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A New Version of Forward-Backward-Forward Splitting Algorithm For Monotone Inclusions

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Abstract
This paper considers a new version of the forward-backward-forward splitting algorithm to find a zero of the sum of two monotone operators in real Hilbert spaces. Our proposed method comprises a forward-backward-forward algorithm, an inertial term, correction terms, and a self-adaptive stepsize. Under certain standard conditions, we obtain the proposed method's weak and linear convergence results. Numerical experiments show that our proposed algorithm outperforms other forward-backward-forward splitting algorithms in the literature. Some popular splitting algorithms have also been recovered from our algorithm.

Keywords Forward-backward-forward algorithm · Inertial technique · Correction term · Monotone operators · Self-adaptive stepsize

Mathematics Subject Classification 90C23 · 90C60 · 49M25

1 Introduction

Let H be a real Hilbert space. Let $A : H \rightrightarrows H$ be a maximal monotone set-valued operator. The monotone inclusion problem (MIP) is to find a point $x \in H$ such that

$$0 \in A(x). \tag{1}$$

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The MIP (1) has many important applications in several scientific fields, either directly or using a suitable reformulation. A variant of (1) is the two operator inclusion

$$\text{find } x \in H \text{ such that } 0 \in (A+B)x, \tag{2}$$

where $A : H \rightrightarrows H$ is maximal monotone and $B : H \to H$ a Lipschitz monotone operator. We denote the solution set of (2) by $(A+B)^{-1}(0)$. The problem (2) is an interesting area of research with vast applications in fields like convex minimization, variational inequality, linear inverse problem, and image restoration, among others (see Combettes and Wajs 2005; Bauschke et al. 2005; Raguet et al. 2013; Reich 1979; Thong and Cholamjiak 2019; Bing and Cho 2022; Barbu 2010 and other references therein).
Several authors have proposed and studied various iterative methods for solving (1). One of the popular methods for solving (1) is the proximal point algorithm (see Moreau 1965; Moudafi and Elisabeth 2003 and other references therein). Given $x_0 \in H$, the proximal point algorithm is defined by

$$x_{k+1} = J_\lambda^A(x_k), \tag{3}$$

where $J_\lambda^A := (I + \lambda A)^{-1},\ \lambda > 0$ is a single valued resolvent operator of A (see Moreau 1965). We note that the classical augmented Lagrangian method, Douglas-Rachford splitting method, and the alternating direction method of multipliers (see Gabay and Mercier 1976; Lions and Mercier 1979; Powell 1968) are all special cases of the proximal point algorithm (3). Several authors have also studied and proposed various methods for solving (2). One of the methods for solving (2) is the forward-backward splitting method (see, e.g., Levitin and Polyak 1966; Lions and Mercier 1979) defined by

$$x_{k+1} = J_{\lambda_k}^A(I - \lambda_k B)x_k, \tag{4}$$

where I is the identity operator on H and $\{\lambda_k\}$ is a sequence of positive real numbers. At each iteration of this method, only one forward evaluation of A and one backward evaluation of B are needed. The method (4) converges weakly to a solution of the problem (2) when B is L^{-1} cocoercive, A is maximal monotone and the solution set $(A+B)^{-1}(0) \neq \emptyset$. Observe that the cocoercive assumption on B is a strict condition. In the case when B is weakened to a Lipschitz continuous monotone operator, Tseng (2000) proposed the following forward-backward-forward splitting method

$$\begin{cases} y_k = J_{\lambda_k}^A(I - \lambda_k B)x_k, \\ x_{k+1} = y_k + \lambda_k(Bx_k - By_k), \quad k \geq 1. \end{cases} \tag{5}$$

Tseng (2000) obtained both weak and linear convergence results of (5) in real Hilbert spaces. Several authors have studied the forward-backward-forward splitting method (see Alves and Geremia 2019; Bot and Csetnek 2016; Cholamjiak et al. 2018; Gibali and Thong 2018; Latafat and Patrinos 2017; Sahu et al. 2015; Shehu 2019; Shehu and Cai 2018; Thong and Vinh 2019 and other references therein).

To improve the speed of convergence of iterative methods for solving MIP and other related optimization problems, Polyak (1964) introduced the inertial techniques, which originated from an implicit discretization method of the second-order dynamical systems for solving the smooth convex minimization problem.
Consequently, several authors have studied and proposed inertial versions forward-backward-forward splitting algorithms (see Bing and Cho 2022; Gibali and Thong 2018; Thong and Cholamjiak 2019; Jun-On and Cholamjiak 2024; Peeyada et al. 2022; Savoye et al. 2024;

Wang and Wang 2018 and other references therein). The inertial forward-backward-forward splitting method is defined as follows;

$$\begin{cases} w_k = x_k + \delta(x_k - x_{k-1}), \\ y_k = J_{\lambda_k}^A(w_k - \lambda_k B w_k), \\ x_{k+1} = (1-\rho)w_k + \rho(y_k - \lambda_k(By_k - Bw_k)), \quad k \geq 1. \end{cases} \tag{6}$$

Bot et al. (2023) obtained a weak convergence result of (6) when $\delta \in [0, 1)$ and $0 < \rho < \frac{2(1-\delta)^2}{(1+\mu)(2\delta^2-\delta+1)}$. Several other authors have studied the weak convergence results of (6) (see Padcharoen et al. 2021; Bot and Csetnek 2016 and other references therein).

To accelerate further the convergence speed of inertial methods numerically, some authors have proposed inertial iterative methods containing correction terms for solving MIP. Recently, Kim (2021) extended the results of Güler (1992) from convex minimization to monotone inclusions. Kim (2021) proposed an accelerated proximal point algorithm, which consists of the proximal point algorithm, an inertial term, and a correction term. In essence, Kim (2021) introduced the following algorithm

$$\begin{cases} x_{k+1} = J_\lambda^A(z_k), \\ z_{k+1} = x_k + \frac{k}{k+2}(x_{k+1} - x_k) + \frac{k}{k+2}(z_{k-1} - x_k), \end{cases} \tag{7}$$

and obtained a worst case convergence rate $\|y_k - w_{k-1}\| = \mathcal{O}(1/k)$. However, Kim (2021) did not obtain the weak convergence of the sequence of iterates $\{x_k\}$ and $\{z_k\}$ generated by (7). Furthermore, Maingé (2021) introduced an accelerated proximal algorithm for solving (1). The proposed algorithm consists of an inertial term, relaxation factors, and a correction term. The proposed algorithm is

$$\begin{cases} x_{k+1} = \frac{1}{1+\beta_k} z_k + \frac{\beta_k}{1+\beta_k} J_{\lambda(1+\beta_k)}^A(z_k), \\ z_{k+1} = x_{k+1} + \alpha_{k+1}(x_{k+1} - x_k) + \delta_{k+1}(z_k - x_{k+1}), \end{cases} \tag{8}$$

where β_k and α_k are given recursively. Maingé (2021) obtained a weak convergence of the iterates $\{x_k\}$ and $\{y_k\}$ generated by (8) with the fast rate $\|x_{k+1} - x_k\| = \mathcal{O}(k^{-1})$ and $\frac{\beta_k}{1-\beta_k} \in (0, 1)$ but no linear convergence result was given.

