University of the Witwatersrand,
Johannesburg

Higher derivative gravity black holes and the attractor mechanism in higher dimensions

Author:
James Junior Mashiyane

Supervisor:
Prof. Kevin Goldstein

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UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

Declaration

I, the undersigned, hereby declare that the work contained in this Ph.D thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. This thesis has not been submitted before for any degree or examination in any other university.

James Junior Mashiyane

February 18, 2020

Black hole solutions in higher-derivative Gravity and the hot attractor mechanism in higher dimensions

James Junior Mashiyane

Abstract

This Ph.D. thesis is made of two major parts, the investigation of black hole solutions in higher derivative gravity and the generalization of the attractor mechanism to hot black holes in arbitrary dimensions. The investigation of black hole solutions in higher derivative gravity was inspired by the possibility that the Schwarzschild black hole may not be the unique spherically symmetric vacuum solution to generalizations of general relativity. We consider black holes in pure fourth order higher derivative gravity treated as an effective theory where the Lagrangian is truncated to terms which are second order powers of the curvature. These higher order terms are of interest to us because they arise in effective descriptions of gravity and proposed theories of quantum gravity like string theory [1]. The solutions of this theory are called non-Schwarzschild black hole solutions, and they have been studied before [1] but we have put earlier results on a firmer footing by finding a systematic asymptotic expansion for the black holes and matching them with known numerical solutions obtained by integrating out from the near horizon region. Such solutions may be of interest in addressing the issue of higher derivative hair or during the later stages of black hole evaporation. These asymptotic expansions can be cast in the form of trans-series expansions which we conjecture will be a generic feature of non-Schwarzschild higher derivative black holes. Although we find a new branch of solutions, as found in earlier work [1], solutions only seem to exist for black holes with large curvatures meaning that one should not generically neglect even higher derivative corrections. This suggests that one effectively recovers the no-hair theorems in this context. We also study these black hole solutions by looking at their structure, and their properties such as their mass, temperature, and entropy.

In the second part of the thesis, we extend the previous work of [2] on hot black hole attractors, by considering higher dimensions and the presence of a scalar potential (which is relevant for asymptotically AdS spaces and gauged supergravity). While, earlier work [2, 3] showed that the product of horizon areas is the geometric mean of the extremal area – we find that in the presence of a scalar potential this property no longer holds. When one complexifies the radial coordinate the non-extremal black hole solutions result in other horizons in addition to the classical event horizon of a black hole and the Cauchy horizons. In this work, we generalize the attractor mechanism for extremal black holes by proving that the product of all black hole horizon areas is independent of variations of the asymptotic moduli [4]. By adding the scalar potential we find that the product of horizon areas is not the geometric mean of the extremal area, which typically is true in gauged supergravity. Finally, we sketch the derivation of horizon invariants for stationary backgrounds.

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Chapter 1

Introduction

In this thesis, we investigate two phenomena, higher derivative theories of gravity and the generalized attractor mechanism in the presence of a scalar potential.

In Einstein gravity, we know that Birkhoff's theorem [5, 6, 7, 8] implies that the unique spherically symmetric, asymptotically flat, vacuum solution of general relativity is the Schwarzschild black hole. Despite apparently having a large entropy, this black hole is completely characterized by just one parameter, namely its mass. Essentially taking $R_{\mu\nu} = 0$; together with spherical symmetry and asymptotic flatness; gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ . \tag{1.1}$$

Recently, it was shown in [1] that in fourth order gravity, the stronger assumption of a static spherically asymptotically flat solution leads to a weaker restriction that the Ricci scalar, R, but *not* the Ricci tensor, $R_{\mu\nu}$, is zero[1]. This, in turn, opens up the exciting possibility that there are so-called non-Schwarzschild black holes that we investigate. Such solutions are interesting since higher derivative terms generically arise when one attempts to find a UV description of gravity. We use the variational principle to derive the equations of motion from the action, and also use the fact that the Gauss-Bonnet action is topologically invariant in 4 dimensional spacetime [1]. It turns out that the equations of motion resulting from higher derivative gravity theories are very complicated to solve analytically because they are a system of four nonlinear, coupled differential equations with fourth order derivatives(which can be simplified to two second order derivatives in our case) and consequently we have to resort to numerical techniques.

For reasons, we will review later, the fourth order equations of motion are numerically unstable (or "stiff") and one can only integrate them for a while before they

[1]It is not hard to see that in general relativity vacuum solutions necessarily have $R = R_{\mu\nu} = 0$

diverge. In [1] it was found that by tuning the initial conditions near the horizon one can successively integrate out further and further approaching asymptotically flat space. This is convincing evidence for the existence of such black hole solutions [1].

The initial conditions are determined by using the shooting method and a polynomial expansion as in [1], where we start integrating very close to the event horizon. We also propose an asymptotic expansion of the solutions up to third order, then match the resulting asymptotic expansion and the numerical solutions. While it was known that these solutions were characterized by just two near horizon parameters, from the asymptotic form of the solution, we explicitly see that in addition to the mass, the solutions are characterized by the size of a massive Yukawa mode associated with the higher derivative terms. Armed with the asymptotic form of the solution we were also able to find a new branch of solutions that were previously missed.

We are interested in studying these solutions for two reasons. Firstly, the fact that Birkhoff's theorem is weakened once we add higher derivatives suggests the possibility of higher derivative hair which could play a role in understanding black hole entropy. Although one finds that these solutions are only characterized by one additional parameter, it does suggest that even higher order terms in the Lagrangian could lead to more hair. Secondly, as a Schwarzschild black hole shrinks by Hawking evaporation, treating gravity as an effective theory, one expects that at some length scale, higher derivative terms may become important. This means that non-Schwarzschild black holes may play some role in the late time evolution of black holes.

The laws of black hole thermodynamics[9, 10, 11] generalize once one adds higher derivative terms. In particular, the Bekenstein-Hawking entropy, given by

$$S_{\mathrm{BH}} = \tfrac{1}{4}A \ , \tag{1.2}$$

where A is the horizon area (in Plank units), is replaced by the Wald entropy which, for a spherical solution, is given by [12]

$$S_{\mathrm{Wald}} = -8\pi \int_{\mathrm{Horizon}} d\theta d\phi \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{tt}g_{rr}} \ , \tag{1.3}$$

with $\mathcal{S}$ the relevant action. We take the non-Schwarzschild black hole's entropy to given by the Wald entropy. It is worth mentioning some features of the thermodynamics of the non-Schwarzschild black holes. Unlike the Schwarzschild black hole, the horizon radius decreases as these black holes get colder. At a fixed horizon size, one has both a Schwarzschild and a non-Schwarzschild solution except for one critical point where the solutions coincide. Below this critical size, the non-Schwarzschild black hole is colder than the Schwarzschild solution, while above the critical size it is hotter than the Schwarzschild solution. We will call the two sets of solutions the cold and hot branches respectively[2].

Like the Schwarzschild black hole, their entropy decreases as one increases the temperature. However, on the cold branch, the black hole's entropy seems to approach a finite value as the temperature approaches zero. Bizarrely on the hot branch, as one increases the temperature, the entropy reaches zero at some point and then becomes negative as the temperature increases further [1]. We cannot see how this makes any physical sense. Finally, it is interesting to note that, at a fixed temperature, both branches have lower entropy than the Schwarzschild solution. Intriguingly, also at a fixed temperature, the cold branch has lower free energy, while the hot branch has higher free energy than the corresponding Schwarzschild solution. In addition to troubling thermodynamics, both branches of the non-Schwarzschild solutions are highly curved which would not generally be consistent with neglecting even higher order curvature terms in the action. One could insist on truncating the theory rather than treating it as an effective theory, but this will come at the cost of introducing ghosts [13]. These nonphysical excitations could be responsible for some of the strange thermodynamic features of the solutions. This suggests that the only physically reasonable black hole one can take is the Schwarzschild eliminating the prospect of higher derivative hair at the first non-trivial order.

On a technical note, the asymptotic expansion we find takes the form of trans-series. We conjecture that this will be a generic feature and may prove to be a useful tool for studying the asymptotic solutions of the higher derivative theories in general.

The second component of the thesis starts with a review of the black hole attractor mechanism followed by new work on hot attractors with the addition of a scalar potential and in higher dimensions. Although a variety of black holes exhibit the attractor mechanism, in this second component of the thesis we limit our investigation of the attractor mechanism to spherically symmetric black holes. Black holes in $\mathcal{N} = 2$ supersymmetric theories including spherically symmetric and extremal black holes with double-zero horizons are known to exhibit a phenomenon called the attractor mechanism [1]. In these solutions, several moduli fields are drawn to fixed values (completely determined by charges carried by the black hole charges) at the horizon regardless of the values they have at asymptotic infinity. This section aims to review the attractor mechanism for non-supersymmetric black hole solutions [14]. The investigation of the attractor mechanism for non-supersymmetric black holes might help in furthering the study of these black holes and their entropy [4].

To obtain the attractor mechanism, the effective potential, V_{eff}, needs to satisfy

[2]Only the hot branch was found in [1]

certain conditions. The function of the moduli fields V_{eff} must have a critical point, $\partial_i V_{eff}(\phi_{i0}) = 0$, and the matrix of second derivatives of the effective potential at the critical point, $\partial_{ij} V_{eff}(\phi_{i0})$, must have only positive eigenvalues, where the moduli fields ϕ_{i0} are the critical values. The entropy of the black hole is proportional to $V(\phi_{i0})$ and therefore, independent of the asymptotic values of the moduli fields. At the critical point of the effective potential, it is possible for the scalars to be constant throughout space-time leading to Reissner-Nordström like solutions. We review a perturbative analysis of black hole attractors around the constant scalar solution. This analysis gave rise to second order linear differential equations which are not difficult to solve, and thus the attractor mechanism can be generalized for any effective potential function as long as the attractor conditions are met.

The equations of the attractor are also solved numerically, with specific forms of the effective potential, but this analysis goes beyond perturbative analysis. Special cases were are also explored, where the equations of motion can be solved exactly, by mapping them to a solvable Toda system. It was found that the attractor phenomenon continues to hold in AdS and in higher dimensions.

Embedded in string theory is the generalized attractor mechanism in $\mathcal{N} = 2$ supergravity and which provides a bridge between extremal and non-extremal black hole solutions due to the existence of a smooth zero temperature limit see [2, 3]. As we explained previously the attractor mechanism shows that the horizon area and thus the entropy is independent of background moduli, and the asymptotic fluctuations due to scalar fields being attenuated by the existence of a AdS_2 throat near the horizon which also fixes the background moduli to its attractor values at the horizon. The attractor mechanism has to happen because as the moduli at infinity can be varied continuously, the entropy (which is a function of quantized charges) jumps discretely.

The on-shell values of the scalar fields at the horizon are fixed by black hole charges by ensuring that the inner throat asymptotic region is decoupled. The attractor mechanism only depends on extremality and not on supersymmetry [2, 4]. Unlike extremal black holes, non-extremal black holes can have two or more horizons, the inner horizon(which is the Cauchy horizon) and outer horizons(which is the event horizon if there are only two horizons). In the case of only two horizons, in $\mathcal{N} = 2$ supergravity the scalar field values at the horizons depend on background moduli at infinity. The generalized attractor mechanism involves invariants constructed from data on multiple horizons and averaging the equations of motion between horizons. In the simplest cases one finds the geometric mean of the areas of the inner horizon and the outer horizon is equal to the area of the extremal horizon:

$$A_+ A_- = A_{ext}^2, \tag{1.4}$$

which was initially observed phenomenologically [15, 16, 17]. Since the right-hand side is the area of an extremal black hole, then it only depends on quantized charges, and thus there exists an underlying microscopic explanation of the moduli independent of the horizon products [16]. By proving equation (1.4), the attractor mechanism is established and consequently generalizes the extremal attractor mechanism to non-extremal black holes. The quantities that vanish at the horizon on extremal black holes due to the attractor mechanism, now vanish when they are integrated between the inner and outer horizon ($r_- \to r_+$). This suggests that the inner horizon, which is decoupled from infinity, may explicate the entropy for these non-extremal black-holes [2]. It was shown in [18] that using the Harrison transformation, then fixing the scalar fields to their attractor values, non-extremal Reissner-Nordström black holes can be lifted to BTZ black holes in $AdS_3 \times S^2$. If one takes a zero temperature limit, a mass gap develops and the $SL(2, \mathcal{R})$ symmetry in the moduli background of the scalar fields near their attractor values becomes the AdS_2 symmetry in spacetime [4].

A summary of the rest of the thesis is outlined below.

In chapter 2 we derive the equations of motion of higher derivative gravity theories using the variational principle, these equations of motion are conformally invariant and contain a Bach tensor term which is trace-less and divergence-less. In chapter 3 we linearize the equations of motion around a flat background, to understand the behavior of the black hole at large values of the radial coordinate r. Linearizing the equations of motion demonstrates the presence of a massive spin-0 and a massive spin-2 Yukawa modes in addition to an expected massless spin-2 mode corresponding to the graviton [19]. In chapter 4 we numerically demonstrate the existence of two families of non-Schwarzschild solutions, which we dub the hot and cold branches, using MATHEMATICA. In chapter 5 we investigate the thermodynamics of these black hole families. In chapter 6 we derive a trans-series asymptotic expansion, up to third order, for the non-Schwarzschild solutions and show that as r increases, these solutions approach the Schwarzschild black hole. In chapter 7 we discuss numerically matching a near-horizon polynomial expansion with our large r asymptotic solution. The results of the investigations on chapter 4-6 are based on our paper [20]. In chapter 8 we review the hot attractor mechanism in non-supersymmetric black hole solutions in four dimensions by summarizing the paper [14]. In chapter 9 we generalize the hot attractor by extending it to non-extremal black holes in arbitrary dimensions and with an addition of a scalar potential and the results obtained are largely based on the paper [4]. As stated before, chapters 2-7 form the first component of the thesis on higher derivative black hole solutions, and chapters 8-9 concludes the thesis with the investigation of the generalized hot attractor mechanism.

Chapter 2

Equations of motion for Weyl-Einstein gravity

2.1 Weyl-Einstein gravity equations of motion

The convention we use throughout the thesis is $\eta_{\mu\nu} = [-,+,+,+]$, and for now we work in 4-dimensional spacetime, meaning, μ and ν run from $0,1,2,3$. We start with the Einstein-Weyl action [1],

$$
\begin{aligned}
S &= \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2), \\
&= \frac{1}{8\pi L_P^2}\int d^4x\sqrt{-g}(R - L_P^2\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + L_P^2\beta R^2)
\end{aligned}
\tag{2.1}
$$

where $16\pi G_N = 1$, L_P is the Planck scale, α and β are constants with dimensions $[L^2]$ to be fixed as we proceed. Without loss in generality, we take all constants to be dimensionless,

$$
\begin{aligned}
C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\mu\sigma}R_{\nu\rho} + g_{\nu\rho}R_{\mu\sigma} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho}) \\
&\quad + \frac{1}{6}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu})R,
\end{aligned}
\tag{2.2}
$$

is the Weyl tensor in 4 dimensions, and R is the Ricci scalar. As shown in appendix A, the variation of the action with respect to the inverse metric gives the following equations of motion,

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 4\alpha B_{\mu\nu} + 2\beta R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) + 2\beta(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R) = 0, \tag{2.3}
$$

where we define $\Box = \nabla^2$ as the Laplacian in 4-dimensions, and $B_{\mu\nu}$ is the Bach tensor given by,

$$\begin{aligned} B_{\mu\nu} &= \left(\nabla^\rho\nabla^\sigma + \frac{1}{2}R^{\rho\sigma}\right)C_{\mu\rho\nu\sigma} \\ &= -\frac{1}{4}g_{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) - \frac{1}{6}\nabla_\mu\nabla_\nu R + \frac{1}{2}\Box R_{\mu\nu} \\ &- \frac{1}{12}g_{\mu\nu}\Box R + R_{\mu\rho}R^\rho_\nu - \frac{2}{6}RR_{\mu\nu}, \end{aligned} \tag{2.4}$$

which is traceless $g^{\mu\nu}B_{\mu\nu} = 0$, and divergence-less $\nabla^\mu B_{\mu\nu} = 0$ [21] (see appendix A). The first two terms of the equations of motion come from varying the Ricci scalar and one can see that by setting α and β to zero, then we would have the Einstein-Hilbert action that results in pure Einstein equations of motion in gravity. The second term of the equation, which is the Bach tensor, can be obtained from varying the Weyl squared term and the last two terms of the equations of motion come from varying the Ricci squared term. This theory governs a system with a massive spin-2 mode of mass $m_2^2 = \frac{1}{2\alpha}$, and a massive spin-0 mode with $m_0^2 = \frac{1}{6\beta}$, and a massless graviton [1], which will be shown in chapter 3. These massive modes are associated with the rising and falling Yukawa modes of the form $\frac{1}{r}e^{\pm m_0 r}$ and $\frac{1}{r}e^{\pm m_2 r}$ [1]. Taking $m_0, m_2 > 0$, then by taking the "positive" sign, as r increases, the solutions blow up [19]. We are interested in stable and converging black hole solutions thus we are going choose initial conditions that will stabilize these black hole solution. The aim of this research is to study the non-Schwarzschild black hole solutions. In the first component of the thesis, we follow [1] and focus only on spherically symmetric black holes which satisfy the condition R = 0 [1], and in this case, one sees from (2.3) that, with out loss of generality, one can set $\beta = 0$, leaving us with the equations of motion,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 4\alpha B_{\mu\nu} = 0, \tag{2.5}$$

$$\begin{aligned} \implies R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2\alpha[-\frac{1}{2}g_{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) - \frac{1}{3}\nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} \\ -\frac{1}{6}g_{\mu\nu}\Box R + 2R_{\mu\rho}R^\rho_\nu - \frac{2}{3}RR_{\mu\nu}] = 0. \end{aligned} \tag{2.6}$$

These equations of motion are very complicated to solve analytically, since they are nonlinear and, as there are double derivatives of the Ricci tensor, we have fourth order derivatives of the metric. To understand the behavior of this theory we linearize the equations of motion on a Schwarzschild-like spherical symmetric background to understand its asymptotic behavior and hope to be able to obtain analytical solutions. The linearization of gravity equations is very important to understand the physics of spacetime close to a known background. For example, another important application is the study of gravitational waves (ripples in spacetime) which were detected experimentally by the LIGO experiment [21].

Chapter 3

Linearized Weyl-Einstein gravity

In this chapter, we investigate the physics of the system governed by the equation of motion we obtained after varying the action. To do that we expand the metric around the Minkowski background up to order 2 ($O(2)$) in the metric, then only keep linear terms, using perturbation theory [22].

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

where $|h_{\mu\nu}| << 1$.

$$h^{\mu}_{\nu} = \eta^{\mu\alpha} h_{\alpha\nu}, \qquad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}. \tag{3.2}$$

Focusing only on spherically symmetric solutions, $g_{\theta\theta}$, $g_{\phi\phi}$, and the off diagonal components remain fixed and we only need to consider perturbations of g_{tt} , and g_{rr} depending on r, define,

$$g_{tt} = -1 + W(r), \tag{3.3}$$

$$g_{rr} = 1 + V(r), \tag{3.4}$$

$$\tag{3.5}$$

we have defined $h_{tt} = W(r)$, and $h_{rr} = V(r)$. In order to rewrite the equations of motion in linear form, we need to express the Ricci tensor and the Ricci scalar in linear form. To do that we start with calculating the linearized Christoffel connection, the linearized Ricci tensor, and the linearized Ricci scalar by substituting (3.1) to their original definitions [22].

Recall that the Christoffel connection, Ricci tensor, and Ricci scalar, are given by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \tag{3.6}$$

$$R_{\mu\nu} = \nabla_\alpha \Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \Gamma^{\alpha}_{\mu\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha}, \tag{3.7}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{3.8}$$

where ∇_α is the covariant derivative in general relativity defined as, $\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^{\beta}_{\alpha\lambda}V^\lambda$ for any spacetime vector [23].

To linearize (3.6), (3.7), and (3.8), we use (3.1) and simplify keeping only terms of first order in $h_{\mu\nu}$. Therefore, the linearized Christoffel connection is given by,

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \\ &= \frac{1}{2}(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^\lambda h_{\mu\nu}).\end{aligned} \tag{3.9}$$

To first order in the metric, the third and the fourth terms of the Ricci tensor (3.7) go to zero since they contain second order terms, thus the linearized Ricci tensor becomes,

$$\begin{aligned}R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} \\ &= \partial_\alpha(\frac{1}{2}(-\partial^\alpha h_{\mu\nu} + \partial_\nu h^{\alpha}_{\mu} + \partial_\mu h^{\alpha}_{\nu})) - \partial_\nu(\frac{1}{2}(-\partial^\alpha h_{\mu\alpha} + \partial_\alpha h^{\alpha}_{\mu} + \partial_\mu h^{\alpha}_{\alpha})) \\ &= \frac{1}{2}(-\partial_\alpha\partial^\alpha h_{\mu\nu} + \partial_\alpha\partial_\mu h^{\alpha}_{\nu} + \partial_\nu\partial^\alpha h_{\mu\alpha} - \partial_\nu\partial_\mu h).\end{aligned} \tag{3.10}$$

In this chapter, we can denote the flat-space Laplacian as,

$$\Box = \partial_\alpha\partial^\alpha, \tag{3.11}$$

so that the linear Ricci tensor becomes,

$$R_{\mu\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + \partial_\alpha\partial_\mu h^{\alpha}_{\nu} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial_\nu\partial_\mu h), \tag{3.12}$$

and using the results of the linearized Ricci tensor (3.12), the linearized Ricci scalar is given by,

$$\begin{aligned}R = g^{\mu\nu}R_{\mu\nu} &= \eta^{\mu\nu}R_{\mu\nu} + h^{\mu\nu}R_{\mu\nu} \\ &= \frac{1}{2}(-\Box h + \partial_\alpha\partial_\mu h^{\alpha\mu} - \Box h + \partial^\alpha\partial^\mu h_{\mu\alpha}) \\ &= \partial_\alpha\partial_\mu h^{\alpha\mu} - \Box h.\end{aligned} \tag{3.13}$$

To obtain the solution $h_{\mu\nu}$, we separate the calculation into two cases. The first case we consider a (i),non-Ricci flat space($R \neq 0$), and set $\alpha = 0$ while keeping $\beta \neq 0$. To solve this case we first take the trace of the equations of motion in order to further simplify these equations, and then for the second case (ii), we consider Ricci-flat space($R = 0$) [1] and set $\beta = 0$,$\alpha \neq 0$. We will not consider the $\alpha \neq \beta \neq 0$ because we would have to resort to numerical solutions since the resulting equations are also complicated, and we won't be able to understand the analytical behaviour of the solutions.

Case 1: $\alpha = 0$, $R \neq 0$, $\beta \neq 0$.

In this case,the equations of motion (2.3) become,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 2\beta R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) + 2\beta(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R) = 0, \tag{3.14}$$

now taking the trace of (3.14), we get the following,

$$\begin{aligned}
&g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} + 2\beta g^{\mu\nu}R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) + 2\beta(g^{\mu\nu}g_{\mu\nu}\nabla^2 R - g^{\mu\nu}\nabla_\mu\nabla_\nu R) = 0,\\
&\implies R - 2R + 2\beta R(R - R) + 2\beta(4\Box R - \Box R) = 0,\\
&\implies (6\beta\Box - 1)R = 0. \qquad (3.15)
\end{aligned}$$

Substituting (3.13) to (3.14), we get,

$$\begin{aligned}
&(6\beta\Box - 1)(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h) = 0, &(3.16)\\
\implies &(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - m_0^2)(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h) = 0, &(3.17)\\
\implies &(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h) = R = C_1\frac{e^{-m_0 r}}{r} + \frac{C_2}{2m_0}\frac{e^{m_0 r}}{r}, &(3.18)
\end{aligned}$$

where we have defined,

$$m_0^2 = \frac{1}{6\beta L_P^2} = \frac{1}{6\beta}, \tag{3.19}$$

and C_1, C_2 are dimensionless integration constants. Now going back to the equations of motion (3.14) to solve for $h_{\mu\nu}$ and only keep the first order terms of the metric. Before we solve for (3.14), we first simplify it term by term, and use the results we obtained for the Ricci scalar($\Box R = \frac{1}{6\beta}R = m_0^2 R$). The first term of (3.14) is $R_{\mu\nu}$, which we have linearized, see (3.12). The second term of (3.14) is $-\frac{1}{2}Rg_{\mu\nu}$, and substituting (3.1) it can be written as ,

$$-\frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}R(\eta_{\mu\nu} + h_{\mu\nu}) = -\frac{1}{2}R\eta_{\mu\nu} - \frac{1}{2}Rh_{\mu\nu}, \tag{3.20}$$

note that $-\frac{1}{2}Rh_{\mu\nu} = -\frac{1}{2}(\partial^\mu\partial^\nu h_{\mu\nu} - \Box h)h_{\mu\nu} \to 0$ because it contains second order powers of h, and as stated at the begining of this chapter, we are only interested in linear order solutions of h ($|h_{\mu\nu}| << 1$). Therefore, (3.20) becomes,

$$-\frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}R\eta_{\mu\nu}. \tag{3.21}$$

Similarly the third term of (3.14), $2\beta R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) \to 0$, because we can see that substituting (3.1), (3.12), and (3.13) into :

$$R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) = RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu}, \tag{3.22}$$

will lead to only second and third orders in h. Finally, the term $(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R) \to (\eta_{\mu\nu}\Box R - \partial_\mu\partial_\nu R)$, using the same arguments as above. Therefore, simplifying the equations of motion (3.14) using the results we just obtained, we get

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} + 2\beta(\eta_{\mu\nu}\Box R - \partial_\mu\partial_\nu R) = 0, \\ \implies R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} + 2\beta(\eta_{\mu\nu}\frac{1}{6\beta}R - \partial_\mu\partial_\nu R) = 0, \\ \implies R_{\mu\nu} = \frac{1}{6}R\eta_{\mu\nu} + \frac{1}{3m_0^2}\partial_\mu\partial_\nu R. \end{aligned} \tag{3.23}$$

Now to fully solve equation (3.23), we look at its components explicitly. Starting with the tt component leads to

$$R_{tt} = \frac{1}{6}R, \tag{3.24}$$

and substituting (3.12)-(3.13) into (3.24) gives

$$\frac{\partial^2}{\partial r^2}h_{tt} + \frac{2}{r}\frac{\partial}{\partial r}h_{tt} = \frac{1}{3}\left[C_1\frac{e^{-m_0 r}}{r} + \frac{C_2}{2m_0}\frac{e^{m_0 r}}{r}\right]. \tag{3.25}$$

Solving (3.24), we get the solution

$$h_{tt} = \frac{C_1}{3m_0^2}\frac{e^{-m_0 r}}{r} + \frac{C_2}{6m_0^3}\frac{e^{m_0 r}}{r} - \frac{C_3}{r} + C_4, \tag{3.26}$$

where similarly C_1, C_2, C_3, C_4, are dimensionless integration constants. We note that, four integration constants are expected given that we are looking at a 4-derivative theory. We can set $C_4 = 0$ since it just corresponds to a re-scaling to the time coordinate. By examining the solutions at large r, we can identify that $C_3 = 2M$,

$$h_{tt} = \frac{2M}{r} + \frac{C_1}{3m_0^2}\frac{e^{-m_0 r}}{r} + \frac{C_2}{6m_0^3}\frac{e^{m_0 r}}{r}, \tag{3.27}$$

since we do not want the solution to explode as r goes to infinity, we set $C_2 = 0$, to get,

$$g_{tt} = -1 + \frac{2M}{r} + \frac{C_1}{3m_0^2}\frac{e^{-m_0 r}}{r}. \tag{3.28}$$

C_1 can be chosen by using the boundary condition $g_{tt}(r = r_0) = 0$ at the horizon or $g_{tt}(r = r_{large}) = -1$ at large r.

Now we look at the rr component of the linear equations of motion (3.23),

$$R_{rr} = \frac{1}{6}R + \frac{1}{3m_0^2}\partial_r\partial_r R, \tag{3.29}$$

substituting the rr component of the Ricci tensor:

$$R_{rr} = \frac{1}{2}(\Box h_{rr} - \partial_r\partial_r(h_{tt} + h_{rr})), \tag{3.30}$$

into equation (3.29) and simplifying, we get

$$\begin{aligned} \frac{1}{2}(\Box h_{rr} - \partial_r\partial_r(h_{tt} + h_{rr})) &= \frac{1}{6}R + \frac{1}{3m_0^2}\partial_r\partial_r R, \\ \implies \frac{\partial}{\partial r}h_{rr} &= \frac{\partial}{\partial r}h_{tt} + \frac{1}{3}rR + \frac{1}{3m_0^2}r\partial_r\partial_r R. \end{aligned} \tag{3.31}$$

Finally integrating both sides of (3.31) with respect to r,

$$\begin{aligned} h_{rr} &= h_{tt} + \frac{1}{3}\int^r sR(s)ds + \frac{1}{3m_0^2}\int^r s\partial_s\partial_s R(s)ds, \\ &= h_{tt} + \frac{1}{3}\left(-\frac{C_1}{m_0}e^{-m_0 r}\right) \\ &+ \frac{1}{3m_0^2}(-C_5 m_0 e^{-m_0 r}) + (\frac{1}{3m_0^2} - 1)R. \end{aligned} \tag{3.32}$$

In the calculation above we have used integration by parts on the last term and substituted in the explicit form of R. C_5 is a dimensionless constant so it can be absorbed into C_1 and the result becomes,

$$h_{rr} = \frac{2M}{r} - \left(\frac{2r}{3m_0} + 1\right)C_1\frac{e^{-m_0 r}}{r}. \tag{3.33}$$

The rr component of the metric is then given by,

$$\begin{aligned} g_{rr} &= 1 + h_{rr} \\ g_{rr} &= 1 + \frac{2M}{r} - \left(\frac{2r}{3m_0} + 1\right)C_1\frac{e^{-m_0 r}}{r}, \end{aligned} \tag{3.34}$$

we can solve for C_1 by using the boundary condition $g_{rr}(r = r_0) = 0$ at the horizon or $g_{rr}(r = r_{large}) = 1$ at large r. This solution definitely approximates the Schwarzschild metric at infinity. Note that the constants of integration obtained C_1 from the solutions of g_{tt} and g_{rr} are related by a scaling factor. Thus C_1 is a free parameter for both solutions.

Case 2: $\alpha \neq 0$, $\beta = 0$.

In this case, taking the trace of the equations of motion, we see that $R = 0$. This theory gives a massive spin-2 mode as expected (see[1]). Using the condition $R = 0$, to simplify the equations of motion (2.3),

$$\begin{aligned} R_{\mu\nu} - 2\alpha(\frac{1}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma} + \Box R_{\mu\nu} + 2R_{\mu\rho}R^{\rho}_{\nu}) &= 0 \\ \implies (\Box - m_2^2)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma} + 2R_{\mu\rho}R^{\rho}_{\nu} &= 0, \end{aligned} \tag{3.35}$$

where we have defined, $m_2^2 = \frac{1}{2\alpha}$.

Linearizing equation (3.35), we obtain the following,

$$\begin{aligned} (\Box - m_2^2)R_{\mu\nu} &= 0, \\ R_{\mu\nu} &= \left(C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{r}\right)\delta_{\mu\nu}, \end{aligned} \tag{3.36}$$

from the tt component of equation (3.36) and using equation (3.12) we get,

$$\begin{aligned} R_{tt} &= C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{r}, \\ \implies \Box h_{tt} &= -2\left(C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{r}\right), \\ \implies h_{tt} &= -\frac{2}{m_2^2}\left(C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{r}\right) - \frac{C_3}{r} + C_4, \\ \implies h_{tt} &= -\frac{2M}{r} - \frac{2}{m_2^2}C_1\frac{e^{-m_2 r}}{r}. \end{aligned} \tag{3.37}$$

In this case set, $C_4 = 1$, $C_2 = 0$, and $C_3 = 2M$ and get,

$$g_{tt} = 1 - \frac{2M}{r} - \frac{2}{m_2^2}C_1\frac{e^{-m_2 r}}{r}. \tag{3.38}$$

Now the rr component leads to,

$$\begin{aligned}
(\Box - m_2^2)R_{\mu\nu} = 0,\\
\implies (\Box - m_2^2)(2\partial^\rho\partial_\mu h_{\rho\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) = 0,\\
\implies 2\Box h_{rr} - \partial_r\partial_r(h_{tt} + h_{rr}) - \Box h_{rr} = C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{2m_2 r},\\
\implies \frac{2}{r}\partial_1 h_{rr} = \partial_r^2 h_{tt} + C_1\frac{e^{-m_2 r}}{r} + C_2\frac{e^{m_2 r}}{2m_2 r},\\
\implies h_{rr} = C_4 + \frac{C_3}{r} - \frac{(1+m_2 r)C_1}{m_2^2}\frac{e^{-m_2 r}}{r} + \frac{(-1+m_2 r)C_2}{2m_2^3}\frac{e^{m_2 r}}{r},\\
\implies h_{rr} = \frac{2M}{r} - \frac{(1+m_2 r)C_1}{m_2^2}\frac{e^{-m_2 r}}{r}.
\end{aligned} \tag{3.39}$$

In this calculation we set $C_4 = 1$, $C_2 = 0$, and $C_3 = 2M$ and thus the rr component of the metric becomes,

$$g_{rr} = 1 + \frac{2M}{r} - \frac{(1+m_2 r)C_1}{m_2^2}\frac{e^{-m_2 r}}{r}. \tag{3.40}$$

In this chapter, we have analysed the linearized equation of motion in fourth order gravity, finding two Yukawa modes: with mass $m_0^2 = \frac{1}{6\beta L_P^2}$, and $m_2^2 = \frac{1}{2\alpha L_P^2}$. These modes correspond to spin-0 and spin-2 excitations respectively [1, 22].

Chapter 4

Higher derivative black hole numerical solutions

In this chapter we are going to solve the equations of motion (2.3) numerically using MATHEMATICA. We start by rewriting the action (2.1) in the alternate form [20],

$$S=\frac{1}{8\pi L_P^2}\int d^4x\sqrt{-g}\ \left(R+L_P^2\left(\beta R^2-\alpha C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu}\right)+\mathcal{O}\left(\frac{L_P^3}{L_C^3}\right)+\dots\right)\ , \tag{4.1}$$

with the following equations of motion.

$$\begin{aligned} E_{\mu\nu}=0 \quad &= \quad R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R-4\alpha L_P^2\left(\nabla^\alpha\nabla^\beta+\tfrac{1}{2}R^{\alpha\beta}\right)C_{\mu\alpha\nu\beta} \\ &\quad +2\beta L_P^2\left(R\left(R_{\mu\nu}+\tfrac{1}{4}g_{\mu\nu}R\right)+\left(g_{\mu\nu}\nabla^\alpha\nabla_\alpha-\nabla_\mu\nabla_\nu\right)R\right)\ , \end{aligned} \tag{4.2}$$

where α and β are dimensionless constants[1]. L_P is the Planck scale and L_C is some length scale associated with the curvature scale of our solution at the horizon – for a Schwarzschild black hole $L_C\sim M^{-1}$. Up to $\mathcal{O}\left(\frac{L_P^2}{L_C^2}\right)$. From (4.1) we can get back to (2.1) by scaling,

$$\begin{aligned} \frac{1}{8\pi L_p^2}&\to 1,\\ \frac{\beta}{8\pi}&\to\beta,\\ \frac{\alpha}{8\pi}&\to\alpha. \end{aligned} \tag{4.3}$$

This is done in order to simplify some calculations and plots, as we will see later in the chapter. In [22] and chapter 3, it was shown that when one studies linear fluctuations of (2.3) and (4.2) about flat space, in addition to the usual massless spin-2

[1]One can add a Gauss-Bonnet term which, in four dimensions, does not affect the equations of motion, but can shift the Wald entropy.

graviton mode, there is are massive spin-0 (with $m_0^2 = 1/(6\beta L_P^2)$ or $m_0^2 = 1/(6\beta)$ and spin-2 (with $m_2^2 = 1/(2\alpha L_P^2)$ or $m_2^2 = 1/(2\alpha)$) modes depending on the units of our theory. The massive modes will introduce Yukawa like behavior, $e^{\pm mr}/r$, in the asymptotic form of the metric. This means that to obtain asymptotically flat solutions, one needs to finely tune conditions in the bulk of spacetime to avoid the growing Yukawa mode. Furthermore, the growing mode makes the numerical integration of the equations tricky since rounding errors can inadvertently excite the growing mode leading to so-called stiff equations which easily blow up. Such Yukawa modes could also have implications for the stability of such solutions against perturbations which we do not consider at this stage. Also, notice that one needs to take α and β positive to avoid tachyon-like modes. In [1] it was shown that any asymptotically flat black hole solution to (2.3) and (4.2) must have $R = 0$ but not necessarily $R_{\mu\nu} = 0$. As is well known [5, 6, 7, 8], spherical symmetry and $R_{\mu\nu} = 0$ implies that the only black hole solution is the Schwarzschild one. With the weaker requirement that $R = 0$, there is the possibility of other solutions. Given the complexity of (2.3) and (4.2), it is challenging finding analytic solutions, but one can resort to numerical and perturbative expansions. The matching of a near horizon expansion with numerical integration was studied in [1] – we have further studied an expansion about flat space which complements this earlier work.

Following on from the discussion in the previous paragraph, we will only be interested in solutions with $R = 0$ from now on. This means that the value of β will not affect the equation of motion so we will ignore it. Furthermore, once we neglect the terms involving β, it is not hard to see that the trace of the equations of motion (2.3) and (4.2) leads to $R = 0$.

We now consider the near horizon form of the metric which give the boundary conditions for numerical integration of the equations of motion. Given that we can neglect the terms involving β, the only length scale in (4.2) is $L_\alpha = \sqrt{2\alpha}L_P$[2]. It is convenient to scale out this dependence by defining a dimensionless radial coordinate

$$\rho = m_2 r = \frac{r}{L_\alpha}. \tag{4.4}$$

To solve the equations of motion we assume the following ansatz

$$\begin{aligned} d^2s &= -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -h(\rho)dt^2 + L_\alpha^2 \frac{d\rho^2}{f(\rho)} + L_\alpha^2\rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \end{aligned} \tag{4.5}$$

[2] We put in a factor of $\sqrt{2}$ into the definition of L_α for later convenience.

using (4.5) In matrix form the metric given by,

$$g_{\mu\nu} = \begin{bmatrix} -h(\rho) & 0 & 0 & 0 \\ 0 & \frac{L_\alpha^2}{f(\rho)} & 0 & 0 \\ 0 & 0 & L_\alpha^2\rho^2 & 0 \\ 0 & 0 & 0 & L_\alpha^2\rho^2\sin^2\theta \end{bmatrix} .$$

By using (4.2) with this ansatz (4.5), the trace of the equations of motion, $g^{\mu\nu}E_{\mu\nu} = 0$ [3], gives

$$4h^2(f-1) + 4\rho h\,(hf)' + \rho^2\left(h\left(f'h' + 2fh''\right) - fh'^2\right) = 0 \ , \tag{4.6}$$

where, $'$, denotes derivatives with respect to ρ, and the $(\rho\rho)$ equation of motion, $E_{\rho\rho} = 0$, gives

$$\begin{aligned} 0 = \\ & 16h^4(1-f^2) + 32\rho f^2h^3h' + 4\rho^2h^2\left(2fh\left(2h\left(f''+3\right) - f'h'\right) - h^2\left(f'^2+12\right) - f^2\left(8hh''+h'^2\right)\right) \\ & +4\rho^3\left(h^3f'^2h' + 4fh\left(hf'h'^2 - h^2\left(\left(f''-3\right)h' + f'h''\right)\right) - f^2\left(4h'''h^2 + 5h'^3 - 12hh'h''\right)\right) \\ & +\rho^4\left(2fhh'\left(2h\left(f''h' + 2f'h''\right) - 3f'h'^2\right) - h^2f'^2h'^2 + f^2\left(7h'^4 - 12hh'^2h'' + 4h^2\left(2h'''h' - h''^2\right)\right)\right) \ . \end{aligned} \tag{4.7}$$

Equations (4.6) and (4.7) are intractable to solve analytically thus we focus on numerical solutions. We start with the assumption that there exists a black hole with the horizon radius $r = r_0$ or $\rho = \rho_0$, where the functions h, and f vanish. Then we take the Taylor expansion of the functions h, and f such that near the horizon, and get

$$h = c\left[(\rho-\rho_0) + h_2(\rho-\rho_0)^2 + h_3(\rho-\rho_0)^3 + \cdots\right] \ , \tag{4.8}$$

$$f = f_1(\rho-\rho_0) + f_2(\rho-\rho_0)^2 + f_3(\rho-\rho_0)^3 + \cdots \ . \tag{4.9}$$

Note that the constant c can be shifted by a trivial rescaling of t and consequently does not enter into the equations of motion (although c does need to be tuned if one demands that h asymptote to 1 at the end). One can show that $\rho_0 = 1/f_1$ corresponds to the Schwarzschild solution so that the parameter,

$$\delta = f_1\rho_0 - 1 \ ,$$

Systematically solving (4.8) and (4.9) for f_i and h_i at each order in $(\rho-\rho_0)^i$, one finds that there are two free parameters which can be taken to be δ and ρ_0. For instance at second order one finds

$$f_2 = -\frac{1 + (3 - \frac{3}{4}\rho_0^2)\delta + 2\delta^2}{(1+\delta^2)\rho_0^2} \ , \qquad h_2 = -\frac{1 + (3 + \frac{1}{4}\rho_0^2)\delta + 2\delta^2}{(1+\delta^2)\rho_0^2} \ . \tag{4.10}$$

[3] As mentioned in the beginning of this section, the trace of the equations of motion, reduces to $R = 0$ in this case.

δ is a deflection parameter from the Schwarzschild solution, since we used perturbation expansions in order to obtain non-Schwarzschild black hole solutions. After obtaining these coefficients, we then use these near horizon solutions as initial conditions in order to integrate the equations of motion for values of $r > r_0$ or $\rho > \rho_0$ using the shooting method as stated in [1]. The non-trivial solutions(the non-Schwarzschild black holes) can be obtained by setting the two parameters r_0, f_1 to correct values. r_0 or ρ_0 is the event horizon of the black hole [1]. All the coefficients h_i, f_i's depend on these two parameters r_0 or ρ_0, f_1. In order to find the values of r_0 and δ we fix the value of r_0 and check for the corresponding value of δ that gives no singularity on the solutions as we perform numerical integration. The numerical procedure to find δ was performed up to nine significant digits. If we chose the wrong value of δ the solutions rapidly become singular but the special value of $\delta = \delta^*$ that does not give any singularity of the solutions we see that the graphs of $f(r)$, and $h(r)$ approach the value 1 if we integrate out to larger values of r. This behavior is expected since we already observed it on the linear gravity case where the rising and falling $\frac{e^{\pm m_2 r}}{r}$ Yukawa terms. We are only interested in the falling Yukawa terms which they become stable and converge to 1 where the singularity is not observed. Without loss of generality we set $\alpha = \frac{1}{2}$. After finding the δ^*'s, we obtain two ($\delta > 0$ and $\delta < 0$) families of non-Schwarzschild black hole solutions, which represents the "hot branch" and the "cold branch" respectively. We also observe that there is a minimum horizon radius [1] for the "the hot branch" of which turns out to be the maximum horizon radius for the "the cold branch" which is $r_0 = r^{min}_{0_{\delta>0}} = r^{max}_{0_{\delta<0}}$. As we integrate from $r_0 > r^{min}_{0_{\delta>0}}$ to larger r in the "the hot branch" and integrate towards $r_0 < r^{max}_0$ "the cold branch", there is a special $\delta = \delta^* > 0$ for the the "the hot branch" of black hole solutions and another special $\delta = \delta^{*'} < 0$ for "the cold branch" of black hole solutions.

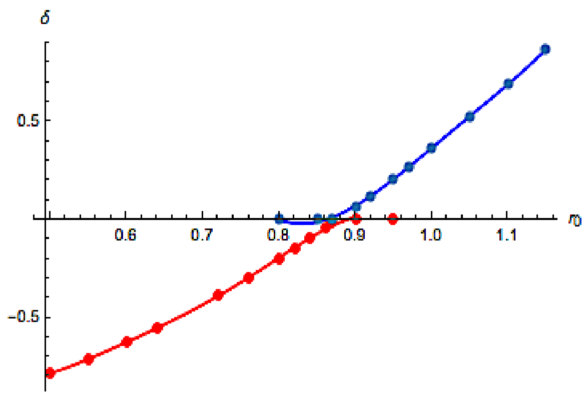

Figure 4.1: A plot of the parameter δ as a function of r_0 . The red and blue lines correspond to the hot and cold branches respectively.

Figure 4.1 represents the function of $\delta(r_0)$ for two families of non-Schwarzschild black hole solutions, $\delta > 0$ which is the "hot" branch (the blue curve), and $\delta < 0$ which is the "cold" branch (the red curve). Each dot in both curves when $\delta \neq 0$ is the black hole solution obtained in our numerical computation. We also observe that in the "cold" branch, δ increases from $\delta \approx -0.8$ (where $r_0 \approx 0$) to $\delta = 0$ (where $r_0 = r_{0_{\delta<0}}^{max} \approx 0.876$). Then in the "hot" branch, δ increases from $\delta = 0$ (where $r_0 = r_{0_{\delta>0}}^{min} \approx 0.876$) to $\delta \approx 0.9$ [1]. The value $r_0 \approx 0.876$, is special, since it is the horizon radius for the Schwarzschild black hole, and the non-Schwarzschild black holes for both families, it is the point where these three black hole solutions coincide. We also observe that for the "hot" branch, if you try to find a black hole, with $r_0 < 0.876$, you obtain a δ value that is very small and the solution is a Schwarzschild black hole, similarly for the "cold" if one tries to find a black hole at $r_0 > 0.876$, one also obtains a very small value of delta, which corresponds to the Schwarzschild black hole.

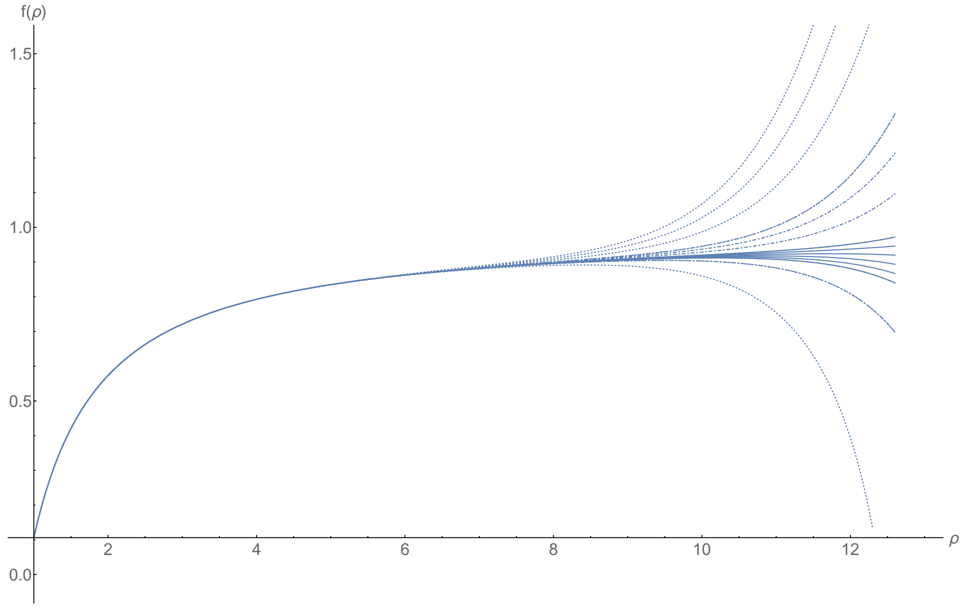

Figure 4.2: Numerical plot of $f(\rho)$ (with ρ defined in (4.4) $\rho_0 = 0.9$) showing the procedure for tuning δ and extending how far one can integrate before the solution diverges. The dotted plots correspond to $0.06584 \leqslant \delta \leqslant 0.6616$. The dashed plots show the next step with $0.066032 \leqslant \delta \leqslant 0.66096$ and the solid plots show a further step with $0.0660704 \leqslant \delta \leqslant 0.0660832$.

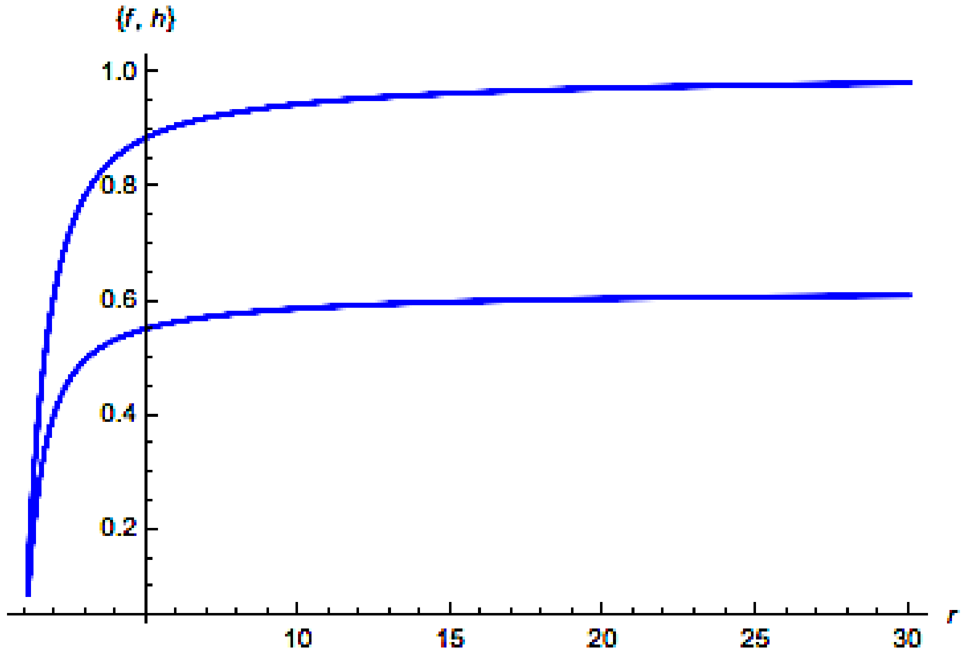

Figure 4.3: Numerical plot of $f(r)$, and $h(r)$ (with $r_0 = 1$), we get $\delta = 0.3633022446966598$, and the mass of the black hole is $M = 0.281794$ for the first family/hot branch of black holes.

In practice, we scan over a range of guesses for δ, successively tuning it so that the integration can extend further and further –this process is shown in Figure 4.2 and 4.3. It was claimed in [1], that they were able to extend the numerical integration as far as $\rho = 60$. While this is good evidence for the existence of non-Schwarzschild solutions, there is no guarantee that one will eventually hit some sort of barrier. We found that one does not have to integrate too far before the asymptotic expansion, which also involves only three parameters, M, (λ or m_2) and (h_∞ or c), matches very well with numerical integration[4]. This gives even stronger evidence for the non-Schwarzschild solutions.

We only plotted the first family ("the hot branch") of solutions since the curves for $h(r)$, and $f(r)$, look the same, for the second family("the cold branch") also. The simulations were run up to $r = 30$, unlike on [1] where they ran them up to $r = 60$, we ran them up to $r = 30$ because the computer we were using is not as powerful, but still we got reasonable results, because our value for δ is accurate up to the 17th decimal, whereas at [1], is up to the 20th. The fact that we only did the numerical integration up to the 17th decimal place does not compromise the accuracy of the value of mass of the black hole since we take its accuracy up to the 5th decimal place, and since the mass value's accuracy is not compromised then we can be sure that the values of the thermodynamic properties of the black hole will be accurate enough for our investigation. By doing these calculations we do observe that definitely there exist a black hole with the special value of δ such that the numerical integrations become smoother as we integrate at larger values of r. From Figure 4.3 we can see that the curve of f converges to 1 while the function h converges to 0.6 not one since we scaled the function h, with the scaling factor $c = 1$, but we can scale the function by choosing the constant value c such that $h(r)$ also converges to 1.

By using these solutions on the table on the appendix C and appendix F we can see that the entropy of these black holes goes to zero and end up being negative and therefore, the black holes become colder and colder as the area of the even horizon increases. We plot the values of the black hole's event horizons and their corresponding black hole mass [1].

[4] We used a non-linear fit to match the numerical results with the asymptotic expansion.

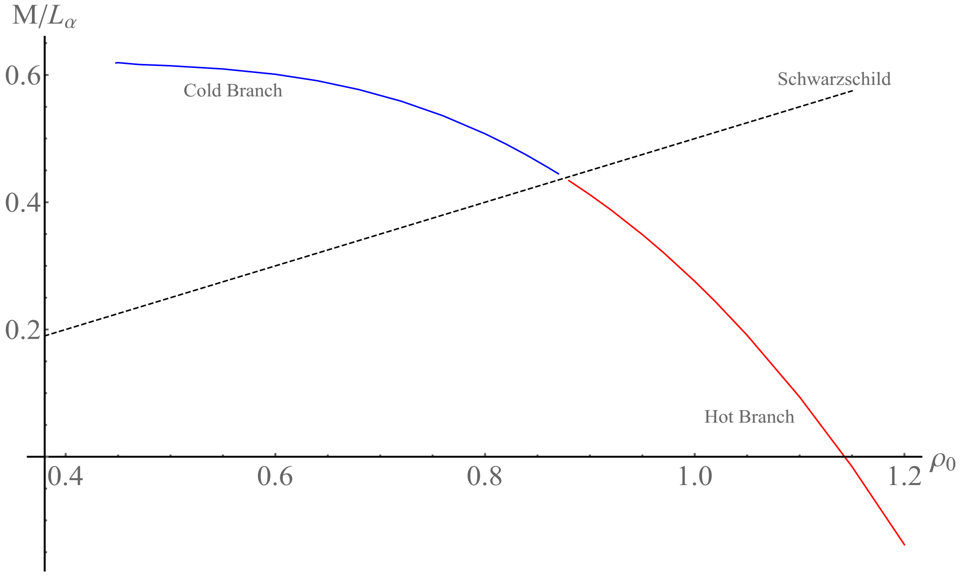

Figure 4.4: The non-Schwarzschild black hole mass against ρ_0. The blue curve is for the first family of solutions ($\delta > 0$), and the red curve is for the second family of solutions($\delta < 0$)

Now, looking at Figure 4.4, which is a plot of mass as a function of ρ_0, we start to see some more peculiar features of these non-Schwarzschild solutions. The dashed line corresponds is the Schwarzschild black hole with the hot and cold non-Schwarzschild branches represented by solid red and blue lines respectively. In contrast to the Schwarzschild solution, the mass of the non-Schwarzschild solutions increases as ρ_0 decreases. On the cold branch, the mass seems to saturate at around $0.62L_\alpha$ as the temperature approaches zero. On the hot branch, the mass actually becomes negative at some point, continuing to become smaller as ρ_0 increases without any apparent lower bound. For Schwarzschild black holes, a negative mass is associated with a naked singularity, but the negative mass non-Schwarzschild do not seem to violate cosmic censorship [24]. In this plot, we used the fact that the mass of the Schwarzschild black hole can be determined by $\frac{M}{L_\alpha} = \frac{1}{2}\rho_0$. To obtain the mass of the non-Schwarzschild black hole we took $\frac{1}{2}g_{tt}$, as $\rho \to \infty$. This is how you determine the mass of an asymptotic black hole. The "hot branch" solutions are valid at the values of $\rho_{0_{\delta>0}}^{min} \approx 0.876 < \rho_0 < 1.145$, where $M > 0$, we also observe that $M = 0$, at $\rho_0 = 1.145$, for the "cold" branch, the solutions are valid at values of $\rho_0 < \rho_{0_{\delta<0}}^{max}$ and the mass of those black hole solutions do not become negative. This graph shows that if one takes values of $\delta < 0 \to \delta > 0$, one obtains one curve starting from the maximum number of mass at $\rho_0 \approx 0$ to zero at $\rho_0 = 1.149$.

