeomorphism and h is the identity map in M [13].*

The class of $\mathbb{F}$-bundles together with $\mathbb{F}$-morphisms form a category that we shall call the **$\mathbb{F}$-bundles category**, where $\mathbb{F}$-bundles are objects and the $\mathbb{F}$-bundle morphisms are arrows.

2.3 Pullback and Whitney sum of Frölicher bundles

Proposition 2.3.1 *Let $\xi' := (E', \pi', M')$ be an $\mathbb{F}$-bundle and h be an $\mathbb{F}$-smooth map from an $\mathbb{F}$-space M into M'. If E is a subspace of $M \times E'$,* *which consists of points (b, x) such that $h(b) = \pi'(x)$, and the maps*

$$\pi : E \to M \text{ defined by } \pi(b, x) = b \text{ ; } H : E \to E' \text{ defined by } H(b, x) = x,$$

then the triple $\xi := (E, \pi, M)$ is an $\mathbb{F}$-bundle and H is $\mathbb{F}$-smooth such that $\pi' \circ H = h \circ \pi$ i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "H"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "h"] & M' \end{tikzcd}$$

Proof. Note that the triple $(M', \mathcal{C}_{M'}, \mathcal{F}_{M'})$ is an $\mathbb{F}$-space. Also the set

$$\mathcal{F}_{oM} := \{f \circ h | \, f \in \mathcal{F}_{M'}\}$$

generates an $\mathbb{F}$-structure on M making $\mathbb{F}$-smooth the map h. By Proposition 1.4.1, the product $M \times E'$ is an $\mathbb{F}$-space. Moreover, according to Proposition 1.3.1, the subset $E \subseteq M \times E'$ has the induced $\mathbb{F}$-structure. Now, the map

$$\pi : E \to M \text{ defined by } \pi(b, x) = b$$

is a projection of E on M. Then, according to Proposition 1.4.1, π is a surjective $\mathbb{F}$-smooth map. Thus, by Definition 2.1.1, (E, π, M) is an $\mathbb{F}$-bundle. The map $H : E \to E'$ defined by $H(b, x) = x$ is $\mathbb{F}$-smooth since it is a projection of E on E'. Let $(b, x) \in E$. Then,

$$(\pi' \circ H)(b, x) = \pi'(H(b, x)) = \pi'(x) = h(b) = h(\pi(b, x)) = (h \circ \pi)(b, x)$$

refering to the definitions of the set E and the maps H and π above. Therefore, $\pi' \circ H = h \circ \pi$. $\square$

Definition 2.3.1 *The $\mathbb{F}$-bundle $\xi = (E, \pi, M)$ built in Proposition 2.3.1 is called the* **pullback** *of $\xi' = (E', \pi', M')$ by h and denoted by $h^*(\xi')$ [13].*

Remark 2.3.1 *By Proposition 2.3.1, one can see that the pullback of an $\mathbb{F}$-bundle is also an $\mathbb{F}$-bundle and is called the* **pullback $\mathbb{F}$-bundle**.

Proposition 2.3.2 *Let $\xi := (E, \pi, M)$ be an $\mathbb{F}$-bundle. Let N be a subspace of M and $\iota : N \hookrightarrow M$ the inclusion map. Then the $\mathbb{F}$-bundles $(\pi^{-1}(N), \pi_{|_{\pi^{-1}(N)}}, N)$ and $\iota^*(\xi)$ are equivalent.*

Proof. From Proposition 2.1.3, it follows that the triple

$$\xi_{|_N} := (\pi^{-1}(N), \pi_{|_{\pi^{-1}(N)}}, N)$$

is an $\mathbb{F}$-subbundle of ξ since, by assumption, $N \subseteq M$ and ξ is an $\mathbb{F}$-bundle. Now, combining Definition 2.3.1 and Remark 2.3.1, one can observe that $\iota^*(\xi)$ is the pullback $\mathbb{F}$-bundle of ξ, since the canonical injection ι is $\mathbb{F}$-smooth. We define a map

$$H : \xi_{|_N} \to \iota^*(\xi) \text{ by setting } H(x) := (\pi(x), x),$$

where $\pi(x) \in N$ for any $x \in E$ and π is the projection of ξ. For each $x \in \pi^{-1}(N) \subseteq E$, we have $\pi(x) \in N$ and $x \in E$. That is, the image of $x \in N$ by the map H is

$$(\pi(x), x) \in N \times \pi^{-1}(N) \subseteq N \times E$$

in a unique way with π an $\mathbb{F}$-smooth. Thus, H is a bijective $\mathbb{F}$-smooth map. Let us define an $\mathbb{F}$-smooth map

$$\tilde{H}_b : \iota^*(N) \to \pi^{-1}(N) \text{ by } \ \tilde{H}_b(x) = x,$$

where $\pi(b) = x$. Therefore, $H^{-1}(b, x) := \tilde{H}_b(x)$ and H^{-1} is $\mathbb{F}$-smooth. Hence, by Definition 1.2.2, H is an $\mathbb{F}$-diffeomorphism. According to Definition 2.2.3, the proof is done since $\xi_{|_N}$ and $\iota^*(\xi)$ are both $\mathbb{F}$-bundles over N. □

Proposition 2.3.3 *Let $\xi := (E, \pi, M)$ and $\xi' := (E', \pi', M')$ be two $\mathbb{F}$-bundles. The triple $\xi \times \xi' := (E \times E', \pi \times \pi', M \times M')$ is also an $\mathbb{F}$-bundle.*

Proof. Let E, E', M and M' be $\mathbb{F}$-spaces. It follows, from Proposition 1.4.1, that $E \times E'$ and $M \times M'$ are $\mathbb{F}$-spaces. We need to show that $\pi \times \pi'$ is a surjective $\mathbb{F}$-smooth map. Consider the following natural projections p of $E \times E'$ onto E, p' of $E \times E'$ onto E', q of $M \times M'$ onto M and q' of $M \times M'$ onto M'.

$$\begin{tikzcd}
E \arrow[dd, "\pi"'] \arrow[dr, "f"] & & E \times E' \arrow[ll, "p"'] \arrow[r, "p'"] \arrow[dd, "\pi \times \pi'"] & E' \arrow[dd, "\pi'"] \\
& \mathbb{R} & & \\
M \arrow[ur, "g"] & & M \times M' \arrow[ll, "q"] \arrow[ul, "h"'] \arrow[r, "q'"] & M'
\end{tikzcd}$$

Let us mention that $g \in \mathcal{F}_M$ and $g' \in \mathcal{F}'_M$. The set

$$\mathcal{F}_{E \times E'} := \{f \circ p \,|\, f \in \mathcal{F}_E\} \cup \{f' \circ p' \,|\, f' \in \mathcal{F}_{E'}\}$$

and the set

$$\mathcal{F}_{M\times M'} := \{g\circ q \,|\, g\in\mathcal{F}_M\}\cup\{g'\circ q' \,|\, g'\in\mathcal{F}_{M'}\}$$

respectively generate $\mathbb{F}$-structures on $E\times E'$ and $M\times M'$ (see the proof of Propositon 1.4.1). For any $h\in\mathcal{F}_{M\times M'}$ that is $h=g\circ q$ and any $v:=(x,x')\in E\times E'$, we have

$$\begin{aligned}(h\circ(\pi\times\pi'))(v) &= h((\pi\times\pi')(x,x'))\\ &= h(\pi\times\pi'(p(v),p'(v))\\ &= h(\pi(p(v)),\pi'(p'(v)))\\ &= (g\circ q)(\pi(p(v)),\pi'(p'(v)))\\ &= g(q(\pi(p(v)),\pi'(p'(v))))\\ &= g(\pi(p(v)))\\ &= ((g\circ\pi)\circ p)(v),\end{aligned}$$

where q is the projection of $M\times M'$ on M. Then, $h\circ(\pi\times\pi')=(g\circ\pi)\circ p$. Since $g\circ\pi\in\mathcal{F}_E$, as from the proof of Proposition 1.4.1, one concludes that $(g\circ\pi)\circ p\in\mathcal{F}_{E\times E'}$. That is, $h\circ(\pi\times\pi')\in\mathcal{F}_{E\times E'}$. Thus, by Definition 1.2.1(2), $\pi\times\pi'$ is an $\mathbb{F}$-smooth map. By assumption, the map $\pi\times\pi'$ is a surjective map since π and π' are surjective. The map $\pi\times\pi'$ is then a surjective $\mathbb{F}$-smooth map of $E\times E'$ onto $M\times M'$. Hence, by Definition 2.1.1, $\xi\times\xi'$ is an $\mathbb{F}$-bundle. □

Definition 2.3.2 *The* **product** *of two $\mathbb{F}$-bundles ξ and ξ' is the $\mathbb{F}$-bundle $\xi\times\xi'$ as built in Proposition 2.3.3.*

Remark 2.3.2 *From Proposition 2.3.3, one has:*

1. *The category of $\mathbb{F}$-bundles is closed under the product of $\mathbb{F}$-bundles. So the product of $\mathbb{F}$-bundles will be called the* **product $\mathbb{F}$-bundle**.
2. *The operation of product as a functor in the category of $\mathbb{F}$-bundles is defined in the same way as for the usual structured spaces.*
3. *The fiber of $\xi\times\xi'$ over a point $(x,x')\in M\times M'$ is exactly the product of the fibers $\pi^{-1}(x)$ and $\pi'^{-1}(x')$ respectively over $x\in M$ and over $x'\in M'$.*

Definition 2.3.3 *(See [13]) Let $\xi:=(E,\pi,M)$ and $\xi':=(E',\pi',M)$ be $\mathbb{F}$-bundles over M. The* **Whitney sum** *of ξ and ξ', denoted by $\xi\oplus\xi'$, is the pullback of $\xi\times\xi'$ by the diagonal map*

$$h: M\to M\times M \text{ defined by } h(b)=(b,b).$$

The fiber of $\xi\oplus\xi'$ at b is the product set $\pi^{-1}(b)\times\pi'^{-1}(b)\subset E\times E'$.

Remark 2.3.3 *From Definition 2.3.3, we have:*

1. *The Whitney sum of* $\mathbb{F}$*-bundles is an* $\mathbb{F}$*-bundle. We call it the* **Whitney sum** $\mathbb{F}$**-bundle**.
2. *The category of* $\mathbb{F}$*-bundles is closed under taking Whitney sum of* $\mathbb{F}$*-bundles.*
3. *The Whitney sum is a functor on the subcategory of* $\mathbb{F}$*-bundles over a set.*

Proposition 2.3.4 *The* $\mathbb{F}$*-smooth cross sections of the* $\mathbb{F}$*-bundle* $\xi \oplus \xi'$ *are of the form* $(s(b), s'(b))$*, where* s *and* s' *are respectively the* $\mathbb{F}$*-smooth cross sections of the* $\mathbb{F}$*-bundles* ξ *and* ξ' *over* M *and* $b \in M$*.*

Proof. See [13]. The map defined by $(s \times s')(b) := (s(b), s'(b))$ with $b \in M$ is a cross section of $\xi \oplus \xi'$. Now, let us show that $s \times s'$ is $\mathbb{F}$-smooth. Since s and s' are $\mathbb{F}$-smooth maps, $s \times s'$ is an $\mathbb{F}$-smooth map (see the proof of Proposition 2.3.3). Thus, the map $s \times s'$ is an $\mathbb{F}$-smooth cross section of $\xi \oplus \xi'$. □

2.4 Fiber Frölicher bundles

It must be recalled that our $\mathbb{F}$-spaces are balanced. That is, the two underlying topologies respectively induced by structure curves and structure functions coincide. Moreover, these $\mathbb{F}$-spaces are compact Hausdorff as mentioned in the first chapter. They are endowed with the same topological structure with their subspaces.

Definition 2.4.1 *The quadruple* $\xi := (E, \pi, M, F)$ *is a* **fiber** $\mathbb{F}$**-bundle** *if* E*,* M*,* F *are* $\mathbb{F}$*-spaces and* $\pi : E \to M$ *a surjective* $\mathbb{F}$*-smooth map satisfying the following properties: for every* $x \in M$*,* $\pi^{-1}(x)$ *is* $\mathbb{F}$*-diffeomorphic to* F *and for every* $b \in M$ *there exists an open neighborhood* $U \subseteq M$ *of* b *and an* $\mathbb{F}$*-diffeomorphism*

$$\phi : \pi^{-1}(U) \to U \times F \text{ such that } p_1 \circ \phi = \pi,$$

where p_1 *is the projection of* $U \times F$ *on* U *making the following diagram commutative:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

As mentioned at Definition 2.1.1, E is called the **total space** of ξ, M the **base space** or the **basis** of ξ, π the **projection** of ξ and F the **standard fiber** or the **typical fiber** of ξ. The pair (U, ϕ) above is called a **chart** of ξ (see [10],[13] and [16]).

Proposition 2.4.1 *Let $\xi := (E, \pi, M, F)$ be a fiber $\mathbb{F}$-bundle and (U, ϕ) a chart of ξ. The $\mathbb{F}$-bundles $(\pi^{-1}(U), \pi_{|_{\pi^{-1}(U)}}, U)$ and $(U \times F, \pi_1, U)$ are equivalent.*

Proof. Since ξ is an $\mathbb{F}$-bundle and U a subset of M, according to Proposition 2.1.3, $(\pi^{-1}(U), \pi_{|_{\pi^{-1}(U)}}, U)$ is an $\mathbb{F}$-subbundle of ξ. Moreover, U and F are $\mathbb{F}$-spaces and π_1 is a surjective $\mathbb{F}$-smooth map as a natural projection of $U \times F$ onto U. Thus, $(U \times F, \pi_1, U)$ is a trivial $\mathbb{F}$-bundle (see Example 8(1)). By Definition 2.4.1, there exists an $\mathbb{F}$-diffeomorphism ϕ over U such that $p_1 \circ \phi = \pi$. That is, according to Definition 2.2.3, the $\mathbb{F}$-bundles $(\pi^{-1}(U), \pi_{|_{\pi^{-1}(U)}}, U)$ and $(U \times F, \pi_1, U)$ are equivalent. $\square$

Remark 2.4.1 *According to Definition 2.4.1 and Proposition 2.4.1, one can consider the following results:*

1. *Every fiber $\mathbb{F}$-bundle $\xi := (E, \pi, M, F)$ is locally trivial and is also called a* **locally trivial $\mathbb{F}$-bundle** *and the map ϕ a* **local trivialization** *of ξ since it identifies $\pi^{-1}(U) \subseteq E$ with $U \times F$ (see [13]).*

2. *If (U, ϕ) and (V, ψ) are two charts of ξ such that $U \cap V \neq \emptyset$, we can write*
$$(\psi \circ \phi^{-1})(b, f) = (b, g(b)(f)),$$
with $(b, f) \in (U \cap V) \times F$, where g is an $\mathbb{F}$-smooth map from $U \cap V$ to the group G of $\mathbb{F}$-diffeomorphisms of F. The map g is called a **transition function** *and the group G the* **structural group** *[10], [16]. One can observe that g is an $\mathbb{F}$-diffeomorphism as a composite of two $\mathbb{F}$-diffeomorphisms ϕ and ψ.*

Remark 2.4.2 *Let us mention that the notion of $\mathbb{F}$-smooth cross section of a fiber $\mathbb{F}$-bundle is similar to that of $\mathbb{F}$-bundle.*

1. *A special case of an $\mathbb{F}$-smooth cross section is given by the trivial (fiber) $\mathbb{F}$-bundle $(B \times F, \pi_1, B)$ in Example 9.*
2. *When the fiber $\mathbb{F}$-bundle $\xi := (E, \pi, M, F)$ is not trivial, it will help to think of $\mathbb{F}$-smooth cross section as a sort of twisted map from M to F.*
3. *If $\phi : \pi^{-1}(U) \to U \times F$ is a local trivialization of ξ, then we obtain the following diagram:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] & & U \times F \\ & U \arrow[ul, "s"] \arrow[ur, "\phi \circ s = s_U"'] & \end{tikzcd}$$

And, like in the case of (fiber) trivial $\mathbb{F}$-bundle, the local description of the $\mathbb{F}$-smooth cross section is a map $U \to F$ (see [4]). We may then speak of a **local $\mathbb{F}$-smooth cross section** *or* **local section** *if there is no risk of confusion. A local section is defined only over an open set $U \subset M$. Then, the set of local $\mathbb{F}$-smooth cross sections is denoted by $\Gamma(U, F)$ [16].*

Remark 2.4.3 *Let $\xi := (E, \pi, M, F)$ and $\xi' := (E', \pi', M', F')$ be two fiber $\mathbb{F}$-bundles. The morphism (H, h) between ξ and ξ' are defined in Definition 2.2.1. Let now (U, ϕ) and (U', ϕ') be the charts of ξ and ξ'. If $h(U) \cap U' \neq \emptyset$, we can write*

$$\phi' H \phi^{-1}(b, f) = (h(b), g(b)(f)), \quad \textit{with } (b, f) \in (U \cap h^{-1}(U')) \times F,$$

where g is an $\mathbb{F}$-smooth map of $U \cap h^{-1}(U')$ into the set of $\mathbb{F}$-diffeomorphisms of F onto F'. One can observe that g is an $\mathbb{F}$-smooth map because of the $\mathbb{F}$-smoothness of ϕ', H and ϕ^{-1}.

Example 11

1. The trivial $\mathbb{F}$-bundle $\xi = (B \times F, \pi_1, B)$ defined at Example 9 is a fiber $\mathbb{F}$-bundle with the base space B, the typical fiber F and the projection π_1. For example, $(S^1 \times [-1, +1], \pi_1, S^1)$ is a fiber $\mathbb{F}$-bundle with a typical fiber $F = [-1, +1]$.

2. Let D be the $\mathbb{F}$-bundle $\mathbb{R}\times[0,+1] \subset \mathbb{R}^2$ and E the quotient space of D under the identification of $(x,y)\in D$ with $(x+1,1-y)$. Let denote by w the projection of D onto E. According to Proposition 1.3.1, $D\subset\mathbb{R}^2$ is an $\mathbb{F}$-space. Then, according to Proposition 1.8.1, there exists an $\mathbb{F}$-structure on E making w $\mathbb{F}$-smooth. Let us show that the triple $\xi=(E,p,S^1)$ is a fiber $\mathbb{F}$-bundle with

$$p:(x,y)\in E\mapsto e^{2\pi ix}\in S^1,$$

S^1 the standard circle and $[0,+1]$ the typical fiber. Let $b\in S^1$. There exists $u\in\mathbb{R}$ such that $b=e^{2\pi iu}$. Observe that $U:=S^1\setminus\{-b\}$ is an open neighborhood of b and $W:=(u-\frac{1}{2},u+\frac{1}{2})\times[0,+1]$ is an open set in D. The map

$$\phi: w(u,v)\mapsto(e^{2\pi iu},v)$$

is an $\mathbb{F}$-diffeomorphism from $p^{-1}(U):=w(W)$ into $U\times[0,+1]$. The inverse of ϕ on $U\times[0,+1]$ is given by

$$(x,y)\mapsto w(a\,ln\,x,y)$$

where $a=\frac{1}{2}\pi i$.

$$\begin{tikzcd}
W\subset D \arrow[r, "w"] & p^{-1}(U)\subset E \arrow[r, "\phi"] \arrow[dr, "p"'] & U\times[0,+1] \arrow[d, "\pi_1"] \\
 & & U
\end{tikzcd}$$

The pair (U,ϕ) then gives a chart in the neighborhood of b. Let $(u,v)\in W$. We have

$$\begin{aligned}(\pi_1\circ\phi)(w(u,v)) &= (\pi_1(\phi(w(u,v))))\\ &= (\pi_1(e^{2\pi iu},v))\\ &= e^{2\pi iu}\\ &= p(w(u,v)),\end{aligned}$$

by the definition of p. That is, $\pi_1\circ\phi=p$. Thus, ξ is a fiber $\mathbb{F}$-bundle called *Möbius strip* [10], [16].

