UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

MSc Research

Symmetries and Exact Solutions of Systems of Ordinary Difference Equations

Kgatliso Mkhwanazi, 1477029

Supervisor: Dr Mensah Folly-Gbetoula

January 22, 2022

Declaration

I, the undersigned, declare that the work in this dissertation is my original work, and that any previous work done by others or by myself has been acknowledged and referenced appropriately. It is being submitted for a Master of Science degree at the University of the Witwatersrand, Johannesburg. It has never been submitted to any other University for any degree or examination.

K.Mkhwanazi

Kgatliso Mkhwanazi, 2022.

Dedication

To my mother Nomsa Mkhwanazi and my siblings Kgomotso, Tokollo, Lesego and Boitumelo Mkhwanazi.

Acknowledgements

I would firstly like to thank my mother Nomsa Mkhwanazi who has raised me and my siblings as a single parent, she is the best and I would have never reached this point in life if it was not for her sacrifices, encouragement, love and faith in me, so thank you mom.

I want to thank my supervisor, Dr Folly-Gbetoula, for his continued assistance, advice and guidance in this dissertation, I could have never asked for a better supervisor as he is an exceptional supervisor who was always available whenever I needed help or advice from him. He always availed himself whenever I asked for a meeting and he gave the best explanations in such a way that I never had to ask more than once for a specific question, it has really been an honour to be supervised by him.

I am also grateful to the Gauteng City Region Academy(GCRA) for their continued financial support throughout my studies as their support has helped me tremendously and afforded me the opportunity to pursue my studies.

Abstract

In this dissertation, we will generalize the work done by Almatrafi and Elsayed to a general order. We will do this by performing a complete Lie analysis for the following system of $(k+2)^{th}$ order difference equations

$$\begin{aligned} u_{n+1} =& \frac{u_{n-1}v_{n-k-1}}{v_{n-k+1}(A_n + B_n u_{n-1}v_{n-k-1})}, \\ v_{n+1} =& \frac{u_{n-k-1}v_{n-1}}{u_{n-k+1}(C_n + D_n u_{n-k-1}v_{n-1})}, \end{aligned} \tag{1}$$

where A_n, B_n, C_n and D_n are real sequences. We will find the symmetries of (1) in terms of k. We will use these symmetries to find exact formulas for the solutions in terms of k. These exact formulas will be used to show that the results by Almatrafi and Elsayed are special cases of our findings.

Keywords: Lie analysis; initial conditions; difference equations; symmetries; exact solutions; periodicity; invariant solutions; reduction.

Contents

Publications Associated with this Dissertation

Kgatliso Mkhwanazi and Mensah Folly-Gbetoula. 2022. *Symmetries and Solvability of a Class of Higher Order Systems of Ordinary Difference Equations* Symmetry 14(1), 108.

Chapter 1

Introduction

A Norwegian mathematician Marius Sophus Lie (1842 - 1899) is responsible for the discovery of the transformations of variables and its properties. Sophus Lie began by investigating the continuous groups of transformations that would leave the differential equations invariant and his work created what is now known as symmetry analysis of differential equations. In 1869, Sophus Lie published his first mathematical work [1]. Lie's original aim was to set a general theory for the integration of ordinary differential equations which was similar to the work that was done by Galois Abel [2] on algebraic equations in 1888. During the nineteenth century Lie developed and applied the symmetry analysis of differential equations [3, 4]. The theory that he developed enabled one to derive the solutions of differential equations in an algorithmic way that didn't require any special guesses. In 1891, Sophus Lie gave a lecture about differential equations (differentialgleichungen) in which he pointed out the prominent problem that he had faced with the theory that he came up with. There has been a lot of interest in the way Lie approached differential equations and notable mentions include the work done by Sedov Leonid Ivanovich [5] and Garrett Birkhoff [6] on the dimensional analysis where their individual work proved to be important in the understanding of Lie's approach to differential equations because their work showed that Lie's theory gave pertinent results in applied problems. Before this, there was a German mathematician Hermann Weyl (1885-1955) who in 1928 took interest in the abstract theory of Lie groups and the term Lie group was coined by him in the same year. There has been a restoration of interest in Lie's theory in recent decades and during these decades there has been a lot of crucial progress that was made from an applied point of view or a theoretical one. Lie's theory is very important, adaptable and elementary in the development of the coherent approach that enables us to generalize higher order ordinary differential equations into a general lower order so that it can be easier to solve as opposed to when it is of higher order. Lie's theory involves a lot of tedious and cumbersome calculations, like when we want to find the symmetries of the system of ordinary difference equations (OΔEs), ordinary differential equations (ODEs) or even the partial differential equations (PDEs) this at times leads

one to have to handle a number of equations just to find only one solution. With the help of the powerful computer programs that we have today, we can mostly avoid these tedious and cumbersome calculations but these computer programs only work for the differential equations and not for the difference equations that we will be dealing with in this dissertation. Hence, every calculation will be done by hand. Invariant solutions are very crucial in Lie's theory because of the transformation of independent and dependent variables in the symmetry analysis of differential equations. Lie group analysis has been and is still an essential tool in so many various fields like physics, number theory, differential equations, differential geometry, analysis and more. Since in this study, we want to generalize the work that was done in the articles [7, 8], we have to do a lot of tedious and cumbersome calculations to obtain the symmetries for the OΔEs (1) and eventually the exact formulas.
In [7], the authors solved and dealt with the recursive equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1 + x_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(\pm 1 \pm y_{n-1}x_{n-3})}, \quad (2)$$

where the real numbers $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are the initial conditions. Also in [8], the author determined and formulated the analytical solutions of the rational recursive equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(\pm 1 \mp x_{n-1}y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(\mp 1 \pm y_{n-1}x_{n-3})}, \quad (3)$$

here $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are the initial conditions which are arbitrary non-zero real numbers. Since in this dissertation we are performing a complete lie analysis for the $(k+2)^{th}$ order (1), we will look closely into the work done by Folly-Gbetoula and Nyirenda [9], where the exact formulas were found for the system of $(k+1)^{th}$ order difference equations

$$x_{n+1} = \frac{x_{n-k+1}y_{n-k}}{y_n(a_n + b_n x_{n-k+1}y_{n-k})}, \ y_{n+1} = \frac{x_{n-k}y_{n-k+1}}{x_n(c_n + d_n x_{n-k}y_{n-k+1})}. \quad (4)$$

Here, the non-zero real sequences are a_n, b_n, c_n and d_n.
Understanding Lie groups and being able to use them is important because there is a lot that can be done with Lie groups, for example, we can get the Lie algebra action for a linear object by getting the derivative of the Lie group action. This is useful because when it comes to linear objects, it is much simpler to work with a Lie algebra than directly working with a Lie group. This is just one

example of the many useful ways Lie groups can be used. In this dissertation, we will use Lie groups.

Outline of Chapters

1. Chapter 1 has the introduction to the study.

2. Chapter 2 provides a mathematical description of the basic concepts of symmetry analysis in reference to difference equations.

3. Chapter 3 has the main results of this dissertation, the symmetries and exact solutions of the mathematical system are presented.

4. Chapter 4 concludes and summarizes the study.

Chapter 2

Preliminaries

Consider the general form of the difference equation

$$u_{n+k} = \omega(n, u_n, u_{n+1}, ..., u_{n+k-1}), \tag{5}$$

for some function ω.

Definition 1. *[10] A forward OΔE has the form*

$$u_k = \omega(n, u, u_1, \ldots, u_{k-1}), \quad n \in D, \tag{6}$$

and a backward OΔE has the form

$$u = \Omega(n, u_1, u_2, \ldots, u_k), \quad n \in D, \tag{7}$$

here $n \in D$, $D \subset \mathbb{Z}$.

The domain D of the forward OΔE is said to be a regular domain if

$$\omega_{,1}(n, u, \ldots, u_{k-1}) \neq 0, \quad n \in D. \tag{8}$$

Similarly, the backward OΔE's domain D is a regular domain if $\Omega_k \neq 0$. Here, the notation $\omega_{,1}$ stands for $\frac{\partial \omega}{\partial u}$ and we also have that $\omega_{,k+1}$ stands for $\frac{\partial \omega}{\partial u_k}$.

A difference equation is formed by eliminating the arbitrary constants from a given relation. Difference equations relate to differential equations as discrete mathematics relate to continuous mathematics. This means that difference equations are the discrete equivalent of differential equations and arise whenever an independent variable can have only discrete values.

Definition 2. *[10] A parametrized set of point transformations,*

$$\Gamma_\varepsilon : \boldsymbol{x} \mapsto \hat{\boldsymbol{x}}(\boldsymbol{x}; \varepsilon), \quad \varepsilon \in (\varepsilon_0, \varepsilon_1), \tag{9}$$

where $\varepsilon_0 < 0 < \varepsilon_1$, is a one-parameter local Lie group if the following conditions are satisfied:

1. *Γ_0 is the identity map, so that $\hat{\boldsymbol{x}} = \boldsymbol{x}$ when $\varepsilon = 0$.*
2. *$\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$ for every δ, ε sufficiently close to zero.*
3. *Each $\hat{x}_\alpha$ can be represented as Taylor series in ε (in a neighborhood of $\varepsilon = 0$ that is determined by $\boldsymbol{x}$), and therefore*

$$\hat{x}_\alpha(\boldsymbol{x}; \varepsilon) = x_\alpha + \varepsilon \xi_\alpha(\boldsymbol{x}) + O(\varepsilon^2), \ \alpha = 1, \ldots, N. \tag{10}$$

Definition 3. *[11] The one-parameter Lie group of transformations is equivalent to*

$$\tilde{U} = e^{\varepsilon\hat{U}}U = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!}\hat{U}^k U. \tag{11}$$

The expression (11) is called the *Lie series.*

Theorem 1. *[11] The function $f(u)$ is invariant under the one-parameter Lie group of transformations $\boldsymbol{F}$ if and only if*

$$\tilde{U}f(u) = 0. \tag{12}$$

Defined below is the forward shift operator.

Definition 4. *[12]*

$$S : n \mapsto n+1, \qquad S^j x_n = x_{n+j} \tag{13}$$

where x_{n+j} denotes the j^{th} shift of x_n.

The shift operator is also referred to as a transition operator which is an operator that takes a function $u \mapsto f(u)$ to its transition $u \mapsto f(u+a)$.
A symmetry is a change of transformation that leaves the object invariant. Here, the individual points of an object may be mapped to different points, but the object as a whole is unchanged.

