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TWO-STEP INERTIAL TSENG'S EXTRAGRADIENT METHOD FOR SOLVING QUASIMONOTONE VARIATIONAL INEQUALITIES

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Abstract. We focus on quasimonotone variational inequality problems with non-Lipschitz operator and combine some operator theory techniques to establish a weak convergent fixed point iterative method to approximate the solution of the presented problems. The convergence analysis assumes very mild conditions and strictly weaker assumption on the cost operator associated with the problems and thus generalizes and extend recent related results in the literature. Finally, several numerical examples illustrate the potential and advantages of the proposed algorithm over some existing methods are presented.

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Key words: Quasimonotone variational inequalities, Tseng extragradient method, inertial extrapolation, Armijo-like step size technique, Hilbert spaces.

1. Introduction. Let H be a real Hilbert space with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ as inner product and induced norm respectively. Suppose $C \subset H$ is nonempty, closed and convex. In this work, we focus on the following variational inequality problem (VIP): Find $x^* \in C$ such that

$$\langle Fx^*,\ z - x^* \rangle \geq 0,\ \forall z \in C, \tag{1}$$

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where $F : H \to H$ is an operator. The solution set of VIP (1) will be denoted by S.

It is well known that many problems in nonlinear analysis and optimization can be formulated as VIP (1). The two notable methods of solving VIPs are the regularized method and projection method. Numerous iterative algorithms for solving VIPs have been studied and proposed in these directions (see, for example, [1, 13, 12, 19, 24, 30, 34, 35, 38, 48, 54, 56] and the references therein). The fundamental concept is to broaden the projected gradient method for solving the minimization problem $\min f(x)$ subject to $x \in C$ defined by:

$$x_{n+1} = P_C(x_n - \alpha_n \nabla f(x_n)), \quad n \geq 0. \tag{2}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions and P_C is the metric projection onto C. Alber [4] showed the convergence properties of this method for the case where $f : \mathbb{R}^2 \to \mathbb{R}$ is convex and differentiable function. The projected gradient method for optimization problems is a direct extension of method (2) where the gradient, ∇f is replaced by the operator F, so that we generate a sequence $\{x_n\}$ as follows:

$$x_{n+1} = P_C(x_n - \alpha_n F x_n), \quad n \geq 0. \tag{3}$$

It was shown in [57] that the convergence of (3) requires stringent assumptions that the cost operator F is either strongly monotone or inverse strongly monotone. So, it becomes necessary to relax the condition on F to solve the VIP.

To overcome the above drawback, Korpelevich [29] succeeded in relaxing the condition by proposing the extragradient method (EGM) in Euclidean spaces and Hilbert spaces when the operator F is monotone and Lipchitz continuous. This method is given by:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = P_C(x_n - \lambda F y_n), \quad n \geq 1, \end{cases} \tag{4}$$

where $\lambda \in (0, \frac{1}{L})$ and P_C is the metric projection from H onto C. The limitation of the EGM is the fact that it requires two projections onto the closed and convex set. Computing two projections per iteration onto the closed and convex set C could be computationally expensive. In such situation, the efficiency of this method is affected.

In recent years, many researchers have tried to improve on the EGM (see, for example, [13, 10, 33, 35, 36, 45, 53] and the refrences therein). To improve on the EGM, Censor et al. [11] introduced a new method called the subgradient extragradient method (SEGM) in which the second projection onto C in the EGM is replaced by a projection onto a constructible half-space, which is less computationally expensive and this makes it more efficient than the EGM proposed by

Korpelevich [29]. The SEGM is defined as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H | \langle x_n - \lambda F x_n - y_n,\ w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n),\ n \geq 1, \end{cases} \tag{5}$$

where $\lambda \in (0, \frac{1}{L})$. The weak convergence of the algorithm was established.

Subsequently, Tseng [53] introduced another method called Tseng's extragradient method (TEGM) for solving the VIP. In recent years, many authors have employed this method in their works (see, for example, [7, 48]). The TEGM is presented as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = y_n - \lambda(F y_n - F x_n),\ n \geq 1, \end{cases} \tag{6}$$

where $\lambda \in (0, \frac{1}{L})$.

Based on the heavy ball method of the second-order dynamical system with friction, Polyak [41] first proposed an inertial extrapolation as an acceleration process for solving the smooth convex minimization problem. The inertial technique can be seen as a means of speeding up the convergence properties of iterative algorithms. Several authors have employed the inertial type algorithm due to its important application in accelerating the convergence rates of iterative algorithms (see, for example, [2, 3, 5, 8, 14, 18, 20, 26, 47, 55] and the references therein).

In [6], Alvarez and Attouch converted the idea of the heavy ball method to the setting of a general maximal monotone operator using the proximal point algorithm. The proposed algorithm which they called inertial proximal point algorithm is defined as:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n F)^{-1} y_n, \quad n \geq 1, \end{cases} \tag{7}$$

where $(I + \lambda_n F)^{-1}$ is the resolvent of the maximal monotone operator F with parameter λ_n. They established that the sequence $\{x_n\}$ generated by (7) converges weakly to a zero of F with the conditions that $\{\lambda_n\}$ is non-decreasing and $\theta_n \in [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < +\infty. \tag{8}$$

To improve the result of Tseng [53] and to obtain algorithm with better performance, Thong and Hieu [51] introduced an inertial Tseng's extragradient method (ITEGM) for solving the monotone VIP (1) in real Hilbert spaces. They proposed the following algorithm:

Algorithm 1.1. *An Inertial Tseng's Extragradient Method*

Initialization: *Given $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$. Let $x_0,\ x_1 \in H$ be arbitrary points.*

Iterative steps: *Calculate x_{n+1} as follows:*

Step 1: *Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute*

$$y_n = P_C(w_n - \lambda_n F w_n),$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Fw_n - Fy_n\| \leq \mu \|w_n - y_n\|. \tag{9}$$

If $y_n = w_n$ then stop and y_n is a solution of the problem (VIP). Otherwise,

Step 2: *Compute*

$$x_{n+1} = y_n - \lambda_n(Fy_n - Fw_n).$$

Set $n := n+1$ and go to **step 1**.

where F is monotone and Lipschitz continuous. They obtained weak convergence result for Algorithm 1.1 with adoption of Armijo-like step size rule and one-step inertial extrapolation.

In [44], Poon and Liang discussed some limitations of inertial Douglas-Rachford splitting method and inertial ADMM. For example, consider the following feasibility problem in $\mathbb{R}^2$.

Example 1.2. Let $T_1,\ T_2 \subset \mathbb{R}^2$ be two subspaces such that $T_1 \cap T_2 \neq \emptyset$. Find $x \in \mathbb{R}^2$ such that $x \in T_1 \cap T_2$.

It was shown in [44, Section 4] that two-step inertial Douglas-Rachford splitting method, where

$$x_{n+1} = F(x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2})) \tag{10}$$

converges faster than one-step inertial Douglas-Rachford splitting method

$$x_{n+1} = F(x_n + \theta(x_n - x_{n-1})) \tag{11}$$

for Example 1.2. In fact, it was shown using Example 1.2 that one-step inertial Douglas-Rachford splitting method (11) converges slower than the Douglas-Rachford splitting method

$$x_{n+1} = F(x_n), \tag{12}$$

where

$$F := \frac{1}{2}\Big(I + (2P_{T_1} - I)(2P_{T_2} - I)\Big)$$

is the Douglas-Rachford splitting operator. Moreover, this example shows that the one-step inertial extrapolation $w_n = x_n + \theta(x_n - x_{n-1}),\ \theta \in [0,1)$ may fail to provide acceleration. In [31, Chapter 4], it was remarked that using two-step inertial extrapolation with more than two points $x_n,\ x_{n-1}$ could give acceleration, for example,

$$w_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}). \tag{13}$$

with $\theta > 0$ and $\delta < 0$.

Recently, some authors (see, for example,[15]) have studied and shown some results on multi-step inertial methods. The failure of one-step inertial acceleration of ADMM was discussed in [43, section 3] and adaptive acceleration for ADMM was proposed instead. Further, Polyak [42] pointed out that the multi-step inertial method can boost the speed of optimization methods. Consequently, for some cases, the use of inertial of more than two points could give better result.

In this paper, inspired by the work of Thong and Hieu [51] and many other results in this direction, we propose a two-step inertial Tseng's extragradient method for solving the VIP (1) with quasimonotone and non-Lipschitz operator in the framework of Hilbert spaces. We employ the two-step inertial extrapolation to provide acceleration for our algorithm. Moreover, our algorithm involves only one projection per iteration onto the closed convex set and our method uses Armijo-like step size rule. In comparison with the methods proposed in [32, 35, 51, 53], our contributions in this work are:

(i) We propose a two-step inertial Tseng's extragradient method to solve the VIP (1). Thus, overcoming the shortcomings of the one-step inertial extragradient method proposed in [49, 51], subgradient extragradient in [11, 59] and one-step inertial subgradient extragradient method in [17, 52, 58] for both monotone and pseudomonotone cost operators.

Consequently, we have introduced an algorithm with better convergence properties which can easily be implemented for solving the VIP (1).

(ii) Our proposed method solves the VIP (1) for a larger class of mappings. It is applicable when the cost operator F is quasimonotone and uniformly continuous.

(iii) A weak convergence result is obtained in Hilbert spaces under very mild conditions on the control parameters.

(iv) We present several numerical experiments to verify the feasibility of our algorithm and compare with the methods in [32, 35, 51, 53].