3. Let S^3 be the standard 3-sphere described by,

$$S^3=\{(z_1,z_2)\in\mathbb{C}^2\cong\mathbb{R}^4\ :\ |z_1|^2+|z_2|^2=1\},$$

S^2 the standard 2-sphere and the complex projective space $\mathbb{C}P^1$, the set of all equivalence classes $[z_1, z_2]$ of $(z_1, z_2) \in \mathbb{C}^2 - \{(0,0)\}$ under the equivalence relation

$$(z_1, z_2) \sim (w_1, w_2) \text{ if } (z_1, z_2) = \lambda(w_1, w_2)$$

for some nonzero $\lambda \in \mathbb{C}$. There is a smooth identification of S^2 with $\mathbb{C}P^1$ using stereographic projection as follows. From the north pole $N \in S^2$, identify $p_N : S^2 - \{N\} \to \mathbb{R}^2 \approx \mathbb{C}$ and also to identify $\mathbb{C}P^1 - \{[0,1]\}$ with $\mathbb{C}$ via the map $\varphi : [z_1, z_2] \mapsto \frac{z_2}{z_1}$. So, the map that we want is obtained through the following diagram

$$\begin{tikzcd}
S^2 - \{N\} \arrow[rr, "p_N"] & & \mathbb{R}^2 \approx \mathbb{C} \\
& \mathbb{C}P^1 - \{[0,1]\} \arrow[ul, "f"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where $f := p_N^{-1} \circ \varphi$. That is

$$f : [z_1, z_2] \longmapsto \begin{cases} p_N^{-1}(\frac{z_2}{z_1}) & \text{if } z_1 \neq 0 \\ N & \text{if } z_1 = 0. \end{cases}$$

Define a map

$$\pi : S^3 \to S^2 \text{ by } (z_1, z_2) \mapsto [z_1, z_2].$$

According to Proposition 1.3.1, the spheres S^3 and S^2 are $\mathbb{F}$-spaces since they are respectively subspaces of $\mathbb{F}$-spaces $\mathbb{R}^4$ and $\mathbb{R}^3$. Also, according to Proposition 1.8.1, the map π is a well defined $\mathbb{F}$-smooth surjection. Thus the triple (S^3, π, S^2) is an $\mathbb{F}$-bundle called the *Hopf map* or *Hopf bundle* [4], [16]. The fiber over any point of S^2 is

$$\pi^{-1}([z_1, z_2]) = \{\lambda(z_1, z_2) \,/\, \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\}.$$

In particular

$$\pi^{-1}([1, 0]) = \{\lambda(1, 0) \,/\, \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\},$$

which is $\mathbb{F}$-diffeomorphic to S^1, the circle of unit radius. Thus $\pi^{-1}([z_1, z_2]) \cong S^1$. Let define now the local trivializations. Observe that

$$U_i = \{[z_0, z_1] \,/\, z_i \neq 0\}\,, (i = 0, 1)\,,$$

which is an open set of S^2 since U_0 can be identified to $S^2-\{N\}$ and U_1 can be identified to $S^2-\{S\}$ by the $\mathbb{F}$-smooth map f defined above. Then $\pi^{-1}(U_i)$ is $\mathbb{F}$-diffeomorphic to $U_i \times S^1$ via

$$(z_0,z_1) \longmapsto \left([z_0,z_1], \frac{z_i}{|z_i|}\right) = \begin{cases} ([1,\frac{z_1}{z_0}], \frac{z_0}{|z_0|}) & \text{over } U_0 \\ ([\frac{z_0}{z_1},1], \frac{z_1}{|z_1|}) & \text{over } U_1. \end{cases}$$

The inverse map on $U_0 \times S^1$ is given by

$$([1,z],\lambda) \longmapsto \left(\frac{\lambda}{\sqrt{1+|z|^2}}, \frac{\lambda z}{\sqrt{1+|z|^2}}\right),$$

where $z=\frac{z_1}{z_0}$ and $\lambda=\frac{z_0}{|z_0|}$. And the inverse map on $U_1 \times S^1$ is given by

$$([z,1],\lambda) \longmapsto \left(\frac{\lambda z}{\sqrt{1+|z|^2}}, \frac{\lambda}{\sqrt{1+|z|^2}}\right),$$

where $z=\frac{z_0}{z_1}$ and $\lambda=\frac{z_1}{|z_1|}$. The local trivializations are well defined on each chart above. The triple (S^3,π,S^2) is then a fiber $\mathbb{F}$-bundle, the typical fiber of which is S^1 .

4. Consider the $\mathbb{F}$-bundle $(G,\pi,G/H)$ defined at Example 8(3), where H is a closed $\mathbb{F}$-Lie subgroup of the $\mathbb{F}$-Lie group G and π the projection of

$$G \text{ into } G/H \text{ defined by } g \mapsto [g] = \{gh \,/\, h \in H\}.$$

For any $[g] \in G/H$, with $g \in G$, the fiber over $[g]$ can be constructed as

$$\pi^{-1}([g]) = \{ga \,/\, a \in H\}$$

since H acts transitively on G. Then, if we consider the unit element e of G, $\pi^{-1}([e]) = H$. To define the local trivializations, we need to consider a map $f_i : U_i \to H$, where each U_i is an open set of G/H. Let s_i be a local $\mathbb{F}$-smooth cross-section over U_i and $g \in \pi^{-1}([g])$. There exists a unique element $h_g \in H$ such that $g = s_i([g])h_g$. Let us define $f_i(g) = (s_i([g]))^{-1}g$. This is a well defined $\mathbb{F}$-diffeomorphism on U_i since $f_i(g) := h_g \in H$ for $g \in \pi^{-1}(U_i)$. Then we define the local trivializations

$$\phi_i : \pi^{-1}(U_i) \to U_i \times H \text{ by } \phi_i(g) = ([g], f_i(g)).$$

Thus $(G,\pi,G/H)$ is fiber $\mathbb{F}$-bundle with H the typical fiber.
One can consider a useful particular example, given in [4], the *Homogeneous bundle* $(O(n),\pi,O(n)/O(n-1))$, where the homogeneous space

$O(n)/O(n-1)$ is $\mathbb{F}$-diffeomorphic to the standard $(n-1)$-sphere S^{n-1} by the map

$$[A] \mapsto A.e_1 \text{ with } e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $O(n-1)$ (the isotropy subgroup of e_1) is the typical fiber and

$$\pi : O(n) \to O(n)/O(n-1)$$

the projection of the bundle.

Remark 2.4.4 *Let $\xi = (E, \pi, M, F)$ be a fiber $\mathbb{F}$-bundle and $h : N \to M$ a given $\mathbb{F}$-smooth map. The* **pullback** *$h^*(\xi)$ of ξ by h is defined at Definition 2.3.1. The fiber F_x of $h^*(\xi)$ over $x \in M$ is just a copy of the fiber $F_{h(x)}$ of ξ.*

In what follows, we show that the pullback of a fiber $\mathbb{F}$-bundle is also a fiber $\mathbb{F}$-bundle.

Proposition 2.4.2 *The pullback $h^*(\xi)$ of the fiber $\mathbb{F}$-bundle $\xi = (E, \pi, M, F)$ is a fiber $\mathbb{F}$-bundle.*

Proof. According to Proposition 2.3.1, $h^*(\xi)$ is an $\mathbb{F}$-bundle. Then, let us show that $h^*(\xi)$ is locally trivial. Let x be an element of N, and let (U, ϕ) be a chart of ξ with $\phi(u) = (h(x), f)$, where $u \in \pi^{-1}(U)$ such that $\pi(u) = h(x) \in U$ and f an $\mathbb{F}$-diffeomorphism of $\pi^{-1}(U)$ onto F. Obviously, $h^{-1}(U)$ is an open set of N because $h : N \to M$ is an $\mathbb{F}$-smooth map. Let π' be the projection of $h^*(\xi)$. The map $\psi : (x, u) \mapsto (x, f)$ is an $\mathbb{F}$-diffeomorphism of $\pi'^{-1}(h^{-1}(U))$ into $h^{-1}(U) \times F$ since ψ is a composition of $\mathbb{F}$-smooth maps ev, ins and $comp$ (see Corollary 1.5.1) and for the same reason its inverse $\psi^{-1} : (x, f) \mapsto (x, \phi^{-1}(h(x), f))$ is $\mathbb{F}$-smooth. Thus, $h^*(\xi)$ is a fiber $\mathbb{F}$-bundle with the typical fiber F. $\square$

2.5 Vector Frölicher bundles

Definition 2.5.1 *A* **vector $\mathbb{F}$-bundle** *$\xi = (E, \pi, M, F)$ is a fiber $\mathbb{F}$-bundle ξ whose typical fiber F is a linear $\mathbb{F}$-space [16].*

Example 12 The trivial $\mathbb{F}$-bundle $E = M \times F$, where M is an $\mathbb{F}$-space and F a linear $\mathbb{F}$-space, is a vector $\mathbb{F}$-bundle.

Some additional examples of vector bundles such as tangent and cotangent bundles on a finite-dimensional pseudomanifold will be given in the next chapter.

Proposition 2.5.1 *Let $\pi : E \to M$ be a surjective map of a set E to an $\mathbb{F}$-space M. If there exists a linear $\mathbb{F}$-space F such that, for every $b \in M$, there exists an open neighborhood $U \subseteq M$ containing b and a bijective map $\varphi : \pi^{-1}(U) \to U \times F$ then there exists on E an $\mathbb{F}$-structure of vector bundle with basis M, the typical fiber is F and the projection is π.*

Proof. In effect, the set

$$\{f \circ \pi \mid f \in \mathcal{F}_M\}$$

generates an $\mathbb{F}$-structure on E making π $\mathbb{F}$-smooth. Also, for every $U \subseteq M$, $\pi^{-1}(U) \subseteq E$. Then, according to Proposition 1.3.1, we have that U and $\pi^{-1}(U)$ have an $\mathbb{F}$-structure. According to Proposition 1.4.1, we have also that the product $U \times F$ has an $\mathbb{F}$-structure. Thus, $\pi^{-1}(U)$ and $U \times F$ are $\mathbb{F}$-spaces.

$$\begin{tikzcd}
\pi^{-1}(U) \subseteq E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\
& U \subseteq M &
\end{tikzcd}$$

Now, we need to show that φ is an $\mathbb{F}$-diffeomorphism. For any $g \in \mathcal{F}_{U\times F}$, we have $g = \tilde{g} \circ \iota \circ p_1$, where $\tilde{g} \in \mathcal{F}_M$, ι the canonical injection of U into M and p_1 the natural projection on the first factor of $U \times F$. Thus, by the associativity of the composition of maps and the commutativity of the diagram above, we have

$$\begin{aligned} g \circ \varphi &= (\tilde{g} \circ \iota \circ p_1) \circ \varphi \\ &= \tilde{g} \circ (\iota \circ p_1 \circ \varphi) \\ &= \tilde{g} \circ \pi_{|_{\pi^{-1}(U)}}. \end{aligned}$$

Since $\tilde{g} \in \mathcal{F}_M$ and $\pi_{|_{\pi^{-1}(U)}}$ is the restriction of π to $\pi^{-1}(U) \subseteq E$, then $\tilde{g} \circ \pi_{|_{\pi^{-1}(U)}}$ is an element of the generating set of structure functions on $\pi^{-1}(U)$. That is, $\tilde{g} \circ \pi_{|_{\pi^{-1}(U)}} \in \mathcal{F}_{\pi^{-1}(U)}$. According to Definition 1.2.1(2), φ is $\mathbb{F}$-smooth. Similary, for any $h \in \mathcal{F}_{\pi^{-1}(U)}$, we have $h = f \circ \pi \circ j$, where $f \in \mathcal{F}_M$ and j is the canonical injection of $\pi^{-1}(U)$ into E. Thus, as above, we have

$$\begin{aligned} h \circ \varphi^{-1} &= (f \circ \pi \circ j) \circ \varphi^{-1} \\ &= f \circ (\pi \circ j \circ \varphi^{-1}) \\ &= f \circ (\iota \circ p_1) \\ &= (f \circ \iota) \circ p_1 \\ &= f_{|U} \circ p_1, \end{aligned}$$

where ι is the canonical injection of U into M, p_1 the natural projection of $U \times F$ on the first factor and $f_{|U}$ is the restriction of f to $U \subseteq M$. Since $f_{|U} \in \mathcal{F}_U$, then $f_{|U} \circ p_1$ is an element of the generating set of structure functions on $U \times F$. Thus, $h \circ \varphi^{-1} \in \mathcal{F}_{U\times F}$. That is, according to Definition 1.2.1(2), φ^{-1} is $\mathbb{F}$-smooth. Briefly, φ is bijective such that φ and φ^{-1} are $\mathbb{F}$-smooth. Hence, according to Definition 1.2.2, φ is an $\mathbb{F}$-diffeomorphism. And we can carry the linear structure of F to each fiber $F_x = \pi^{-1}(x)$ over $x \in M$. Finally, the surjective $\mathbb{F}$-smooth map $\pi : E \to M$ is locally given by $\pi = p_1 \circ \varphi$. Hence, according to Definition 2.5.1, (E, π, M, F) is a vector $\mathbb{F}$-bundle. □

Definition 2.5.2 *Let $\xi = (E, \pi, M, F)$ be a vector $\mathbb{F}$-bundle. Sections of ξ always exist and are also called C^∞-***sections***. They are similar to those of fiber $\mathbb{F}$-bundle [4]. Each fiber $F_x := \pi^{-1}(x)$ ($x \in M$) is a linear $\mathbb{F}$-space. The $\mathbb{F}$-smooth map $\theta : M \to E$ given by $\theta(x) = 0 \in F_x$ for all $x \in M$ (0 is the null vector) is called the* **zero section** *or the* **null section** *[14], [16].*

Obviously $\theta \in \Gamma(M, E)$. The vector addition and the scalar multiplication in $\Gamma(M, E)$ are pointwise. They are defined as follows

- $(s + s')(x) = s(x) + s'(x)$
- $(f.s)(x) = f(x).s(x)$,

where $x \in M$ and $f \in \mathcal{F}_M$ (see [16]). Notice that there is no risk of confusion using the common notations for the vector addition and the scalar multiplication in $\Gamma(M, E)$ and for those (fiberwise) in E.

Remark 2.5.1 *From Definition 2.5.2 and Proposition 2.5.1, we have that:*

1. *The set $\Gamma(M, E)$ takes on the structure of a module over $\mathcal{F}_M$.*
2. *Let $\xi = (E, \pi, M, F)$ and $\xi' = (E', \pi', M', F')$ be two vector $\mathbb{F}$-bundles. The morphism (H, h) between ξ and ξ' is defined as at Definition 2.2.1 [16]. Obviously, since $F_x = \pi^{-1}(x)$ and $F'_{h(x)} = \pi'^{-1}(h(x))$ are linear $\mathbb{F}$-spaces, the restriction to H on each fiber F_x is a linear $\mathbb{F}$-smooth map $F_x \to F'_{h(x)}$ [4].*

Definition 2.5.3 *(See [16]) Let $\xi = (E, \pi, M, F)$ and $\xi' = (E', \pi', M', F')$ be two vector $\mathbb{F}$-bundles. The* **product $\mathbb{F}$-bundle**

$$\xi \times \xi' = (E \times E', \pi \times \pi', M \times M')$$

of ξ and ξ' is a fiber $\mathbb{F}$-bundle whose typical fiber is $F \oplus F'$.

If $\pi(u) = p$ and $\pi'(u') = p'$, the image of $(u, u') \in E \times E'$ by the projection $\pi \times \pi'$ is given by $(\pi \times \pi')(u, u') = (p, p')$. The fiber at $(p, p') \in M \times M'$ is $F_p \oplus F'_{p'}$. See again [16].

Proposition 2.5.2 *The product $\mathbb{F}$-bundle of two vector $\mathbb{F}$-bundles is a vector $\mathbb{F}$-bundle.*

Proof. Let ξ and ξ' be two vector $\mathbb{F}$-bundles, the typical fibers of which are F and F' respectively. By Definition 2.5.3 above, the product $\mathbb{F}$-bundle $\xi \times \xi'$ is a fiber $\mathbb{F}$-bundle with typical fiber $F \oplus F'$. Since F and F' are linear $\mathbb{F}$-spaces, $F \oplus F'$ is also a linear $\mathbb{F}$-space. Therefore, according to Definition 2.5.1, $\xi \times \xi'$ is a vector $\mathbb{F}$-bundle. $\square$

Remark 2.5.2 *We have:*

1. *If $\xi_i = (E_i, \pi_i, M_i, F_i)$, $i = 1, 2, \ldots, n$, are vector $\mathbb{F}$-bundles, then $\xi_1 \times \xi_2 \times \cdots \times \xi_n$ is a vector $\mathbb{F}$-bundle with $E_1 \times E_2 \times \cdots \times E_n$ the total space, $p_1 \times p_2 \times \cdots \times p_n$ the projection and $F_1 \oplus F_2 \oplus \cdots \oplus F_n$ the typical fiber.*

2. *If ξ is a vector $\mathbb{F}$-bundle, then ξ^n is the product $\mathbb{F}$-bundle of n equivalent vector $\mathbb{F}$-bundles.*

3. *The pullback $\mathbb{F}$-bundle is defined like in Definition 2.3.1.*

Definition 2.5.4 *(See [16]) Let $\xi = (E, \pi, M, F)$ and $\xi' = (E', \pi', M', F')$ be two vector $\mathbb{F}$-bundles. The* **Whitney Sum** *$\xi \oplus \xi'$ of ξ and ξ' is a pullback $\mathbb{F}$-bundle of $\xi \times \xi'$ by the diagonal map*

$$h : M \to M \times M \text{ defined by } h(b) = (b, b).$$

$$\begin{tikzcd} E \oplus E' \arrow[r, "H"] \arrow[d, "(\pi \times \pi')^*"'] & E \times E' \arrow[d, "\pi \times \pi'"] \\ M \arrow[r, "h"] & M \times M \end{tikzcd}$$

Let us recall that $E \oplus E' = \{(b, u, u') \in M \times E \times E' \,|\, \pi(u) = \pi'(u') = b\}$. It can be seen that $E \oplus E'$ is $\mathbb{F}$-diffeomorphic to the set $\{(u, u') \in E \times E' \,|\, (\pi \times \pi')(u, u') = (b, b)\}$.

Proposition 2.5.3 *Let ξ and ξ' be two vector $\mathbb{F}$-bundles. The Whitney Sum $\xi \oplus \xi'$ is a vector $\mathbb{F}$-bundle.*

Proof. By Definition 2.5.3, $\xi \times \xi'$ is a fiber $\mathbb{F}$-bundle whose the typical fiber is $F \oplus F'$. But, we have that $F \oplus F'$ is a linear $\mathbb{F}$-space since F and F' are $\mathbb{F}$-spaces, by assumption. We have also that $\xi \oplus \xi' := h^*(\xi \times \xi')$. The map

$$h : M \to M \times M \text{ given by } h(b) = (b, b)$$

is $\mathbb{F}$-smooth since $h(b) = ins(ev(id_M, b), ev(id_M, b))$ and M is an $\mathbb{F}$-space (see Corollary 1.5.1). Thus, according to Proposition 2.4.2 and Definition 2.5.4, $\xi \oplus \xi'$ is a vector $\mathbb{F}$-bundle whose the typical fiber is $F \oplus F'$. $\square$

Definition 2.5.5 *(See [16]) Let $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$ be two vector $\mathbb{F}$-bundles over M, where E, E' and M are all n- dimensional pseudo-manifolds with F_p and F'_p respectively their fibers at $p \in M$. The* **tensor product $\mathbb{F}$-bundle** *$E \otimes E'$ is obtained by assigning the tensor product $F_p \otimes F'_p$ at each point $p \in M$ (see Proposition 1.10.3). In the same way, we have the* **wedge product $\mathbb{F}$-bundle** *$\bigwedge^k E$ by assigning $\bigwedge^k F_p$ at each $p \in M$ (see Proposition 1.10.4).*

2.6 Principal Frölicher bundles

Let us define the notion of an action on an $\mathbb{F}$-Lie group of an $\mathbb{F}$-space before introducing those of principal and associated bundles in the category $\underline{\mathcal{FRL}}$. The notion of $\mathbb{F}$-Lie group can be found at Section 1.6 of this work.

Definition 2.6.1 *A* **right action** *of an $\mathbb{F}$-Lie group G on an $\mathbb{F}$-space E is an $\mathbb{F}$-smooth map*

$$\sigma : E \times G \to E, \text{ defined by } (x, g) \mapsto \sigma(x, g) := x.g,$$

satisfying the conditions:

1. *$\sigma(x, e) = x$ for any $x \in E$, where e is the unit element of G*
2. *$\sigma(\sigma(x, g_1), g_2) = \sigma(x, g_1 \circ g_2)$ for $x \in E$ and $g_1, g_2 \in G$, where $\circ$ is the composition of maps.*

The quotient $\mathbb{F}$-space E by G is called the **right G-space**, *see [13], and we will call it* **Frölicher G-space** *or for short* **$\mathbb{F}G$-space** *(see [2]).*

One can define similarly the notion of a *left action* by setting $G \times E$ instead of $E \times G$. Since there exists a bijective correspondence between left and right G-structures (see [13] and [16]), we will restrict ourselves to the case of a right action.