Chapter 5

Higher derivative black hole thermodynamics

In this chapter, we are going to study the thermodynamics of the non-Schwarzschild black hole solutions, such their mass, entropy, the temperature and also show that these black hole solutions follow the first law of thermodynamics. For calculating the black hole Entropy we are going to use the results of Wald formalism [12, 25, 20]. It can be shown from the appendix D that the entropy and temperature of the non-Schwarzschild black holes in the case of $\beta = 0$ are given by [12, 25]

$$S = \pi r_0^2 - 4\pi\alpha\delta^* = S_{\rm BH} - (2\pi\delta^*)L_\alpha^2 \ , \tag{5.1}$$

where as usual the Bekenstein-Hawking entropy, $S_{\rm BH}$, is one quarter the horizon area. It will turn out, for a given mass, the Schwarzschild and non-Schwarzschild black holes have a different horizon area Figure 4.4. The temperature as seen in the appendix D is given by,

$$\begin{aligned} T = \frac{1}{4\pi}\sqrt{h'(r_0)f'(r_0)} = \frac{1}{4\pi}\sqrt{cf_1} &= \frac{1}{4\pi}\sqrt{\frac{c(1+\delta)}{r_0}} \\ &= \frac{1}{2\pi L_\alpha}\sqrt{cf_1} \ , \end{aligned} \tag{5.2}$$

where c is the scaling factor from the near horizon expansion function $h(r)$. From the equation of the entropy we must recall that one is free to add any constant which is a multiple of the Gauss-Bonnet action. The Gauss-Bonnet term as we stated is invariant to the action and therefore, does not affect the equations of motion[1]. This form of the entropy expression was chosen such that the mass of the Schwarzschild black hole vanishes if the entropy vanishes [1]. Now we look at the relationship between the mass of the non-Schwarzschild black hole and it's corresponding entropy. We used a nonlinear fit using the data on the appendix C to show that the mass and temperature against entropy takes the form,

$$M(S) \approx -0.00014S^3 - 0.0075S^2 + 0.13S + 0.17, \tag{5.3}$$
$$T(S) \approx -0.00043S^2 - 0.015S + 0.13. \tag{5.4}$$

These results were obtained by plotting the data on the appendix C for the mass against entropy and the temperature against entropy, then we fitted a cubic polynomial for the Mass against entropy and a quadratic for the temperature against entropy. The resulting polynomials are shown in equation (5.3), and (5.4).These results for the first family of black hole solutions shows that these solutions indeed obey the first law of Thermodynamics since $T(S) \approx \frac{\partial}{\partial S}M(s)$ [1]. For the second family of black hole solutions, we could not obtain a polynomial fit that obeys the first law of Thermodynamics, and that was reserved for future investigations. This shows that even though we integrated the equations of motion to $r = 30$, we still obtained the correct mass values which in turn resulted in correct values for the entropy and temperature of the non-Schwarzschild black holes.

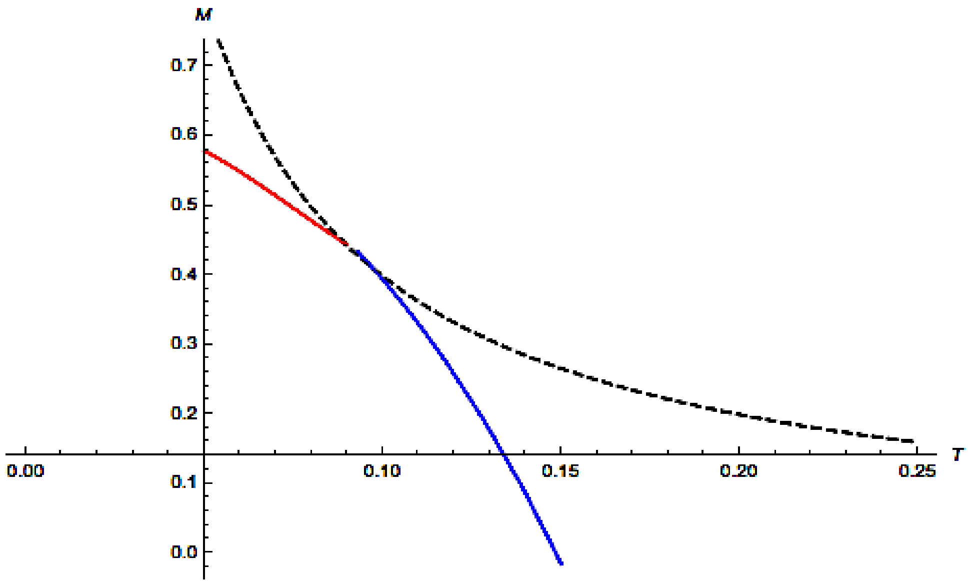

Figure 5.1: The non-Schwarzschild black hole Mass against Temperature, and Schwarzschild Mass against Hawking Temperature. The blue curve is for the first family of solutions and the red curve is for the second family of solutions.

Figure 5.1 shows us that the mass of the Schwarzschild black hole asymptotically goes to zero as the temperature is increased, meaning that it evaporates, while the mass of the non-Schwarzschild black holes eventually becomes negative as the temperature increases for the first family of solution (blue curve). The maximum value of the mass that was observed can be determined from the minimum value of horizon radius r_0, using the plot we observed that the maximum value the non-Schwarzschild black hole (hot branch) can have is $M_{max} = \frac{1}{2}r_0^{min} \approx 0.435$ [1]. For the second family of solutions (red curve), the mass starts from maximum, then decrease to their minimum mass, of which we observe that is $M^{min}_{\delta<0} = M^{max}_{\delta>0} \approx 0.435$ as the temperature is increased. These results are interesting but not shocking, since we observed the same behavior when we plotted δ against r_0, and the mass M against r_0. This shows that for $\delta < 0$ one obtains larger mass black holes, with lower temperatures and for $\delta > 0$, we obtain smaller mass values, with higher temperatures. And the critical mass is $M^{min}_{\delta<0} = M^{max}_{\delta>0} \approx 0.435$, at this point the mass of the non-Schwarzschild black holes coincide with the Schwarzschild black hole.

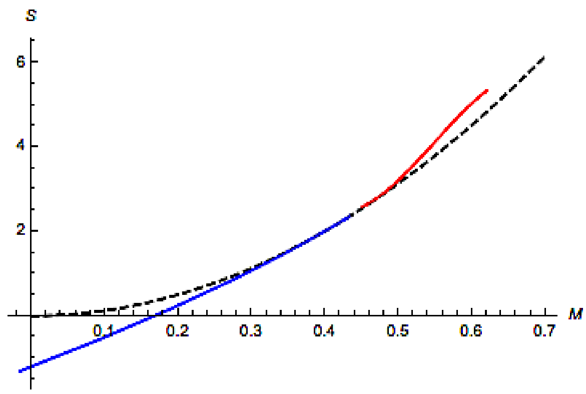

Figure 5.2: The non-Schwarzschild black hole entropy against mass. The blue curve first family of black hole solutions and the red curve is the second family of black hole solutions, the dashed curve is the Schwarzschild black hole.

For the Schwarzschild black hole, the mass and Hawking temperature and entropy, are related by,

$$M = \sqrt{\frac{S}{4\pi}}, \tag{5.5}$$

$$T = \frac{1}{4}\frac{1}{\sqrt{\pi S}}, \tag{5.6}$$

where S is the entropy of the black hole, M is the mass, and T is the temperature. Figure 5.2 shows us that, as the mass of the first family of black holes increases, it's entropy also increases, but it is less than the entropy of the Schwarzschild black hole. While the entropy of the second family of solutions is higher than the entropy of the Schwarzschild black hole. The entropy of the first family converges to the entropy of the Schwarzschild entropy [1] while the second family solutions entropy diverges away. We also observe a similar behavior as before, that the maximum mass of the first family of solutions is the same as the minimum mass of the second family of solutions.

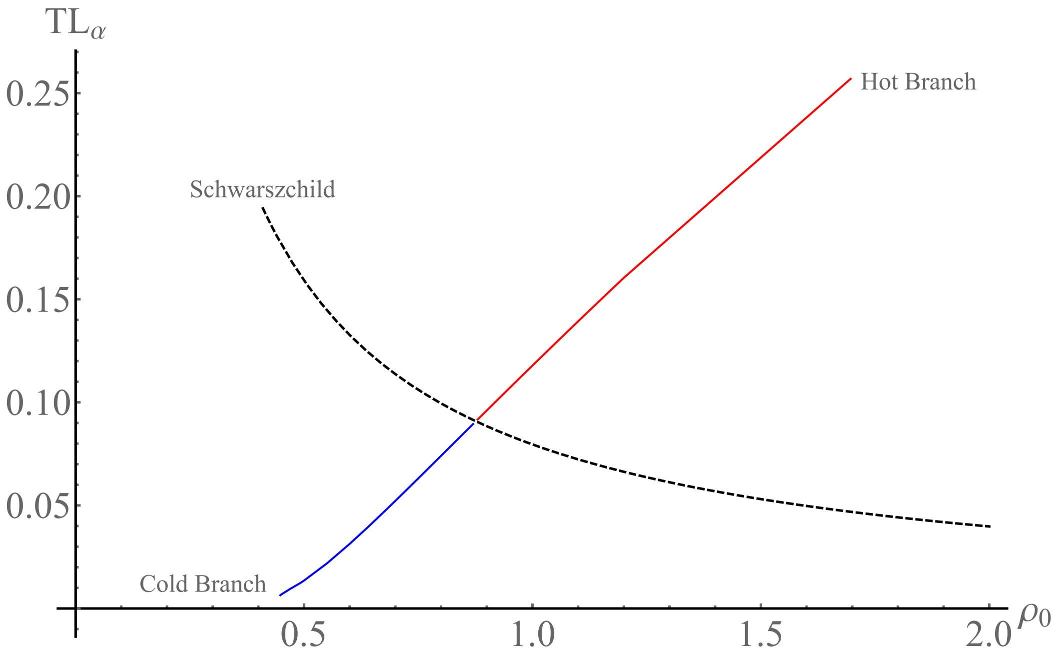

Figure 5.3: Plot of temperature, T, (in units of $1/L_\alpha$) against horizon radius, ρ_0, for the Schwarzschild and non-Schwarzschild black holes. The dashed line denotes the Schwarzschild solutions whereas the red and blue solid lines denote the hot and cold non-Schwarzschild branches respectively.

We also consider Figure 5.3 which is a plot of temperature as a function of ρ_0 (recall that ρ_0 corresponds to the length scale of the horizon in units of L_α). The solid line denotes the non-Schwarzschild solution while the dashed line corresponds to the Schwarzschild one. As mentioned in the introduction, above a certain critical point $\rho_0 \approx 0.876$ (already found in [1] the non-Schwarzschild solution is hotter than the Schwarzschild one whereas below it the non-Schwarzschild one is colder. We call these two branches the hot (red line) and cold branches (blue line) respectively. Notice that, unlike the Schwarzschild case, the temperature of the non-Schwarzschild decreases as ρ_0 decreases. Extrapolating the cold branch to smaller ρ_0, it would appear as if one approaches a zero temperature solution – unfortunately, the numerical ingratiation becomes increasingly unstable and we were not able to investigate this limit.

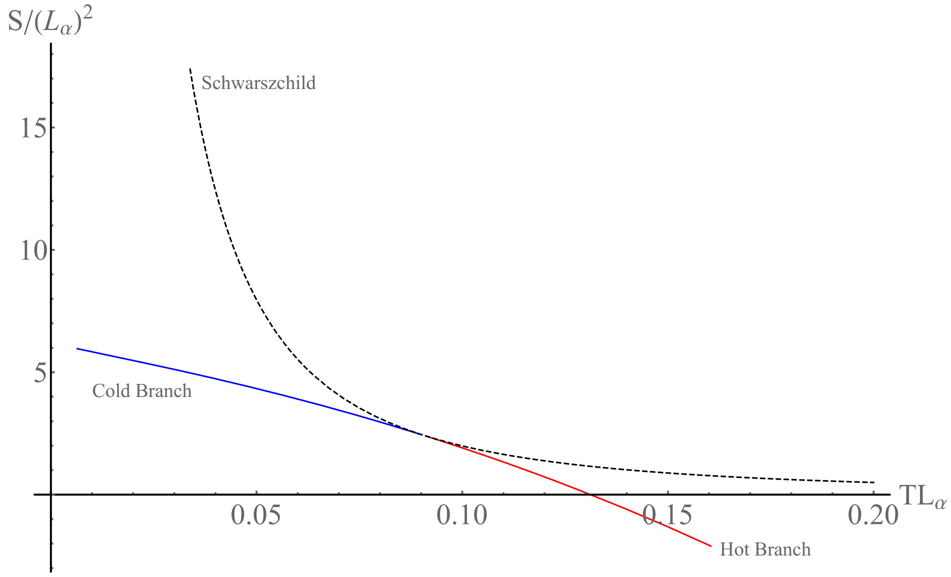

Figure 5.4: Plot of Wald entropy, S, (in units of L_α^2) against temperature, T, (in units of $1/L_\alpha$), for the Schwarzschild and non-Schwarzschild black holes. The dashed line denotes the Schwarzschild solutions whereas the red and blue solid lines denote the hot and cold non-Schwarzschild branches respectively.

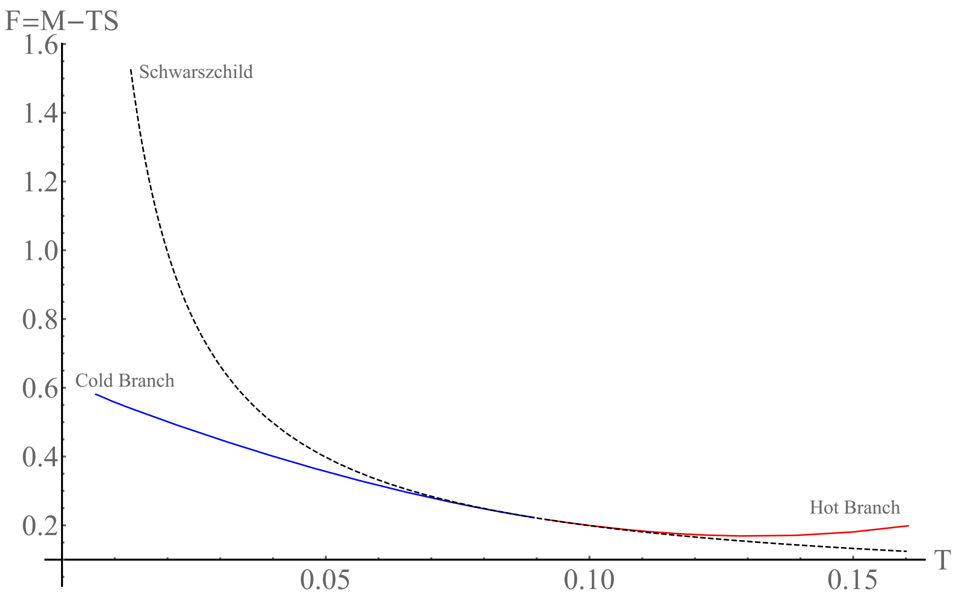

Figure 5.5: Plot of free energy, $F = M - TS$, against temperature, T, (in units $L_\alpha = 1$), for the Schwarzschild and non-Schwarzschild black holes. The dashed line denotes the Schwarzschild solutions whereas the red and blue solid lines denote the hot and cold non-Schwarzschild branches respectively.

While negative mass solutions may be a bit troubling, Figure 5.4, which is a plot of entropy as a function of temperature, shows a severe pathology of the hot branch. Like the Schwarzschild solution, the entropy of the non-Schwarzschild solutions decreases as the temperature increases but there are important differences. The entropy of the Schwarzschild black hole decreases as they become small and hot, with the entropy approaching 0 as the temperature approaches infinity and the size approaches zero. On the other hand, the entropy of the hot branch of the Schwarzschild solution continues to decrease below zero as the temperature and size of the black hole increase. A negative entropy, which one would ultimately like to identify with the logarithm of the number of microstates, does not make any sense. Bizarre features aside, we note that on the cold branch, the entropy seems to approach a finite value as we approach zero temperature and furthermore the non-Schwarzschild solutions always seem to have lower entropy than the Schwarzschild ones.

Finally, in Figure 5.5, we plot the free energy, $F = M - TS$, of our solutions. Notably, the free energy of the cold branch is lower than the corresponding Schwarzschild black hole with the same temperature were as the hot branch has higher free energy than its Schwarzschild counterpart. Like the Schwarzschild black hole, the free energy of the cold branch increases as one decreases the temperature. Unlike the Schwarzschild solution, it appears to tend to a finite value as one approaches zero temperature. Unlike the Schwarzschild black hole, whose free energy decreases monotonically, the hot branch's free energy initial decreases but then starts increasing, with increasing temperature.

Chapter 6

Asymptotic Expansion

In this chapter we consider the asymptotic expansion of non-Schwarzschild black holes. This work was also shown in [20].

6.1 Derivation of the asymptotic expansion

To derive the asymptotic expansion we consider the following metric in 4 dimensions,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2. \tag{6.1}$$

We consider perturbative expansions of these functions up to second order so that we can use the numerical solutions and try to match their analytical functions with the numerical solutions we obtained in the previous sections. There are only two parameters in these expansions, the mass M of the black hole, and the parameter δ. These analytical perturbative solutions will also be determined by two parameters, and if we can set those two parameters from the analytical solutions equal to the parameters obtained from numerical solutions(δ and M) then we can can show that there is no need to do long numerical integrations if one can obtained perturbative expansions of these black holes. We will be using MATHEMATICA in order to determine such expansions since they are very complex expressions.

It turns out that the asymptotic expansion can be expanded to any order, we will start with the first order expansion, then do the second order expansion, we then try to match the parameters obtained in those solutions to the parameters of the numerical solutions that were done previously and check if those two solutions agree. In the case where we used the numerical methods to solve the differential equations, we use the following ansatz [20],

$$ds^2 = -h(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2. \tag{6.2}$$

It turns out that for the asymptotic expansion to be computationally simple we have to use the following ansatz,

$$ds^2 = -B(\rho)dt^2 + L_\alpha^2 A(\rho)d\rho^2 + L_\alpha^2 \rho^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \tag{6.3}$$

where we have defined a change in coordinate, $\rho = m_2 r$. We can see that we can use two metric ansatz, one that is scaled with the Planck length L_α, and another that is not scaled. As we did in chapter 4, we will derive results for one ansatz and show results for the other. We consider an expansion around a flat background,

$$B(\rho) = 1 + \epsilon h_{(1)}(\rho) + \epsilon^2 h_{(2)}(\rho) + \cdots \ , \tag{6.4}$$

$$A(\rho) = 1 + \epsilon A_{(1)}(\rho) + \epsilon^2 A_{(2)}(\rho) + \cdots \ , \tag{6.5}$$

with the two linear combination of the equations of motion, that we obtained in chapter 3,

$$g^{\mu\nu} I_{\mu\nu} = 6\beta \Box R - R = 0, \tag{6.6}$$

where $I_{\mu\nu}$ is equations of motion (2.3), and $\mu, \nu, \in \{0, 1, 2, 3\}$. Recall that in this section we still work with $\beta = 0 \rightarrow R = 0$.

6.1.1 First order expansion

The first order expansion we work on is given by,

$$A(\rho) = 1 + \epsilon A_{(1)}(\rho) + O\left(\epsilon^2\right), \tag{6.7}$$

$$B(\rho) = 1 + \epsilon h_{(1)}(\rho) + O\left(\epsilon^2\right). \tag{6.8}$$

The equations of motion in matrix form can be written as,

$$E_{\mu\nu} = \begin{bmatrix} E_{tt} & 0 & 0 & 0 \\ 0 & E_{\rho\rho} & 0 & 0 \\ 0 & 0 & E_{\theta\theta} & 0 \\ 0 & 0 & 0 & E_{\phi\phi} \end{bmatrix} = 0.$$

To simplify the equations, we substitute (6.7), into into the equations of motion $E_{\mu\nu}$, and we find that,

$$
\begin{aligned}
E_{tt} =& \left(-\frac{2A_{(1)}(\rho)}{3\rho^4}+\frac{2A'_{(1)}(\rho)}{3\rho^3}-\frac{A''_{(1)}(\rho)}{3\rho^2}+\frac{A_{(1)}(\rho)}{\rho^2}+\frac{A'_{(1)}(\rho)}{\rho}-\frac{A_{(1)}{}^{(3)}(\rho)}{3\rho}\right)\epsilon \\
&-\left(\frac{4h_{(1)}{}^{(3)}(\rho)}{3\rho}-\frac{1}{3}h_{(1)}{}^{(4)}(\rho)\right)\epsilon+O\left(\epsilon^2\right), \\
E_{\rho\rho} =& \left(\frac{2A_{(1)}(\rho)}{3\rho^4}-\frac{A''_{(1)}(\rho)}{3\rho^2}-\frac{A_{(1)}(\rho)}{\rho^2}+\frac{2h'_{(1)}(\rho)}{3\rho^3}-\frac{2h''_{(1)}(\rho)}{3\rho^2}+\frac{h'_{(1)}(\rho)}{\rho}-\frac{h_{(1)}{}^{(3)}(\rho)}{3\rho}\right)\epsilon+O\left(\epsilon^2\right), \\
E_{\theta\theta} =& \left(-\frac{2A_{(1)}(\rho)}{3\rho^2}-\frac{1}{2}\rho A'_{(1)}(\rho)+\frac{A'_{(1)}(\rho)}{3\rho}-\frac{1}{6}\rho A_{(1)}{}^{(3)}(\rho)+\frac{1}{2}\rho^2 h''_{(1)}(\rho)\right)\epsilon \\
&-\left(\frac{1}{6}\rho^2 h_{(1)}{}^{(4)}(\rho)+\frac{1}{2}\rho h'_{(1)}(\rho)-\frac{h'_{(1)}(\rho)}{3\rho}+\frac{1}{3}h''_{(1)}(\rho)-\frac{1}{2}\rho h_{(1)}{}^{(3)}(\rho)\right)\epsilon+O\left(\epsilon^2\right), \\
E_{\phi\phi} =& \left(-\frac{2A_{(1)}(\rho)}{3\rho^2}-\frac{1}{2}\rho A'_{(1)}(\rho)+\frac{A'_{(1)}(\rho)}{3\rho}-\frac{1}{6}\rho A_{(1)}{}^{(3)}(\rho)+\frac{1}{2}\rho^2 h''_{(1)}(\rho)\right)\sin^2(\theta)\epsilon \\
&\left(-\frac{1}{6}\rho^2 h_{(1)}{}^{(4)}(\rho)+\frac{1}{2}\rho h'_{(1)}(\rho)-\frac{h'_{(1)}(\rho)}{3\rho}+\frac{1}{3}h''_{(1)}(\rho)-\frac{1}{2}\rho h_{(1)}{}^{(3)}(\rho)\right)\sin^2(\theta)\epsilon+O\left(\epsilon^2\right),
\end{aligned}
\tag{6.9}
$$

we set $\theta=\frac{\pi}{2}\rightarrow sin(\theta)=1$, then take the linear combination,

$$
\begin{aligned}
\frac{m_2^2}{A(\rho)}E_{\rho\rho}+2\frac{m_2^2}{\rho^2}E_{\theta\theta}-\frac{1}{B(\rho)}E_{tt}=0, \\
\frac{m_2^2}{A(\rho)}E_{\rho\rho}+2\frac{m_2^2}{\rho^2}E_{\theta\theta}+\frac{1}{B(\rho)}E_{tt}=0,
\end{aligned}
\tag{6.10}
$$

substitute B, and A, to (6.9) and (6.10) up to $O\left(\epsilon^2\right)$ in ϵ, we get,

$$
\begin{aligned}
&-2\rho A'_{(1)}(\rho)-2A_{(1)}(\rho)+\rho^2 h''_{(1)}(\rho)+2\rho h'_{(1)}(\rho)=0, \\
&2\rho^3 A_{(1)}{}^{(3)}(\rho)+2\rho^2 A''_{(1)}(\rho)-4\rho A'_{(1)}(\rho)+4A_{(1)}(\rho) \\
&-3\rho^4 h''_{(1)}(\rho)+2\rho^4 h_{(1)}{}^{(4)}(\rho)-6\rho^3 h'_{(1)}(\rho)+8\rho^3 h_{(1)}{}^{(3)}(\rho)=0.
\end{aligned}
\tag{6.11}
$$

To decouple these equations of motion (6.11), we perform the following substitution,

$$
\begin{aligned}
Y_{(1)}(\rho) &= \frac{1}{\rho}\left(\rho A_{(1)}\right)', \\
Z_{(1)}(\rho) &= \left(\rho^2 h'_{(1)}\right)',
\end{aligned}
\tag{6.12}
$$

solving for $A_{(1)}$, and $h_{(1)}$, with their derivatives from the substitution (6.12), we get,

$$\begin{aligned} A'_{(1)}(\rho) &= \frac{\rho Y_{(1)}(\rho) - A_{(1)}(\rho)}{\rho}, \\ A''_{(1)}(\rho) &= \frac{-\rho A'_{(1)}(\rho) + A_{(1)}(\rho) + \rho^2 Y'_{(1)}(\rho)}{\rho^2}, \\ {A_{(1)}}^{(3)}(\rho) &= \frac{-\rho^2 A''_{(1)}(\rho) + 2\rho A'_{(1)}(\rho) - 2A_{(1)}(\rho) + \rho^3 Y''_{(1)}(\rho)}{\rho^3}, \\ h''_{(1)}(\rho) &= \frac{Z_{(1)}(\rho) - 2\rho h'_{(1)}(\rho)}{\rho^2}, \\ {h_{(1)}}^{(3)}(\rho) &= \frac{-2h'_{(1)}(\rho) - 4\rho h''_{(1)}(\rho) + Z'_{(1)}(\rho)}{\rho^2}, \\ {h_{(1)}}^{(4)}(\rho) &= \frac{-6h''_{(1)}(\rho) - 6\rho {h_{(1)}}^{(3)}(\rho) + Z''_{(1)}(\rho)}{\rho^2}, \end{aligned} \tag{6.13}$$

then substitute (6.13) to (6.11), then we get,

$$\begin{aligned} &\frac{m_2^2\left(Z_{(1)}(\rho) - 2\rho Y_{(1)}(\rho)\right)}{\rho^2} = 0 \implies Z_{(1)}(\rho) = 2\rho Y_{(1)}(\rho), \\ \implies &\frac{2m_2^2\left(Y_{(1)}(\rho) - Y''_{(1)}(\rho)\right)}{\rho} = 0 \implies Y_{(1)}(\rho) - Y''_{(1)}(\rho) = 0, \end{aligned} \tag{6.14}$$

using back substitution (6.12) to solve (6.14) solve for $A_{(1)}$ and,

$$A_{(1)}(\rho) = \frac{c_1}{\rho} + \frac{c_2 e^{-\rho}(-\rho - 1)}{\rho} + \frac{c_3 e^{\rho}(\rho - 1)}{2\rho} \tag{6.15}$$

where c_1, c_2 and c_3 are integration constants. Now we solve for $h_{(1)}(\rho)$, using the substitution $Z_{(1)} = 2\rho Y_{(1)}(\rho)$, and it's derivatives, to get,

$$\begin{aligned} h_{(1)}(\rho) &= \int \frac{\int \left(Z_{(1)}(\rho) Y_{(1)}(\rho)\right) d\rho + c_4}{\rho^2} d\rho + c_5, \\ h_{(1)}(\rho) &= \frac{2c_2 e^{-\rho}}{\rho} + \frac{c_3 e^{\rho}}{\rho} - \frac{c_4}{\rho} + c_5 \end{aligned} \tag{6.16}$$

As discussed in [1], we expect our solution to have 4 free parameters and indeed, substituting (6.16 back into the equations of motions shows that we must have $c_1 = c_4$. Furthermore, requiring that our solution asymptote to flat space means one must eliminate the growing Yukawa mode and we set $c_3 = 0$. The constant c_5 corresponds to a trivial rescaling of the time coordinate and requiring $\lim_{r\to\infty} h(r) = 1$ sets $c_5 = 0$. So finally the first order solution for the metric components is

$$A = 1 + \epsilon\left(\frac{c_1}{\rho} - \frac{c_2 e^{-\rho}(\rho+1)}{\rho}\right) + \ldots \tag{6.17}$$

$$h = 1 + \epsilon\left(-\frac{c_1}{\rho} + 2c_2\frac{e^{-\rho}}{\rho}\right) + \ldots \tag{6.18}$$

which is a special case of results in [19]. Given the boundary conditions we have imposed, at first order, there are two independent parameters in this expansion, c_1 and c_2. The parameter c1 is related to the mass, while c_2 is related to the strength of the Yukawa mode $e^{-m_2 r}/r$. Rescaling that, $\rho = m_2 r = r/L_\alpha$, and letting

$$\epsilon L_\alpha c_1 = \lambda, \tag{6.19}$$

$$\epsilon L_\alpha c_2 = 2M. \tag{6.20}$$

The (6.19) and (6.20) shows the results after scaling with L_α. We see that we can consider an expansion in M and λ of the form,

$$\begin{aligned}
B(\rho) &= 1 + \sum_{\substack{n=1,2,\ldots \\ 0\le i\le n}} M^i \lambda^{n-i} h_{(n,i)}(\rho), \\
\implies B(\rho) &= 1 + \sum_{\substack{n=1,2,\ldots \\ 0\le i\le n}} \left(\frac{M}{L_\alpha}\right)^i \left(\frac{\lambda}{L_\alpha}\right)^{n-i} h_{(n,i)}(\rho), \\
A(\rho) &= 1 + \sum_{\substack{n=1,2,\ldots \\ 0\le i\le n}} \lambda^{n-i} A_{(n,i)}(\rho), \\
\implies A(\rho) &= 1 + \sum_{\substack{n=1,2,\ldots \\ 0\le i\le n}} \left(\frac{M}{L_\alpha}\right)^i \left(\frac{\lambda}{L_\alpha}\right)^{n-i} A_{(n,i)}(\rho),
\end{aligned} \tag{6.21}$$

with $h_{n,i}(\rho)$ and $A_{n,i}(\rho)$ dimensionless. To get a feel for the parameter λ, we note that the effective Newtonian potential seen by a particle far away from the black hole is

$$\phi_N = \tfrac{1}{2}|g_{00}| \approx 1 - \frac{M}{r} + \frac{\lambda e^{-r/L_\alpha}}{r}\ . \tag{6.22}$$

In practice, since the equations are linear at each order of ϵ, one can also use the expansion (6.21) and easily read off powers of M and λ from (6.16). Notice that near the horizon (at $r = r_0$) one might roughly expect

$$-\frac{\epsilon c_1}{r} = \frac{2M}{r_0} \sim 1\ , \tag{6.23}$$

even for small black holes. Similarly, for the other part of g_{00},

$$-\frac{\epsilon c_2 e^{-r/L_\alpha}}{r} = \frac{2\lambda e^{-r_0/L_\alpha}}{r_0} \sim \frac{\lambda e^{-2M/L_\alpha}}{M}, \tag{6.24}$$

which is, a priori, difficult to estimate. Given that (6.23) and (6.24) are not necessarily small near the horizon, one should not generically expect the expansion, (6.4), to be vaid there. On the other hand, if $h_{(n,i)}(\rho)$ and $A_{(n,i)}(\rho)$ fall off fast enough,the expansion should become good some distance from the horizon as these terms become small. In fact, we will find that as $|\lambda|$ increases, our expansion becomes worse and worse very near the horizon but rapidly converges as one moves just a small distance away from the horizon (measured in units of L_α).

Finally, we have a technical aside. We find it convenient to introduce a third parameter in the asymptotic expansion, h_∞, so that

$$h(r) \quad = \quad h_\infty \left(1 + \sum_{\substack{n=1,2,\ldots \\ 0\le i\le n}} \left(\frac{M}{L_\alpha}\right)^i \left(\frac{\lambda}{L_\alpha}\right)^{n-i} h_{(n,i)}(\rho) \right), \tag{6.25}$$

As discussed earlier, one is free to scale h as long as one scales time to compensate. We would like to work in units for t which give $h_\infty = 1$, but one does not initially know how the near horizon expression for h, (6.3), should be scaled to achieve this. For convenience, when setting the boundary conditions for our numerical integration, we fix $c = 1$ in (4.8). Fitting a value for h_∞ to our numerical results then allows us to determine how we should rescale t to ensure that $g_{00}|_{\rho\to\infty} = -1$. Scaling $t \to t/\sqrt{h_\infty}$, and using $c = 1$, gives the modified form of (5.2) we will be using

$$T = \frac{1}{2\pi L_\alpha}\sqrt{\frac{f_1}{h_\infty}}. \tag{6.26}$$

6.1.2 Second order expansion

The second order expansion we work on is given by,

$$A(\rho) = 1 + \epsilon A_{(1)}(\rho) + \epsilon^2 A_{(2)}(\rho) + O\left(\epsilon^3\right), \tag{6.27}$$
$$B(\rho) = 1 + \epsilon h_{(1)}(\rho) + \epsilon^2 h_{(2)}(\rho) + O\left(\epsilon^3\right). \tag{6.28}$$

In this expansion we use the same methods we have used before in order to simplify and solve for the second coefficients of the solutions. The equations of motion up to $O\left(\epsilon^3\right)$ excluding the first order terms, are given by,

$$
\begin{aligned}
E_{tt} = {} & \frac{m^2 e^{-2\rho}}{12\rho^6}(-4e^{2\rho}\rho^5 {A_{(2)}}^{(3)}(\rho) - 4e^{2\rho}\rho^4 A''_{(2)}(\rho) + 4e^{2\rho}\left(3\rho^2-2\right)\rho^2 A_{(2)}(\rho) + 4e^{2\rho}\left(3\rho^2+2\right)\rho^3 A'_{(2)}(\rho) \\
& - 4e^{2\rho}\rho^6 {h_{(2)}}^{(4)}(\rho) - 16e^{2\rho}\rho^5 {h_{(2)}}^{(3)}(\rho) - 74c_2^2\rho^5 - 25c_2^2\rho^4 - 2c_1c_2e^{\rho}\rho^4 + 14c_2^2\rho^3 + 14c_1c_2e^{\rho}\rho^3 \\
& + 12c_1^2e^{2\rho}\rho^2 + 91c_2^2\rho^2 + 26c_1c_2e^{\rho}\rho^2 + 126c_2^2\rho + 36c_1c_2e^{\rho}\rho - 48c_1^2e^{2\rho} + 36c_1c_2e^{\rho} + 63c_2^2)\epsilon^2 + O\left(\epsilon^3\right), \\
E_{\rho\rho} = {} & \frac{e^{-2\rho}}{12\rho^6}(-4e^{2\rho}\rho^4 A''_{(2)}(\rho) - 4e^{2\rho}\left(3\rho^2-2\right)\rho^2 A_{(2)}(\rho) - 4e^{2\rho}\rho^5 {h_{(2)}}^{(3)}(\rho) - 8e^{2\rho}\rho^4 h''_{(2)}(\rho) \\
& + 4e^{2\rho}\left(3\rho^2+2\right)\rho^3 h'_{(2)}(\rho) + 43c_2^2\rho^4 + 82c_2^2\rho^3 - 40c_1c_2e^{\rho}\rho^3 + 12c_1^2e^{2\rho}\rho^2 + 29c_2^2\rho^2 \\
& - 68c_1c_2e^{\rho}\rho^2 - 42c_2^2\rho - 12c_1c_2e^{\rho}\rho + 16c_1^2e^{2\rho} - 12c_1c_2e^{\rho} - 21c_2^2)\epsilon^2 + O\left(\epsilon^3\right), \\
E_{\theta\theta} = {} & -\frac{1}{12\rho^4}(2\rho^5 {A_{(2)}}^{(3)}(\rho) + 8\rho^2 A_{(2)}(\rho) + \left(6\rho^5-4\rho^3\right)A'_{(2)}(\rho) - 6\rho^6 h''_{(2)}(\rho) + 2\rho^6 {h_{(2)}}^{(4)}(\rho) \\
& - 6\rho^5 h'_{(2)}(\rho) + 6\rho^5 {h_{(2)}}^{(3)}(\rho) - 4\rho^4 h''_{(2)}(\rho) + 4\rho^3 h'_{(2)}(\rho) + 43c_2^2e^{-2\rho}\rho^5 + 82c_2^2e^{-2\rho}\rho^4 \\
& - 20c_1c_2e^{-\rho}\rho^4 + 70c_2^2e^{-2\rho}\rho^3 - 54c_1c_2e^{-\rho}\rho^3 + 12c_1^2\rho^2 - 13c_2^2e^{-2\rho}\rho^2 - 74c_1c_2e^{-\rho}\rho^2 \\
& - 84c_2^2e^{-2\rho}\rho - 24c_1c_2e^{-\rho}\rho - 42c_2^2e^{-2\rho} - 24c_1c_2e^{-\rho} + 32c_1^2)\epsilon^2 + O\left(\epsilon^3\right), \\
E_{\phi\phi} = {} & \frac{e^{-2\rho}\sin^2(\theta)}{12\rho^4}(-2e^{2\rho}\rho^5 {A_{(2)}}^{(3)}(\rho) - 8e^{2\rho}\rho^2 A_{(2)}(\rho) - 2e^{2\rho}\left(3\rho^2-2\right)\rho^3 A'_{(2)}(\rho) + 6e^{2\rho}\rho^6 h''_{(2)}(\rho) \\
& - 2e^{2\rho}\rho^6 {h_{(2)}}^{(4)}(\rho) + 6e^{2\rho}\rho^5 h'_{(2)}(\rho) - 6e^{2\rho}\rho^5 {h_{(2)}}^{(3)}(\rho) + 4e^{2\rho}\rho^4 h''_{(2)}(\rho) - 4e^{2\rho}\rho^3 h'_{(2)}(\rho) \\
& - 43c_2^2\rho^5 - 82c_2^2\rho^4 + 20c_1c_2e^{\rho}\rho^4 - 70c_2^2\rho^3 + 54c_1c_2e^{\rho}\rho^3 - 12c_1^2e^{2\rho}\rho^2 + 13c_2^2\rho^2 \\
& + 74c_1c_2e^{\rho}\rho^2 + 84c_2^2\rho + 24c_1c_2e^{\rho}\rho - 32c_1^2e^{2\rho} + 24c_1c_2e^{\rho} + 42c_2^2)\epsilon^2 + O\left(\epsilon^3\right).
\end{aligned}
\tag{6.29}
$$

Now using the linear combination (6.10), we get the following equations to solve,

$$
\begin{aligned}
& -\frac{m_2^2e^{-2\rho}}{2\rho^4}(4e^{2\rho}\rho^3 A'_{(2)}(\rho) + 4e^{2\rho}\rho^2 A_{(2)}(\rho) - 2e^{2\rho}\rho^4 h''_{(2)}(\rho) - 4e^{2\rho}\rho^3 h'_{(2)}(\rho) + 2c_2^2\rho^3 \\
& + 16c_2^2\rho^2 - 7c_1c_2e^{\rho}\rho^2 + 12c_2^2\rho - 9c_1c_2e^{\rho}\rho + 4c_1^2e^{2\rho} - 9c_1c_2e^{\rho} + 6c_2^2) = 0, \\
& \frac{m_2^2e^{-2\rho}}{6\rho^6}(-4e^{2\rho}\rho^5 {A_{(2)}}^{(3)}(\rho) - 4e^{2\rho}\rho^4 A''_{(2)}(\rho) + 8e^{2\rho}\rho^3 A'_{(2)}(\rho) - 8e^{2\rho}\rho^2 A_{(2)}(\rho) \\
& + 6e^{2\rho}\rho^6 h''_{(2)}(\rho) - 4e^{2\rho}\rho^6 {h_{(2)}}^{(4)}(\rho) + 12e^{2\rho}\rho^5 h'_{(2)}(\rho) - 16e^{2\rho}\rho^5 {h_{(2)}}^{(3)}(\rho) - 80c_2^2\rho^5 \\
& - 73c_2^2\rho^4 + 19c_1c_2e^{\rho}\rho^4 - 22c_2^2\rho^3 + 41c_1c_2e^{\rho}\rho^3 + 73c_2^2\rho^2 + 53c_1c_2e^{\rho}\rho^2 + 126c_2^2\rho \\
& + 36c_1c_2e^{\rho}\rho - 48c_1^2e^{2\rho} + 36c_1c_2e^{\rho} + 63c_2^2) = 0.
\end{aligned}
\tag{6.30}
$$

Using the same substitution method,

$$
\begin{aligned}
Y_{(2)}(\rho) &= \frac{1}{\rho}\left(\rho A_{(2)}\right)', \\
Z_{(2)}(\rho) &= \left(\rho^2 h'_{(2)}\right)',
\end{aligned}
\tag{6.31}
$$

solving for $Z_{(2)}$, we get,

$$Z_{(2)} = 2\rho Y_{(2)}(\rho) + F_2(\rho), \tag{6.32}$$

where,

$$\begin{aligned} F_{(2)} = &\frac{3c_2^2 e^{-2\rho}}{\rho^2} - \frac{9c_1c_2e^{-\rho}}{2\rho^2} + \frac{2c_1^2}{\rho^2} + 8c_2^2e^{-2\rho} + c_2^2e^{-2\rho}\rho \\ &+ \frac{6c_2^2e^{-2\rho}}{\rho} - \frac{7}{2}c_1c_2e^{-\rho} - \frac{9c_1c_2e^{-\rho}}{2\rho}. \end{aligned} \tag{6.33}$$

Now using $Z_{(2)}$ and it's derivatives to solve for $A_{(1)}$ and $h_{(1)}$, we get

$$Y''_{(2)}(\rho) - Y_{(2)}(\rho) = J_2(\rho). \tag{6.34}$$

where,

$$\begin{aligned} J_2(\rho) = &\frac{e^{-2\rho}}{4\rho^5}(-30c_2^2\rho^5 - 51c_2^2\rho^4 + 4c_2c_1e^{\rho}\rho^4 - 70c_2^2\rho^3 \\ &+ 20c_2c_1e^{\rho}\rho^3 - 71c_2^2\rho^2 + 4c_1^2e^{2\rho}\rho^2 + 48c_2c_1e^{\rho}\rho^2 - 54c_2^2\rho \\ &+ 84c_2c_1e^{\rho}\rho - 48c_1^2e^{2\rho} + 84c_2c_1e^{\rho} + 27c_2^2). \end{aligned} \tag{6.35}$$

Now solving (6.34) for $Y_{(2)}$, we use the Green's function shown in the appendix E, and the solution of the second order expansion is given by,

$$\begin{aligned} h_{(2)}(\rho) = &c_2^2\left(-\frac{69e^{\rho}\text{Ei}(-3\rho)}{32\rho} + \frac{3\text{Ei}(-2\rho)}{2} + \frac{69e^{-\rho}\text{Ei}(-\rho)}{32\rho} + \frac{15e^{-2\rho}}{16\rho^2} - \frac{e^{-2\rho}}{4\rho}\right) \\ &+ c_1c_2\left(\frac{e^{\rho}\text{Ei}(-2\rho)}{\rho} - \frac{e^{-\rho}}{2\rho^2} - \frac{e^{-\rho}\log(\rho)}{\rho}\right), \end{aligned} \tag{6.36}$$

$$\begin{aligned} A_{(2)}(\rho) = &\frac{c_1^2}{\rho^2} + \\ &+ c_2c_1\left(\frac{1}{2}e^{\rho}\text{Ei}(-2\rho) - \frac{e^{\rho}\text{Ei}(-2\rho)}{2\rho} - \frac{7e^{-\rho}}{4\rho^2} - \frac{3e^{-\rho}}{2\rho} + \frac{1}{2}e^{-\rho}\log(\rho) + \frac{e^{-\rho}\log(\rho)}{2\rho}\right) \\ &+ c_2^2\left(-\frac{69}{64}e^{\rho}\text{Ei}(-3\rho) + \frac{69e^{\rho}\text{Ei}(-3\rho)}{64\rho} - \frac{69}{64}e^{-\rho}\text{Ei}(-\rho) - \frac{69e^{-\rho}\text{Ei}(-\rho)}{64\rho}\right) \\ &+ c_2^2\left(\frac{9e^{-2\rho}}{16\rho^2} + \frac{5e^{-2\rho}}{4} + \frac{21e^{-2\rho}}{16\rho}\right) \end{aligned} \tag{6.37}$$

where Ei(-x) s the exponential integral defined by,

$$\text{Ei}\,(x) = -\int_x^{\infty}\frac{e^{-t}}{t}dt \sim -\frac{e^{-x}}{x} \quad \text{(for large x)}. \tag{6.38}$$

To get some intuition for the terms involving Ei appearing in (6.36) and (6.37), note that for large x,

$$- e^x \mathrm{Ei}(-x) \sim \frac{1}{x} + O\left(x^{-2}\right), \tag{6.39}$$

whereas, for small x,

$$- e^x \mathrm{Ei}(-x) \sim \gamma + log(x) + O\left(x\right), \tag{6.40}$$

where γ is Euler's constant. The integrations in (6.36) produce two constants of integration which we neglected.One of the constants can be absorbed into the mass and the other corresponds to are scaling of the time coordinate so that no new degrees of freedom are introduced at this order. For more details of the calculation see appendix E.

6.1.3 Higher order expansion

We now look at higher order expansions, and generalize the substitution up to order n,

$$\begin{aligned} Y_{(n)} \frac{1}{\rho}\left(\rho A_{(n)}\right)' \\ Z_{(n)} &= \left(\rho^2 h'_{(n)}\right)', \end{aligned} \tag{6.41}$$

as we have seen from the first order and the second order expansions, we can also generalize the following results as,

$$\begin{aligned} Z_{(n)} &= 2\rho Y_{(n)} + F_{(n)}, \\ Y_{(n)} - Y''_{(n)} &= J_n, \end{aligned} \tag{6.42}$$

for some functions $F_{(n)}$ and $J_{(n)}$. As at second order, we find solutions of the form,

$$\begin{aligned} A_{(n)} &= \frac{e^{-\rho}(\rho+1)}{2\rho}\int^{\rho} dse^{s} J_{(n)}(s) - \frac{e^{-\rho}(\rho-\rho)}{2\rho}\int^{\rho} dse^{-s} J_{(n)}(s) \\ &- \frac{1}{\rho}\int^{\rho} dssJ_{(n)}(s), \end{aligned} \tag{6.43}$$

$$h_{(n)} = \int d\rho \left(\frac{2\rho A_{(n)} + \int d\rho F_{(n)}}{\rho^2}\right). \tag{6.44}$$

We note that, as for the second order case, no new degrees of freedom are introduced at each order. Unfortunately, in practice, the integrals become more and more difficult to evaluate because of nontrivial integrals involving Ei and Meijer-G functions, although in principle, it seems one could evaluate up to arbitrary order using numerical integration. More details, including the third order solutions, are given in appendix E or appendix A of [20].

Chapter 7

Matching Numerical Solutions and the Asymptotic Expansion

7.1 Features of the asymptotic expansion

We would like to investigate some features of the asymptotic expansion. When all the dust has settled, we find that the expression for the metric components to second order in M and λ is

$$
\begin{aligned}
h = 1 &- \frac{2Mm_2}{\rho} + \frac{2m_2\lambda e^{-\rho/m_2 L_\alpha}}{\rho} + \left(\frac{M\lambda}{L_\alpha^2}\right) e^{-\rho}\left(\frac{2e^{2\rho}\mathrm{Ei}(-2\rho)}{\rho} - \frac{2log(\rho)}{\rho} - \frac{1}{\rho^2}\right) \\
&+ \left(\frac{\lambda^2}{L_\alpha^2}\right)\left(-\frac{69e^{\rho}\mathrm{Ei}(-3\rho)}{32\rho} + \frac{3\mathrm{Ei}(-2\rho)}{2} + \frac{69e^{-\rho}\mathrm{Ei}(-\rho)}{32\rho} + \frac{15e^{-2\rho}}{16\rho^2} - \frac{e^{-2\rho}}{4\rho}\right) + \dots
\end{aligned}
\tag{7.1}
$$

$$
\begin{aligned}
A = 1 &+ \frac{2Mm_2}{\rho} + \frac{4M^2m_2^2}{\rho^2} - \frac{\lambda}{L_\alpha}e^{-\rho}\left(1 + \frac{1}{\rho}\right) \\
&+ \left(\frac{M\lambda}{L_\alpha^2}\right)\left(\frac{1}{2}e^{\rho}\mathrm{Ei}(-2\rho) - \frac{e^{\rho}\mathrm{Ei}(-2\rho)}{2\rho} - \frac{7e^{-\rho}}{4\rho^2} - \frac{3e^{-\rho}}{2\rho} + \frac{1}{2}e^{-\rho}\log(\rho) + \frac{e^{-\rho}\log(\rho)}{2\rho}\right) \\
&+ \left(\frac{\lambda^2}{L_\alpha^2}\right)\left(-\frac{69}{64}e^{\rho}\mathrm{Ei}(-3\rho) + \frac{69e^{\rho}\mathrm{Ei}(-3\rho)}{64\rho} - \frac{69}{64}e^{-\rho}\mathrm{Ei}(-\rho) - \frac{69e^{-\rho}\mathrm{Ei}(-\rho)}{64\rho}\right) \\
&+ \left(\frac{\lambda^2}{L_\alpha^2}\right)\left(\frac{9e^{-2\rho}}{16\rho^2} + \frac{5e^{-2\rho}}{4} + \frac{21e^{-2\rho}}{16\rho}\right) + \dots
\end{aligned}
\tag{7.2}
$$

Notice that with $\lambda = 0$, it is easy to check that we recover the Schwartzchild solution (up to $O(M^2)$):

$$h_{Schwartzchild} = 1 - \frac{2M}{r}, \tag{7.3}$$

$$A_{Schwartzchild} = \left(1 - \frac{2M}{r}\right)^{-1} = 1 + \frac{2M}{r} + \frac{4M^2}{r^2} + \frac{8M^3}{r^3} + \dots \tag{7.4}$$

At large r, we find the slightly simpler expressions

$$\begin{aligned} h &= 1 - \frac{2Mm_2}{\rho} + \frac{2m_2\lambda e^{-\rho/m_2 L_\alpha}}{\rho} + \left(\frac{M\lambda}{L_\alpha^2}\right) e^{-\rho}\left(-\tfrac{2\log(\rho)}{\rho} - \tfrac{2}{\rho^2} + \mathcal{O}(\rho^{-3})\right) \\ &\quad + \left(\frac{\lambda^2}{L_\alpha^2}\right) e^{-2\rho}\left(-\tfrac{1}{\rho} - \tfrac{1}{8\rho^2} + \mathcal{O}(\rho^{-3})\right) + \cdots , \end{aligned} \tag{7.5}$$

$$\begin{aligned} A &= 1 + \frac{2Mm_2}{\rho} + \frac{4M^2m_2^2}{\rho^2} - \left(\frac{\lambda}{L_\alpha}\right) e^{-\rho}\left(1 + \tfrac{1}{\rho}\right) \\ &\quad + \left(\frac{M\lambda}{L_\alpha^2}\right) e^{-\rho}\left(\log(\rho)\left(1 + \tfrac{1}{\rho}\right) - \tfrac{7}{2\rho} - \tfrac{11}{4\rho^2} + \mathcal{O}(\rho^{-3})\right) \\ &\quad + \left(\frac{\lambda^2}{L_\alpha^2}\right) e^{-2\rho}\left(\tfrac{5}{4} + \tfrac{11}{4\rho} + \tfrac{1}{12\rho^2} + \mathcal{O}(\rho^{-3})\right) \\ &\quad + \cdots . \end{aligned} \tag{7.6}$$

We note that the expansions (7.5) and (7.6) look like trans-series (see [26] for a nice introduction to trans-series). Perhaps starting with an appropriate trans-series ansatz would be an easier way to approach this problem which we hope to look at in the future. Given the appearance of Yukawa modes at first order, it is perhaps not surprising that we end up with a trans-series expansion & and we conjecture that this feature will be a common characteristic of the asymptotical solutions of other higher derivative black holes.

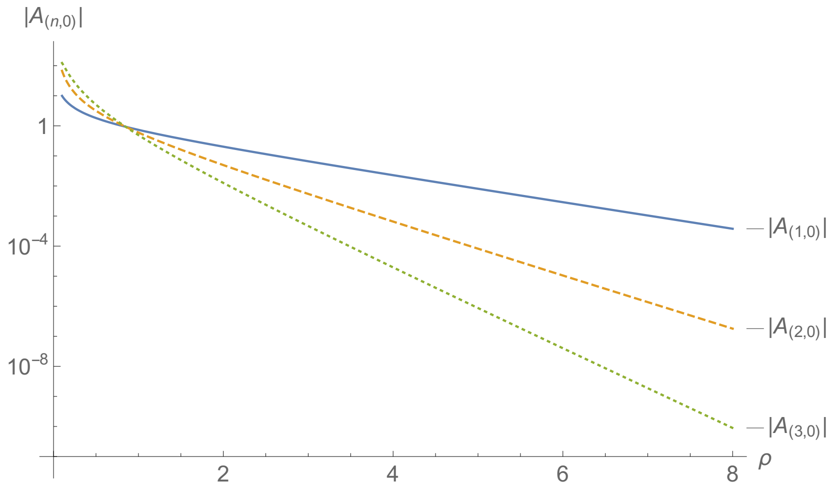

Figure 7.1: Log plot of $|A_{(n,0)}|$ for $n = 1, 2, 3$. In the expansion (7.6), $A_{(n,0)}$ is multiplied by a factor $M^0(\lambda/L_\alpha)^n$, i.e. $|A_{(n,0)}|$ are the pieces of the metric which depend only on λ and not M. The solid line is $|A_{(1,0)}|$, the dashed line is $|A_{(2,0)}|$ and the dotted line is $|A_{(3,0)}|$.

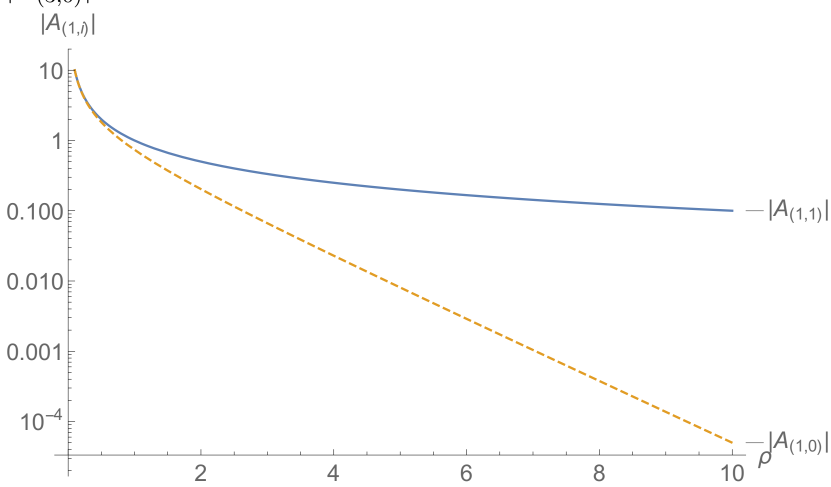

Figure 7.2: Log plot of the first order terms $|A_{(1,i)}|$ for $i = 0, 1$. In the expansion, (7.6), $A_{(1,i)}$ is multiplied by a factor $M^i\lambda^{(1-i)}$. The solid line is $|A_{(1,1)}|$ and the dotted line is $|A_{(1,0)}|$. For the Schwarzschild black hole, one only has the term $A_{(1,1)}$ at this order.

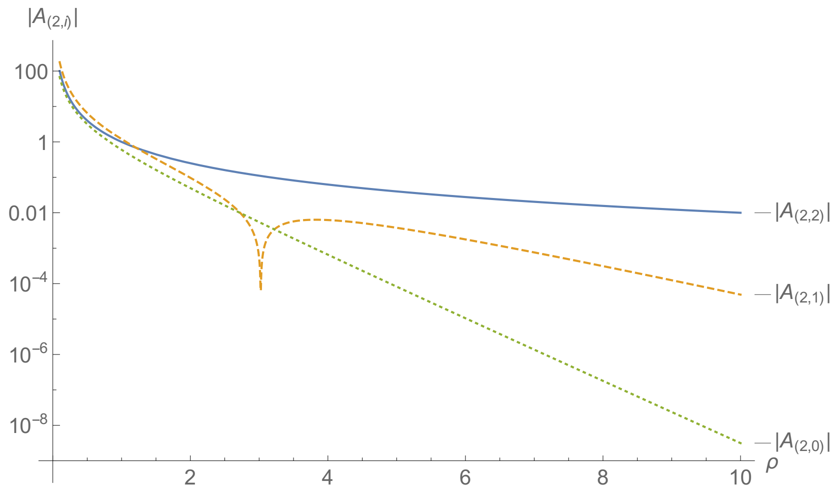

Figure 7.3: Log plot of the second order terms $|A_{(2,i)}|$ for $i = 0, 1, 2$. In the expansion, (7.6), $A_{(2,i)}$ is multiplied by a factor $M^i\lambda^{(2-i)}$. The solid line is $|A_{(2,2)}|$, the dashed line is $|A_{(2,1)}|$ and the dotted line is $|A_{(2,0)}|$. For the Schwarzschild black hole, one only has the term, $A_{(2,2)} = \frac{4}{\rho^2}$, at this order. The funny blip in $|A_{(2,1)}|$ corresponds to a sign change in $A_{(2,1)}$.