2. Preliminaries.

Definition 2.1. ([39]) Let H be a real Hilbert space. An operator $F : H \to H$ is said to be:

(i) *Lipschitz continuous* on H, if there exist a constant $L > 0$ such that

$$\|Fx - Fy\| \le L\|x - y\|,\ \forall x, y \in H.$$

If $L \in [0, 1)$, Then F is said to be a contraction mapping;

(ii) *uniformly continuous*, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that

$$\|Fx - Fy\| < \epsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in H;$$

(iii) *sequentially weakly-strongly continuous*, if for each sequence $\{x_n\}$, we have $x_n \rightharpoonup x \in H$ implies that $Fx_n \to Fx \in H$;

(iv) *sequentially weakly continuous*, if for each sequence $\{x_n\}$, we have $x_n \rightharpoonup x \in H$ implies that $Fx_n \rightharpoonup Fx \in H$;

(v) *monotone*, if
$$\langle Fx - Fy,\ x - y\rangle \geq 0,\ \forall x, y \in H;$$

(vi) α-*strongly pseudomonotone*, if there exists $\alpha > 0$ such that

$$\langle Fx, y - x\rangle \geq 0 \implies \langle Fy, y - x\rangle \geq \alpha\|x - y\|^2 \ \ \forall x, y \in H;$$

(vii) *pseudomonotone*, if

$$\langle Fy,\ x - y\rangle \geq 0 \implies \langle Fx,\ x - y\rangle \geq 0,\ \forall x, y \in H;$$

(viii) *quasimonotone*, if

$$\langle Fy,\ x - y\rangle > 0 \implies \langle Fx,\ x - y\rangle \geq 0,\ \forall x, y \in H.$$

From the definition above, the following implications hold: $(v) \implies (vii) \implies (viii)$ but the converse is not true in general. We also note that uniform continuity is a weaker notion than Lipschitz continuity.

Furthermore, it is well known that if D is a convex subset of H, then $F : D \to H$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $K < +\infty$ such that

$$\|Fx - Fy\| \leq K\|x - y\| + \epsilon \quad \forall x, y \in D. \tag{14}$$

Let S_D be the solution set of the dual formulation of the VIP (1) defined as: find $x^* \in C$ such that

$$\langle Fz,\ z - x^*\rangle \geq 0,\ \forall z \in C. \tag{15}$$

Then, S_D is a closed and convex subset of C, and since F is continuous and C is convex, we have that $S_D \subset S$.

We have the following result on the solution set of the dual VIP (15).

DEFINITION 2.2. ([40]) Let $\{x_n\}$ be a sequence in H.

(i) $\{x_n\}$ is said to converge R-linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \leq c\rho^n, \quad \forall n \in \mathbb{N}.$$

(ii) $\{x_n\}$ is said to converge Q-linearly to x^* with rate $\rho \in [0,1)$ if

$$\|x_{n+1} - x^*\| \leq \rho\|x_n - x^*\|, \ \forall n \in \mathbb{N}.$$

LEMMA 2.3. ([60]) *Suppose either*

(a) F is pseudomonotone on C and $S \neq \emptyset$;

(b) F is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$ and attains its global minimum on C;

(c) F is quasimonotone on C, $F \neq 0$ on C and C is bounded;

(d) A is quasimonotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle Fx, \ y - x\rangle \leq 0$;

(e) F is quasimonotone on C, int $C \neq \emptyset$ and there exists $x^* \in S$ such that $Fx^* \neq 0$,

then S_D is nonempty.

For a nonempty, closed and convex subset C of H, the metric projection $P_C : H \to C$ (see [21, 27] is defined, for every $v \in H$ as the unique element $P_C v \in C$ such that

$$\|v - P_C v\| = \inf\{\|v - y\| : y \in C\}.$$

It is well known that P_C is characterized by the following inequality:

$$||P_C v - y||^2 + \|v - P_C v\|^2 \leq \|v - y\|^2, \quad v \in H, y \in C.$$

LEMMA 2.4. ([22]) *Let C be a nonempty closed subset of a real Hilbert Space H. For any $v \in H$ and $z \in C$, we have*

$$z = P_C v \iff \langle v - z, \ z - y\rangle \geq 0, \ \forall y \in C. \tag{16}$$

LEMMA 2.5. ([37]) *The following identities hold for all $u, v \in H$:*

$$2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2.$$

LEMMA 2.6. *Let $x, y, z \in H$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned}
&\|(1+a)x - (a-b)y - bz\|^2 \\
&\qquad = (1+a)\|x\|^2 - (a-b)\|y\|^2 - b\|z\|^2 + (1+a)(a-b)\|x - y\|^2 \\
&\qquad + b(1+a)\|x - z\|^2 - b(a-b)\|y - z\|^2.
\end{aligned}$$

LEMMA 2.7. ([16]) *Let H be a Hilbert space and $F : H \to H$ be a uniformly continuous operator. Suppose $x \in H$ and $\psi \geq \sigma > 0$. The following inequality holds:*

$$\frac{\|x - P_C(x - \psi Ax)\|}{\psi} \leq \frac{\|x - P_C(x - \sigma Ax)\|}{\sigma}.$$

3. Proposed method. We start with the following assumptions under which our weak convergent result is established.

Assumption 3.1.

(i) $S_D \neq \emptyset$.

(ii) F is uniformly continuous on H.

(iii) F satisfies the following condition: whenever $\{x_n\} \subset C$ and $x_n \rightharpoonup v^*$, one has $\|Fv^*\| \leq \liminf_{n\to\infty} \|Fx_n\|$.

(iv) F is quasimonotone on H.

(v) The set $\{v \in C : Fv = 0\} \setminus S_D$ is a finite set.

Assumption 3.2.

Suppose that θ and β satisfy the following conditions:

(a) $0 \leq \theta < \frac{1-\mu}{3+\mu}$;

(b) $\max\left\{2\theta\left(\frac{1+\mu}{1-\mu}\right) - (1-\theta),\ \frac{1}{2}\left[\theta(1+\mu) - \frac{(1-\mu)(1-\theta)^2}{1+\theta}\right]\right\} < \beta \leq 0$;

(c) $2\theta^2\mu - (1-3\theta) + \mu(1-\theta) - \beta(4\theta+3-\mu) + 2\mu\beta^2 < 0$.

We now present our proposed method for solving VIP (1)

Algorithm 3.3.

Initialization: *Given* $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$. *Let* x_{-1}, x_0, $x_1 \in H$ *be arbitrary points and given* x_{n-2}, x_{n-1}, x_n.

Iterative steps: *Calculate* x_{n+1} *as follows*

Step 1: *Set* $w_n = x_n + \theta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2})$ *and compute*

$$y_n = P_C(w_n - \lambda_n F w_n),$$

where $\lambda_n = \gamma l^{m_n}$ *and* m_n *is the smallest nonnegative integer* m *such that*

$$\lambda_n \|Fw_n - Fy_n\| \leq \mu \|w_n - y_n\|. \tag{17}$$

If $y_n = w_n$ *then stop:* y_n *is a solution of the problem (VIP). Otherwise,*

Step 2: *Compute*

$$x_{n+1} = y_n - \lambda_n(Fy_n - Fw_n),$$

Set $n := n+1$ *and go to* **step 1**.

Remark 3.4.

(a) Assumption 3.1(iii) is strictly weaker than the sequentially weakly continuous assumption in [32] and other recent papers for solving pseudomonotone VIPs. An example of an operator satisfying Assumption 3.1(iii) but not sequentially weakly continuous is $Fv = v\|v\|$, $\forall v \in C$, where $F : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$.

In this section, we analyze the convergence of our proposed algorithm. First, we establish some lemmas required to prove our weak convergence theorem.

4. Convergence analysis.

Lemma 4.1. *The Armijo-like search rule* (17) *is well defined. Moreover, we have* $\lambda_n \leq \gamma$.

Proof. If $w_n \in S$, then $w_n = P_C(w_n - \lambda_n F w_n)$, and this implies that $y_n = w_n$. Hence, $m_n = 0$. If $w_n \notin S$, we assume the contrary of (17) holds, that is, for all m we have

$$\gamma\ell^m \|Fy_n - Fw_n\| > \mu\|y_n - w_n\|, \tag{18}$$

which implies that

$$\|FP_C(w_n - \gamma\ell^m Fw_n) - Fw_n\| > \mu\frac{\|P_C(w_n - \gamma\ell^m Fw_n) - w_n\|}{\gamma\ell^m}. \tag{19}$$

We consider two cases of w_n. First, suppose $w_n \in C$. Since F and P_C are continuous, we have

$$\lim_{m\to\infty} \|w_n - P_C(w_n - \gamma\ell^m Fw_n)\| = 0. \tag{20}$$

By the uniform continuity of F, it follows that

$$\lim_{m\to\infty} \|Fw_n - FP_C(w_n - \gamma\ell^m Fw_n)\| = 0. \tag{21}$$

From (19) and (21), we obtain

$$\lim_{m\to\infty} \frac{\|w_n - P_C(w_n - \gamma\ell^m Fw_n)\|}{\gamma\ell^m} = 0 \tag{22}$$

Let $q_m = P_C(w_n - \gamma\ell^m Fw_n)$. By Lemma 2.4, we have

$$\langle q_m - w_n + \gamma\ell^m Fw_n, x - q_m\rangle \geq 0, \quad \forall\, x \in C,$$

which implies

$$\frac{1}{\gamma\ell^m}\langle q_m - w_n, x - q_m\rangle + \langle Fw_n, w_n - q_m\rangle + \langle Fw_n, x - w_n\rangle \geq 0, \quad \forall\, x \in C. \tag{23}$$

Taking the limit $m \to \infty$ in (23), and applying (20) and (22), we obtain

$$\langle Fw_n, x - w_n\rangle \geq 0, \quad \forall\, x \in C.$$

This implies that $w_n \in S$, which contradicts the hypothesis.