Definition 5. *[13] Let x be a real number. Then, $\lfloor x \rfloor$ is defined to be the unique integer n that satisfies $n \leq x < n+1$. This integer $\lfloor x \rfloor$ is called the floor of x, or the integer part of x.*

Definition 6. *[9] Let r be a non-negative integer and $\tau(r)$ denotes the remainder when r is divided by k. Thus, we have the following representation*

$$r = k\left\lfloor \frac{r}{k} \right\rfloor + \tau\left(\frac{r}{k}\right) = k\left\lfloor \frac{r}{k} \right\rfloor + \tau\left(r\right), \text{ here } 0 \leq \tau(r) \leq r-1. \tag{14}$$

Definition 7. *[14] (Infinitesimal transformations).*
Consider $\tilde{u} = U(u,\varepsilon)$, with $\varepsilon = 0$, expanded in Taylor series,

$$\tilde{u} = u + \varepsilon\left(\left.\frac{\partial U}{\partial \varepsilon}\right|_{\varepsilon=0}\right) + \frac{\varepsilon^2}{2!}\left(\left.\frac{\partial^2 U}{\partial \varepsilon^2}\right|_{\varepsilon=0}\right) + \ldots \tag{15}$$

up to first order, we have

$$\tilde{u} = u + \varepsilon\left(\left.\frac{\partial U}{\partial \varepsilon}\right|_{\varepsilon=0}\right) + O(\varepsilon^2). \tag{16}$$

In this dissertation, we define Lie symmetries as the symmetries that belong to a one-parameter local Lie group and the Lie point transformations as the point transformations that belong to a one-parameter local Lie group [10], this is because we are working with O∆Es (1). Hence, we consider the system of ordinary difference equations of the form

$$u_{n+k+2} = \omega_1(u_{n+k}, v_n, v_{n+2}), \ v_{n+k+2} = \omega_2(u_n, u_{n+2}, v_{n+k}), \tag{17}$$

where the independent variable here is denoted by n and the dependent variables are denoted by u_n, v_n and their shifts.
We will look at Lie point symmetries, which are symmetries that act on the space of dependent and independent variables. One of the many uses of symmetries is to reduce the order of the ODE/O∆E. Symmetry reduction can also be applied to boundary value problems for ODEs/O∆Es.

We now consider the group of transformations

$$\begin{aligned}(n, u_n, v_n) \mapsto (n, \tilde{u}_n &= u_n + \varepsilon Q_1(n, u_n) \\ &+ O(\varepsilon^2), \tilde{v}_n = v_n + \varepsilon Q_2(n, v_n) + O(\varepsilon^2)).\end{aligned} \tag{18}$$

In (18), the characteristic of the group of transformations is $Q = (Q_1, Q_2)$. The infinitesimal generator corresponds to

$$U = Q_1 \frac{\partial}{\partial u_n} + Q_2 \frac{\partial}{\partial v_n}. \tag{19}$$

This infinitesimal generator is very important because we are going to use it to find the symmetries which will in turn be used to find the exact solutions for the system of ordinary difference equations (1).

Definition 8. *[15] A symmetry generator is denoted by U and is given by*

$$U = Q \frac{\partial}{\partial u_n} + SQ \frac{\partial}{\partial u_{n+1}} + \cdots + S^{k-1} Q \frac{\partial}{\partial u_{n+k-1}}. \tag{20}$$

In this dissertation, we are working with the $(k+2)^{th}$ O∆Es (1), so we need the k^{th} extension

$$\begin{aligned}U^{[k]} = Q_1 \frac{\partial}{\partial u_n} + Q_2 \frac{\partial}{\partial v_n} + S^2 Q_1 \frac{\partial}{\partial u_{n+2}} + S^2 Q_2 \frac{\partial}{\partial v_{n+2}} \\ + S^k Q_1 \frac{\partial}{\partial u_{n+k}} + S^k Q_2 \frac{\partial}{\partial v_{n+k}}\end{aligned} \tag{21}$$

of (19).
For the set of solutions of (17) to be mapped to itself, the following linearised symmetry conditions

$$S^{(k+2)}Q_1 - U^{[k]}\omega_1 = 0 \tag{22a}$$

and

$$S^{(k+2)}Q_2 - U^{[k]}\omega_2 = 0, \tag{22b}$$

whenever (14) is true, must be satisfied.
If the above conditions, that is,

$$Q_j(n+k+2,\Omega_j) - U^{[k]}\Omega_j = 0, \qquad j = 1,2, \tag{23}$$

are satisfied, then the group of transformations (15) is a symmetry group. We will solve the equations that arise from equation (21) for Q_j's together with all the canonical equations

$$s^k = \int \frac{dx^k}{Q_j(n,x^k)}, \qquad j = 1,2. \tag{24}$$

Chapter 3

Solvability of a Class of Higher Order Systems of Ordinary Difference Equations

3.1 Symmetries of the System of Difference Equations (1)

Equivalently, equation (1) can be written as

$$\begin{aligned} u_{n+k+2} =&\omega_1 = \frac{u_{n+k}v_n}{v_{n+2}(a_n + b_n u_{n+k}v_n)}, \\ v_{n+k+2} =&\omega_2 = \frac{u_n v_{n+k}}{u_{n+2}(c_n + d_n u_n v_{n+k})}, \end{aligned} \tag{25}$$

where a_n and b_n are real sequences. Applying (22) onto (25) yields,

$$\begin{aligned} -S^{k+2}Q_1 + \frac{a_n u_{n+k} Q_2}{v_{n+2}(a_n + b_n u_{n+k} v_n)^2} - \\ \frac{u_{n+k} v_n S^2 Q_2}{v_{n+2}^2 (a_n + b_n u_{n+k} v_n)} + \frac{a_n v_n S^k Q_1}{v_{n+2}(a_n + b_n u_{n+k} v_n)^2} = 0, \end{aligned} \tag{26a}$$

$$\begin{aligned} -S^{k+2}Q_2 + \frac{c_n v_{n+k} Q_1}{u_{n+2}(c_n + d_n u_n v_{n+k})^2} - \\ \frac{u_n v_{n+k} S^2 Q_1}{u_{n+2}^2 (c_n + d_n u_n v_{n+k})} + \frac{c_n u_n S^k Q_2}{u_{n+2}(c_n + d_n u_n v_{n+k})^2} = 0. \end{aligned} \tag{26b}$$

To find the differential operators for the system (25), we respectively use the following equations, namely:

$$\frac{\partial}{\partial u_{n+k}} - \left(\frac{\omega_1, u_{n+k}}{\omega_1, v_n} \right) \frac{\partial}{\partial v_n}, \tag{27a}$$

and

$$\frac{\partial}{\partial u_n} - \left(\frac{\omega_2, u_n}{\omega_2, v_{n+k}} \right) \frac{\partial}{\partial v_{n+k}}. \tag{27b}$$

Applying (27a) onto (26a) leads to the following,

$$-v_n \left(s^k Q_1 \right)' + v_n Q_2' + \frac{v_n}{u_{n+k}} s^k Q_1 - Q_2 = 0. \tag{28}$$

Note that $'$ denotes the derivative with respect to the continuous variable.
Similarly, when we apply (27b) onto (26b), we get

$$v_{n+k}\left(S^kQ_2\right)' - v_{n+k}Q_1' + \frac{v_{n+k}}{u_n}Q_1 - S^kQ_2 = 0. \tag{29}$$

We now find the characteristics $Q_1(n,u_n)$ and $Q_2(n,v_n)$, so dividing both sides of (28) by v_n and differentiating with respect to v_n yields

$$-\left(S^kQ_1\right)' + Q_2' + \frac{S^kQ_1}{u_{n+k}} - \frac{Q_2}{v_n} = 0. \tag{30}$$

Differentiating (30) with respect to v_n gives

$$\left[Q_2'(n,v_n) - \frac{1}{v_n}Q_2(n,v_n)\right]' = 0. \tag{31}$$

The general solution of (31) is given by

$$Q_2(n,v_n) = \mu_n v_n \ln v_n + \lambda_n v_n. \tag{32a}$$

Similarly, when dividing both sides of (29) by v_{n+k} and differentiating with respect to u_n and also solving the resulting differential equations we get

$$Q_1(n,u_n) = \beta_n u_n \ln u_n + \alpha_n u_n. \tag{32b}$$

Next, substitute (32) into equation (26a). Here the coefficients of the products of shifts of u_n and v_n are set to zero to obtain the system below:

$$u_{n+k}v_n \ln u_{n+k+2}v_{n+2} \quad \text{terms}: \quad a_n\beta_{n+k+2} = 0, \tag{33a}$$

$$u_{n+k}^2v_n^2 \ln u_{n+k+2}v_{n+2} \quad \text{terms}: \quad b_n\beta_{n+k+2} = 0, \tag{33b}$$

$$u_{n+k}v_nv_{n+2} \quad \text{terms}: \quad a_n\alpha_{n+k+2} - a_n\lambda_n + a_n\lambda_{n+2} - a_n\alpha_{n+k} = 0, \tag{33c}$$

$$u_{n+k}^2v_n^2v_{n+2} \quad \text{terms}: \quad b_n\alpha_{n+k+2} + b_n\lambda_{n+2} = 0, \tag{33d}$$

$$u_{n+k}v_n \ln v_nv_{n+2} \quad \text{terms}: \quad -a_n\mu_n = 0, \tag{33e}$$

$$u_{n+k}v_nv_{n+2} \ln v_{n+2} \quad \text{terms}: \quad a_n\mu_{n+2} = 0, \tag{33f}$$

$$u_{n+k}^2v_n^2v_{n+2} \ln v_{n+2} \quad \text{terms}: \quad b_n\mu_{n+2} = 0, \tag{33g}$$

$$u_{n+k}v_nv_{n+2} \ln u_{n+k} \quad \text{terms}: \quad -a_n\beta_{n+k} = 0. \tag{33h}$$

Similarly, substituting (32) into equation (26b), and setting the coefficients of the products of shifts of u_n and v_n to zero, leads to the following system:

$$
\begin{aligned}
u_n v_{n+k} u_{n+2} \ln v_{n+k+2} \quad &\text{terms :} \quad c_n \mu_{n+k+2} = 0, && \text{(34a)}\\
u_n^2 v_{n+k}^2 u_{n+2} \ln v_{n+k+2} \quad &\text{terms :} \quad d_n \mu_{n+k+2} = 0, && \text{(34b)}\\
u_n u_{n+2} v_{n+k} \quad &\text{terms :} \quad c_n \lambda_{n+k+2} - c_n \alpha_n \\
& \qquad + c_n \alpha_{n+2} - c_n \lambda_{n+k} = 0, && \text{(34c)}\\
u_n^2 v_{n+k}^2 u_{n+2} \quad &\text{terms :} \quad d_n \lambda_{n+k+2} + d_n \alpha_{n+2} = 0, && \text{(34d)}\\
u_n u_{n+2} \ln u_n v_{n+k} \quad &\text{terms :} \quad -c_n \beta_n = 0, && \text{(34e)}\\
u_n u_{n+2} \ln u_{n+2} v_{n+k} \quad &\text{terms :} \quad c_n \beta_{n+2} = 0, && \text{(34f)}\\
u_n^2 v_{n+k}^2 u_{n+2} \ln u_{n+2} \quad &\text{terms :} \quad d_n \beta_{n+2} = 0, && \text{(34g)}\\
u_n u_{n+2} v_{n+k} \ln v_{n+k} \quad &\text{terms :} \quad -c_n \mu_{n+k} = 0. && \text{(34h)}
\end{aligned}
$$

