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Risk Measures via Vector Lattice Theory [1]

Ur A KOUMBA

Under the supervision of

Prof. Coenraad C.A LABUSCHAGNE and Dr Olivier M. PAMEN

School of Mathematics

University of the Witwatersrand

Private Bag 3, PO WITS 2050, South Africa

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Ur A KOUMBA

This _____ day of October 2010, at Johannesburg, South Africa.

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Chapter 1

Introduction

The term risk plays a major role in the literature on economic, technological, social and political issues. In the period 1993-1996, the market was affected by numerous derivative losses and bankruptcies such as Orange County (1.7 billion US dollars), Metallgesellshaft (1.3 billion US dollars) and Barings (1 billion US dollars). Moreover, the failure of the Long-Term Capital Management (LTCM) in 1998 nearly exploded the world's financial system. Although the causes of those catastrophic events originated from the mismanagement of risk, researchers agreed on further empowering this financial tool considered to be one of the greatest innovations of the 20th century. Luminant results inevitably enhanced the building of some practical families of financial risk (market risk, credit risk, operational risk, liquidity risk, risk aversion, risk neutral) used nowadays by investment agents and institutions.

Risk measurement (and management) played a crucial role amongst the real causes of the ongoing global financial crisis that started in 2008. Financial experts agree that one of the biggest mistakes they made throughout was to focus more on the sophistication of margining systems instead of emphasizing on their robustness. This direction was extremely dangerous to use over the short time series of market prices. The most pertinent lesson is that the quality of risk measurement models matters.

The question that arises is: how to measure risk in a proper way ?

Two well known risk measures that are widely used in practice are mean value and the very-well established capital requirement called Value at Risk (VaR). However, both of them have multiple drawbacks. Variance for instance penalizes high profits in the same way as high losses. VaR takes only into account the quantile of the distribution without considering what is happening to the left and the right of this quantile. In other words, Var is only concerned with the probability of the loss but not with the actual size of that loss. Moreover, VaR is not sub-additive. The financial implication of this result is that VaR does not promote diversification, that is, VaR severely penalizes the increase of the probability that something goes wrong, without giving credit to the considerable reduction of expected loss conditional on the event of default. This turns out to be a big contradiction between VaR and portfolio theory. Further criticism of Variance and VaR can be found in [5] as well as numerous discussions in financial journals. Henceforth, risk mismanagement is undoubtedly a very crucial consideration since it exposes the economy to enormous financial difficulties. Numerous works on financial risk measures have been carried through general probability spaces, through assumptions that risk can be quantified on the basis of a random variable X, therefore deciding their acceptability. This random variable may present, for instance the future net worth of a position, the relative or absolute changed values of an investment or the accumulated claims over a period for a collective of insureds.

In fact, risk is generally taken as random profit/loss of a financial position, known for this purpose as X. X can be positive or negative and respectively interpreted as gains or losses. Mathematically speaking, a risk measure is a mapping ρ from a class $\mathcal{X}$ of random variables defined on some measurable spaces to the (extended) real line $\mathbb{R}$. We consider $\rho(X)$ to be the minimum extra cash (capital) added to X to make the financial position acceptable for the holder or a regulator. In 1999, Artzner et al in [5] introduced the concept of coherent risk measure in finite probability space as a

new way of measuring risk. These results were extended in general probability space by Delbaen in [9]. He proved that under weak continuity assumption, every coherent risk measure can be represented as worst expected loss with respect to a given set of probability. Relaxing coherence axioms, Föllmer and Schied in [12],[13], Frittelli and Rosazza Gianin in [15],[16] proposed independently the more general class of convex risk measures. They also extended Delbaen's representation result incorporating a penalty function defined on a set probability model. One of the common feature of all these approaches is that the authors do not use stochastic ordering. Insightful, advanced results have been released using the notion of *partial ordering*. This is not surprising since ordered structures occur naturally in many examples such as spaces of measurable functions or spaces of real continuous functions. Lattice ordered vector spaces are called vector lattices or Riesz spaces.

In order to empower this notion into a more generalized mathematical scene, this study will design the way financial risk can be quantified into vector lattices.
We will restrict this literature study to single-period framework with no interest rates, meaning that we will only deal with two dates (0 and T for today and tomorrow respectively) where no trading will interfere in between.

Extending risk measures to different dimensions will not be surprising since some of the research related to Riesz spaces leads to interesting connections between properties and axioms established in those spaces and Riesz space. However readers will be amazed to realize how simple it is to work in the ordering vector space of vector lattices. The aim of this dissertation is to focus on the possibility of translating risk measurement problems using a pure mathematical approach such as functional analysis, stochastic ordering, vector lattice.