On the other hand, if $w_n \notin C$, then we have

$$\lim_{m\to\infty} \|w_n - P_C(w_n - \gamma\ell^m Fw_n)\| = \|w_n - P_C w_n\| > 0. \tag{24}$$

and

$$\lim_{m\to\infty} \gamma\ell^m \|Fw_n - FP_C(w_n - \gamma\ell^m Fw_n)\| = 0. \tag{25}$$

Combining (24) and (25) with (18), we have a contradiction, and this completes the proof. □

Lemma 4.2. *Suppose Assumption 3.1 (i) and (ii) and Assumption 3.2 hold. Then, the sequence $\{x_n\}$ generated by Algorithm 3.3 is bounded.*

Proof.
Let $p \in S_D$, then

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|y_n - \lambda_n(Fy_n - Fw_n) - p\|^2 \\
&\quad = \|y_n - p\|^2 + \lambda_n^2\|Fy_n - Fw_n\|^2 - 2\lambda_n\langle y_n - p,\ Fy_n - Fw_n\rangle \\
&\quad = \|w_n - p\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n,\ w_n - p\rangle + \lambda_n^2\|Fy_n - Fw_n\|^2 \\
&\qquad - 2\lambda_n\langle y_n - p,\ Fy_n - Fw_n\rangle \\
&\quad = \|w_n - p\|^2 + \|w_n - y_n\|^2 - 2\langle y_n - w_n,\ y_n - w_n\rangle + 2\langle y_n - w_n,\ y_n - p\rangle \\
&\qquad + \lambda_n^2\|Fy_n - Fw_n\|^2 - 2\lambda_n\langle y_n - p,\ Fy_n - Fw_n\rangle \\
&\quad = \|w_n - p\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n,\ y_n - p\rangle + \lambda_n^2\|Fy_n - Fw_n\|^2 \\
&\qquad - 2\lambda_n\langle y_n - p,\ Fy_n - Fw_n\rangle.
\end{aligned} \tag{26}$$

From $y_n = P_C(w_n - \lambda_n Fw_n)$ and by Property (16), and since $p \in S_D \subset C$ we get

$$\langle w_n - \lambda_n Fw_n - y_n,\ y_n - p\rangle \geq 0,$$

which implies that

$$\langle y_n - w_n,\ y_n - p\rangle \leq -\lambda_n\langle Fw_n,\ y_n - p\rangle. \tag{27}$$

From (26) and (27), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\lambda_n\langle Fw_n,\ y_n - p\rangle + \lambda_n^2\|Fy_n - Fw_n\|^2 \\
&\quad - 2\lambda_n\langle y_n - p,\ Fy_n - Fw_n\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda_n^2\|Fy_n - Fw_n\|^2
\end{aligned}$$

$$- 2\lambda_n\langle Fy_n,\ y_n - p\rangle. \tag{28}$$

Since $p \in S_D$, $\lambda_n > 0$ and $y_n \in C$, we have

$$\lambda_n\langle Fy_n,\ y_n - p\rangle \geq 0. \tag{29}$$

Hence, using (17) and (29) in (28), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda_n^2\|Fw_n - Fy_n\|^2\\ &\leq \|w_n - p\|^2 - \|w_n - y_n\|^2 + \mu^2\|w_n - y_n\|^2\\ &= \|w_n - p\|^2 - (1-\mu^2)\|w_n - y_n\|^2.\end{aligned} \tag{30}$$

Also,

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|y_n - \lambda_n(Fy_n - Fw_n) - y_n\|\\ &= \lambda_n\|Fy_n - Fw_n\|\\ &\leq \mu\|w_n - y_n\|,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - w_n\| &\leq \|x_{n+1} - y_n\| + \|w_n - y_n\|\\ &\leq \mu\|w_n - y_n\| + \|w_n - y_n\|\\ &= (1+\mu)\|w_n - y_n\|\end{aligned}$$

Therefore,

$$-\|w_n - y_n\|^2 \leq -\frac{1}{(1+\mu)^2}\|x_{n+1} - w_n\|^2. \tag{31}$$

Applying (31) in (30), we get

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \Big(\frac{1-\mu}{1+\mu}\Big)\|x_{n+1} - w_n\|^2 \tag{32}$$

Observe that

$$\begin{aligned}w_n - p &= x_n + \theta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}) - p\\ &= (1+\theta)(x_n - p) - (\theta - \beta)(x_{n-1} - p) - \beta(x_{n-2} - p).\end{aligned}$$

Hence, by Lemma 2.6, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|(1+\theta)(x_n - p) - (\theta-\beta)(x_{n-1} - p) - \beta(x_{n-2} - p)\|^2\\ &= (1+\theta)\|x_n - p\|^2 - (\theta-\beta)\|x_{n-1} - p\|^2 - \beta\|x_{n-2} - p\|^2\\ &\quad + (1+\theta)(\theta-\beta)\|x_n - x_{n-1}\|^2 + \beta(1+\theta)\|x_n - x_{n-2}\|^2\\ &\quad - \beta(\theta-\beta)\|x_{n-1} - x_{n-2}\|^2.\end{aligned} \tag{33}$$

Furthermore,

$$
\begin{aligned}
&\|x_{n+1}-w_n\|^2\\
&=\|x_{n+1}-[x_n+\theta(x_n-x_{n_1})+\beta(x_{n-1}-x_{n-2})]\|^2\\
&=\|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\beta(x_{n-1}-x_{n-2})\|^2\\
&=\|x_{n+1}-x_n\|^2-2\theta\langle x_{n+1}-x_n,\ x_n-x_{n-1}\rangle-2\beta\langle x_{n+1}-x_n,\ x_{n-1}-x_{n-2}\rangle\\
&\quad+\theta^2\|x_n-x_{n-1}\|^2+2\beta\theta\langle x_n-x_{n-1},\ x_{n-1}-x_{n-2}\rangle+\beta^2\|x_{n-1}-x_{n-2}\|^2\\
&\geq\|x_{n+1}-x_n\|^2-\theta\|x_{n+1}-x_n\|^2-\theta\|x_n-x_{n-1}\|^2+\theta^2\|x_n-x_{n-1}\|^2\\
&\quad-|\beta|\theta\|x_n-x_{n-1}\|-|\beta|\theta\|x_{n-1}-x_{n-2}\|^2+\beta^2\|x_{n-1}-x_{n-2}\|^2\\
&\quad-|\beta|\|x_{n+1}-x_n\|^2-|\beta|\|x_{n-1}-x_{n-2}\|^2\\
&=(1-|\beta|-\theta)\|x_{n+1}-x_n\|^2+(\theta^2-\theta-|\beta|\theta)\|x_n-x_{n-1}\|^2\\
&\quad+(\beta^2-|\beta|-|\beta|\theta)\|x_{n-1}-x_{n-2}\|^2. \qquad (34)
\end{aligned}
$$

Substituting (33) and (34) in (32) with $\beta\leq 0$ we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq(1+\theta)\|x_n-p\|^2-(\theta-\beta)\|x_{n-1}-p\|^2-\beta\|x_{n-2}-p\|^2\\
&\quad+(1+\theta)(\theta-\beta)\|x_n-x_{n-1}\|^2+\beta(1+\theta)\|x_n-x_{n-2}\|^2\\
&\quad-\beta(\theta-\beta)\|x_{n-1}-x_{n-2}\|^2-\Big(\frac{1-\mu}{1+\mu}\Big)(1-|\beta|-\theta)\|x_{n+1}-x_n\|^2\\
&\quad-\Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2-\theta-|\beta|\theta)\|x_n-x_{n-1}\|^2-\Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2-|\beta|-|\beta|\theta)\|x_{n-1}-x_{n-2}\|^2\\
&\leq(1+\theta)\|x_n-p\|^2-(\theta-\beta)\|x_{n-1}-p\|^2-\beta\|x_{n-2}-p\|^2+[(1+\theta)(\theta-\beta)\\
&\quad-\Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2-\theta+\beta\theta)]\|x_n-x_{n-1}\|^2\\
&\quad-[\beta(\theta-\beta)+\Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2+\beta+\beta\theta)]\|x_{n-1}-x_{n-2}\|^2\\
&\quad-\Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_{n+1}-x_n\|^2.
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
&\|x_{n+1}-p\|^2-\theta\|x_n-p\|^2-\beta\|x_{n-1}-p\|^2+\Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_{n+1}-x_n\|^2\\
&\leq\|x_n-p\|^2-\theta\|x_{n-1}-p\|^2-\beta\|x_{n-2}-p\|^2+\Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_n-x_{n-1}\|^2\\
&\quad+[(1+\theta)(\theta-\beta)-(\frac{1-\mu}{1+\mu})(\theta^2-2\theta+\beta\theta+\beta+1)]\|x_n-x_{n-1}\|^2\\
&\quad-[\beta(\theta-\beta)+\Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2+\beta+\beta\theta)]\|x_{n-1}-x_{n-2}\|^2. \qquad (35)
\end{aligned}
$$

Define

$$\Gamma_n=\|x_n-p\|^2-\theta\|x_{n-1}-p\|^2-\beta\|x_{n-2}-p\|^2+\Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_n-x_{n-1}\|^2.$$

Now, (35) can be written as

$$\Gamma_{n+1} \le \Gamma_n + [(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1)]\|x_n - x_{n-1}\|^2 \\ - [\beta(\theta-\beta) + \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 + \beta + \beta\theta)]\|x_{n-1} - x_{n-2}\|^2. \tag{36}$$

Next, we show that $\Gamma_n \ge 0,\ \forall n \ge 1$. Now,

$$\begin{aligned}\Gamma_n &= \|x_n - p\|^2 - \theta\|x_{n-1} - p\|^2 - \beta\|x_{n-2} - p\|^2 + \Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_n - x_{n-1}\|^2 \\ &\ge \|x_n - p\|^2 - 2\theta\|x_n - x_{n-1}\|^2 - 2\theta\|x_n - p\|^2 - \beta\|x_{n-2} - p\|^2 \\ &\quad + \Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta)\|x_n - x_{n-1}\|^2 \\ &= (1-2\theta)\|x_n - p\|^2 + \Big[\Big(\frac{1-\mu}{1+\mu}\Big)(1+\beta-\theta) - 2\theta\Big]\|x_n - x_{n-1}\|^2 \\ &\quad - \beta\|x_{n-2} - p\|^2. \end{aligned} \tag{37}$$

Since $\theta < \frac{1}{2},\ \beta \le 0,\ \frac{\theta(3+\mu)}{1-\mu} - 1 \le \beta$ and $0 \le \theta \le \frac{1-\mu}{3+\mu}$, it follows from (37) that $\Gamma_n \ge 0,\ \ \forall n \ge 1$.