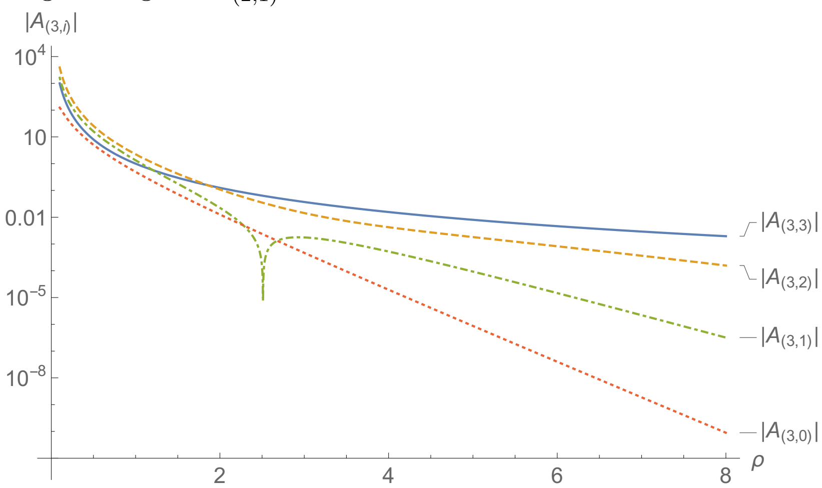

Figure 7.4: Log plot of the third order terms $|A_{(3,i)}|$ for $i = 0, 1, 2, 3$. In the expansion, (7.6), $A_{(3,i)}$ is multiplied by a factor $M^i\lambda^{(3-i)}$. The solid line is $|A_{(3,3)}|$, the dashed line is $|A_{(3,2)}|$, the dash-dotted line is $|A_{(3,1)}|$ and the dotted line is $|A_{(3,0)}|$. For the Schwarzschild black hole, one only has the term $A_{(3,3)} = \frac{8}{\rho^3}$ at this order. The funny blip in $|A_{(3,1)}|$ corresponds to a sign change in $A_{(3,1)}$.

To get a feel for expansion we have numerically plotted various terms in the expansion of A (7.6) or (3.19) of [20], in Figure 7.1-Figure 7.4. There are 3 notable features of these plots we would like to emphasize.

1. Figure 7.1 is a log plot of $A_{(1,0)}$, $A_{(2,0)}$, and $A_{(3,0)}$ as a function of $\rho = r/L_\alpha$. These are terms in (7.6) or (3.19) of [20] with coefficients λ, λ^2 and λ^3 that do not involve the mass M. It is clear from the plot that as ρ increases, successive terms become exponentially smaller at higher orders in the expansion. There is similar behavior when one compares other higher order terms in the expansion to the lower order term which effectively act as sources for them. This is good evidence that the expansion will converge at least at for sufficiently large ρ – we leave a more rigorous examination of convergence for possible future work. The plots in Figure 7.1 all approach straight lines indicating exponential fall off on a log plot.

2. Figure 7.2-Figure 7.4 are plots of different terms at the same order in the asymptotic expansion. What is clear is that for $\rho >> 1$, the terms $A_{(n,n)}$, which correspond to the pure Schwarzschild black hole, dominate for large ρ. Far from the horizon, the non-Schwarzschild terms become negligible. The non-Schwarzschild terms also become linear at large ρ once again indicating exponential fall off.

3. From Figure 7.1, one see that for small ρ, higher order terms in the expansion are larger than the lower order terms. This suggests that the expansion breaks down at some point. We know that the expansion for the Schwarzschild black hole only breaks down at the horizon in these coordinates (which would be at $\rho = 1$). We calculate the asymptotic expansion up to third order and our numerical investigations show that as λ becomes large, the asymptotic solution does not match well with the near horizon expansion. Consequently, we will need numerical integration to match the near horizon solution with the asymptotic expansion which is what we cover in the next section.

7.2 Matching Conditions

To match these solutions we will have to obtain matching conditions which are the two parameters $c_1 \to 2M$, and $c_2 \to \lambda$. The nonlinear model fit is obtained using MATHEMATICA, and it will help us obtain the mass M, and the Yukawa mode mass m_2. These results must somewhat agree with the mass up to some error with the mass obtained in the numerical solutions. We start with the differential equations in Chapter 4, and the second order asymptotic solutions we obtained in chapter 5 written in terms of r, instead of ρ, we recall that $\rho = m_2 r = \lambda r$.

We start with a similar procedure to the one outlined in [1]. For a fixed ρ_0, one uses the near horizon expansion to set the boundary conditions for numerical integration of the $\rho\rho$ and the $R = 0$ equation, starting a little bit away from the horizon. One then has to find δ so that the positive Yukawa mode, which would cause the solution to blow up, is not excited.

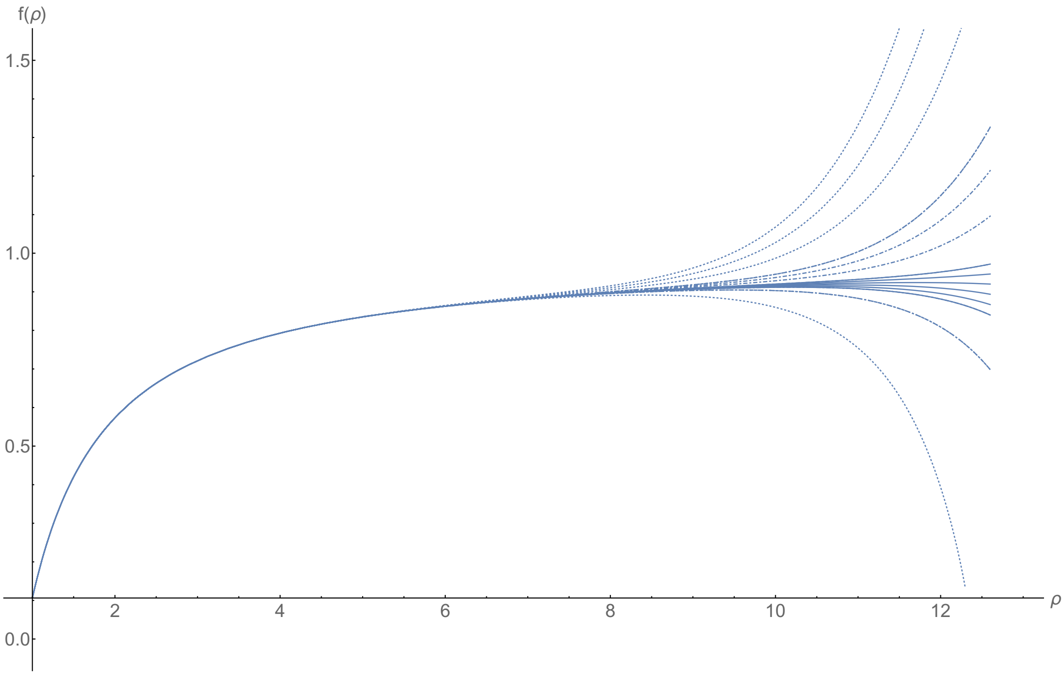

Figure 7.5: Numerical plot of $f(r)$ (with $\rho_0 = 0.9$) showing the procedure for tuning δ and extending how far one can integrate before the solution diverges. The dotted plots correspond to $0.06584 \leqslant \delta \leqslant 0.6616$. The dashed plots show the next step with $0.066032 \leqslant \delta \leqslant 0.66096$ and the solid plots show a further step with $0.0660704 \leqslant \delta \leqslant 0.0660832$.

In practice, we scan over a range of guesses for δ, successively tuning it so that the integration can extend further and further – this process is shown in Figure 7.5 as showed in chapter 4.

7.2.1 Convergence and Asymptotic

We would like to discuss some investigations of the near horizon and asymptotic expansions as well as our matching procedure.

In Figure 7.6 we show a numerical plot for $\rho_0 = 1.02$ and $\delta \approx 0.4266\ldots$, together with the near-horizon expansion at various orders. It is clear from the plot that the near horizon expansion breaks down pretty quickly, namely before $2\rho_0$, even at high order. We see that one does not greatly extend the expansion's domain of validity each time one increases the order of the expansion. We found this behavior was universal, overall values of ρ_0 investigated, with the near horizon expansion, even up to $\mathcal{O}(18)$, not extending far.

Given the computing power at our disposal, we did not extend our numerical integration much beyond 40-50 ρ_0. As mentioned, and as can be seen from Figure 7.1 - Figure 7.4, terms in our asymptotic expansion involving λ fall of exponentially fast. This means that our solution is soon very well approximated by the Schwarzschild making it relatively easy to find the mass, M, and h_∞. Using a non-linear fitting procedure we extract the parameters by finding the best fit between our numerical and asymptotic solutions (over some appropriately chosen range of ρ).

In Figure 7.7 we superimpose our fit of the $\mathcal{O}(2)$ and $\mathcal{O}(3)$ asymptotic expansions onto Figure 7.6 [1]. We see that there is a good match even close to the horizon. Furthermore, although it is not too easy to see on this graph, the $\mathcal{O}(3)$ plot looks like a better fit. Unfortunately, we also found that the asymptotic expansion is generally not good in the limited domain of validity of the near horizon expansion.

[1] The asymptotic expansions were fitted to the numerical results over the range $2.5\rho_0 < \rho < 28\rho_0$.

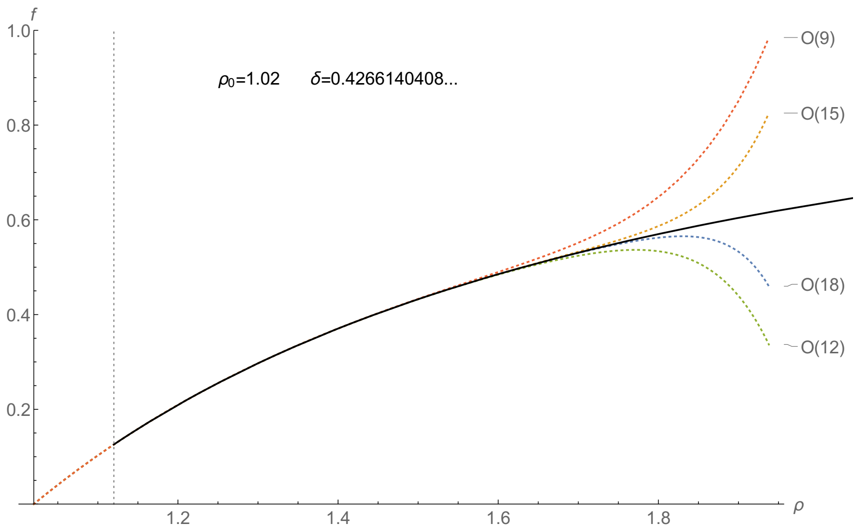

Figure 7.6: Plots of the near horizon expansion (4.9) for $f(\rho)$ compared with results of the numerical integration with $\rho_0 = 1.02$ and $\delta = 0.4266\ldots$. The numerical plot is shown as a solid line and the dashed lines are plots of the near horizon expansion up to orders 9,12,15 and 18 as labelled. The $\mathcal{O}(18)$ expansion was used to define boundary conditions for the numerical integration which started at $\rho = 1.12$. The point at which we started the numerical integration is denoted by a dashed grey vertical line in the plot.

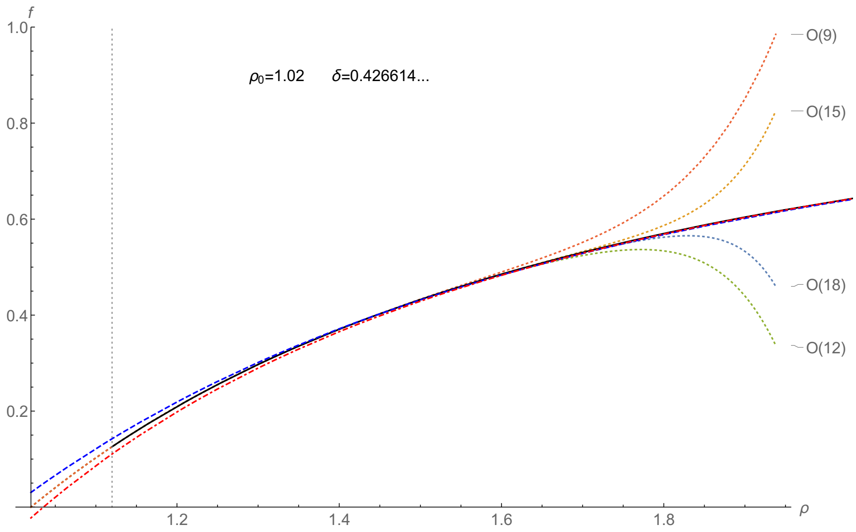

Figure 7.7: A repeat of Figure 7.6 with the fit of the $\mathcal{O}(2)$ (blue line with larger dashes) and $\mathcal{O}(3)$ (red dot-dashed line) asymptotic expansions added.

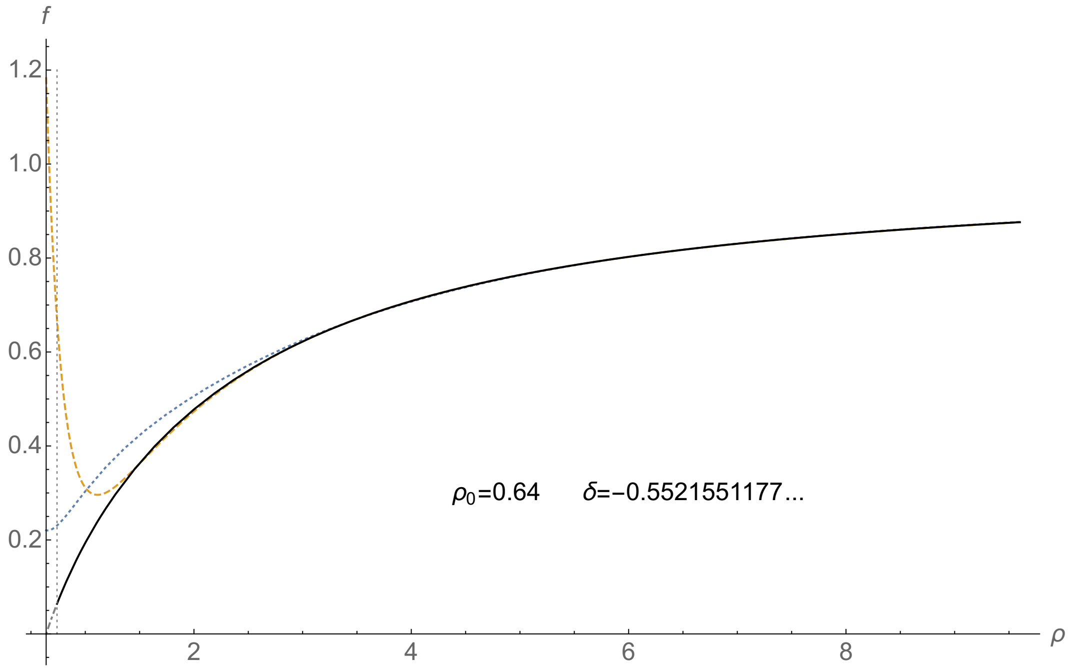

Figure 7.8: Plot of the numerical integration of $f(\rho)$ against fits of the second and third order asymptotic expansion. The numerical integration is denoted by a solid back line, the third order fit is a mustard dashed line and the second order fit is a blue dotted line. A vertical dotted line denotes where we started the numerical integration and the dash-dotted grey line is the near horizon expansion. The horizon radius was $\rho_0 = 0.64$ and we found $\delta = -0.552155\ldots$. The fitted values for λ at second and third order where, $\lambda_{\mathcal{O}(2)} = 1.48\ldots$ and $\lambda_{\mathcal{O}(3)} = 0.42\ldots$, respectively. The asymptotic expansions were fitted to the numerical results over the range $\rho_* = 4\rho_0$ to $\rho_f = 40\rho_0$.

This is evident in Figure 7.8 where we've plotted numerical results for $\rho_0 = 0.64$ together with fits for the asymptotic expansions. We see that the $O(3)$ expansion diverges more from the near horizon solutions than the $O(2)$ expansion but on the other hand, the $O(3)$ expansion converges more rapidly to the numerical results as ρ increases. We find similar results for other values of ρ_0 with the divergence between the asymptotic and near horizon expansion increasing as $|\lambda|$ increases (where λ is obtained from our best fit procedure). This behavior is not too surprising if we look back at $Ai's$ figures, where terms which appear with a coefficient of λ can be very large for small ρ but fall off very fast. Putting this all together strongly suggests that expansion (7.6) or (3.19) of [20] becomes good for ρ sufficiently large. On the other hand, this means that in general, to find the relationship between near horizon and asymptotic data some numerical integration is required.

Lastly, we would like to note that while the values obtained for M and h_∞ by fitting the $\mathcal{O}(2)$ or $\mathcal{O}(3)$ asymptotic expansions to the numerical results only differ by 1%

or less, the values obtained for λ can have much larger variation. For example, for the case shown in Figure 7.8, fitting the second order expansion gave $\lambda_{\mathcal{O}(2)} \approx 1.48$ while fitting the third expansion gave $\lambda_{\mathcal{O}(3)} \approx 0.42$. Once again, we can understand this from the fact that terms involving λ fall off very quickly leaving an essentially Schwarzschild like solution. While we can see from near horizon perturbation figures, that the numerical and asymptotic results agree well beyond a few multiples of ρ_0, to get a better feel for our results we now switch to log-log plots to make small differences at large ρ more visible.

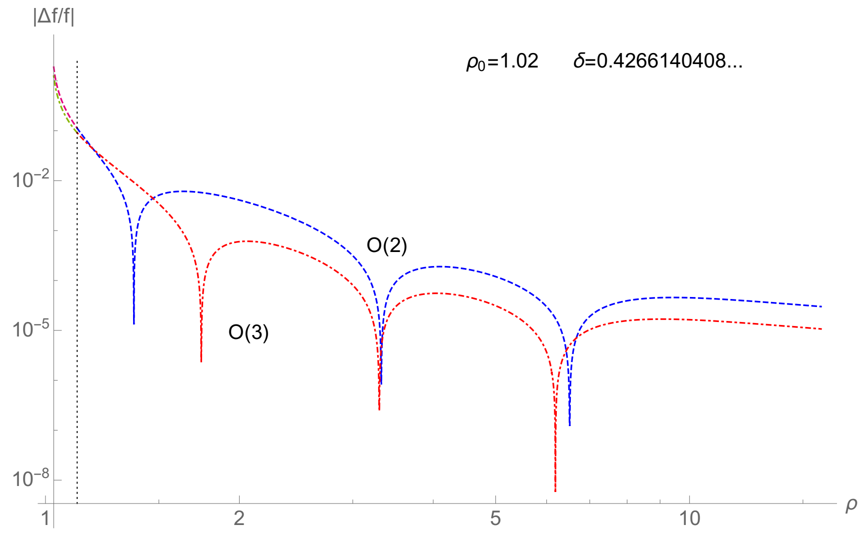

Figure 7.9: Log-log plots of $|\Delta f|/f$ against ρ for the second (blue dashed line) and third order (red dot-dashed line) asymptotic expansions (see (7.7)-(7.8) for the definition of $|\Delta f|/f$). The fits for λ where, $\lambda_{\mathcal{O}(2)} = 0.79\ldots$ and $\lambda_{\mathcal{O}(3)} = 0.71\ldots$, over the range $\rho_* = 2.5\rho_0$ to $\rho_f = 40\rho_0$.

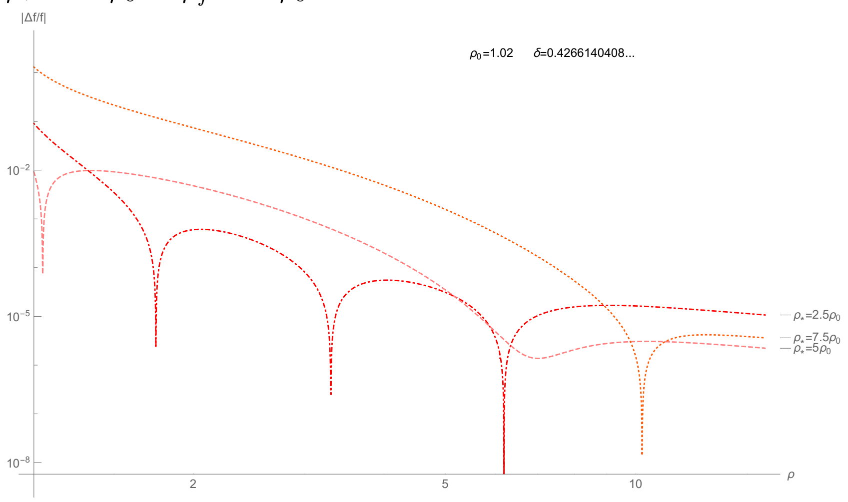

Figure 7.10: Log-log plots of $|\Delta f|/f$ against ρ for the third order asymptotic expansion fitted over the range, ρ_* to $\rho_f = 40\rho_0$, for various values of ρ_*. The fitted values for λ where, $\lambda_{\rho_*=2.5} = 0.79\ldots$(red dot-dashed line), $\lambda_{\rho_*=5.0} = 0.70\ldots$(pink dashed line) and $\lambda_{\rho_*=7.5} = 0.45\ldots$(orange dotted line).

The above figures are log-log plots of

$$\frac{|\Delta f|}{\tilde{f}} = \frac{|\tilde{f} - f_{\text{Asymptotic}}|}{\tilde{f}} \,, \tag{7.7}$$

where

$$\tilde{f} = \begin{cases} \text{Near horizon expansion for } f(\rho) \text{ for } \rho_0 \leqslant \rho \leqslant \rho_0 + \frac{1}{10} \\ \text{Numerical integration of } f(\rho) \text{ for } \rho \geqslant \rho_0 + \frac{1}{10} \end{cases} . \tag{7.8}$$

The plot in Figure 7.9 has the same δ as Figure 7.7 but it makes the difference between our fits and the numerical result more visible. The dotted vertical line in the plot corresponds to the point at which we set the boundary conditions for the numerical integration using the near horizon result. It is clear that the expansions are good fits for the numerical data and that increasing the order of the expansion improves the fit. In Figure 7.10, we wanted to check the effect of changing the range of ρ over which we fit the asymptotic expansion. We shifted the starting point, ρ_*, for the fit, keeping the end point fixed. In this case, the best fit was obtained at $\rho_* = 5\rho_0$. The different values we took for ρ_* had very little effect on the best fit for M (and h_∞) but λ changed considerably. Ideally, one should try to optimise the value of ρ_* to obtain the best fit since a priori one does not know at what radius one can really start to trust the asymptotic expansion. We did not bother doing that since it had little effect on the mass extracted from the solution which is what we were primarily interested in.

Chapter 8

Review of non-supersymmetric attractors

In this chapter we review the attractor mechanism extended to non-supersymmetric black hole solutions which largely rely on the work done by [14]. The attractor mechanism was initially studied in black hole solutions in $\mathcal{N} = 2$ supersymmetric gravity theories [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37], where it was found that, for fixed charges, moduli fields assume fixed values at the horizon regardless of their values at asymptotic infinity. Supergravity is a low-energy approximation of string theory [38, 39]. Specifically compactifying the low energy action of type II string theory on a 3 dimensional Calabi-Yau manifold one obtains a four dimensional $\mathcal{N} = 2$ supersymmetric gravity theory, also one can obtain $\mathcal{N} = 2$ supergravity from M-theory on a 7-dimensional manifold with $SU(3)$ algebra. $\mathcal{N} = 2$ supergravity is made up of the gravity component(the metric and a vector field), the vector component (which contains a vector field and a complex scalar), and a scalar component (four real scalar fields) and it is a theory that admits superconformal symmetries and a subgroup of the Poincaré algebra [40, 41]. The attractor is is a fixed point near the horizon of a black hole, where the moduli fields only depend on the electric and magnetic charges of the black hole irregardless of the moduli field values at asymptotic infinity. For black hole solutions to exhibit the attractor mechanism they must satisfy two conditions. These conditions relate to an effective potential, V_{eff} which governs the equations of motion. The first condition states that there exists a set of scalar fields such that the effective potential V_{eff} is a minimum, $\partial_i V_{eff}(\phi_i = \phi_{i0}) = 0$. The second condition is that at the point $\phi_i = \phi_{i0}$, $\partial_{ij} V_{eff}(\phi_i = \phi_{i0})$ must be positive definite. This effective potential is proportional to the energy density of the electric and magnetic fields which depends on the charges of the black hole.

8.1 Attractor in Four-Dimensional asymptotically Flat Space

To demonstrate the attractor mechanism, in section 2 of [14] they considered Einstein-Hilbert gravity theory in 4 dimensions coupled with $U(1)$ gauge fields and scalar fields. Since in gauged supergravity, the scalar fields have a potential [42], then we consider the action is of the form,

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu})\right), \tag{8.1}$$

where we take to be dimensionless, $\kappa = 1$, i is the index of the different scalar fields, a, b are the indices of the different gauge fields and $F^a_{\mu\nu}$ is the fields strength of the gauge field and f_{ab} is the gauge couplings. Varying the action with respect to the metric, scalar fields, and gauge fields one obtains the equations of motion,

$$R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i = 2f_{ab}(\phi_i)F^a_{\mu\lambda}F^{b\lambda}_{\nu} + \frac{1}{2}G_{\mu\nu}L, \tag{8.2}$$

$$\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) = \frac{1}{4}\partial_i f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu}, \tag{8.3}$$

$$\partial_\mu(\sqrt{-G}f_{ab}(\phi_i)F^{b\mu\nu}) = 0. \tag{8.4}$$

for simplicity there is no lose in generality if we only consider the case where the gauge fields have only magnetic charge. Since we are interested in spherically symmetric and magnetically charged black hole solutions, the metric and gauge fields can then be written as,

$$ds^2 = -a(r)^2dt^2 + a(r)^2dr^2 + b(r)^2d\Omega^2, \tag{8.5}$$

$$F^a = Q^a_m \sin\theta d\theta \wedge d\phi, \tag{8.6}$$

simplifying (8.2) we get,

$$R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i = f_{ab}(\phi_i)\left(2F^a_{\mu\lambda}F^{b\lambda}_{\nu} - \frac{1}{2}G_{\mu\nu}F^a_{\kappa\lambda}F^{b\kappa\lambda}\right). \tag{8.7}$$

From the equations of motion (8.2), one can calculate the components of the Ricci tensor, as shown in appendix H, and obtain,

$$R_{tt} = \frac{a^2(r)}{b^4(r)}V_{eff}(\phi_i), \tag{8.8}$$

$$R_{\theta\theta} = \frac{1}{b^2(r)}V_{eff}(\phi_i), \tag{8.9}$$

$$R_{rr} - 2(\partial_r\phi_i)^2 = -\frac{1}{a^2b^4}V_{eff}(\phi_i), \tag{8.10}$$

where $V_{eff}(\phi_i) = f_{ab}(\phi_i)Q_m^a Q_m^b$ is the effective potential. These results obtained from substituting the ansatz to the equations of motion (8.2), and assigned everything that depended on the scalar fields and the gauge fields or charges as the effective potential. After some algebraic manipulations, one obtains the equations of motion in terms of the metric components.

$$
\begin{aligned}
(a^2b^2)'' &= 2, &(8.11)\\
\frac{b''}{b} &= -(\partial_r\phi_i)^2, &(8.12)\\
-1 + a^2b'^2 + \frac{(a')^2(b')^2}{2} &= -\frac{1}{b^2}V_{eff}(\phi_i) + a^2b^2(\phi_i^{'})^2, &(8.13)\\
\partial_r(a^2b^2\partial_r\phi_i) &= \frac{\partial_i V_{eff}}{2b^2}. &(8.14)
\end{aligned}
$$

For a case where we have turned off the electric charge, we have obtained equations of motion and the energy constraint, in term of the metric components and effective potential. This is important to move forward in applying the attractor mechanism conditions which depend on the analysis of the effective potential and deriving the black hole solutions which satisfy such conditions, but before we describe that, we also point out that even if one turns on both the electric and magnetic charges, we can still obtain equations of motion in terms of metric components and the corresponding effective potential. The theory considered was of the form,

$$
S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu} - \frac{1}{2}\tilde{f}_{ab}(\phi_i)F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\right). \tag{8.15}
$$

The equations of motion for the action (8.15) are given by,

$$
\begin{aligned}
&R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i - 2f_{ab}(\phi_i)F^a_{\mu\beta}G^{\beta\gamma}F^b_{\nu\gamma} - \frac{1}{2}G_{\mu\nu}(R - 2(\partial\phi_i)^2\\
&- f_{ab}(\phi_i)\left(G^{\alpha\alpha'}G^{\beta\beta'}F^a_{\alpha\beta}F^b_{\alpha'\beta'}\right)\\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right))\\
&- \frac{1}{2}\tilde{f_{ab}}(F^a_{\mu\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\nu\beta'\rho'\sigma'} + F^a_{\alpha\mu}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\nu\rho'\sigma'}\\
&+ F^a_{\alpha\beta}F^b_{\mu\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\nu\sigma'} + F^a_{\alpha\beta}F^b_{\rho\mu}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho'\nu}))\\
&= 0.
\end{aligned} \tag{8.16}
$$

Taking the trace of the equations of motion (8.16) we get,

$$
R - 2(\partial\phi_i)^2 + \tilde{f_{ab}}(\phi_i)F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'} = 0. \tag{8.17}
$$

We notice that since we have turned on both electric and magnetic charges in our theory, then the Ricci scalar depends on both the scalar fields and the charges, unlike in the previous theory where the Ricci scalar just depended on the scalar fields.

Similarly as we did in the previous theory we calculate the metric component of the equations of motion and they are similar to equation (8.11-8.14). See details in [14] and appendix H, while the effective potential is given by,

$$V_{eff}(\phi_i) = f_{ab}(\phi_i)Q^a_m Q^b_m + f^{ab}(\phi_i)\left(Q_{ea} - \tilde{f}_{ac}Q^c_m\right)\left(Q_{eb} - \tilde{f}_{bd}Q^d_m\right). \tag{8.18}$$

We have derived all the equations needed to understand the attractor mechanism and the effective potential, which will be crucial and has significance in understanding the entropy of these black hole solutions.

8.2 Conditions of an attratractor

As stated in [14], for the attractor mechanism to exist, the effective potential must satisfy the two following conditions.
Condition 1:

$$\partial_i V_{eff}(\phi_{i0}) = 0. \tag{8.19}$$

Condition 2:

$$M_{ij} = \frac{1}{2}\partial_i\partial_j V_{eff}(\phi_{i0}) > 0. \tag{8.20}$$

where ϕ_{i0} are the critical or attractor values for the scalar.
We noted that if we set the scalar fields to be constants, their attractor values, and using equation (8.14), the energy constraint (8.13) gives the horizon radius as

$$b_H^2 = V_{eff}(\phi_{i0}). \tag{8.21}$$

This is important because we know that from the Bekenstein-Hawking entropy, the entropy of the black hole is given by,

$$S_{BH} = \frac{1}{4}A = \pi b_H^2. \tag{8.22}$$

We see that the attractor conditions imply that the entropy depends only on the effective potential evaluated at a minimum. Now we investigate which black hole solutions obey such assumptions. Setting $\phi_i = \phi_{i0}$ for all values of r one obtains an extremal Reissner-Nordström black hole, which has a double-zero horizon. In this solution the scalar fields are constant, thus $\partial_r\phi_i = 0$, and a, and b are given by,

$$a_0(r) = \left(1 - \frac{r_H}{r}\right), \quad b_0(r) = r, \tag{8.23}$$

where r_H is the horizon radius. From (8.23) we see that, a_0^2, $(a^2)'$ vanish at the horizon, r_H, while, b_0, and b_0' are finite. If one substitutes a, and b to the equations of motion and the energy constraint, one indeed gets the horizon radius given by,

$$r_H^2 = b_H^2 = V_{eff}(\phi_{i0}). \tag{8.24}$$

In [14], by studying both, particular effective potentials where one can solve the equations analytically, and perturbations around the constant scalar solutions, it was found for extremal black holes with non-constant scalars, (8.22) still holds, so that the entropy is determined by the minimum of the effective potential.

8.3 Perturbation theory

In this section, we discuss a perturbative demonstration of the attractor mechanism. Starting with the constant scalar solution, and introducing a small position dependent scalar perturbation, one can systematically expand and solve the equations of motion at higher and higher orders. In [14], it was found that, in this perturbative approach, one can preserve the asymptotic structure of the metric. For supersymmetric black holes, the attractor equations follow from first order equations, but in [14] it was shown that the attractor mechanism arises from the second order equations of motion without having to invoke supersymmetry. We explain how this works below.

8.3.1 First Order Solution

We start with the first order perturbation in the scalars,

$$\delta\phi_i = \phi_i - \phi_{i0} = \epsilon\phi_{i1}, \tag{8.25}$$

where, $\epsilon << 1$, ϕ_i are chosen to be eigenvectors of the second derivative matrix. substituting this transformation to equations of motion (8.11), (8.12), and (8.14), one can see that there are no first order corrections to the metric components, a, and $b(a_1 = b_1 = 0)$. They start having corrections in second order. This tells us that first order correction on the scalar, only leads to second order correction and above in the metric components. Therefore, we can define the metric components as follows, and we chose our black hole background is the extremal Reissner-Nordström background (8.23),

$$\begin{aligned} a &= a_0, \\ a^2 &= a_0^2, \\ b^2 &= b_0^2. \end{aligned} \tag{8.26}$$

The equations of motion for the scalars (8.14) become, and using the first attractor condition, $\partial_i V_{eff}(\phi_{i0}) = 0$,

$$\partial_r \left(a_0^2 b_0^2 \partial_r \phi_{i1}\right) = \frac{\beta_i^2}{b_0^2}\phi_{i1}, \tag{8.27}$$

the second condition of the attractor, requires that, $\beta_i^2 > 0$, which holds. Now we solve for the first order scalar solution ϕ_{i1}, on the extremal Reissner-Nordström background and this leads to a second order differential equation in the first order perturbation scalar,

$$\begin{aligned} \partial_r \left((r - r_H)^2 \, \partial_r(\phi_{i1})\right) &= \frac{\beta_i^2}{r^2 \phi_{i1}}, \\ \implies (r - r_H)^2 \partial_r^2 \phi_{i1} + 2(r - r_H)\partial_r \phi_{i1} - \frac{\beta_i^2 \phi_{i1}}{r^2} &= 0, \end{aligned} \tag{8.28}$$

which has a solution, where we are only interested in well defined solutions, we take the positive sign,

$$\phi_{i1} = c_{1i}\left(\frac{r - r_H}{r}\right)^{\gamma_i}, \qquad \gamma_i = \frac{1}{2}\left(\sqrt{1 + \frac{4\beta_i^2}{r_H}} - 1\right). \tag{8.29}$$

Now we analyse the behaviour of the solution near the horizon, and at asymptotic infinity. In the vicinity of the horizon, $r \to r_H$

$$\lim_{r \to r_H} \phi_{1i} = c_{1i}(1 - 1)^{\gamma_i} \to 0. \tag{8.30}$$

Near the horizon, the first order scalar vanishes, which is good, because it means that at the horizon, all the information that depends on the scalars in the black hole will be purely defined by the attractor values(ϕ_{i0} scalars) which are constant. Asymptotically, $r \to \infty$,

$$\begin{aligned} \phi_{i1} &= c_{1i}\left(\frac{r - r_H}{r}\right)^{\gamma_i} \\ &\approx c_{1i}\left(1 - \frac{\gamma_i r_H}{r}\right) \\ \implies \lim_{r \to \infty} \phi_{i1} &= constant. \end{aligned} \tag{8.31}$$

We see that the first order perturbative expansion is fixed at its attractor values, at infinity and vanishes at the horizon. This shows that the attractor mechanism works to first order in perturbation theory. Thus we can conclude that if one performs a first order perturbation in the scalars which is a small deviation from the attractor values, such values become constants as we increase r, and they do not contribute in changes in the system's whole scalar fields.

8.3.2 Second Order Solution

We now look at the second order perturbation using similar methods. By defining the second order expansion of the metric components and the scalars which lead to ϕ_{i1}, ϕ_{i2}, and $a_1 = b_1 = 0$, and a_2, b_2, then substitute these to the equations of motion (8.11), (8.12), and (8.14), and use the first and second condition of the attractor and see if they hold, then derive the solutions, as we stated that this analysis was performed by [14], we can just state the solutions and describe their near horizon and asymptotic behaviour. For second order perturbation, these are the solutions,

$$
\begin{aligned}
a_2 &= \frac{-2(1-\frac{r_H}{r})}{r} b_2, &(8.32)\\
b_2(r) &= -\sum_i \frac{c_{1i}^2\gamma_i}{2(2\gamma_i-1)} r \left(\frac{r-r_H}{r}\right)^{2\gamma_i} + A_1 r + A_2 r_H, &(8.33)\\
&= -\sum_i \frac{c_{1i}^2\gamma_i}{2(2\gamma_i-1)} r \left(\frac{r-r_H}{r}\right)^{2\gamma_i} &(8.34)
\end{aligned}
$$

where we have set the integration constants, $A_1 = A_2 = 0$. The near horizon analysis, $r \to r_H$, $(a_2), b_2, (a_2)', (b_2)' \to 0$, and as it was shown in $\phi_{i1} \to 0$, and $\phi_{i2} \to 0$, the attractor mechanism works for this analysis since these functions all vanish near the horizon, now, asymptotically, as $r \to \infty$,

$$
\begin{aligned}
-\sum_i \frac{c_{i1}^2\gamma_i}{2(2\gamma_i-1)} \lim_{r\to\infty} r \left(\frac{r-r_H}{r}\right)^{2\gamma_i} &\approx -\sum_i \frac{c_{i1}^2\gamma_i}{2(2\gamma_i-1)} r \left(\frac{r-2\gamma_i r_H}{r}\right)\\
&\approx -\sum_i \frac{c_{i1}^2\gamma_i}{2(2\gamma_i-1)} (r - 2r_H\gamma_i)\\
&\Longrightarrow b_2(r) \to -c \times r, &(8.35)
\end{aligned}
$$

now for a_2,

$$
\begin{aligned}
a_2 &\to -2\left(1-\frac{r_H}{r}\right)^2 \frac{c\times r}{r}\\
&\to -2\left(1-\frac{2r_H}{r}\right) c\\
&\to -2 \times c. &(8.36)
\end{aligned}
$$

In the vicinity of the horizon, $r \to r_H$, $b_2(r) \to 0$, and $a_2(r) \to 0$. Thus the solution is well defined close to the horizon, in fact b_2, and a_2 vanishes and it continues to be asymptotically flat at second order. Therefore, the second order solution continues to be a double-zero horizon black hole with vanishing surface gravity. And as $r \to \infty$, these terms continue to become constants, thus as we perturb the system to second order, asymptotically the terms remain constant, which leads to the conclusion that the attractor mechanism works up to second order. In the next subsection we investigate the expansion of the ansatz to all orders, and demonstrate that the attractor mechanism still works.

8.3.3 An ansatz to all orders

Recall that the analysis being performed works for extremal black holes as explained in [14], meaning at the horozon, the metric components should vanish, we start with,

$$a^2 b^2 = (r - r_H)^2, \tag{8.37}$$

expanding the solutions b, ϕ, and a in a power series in ϵ,

$$b = b_0 + \sum_{n=1}^{\infty} \epsilon^2 b_n, \tag{8.38}$$

$$\phi = \phi_0 + \sum_{n=1}^{\infty} \epsilon^n \phi_n, \tag{8.39}$$

$$a^2 = a_0^2 + \sum_{n=1}^{\infty} \epsilon^n a_n. \tag{8.40}$$

The ansatz that works to all orders is given by,

$$\begin{aligned} \phi_n(r) &= c_n \left(\frac{r - r_H}{r} \right)^{n\gamma}, \\ b_n(r) &= d_n r \left(\frac{r - r_H}{r} \right)^{n\gamma}, \\ a_n &= e_n \left(\frac{r - r_H}{r} \right)^{n\gamma + 2}, \end{aligned} \tag{8.41}$$

The analysis above agrees with the first order and the second order perturbation analysis, we see that $b_1 = 0$, and b_2 also agrees with equation (8.33), we also found that $a_1 = 0$, and a_2 is given by (8.32). This tells us that we can be able to verify the ansatz for all orders of n. We also notices that each term of order ϵ^n contains the term $\left(\frac{(r-r_H)}{r}\right)^{2\gamma n}$. Which makes it easier to calculate the constants, as it was shown in the appendix H how one can calculate c_2 from the higher order expansion. It was shown that b_2 goes to infinity when $\gamma = \frac{1}{2}$, similarly it was shown that b_n goes to infinity when $\gamma = \frac{1}{n}$, where n is an integer. This shows that for all values of $n > 1$, the perturbation expansion does not work.

To conclude this section, we have studied how to derive the equations of motion (8.2)-(8.4) from the action (8.1) in four dimensions with the Einstein-Hilbert action, and a scalar and $U(1)$ gauge fields, by first only turning the magnetic charges, then also derived equations of motion by turning both the magnetic and electric charges. After deriving the equations of motion, the attractor mechanism was defined by its two conditions and found solutions such that the two conditions hold. They solved simpler versions of the effective potential which leads to exact black hole solutions and also solved for the metric components using perturbation theory. These black holes solved using perturbation theory all have an extremal Reissner-Nordström background. Thus the family of black hole solutions are extremal, but they are non-supersymmetric.

In this chapter, we have introduced the concept of the attractor mechanism. We note that the main goal of this second component of the thesis was to demonstrate the attractor mechanism in arbitrary dimensions, with the addition of a potential function. The last sections of [14] also attempts to derive the attractor mechanism in higher dimensions but we will not describe it here since most of the calculations will be shown in the next chapter. In this chapter we just analyzed it in four dimensions and what remains in higher dimensions of [14] is an extension where they use the extremal Reissner-Nordström black hole as a background and also use perturbation theory to second order. These solutions also follow directly from the four dimensional case. There is little to explain these phenomena in those sections since it is are rigorously detailed in [4] which also forms part of our thesis. In the next chapter, we will extend the notion of the attractor mechanism, to arbitrary dimensions and the invariants that can be derived because of this phenomenon which can also be found on [4].

Chapter 9

Generalized Attractors

In this chapter, we review the attractor mechanism for non-extremal black holes in four dimensions, as discussed in [2, 3]. We then consider these "hot attractors" in higher dimensions and with the addition of a scalar potential, which was originally presented in [4].

9.1 Review: Hot and cold attractors

In this section, we start by looking at a 4-dimensional gravity theory coupled to $U(1)$ gauge fields with the inclusion of a scalar potential.

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(R - 2g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - f_{ab}F^a_{\kappa\beta}F^{b\kappa\beta}\frac{1}{2}\tilde{f}_{ab}(\phi)F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - 2V_g(\phi)\right), \tag{9.1}$$

where $G_{\mu\nu}$ is the space-time metric, g_{ij} is the metric in field space, and $V_g(\phi)$ is a scalar potential. By taking choices of g_{ij}, f_{ab}, $\tilde{f}_{ab}$, and V_g corresponds to the bossonic part of an $\mathcal{N} = 2$ gauged supergravity. This theory is similar to what was studied in [14], except that last term in the action was zero. We assume the following ansatz,

$$\begin{aligned}
ds^2 &= -a(r)^2dt^2 + a(r)^{-2} + b(r)^2d\Omega_2^2,\\
F^a &= f^{ab}(\phi)(Q_{eb} - \tilde{f}_{bc}(\phi)Q^c_m)\frac{1}{b^2}dt\wedge dr + Q^a_m\sin\theta d\theta\wedge d\phi,\\
\phi^i &= \phi^i(r),
\end{aligned} \tag{9.2}$$

which leads to the equations of motion,

$$\frac{\partial^2}{\partial r^2}\left(a^2b^2\right) = 2 - 4V_g^2, \tag{9.3}$$

$$\frac{1}{b}\frac{\partial^2}{\partial r^2}b = -\sum_i\left(\frac{\partial}{\partial r}\phi^i\right)^2, \tag{9.4}$$

$$\frac{\partial}{\partial r}\left(a^2b^2\frac{\partial}{\partial r}\phi^i\right) = \frac{1}{2}\left(\frac{1}{b^2}\frac{\partial}{\partial\phi^i}V_{eff} + \frac{\partial}{\partial\phi^i}V_g b^2\right), \tag{9.5}$$

where the the effective potential, V_{eff}, is given in (8.18). In addition to the second order equations (9.3)-(9.5), we have the first order hamiltonian constraint,

$$-1 + a^2b'^2 + \frac{1}{2}\frac{\partial}{\partial r}(a^2)\frac{\partial}{\partial r}(b^2) = -\frac{V_{eff}(\phi)}{b^2} - b^2V_g + a^2b^2\sum_i\left(\frac{\partial}{\partial r}\phi^i\right)^2. \tag{9.6}$$

Notice that, the only difference between the constraint (9.6) and, the previous constraint, (8.13), is the addition of a term involving the scalar potential V_g.

Both extremal and non-extremal spherically symmetric solutions satisfy (9.3)–(9.6), albeit with different boundary conditions at the horizon. Examining (9.6) at large r allows one to extract a relationship between the mass and the black hole charges using the extremality condition. As in the previous section, we emphasize that supersymmetric solutions are extremal, but extremal solutions are not necessarily supersymmetric. Supersymmetric solutions are of course extremal, but non-supersymmetric extremal solutions also exist and our discussion applies to them as well. In (9.3) when we set $V_g = 0$, we can integrate with respect to the radial coordinate, and the solution is $a^2b^2 = (r - r_-)(r - r_+)$, where $r_\pm$ are integration constants, which also represents the inner and outer horizon of the non-extremal balck hole solutions. The warp factor $b(r_\pm)$ tells us the radius of the S^2 at the horizons, if the roots $r_\pm$ are equal, then we have an extremal solution. For extremal solutions, because the presence of a double horizon leads to cancelation of terms in (9.6), then the Hamiltonian constraint becomes

$$b_{ext}^2 = V_{eff,min}, \tag{9.7}$$

where we use the subscript “ext” to denote quantities evaluated at the double horizon and the subscript “min” to denote quantities evaluated at a minimum of a potential. In this case, since the near horizon geometry is $AdS_2 \times S^2$, the attractor mechanism ensures the moduli fields at infinity fixes the values ϕ_0^i at the horizon of which is consistent with the “no hair” theorem, are determined by the charges [43]. From (9.7), the zero temperature solutions have an event horizon whose area is determined by the effective potential evaluated on the attractor values of the moduli. For non-extremal black holes, the scalars at the inner and the outer horizons vary with the asymptotic values of the moduli. Nevertheless, in [2, 3] we show that from the existence of a smooth zero temperature limit that connects non-extremal black holes to extremal solutions, there are invariants. In particular, the geometric mean of the areas of the inner and outer horizons of the finite temperature black hole is the area of the event horizon of the zero temperature black hole. Also, for non-extremal solutions

$$\int_{r_-}^{r_+} dr \frac{\partial_i V_{eff}(\phi)}{b^2} = 0, \tag{9.8}$$

$$\int_{r_-}^{r_+} dr \left(\frac{V_{eff}(\phi)}{b^2} - 1 \right) = 0. \tag{9.9}$$

For extremal solutions, the integrands of (9.8) and (9.9) are zero since the moduli fields are fixed at the horizon to the attractor values at the minimum of the scalar potential, whereas in non-extremal black hole solutions, the integrals vanish over Region 2 [4]. If one averages over Region 2, one is able to obtain attractor equations for extremal solutions.

We now consider the case of $V_g \neq 0$. In this case the equation (9.3) is much more complicated to integrate, and needs to be solved numerically, but we can consider a simpler case such as only restricting our analysis to solutions with constant scalars, which were described in [44]. When we consider constant scalars, equation (9.4) becomes,

$$\begin{aligned} \frac{d^2}{dr^2} b &= 0, \\ \implies b(r) &= r, \end{aligned} \tag{9.10}$$

and the right hand side of (9.5) has different powers of r, while the left hand side is zero. We are in interested in solutions such that,

$$\partial_i V_{eff} = 0, \tag{9.11}$$

$$\partial_i V_g = 0. \tag{9.12}$$

In (9.11) and (9.12), suppose there are n scalar fields, then we get $2n$ equations of motion in n variables, so this is an over-determined system of equations. There can be instances were the system is under-determined but we can still solve these equations. Using the discussions in [14] and required minimum of the potential in (9.11) and (9.12), and substituting the solution of $b(r)$ given in (9.10), we can integrate the equation and get,

$$
\begin{aligned}
a^2b^2 &= \frac{(-V_{g,min})}{3}r^4 + r^2 - c_1 r + c_2 \\
&= \frac{(-V_{g,min})}{3}(r-r_+)(r-r_-)(r-\lambda)(r-\tilde{\lambda}),
\end{aligned} \tag{9.13}
$$

where, c_1, and c_2 are integration constants, which can be solved by expanding the right hand side of (9.13) in orders of powers of r, and match the coefficients, we find that,

$$
c_1 = \frac{1}{3}(r_+ + r_-)\left(3 + (-V_{g,min})(r_+^2 + r_-^2)\right) = 2GM, \tag{9.14}
$$

$$
c_2 = \frac{1}{3}r_+ r_-\left(3 + (-V_{g,min})(r_+^2 + r_+ r_- + r_-^2)\right) = V_{eff,min}, \tag{9.15}
$$

the constants also depend on the mass and the charges of the solution.
We can also solve for λ, and $\tilde{\lambda}$, as

$$
\lambda = -\frac{r_+ + r_-}{2} + \frac{i}{2}\sqrt{\Xi}, \tag{9.16}
$$

where,

$$
\Xi = (r_+ + r_-)^2 + 2(r_+^2 + r_-^2) + \frac{12}{-V_{g,min}}. \tag{9.17}
$$

In equation (9.13), requiring that g_{tt} be timelike at infinity, we need $V_{g,min}$ to be negative, this implies that the solution for ($V_g \neq 0$) and constant scalars are AdS Reissner-Nordström black hole solutions. Since we are taking $V_{g,min} < 0$, equation (9.16) and (9.17) implies that λ is required to be complex if we take $r_\pm$ to be real. In the case where $V_g \neq 0$ an non-constant scalars, extremal black hole solutions, the horizons $r_\pm$, are $r_+ = r_- = r_H$, meaning that $a = a' = 0$ at the horizon, using these constrains in the energy constrain equation (9.6) we get,

$$
\frac{\partial_i V_{eff}}{b_{ext}^2} + \partial_i V_g b_{ext}^2 = 0, \tag{9.18}
$$

$$
\implies \partial_i V_{eff} + \partial_i V_g b_{ext}^4 = 0, \tag{9.19}
$$

$$
\implies b_{ext}^2 = \pm\sqrt{-\frac{\partial_i V_{eff}}{\partial_i V_g}} = \sqrt{-\frac{\partial_i V_{eff}}{\partial_i V_g}}. \tag{9.20}
$$

We are not aware of any solutions to gauged supergravity that solves the attractor equations through the relation in (9.20). In this thesis we restrict ourselves to solutions that satisfy (9.11) and (9.12) at the horizon. Now, evaluating the energy constraint (9.6) at the horizon, we get,

$$\frac{V_{eff,min}}{b_{ext}^2} + b_{ext}^2 V_{g,min} = 1, \tag{9.21}$$

which implies that,

$$b_{ext}^2 = \frac{\sqrt{1 + 4V_{eff,min}(-V_{g,min})} - 1}{2(-V_{g,min})}. \tag{9.22}$$

If the solutions are non-extremal, we can integrate between the inner and outer horizon, then we can generalize (9.8) (9.9) to give,

$$\int_{r_-}^{r_+} dr \left(\frac{\partial_i V_{eff}}{b^2} + b^2 \partial_i V_g \right) = 0, \tag{9.23}$$

$$\int_{r_-}^{r_+} dr \left(\frac{V_{eff}}{b^2} + b^2 V_g^{-1} \right) = 0, \tag{9.24}$$

which once again averages the attractor equations (9.18) and (9.21) over Region 2.

9.2 Attractors in general dimensions

In the previous section we have already establised the attractor equations, now we extend it to general dimensions. To do that we first consider the action of the form,

$$S = \frac{1}{\kappa^2} \int d^d x \sqrt{-G} \left(R - 2(\partial\phi_i) - f_{ab} F^a_{\kappa\beta} F^{b\kappa\beta} - 2V_g(\phi) \right). \tag{9.25}$$

For us to obtain spherically symmetric solutions, we only turn on magnetic charges, where the field strengths F^a are $(d-2)$-forms. Varying the action with respect to to the metric leads to (see appendix I),

$$\begin{aligned} R_{\mu\nu} - 2g_{ij}(\phi)\partial_\alpha\phi^i\partial_\alpha\phi^j &= 2f_{ab}(\phi)G^{\beta\tau}F^a_{\mu\beta}F^b_{\nu\tau} - \frac{1}{d-2}G_{\mu\nu}f_{ab}(\phi_i)F^a_{\sigma\rho}F^{b\sigma\rho} \\ &+ \frac{2}{d-2}G_{\mu\nu}V_g(\phi). \end{aligned} \tag{9.26}$$

Substituting the ansatz,

$$ds^2 = -a(r)^2 dt^2 + a(r)^{-2} dr^2 + b(r)^2 d\Omega^2_{d-2}, \tag{9.27}$$

$$F^a = Q^a \sin^{d-3}\theta \sin^{d-4}\phi \cdots d\theta \wedge d\phi \wedge \cdots, \tag{9.28}$$

to (9.26) we get,

$$R_{\theta\theta} - \frac{(d-3)!}{b(r)^{2(d-3)}}V_{eff}(\phi) + \frac{2}{d-2}G_{\theta\theta}V_g(\phi) = 0, \tag{9.29}$$

$$R_{tt} - \frac{(d-3)(d-3)!a(r)^2}{b(r)^{2(d-2)}}V_{eff}(\phi) + \frac{2}{d-2}G_{tt}V_g(\phi) = 0, \tag{9.30}$$

$$R_{rr} - 2(\partial_r\phi)^2 + \frac{(d-3)(d-3)!}{b(r)^{2(d-2)}a(r)^2}V_{eff}(\phi_i) + \frac{2}{d-2}G_{rr}V_g(\phi) = 0, \tag{9.31}$$

where V_{eff} is given by,

$$V_{eff}(\phi) = f_{ab}(\phi)Q^a_m Q^b_m. \tag{9.32}$$

Now taking the combination, $\frac{1}{2}(R_{rr} - \frac{G_{rr}}{G_{tt}}R_{tt})$, gives,

$$\sum_i (\phi^{i\prime})^2 = \frac{(d-2)}{2b}\frac{\partial^2 b}{\partial r^2}, \tag{9.33}$$

and also using the relation $R_{tt} = (d-3)\frac{a^2}{b^2}R_{\theta\theta}$, we get,

$$(d-3)^2(-1+a^2b'^2) + b^2(a'^2 + aa'') + aba'b'(3d-8) + (d-3)a^2bb'' = -2b^2V_g(\phi), \tag{9.34}$$

finally if we vary the action, with respect to the scalars, we get,

$$\frac{\partial}{\partial r}\left(a^2b^{d-2}\phi^{i'}\right) = \frac{(d-2)!\partial_i V_{eff}(\phi)}{4b^{d-3}} + \frac{1}{2}b^{d-2}\partial_i V_g(\phi). \tag{9.35}$$

we can also use the R_{rr} in (9.31) component to obtain the energy constraint as,

$$\begin{aligned} -(d-2)[(d-3) - ab'(2a'b + (d+3)ab')] = 2\sum_i \left(\phi^{i'}\right)^2 a^2b^2 \\ - \frac{(d-2)!V_{eff}(\phi)}{b^{2(d-3)}} - 2b^2V_g(\phi). \end{aligned} \tag{9.36}$$

We notice that adding the scalar potential to the d dimensional theory in [14] and following the analysis most of the equations derived also include an additional term that depends on the scalar potential V_g. These equations will help us build an attractor mechanism in general dimensions which we show in the next section.