From (33c) and (33d), we respectively have:

$$\alpha_{n+k+2} + \lambda_{n+2} - \lambda_n - \alpha_{n+k} = 0, \tag{35}$$

and

$$\alpha_{n+k+2} + \lambda_{n+2} = 0. \tag{36}$$

Substitute (36) into (35) to get:

$$\lambda_n + \alpha_{n+k} = 0. \tag{37}$$

From (34c) and (34d), we perform the same computations as above to obtain

$$\alpha_n + \lambda_{n+k} = 0. \tag{38}$$

Thus,

$$\alpha_n + \lambda_{n+k} = 0 \text{ and } \lambda_n + \alpha_{n+k} = 0. \tag{39}$$

Using (39), we get

$$\alpha_{n+2k} - \alpha_n = 0. \tag{40}$$

Solving (40) yields

$$\alpha_n = \left(e^{2i\pi(p+1)/k}\right)^n \text{ or } \alpha_n = \left(e^{i\pi(2p+1)/k}\right)^n. \tag{41}$$

Since from (39) we saw that $\lambda_n + \alpha_{n+k} = 0$, so $\lambda_n = -\alpha_{n+k}$, thus using this with (41), we have:

When $\alpha_n = \left(e^{2i\pi(p+1)/k}\right)^n$, we get

$$\lambda_n = -\left(e^{2i\pi(p+1)/k}\right)^{n+k}. \tag{42a}$$

Also when $\alpha_n = \left(e^{i\pi(2p+1)/k}\right)^n$ we get

$$\lambda_n = -\left(e^{i\pi(2p+1)/k}\right)^{n+k}. \tag{42b}$$

Since from (33e), $\mu_n = 0$ and also from (34e), $\beta_n = 0$, now respectively substituting these into (32a) and (32b), leads to the following:

$$Q_1(n,u_n) = \alpha_n u_n \text{ and } Q_2(n,v_n) = \lambda_n v_n. \tag{43}$$

To get the characteristics, we substitute (42) into (43), which gives

$$Q_1(n,u_n) = \left(e^{2i\pi(p+1)/k}\right)^n u_n \text{ and } Q_2(n,v_n) = -\left(e^{2i\pi(p+1)/k}\right)^{n+k} v_n, \tag{44a}$$

$$Q_1(n,u_n) = \left(e^{i\pi(2p+1)/k}\right)^n u_n \text{ and } Q_2(n,v_n) = -\left(e^{i\pi(2p+1)/k}\right)^{n+k} v_n. \tag{44b}$$

Therefore, we have the following symmetry generators:

$$U = \left(e^{2i\pi(p+1)/k}\right)^n u_n \frac{\partial}{\partial u_n} - \left(e^{2i\pi(p+1)/k}\right)^{n+k} v_n \frac{\partial}{\partial v_n}, \tag{45a}$$

$$U = \left(e^{i\pi(2p+1)/k}\right)^n u_n \frac{\partial}{\partial u_n} - \left(e^{i\pi(2p+1)/k}\right)^{n+k} v_n \frac{\partial}{\partial v_n}. \tag{45b}$$

3.2 Reductions and Exact Solutions

We use (24) and (43) to find the canonical coordinates as follows:

For $Q_1(n,u_n) = \alpha_n u_n$:

$$s_n^1 = \int \frac{du_n}{Q_1(n,u_n)} = \int \frac{du_n}{\alpha_n u_n} = \frac{1}{\alpha_n}\int \frac{du_n}{u_n} = \frac{1}{\alpha_n}\ln|u_n| \tag{46}$$

$$\Rightarrow s_n^1 = \tfrac{1}{\alpha_n}\ln|u_n|.$$

For $Q_2(n,v_n) = \lambda_n v_n$:

$$s_n^2 = \int \frac{dv_n}{Q_2(n,v_n)} = \int \frac{dv_n}{\lambda_n v_n} = \frac{1}{\lambda_n}\int \frac{dv_n}{v_n} = \frac{1}{\lambda_n}\ln|v_n| \tag{47}$$

$\Rightarrow s_n^2 = \frac{1}{\lambda_n} \ln |v_n|$.

Forward shifting $\alpha_n s_n^1 = \ln |u_n|$ and $\lambda_n s_n^2 = \ln |v_n|$ by k, we get

$\alpha_{n+k} s_{n+k}^1 = \ln |u_{n+k}|$ and $\lambda_{n+k} s_{n+k}^2 = \ln |v_{n+k}|$. Now motivated by (39), we have

$\widetilde{Y}_n = \alpha_n s_n^1 + \lambda_{n+k} s_{n+k}^2 = \ln |u_n| + \ln |v_{n+k}| = \ln |u_n v_{n+k}|$ We also have $\widetilde{X}_n = \lambda_n s_n^2 + \alpha_{n+k} s_{n+k}^1 = \ln |v_n| + \ln |u_{n+k}| = \ln |v_n u_{n+k}|$. Thus we get

$$\widetilde{Y}_n = \ln |u_n v_{n+k}| \text{ and } \widetilde{X}_n = \ln |v_n u_{n+k}|. \tag{48}$$

For a better understanding, we use $X_n = \exp\{-\widetilde{X}_n\}$ and $Y_n = \exp\{-\widetilde{Y}_n\}$,
So

$$\begin{aligned} X_n &= \exp\{-\widetilde{X}_n\} = \exp\{-\ln |v_n u_{n+k}|\} = \pm \frac{1}{v_n u_{n+k}}, \\ Y_n &= \exp\{-\widetilde{Y}_n\} = \exp\{-\ln |u_n v_{n+k}|\} = \pm \frac{1}{u_n v_{n+k}}. \end{aligned} \tag{49}$$

When we use the plus sign in (49), we get the invariants as follows:

$$X_n = \frac{1}{v_n u_{n+k}}, \tag{50a}$$

$$Y_n = \frac{1}{u_n v_{n+k}}. \tag{50b}$$

Here, (50) is invariant under the group transformations of (1).
Using (50), (1) is reduced to a second-order difference equations:

$$X_{n+2} = a_n X_n + b_n \text{ and } Y_{n+2} = c_n Y_n + d_n. \tag{51}$$

The closed form solutions of (51) are respectively as follows:

$$X_{2n+i} = X_i \left(\prod_{k_1=0}^{n-1} a_{2k_1+i} \right) + \sum_{l=0}^{n-1} \left(b_{2l+i} \prod_{k_2=l+1}^{n-1} a_{2k_2+i} \right), \tag{52a}$$

$$Y_{2n+i} = Y_i \left(\prod_{k_1=0}^{n-1} c_{2k_1+i} \right) + \sum_{l=0}^{n-1} \left(d_{2l+i} \prod_{k_2=l+1}^{n-1} c_{2k_2+i} \right), \tag{52b}$$

for $i = 0, 1$. From (50), we get that

$$u_{n+2k} = \frac{Y_n}{X_{n+k}} u_n \text{ and } v_{n+2k} = \frac{X_n}{Y_{n+k}} v_n. \tag{53}$$

Performing iterations on (53) respectively, leads to the following:

$$u_{2kn+j} = u_j \prod_{m=0}^{n-1} \frac{Y_{2km+j}}{X_{2km+k+j}} \text{ and } v_{2kn+j} = v_j \prod_{m=0}^{n-1} \frac{X_{2km+j}}{Y_{2km+k+j}}, \tag{54}$$

where $j = 0, 1, 2, \ldots, 2k-1$.
The subscript of X and Y in (52) is $2n+i$, where $i = 0, 1$ but the ones of X and Y in (54) are $2km+j$ and $2km+k+j$, where $j = 0, 1, 2, \ldots, 2k-1$. So we want to write $2km+j$ and $2km+k+j$ in the form similar to $2n+i$, we do this as follows:

For $2km + j$:

$$j = 2\left\lfloor \frac{j}{2} \right\rfloor + \tau(j), \text{ so } 2km + j = 2km + 2\left\lfloor \frac{j}{2} \right\rfloor + \tau(j), \tag{55a}$$

$$\Rightarrow 2km + j = 2\left(km + \left\lfloor \frac{j}{2} \right\rfloor\right) + \tau(j). \tag{55b}$$

For $2km + k + j$:

$$k + j = 2\left\lfloor \frac{k+j}{2} \right\rfloor + \tau(k+j), \tag{55c}$$

$$\text{so } 2km + k + j = 2km + 2\left\lfloor \frac{k+j}{2} \right\rfloor + \tau(k+j), \tag{55d}$$

$$\Rightarrow 2km + k + j = 2\left(km + \left\lfloor \frac{k+j}{2} \right\rfloor\right) + \tau(k+j). \tag{55e}$$

Now, (55) is of the form similar to $2n+i$, as $\tau(j)$ and $\tau(k+j)$ will either be 0 or 1. Substituting (55) into (54) leads to the following:

$$u_{2kn+j} = u_j \prod_{m=0}^{n-1} \frac{Y_{2(km+\lfloor \frac{j}{2} \rfloor)+\tau(j)}}{X_{2(km+\lfloor \frac{k+j}{2} \rfloor)+\tau(k+j)}} \tag{56a}$$

and

$$v_{2kn+j} = v_j \prod_{m=0}^{n-1} \frac{X_{2(km+\lfloor \frac{j}{2} \rfloor)+\tau(j)}}{Y_{2(km+\lfloor \frac{k+j}{2} \rfloor)+\tau(k+j)}}. \tag{56b}$$

Using (52) and (56), we get the following:

$$
\begin{aligned}
&u_{2kn+j} \\
&= u_j \prod_{m=0}^{n-1} \frac{Y_{\tau(j)} \left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_1+\tau(j)} \right) + \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1}}{X_{\tau(k+j)} \left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_1+\tau(k+j)} \right) + \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1}} \\
&\qquad \frac{\left(d_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_2+\tau(j)} \right)}{\left(b_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_2+\tau(k+j)} \right)}, \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
&= u_j \prod_{m=0}^{n-1} \frac{Y_{\tau(j)}}{X_{\tau(k+j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_1+\tau(j)} \right) + \frac{1}{Y_{\tau(j)}} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1}}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_1+\tau(k+j)} \right) + \frac{1}{X_{\tau(k+j)}} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1}} \\
&\qquad \frac{\left(d_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_2+\tau(j)} \right)}{\left(b_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_2+\tau(k+j)} \right)}, \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
&= u_j \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)+k} v_{\tau(k+j)}}{u_{\tau(j)} v_{\tau(j)+k}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_1+\tau(j)} \right) + u_{\tau(j)} v_{\tau(j)+k}}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_1+\tau(k+j)} \right) + u_{\tau(k+j)+k} v_{\tau(k+j)}} \\
&\qquad \frac{\sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(d_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} c_{2k_2+\tau(j)} \right)}{\sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(b_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} a_{2k_2+\tau(k+j)} \right)}. \qquad (59)
\end{aligned}
$$