Now, refering to [4] and [16], let us define the notion of a principal bundle in the category $\underline{\mathcal{F}RL}$.

Definition 2.6.2 *Let (E, π, M, G) be a vector $\mathbb{F}$-bundle. Let G be an $\mathbb{F}$-Lie group which has a right action on E such that*

1. *The action is free (that is, $x.e = x$ if and only if e is the identity element of G)*
2. *The action preserves the fibers of E.*

We then call (E, π, M, G) a **principal Frölicher bundle** *or for short* **$p\mathbb{F}$ G-bundle** *[17].*

Example 13 The Hopf bundle, defined at Example 11(3), is an example of a $p\mathbb{F}$ S^1-bundle and the homogeneous bundle bundle $O(n) \to O(n)/O(n-1)$, given at Example 11(4), is a $p\mathbb{F}$ $O(n-1)$-bundle.

In order to introduce the notion of an associated bundle in the category $\underline{\mathcal{F}RL}$, we refer to [14] and [16].

Definition 2.6.3 *Let $\xi = (E, \pi, M, G)$ be a vector $p\mathbb{F}$ G-bundle, and let G act on an $\mathbb{F}$-space on the left. We consider the right action*

$$\tau : (E \times P) \times G \to E \times P, \text{ given by } \tau((x, y), g) = (x.g, g^{-1}.y).$$

Then, the **associate Frölicher bundle** *or for short* **associate $\mathbb{F}$-bundle** *of ξ is $(E \times P/G, \tilde{\pi}, M, G)$, where $E \times P/G$ is the set of equivalence classes in which two elements (x, y) and $(x.g, g^{-1}.y)$ are identified and*

$$\tilde{\pi} : E \times P/G \to M$$

is the projection defined by $\tilde{\pi}([(x, y)]) = \pi(x)$.

$$\begin{tikzcd}
E \times P \arrow[r, "q"] \arrow[d, "\pi_1"'] & E \times P/G \arrow[d, "\tilde{\pi}"] \\
E \arrow[r, "\pi"] & M
\end{tikzcd}$$

Remark 2.6.1 *We have the following:*

1. *The space $E \times P/G$ carries an $\mathbb{F}$-structure making the quotient map*
$$q : E \times P \to E \times P/G$$
$\mathbb{F}$*-smooth, according to Proposition 1.4.1 and Proposition 1.8.1.*
2. *The projection*
$$\tilde{\pi} : E \times P/G \to M,$$
is well defined since $\pi(x) = \pi(x.g)$ implies
$$\tilde{\pi}([(x.g, g^{-1}.y)]) = \pi(x.g) = \pi(x) = \tilde{\pi}([(x, y)]).$$
3. *The quadruple $(E \times P, q, E \times P/G, G)$ is a p$\mathbb{F}$ G-bundle with the right action τ.*

We defined the notion of bundles and morphism between bundles in the category of Frölicher spaces following the common approach in other categories. We proved that a fiber of a Frölicher bundle has a Frölicher structure. We noticed that a section of a Frölicher bundle is a smooth map. We reviewed some types Frölicher bundles. Some detailed examples are given. We will be interested in the vector Frölicher bundles (in the next part of this work). We proved as a first example that the product of any Frölicher space with a linear Frölicher space is a vector Frölicher bundle (called the trivial vector Frölicher bundle). In the next chapter, we study two more examples of vector Frölicher bundles: the tangent and the cotangent bundles on a finite-dimensional pseudomanifold.

Chapter 3

Tangent structures on Frölicher spaces

Before embarking on the familiar examples of vector bundles in the category $\underline{\mathcal{FRL}}$, the *tangent* and *cotangent bundles*, we shall define some basic notions related to the tangent structures on an $\mathbb{F}$-space as developped in [2], [6] and [7]. There exists two kinds of tangent vector on an $\mathbb{F}$-space: the one from the structure functions called the *operational tangent vector* and the other from the structure curves called the *kinematic tangent vector*.

3.1 Operational tangent vectors

The following lemma will ensure us that the set of structure functions on an $\mathbb{F}$-space is closed under pointwise multiplication. We define the notion of a smooth derivation on the algebra of structure functions (see Proposition 1.5.2).

Definition 3.1.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be an* $\mathbb{F}$*-space. The* **operational tangent vector** *at* $x \in M$ *is a linear map* $v_x : \mathcal{F}_M \to \mathbb{R}$ *satisfying the Leibniz rule on* $\mathcal{F}_M$ *at the point* $x \in M$*:*

$$v_x(f.g) = g(x)v_x(f) + f(x)v_x(g).$$

The map v_x is also called a *smooth derivation* at $x \in M$. The set of all operational tangent vectors to M at $x \in M$ is denoted by T_xM and is called the *operational tangent space* to M at $x \in M$ [2], [6], [7] and [17]. We will denote by $\mathcal{D}_M$ (see [6] and [7]) the set of operational tangent vectors at the points of M.

Remark 3.1.1 *According to Definition 3.1.1 and the note above, one has* $T_xM \subseteq C^\infty(\mathcal{F}_M, \mathbb{R})$ *for* $x \in M$ *and* $\mathcal{D}_M \subseteq C^\infty(\mathcal{F}_M, \mathbb{R})$.

Proposition 3.1.1 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be an* $\mathbb{F}$*-space and* $x \in M$. *The set* T_xM *is a linear* $\mathbb{F}$*-space.*

Proof. From Remark 3.1.1, Proposition 1.5.1 and Corollary 1.5.2, we have $C^\infty(\mathcal{F}_M, \mathbb{R})$ has an $\mathbb{F}$-structure and $T_xM \subseteq C^\infty(\mathcal{F}_M, \mathbb{R})$ since M is an $\mathbb{F}$-space. Hence, according to Proposition 1.3.1, there exists an $\mathbb{F}$-structure on T_xM. At every element of M, the map

$$\theta : \mathcal{F}_M \to \mathbb{R} \text{ given by } \theta(f) = 0$$

satisfies the Leibniz rule. So, at $x \in M$, we have that $\theta \in T_xM$. That is, $T_xM \neq \emptyset$. Let us show that the addition of operational tangent vectors and the scalar multiplication of operational tangent vectors by a real number at a point $x \in M$ are internal operations. Let $v_x, v'_x \in T_xM$ and $f, g \in \mathcal{F}_M$. In effect, by Definition 3.1.1 and the properties of the addition and the multiplication of real numbers, we have

$$\begin{aligned}
(v_x + v'_x)(f.g) &= v_x(f.g) + v'_x(f.g) \\
&= (g(x)v_x(f) + f(x)v_x(g)) + (g(x)v'_x(f) + f(x)v'_x(g)) \\
&= g(x)v_x(f) + f(x)v_x(g) + g(x)v'_x(f) + f(x)v'_x(g) \\
&= g(x)v_x(f) + g(x)v'_x(f) + f(x)v_x(g) + f(x)v'_x(g) \\
&= g(x)(v_x(f) + v'_x(f)) + f(x)(v_x(g) + v'_x(g)) \\
&= g(x)(v_x + v'_x)(f) + f(x)(v_x + v'_x)(g).
\end{aligned}$$

Thus, the sum $v_x + v'_x$ satisfies the Leibniz rule for all $v_x, v'_x \in T_xM$. That is, by Definition 3.1.1, $v_x + v'_x \in T_xM$. Let $v_x \in T_xM$ and $\lambda \in \mathbb{R}$. Let $f, g \in \mathcal{F}_M$. By Definition 3.1.1 and the properties of the multiplication of real numbers, we have

$$\begin{aligned}
(\lambda.v_x)(f.g) &= \lambda.v_x(f.g) \\
&= \lambda(g(x)v_x(f) + f(x)v_x(g)) \\
&= \lambda g(x)v_x(f) + \lambda f(x)v_x(g) \\
&= g(x)\lambda v_x(f) + f(x)\lambda v_x(g) \\
&= g(x)(\lambda.v_x)(f) + f(x)(\lambda.v_x)(g).
\end{aligned}$$

Hence, $\lambda.v_x$ satisfies the Leibniz rule. That is, by Definition 3.1.1, $\lambda.v_x \in T_xM$ for all $\lambda \in \mathbb{R}$ and $v_x \in T_xM$. □

Proposition 3.1.2 *Let* $(M, \mathcal{C}_M, \mathcal{F}_M)$ *be an* $\mathbb{F}$*-space and* $x \in M$. *The set* $\mathcal{D}_M$ *has an* $\mathbb{F}$*-structure.*

Proof. Again from Remark 3.1.1, Proposition 1.5.1 and Corollary 1.5.2, $C^\infty(\mathcal{F}_M, \mathbb{R})$ has an $\mathbb{F}$-structure and $\mathcal{D}_M \subseteq C^\infty(\mathcal{F}_M, \mathbb{R})$. Hence, according to Proposition 1.3.1, there exists an $\mathbb{F}$-structure on $\mathcal{D}_M$. $\square$

Example 14

1. The set $\mathbb{Q}$ of rational numbers has an $\mathbb{F}$-structure as it is a subspace of the canonical Euclidian $\mathbb{F}$-space $\mathbb{R}$. Let $q \in \mathbb{Q}$. The function

$$f(x) = \begin{cases} 1 & , \text{ if } x = q \\ 0 & , \text{ if } x \neq q \end{cases}$$

belongs to $\mathcal{F}_{\mathbb{Q}}$, since f and the curves in $\mathbb{Q}$ are constant. One has $f^2(x) = (f(x))^2 = f(x)$ for $x = q$ or $x \neq q$. That is, $f^2 = f$. Let $v_q \in T_q\mathbb{Q}$. Then,

$$\begin{aligned} v_q(f^2) &= v_q(f.f) \\ &= f(q)v_q(f) + f(q)v_q(f) \\ &= 2f(q)v_q(f) \\ &= 2v_q(f) \end{aligned}$$

since v_q satisfies Leibniz rule and $f(q) = 1$. From $v_q(f^2) = v_q(f)$ (since $f^2 = f$) and $v_q(f^2) = 2v_q(f)$, we have $v_q(f) = 2v_q(f)$. That is, $v_q(f) = 0$. Let $g \in \mathcal{F}_{\mathbb{Q}}$ and $\tilde{g} = g.f \in \mathcal{F}_{\mathbb{Q}}$. Then,

$$v_q(\tilde{g}) = f(q)v_q(g) + g(q)v_q(f) = v_q(g),$$

since $v_q(f) = 0 = \theta(f)$ and $f(q) = 1$. Thus, we have $T_q\mathbb{Q} = \{\theta\}$ since $f \in \mathcal{F}_{\mathbb{Q}}$ and $v_q(f) = 0$.

2. Let $D := \{(x, y) \in \mathbb{R}^2 \,|\, xy = 0\}$. The set D has an $\mathbb{F}$-structure as a subspace of the $\mathbb{F}$-space $\mathbb{R}^2$, see Proposition 1.3.1. Note that $D = D_1 \cup D_2$, where $D_1 = \{(x, 0) \,|\, x \in \mathbb{R}\}$ and $D_2 = \{(0, y) \,|\, y \in \mathbb{R}\}$. We have $\mathcal{F}_D = \{f : D \to \mathbb{R} \,|\, f := g_{|D} \,;\, g \in \mathcal{F}_{\mathbb{R}^2}\}$, where

$$f(x, y) = \begin{cases} f_1(x, 0) & \text{along } D_1 \\ f_2(0, y) & \text{along } D_2 \end{cases}$$

for f_1 and f_2 are $\mathbb{F}$-smooth real functions that intersect at the origin $(0, 0)$. An operational tangent vector on D is a linear derivation along D_1 or along D_2. That is, $V = \frac{\partial f}{\partial x} = \frac{df_1}{dx}$ or $V = \frac{\partial f}{\partial y} = \frac{df_2}{dy}$. We can find a non-zero vector in the direction D_1, which generates a one-dimensional tangent space at a point $p_1 = (x, 0)$, where $x \neq 0$. Similarly, we have a one-dimensional tangent space at a point $p_2 = (0, y)$, where $y \neq 0$.

Thus, for $p \neq (0,0), dim\, T_pD = 1$. But at the origin $(0,0)$, one can obtain two linearly independent operational tangent vectors given by

$$\frac{\partial f}{\partial x}|_{(0,0)} \text{ and } \frac{\partial f}{\partial y}|_{(0,0)}, \text{ for all } f \in \mathcal{F}_{\mathbb{R}^2}.$$

Thus, $T_{(0,0)}D$ is isomorphic $\mathbb{R}^2$ and $dim\, T_{(0,0)}D = 2$.

Definition 3.1.2 *Let M be an $\mathbb{F}$-space. The* **operational cotangent space** *at $p \in M$, denoted T_p^*M, is the algebraic dual of T_pM. The elements of T_p^*M are called* **covariant vectors** *or* **covectors** *[2].*

Remark 3.1.2 *Let M be an $\mathbb{F}$-space.*

1. *According to Propostion 1.10.2, the operational cotangent space T_p^*M at $p \in M$ has an $\mathbb{F}$-structure.*

2. *T_p^*M is a linear $\mathbb{F}$-space if T_pM is a finite dimensional $\mathbb{F}$-space and then $dim\, T_pM = dim\, T_p^*M$, see [2].*

3.2 Kinematic tangent vectors

Definition 3.2.1 *Let M be an $\mathbb{F}$-space. Let us denote by $C_M^{a,x}$ the set of all structure curves*

$$c : \mathbb{R} \to M \text{ such that } c(a) = x,$$

that is, the curves passing through $x \in M$, with the footpoint $a \in \mathbb{R}$. One calls **kinematic tangent vector** *to the space M with footpoint $a \in \mathbb{R}$ a derivation defined by*

$$f \longmapsto \frac{d(f \circ c)(t)}{dt}|_{t=a},$$

where $c \in C_M^{a,x}$ and $f \in \mathcal{F}_M$. The set of all kinematic tangent vectors at $x \in M$ is denoted T_xCM and called the **tangent cone space** *to M at $x \in M$, see [2], [6] and [7].*

Example 15

1. Let us consider the kinematic tangent vector to the set $\mathbb{Q}$ of rationals. The set $\mathcal{C}_{\mathbb{Q}}$ consists of the constant maps of $\mathbb{R}$ to $\mathbb{Q}$ and $\mathcal{F}_{\mathbb{Q}}$ consists of functions of $\mathbb{Q}$ to $\mathbb{R}$. Let $q \in \mathbb{Q}$ such that $c(a) = q$.

$$\frac{d(f(c(t)))}{dt}|_{t=a} = 0,$$

since curves in $\mathbb{Q}$ are constant maps. Hence, similar to Example 14(1), $T_qCM = \{\theta\}$.

2. Let us consider again the $\mathbb{F}$-space $D := \{(x, y) \in \mathbb{R}^2 \,|\, xy = 0\}$. For every $c \in \mathcal{C}_D$ and $t \in \mathbb{R}$, we have $(f \circ c)(t) = f(x, 0)$ or $(f \circ c)(t) = f(0, y)$. Thus,
$$\frac{d(f \circ c)(t)}{dt}|_{t=a} = \frac{\partial f}{\partial x}\frac{dc(t)}{dt}|_{t=a} \text{ a long the } x-\text{axis},$$
and
$$\frac{d(f \circ c)(t)}{dt}|_{t=a} = \frac{\partial f}{\partial y}\frac{dc(t)}{dt}|_{t=a} \text{ a long the } y-\text{axis}.$$
It turns out that, for $p \neq (0, 0)$, the derivation does not vanish in both cases. Like for the operational tangent space, it follows that T_pCD is a straight line on the x-axis or on the y-axis. But for $p = (0, 0)$, T_pCD lies either on the x-axis or on the y-axis which is in fact D.

3.3 Tangent linear maps

Proposition 3.3.1 *Let $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \to (N, \mathcal{C}_N, \mathcal{F}_N)$ be an $\mathbb{F}$-smooth map and $v_p \in T_pM$. The pair (v_p, φ) induces on N an operational tangent vector in the neighborhood of $\varphi(p) \in N$.*

Proof. According to Definition 1.2.1(2), for all $g \in \mathcal{F}_N$, the composite $g \circ \varphi$ is a structure function on M since φ is $\mathbb{F}$-smooth. In the neighborhood of $\varphi(p) \in N$, for $v_p \in T_pM$, define the map $\varphi_{*p}(v_p) : \mathcal{F}_N \to \mathbb{R}$ by
$$\varphi_{*p}(v_p)(g) = v_p(g \circ \varphi).$$
Let us show that $\varphi_{*p}(v_p)$ is a linear map satisfying the Leibniz rule. Let $\lambda \in \mathbb{R}$ and $g \in \mathcal{F}_N$. By the definition of $\varphi_{*p}(v_p)$ and since v_p is a linear map, we have
$$\begin{aligned}\varphi_{*p}(v_p)(\lambda \cdot g) &= v_p((\lambda \cdot g) \circ \varphi)\\ &= v_p(\lambda \cdot (g \circ \varphi))\\ &= \lambda v_p(g \circ \varphi)\\ &= \lambda \varphi_{*p}(v_p)(g).\end{aligned}$$
For the same reasons, if $f \in \mathcal{F}_N$ and $g \in \mathcal{F}_N$, we obtain
$$\begin{aligned}\varphi_{*p}(v_p)(f + g) &= v_p((f + g) \circ \varphi)\\ &= v_p((f \circ \varphi) + (g \circ \varphi))\\ &= v_p(f \circ \varphi) + v_p(g \circ \varphi)\\ &= \varphi_{*p}(v_p)(f) + \varphi_{*p}(v_p)(g).\end{aligned}$$

Now let $f \in \mathcal{F}_N$, $g \in \mathcal{F}_N$ and $q := \varphi(p) \in N$. By the definition of $\varphi_{*p}(v_p)$ and since v_p satifies the Leibniz rule on $\mathcal{F}_M$, it follows that

$$\begin{array}{rcl} \varphi_{*p}(v_p)(f \cdot g) & = & v_p((f \cdot g) \circ \varphi) \\ & = & v_p((f \circ \varphi) \cdot (g \circ \varphi)) \\ & = & (g \circ \varphi)(p) v_p(f \circ \varphi) + (f \circ \varphi)(p) v_p(g \circ \varphi) \\ & = & g(\varphi(p)) \varphi_{*p}(v_p)(f) + f(\varphi(p)) \varphi_{*p}(v_p)(g) \\ & = & g(q) \varphi_{*p}(v_p)(f) + f(q) \varphi_{*p}(v_p)(g). \end{array}$$

Hence, $\varphi_{*p}(v_p)$ is an induced operational tangent vector on N at $q = \varphi(p) \in N$. □

Definition 3.3.1 *(see [2] and [14]) Let $\varphi : M \to N$ be an $\mathbb{F}$-smooth map. Let $p \in M$ and $v_p \in T_pM$. The $\mathbb{F}$-smooth linear map*

$$\varphi_{*p} : T_pM \to T_{\varphi(p)}N \text{ defined by } \varphi_{*p}(v_p)(g) = v_p(g \circ \varphi)$$

is called the **tangent linear map** *associated with φ at $p \in M$.*

Let us recall that, in some sense, the tangent linear map φ_{*p} is the best linear approximation of φ near p.

The tangent linear map associated with φ at $p \in M$ is also called, by various authors, the **push forward**, or **total derivative**, or simply **derivative**, or again **differential** of φ at $p \in M$ and is frequently expressed using a variety of other notations such as $T_p\varphi, d\varphi_p, D\varphi_p, \varphi'(p)$ or $D_p\varphi$ (see [14]).