Very recently, Maingé (2021) proposed and studied a corrected, relaxed inertial forward-backward algorithm for solving (2) in a real Hilbert space. This method can be referred to as a preconditioned and relaxed variant of the classical forward-backward algorithm, which consists of the forward-backward algorithm, a constant relaxation term, an inertial term, and a correction term. Maingé (2021) introduced the algorithm as

$$\begin{cases} z_k = x_k + \alpha_k(x_k - x_{k-1}) + \delta_k(z_{k-1} - x_k), \\ x_{k+1} = (1-\rho)z_k + \rho J_{\lambda M^{-1}A}(z_k - \lambda M^{-1} B z_k), \end{cases} \tag{9}$$

where $J_{\lambda M^{-1}A} := (1 + \lambda M^{-1}A)^{-1}$ denotes the resolvent of $M^{-1}A$. Under certain standard conditions, Maingé (2021) obtained the weak convergence with worst-case rates of $\mathcal{O}(k^{-2})$ in terms of the discrete velocity and the fixed point residual, but no linear convergence was obtained. The limitation of this method is that it requires the coerciveness assumption on the operator B and the strong monotonicity of $A + B$.

Motivated by the works mentioned above in the literature, we propose and study a fast convergent forward-backward-forward splitting algorithm to solve (2) without the coerciveness assumption. Thus, we introduce a new fast algorithm of the forward-backward-forward splitting type for finding a zero of the sum of two monotone operators. Our proposed version of the forward-backward-forward algorithm consists of the inertial term, two correction terms, and a self-adaptive stepsize. We present the weak and linear convergence analysis of our proposed method. The variable stepsize used is self-adaptive; hence, it does not depend on the Lipschitz constant of the involved operator. We also present numerical experiments of our proposed method in comparison to other methods in the literature to illustrate the applicability of our proposed method.

The rest of the paper is organized as follows: In Section 2, we present some basic definitions, lemmas, and results needed for our convergence analysis. In Section 3, we present our assumptions and proposed method. In Section 4, we investigate the weak and linear convergence of our proposed method. In Section 5, we present numerical experiments to illustrate the applicability of our algorithm, and then we give some conclusions in Section 6.

2 Preliminaries

Definition 2.1 A mapping $B : H \to H$ is said to be

(i) *strongly monotone* with $L > 0$ on H if

$$\langle Bx - By,\ x - y\rangle \geq L\|x - y\|^2, \quad \forall\, x, y \in H,$$

(ii) monotone, if

$$\langle Bx - By,\ \ x - y\rangle \geq 0, \ \ \forall\, x, y \in H,$$

(iii) *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Bx - By\| \leq L\|x - y\|, \ \ \forall\, x, y \in H.$$

The mapping B is called a contraction if $L \in [0, 1)$.

Definition 2.2 A multivalued operator $A : H \to 2^H$ is said to be monotone if for any $x, y \in D(A), \bar{x} \in Ax$ and $\bar{y} \in Ay$,

$$\langle x - y,\ \bar{x} - \bar{y}\rangle \geq 0.$$

The monotone operator A is called maximal if the graph $G(A)$ of A, defined by

$$G(A) := \{(v, \bar{v}) \in H \times H : \bar{v} \in Av\},$$

is not properly contained in the graph of any other monotone operator.
Let A be a multivalued operator. Then, the resolvent associated with A is the mapping $J_\lambda^A : H \to 2^H$ defined by

$$J_\lambda^A(v) := (I + \lambda A)^{-1}(v), \ \ v \in H,\ \lambda > 0.$$

Lemma 2.3 *The following equalities are true:*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2,\ \forall\, x, y \in H,$
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2,\ \ \forall\, x, y \in H,\ \ \alpha \in \mathbb{R};$
(iii) $\|\alpha x + \beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x - y\|^2,\ \ \forall\, x, y \in H,\ \forall \alpha, \beta \in \mathbb{R}$ *and* $\alpha + \beta = 1.$

Lemma 2.4 (Opial 1967) *Let C be a nonempty set of H and $\{x_k\}$ be a sequence in H such that the following two conditions hold:*

(i) for any $x \in C$, $\lim_{k\to\infty} \|x_k - x\|$ exists;
(ii) every sequential weak cluster point of $\{x_k\}$ is in C.

Then $\{x_k\}$ converges weakly to a point in C.

Definition 2.5 Suppose that the sequence $\{x_k\}$ in H converges in norm to $\bar{x} \in H$. We have that $\{x_k\}$ converges to $\bar{x}$ R-linearly if $\limsup_{k\to\infty} \|x_k - \bar{x}\|^{\frac{1}{k}} < 1$. We say that $\{x_k\}$ converges to $\bar{x}$ Q-linearly if there exists $\mu \in (0, 1)$ such that $\|x_{k+1} - \bar{x}\| \leq \mu\|x_k - \bar{x}\|$ for all sufficiently large k. It is well known that Q-linear convergence implies R-linear convergence, but the reverse implication fails.

3 Proposed Algorithm

In this section, we present our proposed method. We begin with the following assumptions under which our convergence result is obtained.

Assumption 3.1 Suppose that the following assumptions hold:

(i) $A : H \rightrightarrows H$ is maximal monotone and $B : H \to H$ is Lipschitz continuous and monotone.
(ii) $(A + B)^{-1}(0) \neq \emptyset$.
(iii) $\alpha_k \in (-1, 0]$, $\sum_{k=1}^{\infty} d_k < \infty$.
(iv) $0 < \liminf_{k\to\infty} \rho_k \leq \limsup_{k\to\infty} \rho_k < \frac{2}{1+\mu}$.

Algorithm 3.2 *New fast inertial forward-backward-forward algorithm*

Step 1: *Choose sequences $\{\alpha_k\}$ and $\{\rho_k\}$ such that the conditions from Assumption 3.1 hold, take $\delta_1 > 0$, $\delta \in [0, 1)$ and $\mu \in (0, 1)$. Let $x_1, x_0, z_0, z_{-1} \in H$ be a given starting point. Set* $k := 1$.

Step 2: *Given $x_k, x_{k-1}, z_{k-1}, z_{k-2}$, compute*

$$\begin{cases} z_k = x_k + \alpha_k(x_k - x_{k-1}) + \delta(1 + \alpha_k)(z_{k-1} - x_k) - \alpha_k\delta(z_{k-2} - x_{k-1}), \\ y_k = J_{\lambda_k A}(I - \lambda_k B)z_k, \\ x_{k+1} = (1 - \rho_k)z_k + \rho_k(y_k - \lambda_k(By_k - Bz_k)), \end{cases}$$

where

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k + d_k, \ \frac{\mu\|y_k - z_k\|}{\|By_k - Bz_k\|}\right\}, & By_k - Bz_k \neq 0 \\ \lambda_k + d_k, & \textit{otherwise}. \end{cases} \tag{10}$$

Step 3: *Set $k \leftarrow k + 1$, and* **return to Step 2.**

Remark 3.3 In the case where $\alpha_k = 0 = \delta$ and $\rho_k = 1$, Algorithm 3.2 reduces to the forward-backward-forward algorithm proposed by Tseng (2000). Also, in the case where $\delta = 0$, Algorithm 3.2 reduces to the inertial forward-backward-forward algorithm which have been studied in Bing and Cho (2022), Gibali and Thong (2018), Thong and Cholamjiak (2019), Wang and Wang (2018). In our convergence analysis, we do not consider the case when $\delta = 0$. This is because Tseng (2000) already considered this case.

Remark 3.4 The inertial forward-backward-forward algorithm (6) consists of the forward-backward-forward algorithm and the inertial technique, while Algorithm 3.2 contains the forward-backward-forward algorithm, an inertial technique, and two correction terms. The presence of the inertial technique and the correction terms in the forward-backward-forward algorithm helps to improve numerically the convergence speed of the forward-backward-forward algorithm. Also, the inertial factor in Algorithm 3.2 is not restricted to $[0, \frac{1}{3})$ which has been assumed by several authors (Bing and Cho 2022; Gibali and Thong 2018; Thong and Cholamjiak 2019; Wang and Wang 2018).