Back-shifting the equation above by k, yields

$$u_{2kn+j-k} = u_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)} v_{\tau(k+j)-k}}{u_{\tau(j)-k} v_{\tau(j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} C_{2k_1+\tau(j)} \right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A_{2k_1+\tau(k+j)} \right)}$$

$$\frac{+u_{\tau(j)+(-k)} v_{\tau(j)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(D_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} C_{2k_2+\tau(j)} \right)}{+u_{\tau(k+j)} v_{\tau(k+j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(B_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A_{2k_2+\tau(k+j)} \right)}. \tag{60}$$

We also have the following:

$$v_{2kn+j}$$

$$= v_j \prod_{m=0}^{n-1} \frac{X_{\tau(j)} \left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_1+\tau(j)} \right) + \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1}}{Y_{\tau(k+j)} \left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_1+\tau(k+j)} \right) + \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1}}$$

$$\frac{\left(b_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_2+\tau(j)} \right)}{\left(d_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_2+\tau(k+j)} \right)}, \tag{61}$$

$$= v_j \prod_{m=0}^{n-1} \frac{X_{\tau(j)}}{Y_{\tau(k+j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_1+\tau(j)}\right) + \frac{1}{X_{\tau(j)}} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1}}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_1+\tau(k+j)}\right) + \frac{1}{Y_{\tau(k+j)}} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1}}$$

$$\frac{\left(b_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_2+\tau(j)}\right)}{\left(d_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_2+\tau(k+j)}\right)}, \tag{62}$$

$$= v_j \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)} v_{\tau(k+j)+k}}{u_{\tau(j)+k} v_{\tau(j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_1+\tau(j)}\right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_1+\tau(k+j)}\right)}$$

$$\frac{+u_{\tau(j)+k} v_{\tau(j)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(b_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} a_{2k_2+\tau(j)}\right)}{+u_{\tau(k+j)} v_{\tau(k+j)+k} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(d_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} c_{2k_2+\tau(k+j)}\right)}. \tag{63}$$

Back-shifting the equation above by k, yields

$$v_{2kn+j-k} = v_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)-k} v_{\tau(k+j)}}{u_{\tau(j)} v_{\tau(j)-k}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} A_{2k_1+\tau(j)}\right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C_{2k_1+\tau(k+j)}\right)}$$

$$\frac{+u_{\tau(j)} v_{\tau(j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(B_{2l+\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} A_{2k_2+\tau(j)}\right)}{+u_{\tau(k+j)+(-k)} v_{\tau(k+j)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(D_{2l+\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C_{2k_2+\tau(k+j)}\right)}. \tag{64}$$

Hence, the following is the solution of (1) where $j = 0, 1, 2, \ldots, 2k-1$

$$u_{2kn+j-k} = u_{j-k}\prod_{m=0}^{n-1}\frac{u_{\tau(k+j)}v_{\tau(k+j)-k}}{u_{\tau(j)-k}v_{\tau(j)}}\frac{\left(\prod_{k_1=0}^{km+\lfloor\frac{j}{2}\rfloor-1}C_{2k_1+\tau(j)}\right)}{\left(\prod_{k_1=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}A_{2k_1+\tau(k+j)}\right)}$$
$$\frac{+u_{\tau(j)+(-k)}v_{\tau(j)}\sum_{l=0}^{km+\lfloor\frac{j}{2}\rfloor-1}\left(D_{2l+\tau(j)}\prod_{k_2=l+1}^{km+\lfloor\frac{j}{2}\rfloor-1}C_{2k_2+\tau(j)}\right)}{+u_{\tau(k+j)}v_{\tau(k+j)+(-k)}\sum_{l=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}\left(B_{2l+\tau(k+j)}\prod_{k_2=l+1}^{km+\lfloor\frac{k+j}{2}\rfloor-1}A_{2k_2+\tau(k+j)}\right)}, \tag{65a}$$

and

$$v_{2kn+j-k} = v_{j-k}\prod_{m=0}^{n-1}\frac{u_{\tau(k+j)-k}v_{\tau(k+j)}}{u_{\tau(j)}v_{\tau(j)-k}}\frac{\left(\prod_{k_1=0}^{km+\lfloor\frac{j}{2}\rfloor-1}A_{2k_1+\tau(j)}\right)}{\left(\prod_{k_1=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}C_{2k_1+\tau(k+j)}\right)}$$
$$\frac{+u_{\tau(j)}v_{\tau(j)+(-k)}\sum_{l=0}^{km+\lfloor\frac{j}{2}\rfloor-1}\left(B_{2l+\tau(j)}\prod_{k_2=l+1}^{km+\lfloor\frac{j}{2}\rfloor-1}A_{2k_2+\tau(j)}\right)}{+u_{\tau(k+j)+(-k)}v_{\tau(k+j)}\sum_{l=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}\left(D_{2l+\tau(k+j)}\prod_{k_2=l+1}^{km+\lfloor\frac{k+j}{2}\rfloor-1}C_{2k_2+\tau(k+j)}\right)}. \tag{65b}$$

3.2.1 The case when A_n ,B_n ,C_n and D_n are 2- periodic

Letting,

$$\begin{aligned}
&A_{2k_1+\tau(j)} = A_{\tau(j)}, A_{2k_2+\tau(j)} = A_{\tau(j)}, B_{2l+\tau(j)} = B_{\tau(j)},\\
&C_{2k_1+\tau(j)} = C_{\tau(j)}, C_{2k_2+\tau(j)} = C_{\tau(j)}, D_{2l+\tau(j)} = D_{\tau(j)},\\
&A_{2k_1+\tau(k+j)} = A_{\tau(k+j)}, A_{2k_2+\tau(k+j)} = A_{\tau(k+j)}, B_{2l+\tau(k+j)} = B_{\tau(k+j)},\\
&C_{2k_1+\tau(k+j)} = C_{\tau(k+j)}, C_{2k_2+\tau(k+j)} = C_{\tau(k+j)}, D_{2l+\tau(k+j)} = D_{\tau(k+j)},
\end{aligned}$$

simplifies (65) as follows:

$$u_{2kn+j-k} = u_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)} v_{\tau(k+j)-k}}{u_{\tau(j)-k} v_{\tau(j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} C_{\tau(j)} \right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A_{\tau(k+j)} \right)}$$

$$\frac{+u_{\tau(j)+(-k)} v_{\tau(j)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(D_{\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} C_{\tau(j)} \right)}{+u_{\tau(k+j)} v_{\tau(k+j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(B_{\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A_{\tau(k+j)} \right)}, \tag{66}$$

$$v_{2kn+j-k} = v_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)-k} v_{\tau(k+j)}}{u_{\tau(j)} v_{\tau(j)-k}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} A_{\tau(j)} \right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C_{\tau(k+j)} \right)}$$

$$\frac{+u_{\tau(j)} v_{\tau(j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(B_{\tau(j)} \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} A_{\tau(j)} \right)}{+u_{\tau(k+j)+(-k)} v_{\tau(k+j)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(D_{\tau(k+j)} \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C_{\tau(k+j)} \right)}. \tag{67}$$

Thus

$$u_{2kn+j-k} = u_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)} v_{\tau(k+j)-k}}{u_{\tau(j)-k} v_{\tau(j)}} \frac{C_{\tau(j)}^{km+\lfloor \frac{j}{2} \rfloor}}{A_{\tau(k+j)}^{km+\lfloor \frac{k+j}{2} \rfloor}}$$

$$\frac{+D_{\tau(j)} u_{\tau(j)+(-k)} v_{\tau(j)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} C_{\tau(j)}^{l}}{+B_{\tau(k+j)} u_{\tau(k+j)} v_{\tau(k+j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A_{\tau(k+j)}^{l}}, \tag{68a}$$

and

$$v_{2kn+j-k} = v_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)-k} v_{\tau(k+j)}}{u_{\tau(j)} v_{\tau(j)-k}} \frac{A_{\tau(j)}^{km+\lfloor \frac{j}{2} \rfloor}}{C_{\tau(k+j)}^{km+\lfloor \frac{k+j}{2} \rfloor}}$$

$$\frac{+B_{\tau(j)} u_{\tau(j)} v_{\tau(j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} A_{\tau(j)}^{l}}{+D_{\tau(k+j)} u_{\tau(k+j)+(-k)} v_{\tau(k+j)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C_{\tau(k+j)}^{l}}. \tag{68b}$$

where $j = 0, 1, 2, \ldots, 2k-1$.

3.2.2 The case when A_n ,B_n ,C_n and D_n are constant sequences

Letting $A_n = A, B_n = B, C_n = C$ and $D_n = D$ for all n i.e,

$$\begin{aligned} &A_{2k_1+\tau(j)} = A, A_{2k_2+\tau(j)} = A, B_{2l+\tau(j)} = B, C_{2k_1+\tau(j)} = C, \\ &C_{2k_2+\tau(j)} = C, D_{2l+\tau(j)} = D, A_{2k_1+\tau(k+j)} = A, A_{2k_2+\tau(k+j)} = A, \\ &B_{2l+\tau(k+j)} = B, C_{2k_1+\tau(k+j)} = C, C_{2k_2+\tau(k+j)} = C, D_{2l+\tau(k+j)} = D, \end{aligned}$$

simplifies (65) as follows:

$$u_{2kn+j-k} = u_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)} v_{\tau(k+j)-k}}{u_{\tau(j)-k} v_{\tau(j)}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} C \right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A \right)}$$

$$\frac{+u_{\tau(j)+(-k)} v_{\tau(j)} \sum_{l=0}^{km+\lfloor \frac{j}{2} \rfloor -1} \left(D \prod_{k_2=l+1}^{km+\lfloor \frac{j}{2} \rfloor -1} C \right)}{+u_{\tau(k+j)} v_{\tau(k+j)+(-k)} \sum_{l=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} \left(B \prod_{k_2=l+1}^{km+\lfloor \frac{k+j}{2} \rfloor -1} A \right)}, \tag{69}$$

$$v_{2kn+j-k} = v_{j-k} \prod_{m=0}^{n-1} \frac{u_{\tau(k+j)-k} v_{\tau(k+j)}}{u_{\tau(j)} v_{\tau(j)-k}} \frac{\left(\prod_{k_1=0}^{km+\lfloor \frac{j}{2} \rfloor -1} A \right)}{\left(\prod_{k_1=0}^{km+\lfloor \frac{k+j}{2} \rfloor -1} C \right)}$$

$$\frac{+u_{\tau(j)}v_{\tau(j)+(-k)}\sum_{l=0}^{km+\lfloor\frac{j}{2}\rfloor-1}\left(B\prod_{k_2=l+1}^{km+\lfloor\frac{j}{2}\rfloor-1}A\right)}{+u_{\tau(k+j)+(-k)}v_{\tau(k+j)}\sum_{l=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}\left(D\prod_{k_2=l+1}^{km+\lfloor\frac{k+j}{2}\rfloor-1}C\right)}. \tag{70}$$

Thus

$$u_{2kn+j-k}=u_{j-k}\prod_{m=0}^{n-1}\frac{u_{\tau(k+j)}v_{\tau(k+j)-k}}{u_{\tau(j)-k}v_{\tau(j)}}\frac{C^{km+\lfloor\frac{j}{2}\rfloor}}{A^{km+\lfloor\frac{k+j}{2}\rfloor}}$$

$$\frac{+Du_{\tau(j)+(-k)}v_{\tau(j)}\sum_{l=0}^{km+\lfloor\frac{j}{2}\rfloor-1}C^l}{+Bu_{\tau(k+j)}v_{\tau(k+j)+(-k)}\sum_{l=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}A^l}, \tag{71a}$$

and

$$v_{2kn+j-k}=v_{j-k}\prod_{m=0}^{n-1}\frac{u_{\tau(k+j)-k}v_{\tau(k+j)}}{u_{\tau(j)}v_{\tau(j)-k}}\frac{A^{km+\lfloor\frac{j}{2}\rfloor}}{C^{km+\lfloor\frac{k+j}{2}\rfloor}}$$

$$\frac{+Bu_{\tau(j)}v_{\tau(j)+(-k)}\sum_{l=0}^{km+\lfloor\frac{j}{2}\rfloor-1}A^l}{+Du_{\tau(k+j)+(-k)}v_{\tau(k+j)}\sum_{l=0}^{km+\lfloor\frac{k+j}{2}\rfloor-1}C^l}. \tag{71b}$$

where $j=0,1,2,\dots,2k-1$.