Furthermore, we derive from (36) that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n &\le [(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1)]\|x_n - x_{n-1}\|^2 - [\beta(\theta-\beta) \\ &\quad + \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 + \beta + \beta\theta)]\|x_{n-1} - x_{n-2}\|^2 \\ &= -[(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1)](\|x_{n-1} - x_{n-2}\|^2 - \|x_n - x_{n-1}\|^2) \\ &\quad + [(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1) - \beta(\theta-\beta) \\ &\quad - \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 + \beta + \beta\theta)]\|x_{n-1} - x_{n-2}\|^2. \end{aligned}$$

From the last inequality, we obtain

$$\Gamma_{n+1} - \Gamma_n \le k_1(\|x_{n-1} - x_{n-2}\|^2 - \|x_n - x_{n-1\|^2}) - k_2\|x_{n-1} - x_{n-2}\|^2, \tag{38}$$

where

$$\begin{aligned} k_1 &:= -[(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1)]; \\ k_2 &:= -[(1+\theta)(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\theta^2 - 2\theta + \beta\theta + \beta + 1) \\ &\qquad - \beta(\theta-\beta) - \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 + \beta + \beta\theta)]. \end{aligned}$$

By Assumption 3.2 (b), we get

$$\frac{1}{2}\Big[\theta(1+\mu) - \frac{(1-\mu)(1-\theta)^2}{1+\theta}\Big] < \beta. \tag{39}$$

As a result $k_1 > 0$ and $k_2 > 0$ by Assumption 3.2 (c). Moreover, from (38) we get

$$\Gamma_{n+1} + k_1\|x_n - x_{n-1}\|^2 \leq \Gamma_n + k_1\|x_{n-1} - x_{n-2}\|^2 - k_2\|x_{n-1} - x_{n-2}\|. \tag{40}$$

Letting $\Gamma_n^{'} = \Gamma_n + k_1\|x_{n-1} - x_{n-2}\|^2$. Then $\Gamma_n^{'} \geq 0$ for all $n \geq 1$. Also, it follows from (40) that

$$\Gamma_{n+1}^{'} \leq \Gamma_n^{'}. \tag{41}$$

By (41), it implies that the sequence $\{\Gamma_n^{'}\}$ is decreasing and bounded from below and thus $\lim\limits_{n\to\infty} \Gamma_n^{'}$ exists. Consequently, we get from (40) that

$$\lim_{n\to\infty} k_2\|x_{n-1} - x_{n-2}\|^2 = 0.$$

Hence,

$$\lim_{n\to\infty} \|x_{n-1} - x_{n-2}\| = 0. \tag{42}$$

Consequently,

$$\begin{aligned}\|x_{n+1} - w_n\| &= \|x_{n+1} - x_n - \theta(x_n - x_{n-1}) - \beta(x_{n-1} - x_{n-2})\| \\ &\leq \|x_{n+1} - x_n\| + \theta\|x_n - x_{n-1}\| + \beta\|x_{n-1} - x_{n-2}\| \to 0,\ n\to\infty;\end{aligned} \tag{43}$$

and

$$\begin{aligned}\|x_n - w_n\| &= \|x_n - x_n - \theta(x_n - x_{n-1}) - \beta(x_{n-1} - x_{n-2})\| \\ &\leq \theta\|x_n - x_{n-1}\| + \beta\|x_{n-1} - x_{n-2}\| \to 0,\ n\to\infty\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - w_n\| &= \|y_n - \lambda_n(Fy_n - Fw_n) - w_n\| \\ &\geq \|y_n - w_n\| - \lambda_n\|Fy_n - Fw_n\| \\ &\geq \|y_n - w_n\| - \mu\|y_n - w_n\| \\ &\geq (1-\mu)\|y_n - w_n\|.\end{aligned} \tag{44}$$

It follows from (43) and (44) that

$$\|w_n - y_n\| \to 0,\ n\to\infty; \tag{45}$$

and

$$\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| \to 0,\ n\to\infty.$$

By (42) and the existence of $\lim_{n\to\infty} \Gamma'_n$, we have that $\lim_{n\to\infty} \Gamma_n$ exists and hence $\{\Gamma_n\}$ is bounded. Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have from the definition of Γ_n that

$$\lim_{n\to\infty} (\|x_n - p\|^2 - \theta\|x_{n-1} - p\|^2 - \beta\|x_{n-2} - p\|^2) \tag{46}$$

exists. Using the boundedness of $\{\Gamma_n\}$, we obtain from (37) that $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ and $\{w_n\}$ are bounded. □

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 such that Assumption 3.1 (i)-(iv) and Assumption 3.2 (a)-(c) are satisfied, and suppose $\lim_{n\to\infty} \|w_n - y_n\| = 0$. If v^* is one of the weak cluster points of $\{y_n\}$, then we have at least one of the following : $v^* \in S_D$ or $Fv^* = 0$.*

Proof. By Lemma 4.2, $\{y_n\}$ is bounded. Hence, let v^* be a weak cluster point of $\{y_n\}$. Then, we can choose a subsequence of $\{y_n\}$, denoted by $\{y_{n_k}\}$ such that $y_{n_k} \rightharpoonup v^* \in C$.

We consider the following two possible cases.

CASE 1 Suppose that $\limsup_{k\to\infty} ||Fy_{n_k}|| = 0$. Then, $\lim_{k\to\infty} ||Fy_{n_k}|| = \liminf_{k\to\infty} ||Fy_{n_k}|| = 0$.

Since F satisfies Assumption 3.1 (iii) and $y_{n_k} \rightharpoonup v^* \in C$, we have

$$0 \leq ||Fv^*|| \leq \liminf_{k\to\infty} ||Fy_{n_k}|| = 0. \tag{47}$$

This implies that $Fv^* = 0$.

CASE 2 Suppose that $\limsup_{k\to\infty} ||Fy_{n_k}|| > 0$. Then, without loss of generality, we can choose a subsequence of $\{Fy_{n_k}\}$ still denoted by $\{Fy_{n_k}\}$ such that $\lim_{k\to\infty} ||Fy_{n_k}|| = M_1 > 0$. Also, by the characteristics property of P_C, we obtain for all $x \in C$ that

$$\langle w_{n_k} - \lambda_{n_k} F w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0.$$

This implies that

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Fw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Fw_{n_k}, x - w_{n_k}\rangle. \tag{48}$$

Next, we show that $\liminf_{k\to\infty}\langle Fw_{n_k},\ x - w_{n_k}\rangle \geq 0, \quad \forall\, x \in C$. We consider the following two cases for λ_{n_k}:

Case I: Suppose that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. Since $\{w_{n_k}\}$ is bounded and F is uniformly continuous, it follows that $\{Fw_{n_k}\}$ is bounded. Thus, by (45) and the fact that $\liminf_{k\to\infty} \lambda_{n_k} > 0$, we have

$$0 \leq \liminf_{k\to\infty}\langle Fw_{n_k}, x - w_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Fw_{n_k}, x - w_{n_k}\rangle < +\infty, \quad \forall x \in C.$$

Case II: Suppose that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Let $t_{n_k} = P_C(w_{n_k} - \lambda_{n_k} l^{-1} F w_{n_k})$. From Lemma 2.7 and the fact that $\lambda_{n_k} l^{-1} > \lambda_{n_k}$ we have,

$$l\|w_{n_k} - t_{n_k}\| \leq \|w_{n_k} - y_{n_k}\| \to 0, \quad \text{as} \quad k \to \infty.$$

Therefore, it follows that $t_{n_k} \rightharpoonup v^* \in C$ and $\{t_{n_k}\}$ is bounded.
Since $\lim_{k\to\infty} \|w_{n_k} - t_{n_k}\| = 0$ and F is uniformly continuous, we have

$$\lim_{k\to\infty} \|Fw_{n_k} - Ft_{n_k}\| = 0. \tag{49}$$

From (17), we obtain

$$\lambda_{n_k} l^{-1} \|Fw_{n_k} - FP_C(w_{n_k} - \lambda_{n_k} l^{-1} F w_{n_k})\| > \mu \|w_{n_k} - P_C(w_{n_k} - \lambda_{n_k} l^{-1} F w_{n_k}).$$

From this, we have

$$\frac{1}{\mu}\|Fw_{n_k} - Ft_{n_k}\| > \frac{1}{\lambda_{n_k} l^{-1}}\|w_{n_k} - t_{n_k}\|. \tag{50}$$

Applying (49) to (50) we have

$$\lim_{k\to\infty} \frac{1}{\lambda_{n_k} l^{-1}}\|w_{n_k} - t_{n_k}\| = 0.$$

From the definition of t_{n_k} and by (16), we have

$$0 \leq \left\langle t_{n_k} - w_{n_k} + \lambda_{n_k} l^{-1} F w_{n_k},\ x - t_{n_k} \right\rangle.$$

From this, we obtain

$$\frac{1}{\lambda_{n_k} l^{-1}} \langle w_{n_k} - t_{n_k},\ x - t_{n_k} \rangle + \langle F w_{n_k},\ t_{n_k} - w_{n_k} \rangle \leq \langle F w_{n_k},\ x - w_{n_k} \rangle, \quad \forall \ x \in C.$$

Taking the limit as $k \to \infty$ in the last inequality, we obtain

$$0 \leq \liminf_{k\to\infty} \langle F w_{n_k}, x - w_{n_k} \rangle \leq \limsup_{k\to\infty} \langle F w_{n_k}, x - w_{n_k} \rangle < +\infty, \quad \forall x \in C. \tag{51}$$

which is the desired result.
Now, note that

$$\begin{aligned} \langle F y_{n_k}, x - y_{n_k} \rangle &= \langle F y_{n_k} - F w_{n_k}, x - w_{n_k} \rangle + \langle F w_{n_k}, x - w_{n_k} \rangle \\ &\quad + \langle F y_{n_k}, w_{n_k} - y_{n_k} \rangle. \end{aligned} \tag{52}$$

Since F is uniformly continuous on C, we obtain from (45) that $\lim_{k\to\infty} \|F w_{n_k} - F y_{n_k}\| = 0$. Hence, we obtain from (45), (51) and (52) that

$$0 \leq \liminf_{k\to\infty} \langle F y_{n_k}, x - y_{n_k} \rangle \leq \limsup_{k\to\infty} \langle F y_{n_k}, x - y_{n_k} \rangle < +\infty, \quad \forall x \in C. \tag{53}$$

Based on (53), we consider the following cases under **Case II,** as follows:

CASE A Suppose that $\limsup_{k\to\infty}\langle Fy_{n_k}, x-y_{n_k}\rangle > 0, \forall x \in C$. Then we can choose a subsequence of $\{y_{n_k}\}$ denoted by $\{y_{n_{k_j}}\}$ such that $\lim_{j\to\infty}\langle Fy_{n_{k_j}}, x-y_{n_{k_j}}\rangle > 0$. Thus there exist $j_0 \geq 1$ such that

$$\langle Fy_{n_{k_j}}, x - y_{n_{k_j}}\rangle > 0, \quad \forall j \geq j_0,$$

which by the quasimonotonicity of F on C, implies that

$$\langle Fx, x - y_{n_{k_j}}\rangle \geq 0, \quad \forall x \in C, j \geq j_0.$$

Hence, letting $j \to \infty$, we get that

$$\langle Fx, x - v^*\rangle \geq 0, \quad \forall x \in C.$$

Therefore, $v^* \in S_D$.