Proposition 3.3.2 *Given any two $\mathbb{F}$-smooth maps $\varphi : M \to N$ and $\psi : N \to P$ between $\mathbb{F}$-spaces, for any $p \in M$, we have (the chain rule)*

$$(\psi \circ \varphi)_{*p} = \psi_{*\varphi(p)} \circ \varphi_{*p}$$

Proof. Since φ and ψ are $\mathbb{F}$-smooth, according to Proposition 1.2.2, the composite $\psi \circ \varphi$ is $\mathbb{F}$-smooth. Let $p \in M$ and $v_p \in T_pM$. By Proposition 3.3.1, the pair $(v_p, \psi \circ \varphi)$ induces on P an operational tangent vector in the neighborhood of $(\psi \circ \varphi)(p) \in P$. Let $h \in \mathcal{F}_P$. By Definition 3.3.1 and the associativity of the composition of maps, we have

$$\begin{array}{rcl} (\psi \circ \varphi)_{*p}(v_p)(h) & = & v_p(h \circ (\psi \circ \varphi)) \\ & = & v_p((h \circ \psi) \circ \varphi) \\ & = & \varphi_{*p}(v_p)(h \circ \psi) \\ & = & \psi_{*\varphi(p)}(\varphi_{*p}(v_p))(h), \end{array}$$

where $\varphi_{*p}(v_p) \in T_{\varphi(p)}N$ and $\psi_{*\varphi(p)}(\varphi_{*p}(v_p)) \in T_{\psi \circ \varphi(p)}P$. Hence,

$$\begin{array}{rcl} (\psi \circ \varphi)_{*p}(v_p) & = & \psi_{*\varphi(p)}(\varphi_{*p}(v_p)) \\ & = & (\psi_{*\varphi(p)} \circ \varphi_{*p})(v_p), \end{array}$$

for every $v_p \in T_pM$. Thus, $(\psi \circ \varphi)_{*p} = \psi_{*\varphi(p)} \circ \varphi_{*p}$. □

3.4 Tangent bundles

In the following, we will be interested in the operational tangent bundles, or tangent bundles for short. We will investigate the notion of tangent bundles in the category $\underline{FRL}$. Let M be an $\mathbb{F}$-space and let $x \in M$, T_xM and $\mathcal{D}_M$ are defined at Definition 3.1.1.

Proposition 3.4.1 *Let M be an $\mathbb{F}$-space. Then $M \times \mathcal{D}_M$ is an $\mathbb{F}$-space.*

Proof. By assumption, M is an $\mathbb{F}$-space. From Proposition 3.1.2, $\mathcal{D}_M$ has an $\mathbb{F}$-structure. Hence, according to Proposition 1.4.1, the product set $M \times \mathcal{D}_M$ is an $\mathbb{F}$-space. $\square$

Definition 3.4.1 *(See [2], [6] and [7]) The* **tangent bundle** *TM on an $\mathbb{F}$-space M is a subspace of $M \times \mathcal{D}_M$ consisting of all (p, v_p) such that v_p is an operational tangent vector at $p \in M$. The projection map $\pi : TM \to M$ is the $\mathbb{F}$-smooth map sending (p, v_p) to p and $\pi^{-1}(p) := T_pM$ is the fiber over $p \in M$.*

Remark 3.4.1 *According to Proposition 1.3.1 and Proposition 3.4.1, $TM \subseteq M \times \mathcal{D}_M$ has an induced $\mathbb{F}$-structure. We will call TM the tangent $\mathbb{F}$-bundle. In the following, (TM, π, M) or TM for short will be called the tangent $\mathbb{F}$-bundle on M.*

Proposition 3.4.2 *Let M be the n-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^n$. The tangent $\mathbb{F}$-bundle TM is $\mathbb{F}$-diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. An element of TM is specified by a point $p \in M = \mathbb{R}^n$ and an operational tangent vector

$$v_p = \sum_{i=1}^{n} v_{i,p} \frac{\partial}{\partial x_i}|_p$$

at p, where $v_{i,p} \in \mathbb{R}$ and $(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p)$ is the natural basis of T_pM. Then, there exists a bijection

$$\delta : TM \to \mathbb{R}^n \times \mathbb{R}^n, \text{ defined by } \delta(p, v_p) = (p, (v_{i,p})).$$

Now we have to show that δ and δ^{-1} are $\mathbb{F}$-smooth. Consider a structure curve

$$\zeta : \mathbb{R} \to TM, \text{ given by } \zeta(t) = (c(t), \sum_{i=1}^{n} v_{i,c(t)} \frac{\partial}{\partial x_i}|_{c(t)}),$$

on TM, with $c : \mathbb{R} \to M = \mathbb{R}^n$ a structure curve in M. Since

$$TM \subseteq M \times \mathcal{D}_M \subseteq M \times C^\infty(\mathcal{F}_M, \mathbb{R}),$$

by Definition 3.4.1 and Remark 3.1.1,

$$\iota : TM \to M \times C^\infty(\mathcal{F}_M, \mathbb{R})$$

is an inclusion map such that

$$\iota \circ \zeta : \mathbb{R} \to M \times C^\infty(\mathcal{F}_M, \mathbb{R})$$

is a structure curve in $M \times C^\infty(\mathcal{F}_M, \mathbb{R})$. Thus, ζ is $\mathbb{F}$-smooth if and only if

$$c : \mathbb{R} \to M \text{ and } p_2 \circ \iota \circ \zeta : \mathbb{R} \to C^\infty(\mathcal{F}_M, \mathbb{R})$$

are $\mathbb{F}$-smooth, where p_2 is the projection on the second factor. But, $p_2 \circ \iota \circ \zeta$ is $\mathbb{F}$-smooth if and only if the map

$$\mu : \mathbb{R} \times \mathcal{F}_M \to \mathbb{R} \text{ given by } (t, f) \mapsto \sum_{i=1}^{n} v_{i,c(t)} \frac{\partial f}{\partial x_i}(c(t))$$

is $\mathbb{F}$-smooth. Since $f \in \mathcal{F}_M$ and $M = \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{F}$-smooth. One can take $f = x_i$ and obtain

$$\mu(t, x_i) = v_{i,c(t)}, i = 1, \ldots, n, \text{ for every } t \in \mathbb{R}.$$

In other words,

$$(v_i \circ c)(\ ) = \mu(\ \ , x_i) \text{ for each } i = 1, \ldots, n.$$

Since μ and x_i are $\mathbb{F}$-smooth and by the Cartesian closedness of $\underline{\mathcal{F}RL}$, i.e.

$$C^\infty(\mathbb{R} \times \mathcal{F}_M, \mathbb{R}) \cong C^\infty(\mathbb{R}, C^\infty(\mathcal{F}_M, \mathbb{R})),$$

it follows that each $v_{i,p}$ is $\mathbb{F}$-smooth. Briefly, ζ is $\mathbb{F}$-smooth since $c(t)$ and $v_{i,c(t)}$, $i = 1, \ldots, n$, are $\mathbb{F}$-smooth. That is,

$$\delta(c(t), v_p) = (c(t), (v_{i,c(t)}))$$

is $\mathbb{F}$-smooth. Hence,

$$\delta \circ \zeta : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$$

is a structure curve in $\mathbb{R}^n \times \mathbb{R}^n$. Thus, according to Definition 1.2.1(1), δ is $\mathbb{F}$-smooth. Obviously,

$$\delta^{-1} : \mathbb{R}^n \times \mathbb{R}^n \to TM$$

the inverse of δ and

$$\delta \circ \zeta : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$$

the structure curve in $\mathbb{R}^n \times \mathbb{R}^n$ are such that

$$\delta^{-1} \circ (\delta \circ \zeta) = (\delta^{-1} \circ \delta) \circ \zeta = id_{TM} \circ \zeta = \zeta,$$

where id_{TM} is the identity on TM. But ζ is $\mathbb{F}$-smooth. That is,

$$\delta^{-1} \circ (\delta \circ \zeta) : \mathbb{R} \to TM$$

is a structure curve in TM. Thus, δ^{-1} is $\mathbb{F}$-smooth and the conclusion sought follows by Definition 1.2.2. $\square$

Proposition 3.4.3 *Let M be an n-dimensional pseudomanifold and $\pi : TM \to M$ be its tangent $\mathbb{F}$-bundle. If U is an open subset of M, then $\pi^{-1}(U)$ is $\mathbb{F}$-diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ of constant dimension and nonempty interior.*

Proof. Note that, according to Definition 1.6.1, U is $\mathbb{F}$-diffeomorphic to a closed subset $\varphi(U)$ of $\mathbb{R}^n$ by an $\mathbb{F}$-diffeomorphism φ. Thus, an element of $TU := \pi^{-1}(U)$ is given by $x = \varphi(p) \in \mathbb{R}^n$, where $p \in U$, and an operational tangent vector

$$v_p = \sum_{i=1}^{n} v_{i,p} \varphi_{*p}^{-1} (\frac{\partial}{\partial x_i}|_{\varphi(p)}),$$

where φ_{*p} is an isomorphism (the tangent linear map associated with φ at p) of T_pU to $T_{\varphi(p)}\varphi(U)$. Then, there exists a bijection

$$\delta : TU \to \mathbb{R}^n \times \mathbb{R}^n \text{ defined by } \delta(p, v_p) = (\varphi(p), (v_{i,p})).$$

One can prove, as at Proposition 3.4.2, δ and δ^{-1} are $\mathbb{F}$-smooth. $\square$

Remark 3.4.2 *From Proposition 3.4.2 and Proposition 3.4.3, we have:*

1. *The notion of tangent $\mathbb{F}$-bundle on pseudomanifold is like that of tangent bundle on manifold. That is, the tangent $\mathbb{F}$-bundle TM on a pseudomanifold M is a disjoint union of all the tangent spaces:*

$$TM := \bigsqcup_{p \in M} T_pM$$

(see Proposition 1.9.1) or again

$$TM := \bigcup \{p\} \times T_pM \subset M \times \mathcal{D}_M$$

(see [2] and [16]).

2. *The tangent $\mathbb{F}$-bundle TM on a n-dimensional pseudomanifold M is a vector $\mathbb{F}$-bundle over M, since it is locally $\mathbb{F}$-diffeomorphic to the trivial $\mathbb{F}$-bundle $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$ is a linear $\mathbb{F}$-space (see Definition 2.4.1, Definition 2.5.1 and Example 12). Thus, $\dim TM = 2n$.*

Proposition 3.4.4 *If $\varphi : M \to N$ is an $\mathbb{F}$-smooth map, the map*

$$\varphi_* : TM \to TN \text{ defined by } \varphi_*(p, v_p) = (\varphi(p), \varphi_{*p}(v_p))$$

is $\mathbb{F}$-smooth.

Proof.

$$\begin{tikzcd}
 & TM \arrow[r, "\varphi_*"] \arrow[dd, "\pi_M"] & TN \arrow[dd, "\pi_N"] \\
\mathbb{R} \arrow[ur, "\zeta_M"] \arrow[urr, "\zeta_N"] \arrow[dr, "c_M"'] \arrow[drr, "c_N"] & & \\
 & M \arrow[r, "\varphi"'] & N
\end{tikzcd}$$

Let the map

$$\zeta_M : \mathbb{R} \to TM \text{ given by } \zeta_M(t) = (c_M(t), v_{c_M(t)})$$

be a structure curve in TM, where $c_M : \mathbb{R} \to M$ is a structure curve in M. Therefore, by the diagram and the definition of φ_* above, we have

$$\begin{aligned}
(\varphi_* \circ \zeta_M)(t) &= \varphi_*(\zeta_M(t)) \\
&= \varphi_*(c_M(t), v_{c_M(t)}) \\
&= (\varphi(c_M(t)), \varphi_{*\, c_M(t)}(v_{c_M(t)})) \\
&= ((\varphi \circ c_M)(t), \varphi_{*\, c_M(t)}(v_{c_M(t)})).
\end{aligned}$$

Since $\varphi : M \to N$ is $\mathbb{F}$-smooth, we have that $c_N := \varphi \circ c_M$ is a structure curve in N (see Definition 1.2.1(1)). Let $g \in \mathcal{F}_N$. In fact, by Definition 3.3.1, it turns out that

$$\varphi_{*\, c_M(t)}(v_{c_M(t)})(g) = v_{c_M(t)}(g \circ \varphi) = v_{(\varphi \circ c_M)(t)}(g) = v_{c_N(t)}(g).$$

That is, $\varphi_{*\, c_M(t)}(v_{c_M(t)})$ is an operational tangent vector on N at $c_N(t) := \varphi(c_M(t))$. Thus,

$$\zeta_N := \varphi_* \circ \zeta_M : \mathbb{R} \to TN$$

is a structure curve in TN. Hence, according to Definition 1.2.1(1), φ_* is $\mathbb{F}$-smooth. $\square$

Definition 3.4.2 *Let $\varphi : M \to N$ be an $\mathbb{F}$-smooth map. The $\mathbb{F}$-smooth map*

$$\varphi_* : TM \to TN \text{ defined by } \varphi_*(p, v_p) = (\varphi(p), \varphi_{*p}(v_p))$$

is called the **tangent map** *to φ [2], [14].*

The tangent map φ_* is sometimes denoted by $D\varphi$ or again by $T\varphi$ [10],[14].

Proposition 3.4.5 *Let $\varphi : M \to N$ be an $\mathbb{F}$-smooth map. The tangent map $\varphi_* : TM \to TN$ is an $\mathbb{F}$-bundle morphism over φ.*

Proof. By Definition 3.4.1, (TM, π_M, M) and (TN, π_N, N) are $\mathbb{F}$-bundles.

$$\begin{tikzcd} TM \arrow[r, "\varphi_*"] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

By assumption, φ is $\mathbb{F}$-smooth. According to Proposition 3.4.4, the map φ_* is $\mathbb{F}$-smooth. For any $(p, v_p) \in TM$,

$$(\varphi \circ \pi_M)(p, v_p) = \varphi(\pi_M(p, v_p)) = \varphi(p)$$

and

$$(\pi_N \circ \varphi_*)(p, v_p) = \pi_N(\varphi_*(p, v_p)) = \pi_N(\varphi(p), \varphi_{*p}(v_p)) = \varphi(p).$$

Thus, by the definition of φ_*,

$$(\varphi \circ \pi_M)(p, v_p) = (\pi_N \circ \varphi_*)(p, v_p)$$

for all $(p, v_p) \in TM$. Hence, $\varphi \circ \pi_M = \pi_N \circ \varphi_*$. According to Definition 2.2.1, φ_* is an $\mathbb{F}$-bundle morphism over φ. □

Proposition 3.4.6 *If φ is the identity map on an $\mathbb{F}$-space M, then φ_* is the identical morphism of TM.*

Proof. According to Definition 2.2.1, Example 10(1) and Proposition 3.4.5, one has for the identity map φ on M and the $\mathbb{F}$-bundle (TM, π_M, M) that

$$\varphi \circ \pi_M = \pi_M \circ \varphi_* \text{ implies } \pi_M = \pi_M \circ \varphi_*.$$

Therefore, obviously φ_* is the identity map on TM. Otherwise, according Definition 3.4.2, φ is not an identity map. □

Proposition 3.4.7 *If $\varphi : M \to N$ and $\psi : N \to P$ are $\mathbb{F}$-smooth then*

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$$

Proof. According to Proposition 1.2.2, $\psi \circ \varphi$ is $\mathbb{F}$-smooth. For any $(p, v_p) \in TM$, by Definition 3.4.2 and Proposition 3.3.2, we have

$$\begin{aligned}(\psi \circ \varphi)_*(p, v_p) &= ((\psi \circ \varphi)(p), (\psi \circ \varphi)_{*p}(v_p)) \\ &= (\psi(\varphi(p)), (\psi_{*\varphi(p)} \circ \varphi_{*p})(v_p)) \\ &= (\psi(\varphi(p)), \psi_{*\varphi(p)}(\varphi_{*p}(v_p))) \\ &= \psi_*(\varphi(p), \varphi_{*p}(v_p)) \\ &= \psi_*(\varphi_*(p, v_p)) \\ &= (\psi_* \circ \varphi_*)(p, v_p).\end{aligned}$$

Thus, $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. □

Proposition 3.4.8 *If φ is an $\mathbb{F}$-diffeomorphism of M to N then φ_* is an $\mathbb{F}$-diffeomorphism of TM to TN and $(\varphi^{-1})_* = \varphi_*^{-1}$.*

Proof. According to Definition 1.2.2, if φ is an $\mathbb{F}$-diffeomorphism then φ^{-1} is $\mathbb{F}$-smooth. We have

$$\varphi \circ \varphi^{-1} = id_N \text{ and } \varphi^{-1} \circ \varphi = id_M.$$

Therefore

$$(\varphi \circ \varphi^{-1})_* = id_{TN} \text{ and } (\varphi^{-1} \circ \varphi)_* = id_{TM}.$$

But

$$(\varphi \circ \varphi^{-1})_* = \varphi_* \circ (\varphi^{-1})_* \text{ and } (\varphi^{-1} \circ \varphi)_* = (\varphi^{-1})_* \circ \varphi_* \, , (\Delta)$$

in accordance with Proposition 3.4.7 above. Hence

$$\varphi_* \circ (\varphi^{-1})_* = id_{TN} \text{ and } (\varphi^{-1})_* \circ \varphi_* = id_{TM} \text{ implies that } (\varphi^{-1})_* = \varphi_*^{-1}.$$

Now we have shown that φ^{-1} exists and is smooth, that φ_*^{-1} is a bijective map by (Δ), and an $\mathbb{F}$-smooth map according to Proposition 3.4.4. Thus, φ_* is an $\mathbb{F}$-diffeomorphism of TM onto TN. □

Remark 3.4.3 *From Proposition 3.4.2 and Definition 3.4.2, we have:*

1. *If $c : \mathbb{R} \to M$ is a structure curve in an $\mathbb{F}$-space M, the tangent map $c_* : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to TM$; $c_*(t, v_t) := c'(t)$ is a kinematic tangent vector on M at $c(t)$ as defined in Definition 3.2.1.*

2. *Let M be an $\mathbb{F}$-space. If f is a structure function on M, one has $f_* : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$.*

Definition 3.4.3 *Let M be an $\mathbb{F}$-space and $f \in \mathcal{F}_M$. The second component of f_*, that is $df : TM \to \mathbb{R}$ defined by*

$$df(x, v_x) = v_x(f)$$

is called the **differential** *of f (see [10]).*

Remark 3.4.4 *One has the following diagram, for any $\mathbb{F}$-space $(M, \mathcal{C}_M, \mathcal{F}_M)$ with $c \in \mathcal{C}_M$ and $f \in \mathcal{F}_M$,*

$$\begin{tikzcd}
 & M \arrow[dr, "f"] & \\
\mathbb{R} \arrow[ur, "c"] \arrow[rr, "f \circ c"] \arrow[dr, "c'"] & & \mathbb{R} \\
 & TM \arrow[uu, "\pi_M"] \arrow[ur, "df"] \arrow[dr, "f_*"] & \\
T\mathbb{R} = \mathbb{R} \times \mathbb{R} \arrow[uu, "\pi_{\mathbb{R}}"] \arrow[ur, "c_*"] \arrow[rr, "f_* \circ c_*"] & & T\mathbb{R} = \mathbb{R} \times \mathbb{R} \arrow[uu, "\pi_{\mathbb{R}}"']
\end{tikzcd}$$

Proposition 3.4.9 *If $\varphi : M \to N$ is an $\mathbb{F}$-smooth map between two $\mathbb{F}$-spaces and $f \in \mathcal{F}_N$ then*

$$d(f \circ \varphi) = (df) \circ \varphi_*.$$

Proof. For any $(x, v_x) \in TM$, by Remark 3.4.3(2) and Definition 3.3.1, on the one hand we have

$$d(f \circ \varphi)(x, v_x) = v_x(f \circ \varphi) = \varphi_{*x}(v_x)(f)$$

and in the other hand

$$(df) \circ \varphi_*(x, v_x) = (df)(\varphi(x), \varphi_{*x}(v_x)) = \varphi_{*x}(v_x)(f).$$

That is, for any $(x, v_x) \in TM$,

$$d(f \circ \varphi)(x, v_x) = \varphi_{*x}(v_x)(f) = (df) \circ \varphi_*(x, v_x).$$

So, $d(f \circ \varphi) = (df) \circ \varphi_*$. $\square$

3.5 Cotangent bundles

Let $\pi : TM \to M$ be a tangent $\mathbb{F}$-bundle. For every $p \in M$, $\pi^{-1}(p) := T_pM$ is a linear $\mathbb{F}$-space (see Proposition 3.1.1). We denote by $LIN(TM, \mathbb{R})$ the $\mathbb{F}$-subspace of $C^\infty(TM, \mathbb{R})$ consisting of all linear $\mathbb{F}$-smooth maps restricted to T_pM (see [6] and [7]). Let us define a relation $\simeq$ in

$$\underline{LIN} := M \times LIN(TM, \mathbb{R})$$

by setting $(p, f) \simeq (q, g)$ if and only if $p = q$ and, for some neighborhood U of p, $f_{|_{\pi^{-1}(U)}} = g_{|_{\pi^{-1}(U)}}$.