3.3 A Detailed Study of the Case $k=2$

The aim of this section is to retrieve the results in [1, 2, 16]. To achieve this, we substitute $k=2$ into (71). It yields the following:

$$u_{4n+j-2}=u_{j-2}\prod_{m=0}^{n-1}\frac{u_{\tau(2+j)}v_{\tau(2+j)-2}}{u_{\tau(j)-2}v_{\tau(j)}}\frac{C^{2m+\lfloor\frac{j}{2}\rfloor}}{A^{2m+\lfloor\frac{2+j}{2}\rfloor}}$$

$$\frac{+Du_{\tau(j)+(-2)}v_{\tau(j)}\sum_{l=0}^{2m+\lfloor\frac{j}{2}\rfloor-1}C^l}{+Bu_{\tau(2+j)}v_{\tau(2+j)+(-2)}\sum_{l=0}^{2m+\lfloor\frac{2+j}{2}\rfloor-1}A^l}, \tag{72a}$$

and

$$v_{4n+j-2}=v_{j-2}\prod_{m=0}^{n-1}\frac{u_{\tau(2+j)-2}v_{\tau(2+j)}}{u_{\tau(j)}v_{\tau(j)-2}}\frac{A^{2m+\lfloor\frac{j}{2}\rfloor}}{C^{2m+\lfloor\frac{2+j}{2}\rfloor}}$$

$$\frac{+Bu_{\tau(j)}v_{\tau(j)+(-2)}\sum_{l=0}^{2m+\lfloor\frac{j}{2}\rfloor-1}A^l}{+Du_{\tau(2+j)+(-2)}v_{\tau(2+j)}\sum_{l=0}^{2m+\lfloor\frac{2+j}{2}\rfloor-1}C^l}. \tag{72b}$$

where $j=0,1,2,3$.
For the sake of clarity, we can rewrite (72) in expanded forms as follows:

$$u_{4n-2}=u_{-2}\prod_{m=0}^{n-1}\frac{u_0v_{-2}}{u_{-2}v_0}\frac{C^{2m}+Du_{-2}v_0\sum_{l=0}^{2m-1}C^l}{A^{2m+1}+Bu_0v_{-2}\sum_{l=0}^{2m}A^l}, \tag{73a}$$

$$u_{4n-1}=u_{-1}\prod_{m=0}^{n-1}\frac{u_1v_{-1}}{u_{-1}v_1}\frac{C^{2m}+Du_{-1}v_1\sum_{l=0}^{2m-1}C^l}{A^{2m+1}+Bu_1v_{-1}\sum_{l=0}^{2m}A^l}, \tag{73b}$$

$$u_{4n}=u_0\prod_{m=0}^{n-1}\frac{u_0v_{-2}}{u_{-2}v_0}\frac{C^{2m+1}+Du_{-2}v_0\sum_{l=0}^{2m}C^l}{A^{2m+2}+Bu_0v_{-2}\sum_{l=0}^{2m+1}A^l}, \tag{73c}$$

$$u_{4n+1}=u_1\prod_{m=0}^{n-1}\frac{u_1v_{-1}}{u_{-1}v_1}\frac{C^{2m+1}+Du_{-1}v_1\sum_{l=0}^{2m}C^l}{A^{2m+2}+Bu_1v_{-1}\sum_{l=0}^{2m+1}A^l}, \tag{73d}$$

$$v_{4n-2} = v_{-2} \prod_{m=0}^{n-1} \frac{u_{-2}v_0}{u_0v_{-2}} \frac{A^{2m} + Bu_0v_{-2} \sum\limits_{l=0}^{2m-1} A^l}{C^{2m+1} + Du_{-2}v_0 \sum\limits_{l=0}^{2m} C^l}, \tag{73e}$$

$$v_{4n-1} = v_{-1} \prod_{m=0}^{n-1} \frac{u_{-1}v_1}{u_1v_{-1}} \frac{A^{2m} + Bu_1v_{-1} \sum\limits_{l=0}^{2m-1} A^l}{C^{2m+1} + Du_{-1}v_1 \sum\limits_{l=0}^{2m} C^l}, \tag{73f}$$

$$v_{4n} = v_0 \prod_{m=0}^{n-1} \frac{u_{-2}v_0}{u_0v_{-2}} \frac{A^{2m+1} + Bu_0v_{-2} \sum\limits_{l=0}^{2m} A^l}{C^{2m+2} + Du_{-2}v_0 \sum\limits_{l=0}^{2m+1} C^l}, \tag{73g}$$

$$v_{4n+1} = v_1 \prod_{m=0}^{n-1} \frac{u_{-1}v_1}{u_1v_{-1}} \frac{A^{2m+1} + Bu_1v_{-1} \sum\limits_{l=0}^{2m} A^l}{C^{2m+2} + Du_{-1}v_1 \sum\limits_{l=0}^{2m+1} C^l}. \tag{73h}$$

Theorem 2. *The following system of equations*

$$u_{n+1} = \frac{u_{n-1}v_{n-3}}{v_{n-1}(A + Bu_{n-1}v_{n-3})} \quad \text{and} \quad v_{n+1} = \frac{u_{n-3}v_{n-1}}{u_{n-1}(C + Du_{n-3}v_{n-1})}, \tag{74}$$

has a 2-periodic solution if $u_{-3} = u_{-1}, v_{-3} = v_{-1}$ *and* $u_{-3}v_{-3} = u_{-2}v_{-2} = \frac{(1-A)}{B}$, *here* $A = C \neq -1, B = D \neq -1$ *and also* $B \neq 0,\ D \neq 0$.

Proof. Let $u_{-3} = u_{-1}, v_{-3} = v_{-1}$ and $u_{-3}v_{-3} = u_{-2}v_{-2} = \frac{(1-A)}{B}$, where $A = C \neq -1, B = D \neq -1$ and also $B \neq 0,\ D \neq 0$ in the exact equation (73a). Then

$$u_{4n-2} = u_{-2} \prod_{m=0}^{n-1} \frac{C^{2m} + D\left(\frac{1-C}{D}\right) \sum\limits_{l=0}^{2m-1} C^l}{A^{2m+1} + B\left(\frac{1-A}{B}\right) \sum\limits_{l=0}^{2m} A^l} = u_{-2} \prod_{m=0}^{n-1} \frac{C^{2m} + (1-C) \sum\limits_{l=0}^{2m-1} C^l}{A^{2m+1} + (1-A) \sum\limits_{l=0}^{2m} A^l}. \tag{75}$$

We have $\frac{1-A}{B} = \frac{1-C}{D}$ since $A = C$ and $B = D$. So,

$$\begin{aligned} u_{4n-2} =& u_{-2} \prod_{m=0}^{n-1} \frac{A^{2m} + (1-A) \sum_{l=0}^{2m-1} A^l}{A^{2m+1} + (1-A) \sum_{l=0}^{2m} A^l} \\ =& u_{-2}. \end{aligned} \tag{76}$$

Following the same procedure as above on equations (73b) through (73h) gives $u_{4n-1} = u_{-1}; u_{4n} = u_0; u_{4n+1} = u_1$ and $v_{4n-1} = v_{-1}; v_{4n} = v_0; v_{4n+1} = v_1$.

□

Some Numerical Examples

Below are 2 graphs that illustrate Theorem 2, Figure 1 is when we let $A = C = 2,\ B = D = -0.125, u_{-3} = u_{-1} = 4, u_{-2} = u_0 = 1, v_{-3} = v_{-1} = 2$ and $v_{-2} = v_0 = 8$ into the system of equations (74). In Figure 2 we let $A = B = C = D = 1/13$ and use different values for u_n and v_n in the difference equations (74) as follows: $u_{-3} = u_{-1} = 3, u_{-2} = u_0 = 6, v_{-3} = v_{-1} = 4$ and $v_{-2} = v_0 = 2$. On both figures, the solid line represents u_n and the dash dotted line represents v_n.

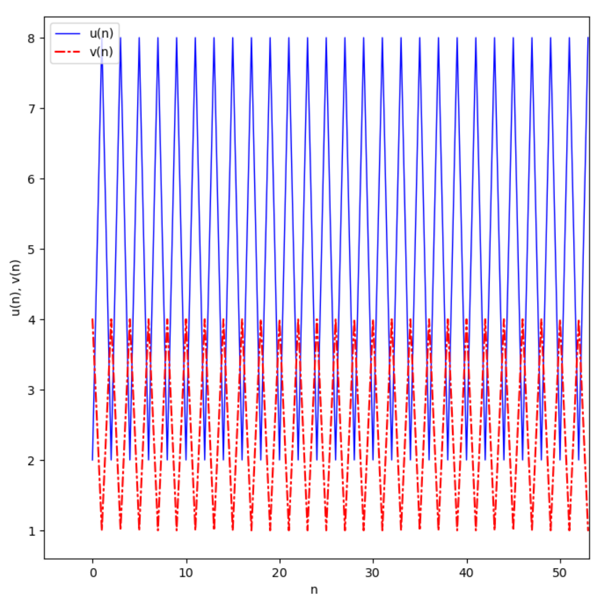

Figure 1

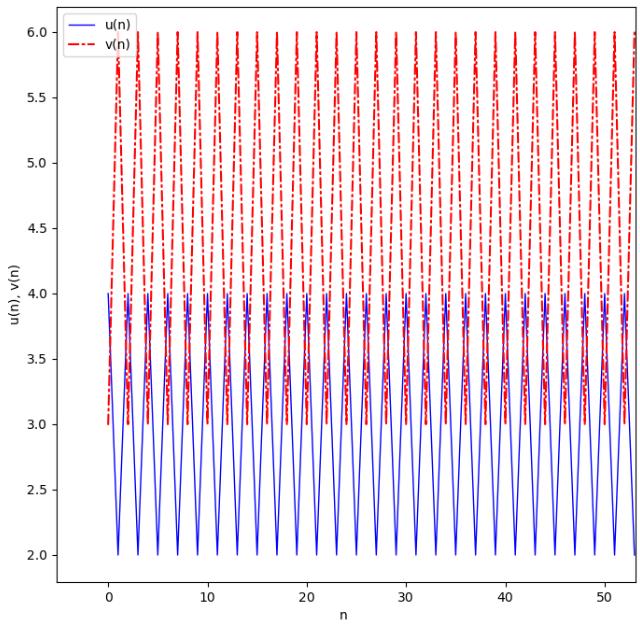

Figure 2

3.3.1 The case when $k = 2, A = B = C = 1$ and $D = -1$

Here we use (73), the solution is as follows:

$$
\begin{aligned}
u_{4n-2} =& u_{-2} \prod_{m=0}^{n-1} \frac{u_0 v_{-2}}{u_{-2} v_0} \frac{(1)^{2m} - u_{-2} v_0 \sum_{i=1}^{2m} (1)^i}{(1)^{2m+1} + u_0 v_{-2} \sum_{i=1}^{2m+1} (1)^i} \\
=& u_{-2} \prod_{m=0}^{n-1} \frac{(-((2m) u_{-2} v_0 - 1))}{1 + (2m+1) u_0 v_{-2}} \times \frac{u_0 v_{-2}}{u_{-2} v_0} \\
=& \frac{(-1)^n (u_0)^n (v_{-2})^n \prod_{m=0}^{n-1} [(2m) u_{-2} v_0 - 1]}{(u_{-2})^{n-1} (v_0)^n \prod_{m=0}^{n-1} [(2m+1) u_0 v_{-2} + 1]}.
\end{aligned} \tag{77a}
$$

Just as above, we similarly obtain the following:

$$u_{4n-1} = \frac{(-1)^n (u_{-1})^{n+1} (v_{-3})^n \prod_{m=0}^{n-1} [(2m+1)u_{-3}v_{-1} - 1]}{(u_{-3})^n (v_{-1})^n \prod_{m=0}^{n-1} [(2m+2)u_{-1}v_{-3} + 1]}, \tag{77b}$$

$$u_{4n} = \frac{(-1)^n (u_0)^{n+1} (v_{-2})^n \prod_{m=0}^{n-1} [(2m+1)u_{-2}v_0 - 1]}{(u_{-2})^n (v_0)^n \prod_{m=0}^{n-1} [(2m+2)u_0v_{-2} + 1]}, \tag{77c}$$

$$u_{4n+1} = \frac{(-1)^n (u_1)^{n+1} (v_{-1})^n \prod_{m=0}^{n-1} [(2m+1)u_{-1}v_1 - 1]}{(u_{-1})^n (v_1)^n \prod_{m=0}^{n-1} [(2m+2)u_1v_{-1} + 1]}, \tag{77d}$$

$$v_{4n-2} = \frac{(-1)^n (u_{-2})^n (v_0)^n \prod_{m=0}^{n-1} [(2m)u_0v_{-2} + 1]}{(u_0)^n (v_{-2})^n \prod_{m=0}^{n-1} [(2m+1)u_{-2}v_0 - 1]}, \tag{77e}$$

$$v_{4n-1} = \frac{(-1)^n (u_{-3})^n (v_{-1})^{n+1} \prod_{m=0}^{n-1} [(2m+1)u_{-1}v_{-3} + 1]}{(u_{-1})^n (v_{-3})^n \prod_{m=0}^{n-1} [(2m+2)u_{-3}v_{-1} - 1]}, \tag{77f}$$

$$v_{4n} = \frac{(-1)^n (u_{-2})^n (v_0)^{n+1} \prod_{m=0}^{n-1} [(2m+1)u_0v_{-2} + 1]}{(u_0)^n (v_{-2})^n \prod_{m=0}^{n-1} [(2m+2)u_{-2}v_0 - 1]}, \tag{77g}$$

$$v_{4n+1} = \frac{(-1)^n (u_{-1})^n (v_1)^{n+1} \prod_{m=0}^{n-1} [(2m+1)u_1v_{-1} + 1]}{(u_1)^n (v_{-1})^n \prod_{m=0}^{n-1} [(2m+2)u_{-1}v_1 - 1]}, \tag{77h}$$

Figure 3 below is obtained by letting the initial conditions to be $u_{-3} = 0.6,\ u_{-2} = 2,\ u_{-1} = 0.4,\ u_0 = 5,\ v_{-3} = 4,\ v_{-2} = -2,\ v_{-1} = 6$ and $v_0 = -1$ in the following difference system

$$\begin{aligned} u_{n+1} =& \frac{u_{n-1}v_{n-3}}{v_{n-1}\left(1 + u_{n-1}v_{n-3}\right)}, \\ v_{n+1} =& \frac{u_{n-3}v_{n-1}}{u_{n-1}\left(1 - u_{n-3}v_{n-1}\right)}. \end{aligned} \tag{78}$$

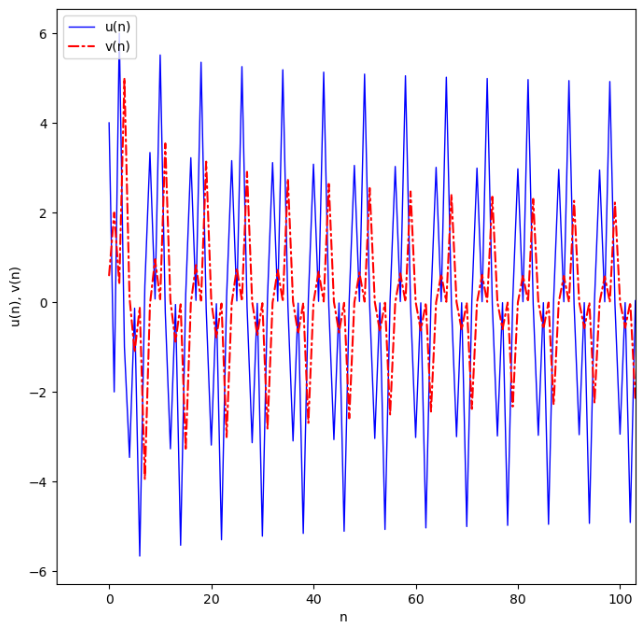

Figure 3

3.3.2 The case when $k = 2, A = B = C = 1$ and $D = 1$

If we let $A = B = C = D = 1$, then the equations in (73) reduce to Theorem 1 in [7] as shown below:

$$u_{4n-2} = u_{-2} \prod_{m=0}^{n-1} \frac{u_0 v_{-2}}{u_{-2} v_0} \frac{(1)^{2m} + (1)u_{-2}v_0 \sum_{i=1}^{2m} (1)^{i+1}}{(1)^{2m+1} + (1)u_0 v_{-2} \sum_{i=1}^{2m+1} (1)^{i+1}}. \tag{79}$$

In the above equation, we performed index shifting. Now,

$$u_{4n-2} = u_{-2}\prod_{m=0}^{n-1}\frac{u_0v_{-2}}{u_{-2}v_0}\frac{1+(2m)u_{-2}v_0}{1+(2m+1)u_0v_{-2}}$$

$$= u_{-2}\prod_{m=0}^{n-1}\frac{1+(2m)u_{-2}v_0}{1+(2m+1)u_0v_{-2}} \times \frac{u_0v_{-2}}{u_{-2}v_0}.$$

Thus, (73a) simplifies to

$$u_{4n-2} = \frac{(u_0)^n(v_{-2})^n\prod\limits_{m=0}^{n-1}[(2m)u_{-2}v_0+1]}{(u_{-2})^{n-1}(v_0)^n\prod\limits_{m=0}^{n-1}[(2m+1)u_0v_{-2}+1]}. \tag{80a}$$

Following the same procedure as above, equations (73b) to (73h), respectively, result to the following equations:

$$u_{4n-1} = \frac{(u_{-1})^{n+1}(v_{-3})^n\prod\limits_{m=0}^{n-1}[(2m+1)u_{-3}v_{-1}+1]}{(u_{-3})^n(v_{-1})^n\prod\limits_{m=0}^{n-1}[(2m+2)u_{-1}v_{-3}+1]}, \tag{80b}$$

$$u_{4n} = \frac{(u_0)^{n+1}(v_{-2})^n\prod\limits_{m=0}^{n-1}[(2m+1)u_{-2}v_0+1]}{(u_{-2})^n(v_0)^n\prod\limits_{m=0}^{n-1}[(2m+2)u_0v_{-2}+1]}, \tag{80c}$$

$$u_{4n+1} = \frac{(u_{-1})^{n+1}(v_{-3})^n\prod\limits_{m=0}^{n-1}[(2m+1)u_{-3}v_{-1}+1]}{(u_{-3})^n(v_{-1})^n\prod\limits_{m=0}^{n-1}[(2m+2)u_{-1}v_{-3}+1]}, \tag{80d}$$

$$v_{4n-2} = \frac{(u_{-2})^n(v_0)^n\prod\limits_{m=0}^{n-1}[(2m)u_0v_{-2}+1]}{(u_0)(v_{-2})^{n-1}\prod\limits_{m=0}^{n-1}[(2m+1)u_{-2}v_0+1]}, \tag{80e}$$

$$v_{4n-1} = \frac{(u_{-3})^n(v_{-1})^{n+1}\prod\limits_{m=0}^{n-1}[(2m+1)u_{-1}v_{-3}+1]}{(u_{-1})^n(v_{-3})^n\prod\limits_{m=0}^{n-1}[(2m+2)u_{-3}v_{-1}+1]}, \tag{80f}$$

$$v_{4n} = \frac{(u_{-2})^n (v_0)^{n+1} \prod_{m=0}^{n-1} [(2m+1)u_0 v_{-2} + 1]}{(u_0)^n (v_{-2})^n \prod_{m=0}^{n-1} [(2m+2)u_{-2} v_0 + 1]}, \tag{80g}$$

$$v_{4n+1} = \frac{(u_{-3})^n (v_{-1})^{n+1} \prod_{m=0}^{n-1} [(2m+1)u_{-1} v_{-3} + 1]}{(u_{-1})^n (v_{-3})^n \prod_{m=0}^{n-1} [(2m+2)u_{-3} v_{-1} + 1]}. \tag{80h}$$

Thus from the equations in (80), we can see that equations in (73) are reduced to Theorem 1 in [7] when $A = B = C = D = 1$. Figure 4 below is an illustration of when $u_{-3} = u_{-1} = 8,\ u_{-2} = u_0 = 2,\ v_{-3} = v_{-1} = 4$ and $v_{-2} = v_0 = 1$ are the initial conditions in

$$\begin{aligned} u_{n+1} =& \frac{u_{n-1} v_{n-3}}{v_{n-1}\left(1 + u_{n-1} v_{n-3}\right)}, \\ v_{n+1} =& \frac{u_{n-3} v_{n-1}}{u_{n-1}\left(1 + u_{n-3} v_{n-1}\right)}. \end{aligned} \tag{81}$$

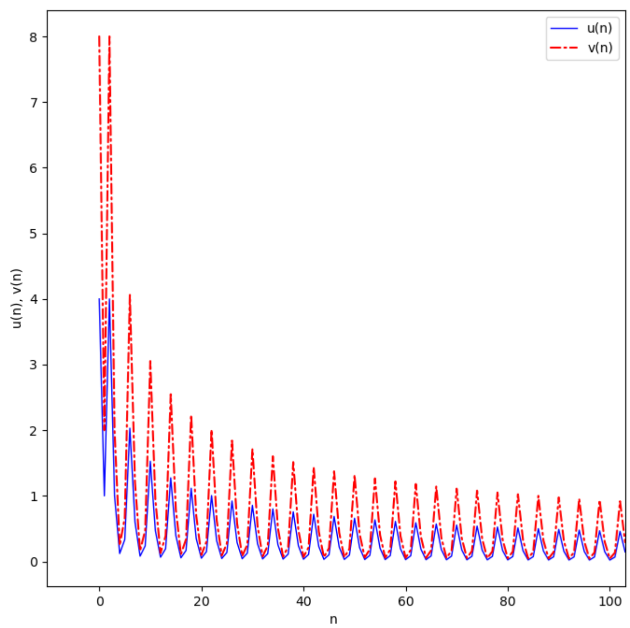

Figure 4

Both of the examples below show us some interesting numerical examples for $k = 2$ in the OΔEs (25).