9.2.1 Change of coordinates

In order to write down the attractor equations we need to transform the equations of motion to a form we have used before in the previous section. Using the transformation,

$$\partial_r = (d-3)b^{d-4}\partial_\rho, \tag{9.37}$$

the units of ρ are given by, $[L]^{d-3}$. In these new coordinates, in order to obtain the equations of motion we invert the coordinate transformation (9.37) to,

$$\partial_\rho = \frac{1}{(d-3)}b^{-(d-4)}\partial_r, \tag{9.38}$$

then relation (9.34) becomes,

$$\partial^2_\rho(a^2b^{2(d-3)}) = 2 - \frac{4b^2}{(d-3)}V_g, \tag{9.39}$$

which is analogous to (9.3). In order to obtain the energy constrain equation we start by calculating the term,

$$\partial_\rho\left(a^2\partial_\rho(b^2(d-3))\right). \tag{9.40}$$

Using the relation

$$\partial_\rho(b^2(d-3)) = 2b^{d-3}b', \tag{9.41}$$

we obtain the following,

$$\partial_\rho\left(a^2\partial_\rho(b^{2(d-3)})\right) = \frac{2}{(d-3)}\left(2aa'b'b + (d-3)a^2b'^2 + a^2bb''\right), \tag{9.42}$$

using the energy constraint (9.36) and the relation (9.42), we can rewrite it as the following,

$$\begin{aligned}
-\frac{(d-2)!V_{eff}(\phi)}{b^{2(d-3)}} - 2b^2V_g(\phi) &= -(d-2)(d-3) + (d-2)\left(2aba'b' + (d-3)a^2b'^2 + a^2bb''\right)\\
&= -(d-2)(d-3) + (d-2)\left(\frac{d-3}{2}\partial_\rho(a^2\partial_\rho(b^{2(d-3)}))\right).
\end{aligned} \tag{9.43}$$

Finally using (9.42) and (9.43) the energy constraint equation becomes,

$$\begin{aligned}
\frac{1}{2}(d-2)(d-3)\partial_\rho(a^2\partial_\rho(b^{2(d-3)})) &= (d-2)(d-3) - \frac{(d-2)!V_{eff}(\phi)}{b^{2(d-3)}} - 2b^2V_g(\phi),\\
\implies \frac{1}{2}\partial_\rho\left(a^2\partial_\rho(b^{2(d-3)})\right) &= -1 + \frac{(d-4)!V_{eff}(\phi)}{b^{2(d-3)}} + \frac{2b^2}{(d-2)(d-3)}V_g(\phi).
\end{aligned} \tag{9.44}$$

Now we derive the equations of motion for scalars by first calculating the relation,

$$\partial_\rho \phi^i = \frac{1}{(d-3)} b^{-(d-4)} \partial_r \phi^i = \frac{b^{-(d-4)}}{(d-3)} \partial_r \phi^i, \tag{9.45}$$

then we use (9.37) and (9.35) to obtain

$$\begin{aligned}\partial_\rho \left(a^2 b^{2(d-3)} \partial_\rho \phi^i\right) &= \frac{1}{(d-3)} b^{-(d-4)} \partial_r \left(a^2 b^{d-2} \partial_r \phi^i\right) \\ &= \frac{1}{(d-3)^2} b^{-(d-4)} \left(\frac{(d-2)! \partial_i V_{eff}(\phi)}{4 b^{(d-2)}} + \frac{1}{2} b^{(d-2)} \partial_i V_g\right),\end{aligned} \tag{9.46}$$

therefore, the equation of motion for the scalars in the form of (9.35),

$$\partial_\rho \left(a^2 b^{2(d-3)} \partial_\rho \phi^i\right) = \frac{(d-2)!}{4(d-3)^2} \frac{\partial_i V_{eff}(\phi)}{b^{2(d-3)}} + \frac{1}{2(d-3)^2} b^2 \partial_i V_g. \tag{9.47}$$

Now we derive the equation analogous to (9.33) using (9.37). Using, (9.45) and (9.33), we have,

$$\sum_i (\partial_\rho \phi^i)^2 = -\frac{1}{(d-3)^2} b^{-(d-4)} \frac{(d-2)\partial_r^2 b}{2b}, \tag{9.48}$$

we can write the second derive of b in terms of the new coordinates as,

$$\begin{aligned}\partial_\rho^2 b^{d-3} &= \partial_\rho \left(\frac{1}{(d-3)} b^{-(d-4)} \partial_r b^{d-3}\right) \\ &= \frac{1}{(d-3)} b^{-(d-4)} \partial_r^2 b,\end{aligned} \tag{9.49}$$

this allows us to finally write the equation of motion analogous to (9.33) as,

$$\sum_i (\partial_\rho \phi^i)^2 = -\frac{(d-2)}{2(d-3)} \frac{\partial_\rho^2 \left(b^{d-3}\right)}{b^{d-3}}. \tag{9.50}$$

It is straightforward to check that in 4 dimensions, we recover the equations of motion of the attractor mechanism as before.

For extremal horizon black hole solutions with $a = a' = 0$, (9.35) at the horizon, becomes,

$$\begin{aligned}&b_{ext}^{-d+2} \frac{(d-2)! \partial_i V_{eff}(\phi)}{4 b_{ext}^{d-2}} + \frac{1}{2} \partial_i V_g(\phi) = 0, \\ \implies &\frac{(d-2)! \partial_i V_{eff}(\phi)}{2 b_{ext}^{2(d-3)}} + b_{ext}^2 \partial_i V_g(\phi) = 0\end{aligned} \tag{9.51}$$

We only restrict ourselves to solutions of (9.51), which satisfy the attractor condition, and the following,

$$\partial_i V_{eff}(\phi) = 0, \partial_i V_g = 0. \tag{9.52}$$

At the horizon, while considering extremal solutions($a = a' = 0$), the energy constraint becomes,

$$\begin{aligned}(d-2)(d-3)b_{ext}^{2(d-3)} - (d-2)!V_{eff}(\phi) - 2b_{ext}^{2+2(d-3)}V_g(\phi) = 0,\\ \implies (d-4)!V_{eff,min} - \frac{2b_{ext}^{2(d-3)}}{(d-2)(d-3)}(-V_{g,min}(\phi)) - b_{ext}^{2(d-3)} = 0.\end{aligned} \tag{9.53}$$

Given $V_{eff,min} > 0$ and $V_{g,min} > 0$ we can see that the polynomial in (9.53) crosses the b_{ext}^2 axis exactly once. Thus the solution is a unique real valued solution for b_{ext} which solely a function, of the minimum values of $V_{eff,min}$, and $V_{g,min}$. This is only true for extremal solutions. In the case the solutions do not have extremal solutions, we can integrate the energy constraint (9.44), and the equation of motion for scalars (9.35) between the double horizons $\rho_\pm$, which corresponds to the zeros a, a', and give the average attractor equations.

$$\int_{\rho_-}^{\rho_+} d\rho \left(\frac{(d-2)!\partial_i V_{eff}(\phi)}{2b^{2(d-3)}} + b^2 \partial_i V_g(\phi)\right) = 0, \tag{9.54}$$

$$\int_{\rho_-}^{\rho_+} \left(\frac{(d-2)!\partial_i V_{eff}(\phi)}{b^{2(d-3)}} + \frac{2b^2}{(d-2)(d-3)} V_g(\phi) - 1\right) = 0. \tag{9.55}$$

These integrands vanish at zero temperature in d dimensions. These are the attractor equations that we will be paying attention to, we note that if $V_g = 0$, and $\rho_+ = \rho_-$, then we get back the attractor equations we obtained in the previous analysis. Next, we are going to use these equations in different cases of V_g, and show the type of black hole solutions we obtain.

9.2.2 $V_g = 0$, and a Reissner-Nordström-like solution

Setting $V_g = 0$, and integrating the equation of motion (9.39) we get,

$$a^2 b^{2(d-3)} = (\rho - \rho_+)(\rho - \rho_-), \tag{9.56}$$

assuming the horizon area is non-zero, we obtain two zeros corresponding to the double horizon, $\rho_\pm$. If we also assume constant scalars that satisfy the attractor conditions, we obtain a Reissner-Nordström-like solution. Substituting $b_0 = r$ to (9.56), and using the transformation (9.37) , we get,

$$a_0^2 = \frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2} = \frac{(r^{d-3} - r_+^{d-3})(r^{d-3} - r_-^{d-3})}{r^{2(d-3)}}, \tag{9.57}$$

where, $\rho_\pm$ and $r_\pm$. Substituting (9.56) to the energy constraint (9.44), and using the fact that,

$$(a_0^2)' = \frac{(\rho_+ + \rho_-)}{\rho^2} + \frac{\rho_+\rho_-}{\rho^3}, \tag{9.58}$$

the energy constraint (9.44), becomes,

$$\begin{aligned} \frac{(\rho_+ + \rho_-)}{\rho} - \frac{2\rho_+\rho_-}{\rho^2} + 1 - \frac{(\rho_+ + \rho_-)}{\rho} + \frac{\rho_+\rho_-}{\rho^2} &= 1 - (d-4)! \frac{V_{eff}(\phi_0)}{\rho^2}, \\ \implies V_{eff}(\phi_0) = \frac{\rho_+\rho_-}{(d-4)!} &= \frac{\rho_{ext}^2}{(d-4)!}. \end{aligned} \tag{9.59}$$

This relation is in agreement to the area law, and the relation $A_+ A_-$ follows immediately from $\rho_+\rho_- = \rho_{ext}^2$, which we just showed in the above equation. The near horizon geometry of the extremal Reissner-Nordström solution is $AdS_2 \times S^{d-2}$ and is also a string theory solution. For non-extremal solutions, the Harrison transformation [18] enables us to transform the flat space Reissner-Nordström solution to a black hole with $AdS_2 \times S^{d-2}$ asymptopia. This also provides a CFT interpretation of the solutions which was explained in detail in [4].

9.2.3 $V_g \neq 0$, constant scalars

Now we consider the case of on non-negative scalar potential. Using the equation of motion for the scalars (9.50) and the transformation (9.37), while assuming constant scalars, we obtain,

$$\sum_i (\partial_\rho \phi^i)^2 = -\frac{d-2}{2(d-3)} \frac{\partial_\rho^2 (b^{d-3})}{b^{d-3}} = 0,$$
$$\implies \partial_\rho^2 (b^{d-3}) = 0, \tag{9.60}$$

this gives the solution for for the warp factor b, as,

$$b^{d-3} = c_1 \rho + c_2. \tag{9.61}$$

Choosing $c_2 = 0$, and $c_1 = 1$, and using the fact that ρ has units of $[L]^{d-3}$, we show that the warp factor becomes,

$$b^{d-3} = \rho = r^{d-3}. \tag{9.62}$$

Substituting this relation (9.62) to the equation of motion (9.35) setting the scalars to their attractor values since we assumed constant scalars, we see that (9.11)-(9.12) are satisfied. Now substituting the relation (9.62) to the equation of motion (9.39) and integrating it twice while keeping the scalars constant we get,

$$\partial_\rho^2 (a^2 \rho^2) = 2 - \frac{4}{(d-3)^2} \rho^{\frac{2}{d-3}} V_{g,min},$$
$$\implies a^2 \rho^2 = \rho^2 - \frac{4}{(d-2)^2} \frac{1}{\frac{2}{d-3}+1} \frac{1}{\frac{2}{d-3}+2} \rho^{\frac{2}{d-3}+1} V_{g,min} + c_1 \rho + c_2,$$
$$\implies a^2 b^{2(d-3)} = r^{2(d-3)} - \frac{2V_{g,min}}{(d-1)(d-2)} r^{2(d-2)} + c_1 r^{d-3} + c_2, \tag{9.63}$$

where $V_{g,min} = V_g(\phi = \phi_0)$, with the attractor conditions being satisfied. Evaluating (9.63) at the horizon, with $b = b_0 = r$ and taking $c_1 = -c_1$ without loss of generality, we get,

$$a_0^2 = \frac{2(-V_{g,min})}{(d-1)(d-2)} r^2 + 1 - c_1 r^{-(d-3)} + c_2 r^{-2(d-3)}. \tag{9.64}$$

It is easy to show that if we set $c_2 = 0$, and $d = 4$, we get,

$$a_0^2 = 1 - \frac{c_1}{r} - \frac{V_{g,min}}{3} r^2, \tag{9.65}$$

which is an AdS Schwarzschild solution, with $V_{g,min} = \Lambda < 0$, and $c_1 = 2G_N M$, and for larger r, one requires a to be negative as a metric component. Going back to the solution (9.63), we see that as we take r to larger values, the negativity of g_{tt}

requires $V_{g,min} < 0$. We also see that from [45],

$$c_1 = \frac{16\pi G_d M}{(d-2)\omega_{d-2}}, \tag{9.66}$$

$$V_{g,min} = \Lambda_{eff}, \tag{9.67}$$

where ω_{d-2} is the $(d-2)$ unit's sphere volume, M is the ADM mass and Λ_{eff} is the effective cosmological constant which remains negative. Substituting (9.62) to (9.63) we obtain the g_{tt} component of the metric,

$$a^2 = 1 + \frac{2(-V_{g,min})}{(d-1)(d-2)} r^2 - c_1 r^{-(d-3)} + c_2 r^{-2(d-3)}, \tag{9.68}$$

and then substituting (9.65) to (9.68) and also ((9.62)), and (9.68) to the energy constraint (9.36) and match with respect to the corresponding powers of r on both sides, we can solve for c_2 and obtain,

$$c_2 = (d-4)! V_{eff}. \tag{9.69}$$

Thus (9.68) finally becomes,

$$a^2 = 1 + \frac{2(-V_{g,min})}{(d-1)(d-2)} r^2 - \frac{16\pi G_d M}{(d-2)\omega_{d-2}} r^{-(d-3)} + (d-4)! V_{eff} r^{-2(d-3)}, \tag{9.70}$$

from the second term of (9.70) multiplied by $b^{2(d-3)}$, we can see that with $V_{g,min} < 0$, and $M > 0$, when $d > 3$ (9.70) has at most two real strictly positive roots. This can be seen by taking small values of r and $d > 3$, the second term of (9.70) dominates, and due to the term having a negative coefficient, it means, the polynomial is decreasing. As r increases, the first two terms will dominate so that the the polynomial will eventually curve upwards and keep increasing. We conclude that, depending on the value of c_2.

9.2.4 Perturbation about the attractor point

In this subsection we consider a perturbation analysis at zero temperature, around the Reissner-Nordström background, this is a solution with $\rho_\pm = \rho_{ext}$. The first order expansion of the scalars,

$$\phi^i = \phi^i_0 + \epsilon\phi^i_1. \tag{9.71}$$

Substituting the first order expansion to the equation of motion for the scalars (9.47) we get,

$$\partial_\rho \left(a_0^2 b_0^{2(d-3)} \partial_\rho \phi_1^i \right) = \frac{(d-2)}{4(d-3)^2} \frac{\partial_i^2 V_{eff}(\phi_0^i)}{b^{b^{2(d-3)}}} = \frac{m_i^2 \phi_1^i}{b_0^{2(d-3)}}, \tag{9.72}$$

where,

$$m_i^2 = \frac{(d-2)}{4(d-3)^2} \partial_i^2 V_{eff}(\phi_0), \tag{9.73}$$

as before, we used the fact that, the first order expansion in the scalars does not contribute to the correction of the metric components. As we saw in Section 8.3.1, that the solution is of the form,

$$\phi_1^i = c_1^i \left(1 - \frac{\rho_{ext}}{\rho} \right)^{\gamma_i}, \tag{9.74}$$

with

$$\gamma_i = \frac{1}{2} \left(\pm\sqrt{1 + \frac{4m_i^2}{\rho_{ext}^2}} - 1 \right), \tag{9.75}$$

we only take the positive sign, due to convergence without blowing blowing up at the horizon. Using the change in coordinates,

$$z = \frac{\rho_+(\rho - \rho_-) + \rho_-(\rho - \rho_+)}{\rho(\rho_+ - \rho_-)}, \tag{9.76}$$

we obtain, the Klein-Gordon equation.

$$\partial_z \left(\rho_{ext}^2 (z^2 - 1) \partial_z \phi_1^i \right) = m_i^2 \phi_1^i, \tag{9.77}$$

if scalars are taken as static with mass m_i we get a AdS_2 solution with metric,

$$ds^2 = -\rho_{ext}^2 (z^2 - 1) dt^2 + \frac{l^2 dz^2}{\rho_{ext}^2 (z^2 - 1)}, \tag{9.78}$$

this solution suggests the existance of an underlying SL(2,$\mathbb{R}$) symmetry thus this indirectly suggests an AdS/CFT description of the black hole's entropy. We know that the first order expansion in the scalars, gives rise to the second order expansion in the metric components [14]. Thus we can write,

$$\begin{aligned} b^{d-3} &= b_0^{d-3} + \epsilon^2 b_2, \\ a^2 &= a_0^2 + \epsilon^2 a_2, \end{aligned} \tag{9.79}$$

we can expand (9.56) up to second order and obtain the expansion coefficients b_2 and a_2, as,

$$a_0^2 b_0^{2(d-3)} + \left(2a_0^2 b^{d-3} b_2 + b_0^{2(d-3)} a_2\right)\epsilon^2 + \left(a_0^2 b_0^2 + 2a_2 b_2 b_0^{d-3}\right)\epsilon^4 + a_2 b_2^2 \epsilon^6 = (\rho-\rho_+)(\rho-\rho_-). \tag{9.80}$$

From (9.80) we can deduce that for $O(\epsilon^0)$, we get,

$$a_0^2 b_0^{2(d-3)} = (\rho-\rho_+)(\rho-\rho_-), \tag{9.81}$$

which is the Reissner-Nordström-like solution we obtained in Section 8.3.1. For $O(\epsilon^2)$, we can solve for a_2, and by using $\rho_\pm = \rho_{ext}$

$$\begin{aligned} 2a_0^2 b^{d-3} b_2 + b_0^{2(d-3)} a_2 &= 0, \\ \implies a_2 &= -\frac{2}{\rho}\left(1 - \frac{(\rho_+ + \rho_-)}{\rho} + \frac{\rho_+\rho_-}{\rho^2}\right) b_2, \qquad (9.82) \\ \implies a_2 &= -\frac{2}{\rho}\left(1 - \frac{\rho_{ext}}{\rho}\right)^2 b_2. \qquad (9.83) \end{aligned}$$

To solve for b_2, we use the equation of motion (9.50) and substitute (9.71), and get,

$$\begin{aligned} \sum_i \left(\partial_\rho(\phi_0^i + \epsilon\phi_1^i)\right)^2 (b_0^{d-3} + \epsilon^2 b_2) &= -\frac{d-2}{2(d-3)}\partial_\rho^2(b_0^{d-3} + \epsilon^2 b_2) \\ \implies \epsilon^2 \sum_i (\partial_\rho \phi_1^i)^2 b_0^{d-3} &= -\frac{d-2}{2(d-3)}\partial_\rho^2 b_0^{d-3} - \epsilon^2 \frac{(d-2)}{2(d-3)}\partial_\rho^2 b_2. \end{aligned} \tag{9.84}$$

For $O(\epsilon^0)$, we get

$$-\frac{d-2}{2(d-3)}\partial_\rho^2 b_0^{d-3} = 0 \to \partial_\rho^2 b_0^{d-3} = 0, \tag{9.85}$$

which is valid, since $b_0 = r$.

For $O(\epsilon^2)$, we get,

$$\partial_\rho^2 b_2 = -\frac{2(d-3)}{(d-2)}\sum_i (\partial_\rho \phi_1^i)^2 b_0^{d-3}, \tag{9.86}$$

integrating (9.86) twice to solve for b_2, we obtain,

$$b_2 = \sum_i d_2^i \rho \left(1 - \frac{\rho_{ext}}{\rho}\right)^{2\gamma_i}, \tag{9.87}$$

where,

$$d_2^i = -\frac{(c_1^i)^2(d-3)\gamma_i}{(d-2)(2\gamma_i - 1)}. \tag{9.88}$$

Thus the metric components are given by,

$$\begin{aligned} a^2 &= a_0^2 + \epsilon^2 a_2, \\ &= \frac{(\rho-\rho_+)(\rho-\rho_-)}{b_0^{2(d-3)}} + \epsilon^2 \left(-\frac{2}{\rho}\left(1-\frac{\rho_{ext}}{\rho}\right)^2 \sum_i d_2^i \rho \left(1 - \frac{\rho_{ext}}{\rho}\right)^{2\gamma_i}\right), \end{aligned} \tag{9.89}$$

in the vicinity of the horizon, $a^2 \to 0$, and $b_2 \to \sum_i d_2^i \rho \left(1 - \frac{2\gamma_i \rho_{ext}}{\rho}\right) \to constant$ and as we take $\rho \to \infty$, we see that $a \to constant$, and $b \to 0$, which is in agreement with the attractor equations.

9.2.5 Exact solution for a single scalar

In this section we consider a simple example of constructing an exact solution by following [43, 14]. The simple case consists of a single scalar coupled to a pair of gauge fields with $V_g = 0$ and we consider a magnetically charged solution. This leads to the potential of the form,

$$V_{eff}(\phi) = \sum_{a=1}^{2} e^{\alpha_a \phi}(Q_m^a)^2, \tag{9.90}$$

for this special case in the appendix I or appendix A of [4] we shown that,

$$\alpha_1 = -\alpha_2 = \alpha = \sqrt{\frac{8(d-3)}{d-2}}, \tag{9.91}$$

using this relation, we show that the metric components and the equation for the scalars are given by,

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\frac{(\rho + \frac{1}{2}\alpha\Sigma)}{(\rho - \frac{1}{2}\alpha\Sigma)}, \tag{9.92}$$

$$a^2 = \frac{(\rho-\rho_+)(\rho-\rho_-)}{b^{2(d-3)}}, \tag{9.93}$$

$$b^{2(d-3)} = \rho^2 - \frac{1}{4}\alpha^2\Sigma^2, \tag{9.94}$$

where σ is the scalar charge from the equation,

$$\phi = \phi_\infty + \frac{\Sigma}{r^{d-3}} + ... = \phi_\infty + \frac{\Sigma}{\rho} + ... \tag{9.95}$$

if we write the expression of a^2 in an asymptotic exapantion form, we can read off the mass,

$$M = \frac{\rho_+ + \rho_-}{2}, \tag{9.96}$$

we also show that the scalar charge depends on the mass term as follows,

$$\Sigma = \frac{(d-4)!\left((\bar{Q}_m^2)^2 - (\bar{Q}_m^1)^2\right)}{\alpha M}, \tag{9.97}$$

where,

$$\bar{Q}_m^a = e^{\frac{1}{2}\alpha_i \phi_\infty} Q_m^2. \tag{9.98}$$

We can obtain the relationship between ρ, and r, by defining,

$$\theta = \frac{(\rho - \frac{1}{2}\alpha|\Sigma|)}{\alpha|\Sigma|}. \tag{9.99}$$

Substituting this relation to the warp factor equation, (9.94) and the equation for the scalars (9.92), we get,

$$b = |\alpha\Sigma|^{\frac{1}{d-3}} \left(\theta(\theta+1)\right)^{\frac{1}{2(d-3)}}, \tag{9.100}$$

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\left(1 + \frac{1}{\theta}\right)^{sign(\Sigma)}. \tag{9.101}$$

Now if we integrate the coordinate transformation (9.37), we obtain the following relationship,

$$r = \frac{|\alpha\Sigma|}{d-3}\int_0^\theta d\bar{\theta} b^{4-d}(\bar{\theta}), \tag{9.102}$$

$$= \frac{|\alpha\Sigma|^{\frac{1}{d-3}}}{\frac{1}{2}(d-2)}\theta^{\left(\frac{1}{2}+\frac{1}{2(d-3)}\right)} \times_2 F_1\left(\frac{1}{2} - \frac{1}{2(d-3)}, \frac{1}{2} + \frac{1}{2(d-3); \frac{3}{2} + \frac{1}{2(d-3)}; -\theta}\right), \tag{9.103}$$

we take the initial condition of the integral as, $b(r) = 0|_{r=0}$. Taking the near extremal limit($\rho_\pm \to \rho_{ext}^2$), we find that the scalar charge and the mass term becomes,

$$\frac{1}{2}\alpha\Sigma = \sqrt{\frac{(d-4)!}{2}}(\bar{Q^2_m} - \bar{Q^1_m}), \tag{9.104}$$

$$M = \sqrt{\frac{(d-4)!}{2}}(\bar{Q^2_m} + \bar{Q^1_m}). \tag{9.105}$$

9.2.6 Attractors in infinite dimensions

In d dimensional spacetime, we usually expect the Newtonian gravitational potential of a single point source to fall like r^{d-3}, this agrees with the two dimensional space where the potential falls like $r^{3-2} = r$. If we take d to approach infinity, $r^{3-d} \to \frac{r^3}{r^d} \to 0$. This feature extends to general relativity[46]. This means that if the attractor mechanism is true as $d \to \infty$, then we would expect that the scalar fields start approaching $\phi_0' s$(the attractor values) close to the horizon. This suggests that away from the horizon, one expect scalars to not flow to their attractor values, by maintaining asymptotic values but they suddenly jump to their attractor values at the horizon. We confirmed this behavior by analyzing the exact extremal solutions found in the previous section. We notice that given an asymptotic value, the flow starts approaching a step function as the number of spatial dimensions increases.[4]

9.2.7 Horizon Invariants

This subsection discusses the existence of the products of horizon areas that form invariants in the attractor mechanism.

Higher dimensional area law

To simplify our analysis and discussion we set $V_g = 0$ and generalize the area law [2] using the results we obtained in Appendix D, we know that temperature can be calculated from computing the periodicity of the Euclidean time direction at two real horizons, which gives

$$T_{\pm} = \frac{(a^2)'_{\pm}}{4\pi}, \tag{9.106}$$

where $(\pm)$ represents the inner and outer horizons, from (9.37) and (9.56) we calculate the derivative of the metric component $(a^2)'$ in order to give different representations of the temperature. We start with,

$$\begin{aligned}(a^2)' &= \partial_r \left(\frac{(\rho - \rho_+)(\rho - \rho_-)}{b^{2(d-3)}} \right) \\ &= (d-3)b^{-(d-2)} \left(2\rho - (\rho_+ + \rho_-) - 2(d-3)(\rho - \rho_+)(\rho - \rho_-)b^{-1}\partial_\rho b\right).\end{aligned} \tag{9.107}$$

Evaluating the results of (9.107) at the horizon (+) we get,

$$(a^2)'_+ = (d-3)b^{-(d-2)} \left(2\rho_+ - (\rho_+ + \rho_-) - 0\right) = \frac{2(d-3)\Delta}{b_+^{d-2}}. \tag{9.108}$$

Similarly at the $(-)$ horizon,

$$(a^2)'_- = -\frac{2(d-3)\Delta}{b_-^{d-2}}. \tag{9.109}$$

Thus the temperature can be written as,

$$T_{\pm} = \frac{(a^2)'_{\pm}}{4\pi} = \pm\frac{2(d-3)\Delta}{b_{\pm}^{d-2}}. \tag{9.110}$$

Where we have defined $\Delta = \frac{1}{2}(\rho_+ - \rho_-)$ as the non-extremality measure. We notice from (9.106) that the extremal case is when $\Delta = 0$, therefore, as $\Delta \to 0$, the solution approaches an extremal solution, we also notice that the extremality is proportional to the temperature. The inner horizon has a negative temperature. Thus as Δ increases, the area of the inner horizon becomes smaller. Interpreting the area as entropy with respect to the black hole mechanics, it means the inner horizon represents fewer degrees of freedom the more non-extremal the solution becomes.

Using (9.106) we can write,

$$\frac{T_+}{T_-} = \frac{\frac{2(d-3)\Delta}{b_+^{d-2}}}{\frac{-2(d-3)\Delta}{b_-^{d-2}}} = -\frac{b_-^{d-2}}{b_+^{d-2}},$$

$$\implies T_+ b_+^{d-2} + T_- b_-^{d-2} = 0, \tag{9.111}$$

since in d dimensions, the area of the horizon is given by $A \to b^{d-2}$, therefore,

$$A_+T_+ + A_-T_- = 0, \tag{9.112}$$

and if the entropy scales with the area, we can deduce that,

$$\frac{\partial}{\partial M} S_+S_-|_{Q_m^a} = \frac{S_+T_+ + S_+T_-}{T_+T_-} = 0. \tag{9.113}$$

If we have an extremal solution, the product of the entropies and the areas are mass indipendent.

$$S_+S_- = S_{ext}^2 \to A_+A_- = A_{ext}^2. \tag{9.114}$$

When $V_g \neq 0$, and taking $d = 4$ with constant scalars, we observe that from (9.13), (9.14) and (9.15) the product of all the horizons is given by,

$$r_+r_-|\lambda|^2 = \frac{3V_{eff,min}}{(-V_{g,min})}. \tag{9.115}$$

We notice that this case is doesn't include our assumption of only focusing on only real positions of the horizons. Keeping $V_g \neq 0$, leads to non real or complex horizon positions. Now if we set $b(r) = r$, then from (9.22) we can get,

$$r_{ext}^4 = \left(\frac{\sqrt{1 + 4V_{eff,min}(-V_{g,min})} - 1}{2(-V_{g,min})}\right)^2, \tag{9.116}$$

from this analysis one can see that unlike in the case of $V_g = 0$, the geometric mean of the two horizon areas is not equal to the extremal area.

9.2.8 Only two real zeros

In this subsection we start with a simple case of calculating product of horizon invariants, buy setting $V_g = 0$, and setting $d = 4$. We first start by integrating the equation of motion (9.3), and using $a^2(r_\pm) = 0$ (the Reissner-Nordström like solution), we get

$$\begin{aligned} \int_{r_-}^{r_+} (a^2 b^2)'' dr &= 2\int_{r_-}^{r_+} dr, \\ \implies (a^2)' b^2|_{r=r_-,r_+} + a^2 (b^2)'|_{r=r_-,r_+} &= 2(r_+ - r_-) \\ \implies (a^2)' b^2|_{r=r_\pm} = 2(r_+ - r_-) &= 4\Delta = \alpha(r_-, r_+). \end{aligned} \tag{9.117}$$

Thus from (9.3) we can see that there exists l^- and l^+, such that $l^+ - l^- = 1$(if we choose, $\alpha = \frac{1}{2(r_+ - r_-)}$(without loss of generality), which gives,

$$\begin{aligned} (a^2)' b^2|_{r=r_-} &= \alpha l^-, \\ (a^2)' b^2|_{r=r_+} &= \alpha l^+. \end{aligned} \tag{9.118}$$

We have now calculated the constraint that the equations of motion must satisfy, we now consider the one dimensional on-shell static spherically symmetric action with a single scalar is given by,

$$S_{d=1} = \int dr \left((a^2 b^2)' - a^2 b^2 (\phi')^2 - \frac{V_{eff}(\phi)}{b^2} + totalderivative \right). \tag{9.119}$$

We vary this action with respect to the degrees of freedom a^2, b^2, and ϕ, and impose the following asymptotic boundary conditions,

$$a^2 = a - \frac{2M}{r} + \cdots , \tag{9.120}$$

$$b^2 = r^2 + constant + \cdots , \tag{9.121}$$

$$\phi(r) = \phi_\infty + \frac{\Sigma}{r} + \cdots . \tag{9.122}$$

Varying this action gives the equations of motion we obtained before, (9.3)- (9.5). By performing the variation of the on-shell field configurations only, the volume term about the extremum vanishes. Therefore, only the derivative of the surface terms survive. If we let Φ represent some degree of freedom for a general action its variation is solved by the Lagrange equations of motion. Before we consider the general case, to illustrate our method, we initially set $V_g = 0$ and work in four dimensions and recover known results. Since the effective action (9.119) satisfies the equations of motion, then the solutions $a(r)$, $b(r)$, and $\phi(r)$, also satisfy the energy constraint (9.6). We also parametrize our solution space by $(a^2)'_\pm$. The variation we perform in this example are only on-shell, thus we keep electric and magnetic charges constant through out. The solution of the action on-shell is given by the Euler-Lagrange equations,

$$\frac{\delta L}{\delta\Phi} - \frac{d}{dr}\left(\frac{\delta L}{\delta(\partial\Phi)}\right) = 0. \tag{9.123}$$

If we integrate this equation, the terms arising from the volum term vanishes about the extremum, and the only term left with is the total derivative or the surface term. If we integrate by parts on the term, $\frac{\delta L}{\delta(\partial\Phi)}\delta(\partial\Phi)$, we get,

$$\int_{on-shell} dr \frac{\delta L}{\delta(\partial\Phi)}\delta(\partial\Phi), \tag{9.124}$$

set, $u = \frac{\delta L}{\delta(\partial\Phi)}$, $dv = \delta(\partial\phi) = \partial(\delta\phi)$, then using integration by parts, we have,

$$\int_{on-shell} dr \frac{\delta L}{\delta(\partial\Phi)}\delta(\partial\Phi) = \frac{\delta L}{\delta(\partial\Phi)}\delta(\Phi)|_{r_-}^{r_+} - \int_{on-shell} dr\delta\Phi\partial\frac{\delta L}{\delta(\partial\Phi)}, \tag{9.125}$$

$$\implies 2\int_{on-shell} dr \frac{\delta L}{\delta(\partial\Phi)}\delta(\partial\Phi) = \frac{\delta L}{\delta(\partial\Phi)}\delta(\Phi)|_{r_-}^{r_+} - \int_{on-shell} dr\partial\left(\frac{L}{\delta(\partial\Phi)}\delta\Phi\right) = 0, \tag{9.126}$$

this tells that

$$\int_{r_-}^{r_+} dr\frac{d}{dr}\left(\frac{\delta L}{\delta(\partial\Phi)}\delta(\Phi)\right) = \frac{\delta L}{\delta(\partial\Phi)}\delta(\Phi)|_{r_-}^{r_+} = 0. \tag{9.127}$$

It is clear that the surface term (9.127) does not receive any contribution from the potential term, $\frac{\partial L}{\partial\Phi}$, and applying this to our metric components keeping the boundary at the horizons, $a^2 = 0$ and $\delta a^2 = 0$, we can re write (9.127) as,

$$\frac{\delta L}{\delta(\partial\Phi)}\delta\Phi|_{r_-}^{r_+} = \left(\frac{\delta L}{\delta((a^2)')}\delta a^2 + \frac{\delta L}{\delta((b^2)')}\delta b^2 + \frac{\delta L}{\delta((\phi^i)')}\delta\phi^i\right)|_{r_-}^{r_+}, \tag{9.128}$$

$$= bb'\delta a^2|_{r_-}^{r_+} + (a^2)'\delta b^2|_{r_-}^{r_+} - 2a^2b^2(\phi^i)'|_{r_-}^{r_+}, \tag{9.129}$$

$$= (a^2)'\delta b^2|_{r_-}^{r_+} = 0. \tag{9.130}$$

Now using the results obtained at (9.118) to rewrite (9.128), we have,

$$
\begin{aligned}
&(a^2)'\delta b^2(r_+) - (a^2)'\delta b^2(r_-) = 0, &(9.131)\\
\implies\ &\alpha l^+ \frac{\delta b^2(r_+)}{b^2(r_+)} - \alpha l^- \frac{\delta b^2(r_-)}{b^2(r_-)} = 0, &(9.132)\\
\implies\ &\alpha\left(l^+\delta(\log(b^2(r_+))) - l^-\delta(\log(b^2(r_-)))\right) = 0, &(9.133)\\
\implies\ &l^+\delta(\log(b^2(r_+)) = l^-\delta(\log(b^2(r_-)). &(9.134)
\end{aligned}
$$

As we stated before that the variation is performed in a space of on-shell solutions, at fixed charges denoted by Q, and every point is characterized by constant scalars fixed at asymptotic values. These assumptions satisfy the analysis of the attractor mechanism. We know that for extremal solutions, the scalar fields flow to the point where the effective potential is minima. Preserving this behavior, we assume a smooth zero temperature limit to recover this behavior. Now we consider a variation around a point in some solution space $\star$ to another point in some solution space $\star\star$, moving from $\star$ to $\star\star$ is determined by by the infinitesimal transformation,

$$\log b^2_{\star\star} = \log b^2_{\star} + \delta \log b^2. \tag{9.135}$$

Therefore, the constraint (9.134) can be rewritten as in forward and reverse directions,

$$
\begin{aligned}
l^+_\star\delta(\log(b^2(r_+))) &= l^-_\star\delta(\log(b^2(r_-))), &(9.136)\\
l^+_{\star\star}\left(-\delta(\log(b^2(r_+)))\right) &= l^-_{\star\star}\left(-\delta(\log(b^2(r_-)))\right). &(9.137)
\end{aligned}
$$

If we take the ratio of (9.136) and (9.137), we get,

$$\frac{l^+_{\star\star}}{l^+_\star} = \frac{l^-_{\star\star}}{l^-_\star}. \tag{9.138}$$

This is true for any two points connected in the solution spaces with an infinitesimal transformation. If we also assume that the solution space is connected and smooth, then we can conclude that,

$$\frac{l^-}{l^+} = \kappa = constant. \tag{9.139}$$

This is true at any point in the space of solutions. Since we have identified the constant, we can rewrite the constraint (9.134) at any point in the space of solutions as,

$$
\begin{aligned}
& l^{+}\delta(\log(b^2(r_+)) = l^{-}\delta(\log(b^2(r_-)), && (9.140)\\
\Longrightarrow\ & l^{+}\delta(\log(b^2(r_+)) = \kappa l^{+}\delta(\log(b^2(r_-)), && (9.141)\\
\Longrightarrow\ & \delta(\log b(r_+)) - \delta(\kappa \log b(r_-)) = 0, && (9.142)\\
\Longrightarrow\ & \delta\left(\log \frac{b(r_+)}{b^{\kappa}(r_-)}\right) = 0, && (9.143)\\
\Longrightarrow\ & \log \frac{b(r_+)}{b^{\kappa}(r_-)} = \tilde{\kappa}, && (9.144)\\
\Longrightarrow\ & b(r_+) b^{-\kappa}(r_-) = e^{\tilde{\kappa}}, && (9.145)\\
\Longrightarrow\ & b(r_+) b^{-l^-/l^+}(r_-) = constant, && (9.146)
\end{aligned}
$$

where κ, and $\tilde{\kappa}$ are constants. Recall that the variation was done on-shell, with constant charges, using that assumption we can rewrite the right-hand side of (9.146) as a function of constant charges,

$$b(r_+) b^{-l^-/l^+}(r_-) = F(Q). \tag{9.147}$$

Now we determine the function $F(Q)$. To do this, we recall that in the solution space there exist a point that is purely characterized by constant scalars, and we also recall that, at this point, we obtained a Reissner-Nordström solution with the property,

$$r_+ r_- = F(Q) = r_{ext}^2. \tag{9.148}$$

Using this analysis, we can see that $l^+ = l^-$, and thus, the invariant constraint is given by,

$$b^2(r_+) b^2(r_-) = constant. \tag{9.149}$$

Using this property we can see that we can directly derive the area law by multiplying both sides with $(4\pi^2)$, which gives,

$$4\pi b^2(r_+) 4\pi b^2(r_-) = A_+ A_- = A_{ext}^2. \tag{9.150}$$

Which is an invariant as we saw before.

9.2.9 N complex zeros

In the previous subsection we found invariants, but we only focused on two real roots of $a(r)$. We generalize this technique where $a(r)$ has N roots. Suppose that the roots of $a(r)$ lie on the complex plane $\mathcal{C}$ and are positioned at r_i, $1 \leq i \leq N$. This implies that each r_i is a boundary for variational problem. Following the procedure as in the previous subsection, we start by considering the effective one dimensional action [4],

$$\begin{aligned} S_{1d} = & \int dr\Big((d-3)(d-2)b^{d-4}(1+a^2b'^2) + (d-2)b^{d-3}(a^2)'b' \\ & - 2a^2b^{d-2}(\partial_r\phi)^2 - \frac{(d-2)!}{b^{d-2}}V_{\text{eff}} - 2b^{d-2}V_g\Big) \qquad (9.151) \\ = & \int d\rho\Big((d-2)\left(1 + a^2(\partial_\rho b^{d-3})^2 + \tfrac{1}{2}\partial_\rho(a^2)\partial_\rho[(b^{d-3})^2]\right) \\ & - 2(d-3)a^2(b^{d-3})^2(\partial_\rho\phi)^2 - \frac{(d-2)!}{(d-3)(b^{d-3})^2}V_{\text{eff}} - \frac{2}{(d-3)}b^2V_g\Big), \qquad (9.152) \end{aligned}$$

as a line integral along a contour $\mathcal{C}$ in the complex ρ-plane. This one dimensional effective action is obtained from integrating out the gauge charges from the equations of motion. Unless all of the roots lie along a line, we can always form a simple polygon on the complex plane with the roots at the vertices. We define $\mathcal{C}$ as this N-gon and order the vertices $\rho_1, \cdots, \rho_N$. Suppose we consider the variation of an on-shell configuration. The variation remains on-shell along any real line. (This line may connect, for example, the horizon to the asymptotic boundary.) Along the real line the Euler-Lagrange equation for a degree of freedom Φ is

$$\frac{\partial L}{\partial \Phi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial(\partial_\rho \Phi)}\right) = 0. \qquad (9.153)$$

Here, we wish to vary the configuration evaluated on the contour $\mathcal{C}$ in the complex plane. On the i-th edge of the N-gon, the Euler-Lagrange equation becomes

$$\frac{\partial L}{\partial \Phi} - \frac{ds_i}{d\rho}\frac{d}{ds_i}\left(\frac{\partial L}{\partial\left(\frac{ds_i}{d\rho}\frac{\partial \Phi}{\partial s_i}\right)}\right) = 0, \qquad (9.154)$$

where $\rho = \rho(s_i)$ and s_i is an affine parameter with $\frac{d\rho}{ds_i} = constant$. These Euler-Lagrange equations are valid on the $i-th$ edge of the N-gon. To make sure that the configurations remain on-shell, we put

$$\rho(s_i) = \rho_i + (\rho_{i+1} - \rho_i)s_i, 0 \leq s_i \leq 1, \tag{9.155}$$

to linearly interpolate between adjacent roots. For on-shell configurations along $\mathcal{C}$, following the arguments of the previous subsection, integrating along the i-th edge gives

$$(a^2)'\delta b^{2(d-3)}|_{\rho_i}^{\rho_{i+1}} = \frac{1}{D_i}\frac{d}{ds_i}(a^2)\delta b^{2(d-3)}|_{\rho_i}^{\rho_{i+1}} = 0, \tag{9.156}$$

with $D_i = \frac{d\rho}{ds_i} = \rho_{i+1} + \rho_i$. to obtain the constraint as before, we also apply (9.117), by integrating the equation of motion to get,

$$\int_{\rho_i}^{\rho_{i+1}} d\rho\partial_\rho^2\left(a^2 b^{2(d-3)}\right) = 2(\rho_{i+1} - \rho_i) - \frac{4}{(d-3)^2}\int_{\rho_i}^{\rho_{i+1}} V_g b^2, \tag{9.157}$$

$$\partial_\rho(a^2)b^{2(d-3)}|_{\rho_i}^{\rho_{i+1}} + a^2\partial_\rho(b^{2(d-3)})|_{\rho_i}^{\rho_{i+1}} = 2(\rho_{i+1} - \rho_i), \tag{9.158}$$

recall that $a^2(\rho_{i+1}) = a^2(\rho_i) = 0$, thus,

$$\frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)b^{2(d-3)}|_{\rho_i}^{\rho_{i+1}} = \alpha_i(\rho_i, \rho_{i+1}). \tag{9.159}$$

As before, this tells us that we can find $2N$ real numbers, $l_i^\pm$ such that,

$$\alpha_i l_i^+ = \frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)b^{2(d-3)}|_{\rho_i}, \tag{9.160}$$

$$\alpha_i l_i^- = \frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)b^{2(d-3)}|_{\rho_{i+1}}, \tag{9.161}$$

with $l_i^+ - l_i^- = 1$. Using (9.160)-(9.161) we can rewrite (9.156) as

$$\frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)\delta b^{2(d-3)}|_{\rho_i}^{\rho_{i+1}} = 0, \tag{9.162}$$

$$\implies \frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)b^{2(d-3)}|_{\rho_{i+1}} - \frac{1}{D_i}\left(\frac{d(a^2)}{ds_i}\right)b^{2(d-3)}|_{\rho_i} = 0, \tag{9.163}$$

which implies,

$$\frac{\alpha_i l_i^+}{b^{2(d-3)}}\delta b^{2(d-3)}|_{\rho_{i+1}} - \frac{\alpha_i l_i^-}{b^{2(d-3)}}\delta b^{2(d-3)}|_{\rho_i} = 0, \tag{9.164}$$

$$\implies \alpha_i\left(l^+\delta\log b_{i+1}^{2(d-3)} - l^-\delta\log b_i^{2(d-3)}\right) = 0. \tag{9.165}$$

As we did in the "two zero only" case, we can conclude that $\frac{l_i^-}{l_i^+} = constant$ and thus substituting this constant to the previous equation we get,

$$\delta\left(\log\left(b_{i+1}^{2(d-3)} b_i^{2(d-3)\left(-\frac{l_i^-}{l_i^+}\right)}\right)\right) = 0. \tag{9.166}$$

Now combining all the edges in the N-gon, we have,

$$\delta\left(\log\prod_{i=1}^{N} b_i^{2(d-3)\left(1-\frac{l_i^-}{l_i^+}\right)}\right) = 0, \tag{9.167}$$

and integrating we get,

$$\log\prod_{i=1}^{N} b_i^{2(d-3)\left(1-\frac{l_i^-}{l_i^+}\right)} = \kappa = constant, \tag{9.168}$$

$$\implies \prod_{i=1}^{N} b_i^{\left(1-\frac{l_i^-}{l_i^+}\right)} = e^{\kappa} = constant. \tag{9.169}$$

Again, we emphasize that in obtaining this equation, we have considered variations in the space of solutions for which the gauge charges are held fixed. Going into this are the assumptions that the solution space is continuous and is characterized by the temperatures at each of the horizons and the moduli at infinity. A special point in the space of solutions is the extremal black hole spacetime. Here, the hot attractor mechanism we are studying reduces to the usual attractor mechanism. The temperature vanishes at the extremal horizon and the scalars flow from infinity to attractor values. From our knowledge of what happens at the extremal point in solution space, we may deduce the constant on the right hand side. Now to derive this constant in the right hand side of (9.168), we consider a point in solution space where we set the moduli at infinity to their attractor values, as we noticed in Section 9.2.1, $a^2b^{2(d-3)}$ is a $2(d-2)$ degree polynomial in r. Thus using (9.63) and (9.63), Vieta's formula tell us that the product of the roots is

$$\prod_{i=1}^{2(d-2)} r_i = \frac{(d-1)!}{2(d-3)}\frac{V_{eff,min}}{(-V_{g,min})}. \tag{9.170}$$

The roots specify the locations of the horizons on the complex r-plane. From (9.170), the product of the positions of the horizons depends only on $V_{g,min}$ and $V_{eff,min}$. The potential V_g for the scalars appears in the action (9.25) as a cosmological term. We recall from (9.32) that V_{eff} is a function of the scalars and the charges. The right hand side of (9.170) is therefore determined completely by the charges and the attractor values of the moduli. Let us hold the moduli fixed at the attractor values and consider variations of the solution that change, for instance, the mass or the temperature of the black hole. Under such variations, we have shown that

$$\delta\left(\prod_{i=1}^{2(d-2)} r_i\right) = \delta\left(\prod_{i=1}^{2(d-2)} b_i\right) = 0, \tag{9.171}$$

where the first equality uses the identification of coordinates in (9.62). Comparing to (9.168), we conclude from this that

$$\frac{l_i^-}{l_i^+} = \frac{l_j^-}{l_j^+}, \tag{9.172}$$

for all i and j. Hence, analysis of the solution at the special point in moduli space where the scalars assume the attractor values tells us that the general on-shell 2(d 2)-horizon solution space has an invariant defined by $\delta\left(\prod_{i=1}^{2(d-2)} b_i\right)$ and this invariant is

$$\prod_{i=1}^{2(d-2)} b_i = \frac{(d-1)!}{2(d-3)} \frac{V_{eff,min}}{(-V_{g,min})}. \tag{9.173}$$

The left hand side applies to any d dimensional spherically symmetric solution of the equations of motion, in particular to a general non-extremal black hole with horizons at complex radii. The right hand side is fully determined by the behavior of the potential when the scalars are fixed to attractor values corresponding to the extremal black hole. The former solution is related to the latter by taking the zero temperature limit.

9.2.10 Invariants for stationary backgrounds

In this second component of the thesis, we have so far analyzed static spherically symmetric black hole backgrounds in arbitrary two derivative theories of Einstein-Hilbert gravity coupled to matter in arbitrary dimensions. The analysis was dramatically simplified by the existence of a universal ansatz for these black hole backgrounds which enabled us to focus on a 1 + 1 dimensional effective non-linear sigma model and identify the boundary terms that are required to be invariant under a perturbation from one point to another in solution space. One observes empirically that for known rotating solutions such as the Kerr-Newman solution, there exists a similar invariant constraint on the solution space. In this case, however, we do not have a universal ansatz for a stationary black hole background, and hence, in order to explore the existence of an invariant in these backgrounds, we adopt a more abstract though generalized analysis. We sketch the argument below. In order to restrict to a single angular momentum, we will specialize to the four dimensional case. As we have seen in Iyer and Wald [11] showed that for any black hole backgrounds with Killing vector χ in a diffeomorphism invariant theory of gravity, under a perturbation in solution space that keeps the Killing vector invariant, the contribution to the change in the conserved quantity $W = \kappa S$ at each Killing horizon is given by

$$\delta W = \kappa \delta S, \tag{9.174}$$

Here, $\kappa = \chi \cdot \nabla \chi$ evaluated at the Killing horizon, and S is a quantity defined in terms of the local fields evaluated at the horizon and is obtained from variation of the Lagrangian with respect to the Riemann tensor. At the Schwarzschild horizon, S is the Wald entropy of the black hole background. The Wald formula can be used to evaluate S at each of the other Killing horizons in the spacetime, except that it no longer carries the interpretation of entropy. Therefore, given n Killing horizons, we can define the boundary term at each horizon, which depends on the local values of the fields at the i-th boundary and the norm of the gradient of the Killing vector at that boundary as

$$W_i = \kappa_i S_i = \alpha_i l_i(\phi), \tag{9.175}$$

where $l_i(\phi)$ is a function of all the fields evaluated at the i-th Killing horizon. This equation is analogous to (9.118) and if we perform on-shell variations and making sure that the boundary terms are zero, akin to the Iyer-Wald perturbation in solution space that gave rise to (9.174), we get the constraint,

$$\sum_i \delta W_i = \sum_i \kappa_i \delta S_i = \sum_i \alpha_i l_i(\phi) \delta(\log S_i) = 0. \tag{9.176}$$

This expression is analogous to what we did in going from (9.128) to (9.133) using (9.118) or from (9.160,9.161) to (9.164) using (9.156). Notice that this is essentially a generalization of the null variation of boundary terms at the Killing horizons for on-shell variations for static black backgrounds, as we have explicated in subsection 9.2.7. It may, however, be useful to step back and examine the inputs that inform (9.176). We have assumed the validity of the first law of black hole mechanics based on the general assumptions of [11]. We expect that there is a well defined, continuous space of solutions that includes the extremal point and that the variational problem makes sense on this geometry. To make contact with our discussion of Reissner-Nordström-like hot attractors, we as well keep any gauge charges fixed when we vary below from one solution to another. Motivated by the "no hair" theorem, we should also keep angular momenta fixed. Depending on the circumstances under consideration, these assumptions may not be the most general or may be too restrictive. For our purposes, these are minimal assumptions, and we will nevertheless proceed with these caveats in mind. Now, in analogy to (9.136)-(9.138), we consider an on-shell variation between two in- finitesimally separated points in solution space, $\star$ and $\star\star$. Under a variation from the former to the latter, we obtain the constraint,

$$\sum_i \alpha_i l_{i,\star}(\phi)\delta(\log S_i) = 0, \tag{9.177}$$

whereas a reverse variation from $\star\star$ to $\star$ yields,

$$\sum_i \alpha_i l_{i,\star\star}(\phi)\delta(\log S_i) = 0. \tag{9.178}$$

The two variations (9.177) and (9.178) are compatible if and only if,

$$\frac{l_{i,\star}(\phi)}{l_{i,\star\star}(\phi)} = \frac{l_{j,\star}(\phi)}{l_{j,\star\star}(\phi)}, \tag{9.179}$$

where $1 \leq i, j \leq n$. Hence, the $l_i(\phi)$ are constant throughout the space of stationary black hole backgrounds assuming that this solution space is connected. The on-shell variational constraint then reduces to

$$\sum_i \delta(\log S_i^{l_i(\phi)}) = 0. \tag{9.180}$$

This implies that in theories of gravity in arbitrary dimensions, with smoothly connected solution spaces, determining the $l_i(\phi)$ at any given point fixes them throughout the space. In particular, if this space contains a point corresponding to a constant scalar field background that resembles the Kerr-Newman spacetime, then an empirical observation of this solution shows that $l(\phi) = |l_i(\phi)|$ for each Killing horizon i. We therefore arrive at the general result that under an on-shell variation of stationary black hole backgrounds with n Killing horizons in two derivative theories of gravity, in arbitrary dimensions, with smoothly connected solution-spaces, containing a Kerr-Newman background, there exists a conserved quantity given by

$$\prod_i^n S_i = f(Q), \tag{9.181}$$

where $f(Q)$ represents a function of all of the non-mass parameters (such as U(1) charges and angular momenta) of the black hole background that are fixed under the on-shell variation. We note that the statement (9.181), though a stronger statement than (9.180), can only be made in the context of two derivative theories where empirical constant scalar black hole backgrounds can be explicitly written down, while (9.180) holds in greater generality. If a constant scalar Kerr-Newman-like solution exists in a higher derivative theory, we can make the statement (9.181) in this context as well. We conclude this section by sketching a heuristic motivation for the area law for ro- tating, charged black holes in four dimensional ungauged supergravity, and make some contextual comments on the logic of our calculations. Given an asymptotically flat black hole background with a specific set of charges and angular momenta and with two hori- zons corresponding to two length scales, if the extremal limit exists, then the non-extremal solution can be regarded as a non-extremal excitation of the extremal background that is characterized by the warp factor $b_{ext}(r,\theta)$ and the non-extremality parameter Δ. This is an uncontroversial observation about the near extremal solutions, but the statement applies more broadly to any non-extremal black hole which admits a smooth zero temperature limit in $\mathcal{N} = 2$ supergravity. The claim is justified by the attractor mechanism. We note that (9.181) tells us that the product of the S_i is an invariant. Each of the S_i is characterized by a length scale, b_i. In the case where there are two Killing horizons, we have $b_+(r,\theta)$ and $b_-(r,\theta)$, which, respectively, corresponding to the event and Cauchy horizons. On dimensional grounds, the two parameterizations are related by

$$b_+b_- = \sum_n c_n b_{ext}^n \Delta^{2-n}. \tag{9.182}$$

Here, we assume that a nice expansion exists for which c_n are system constants; by virtue of extremality, b_{ext} is mass independent. If the double extremal limit exists as a Kerr- Newman solution, we see that $c_n = \delta_{n,2}$. This motivates the area law for this specific set of black hole backgrounds.

Notice that both the above motivational argument for solutions in the Kerr-Newman class as well as the rigorous discussions for solutions in the Reissner-Nordström class are motivated by an exploration of the attractor mechanism for black holes. This mechanism, originally discovered for extremal black hole backgrounds with neutral scalar fields, fixes the scalars at the extremal horizon in terms of the charges of the black hole, irrespective of their asymptotic values. The on-shell variation in the solution space is therefore naturally performed at fixed charges and with varying values of asymptotic scalar moduli. The entropy of the black hole, which is essentially the horizon length scale is therefore completely independent of all asymptotic data and is purely a function of the charges and the angular momenta. This is the area law for extremal black holes — a length scale defined purely by non-mass parameters of the black hole. We identified a generalized version of the attractor mechanism under a similar variation for non-extremal charged black holes, and proved a corresponding area law, a product of the horizon length scales which again proved to be defined purely in terms of the non-mass parameters of the black hole. The on-shell variation involves modulating the asymptotic data, which includes the scalar field asymptotia at fixed charges and angular momenta. The ADM mass of the black hole, in the event, that it is defined is purely a function of the scalar asymptotia at fixed charges and angular momenta and its variation is fully specified by the variation in the scalar field data. All the results in this paper are based on this variation, which is natural to the attractor mechanism. Under a different variation in solution space, with say varying charges, one might potentially find different invariants. However, that is an open question at this point in time.

Chapter 10

Discussion and Conclusions

In chapter 7 we observed that we were able to match the numerical solutions of the non-Schwarzschild black holes and the asymptotic expansion solution, the matching of these two solutions also give accurate results of the mass of the black hole up to two decimal places. We also showed that as we increase the order of the asymptotic expansion, higher order corrections rapidly decrease as the radial component of the black hole increases. We observed that, instead of one using numerical methods for integrating the equations of motion in order to find black holes, one can just use the near horizon expansion, which can be expanded at any order greater than $O(9)$, then patch that near horizon expansion with the asymptotic expansion to obtain full solutions. In this component of the thesis we used $O(9)$, $O(12)$, $O(15)$, and $O(18)$, and the order of the near horizon expansion is increased the more accurate the solution meaning the more it is valid for larger values of r away from the event horizon.

We found convincing further evidence for the existence of non-Schwarzschild black holes as mathematical solutions to (**??**) and (**??**). We showed that the numerical solutions matched well will our asymptotic expansion and that the expansions seemed to work well at large distances from the horizon. Unfortunately, this class of black holes does not seem to be physically reasonable. The fact that only small (on the length scale of L_α) solutions seem to exist[1], invalidates our original assumptions about treating gravity as some effective theory and neglecting even higher order corrections. Truncating (4.1) at $\mathcal{O}\left(L_P^2/L_C^2\right)$, and quantizing, leads to ghosts [13]. Perhaps, these ghosts are responsible for some of the strange features, like negative entropy and mass, of the hot branch of non-Schwarzschild black holes. It would appear, that in the context of effective theory, that there is no higher derivative hair at $\mathcal{O}\left(L_P^2/L_C^2\right)$, in asymptotically flat space – hence the title of our paper[20]. We did try to search for other branches of non-Schwarzschild solutions with more physically reasonable solutions. Although we did not succeed, we may just have missed

[1] One can construct large solutions with negative entropy but these are not physical.

them just as the cold branch was missed in [1] – further study would be required to completely exclude this possibility.