Remark 3.5 By (10), $\lim_{k\to\infty} \lambda_k = \bar{\lambda}$, where $\bar{\lambda} \in [\min\{\mu \mathsf{L}^{-1}, \lambda_1\},\ \lambda_1 + d]$ with $d = \sum_{k=1}^{\infty} d_k$ (see Liu and Yang 2020). If $d_k = 0$, then the stepsize λ_k defined in (10) is similar to the one in Hieu et al. (2021).

4 Convergence result

In this section, we present the weak and linear convergence results of our proposed algorithm.

4.1 Weak convergence

Lemma 4.1 *Let $\{x_k\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then, $\{x_k\}$ is bounded when $-1 < \alpha_k \leq \alpha_{k+1} \leq 0$.*

Proof Let $z \in (A+B)^{-1}(0)$ and $u_k := y_k - \lambda_k(By_k - Bz_k)$. Then,

$$\begin{aligned}
\|z_k - z\|^2 &= \|z_k - y_k + y_k - u_k + u_k - z\|^2 \\
&= \|z_k - y_k\|^2 - \|y_k - u_k\|^2 + \|u_k - z\|^2 + 2\langle z_k - u_k,\ y_k - z\rangle \\
&= \|z_k - y_k\|^2 - \lambda_k^2\|By_k - Bz_k\|^2 + \|u_k - z\|^2 + 2\langle z_k - u_k,\ y_k - z\rangle \\
&\geq \|z_k - y_k\|^2 - \frac{\mu^2\lambda_k^2}{\lambda_{k+1}^2}\|y_k - z_k\|^2 \qquad (11) \\
&\quad + \|u_k - z\|^2 + 2\langle z_k - u_k,\ y_k - z\rangle.
\end{aligned}$$

From $y_k = (I + \lambda_k A)^{-1}(z_k - \lambda_k Bz_k)$, we obtain

$$(I - \lambda_k B)z_k \in (I + \lambda_k A)y_k$$

and so

$$\begin{aligned}
z_k \in y_k + \lambda_k Ay_k + \lambda_k Bz_k &= y_k - \lambda_k(By_k - Bz_k) + \lambda_k(A+B)y_k \\
&= u_k + \lambda_k(A+B)y_k.
\end{aligned}$$

Thus,

$$\frac{1}{\lambda_k}(z_k - u_k) \in (A+B)y_k.$$

Noting that $0 \in (A+B)z$ and $A+B$ is monotone, we obtain

$$\langle z_k - u_k, y_k - z\rangle \geq 0. \qquad (12)$$

Using (12) in (11), we have

$$\|z_k - z\|^2 \geq \|z_k - y_k\|^2 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2} \|y_k - z_k\|^2 + \|u_k - z\|^2$$

and thus

$$\|u_k - z\|^2 \leq \|z_k - z\|^2 - \left(1 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right) \|y_k - z_k\|^2. \tag{13}$$

Consequently, from Lemma 2.3 we have

$$\begin{aligned}
\|x_{k+1} - z\|^2 &= \left\|(1-\rho_k)z_k + \rho_k u_k - z\right\|^2 \\
&= \left\|(1-\rho_k)(z_k - z) + \rho_k(u_k - z)\right\|^2 \\
&= (1-\rho_k)\|z_k - z\|^2 + \rho_k\|u_k - z\|^2 - \rho_k(1-\rho_k)\|u_k - z_k\|^2 \\
&\leq (1-\rho_k)\|z_k - z\|^2 + \rho_k\|z_k - z\|^2 - \rho_k\left(1 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k - z_k\|^2 \\
&\quad - \frac{(1-\rho_k)}{\rho_k}\|x_{k+1} - z_k\|^2 \\
&= \|z_k - z\|^2 - \rho_k\left(1 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k - z_k\|^2 \\
&\quad - \frac{(1-\rho_k)}{\rho_k}\|x_{k+1} - z_k\|^2.
\end{aligned} \tag{14}$$

Since $\lambda_{k+1} \leq \frac{\mu\|y_k - z_k\|}{\|By_k - Bz_k\|}$, we obtain

$$\|By_k - Bz_k\| \leq \frac{\mu}{\lambda_{k+1}}\|y_k - z_k\|$$

and thus

$$\begin{aligned}
\frac{1}{\rho_k}\|x_{k+1} - z_k\| &= \|u_k - z_k\| \\
&\leq \|u_k - y_k\| + \|y_k - z_k\| \\
&= \lambda_k\|By_k - Bz_k\| + \|y_k - z_k\| \\
&\leq \left(1 + \mu \frac{\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|.
\end{aligned} \tag{15}$$

Since $\lim_{k\to\infty} \lambda_k$ exists, we have

$$\lim_{k\to\infty}\left(1 - \mu \frac{\lambda_k}{\lambda_{k+1}}\right) = 1 - \mu > 0.$$

Hence, there exists $k_1 \in \mathbb{N}$ such that

$$1 - \mu \frac{\lambda_k}{\lambda_{k+1}} > 0, \quad \forall k \geq k_1.$$

Thus, for all $k \geq k_1$, we obtain from (15) that

$$\frac{\left(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}\|x_{k+1}-z_k\|^2 \leq \rho_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2.$$

Therefore, (14) becomes $\forall k \geq k_1$,

$$\begin{aligned}\|x_{k+1}-z\|^2 &\leq \|z_k-z\|^2-\left(\frac{\left(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}+\frac{1-\rho_k}{\rho_k}\right)\|x_{k+1}-z_k\|^2\\ &= \|z_k-z\|^2-\left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}-1\right)\|x_{k+1}-z_k\|^2. \qquad (16)\end{aligned}$$

Let us define $w_k := x_k+\delta(z_{k-1}-x_k)$. Then, we have from Algorithm 3.2 that $z_k = (1+\alpha_k)w_k-\alpha_k w_{k-1}$. Consequently, we can obtain x_k from w_k as $x_k = \frac{1}{1-\delta}w_k-\frac{\delta}{1-\delta}z_{k-1}$. By Lemma 2.3, we have

$$\begin{aligned}\|x_k-z\|^2 &= \left\|\frac{1}{1-\delta}(w_k-z)-\frac{\delta}{1-\delta}(z_{k-1}-z)\right\|^2\\ &= \frac{1}{1-\delta}\|w_k-z\|^2-\frac{\delta}{1-\delta}\|z_{k-1}-z\|^2+\frac{\delta}{(1-\delta)^2}\|w_k-z_{k-1}\|^2\\ &= \frac{1}{1-\delta}\|w_k-z\|^2-\frac{\delta}{1-\delta}\|z_{k-1}-z\|^2+\delta\|x_k-z_{k-1}\|^2. \qquad (17)\end{aligned}$$

If we combine (16) and (17), we get $\forall k \geq k_1$,

$$\begin{aligned}\|x_{k+1}-z\|^2-\|x_k-z\|^2 \leq\ & \|z_k-z\|^2-\frac{1}{1-\delta}\|w_k-z\|^2+\frac{\delta}{1-\delta}\|z_{k-1}-z\|^2\\ &-\delta\|x_k-z_{k-1}\|^2\\ &-\left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}-1\right)\|x_{k+1}-z_k\|^2. \qquad (18)\end{aligned}$$