Example 1. *Here we define the initial conditions of the difference system*

$$u_{n+1} = \frac{u_{n-1}v_{n-3}}{v_{n-1}(1 - u_{n-1}v_{n-3})} \quad \text{and} \quad v_{n+1} = \frac{u_{n-3}v_{n-1}}{u_{n-1}(1 + u_{n-3}v_{n-1})}$$

as $u_{-3} = -3$, $u_{-2} = -0.2$, $u_{-1} = 5$, $u_0 = -1$, $v_{-3} = -3$, $v_{-2} = 3$, $v_{-1} = 1$ *and* $v_0 = -4$, *see Figure 5 for the resulting plot.*

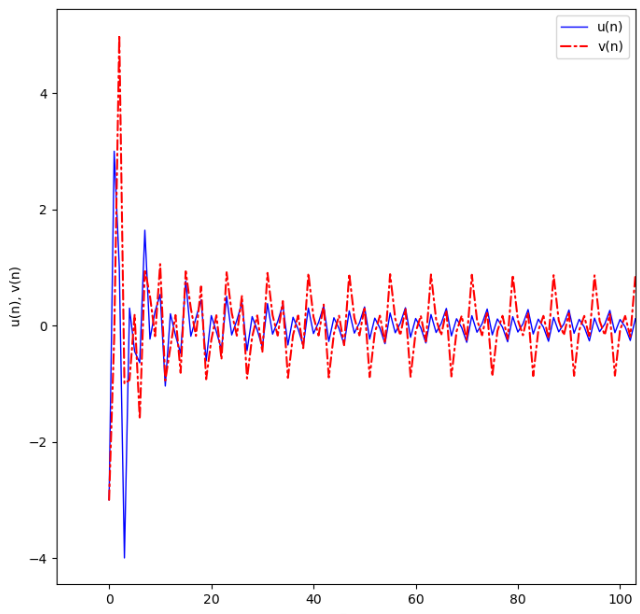

Figure 5

Example 2. *We define the initial conditions* $u_{-3} = 0.32$, $u_{-2} = 0.15$, $u_{-1} = 0.65$, $u_0 = 0.07$, $v_{-3} = 0.54$, $v_{-2} = 0.37$, $v_{-1} = 0.06$ *and* $v_0 = 0.51$ *in the difference system*

$$u_{n+1} = \frac{u_{n-1}v_{n-3}}{v_{n-1}(1 - u_{n-1}v_{n-3})} \quad \text{and} \quad v_{n+1} = \frac{u_{n-3}v_{n-1}}{u_{n-1}(1 - u_{n-3}v_{n-1})}$$

to get the graph in Figure 6.

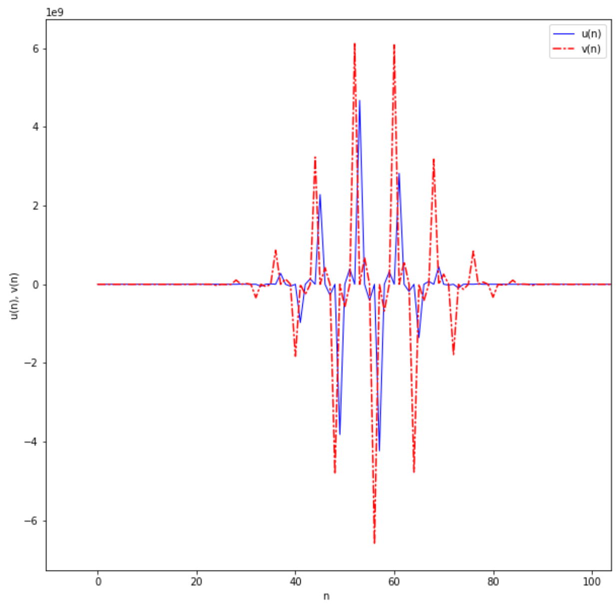

Figure 6

3.4 A Detailed Study of the Case $k = 3$

Substituting $k = 3$ into (71), leads to the following

$$u_{6n+j-3} = u_{j-3} \prod_{m=0}^{n-1} \frac{u_{\tau(3+j)} v_{\tau(3+j)-3}}{u_{\tau(j)-3} v_{\tau(j)}} \frac{C^{3m+\lfloor \frac{j}{2} \rfloor} + D u_{\tau(j)+(-3)} v_{\tau(j)} \sum_{l=0}^{3m+\lfloor \frac{j}{2} \rfloor -1} C^l}{A^{3m+\lfloor \frac{3+j}{2} \rfloor} + B u_{\tau(3+j)} v_{\tau(3+j)+(-3)} \sum_{l=0}^{3m+\lfloor \frac{3+j}{2} \rfloor -1} A^l}, \tag{82a}$$

and

$$v_{6n+j-3} = v_{j-3} \prod_{m=0}^{n-1} \frac{u_{\tau(3+j)-3} v_{\tau(3+j)}}{u_{\tau(j)} v_{\tau(j)-3}} \frac{A^{3m+\lfloor \frac{j}{2} \rfloor} + B u_{\tau(j)} v_{\tau(j)+(-3)} \sum_{l=0}^{3m+\lfloor \frac{j}{2} \rfloor -1} A^l}{C^{3m+\lfloor \frac{3+j}{2} \rfloor} + D u_{\tau(3+j)+(-3)} v_{\tau(3+j)} \sum_{l=0}^{3m+\lfloor \frac{3+j}{2} \rfloor -1} C^l}. \tag{82b}$$

where $j = 0, 1, 2, 3, 4, 5$.
More explicitly

$$u_{6n-3} = u_{-3} \prod_{m=0}^{n-1} \frac{u_0 v_{-3}}{u_{-3} v_0} \frac{C^{3m} + D u_{-3} v_0 \sum\limits_{l=0}^{3m-1} C^l}{A^{3m+1} + B u_0 v_{-3} \sum\limits_{l=0}^{3m} A^l}, \tag{83a}$$

$$u_{6n-2} = u_{-2} \prod_{m=0}^{n-1} \frac{u_1 v_{-2}}{u_{-2} v_1} \frac{C^{3m} + D u_{-2} v_1 \sum\limits_{l=0}^{3m-1} C^l}{A^{3m+2} + B u_1 v_{-2} \sum\limits_{l=0}^{3m+1} A^l}, \tag{83b}$$

$$u_{6n-1} = u_{-1} \prod_{m=0}^{n-1} \frac{u_2 v_{-1}}{u_{-1} v_2} \frac{C^{3m+1} + D u_{-1} v_2 \sum\limits_{l=0}^{3m} C^l}{A^{3m+2} + B u_2 v_{-1} \sum\limits_{l=0}^{3m+1} A^l}, \tag{83c}$$

$$u_{6n} = u_0 \prod_{m=0}^{n-1} \frac{u_0 v_{-3}}{u_{-3} v_0} \frac{C^{3m+1} + D u_{-3} v_0 \sum\limits_{l=0}^{3m} C^l}{A^{3m+3} + B u_0 v_{-3} \sum\limits_{l=0}^{3m+2} A^l}, \tag{83d}$$

$$u_{6n+1} = u_1 \prod_{m=0}^{n-1} \frac{u_1 v_{-2}}{u_{-2} v_1} \frac{C^{3m+2} + D u_{-2} v_1 \sum\limits_{l=0}^{3m+1} C^l}{A^{3m+3} + B u_1 v_{-2} \sum\limits_{l=0}^{3m+2} A^l}, \tag{83e}$$

$$u_{6n+2} = u_2 \prod_{m=0}^{n-1} \frac{u_2 v_{-1}}{u_{-1} v_2} \frac{C^{3m+2} + D u_{-1} v_2 \sum\limits_{l=0}^{3m+1} C^l}{A^{3m+4} + B u_2 v_{-1} \sum\limits_{l=0}^{3m+3} A^l}, \tag{83f}$$

$$v_{6n-3} = v_{-3} \prod_{m=0}^{n-1} \frac{u_{-3} v_0}{u_0 v_{-3}} \frac{A^{3m} + B u_0 v_{-3} \sum\limits_{l=0}^{3m-1} A^l}{C^{3m+1} + D u_{-3} v_0 \sum\limits_{l=0}^{3m} C^l}, \tag{83g}$$

$$v_{6n-2} = v_{-2} \prod_{m=0}^{n-1} \frac{u_{-2} v_1}{u_1 v_{-2}} \frac{A^{3m} + B u_1 v_{-2} \sum\limits_{l=0}^{3m-1} A^l}{C^{3m+2} + D u_{-2} v_1 \sum\limits_{l=0}^{3m+1} C^l}, \tag{83h}$$

$$v_{6n-1} = v_{-1} \prod_{m=0}^{n-1} \frac{u_{-1}v_2}{u_2v_{-1}} \frac{A^{3m+1} + Bu_2v_{-1} \sum_{l=0}^{3m} A^l}{C^{3m+2} + Du_{-1}v_2 \sum_{l=0}^{3m+1} C^l}, \tag{83i}$$

$$v_{6n} = v_0 \prod_{m=0}^{n-1} \frac{u_{-3}v_0}{u_0v_{-3}} \frac{A^{3m+1} + Bu_0v_{-3} \sum_{l=0}^{3m} A^l}{C^{3m+3} + Du_{-3}v_0 \sum_{l=0}^{3m+2} C^l}, \tag{83j}$$

$$v_{6n+1} = v_1 \prod_{m=0}^{n-1} \frac{u_{-2}v_1}{u_1v_{-2}} \frac{A^{3m+2} + Bu_1v_{-2} \sum_{l=0}^{3m+1} A^l}{C^{3m+3} + Du_{-2}v_1 \sum_{l=0}^{3m+2} C^l}, \tag{83k}$$

$$v_{6n+2} = v_2 \prod_{m=0}^{n-1} \frac{u_{-1}v_2}{u_2v_{-1}} \frac{A^{3m+2} + Bu_2v_{-1} \sum_{l=0}^{3m+1} A^l}{C^{3m+4} + Du_{-1}v_2 \sum_{l=0}^{3m+3} C^l}. \tag{83l}$$