Recently, neglecting the $\mathcal{O}\left(L_P^2/L_C^2\right)$ order terms and going to the next order, have constructed black holes in pure gravity. These solutions seem to be more reasonable than the ones discussed in this paper. We hope that generalizing our results to higher dimensions and asymptotically AdS space-times will lead to new non-pathological solutions without having to resort to the complexity of the next order. In particular, adding a cosmological constant, and working in asymptotically AdS space, it was shown in [1], that one necessarily has $R = \Lambda$ everywhere. This means that one can not neglect terms in the equations of motion involving β as we did in flat space, this may lead to a richer solution space in AdS. In appendix D we have shown how one derives the equations of motion and linearizing them and calculate the entropy with the inclusion of the cosmological constant, but we decided not to pursue any further investigations such as finding numerical solutions, asymptotic expansion and matching conditions for such a theory, so we left it as an exercise in this thesis.

$\mathcal{N} = 2$ supergravity supplies a powerful framework for studying black holes. In particular, in four and five dimensions, the attractor mechanism explains the dynamics of moduli in extremal geometries. The presence of an AdS_2 throat fixes scalars at the horizon. For non- extremal black holes, our previous work [2, 3] derives conserved quantities associated to the product of the areas of the outer and inner horizon and the extremal attractor equations integrated over the inter-horizon domain. Let us review what we have done in this component of the thesis. A change of coordinates (9.37) recasts the equations of motion into a form analogous to what we encounter in four dimensions. This enables us to establish averaged attractor equations (9.54) and (9.55) in higher dimensional settings. The $d \to \infty$ limit of extremal solutions from subsection 9.2.4 shows that the scalar flow follows a step function that interpolates between the asymptotic moduli and the attractor value at the horizon. We consider fluctuations about the attractor point in subsection 9.2.2 and demonstrate the existence of an AdS_2 in the space of solutions. Section 9.2.3 and I constructs an exact non-extremal solution with a single scalar field that couples to a pair of gauge fields in arbitrary dimension. In addition to ungauged supergravity, we can introduce Fayet-Iliopoulos terms to perform an Abelian gauging. At the level of the low energy effective action, the Fayet-Iliopoulos parameters introduce a potential $V_g(\phi)$ for the scalars that acts like a negative cosmological term (9.67). We write the attractor equations in the presence of this term. Structurally, (9.33)-(9.35) are similar to the equations in the four dimensional case. The solutions to these equations that we study are the simplest ones: the scalars are fixed to their attractor values. In subsection 9.2.1 we demonstrate that there are at most two real strictly positive roots where horizons are situated.

In subsection 9.2.5, we closely investigated the horizon invariants. In ungauged supergravity, following [2], it is straightforward to extend the area law to higher dimensions. With $V_g(\phi) \neq 0$, for constant scalar solutions, the product of the positions of the horizons of a non-extremal black hole is an invariant expressed in terms of the potentials, but the extremal horizon area is not the geometric mean of the areas of the non-extremal horizons. We establish the form of the invariant (9.173) in arbitrary dimensions. This relation applies to the case of N horizons positioned at complex values of the radial coordinate. The logic used in deriving this expression generalizes to the case of rotating Kerr-Newman-like black holes.

In subsection 9.2.5, we sketch this argument in four dimensional ungauged supergravity. Promoting black hole thermodynamics to black hole statistical physics, we expect that the entropy, which at leading order is computed by the area of the event horizon in Planck units, counts the microstates of the black hole in quantum gravity. For non-extremal black holes in four dimensions, there are three distinct regions in the spacetime: Region 1 in which $r \in [r_+, \infty]$, Region 2 in which $r \in [r_-, r_+]$, and Region 3 in which $r \in [0, r_-]$. In each of the three regions, the time coordinate $t \in (-\infty, \infty)$, but as one crosses the horizons, the temporal and radial motion exchange roles in the spacetime metric. In [2] it was argued that the decoupling of Region 2 from the asymptopia is in analogy to what happens when we take a near horizon limit of the extremal solution. The AdS_2 factor actor that gets identified in this limit explains how moduli are fixed by the attractor mechanism. It is telling that the same quantities that vanish by virtue of the attractor equations in the extremal case also vanish when averaged over Region 2 of non-extremal solutions. The area of the event horizon is determined by r_+ and fixes the entropy of the black hole. The area of the Cauchy horizon is likewise determined by r_-. Using Bekenstein-Hawking, the area law gives,

$$S_- = \frac{S_{ext}^2}{S_+} \to S_{ext}^2 = S_- S_+, \tag{10.1}$$

Since the two are inversely proportional, as we become more and more non-extremal, S_+ grows and S correspondingly shrinks. Moreover, as the solution becomes increasingly non-extremal, the reduction in S may suggest that there is a universality to states enumerated by the inner horizon. If the entropy of the Cauchy horizon isolates a particular subset of the states captured by S_+, it would be interesting to learn which ones and what selection criteria to apply when restricting to these states. If Region 2 and Region 3 in the interior of the black hole are decoupled, the additivity of entropy suggests that the degrees of freedom in Region 2 are captured by the difference in the entropies of the horizons: $S_2 = S_+ - S_- = \frac{S_+^2 - S_{ext}^2}{S_+}$. Note that any association of thermodynamic quantities with the Cauchy horizon S_+ in supergravity is na ıvely suspicious as states in the inter-horizon region cannot see the asymptotic vacuum. We take our cue from the holographic picture of the black hole as follows. Since scalar excitations in this region satisfy the Klein-Gordon equation in an AdS_2 spacetime, which can be lifted up to a BTZ black hole, in the holographically dual CFT, we associate the right movers to the extremal solution and assume without loss of generality that $0 < T_L \leq T_R$. In this case, from

$$S_{\pm} = \frac{\pi r_{\pm}}{2G_3} = \frac{\pi^2 L}{3}(c_R T_R \pm c_L T_L), \tag{10.2}$$

$$T_{\pm}^{-1} = \pm\frac{2\pi r_{\pm} L^2}{r_+^2 - r_-^2} = \frac{1}{2}\left(\frac{1}{T_R \pm \frac{1}{T_L}}\right), \tag{10.3}$$

we have

$$S_2 = \frac{\pi\Delta}{G_3} = \frac{2}{3}\pi^2 L c_L T_L = -\frac{2}{3}\pi^2 L c_L\left(\frac{T_+ T_-}{T_+ - T_-}\right), \tag{10.4}$$

where we use the Brown-Henneaux formula to write the central charge $c_L = c_R = \frac{3L}{2G_3}$, where G_3 is the gravitational constant in AdS scale. We may then associate the modes of one set of oscillators — *viz.*, the ones at a lower temperature — to the inter-horizon region. This is consistent with the idea that exciting the second set of oscillators above the ground state is what takes us away from extremality. While (10.4) is suggestive, we note that macroscopically, Region 2 and Region 3 are not decoupled. By this we mean that, we cannot find near horizon limits of each horizon which are independently solutions of the equations of motion. Moreover, in Region 2, an observer is forced to move radially — i.e., toward the inner horizon — so it cannot be said that Region 2 and Region 3 are independent. Because of entanglement, Hilbert spaces do not factorize across horizons. While we cannot explain these tensions, perhaps they are artifacts of coarse graining microscopic solutions. In any case, because the Cauchy horizon suffers a classical instability, how seriously one should take the notion of the inner horizon's entropy is unclear. Even more mysteriously, one can apply the Bekenstein-Hawking formula to compute the entropy of horizons at the roots of $a(r)$ on the complex r-plane and write a first law for these complex horizons. The interpretation of the complex areas in semiclassical or quantum gravity of course remains elusive.

In addition to (9.170), one can use Vieta's formula to extract other interesting horizon relationships from (**??**) for AdS Reissner-Nordström black holes. For instance, one finds,

$$
\begin{aligned}
\sum_i r_i &= 0, \\
\sum_{i<j} r_i r_j &= \frac{(d-1)(d-2)}{2(-V_{g,min})}, \\
\sum_{i<j<k} r_i r_j r_k &= 0, \cdots \\
\sum_{1\leq i_1 \leq i_2 < \ldots < i_{d-1} \leq N} r_{i_1 \cdots r_{i_{d-1}}} &= (-1)^d \frac{8\pi G_d M(d-1)}{\omega_{d-2}(-V_{g,min})}, \cdots.
\end{aligned}
\tag{10.5}
$$

Appendix A

Derivation of the equations of motion

A.1 Variation of the Einstein-Hilbert term

In this section we show a detailed calculation of varying the action with respect to the inverse metric $g^{\mu\nu}$, we start with the Einstein-Hilbert action.

$$\begin{aligned}
\delta S1 &= \delta \int d^4x\sqrt{-g}R \\
&= \int d^4x\delta(\sqrt{-g}R) \\
&= \int d^4x(R\delta\sqrt{-g} + \sqrt{-g}\delta R) \\
&= \int d^4x(R(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}) + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}) \\
&= \int d^4x\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}R\delta g^{\mu\nu} + (R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})) \\
&= \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \qquad \text{(A.1)}
\end{aligned}$$

The Riemann tensor is given by,

$$R^{\rho}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}, \qquad \text{(A.2)}$$

$\Gamma^{\rho}_{\mu\nu}$ are the Christoffel symbols, and if we replace

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu}, \qquad \text{(A.3)}$$

where $\delta\Gamma^{\rho}_{\nu\mu}$ is the difference between two tensors, thus it is also a tensor, thus if we take it's covariant derivative we get,

$$\nabla_{\lambda}(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_{\lambda}(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}. \tag{A.4}$$

Now we vary the Riemann tensor

$$\begin{aligned}
\delta R^{\rho}_{\mu\lambda\nu} &= \partial_{\lambda}\delta\Gamma^{\rho}_{\nu\mu} + \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \partial_{\nu}\delta\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} \\
&- \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} \\
&= (\partial_{\lambda}\delta\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu}) \\
&- (\partial_{\nu}\delta\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu}) \\
&= \nabla_{\lambda}(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\lambda\mu}).
\end{aligned} \tag{A.5}$$

Now if we put $\rho = \lambda$, then we get the variation of the Ricci tensor as,

$$\delta R^{\lambda}_{\mu\lambda\nu} = \delta R_{\mu\nu} = \nabla_{\lambda}(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\lambda\mu}), \tag{A.6}$$

therefore,

$$\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\nabla_{\lambda}(\Gamma^{\lambda}_{\nu\mu}) - g^{\mu\nu}\nabla_{\nu}(\Gamma^{\lambda}_{\lambda\mu}) \\
&= \nabla_{\lambda}(g^{\mu\nu}\Gamma^{\lambda}_{\nu\mu}) - \nabla_{\nu}(g^{\mu\nu}\Gamma^{\lambda}_{\lambda\mu}) \\
&= \nabla_{\alpha}(g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\lambda\mu}).
\end{aligned} \tag{A.7}$$

Finally the variation of the Einstein-Hilbert term is given by,

$$\delta S1 = \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \int d^4x\sqrt{-g}\nabla_{\alpha}(g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\mu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\lambda\mu}). \tag{A.8}$$

A.2 The Weyl Conformal gravity term

The Weyl tensor in 4 dimensions is given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\gamma}R_{\delta\alpha} - g_{\alpha\gamma}R_{\delta\beta} - g_{\beta\delta}R_{\gamma\alpha}) + \frac{1}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R.$$

The Weyl squared term is then given by

$$\begin{aligned}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = {}&(R^{\alpha\beta\gamma\delta} + \frac{1}{2}(g^{\alpha\delta}R^{\gamma\beta} + g^{\beta\gamma}R^{\delta\alpha} - g^{\alpha\gamma}R^{\delta\beta} - g^{\beta\delta}R^{\gamma\alpha})\\ &+ \frac{1}{6}(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta})R)\times\\ &(R_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\gamma}R_{\delta\alpha} - g_{\alpha\gamma}R_{\delta\beta} - g_{\beta\delta}R_{\gamma\alpha})\\ &+ \frac{1}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R).\end{aligned} \tag{A.9}$$

A.2.1 The first term of Weyl tensor squared

$$\begin{aligned}T1 = {}&R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\\ &+ \frac{1}{2}(R^{\alpha\beta\gamma\delta}g_{\alpha\gamma}R_{\gamma\beta} + R^{\alpha\beta\gamma\delta}g_{\beta\gamma}R_{\delta\alpha} - R^{\alpha\beta\gamma\delta}g_{\alpha\gamma}R_{\delta\beta} - R^{\alpha\beta\gamma\delta}g_{\beta\delta}R_{\gamma\alpha})\\ &+ \frac{1}{6}(R^{\alpha\beta\gamma\delta}g_{\alpha\gamma}g_{\delta\beta} - R^{\alpha\beta\gamma\delta}g_{\alpha\delta}g_{\gamma\beta})R\\ = {}&R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\\ &+ \frac{1}{2}(-R^{\alpha\beta\delta\gamma}g_{\alpha\delta}R_{\gamma\beta} - R^{\beta\alpha\gamma\delta}g_{\beta\gamma}R_{\delta\alpha} - R^{\alpha\beta\gamma\delta}g_{\alpha\gamma}R_{\delta\beta} + R^{\beta\alpha\gamma\delta}g_{\beta\delta}R_{\gamma\alpha})\\ &+ \frac{1}{6}(g_{\delta\beta}R^{\alpha\beta\gamma\delta}R + g_{\alpha\delta}g_{\gamma\delta}R^{\alpha\beta\delta\gamma}R)\\ = {}&R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + \frac{1}{2}(-R^{\beta\gamma}R_{\beta\gamma} - R^{\alpha\delta}R_{\alpha\delta} - R^{\beta\delta}R_{\beta\delta} - R^{\alpha\gamma}R_{\alpha\gamma})\\ &+ \frac{1}{6}(g_{\delta\beta}R^{\beta\delta}R + g_{\gamma\beta}R^{\beta\gamma}R)\\ = {}&R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2.\end{aligned} \tag{A.10}$$

A.2.2 The second term of Weyl tensor squared

$$
\begin{aligned}
T2 = &\frac{1}{2}(g^{\alpha\delta}R^{\gamma\beta} + g^{\beta\gamma}R^{\delta\alpha} - g^{\alpha\gamma}R^{\delta\beta} - g^{\beta\delta}R^{\gamma\alpha})R_{\alpha\beta\gamma\delta} + \\
&\frac{1}{4}(g^{\alpha\delta}R^{\gamma\beta} + g^{\beta\gamma}R^{\delta\alpha} - g^{\alpha\gamma}R^{\delta\beta} - g^{\beta\delta}R^{\gamma\alpha}) \\
&\times (g_{\alpha\gamma}R_{\gamma\beta} + g_{\beta\gamma}R_{\delta\alpha} - g_{\alpha\gamma}R_{\delta\beta} - g_{\beta\delta}R_{\gamma\alpha}) \\
&+ \frac{1}{12}(g^{\alpha\delta}R^{\gamma\beta} + g^{\beta\gamma}R^{\delta\alpha} - g^{\alpha\gamma}R^{\delta\beta} - g^{\beta\delta}R^{\gamma\alpha})(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta})R) \\
&= \frac{1}{2}(-g^{\alpha\delta}R_{\alpha\beta\delta\gamma}R^{\gamma\beta} + g^{\beta\gamma}R_{\beta\alpha\gamma\delta}R^{\delta\alpha} - g^{\alpha\gamma}R_{\alpha\beta\gamma\delta}R^{\delta\beta} + g^{\beta\delta}R_{\beta\alpha\delta\gamma}R^{\gamma\alpha}) \\
&+ \frac{1}{4}(4R^{\gamma\beta}R_{\gamma\beta} + \delta^{\alpha\delta}_{\beta\gamma}R^{\gamma\beta}R_{\delta\alpha} - \delta^{\delta}_{\gamma}R^{\gamma\beta}R_{\delta\beta} - \delta^{\alpha}_{\beta}R^{\gamma\beta}R_{\gamma\alpha} \\
&+ \delta^{\beta\gamma}_{\alpha\delta}R^{\delta\alpha}R_{\gamma\beta} + 4R^{\delta\alpha}R_{\delta\alpha} - \delta^{\beta}_{\alpha}R^{\delta\alpha}R_{\delta\beta} - \delta^{\gamma}_{\delta}R^{\delta\alpha}R_{\gamma\alpha} \\
&+ \delta^{\gamma}_{\alpha}R^{\delta\beta}R_{\gamma\beta} - \delta^{\alpha}_{\beta}R^{\delta\beta}R_{\delta\alpha} + 4R^{\delta\beta}R_{\delta\beta} + \delta^{\alpha\gamma}_{\beta\delta}R^{\delta\beta}R_{\gamma\alpha} \\
&- \delta^{\beta}_{\alpha}R^{\gamma\alpha}R_{\gamma\beta} - \delta^{\delta}_{\gamma}R^{\gamma\alpha}R_{\delta\alpha} + \delta^{\beta\delta}_{\alpha\gamma}R^{\gamma\alpha}R_{\delta\beta} + 4R^{\gamma\alpha}R_{\gamma\alpha}) \\
&+ \frac{1}{12}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R(g^{\alpha\delta}R^{\gamma\beta} + g^{\beta\gamma}R^{\delta\alpha} - g^{\alpha\gamma}R^{\delta\beta} - g^{\beta\delta}R^{\gamma\alpha}) \\
&= \frac{1}{2}(-R_{\beta\gamma}R^{\gamma\beta} + R_{\alpha\delta}R^{\delta\alpha} - R_{\beta\delta}R^{\delta\beta} + R_{\alpha\gamma}R^{\gamma\alpha}) \\
&+ \frac{1}{4}(4R^{\gamma\beta}R_{\gamma\beta} + 4R^{\gamma\beta}R_{\gamma\beta} - 4R^{\gamma\beta}R_{\gamma\beta} - 4R^{\gamma\beta}R_{\gamma\beta} \\
&+ 4R^{\delta\alpha}R_{\delta\alpha} + 4R^{\delta\alpha}R_{\delta\alpha} - 4R^{\delta\alpha}R_{\delta\alpha} - 4R^{\delta\alpha}R_{\delta\alpha} \\
&- 4R^{\delta\beta}R_{\delta\beta} - 4R^{\delta\beta}R_{\delta\beta} + 4R^{\delta\beta}R_{\delta\beta} + 4R^{\delta\beta}R_{\delta\beta} \\
&- 4R^{\gamma\alpha}R_{\gamma\alpha} - 4R^{\gamma\alpha}R_{\gamma\alpha} + 4R^{\gamma\alpha}R_{\gamma\alpha} + 4R^{\gamma\alpha}R_{\gamma\alpha}) \\
&+ \frac{1}{12}(\delta^{\delta}_{\gamma}g_{\delta\beta}RR^{\gamma\beta} + \delta^{\beta}_{\alpha}g_{\delta\beta}RR^{\delta\alpha} - 4g_{\delta\beta}RR^{\delta\beta} - 4g_{\alpha\gamma}RR^{\gamma\alpha} \\
&- 4R^2 - 4R^2 + \delta^{\gamma}_{\gamma}g_{\gamma\beta}RR^{\gamma\beta} + \delta^{\beta}_{\alpha}g_{\gamma\beta}RR^{\gamma\alpha}) \\
&= 0.
\end{aligned} \tag{A.11}
$$

A.2.3 The third term of Weyl tensor squared

$$
\begin{aligned}
T3 &= \frac{1}{6}(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta})RR_{\alpha\beta\gamma\delta} \\
&+ \frac{1}{12}(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta})R(g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\gamma}R_{\delta\alpha} - g_{\alpha\gamma}R_{\delta\beta} - g_{\beta\delta}R_{\gamma\alpha}) \\
&+ \frac{1}{36}R^2(g^{\alpha\gamma}g^{\delta\beta} - g^{\alpha\delta}g^{\gamma\beta})(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}) \\
&= \frac{1}{6}(R^2 - R^2) \\
&+ \frac{1}{12}R(\delta^{\gamma}_{\delta}g^{\delta\beta}R_{\gamma\beta} - \delta^{\alpha}_{\beta}g^{\delta\beta}R_{\delta\alpha} - 4R - 4R - 4R - 4R + \delta^{\delta}_{\gamma}g^{\gamma\beta}R_{\delta\beta} + \delta^{\alpha}_{\beta}g^{\gamma\beta}R_{\gamma\alpha}) \\
&+ \frac{1}{36}R^2(16 - \delta^{\gamma}_{\delta}\delta^{\delta}_{\gamma} - \delta^{\gamma}_{\delta}\delta^{\delta}_{\gamma} + 16) \\
&= \frac{1}{12}(4R^2 + 4R^2 - 4R^2 - 4R^2 - 4R^2 + 4R^2 - 4R^2 + 4R^2) \\
&= 0. \qquad (A.12)
\end{aligned}
$$

Thus the results are

$$C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = T1 + T2 + T3, \qquad (A.13)$$

$$
\begin{aligned}
C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} &= R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2, \\
C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2, \qquad (A.14)
\end{aligned}
$$

we have changed the dummy indices for consistency.

A.2.4 Variation of the Weyl action

$$\delta S2 = -\alpha\delta\int d^4x\sqrt{-g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \qquad (A.15)$$

from the previous calculation we know that we can simplify this expression to be

$$S2 = -\alpha\int d^4x\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2). \qquad (A.16)$$

We can also simplify this action by using the Gauss-Bonnet theorem which states that the action is topological invariant[47],

$$I = \int d^4\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2), \qquad (A.17)$$

therefore, if we subtract this term called the Gauss-Bonnet term[47] from our action we are still going to get the same results, and this method provides the simplification, thus

$$\begin{aligned}
S2 &= \int d^4x\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2) \\
&- \int d^4\sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \\
&= 2\int d^4\sqrt{-g}(R^{\mu\nu}R_{\mu\nu} - \frac{1}{3}R^2),
\end{aligned} \tag{A.18}$$

therefore, the variation with respect to the metric becomes

$$\begin{aligned}
\delta S2 &= 2\int d^4x\delta(\sqrt{-g}(R^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R^2)) \\
&= 2\int d^4x(\frac{1}{2}g^{\mu\nu}\sqrt{-g}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2)\delta g_{\mu\nu} \\
&+ \sqrt{-g}(R_{\mu\nu}\delta R^{\mu\nu} + R^{\mu\nu}\delta R_{\mu\nu} - \frac{2}{3}R\delta R)).
\end{aligned} \tag{A.19}$$

Now let's start by considering this term

$$\begin{aligned}
R_{\mu\nu}\delta R^{\mu\nu} + R^{\mu\nu}\delta R_{\mu\nu} &= R_{\mu\nu}\delta(g^{\rho\mu}g^{\sigma\nu}R_{\rho\sigma}) + R^{\mu\nu}\delta R_{\mu\nu} \\
&= R_{\mu\nu}(g^{\rho\mu}g^{\sigma\nu}\delta R_{\rho\sigma} + R_{\rho\sigma}g^{\sigma\nu}\delta g^{\rho\mu} + g^{\rho\mu}R_{\rho\sigma}\delta g^{\sigma\nu}) \\
&+ R^{\mu\nu}\delta R_{\mu\nu} \\
&= R^{\rho\sigma}\delta R_{\rho\sigma} + R_{\mu\nu}R^{\nu}_{\rho}\delta g^{\rho\mu} + R_{\mu\nu}R^{\mu}_{\sigma}\delta g^{\sigma\nu} + R^{\mu\nu}\delta R_{\mu\nu} \\
&= 2R^{\mu\nu}\delta R_{\mu\nu} + 2R_{\mu\nu}R^{\mu}_{\sigma}\delta g^{\sigma\nu}.
\end{aligned} \tag{A.20}$$

Then the variation of the Ricci tensor then becomes[47]

$$\begin{aligned}
\delta R_{\mu\nu} &= \nabla_\lambda(\frac{1}{2}g^{\lambda\rho}(\nabla_\nu\delta g_{\mu\rho} + \nabla_\mu\delta g_{\nu\rho} - \nabla_\rho\delta g_{\mu\nu})) \\
&- \nabla_\nu(\frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\lambda\rho} + \nabla_\lambda\delta g_{\mu\rho} - \nabla_\rho\delta g_{\lambda\mu})) \\
&= \frac{1}{2}(\nabla^\rho\nabla_\nu\delta g_{\mu\rho} + \nabla^\rho\nabla_\mu\delta g_{\nu\rho} - \nabla^\rho\nabla_\rho\delta g_{\mu\nu}) \\
&- \frac{1}{2}(\nabla_\nu\nabla_\mu g^{\lambda\rho}\delta_{\lambda\rho} + \nabla_\nu\nabla^\rho\delta g_{\mu\rho} - \nabla_\nu\nabla^\lambda\delta g_{\lambda\mu}).
\end{aligned} \tag{A.21}$$

Substituting (A.21) to (A.20), we get,

$$\begin{aligned}
2R^{\mu\nu}\delta R_{\mu\nu} + 2R_{\mu\nu}R^{\mu}_{\sigma}\delta g^{\sigma\nu} &= (R^{\mu\nu}\nabla^{\rho}\nabla_{\nu}\delta g_{\mu\rho} + R^{\mu\nu}\nabla^{\rho}\nabla_{\mu}\delta g_{\nu\rho} - R^{\mu\nu}\nabla^{\rho}\nabla_{\rho}\delta g_{\mu\nu}) \\
&- (R^{\mu\nu}\nabla_{\nu}\nabla_{\mu}g^{\lambda\rho}\delta_{\lambda\rho} + R^{\mu\nu}\nabla_{\nu}\nabla^{\rho}\delta g_{\mu\rho} - R^{\mu\nu}\nabla_{\nu}\nabla^{\lambda}\delta g_{\lambda\mu}) \\
&- 2R_{\mu\nu}R^{\mu}_{\sigma}g^{\alpha\sigma}g^{\beta\nu}\delta g_{\alpha\beta} \\
&= (\nabla^{\rho}\nabla_{\nu}R^{\mu\nu}\delta g_{\mu\rho} + \nabla^{\rho}\nabla_{\mu}R^{\mu\nu}\delta g_{\nu\rho} - \nabla^{\rho}\nabla_{\rho}R^{\mu\nu}\delta g_{\mu\nu}) \\
&- (\nabla_{\nu}\nabla_{\mu}R^{\mu\nu}g^{\lambda\rho}\delta_{\lambda\rho} + \nabla_{\nu}\nabla^{\rho}R^{\mu\nu}\delta g_{\mu\rho} - \nabla_{\nu}\nabla^{\lambda}R^{\mu\nu}\delta g_{\lambda\mu}) \\
&- 2R^{\beta}_{\mu}R^{\mu\alpha}\delta g_{\alpha\beta} \\
&= (\nabla^{\nu}\nabla_{\rho}R^{\mu\rho}\delta g_{\mu\nu} + \nabla^{\mu}\nabla_{\rho}R^{\rho\nu}\delta g_{\mu\nu} - \nabla^{2}R^{\mu\nu}\delta g_{\mu\nu}) \\
&- (\nabla_{\rho}\nabla_{\lambda}R^{\lambda\rho}g^{\mu\nu}\delta_{\mu\nu} + \nabla_{\rho}\nabla^{\nu}R^{\mu\rho}\delta g_{\mu\nu} - \nabla_{\lambda}\nabla^{\nu}R^{\mu\lambda}\delta g_{\mu\nu}) \\
&- 2R^{\nu}_{\lambda}R^{\lambda\mu}\delta g_{\mu\nu} \\
&= (\nabla^{\mu}\nabla_{\rho}R^{\rho\nu} - \nabla^{2}R^{\mu\nu} - \nabla_{\rho}\nabla_{\lambda}R^{\lambda\rho}g^{\mu\nu} + \nabla_{\lambda}\nabla^{\nu}R^{\mu\lambda})\delta g_{\mu\nu} \\
&- 2R^{\nu}_{\lambda}R^{\lambda\mu}\delta g_{\mu\nu}. \qquad \text{(A.22)}
\end{aligned}$$

Now we consider the variation of the Ricci scalar explicitly

$$\begin{aligned}
\delta R &= \delta(g^{\mu\nu}R_{\mu\nu}) \\
&= g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} \\
&= \frac{1}{2}g^{\mu\nu}(\nabla^{\rho}\nabla_{\mu}\delta g_{\nu\rho} - \nabla^{2}\delta g_{\mu\nu} - \nabla_{\nu}\nabla_{\mu}g^{\lambda\rho}\delta g_{\lambda\rho} + \nabla_{\nu}\nabla^{\lambda}\delta g_{\lambda\mu}) + R_{\mu\nu}\delta g^{\mu\nu} \\
&= \frac{1}{2}(\nabla^{\rho}\nabla^{\nu}\delta g_{\nu\rho} - \nabla^{2}g^{\mu\nu}\delta g_{\mu\nu} - \nabla^{2}g^{\lambda\rho}\delta g_{\lambda\rho} + \nabla^{\mu}\nabla_{\lambda}\delta g_{\lambda\mu}) - R_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}\delta g_{\alpha\beta} \\
&= \frac{1}{2}(\nabla^{\nu}\nabla^{\mu}\delta g_{\mu\nu} - 2\nabla^{2}g^{\mu\nu}\delta g_{\mu\nu} + \nabla^{\nu}\nabla^{\mu}\delta g_{\mu\nu}) - R^{\mu\nu}\delta g_{\mu\nu} \\
&= \nabla^{\nu}\nabla^{\mu}\delta g_{\mu\nu} - \nabla^{2}g^{\mu\nu}\delta g_{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu}, \qquad \text{(A.23)}
\end{aligned}$$

therefore, the last term of (A.19) becomes,

$$-\frac{2}{3}R\delta R = -\frac{2}{3}(\nabla^{\nu}\nabla^{\mu}R - \nabla^{2}Rg^{\mu\nu} - RR^{\mu\nu})\delta g_{\mu\nu}. \qquad \text{(A.24)}$$

The variation of the Weyl term (A.15) becomes,

$$\begin{aligned}
\delta S2 &= \int d^4x\delta(\sqrt{-g}(R^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R^2)) \\
&= \int d^4x(\frac{1}{2}g^{\mu\nu}\sqrt{-g}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2)\delta g_{\mu\nu} \\
&+ \sqrt{-g}(R_{\mu\nu}\delta R^{\mu\nu} + R^{\mu\nu}\delta R_{\mu\nu} - \frac{2}{3}R\delta R)) \\
&= \int d^4x(\frac{1}{2}g^{\mu\nu}\sqrt{-g}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2)\delta g_{\mu\nu} \\
&+ \sqrt{-g}(\nabla^\mu\nabla_\rho R^{\rho\nu} - \nabla^2 R^{\mu\nu} - \nabla_\rho\nabla_\lambda R^{\lambda\rho}g^{\mu\nu} + \nabla_\lambda\nabla^\nu R^{\mu\lambda})\delta g_{\mu\nu} \\
&- 2R^\nu_\lambda R^{\lambda\mu}\sqrt{-g}\delta g_{\mu\nu} - \frac{2}{3}\sqrt{-g}(\nabla^\nu\nabla^\mu R - \nabla^2 R g^{\mu\nu} - RR^{\mu\nu})\delta g_{\mu\nu} \\
&= \int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) \\
&+ \nabla^\nu\frac{1}{2}\nabla^\mu R + \nabla^\nu\frac{1}{2}\nabla^\mu R - \nabla^2 R^{\mu\nu} - \nabla^2\frac{1}{2}Rg^{\mu\nu} \\
&- 2R^\nu_\rho R^{\mu\rho} + \frac{2}{3}RR^{\mu\nu} - \frac{2}{3}\nabla^\nu\nabla^\mu R + \frac{2}{3}\nabla^2 g^{\mu\nu}R)\delta g_{\mu\nu} \\
&= \int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{1}{3}\nabla^\nu\nabla^\mu R - \nabla^2 R^{\mu\nu} \\
&+ \frac{1}{6}\nabla^2 g^{\mu\nu}R - 2R^\nu_\rho R^{\mu\rho} + \frac{2}{3}RR^{\mu\nu})\delta g_{\mu\nu},
\end{aligned} \tag{A.25}$$

for simplification of the terms we used

$$\nabla_\lambda R^{\lambda\mu} = \frac{1}{2}\nabla^\mu R. \tag{A.26}$$

If we change the variation of (A.25) from the variation with respect to the metric to with respect to the inverse metric defined by,

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}, \tag{A.27}$$

we get,

$$\begin{aligned}\delta S2 = &- \int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{1}{3}\nabla^\nu\nabla^\mu R - \nabla^2 R^{\mu\nu} \\ &+ \frac{1}{6}\nabla^2 g^{\mu\nu}R - 2R^\nu_\rho R^{\mu\rho} + \frac{2}{3}RR^{\mu\nu})g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta} \\ &= - \int d^4x\sqrt{-g}(\frac{1}{2}\delta^\nu_\alpha g_{\beta\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{1}{3}\nabla_\beta\nabla_\alpha R - \nabla^2 R_{\alpha\beta} \\ &+ \frac{1}{6}\nabla^2\delta^\nu_\alpha g_{\beta\nu}R - 2R_{\rho\beta}R^\rho_\alpha + \frac{2}{3}RR_{\alpha\beta})\delta g^{\alpha\beta} \\ &= - \int d^4x\sqrt{-g}(\frac{1}{2}g_{\alpha\beta}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{1}{3}\nabla_\beta\nabla_\alpha R - \nabla^2 R_{\alpha\beta} \\ &+ \frac{1}{6}\nabla^2 g_{\alpha\beta}R - 2R_{\rho\beta}R^\rho_\alpha + \frac{2}{3}RR_{\alpha\beta})\delta g^{\alpha\beta} \\ &= - \int d^4x\sqrt{-g}(\frac{1}{2}g_{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{1}{3}\nabla_\nu\nabla_\mu R - \nabla^2 R_{\mu\nu} \\ &+ \frac{1}{6}\nabla^2 g_{\mu\nu}R - 2R_{\rho\nu}R^\rho_\mu + \frac{2}{3}RR_{\mu\nu})\delta g^{\mu\nu}.\end{aligned} \tag{A.28}$$

Proof that (A.25) is equivalent to the Bach tensor.

The Bach tensor is defined as (see [1])

$$B_{\alpha\gamma} = (\nabla^\beta\nabla^\delta + \frac{1}{2}R^{\beta\delta})C_{\alpha\beta\gamma\delta}. \tag{A.29}$$

In this proof we use the contracted Bianchi identity given by,

$$\nabla^\delta R_{\alpha\beta\gamma\delta} = -\nabla_\alpha R_{\beta\gamma} + \nabla_\beta R_{\alpha\gamma}, \tag{A.30}$$

and the contracted Hessian of the Ricci tensor given by,

$$\nabla^\delta\nabla_\alpha R_{\delta\gamma} = \nabla_\alpha\nabla^\delta R_{\delta\gamma} + R_{\alpha\delta}R^\delta_\gamma - R_{\alpha\beta\gamma\delta}R^{\beta\delta}, \tag{A.31}$$

$$\nabla^\delta R_{\beta\delta} = \frac{1}{2}\nabla_\beta R. \tag{A.32}$$

The first term of (A.29) becomes,

$$\begin{aligned}
\nabla^\beta\nabla^\delta C_{\alpha\beta\gamma\delta} &= \nabla^\beta\nabla^\delta(R_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\gamma}R_{\delta\alpha} - g_{\alpha\gamma}R_{\delta\beta} - g_{\beta\delta}R_{\gamma\alpha}) \\
&+ \frac{1}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R) \\
&= \nabla^\beta\nabla^\delta R_{\alpha\beta\gamma\delta} + \frac{1}{2}(\nabla^\beta\nabla^\delta g_{\alpha\delta}R_{\gamma\beta} + \nabla^\beta\nabla^\delta g_{\beta\gamma}R_{\delta\alpha} \\
&- \nabla^\beta\nabla^\delta g_{\alpha\gamma}R_{\delta\beta} - \nabla^\beta\nabla^\delta g_{\beta\delta}R_{\gamma\alpha}) \\
&+ \frac{1}{6}(\nabla^\beta\nabla^\delta g_{\alpha\gamma}g_{\delta\beta} - \nabla^\beta\nabla^\delta g_{\alpha\delta}g_{\gamma\beta})R \\
&= \nabla^\beta(-\nabla_\alpha R_{\beta\gamma} + \nabla_\beta R_{\alpha\gamma}) \\
&+ \frac{1}{2}(\nabla^\beta\nabla_\alpha R_{\gamma\beta} + \nabla_\gamma\nabla^\delta R_{\delta\alpha} - g_{\alpha\gamma}\nabla^\beta\nabla^\delta R_{\delta\beta} - \nabla^2 R_{\gamma\alpha}) \\
&+ \frac{1}{2}(g_{\alpha\gamma}\nabla^2 R - \nabla_{\gamma\alpha}R) \\
&= \nabla^2 R_{\alpha\gamma} - \nabla^\beta\nabla_\alpha R_{\beta\gamma} + \frac{1}{2}\nabla^\beta\nabla_\alpha R_{\gamma\beta} + \frac{1}{2}\nabla_\gamma\nabla^\delta R_{\delta\alpha} - \frac{1}{2}g_{\alpha\gamma}\nabla^\beta\nabla^\delta R_{\delta\beta} \\
&- \frac{1}{2}\nabla^2 R_{\gamma\alpha} + \frac{1}{6}g_{\alpha\gamma}\nabla^2 R - \frac{1}{6}\nabla_\gamma\nabla_\alpha R \\
&= \frac{1}{2}\nabla^2 R_{\alpha\gamma} - \frac{1}{2}\nabla^\beta\nabla_\alpha R_{\beta\gamma} + \frac{1}{2}\nabla_\gamma\nabla^\delta R_{\delta\alpha} \\
&- \frac{1}{2}g_{\alpha\gamma}\nabla^\beta\nabla^\delta R_{\delta\beta} + \frac{1}{6}g_{\alpha\gamma}\nabla^2 R - \frac{1}{2}\nabla_\gamma\nabla_\alpha R,
\end{aligned} \tag{A.33}$$

and the second term of (A.29) becomes,

$$\begin{aligned}
\frac{1}{2}R^{\beta\delta}C_{\alpha\beta\gamma\delta} &= \frac{1}{2}R^{\beta\delta}R_{\alpha\beta\gamma\delta} \\
&+ \frac{1}{4}(R^{\beta\delta}g_{\alpha\delta}R_{\gamma\beta} + R^{\beta\delta}g_{\beta\gamma}R_{\delta\alpha} - R^{\beta\delta}g_{\alpha\gamma}R_{\delta\beta} - R^{\beta\delta}g_{\beta\delta}R_{\gamma\alpha}) \\
&+ \frac{1}{12}(R^{\beta\delta}g_{\alpha\gamma}g_{\delta\beta}R - R^{\beta\delta}g_{\alpha\delta}g_{\gamma\beta}R) \\
&= \frac{1}{2}(-\nabla^\delta\nabla_\alpha R_{\delta\gamma} + \nabla_\alpha\nabla^\delta R_{\delta\gamma} + R_{\alpha\delta}R^\delta_\gamma) + \frac{1}{4}R_{\gamma\beta}R^\beta_\alpha \\
&+ \frac{1}{4}R_{\delta\alpha}R^\delta_\gamma - \frac{1}{4}g_{\alpha\gamma}R^{\beta\delta}R_{\beta\delta} - \frac{1}{4}R_{\gamma\alpha}R + \frac{1}{12}g_{\alpha\gamma}R^2 - \frac{1}{12}R_{\gamma\alpha}R.
\end{aligned} \tag{A.34}$$

Therefore, adding both terms (A.33) and (A.34) we get the following results

$$
\begin{aligned}
(\nabla^{\beta}\nabla^{\delta}+\frac{1}{2}R^{\beta\delta})C_{\alpha\beta\gamma\delta} &= \frac{1}{2}\nabla^2 R_{\alpha\gamma}-\frac{1}{2}\nabla^{\beta}\nabla_{\alpha}R_{\beta\gamma}+\frac{1}{2}\nabla_{\gamma}\nabla^{\delta}R_{\delta\alpha}-\frac{1}{2}g_{\alpha\gamma}\nabla^{\beta}\nabla^{\delta}R_{\delta\beta}\\
&+\frac{1}{6}g_{\alpha\gamma}\nabla^2 R-\frac{1}{2}\nabla_{\gamma}\nabla_{\alpha}R\\
&+\frac{1}{2}(-\nabla^{\delta}\nabla_{\alpha}R_{\delta\gamma}+\nabla_{\alpha}\nabla^{\delta}R_{\delta\gamma}+R_{\alpha\delta}R^{\delta}_{\gamma})+\frac{1}{4}R_{\gamma\beta}R^{\beta}_{\alpha}\\
&+\frac{1}{4}R_{\delta\alpha}R^{\delta}_{\gamma}-\frac{1}{4}g_{\alpha\gamma}R^{\beta\delta}R_{\beta\delta}-\frac{1}{4}R_{\gamma\alpha}R+\frac{1}{12}g_{\alpha\gamma}R^2-\frac{1}{12}R_{\gamma\alpha}R\\
&=-\frac{1}{4}\nabla_{\alpha}\nabla_{\gamma}R+\frac{1}{4}\nabla_{\gamma}\nabla_{\alpha}R+\frac{1}{2}\nabla^2 R_{\alpha\gamma}-\frac{1}{2}\nabla^{\beta}\nabla^{\delta}g_{\alpha\gamma}R_{\delta\beta}\\
&+\frac{1}{6}\nabla^2 g_{\alpha\gamma}R-\frac{1}{6}\nabla_{\gamma}\nabla_{\alpha}R\\
&+R_{\alpha\delta}R^{\delta}_{\gamma}-\frac{1}{4}g_{\alpha\gamma}R^{\beta\delta}R_{\beta\delta}-\frac{1}{3}RR_{\gamma\alpha}+\frac{1}{12}g_{\alpha\gamma}R^2\\
&=\frac{1}{2}\nabla^2 R_{\alpha\gamma}-\frac{1}{2}\nabla^{\beta}\nabla^{\delta}g_{\alpha\gamma}R_{\delta\beta}+\frac{1}{6}\nabla^2 g_{\alpha\gamma}R-\frac{1}{6}\nabla_{\gamma}\nabla_{\alpha}R\\
&+R_{\alpha\delta}R^{\delta}_{\gamma}-\frac{1}{4}g_{\alpha\gamma}R^{\beta\delta}R_{\beta\delta}-\frac{1}{3}RR_{\gamma\alpha}+\frac{1}{12}g_{\alpha\gamma}R^2, \qquad \text{(A.35)}
\end{aligned}
$$

multiplying the (A.35) by 2 we get the equations of motion for the Weyl conformal gravity term becomes,

$$
\begin{aligned}
2(\nabla^{\beta}\nabla^{\delta}+\frac{1}{2}R^{\beta\delta})C_{\alpha\beta\gamma\delta} &= -(-\nabla^2 R_{\alpha\gamma}+\nabla^{\beta}\nabla^{\delta}g_{\alpha\gamma}R_{\delta\beta}-\frac{1}{3}\nabla^2 g_{\alpha\gamma}R+\frac{1}{3}\nabla_{\gamma}\nabla_{\alpha}R\\
&-2R_{\alpha\delta}R^{\delta}_{\gamma}+\frac{1}{2}g_{\alpha\gamma}R^{\beta\delta}R_{\beta\delta}+\frac{2}{3}RR_{\gamma\alpha}-\frac{1}{6}g_{\alpha\gamma}R^2)\\
&=-(\frac{1}{2}g_{\alpha\gamma}(R^{\beta\delta}R_{\beta\delta}-\frac{1}{3}R^2)+\frac{1}{3}\nabla_{\gamma}\nabla_{\alpha}R-\nabla^2 R_{\alpha\gamma}\\
&+\frac{1}{6}\nabla^2 g_{\alpha\gamma}R-2R_{\alpha\delta}R^{\delta}_{\gamma}+\frac{2}{3}RR_{\alpha\gamma}). \qquad \text{(A.36)}
\end{aligned}
$$

Finally the Weyl term (A.15) is given by

$$
\begin{aligned}
\delta S2 &= \alpha\delta 2\int d^4x\sqrt{-g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, && \text{(A.37)}\\
&=4\alpha\int d^4x\sqrt{-g}(\nabla^{\rho}\nabla^{\sigma}+\frac{1}{2}R^{\rho\sigma})C_{\mu\rho\nu\sigma}\delta g^{\mu\nu}, && \text{(A.38)}\\
&=\int d^4x\sqrt{-g}4\alpha B_{\mu\nu}\delta g^{\mu\nu}, && \text{(A.39)}
\end{aligned}
$$

where the Bach tensor $B_{\mu\nu}$ is given by,

$$B_{\mu\nu} = \frac{1}{2}[-\frac{1}{2}g_{\mu\nu}(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) - \frac{1}{3}\nabla_\mu\nabla_\nu R + \Box R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}\Box R + 2R_{\mu\rho}R^\rho_\nu - \frac{2}{3}RR_{\mu\nu}]. \tag{A.40}$$

Taking the trace of the Bach tensor (A.40),

$$\begin{aligned} g^{\mu\nu}B_{\mu\nu} &= \frac{1}{2}\left(-\frac{1}{2}4(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) - \frac{1}{3}\Box R + \Box R - \frac{4}{6}\Box R + 2R^\nu_\rho R^\rho_\nu - \frac{2}{3}R^2\right) \\ &= \frac{1}{2}\left(-2(R^{\rho\sigma}R_{\rho\sigma} - \frac{1}{3}R^2) + \frac{2}{3}\Box R - \frac{2}{3}\Box R + 2R_{\nu\rho}R^{\nu\rho} - \frac{2}{3}R^2\right) \\ &= \frac{1}{2}\times 0 \\ &= 0, \end{aligned} \tag{A.41}$$

and taking the divergence of the Bach tensor (A.40),

$$\nabla^\mu B_{\mu\nu} = \frac{1}{2}\nabla_\nu B = 0, \tag{A.42}$$

$B = g^{\mu\nu}B_{\mu\nu} = 0$, as we just showed. Therefore, we showed that the Bach tensor is traceless and divergenceless.

A.2.5 Variation of the Ricci tensor squared

In this subsection we are going to use the results obtained from the previous subsections to obtain the variation of the Ricci squared tensor. varying the third term of the action (2.1), we get,

$$\begin{aligned} \delta S3 &= \int d^4x\delta(\sqrt{-g}\beta R^2) \\ &= \beta\int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}R^2\delta_{\mu\nu} + 2R\delta R) \\ &= \beta\int d^4x\sqrt{-g}(\frac{1}{2}g^{\mu\nu}R^2 + 2\nabla^\nu\nabla^\mu R - 2\nabla^2 Rg^{\mu\nu} - 2RR^{\mu\nu})\delta g_{\mu\nu} \\ &= \beta\int d^4x\sqrt{-g}(-2R(R^{\mu\nu} - \frac{1}{4}Rg^{\mu\nu}) - 2(g^{\mu\nu}\nabla^2 R - \nabla^\nu\nabla^\mu R))\delta g_{\mu\nu} \\ &= \beta\int d^4x\sqrt{-g}(2R(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}) + 2(g_{\mu\nu}\nabla^2 R - \nabla_\nu\nabla_\mu R))\delta g^{\mu\nu}. \end{aligned} \tag{A.43}$$

A.2.6 Variation of the Christoffel connection

The Christoffel connection is given by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{A.44}$$

Now we define a three indices tensor as,

$$\Gamma_{\mu\nu\alpha} = g_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}), \tag{A.45}$$

thus the variation of the Christoffele connection (A.44) is given by,

$$\begin{aligned}
\delta\Gamma^{\lambda}_{\mu\nu} &= \delta(g^{\lambda\alpha}\Gamma_{\mu\nu\alpha}) \\
&= \delta g^{\lambda\delta}\Gamma_{\mu\nu\alpha} + g^{\lambda\alpha}\delta\Gamma_{\mu\nu\alpha} \\
&= -g^{\rho\lambda}g^{\sigma\alpha}\delta g_{\rho\sigma}\Gamma_{\mu\nu\alpha} + g^{\lambda\rho}\delta\Gamma_{\mu\nu\rho} \\
&= -g^{\rho\lambda}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + g^{\lambda\rho}\delta\Gamma_{\mu\nu\rho} \\
&= -g^{\rho\lambda}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + g^{\lambda\rho}\frac{1}{2}(\partial_\nu\delta g_{\mu\rho} + \partial_\mu\delta g_{\nu\rho} - \partial_\rho\delta g_{\mu\nu}) \\
&= \frac{1}{2}g^{\lambda\rho}(\partial_\nu\delta g_{\mu\rho} + \partial_\mu\delta g_{\nu\rho} - \partial_\rho\delta g_{\mu\nu} - 2\Gamma^{\sigma}_{\mu\nu}\delta g_{\rho\sigma}),
\end{aligned} \tag{A.46}$$

(A.46) can be re-written as,

$$\begin{aligned}
\delta\Gamma^{\lambda}_{\mu\nu} =& \frac{1}{2}g^{\lambda\rho}[(\partial_\nu\delta g_{\mu\rho} - \Gamma^{\sigma}_{\mu\nu}\delta g_{\sigma\rho} - \Gamma^{\sigma}_{\nu\rho}\delta g_{\sigma\mu}) \\
&+ (\partial_\mu\delta g_{\nu\rho} - \Gamma^{\sigma}_{\nu\mu}\delta g_{\sigma\rho} - \Gamma^{\sigma}_{\rho\mu}\delta g_{\sigma\nu}) \\
&- (\partial_\rho\delta g_{\mu\nu} - \Gamma^{\sigma}_{\mu\rho}\delta g_{\sigma\nu} - \Gamma^{\sigma}_{\nu\rho}\delta g_{\sigma\mu})] \\
&= \frac{1}{2}g^{\lambda\rho}(\nabla_\nu\delta g_{\mu\rho} + \nabla_\mu\delta g_{\nu\rho} - \nabla_\rho\delta g_{\mu\nu}).
\end{aligned} \tag{A.47}$$

Appendix B

Linearized Bach tensor

The only contributing terms of linear components of the Bach tensor are,

$$B_{\mu\nu}^{linear} \to \nabla^{\rho}\nabla^{\sigma}C_{\mu\rho\nu\sigma}, \tag{B.1}$$

and recall that the Weyl tensor is given by

$$\begin{aligned} C_{\mu\rho\nu\sigma} &= R_{\mu\rho\nu\sigma} + \frac{1}{2}(-R_{\mu\nu}g_{\rho\sigma} + R_{\mu\sigma}g_{\rho\nu} + R_{\rho\nu}g_{\mu\sigma} - R_{\rho\sigma}g_{\mu\nu}) \\ &\quad + \frac{1}{6}R(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\rho\nu}), \end{aligned} \tag{B.2}$$

and the linear Riemann tensor is given by,

$$R_{\mu\rho\nu\sigma}^{linear} = -\frac{1}{2}(\partial_{\mu}\partial_{\nu}h_{\rho\sigma} + \partial_{\rho}\partial_{\sigma}h_{\mu\nu} - \partial_{\mu}\partial_{\sigma}h_{\rho\nu} - \partial_{\rho}\partial_{\nu}h_{\mu\sigma}), \tag{B.3}$$

therefore, using (B.2) and (B.3), we can re-write (B.1) as,

$$\begin{aligned} B_{\mu\nu}^{linear} \to \partial^{\rho}\partial^{\sigma}(&R_{\mu\rho\nu\sigma}^{linear} \\ &+ \frac{1}{2}(-R_{\mu\nu}^{linear}g_{\rho\sigma}^{linear} + R_{\mu\sigma}^{linear}g_{\rho\nu}^{linear} + R_{\rho\nu}^{linear}g_{\mu\sigma}^{linear} - R_{\rho\sigma}^{linear}g_{\mu\nu}^{linear}) \\ &+ \frac{1}{6}R^{linear}(g_{\mu\nu}^{linear}g_{\rho\sigma}^{linear} - g_{\mu\sigma}^{linear}g_{\rho\nu}^{linear})). \end{aligned} \tag{B.4}$$

We substitute the linear definition of the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R to (B.4) in chapter 3 anf [22], and simplify it by breaking it down term by term. From now on we drop the superscript "*linear*" in the tensors. Now the first term of (B.4) is given by,

$$\begin{aligned}\partial^\rho\partial^\sigma R_{\mu\rho\nu\sigma} &= -\frac{1}{2}(\partial^\rho\partial^\sigma\partial_\mu\partial_\nu h_{\rho\sigma} + \partial^\rho\partial^\sigma\partial_\rho\partial_\sigma h_{\mu\nu} - \partial^\rho\partial^\sigma\partial_\mu\partial_\sigma h_{\rho\nu} - \partial^\rho\partial^\sigma\partial_\rho\partial_\nu h_{\mu\sigma}) \\ &= -\frac{1}{2}(\partial_\mu\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma} + \Box\Box h_{\mu\nu} - \Box\partial^\rho\partial_\mu h_{\rho\nu} - \Box\partial^\sigma\partial_\nu h_{\mu\sigma}).\end{aligned} \tag{B.5}$$

The second term of (B.4) is given by,

$$\begin{aligned}&\frac{1}{2}(-\partial^\rho\partial^\sigma R_{\mu\nu}\eta_{\rho\sigma} + \partial^\rho\partial^\sigma R_{\mu\sigma}\eta_{\rho\nu} + \partial^\rho\partial^\sigma R_{\rho\nu}\eta_{\mu\sigma} - \partial^\rho\partial^\sigma R_{\rho\sigma}\eta_{\mu\nu}) \\ &= -\frac{1}{2}(\Box R_{\mu\nu} + \partial_\mu\partial^\rho R_{\mu\sigma} + \partial_\mu\partial^\rho R_{\rho\nu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma R_{\rho\sigma}),\end{aligned} \tag{B.6}$$

simplifying (B.6), we get,

$$\begin{aligned}-\Box R_{\mu\nu} &= -\frac{1}{2}\Box(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \\ &= -\frac{1}{2}\Box(\partial_\sigma\partial_\nu\eta^{\rho\sigma}h_{\rho\mu} + \partial_\sigma\partial_\mu\eta^{\rho\sigma}h_{\rho\nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}),\end{aligned} \tag{B.7}$$

the second and third term of (B.6) are equal using symmetry of the Ricci tensor, and are both given by,

$$\begin{aligned}\partial_\mu\partial^\rho R_{\rho\nu} &= \frac{1}{2}\partial_\mu\partial^\rho(\partial_\sigma\partial_\nu h^\sigma_\rho + \partial_\sigma\partial_\rho h^\sigma_\nu - \partial_\rho\partial_\nu h - \Box h_{\rho\nu}) \\ &= \frac{1}{2}(\partial_\mu\partial_\nu\partial_\sigma\partial^\rho h^\sigma_\rho + \partial_\mu\partial_\sigma\Box h^\sigma_\nu - \partial_\mu\Box\partial_\nu h - \partial_\mu\Box\partial^\rho h_{\rho\nu}) \\ &= \frac{1}{2}(\partial_\mu\partial_\nu\partial_\sigma\partial^\rho\eta^{\beta\sigma}h_{\beta\rho} + \partial_\mu\partial_\sigma\Box\eta^{\beta\sigma}h_{\beta\nu} - \Box\partial_\mu\partial_\nu h - \Box\partial_\mu\partial^\rho h_{\rho\nu}) \\ &= \frac{1}{2}(\partial_\mu\partial_\nu\partial^\sigma\partial^\rho h_{\sigma\rho} + \Box\partial_\mu\partial^\rho h_{\rho\nu} - \Box\partial_\mu\partial_\nu h - \Box\partial_\mu\partial^\rho h_{\rho\nu}) \\ &= \frac{1}{2}\partial_\mu\partial_\nu(\partial^\sigma\partial^\rho h_{\sigma\rho} - \Box h),\end{aligned} \tag{B.8}$$

and,

$$\begin{aligned}-\eta_{\mu\nu}\partial^\rho\partial^\sigma R_{\rho\sigma} &= -\frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma(\partial_\beta\partial_\sigma h^\beta_\rho + \partial_\beta\partial_\rho h^\beta_\sigma - \partial_\rho\partial_\sigma h - \Box h_{\rho\sigma}) \\ &= -\frac{1}{2}\eta_{\mu\nu}(\Box\partial_\beta\partial^\rho h^\beta_\rho + \Box\partial_\beta\partial^\sigma h^\beta_\sigma - \Box\Box h - \Box\partial^\rho\partial^\sigma h_{\rho\sigma}) \\ &\quad -\frac{1}{2}\eta_{\mu\nu}\Box(2\partial^\rho\partial^\sigma h_{rho\sigma} - \Box h - \partial^\rho\partial^\sigma h_{\rho\sigma}) \\ &= -\frac{1}{2}\eta_{\mu\nu}\Box(\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h),\end{aligned} \tag{B.9}$$

thus combining the above, the second term of (B.4) becomes,

$$
\begin{aligned}
&-\frac{1}{2}(\Box R_{\mu\nu}+\partial_\mu\partial^\rho R_{\mu\sigma}+\partial_\mu\partial^\rho R_{\rho\nu}-\eta_{\mu\nu}\partial^\rho\partial^\sigma R_{\rho\sigma})\\
&=\frac{1}{2}(-\frac{1}{2}\Box(\partial_\sigma\partial_\nu\eta^{\rho\sigma}h_{\rho\mu}+\partial_\sigma\partial_\mu\eta^{\rho\sigma}h_{\rho\nu}-\partial_\mu\partial_\nu h-\Box h_{\mu\nu})\\
&+\partial_\mu\partial_\nu(\partial^\sigma\partial^\rho h_{\sigma\rho}-\Box h)-\frac{1}{2}\eta_{\mu\nu}\Box(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h)). \qquad (B.10)
\end{aligned}
$$

The third term of (B.4) is given by,

$$
\frac{1}{2}\partial^\rho\partial^\sigma(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\sigma}\eta_{\rho\nu})R=\frac{1}{6}(\Box\eta_{\mu\nu}-\partial_\mu\partial_\nu)(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h). \qquad (B.11)
$$

Finally combining (B.5), (B.10), and (B.11), we get the linearized Bach tensor given by,

$$
\begin{aligned}
B_{\mu\nu}=\partial^\rho\partial^\sigma C_{\mu\rho\nu\sigma}=&-\frac{1}{2}(\partial_\mu\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma}+\Box\Box h_{\mu\nu}-\Box\partial^\rho\partial_\mu h_{\rho\nu}-\Box\partial^\sigma\partial_\nu h_{\mu\sigma})\\
&+\frac{1}{2}(-\frac{1}{2}\Box(2\partial^\rho\partial_\nu h_{\rho\mu}-\partial_\mu\partial_\nu h-\Box h_{\mu\nu})\\
&+\partial_\mu\partial_\nu(\partial^\sigma\partial^\rho h_{\sigma\rho}-\Box h)-\frac{1}{2}\eta_{\mu\nu}\Box(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h))\\
&+\frac{1}{6}(\Box\eta_{\mu\nu}-\partial_\mu\partial_\nu)(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h). \qquad (B.12)
\end{aligned}
$$

The linearized equations of motion of (2.5) then are then given by

$$
\begin{aligned}
R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-4\alpha B_{\mu\nu}=&\frac{1}{2}(-\Box h_{\mu\nu}+\partial_\alpha\partial_\mu h^\alpha_\nu+\partial^\alpha\partial_\nu h_{\mu\alpha}-\partial_\nu\partial_\mu h)\\
&-\frac{1}{2}(\partial_\mu\partial_\nu h^{\mu\nu}-\Box h)\eta_{\mu\nu}\\
&-4\alpha(-\frac{1}{2}(\partial_\mu\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma}+\Box\Box h_{\mu\nu}-\Box\partial^\rho\partial_\mu h_{\rho\nu}\\
&-\Box\partial^\sigma\partial_\nu h_{\mu\sigma})\\
&+\frac{1}{2}(-\frac{1}{2}\Box(2\partial^\rho\partial_\nu h_{\rho\mu}-\partial_\mu\partial_\nu h-\Box h_{\mu\nu})\\
&+\partial_\mu\partial_\nu(\partial^\sigma\partial^\rho h_{\sigma\rho}-\Box h)-\frac{1}{2}\eta_{\mu\nu}\Box(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h))\\
&+\frac{1}{6}(\Box\eta_{\mu\nu}-\partial_\mu\partial_\nu)(\partial^\rho\partial^\sigma h_{\rho\sigma}-\Box h)), \qquad (B.13)
\end{aligned}
$$

simplifying these equations of motion (B.13) we get,

$$\begin{aligned}&(\frac{1}{2} - \alpha\Box)(2\partial^{\rho}\partial_{\nu}h_{\mu\rho} - \partial_{\mu}\partial_{\nu}h - \Box h_{\mu\nu})\\+&(-\frac{1}{2}\eta_{\mu\nu} - \frac{2}{3}\alpha\partial_{\mu}\partial_{\nu} + \frac{1}{3}\alpha\eta_{\mu\nu}\Box)(\partial^{\rho}\partial^{\sigma}h_{\rho\sigma} - \Box h) = 0.\end{aligned} \tag{B.14}$$

Appendix C

Numerical results

The following table gives data obtained for various black holes obtained by numerically solving the equations of motion (2.5).

r_0	δ	M	T	S
0.8	1.576219994076353^{-6}	0.399981	0.0989096	2.01061
0.85	3.3870094642052238^{-6}	0.425128	0.0945213	2.26978
0.87	0.0000126094	0.435274	0.0926755	2.37779
0.9	0.066077	0.411658	0.0969034	2.12952
0.92	0.122934	0.387555	0.100988	1.88663
0.95	0.210651	0.348439	0.107195	1.51173
0.97	0.270745	0.320309	0.111379	1.25478
1	0.363302	0.274902	0.117715	0.858897
1.05	0.523988	0.190219	0.128401	0.171293
1.1	0.692677	0.0935836	0.139189	-0.55089
1.15	0.869345	-0.0157767	0.150012	-1.3075

r_0	δ	M	S	T
0.5	-0.783765	0.611191	5.70994	0.013509
0.55	-0.710342	0.607143	5.41354	0.0218717
0.6	-0.626343	0.610886	5.0664	0.0313805
0.64	-0.552158	0.615764	4.75611	0.0395127
0.72	-0.386417	0.553211	4.05653	0.0565589
0.76	-0.295212	0.528545	3.66946	0.0652869
0.8	-0.198607	0.500704	3.2585	0.0740715
0.82	-0.148305	0.48552	3.04423	0.0784725
0.84	-0.0966796	0.469456	2.82416	0.0828742
0.86	-0.0437403	0.452482	2.59835	0.0872736
0.87	-0.0167852	0.444501	2.48334	0.0894713
0.9	$-2.665895877240394(024125) \times 10^{-6}$	0.450001	2.5447	0.0839
0.95	$-6.116994197900739(369571) \times 10^{-7}$	0.475	2.8353	0.0816

Appendix D

Higher Derivative Black Hole Entropy and Temperature

Deriving the entropy of the black hole we use the results obtained by [21] from the Wald formalism.