CASE B Suppose that $\limsup_{k\to\infty}\langle Fy_{n_k}, x-y_{n_k}\rangle = 0, \forall\, x \in C$. Then by (53), we get

$$\lim_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle = 0, \quad \forall \quad x \in C, \tag{54}$$

from which we get

$$\langle Fy_{n_k}, x - y_{n_k}\rangle + |\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1} > 0 \quad , \forall \quad x \in C. \tag{55}$$

Also, since $\lim_{k\to\infty}\|Fy_{n_k}\| = M_1 > 0$, we can find $k_0 \geq 1$ such that $||Fy_{n_k}|| > \frac{M_1}{2}$ $\forall k \geq k_0$. Hence, we can set $b_{n_k} = \frac{Fy_{n_k}}{\|Fy_{n_k}\|^2}$ $\forall\, k \geq k_0$. Thus $\langle Fy_{n_k}, b_{n_k}\rangle = 1\ \forall\ k \geq k_0$. Therefore, by (55) we get

$$\left\langle Fy_{n_k}, x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1} - y_{n_k}\right]\right\rangle > 0,$$

and using the quasimonotonicity of F on H, we obtain

$$\begin{aligned}\Bigg\langle F\left(x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right]\right)&,\\ x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\Bigg\rangle &\geq 0.\end{aligned}$$

Now, by applying (14) and the last inequality we have

$$\begin{aligned}&\langle Fx, x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\rangle\\ &\geq \langle Fx - F(x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right]),\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\rangle.\\
&\geq -\|Fx - F(x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right])\|\\
&\quad \times \|x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\|\\
&\geq -(K+\epsilon_1)\|b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right]\|.\|\\
&\qquad\qquad\qquad x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\|\\
&= \frac{-(K+\epsilon_1)}{\|Fy_{n_k}\|}\left(|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right)\|\\
&\qquad\qquad\qquad x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\|\\
&\geq \frac{-2(K+\epsilon_1)}{M_1}\left(|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right)M_2, \qquad (56)
\end{aligned}
$$

where $\epsilon = \epsilon_1\|b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right]\|$ for some $\epsilon_1 \in (0,1), K < +\infty$ and for some $M_2 > 0$, where the existence of M_2 is from the boundedness of $\{x + b_{n_k}\left[|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right] - y_{n_k}\}$. Now, we observe that (54) implies that
$\lim_{k\to\infty}\left(|\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}\right) = 0$. Thus, as $k \to \infty$ in (56), we get that

$$\langle Fx, x - v^*\rangle \geq 0, \quad \forall x \in C.$$

Therefore, $v^* \in S_D$. □

THEOREM 4.4. *Suppose $\{x_n\}$ is a sequence generated by Algorithm 3.3. Then under Assumptions 3.1 and Assumptions 3.2 and $Fx \neq 0, \forall x \in C$. Then $\{x_n\}$ converges weakly to an element of $S_D \subset S$.*

Proof. Suppose $w_\omega(x_n)$ is the set of weak cluster points of $\{x_n\}$. We show that

$$w_\omega(x_n) \subset S_D.$$

Take $v^* \in w_\omega(x_n)$. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup v^*,\ k \to \infty$. Since C is weakly closed, we have that $v^* \in C$. Since $Fx \neq 0,\ \forall x \in C$, we have $Fv^* \neq 0$. By Lemma, we get $v^* \in S_D$. Therefore, $w_\omega(x_n) \subset S_D$. Since $\lim_{n\to\infty}\Gamma_n$ exists and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have that

$$\lim_{n\to\infty}[\|x_n - p\|^2 - \theta\|x_{n-1} - p\|^2 - \beta\|x_{n-2} - p\|^2]$$

exists for all $p \in S_D$.

We now show show that $x_n \rightharpoonup x^* \in S_D$. Suppose now that there exists $\{x_{n_j}\} \subset \{x_n\}$ and $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup v^*, j \to \infty$ and $x_{n_m} \rightharpoonup x^*, m \to \infty$. we show that $v^* = x^*$ Observe that

$$2\langle x_n, x^* - v^* \rangle = \|x_n - v^*\|^2 - \|x_n - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2; \tag{57}$$

$$2\langle x_{n-1}, x^* - v^* \rangle = \|x_{n-1} - v^*\|^2 - \|x_{n-1} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2;$$

and

$$2\langle x_{n-2}, x^* - v^* \rangle = \|x_{n-2} - v^*\|^2 - \|x_{n-2} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2.$$

Therefore,

$$2\langle -\theta x_{n-1}, x^* - v^* \rangle = -\theta\|x_{n-1} - v^*\|^2 + \theta\|x_{n-1} - x^*\|^2 + \theta\|v^*\|^2 - \theta\|x^*\|^2. \tag{58}$$

and

$$\begin{aligned} 2\langle -\beta x_{n-2}, x^* - v^* \rangle &= -\beta\|x_{n-2} - v^*\|^2 + \beta\|x_{n-2} - x^*\|^2 \\ &\quad +\beta\|v^*\|^2 - \beta\|x^*\|^2. \end{aligned} \tag{59}$$

Addition of (57), (58) and (59) gives

$$\begin{aligned} 2\langle x_n - \theta x_{n-1} - \beta x_{n-2}, x^* - v^* \rangle &= \big(\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \beta\|x_{n-2} - v^*\|^2\big) \\ &\quad - (\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 \\ &\quad - \beta\|x_{n-2} - x^*\|^2) \\ &\quad + (1 - \theta - \beta)(\|x^*\| - \|v^*\|^2). \end{aligned} \tag{60}$$

Observe that by (46), we know that

$$\lim_{n\to\infty} [\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \beta\|x_{n-2} - x^*\|^2] \tag{61}$$

exists and

$$\lim_{n\to\infty} [\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \beta\|x_{n-2} - v^*\|^2] \tag{62}$$

also exists. This implies from (60) that $\lim\limits_{n\to\infty} \langle x_n - \theta x_{n-1} - \beta x_{n-2}, x^* - v^* \rangle$ exists. Consequently,

$$\begin{aligned} \langle v^* - \theta v^* - \beta v^*, x^* - v^* \rangle &= \lim_{j\to\infty} \langle x_{n_j} - \theta x_{n_{j-1}} - \beta x_{n_{j-2}}, x^* - v^* \rangle \\ &= \lim_{n\to\infty} \langle x_n - \theta x_{n-1} - \beta x_{n-2}, x^* - v^* \rangle \\ &= \lim_{m\to\infty} \langle x_{n_m} - \theta x_{n_{m-1}} - \beta x_{n_{m-2}}, x^* - v^* \rangle \\ &= \langle x^* - \theta x^* - \beta x^*, x^* - v^* \rangle. \end{aligned}$$

Hence,

$$(1 - \theta - \beta)\|x^* - v^*\|^2 = 0.$$

Since $\beta \leq 0 < 1 - \theta$, we obtain that $x^* = v^*$. Hence, $\{x_n\}$ converges weakly to a point in S_D. This completes the proof. $\square$

5. Linear convergence. In this section, we give the linear convergence of our Proposed Algorithm 3.3, when the cost operator F in (1) is α-strongly pseudomonotone and $\beta = 0$. It was shown in [28] that whenever F is α-strongly pseudomonotone, then the VIP (1) has a unique solution.

Theorem 5.1. *Assume that F is $L-$ Lipschitz continuous on H and α-strongly pseudomonotone. Then with $\beta = 0$, Algorithm 3.3 reduced to*

Algorithm 5.2.

Initialization: *Given $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$. Let x_{-1}, x_0, $x_1 \in H$ be arbitrary points and given x_{n-1}, x_n.*

Iterative steps: *Calculate x_{n+1} as follows*

Step 1: *Set $w_n = x_n + \theta(x_n - x_{n-1})$ and compute*

$$y_n = P_C(w_n - \lambda_n F w_n),$$

where $\lambda_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\lambda_n \|Fw_n - Fy_n\| \leq \mu \|w_n - y_n\|. \tag{63}$$

If $y_n = w_n$ then stop: y_n is a solution of the problem (VIP). Otherwise,
Step 2: *Compute*

$$x_{n+1} = y_n - \lambda_n(Fy_n - Fw_n),$$

Set $n := n + 1$ and go to **step 1**.

Let $\omega \in (0,1)$ and θ be such that

$$0 \leq \theta \leq \min\left\{ \frac{\psi}{2+\psi}, \frac{\sqrt{(1+\omega\psi)^2 + 4\omega\psi} - (1+\omega\psi)}{2}, \right.$$
$$\left. (1-\omega)\left(1 - \frac{(1-\psi)(1-\mu)}{2(1+\mu)}\right) \right\}, \tag{64}$$

where $\psi := \delta \frac{(1-\mu)}{(1+\mu)}$. Then the sequence $\{x_n\}$ generated by Algorithm 5.2 converges R-linearly to the unique solution p of (1).