3.4.1 The case when $k = 3, A = B = C = 1$ and $D = -1$

When using (83a), we get the following solution

$$\begin{aligned} u_{6n-3} =& u_{-3} \prod_{m=0}^{n-1} \frac{u_0v_{-3}}{u_{-3}v_0} \frac{(1)^{3m} - u_{-3}v_0 \sum_{l=0}^{3m-1} (1)^l}{(1)^{3m+1} + u_0v_{-3} \sum_{l=0}^{3m} (1)^l} \\ =& u_{-3} \prod_{m=0}^{n-1} \frac{1 - (3m)u_{-3}v_0}{1 + (3m+1)u_0v_{-3}} \times \frac{u_0v_{-3}}{u_{-3}v_0} \end{aligned}$$

$$u_{6n-3} = \frac{(-1)^n (u_0)^n (v_{-3})^n \prod_{m=0}^{n-1} [(3m)u_{-3}v_0 - 1]}{(u_{-3})^{n-1} (v_0)^n \prod_{m=0}^{n-1} [(3m+1)u_0v_{-3} + 1]}. \tag{84a}$$

We obtain the equations below when following the same procedure as above on the equations (83b) through (83h).

$$u_{6n-2} = \frac{(-1)^n (u_1)^n (v_{-2})^n \prod_{m=0}^{n-1} [(3m)u_{-2}v_1 - 1]}{(u_{-2})^{n-1}(v_1)^n \prod_{m=0}^{n-1} [(3m+2)u_1 v_{-2} + 1]}, \tag{84b}$$

$$u_{6n-1} = \frac{(-1)^n (u_2)^n (v_{-1})^n \prod_{m=0}^{n-1} [(3m+1)u_{-1}v_2 - 1]}{(u_{-1})^{n-1}(v_2)^n \prod_{m=0}^{n-1} [(3m+2)u_2 v_{-1} + 1]}, \tag{84c}$$

$$u_{6n} = \frac{(-1)^n (u_0)^{n+1} (v_{-3})^n \prod_{m=0}^{n-1} [(3m+1)u_{-3}v_0 - 1]}{(u_{-3})^{n}(v_0)^n \prod_{m=0}^{n-1} [(3m+3)u_0 v_{-3} + 1]}, \tag{84d}$$

$$u_{6n+1} = \frac{(-1)^n (u_1)^{n+1} (v_{-2})^n \prod_{m=0}^{n-1} [(3m+2)u_{-2}v_1 - 1]}{(u_{-2})^{n}(v_1)^n \prod_{m=0}^{n-1} [(3m+3)u_1 v_{-2} + 1]}, \tag{84e}$$

$$u_{6n+2} = \frac{(-1)^n (u_2)^{n+1} (v_{-1})^n \prod_{m=0}^{n-1} [(3m+2)u_{-1}v_2 - 1]}{(u_{-1})^{n}(v_2)^n \prod_{m=0}^{n-1} [(3m+4)u_2 v_{-1} + 1]}, \tag{84f}$$

$$v_{6n-3} = \frac{(u_{-3})^n (v_0)^n \prod_{m=0}^{n-1} [(3m)u_0 v_{-3} + 1]}{(-1)^n (u_0)^n (v_{-3})^{n-1} \prod_{m=0}^{n-1} [(3m+1)u_{-3}v_0 - 1]}, \tag{84g}$$

$$v_{6n-2} = \frac{(u_{-2})^n (v_1)^n \prod_{m=0}^{n-1} [(3m)u_1 v_{-2} + 1]}{(-1)^n (u_1)^n (v_{-2})^{n-1} \prod_{m=0}^{n-1} [(3m+2)u_{-2}v_1 - 1]}, \tag{84h}$$

$$v_{6n-1} = \frac{(u_{-1})^n (v_2)^n \prod_{m=0}^{n-1} [(3m+1)u_2 v_{-1} + 1]}{(-1)^n (u_2)^n (v_{-1})^{n-1} \prod_{m=0}^{n-1} [(3m+2)u_{-1}v_2 - 1]} \tag{84i}$$

$$v_{6n} = \frac{(u_{-3})^n (v_0)^{n+1} \prod_{m=0}^{n-1} [(3m+1)u_0 v_{-3} + 1]}{(-1)^n (u_0)^n (v_{-3})^n \prod_{m=0}^{n-1} [(3m+3)u_{-3} v_0 - 1]}, \tag{84j}$$

$$v_{6n+1} = \frac{(u_{-2})^n (v_1)^{n+1} \prod_{m=0}^{n-1} [(3m+2)u_1 v_{-2} + 1]}{(-1)^n (u_1)^n (v_{-2})^n \prod_{m=0}^{n-1} [(3m+3)u_{-2} v_1 - 1]}, \tag{84k}$$

$$v_{6n+2} = \frac{(u_{-1})^n (v_2)^{n+1} \prod_{m=0}^{n-1} [(3m+2)u_2 v_{-1} + 1]}{(-1)^n (u_2)^n (v_{-1})^n \prod_{m=0}^{n-1} [(3m+4)u_{-1} v_2 - 1]}. \tag{84l}$$

Some Numerical Examples

When $k = 3$ in (25), the resulting OΔEs are

$$u_{n+1} = \frac{u_{n-1} v_{n-4}}{v_{n-2}(a + b u_{n-1} v_{n-4})} \quad \text{and} \quad v_{n+1} = \frac{u_{n-4} v_{n-1}}{u_{n-2}(c + d u_{n-4} v_{n-1})}. \tag{85}$$

Example 3. *We get the illustration below (see Figure 7) when we let* $a = c = -1,\ b = d = 1,\ u_{-4} = 7,\ u_{-3} = 5,\ u_{-2} = -0.0971,\ u_{-1} = 7,\ u_0 = 1,\ v_{-4} = 3,\ v_{-3} = 9,\ v_{-2} = 0.01,\ v_{-1} = -3$ *and* $v_0 = 2$ *in* (85).

Example 4. *Setting* $a = c = 1,\ b = d = -1,\ u_{-4} = 0.88,\ u_{-3} = 0.35,\ u_{-2} = 0.79,\ u_{-1} = 0.95,\ u_0 = 0.31,\ v_{-4} = 0.77,\ v_{-3} = 0.33,\ v_{-2} = 0.16,\ v_{-1} = 0.35$ *and* $v_0 = 0.75$ *in* (85) *yields Figure 8.*

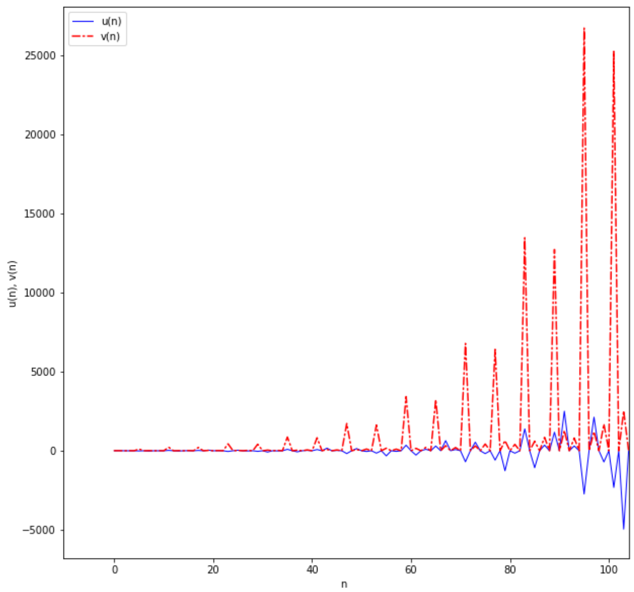

Figure 7

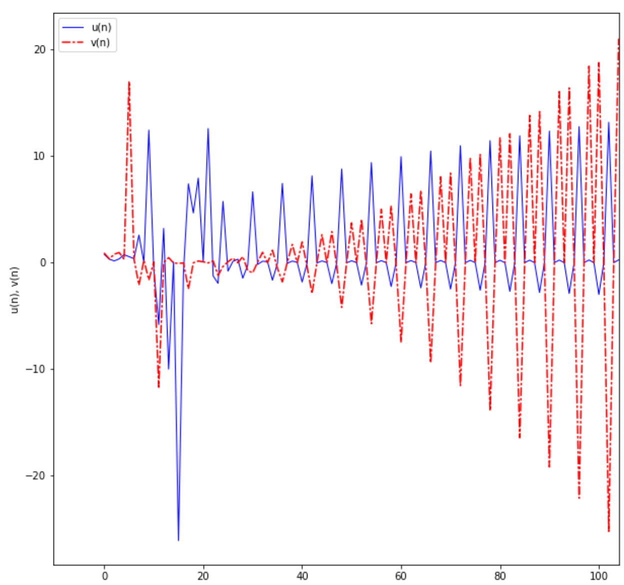

Figure 8

Chapter 4

Conclusion

In this dissertation, we looked at a system of $(k+2)^{th}$ order ordinary difference equations. We investigated these equations by finding the symmetry generators (45a) and (45b). We then used the canonical coordinates to find the invariants in (50) of which led us to get the closed form solutions (52a) and (52b). Performing iterations resulted in (54), where we used the floor function definition and back-shifted the resulting equations to obtain the equations in (65). We performed a detailed study of the case where $k = 2$, the reason for this is that we wanted to show that the work done in the articles [7, 8] are special cases of our results. In fact, for different combinations of values of A, B, C and D which we get from the articles [7, 8], we obtain the following important results:

- Equations in (73) are reduced to equations in Theorem 2 in [7] when we set $A = B = 1$ and $C = D = -1$.
- Equations in (73) are reduced to equations in Theorem 3 in [7] when we set $A = B = D = 1$ and $C = -1$.
- Equations in (73) are reduced to equations in Theorem 4 in [7] when we set $A = B = C = 1$ and $D = -1$.
- Equations in (73) are reduced to equations in Theorem 1 in [8] when we set $A = C = 1$ and $B = D = -1$.
- Equations in (73) are reduced to equations in Theorem 2 in [8] when we set $A = C = -1$ and $B = D = 1$.
- Equations in (73) are reduced to equations in Theorem 3 in [8] when we set $A = D = 1$ and $B = C = -1$.
- Equations in (73) are reduced to Theorem 4 in [8] when we set $A = D = -1$ and $B = C = 1$.

We also stated and provided the proof for the existence of 2-periodic solutions. Graphs that give verification and confirmation to the theoretical discussions were included. Finally, we gave a detailed study of the case where $k = 3$, we included the numerical analysis with the illustrations of the plots.

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