The action in [21] is

$$S_{action} = \int d^n x\sqrt{-g}(\kappa_0 R + \alpha_0 R^2 + \beta_0 R_{\mu\nu}R^{\mu\nu} + \gamma_0 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \tag{D.1}$$

The entropy is given by[12]

$$\begin{aligned} S = &\frac{1}{4}A[\kappa_0 - (n-2)(4\alpha_0 + \beta_0)\frac{f'(r_0)}{r_0} + \frac{2k(n-2)(n-3)\alpha_0}{r_0^2} \\ &- \frac{1}{4}(2\alpha_0 + \beta_0 + 2\gamma_0)(f''(r_0) + \frac{3kf'(r_0)h''(r_0)}{h'(r_0)})], \end{aligned} \tag{D.2}$$

$n = 4$ is the dimension we are working in, $k = 1$, since we model the black hole as a 2 sphere[12]. We can match the constants of our action (2.1) with the action in [21], then use their results of their entropy to get the entropy that is valid for our theory. To do that we we can rewrite (2.1) in the form of (D.1),

$$\begin{aligned} I &= \int d^4x\sqrt{-g}(\gamma R - \alpha(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}R^2) + \beta R^2), \\ &= \int d^4x\sqrt{-g}(\gamma R - \alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 2\alpha R_{\mu\nu}R^{\mu\nu} - \frac{\alpha}{3}R^2 + \beta R^2), \\ &= \int d^4x\sqrt{-g}(\gamma R + (-\alpha)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + (\beta - \frac{\alpha}{3})R^2 + 2\alpha R_{\mu\nu}R^{\mu\nu}), \end{aligned} \tag{D.3}$$

for $I = S_{action}$ we see that,

$$\kappa_0 = \gamma = 1, \tag{D.4}$$
$$\gamma_0 = -\alpha, \tag{D.5}$$
$$\beta_0 = \beta - \frac{\alpha}{3}, \tag{D.6}$$

thus the entropy becomes

$$\begin{aligned}
S = & \frac{1}{4}A[1-(2)(4(\beta-\frac{\alpha}{3})+2\alpha)\frac{f'(r_0}{r_0}+\frac{2(2)(1)(\beta-\frac{\alpha}{3})}{r_0^2} \\
& -\frac{1}{4}(2(\beta-\frac{\alpha}{3})+2\alpha-2\alpha)(f''(r_0)+\frac{3f'(r_0)h''(r_0)}{h'(r_0)})] \\
& =\frac{1}{4}A[1-2(4(\beta-\frac{\alpha}{3})+2\alpha)\frac{f'(r_0)}{r_0} \\
& +\frac{4(\beta-\frac{\alpha}{3})}{r_0^2}-\frac{1}{4}(2\beta-\frac{2\alpha}{3})(2f_2+\frac{3f_1 2ch_2}{c})], \\
& =\frac{1}{4}A[1-2(4\beta+\alpha(2-\frac{4}{3}))\frac{f_1}{r_0}+\frac{4\beta}{r_0^2}-\frac{4}{3}\frac{\alpha}{r_0^2}-\frac{1}{4}(2\beta-\frac{2}{3}\alpha)(2f_2+6f_1h_2)].
\end{aligned} \tag{D.7}$$

To further simplify the entropy (D.7), lets look at the term

$$\begin{aligned}
2f_2+6f_1h_2 &= 2\frac{(1-2f_1r_0)}{r_0^2}-6\frac{(1-f_1r_0)}{8\alpha f_1r_0}+6f_1\frac{(1-2f_1r_0)}{f_1r_0r_0^2}+6f_1\frac{(1-f_1r_0)}{8\alpha f_1^2r_0} \\
&= 2\frac{(1-2f_1r_0)}{r_0^2}-6\frac{(1-f_1r_0)}{8\alpha f_1r_0}+6\frac{(1-2f_1r_0)}{r_0^2}+6\frac{(1-f_1r_0)}{8\alpha f_1r_0} \\
&= 8\frac{(1-2f_1r_0)}{r_0^2}.
\end{aligned} \tag{D.8}$$

Using (D.8), (D.7) becomes,

$$\begin{aligned}
S &= \frac{1}{4}A[1-2(4\beta+\alpha(2-\frac{4}{3}))\frac{f_1}{r_0}+\frac{4\beta}{r_0^2}-\frac{4}{3}\frac{\alpha}{r_0^2}-\frac{1}{4}(2\beta-\frac{2}{3}\alpha)(8\frac{(1-2f_1r_0)}{r_0^2})] \\
&= \frac{1}{4}A[1+(\frac{4}{r_0^2}-\frac{8f_1}{r_0}-\frac{4(1-2f_1r_0)}{r_0^2})\beta+(\frac{4}{3}\frac{(1-2f_1r_0)}{r_0^2}-\frac{4}{3}\frac{1}{r_0^2}-\frac{4}{3}\frac{f_1}{r_0})\alpha] \\
&= \frac{1}{4}A[1+(\frac{4}{r_0^2}(1-2f_1r_0)-\frac{4}{r_0^2}(1-2f_1r_0))\beta \\
&+(\frac{4}{3}\frac{(1-2f_1r_0)}{r_0^2}+\frac{4}{3}\frac{(1+f_1r_0)}{r_0^2})\alpha] \\
&= \frac{1}{4}A(1+\frac{4}{3}\frac{1}{r_0^2}(-3f_1r_0)\alpha) \\
&= \frac{1}{4}4\pi r_0^2(1-\frac{4\alpha f_1}{r_0}) \\
&= \pi r_0^2-4\pi\alpha f_1r_0.
\end{aligned} \tag{D.9}$$

Since the entropy is defined up to a constant, then we can add the constant Gauss-Bonnet term $4\pi\alpha$ to get the same entrpy as in [1]

$$\begin{aligned}
S &= \pi r_0^2-4\pi\alpha f_1r_0 \\
&= \pi r_0^2-4\pi\alpha-4\pi\delta^* \\
&= \pi r_0^2-4\pi\alpha\delta^*.
\end{aligned} \tag{D.10}$$

D.1 Higher Derivative Black Hole Temperature

To calculate the temperature of the black hole, we use a similar method used in [25]. Recall that the metric is given by,

$$ds^2=-h(r)d^2t+\frac{1}{f(r)}d^2r+r^2(\sin^2\theta+d^2\phi), \tag{D.11}$$

we first write the metric in Euclidean space, by using the time reversal $\tau\to -t$. Then the metric becomes[25],

$$d^2s=h(r)d^2\tau+\frac{1}{f(r)}d^2r+r^2(\sin^2\theta+d^2\phi), \tag{D.12}$$

since the functions $f(r)$, and $h(r)$, have these expansions,

$$\begin{aligned} h(r) &= c((r-r_0) + h_2(r-r_0)^2 + h_3(r-r_0)^3 + h_4(r-r_0)^4 + h_4(r-r_0)^4 \\ &\quad + h_5(r-r_0)^5 + h_6(r-r_0)^6 + h_7(r-r_0)^7 + h_8(r-r_0)^8 + h_9(r-r_0)^9 + \cdots), \end{aligned} \tag{D.13}$$

$$\begin{aligned} f(r) &= f_1(r-r_0) + f_2(r-r_0)^2 + f_3(r-r_0)^3 + f_4(r-r_0)^4 + f_4(r-r_0)^4 \\ &\quad + f_5(r-r_0)^5 + f_6(r-r_0)^6 + f_7(r-r_0)^7 + f_8(r-r_0)^8 + f_9(r-r_0)^9 + \cdots. \end{aligned} \tag{D.14}$$

We can rewrite these functions as

$$h(r) = (r-r_0)H(r), \tag{D.15}$$

$$f(r) = (r-r0)F(r), \tag{D.16}$$

and we can see that $H(r)$, and $F(r)$ are non singular as long as $r > r_0$ and r_0 is the event horizon. We consider the behaviour of the metric close to the horizon as,

$$r - r_0 = \rho\sigma^2, \tag{D.17}$$

with ρ as a constant to be determined, and

$$r - r_0 = \rho\sigma^2, \tag{D.18}$$

$$dr = 2\rho\sigma d\sigma, \tag{D.19}$$

$$dr^2 = 4\rho^2\sigma^2 d\sigma^2. \tag{D.20}$$

The metric (D.12) becomes[25]

$$\begin{aligned} ds^2 &= (r-r_0)H(r)d\tau^2 + \frac{1}{(r-r_0)F(r)}dr^2 + r^2(\sin^2\theta + d^2\phi) \\ &= \rho\sigma^2 H(r_0)d\tau^2 + \frac{1}{\rho\sigma^2 F(r_0)}4\rho^2\sigma^2 d\sigma^2 + (\rho\sigma^2 + r_0)(\sin^2\theta + d^2\phi) \\ &= \rho\sigma^2 H(r_0)d\tau^2 + \frac{4\rho}{F(r_0)}d\sigma^2 + (\rho\sigma^2 + r_0)(\sin^2\theta + d^2\phi). \end{aligned} \tag{D.21}$$

Putting $\rho = \frac{F(r_0)}{4}$ and substituting to (D.21) we get,

$$ds^2 = \frac{F(r_0)}{4}\sigma^2 H(r_0)d\tau^2 + d\sigma^2 + (\frac{F(r_0)}{4}\sigma^2 + r_0^2)(\sin^2\theta + d^2\phi). \tag{D.22}$$

Then close to the horizon, $r \to r_0$ meaning $\sigma^2 << 1$,

$$ds^2 \approx \sigma^2\frac{F(r_0)H(r_0)}{4}d\tau^2 + d\sigma^2 + r_0^2(\sin^2\theta + d^2\phi), \tag{D.23}$$

we can see that the metric has broken into $M_2 \times S^2$, a flat two dimensional metric M_2, σ is the polar axis and τ is the angular coordinate, and a two sphere with constant radius r_0, S^2

$$ds^2 \approx d\sigma^2 + \sigma^2 d(\frac{\sqrt{F(r_0)H(r_0)}}{2}\tau)^2 + r_0^2(\sin^2\theta + d^2\phi), \tag{D.24}$$

now we know that M_2 is analytic if and only if

$$\frac{\sqrt{F(r_0)H(r_0)}}{2}\tau \rightarrow \frac{\sqrt{F(r_0)H(r_0)}}{2}\tau + 2\pi, \tag{D.25}$$

that is

$$\tau \rightarrow \tau + \frac{4\pi}{\sqrt{F(r_0)H(r_0)}} = \tau + \beta, \tag{D.26}$$

and β is the τ-period that allows smooth metric on $\mathbb{R}^2$. Since the temperature is given by if we assume the Hawking temperature,

$$T = \frac{1}{\beta}, \tag{D.27}$$

with $k_B = 1$, and since

$$T = \frac{\kappa}{2\pi}, \tag{D.28}$$

κ is the surface gravity of the black hole.

$$\begin{aligned} \frac{\kappa}{2\pi} = \frac{1}{\beta} &= \frac{\sqrt{F(r_0)H(r_0)}}{4\pi} \\ \implies \kappa &= \frac{\sqrt{F(r_0)H(r_0)}}{2}. \end{aligned} \tag{D.29}$$

Thus the temperature is given by[25],

$$\begin{aligned} T = \frac{\sqrt{h'(r_0)f'(r_0)}}{4\pi} \\ = \frac{1}{4\pi}\sqrt{cf_1} \\ = \frac{1}{4\pi}\sqrt{\frac{c(1+\delta^*)}{r_0}}. \end{aligned} \tag{D.30}$$

Appendix E

Second and third order expansion

E.1 Solving second order asymptotic expansions using a D operator

in this section we show how one can solve for A, and B, using the operator $\mathcal{D}$. Using (6.12), in terms of $A_{(1)}$, (6.11) becomes,

$$\mathcal{D}A_{(1)} = 0 \ , \tag{E.1}$$

where,

$$\mathcal{D} = \left(\frac{1}{\rho} - \frac{2}{\rho^3}\right) + \left(\frac{2}{\rho^2} + 1\right)\frac{d}{d\rho} - \frac{1}{\rho}\frac{d^2}{d\rho^2} - \frac{d^3}{d\rho^3} \ . \tag{E.2}$$

$\mathcal{D}$ can be inverted leading to

$$A_{(1)} = \mathcal{D}^{-1}0 = \frac{c_1}{\rho} + \frac{c_2 e^{-\rho}(-\rho - 1)}{\rho} + \frac{c_3 e^{\rho}(\rho - 1)}{2\rho} \ . \tag{E.3}$$

Similar can be used to solve for the second, and third order expansions of A, and B.

E.1.1 Solving second and third order expansions using Green's Function

In order to solve (6.34), we first realize that it is a wave equation, that is non homogeneous. We already have the solution for it's homogeneous part,

$$Y''_{(2)}(\rho) - Y_{(2)}(\rho) = 0, \tag{E.4}$$

the solution is given by,

$$Y_{(2)}(\rho) = c_4 e^{\rho} + c_5 e^{-\rho}. \tag{E.5}$$

No we need to find the particular solution of (6.34). To do that we consider the equation,

$$y''(\rho) - y(\rho) = J(\rho). \tag{E.6}$$

In order to solve (E.6) we first realize that it is a wave equation, that is non-homogeneous. We already have the solution for it's homogeneous part,

$$y''(\rho) - y(\rho) = 0, \tag{E.7}$$

the solution is given by, which is also the complementary solution of the in-homogeneous solution

$$y_c(\rho) = c_4 e^{\rho} + c_5 e^{-\rho}. \tag{E.8}$$

The second order differential equation (E.6) can be written as in Sturm-Liouvile form[9, 10, 11],

$$L(\rho)y(\rho) = J(\rho), \tag{E.9}$$

$$L(\rho) = \frac{d^2}{d\rho^2} - 1. \tag{E.10}$$

now we assume that the particular solution of (E.6) can be written as,

$$y_p(\rho) = \int_a^b d\rho' G(\rho, \rho') J(\rho'), \tag{E.11}$$

$G(\rho, \rho')$ is the Green's function. Substituting the particular solution (E.11) on the Sturm-Louiville form (E.9) we get,

$$\int_a^b d\rho' [L(\rho)G(\rho, \rho')] J(\rho') = J(\rho). \tag{E.12}$$

From this equation (E.12) we can see that,

$$L(\rho)G(\rho, \rho') = \delta(\rho - \rho'), \tag{E.13}$$

$$\implies \frac{d^2}{d\rho^2} G(\rho, \rho') - G(\rho, \rho') = \delta(\rho - \rho'). \tag{E.14}$$

We first integrate (E.14) $G(\rho, \rho')$, with respect to ρ, around the neighbourhood, of ρ'.

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} \left(\frac{d^2}{d\rho^2} G(\rho, \rho') - G(\rho, \rho') \right) d\rho = 1,$$

$$\implies \left[\frac{d}{d\rho} G(\rho, \rho') \right]_{\rho'-\varepsilon}^{\rho'+\varepsilon} + \varepsilon G(\rho, \rho')_{|\rho-\rho'|<\varepsilon} = 1, \tag{E.15}$$

If we take the limit as $\varepsilon \to 0$ on (E.15), we have,

$$lim_{\varepsilon\to 0^+}\frac{d}{d\rho}G(\rho,\rho') - lim_{\varepsilon\to 0^-}\frac{d}{d\rho}G(\rho,\rho') = 1, \tag{E.16}$$

the equation (E.16) is evaluated at $\rho = \rho'+\varepsilon$. Now for $\rho \neq \rho'$. Recall that $G(\rho,\rho')$ also satisfies the homogeneous equation (E.7), with two linearly independent solutions. $g_1 e^{\rho} + g_2 e^{-\rho}$.Now we can choose,

$$G(\rho,\rho') = g_1 e^{\rho}, \tag{E.17}$$

for $\rho < \rho'$, and

$$G(\rho,\rho') = g_2 e^{-\rho}, \tag{E.18}$$

for $\rho > \rho'$. Now we use the continuity of $G(\rho,\rho')$, and the discontinuity of $G'(\rho,\rho')$ at $\rho = \rho'$. Continuity of G gives,

$$g_2 e^{-\rho'} - g_1 e^{\rho'} = 0. \tag{E.19}$$

And the discontinuity gives,

$$- g_2 e^{-\rho'} - g_1 e^{\rho'} = 1. \tag{E.20}$$

From these equations we need to solve for g_1, and g_2 using (E.19) and (E.20), and get,

$$\begin{aligned}
& - g_1 e^{\rho'} - g_1 e^{\rho'} = 1, \\
& \implies -2 g_1 e^{\rho'} = 1 \\
& \implies g_1 = -\frac{1}{2} e^{-\rho'},
\end{aligned} \tag{E.21}$$

then from (E.19), we have,

$$g_2 = -\frac{1}{2} e^{\rho'}. \tag{E.22}$$

Therefore, $G(\rho,\rho')$ becomes,

$$G(\rho,\rho') = -\frac{1}{2}\left(e^{-(\rho'-\rho)}\theta(\rho'-\rho) + e^{-(\rho-\rho')}\theta(\rho-\rho')\right). \tag{E.23}$$

Therefore, the particular solution (E.11) is given by

$$\begin{aligned}
y_p &= \int d\rho' G(\rho,\rho') J(\rho') \\
&= -\frac{1}{2}\left(-e^{\rho}\int^{\rho} d\rho'\left(e^{-\rho'} J(\rho')\right) + e^{-\rho}\int^{\rho} d\rho'\left(e^{\rho'} J(\rho')\right)\right),
\end{aligned} \tag{E.24}$$

If we apply (E.24) to the equation (E.6), then $J(\rho)$, is the right hand side.Thus the first term of this equation becomes, using Mathematica to integrate,

$$\begin{aligned}\frac{1}{2}\int^{\rho} d\rho' \left(e^{-\rho'}J(\rho')\right) &= \frac{69}{64}c_2^2 e^{\rho}\mathrm{Ei}(-3\rho) - \frac{1}{2}c_3c_2e^{\rho}\mathrm{Ei}(-2\rho) - \frac{27c_2^2e^{-2\rho}}{32\rho^4} \\ &+ \frac{21c_3c_2e^{-\rho}}{8\rho^4} - \frac{3c_3^2}{2\rho^4} - \frac{45c_2^2e^{-2\rho}}{32\rho^3} + \frac{7c_3c_2e^{-\rho}}{4\rho^3} + \frac{c_3^2}{2\rho^3} \\ &- \frac{149c_2^2e^{-2\rho}}{64\rho^2} + \frac{5c_3c_2e^{-\rho}}{4\rho^2} - \frac{5}{4}c_2^2e^{-2\rho} - \frac{113c_2^2e^{-2\rho}}{64\rho},\end{aligned} \tag{E.25}$$

and the right hand side is,

$$\begin{aligned}-\frac{1}{2}\int^{\rho} d\rho' \left(e^{\rho'}J(\rho')\right) &= -\frac{69}{64}c_2^2e^{-\rho}\mathrm{Ei}(-\rho) + \frac{27c_2^2e^{-2\rho}}{32\rho^4} - \frac{21c_3c_2e^{-\rho}}{8\rho^4} + \frac{3c_3^2}{2\rho^4} \\ &+ \frac{63c_2^2e^{-2\rho}}{32\rho^3} - \frac{7c_3c_2e^{-\rho}}{2\rho^3} + \frac{c_3^2}{2\rho^3} + \frac{221c_2^2e^{-2\rho}}{64\rho^2} - \frac{3c_3c_2e^{-\rho}}{\rho^2} \\ &+ \frac{15}{4}c_2^2e^{-2\rho} + \frac{339c_2^2e^{-2\rho}}{64\rho} - \frac{5c_3c_2e^{-\rho}}{2\rho} + \frac{1}{2}c_3c_2e^{-\rho}\log(\rho),\end{aligned} \tag{E.26}$$

Then the particular solution is given by,

$$\begin{aligned}y_p(r) = &\frac{69}{64}c_2^2e^{\rho}\mathrm{Ei}(-3\rho) - \frac{1}{2}c_3c_2e^{\rho}\mathrm{Ei}(-2\rho) - \frac{27c_2^2e^{-2\rho}}{32\rho^4} \\ &+ \frac{21c_3c_2e^{-\rho}}{8\rho^4} - \frac{3c_3^2}{2\rho^4} - \frac{45c_2^2e^{-2\rho}}{32\rho^3} + \frac{7c_3c_2e^{-\rho}}{4\rho^3} + \frac{c_3^2}{2\rho^3} \\ &- \frac{149c_2^2e^{-2\rho}}{64\rho^2} + \frac{5c_3c_2e^{-\rho}}{4\rho^2} - \frac{5}{4}c_2^2e^{-2\rho} - \frac{113c_2^2e^{-2\rho}}{64\rho} \\ &- \frac{69}{64}c_2^2e^{-\rho}\mathrm{Ei}(-\rho) + \frac{27c_2^2e^{-2\rho}}{32\rho^4} - \frac{21c_3c_2e^{-\rho}}{8\rho^4} + \frac{3c_3^2}{2\rho^4} \\ &+ \frac{63c_2^2e^{-2\rho}}{32\rho^3} - \frac{7c_3c_2e^{-\rho}}{2\rho^3} + \frac{c_3^2}{2\rho^3} + \frac{221c_2^2e^{-2\rho}}{64\rho^2} - \frac{3c_3c_2e^{-\rho}}{\rho^2} \\ &+ \frac{15}{4}c_2^2e^{-2\rho} + \frac{339c_2^2e^{-2\rho}}{64\rho} - \frac{5c_3c_2e^{-\rho}}{2\rho} + \frac{1}{2}c_3c_2e^{-\rho}\log(\rho).\end{aligned} \tag{E.27}$$

Recall that the complete solution is given by,

$$Y_{(2)}(\rho) = c_4e^{\rho} + c_5e^{-\rho} + J_2(\rho), \tag{E.28}$$

using,

$$\rho Y_{(2)}(\rho) = \frac{d}{d\rho}(\rho A_{(2)}(\rho)), \tag{E.29}$$

then we can solve for $A_{(2)}(\rho)$,

$$\frac{d}{d\rho}(\rho A_{(2)}(\rho)) = \rho J_2(\rho),$$
$$\Longrightarrow A_{(2)}(\rho) = -\frac{69}{64}c_2^2 e^{\rho}\text{Ei}(-3\rho) + \frac{69c_2^2 e^{\rho}\text{Ei}(-3\rho)}{64\rho} - \frac{69}{64}c_2^2 e^{-\rho}\text{Ei}(-\rho) - \frac{69c_2^2 e^{-\rho}\text{Ei}(-\rho)}{64\rho} + \frac{1}{2}c_3 c_2 e^{\rho}\text{Ei}(-2\rho) - \frac{c_3 c_2 e^{\rho}\text{Ei}(-2\rho)}{2\rho} + \frac{9c_2^2 e^{-2\rho}}{16\rho^2} - \frac{7c_3 c_2 e^{-\rho}}{4\rho^2} + \frac{c_3^2}{\rho^2} + \frac{5}{4}c_2^2 e^{-2\rho} + \frac{21c_2^2 e^{-2\rho}}{16\rho} - \frac{3c_3 c_2 e^{-\rho}}{2\rho} + \frac{1}{2}c_3 c_2 e^{-\rho}\log(\rho) + \frac{c_3 c_2 e^{-\rho}\log(\rho)}{2\rho}, \tag{E.30}$$

Solving for, A and B by integrating using MATHEMATICA, and the substitution, we get,

$$A(\rho) = 1 + \left(c_2\left(-e^{-\rho}\right) - \frac{c_2 e^{-\rho}}{\rho} + \frac{c_3}{\rho}\right)\epsilon \\ (-\frac{69}{64}c_2^2 e^{\rho}\text{Ei}(-3\rho) + \frac{69c_2^2 e^{\rho}\text{Ei}(-3\rho)}{64\rho} - \frac{69}{64}c_2^2 e^{-\rho}\text{Ei}(-\rho) - \frac{69c_2^2 e^{-\rho}\text{Ei}(-\rho)}{64\rho} + \frac{1}{2}c_3 c_2 e^{\rho}\text{Ei}(-2\rho) - \frac{c_3 c_2 e^{\rho}\text{Ei}(-2\rho)}{2\rho} + \frac{9c_2^2 e^{-2\rho}}{16\rho^2} - \frac{7c_3 c_2 e^{-\rho}}{4\rho^2} + \frac{c_3^2}{\rho^2} + \frac{5}{4}c_2^2 e^{-2\rho} + \frac{21c_2^2 e^{-2\rho}}{16\rho} - \frac{3c_3 c_2 e^{-\rho}}{2\rho} + \frac{1}{2}c_3 c_2 e^{-\rho}\log(\rho) + \frac{c_3 c_2 e^{-\rho}\log(\rho)}{2\rho})\epsilon^2 + O\left(\epsilon^3\right), \tag{E.31}$$

in this solution we assigned, $c_4 \to 0, c_5 \to c_2, c_6 \to c_3$. Using the substitution for $h_{(2)}(\rho)$, we can solve,

$$h_{(2)}(\rho) = \int \frac{\int \left(Z_{(2)}(\rho)Y_{(2)}(\rho)\right)d\rho + c_7}{\rho^2}d\rho + c_8 \\ = -\frac{69c_2^2 e^{\rho}\text{Ei}(-3\rho)}{32\rho} + \frac{3}{2}c_2^2\text{Ei}(-2\rho) + \frac{69c_2^2 e^{-\rho}\text{Ei}(-\rho)}{32\rho} + \frac{c_3 c_2 e^{\rho}\text{Ei}(-2\rho)}{\rho} + \frac{15c_2^2 e^{-2\rho}}{16\rho^2} - \frac{c_3 c_2 e^{-\rho}}{2\rho^2} - \frac{c_2^2 e^{-2\rho}}{4\rho} + \frac{2c_4 e^{\rho}}{\rho} + \frac{2c_5 e^{-\rho}}{\rho} - \frac{c_7}{\rho} - \frac{c_3 c_2 e^{-\rho}\log(\rho)}{\rho} + c_8, \tag{E.32}$$

then finally we have,

$$B(\rho) = 1 + \left(\frac{2c_2 e^{-\rho}}{\rho} - \frac{c_3}{\rho}\right)\epsilon \\ + (-\frac{69c_2^2 e^{\rho}\text{Ei}(-3\rho)}{32\rho} + \frac{3}{2}c_2^2\text{Ei}(-2\rho) + \frac{69c_2^2 e^{-\rho}\text{Ei}(-\rho)}{32\rho} + \frac{c_3 c_2 e^{\rho}\text{Ei}(-2\rho)}{\rho} + \frac{15c_2^2 e^{-2\rho}}{16\rho^2} - \frac{c_3 c_2 e^{-\rho}}{2\rho^2} - \frac{c_2^2 e^{-2\rho}}{4\rho} - \frac{c_3 c_2 e^{-\rho}\log(\rho)}{\rho})\epsilon^2 + O\left(\epsilon^3\right), \tag{E.33}$$

in this solution we have assigned, $c_6 \to c_7, c_8 \to 0, c_4 \to 0$, and $c_5 \to c_2, c_7 \to c_3$. Now we simplify the results and writing them in component form,

$$\begin{aligned}
A_{(2,2)} &= c_3^2\frac{1}{\rho^2}, && \text{(E.34)}\\
A_{(2,1)} &= c_3c_2\Big(\frac{1}{2}e^{\rho}\mathrm{Ei}(-2\rho) - \frac{e^{\rho}\mathrm{Ei}(-2\rho)}{2\rho} - \frac{7e^{-\rho}}{4\rho^2} - \frac{3e^{-\rho}}{2\rho} + \frac{1}{2}e^{-\rho}\log(\rho)\\
&\quad + \frac{e^{-\rho}\log(\rho)}{2\rho}\Big), && \text{(E.35)}\\
A_{(0,2)} &= c_2^2\Big(-\frac{69}{64}e^{\rho}\mathrm{Ei}(-3\rho) + \frac{69e^{\rho}\mathrm{Ei}(-3\rho)}{64\rho} - \frac{69}{64}e^{-\rho}\mathrm{Ei}(-\rho) - \frac{69e^{-\rho}\mathrm{Ei}(-\rho)}{64\rho}\\
&\quad + \frac{92e^{-2\rho}}{16\rho^2} + \frac{5}{4}e^{-2\rho} + \frac{21e^{-2\rho}}{16\rho}\Big). && \text{(E.36)}
\end{aligned}$$

E.2 Third order expansion

We now look at the third order expansion given by,

$$\begin{aligned}
&-128e^{3\rho}A_{(3)}{}^{(3)}(\rho)\rho^6 - 128e^{3\rho}A''_{(3)}(\rho)\rho^5 + 128e^{3\rho}\left(\rho^2+2\right)A'_{(3)}(\rho)\rho^4\\
&+4992c_2^3\rho^6 - 2070e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho^6 + 960e^{\rho}c_1c_2^2\log(\rho)\rho^6 + 13354c_2^3\rho^5\\
&-4160e^{\rho}c_1c_2^2\rho^5 - 69e^{3\rho}c_2^3\mathrm{Ei}(-3\rho)\rho^5 - 138e^{4\rho}c_1c_2^2\mathrm{Ei}(-3\rho)\rho^5 + 32e^{3\rho}c_1c_2^2\mathrm{Ei}(-2\rho)\rho^5\\
&+64e^{4\rho}c_1^2c_2\mathrm{Ei}(-2\rho)\rho^5 - 3519e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho^5 + 138e^{2\rho}c_1c_2^2\mathrm{Ei}(-\rho)\rho^5 + 1632e^{\rho}c_1c_2^2\log(\rho)\rho^5\\
&-64e^{2\rho}c_1^2c_2\log(\rho)\rho^5 + 18432c_2^3\rho^4 - 11556e^{\rho}c_1c_2^2\rho^4 + 448e^{2\rho}c_1^2c_2\rho^4\\
&+690e^{4\rho}c_1c_2^2\mathrm{Ei}(-3\rho)\rho^4 - 320e^{4\rho}c_1^2c_2\mathrm{Ei}(-2\rho)\rho^4 - 4830e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho^4\\
&+690e^{2\rho}c_1c_2^2\mathrm{Ei}(-\rho)\rho^4 + 2240e^{\rho}c_1c_2^2\log(\rho)\rho^4 - 320e^{2\rho}c_1^2c_2\log(\rho)\rho^4\\
&+16690c_2^3\rho^3 - 20392e^{\rho}c_1c_2^2\rho^3 + 2464e^{2\rho}c_1^2c_2\rho^3\\
&+1173e^{3\rho}c_2^3\mathrm{Ei}(-3\rho)\rho^3 - 1656e^{4\rho}c_1c_2^2\mathrm{Ei}(-3\rho)\rho^3 - 544e^{3\rho}c_1c_2^2\mathrm{Ei}(-2\rho)\rho^3\\
&+768e^{4\rho}c_1^2c_2\mathrm{Ei}(-2\rho)\rho^3 - 4899e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho^3 + 1656e^{2\rho}c_1c_2^2\mathrm{Ei}(-\rho)\rho^3\\
&+2272e^{\rho}c_1c_2^2\log(\rho)\rho^3 - 768e^{2\rho}c_1^2c_2\log(\rho)\rho^3 + 128e^{3\rho}\left(\rho^2-2\right)A_{(3)}(\rho)\rho^3 + 256e^{3\rho}c_1^3\rho^2\\
&+8650c_2^3\rho^2 - 24876e^{\rho}c_1c_2^2\rho^2 + 8128e^{2\rho}c_1^2c_2\rho^2 + 2898e^{4\rho}c_1c_2^2\mathrm{Ei}(-3\rho)\rho^2\\
&-1344e^{4\rho}c_1^2c_2\mathrm{Ei}(-2\rho)\rho^2 - 3726e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho^2 + 2898e^{2\rho}c_1c_2^2\mathrm{Ei}(-\rho)\rho^2 + 1728e^{\rho}c_1c_2^2\log(\rho)\rho^2\\
&-1344e^{2\rho}c_1^2c_2\log(\rho)\rho^2 + 84c_2^3\rho - 14696e^{\rho}c_1c_2^2\rho + 11872e^{2\rho}c_1^2c_2\rho + 1863e^{3\rho}c_2^3\mathrm{Ei}(-3\rho)\rho\\
&-2898e^{4\rho}c_1c_2^2\mathrm{Ei}(-3\rho)\rho - 864e^{3\rho}c_1c_2^2\mathrm{Ei}(-2\rho)\rho + 1344e^{4\rho}c_1^2c_2\mathrm{Ei}(-2\rho)\rho - 1863e^{\rho}c_2^3\mathrm{Ei}(-\rho)\rho\\
&+2898e^{2\rho}c_1c_2^2\mathrm{Ei}(-\rho)\rho + 864e^{\rho}c_1c_2^2\log(\rho)\rho - 1344e^{2\rho}c_1^2c_2\log(\rho)\rho - 5120e^{3\rho}c_1^3\\
&-800c_2^3 - 6400e^{\rho}c_1c_2^2 + 12160e^{2\rho}c_1^2c_2\\
&=0, \qquad \text{(E.37)}
\end{aligned}$$

the differential equation can be written as,

$$\left(\frac{2}{\rho^3}-\frac{1}{\rho}\right)A_{(3)}(\rho)+\left(-\frac{2}{\rho^2}-1\right)A'_{(3)}(\rho)+\frac{A''_{(3)}(\rho)}{\rho}+{A_{(3)}}^{(3)}(\rho)=J_{(3)}, \tag{E.38}$$

the right hand side can be broken into powers of c_1, and c_2,

$$J_{(3)}=J_{(3,0)}+J_{(3,1)}+J_{(3,2)}, \tag{E.39}$$

and,

$$\begin{aligned}J_{(3,0)}=c_2^3\frac{1}{128e^\rho\rho^6}&(e^\rho\left(-69\rho^5+1173\rho^3+1863\rho\right)\mathrm{Ei}(-3\rho)+84e^{-2\rho}\rho-800e^{-2\rho}\\&+e^{-\rho}\left(-2070\rho^6-3519\rho^5-4830\rho^4-4899\rho^3-3726\rho^2-1863\rho\right)\mathrm{Ei}(-\rho)\\&+4992e^{-2\rho}\rho^6+13354e^{-2\rho}\rho^5+18432e^{-2\rho}\rho^4+16690e^{-2\rho}\rho^3+8650e^{-2\rho}\rho^2),\end{aligned} \tag{E.40}$$

$$\begin{aligned}J_{(3,1)}=c_2^2c_1\frac{1}{128e^\rho\rho^6}&(138\rho^5\mathrm{Ei}(-\rho)+690\rho^4\mathrm{Ei}(-\rho)+1656\rho^3\mathrm{Ei}(-\rho)+2898\rho^2\mathrm{Ei}(-\rho)\\&+e^\rho\left(32\rho^5-544\rho^3-864\rho\right)\mathrm{Ei}(-2\rho)+e^{2\rho}\left(-138\rho^5+690\rho^4-1656\rho^3+2898\rho^2-2898\rho\right)\\&\times\mathrm{Ei}(-3\rho)+2898\rho\mathrm{Ei}(-\rho)\\&-4160e^{-\rho}\rho^5-11556e^{-\rho}\rho^4-20392e^{-\rho}\rho^3-24876e^{-\rho}\rho^2-14696e^{-\rho}\rho-6400e^{-\rho}\\&+\left(960e^{-\rho}\rho^6+1632e^{-\rho}\rho^5+2240e^{-\rho}\rho^4+2272e^{-\rho}\rho^3+1728e^{-\rho}\rho^2+864e^{-\rho}\rho\right)\log(\rho)),\end{aligned} \tag{E.41}$$

$$\begin{aligned}J_{(3,2)}=c_2c_1^2\frac{1}{128e^\rho\rho^6}&(e^\rho\left(-\rho^5+17\rho^3+27\rho\right)\mathrm{Ei}(-\rho)\\&+e^{2\rho}\left(64\rho^5-288\rho^4+896\rho^3-1824\rho^2+1824\rho\right)\mathrm{Ei}(-2\rho)\\&+e^{-\rho}\left(-30\rho^6-51\rho^5-70\rho^4-71\rho^3-54\rho^2-27\rho\right)\mathrm{Ei}(\rho)\\&+30\rho^5+496\rho^4+2358\rho^3+9070\rho^2+16728\rho+20640\\&+\left(-64\rho^5-288\rho^4-896\rho^3-1824\rho^2-1824\rho\right)\log(\rho)).\end{aligned} \tag{E.42}$$

$F_{(3)}$, is given by,

$$
\begin{aligned}
F_{(3)} = 2(&-\frac{9c_2c_1^2e^{\rho}\text{Ei}(-2\rho)}{4\rho^2}+\frac{621c_2^2c_1e^{\rho}\text{Ei}(-3\rho)}{128\rho^2}+\frac{3c_2^2c_1\text{Ei}(-2\rho)}{\rho^2}-\frac{621c_2^2c_1e^{-\rho}\text{Ei}(-\rho)}{128\rho^2}\\
&-\frac{207c_2^3\text{Ei}(-3\rho)}{32\rho^2}+\frac{207c_2^3e^{-2\rho}\text{Ei}(-\rho)}{32\rho^2}-\frac{7}{4}c_2c_1^2e^{\rho}\text{Ei}(-2\rho)+\frac{9c_2c_1^2e^{\rho}\text{Ei}(-2\rho)}{4\rho}\\
&+\frac{483}{128}c_2^2c_1e^{\rho}\text{Ei}(-3\rho)-\frac{621c_2^2c_1e^{\rho}\text{Ei}(-3\rho)}{128\rho}+2c_2^2c_1\text{Ei}(-2\rho)-\frac{483}{128}c_2^2c_1e^{-\rho}\text{Ei}(-\rho)\\
&-\frac{621c_2^2c_1e^{-\rho}\text{Ei}(-\rho)}{128\rho}-\frac{69}{16}c_2^3\text{Ei}(-3\rho)+3c_2^3e^{-\rho}\rho\text{Ei}(-2\rho)+\frac{69}{4}c_2^3e^{-2\rho}\text{Ei}(-\rho)\\
&+\frac{69}{32}c_2^3e^{-2\rho}\rho\text{Ei}(-\rho)+\frac{207c_2^3e^{-2\rho}\text{Ei}(-\rho)}{16\rho}+\frac{4c_1^3}{\rho^3}-\frac{43c_2c_1^2e^{-\rho}}{4\rho^3}+\frac{8c_2^2c_1e^{-2\rho}}{\rho^3}\\
&-\frac{11c_2^3e^{-3\rho}}{16\rho^3}-\frac{73c_2c_1^2e^{-\rho}}{8\rho^2}+\frac{471c_2^2c_1e^{-2\rho}}{32\rho^2}-\frac{57c_2^3e^{-3\rho}}{16\rho^2}+\frac{9c_2c_1^2e^{-\rho}\log(\rho)}{4\rho^2}\\
&-\frac{3c_2^2c_1e^{-2\rho}\log(\rho)}{\rho^2}-\frac{29c_2c_1^2e^{-\rho}}{4\rho}+10c_2^2c_1e^{-2\rho}+\frac{1711c_2^2c_1e^{-2\rho}}{64\rho}-\frac{631}{32}c_2^3e^{-3\rho}\\
&-3c_2^3e^{-3\rho}\rho-\frac{615c_2^3e^{-3\rho}}{32\rho}+\frac{7}{4}c_2c_1^2e^{-\rho}\log(\rho)+\frac{9c_2c_1^2e^{-\rho}\log(\rho)}{4\rho}-8c_2^2c_1e^{-2\rho}\log(\rho)\\
&-c_2^2c_1e^{-2\rho}\rho\log(\rho)-\frac{6c_2^2c_1e^{-2\rho}\log(\rho)}{\rho}).
\end{aligned}
\tag{E.43}
$$

These sources need to be plugged into (6.43) and (6.44) and integrated. Given the fact that $J_{(3)}$ and $F_{(3)}$ are quite large, we found it more managable to use the linearity of (6.43)-(6.44), and calculate the pieces, $A_{(3,i)}$ and $h_{(3,i)}$, independently. For the first term (which is what one would have for the Schwarzschild black hole) one finds

$$
\begin{aligned}
(M^3m_2^3)A_{(3,3)} &= \frac{8M^3}{r^3},\\
h_{(3,3)} &= 0,
\end{aligned}
\tag{E.44}
$$

as expected. For the other parts of the metric, we were unable to express all the integrals in terms of known special functions. In fact we found in easier to just calculate $h'_{(3)}$ rather than just h_3 - fortunately this is sufficient for matching the numerical data and asymptotic expressions. In fact, since h (as opposed to its derivatives) does not appear in the equations of motion, (6.43)-(6.44), one could even proceed to the next order without calculating h. For completeness,we present the rest of our results at third order order below:

$$\begin{aligned}
A_{(3,0)} = c_2^3(&\frac{138897e^{\rho}X^{1,3}}{1024} - \frac{138897e^{\rho}X^{1,3}(\rho)}{1024\rho} + \frac{4761e^{-\rho}X^{-1,3}(\rho)}{2048} \\
&+ \frac{4761e^{-\rho}X^{-1,3}(\rho)}{2048\rho} + \frac{1863X^{1,2}(\rho)}{16\rho} + \frac{621e^{-2\rho}\mathrm{Ei}(-\rho)}{512\rho^2} - \frac{621\mathrm{Ei}(-3\rho)}{512\rho^2} \\
&- \frac{14697e^{-\rho}\mathrm{Ei}(-\rho)^2}{4096} - \frac{14697e^{-\rho}\mathrm{Ei}(-\rho)^2}{4096\rho} + \frac{345}{128}e^{-2\rho}\mathrm{Ei}(-\rho) \\
&- \frac{273033e^{\rho}\mathrm{Ei}(-3\rho)\mathrm{Ei}(-\rho)}{2048} + \frac{273033e^{\rho}\mathrm{Ei}(-3\rho)\mathrm{Ei}(-\rho)}{2048\rho} \\
&- \frac{1863\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho)}{16\rho} - \frac{23391e^{-2\rho}\mathrm{Ei}(-\rho)}{512\rho} - \frac{101581}{256}e^{\rho}\mathrm{Ei}(-4\rho) \\
&+ \frac{101581e^{\rho}\mathrm{Ei}(-4\rho)}{256\rho} - \frac{4899\mathrm{Ei}(-3\rho)}{16\rho} + \frac{69\mathrm{Ei}(-3\rho)}{128} - \frac{16933}{512}e^{-\rho}\mathrm{Ei}(-2\rho) \\
&- \frac{16933e^{-\rho}\mathrm{Ei}(-2\rho)}{512\rho} + \frac{5e^{-3\rho}}{32\rho^3} - \frac{147e^{-3\rho}}{128\rho^2} - \frac{13e^{-3\rho}}{8} - \frac{60633e^{-3\rho}}{512\rho}),
\end{aligned} \tag{E.45}$$

$$\begin{aligned}
A_{(3,1)} = c_1c_2^2(&-\frac{69}{128}e^{-\rho}G^{3,0}_{2,3}\left(\rho\,\middle|\,\begin{matrix}0,0\\-1,-1,-1\end{matrix}\right) + \frac{621}{128}e^{\rho}G^{3,0}_{2,3}\left(3\rho\,\middle|\,\begin{matrix}0,0\\-1,-1,-1\end{matrix}\right) \\
&- \frac{69}{128}e^{-\rho}G^{3,0}_{2,3}\left(\rho\,\middle|\,\begin{matrix}1,1\\0,0,0\end{matrix}\right) - \frac{207}{128}e^{\rho}G^{3,0}_{2,3}\left(3\rho\,\middle|\,\begin{matrix}1,1\\0,0,0\end{matrix}\right) \\
&- \frac{69}{128}e^{-\rho}X^{-2,3}(\rho) - \frac{69e^{-\rho}X^{-2,3}(\rho)}{128\rho} \\
&+ \frac{69}{128}e^{\rho}\left(\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho) - X^{1,2}(\rho)\right) - \frac{69e^{\rho}\left(\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho) - X^{1,2}(\rho)\right)}{128\rho} \\
&- \frac{69}{64}e^{\rho}X^{1,2}(\rho) - \frac{69e^{\rho}X^{1,2}}{64\rho} + \frac{69}{64}e^{-\rho}X^{-1,2}(\rho) - \frac{69e^{-\rho}X^{-1,2}(\rho)}{64\rho} \\
&+ \frac{483e^{\rho}\mathrm{Ei}(-3\rho)}{256\rho^2} + \frac{9\mathrm{Ei}(-2\rho)}{16\rho^2} - \frac{483e^{-\rho}\mathrm{Ei}(-\rho)}{256\rho^2} + \frac{675}{256}e^{\rho}\mathrm{Ei}(-3\rho) \\
&- \frac{1089e^{\rho}\mathrm{Ei}(-3\rho)}{256\rho} - \frac{1}{4}\mathrm{Ei}(-2\rho) + \frac{225}{256}e^{-\rho}\mathrm{Ei}(-\rho) - \frac{189e^{-\rho}\mathrm{Ei}(-\rho)}{256\rho} \\
&+ \frac{69}{64}e^{\rho}\mathrm{Ei}(-3\rho)\log(\rho) - \frac{69e^{\rho}\mathrm{Ei}(-3\rho)\log(\rho)}{64\rho} + \frac{69}{64}e^{-\rho}\mathrm{Ei}(-\rho)\log(\rho) \\
&+ \frac{69e^{-\rho}\mathrm{Ei}(-\rho)\log(\rho)}{64\rho} + \frac{5e^{-2\rho}}{4\rho^3} + \frac{239e^{-2\rho}}{64\rho^2} - \frac{9e^{-2\rho}\log(\rho)}{16\rho^2} + \frac{609e^{-2\rho}}{128\rho} \\
&- \frac{5}{4}e^{-2\rho}\log(\rho) - \frac{21e^{-2\rho}\log(\rho)}{16\rho}),
\end{aligned} \tag{E.46}$$

$$
\begin{aligned}
A_{(3,2)} = {} & c_2 c_1^2 (-e^{\rho} G_{2,3}^{3,0}\left(2\rho \,\middle|\, \begin{matrix} 0,0 \\ -1,-1,-1 \end{matrix}\right) + \frac{1}{2} e^{\rho} G_{2,3}^{3,0}\left(2\rho \,\middle|\, \begin{matrix} 1,1 \\ 0,0,0 \end{matrix}\right) \\
& + \frac{69 e^{-\rho} G_{2,3}^{3,1}\left(\rho \,\middle|\, \begin{matrix} -1,0 \\ -1,-1,-1 \end{matrix}\right)}{1024} + \frac{1}{2} e^{-\rho} G_{2,3}^{3,1}\left(2\rho \,\middle|\, \begin{matrix} -1,0 \\ -1,-1,-1 \end{matrix}\right) \\
& + \frac{69 e^{-\rho} G_{2,3}^{3,1}\left(\rho \,\middle|\, \begin{matrix} 0,1 \\ 0,0,0 \end{matrix}\right)}{1024} + \frac{1}{4} e^{-\rho} G_{2,3}^{3,1}\left(2\rho \,\middle|\, \begin{matrix} 0,1 \\ 0,0,0 \end{matrix}\right) \\
& + \left(\frac{69 e^{\rho}}{2048\rho} - \frac{69 e^{\rho}}{2048}\right) \Xi(3,-1,\rho) - \frac{9\mathrm{Ei}(-\rho)}{512\rho^2} \\
& + e^{-2\rho}\left(\frac{9}{512\rho^2} + \frac{21}{512\rho} + \frac{5}{128}\right) \mathrm{Ei}(\rho) \\
& + e^{\rho}\mathrm{Ei}(-2\rho)\left(-\frac{19}{16\rho^2} + \frac{2483}{1536\rho} + \left(\frac{1}{4\rho} - \frac{1}{4}\right)\log(\rho) - \frac{1523}{1536}\right) \\
& + \frac{69 e^{\rho}\mathrm{Ei}(-\rho)^2}{4096} - \frac{69 e^{\rho}\mathrm{Ei}(-\rho)^2}{4096\rho} + e^{-\rho}\left(-\frac{69\mathrm{Ei}(\rho)}{2048\rho} - \frac{69\mathrm{Ei}(\rho)}{2048}\right)\mathrm{Ei}(-\rho) \\
& + \frac{15\mathrm{Ei}(-\rho)}{128\rho} + \frac{\mathrm{Ei}(-\rho)}{128} - \frac{129 e^{-\rho}}{32\rho^3} \\
& - \frac{95 e^{-\rho}}{64\rho^2} + \left(\frac{19 e^{-\rho}}{16\rho^2} + \frac{5 e^{-\rho}}{8\rho}\right)\log(\rho) - \frac{1681 e^{-\rho}}{1536\rho} + \left(-\frac{e^{-\rho}}{8} - \frac{e^{-\rho}}{8\rho}\right)\log^2(\rho)).
\end{aligned} \tag{E.47}
$$

Using the substitution (6.12), we recognize that, $h_{(3)}$ is much more complicated to solve, but we can at least solve for its derivative, which is given by,

$$
h'_{(3)} = \frac{2\rho A_{(3)} + \int d\rho F_{(3)}}{\rho^2}. \tag{E.48}
$$

$$\begin{aligned}
h'_{(3,0)}(\rho) =& \frac{4761e^{-\rho}X^{-1,3}(\rho)}{1024\rho^2} + \frac{1863X^{1,2}(\rho)}{8\rho^2} - \frac{138897e^{\rho}X^{1,3}(\rho)}{512\rho^2} \\
&+ \frac{4761e^{-\rho}X^{-1,3}(\rho)}{1024\rho} + \frac{138897e^{\rho}X^{1,3}(\rho)}{512\rho} - \frac{2691e^{-2\rho}\mathrm{Ei}(-\rho)}{256\rho^3} \\
&+ \frac{2691\mathrm{Ei}(-3\rho)}{256\rho^3} - \frac{14697e^{-\rho}\mathrm{Ei}(-\rho)^2}{2048\rho^2} + \frac{273033e^{\rho}\mathrm{Ei}(-3\rho)\mathrm{Ei}(-\rho)}{1024\rho^2} \\
&- \frac{1863\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho)}{8\rho^2} - \frac{28083e^{-2\rho}\mathrm{Ei}(-\rho)}{256\rho^2} + \frac{101581e^{\rho}\mathrm{Ei}(-4\rho)}{128\rho^2} \\
&- \frac{39123\mathrm{Ei}(-3\rho)}{64\rho^2} - \frac{18469e^{-\rho}\mathrm{Ei}(-2\rho)}{256\rho^2} - \frac{14697e^{-\rho}\mathrm{Ei}(-\rho)^2}{2048\rho} \\
&- \frac{273033e^{\rho}\mathrm{Ei}(-3\rho)\mathrm{Ei}(-\rho)}{1024\rho} + \frac{207e^{-2\rho}\mathrm{Ei}(-\rho)}{64\rho} - \frac{101581e^{\rho}\mathrm{Ei}(-4\rho)}{128\rho} \\
&- \frac{483\mathrm{Ei}(-3\rho)}{64\rho} - \frac{18469e^{-\rho}\mathrm{Ei}(-2\rho)}{256\rho} + \frac{e^{-3\rho}}{\rho^4} + \frac{177e^{-3\rho}}{64\rho^3} \\
&- \frac{58529e^{-3\rho}}{256\rho^2} - \frac{5e^{-3\rho}}{4\rho},
\end{aligned} \tag{E.49}$$

$$\begin{aligned}
h'_{(3,1)}(\rho) =& -\frac{69}{64}e^{-\rho}G^{3,0}_{2,3}\left(\rho \,\middle|\, \begin{matrix} -1,-1 \\ -2,-2,-2 \end{matrix}\right) + \frac{1863}{64}e^{\rho}G^{3,0}_{2,3}\left(3\rho \,\middle|\, \begin{matrix} -1,-1 \\ -2,-2,-2 \end{matrix}\right) \\
&- \frac{69}{64}e^{-\rho}G^{3,0}_{2,3}\left(\rho \,\middle|\, \begin{matrix} 0,0 \\ -1,-1,-1 \end{matrix}\right) - \frac{621}{64}e^{\rho}G^{3,0}_{2,3}\left(3\rho \,\middle|\, \begin{matrix} 0,0 \\ -1,-1,-1 \end{matrix}\right) \\
&- \frac{69e^{-\rho}X^{-2,3}(\rho)}{64\rho^2} - \frac{69e^{-\rho}X^{-1,2}(\rho)}{32\rho^2} - \frac{621X^{-1,3}(\rho)}{64\rho^2} + \frac{207e^{\rho}X^{1,2}(\rho)}{64\rho^2} \\
&- \frac{69e^{-\rho}X^{-2,3}(\rho)}{64\rho} - \frac{69e^{-\rho}X^{-1,2}(\rho)}{32\rho} - \frac{207e^{\rho}X^{1,2}(\rho)}{64\rho} + \frac{483e^{\rho}\mathrm{Ei}(-3\rho)}{128\rho^3} \\
&- \frac{39\mathrm{Ei}(-2\rho)}{8\rho^3} + \frac{759e^{-\rho}\mathrm{Ei}(-\rho)}{128\rho^3} - \frac{1089e^{\rho}\mathrm{Ei}(-3\rho)}{128\rho^2} + \frac{1917\mathrm{Ei}(-2\rho)}{64\rho^2} \\
&- \frac{69e^{\rho}\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho)}{64\rho^2} + \frac{777e^{-\rho}\mathrm{Ei}(-\rho)}{128\rho^2} - \frac{69e^{\rho}\mathrm{Ei}(-3\rho)\log(\rho)}{32\rho^2} \\
&+ \frac{69e^{-\rho}\mathrm{Ei}(-\rho)\log(\rho)}{32\rho^2} + \frac{675e^{\rho}\mathrm{Ei}(-3\rho)}{128\rho} + \frac{7\mathrm{Ei}(-2\rho)}{2\rho} \\
&+ \frac{69e^{\rho}\mathrm{Ei}(-2\rho)\mathrm{Ei}(-\rho)}{64\rho} + \frac{225e^{-\rho}\mathrm{Ei}(-\rho)}{128\rho} + \frac{69e^{\rho}\mathrm{Ei}(-3\rho)\log(\rho)}{32\rho} \\
&+ \frac{69e^{-\rho}\mathrm{Ei}(-\rho)\log(\rho)}{32\rho} - \frac{11e^{-2\rho}}{2\rho^4} + \frac{239e^{-2\rho}}{64\rho^3} + \frac{39e^{-2\rho}\log(\rho)}{8\rho^3} \\
&+ \frac{129e^{-2\rho}}{64\rho^2} + \frac{47e^{-2\rho}\log(\rho)}{8\rho^2} - \frac{3e^{-2\rho}\log(\rho)}{2\rho},
\end{aligned} \tag{E.50}$$

$$
\begin{aligned}
h'_{(3,2)}(\rho) = & -4e^{\rho}G^{3,0}_{2,3}\left(2\rho \middle| \begin{matrix} -1,-1 \\ -2,-2,-2 \end{matrix}\right) + 2e^{\rho}G^{3,0}_{2,3}\left(2\rho \middle| \begin{matrix} 0,0 \\ -1,-1,-1 \end{matrix}\right) \\
& + 2e^{-\rho}G^{3,1}_{2,3}\left(2\rho \middle| \begin{matrix} -2,-1 \\ -2,-2,-2 \end{matrix}\right) + e^{-\rho}G^{3,1}_{2,3}\left(2\rho \middle| \begin{matrix} -1,0 \\ -1,-1,-1 \end{matrix}\right) \\
& + \frac{11e^{\rho}\mathrm{Ei}(-2\rho)}{4\rho^3} - \frac{5e^{\rho}\mathrm{Ei}(-2\rho)}{6\rho^2} + \frac{e^{\rho}\mathrm{Ei}(-2\rho)\log(\rho)}{2\rho^2} - \frac{7e^{\rho}\mathrm{Ei}(-2\rho)}{6\rho} \\
& - \frac{e^{\rho}\mathrm{Ei}(-2\rho)\log(\rho)}{2\rho} + \frac{6e^{-\rho}}{\rho^4} + \frac{25e^{-\rho}}{8\rho^3} - \frac{11e^{-\rho}\log(\rho)}{4\rho^3} - \frac{19e^{-\rho}}{12\rho^2} \\
& - \frac{e^{-\rho}\log^2(\rho)}{4\rho^2} - \frac{2e^{-\rho}\log(\rho)}{\rho^2} - \frac{e^{-\rho}\log^2(\rho)}{4\rho},
\end{aligned}
\tag{E.51}
$$

where

$$
G^{m,n}_{p,q}\left(\begin{smallmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{smallmatrix} \middle| z\right)
$$

is the Meijer G-function (see [20]) and the function X is short hand for the only integral we could not easily evaluate:

$$
X^{a,b}(\rho) = \int_1^{\rho} du \frac{e^{au}\mathrm{Ei}(\rho)}{u} - X^{a,b}. \tag{E.52}
$$

where $X^{a,b}$ are integration constants. Comparing the form of (6.15) with the terms multiplying X in the expression of A_3 and h_3, we see that the constants $X^{a,b}$ can be absorbed into the constants arising at first order, namely c_1, c_2 and c_3. This effectively means that the constants $X^{a,b}$ can be chosen so that $X^{a,b}(\rho) \to 0$ as $\rho \to \infty$ - we numerically calculated the required constants

a	b	$X^{1,b}$
1	3	-0.00086087
-1	3	-0.0102323
1	2	-0.00400426
-1	2	-0.0574777
-2	3	-0.0410508

Table E.1: Summary of first family of non-Schwarzchild black holes obtain by numerically matching near horizon data with asymptotic parameters.