Lemma 3.5.1 *The relation $\simeq$ is an equivalence relation in $\underline{LIN}$.*

Proof. Clearly, the reflexivity and the symmetry of $\simeq$ hold. Let us show the transivity property of $\simeq$ in $\underline{LIN}$. Assume

$$(x, f) \simeq (y, g) \text{ and } (y, g) \simeq (z, h)$$

in $\underline{LIN}$. By definition of $\simeq$, on the one hand $x = y$ and there is a neighborhood U of x such that $f_{|_{\pi^{-1}(U)}} = g_{|_{\pi^{-1}(U)}}$. Also, on the other hand $y = z$ and there is a neighborhood V of y such that $g_{|_{\pi^{-1}(V)}} = h_{|_{\pi^{-1}(V)}}$. From $x = y$ and $y = z$, we obtain $x = z$. And $W := U \cap V$ is a neighborhood of x and y. Thus,

$$f_{|_{\pi^{-1}(W)}} = g_{|_{\pi^{-1}(W)}} \text{ and } g_{|_{\pi^{-1}(W)}} = h_{|_{\pi^{-1}(W)}}$$

since $W \subseteq U$ and $W \subseteq V$. Then, we obtain $f_{|_{\pi^{-1}(W)}} = h_{|_{\pi^{-1}(W)}}$. Therefore, $x = z$ and W is a neighborhood of x such that $f_{|_{\pi^{-1}(W)}} = h_{|_{\pi^{-1}(W)}}$. That is, $(x, f) \simeq (z, h)$. The transitivity is done. Hence, the relation $\simeq$ is an equivalence relation in $\underline{LIN}$. $\square$

Corollary 3.5.1 *The quotient space $\underline{LIN}/\simeq$ has an $\mathbb{F}$-structure.*

Proof. By the definition of $\underline{LIN}$ above,

$$\underline{LIN} = M \times LIN(TM, \mathbb{R}),$$

where M and $LIN(TM, \mathbb{R})$ are $\mathbb{F}$-spaces (see Proposition 1.5.2). Thus, $\underline{LIN}$ has an $\mathbb{F}$-structure, according to Proposition 1.4.1. It follows from Lemma 3.5.1 that the relation $\simeq$ is an equivalence relation in $\underline{LIN}$. Therefore, by Proposition 1.8.1, $\underline{LIN}/\simeq$ has an $\mathbb{F}$-structure. $\square$

Definition 3.5.1 *The* **cotangent bundle** T^*M *to an* $\mathbb{F}$*-space* M *is the quotient* $\mathbb{F}$*-space* $\underline{LIN}/\simeq$*. The projection map* $\tau : T^*M \to M$ *is the* $\mathbb{F}$*-smooth map induced from the projection of* $\underline{LIN}$ *onto its first factor and* $\tau^{-1}(p) := T_p^*M$ *is the fiber over* $p \in M$ *[2], [6], [7].*

In the following, (T^*M, τ, M) or T^*M will be called the *cotangent* $\mathbb{F}$*-bundle* on M.

Proposition 3.5.1 *Let* $M = \mathbb{R}^n$*, the* n*-dimensional Euclidian canonical* $\mathbb{F}$*-space. The cotangent* $\mathbb{F}$*-bundle* T^*M *to* $M = \mathbb{R}^n$ *is* $\mathbb{F}$*-diffeomorphic to* $\mathbb{R}^n \times \mathbb{R}^n$*.*

Proof. Let $M = \mathbb{R}^n$ and $q : \underline{LIN} \to T^*M$ be the quotient map. According to Proposition 1.8.1, the map q is $\mathbb{F}$-smooth. Take $[(p, f)] \in T^*M$, where $p \in M = \mathbb{R}^n$ and $f \in LIN(TM, \mathbb{R})$. We have that

$$f_{|_{\pi^{-1}(p)}} = \sum_{i=1}^{n} a_{i,p} dx^i|_p,$$

where $\pi^{-1}(p) = T_pM$, $a_{i,p} \in \mathbb{R}$ and $(dx^i|_p)$ the basis of T_p^*M, the dual of the basis $(\frac{\partial}{\partial x_i}|_p)$ of T_pM. Then, there is a bijection

$$\varphi : T^*M \to \mathbb{R}^n \times \mathbb{R}^n, \text{ given by } \varphi([(p, f)]) = (p, (a_{i,p})).$$

We need show that φ is an $\mathbb{F}$-diffeomorphism. Let $c : \mathbb{R} \to T^*M$ be a structure curve in T^*M and denote by

$$\varphi(t) := (\varphi \circ c)(t) = (p_t, (a_{i,p_t})).$$

One can see that the $\mathbb{F}$-smooth map $\tau : T^*M \to M$ is such that $\pi^* \circ q$ is the projection onto the first factor of $\underline{LIN}$. Since $(\tau \circ q)(p_t, f_t) = p_t$ is $\mathbb{F}$-smooth on t and $\tau = p_1 \circ \varphi$, where p_1 is the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the first factor, φ is $\mathbb{F}$-smooth in its first coordinate. Let us define

$$\theta_i : \underline{LIN} \to \mathbb{R} \text{ by setting } \theta_i(p, f) = a_{i,p}.$$

Again, there is a structure function $\rho_i : T^*M \to \mathbb{R}$ such that $\rho_i \circ q = \theta_i$. First, we wish to show that θ_i is $\mathbb{F}$-smooth. Let $d : \mathbb{R} \to \underline{LIN}$ be a structure curve in $\underline{LIN}$ and $d(t) = (p_t, f_t)$. Then, f_t is $\mathbb{F}$-smooth in t if and only if the associated map

$$\hat{f} : \mathbb{R} \times TM \to \mathbb{R} \text{ defined by } \hat{f}(t, V) = f_t(V)$$

is $\mathbb{F}$-smooth. Identifying TM with $\mathbb{R}^n \times \mathbb{R}^n$ (see Proposition 3.4.2), one can write $\hat{f}(t, p, \omega) = A(t, p) \cdot \omega$, where $A(t, p)$ is a matrix with entries which are

$\mathbb{F}$-smooth functions of t and $p \in M = \mathbb{R}^n$, $\omega \in M = \mathbb{R}^n$ is written as a column vector and $\cdot$ is the matrix multiplication. Let $\omega = \frac{\partial}{\partial x_i}$. Then, with ω viewed as a column vector,

$$a_{i,p_t}(t) = (\theta_i \circ d)(t) = A(t, p_t) \cdot \omega \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Hence, according to Definition 1.2.1(1), θ_i is $\mathbb{F}$-smooth as a structure function on $\underline{LIN}$. We also have that the ρ_i are $\mathbb{F}$-smooth (see the proof of Proposition 1.8.1). Since $\rho_i = p_{n+i} \circ \varphi$ and p_{n+i} is the projection on the $(n+i)$-th coordinate of $\mathbb{R}^n \times \mathbb{R}^n$ then φ must be $\mathbb{F}$-smooth, according to Definition 1.3.1(2). Conversely, let us suppose that the a_{i,p_t} for $i = 1, 2, \ldots, n$ and p_t are $\mathbb{F}$-smooth functions. Let

$$\beta : \mathbb{R} \to \underline{LIN} \text{ such that } \beta(t) = (p_t, \sum_{i=1}^{n} a_{i,p_t} dx^i|_{p_t}).$$

The second component of β is $\mathbb{F}$-smooth since the induced map sending $(t, (p_t, b_i))$ to $\sum a_{i,p_t} b_i$ is $\mathbb{F}$-smooth, by the Cartesian closedness of $\underline{\mathcal{F}RL}$ and the identification of TM with $\mathbb{R}^n \times \mathbb{R}^n$ (see Proposition 3.4.2). Since β is $\mathbb{F}$-smooth, so is $q \circ \beta$. Hence, by Definition 1.2.2, φ^{-1} is $\mathbb{F}$-smooth and the conclusion sought follows. □

Proposition 3.5.2 *Let M be an n-dimensional pseudomanifold and $\tau : T^*M \to M$ its cotangent $\mathbb{F}$-bundle. If U is an open subset of M then $(\tau^{-1})(U) := T^*U$ is $\mathbb{F}$-diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Let $U \subseteq M$. By Definition 1.6.1, we have that U is $\mathbb{F}$-diffeomorphic to a closed subset $\varphi(U) \subseteq \mathbb{R}^n$ of constant dimension and nonempty interior, where φ is an $\mathbb{F}$-diffeomorphism, since M is an n-dimensional pseudomanifold. Let $[(p, f)] \in T^*U,$ with $p \in U$ and $f \in LIN(TU, \mathbb{R})$. We have

$$f_{|_{(\pi^{-1})(p)}} = \sum_{i=1}^{n} a_{i,p} \bar{\varphi}_p^{-1}(dx^i|_{\varphi(p)}),$$

where π is the projection of TM onto M (see Proposition 3.4.3), the map $\bar{\varphi}_p$ is an isomorphism of T_p^*U to $T^*_{\varphi(p)}\varphi(U)$ and $a_{i,p} \in \mathbb{R}$, with $a_i \in \mathcal{F}_U$ for $i = 1, 2, \ldots, n$. The map $\bar{\varphi}_p$ induces the bijection

$$\psi : T^*U \to \mathbb{R}^n \times \mathbb{R}^n, \ [(p, f)] \mapsto (\varphi(p), (a_{i,p})).$$

Like at Proposition 3.5.1, we prove that ψ and ψ^{-1} are $\mathbb{F}$-smooth. The proof is then completed. □

Remark 3.5.1 *From Proposition 3.5.1 and Proposition 3.5.2, we have:*

1. *The notion the cotangent $\mathbb{F}$-bundle on a pseudomanifold is similar to that of the cotangent bundle on a manifold. So, if M is a pseudomanifold then*
$$T^*M := \bigsqcup_{p\in M} T_p^*M$$
(see Proposition 1.9.1) or
$$T^*M := \bigcup \{p\} \times T_p^*M \subset \underline{LIN}.$$

2. *The cotangent $\mathbb{F}$-bundle T^*M on a n-dimensional pseudomanifold M is a vector $\mathbb{F}$-bundle over M since it is locally $\mathbb{F}$-diffeomorphic to the trivial $\mathbb{F}$-bundle $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n$ is linear $\mathbb{F}$-space (see Definition 2.4.1, Definition 2.5.1 and Example 12). Thus, dim $T^*M = 2n$.*

3.6 Vector fields and forms

Definition 3.6.1 *Let M be an n-dimensional pseudomanifold. A* **(tangent) vector field** *on M is a section of the tangent $\mathbb{F}$- bundle (TM, π, M) [2], [6], [7], [16].*

Remark 3.6.1 *The set $\mathfrak{X}(M) := \Gamma(M, TM)$ of all tangent vector fields over M is a module over the algebra $\mathcal{F}_M$ (see Remark 2.5.1(1)).*

Proposition 3.6.1 *Let M and N be two pseudomanifolds of finite dimensions, and let $\varphi : M \to N$ be an $\mathbb{F}$-diffeomorphism. If X is a vector field on M then $Y := \varphi_* \circ X \circ \varphi^{-1}$ is a vector field on N.*

Proof. Let $\varphi : M \to N$ be an $\mathbb{F}$-diffeomorphism between two finite-dimensional pseudomanifolds. Let X be a vector field on M. We have the following commutative diagram:

$$\begin{tikzcd}
TM \arrow[r, "\varphi_*"] \arrow[d, shift left, "\pi_M"] & TN \arrow[d, "\pi_N"] \\
M \arrow[r, "\varphi"] \arrow[u, shift left, "X"] & N
\end{tikzcd}$$

In effect, according to Proposition 1.2.6, $Y := \varphi_* \circ X \circ \varphi^{-1}$ is $\mathbb{F}$-smooth as a composite of $\mathbb{F}$-smooth maps. Thus, by the properties of the composition of maps, by the commutativity of the the diagram above and by the fact that φ is $\mathbb{F}$-diffeomorphism, Y is an $\mathbb{F}$-smooth map of N to TN satifying the following equalities for all $y \in N$:

$$\begin{aligned}
(\pi_N \circ Y)(y) &= [\pi_N \circ (\varphi_* \circ X \circ \varphi^{-1})](y) \\
&= [(\pi_N \circ \varphi_*) \circ X \circ \varphi^{-1}](y) \\
&= [(\varphi \circ \pi_M) \circ X \circ \varphi^{-1}](y) \\
&= [\varphi \circ (\pi_M \circ X) \circ \varphi^{-1}](y) \\
&= (\varphi \circ id_M \circ \varphi^{-1})(y) \\
&= (\varphi \circ \varphi^{-1})(y) \\
&= id_N(y),
\end{aligned}$$

where id_M and id_N are the identities maps on M and N respectively. That is, $\pi_N \circ Y = id_N$. Thus, according to Definition 2.1.3 and Definition 3.6.1, Y is a vector field on N. □

Proposition 3.6.2 *Let X be a vector field on a finite-dimensional pseudomanifold M. For all structure functions $f \in \mathcal{F}_M$, the map*

$$X.f : M \to \mathbb{R} \text{ defined by } x \mapsto df(X(x))$$

is $\mathbb{F}$-smooth.

Proof. Let X be a vector field on a finite-dimensional pseudomanifold M. By Definition 3.6.1, X is an $\mathbb{F}$-smooth map such that if $x \in M$ then $X(x) \in TM$. According to Definition 3.4.3, we have

$$X.f(x) = df(X(x)) = (df \circ X)(x).$$

Hence, according to Proposition 1.2.2, the map $X.f$ is $\mathbb{F}$-smooth since df and X are $\mathbb{F}$-smooth (see Definition 3.4.3 and Definition 3.6.1). □

Definition 3.6.2 *The $\mathbb{F}$-smooth map $X.f$ is called the* **derivative** *of f with respect to X [10].*

Remark 3.6.2 *The derivative $X.f$ satisfies the same properties as the differential df.*

Definition 3.6.3 *Let M be a finite-dimensional pseudomanifold. A* **derivation** *of the algebra $\mathcal{F}_M$ is an $\mathbb{F}$-smooth map $D : \mathcal{F}_M \to \mathcal{F}_M$ satisfying the following properties:*

1. *$D(f + g) = D(f) + D(g)$;*

2. *$D(f.g) = D(f)g + fD(g)$;*

3. *$D(f) = 0$, if f is a constant on M;*

4. *$D(af) = aD(f)$, $a \in \mathbb{R}$.*

Remark 3.6.3 *From Definition 3.6.3 and Remark 3.6.1, we have:*

1. *The set $\mathcal{D}er(M)$ of derivations of the algebra $\mathcal{F}_M$ is a module on $\mathcal{F}_M$.*

2. *A vector field X on M induces a derivation $f \mapsto X.f$ of $\mathcal{F}_M$ since $f \in \mathcal{F}_M$ implies $X.f \in \mathcal{F}_M$.*

3. *In the following, we shall identify $\mathfrak{X}(M)$ with the module $\mathcal{D}er(M)$.*

Definition 3.6.4 *Let M be a finite-dimensional pseudomanifold. A* **1-form** *or a* **Pfaffian form** *on M is a section of the cotangent $\mathbb{F}$-bundle (T^*M, π^*, M) [2], [6], [7], [10], [16].*

The set of all 1-forms on M is denoted by $\Omega^1(M) := \Gamma(M, T^*M)$ [2], [16].

Definition 3.6.5 *Let M be a n-dimensional pseudomanifold. A* **k-form** *on M or a* **form of degree** *k is a section of the $\mathbb{F}$-bundle*

$$\bigwedge^k(T^*M) = \bigsqcup_{p\in M} \bigwedge^k(T_p^*M)$$

of base space M and of fibers $\bigwedge^k(T_p^*M)$.

The set of all k-forms on M is denoted by $\Omega^k(M) := \Gamma(M, \bigwedge^k(T^*M))$ [2], [16].

Remark 3.6.4 *We have the following:*

1. *According to Proposition 1.8.1 and Proposition 1.9.4, the bundle $\bigwedge^k(T^*M)$ over the finite-dimensional pseudomanifold M has an $\mathbb{F}$-structure since T_p^*M is a linear $\mathbb{F}$-space.*

2. *According to Definition 2.1.3 and Proposition 2.1.4, a section of an* $\mathbb{F}$*-bundle is* $\mathbb{F}$*-smooth. So, a vector field, a* 1*-form and a* k*-form are* $\mathbb{F}$*-smooth.*

3. *From Definition 3.6.3,*

$$\bigwedge^{0}(T_p^*M) = \mathbb{R},\ \Omega^0(M) = \mathcal{F}_M \ \textit{and} \ \bigwedge^{1}(T_p^*M) = T_p^*M.$$

4. *The set* $\Omega^k(M)$ *of* k*-forms on* M *is a module on the algebra* $\mathcal{F}_M$*.*

Proposition 3.6.3 *Let* M *be a finite-dimensional pseudomanifold, and let* $\xi_k = (D^k, \pi, M)$ *be the Whitney sum of* k *copies of the tangent* $\mathbb{F}$*-bundle* TM*. The module* $\Omega^k(M)$ *of* k*-forms on* M *is isomorphic to the module of* $\mathbb{F}$*-smooth functions* $\sigma : D^k \to \mathbb{R}$*, the restriction of which to each fiber* $(T_xM)^k$ *over* $x \in M$ *is an exterior* k*-form on* T_xM*.*

Proof. Let M be a finite-dimensional pseudomanifold. A k-form $\omega \in \Omega^k(M)$ is an $\mathbb{F}$-smooth map $\omega : M \to \bigwedge^{k}(T^*M)$ such that

$$\omega_x := s_1(x) \wedge s_2(x) \wedge \cdots \wedge s_k(x),$$

where $s_i(x) \in T_x^*M$ for each $i = 1, 2, \ldots, k$. Let us define a function σ of $(T_xM)^k$ to $\mathbb{R}$ by

$$\sigma(v_1, v_2, \ldots, v_k) := \omega_x(v_1, v_2, \ldots, v_k).$$

By Corollary 1.5.1(1) and Remark 1.5.1, the function σ is $\mathbb{F}$-smooth since ω_x is $\mathbb{F}$-smooth. Since ω_x is an exterior k-form on T_xM, so is the map σ. Conversely, if we define

$$\omega : M \to \bigwedge^{k}(T_x^*M) \text{ by } \omega_x := alt \circ \sigma,$$

where σ is an $\mathbb{F}$-smooth function of D^k into $\mathbb{R}$ such that the restriction to $(T_xM)^k$ is an exterior k-form and *alt* the alternator map (see the proof of Proposition 1.10.4). Then, ω is $\mathbb{F}$-smooth since σ is $\mathbb{F}$-smooth. Let us consider the projection τ of the $\mathbb{F}$-bundle $\bigwedge^{k}(T^*M)$ onto M. For $x \in M$,

$$(\tau \circ \omega)(x) = \tau(\omega_x) = \tau(alt \circ \sigma) = \tau(alt(\sigma)) = x$$

since x is the footpoint of $alt(\sigma)$. Thus, according to Definition 3.6.5, ω is a k-form on M. □

Remark 3.6.5 *In the following, we shall identify a k-form on a finite-dimensional pseudomanifold M with the corresponding $\mathbb{F}$-smooth function on D^k.*

Corollary 3.6.1 *Let M be a finite-dimensional pseudomanifold. The differential df of an $\mathbb{F}$-smooth function $f \in \mathcal{F}_M$ is a Pfaffian form on M.*

Proof. Let M be a finite-dimensional pseudomanifold. In effect, by Proposition 3.6.3 for $k = 1$, we identify the module $\Omega^1(M)$ to the module of $\mathbb{F}$-smooth function of TM to $\mathbb{R}$,the restriction of which to each fiber T_xM over $x \in M$ is linear. So, we identify df (see Definition 3.4.3) with a 1-form on M (see Definition 3.6.4). □

Corollary 3.6.2 *Let ω be a k-form on a finite-dimensional pseudomanifold M. Let $x \in M$. If $X_1, X_2, \ldots, X_k$ are k vector fields on M, the function $\omega(X_1, X_2, \ldots, X_k) : M \to \mathbb{R}$ defined by*

$$\omega(X_1, X_2, \ldots, X_k)(x) := \omega_x(X_1(x), X_2(x), \ldots, X_k(x)),$$

where $\omega_x := \omega(x)$ for $x \in M$, is $\mathbb{F}$-smooth on M.