In view of the above discussion, it is useful to present a chronological structure of the study.

Chapter 2 addresses a brief review of fundamental structures (related to this topic) in Riesz space. To be efficient, a risk measure needs to possess some qualities characterized by rationality, additivity and technical axioms. The first type is essentially satisfied by most, the second one deals with the sum of risks by encouraging risks to be diversified. Finally the technical part preserves the continuity conditions.

In that regard, Chapter 3 deals with risk measures in a static way (one single period) under specific probability and emphasizes two fundamental aspects characterized by the notion of coherency and convexity. The continuity properties on compact and non-compact sets is pointed out with some given theorems, examples and remarks. Obviously any risk measure $\rho : \mathcal{X} \to \mathbb{R}$ induces a set of acceptance in which special attention should arise. Given some known risk measures, is it possible to generate a new risk measure? Moreover, the strategy in this chapter will be to discuss the properties of the acceptance space of coherent and convex risk measures. A risk measure ρ is said to be acceptable if $\rho(X) \leq 0$ and not acceptable otherwise.

Chapter 4 expresses this dedicated notion of ordering by considering losses X and Y. Let $\preceq$ indicate preference under partial ordering, so that $X \preceq Y$ means that X is preferred to Y. It is imperative to indicate that risk measures $\rho(\cdot)$ preserve a stochastic ordering if $X \preceq Y \Rightarrow \rho(X) \preceq \rho(Y)$. Let the set $\mathcal{X}$ be the space of any financial positions. We endow an ordering $\preceq$ with $\mathcal{X}$ considered to be a vector space. The ordering $\preceq$ can be thought of as strengthening the partial preferences pre-ordered on $\mathcal{X}$ and defined as :

$$\text{“X is at least as good as Y”} \quad \Leftrightarrow \quad \rho(X - Y) \leq 0.$$

For this class of risk measures, the problem of consistency with respect to a given stochastic order arises naturally. Roughly speaking a risk measure ρ is said to be consistent with a given stochastic order $\preceq$ if:

$$X \preceq Y \Rightarrow \rho(X) \leq \rho(Y).$$

This means that evaluating risk by means of the risk measure ρ does not contradict evaluating risks by means of the stochastic order $\preceq$. $(\mathcal{X}, \preceq)$ is said to be a vector lattice often called Riesz space. Several works based on the financial area of risk measurement were done in stochastic process and probability spaces. It is important to note that all of the above measures of risk are based on certain characteristics of the distribution of a random loss under a fixed, and static (single-stage), portfolio allocation. A more concrete property called *Law-invariance* is presented. It is proven to be a crucial point when applying risk measures in the financial industry. On the contrary, coherent risk measures ρ which are not *law-invariant* cannot have an approximation via empirical data.

Chapter 2

Fundamental review in Riesz Space

The purpose of this chapter is to outline the terminology and notations we will be using throughout the dissertation. We give a brief overview of the notion of partially order sets and cones so that we can introduce the basic framework in which we are going to consider risk measures.

2.1 Ordered Sets and lattices

In this section, we recall some elementary notions and concepts concerning order and lattices. For more details, we refer to [32], [36] and [37].

Definition 2.1 A *partial order* (often referred to as an *order* or *ordering*) is a relation $\leq$ in a set $\mathcal{X}$ that satisfies the following three properties :

- $a \leq a, \forall a \in \mathcal{X}$ (reflexivity).
- $a \leq b$ and $b \leq a, \forall a, b \in \mathcal{X} \quad \Rightarrow \quad a = b$ (antisymmetry).
- $a \leq b$ and $b \leq c, \forall a, b, c \in \mathcal{X} \quad \Rightarrow \quad a \leq c$ (transitivity).

The set $\mathcal{X}$ equipped with a partial ordering, is called a *partial ordered set*. A partial order is called a *total order* (or a *linear order*, or a *chain*) if it satisfies the following 4th property:

- $\forall a, b \in \mathcal{X}$ either $a \leq b$ or $b \leq a$ (comparability).

Definition 2.2 The parent entry defines a *lattice* as a relational structure (a partial ordered set i.e poset) satisfying the condition that every pair of elements has a supremum and an infimum.

Alternatively and equivalently, a *lattice* $\mathcal{L}$ can be defined directly as an algebraic structure with two binary operations called meet and join (respectively $\wedge$ and $\vee$) satisfing the following conditions :

- $\forall a \in \mathcal{L}, \quad a \vee a = a \wedge a = a$ (idempotency of $\wedge$ and $\vee$).
- $\forall a, b \in \mathcal{L}, \quad a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (commutativity of $\wedge$ and $\vee$).
- $\forall a, b \in \mathcal{L}, \quad a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ (absorption of $\wedge$ and $\vee$).
- $\forall a, b, c \in \mathcal{L}, \quad a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associativity of $\wedge$ and $\vee$).