Proof. Under the assumption made, (1) has a unique solution, see [28]. From the α-stronly pseudomonotonicity of F, we have

$$\langle Fy_n, y_n - p\rangle \geq \alpha \|y_n - p\|^2. \tag{65}$$

Thus, we get

$$-2\langle \lambda_n F y_n, y_n - p\rangle \leq -2\alpha\lambda_n \|y_n - p\|^2. \tag{66}$$

We obtain from (28), (30), (31) and (66) that

$$\|x_{n+1} - p\|^2 = \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda_n^2 \|Fy_n - Fw_n\|^2 - 2\alpha\lambda_n \|y_n - p\|^2$$

$$\leq \quad \|w_n - p\|^2 - \frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 - 2\alpha\lambda_n\|y_n - p\|^2. \qquad (67)$$

Now, we show that $\lambda_n > \frac{\mu l}{L}$ for all n. Indeed, by the search rule (63), we know that $\frac{\lambda_n}{l}$ must violate inequality (63) i.e.

$$\|Fy_n - Fw_n\| > \frac{\mu}{\lambda_n/l}\|y_n - w_n\|.$$

This follows that

$$L\|y_n - w_n\| > \frac{\mu}{\lambda_n/l}\|y_n - w_n\|.$$

Thus,

$$L > \frac{\mu}{\lambda_n/l}.$$

That is

$$\lambda_n > \frac{\mu l}{L} \ \forall \ n. \qquad (68)$$

Combining (67) and (68), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|w_n - p\|^2 - \frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 - 2\alpha\frac{\mu l}{L}\|y_n - p\|^2 \\ &= \|w_n - p\|^2 - \delta\frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 - (1-\delta)\frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 \\ &\quad - 2\alpha\frac{\mu l}{L}\|y_n - p\|^2. \end{aligned} \qquad (69)$$

Now,

$$\begin{aligned}\|x_{n+1} - w_n\| &= \|y_n - \lambda_n(Fy_n - Fw_n) - w_n\| \\ &\geq \|y_n - w_n\| - \lambda_n\|Fy_n - Fw_n\| \\ &\geq \|y_n - w_n\| - \mu\|y_n - w_n\| \\ &\geq (1-\mu)\|y_n - w_n\|.\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \delta\frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 - (1-\delta)\frac{(1-\mu)^2}{(1+\mu)}\|y_n - w_n\|^2 \\ &\quad - 2\alpha\frac{\mu l}{L}\|y_n - p\|^2 \\ &\leq \|w_n - p\|^2 - \delta\frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2 - 2\Lambda(\|w_n - y_n\|^2 + \|y_n - p\|^2) \\ &= (1-\Lambda)\|w_n - p\|^2 - \delta\frac{(1-\mu)}{(1+\mu)}\|x_{n+1} - w_n\|^2, \end{aligned} \qquad (70)$$

where $\Lambda := \frac{1}{2}\min\left\{(1-\psi)\frac{(1-\mu)^2}{(1+\mu)}; \frac{2\alpha\mu l}{L}\right\}$. Let $\varphi = (1-\Lambda)$. Then $\psi := \delta\frac{(1-\mu)}{(1+\mu)}$. From (70) we get

$$\|x_{n+1} - p\|^2 \leq \varphi\|w_n - p\|^2 - \psi\|x_{n+1} - w_n\|^2. \tag{71}$$

We have

$$\begin{aligned} \|w_n - p\|^2 &= \|(1+\theta)(x_n - p) - \theta(x_{n-1} - p)\|^2 \\ &\leq (1+\theta)\|x_n - p\|^2 - \theta\|x_{n-1} - p\|^2 + \theta(1+\theta)\|x_n - x_{n-1}\|^2 \end{aligned} \tag{72}$$

and

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \theta(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - w_n\|^2 + \theta^2\|x_n - x_{n-1}\|^2 - 2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\geq \|x_{n+1} - x_n\|^2 + \theta^2\|x_n - x_{n-1}\|^2 - 2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\geq \|x_{n+1} - x_n\|^2 + \theta^2\|x_n - x_{n-1}\|^2 - \theta\|x_{n+1} - x_n\|^2 - \theta\|x_n - x_{n-1}\|^2 \\ &\geq (1-\theta)\|x_{n+1} - x_n\|^2 - \theta(1-\theta)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{73}$$

Substituting the inequalities (72) and (73) into (71), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 \quad \leq \quad & \varphi(1+\theta)\|x_n - p\|^2 - \varphi\theta\|x_{n-1} - p\|^2 + \varphi\theta(1+\theta)\|x_n - x_{n-1}\|^2 \\ & -\psi(1-\theta)\|x_{n+1} - x_n\|^2 + \psi\theta\|x_n - x_{n-1}\|^2, \end{aligned}$$

or equivalently

$$\begin{gathered} \|x_{n+1} - p\|^2 - \varphi\theta\|x_n - p\|^2 + \psi(1-\theta)\|x_{n+1} - x_n\|^2 \\ \varphi[\|x_n - p\|^2 - \theta\|x_{n-1} - p\|^2 + \psi(1-\theta)\|x_n - x_{n-1}\|^2] \\ -(\varphi\psi(1-\theta) - \varphi\theta(1+\theta) - \psi\theta(1-\theta))\|x_n - x_{n-1}\|^2 \end{gathered}$$

Setting

$$a_n := \|s_n - p\|^2 - \theta\|x_{n-1} - p\|^2 + \psi(1-\theta)\|x_n - x_{n-1}\|^2,$$

since $\varphi \in (0,1)$, we can write

$$\begin{aligned} a_{n+1} &\leq \|x_{n+1} - p\|^2 - \varphi\theta\|x_n - p\|^2 + \psi(1-\theta)\|x_{n+1} - x_n\|^2 \\ &\leq \varphi a_n - (\varphi\psi(1-\theta) - \varphi\theta(1+\theta) - \psi\theta(1-\theta))\|x_n - x_{n-1}\|^2. \end{aligned}$$

Note that from (64), we have

$$\begin{aligned} \theta &\leq (1-\omega)\left(1 - \frac{(1-\psi)(1-\mu)}{2(1+\mu)}\right) \\ &\leq (1-\omega)(1-\Lambda) = (1-\omega)\varphi, \end{aligned}$$

which implies

$$\psi\theta(1-\theta) \leq (1-\omega)\varphi\psi(1-\theta). \tag{74}$$

Since

$$0 \leq \frac{\sqrt{(1+\omega\psi)^2 + 4\omega\psi} - (1+\omega\psi)}{2},$$

it holds

$$\theta^2 + (1+\omega\psi)\theta - \omega\psi \leq 0,$$

or equivalently

$$\theta(1+\theta) \leq \omega\psi(1-\theta).$$

Hence

$$\varphi\theta(1+\theta) \leq \omega\varphi\psi(1-\theta). \tag{75}$$

From (74) and (75), we deduce

$$\varphi\psi(1-\theta) - \varphi\theta(1+\theta) - \psi\theta(1-\theta) \geq 0. \qquad \square$$

Moreover, since $\theta \leq \frac{\psi}{2+\psi}$, we have $\theta \leq \frac{\psi(1-\theta)}{2}$, which implies

$$\begin{aligned} a_n &= (1-\psi(1-\theta))\|x_n - p\|^2 + \psi(1-\theta)(\|x_n - p\|^2 + \|x_n - x_{n-1}\|^2) - \theta\|x_{n-1} - p\|^2 \\ &\geq (1-\psi(1-\theta))\|x_n - p\|^2 + \frac{\psi(1-\theta)}{2}\|x_{n-1} - p\|^2 - \theta\|x_{n-1} - p\|^2 \\ &\geq (1-\psi(1-\theta))\|x_n - p\|^2 \geq 0. \end{aligned}$$

Therefore, we deduce

$$a_{n+1} \leq \varphi a_n \leq \cdots \leq \varphi^n a_1.$$

This follows that

$$\|x_n - p\|^2 \leq \frac{a_1}{\varphi(1-\psi(1-\theta))}\varphi^n,$$

which implies that $\{x_n\}$ converges R-linearly to p.

6. Numerical examples.

In this section, we provide some numerical examples to demonstrate the performance of our Algorithm (Proposed Alg.) 3.3 in comparison with Appendix 7.1 proposed by Liu & Yang (Liu & Yang), Appendix 7.2 by Ogwo *et al.* (Ogwo *et al.* Alg.), Appendix 7.3 by Ye & He (Ye & He), Appendix 7.4 by Yin & Hussain (Yin & Hussain Alg.) and Appendix 7.5 by Izuchukwu *et al.* (Izuchukwu *et al.* Alg.). All numerical computations were carried out using Matlab version R2021(b) and we use $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion.

EXAMPLE 6.1. ([32]) Let $C = [-1, 1]$

$$Ft = \begin{cases} 2t-1, & t > 1, \\ t^2, & t \in [-1,1], \\ -2t-1, & t < -1. \end{cases}$$

Given that F is quasimonotone and Lipschitz continuous with $S_D = \{-1\}$ and $S = \{-1, 0\}$.

We choose different initial values x_{-1}, x_0 and x_1 as follows:
Case 1: $x_{-1} = 0.9, x_0 = 0.8,\ x_1 = -1.0,$
Case 2: $x_{-1} = 0.4, x_0 = 0.5,\ x_1 = -1.0,$
Case 3: $x_{-1} = 1.0, x_0 = 1.0,\ x_1 = -1.0,$
Case 4: $x_{-1} = 0.7, x_0 = 0.5,\ x_1 = -1.0$

Table 1: Methods Parameters for Examples 6.1- 6.3

Liu & Yang Alg.	$\mu = 0.7$	$\lambda = 0.11$	$\rho_n = \dfrac{1000}{(n+1)^5}$		
Ogwo *et al. Alg.*	$\mu = 0.7$	$\lambda = 0.11$	$\tau = 0.09$	$\rho_n = \dfrac{1000}{(n+1)^5}$	$\psi_n = \dfrac{n+1}{5(n+3)}$
Ye & He Alg.	$\lambda = 0.11$	$\sigma = 0.54$	$l = 0.50$		
Yin & Hussain Alg.	$\lambda = 0.11$	$\mu = 0.70$	$\tau = 0.09$		
Izuchukwu *et al. Alg.*	$\lambda_0 = 0.25$	$\lambda_1 = 0.11$	$\eta = 0.70$	$\rho_n = \dfrac{1000}{(n+1)^5}$	
Proposed Alg. 3.3	$\mu = 0.10$	$\theta = 0.04$	$\beta = -0.02$	$l = 0.5$	

Table 2: Numerical Results for Example 6.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	296	0.0140	282	0.0125	297	0.0142	292	0.0131
Ogwo *et al. Alg.*	830	0.0592	766	0.0591	856	0.0067	754	0.0484
Ye & He Alg.	405	0.0042	380	0.0042	407	0.0044	399	0.0041
Yin & Hussain Alg.	933	0.0048	733	0.0045	949	0.0051	894	0.0042
Izuchukwu *et al. Alg.*	302	0.0038	292	0.0040	305	0.0040	295	0.0037
Proposed Alg. 3.3	6	0.0033	6	0.0037	6	0.0035	6	0.0051