Proof. Let X_i for $i = 1, 2, \ldots, k$ be the vector fields on a finite-dimensional pseudomanifold M. If $x \in M$ then, by Definition 3.6.1, $X_i(x) \in TM$, more precisely it lies in T_xM. So, according to Proposition 3.6.3, $\omega(X_1, X_2, \ldots, X_k)$ is an $\mathbb{F}$-smooth function on M since ω is a k-form. □

Remark 3.6.6 *See [10].*

1. *A k-form ω on a finite-dimensional pseudomanifold M induces an exterior k-form on the $\mathcal{F}_M$-module $\mathfrak{X}(M)$ (see Corollary 3.6.2).*

2. *In the following, we shall identify $\Omega^k(M)$ with a module of exterior k-forms on $\mathfrak{X}(M)$; and the algebra*

$$\Omega(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)$$

of forms on M to the algebra of exterior forms on $\mathfrak{X}(M)$.

Corollary 3.6.3 *Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds and let ω be a k-form on N ($k > 0$). The function $\varphi^*\omega : D^k \to \mathbb{R}$ given by*

$$\varphi^*\omega(v_1, v_2, \ldots, v_k) = \omega_{\varphi(x)}(\varphi_{*x}(v_1), \varphi_{*x}(v_2), \ldots, \varphi_{*x}(v_k)),$$

where $x \in M$ and D^k the Whitney sum of k copies of TM, induces a k-form on M.

Proof. Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds and let ω be a k-form on N ($k > 0$). The restriction of φ_* to each fiber T_xM over $x \in M$ gives us the tangent vectors $\varphi_{*x}(v_i) \in T_{\varphi(x)}N$ for $i = 1, 2, \ldots, k$ (see Proposition 3.3.1). Then, according to Proposition 3.6.3, the function $\varphi^*\omega : D^k \to \mathbb{R}$ defined by

$$\varphi^*\omega(v_1, v_2, \ldots, v_k) = \omega_{\varphi(x)}(\varphi_{*x}(v_1), \varphi_{*x}(v_2), \ldots, \varphi_{*x}(v_k)),$$

restricted to $(T_xM)^k$, is $\mathbb{F}$-smooth and induces a k-form on M since ω is a k-form on N. $\square$

Definition 3.6.6 *Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds and let ω be a k-form on N ($k > 0$). The $\mathbb{F}$-smooth function $\varphi^*\omega$ (see Corollary 3.6.3) is called the* **reciprocal image form** *by φ [10].*

Remark 3.6.7 *Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds and let ω be a k-form on N ($k > 0$).*

1. *We will denote the reciprocal image form $\varphi^*\omega$ by $\omega \circ \varphi_*$ (see Corollary 3.6.3).*
2. *For a form of degree zero that is $f \in \mathcal{F}_N$, we have*

$$\varphi^* f = f \circ \varphi.$$

Proposition 3.6.4 *Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds and $f \in \mathcal{F}_N$. We have*

$$\varphi^*(df) = d(\varphi^* f).$$

Proof. Let $f \in \mathcal{F}_N$. Using Definition 3.4.3, Proposition 3.4.9 and Corollary 3.6.3, we have

$$\varphi^*(df) = df \circ \varphi_* = d(f \circ \varphi) = d(\varphi^* f).$$

$\square$

Proposition 3.6.5 *Let $\varphi : M \to N$ be an an $\mathbb{F}$-smooth map between two finite-dimensional pseudomanifolds. The map $\varphi^* : \Omega(N) \to \Omega(M)$ is a morphism of algebras.*

Proof. Let $\omega_1, \omega_2 \in \Omega(N)$. We have, by Definition 3.6.6,

$$\varphi^*(\omega_1 \cdot \omega_2) = (\omega_1 \cdot \omega_2) \circ \varphi_* = (\omega_1 \circ \varphi_*) \cdot (\omega_2 \circ \varphi_*) = (\varphi^*\omega_1) \cdot (\varphi^*\omega_2).$$

$\square$

Corollary 3.6.4 *If φ is the identity map of a finite-dimensional pseudomanifold M then φ^* is the identity of $\Omega(M)$.*

Proof. Let $\omega \in \Omega(M)$ and $\varphi := id_M$ be the identity map of M. According to Proposition 1.2.4, id_M is an $\mathbb{F}$-diffeomorphism of M. By Definition 3.6.6 and Proposition 3.4.6, we have

$$\varphi^*\omega = \omega \circ \varphi_* = \omega \circ (id_M)_* = \omega \circ id_{TM} = \omega,$$

where id_{TM} is the identity map of TM. Thus, $\varphi^* := id_{\Omega(M)}$ is the identity morphism of $\Omega(M)$. $\square$

Corollary 3.6.5 *If $\varphi : M \to N$ and $\psi : N \to P$ are $\mathbb{F}$-smooth maps of finite-dimensional pseudomanifolds.*

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

Proof. According to Proposition 1.2.2, the composite $\psi \circ \varphi$ is $\mathbb{F}$-smooth since ψ and φ are $\mathbb{F}$-smooth. For all $\omega \in \Omega(M)$, by Definition 3.6.6, Proposition 3.4.7 and the properties of the composition of maps, we have

$$\begin{aligned}
(\psi \circ \varphi)^*(\omega) &= \omega \circ (\psi \circ \varphi)_* \\
&= \omega \circ (\psi_* \circ \varphi_*) \\
&= (\omega \circ \psi_*) \circ \varphi_* \\
&= \varphi^*(\omega \circ \psi_*) \\
&= \varphi^*(\psi^*\omega) \\
&= (\varphi^* \circ \psi^*)(\omega).
\end{aligned}$$

Hence, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. $\square$

Corollary 3.6.6 *Let M and N be two finite-dimensional pseudomanifolds. If φ is an $\mathbb{F}$-diffeomorphism of M to N then φ^* is an isomorphism of $\Omega(N)$ to $\Omega(M)$ and $(\varphi^{-1})^* = (\varphi^*)^{-1}$.*

Proof. Let M and N be two finite-dimensional pseudomanifolds and $\varphi : M \to N$ an $\mathbb{F}$-diffeomorphism. By Definition 1.2.2, φ is a bijective $\mathbb{F}$-smooth map and φ^{-1} is $\mathbb{F}$-smooth. By Definition 3.6.6, Proposition 3.4.8 and Corollary 3.6.5, if $\psi := \varphi^{-1}$, we have that

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \text{ implies } (\varphi^{-1} \circ \varphi)^* = \varphi^* \circ \psi^*.$$

That is,

$$id_{\Omega(M)} = (id_M)^* = \varphi^* \circ \psi^*,$$

where $id_{\Omega(M)}$ is the identity isomorphism of $\Omega(M)$ and id_M is the identity map of M. We have also that,

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^* \text{ implies } (\varphi \circ \varphi^{-1})^* = \psi^* \circ \varphi^*.$$

That is,

$$id_{\Omega(N)} = (id_N)^* = \psi^* \circ \varphi^*,$$

where $id_{\Omega(N)}$ is the identity isomorphism of $\Omega(N)$ and id_N is the identity map of N. Hence, $\psi^* := (\varphi^*)^{-1}$ and φ^* is an isomorphism of $\Omega(N)$ to $\Omega(M)$. □

3.7 Lie bracket

Definition 3.7.1 *Let M be an $\mathbb{F}$-space. The operator*

$$[\ ,\ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \text{ defined by } [X, Y] := XY - YX,$$

is an $\mathbb{F}$-smooth derivation called the **Lie bracket** *of X and Y [2].*

Proposition 3.7.1 *Let X, Y and Z be the tangent vector fields on an $\mathbb{F}$-space M. The Lie bracket has the following properties:*

1. $[X, Y + Z] = [X, Y] + [X, Z]$
2. $[X, fY] = (X.f)Y + f[X, Y]$ *for* $f \in \mathcal{F}_M$
3. $[X, Y] = -[Y, X]$
4. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Proof.

1. Let X, Y and Z be the tangent vector fields on an $\mathbb{F}$-space M. Since $\mathfrak{X}(M)$ is an $\mathcal{F}_M$-module, if $Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(M)$ then $Y + Z \in \mathfrak{X}(M)$. So, by Definition 3.7.1 and since $\mathfrak{X}(M)$ is a $\mathcal{F}_M$-module,

$$\begin{aligned}
[X, Y + Z] &= X(Y + Z) - (Y + Z)X \\
&= (XY + XZ) - (YX + ZX) \\
&= XY + XZ - YX - ZX \\
&= XY + (XZ - YX) - ZX \\
&= XY + (-YX + XZ) - ZX \\
&= (XY - YX) + (XZ - ZX) \\
&= [X, Y] + [X, Z].
\end{aligned}$$

Thus, $[X, Y + Z] = [X, Y] + [X, Z]$.

2. Let X and Y be the tangent vector fields on an $\mathbb{F}$-space M. Since $\mathfrak{X}(M)$ is a $\mathcal{F}_M$-module, if $f \in \mathcal{F}_M$ and $Y \in \mathfrak{X}(M)$ then $fY \in \mathfrak{X}(M)$. Let $g \in \mathcal{F}_M$. Therefore, by Definition 3.7.1 and Remark 3.6.3(2), we have

$$\begin{aligned}
[X, fY].g &= (X(fY) - (fY)X).g \\
&= X(fY).g - ((fY)X).g \\
&= X.(f.(Y.g)) - (fY).(X.g) \\
&= ((X.f).(Y.g) + fX.(Y.g)) - (fY)(X.g) \\
&= (X.f).(Y.g) + (fX.(Y.g) - (fY)(X.g)) \\
&= ((X.f)Y).g + f(XY - YX).g \\
&= ((X.f)Y).g + f[X, Y].g \\
&= ((X.f)Y + f[X, Y]).g
\end{aligned}$$

Thus, $[X, fY] = (X.f)Y + f[X, Y]$.

3. Let X and Y be the tangent vector fields on an $\mathbb{F}$-space M. From Definition 3.7.1 and since $\mathfrak{X}(M)$ is a $\mathcal{F}_M$-module,

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= -YX + XY \\
&= -(YX - XY) \\
&= -[Y, X].
\end{aligned}$$

Hence, $[X, Y] = -[Y, X]$.

4. Let X, Y and Z be the tangent vector fields on an $\mathbb{F}$-space M. By Definition 3.7.1 and the properties of the $\mathcal{F}_M$-module $\mathfrak{X}(M)$, we have

$$\begin{aligned}
[X, [Y, Z]] &= X[Y, Z] - [Y, Z]X \\
&= X(YZ - ZY) - (YZ - ZY)X \\
&= (XYZ - XZY) - (YZX - ZYX) \\
&= XYZ - XZY - YZX + ZYX;
\end{aligned}$$

$$\begin{aligned}
[Y, [Z, X]] &= Y[Z, X] - [Z, X]Y \\
&= Y(ZX - XZ) - (ZX - XZ)Y \\
&= (YZX - YXZ) - (ZXY - XZY) \\
&= YZX - YXZ - ZXY + XZY;
\end{aligned}$$

$$\begin{aligned}
[Z, [X, Y]] &= Z[X, Y] - [X, Y]Z \\
&= Z(XY - YX) - (XY - YX)Z \\
&= (ZXY - ZYX) - (XYZ - YXZ) \\
&= ZXY - ZYX - XYZ + YXZ.
\end{aligned}$$

Thus, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

□

Remark 3.7.1 *From Proposition 3.7.1, the Lie bracket is a bilinear form (1) and (2), the Lie bracket is skew-symmetric and the Lie bracket satisfies the Jacobi identities (4). Therefore, we have that,* $\mathfrak{X}(M)$ *together with the Lie bracket* $[\ ,\ ]$ *forms an algebra called the* **Lie algebra** *of vector fields on* M *[10].*

We proved that the (operational) tangent space of an $\mathbb{F}$-space has a linear $\mathbb{F}$-structure. We defined a tangent linear map between two tangent $\mathbb{F}$-spaces. We proved that a tangent linear map between two tangent $\mathbb{F}$-spaces is $\mathbb{F}$-smooth. We investigated and proved with details some properties of the tangent linear maps.

We proved that the tangent and the cotangent bundles of a n-dimensional pseudomanifold have an $\mathbb{F}$-structure and are locally $\mathbb{F}$-diffeomorphic to the even-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^{2n}$. We noticed that those tangent and cotangent $\mathbb{F}$-bundles are vector $\mathbb{F}$-bundles.

We also defined the tangent map between two tangent $\mathbb{F}$-bundles. We proved that a tangent map between two tangent $\mathbb{F}$-bundles is $\mathbb{F}$-smooth. We investigated and proved with details some properties of the tangent maps. We defined the differential of a structure function on any $\mathbb{F}$-space.

We defined the notions of vector fields and forms in the category of $\mathbb{F}$-spaces. Those vector fields and forms are respectively sections of the tangent and the wedge product $\mathbb{F}$-bundles (particularly the cotangent $\mathbb{F}$-bundle) over a finite-dimensional pseudomanifold. Some properties and detailed proofs have followed. We proved that the differential of a structure function is a one form. We investigated the correspondence between the forms of an $\mathbb{F}$-bundle and the exterior form on its base $\mathbb{F}$-space. We defined the reciprocal image form or the pullback form by an $\mathbb{F}$-smooth map between two $\mathbb{F}$-spaces and resulting properties.

We concluded the present chapter with a brief review of the Lie bracket in the category of $\mathbb{F}$-spaces. These are some tools that will enable us introducing the symplectic geometry on these spaces in the following chapter.

Chapter 4

Symplectic structure on cotangent Frölicher bundles

In this chapter, we investigate an introductive notion of symplectic structure on a cotangent $\mathbb{F}$-bundle. All sections exist and are $\mathbb{F}$-smooth, according to Proposition 2.1.4. Before going through, we start by reviewing some basic concepts of symplectic structure on a linear $\mathbb{F}$-space. Our main references will be [2] for the the notion of symplectic $\mathbb{F}$-spaces and [1], [10], [14], [15] for the common approach developped in the other categories.

4.1 Basic concepts

Definition 4.1.1 *Let E be a linear $\mathbb{F}$-space of dimension n and ω a bilinear form on E.*

- *the form ω is* **skew-symmetric** *on E if, for all $e_1, e_2 \in E$,*

$$\omega(e_1, e_2) = -\omega(e_2, e_1);$$

- *the form ω is* **nondegenerate** *on E if,*

$$\omega(e_1, e_2) = 0, \text{ for all } e_2 \in E, \text{ implies } e_1 = 0.$$

It is also said that ω is a **symplectic form** *on E or that ω defines a* **symplectic structure** *on E. In this case, the pair (E, ω) is a* **symplectic linear $\mathbb{F}$-space**.

Lemma 4.1.1 *([1],[15],[18]) Let E be a linear space of dimension n and E^* its dual. Let $\bigwedge^2(E^*)$ denote the space of antisymmetric covariant tensors on E. For any $\omega \in \bigwedge^2(E^*)$ there exists a basis $\{f^{1*}, \ldots, f^{n*}\}$ of E^* such that*

$$\omega = f^{1*} \wedge f^{n*} + \cdots + f^{(2p-1)*} \wedge f^{2p*}.$$

The integer p depends only on ω. That is, p is independent of the choice of the basis.

Proof. ([1],[11],[18]) Let $\{e_1, \ldots, e_n\}$ be a basis of E and $\{e^{1*}, \ldots, e^{n*}\}$ the corresponding dual basis. Then ω can be in terms of the dual basis as

$$\omega = \sum_{i<j} a_{ij} e^{i*} \wedge e^{j*},$$

where $a_{ij} \in \mathbb{R}$. If $\omega \neq 0$ then we may rearrange the basis such that $a_{12} \neq 0$. That is,

$$\omega = (e^{1*} - \frac{a_{23}}{a_{12}} e^{3*} - \cdots - \frac{a_{2n}}{a_{12}} e^{n*}) \wedge (a_{12} e^{2*} + a_{13} e^{3*} + \cdots + a_{1n} e^{n*}) + \omega^1,$$

where ω^1 does not contain any expression involving e^{1*} or e^{2*}. Now, we set

$$f^{1*} = e^{1*} - \frac{a_{23}}{a_{12}} e^{3*} - \cdots - \frac{a_{2n}}{a_{12}} e^{n*}$$

and

$$f^{2*} = a_{12} e^{2*} + a_{13} e^{3*} + \cdots + a_{1n} e^{n*}.$$

Then, $f^{1*}, f^{2*}, e^{3*}, \ldots, e^{n*}$ are linearly independant and

$$\omega = f^{1*} \wedge f^{2*} + \omega^1.$$

If $\omega^1 = 0$, the proof is complete and we shall write ω in the form

$$\omega = f^{1*} \wedge f^{2*}.$$

If $\omega^1 \neq 0$, we resume the process and we shall stop when we get the desired normal form. Now we show that the integer p depends only on ω. Note that ω induces a linear map

$$\omega^\flat : E \to E^*$$

sending $v \in E$ to $\omega^\flat(v) := \omega(v, \cdot)$ which is a linear form on E. If $\{g_1, g_2, \ldots, g_n\}$ is any basis of E, then the linear forms

$$\omega^\flat(g_1), \omega^\flat(g_2), \ldots, \omega^\flat(g_n) \text{ span } \omega^\flat(E) \subset E^*.$$

In particular if $\{f_1, f_2, \ldots, f_n\}$ is a basis of E with dual basis $\{f^{1*}, f^{2*}, \ldots, f^{n*}\}$ in E^* as constructed above, then is clear that

$$\omega^\flat(f_1) = f^{2*}, \omega^\flat(f_2) = -f^{1*}, \ldots, \omega^\flat(f_{2p-1}) = f^{2p*}, \omega^\flat(f_{2p}) = -f^{(2p-1)*}$$

while $\omega^\flat(f_k) = 0$ for $k > 2p$. Thus, $f^{1*}, f^{2*}, \ldots, f^{2p*}$ form a basis for the subspace $\omega^\flat(E)$ and $2p = dim\ \omega^\flat(E)$; which shows that p is only defined in terms of ω and does not depend on any choice of basis. $\square$

Note that the expression of ω defined at Lemma 4.1.1 is called the *canonical form* of ω or the *Darboux form* of ω or again the *normal form* of ω. The integer $2p := dim\ \omega^\flat(E)$ (see the Proof of Lemma 4.1.1) is called the *rank* of ω. See [10].

Remark 4.1.1 *([1]) The nondegeneracy of ω, as defined at Definition 4.1.1, is equivalent to what follows:*

1. *$\omega \wedge \omega \wedge \cdots \wedge \omega$ (n copies of ω), called the* **volume form**, *is nowhere zero;*
2. *$det(\omega_{IJ}) \neq 0$ or $det(\omega^t_{IJ}) \neq 0$, where ω_{IJ} (ω^t_{IJ} its transpose) is the matrix of ω called the* **symplectic matrix** *of ω;*
3. *The rank of ω is maximal;*
4. *The linear map $\omega^\flat : E \to E^*$ defined by $\omega^\flat(e)(e') := \omega(e, e')$ is an isomorphism;*
5. *There is a basis*
$$\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$$
of E, called the **symplectic basis** *or the* **canonical basis** *of E, such that*
$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0 \text{ and } \omega(u_i, v_j) = \delta_{ij},$$
where $i, j \in \{1, 2, \ldots, p\}$ and δ_{ij} is the Kronecker symbol.

Corollary 4.1.1 *The bilinear form ω is a symplectic form on a finite-dimensional real linear space E if and only if the space E is even dimensional.*

Proof. Suppose that ω is nondegenerate. Let us choose a basis of E such that

$$\omega = \sum_{i=1}^{p} \alpha^i \wedge \alpha^{i+p},$$

as defined at Lemma 4.1.1. By Remark 4.1.1(3), the rank of ω is the dimension of E. Hence, we have $dim\ E = 2p$. The converse holds similarly. $\square$

Example 16

1. Consider the bilinear form ω_o on the canonical Euclidian linear $\mathbb{F}$-space $\mathbb{R}^{2n}$ given by

$$\omega_o : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R};\ \omega_o(\bar{x}, \bar{y}) = \sum_{i=1}^{n} (x_{i+n} y_i - x_i y_{i+n}),$$

where

$$\bar{x} := (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$$

and

$$\bar{y} := (y_1, y_2, \ldots, y_n, y_{n+1}, \ldots, y_{2n})$$

are elements of $\mathbb{R}^{2n}$. Let us show that ω_o is skew-symmetric. In effect, for $\bar{x} \in \mathbb{R}^{2n}$ and $\bar{y} \in \mathbb{R}^{2n}$, by the definition of ω_0 and the properties of the addition and the multiplication of real numbers,

$$\begin{aligned}
\omega_o(\bar{x}, \bar{y}) &= \sum_{i=1}^{n} (x_{i+n} y_i - x_i y_{i+n}) \\
&= \sum_{i=1}^{n} (-x_i y_{i+n} + x_{i+n} y_i) \\
&= -\sum_{i=1}^{n} (x_i y_{i+n} - x_{i+n} y_i) \\
&= -\sum_{i=1}^{n} (y_{i+n} x_i - y_i x_{i+n}) \\
&= -\omega_o(\bar{y}, \bar{x}).
\end{aligned}$$

Now, we will show that ω_o is nondegenerate. In fact, the matrix form of $\omega_o(\bar{x}, \bar{y})$ is expressed as

$$\bar{x} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\omega_{o_{IJ}}} \bar{y}^t$$

It follows that $det(\omega_{o_{IJ}}) = 1$. That is, the matrix $\omega_{o_{IJ}}$ of ω_o is nonsingular. Hence, by Definition 4.1.1 and Remark 4.1.1(2), ω_o is a symplectic form on $\mathbb{R}^{2n}$ and $(\mathbb{R}^{2n}, \omega_o)$ is a symplectic linear $\mathbb{F}$-space.