Now, from Lemma 2.3, we have

$$\begin{aligned}\|z_k-z\|^2 &= \|(1+\alpha_k)(w_k-z)-\alpha_k(w_{k-1}-z)\|^2\\ &= (1+\alpha_k)\|w_k-z\|^2-\alpha_k\|w_{k-1}-z\|^2\\ &\quad+\alpha_k(1+\alpha_k)\|w_k-w_{k-1}\|^2 \qquad (19)\end{aligned}$$

and so (18) becomes

$$\begin{aligned}\|x_{k+1}-z\|^2-\|x_k-z\|^2 \leq\ & \left(\alpha_k-\frac{\delta}{1-\delta}\right)\|w_k-z\|^2-\alpha_k\|w_{k-1}-z\|^2\\ &+\alpha_k(1+\alpha_k)\|w_k-w_{k-1}\|^2\\ &+\frac{\delta}{1-\delta}\Big[(1+\alpha_{k-1})\|w_{k-1}-z\|^2\\ &-\alpha_{k-1}\|w_{k-2}-z\|^2+\alpha_{k-1}(1+\alpha_{k-1})\|w_{k-1}-w_{k-2}\|^2\Big]\end{aligned}$$

$$-\delta\|x_k - z_{k-1}\|^2 - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)} - 1\right)\|x_{k+1} - z_k\|^2.$$

Hence,

$$\begin{aligned}
&\|x_{k+1} - z\|^2 - \left(\alpha_k - \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 - \alpha_k(1+\alpha_k)\|w_k - w_{k-1}\|^2 \\
&\le \|x_k - z\|^2 + \left(\frac{\delta(1+\alpha_{k-1})}{1-\delta} - \alpha_k\right)\|w_{k-1} - z\|^2 \\
&\quad + \frac{\delta\alpha_{k-1}(1+\alpha_{k-1})}{1-\delta}\|w_{k-1} - w_{k-2}\|^2 - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-2} - z\|^2 - \delta\|x_k - z_{k-1}\|^2 \\
&\quad - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)} - 1\right)\|x_{k+1} - z_k\|^2 \\
&= \|x_k - z\|^2 + \left(\frac{\delta}{1-\delta} + \frac{\delta\alpha_{k-1}}{1-\delta} - \alpha_k\right)\|w_{k-1} - z\|^2 \\
&\quad + \frac{\delta\alpha_{k-1}(1+\alpha_{k-1})}{1-\delta}\|w_{k-1} - w_{k-2}\|^2 - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-2} - z\|^2 \\
&\quad - \delta\|x_k - z_{k-1}\|^2 - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)} - 1\right)\|x_{k+1} - z_k\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|x_{k+1} - z\|^2 - \left(\alpha_k - \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-1} - z\|^2 \\
&\quad - \alpha_k(1+\alpha_k)\|w_k - w_{k-1}\|^2 \\
&\le \|x_k - z\|^2 - \left(\alpha_k - \frac{\delta}{1-\delta}\right)\|w_{k-1} - z\|^2 - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-2} - z\|^2 \\
&\quad + \frac{\delta\alpha_{k-1}(1+\alpha_{k-1})}{1-\delta}\|w_{k-1} - w_{k-2}\|^2 - \delta\|x_k - z_{k-1}\|^2 \\
&\quad - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)} - 1\right)\|x_{k+1} - z_k\|^2.
\end{aligned}$$

Since $\alpha_{k-1} \le \alpha_k$, we obtain

$$\begin{aligned}
&\|x_{k+1} - z\|^2 - \left(\alpha_k - \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 - \frac{\delta\alpha_k}{1-\delta}\|w_{k-1} - z\|^2 \\
&\quad - \alpha_k(1+\alpha_k)\|w_k - w_{k-1}\|^2 \\
&\le \|x_k - z\|^2 - \left(\alpha_{k-1} - \frac{\delta}{1-\delta}\right)\|w_{k-1} - z\|^2 \\
&\quad - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-2} - z\|^2 - \alpha_{k-1}(1+\alpha_{k-1})\|w_{k-1} - w_{k-2}\|^2 \\
&\quad + \frac{\alpha_{k-1}(1+\alpha_{k-1})}{1-\delta}\|w_{k-1} - w_{k-2}\|^2 - \delta\|x_k - z_{k-1}\|^2
\end{aligned}$$

$$-\left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}-1\right)\|x_{k+1}-z_k\|^2. \tag{20}$$

Define, for all $k \geq k_1$,

$$a_k := \|x_k - z\|^2 - \left(\alpha_{k-1} - \frac{\delta}{1-\delta}\right)\|w_{k-1}-z\|^2 - \frac{\delta\alpha_{k-1}}{1-\delta}\|w_{k-2}-z\|^2$$
$$-\alpha_{k-1}(1+\alpha_{k-1})\|w_{k-1}-w_{k-2}\|^2.$$

Noting that $\alpha_{k-1} \in (-1, 0]$ and $\delta \in [0, 1)$, we have that $a_k \geq 0, \quad \forall\ k \geq k_1$.
We can then re-write (20) as

$$a_{k+1} \leq a_k + \frac{\alpha_{k-1}(1+\alpha_{k-1})}{1-\delta}\|w_{k-1}-w_{k-2}\|^2$$
$$-\delta\|x_k - z_{k-1}\|^2 - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}-1\right)\|x_{k+1}-z_k\|^2. \tag{21}$$

Then (21) implies that $\lim\limits_{k\to\infty} a_k$ exists and $\{a_k\}$ is bounded. Consequently, by $\|x_k - z\|^2 \leq a_k$, and $\{x_k\}$ is bounded. □

Theorem 4.2 *Let $\{x_k\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then, $\{x_k\}$ converges weakly to a point in $(A+B)^{-1}(0)$ if $\alpha_k = 0$ or $-1 < a \leq \alpha_k \leq \alpha_{k+1} \leq b < 0$.*

Proof To establish the proof, we consider two cases:
Case I: Let $\alpha_k = 0,\ \forall k \geq k_1$, then $z_k = w_k,\ \forall\ k \in \mathbb{N}$ and

$$a_k = \|x_k - z\|^2 + \frac{\delta}{1-\delta}\|z_{k-1}-z\|^2,$$

therefore (21) becomes

$$a_{k+1} \leq a_k - \delta\|x_k - z_{k-1}\|^2 - \left(\frac{2}{\rho_k\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)}-1\right)\|x_{k+1}-z_k\|^2.$$

Hence, we have (noting that $\limsup\limits_{k\to\infty}\rho_k < \frac{2}{1+\mu}$ and $\limsup\limits_{k\to\infty}(b_k c_k) = \lim\limits_{k\to\infty} b_k \limsup\limits_{k\to\infty} c_k$ when $\lim\limits_{k\to\infty} b_k$ exists) $\lim\limits_{k\to\infty}\|x_{k+1}-z_k\| = 0$.
Consequently, we obtain from Algorithm 3.2 that

$$\|z_k - x_k\| = \delta\|x_k - z_{k-1}\| \to 0, \quad k \to \infty.$$

We also have from Algorithm 3.2 that

$$\|z_k - u_k\| = \frac{1}{\rho_k}\|x_{k+1}-z_k\| \to 0, k \to \infty,$$

since $\liminf\limits_{k\to\infty}\rho_k > 0$.
Furthermore,

$$\|x_{k+1}-x_k\| \leq \|x_{k+1}-z_k\| + \|z_k - x_k\| \to 0,\ k \to \infty.$$

Now,

$$\begin{aligned}\|u_k - z_k\| &= \|y_k - z_k - \lambda_k(By_k - Bz_k)\| \\ &\geq \|y_k - z_k\| - \lambda_k\|By_k - Bz_k\| \\ &\geq \left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right)\|y_k - z_k\|.\end{aligned}$$