Appendix F

Numerical results of the matching conditions

M/L_α	ρ_0	δ	λ/L_α	h_∞	$S/(L_\alpha)^2$	TL_α
0.619	0.448	−0.8485292670	0.294	52.200	5.960	0.006
0.619	0.450	−0.8456315816	0.178	47.400	5.950	0.007
0.616	0.470	−0.8209536967	2.370	26.500	5.850	0.010
0.615	0.490	−0.7966829072	2.390	17.900	5.760	0.012
0.614	0.500	−0.7835746355	2.330	14.900	5.710	0.014
0.609	0.550	−0.7103444309	1.720	6.980	5.410	0.022
0.601	0.600	−0.6263338362	1.600	4.010	5.070	0.031
0.591	0.640	−0.5521551177	1.480	2.840	4.760	0.040
0.577	0.680	−0.4721128117	1.280	2.140	4.420	0.048
0.559	0.720	−0.3864156113	1.090	1.690	4.060	0.057
0.536	0.760	−0.2952111978	0.838	1.380	3.670	0.065
0.508	0.800	−0.1986058208	0.553	1.160	3.260	0.074
0.491	0.820	−0.1483029212	0.448	1.070	3.040	0.078
0.474	0.840	−0.0966772370	0.254	0.992	2.820	0.083
0.455	0.860	−0.0437354265	0.094	0.924	2.600	0.087
0.445	0.870	−0.0167728192	0.042	0.894	2.480	0.089
0.434	0.880	0.0105358503	−0.022	0.865	2.370	0.092
0.412	0.900	0.0660769651	−0.135	0.813	2.130	0.096
0.388	0.920	0.1229337407	−0.248	0.766	1.890	0.100
0.349	0.950	0.2106509922	−0.416	0.705	1.510	0.107
0.321	0.970	0.2707449680	−0.527	0.669	1.250	0.111
0.276	1.000	0.3633022447	−0.689	0.622	0.859	0.118
0.243	1.020	0.4266143280	−0.794	0.594	0.588	0.122
0.191	1.050	0.5239879218	−0.949	0.556	0.171	0.129
0.094	1.100	0.6926768791	−1.200	0.502	−0.551	0.139
−0.015	1.150	0.8693454710	−1.440	0.458	−1.310	0.150
−0.138	1.200	1.0539754470	−1.690	0.421	−2.100	0.160

Table F.1: Summary of non-Schwarzschild black holes obtain by numerically matching near horizon data with asymptotic parameters.

Appendix G

Higher Derivative Black hole solutions with cosmological constant

In this chapter we demonstrate how to derive equations of motion for the higher derivative theory (2.1) including the cosmological constant. We do not fully solve these equations, but we linearize them and obtain massive and massless modes which also depend on the cosmological constant. This part of the appendix does not contribute to the entire thesis, but we realized that it can be a good exercise to see how these equations look, and differe from (2.3).

G.1 Equations of motion

We start with the following action

$$I = \int \sqrt{-g} d^4x \left(R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2 \right), \tag{G.1}$$

where, Λ is the cosmological constant, the equations of motions are given by,

$$\begin{aligned}
& G_{\mu\nu} + E_{\mu\nu} = 0, && \text{(G.2)} \\
\implies\ & G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}, && \text{(G.3)} \\
& E_{\mu\nu} = -2\alpha \left(R_{\rho\mu\nu\sigma} R^{\rho\sigma} + \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) + 2\beta R \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \\
& \quad + \alpha \left(\Box R_{\mu\nu} + \frac{1}{2} \Box R g_{\mu\nu} - \nabla_\mu \nabla_\nu R \right) + 2\beta \left(g_{\mu\nu} \Box R - \nabla_\mu \nabla_\nu R \right). && \text{(G.4)}
\end{aligned}$$

These equations of motion as expected are similar to (2.3), but include a cosmological constant term on the left hand side.

G.2 Linearized Equations of Motion using an AdS background

Our interest is on Holography, so we consider excitations around AdS_4 and expand the metric around an AdS bachbround,

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} + \delta g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \tag{G.5}$$

under AdS background, we have

$$\begin{aligned} \tilde{R}_{\mu\nu\rho\sigma} &= \frac{\Lambda}{3}(\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} - \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho}), \\ \tilde{R}_{\mu\nu} &= \Lambda \tilde{g_{\mu\nu}}, \\ \tilde{R} &= 4\Lambda. \end{aligned} \tag{G.6}$$

We now vary up to second order expansion.

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu} + O(h^2), \tag{G.7}$$

we start with the Chistoffel connection,

$$\begin{aligned} \delta\Gamma^{\lambda}_{\mu\nu} &= \tilde{\Gamma}^{\lambda}_{\mu\nu} + \delta^{(1)}\Gamma^{\lambda}_{\mu\nu} \\ &= \frac{1}{2}g^{\lambda\rho}(\nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\rho\mu} - \nabla_\rho g_{\mu\nu}) \\ &= \frac{1}{2}(\tilde{g}^{\lambda\rho} - h^{\lambda\rho})(\nabla_\mu(\tilde{g}_{\nu\rho} + h_{\nu\rho}) + \nabla_\nu(\tilde{g}_{\rho\mu} + h_{\rho\mu}) - \nabla_\rho(\tilde{g}_{\mu\nu} + h_{\mu\nu})) \\ &= \tilde{\Gamma}^{\lambda}_{\mu\nu} + \frac{1}{2}\tilde{g}^{\lambda\rho}(\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\rho\mu} - \nabla_\rho h_{\mu\nu}) + O(h^2), \end{aligned} \tag{G.8}$$

where ${}^{(1)}\Gamma^{\lambda}_{\mu\nu}$ is the first order correction of the Chistoffel connection, therefore

$$\delta^{(1)}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\lambda\rho}(\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\rho\mu} - \nabla_\rho h_{\mu\nu}). \tag{G.9}$$

The Ricci tensor

$$\begin{aligned} \delta R_{\mu\nu} &= \nabla_\alpha \Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \Gamma^{\alpha}_{\mu\alpha} + O(h^2) \\ &= \nabla_\alpha(\tilde{\Gamma}^{\alpha}_{\mu\nu} +^{(1)} \Gamma^{\alpha}_{\mu\nu}) - \nabla_\nu(\tilde{\Gamma}^{\alpha}_{\mu\alpha} +^{(1)} \Gamma^{\alpha}_{\mu\alpha}) \\ &= \nabla_\alpha\tilde{\Gamma}^{\alpha}_{\mu\nu} - \nabla_\nu\tilde{\Gamma}^{\alpha}_{\mu\alpha} + \nabla^{(1)}_\alpha \Gamma^{\alpha}_{\mu\nu} - \nabla^{(1)}_\nu \Gamma^{\alpha}_{\mu\alpha}, \end{aligned} \tag{G.10}$$

the last two terms of (G.10)are the ones contributing to the linear Ricci tensor, therefore the linear Ricci tensor becomes,

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \nabla_\alpha\left(\frac{1}{2}\tilde{g}^\alpha_\rho(\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\rho\mu} - \nabla_\rho h_{\mu\nu})\right) - \nabla_\nu\left(\frac{1}{2}\tilde{g}^{\alpha\rho}(\nabla_\mu h_{\alpha\rho} + \nabla_\alpha h_{\rho\mu} - \nabla_\rho h_{\mu\alpha})\right) \\ &= \frac{1}{2}(\nabla^\rho\nabla_\mu h_{\nu\rho} + \nabla^\rho\nabla_\nu h_{\rho\mu} - \Box h_{\mu\nu}) - \frac{1}{2}\nabla_\nu\tilde{g}^{\alpha\rho}\nabla_\mu h_{\alpha\rho} - \frac{1}{2}\nabla_\nu\nabla^\rho h_{\rho\mu} + \frac{1}{2}\nabla_\nu\nabla^\alpha h_{\mu\alpha} \\ &= \frac{1}{2}\left(\nabla^\rho\nabla_\mu h_{\nu\rho} + \nabla^\rho\nabla_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right). \end{aligned} \tag{G.11}$$

The Ricci scalar is given by,

$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= (\tilde{g}^{\mu\nu} - h^{\mu\nu})(\tilde{R}_{\mu\nu} + R^{(1)}_{\mu\nu}) \\ &= \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{g}^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}\tilde{R}_{\mu\nu} + O(h^2) \\ &= \tilde{R} + \frac{1}{2}\nabla^\rho\nabla^\nu h_{\nu\rho} + \frac{1}{2}\nabla^\rho\nabla^\nu h_{\rho\nu} - \frac{1}{2}\Box h - \frac{1}{2}\Box h - h^{\mu\nu}\Lambda\tilde{g}_{\mu\nu} + O(h^2) \\ &= \tilde{R} + \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - \Lambda h. \end{aligned} \tag{G.12}$$

Thus the linear Ricci scalar is given by,

$$R^{(1)} = \nabla^\mu\nabla^\nu h_{\mu\nu} - \Box h - \Lambda h. \tag{G.13}$$

The Einstein tensor is given by,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{G.14}$$

and it's variation is given by,

$$\begin{aligned} \delta G_{\mu\nu} &= \tilde{R}_{\mu\nu} + R^{(1)}_{\mu\nu} - \frac{1}{2}(\tilde{R} + R^{(1)})(\tilde{g}_{\mu\nu} + h_{\mu\nu}) + \Lambda(\tilde{g}_{\mu\nu} + h_{\mu\nu}) \\ &= \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} + \Lambda\tilde{g}_{\mu\nu} + R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}h_{\mu\nu} - \frac{1}{2}\tilde{R}h_{\mu\nu} - \frac{1}{2}R^{(1)}\tilde{g}_{\mu\nu} + \Lambda h_{\mu\nu}, \end{aligned} \tag{G.15}$$

therefore, the linear Einstein tensor is given by,

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\tilde{g}_{\mu\nu} - \Lambda h_{\mu\nu}. \tag{G.16}$$

Now we vary the terms of the equations of motion (G.4) containing the covariant derivative. Starting with $\Box R_{\mu\nu}$, we get,

$$\begin{aligned} \delta(\Box R_{\mu\nu}) &= \delta(\nabla^\alpha\nabla_\alpha R_{\mu\nu}) \\ &= \delta(\nabla^\alpha(\partial_\alpha R_{\mu\nu} - \Gamma^\lambda_{\alpha\mu}R_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu}R_{\mu\lambda})), \end{aligned} \tag{G.17}$$

to simplify, let

$$T_{\alpha\mu\nu} = \partial_\alpha R_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} R_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} R_{\mu\lambda}, \tag{G.18}$$

then substituting (G.18) to $\Box R_{\mu\nu}$ and expanding it we get,

$$\begin{aligned}
\Box R_{\mu\nu} &= g^{\alpha\beta} \nabla_\beta T_{\alpha\mu\nu} \\
&= g^{\alpha\beta} \left(\partial_\beta T_{\alpha\mu\nu} - \Gamma^\lambda_{\beta\alpha} T_{\lambda\mu\nu} - \Gamma^\lambda_{\beta\mu} T_{\alpha\lambda\nu} - \Gamma^\lambda_{\beta\nu} T_{\alpha\mu\lambda} \right) \\
&= g^{\alpha\beta} (\partial_\beta \left(\partial_\alpha R_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} R_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} R_{\mu\lambda} \right) \\
&- \Gamma^\lambda_{\beta\alpha} \left(\partial_\lambda R_{\mu\nu} - \Gamma^\rho_{\lambda\mu} R_{\rho\nu} - \Gamma^\rho_{\lambda\nu} R_{\mu\rho} \right) \\
&- \Gamma^\lambda_{\beta\mu} \left(\partial_\alpha R_{\lambda\nu} - \Gamma^\rho_{\alpha\lambda} R_{\rho\nu} - \Gamma^\rho_{\alpha\nu} R_{\lambda\rho} \right) \\
&- \Gamma^\lambda_{\beta\nu} \left(\partial_\alpha R_{\mu\lambda} - \Gamma^\rho_{\alpha\mu} R_{\rho\lambda} - \Gamma^\rho_{\alpha\lambda} R_{\mu\rho} \right)).
\end{aligned} \tag{G.19}$$

The variation of the first term of (G.19) and using the results of (G.17), we get,

$$\begin{aligned}
&\delta \left(\partial_\beta \left(\partial_\alpha R_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} R_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} R_{\mu\lambda} \right) \right) \\
&= \partial_\beta (\partial_\alpha (\tilde{R}_{\mu\nu} + R^{(1)}_{\mu\nu}) - (\tilde{\Gamma}^\lambda_{\alpha\mu} + \Gamma^{(1)\lambda}{}_{\alpha\mu})(\tilde{R}_{\lambda\nu} + R^{(1)}_{\lambda\nu}) \\
&- (\tilde{\Gamma}^\lambda_{\alpha\nu} + \Gamma^{(1)\lambda}{}_{\alpha\nu})(\tilde{R}_{\mu\lambda} + R^{(1)}_{\mu\lambda})) \\
&= \partial_\beta (\partial_\alpha \tilde{R}_{\mu\nu} - \tilde{\Gamma}^\lambda_{\alpha\mu} \tilde{R}_{\lambda\nu} - \tilde{\Gamma}^\lambda_{\alpha\nu} \tilde{R}_{\mu\lambda} + \partial_\alpha R^{(1)}_{\mu\nu} - \tilde{\Gamma}^\lambda_{\alpha\mu} R^{(1)}_{\lambda\nu} \\
&- \tilde{\Gamma}^\lambda_{\alpha\nu} R^{(1)}_{\mu\lambda} - \Gamma^{(1)\lambda}{}_{\alpha\mu} \tilde{R}_{\lambda\nu} - \Gamma^{(1)\lambda}{}_{\alpha\nu} \tilde{R}_{\mu\lambda}) \\
&= \partial_\beta \left(\tilde{\nabla}_\alpha \tilde{R}_{\mu\nu} + \tilde{\nabla}_\alpha R^{(1)}_{\mu\nu} - (\Gamma^{(1)\lambda}{}_{\alpha\mu} \tilde{R}_{\lambda\nu} + \Gamma^{(1)\lambda}{}_{\alpha\nu} \tilde{R}_{\mu\lambda}) \right) \\
&= \partial_\beta \left(\tilde{\nabla}_\alpha \tilde{R}_{\mu\nu} + \tilde{\nabla}_\alpha R^{(1)}_{\mu\nu} - \frac{\Lambda}{2} \tilde{\nabla}_\alpha (h_{\mu\nu} + h_{\nu\mu}) \right),
\end{aligned} \tag{G.20}$$

variation of the second term of (G.19),

$$\begin{aligned}
&\delta \left(\Gamma^\lambda_{\beta\alpha} \left(\partial_\lambda R_{\mu\nu} - \Gamma^\rho_{\lambda\mu} R_{\rho\nu} - \Gamma^\rho_{\lambda\nu} R_{\mu\rho} \right) \right) \\
&= \Gamma^\lambda_{\beta\alpha} (\partial_\lambda (\tilde{R}_{\mu\nu} + R^{(1)}_{\mu\nu}) \\
&- (\tilde{\Gamma}^\rho_{\lambda\mu} + \Gamma^{(1)\rho}{}_{\lambda\mu})(\tilde{R}_{\rho\nu} + R^{(1)}_{\rho\nu}) - (\tilde{\Gamma}^\rho_{\lambda\nu} + \Gamma^{(1)\rho}{}_{\lambda\nu})(\tilde{R}_{\rho\mu} + R^{(1)}_{\rho\mu})) \\
&= (\tilde{\Gamma}^\lambda_{\beta\alpha} + \Gamma^{(1)\lambda}{}_{\beta\alpha}) \left(\tilde{\nabla}_\lambda \tilde{R}_{\mu\nu} + \tilde{\nabla}_\lambda R^{(1)}_{\mu\nu} - (\Gamma^{(1)\rho}{}_{\lambda\mu} \tilde{R}_{\rho\nu} + \Gamma^{(1)\rho}{}_{\lambda\nu} \tilde{R}_{\mu\rho}) \right) \\
&= (\tilde{\Gamma}^\lambda_{\beta\alpha} + \Gamma^{(1)\lambda}{}_{\beta\alpha}) \left(\tilde{\nabla}_\lambda \tilde{R}_{\mu\nu} + \tilde{\nabla}_\lambda R^{(1)}_{\mu\nu} - \frac{\Lambda}{2} \tilde{\nabla}_\alpha (h_{\mu\nu} + h_{\nu\mu}) \right).
\end{aligned} \tag{G.21}$$

Variation of the third term of (G.19),

$$\begin{aligned}
&\delta \left(\Gamma^\lambda_{\beta\mu} \left(\partial_\alpha R_{\lambda\nu} - \Gamma^\rho_{\alpha\lambda} R_{\rho\nu} - \Gamma^\rho_{\alpha\nu} R_{\lambda\rho} \right) \right) \\
&= (\tilde{\Gamma}^\lambda_{\beta\mu} + \Gamma^{(1)\lambda}{}_{\beta\mu}) \left(\tilde{\nabla}_\alpha \tilde{R}_{\lambda\nu} + \tilde{\nabla}_\alpha R^{(1)}_{\lambda\nu} - \frac{\Lambda}{2} \tilde{\nabla}_\alpha (h_{\lambda\nu} + h_{\nu\lambda}) \right).
\end{aligned} \tag{G.22}$$

Variation of the fourth term of (G.19),

$$\begin{aligned}
&\delta\left(\Gamma^{\lambda}_{\beta\nu}\left(\partial_\alpha R_{\mu\lambda}-\Gamma^{\rho}_{\alpha\mu}R_{\rho\lambda}-\Gamma^{\rho}_{\alpha\lambda}R_{\rho\mu}\right)\right)\\
&=(\tilde{\Gamma}^{\lambda}_{\beta\nu}+\Gamma^{(1)}{}^{\lambda}_{\beta\nu})\left(\tilde{\nabla}_\alpha\tilde{R}_{\mu\lambda}+\tilde{\nabla}_\alpha R^{(1)}_{\mu\lambda}-\frac{\Lambda}{2}\tilde{\nabla}_\alpha(h_{\mu\lambda}+h_{\lambda\mu})\right). \qquad (G.23)
\end{aligned}$$

Finally, combining (G.20),(G.21), (G.22), and (G.23), (G.17) becomes,

$$\begin{aligned}
\delta(\Box R_{\mu\nu})&=(\tilde{g}^{\alpha\beta}-h^{\alpha\beta})(\partial_\beta(\tilde{\nabla}_\alpha\tilde{R}_{\mu\nu}+\tilde{\nabla}_\alpha R^{(1)}_{\mu\nu}-\frac{\Lambda}{2}\tilde{\nabla}_\alpha(h_{\mu\nu}+h_{\nu\mu}))\\
&-(\tilde{\Gamma}^{\lambda}_{\beta\alpha}\tilde{\nabla}_\lambda\tilde{R}_{\mu\nu}+\tilde{\Gamma}^{\lambda}_{\beta\mu}\tilde{\nabla}_\alpha\tilde{R}_{\lambda\nu}+\tilde{\Gamma}^{\lambda}_{\beta\nu}\tilde{\nabla}_\alpha\tilde{R}_{\mu\lambda})\\
&-(\tilde{\Gamma}^{\lambda}_{\beta\alpha}\tilde{\nabla}_\lambda R^{(1)}_{\mu\nu}+\tilde{\Gamma}^{\lambda}_{\beta\mu}\tilde{\nabla}_\alpha R^{(1)}_{\lambda\nu}+\tilde{\Gamma}^{\lambda}_{\beta\nu}\tilde{\nabla}_\alpha R^{(1)}_{\mu\lambda})\\
&+\Lambda\tilde{\Gamma}^{\lambda}_{\beta\alpha}\tilde{\nabla}_\lambda h_{\mu\nu}+\Lambda\tilde{\Gamma}^{\lambda}_{\beta\mu}\tilde{\nabla}_\alpha h_{\lambda\nu}+\Lambda\tilde{\Gamma}^{\lambda}_{\beta\nu}\tilde{\nabla}_\alpha h_{\mu\lambda})\\
&=(\tilde{g}^{\alpha\beta}-h^{\alpha\beta})\left(\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{R}_{\mu\nu}+\tilde{\nabla}_\beta\tilde{\nabla}_\alpha R^{(1)}_{\mu\nu}-\Lambda\tilde{\nabla}_\beta\tilde{\nabla}_\alpha h_{\mu\nu}\right)\\
&=\tilde{\Box}\tilde{R}_{\mu\nu}+\tilde{\Box}R^{(1)}_{\mu\nu}-\Lambda\tilde{\Box}h_{\mu\nu}-h^{\alpha\beta}\tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{R}_{\mu\nu}\\
&=\tilde{\Box}R^{(1)}_{\mu\nu}-\Lambda\tilde{\Box}h_{\mu\nu}. \qquad (G.24)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\delta(\nabla_\mu\nabla_\nu R)&=\tilde{\nabla}_\mu\tilde{\nabla}_\nu R^{(1)},\\
\delta(\Box R)&=\tilde{\Box}R^{(1)}. \qquad (G.25)
\end{aligned}$$

To vary other terms of (G.4), we calculate,

$$\begin{aligned}
R^{(1)}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma}&=\frac{1}{2}(\tilde{\nabla}_\mu\tilde{\nabla}_\rho h_{\sigma\nu}+\tilde{\nabla}_\mu\tilde{\nabla}_\nu h_{\sigma\rho}-\tilde{\nabla}_\mu\tilde{\nabla}_\sigma h_{\rho\nu})\Lambda\tilde{g}^{\rho\sigma}\\
&=\frac{\Lambda}{2}\left(-(\tilde{\nabla}^\sigma\tilde{\nabla}_\mu h_{\sigma\nu}+\tilde{\nabla}^\sigma\tilde{\nabla}_\nu h_{\mu\sigma})+\tilde{\nabla}_\mu\tilde{\nabla}_\nu h+\tilde{\Box}h_{\mu\nu}\right)\\
&=-\Lambda R^{(1)}_{\mu\nu}. \qquad (G.26)
\end{aligned}$$

Now we vary the term $R_{\rho\nu\sigma\mu}R^{\rho\sigma}$ we have,

$$
\begin{aligned}
&\delta(R_{\rho\nu\sigma\mu}R^{\rho\sigma}) = R_{\rho\nu\sigma\mu}g^{\rho\alpha1}g^{\sigma\alpha2}R_{\alpha1\alpha2}\\
&= R_{\rho\nu\sigma\mu}(\tilde{g}^{\rho\alpha1} - h^{\rho\alpha1})(\tilde{g}^{\sigma\alpha2} - h^{\sigma\alpha2})(\tilde{R}_{\alpha1\alpha2} + R^{(1)}_{\alpha1\alpha2})\\
&= R_{\rho\nu\sigma\mu}(\tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2} - \tilde{g}^{\rho\alpha1}h^{\sigma\alpha2} - h^{\rho\alpha1}\tilde{g}^{\sigma\alpha})(\tilde{R}_{\alpha1\alpha2} + R^{(1)}_{\alpha1\alpha2})\\
&= R_{\rho\nu\sigma\mu}(\tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}\tilde{R}_{\alpha1\alpha2} - \tilde{g}^{\rho\alpha1}h^{\sigma\alpha2}\tilde{R}_{\alpha1\alpha2} - h^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}\tilde{R}_{\alpha1\alpha2}\\
&+ \tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}R^{(1)}_{\alpha1\alpha2})\\
&= R_{\rho\nu\sigma\mu}(\tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}\tilde{R}_{\alpha1\alpha2} - \Lambda\tilde{g}^{\rho\alpha1}h^{\sigma\alpha2}\tilde{g}_{\alpha1\alpha2} - h^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}\Lambda\tilde{g}_{\alpha1\alpha2}\\
&+ \tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}R^{(1)}_{\alpha1\alpha2})\\
&= (\tilde{R}_{\rho\nu\sigma\mu} + R^{(1)}_{\rho\nu\sigma\mu})(\tilde{R}^{\rho\sigma} - \Lambda h^{\sigma\rho} - \Lambda h^{\rho\sigma} + \tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}R^{(1)}_{\alpha1\alpha2})\\
&= \tilde{R}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma} - \frac{\Lambda^2}{3}(\tilde{g}_{\rho\sigma}\tilde{g}_{\nu\mu} - \tilde{g}_{\rho\mu}\tilde{g}_{\nu\sigma})h^{\rho\sigma} - \frac{\Lambda^2}{3}(\tilde{g}_{\rho\sigma}\tilde{g}_{\nu\mu} - \tilde{g}_{\rho\mu}\tilde{g}_{\nu\sigma})h^{\sigma\rho}\\
&- \frac{\Lambda}{3}(\tilde{g}_{\rho\sigma}\tilde{g}_{\nu\mu} - \tilde{g}_{\rho\mu}\tilde{g}_{\nu\sigma})\tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2}R^{(1)}_{\alpha1\alpha2}\\
&= \tilde{R}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma} - \frac{2\Lambda^2}{3}(\tilde{g}_{\mu\nu}h - h_{\mu\nu}) + \frac{\Lambda}{3}(\tilde{g}_{\mu\nu}\tilde{g}^{\sigma\alpha2}R^{(1)}_{\sigma\alpha2} - \tilde{g}_{\nu\sigma}\tilde{g}^{\alpha2\sigma}R^{(1)}_{\mu\alpha2})\\
&+ R^{(1)}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma}\\
&= \tilde{R}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma} - \frac{2\Lambda^2}{3}(\tilde{g}_{\mu\nu}h - h_{\mu\nu}) + \frac{\Lambda}{3}(\tilde{g}_{\mu\nu}(R^{(1)} + \Lambda h) - R^{(1)}_{\mu\nu})\\
&+ R^{(1)}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma}\\
&= \tilde{R}_{\rho\nu\sigma\mu}\tilde{R}^{\rho\sigma} - \frac{2\Lambda^2}{3}(\tilde{g}_{\mu\nu}h - h_{\mu\nu}) + \frac{\Lambda}{3}(\tilde{g}_{\mu\nu}(R^{(1)} + \Lambda h) - R^{(1)}_{\mu\nu}) + \Lambda R^{(1)}_{\mu\nu}\\
&= \frac{2}{3}\Lambda R^{(1)}_{\mu\nu} + \frac{\Lambda}{3}\tilde{g}_{\mu\nu}R^{(1)} + \frac{2}{3}\Lambda h^{(1)}_{\mu\nu} - \frac{\Lambda^2}{3}\tilde{g}_{\mu\nu}h\\
&= \frac{2}{3}\Lambda R^{(1)}_{\mu\nu} + \frac{\Lambda}{3}\tilde{g}_{\mu\nu}R^{(1)} + \frac{1}{3}\Lambda h^{(1)}_{\mu\nu},
\end{aligned}
\tag{G.27}
$$

and the term $R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}$,

$$
\begin{aligned}
\delta(R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}) &= (\tilde{g}^{\rho\alpha1} - h^{\rho\alpha1})(\tilde{g}^{\sigma\alpha2} - h^{\sigma\alpha2})(\tilde{R}_{\alpha1\alpha2} + R^{(1)}_{\alpha1\alpha2})\\
&\times (\tilde{R}_{\rho\sigma} + R^{(1)}_{\rho\sigma})(\tilde{g}_{\mu\nu} + h_{\mu\nu})\\
&= (\tilde{g}^{\rho\alpha1}\tilde{g}^{\sigma\alpha2} - \tilde{g}^{\rho\alpha1}h^{\sigma\alpha2} - h^{\rho\alpha1}\tilde{g}^{\sigma\alpha2})\\
&\times (\tilde{R}_{\alpha1\alpha2}\tilde{R}_{\rho\sigma} + \tilde{R}_{\alpha1\alpha2}R^{(1)}_{\rho\sigma} + R^{(1)}_{\alpha1\alpha2}\tilde{R}_{\rho\sigma})(\tilde{g}_{\mu\nu} + h_{\mu\nu})\\
&= (\tilde{R}^{\rho\sigma}\tilde{R}_{\rho\sigma} + 2\tilde{R}^{\rho\sigma}R^{(1)}_{\rho\sigma} - 2\Lambda^2 h)(\tilde{g}_{\mu\nu} + h_{\mu\nu})\\
&= 2\Lambda\tilde{g}_{\mu\nu}(R^{(1)} + \Lambda h) - 2\Lambda^2\tilde{g}_{\mu\nu}h + 4\Lambda^2 h_{\mu\nu}\\
&= 2\Lambda\tilde{g}_{\mu\nu}R^{(1)} + 4\Lambda^2 h_{\mu\nu},
\end{aligned}
\tag{G.28}
$$

then varying the term, $2\beta\delta\left(R\left(R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}\right)\right)$ we get

$$\begin{aligned}
2\beta\delta\left(R\left(R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}\right)\right) &= 2\beta(\tilde{R}+R^{(1)}) \\
&\times\left(\tilde{R}_{\mu\nu}+R^{(1)}_{\mu\nu}-\frac{1}{4}(\tilde{R}+R^{(1)})(\tilde{g}_{\mu\nu}+h_{\mu\nu})\right) \\
&= 2\beta(\tilde{R}+R^{(1)}) \\
&\times\left(\tilde{R}_{\mu\nu}+R^{(1)}_{\mu\nu}-\frac{1}{4}(\tilde{R}\tilde{g}_{\mu\nu}+\tilde{R}h_{\mu\nu}+R^{(1)}\tilde{g}_{\mu\nu})\right) \\
&= 2\beta\tilde{R}(\tilde{R}_{\mu\nu}-\frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}) \\
&+2\beta\tilde{R}\left(R^{(1)}_{\mu\nu}-\frac{1}{4}(\tilde{R}h_{\mu\nu}+R^{(1)}\tilde{g}_{\mu\nu})\right) \\
&+2\beta R^{(1)}\left(\tilde{R}_{\mu\nu}-\frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}\right) \\
&= 2\beta\tilde{R}\left(R^{(1)}_{\mu\nu}-\frac{1}{4}(\tilde{R}h_{\mu\nu}+R^{(1)}\tilde{g}_{\mu\nu})\right),
\end{aligned} \tag{G.29}$$

and finally using the results obtained above we can vary the term, $2\beta\delta\left(g_{\mu\nu}\Box R-\nabla_\mu\nabla_\nu R\right)$ and get,

$$\begin{aligned}
2\beta\delta\left(g_{\mu\nu}\Box R-\nabla_\mu\nabla_\nu R\right) &= 2\beta\left((\tilde{g}_{\mu\nu}+h_{\mu\nu})\Box(\tilde{R}+R^{(1)})-\nabla_\mu\nabla_\nu(\tilde{R}+R^{(1)})\right) \\
&= 2\beta\left(\tilde{g}_{\mu\nu}\tilde{\Box}\tilde{R}-\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{R}\right) \\
&+2\beta\left(\tilde{g}_{\mu\nu}\tilde{\Box}R^{(1)}-\tilde{\nabla}_\mu\tilde{\nabla}_\nu R^{(1)}\right).
\end{aligned} \tag{G.30}$$

Now if we vary the higher derivative part of the equations of motion (G.4) up to first order we get,

$$
\begin{aligned}
\delta(E_{\mu\nu}) &= -2\alpha\left(\delta(R_{\rho\mu\nu\sigma}R^{\rho\sigma}) + \frac{1}{4}\delta(R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu})\right) + 2\beta\delta\left(R\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right)\right)\\
&+ \alpha\delta\left(\Box R_{\mu\nu} + \frac{1}{2}\Box R g_{\mu\nu} - \nabla_\mu\nabla_\nu R\right) + 2\beta\delta\left(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R\right)\\
&= 2\alpha\left(-2(\frac{2}{3}\Lambda R^{(1)}_{\mu\nu} + \frac{\Lambda}{3}\tilde{r}_{\mu\nu}R^{(1)} + \frac{\Lambda^2}{3}h_{\mu\nu}) - \frac{1}{4}(2\Lambda g_{\mu\nu}R^{(1)} + 4\Lambda^2 h_{\mu\nu})\right)\\
&+ 2\beta\tilde{R}(R^{(1)} - \frac{1}{4}\tilde{R}h_{\mu\nu} - \frac{1}{4}R^{(1)}\tilde{g}_{\mu\nu})\\
&+ \alpha(\tilde{\Box}R^{(1)}_{\mu\nu} - \Lambda\tilde{\Box}h_{\mu\nu} + \frac{1}{2}(\tilde{\Box}R^{(1)}(\tilde{g}_{\mu\nu} + h_{\mu\nu})) - \frac{1}{2}\tilde{\nabla}_\mu\tilde{\nabla}_\nu R^{(1)}\\
&+ 2\left(\frac{1}{2}R^{(1)} + \frac{4}{3}\Lambda R^{(1)}_{\mu\nu} - \frac{\Lambda}{3}\tilde{g}_{\mu\nu}R^{(1)} - \frac{4\Lambda^2}{3}h_{\mu\nu}\right)\\
&+ 2\beta\left((\tilde{g}_{\mu\nu} + h_{\mu\nu})\tilde{\Box}R^{(1)} - \tilde{\nabla}_\mu\tilde{\nabla}_\nu R^{(1)}\right))\\
&= 2\Lambda(\alpha + 4\beta)G^{(1)}_{\mu\nu} + \alpha\left((\tilde{\Box} - \frac{2\Lambda}{3})G^{(1)}_{\mu\nu} - \frac{2\Lambda}{3}R^{(1)}\tilde{g}_{\mu\nu}\right)\\
&+ (\alpha + 2\beta)\left(-\tilde{\nabla}_\mu\tilde{\nabla}_\nu + \tilde{g}_{\mu\nu}\tilde{\Box} + \Lambda\tilde{g}_{\mu\nu}\right)R^{(1)}. \qquad \text{(G.31)}
\end{aligned}
$$

Therefore, we have shown that if you expand the equations of motion with an AdS background, up to first order, we get

$$
\begin{aligned}
G^{(1)}_{\mu\nu} &= R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\tilde{g}_{\mu\nu} - \Lambda h_{\mu\nu}, \qquad \text{(G.32)}\\
R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\tilde{\nabla}^\rho\tilde{\nabla}_\mu h_{\nu\rho} + \tilde{\nabla}^\rho\tilde{\nabla}_\nu h_{\mu\rho} - \tilde{\Box}h_{\mu\nu} - \tilde{\nabla}_\mu\tilde{\nabla}_\nu h\right), \qquad \text{(G.33)}\\
R^{(1)} &= \tilde{\nabla}^\mu\tilde{\nabla}^\nu h_{\mu\nu} - \tilde{\Box}h - \Lambda h, \qquad \text{(G.34)}\\
(G^{(1)}_{\mu\nu} + E^{(1)}_{\mu\nu}) &= \left(1 + 2\Lambda(\alpha + 4\beta)\right)G^{(1)}_{\mu\nu} + \alpha\left((\tilde{\Box} - \frac{2\Lambda}{3})G^{(1)}_{\mu\nu} - \frac{2\Lambda}{3}R^{(1)}\tilde{g}_{\mu\nu}\right)\\
&+ (\alpha + 2\beta)\left(-\tilde{\nabla}_\mu\tilde{\nabla}_\nu + \tilde{g}_{\mu\nu}\tilde{\Box} + \Lambda\tilde{g}_{\mu\nu}\right)R^{(1)}. \qquad \text{(G.35)}
\end{aligned}
$$

Now we need to solve these equations of motion (G.35) for $h_{\mu\nu}$, there are different gauge conditions that can be used to simplified these equations to solve them analytically, and one can also use numerical methods to solve for the full set of equations of motion. In the first case we consider choosing parameters such that the massive scalar mode can be eliminated, and such that the massive spin-2 mode becomes massless. Which is a theory studied in [1]. We choose the gauge condition,

$$
\nabla^\mu h_{\mu\nu} = \nabla_\nu h. \qquad \text{(G.36)}
$$

Using this gauge condition (G.36), and dropping the tilde on the derivative of the tensors we find that, (G.32), (G.33), (G.34), and (G.35) becomes,

$$\begin{aligned}
R^{(1)} &= \Box h - \Box h - \Lambda h \\
&= -\Lambda h, & \text{(G.37)}\\
R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(2\nabla_\mu \nabla_\nu h - \Box h_{\mu\nu} - \nabla_\mu \nabla_\nu h\right) \\
&= \frac{1}{2}(\nabla_\mu \nabla_\nu h - \Box h_{\mu\nu}), & \text{(G.38)}\\
G^{(1)}_{\mu\nu} &= \frac{1}{2}(\nabla_\mu \nabla_\nu h - \Box h_{\mu\nu}) - \frac{1}{2}(-\Lambda h)\tilde{g}_{\mu\nu} - \lambda h_{\mu\nu} \\
&= -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\nabla_\mu \nabla_\nu h + \frac{\Lambda}{3} h_{\mu\nu} + \frac{1}{6}\tilde{g}_{\mu\nu} h. & \text{(G.39)}
\end{aligned}$$

Taking the trace of the equations of motion above and simplifying, we get,

$$\tilde{g}^{\mu\nu}\left(G^{(1)}_{\mu\nu} + E^{(1)}_{\mu\nu}\right) = \Lambda(h - 2(\alpha + 3\beta)\Box h) = 0 \tag{G.40}$$

We notice that h describes a propagating massive scalar mode. and from the equation above, we can have two cases.

Case 1: we set, $\alpha = -3\beta$, then we see that it implies, $h = 0$. Then,

$$\begin{aligned}
R^{(1)} &= 0, & \text{(G.41)}\\
G^{(1)}_{\mu\nu} &= -\frac{1}{2}\Box h_{\mu\nu} + \frac{\Lambda}{3} h_{\mu\nu}, & \text{(G.42)}\\
\delta(G_{\mu\nu} + E_{\mu\nu}) &= (1 + 2\Lambda(\alpha + 4\beta))\left(-\frac{1}{2}\Box h_{\mu\nu} + \frac{\Lambda}{3} h_{\mu\nu}\right) \\
&\quad + \alpha\left((\Box - \frac{2\Lambda}{3})\left(-\frac{1}{2}\Box h_{\mu\nu} + \frac{\Lambda}{3} h_{\mu\nu}\right)\right) \\
&= \left((1 + 2\Lambda(\alpha + 4\beta)) + \alpha(\Box - \frac{2\Lambda}{3})\right)\left(-\frac{1}{2}\Box h_{\mu\nu} + \frac{\Lambda}{3} h_{\mu\nu}\right) \\
&= -\frac{1}{2}\left(1 + 4\Lambda\beta - 3\beta\Box\right)\left(\Box - \frac{2\Lambda}{3}\right) h_{\mu\nu} \\
&= \frac{3\beta}{2}\left(\Box - \frac{2\Lambda}{3}\right)\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3\beta}\right) h_{\mu\nu} = 0. & \text{(G.43)}
\end{aligned}$$

In this case we note that $h_{\mu\nu}$ is in a transverse trace-less gauge, thus we can write,

$$(\Box - \frac{2\Lambda}{3})h_{\mu\nu} = 0. \tag{G.44}$$

Where $h_{\mu\nu}$ is a massless graviton, and

$$\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3\beta}\right) h_{\mu\nu} = 0, \tag{G.45}$$

describes a massive spin-2 field. In order to have stable spin-2 modes satisfying $\left(\Box - \frac{2\Lambda}{3} - M^2\right) h_{\mu\nu} = 0$ in the AdS_4 background requires that $M^2 \geq 0$, we have taken $\Lambda < 0$, we must have

$$0 < \beta \geq -\frac{1}{2\Lambda}. \tag{G.46}$$

For the critical theory [19], such that the massive spin-2 modes becomes massless, we set $\beta = -\frac{1}{2\Lambda}$. Then the equations of motion become,

$$\begin{aligned} \frac{3}{2}\left(-\frac{1}{2\Lambda}\right)\left(\Box - \frac{2\Lambda}{3}\right)\left(\Box - \frac{4\Lambda}{3} - \frac{1}{3}\left(-\frac{1}{2\Lambda}\right)\right) h_{\mu\nu} = 0 \\ \implies \left(\Box - \frac{2\Lambda}{3}\right)\left(\Box - \frac{4\Lambda}{3} + \frac{1}{6\Lambda}\right) h_{\mu\nu} = 0. \end{aligned} \tag{G.47}$$

This equation can be solved analytically and the it's in the form of,

$$h_{\mu\nu} = Ae^{-mc_0 r} + Be^{mc_2 r}, \tag{G.48}$$

where mc_0, and mc_2 depend on the cosmological constant.

Case: 2 $\alpha \neq -3\beta$, is much more complicated to solve analytically so we do not comment on it, as we have obtained a massless graviton and massive spin-2 field in the previous case.

G.3 Entropy of higher derivative AdS black holes

In this section we demostrate how one can use the Wald's formalism [9] to calculate the black hole's entropy. To do this we first define [9],

$$Q^{\mu\nu} = -2\mathcal{L}^{\mu\nu\rho\sigma}\nabla_\rho l_\sigma + \cdots. \tag{G.49}$$

where $Q^{\mu\nu}$ is an anti-symmetric tensor, and the terms denoted by ellipses vanish for stationary horizons, and where l_σ is the horizon normal defined below (G.51). In our case,

$$\mathcal{L}^{\mu\nu\rho\sigma} = \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}}, \tag{G.50}$$

for a stationary black hole, the entropy is given by,

$$S = \frac{1}{T}\int_{\mathcal{H}} Q^{\mu\nu} d\Sigma_{\mu\nu}, \tag{G.51}$$

where T is the horizon temperature, $\mathcal{H}$ is the horizon, and l^k is the horizon normal. We first demonstrate how to calculate entropy Einstein gravity, then move on to higher derivative gravity. Now for Einstein gravity, the Lagrangian is given by,

$$L = \frac{1}{2\kappa^2}\sqrt{-g}(R - 2\Lambda), \tag{G.52}$$

where $\kappa^2 = 8\pi G$, where G is the Newton constant. To calculate the Lagrangian density (G.50), we vary the Lagrangian (G.52) with respect to the Riemann tensor we get,

$$\begin{aligned}
\frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} &= \frac{\sqrt{-g}}{2\kappa^2}\frac{\delta R}{\delta R_{\mu\nu\rho\sigma}} \\
&= \frac{\sqrt{-g}}{2\kappa^2}\frac{g^{ac}g^{bd}\delta R_{abcd}}{\delta R_{\mu\nu\rho\sigma}} \\
&= \frac{\sqrt{-g}}{2\kappa^2}\frac{g^{ac}g^{bd}}{2}\frac{\delta(R_{cbad} - R_{bacd})}{\delta R_{\mu\nu\rho\sigma}} \\
&= \frac{\sqrt{-g}}{4\kappa^2} g^{ac}g^{bd}\left(\delta^c_\mu\delta^b_\nu\delta^a_\rho\delta^d_\sigma - \delta^b_\mu\delta^a_\nu\delta^c_\rho\delta^d_\sigma\right) \\
&= \frac{\sqrt{-g}}{4\kappa^2}\left(g^{ac}g^{bd}\delta^c_\mu\delta^b_\nu\delta^a_\rho\delta^d_\sigma - g^{ac}g^{bd}\delta^b_\mu\delta^a_\nu\delta^c_\rho\delta^d_\sigma\right) \\
&= \frac{\sqrt{-g}}{4\kappa^2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right).
\end{aligned} \tag{G.53}$$

Substituting (G.53) to (G.49) we get,

$$Q^{\mu\nu} = -2\frac{\sqrt{-g}}{4\kappa^2}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\nabla_\rho l_\sigma + \cdots = -\frac{\sqrt{-g}}{2\kappa^2}\left(\nabla^\mu l^\nu - \nabla^\nu l^\mu\right),$$

since $Q^{\mu\nu}$ is antisymmetric, we have,

$$\begin{aligned}
2Q^{\mu\nu} &= -\frac{\sqrt{-g}}{2\kappa^2}\left(\nabla^\mu l^\nu - \nabla^\nu l^\mu - \nabla^\mu l^\nu + \nabla^\nu l^\mu\right) \\
&= -\frac{\sqrt{-g}}{\kappa^2}\nabla^\nu l^\mu.
\end{aligned} \tag{G.54}$$

Thus the entropy (G.51) is given by,

$$S_{Einstein} = \frac{1}{\kappa^2 T}\int_{\mathcal{H}} \sqrt{-g}\left(\nabla^\nu l^\mu d\Sigma_{\mu\nu}\right) = \frac{A_h}{4G}, \tag{G.55}$$

with A_h the horizon area, using $l_\nu\nabla^\nu l^\mu = k_h l^\mu$, where $k_h = 2\pi T$ is the surface gravity of the horizon.

Calculating the entropy for the Higher derivative gravity theory we know that the Lagrangian contains the Ricci scalar and Ricci tensor squared terms R^2, and $R^{\mu\nu}R_{\mu\nu}$. So we start with calculating the variation of these terms with respect to the Riemann tensor. The calculations and results are showed below,

$$\frac{\delta R^2}{\delta R_{\mu\nu\rho\sigma}} = 2\frac{R\delta R}{\delta R_{\mu\nu\rho\sigma}} = 2R\frac{\delta(g^{ac}g^{bd}R_{abcd})}{\delta R_{\mu\nu\rho\sigma}} = 2R(g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho}), \tag{G.56}$$

$$\frac{\delta(R^{\mu\nu}R_{\mu\nu})}{\delta R_{\mu\nu\rho\sigma}} = g_{a\mu}g_{b\nu}\frac{\delta(R^{\mu\nu}R^{ab})}{\delta R_{\mu\nu\rho\sigma}}, \tag{G.57}$$

$$\frac{\delta(R^{ab}R^{cd})}{\delta R_{\mu\nu\rho\sigma}} = R^{cd}\frac{\delta R^{ab}}{\delta R_{\mu\nu\rho\sigma}} + R^{ab}\frac{\delta R^{cd}}{\delta R_{\mu\nu\rho\sigma}}, \tag{G.58}$$

$$\frac{\delta R^{ab}}{\delta R_{\mu\nu\rho\sigma}} = g^{a\theta}g^{b\beta}\frac{\delta R_{\theta\beta}}{\delta R_{\mu\nu\rho\sigma}}, \tag{G.59}$$

$$\frac{\delta R_{\theta\beta}}{\delta R_{\mu\nu\rho\sigma}} = \frac{g^{ab}}{2}\frac{\delta(R_{\beta a\theta b} - R_{a\theta\beta b})}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{2}\left(g^{\nu\sigma}\delta^{\beta}_{\mu}\delta^{\theta}_{\rho} - g^{\mu\sigma}\delta^{\theta}_{\nu}\delta^{\beta}_{\rho}\right), \tag{G.60}$$

$$\frac{\delta R^{ab}}{\delta R_{\mu\nu\rho\sigma}} = \frac{g^{a\theta}g^{b\beta}}{2}(g^{\nu\sigma}\delta^{\beta}_{\mu}\delta^{\theta}_{\rho} - g^{\mu\sigma}\delta^{\theta}_{\nu}\delta^{\beta}_{\rho}) = \frac{1}{2}(g^{a\rho}g^{b\mu}g^{\nu\sigma} - g^{a\nu}g^{a\nu}g^{\mu\sigma}), \tag{G.61}$$

$$\frac{\delta R^{cd}}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{2}\left(g^{c\rho}g^{d\mu}g^{\nu\sigma} - g^{c\nu}g^{d\rho}g^{\mu\sigma}\right), \tag{G.62}$$

$$\frac{\delta\left(R^{ab}R^{cd}\right)}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{2}R^{cd}(g^{a\rho}g^{b\mu}g^{\nu\sigma} - g^{a\nu}g^{a\nu}g^{\mu\sigma}) + \frac{1}{2}R^{ab}\left(g^{c\rho}g^{d\mu}g^{\nu\sigma} - g^{c\nu}g^{d\rho}g^{\mu\sigma}\right), \tag{G.63}$$

using the abover results, we can calculate the term $\frac{\delta(R^{\mu\nu}R_{\mu\nu})}{\delta R_{\mu\nu\rho\sigma}}$,

$$\begin{aligned}
\frac{\delta(R^{\mu\nu}R_{\mu\nu})}{\delta R_{\mu\nu\rho\sigma}} &= g_{ac}g_{bd}\frac{\delta(R^{ab}R^{cd})}{\delta R_{\mu\nu\rho\sigma}} \\
&= \frac{1}{2}g_{ac}g_{bd}R^{cd}\left(g^{a\rho}g^{b\mu}g^{\nu\sigma} - g^{a\nu}g^{b\rho}g^{\mu\sigma}\right) \\
&+ \frac{1}{2}g_{ac}g_{bd}R^{ab}\left(g^{c\rho}g^{d\mu}g^{\nu\sigma} - g^{c\nu}g^{d\rho}g^{\mu\sigma}\right) \\
&= \frac{1}{2}\left(R^{cd}\delta^{\rho}_{c}\delta^{\mu}_{d}g^{\nu\sigma} - R^{cd}\delta^{\nu}_{c}\delta^{\rho}_{d}g^{\mu\sigma}\right) \\
&+ \frac{1}{2}\left(R^{ab}\delta^{\rho}_{a}\delta^{\mu}_{b}g^{\nu\sigma} - R^{ab}\delta^{\nu}_{a}\delta^{\rho}_{b}g^{\mu\sigma}\right) \\
&= \left(R^{\rho\mu}g^{\nu\sigma} - R^{\nu\rho}g^{\mu\sigma}\right).
\end{aligned} \tag{G.64}$$

Recall that the Lagrangian of (G.1) is of the form,

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}\left(R - 2\Lambda + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2\right) \tag{G.65}$$

and using the results above, the variation of (G.65) with respect to the Riemann tensor is given by,

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} &= \frac{\delta L_E}{\delta R_{\mu\nu\rho\sigma}} + \frac{\alpha}{2\kappa^2}\sqrt{-g}\frac{\delta(R^{\mu\nu}R_{\mu\nu})}{\delta R_{\mu\nu\rho\sigma}} + \frac{1}{2\kappa^2}\beta\sqrt{-g}\frac{\delta R^2}{\delta R_{\mu\nu\rho\sigma}}\\ &= \frac{\sqrt{-g}}{4\kappa^2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + \frac{\alpha\sqrt{-g}}{2\kappa^2}(R^{\rho\mu}g^{\nu\sigma} - R^{\nu\rho}g^{\mu\sigma})\\ &+ \frac{\beta\sqrt{-g}}{2\kappa^2}R(g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho}),\end{aligned} \tag{G.66}$$

Therefore, substituting (G.66) to (G.49), we get,

$$\begin{aligned}Q^{\mu\nu} &= -2\frac{\sqrt{-g}}{4\kappa^2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\nabla_\rho l_\sigma\\ &- 2\frac{\alpha\sqrt{-g}}{2\kappa^2}(R^{\rho\mu}g^{\nu\sigma} - R^{\nu\rho}g^{\mu\sigma})\nabla_\rho l_\sigma\\ &- 2\frac{\beta\sqrt{-g}}{2\kappa^2}R(g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho})\nabla_\rho l_\sigma\\ &= -\frac{\sqrt{-g}}{2\kappa^2}(\nabla^\mu l^\nu - \nabla^\nu l^\mu)\\ &- \frac{\alpha\sqrt{-g}}{\kappa^2}(R^{\rho\mu}g^{\mu}\nabla_\rho l^\nu - R^{\nu\rho}\nabla^\rho l^\mu)\\ &- \frac{\beta\sqrt{-g}}{\kappa^2}R(\nabla^\mu l^\nu - \nabla^\nu l^\mu).\end{aligned} \tag{G.67}$$

For the AdS bachground, the Ricci tensor is given by, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and thus (G.67) simplifies to,

$$\begin{aligned}Q^{\mu\nu} &= -\frac{\sqrt{-g}}{2\kappa^2}(\nabla^\mu l^\nu - \nabla^\nu l^\mu)\\ &- 2\alpha\Lambda\frac{\sqrt{-g}}{2\kappa^2}(\nabla^\mu l^\nu - \nabla^\nu l^\mu)\\ &- 8\beta\Lambda\frac{\sqrt{-g}}{2\kappa^2}(\nabla^\mu l^\nu - \nabla^\nu l^\mu)\\ &(1 + 2\alpha\Lambda + 8\beta\Lambda)Q^{\mu\nu}_{Einstein},\end{aligned} \tag{G.68}$$

then finally the the entropy of higher derivative black holes with cosmological constant is given by,

$$S = (1 + 2\alpha\Lambda + 8\beta\Lambda)\frac{A_h}{4G}. \tag{G.69}$$

We note that (G.69) is similar to (5.1), if we set $\beta = 0$. This gives us an indication that if we follow the same process we followed in chapter 4 of solving the equations of motion (G.3) and (G.4) numerically and find the "δ^*" for these black holes, one can be able to determine higher derivative non-Schwarzschild *AdS* black holes, and their entropy would be of the form,

$$S = \pi r_0^2 + 2\alpha\Lambda - 4\pi\alpha f(\Lambda)\delta^*, \tag{G.70}$$

where $f(\Lambda)$ can be determined numerically. As mentioned before, that in this thesis we decided not to continue with this investigation. It is left for future research.