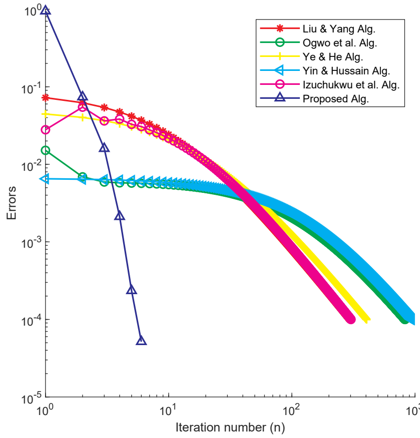

Figure 1: Example 6.1 Case 1

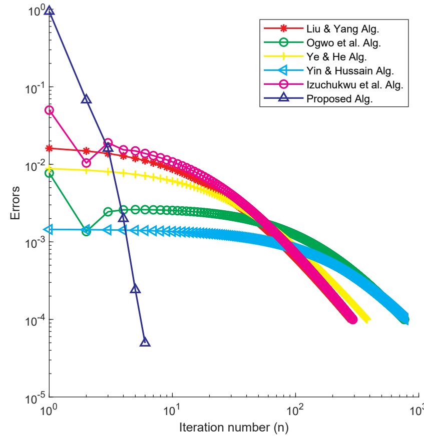

Figure 2: Example 6.1 Case 2

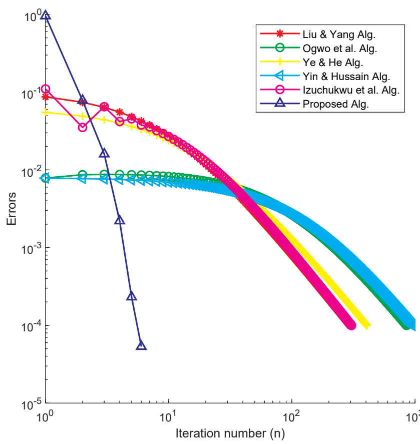

Figure 3: Example 6.1 Case 3

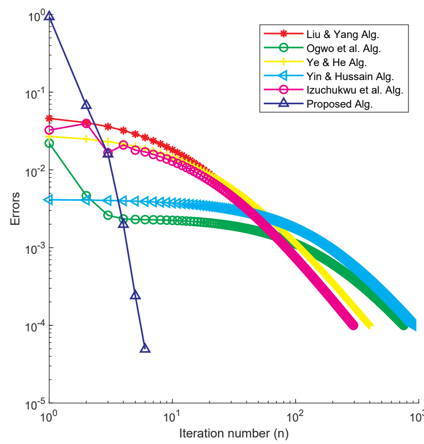

Figure 4: Example 6.1 Case 4

EXAMPLE 6.2. ([32]) Let $C = [0,1]^m$ and $Ft = (h_1 t, h_2 t, ..., h_m t)$, where $h_i t = t_{i-1}^2 + t_i^2 + t_{i-1}t_i + t_i t_{i+1} - 2t_{i-1} + 4t_i + t_{i+1} - 1,\ i = 1,2,...,m,\ t_0 = t_{m+1} = 0$. Then F is quasimonotone.

For this numerical example, we choose $m \in \{30, 60, 90, 120\}$ while x_{-1}, x_0 and x_1 are generated randomly.

Table 3: Numerical Results for Example 6.2

	$m = 30$		$m = 60$		$m = 90$		$m = 120$	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	50	0.0239	51	0.0204	55	0.0204	59	0.0256
Ogwo *et al. Alg.*	80	0.0219	81	0.0221	82	0.1097	82	0.0742
Ye & He Alg.	52	0.0105	49	0.0091	50	0.0218	49	0.0118
Yin & Hussain Alg.	33	0.0084	33	0.0086	36	0.0087	38	0.0092
Izuchukwu *et al. Alg.*	49	0.0093	26	0.0068	20	0.0068	17	0.0079
Proposed Alg. 3.3	23	0.0072	19	0.0057	17	0.0059	16	0.0078

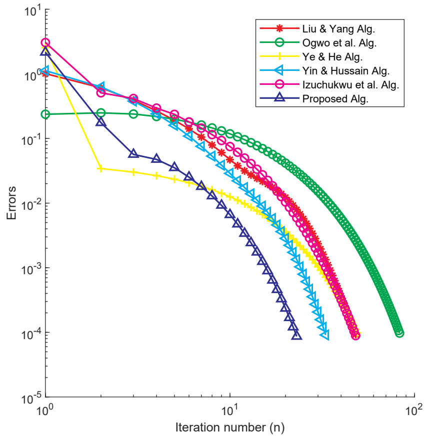

Figure 5: Example 6.2 with $m = 30$

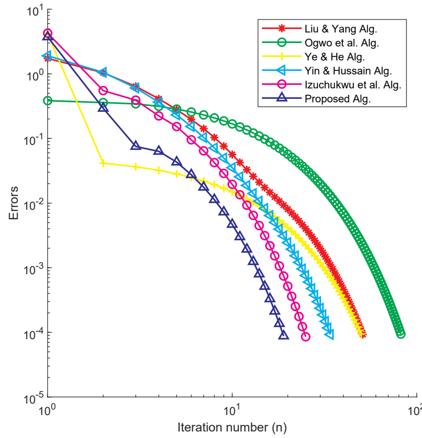

Figure 6: Example 6.2 with $m = 60$

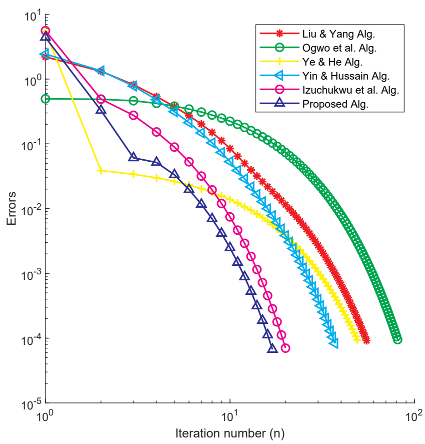

Figure 7: Example 6.2 with $m = 90$

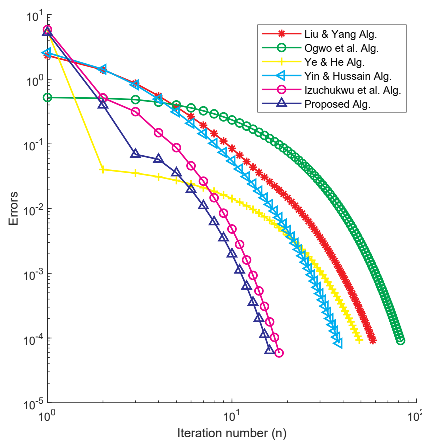

Figure 8: Example 6.2 with $m = 90$

Next, we give an example in an infinite dimensional Hilbert space.

EXAMPLE 6.3. Let $H = \ell_2 := \{x = (x_1, x_2, ..., x_i, ...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. Take $C := \{x \in \ell_2 : \|x\| \leq 3\}$ and $Fx := (x_1 e^{-x_1^2}, 0, 0, ...), x = (x_1, x_2, x_3, ...) \in C$. Then one can see that $S_D = \{0\} \neq \emptyset$. Also, it is easy to show by direct computation that there exists $L > 0$ such that $\|\nabla Fx\| \leq L$, $\forall x \in \ell_2$. Hence, by Mean Value Theorem, F is Lipschitz continuous (see [9]).

Let $A : \ell_2 \to \ell_2$ be defined by $A(x_1, x_2, x_3, ...,) = (x_1, 0, 0, ...); g(x) := \eta(\langle a, x\rangle)$ with $a = (1, 0, 0, ...) \in \ell_2$ and $\eta(t) := e^{-t^2}, t > 0$. Then $Fx = g(x)A(x)$, $x \in \ell_2$ and it is easy to show that A is sequentially weak-to-weak continuous and g is weakly continuous. Hence, for any $\{w_n\} \subset C$ such that $w_n \rightharpoonup v^*$, we have that $\|Fv^*\| \leq \liminf_{n\to\infty}\|Fw_n\|$.

Suppose $x, y \in H$ such that $\langle Fx, y-x\rangle > 0$. Then we have $x_1(y_1-x_1)e^{-x_1^2} > 0$. Since $e^{-x_1^2} > 0$, we have $x_1(y_1-x_1) > 0$. consequently,

$$\begin{aligned}
\langle Fy, y-x\rangle &= \langle (y_1e^{-y_1^2}, 0, 0, ...), (y_1-x_1, y_2-x_2, y_3-x_3, ...)\rangle \\
&= (y_1-x_1)y_1e^{-y_1^2} \\
&\geq e^{-y_1^2}\left(y_1(y_1-x_1) - x_1(y_1-x_1)\right) \\
&= e^{-y_1^2}(y_1-x_1)^2 \geq 0.
\end{aligned}$$

This follows that F is quasimonotone on H. It is easy to see that F is not monotone on H. For example, take $x = (-1, 0, 0, ...)$ and $y = (-2, 0, 0, ...)$. Finally, observe that the set $\{v \in C : Fv = 0\} \setminus S_D = \emptyset$ is a finite set. Hence, Assumption 3.1 is satisfied.

We choose different initial values as follows:

Case 1: $x_{-1} = (1, -0.1, 0.01, ...), x_0 = (-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, ...),\ x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$,
Case 2: $x_{-1} = (-1, \frac{1}{3}, -\frac{1}{9}, ...), x_0 = (-1, 0.1, -0.01, ...),\ x_1 = (\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, ...)$,
Case 3: $x_{-1} = (\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, ...), x_0 = (-1, \frac{1}{3}, -\frac{1}{9}, ...),\ x_1 = (-1, 0.1, -0.01, ...)$,
Case 4: $x_{-1} = (-1, 0.1, -0.01, ...), x_0 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...),\ x_1 = (-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...)$.