Noting that $\lim_{k\to\infty}\left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right) = 1 - \mu > 0$, we have

$$\lim_{k\to\infty}\|y_k - z_k\| = 0.$$

Case II: Suppose that $-1 < a \leq \alpha_k \leq \alpha_{k+1} \leq b < 0$. Then we have from (21) that

$$\lim_{k\to\infty}\|w_{k-1} - w_{k-2}\| = 0,$$

and

$$\lim_{k\to\infty}\|x_{k+1} - z_k\| = 0.$$

By the fact that $\alpha_k \leq \alpha_{k+1}$ and $\alpha_k \in (-1, 0)$ we get that $\lim_{k\to\infty}\alpha_k$ exists. Thus,

$$\|z_k - w_k\| = |\alpha_k|\|w_k - w_{k-1}\| \to 0, \quad k \to \infty,$$

and

$$\|w_k - x_k\| = \delta\|x_k - z_{k-1}\| \to 0, \quad k \to \infty.$$

So

$$\|z_k - x_k\| \leq \|z_k - w_k\| + \|w_k - x_k\| \to 0, \quad k \to \infty.$$

Similarly, one can show that

$$\lim_{k\to\infty}\|z_k - u_k\| = 0, \qquad \lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$$

and

$$\lim_{k\to\infty}\|y_k - z_k\| = 0.$$

This completes the two cases above.

Since $\{x_k\}$ is bounded by Lemma 4.1, $\{x_k\}$ has a weak limit point. Let $x^* \in H$ be a weak limit point of $\{x_k\}$ and $\{x_{k_j}\} \subset \{x_k\}$ such that $x_{k_j} \rightharpoonup x^*, \quad j \to \infty$.

Since $(z_k - x_k) \to 0, \quad k \to \infty$, we have that $z_{k_j} \rightharpoonup x^*, \quad j \to \infty$. Furthermore, $y_{k_j} - z_{k_j} \to 0, \ k \to \infty$ gives $y_{k_j} \rightharpoonup x^*, \quad j \to \infty$. By the definition of $\{y_{k_j}\}$, we have

$$\frac{1}{\lambda_{k_j}}(z_{k_j} - y_{k_j}) + By_{k_j} - Bz_{k_j} \in (A+B)y_{k_j}.$$

Since B is Lipschitz continuous and $z_{k_j} - y_{k_j} \to 0, \quad j \to \infty$, we obtain

$$\left(\frac{1}{\lambda_{k_j}}(z_{k_j} - y_{k_j}) + By_{k_j} - Bz_{k_j}\right) \to 0, \quad j \to \infty.$$

Using the fact that the graph of the maximal monotone operator $A + B$ is sequentially closed with respect to the weak-strong topology of the product space $H \times H$, we obtain $0 \in (A+B)x^*$. Therefore, $x^* \in (A+B)^{-1}(0)$.

Now, take, $\forall k \geq k_1$,

$$\begin{aligned} b_k^* &= 2\langle w_{k-1} - w_k,\ w_k - z\rangle + \|w_{k-1} - w_k\|^2 \\ a_k^* &= 2\langle w_{k-2} - w_k,\ w_k - z\rangle + \|w_{k-2} - w_k\|^2 \\ d_k^* &= \alpha_{k-1}(1 + \alpha_{k-1})\|w_{k-1} - w_{k-2}\|^2. \end{aligned}$$

Observe that

$$\|w_{k-1} - z\|^2 = \|w_{k-1} - w_k\|^2 + 2\langle w_{k-1} - w_k, w_k - z\rangle + \|w_k - z\|^2$$

and

$$\|w_{k-2} - z\|^2 = \|w_{k-2} - w_k\|^2 + 2\langle w_{k-2} - w_k,\ w_k - z\rangle + \|w_k - z\|^2.$$

So,

$$\begin{aligned} a_k + d_k^* + \left(\alpha_{k-1} - \frac{\delta}{1-\delta}\right) b_k^* + \frac{\delta \alpha_{k-1}}{1-\delta} a_k^* &= \|x_k - z\|^2 - \frac{(\alpha_{k-1} - \delta)}{1-\delta}\|w_k - z\|^2 \\ &= \|x_k - w_k\|^2 + 2\langle x_k - w_k, w_k - z\rangle \\ &\quad + \|w_k - z\|^2 - \frac{(\alpha_{k-1} - \delta)}{1-\delta}\|w_k - z\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{(1-\alpha_{k-1})}{1-\delta}\|w_k - z\|^2 &= a_k + d_k^* + \left(\alpha_{k-1} - \frac{\delta}{1-\delta}\right) b_k^* + \frac{\alpha_{k-1}\delta}{1-\delta} a_k^* \\ &\quad - \|x_k - w_k\|^2 - 2\langle x_k - w_k, w_k - z\rangle. \end{aligned} \tag{22}$$

By the fact that $\lim_{k\to\infty} \|w_{k-1} - w_k\| = 0$ and $\{w_k\}$ is bounded, we have $a_k^* \to 0$, $b_k^* \to 0$ and $d_k^* \to 0$.
By the existence of $\lim_{k\to\infty} a_k$, we obtain from (22) that $\lim_{k\to\infty} \|w_k - z\|$ exists for each $z \in (A+B)^{-1}(0)$.
Furthermore,

$$\|x_k - z\|^2 = \|x_k - w_k\|^2 + 2\langle x_k - w_k, w_k - z\rangle + \|w_k - z\|^2$$

implies that $\lim_{k\to\infty} \|x_k - z\|$ exists for each $z \in (A+B)^{-1}(0)$.
By Lemma 2.4, we conclude that $\{x_k\}$ converges weakly to a point in $(A+B)^{-1}(0)$. This completes the proof. □

4.2 Linear convergence

Theorem 4.3 *Let Assumption 3.1 hold with B being γ-strongly monotone and Lipschitz continuous, $\rho_k = \rho$ and $\alpha_k = 0$. Then $\{x_k\}$ generated by Algorithm 3.2 converges linearly to the unique point $z \in (A+B)^{-1}(0)$.*

Proof To establish the linear rate result, we shall consider two cases:
Case I: Suppose that $0 < \rho \leq 1$. Then by the fact that B is γ-strongly monotone and repeating same arguments from (11)-(14), we have

$$\|x_{k+1} - z\|^2 \leq \|z_k - z\|^2 - \rho\left(1 - \mu^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k - z_k\|^2$$

$$
\begin{aligned}
&-\frac{(1-\rho)}{\rho}\|x_{k+1}-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&\le \|z_k-z\|^2-\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&\le \|z_k-z\|^2-\min\left\{\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right),2\gamma\lambda_k\right\}\left(\|y_k-z_k\|^2+\|y_k-z\|^2\right)\\
&\le \|z_k-z\|^2-\frac{1}{2}\min\left\{\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right),2\gamma\lambda_k\right\}\|z_k-z\|^2\\
&=\left(1-\frac{1}{2}\min\left\{\rho\Big(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\Big),2\gamma\lambda_k\right\}\right)\|z_k-z\|^2. \qquad (23)
\end{aligned}
$$