2. Let E be a linear $\mathbb{F}$-space of dimension n and $W := E \times E^*$. If we set
$$\omega((x_1, \alpha^1); (x_2, \alpha^2)) = \alpha^2(x_1) - \alpha^1(x_2),$$
where $x_1, x_2 \in E$ and $\alpha^1, \alpha^2 \in E^*$, then we show that ω is a symplectic on W.
First, the bilinear form ω is skew-symmetric on W. For, if $(x_1, \alpha^1) \in W$ and $(X_2, \alpha^2) \in W$, then by the definition of ω and the properties the addition of reals numbers, we have
$$\begin{aligned} \omega((x_1, \alpha^1); (x_2, \alpha^2)) &= \alpha^2(x_1) - \alpha^1(x_2) \\ &= -\alpha^1(x_2) + \alpha^2(x_1) \\ &= -(\alpha^1(x_2) - \alpha^2(x_1)) \\ &= -\omega((x_2, \alpha^2); (x_1, \alpha^1)). \end{aligned}$$
Now we show that ω is nondegenerate. Let $(x_1, \alpha^1) \in W$ and $(X_2, \alpha^2) \in W$. That is, $x_1 \in E, x_2 \in E, \alpha^1 \in E^*$ and $\alpha^2 \in E^*$. If $\{e_1, e_2, \ldots, e_n\}$ is a basis of E and $\{e_1^*, e_2^*, \ldots, e_n^*\}$ its basis of E^*, then
$$x_k = \sum_{i=1}^{n} x_i^k e_i \text{ and } \alpha^l = \sum_{j=1}^{n} \alpha_j^l e_j^*,$$
where $x_i^k, \alpha_j^l \in \mathbb{R}$ for $k, l \in \{1, 2\}$. Since α^1 and α^2 are linear maps, we have
$$\begin{aligned} \omega((x_1, \alpha^1); (x_2, \alpha^2)) &= \alpha^2(x_1) - \alpha^1(x_2) \\ &= \alpha^2(\sum_{i=1}^{n} x_i^1 e_i) - \alpha^1(\sum_{i=1}^{n} x_i^2 e_i) \\ &= \sum_{i=1}^{n} x_i^1 \alpha^2(e_i) - \sum_{i=1}^{n} x_i^2 \alpha^1(e_i) \\ &= \sum_{i=1}^{n} [x_i^1 \alpha^2(e_i) - x_i^2 \alpha^1(e_i)] \\ &= \sum_{i=1}^{n} [x_i^1 \sum_{j=1}^{n} \alpha_j^2 e_j^*(e_i) - x_i^2 \sum_{j=1}^{n} \alpha_j^1 e_j^*(e_i)] \\ &= \sum_{i=1}^{n} (x_i^1 \sum_{j=1}^{n} \alpha_j^2 \delta_{ij} - x_i^2 \sum_{j=1}^{n} \alpha_j^1 \delta_{ij}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i^1 \alpha_j^2 - x_i^2 \alpha_j^1) \delta_{ij} \\ &= \sum_{i=1}^{n} (x_i^1 \alpha_i^2 - x_i^2 \alpha_i^1) \end{aligned}$$

where δ_{ij} is the Krönecker symbol. Thus,

$$\omega((x_1,\alpha^1);(x_2,\alpha^2)) = \sum_{i=1}^{n}(x_i^1\alpha_i^2 - x_i^2\alpha_i^1).$$

So, the matrix form of $\omega((x_1,\alpha^1);(x_2,\alpha^2))$ is written as follows

$$(x_1,\alpha^1)\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\omega_{IJ}}(x_2,\alpha^2)^t$$

where

$$(x_1,\alpha^1) := (x_1^1, x_2^1, \ldots, x_n^1, \alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1)$$

and

$$(x_2,\alpha^2) := (x_1^2, x_2^2, \ldots, x_n^2, \alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2).$$

Therefore, $\det(\omega_{IJ}) = 1$. That is, ω_{IJ} is the symplectic matrix of ω. Hence, according to Definition 4.1.1 and Remark 4.1.1(2), (W,ω) is a symplectic linear $\mathbb{F}$-space.

Definition 4.1.2 *(See [2]) A linear map $\varphi : (E,\omega) \to (E',\omega')$ between two symplectic linear $\mathbb{F}$-spaces is called* **symplectic $\mathbb{F}$-smooth map** *if*

$$\varphi^*\omega' = \omega \text{ or } \omega(x,y) = \omega'(\varphi(x),\varphi(y))$$

for all $x,y \in E$. A symplectic $\mathbb{F}$-smooth isomorphism on a symplectic linear $\mathbb{F}$-space E is called a **symplectic $\mathbb{F}$-smooth transformation** *or simply a* **symplectic transformation** *or again* **symplectomorphism***.*

The set of all $\mathbb{F}$-smooth transformations on a symplectic linear $\mathbb{F}$-space (E,ω) is denoted by $Sp(E,\omega)$.

Proposition 4.1.1 *Let (E,ω) be a symplectic linear $\mathbb{F}$-space of finite dimension. $Sp(E,\omega)$ is an $\mathbb{F}$-Lie group under the composition of maps.*

Proof. Let (E, ω) be a symplectic linear $\mathbb{F}$-space of finite dimension. It follows that E is a linear $\mathbb{F}$-space. According to Example 5, Diff(E) is an $\mathbb{F}$-Lie group. We have

$$Sp(E, \omega) \subset \mathrm{Diff}(E) \text{ and } Sp(E, \omega) \neq \emptyset$$

since $id_E \in Sp(E, \omega)$ and also by the definition of Diff(E). We only need to show that if $\varphi \in Sp(E, \omega)$ and $\psi \in Sp(E, \omega)$ then $\varphi \circ \psi \in Sp(E, \omega)$ and $\varphi^{-1} \in Sp(E, \omega)$. Let $\varphi \in Sp(E, \omega)$ and $\psi \in Sp(E, \omega)$. That is, by Definition 4.1.2, Definition 1.2.2, Proposition 1.2.2 and Proposition 1.2.5, $\varphi \circ \psi$ and φ^{-1} are $\mathbb{F}$-smooth. In effect, by Corollary 3.6.5 and Definition 4.1.2,

$$(\varphi \circ \psi)^* \omega = (\psi^* \circ \varphi^*)\omega = \psi^*(\varphi^* \omega) = \psi^* \omega = \omega.$$

Thus, by Definition 4.1.2, $\varphi \circ \psi \in Sp(E, \omega)$. We have also, by Corollary 3.6.6 and Definition 4.1.2,

$$(\varphi^{-1})^* \omega = (\varphi^*)^{-1} \omega = (\varphi^*)^{-1} \varphi^* \omega = id_{\Omega(E)} \omega = \omega$$

since $\varphi \in Sp(E, \omega)$ and φ^* is an $\mathbb{F}$-diffeomorphism, where $id_{\Omega(E)}$ is the identity isomorphism of $\Omega(E)$. Thus, by Definition 4.1.2, $\varphi^{-1} \in Sp(E, \omega)$. Hence, according to Definition 1.7.1, $Sp(E, \omega)$ is an $\mathbb{F}$-Lie group. $\square$

Definition 4.1.3 *The $\mathbb{F}$-Lie group $Sp(E, \omega)$ is called the* **symplectic $\mathbb{F}$-Lie group** *of E (see [12] and [20]).*

Proposition 4.1.2 *If (E, ω) and (E', ω') are two symplectic linear $\mathbb{F}$-spaces of the same dimension, then there exists an $\mathbb{F}$-smooth isomorphism $\varphi : E \to E'$ satisfying $\varphi^* \omega' = \omega$.*

Proof. By Remark 4.1.1(5), since (E, ω) and (E', ω') are symplectic linear $\mathbb{F}$-spaces of the same dimension, we can choose a symplectic basis

$$u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p$$

of E and a symplectic basis

$$\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_p, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_p$$

of E' such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = \omega'(\tilde{u}_i, \tilde{u}_j) = \omega(\tilde{v}_i, \tilde{v}_j) = 0$$

and

$$\omega(u_i, v_j) = \omega'(\tilde{u}_i, \tilde{v}_j) = \delta_{ij},$$

where $1 \leq i, j \leq p$ and δ_{ij} is the symbol of Krönecker. Let us define an an $\mathbb{F}$-smooth isomorphism

$$\varphi : E \to E' \text{ by } \varphi(u_i) = \tilde{u}_i \text{ and } \varphi(v_i) = \tilde{v}_i.$$

Let $x \in E$ and $y \in E$. Let $u_i = e_i,\ v_i = e_{i+1},\ \tilde{u}_i = \tilde{e}_i$ and $\tilde{v}_i = \tilde{e}_{i+1}$, for $i = 1, 2, \ldots, p$. That is,

$$x = \sum_{i=1}^{2p} x_i e_i \text{ and } y = \sum_{j=1}^{2p} y_j e_j,$$

where $x_i, y_j \in \mathbb{R}$. We have, by the fact that ω and ω' are bilinear and φ is linear,

$$\begin{aligned}
\varphi^*\omega'(x, y) &= \omega'(\varphi(x), \varphi(y)) \\
&= \omega'(\varphi(\sum_{i=1}^{2p} x_i e_i), \varphi(\sum_{j=1}^{2p} y_j e_j)) \\
&= \omega'(\sum_{i=1}^{2p} x_i \varphi(e_i), \sum_{j=1}^{2p} y_j \varphi(e_j)) \\
&= \omega'(\sum_{i=1}^{2p} x_i \tilde{e}_i, \sum_{j=1}^{2p} y_j \tilde{e}_j) \\
&= \sum_{i=1}^{2p} \sum_{j=1}^{2p} x_i y_j \omega'(\tilde{e}_i, \tilde{e}_j) \\
&= \sum_{i=1}^{2p} \sum_{j=1}^{2p} x_i y_j \omega(e_i, e_j) \\
&= \omega(\sum_{i=1}^{2p} x_i e_i, \sum_{j=1}^{2p} y_j e_j) \\
&= \omega(x, y).
\end{aligned}$$

That is, $\varphi^*\omega' = \omega$. □

Remark 4.1.2 *All symplectic linear $\mathbb{F}$-spaces of the same dimension are isomorphic. That is, they are symplectically indistinguishable. They "look like". See [12].*

Definition 4.1.4 *(See [2]) The* **exterior differentiation** *on a finite-dimensional pseudomanifold M is an operator $\underline{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the following formulas:*

1. *If $\alpha \in \Omega^0(M)$ then $(\underline{d}\alpha)(X) = X(\alpha)$, for any $X \in \mathfrak{X}(M)$.*

2. *If $k \geq 1$ and $\omega \in \Omega^k(M)$ then*

$$\underline{d}\omega(X_1, X_2, \ldots, X_{k+1}) = V + W,$$

where
$$V := \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1, X_2, \ldots, \tilde{X}_i, \ldots, X_{k+1}))$$
and
$$W := \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, X_2, \ldots, \tilde{X}_i, \ldots, \tilde{X}_j, \ldots, X_{k+1}))$$
for any $X_1, X_2, \ldots, X_{k+1} \in \mathfrak{X}(M)$, where $\tilde{X}_i$ and $\tilde{X}_j$ are omitted.

We say that $\underline{d}\omega$ is the **exterior differential** *of the form ω.*

Proposition 4.1.3 *If ω is a Pfaffian form then*

$$\underline{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Proof. A Pfaffian form is a 1-form. Let $k = 1, X_1 = X$ and $X_2 = Y$ and we use Definition 4.1.3. Clearly the proposition holds. □

Corollary 4.1.2 *Let M be a finite-dimensional pseudomanifold and $f \in \mathcal{F}_M$.*

$$\underline{d}(df) = 0 \text{ (i.e. a form of zero degree).}$$

Proof. Let M be a finite-dimensional pseudomanifold and $f \in \mathcal{F}_M$. According to Corollary 3.6.1, df is a Pfaffian form. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)$. We have, by Definition 3.7.1, $[X, Y] \in \mathfrak{X}(M)$. By Proposition 4.1.3, Definition 3.4.3 and Definition 3.7.1, we obtain

$$\begin{aligned}
\underline{d}(df)(X, Y) &= X(df)(Y) - Y(df)(X) - (df)([X, Y]) \\
&= X(df(Y)) - Y(df(X)) - (df([X, Y])) \\
&= X(Y.f) - Y(X.f) - [X, Y].f \\
&= X(Y.f) - Y(X.f) - (XY - YX).f \\
&= X(Y.f) - Y(X.f) - X(Y.f) + Y(X.f) \\
&= (XY - YX - XY + YX).f \\
&= ([X, Y] - [X, Y]).f \\
&= 0.f \\
&= 0
\end{aligned}$$

□

Definition 4.1.5 *Let M be a finite-dimensional pseudomanifold. An $\mathbb{F}$-form $\omega \in \Omega(M)$is said* **closed** *if $\underline{d}\omega = 0$ [10].*

4.2 Symplectic vector Frölicher bundles

Definition 4.2.1 *(See [15]) Let (E, π, M) be a vector $\mathbb{F}$-bundle. A* **symplectic structure** *on E is defined by a section ω of the $\mathbb{F}$-bundle $\bigwedge^2(E^*)$ the such that, for every $x \in M$, the bilinear form $\omega_x := \omega(x)$ is a symplectic form on the fiber E_x over x. A vector $\mathbb{F}$-bundle (E, π, M) endowed with a symplectic structure ω is called a* **symplectic vector $\mathbb{F}$-bundle**, *it is denoted by (E, π, M, ω) or, more briefly, by (E, ω) when there is no risk of confusion.*

Example 17

1. Let M be an arbitrary finite-dimensional pseudomanifold and (E, ω) symplectic vector $\mathbb{F}$-space. The trivial vector $\mathbb{F}$-bundle $M \times E$ over M (see Example 12) is a symplectic vector $\mathbb{F}$-bundle. Indeed, for any $x \in M$, $\omega_x \in \bigwedge^2(E^*)$ is a symplectic form on E. So, the form
$$\alpha \in \bigwedge^2((M \times E)^*) \text{ such that } \alpha((x, v_1); (x, v_2)) := \omega_x(v_1, v_2),$$
where $v_1, v_2 \in E_x$, defines a symplectic structure on the fiber E_x over $x \in M$.

2. Let E be a vector $\mathbb{F}$-bundle with base M and E^* be its dual vector $\mathbb{F}$-bundle. The Whitney sum $W := E \oplus E^*$ possesses a structure of symplectic vector $\mathbb{F}$-bundle since the corresponding form ω is defined by
$$\omega_x((u_1, \alpha_1); (u_2, \alpha_2)) = \alpha_2(u_1) - \alpha_1(u_2)$$
on the fiber $E_x \oplus E_x^*$.

Definition 4.2.2 *(See [15]) Let M be a finite-dimensional pseudomanifold. A* **symplectic structure** *on M is defined by the choice of a 2-form $\omega \in \Omega^2(M)$ satisfying the following properties:*

1. *for every $x \in M$, the bilinear form ω_x is nondegenerate;*

2. *the 2-form ω is closed ($\underline{d}\omega = 0$).*

We will say that the pair (M, ω) is a **symplectic pseudomanifold**, *and ω is a* **symplectic form** *on M.*

Remark 4.2.1 *Refering to [2], the nondegeneracy of the* 2*-form* $\omega \in \Omega^2(M)$ *is equivalent to one the following statements:*

1. $\omega(X, Y) = 0$ *for all* $Y \in \mathfrak{X}(M)$ *implies that* $X = 0$.
2. *For all* $x \in M$*, if* $\omega_x(X_x, Y_x) = 0$ *for all* $Y_x \in T_xM$ *then* $X_x = 0$.
3. *The* $\mathbb{F}$*-smooth map* $\omega^\flat : TM \to T^*M$ *is an isomorphism of vector* $\mathbb{F}$*-bundles.*
4. *The* $\mathbb{F}$*-smooth map* $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ *is an isomorphism of* $\mathcal{F}_M$*-modules.*

Proposition 4.2.1 *Let* (M, ω) *be a symplectic pseudomanifold of dimension* n*. Then for all* $x \in M$*,* (T_xM, ω_x) *is a symplectic linear* $\mathbb{F}$*-space and* n *is an even positive integer.*

Proof. Let M be a pseudomanifold of dimension n and $x \in M$. We have that $dim\ T_xM = n$. Let ω define a symplectic structure on M. It follows that ω_x defines a symplectic structure on T_xM since $\omega \in \Omega^2(M)$ is symplectic and for $x \in M$ implies $\omega_x \in \bigwedge^2(T_x^*M)$ (a canonical symplectic form). So, (T_xM, ω_x) is a symplectic linear $\mathbb{F}$-space. Then, according to Lemma 4.1.1, $dim\ T_xM = 2p$, where p is a positive integer depending only on ω_x. Hence, $n = dim\ T_xM = 2p$. Thus, $dim\ M = 2p$. □

Theorem 4.2.1 *(See [2]). Let* (M, ω) *be a symplectic pseudomanifold of dimension* $2n$*. For every* $x \in M$ *there exists an open neighborhood* U *of* x *and* $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n \in \mathcal{G}_x(U)$*, the germ of* $\mathbb{F}$*-smooth functions at* x *in* U *such that*

$$\omega_{|U} = \sum_{i=1}^{n} dq_i \wedge dp_i$$

(called Darboux form).

Proof. Let M be a symplectic pseudomanifold of dimension $2n$ and $x \in M$. It turns out (see [2] and Proof of Lemma 4.1.1) that there exists an open neighborhood U of x such that

$$W_1, W_2, \ldots, W_n, V_1, V_2, \ldots, V_n \in \mathfrak{X}(U)$$

form a local basis over U and $\mathbb{F}$-smooth functions

$$e^1, e^2, \ldots, e^n, e^{n+1}, \ldots, e^{2n} \in \mathcal{G}_x(U)$$

such that

$$W_{x,i}(e^j) = \delta_{ij},\ W_{x,i}(e^{n+j}) = 0,\ V_{x,i}(e^j) = 0 \text{ and } V_{x,i}(e^{n+j}) = \delta_{ij},$$

where δ_{ij} is the Krönecker symbol. By Definition 3.4.3,

$$de^j(W_{x,i}) = \delta_{ij},\ de^{n+j}(W_{x,i}) = 0,\ de^j(V_{x,i}) = 0 \text{ and } de^{n+j}(V_{x,i}) = \delta_{ij}.$$

Thus,

$$de^1, de^2, \ldots, de^n, de^{n+1}, \ldots, de^{2n}$$

form the dual basis of

$$W_1, W_2, \ldots, W_n, V_1, V_2, \ldots, V_n.$$

According to Lemma 4.1.1, we can choose the $2n$ $\mathbb{F}$-smooth functions

$$q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n \in \mathcal{G}_x(U)$$

such that

$$dq_1, dq_2, \ldots, dq_n, dp_1, dp_2, \ldots, dp_n$$

form the basis in which $\omega_{|_U}$ has the normal form

$$\sum_{i=1}^{n} dq_i \wedge dp_i.$$

□

Corollary 4.2.1 *If U is an open subset of a symplectic pseumanifold (E, ω) of dimension $2n$ then $(U, \omega_{|_U})$ is a symplectic pseumanifold.*

Proof. Let an open $U \subseteq M$ and a symplectic pseumanifold (E, ω) of dimension $2n$. According to Proposition 1.3.1, U is a finite-dimensional pseudomanifold. By Theorem 4.2.1, we can define a symplectic form on U by

$$\omega_{|_U} = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

Thus, $(U, \omega_{|_U})$ is a symplectic pseudomanifold. □

Example 18 The $2n$-dimensional Euclidian canonical $\mathbb{F}$-space $\mathbb{R}^{2n}$ is a symplectic pseudomanifold with respect to the canonical symplectic form

$$\omega_o = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n}$.