Table 4: Numerical Results for Example 6.3

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	119	0.0149	120	0.0130	99	0.0128	119	0.0157
Ogwo *et al.* *Alg.*	581	0.0399	698	0.0571	710	0.0521	579	0.0529
Ye & He Alg.	203	0.0044	204	0.0051	166	0.0048	203	0.0065
Yin & Hussain Alg.	823	0.0055	834	0.0061	600	0.0054	823	0.0110
Izuchukwu *et al.* *Alg.*	93	0.0043	126	0.0045	124	0.0041	94	0.0060
Proposed Alg. 3.3	35	0.0045	35	0.0047	35	0.0049	35	0.0064

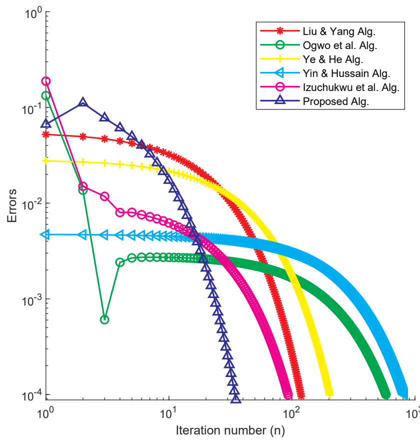

Figure 9: Example 6.3 Case 1

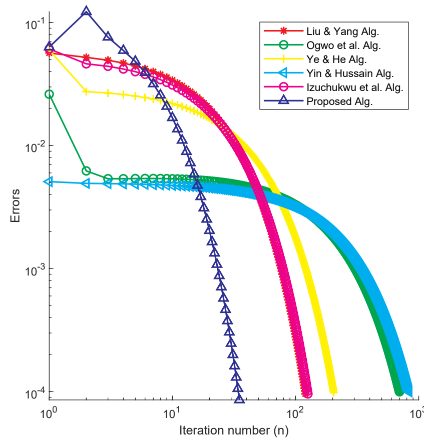

Figure 10: Example 6.3 Case 2

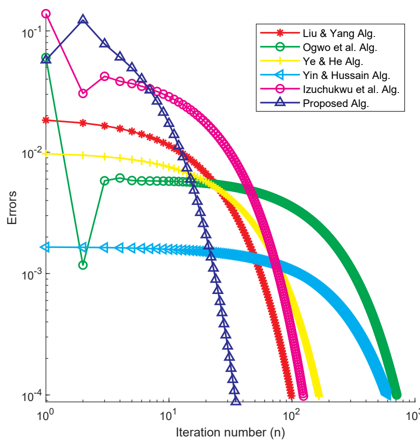

Figure 11: Example 6.3 Case 3

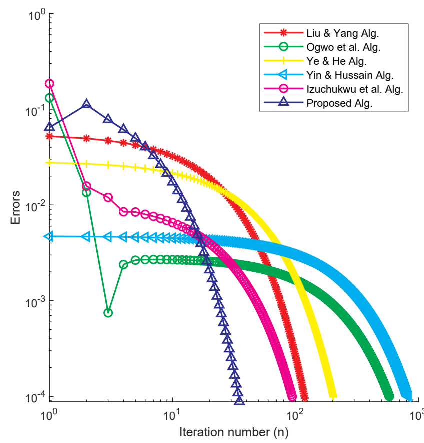

Figure 12: Example 6.3 Case 4

Remark 6.4. By varying the starting points in each of the Examples 6.1 - 6.3, the numerical results shown in Tables 2 - 4 and Figures 1 - 12 were obtained. In addition, we compared our proposed method with the existing methods in Appendices 7.1 - 7.5 in all the examples. The following are noted from the numerical Examples 6.1- 6.3:

- Clearly from Table 2, Table 4 and Figures 1 - 4, 9 - 12, the number of iterations of our proposed method remain consistent, that is, well behaved irrespective of the starting point. For Example 6.2, we note that the number of iterations in our method decreases as m increases. Unfortunately, we did not have similar pattern in any of the existing methods that we compare our method with.

- From Table 2 - 4 and Figures 1 - 12, we can see clearly that in terms of number of iterations and CPU time, our proposed method performs better than these existing methods. Thus, our method is more efficient than these methods.

7. Final remark. In this paper, we have presented a two-step inertial-like algorithm which employs Tseng's extragradient method for solving quasimonotone variational inequalities with non-Lipschitz operator in Hilbert spaces. The algorithm uses Armijo-like step size technique and we established under some standard conditions that the sequence generated by the proposed algorithm converges weakly to a solution of the problem. We established the linear convergence result when the cost operator is strongly pseudo-monotone. Numerical illustrations showed that our method is computationally cheaper than other related methods in the literature.

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Appendix 7.1. *Algorithm 1 of Liu et.al [32]*

Step 0: *Take* $\lambda_0 > 0, \quad x_0 \in H, \quad 0 < \mu < 1$. *Choose a nonnegative real sequence* $\{\rho_n\}$ *such that* $\sum\limits_{n=0}^{\infty} \rho_n < +\infty$.

Step 1: *Given the current iterate* x_n, *compute*

$$y_n = P_C(x_n - \lambda_n F x_n).$$

If $x_n = y_n$, *(or* $Fy_n = 0$*), then stop and* y_n *is a solution of the VIP. Otherwise, go to* **Step 2.**

Step 2: *Compute*

$$x_{n+1} = y_n - \lambda_n(Fy_n - Fx_n),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Fx_n - Fy_n\|}, \lambda_n + \rho_n\right\} & \text{if} \quad Fx_n - Fy_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{76}$$

Set $n := n + 1$ *and go back to* **Step 1**.

Appendix 7.2. *Algorithm 2 of Ogwo et al. [38]*

Step 0: *Choose sequences* $\{\psi_n\}$ *and* $\{\tau_n\}$ *such that* $\psi_n \in [0, 1)$ *and* $\tau_n \in (0, 1]$ *for all* $n \geq 1$. *Let* $\lambda_1 > 0$, $\mu \in (0, 1)$ *and* $x_0, x_1 \in H$ *be given arbitrarily. Choose a nonnegative real sequence* $\{\rho_n\}$ *such that* $\sum\limits_{n=1}^{\infty} \rho_n < \infty$. *Set* $n := 1$.

Step 1: *Given the current iterates* x_{n-1} *and* x_n $(n \geq 1)$, *compute*

$$w_n = x_n + \psi_n(x_n - x_{n-1})$$

and

$$y_n = P_C(w_n - \lambda_n F w_n).$$

If $w_n = y_n$: STOP. Otherwise, go to **Step 2.**
Step 2*: Construct the half-space*

$$T_n = \{x \in H : \langle w_n - \lambda_n F w_n - y_n,\ x - y_n\rangle \leq 0\}.$$

Then, compute

$$z_n = P_{T_n}(w_n - \lambda_n F y_n)$$

and

$$x_{n+1} = (1 - \tau_n) w_n + \tau_n z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right)}{2\langle F w_n - F y_n, z_n - y_n\rangle}, \lambda_n + \rho_n\right\}, & \text{if } \langle F w_n - F y_n, z_n - y_n\rangle > 0, \\ \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{77}$$

Set $n := n + 1$ and return to **Step 1.**

APPENDIX 7.3. *Algorithm 2.1 of Ye et al. [60]*

Step 0: *Choose $x_0 \in C$ as an initial point, $\sigma \in (0, 1)$ and $\gamma \in (0, 1)$. Set $k = 0$ Compute*

$$z_k := P_C(x_k - F(x_k)).$$

Step 1: *Compute*

$$r(x_k) = x_k - z_k.$$

If $r(x_k) = 0$, stop. Otherwise, go to Step 2.
Step 2: *Compute*

$$y_k = x_k - \eta_k r(x_k),$$

where $\eta_k = \gamma^{mk}$, with m_k being the smallest nonnegative integer satisfying

$$\langle F(x_k) - F(x_k - \gamma^m r(x_k)),\ r(x_k)\rangle \leq \sigma \|r(x_k)\|^2.$$

Step 3: *Compute*

$$x_{x+1} = P_{C \cap H_k}(x_k),\ where H_k := \bigcap_{j=0}^{j=k} H_j$$

with $H_j := \{\upsilon : h_j(\upsilon) \leq 0\}$ is a half space defined by the function

$$h_j(\upsilon) := \langle F(y_j),\ \upsilon - y_j\rangle.$$

Let $k = k + 1$ and return to Step 1.

APPENDIX 7.4. *Algorithm 1 of Yin et.al [61]*

Step 0: *Take* $\lambda_0 > 0$, $\mu \in (0,1)$. *Select the starting point* $x_0 \in H$ *and the sequence of relaxation parameters* $\{\tau_n\}_n \geq 0 \subset (0,1]$ *satisfying* $\liminf_{n\to\infty} \tau_n > 0$. *Set* $n = 0$.

Step 1: *Let* x_n, *and* λ_n *be given. Compute*

$$y_n = P_C(x_n - \lambda_n F x_n).$$

If $x_n = y_n$, *then stop.*

Step 2: *Compute*

$$x_{n+1} = \tau_n(y_n + \lambda_n(Fy_n - Fx_n)) + (1-\tau_n)x_n,$$

and

$$\lambda_{n+1} = \begin{cases} min\left\{\frac{\mu\|x_n-y_n\|}{\|Fx_n-Fy_n\|}, \lambda_n\right\} & \text{if} \quad Fx_n - Fy_n \neq 0, \\ \lambda_n, & \text{else.} \end{cases} \tag{78}$$

Update n *to* $n+1$ *and go to* **Step 1**.

APPENDIX 7.5. *Algorithm 3.2 of Izuchukwu et al. [25]*

Step 0: *Let* $\lambda_0,\ \lambda_1,\ \eta \in (\delta, \frac{1-2\delta}{2})$ *with* $\delta \in (0, \frac{1}{4})$. *Choose a nonnegative real sequence* $\{\rho_n\}$ *such that* $\sum_{n=1}^{\infty} \rho_n < \infty$. *Let* $x_0,\ x_1 \in H$ *be given starting points. Set* $n := 1$.

Step 1: *compute*

$$x_{n+1} = P_C(x_n - ((\lambda_n + \lambda_{n-1})Fx_n - \lambda_{n-1}Fx_{n-1})),\ n \geq 1,$$

where

$$\lambda_{n+1} = \begin{cases} min\left\{\frac{\eta\|w_n-y_n\|}{\|Fw_n-Fy_n\|}, \lambda_n + \rho_n\right\} & \text{if} \quad Fx_n - Fy_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{79}$$