Let $\bar{\lambda}<\frac{1}{\gamma}$. Observe that if

$$\delta_k:=1-\frac{1}{2}\min\left\{\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right),2\gamma\lambda_k\right\}$$

then

$$\limsup_{k\to\infty}\delta_k=1-\frac{1}{2}\liminf_{k\to\infty}\min\left\{\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right),2\gamma\lambda_k\right\}\in(0,1),$$

since $\liminf_{k\to\infty}\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)=\rho(1-\mu^2)<1$.
Thus, we can write $\limsup_{k\to\infty}\delta_k=\delta\in(0,1)$. Hence

$$\|x_{k+1}-z\|^2\le\delta_k\|z_k-z\|^2,\quad\forall\, k\ge k_1. \qquad (24)$$

Case II: Suppose that $1\le\rho<\frac{2}{1+\mu}$. We obtain from (23) that

$$
\begin{aligned}
\|x_{k+1}-z\|^2&\le \|z_k-z\|^2-\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2\\
&\quad-\frac{(1-\rho)}{\rho}\|x_{k+1}-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&=\|z_k-z\|^2-\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2\\
&\quad-(1-\rho)\rho\|u_k-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&=\|z_k-z\|^2-\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2\\
&\quad+\rho(\rho-1)\|u_k-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2. \qquad (25)
\end{aligned}
$$

We obtain from (15) and (25) that

$$\|x_{k+1}-z\|^2\le\|z_k-z\|^2-\rho\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y_k-z_k\|^2$$

$$
\begin{aligned}
&+\rho(\rho-1)\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2\|y_k-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&=\|z_k-z\|^2-\rho\left[\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)-(\rho-1)\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2\right]\|y_k-z_k\|^2\\
&\quad-2\gamma\lambda_k\|y_k-z\|^2\\
&=\|z_k-z\|^2-\delta_k^*\|y_k-z_k\|^2-2\gamma\lambda_k\|y_k-z\|^2\\
&\leq\|z_k-z\|^2-\frac{1}{2}\min\left\{\delta_k^*,2\gamma\lambda_k\right\}\|z_k-z\|^2\\
&=\left(1-\frac{1}{2}\min\left\{\delta_k^*,2\gamma\lambda_k\right\}\right)\|z_k-z\|^2, \qquad (26)
\end{aligned}
$$

where

$$\delta_k^*=\rho\left[\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)-(\rho-1)\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2\right].$$

Note that

$$\liminf_{k\to\infty}\left[\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)-(\rho-1)\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)^2\right]>0$$

since $1\leq\rho<\frac{2}{1+\mu}$.

Since $\lim\limits_{k\to\infty}\left(1+\mu\frac{\lambda_k}{\lambda_{k+1}}\right)=1+\mu$, we obtain that

$$
\begin{aligned}
\liminf_{k\to\infty}\delta_k^*&=\rho(1-\mu^2)-\rho(\rho-1)(1+\mu)^2\\
&=\rho(1-\mu^2)-\rho^2(1+\mu)^2+\rho(1+\mu)^2\\
&=-\rho^2(1+\mu)^2+2\rho(1+\mu)<1,
\end{aligned}
$$

if $\left(\rho-\frac{1}{1+\mu}\right)^2>0$, which is true for any $\rho\in\mathbb{R}$. Thus, $\liminf\limits_{k\to\infty}\delta_k^*<1$, for any $\rho\in\mathbb{R}$. Hence (since $\liminf\limits_{k\to\infty}\delta_k^*>0,\ \frac{1}{2}\liminf\limits_{k\to\infty}\delta_k^*<1$)

$$\limsup_{k\to\infty}\theta_k^*=\limsup_{k\to\infty}\left(1-\frac{1}{2}\min\left\{\delta_k^*,2\gamma\lambda_k\right\}\right)\in(0,1),$$

noting that

$$\theta_k^*:=1-\frac{1}{2}\min\left\{\delta_k^*,2\gamma\lambda_k\right\}.$$

From (26), we have

$$\|x_{k+1}-z\|^2\leq\theta_k^*\|z_k-z\|^2. \qquad (27)$$

Therefore, from (24) and (27), we have shown that $\|x_{k+1}-z\|^2\leq\eta_k\|z_k-z\|^2$, where

$$\eta_k=\begin{cases}\delta_k, & \text{if } \ \rho\in(0,1] \ \text{ with} \limsup\limits_{k\to\infty}\eta_k<1\\ \theta_k^*, & \text{if } \ \rho\in[1,\frac{2}{1+\mu}).\end{cases}$$

Now, we have from $w_k = x_k + \delta(z_{k-1} - x_k)$, that $x_{k+1} = \frac{1}{1-\delta} w_{k+1} - \frac{\delta}{1-\delta} z_k$ and so

$$\|x_{k+1} - z\|^2 = \frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \frac{\delta}{1-\delta}\|z_k - z\|^2 + \delta\|x_{k+1} - z_k\|^2.$$

Consequently, (19) implies

$$\begin{aligned}
\frac{1}{1-\delta}\|w_{k+1} - z\|^2 &\leq \left(\eta_k + \frac{\delta}{1-\delta}\right)\|z_k - z\|^2 - \delta\|x_{k+1} - z_k\|^2 \\
&\leq \left(\eta_k + \frac{\delta}{1-\delta}\right)(1+\alpha_k)\|w_k - z\|^2 \\
&\quad -\alpha_k\left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_{k-1} - z\|^2 \\
&\quad +\alpha_k(1+\alpha_k)\left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_k - w_{k-1}\|^2 \\
&\quad -\delta\|x_{k+1} - z_k\|^2. \qquad (28)
\end{aligned}$$

Noting that

$$\begin{aligned}
\|x_{k+1} - z_k\|^2 &= \|(1+\alpha_k)(w_k - x_{k+1}) - \alpha_k(w_{k-1} - x_{k+1})\|^2 \\
&= (1+\alpha_k)\|w_k - x_{k+1}\|^2 - \alpha_k\|w_{k-1} - x_{k+1}\|^2 \\
&\quad +\alpha_k(1+\alpha_k)\|w_k - w_{k-1}\|^2 \\
&\geq (1+\alpha_k)\|w_k - x_{k+1}\|^2 + \alpha_k(1+\alpha_k)\|w_k - w_{k-1}\|^2. \qquad (29)
\end{aligned}$$

By (28) and (29), we get

$$\begin{aligned}
&\frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \alpha_k\left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 \\
&\leq \left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 - \alpha_k\left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_{k-1} - z\|^2 \\
&\quad -\alpha_k(1+\alpha_k)\left(\delta - \left(\eta_k + \frac{\delta}{1-\delta}\right)\right)\|w_k - w_{k-1}\|^2 \\
&\quad -\delta(1+\alpha_k)\|w_k - x_{k+1}\|^2. \qquad (30)
\end{aligned}$$

Observe that

$$\delta - \left(\eta_k + \frac{\delta}{1-\delta}\right) = -\frac{\delta^2}{1-\delta} - \eta_k < 0.$$

Then we have from (30) that

$$\begin{aligned}
&\frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \alpha_k\left(\eta_k + \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 \\
&\leq (\eta_k + \delta(1-\eta_k))\left(\frac{1}{1-\delta}\|w_k - z\|^2 - \frac{\alpha_k}{1-\delta}\|w_{k-1} - z\|^2\right). \qquad (31)
\end{aligned}$$

Since $\alpha_k = 0, \quad \forall k \in \mathbb{N}$. Then (31) becomes

$$\frac{1}{1-\delta}\|w_{k+1} - z\|^2 \leq \frac{\eta_k + \delta(1-\eta_k)}{1-\delta}\|w_k - z\|^2.$$

Observe that since $\limsup_{k\to\infty} \eta_k < 1$ and $\delta \in [0, 1)$, we have $\limsup_{k\to\infty}(\eta_k + \delta(1-\eta_k)) < 1$.