In the following, we will investigate a symplectic structure on the cotangent $\mathbb{F}$-bundle as second example of a symplectic pseudomanifold. Let (T^*M, τ, M) be the cotangent $\mathbb{F}$-bundle on a pseudomanifold M of dimension n. Let $x \in M$ and $\alpha \in T^*M$. According to Definition 3.3.1, there exists a tangent linear map $\tau_{*\alpha}$ defined from $T_\alpha(T^*M)$ into T_xM since $\tau : T^*M \to M$ is $\mathbb{F}$-smooth. One can define a linear map from $T_\alpha(T^*M)$ to $\mathbb{R}$ by the formula

$$\alpha(\tau_{*\alpha}(u)) = (\tau^*(\alpha))(u), \text{ for } u \in T_\alpha(T^*M).$$

We prove in the category $\underline{\mathcal{FRL}}$ the following theorem as mentioned in [10] for the category of differentiable manifolds.

Theorem 4.2.2 *Let M be a finite-dimensional pseudomanifold. The correspondance $\alpha \mapsto \tau^*(\alpha)$ defines a 1-form on T^*M.*

Proof. Let M be a finite-dimensional pseudomanifold and $\alpha \in T^*M$. According to Proposition 3.6.3, α is a 1-form on M. Hence, by Corollary 3.6.3, the map $\tau^*(\alpha)$ induces a 1-form on T^*M since $\tau : T^*M \to M$ is $\mathbb{F}$-smooth. $\square$

Definition 4.2.3 *Let (T^*M, τ, M) be a cotangent $\mathbb{F}$-bundle to a finite-dimensional pseudomanifold M. The 1-form λ such that $\lambda(\alpha) := \tau^*(\alpha)$ is called the* **Liouville form** *on T^*M [10].*

Lemma 4.2.1 *Let M be a finite-dimensional pseudomanifold and (N, ω) be a symplectic pseudomanifold. If $\varphi : M \to N$ is an $\mathbb{F}$-diffeomorphism then $\varphi^*\omega$ is a symplectic form on M.*

Proof. Let M be a finite-dimensional pseudomanifold and (N, ω) be a symplectic pseudomanifold. Let $\varphi : M \to N$ be an $\mathbb{F}$-diffeomorphism. According to Corollary 3.6.3, $\varphi^*\omega$ is a bilinear form on M since ω is a bilinear form on N and φ is $\mathbb{F}$-smooth. Let $X, Y \in \mathfrak{X}(M)$ and $\tilde{X} := \varphi_*(X), \tilde{Y} := \varphi_*(Y)$, $\tilde{X}$ and $\tilde{Y}$ lie in $\mathfrak{X}(N)$.
Let us show that $\varphi^*\omega$ is skew-symmetric. In effect, by Definition 3.6.6 and by skew-symmetry of ω, we have

$$\begin{aligned} \varphi^*\omega(X, Y) &= \omega(\varphi_*(X), \varphi_*(Y)) \\ &= \omega(\tilde{X}, \tilde{Y}) \\ &= -\omega(\tilde{Y}, \tilde{X}) \\ &= -\omega(\varphi_*(Y), \varphi_*(X)) \\ &= -\varphi^*\omega(Y, X). \end{aligned}$$

Thus, $\varphi^*\omega(X,Y) = -\varphi^*\omega(Y,X)$. Hence, the skew-symmetry of $\varphi^*\omega$ holds on M. Now, let us show that $\varphi^*\omega$ is nondegenerate. In effect, by Definition 3.6.4, we have that

$$\varphi^*\omega(X,Y) = 0 \text{ for all } Y \in \mathfrak{X}(M) \text{ implies } \omega(\varphi_*(X), \varphi_*(Y)) = 0$$

for all $Y \in \mathfrak{X}(M)$.Or again,

$$\varphi^*\omega(\tilde{X}, \tilde{Y}) = 0 \text{ for all } \tilde{Y} \in \mathfrak{X}(N)$$

since φ_* is an $\mathbb{F}$-diffeomorphism (see Proposition 3.4.8). By nondegeneracy of ω, it implies that $\tilde{X} = 0$. That is, $\varphi_*(X) = 0$. It yields $X = 0$ since φ_* is an $\mathbb{F}$-diffeomorphism. Thus,

$$\varphi^*\omega(X,Y) = 0 \text{ for all } Y \in \mathfrak{X}(M) \text{ implies } X = 0.$$

Hence, the nondegeneracy of $\varphi^*\omega$ holds on M.
Finally, let us show that $\varphi^*\omega$ is a closed form. In effect, by Definition 3.6.6 and by closedness of ω, we have for $X, Y, Z \in \mathfrak{X}(M)$ such that $\tilde{X} := \varphi_*(X)$, $\tilde{Y} := \varphi_*(Y)$, $\tilde{Z} := \varphi_*(Z) \in \mathfrak{X}(N)$ the following sequence of equalities:

$$\underline{d}\varphi^*\omega(X,Y,Z) = \underline{d}\omega(\varphi_*(X), \varphi_*(Y), \varphi_*(Z)) = \underline{d}\omega(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0.$$

Thus, $\varphi^*\omega$ is closed on M. According to Definition 4.2.2, the bilinear form $\varphi^*\omega$ is a symplectic form on M. □

In the following corollary, we use the Lemma 4.2.1 to show that if M is finite-dimensional pseudomanifold then T^*M is a symplectic pseudomanifold.

Corollary 4.2.2 *Let M be a finite-dimensional pseudomanifold. The exterior differential $\omega := \underline{d}\lambda$ of the Liouville form λ on T^*M defines a symplectic structure.*

Proof. Let M be a finite-dimensional pseudomanifold. By Definition 4.1.3, $\omega := \underline{d}\lambda$ is a 2-form on T^*M since λ is a 1-form on T^*M. According to Proposition 3.5.2, the cotangent $\mathbb{F}$-bundle T^*M is locally $\mathbb{F}$-diffeomorphic to the standard symplectic pseudomanifold $\mathbb{R}^{2n}$. Thus, according to Lemma 4.2.1, the pullback of the canonical symplectic form on $\mathbb{R}^{2n}$ is a symplectic form on T^*M. Let ω be that symplectic form. Hence, (T^*M, ω) is a symplectic pseudomanifold. □

So, according to Corollary 4.2.2, the cotangent $\mathbb{F}$-bundle to any finite-dimensional pseudomanifold is a second example of symplectic pseudomanifold following that of the standard symplectic pseudomanifold given at Example 19.

Definition 4.2.4 *(See [20]) Let (M,ω) and (N,ω') be two symplectic pseudomanifolds of finite-dimension. An $\mathbb{F}$-smooth map $\varphi : M \to N$ is* **symplectic** *if*

$$\varphi^*\omega' = \omega.$$

A symplectic $\mathbb{F}$-diffeomorphism is called a **symplectomorphism**.

Theorem 4.2.3 *(See [1]) Let M be a finite-dimensional pseudomanifold and $\varphi : M \to M$ an $\mathbb{F}$-diffeomorphism; define the lift of φ by*

$$\bar{\varphi} : T^*M \to T^*M;\ \bar{\varphi}(\alpha_q)(v) = \alpha_q(\varphi_*(v)),$$

where $q \in M$, $\alpha_q \in T^*M$ and $v \in T_{\varphi^{-1}(q)}M$. Then, $\bar{\varphi}$ is a symplectomorphism and in fact $\bar{\varphi}^*\theta_o = \theta_o$, where θ_o is the canonical 1-form.

Proof. Let M be a finite-dimensional pseudomanifold and φ an $\mathbb{F}$-diffeomorphism of M. Consider the following commutative diagrams:

$$\begin{tikzcd}
T^*M \arrow[d, "\tau"] & T^*M \arrow[l, "\bar{\varphi}"'] \arrow[d, "\tau"] \\
M \arrow[r, "\varphi"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\alpha_q}(T^*M) \arrow[r, "\bar{\varphi}_*"] \arrow[d] & T_{\alpha_q}(T^*M) \arrow[d] \\
T^*M \arrow[r, "\bar{\varphi}"] & T^*M
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\alpha_q}(T^*M) \arrow[r, "\tau_*"] \arrow[d] & T_qM \arrow[d] \\
T^*M \arrow[r, "\tau"] & M
\end{tikzcd}$$

Let θ_o be a 1-form on T^*M. For $u \in T_{\alpha_q}(T^*M)$, by Definition 3.6.6 and by the definition of $\bar{\varphi}$, we have

$$\begin{aligned}
\bar{\varphi}^*\theta_o(u) &= \theta_o(\bar{\varphi}_*(u))\\
&= \bar{\varphi}(\alpha_q)\tau_*(\bar{\varphi}_*(u))\\
&= \bar{\varphi}(\alpha_q)(\tau_*(\bar{\varphi}_*(u)))\\
&= \bar{\varphi}(\alpha_q)((\tau\circ\bar{\varphi})_*(u))\\
&= \alpha_q(\varphi_*((\tau\circ\bar{\varphi})_*(u)))\\
&= \alpha_q((\varphi\circ\tau\circ\bar{\varphi})_*(u))\\
&= \alpha_q(\tau_*(u))\\
&= \theta_o(u).
\end{aligned}$$

Thus, for all $u \in T_{\alpha_q}(T^*M)$, $\bar{\varphi}^*\theta_o(u) = \theta_o(u)$. That is, $\bar{\varphi}^*\theta_o = \theta_o$. By Definition 4.2.4, $\bar{\varphi}$ is symplectic. For $\alpha_q \in T^*M$ and $v \in T_{\varphi^{-1}(q)}M$, by the definition of $\bar{\varphi}$, we have

$$\bar{\varphi}(\alpha_q)v = \alpha_q(\varphi_*(v)) = \varphi^*(\alpha_q)v.$$

So, $\bar{\alpha}$ can be identified with φ^*. Hence, $\bar{\alpha}$ is an $\mathbb{F}$-diffeomorphism since φ^* is an $\mathbb{F}$-diffeomorphism (see Corollary 3.6.6). By Definition 4.2.4, $\bar{\varphi}$ is a symplectomorphism of T^*M. $\square$

Theorem 4.2.4 *(See [1] and [20]) Let $\varphi : M_1 \to M_2$ be an $\mathbb{F}$-diffeomorphism between finite-dimensional pseudomanifolds M_1 and M_2 and its lift defined by*

$$\bar{\varphi} : T^*M_2 \to T^*M_1;\ \bar{\varphi}(\alpha_{q_1})(v_1) = \alpha_{q_2}(\varphi_*(v_1)),$$

where $q_2 \in M$, $\alpha_{q_2} \in T^*M_2$ and $v_1 \in T_{\varphi^{-1}(q_2)}M_1$. Then, $\bar{\varphi}$ is a symplectomorphism and in fact $\bar{\varphi}^*\omega_1 = \omega_2$, where ω_1 and ω_2 are respectively 1-form on T^*M_1 and T^*M_2.

Proof. Let $\varphi : M_1 \to M_2$ be an $\mathbb{F}$-diffeomorphism between finite-dimensional pseudomanifolds M_1 and M_2. Notice the following commutative diagrams:

$$\begin{tikzcd}
T^*M_1 \arrow[d, "\tau_1"] & T^*M_2 \arrow[l, "\bar{\varphi}"'] \arrow[d, "\tau_2"] \\
M_1 \arrow[r, "\varphi"] & M_2
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\alpha_{q_2}}(T^*M_2) \arrow[r, "\bar{\varphi}_*"] \arrow[d] & T_{\alpha_{q_1}}(T^*M_1) \arrow[d] \\
T^*M_2 \arrow[r, "\bar{\varphi}"] & T^*M_1
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\alpha_{q_1}}(T^*M_1) \arrow[r, "\tau_{1*}"] \arrow[d] & T_{q_1}M_1 \arrow[d] \\
T^*M_1 \arrow[r, "\tau_1"] & M_1
\end{tikzcd}$$

$$\begin{tikzcd}
T_{\alpha_{q_2}}(T^*M_2) \arrow[r, "\tau_{2*}"] \arrow[d] & T_{q_2}M_2 \arrow[d] \\
T^*M_2 \arrow[r, "\tau_2"] & M_2
\end{tikzcd}$$

Let ω_1 be a 1-form on T^*M_1. For $u \in T_{\alpha_{q_2}}(T^*M_2)$, by Definition 3.6.4, by the definition of $\bar{\varphi}$, by the properties of compositions of maps and the fact that φ is an $\mathbb{F}$-diffeomorphism, we have

$$\begin{aligned}
\bar{\varphi}^*\omega_1(u) &= \omega_1(\bar{\varphi}_*(u)) \\
&= \bar{\varphi}(\alpha_{q_2})\tau_{1*}(\bar{\varphi}_*(u)) \\
&= \bar{\varphi}(\alpha_{q_2})((\tau_{1*} \circ \bar{\varphi}_*)(u)) \\
&= \bar{\varphi}(\alpha_{q_2})((\tau_1 \circ \bar{\varphi})_*(u)) \\
&= \alpha_{q_2}(\varphi_*((\tau_1 \circ \bar{\varphi})_*(u))) \\
&= \alpha_{q_2}(\varphi_*(\varphi^{-1} \circ \tau_2)_*(u)) \\
&= \alpha_{q_2}((\varphi \circ \varphi^{-1} \circ \tau_2)_*(u)) \\
&= \alpha_{q_2}((id_{M_2} \circ \tau_2)_*(u)) \\
&= \alpha_{q_2}(\tau_{2*}(u)) \\
&= \omega_2(u),
\end{aligned}$$

where id_{M_2} is the identity map of M_2. Thus, for all $u \in T_{\alpha_q}(T^*M)$, $\bar{\varphi}^*\omega_1(u) = \omega_2(u)$. That is, $\bar{\varphi}^*\omega_1 = \omega_2$. By Definition 4.2.4, $\bar{\varphi}$ is symplectic. For $\alpha_{q_2} \in T^*M_2$ and $v_1 \in T_{\varphi^{-1}(q_1)}M_1$, by the definition of $\bar{\varphi}$, one has

$$\bar{\varphi}(\alpha_{q_2})v_1 = \alpha_{q_2}(\varphi_*(v_1)) = \varphi^*(\alpha_{q_2})v_1.$$

That is, $\bar{\varphi}$ can be identified with φ^*. Hence, $\bar{\varphi}$ is an $\mathbb{F}$-diffeomorphism since φ^* is an $\mathbb{F}$-diffeomorphism (see Corollary 3.6.6). By Definition 4.2.4, $\bar{\varphi}$ is a symplectomorphism of T^*M_2 to T^*M_1. $\square$

We defined the notion of symplectic structure on a vector $\mathbb{F}$-bundle. We gave two examples of a symplectic vector $\mathbb{F}$-bundle. We also defined the notion of symplectic structure on a finite-dimensional pseudomanifold. We reviewed the standard example of a symplectic pseudomanifold or the symplectic trivial vector $\mathbb{F}$-bundle $\mathbb{R}^{2n}$. We defined a 2-form on a fiber of a cotangent $\mathbb{F}$-bundle. Then, we proved the existence of a (local) symplectic structure on a cotangent $\mathbb{F}$-bundle on an n-dimensional pseudomanifold by the pullback of the canonical symplectic form of $\mathbb{R}^{2n}$. We proved that the cotangent $\mathbb{F}$-bundles of two $\mathbb{F}$-diffeomorphic finite-dimensional pseudomanifolds are symplectomorphic.

Conclusion

We showed that on a Frölicher space the topology induced by structure curves is included into the topology generated by structure functions. In order to simplify our structure, we supposed that those two topologies coincide and are compact Hausdorff. We proved that structure curves and structure functions are smooth in the category of Frölicher spaces. We reviewed, with details and examples, the concepts and the properties of Frölicher structure on the set induced from the other Frölicher spaces.

We proved that the fibers of Frölicher bundles have a Frölicher structure as well. We also proved that the sections of Frölicher bundles are smooth maps in the category of Frölicher spaces. We noticed that the set of all sections of vector Frölicher bundles over a Frölicher space, endowed with the addition and the multiplication by a structure function, is a module over the set of structure functions of the Frölicher space. Those $\mathbb{F}$-bundles and $\mathbb{F}$-morphisms between $\mathbb{F}$-bundles form the category of Frölicher bundles.

The concepts of tangent structures on a Frölicher space and linear tangent map between two tangent Frölicher spaces are reviewed with a deep attention and more detailed proofs. We proved that the tangent and the cotangent Frölicher bundles of a finite-dimensional pseudomanifold are locally diffeomorphic to the even-dimensional Euclidian canonical Frölicher space. More properties of tangent maps between tangent Frölicher bundles are given and proved. The differential of a real-valued function on a Frölicher space is that of the structure function. We defined the notions of vector fields and forms of the Frölicher bundles over a finite-dimensional pseudomanifold. We have observed that those vector fields and forms are smooth in the category of Frölicher spaces.

We achieved the investigation of the symplectic structure on a vector Frölicher bundle and some examples are given. The same is done for the symplectic structure on a finite-dimensional pseudomanifold. The standard example of symplectic pseudomanifolds has been given. The existence of a symplectic structure on a cotangent Frölicher bundle has also been given as a second example of a symplectic pseudomanifold. This work ended with the the concept of symplectomorphism between two finite- dimensional pseudomanifolds. We proved that the cotangent Frölicher bundles of two diffeomorphic finite-dimensional pseudomanifolds are symplectomorphic in the category of Frölicher spaces. Those notions can be useful to find out the canonical form (the equivalent of Darboux form) of the symplectic form on a cotangent Frölicher bundle. In our opinion, it could constitue an interesting subject of a future work.

Bibliography

[1] R. Abraham, J.E. Marsden: *Foundation of Mechanics*, Second Edition, The Benjamin/Cummings Publishing Company, Inc., 1978.

[2] A. Batubenge: Symplectic Frölicher Spaces of Constant Dimension,*PhD Thesis, University of Cape Town, 2005 (unpublished).*

[3] A. Batubenge: On Transversal Intersection and Gluing Symplectic Pseudomanifolds, Conference on Symplectic Topology, *Stare Jablonki, Poland*, 3-10 July 2004.

[4] S. Bradlow: The Geometry of Vector Bundles and An Introduction to Gauge Theory, *Class Notes Math433, University of Illinois at Urbana-Champain*, 1998.

[5] A. Cap: K-Theory for Convenient Algebras,*Thesis*, 1993.

[6] P. Cherenack: Applications of Frölicher Spaces To Cosmology, *Annals Univ. Sc. Budapest*, **41**(1998), 63-91

[7] P. Cherenack: Frölicher versus Differential Spaces: A Prelude to Cosmology, G. Brümmer and C. Gilmour (eds), *Papers in Honour of Bernhard Banaschewski*, 2000 Kluwer Academic Publishers, Printed in the Netherlands (1998), 63-91.

[8] Y. Choquet-Bruhat, C. Dewitt-Morette, M. Dillard-Bleick: *Analysis, Manifolds and Physics*, Revised Edition, North-Holland Publishing Company, 1982.

[9] A. Frölicher, A. Kriegl: *Linear Spaces and Differentiation Theory*, Wiley-Interscience, 1971.

[10] C. Godbillon: *Géometrie Différentielle et Mécanique Analytique*, Hermann Paris, Collection Méthodes, 1969.

[11] V. Guillemin, S. Sternberg: *Symplectic Techniques in Physics*, Cambridge University Press, 1984.

[12] H. Hofer, E. Zehnder: *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Verlag, 1994.

[13] D. Husemoller: *Fibre Bundles*, Second Edition, Springer-Verlag, New York, 1966.

[14] A. Kriel, P.W. Michor: *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, Volume 53, American Mathematical Society, 1997.

[15] P. Libermann and C-M. Marle: *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publishing Company, 1987.

[16] M. Nakahara: *Geometry, Topology and Physics*, Institute of Physics Publishing Bristol and Philadelphia, 1990.

[17] P. Ntumba, A. Batubenge: On the Way to Frölicher Lie Groups, Second Edition,*Quaestiones Mathematicae*, **28**(2005),73-74.

[18] S. Sternberg: *Lecture on Differential Geometry*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964.

[19] A. Trafny: Bundles, Linear Bundles and Principal Bundles in the Category of Differential Spaces, *Demonstratio Mathematica*, **XXVI**, no.1 (1993).

[20] A. Weinstein: *Lectures on Symplectic Manifolds*, Regional Conference Series in Mathematics 29, the American Mathematical Society, Providence 1977.