Appendix H

Review of the non-supersymmetric attractors

H.1 Derivation of the equation of motion

Varying the action (8.1) with the metric we get equations of motion (8.2),

$$
\begin{aligned}
S &= \frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu})\right), \\
\implies \delta S &= \delta\frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu})\right), \\
\implies \delta S &= \frac{1}{\kappa^2}\int d^4x\left(-\frac{1}{2}\sqrt{-G}G_{\mu\nu}L\right)\delta G^{\mu\nu} \\
&+ \frac{1}{\kappa^2}\int d^4x\left(\delta R - 2\delta\partial_\alpha\phi_i\partial^\alpha\phi_i - f_{ab}(\phi_i)\delta G^{\alpha\alpha_1}G^{\beta\beta_1}F^a_{\alpha\beta}F^b_{\alpha_1\beta_1}\right) \\
&= \frac{1}{\kappa^2}\int d^4x\left(-\frac{1}{2}\sqrt{-G}G_{\mu\nu}L\right)\delta G^{\mu\nu} \\
&+ \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R_{\mu\nu} + \nabla(\cdots) - 2\delta G^{\alpha\beta}\partial_\alpha\phi_i\partial_\beta\phi_i \\
&- f_{ab}(\phi_i)\delta G^{\alpha\alpha_1}G^{\beta\beta_1}F^a_{\alpha\beta}F^b_{\alpha_1\beta_1}) \\
&= \frac{1}{\kappa^2}\int d^4x\left(-\frac{1}{2}\sqrt{-G}G_{\mu\nu}L\right)\delta G^{\mu\nu} \\
&+ \frac{1}{\kappa^2}\int d^4x((R_{\mu\nu} + \nabla(\cdots))\delta G^{\mu\nu} - \delta^{\alpha\beta}_{\mu\nu}\partial_\alpha\phi_i\partial_\beta\phi_i \\
&- f_{ab}(\phi_i)(\delta^{\alpha\alpha_1}_{\mu\nu}G^{\beta\beta_1}F^a_{\alpha\beta}F^b_{\alpha_1\beta_1} + G^{\alpha\alpha_1}\delta^{\beta\beta_1}_{\mu\nu}F^a_{\alpha\beta}F^b_{\alpha_1\beta_1}) \\
&= -\frac{1}{\kappa^2}\int d^4x\sqrt{-G}\frac{1}{2}G_{\mu\nu}L\delta G^{\mu\nu} \\
&+ \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i - 2f_{ab}(\phi_i)F^a_{\mu\lambda}F^{b\lambda}_{\nu})\delta G^{\mu\nu} = 0. \qquad \text{(H.1)}
\end{aligned}
$$

The trace of the resulting equations of motion (8.2) is given by,

$$\begin{aligned} G^{\mu\nu}R_{\mu\nu} - 2G^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i &= 2f_{ab}(\phi_i)G^{\mu\nu}F^a_{\mu\lambda}F^{b\lambda}_\nu + \frac{1}{2}G^{\mu\nu}G_{\mu\nu}L, \\ \implies R - 2(\partial\phi_i)^2 &= 2f_{ab}(\phi_i F^a_{\mu\nu}F^{b\mu\nu} + 2L, \\ \implies R - 2(\partial\phi_i)^2 &= 2R - 4(\partial\phi_i)^2, \\ \implies R - 2(\partial\phi_i)^2 &= 0. \end{aligned} \tag{H.2}$$

Now we vary the action (8.1) with the scalar fields ϕ_i's,

$$\begin{aligned} \delta S = \frac{1}{\kappa^2}\int d^4x\sqrt{-G}(0 - 2\delta\partial_\mu\phi_i\partial^\mu\phi_i - \partial_i f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu}) = 0, \\ \implies -\partial_\mu\partial^\mu\phi_i = -\partial_i f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu}, \\ \implies \frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}\partial^\mu\phi_i) = \frac{1}{4}\partial_i f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu}. \end{aligned} \tag{H.3}$$

The tt, $\theta\theta$, and rr components of the equations of motion (8.2) are given by,

$$\begin{aligned} R_{tt} &= f_{ab}(\phi_i)\left(2G^{\lambda\beta}F^a_{t\lambda}F^b_{t\beta} - \frac{1}{2}G_{tt}F^a_{\kappa\lambda}F^{b\kappa\lambda}\right) \\ &= f_{ab}(\phi_i)\left(-\frac{1}{2}G_{tt}F^a_{\kappa\lambda}F^{b\kappa\lambda}\right) \\ &= -\frac{1}{2}f_{ab}(\phi_i)G_{tt}\left(F^a_{\theta\phi}F^{b\theta\phi} + F^a_{\phi\theta}F^{b\phi\theta}\right) \\ &= -\frac{1}{2}f_{ab}(\phi_i)G_{tt}\left(2F^a_{\theta\phi}F^{b\theta\phi}\right) \\ &= -f_{ab}(\phi_i)G_{tt}F^a_{\theta\phi}G^{\theta\theta}G^{\phi\phi}F^b_{\theta\phi} \\ &= f_{ab}(\phi_i)\frac{a^2(r)}{b^4(r)}Q^a_mQ^b_m \\ &= \frac{a^2(r)}{b^4(r)}V_{eff}(\phi_i), \end{aligned} \tag{H.4}$$

$$\begin{aligned} R_{\theta\theta} &= f_{ab}(\phi_i)\left(2G^{\lambda\beta}F^a_{\theta\lambda}F^b_{\theta\beta} - \frac{1}{2}G_{\theta\theta}F^a_{\kappa\lambda}F^{b\kappa\lambda}\right) \\ &= f_{ab}(\phi_i)\left(2G^{\phi\phi}F^a_{\theta\phi}F^b_{\theta\phi} - \frac{1}{2}G_{\theta\theta}\frac{2}{b^4(r)}Q^a_mQ^b_m\right) \\ &= \frac{f_{ab}(\phi_i)}{b^2(r)}(2Q^a_mQ^b_m - Q^a_mQ^b_m) \\ &= \frac{1}{b^2(r)}V_{eff}(\phi_i), \end{aligned} \tag{H.5}$$

$$
\begin{aligned}
R_{rr} - 2\partial_r\phi_i\partial_r\phi_i &= f_{ab}(\phi_i)\left(0 - \frac{1}{2}G_{rr}F^a_{\kappa\lambda}F^{b\kappa\lambda}\right) \\
&= -\frac{1}{2}f_{ab}(\phi_i)G_{rr}\frac{2}{b^4}Q^a_mQ^b_m \\
&= -\frac{G_{rr}}{b^4}V_{eff}(\phi_i) \\
\implies R_{rr} - 2(\partial_r\phi_i)^2 &= -\frac{1}{a^2b^4}V_{eff}(\phi_i). \qquad \text{(H.6)}
\end{aligned}
$$

From the metric (8.5), the Ricci components are given by,

$$
R_{tt} = a^2\left(a'^2 + \frac{2aa'b'}{b} + aa''\right), \qquad \text{(H.7)}
$$

$$
R_{rr} = -\frac{1}{a^2b}\left(b(a'^2 + aa'') + 2a(a'b' + ab'')\right), \qquad \text{(H.8)}
$$

$$
R_{\theta\theta} = 1 - 2aba'b' - a^2(b'^2 + bb''). \qquad \text{(H.9)}
$$

The relation $R_{tt} - \frac{a^2}{b^2}R_{\theta\theta}$ gives the equation of motion (8.11),

$$
\begin{aligned}
R_{tt} - \frac{a^2}{b^2}R_{\theta\theta} &= a^2\left(a'^2 + \frac{2aa'b'}{b} + aa''\right) - \frac{a^2}{b^2}\left(1 - 2aba'b' - a^2(b'^2 + bb'')\right), \\
\implies a'^2b^2 &+ 2aba'b' + b^2aa'' + 2aba'b' + a^2(b'^2 + bb'') = 1, \\
\implies 2a'^2b^2 &+ 8aba'b' + 2b^2aa'' + 2a^2(b'^2 + bb'') = 2, \\
\implies (a^2b^2)'' &= 2. \qquad \text{(H.10)}
\end{aligned}
$$

The relation $R_{rr} - \frac{G^{tt}}{G^{rr}}R^{tt}$ and also using the metric components gives the equation of motion (8.12),

$$
\begin{aligned}
R_{rr} - \frac{G^{tt}}{G^{rr}}R^{tt} &= 2(\partial_r\phi_i)^2 - \frac{1}{a^2b^4}V_{eff}(\phi_i) + \frac{1}{a^2b^4}V_{eff}(\phi_i), \\
&= 2(\partial_r\phi_i)^2, \\
\implies 2(\partial_r\phi_i)^2 &= \frac{a^{-2}a^2}{a^2}\left(a'^2 + \frac{2aa'b'}{b} + aa''\right) \\
&\quad - \frac{1}{a^2b}\left(b(a'^2 + aa'') + 2a(a'b' + ab'')\right), \\
\implies 2a^2b(\partial_r\phi_i)^2 &= a'^2b + 2aa'b' + aba'' - b(a'^2 + aa'') - 2a(a'b' + ab''), \\
\implies -2a^2b'' &= 2a^2b(\partial_r\phi_i)^2, \\
\implies \frac{b''}{b} &= -(\partial_r\phi_i)^2. \qquad \text{(H.11)}
\end{aligned}
$$

The R_{rr} component itself gives the energy constraint (8.13),

$$\begin{aligned} R_{rr} - 2(\partial_r\phi_i)^2 &= -\frac{1}{a^2b^4}V_{eff}(\phi_i) \\ \implies -\frac{1}{a^2b}\left(b(a'^2+aa'')+2a(a'b'+ab'')\right) - 2(\partial_r\phi_i)^2 &= -\frac{1}{a^2b^4}V_{eff}(\phi_i) \\ \implies b^2(a'^2+aa'')+2ab(a'b'+ab'') &= -2a^2b^2(\phi_i^{'})^2 + \frac{1}{b^2}V_{eff}(\phi_i), \end{aligned} \tag{H.12}$$

from (8.11), we know that,

$$a'^2b^2 + aa''b^2 + 4aba'b' + a^2b'^2 + a^2bb'' = 1, \tag{H.13}$$

and subtracting (H.12) from (H.13) and using the fact that $\frac{(a')^2(b')^2}{2} = 2aba'b'$ and also eliminating b'' using (8.12), we get,

$$\begin{aligned} 2aba'b' + a^2b'^2 - a^2bb'' &= 1 + 2a^2b^2(\phi_i^{'})^2 - \frac{1}{b^2}V_{eff}(\phi_i) \\ \implies \frac{(a')^2(b')^2}{2} + a^2b'^2 + a^2b^2(\phi_i^{'})^2 &= 1 + 2a^2b^2(\phi_i^{'})^2 - \frac{1}{b^2}V_{eff}(\phi_i) \\ \implies -1 + a^2b'^2 + \frac{(a')^2(b')^2}{2} &= -\frac{1}{b^2}V_{eff}(\phi_i) + a^2b^2(\phi_i^{'})^2, \end{aligned} \tag{H.14}$$

$V_{eff}(\phi_i)$ represents the "effective potential" for the scalar field including both electric and magnetic charges.

Now We consider a theory of the form (8.15).

Varying the action (8.15) with respect to the metric we get,

$$
\begin{aligned}
\delta S = &\frac{1}{\kappa^2}\int d^4x\sqrt{-G}(\delta R - 2\delta(\partial\phi_i)^2 - f_{ab}(\phi_i)\delta(F^a_{\alpha\beta}F^{b\alpha\beta})\\
&-\frac{1}{2}\tilde{f_{ab}}\delta(F^a_{\alpha\beta}F^b_{\rho\sigma}\epsilon^{\alpha\beta\rho\sigma}))\\
&+\frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(-\frac{1}{2}G_{\mu\nu}L\right)\\
&=\frac{1}{\kappa^2}\int d^4x\sqrt{-G}(\delta R - 2\delta\partial_\alpha\phi_i\partial^\alpha\phi_i - f_{ab}(\phi_i)\delta\left(G^{\alpha\gamma}G^{\beta\tau}F^a_{\alpha\beta}F^b_{\gamma\tau}\right)\\
&-\frac{1}{2}\tilde{f_{ab}}\delta\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right))\\
&-\frac{1}{\kappa^2}\int d^4x\sqrt{-G}\left(\frac{1}{2}G_{\mu\nu}L\right)\\
&=\frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R_{\mu\nu} + boundaryterms - 2f_{ab}(\phi_i)F^a_{\mu\beta}G^{\beta\gamma}F^b_{\nu\gamma}\\
&-\frac{1}{2}\tilde{f_{ab}}(F^a_{\alpha\beta}F^b_{\rho\sigma}\delta^{\alpha\alpha'}_{\mu\nu}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'} + F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}\delta^{\beta\beta'}_{\mu\nu}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\\
&+F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}\delta^{\rho\rho'}_{\mu\nu}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'} + F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\delta^{\sigma\sigma'}_{\mu\nu}\epsilon_{\alpha'\beta'\rho'\sigma'}))\\
&-\frac{1}{\kappa^2}\sqrt{-G}\frac{1}{2}G_{\mu\nu}L\\
&=\frac{1}{\kappa^2}\int d^4x\sqrt{-G}(R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i + boundaryterms - 2f_{ab}(\phi_i)F^a_{\mu\beta}G^{\beta\gamma}F^b_{\nu\gamma}\\
&-\frac{1}{2}\tilde{f_{ab}}(F^a_{\mu\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\nu\beta'\rho'\sigma'} + F^a_{\alpha\mu}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\nu\rho'\sigma'}\\
&+F^a_{\alpha\beta}F^b_{\mu\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\nu\sigma'} + F^a_{\alpha\beta}F^b_{\rho\mu}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho'\nu}))\\
&-\frac{1}{\kappa^2}\int d^4x\sqrt{-G}\frac{1}{2}G_{\mu\nu}L\\
&=0.
\end{aligned}
\tag{H.15}
$$

Let's start by looking at the Lagrangian,

$$
\begin{aligned}
L &= R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)F^a_{\mu\nu}F^{b\mu\nu} - \frac{1}{2}\tilde{f_{ab}}(\phi_i)F^a_{\mu\nu}F^b_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\\
&= R - 2(\partial\phi_i)^2 - f_{ab}(\phi_i)\left(G^{\alpha\alpha'}G^{\beta\beta'}F^a_{\alpha\beta}F^b_{\alpha'\beta'}\right)\\
&-\frac{1}{2}\tilde{f_{ab}}(\phi_i)\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right).
\end{aligned}
\tag{H.16}
$$

Therefore, the equations of motion for the metric are given by

$$
\begin{aligned}
&R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i - 2f_{ab}(\phi_i)F^a_{\mu\beta}G^{\beta\gamma}F^b_{\nu\gamma} - \frac{1}{2}G_{\mu\nu}L \\
&- \frac{1}{2}\tilde{f_{ab}}(F^a_{\mu\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\nu\beta'\rho'\sigma'} + F^a_{\alpha\mu}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\nu\rho'\sigma'} \\
&+ F^a_{\alpha\beta}F^b_{\mu\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\nu\sigma'} + F^a_{\alpha\beta}F^b_{\rho\mu}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho'\nu})) \\
&= R_{\mu\nu} - 2\partial_\mu\phi_i\partial_\nu\phi_i - 2f_{ab}(\phi_i)F^a_{\mu\beta}G^{\beta\gamma}F^b_{\nu\gamma} - \frac{1}{2}G_{\mu\nu}(R - 2(\partial\phi_i)^2 \\
&- f_{ab}(\phi_i)\left(G^{\alpha\alpha'}G^{\beta\beta'}F^a_{\alpha\beta}F^b_{\alpha'\beta'}\right) \\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right)) \\
&- \frac{1}{2}\tilde{f_{ab}}(F^a_{\mu\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\nu\beta'\rho'\sigma'} + F^a_{\alpha\mu}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\nu\rho'\sigma'} \\
&+ F^a_{\alpha\beta}F^b_{\mu\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\nu\sigma'} + F^a_{\alpha\beta}F^b_{\rho\mu}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho'\nu})) \\
&= 0.
\end{aligned} \tag{H.17}
$$

Taking the trace of the equations of motion (H.17),

$$
R - 2(\partial\phi_i)^2 + \tilde{f_{ab}}(\phi_i)F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'} = 0. \tag{H.18}
$$

Commenting on the trace,
we notice that since we have turned on both electric and magnetic charges in our theory, then the Ricci scalar depends on both the scalar fields and the charges, unlike in the first theory where the Ricci scalar just depended on the scalar fields.

To caclulate the equations of motion in the form of metric components, we do what we did for the theory (8.2), calculate the $t-t$, $r-r$, and $\theta-\theta$ components of the equations of motion (8.16).

$$
\begin{aligned}
&R_{tt} - 2\partial_t\phi_i\partial_t\phi_i - 2f_{ab}(\phi_i)F^a_{t\beta}G^{\beta\gamma}F^b_{t\gamma} - \frac{1}{2}G_{tt}(R - 2(\partial\phi_i)^2 \\
&- f_{ab}(\phi_i)\left(G^{\alpha\alpha'}G^{\beta\beta'}F^a_{\alpha\beta}F^b_{\alpha'\beta'}\right) \\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right)) \\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)(F^a_{t\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{t\beta'\rho'\sigma'} + F^a_{\alpha t}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha' t\rho'\sigma'} \\
&+ F^a_{\alpha\beta}F^b_{t\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta' t\sigma'} + F^a_{\alpha\beta}F^b_{\rho t}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho' t})) \\
&= R_{tt} - 2f_{ab}(\phi_i)F^a_{t\beta}G^{\beta\gamma}F^b_{t\gamma} \\
&- \frac{1}{2}G_{tt}(2(\partial\phi_i)^2 - \tilde{f_{ab}}(\phi_i)F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'} - 2(\partial\phi_i)^2 \\
&- f_{ab}(\phi_i)\left(G^{\alpha\alpha'}G^{\beta\beta'}F^a_{\alpha\beta}F^b_{\alpha'\beta'}\right) \\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)\left(F^a_{\alpha\beta}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta'\rho'\sigma'}\right)) \\
&- \frac{1}{2}\tilde{f_{ab}}(\phi_i)(F^a_{t\beta}F^b_{\rho\sigma}G^{\beta\beta'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{t\beta'\rho'\sigma'} + F^a_{\alpha t}F^b_{\rho\sigma}G^{\alpha\alpha'}G^{\rho\rho'}G^{\sigma\sigma'}\epsilon_{\alpha' t\rho'\sigma'} \\
&+ F^a_{\alpha\beta}F^b_{t\sigma}G^{\alpha\alpha'}G^{\beta\beta'}G^{\sigma\sigma'}\epsilon_{\alpha'\beta' t\sigma'} + F^a_{\alpha\beta}F^b_{\rho t}G^{\alpha\alpha'}G^{\beta\beta'}G^{\rho\rho'}\epsilon_{\alpha'\beta'\rho' t})) \\
&= R_{tt} + f_{ab}(\phi_i)\left(-2G^{rr}F^a_{tr}F^b_{tr} + G^{rr}F^a_{tr}F^b_{tr} + G_{tt}G^{\theta\theta}G^{\phi\phi}F^a_{\theta\phi}F^b_{\theta\phi}\right) \\
&= R_{tt} + f_{ab}(\phi_i)\left(G_{tt}G^{\theta\theta}G^{\phi\phi}F^a_{\theta\phi}F^b_{\theta\phi} - G^{rr}F^a_{tr}F^b_{tr}\right) \\
&= R_{tt} + f_{ab}(\phi_i)\left(-a^2b^{-2}b^{-2}sin^{-2}\theta Q^a_m Q^{ab}_m sin^2\theta - \frac{a^2}{b^4}f^{ab}(\phi_i)\left(Q_{eb} - \tilde{f_{bc}}Q^c_m\right)f^{bc}(\phi_i)\left(Q_{fc} - \tilde{f_{cd}}Q^c_m\right)\right) \\
&= R_{tt} + f_{ab}(\phi_i)\left(-\frac{a^2}{b^4}f^{ab}(\phi_i)Q^a_m Q^{ab}_m - \frac{a^2}{b^4}f^{ab}(\phi_i)\left(Q_{eb} - \tilde{f_{bc}}Q^c_m\right)f^{bc}(\phi_i)\left(Q_{eb} - \tilde{f_{cd}}Q^c_m\right)\right) \\
&= 0. \qquad \text{(H.19)}
\end{aligned}
$$

Therefore, we see that,

$$R_{tt} = \frac{a^2}{b^4}V_{eff}(\phi_i), \qquad \text{(H.20)}$$

where,

$$V_{eff}(\phi_i) = f_{ab}(\phi_i)Q^a_m Q^{ab}_m + f^{bc}(\phi_i)\left(Q_{eb} - \tilde{f_{bc}}Q^c_m\right)\left(Q_{fc} - \tilde{f_{cd}}Q^c_m\right). \qquad \text{(H.21)}$$

Similarly one can calculate the other components of the Ricci tensor. And all the components come out as befor with the effective potential being given by the above equation.

H.2 Second Order Perturbation Solution

In second order perturbation theory we define,

$$
\begin{aligned}
b &= b_0 + \epsilon^2 b_2 \\
a^2 &= a_0^2 + \epsilon^2 a_2 \\
b^2 &= b_0^2 + 2\epsilon^2 b_2 b_0,
\end{aligned} \tag{H.22}
$$

substituting (H.22) to the equation of motion for scalars (8.14) we get,

$$
\begin{aligned}
&\partial_r(a^2 b^2 \partial_r \partial_i \phi_i) = \frac{\partial_i V_{eff}}{2b^2}, \\
&\Longrightarrow \partial_r\left((a_0^2 + \epsilon^2 a_2)(b_0^2 + 2\epsilon^2 b_2 b_0)\partial_r(\phi_{i0} + \epsilon\phi_{i1} + \epsilon^2\phi_{i2}^2)\right) = \frac{\partial_i V_{eff}}{2b^2}, \\
&\Longrightarrow (b_0^2 + 2\epsilon^2 b_2 b_0)\partial_r\left(a_0^2 b_0^2 \partial_r \phi_{i1}\epsilon + a_0^2 b_0^2 \partial_r \phi_{i2}\epsilon^2\right) = \frac{\partial_i V_{eff}}{2}, \\
&\Longrightarrow b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i1})\epsilon + 2b_2 b_0 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i1})\epsilon^3 + b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i2})\epsilon^2 \\
&\quad = \frac{\partial_i\left(V_{eff}(\phi_{i0} + \epsilon\phi_{i1} + \epsilon^2\phi_{i2}^2)\right)}{2}, \\
&\Longrightarrow b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i1})\epsilon + b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i2})\epsilon^2 = \frac{\partial_i\left(V_{eff}(\phi_{i0} + \epsilon\phi_{i1} + \epsilon^2\phi_{i2}^2)\right)}{2}.
\end{aligned} \tag{H.23}
$$

the right hand side of (H.23) can be simplifies to,

$$
\begin{aligned}
\partial_i\left(V_{eff}(\phi_{i0} + \epsilon\phi_{i1} + \epsilon^2\phi_{i2}^2)\right) &= \partial_i\left(V_{eff}(\phi_{i0} + \epsilon\phi_{i1}) + \partial_j V_{eff}(\phi_{i0} + \epsilon\phi_{i1})\epsilon^2\phi_{i2}\right) \\
&= \partial_i V_{eff}(\phi_{i0} + \epsilon\phi_{i1}) + \partial_i\partial_j V_{eff}(\phi_{i0} + \epsilon\phi_{i1})\epsilon^2\phi_{i2} \\
&= \partial_i\left(V_{eff}(\phi_{i0}) + \epsilon\partial_k V_{eff}(\phi_{i0})\phi_{i1} + \epsilon^2\partial_f V_{eff}(\phi_{i0})\phi_{i1}^2\right) \\
&+ \partial_i\partial_j\left(V_{eff}(\phi_{i0}) + \epsilon\partial_g V_{eff}(\phi_{i0})\phi_{i1} + \epsilon^2\partial_h V_{eff}(\phi_{i0})\phi_{i1}^2\right)\epsilon^2\phi_{i2} \\
&= 0 + \epsilon\partial_i\partial_k V_{eff}(\phi_{i0})\phi_{i1} + \epsilon^2\partial_i\partial_f V_{eff}(\phi_{i0})\phi_{i1}^2 \\
&+ \partial_i\partial_j V_{eff}(\phi_{i0})\epsilon^2\phi_{i2} + O(\epsilon^3),
\end{aligned} \tag{H.24}
$$

the left hand side of (H.23) can be simplified to,

$$
\begin{aligned}
b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i1})\epsilon + b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i2})\epsilon^2 = \frac{1}{2}(&\epsilon\partial_i\partial_k V_{eff}(\phi_{i0})\phi_{i1} \\
&+ (\partial_i\partial_f V_{eff}(\phi_{i0})\phi_{i1}^2 \\
&+ \partial_i\partial_j V_{eff}(\phi_{i0})\phi_{i2})\epsilon^2)),
\end{aligned} \tag{H.25}
$$

for $O(\epsilon)$ we get,

$$
b_0^2 \partial_r(a_0^2 b_0^2 \partial_r \phi_{i1}) = \frac{\beta_i^2 \phi_{i1}}{b_0^2}, \tag{H.26}
$$

and for $O(\epsilon^2)$ we get,

$$\partial_r(a_0^2 b_0^2 \partial_r \phi_{i2}) = \frac{\beta_i^2 \phi_{i1}^2}{b_0^2} + \frac{\beta_j^2}{b_0^2}\phi_{i2}. \tag{H.27}$$

We previously solved for ϕ_{i1}, now we substitute that solution and solve for ϕ_{i2}. Looking at the equations of motion (8.11),

$$\begin{aligned}(a^2b^2)'' &= 2\\ (a^2b^2)' &= 2r + d_1\\ a^2b^2 &= (r - r_H)^2 + d_1 r + d_2,\end{aligned} \tag{H.28}$$

where d_1, d_2, are the integration constants. In this case we are interested in extremal black hole solutions with vanishing surface gravity. At the horizon, b is finite and a^2, and $(a^2)'$ has a double zero. Substituting (H.22), to (H.28), we get,

$$\begin{aligned}&(a_0^2 + \epsilon_2^2)(b_0^2 + 2\epsilon^2 b_2 b_0) = (r - r_H)^2 + +d_1 r + d_2,\\ \implies\ & a_0^2 b_0^2 + (2b_0^2 a_0^2 + a_2 b_0^2)\epsilon^2 = (r - r_H)^2 + +d_1 r + d_2\end{aligned} \tag{H.29}$$

now set, $d_1 = d_2 = 0$, near the horizon,

$$a_0^2 b_0^2 = (r - r_H)^2, \tag{H.30}$$

(H.30) implies,

$$2b_0^2 a_0^2 + a_2 b_0^2 = 0, \tag{H.31}$$

$$\begin{aligned}\implies a_2 &= -\frac{2a_0^2 b_0 b_2}{b_0^2}\\ &= -\frac{2a_0^2}{b_0} b_2\\ &= \frac{-2(1 - \frac{r_H}{r})}{r} b_2\end{aligned} \tag{H.32}$$

To solve for b_2, substitute (H.22) to (8.12),

$$\begin{aligned}
\frac{b''}{b} &= -(\partial_r\phi_i)^2 \\
\implies b'' &= -(\partial_r\phi_i)^2 b \\
&= -(\partial_r(\phi_{i0} + \epsilon\phi_{i1} + \epsilon^2\phi_{i2}))^2 \\
&= -\epsilon^2(\partial_r\phi_{i1} + \epsilon\partial_r\phi_{i2})^2(b_0 + \epsilon^2\epsilon^2 b_2) \\
&= -(\partial_r\phi_{i1} + \epsilon\partial_r\phi_{i2})^2(b_0\epsilon^2) \\
&= -\left((\partial_r\phi_{i1})^2 + 2\partial_r\phi_{i1}\partial_r\phi_{i2}\epsilon + \epsilon^2(\partial_r\phi_{i2})^2\right) b_0\epsilon^2, \\
\implies \epsilon^2 b_2'' &= -(\partial_r\phi_{i1})^2 b_0\epsilon^2,
\end{aligned} \tag{H.33}$$

for $O(\epsilon^2)$

$$\begin{aligned}
b_2'' &= -(\partial_r\phi_{i1})^2 r \\
&= \left(c_{i1}\gamma_i\left(\frac{r-r_H}{r}\right)^{\gamma_i-1}\left(0+\frac{r_H}{r^2}\right)\right) r \\
&= -c_{i1}^2\gamma_i^2\left(\frac{r-r_H}{r}\right)^{2(\gamma_i-1)}\frac{r_H}{r^2} r \\
&= -c_{i1}^2\gamma_i^2\left(\frac{r-r_H}{r}\right)^{2(\gamma_i-1)}\frac{r_H}{r},
\end{aligned} \tag{H.34}$$

solving (H.34) for b_2, we get

$$b_2(r) = -\sum_i \frac{c_{1i}^2\gamma_i}{2(2\gamma_i-1)} r\left(\frac{r-r_H}{r}\right)^{2\gamma_i} + A_1 r + A_2 r_H, \tag{H.35}$$

setting $A_1 = A_2 = 0$.

$$b_2(r) = -\sum_i \frac{c_{1i}^2\gamma_i}{2(2\gamma_i-1)} r\left(\frac{r-r_H}{r}\right)^{2\gamma_i}. \tag{H.36}$$

H.3 Derivation of c_2 in all orders

Starting with a single scalar, and the equations of motion, we derive c_2, directly from Section 2.44 of [14]

$$\frac{b''}{b} = -(\partial_r\phi)^2, \tag{H.37}$$

$$a^2b^2 = (r-r_H)^2 \tag{H.38}$$

using the equation of motion of the scalars (8.14) and considering a single scalar field,

$$\partial_r(a^2b^2\partial_r\phi) = \frac{\partial_i V_{eff}}{2b^2}, \tag{H.39}$$

using the simple example for the effective potential [14],

$$V_{eff} = e^{\alpha_i\phi}Q_i^2 \tag{H.40}$$

we start by considering the left hand side of (H.39),

$$\begin{aligned}
2b^2\partial_r((r-r_H)^2\partial_r\phi) &= 2\left(b_0^2+2\epsilon^2 b_2 b_0\right)\partial_r\left((r-r_H)^2\partial_r(\phi_0+\epsilon\phi_1+\epsilon^2\phi_2)\right)\\
&= 2\left(b_0^2+2\epsilon^2 b_2 b_0\right)\partial_r\left((r-r_H)^2(\epsilon\partial_r\phi_1+\epsilon^2\partial_r\phi_2)\right)\\
&= 2\left(b_0^2+2\epsilon^2 b_2 b_0\right)\epsilon\partial_r\left((r-r_H)^2\partial_r\phi_1+(r-r_H)^2\epsilon\partial_r\phi_2\right)\\
&= 2b_0^2\partial_r((r-r_H)^2\partial_r\phi_1)\epsilon+2b_0^2\partial_r((r-r_H)^2\partial_r\phi_2)\epsilon^2\\
&= e^{\alpha_i\phi}Q_i^2\alpha_i,
\end{aligned} \tag{H.41}$$

and the right hand side of (H.39),

$$\begin{aligned}
e^{\alpha_i\phi}Q_i^2\alpha_i &= e^{\alpha_i\phi_0}e^{\phi_i(\epsilon\phi_1+\epsilon^2\phi_2)}Q_i^2\alpha_i\\
&= e^{\alpha_i\phi_0}e^{\alpha_i\epsilon\phi_1}e^{\alpha_i\epsilon^2\phi_2}Q_i^2\alpha_i\\
&\approx e^{\alpha_i\phi_0}\left(1+\alpha_i\phi_2\epsilon^2+\alpha_i\phi_1\epsilon+\frac{1}{2}\alpha_i^2\phi_1^2\epsilon_1^2\epsilon^2+...\right)Q_i^2\alpha_i\\
&\approx e^{\alpha_i\phi_0}\left(1+\alpha_i\phi_1\epsilon+(\alpha_i\phi_2+\frac{1}{2}\alpha_i^2\alpha_1^2)\epsilon^2\right)Q_i^2\alpha_i,
\end{aligned} \tag{H.42}$$

the $O(\epsilon^2)$ terms of (H.41) and (H.42) gives the following,

$$2b_0^2\partial_r\left((r-r_H)^2\partial_r\phi_2\right) = e^{\alpha_i\phi_0}(\alpha_i\phi_2+\frac{1}{2}\alpha_i^2\phi_1^2)Q_i^2\alpha_i. \tag{H.43}$$

we substitute ϕ_2 and it's derivatives to (H.43) give,

$$\begin{aligned}
\phi_2(r) &= c_2\left(\frac{r-r_H}{r}\right)^{2\gamma}, \\
\implies \partial_r\phi_2 &= 2\gamma c_2\left(\frac{1}{r-r_H}\right)\frac{r_H}{r}\left(\frac{r-r_H}{r}\right)^{2\gamma}, \\
\implies 4b_0^2(r-r_H)\partial_r\phi_2 &= 8\gamma c_2 r_H r\left(\frac{r-r_H}{r}\right)^{2\gamma}, \\
\partial_r^2\phi_2 = 2\gamma c_2\left(\frac{r-r_H}{r}\right)^{2\gamma}&\left(\frac{(2\gamma-1)}{(r-r_H)^2}\frac{r_H^2}{r^2}-\frac{2}{(r-r_H)}\frac{r_H}{r^2}\right), \\
\implies 4b_0^2(r-r_H)^2\gamma\partial_r^2\phi_2 &= 4\gamma c_2(1+2\gamma)r_H^2\left(\frac{r-r_H}{r}\right)^{2\gamma},
\end{aligned} \tag{H.44}$$

again, the right hand side of (H.43) simplifies to,

$$e^{\alpha_i\phi_0}(\alpha_i\phi_2+\frac{1}{2}\alpha_i^2\phi_1^2)Q_i^2\alpha_i = e^{\alpha_i\phi_0}\left(\alpha_i^2 c_2\left(\frac{r-r_H}{r}\right)^{2\gamma}+\frac{1}{2}\alpha_i^3 c_1^2\left(\frac{r-r_H}{r}\right)^{2\gamma}\right)Q_i^2. \tag{H.45}$$

Finally the equation of motion for the scalars (8.14) can be written as,

$$\left(\frac{r-r_H}{r}\right)^{2\gamma}\left(e^{\alpha_i\phi_0}Q_i^2\alpha_i^2 c_2-4\gamma c_2(1+2\gamma)r_H^2+\frac{1}{2}e^{\alpha_i\phi_0}Q_i^2\alpha_i^3 c_1^2\right)=0. \tag{H.46}$$

Now we can finally solve for c_2 [14] gives,

$$c_2=\frac{1}{2}c_1^2(\alpha_1+\alpha_2)\frac{(\gamma+1)}{(3\gamma+1)}. \tag{H.47}$$

Appendix I

Exact solutions

I.1 Equations of motion in general dimensions

Varying the action with respect to the metric we get the following equations of motion,

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R + G_{\mu\nu}g_{ij}(\phi)\partial_\alpha\phi^i\partial^\alpha\phi^j - 2g_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j$$
$$-f_{ab}(\phi_i)\left(2G^{\beta\tau}F^a_{\mu\beta}F^b_{\nu\tau} - \frac{1}{2}G_{\mu\nu}F^a_{\sigma\rho}F^{b\sigma\rho}\right) + G_{\mu\nu}V_g(\phi) = 0, \tag{I.1}$$

we can write the equations of motion (I.1) as,

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = T_{\mu\nu}, \tag{I.2}$$

where,

$$T_{\mu\nu} = 2g_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j - G_{\mu\nu}g_{ij}(\phi)\partial_\alpha\phi^i\partial_\alpha\phi^j$$
$$+ f_{ab}(\phi_i)\left(2G^{\beta\tau}F^a_{\mu\beta}F^b_{\nu\tau} - \frac{1}{2}G_{\mu\nu}F^a_{\sigma\rho}F^{b\sigma\rho}\right) - G_{\mu\nu}V_g(\phi), \tag{I.3}$$

the trace of the equations of motion (I.1) gives us,

$$T = (2-d)g_{ij}(\phi)\partial_\alpha\phi^i\partial_\alpha\phi^j + \left(\frac{4-2}{2}\right)f_{ab}(\phi_i)F^a_{\sigma\rho}F^{b\sigma\rho} - dV_g(\phi), \tag{I.4}$$

and

$$
\begin{aligned}
R - \frac{d}{2}R &= T, \\
\implies \frac{2-d}{2}R &= T, \\
\implies R &= \frac{2}{2-d}T,
\end{aligned} \tag{I.5}
$$

the Ricci scalar is given by,

$$
R = 2g_{ij}(\phi)\partial_\alpha\phi^i\partial_\alpha\phi^j + \frac{4-d}{2-d}f_{ab}F^a_{\sigma\rho}F^{b\sigma\rho} - \frac{2d}{2-d}V_g(\phi) \tag{I.6}
$$

The trace reversal equation of motion (I.1), gives,

$$
\begin{aligned}
R_{\mu\nu} - 2g_{ij}(\phi)\partial_\alpha\phi^i\partial_\alpha\phi^j &= 2f_{ab}(\phi)G^{\beta\tau}F^a_{\mu\beta}F^b_{\nu\tau} - \frac{1}{d-2}G_{\mu\nu}f_{ab}(\phi_i)F^a_{\sigma\rho}F^{b\sigma\rho} \\
&+ \frac{2}{d-2}G_{\mu\nu}V_g(\phi).
\end{aligned} \tag{I.7}
$$

I.1.1 Derivation of (9.34)

Using the relation $R_{tt} = (d-3)\frac{a^2}{b^2}R_{\theta\theta}$, we derive (9.34) as,

$$
\begin{aligned}
&(d-3)\frac{a2(r)}{b^2(r)}R_{\theta\theta} - \frac{(d-3)(d-3)!a(r)^2}{b(r)^{2(d-2)}}V_{eff}(\phi) \\
&+ \frac{(d-3)}{d-2}\frac{a2(r)}{b^2(r)}G_{\theta\theta}V_g(\phi) = 0, \\
\implies &R_{tt} - (d-3)\frac{a2(r)}{b^2(r)}R_{\theta\theta} + (\frac{1}{d-2}G_{tt} - \frac{(d-3)}{d-2}\frac{a2(r)}{b^2(r)}G_{\theta\theta})V_g(\phi) = 0, \\
\implies &R_{tt} - (d-3)\frac{a(r)^2}{b^2(r)}R_{\theta\theta} = -2a^2(r)\frac{(1+d-3)}{d-2}V_g(\phi), \\
\implies &a^2\left(a'^2 + \frac{(d-2)aa'b'}{b} + aa''\right) - (d-3)\frac{a^2}{b^2}\left((d-3) - 2aba'b' - a^2((d-3)b'^2 + bb'')\right) \\
&= -2a^2V_g(\phi), \\
\implies &-(d-3)^2 + 2(d-3)aba'b' + a^2b'^2(d-3)^2 + a'^2b^2 + (d-2)aba'b' + ab^2a'' \\
&= -2b^2V_g(\phi), \\
\implies &(d-3)^2(-1 + a^2b'^2) + b^2(a'^2 + aa'') + aba'b'(3d-8) + (d-3)a^2bb'' = -2b^2V_g(\phi).
\end{aligned} \tag{I.8}
$$

I.2 Exact solution of a single scalar

$$u_1 = \phi, u_2 = \log a, v = \log ab^{d-3}. \tag{I.9}$$

We define,

$$\dot{} = \partial_\tau = e^{2v}\partial_\rho, \tag{I.10}$$

so that,

$$\tau = \int \frac{d\rho}{a^2 b^{2(d-3)}} = \frac{1}{\rho_+ - \rho_-}\log\frac{\rho-\rho_+}{\rho-\rho_-}, \tag{I.11}$$

Solving for ρ in terms of τ, we get,

$$\rho = \frac{\rho_+ - e^{(\rho_+-\rho_-)\tau}\rho_-}{1-e^{(\rho_+-\rho_-)\tau}}. \tag{I.12}$$

Now we want to derive the equation of motion (9.47) in terms of the new variables. From (9.47), we ge we have the relation,

$$e^{2v}\partial_\rho\phi = \partial_\tau u_1, \tag{I.13}$$

using the relation (I.10) we can write the left hand side of (9.47) as,

$$\partial_\rho\left(a^2 b^{2(d-3)}\partial_\rho\phi\right) = \partial_\rho\left(\partial_\tau u_1\right) = e^{-2v}\partial_\tau(\partial_\tau u_1). \tag{I.14}$$

Now the right hand side of (9.47) becomes,

$$\frac{(d-2)!}{4(d-3)^2}\frac{a^2}{e^{2v}}\sum_{a=1}^{2}\alpha_a e^{\alpha_a\phi}(Q_m^a)^2 = e^{-2v}\frac{(d-2)!}{4(d-3)^2}e^{2u_2}\sum_{a=1}^{2}\alpha_a e^{\alpha_a u_1}(Q_m^a)^2, \tag{I.15}$$

and combining (I.14) and (I.15) we get,

$$\partial_\tau^2 u_1 = \frac{(d-2)!}{4(d-3)^2}\sum_{a=1}^{2}\alpha_a e^{2u_2+\alpha_a u_1}(Q_m^a)^2. \tag{I.16}$$

Using the change in coordinates, we can get the following relation,

$$b^{2(d-3)} = e^{2(v-u_2)}, \tag{I.17}$$

which implies that,

$$a^2\partial_\rho(b^{2(d-3)}) = 2\partial_\tau(v-u_2). \tag{I.18}$$

Thus the left hand side of (9.44), becomes,

$$-\frac{1}{2}\partial_\rho\left(a^2\partial_\rho(b^{2(d-3)})\right) = -e^{-2v}\left(\partial_\tau^2 v - \partial_\tau^2 u_2\right), \tag{I.19}$$

and the right hand side of (9.44), is,

$$-1+(d-4)!\frac{V_{eff}(\phi)}{b^{2(d-3)}} = -1+(d-4)!e^{2v}\sum_{a=1}^{2}e^{\alpha_a u_1+2u_2}(Q_m^a)^2, \tag{I.20}$$

combining the left hand side (I.19) and the right hand side (I.20), we finally get,

$$-\partial_\tau^2 v + \partial_\tau^2 u_2 = -e^{2v} + (d-4)!\sum_{a=1}^{2}e^{\alpha_a u_1+2u_2}(Q_m^a)^2, \tag{I.21}$$

which implies that,

$$\partial_\tau^2 v = e^{2v} = (\rho-\rho_+)(\rho-\rho_-), \tag{I.22}$$

and

$$\partial_\tau^2 u_2 = (d-4)!\sum_{a=1}^{2}e^{\alpha_a u_1+2u_2}(Q_m^a)^2. \tag{I.23}$$

we can also show that, (9.39) leads to

$$\partial_\tau^2 v = e^{2v}, \tag{I.24}$$

Finally to derive (9.50), we can show that the left hand side, is given by,

$$(\partial_\rho\phi)^2 = e^{-4v}(\partial_\tau u_1)^2, \tag{I.25}$$

and the right hand side, is given by,

$$\frac{\partial_\rho^2(b^{d-3})}{b^{d-3}} = e^{-4v}\left(-(\partial_\tau v)^2 + (\partial_\tau u_2)^2 + \partial_\tau^2 v - \partial_\tau^2 u_2\right), \tag{I.26}$$

combining the left hand side and the right hand side, we get,

$$-\frac{2(d-3)}{(d-2)}(\partial_\tau u_1)^2 = -(\partial_\tau v)^2 + (\partial_\tau u_2)^2 + \partial_\tau^2 v - \partial_\tau^2 u_2. \tag{I.27}$$

Using (I.16), the expression of v in (I.22) ells us that the left hand side of (I.27) independent of τ:

$$(\partial_\tau v)^2 - \partial_\tau^2 v = \frac{1}{4}(\rho_+ - \rho_-)^2 = \mathcal{E}. \tag{I.28}$$

Now, defining

$$X_a = \sum_b \left(n_{ab}^{-1} u_b + m_{ab}^{-1} \log[(\alpha_1 - \alpha_2)(Q_m^b)^2]\right), \tag{I.29}$$

where,

$$n^{-1} = \begin{pmatrix} (\frac{4(d-3)^2}{(d-2)!} & -\frac{\alpha_2}{(d-4)!} \\ -\frac{4(d-3)^2}{(d-2)!} & \frac{\alpha_1}{(d-4)!} \end{pmatrix}, m_{ab} = \frac{(d-4)!}{4(\alpha_1 - \alpha_2)}\left(8 + \frac{d-2}{d-3}\alpha_a\alpha_b\right), \tag{I.30}$$

we find that (I.16) and (I.23) may be rewritten as

$$\ddot{X}_a = e^{m_{ab}X_b}\ , \tag{I.31}$$

subject to the constraint (I.27)

$$\frac{1}{2}\dot{X}_a m_{ab} \dot{X}_b - \sum_a e^{m_{ab}X_b} = \frac{\alpha_1 - \alpha_2}{(d-4)!}\mathcal{E} \tag{I.32}$$

The τ-evolution of v given in (I.22) decouples from the equations of motion for the X_a. In order to decouple the equations (I.31) for X_a, we take

$$8 + \frac{d-2}{d-3}\alpha_1\alpha_2 = 0. \tag{I.33}$$

Without loss of generality, assume that $\alpha_1 > 0 > \alpha_2$.[1] This implies that

$$m_{ab} = \frac{(d-2)!}{4(d-3)^2}\mathrm{diag}(\alpha_1, -\alpha_2). \tag{I.34}$$

The differential equation (I.31) has a general solution

$$X_a = \frac{1}{|\alpha_a|\kappa}\log\frac{2c_a^2}{|\alpha_a|\kappa\sinh^2(c_a(\tau - d_a))}, \kappa := \frac{(d-2)!}{4(d-3)^2} \tag{I.35}$$

where the c_a and d_a are integration constants.

Now, the scalar field and the warp factors that appear in the metric are written

[1]Plugging in to (9.75), this is the case $\gamma = 1$.

in terms of the charges and the coordinate $\tau(\rho)$ as

$$e^{(\alpha_1-\alpha_2)\phi} = \frac{(Q_m^2)^2}{(Q_m^1)^2} e^{\kappa \sum_a \alpha_a X_a} \ , \tag{I.36}$$

$$a^2 = \exp\left(\frac{2(d-4)!}{\alpha_1-\alpha_2}(X_1+X_2)\right)/\lozenge, \tag{I.37}$$

$$b^{2(d-3)} = \frac{(\rho-\rho_+)(\rho-\rho_-)}{a^2}, \tag{I.38}$$

where,

$$\lozenge = (\alpha_1-\alpha_2)(Q_m^1)^{\frac{-2\alpha_2}{\alpha_1-\alpha_2}}(Q_m^2)^{\frac{2\alpha_1}{\alpha_1-\alpha_2}}. \tag{I.39}$$

Thus, in particular, (I.36) tells us that

$$e^{(\alpha_1-\alpha_2)\phi} = -\frac{\alpha_2 c_1^2 (Q_m^2)^2}{\alpha_1 c_2^2 (Q_m^1)^2} \frac{\sinh^2(c_2(\tau-d_2))}{\sinh^2(c_1(\tau-d_1))}. \tag{I.40}$$

Because $\tau \to -\infty$ as $\rho \to \rho_+$, finiteness of the solution at the horizon implies $c_1 = c_2 =: c$. Using (I.11) and the scaling in the vicinity of the outer horizon, we also see that

$$b^{2(d-3)} \sim \frac{\rho-\rho_+}{a^2} \sim e^{(\rho_+-\rho_-)\tau - 2c\tau}. \tag{I.41}$$

We then discover that $c = \frac{1}{2}(\rho_+ - \rho_-)$ in order for the horizon area to be finite. This is Δ, the non-extremality parameter. It serves to define our solution. Similarly, the flat space asymptotic boundary conditions tell us that

$$\lozenge = \exp\left[-\frac{\alpha_2}{\alpha_1-\alpha_2}\log\left(\frac{-\alpha_2 c^2}{(d-4)!\sinh^2(cd_1)}\right) + \frac{\alpha_1}{\alpha_1-\alpha_2}\log\left(\frac{\alpha_1 c^2}{(d-4)!\sinh^2(cd_2)}\right)\right], \tag{I.42}$$

$$e^{(\alpha_1-\alpha_2)\phi_\infty} = -\frac{\alpha_2 (Q_m^2)^2}{\alpha_1 (Q_m^1)^2} \frac{\sinh^2(cd_2)}{\sinh^2(cd_1)}. \tag{I.43}$$

The first equality arises from the condition $a^2 \to 1$ as $\rho \to \infty$ (or $\tau \to 0$). We may thus solve for the unknown parameters d_a in terms of the charges and the asymptotic value of the scalar, ϕ_∞.

For definiteness, let us restrict to the special case $\alpha_1 = -\alpha_2 = \alpha = \sqrt{\frac{8(d-3)}{d-2}}$. For this solution

$$\sinh cd_1 = \frac{c}{\sqrt{2(d-4)!}\,\bar{Q}^1_m}, \tag{I.44}$$

$$\sinh cd_2 = \frac{c}{\sqrt{2(d-4)!}\,\bar{Q}^2_m}, \tag{I.45}$$

where we recall that $\bar{Q}^a_m$ was defined in (9.98). Thus,

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\frac{\bar{Q}^2_m \sinh\left(\log e^{-cd_2}\left(\frac{\rho-\rho_+}{\rho-\rho_-}\right)^{\frac{1}{2}}\right)}{\bar{Q}^1_m \sinh\left(\log e^{-cd_1}\left(\frac{\rho-\rho_+}{\rho-\rho_-}\right)^{\frac{1}{2}}\right)}. \tag{I.46}$$

Using the identity $\sinh\log X = \frac{X^2-1}{2X}$ and (I.44), we obtain

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\frac{\rho - f_2}{\rho - f_1}\ , \tag{I.47}$$

where

$$f_a = \frac{e^{cd_a}\rho_- - e^{-cd_a}\rho_+}{2\sinh(cd_a)}\ . \tag{I.48}$$

Also,

$$\begin{aligned} b^{2(d-3)} &= a^{-2}(\rho-\rho_+)(\rho-\rho_-) \\ &= \tfrac{2\bar{Q}^2_m\bar{Q}^1_m(d-4)!}{c^2}\sinh\left(\log e^{-cd_2}\left(\frac{\rho-\rho_+}{\rho-\rho_-}\right)^{\frac{1}{2}}\right)\sinh\left(\log e^{-cd_1}\left(\tfrac{\rho-\rho_+}{\rho-\rho_-}\right)^{\frac{1}{2}}\right)(\rho-\rho_+)(\rho-\rho_-) \\ &= (\rho-f_1)(\rho-f_2). \end{aligned} \tag{I.49}$$

Assuming that we are working in an asymptotically flat space, $b \to r$ at spatial infinity, so from (9.37), $\rho \to r^{d-3}$ and from (9.56)

$$a^2 \to \frac{(r^{d-3}-\rho_+)(r^{d-3}-\rho_-)}{r^{2(d-3)}} \approx 1 - \frac{\rho_+ + \rho_-}{r^{d-3}} + \dots\ . \tag{I.50}$$

We are now able to read off the mass parameter from the metric obtaining (9.96) Asymptotically, we also have

$$\partial_\tau \to \rho^2\partial_\rho\ . \tag{I.51}$$

Examining M, this yields

$$M = \dot{a}|_{\tau=0}\ , \tag{I.52}$$

and likewise,

$$\Sigma = -\dot{\phi}|_{\tau=0}\ . \tag{I.53}$$

Using these definitions, (I.27) tells us that,

$$\tfrac{1}{4}\alpha^2\Sigma^2 + M^2 - (d-4)!\left((\bar{Q}^1_m)^2 + (\bar{Q}^2_m)^2\right) = c^2\ . \tag{I.54}$$

Taking the τ derivative of (I.46) and the expression for a^2 from (I.49), we find

$$\begin{aligned} \tfrac{1}{2}\alpha\Sigma &= \tfrac{1}{2}c(\coth(cd_2) - \coth(cd_1)) \ , \qquad (I.55) \\ M &= \tfrac{1}{2}c(\coth(cd_2) + \coth(cd_1)) \ . \qquad (I.56) \end{aligned}$$

Combining (I.55) and (I.56), we then have

$$\alpha M\Sigma = \tfrac{1}{2}c^2\left(\operatorname{csch}^2(cd_2) - \operatorname{csch}^2(cd_1)\right) = (d-4)!((\bar{Q}_m^2)^2 - (\bar{Q}_m^1)^2) \ . \qquad (I.57)$$

Since

$$f_1 = \tfrac{1}{2}\left(\rho_+ + \rho_- - (\rho_+ - \rho_-)\coth(cd_1)\right) = M - c\coth(cd_1) = \tfrac{1}{2}\alpha\Sigma \ , \qquad (I.58)$$

and similarly $f_2 = -\frac{1}{2}\alpha\Sigma$, we get (9.92).

Substituting the expressions for f_a in (I.49), we recast the warp factor in the metric as (9.94).

I.3 Extremal limit

To find the extermal solution, one takes the limit $c \to 0$, and from (I.40), with $\alpha = \alpha_1 = -\alpha_2$, one finds

$$e^{\alpha\phi} = \frac{|Q_m^2|}{|Q_m^1|}\,\frac{\tau - d_2}{\tau - d_1} \ . \qquad (I.59)$$

Taking the $c \to 0$ limit of (I.44) one gets,

$$d_1 = \frac{1}{\sqrt{2(d-4)!}\,\bar{Q}_m^1}, d_2 = \frac{1}{\sqrt{2(d-4)!}\,\bar{Q}_m^2}, \qquad (I.60)$$

so that

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\frac{\sqrt{2(d-4)!}|\bar{Q}_m^2|\tau - 1}{\sqrt{2(d-4)!}|\bar{Q}_m^1|\tau - 1} \ . \qquad (I.61)$$

Note, that with $c = 0$, $\rho_+ = \rho_- = M$, so from (I.11) we obtain

$$\tau = -\frac{1}{\rho - M}, \qquad (I.62)$$

$$e^{\alpha\phi} = e^{\alpha\phi_\infty}\frac{\sqrt{2(d-4)!}|\bar{Q}_m^2| + (\rho - M)}{\sqrt{2(d-4)!}|\bar{Q}_m^1| + (\rho - M)} \ . \qquad (I.63)$$

Taking the $c \to 0$ limit of (I.55) and (I.56)

$$\begin{aligned}
\tfrac{1}{2}\alpha\Sigma &= \tfrac{1}{2}((d_2)^{-1} - (d_1)^{-1}) = \sqrt{\frac{(d-4)!}{2}}\left(\bar{Q}_m^2 - \bar{Q}_m^1\right) \ , && \text{(I.64)}\\
M &= \tfrac{1}{2}((d_2)^{-1} + (d_1)^{-1}) = \sqrt{\frac{(d-4)!}{2}}\left(\bar{Q}_m^2 + \bar{Q}_m^1\right) \ . && \text{(I.65)}
\end{aligned}$$

and we see that in fact (I.63) is the same as (9.92). As a check, we note that taking the $c \to 0$ limit of (I.49) gives,

$$\begin{aligned}
b^{2(d-3)} &= 2Q_m^1 Q_m^2 (d-4)!(\tau - d_1)(\tau - d_2)(\rho - M)^2 && \text{(I.66)}\\
&= (\rho - M - \sqrt{2(d-4)!}|\bar{Q}_m^2|)(\rho - M - \sqrt{2(d-4)!}|\bar{Q}_m^1|) && \text{(I.67)}\\
&= \rho^2 - (\tfrac{1}{2}\alpha\Sigma)^2 && \text{(I.68)}
\end{aligned}$$

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