Therefore $\{w_k\}$ converges R-linearly to z.
From (14), (noting $z_k = w_k$) we have $\{x_k\}$ converges R-linearly to z. □

Remark 4.4 In the case when $\lambda_k \to \lambda \in \left(0, \frac{1}{L}\right)$, $k \to \infty$, where L is the Lipschitz constant of a γ-strongly monotone operator B, we can obtain equivalent results of (23)-(31) as follows:

$$\|x_{k+1} - z\|^2 \le \left(1 - \frac{1}{2}\min\left\{\rho(1-\lambda^2L^2), 2\gamma\lambda\right\}\right)\|z_k - z\|^2$$

when $0 < \rho \le 1$.

Also, we can obtain

$$\|x_{k+1} - z\|^2 \le \left(1 - \frac{1}{2}\min\left\{\rho(1-\lambda^2L^2) - (\rho-1)(1+\lambda L)^2, 2\gamma\lambda\right\}\right)\|z_k - z\|^2$$

when $1 \le \rho < \frac{2}{1+\lambda L}$.
Define

$$\eta = \begin{cases} 1 - \frac{1}{2}\min\left\{\rho(1-\lambda^2L^2), 2\gamma\lambda\right\}, & \text{if } 0 < \rho \le 1, \\ 1 - \frac{1}{2}\min\left\{\rho(1-\lambda^2L^2) - (\rho-1)(1+\lambda L)^2, 2\gamma\lambda\right\}, & \text{if } 1 \le \rho < \frac{2}{1+\lambda L}. \end{cases} \tag{32}$$

Then $\eta \in [0, 1)$. Also (31) becomes

$$\begin{aligned} &\frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \alpha_k\left(\eta + \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 \\ &\quad \le (\eta + \delta(1-\eta))\left(\frac{1}{1-\delta}\|w_k - z\|^2 - \frac{\alpha_k}{1-\delta}\|w_{k-1} - z\|^2\right). \end{aligned}$$

If $\alpha_k(\eta(1-\delta)+\delta) \le \alpha_{k+1}, \quad \forall\, k \in \mathbb{N}$, then

$$\begin{aligned} &\frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \frac{\alpha_{k+1}}{1-\delta}\|w_k - z\|^2 \\ &\quad \le \frac{1}{1-\delta}\|w_{k+1} - z\|^2 - \alpha_k\left(\eta + \frac{\delta}{1-\delta}\right)\|w_k - z\|^2 \\ &\quad \le (\eta + \delta(1-\eta))\left(\frac{1}{1-\delta}\|w_k - z\|^2 - \frac{\alpha_k}{1-\delta}\|w_{k-1} - z\|^2\right) \end{aligned}$$

and thus $\varrho_{k+1} \le (\eta + \delta(1-\eta))\,\varrho_k$, with $\varrho_k := \frac{1}{1-\delta}\|w_k - z\|^2 - \frac{\alpha_k}{1-\delta}\|w_{k-1} - z\|^2$.
This implies that $\{w_k\}$ converges R-linearly to the unique solution $z \in (A+B)^{-1}(0)$. Consequently, both $\{z_k\}$ and $\{x_k\}$ converges R-linearly to z.

We give the summary of the above result in the following statement.

Theorem 4.5 *Let Assumption 3.1 hold with B being γ- strongly monotone and L-Lipschitz continuous with constant $L > 0$. Let $\{x_k\}$ be generated by*

$$\begin{cases} z_k = x_k + \delta(z_{k-1} - x_k) + \alpha_k(1-\delta)(x_k - x_{k-1}) + \alpha_k\delta(z_k - z_{k-1}) \\ y_k = J_{\lambda A}(I - \lambda B)z_k \\ x_{k+1} = (1-\rho)z_k + \rho(y_k - \lambda(By_k - Bz_k)), \end{cases}$$

where $0 < \lambda < \frac{1}{L}$. Assume that $\alpha_k(\eta(1-\delta)+\delta) \le \alpha_{k+1}, \quad \forall\, k \in \mathbb{N}$ where η is as defined in (32). Then $\{x_k\}$ converges R-linearly to the unique solution $z \in (A+B)^{-1}(0)$.

5 Numerical experiment

In this section, we demonstrate the numerical implementations of Algorithm 3.2. In particular, we present numerical comparisons of Algorithm 3.2 with the algorithm in Maingé (2021), denoted as Maingé Alg., and the algorithm in Kim (2021), denoted as Kim Alg. We write our codes in Matlab 2016 (b), which is running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 GB of RAM.

Example 5.1 Let $S, T \in \mathbb{R}^{k\times k}$ be positive definite symmetric matrices. Then, we consider the mixed variational inequality problem: find $x^* \in C$ such that

$$\phi(x) - \phi(x^*) + \langle x - x^*, B(x^*)\rangle \geq 0, \quad \forall x \in C, \tag{33}$$

where $\phi(x) = x^t Sx$ and $\langle x - x^*, B(x^*)\rangle = x^t T(x - x^*)$.
We denote the largest eigenvalue of T by $\rho(T)$ and the smallest eigenvalue of S by $\mu(S)$. By the positive definiteness assumption, we have that both $\rho(T)$ and $\mu(S)$ are positive. In this example, we carry out the numerical analysis of Algorithm 3.2 in comparison with Maingé Alg. Maingé (2021) and Kim Alg. Kim (2021). The numerical results for this example are shown in Table 1 and Figure 1 below.

Example 5.2 This example is taken from (Grad et al. 2023, Example 5.1). Suppose that $h : \mathbb{R}^k \to \mathbb{R}$ is defined by $h(x) := \max\{h_1(x), h_2(x)\}$, where $h_1(x) = \sqrt{\|x\|}$ and $h_2(x) = \|x\|^2 - q$, $q \in \mathbb{N}$. Then h_1 and h_2 are strongly quasiconvex, and thus h, being the maximum of h_1 and h_2 is strongly quasiconvex. However, h is not convex. Also, we have that $\arg\min_{\mathbb{R}^k} h = \{0\}$. In this example, we carry out the numerical analysis of Algorithm 3.2 in comparison with Maingé Alg. Maingé (2021) and Kim Alg. Kim (2021). Numerical results for this example are shown in Table 2 and Figure 2 below. For more information on this example, see (Grad et al. 2023, Example 5.1).

During computations, we randomly generate the starting points x_1, x_0, z_0, z_{-1} for $k = 20, 50, 100$, and 500. We choose our parameters as follows:

- Algorithm 3.2: $\delta_1 = 1$, $\mu = 0.5$, $\delta = 0.3$, $\alpha_k = -\frac{1}{k+1}$, $\rho_k = \frac{2k+1}{3k}$ and $d_k = \frac{1}{k^2}$.
- Maingé Alg. Maingé (2021): $a = 1, c = 2, b = 0.5, a_1 = 0.5, a_2 = 0.9, \bar{c} = 1$ and $\lambda = \frac{0.99}{2\rho(B)}$.
- Kim Alg. Kim (2021): $\lambda = \frac{0.99}{2\rho(B)}$.

We choose the stopping criterion; $\text{TOL}_k := \|y_k - z_k\| < \varepsilon$, where ε is the predetermined error.

In the tables below, "Iter" means the number of iterations. Also, in the tables and figures, Maingé Alg. and Kim Alg. represent the algorithm in Maingé (2021) and the algorithm in Kim (2021), respectively.

6 Conclusion

In this paper, we studied a new version of the forward-backward-forward splitting method for solving the monotone inclusion problems in real Hilbert spaces without the coerciveness assumption. The proposed method consists of the forward-backward-forward algorithm, an inertial term, two correction terms, and a self-adaptive stepsize. Under certain conditions,

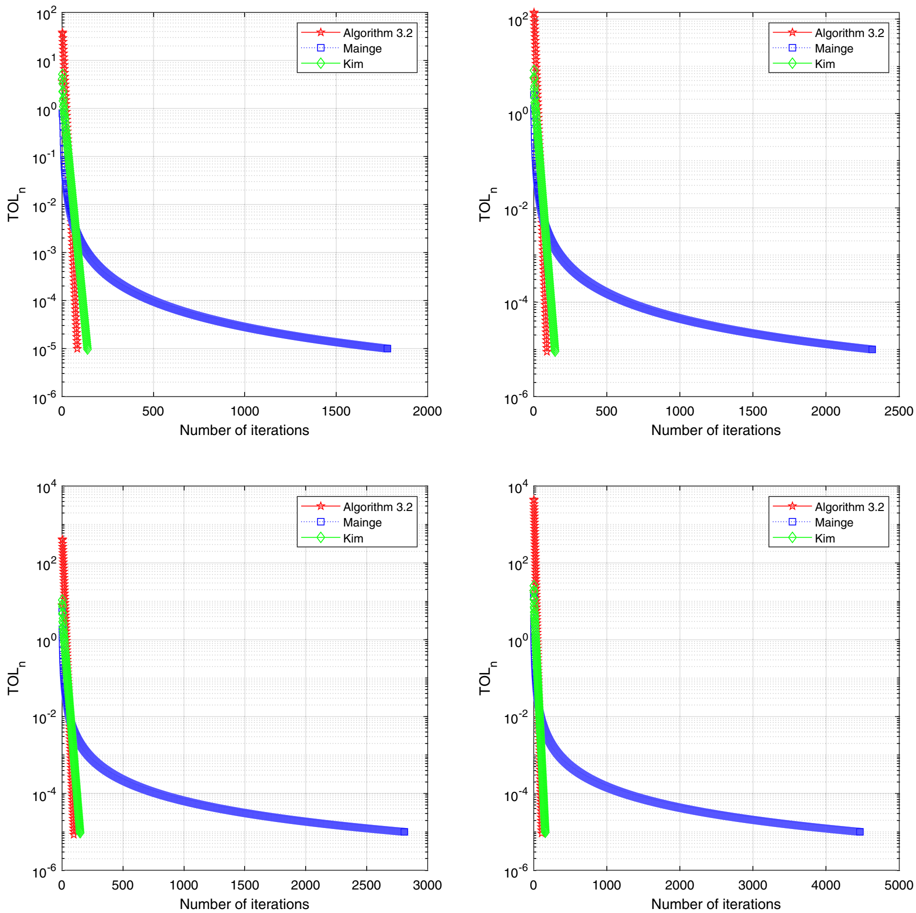

Fig. 1 The behavior of TOL_k with $\epsilon = 10^{-5}$ for Example 5.1: Top Left: $k = 20$; Top Right: $k = 50$; Bottom left: $k = 100$; Bottom Right: $k = 500$

Table 1 Comparison of algorithms with $\epsilon = 10^{-5}$

Algorithms	$k = 20$		$k = 50$		$k = 100$		$k = 500$	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Algorithm 3.2	0.0199	83	0.0168	90	0.0456	96	0.8253	108
Maingé Alg.	0.1170	1780	0.4745	2315	0.3427	2809	33.3197	4462
Kim Alg.	0.0717	139	0.0974	146	0.0914	148	2.2684	158

we obtained the weak and linear convergence results. Finally, we presented numerical experiments of our proposed method in comparison to other methods in the literature. In future research, we aim to extend our results to splitting algorithms for finding a zero of the sum of three monotone operators, where two operators are maximal monotone and the third one is Lipschitz continuous.

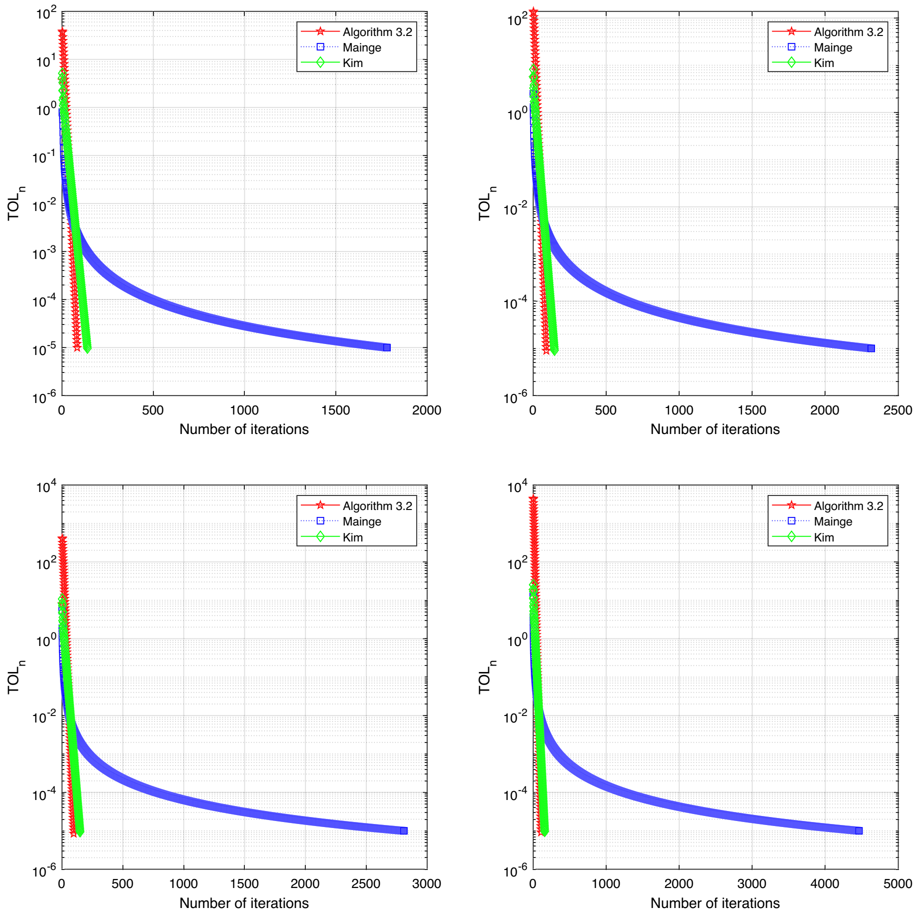

Fig. 2 The behavior of TOL_k with $\epsilon = 10^{-5}$ for Example 5.2: Top Left: $k = 20$; Top Right: $k = 50$; Bottom left: $k = 100$; Bottom Right: $k = 500$

Table 2 Comparison of algorithms with $\epsilon = 10^{-5}$

Algorithms	$k = 20$		$k = 50$		$k = 100$		$k = 500$	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Algorithm 3.2	0.0112	58	0.0058	60	0.0035	68	0.0202	75
Maingé Alg.	0.1843	1460	0.1739	2394	0.1282	2851	0.1895	5043
Kim Alg.	0.0914	117	0.0165	115	0.0170	144	0.0323	479

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Code availability The Matlab codes employed to run the numerical experiments are available on request.

Declarations

Ethical Approval and Consent to participate All the authors gave ethical approval and consent to participate in this article.

Consent for publication All the authors gave consent for the publication of identifiable details in the journal and article.

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