It should not be difficult to realize that this definition is equivalent to the one given in the parent, as follows:
Define a binary relation $\leq$ on $\mathcal{L}$ such that

$$a \leq b \Leftrightarrow a \vee b = b.$$

Then $\leq$ is reflexive by the idempotency of $\vee$. If $a \leq b$ and $b \leq a$, then $a \leq a \vee b = b$. So $\leq$ is anti-symmetric. If $a \leq b$ and $b \leq c$, then we have

$$a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c \Rightarrow a \leq c.$$

So $\leq$ is transitive. This shows that $\leq$ is a partial order on $\mathcal{L}$.

$$a \vee (a \vee b) = (a \vee a) \vee b = a \vee b \text{ so that } a \leq a \vee b.$$

Similarly $b \leq a \vee b$ if $a \leq c$ and $b \leq c$ then

$$(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c.$$

This shows that $a \vee b$ is the supremum of a and b. Similarly $a \wedge b$ is the infimum of a and b. Conversely, if $(\mathcal{L}, \leq)$ is defined as in the parent entry, then by defining $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$, the four above conditions are satisfied.

Definition 2.3 A *lattice* is any non-empty poset $\mathcal{L}$ in which any two elements x and y have a least upper bound, $x \vee y$ and a greatest lower bound, $x \wedge y$.

Definition 2.4 Let $\mathcal{X}$ be a partially ordered set.

- $\mathcal{X}$ is called *order complete* (or *simply complete*) if every non-empty subset of $\mathcal{X}$ has a supremum and an infimum.
- $\mathcal{X}$ is called *Dedekind complete* if every non-empty subset of $\mathcal{X}$ that is bounded above (bounded below) has a supremum (infimum).
- $\mathcal{X}$ is called *Dedekind σ-complete* (or *countably Dedekind complete*) if every non-empty finite or countable subset of $\mathcal{X}$ that is bounded above (bounded below) has a supremum (infimum).

Definition 2.5 Let $\mathcal{X}$ be a real vector space. If $\mathcal{X}$ has a partial ordering so that:

- $f \leq g \Rightarrow f + h \leq g + h, \forall f, g, h \in \mathcal{X}$ and
- $f \leq g \Rightarrow \lambda f \leq \lambda g, \forall \lambda \geq 0$

then $\mathcal{X}$ is called an *ordered vector space*.

Definition 2.6 If the ordered vector space $\mathcal{X}$ determines a lattice structure, then $\mathcal{X}$ is called a *vector lattice* or also a *Riesz space*.

Definition 2.7 Let $\mathcal{X}$ be a real vector space. A non-empty subset C of $\mathcal{X}$ is termed a *cone* if it is closed under multiplication by non-negative scalar, i.e. if $\lambda C \subset C$ for each scalar $\lambda \geq 0$.

Examples

1. In any L^p space, the set $C = \{f \mid f \geq 0\}$ is a cone.
2. In $\mathbb{R}$, the non-negative numbers form a cone.
3. In l^p spaces, the set of non-negative sequences form a cone.
4. In $\mathbb{R}^2$, any wedge which extends to infinity from the origin is a cone.

Remark 2.8 A cone C is said to be *pointed* if $0 \in C$.

Definition 2.9 A pointed cone C is termed *salient* if C contains no 1-dimensional vector subspace of $\mathcal{X}$ (i.e. C contains no lines). Alternatively, we define a *proper cone* C also called *salient* by $C \cap (-C) = \{0\}$.

Remark 2.10 A cone C that is also a *convex* subset of $\mathcal{X}$ is termed a *convex cone*.

Thus a subset C of $\mathcal{X}$ is a *convex cone* if and only if $\lambda C \subset C$ for each scalar $\lambda \geq 0$ and $C + C \subset C$. In this case, the vector subspace of $\mathcal{X}$ generated by C is simply $C - C$ and if C is *pointed*, then $C \cap (-C)$ is the largest vector subspace of $\mathcal{X}$ contained in C, so that a pointed convex cone C is *salient* if and only if $C \cap (-C) = \{0\}$.

Definition 2.11 We say that $y \leq x$ if and only if $x - y \in C$.

Definition 2.12 For an ordered vector space V, the set $V^+ := \{v \in V \mid 0 \leq v\}$ is called the *positive cone* of V.

Remark 2.13 If the vector space $\mathcal{X}$ is endowed with a partial order, written "$x \leq y$" (or equivalently "$y \geq x$"), and the vector space structure with the partial order are such that

1. $x \leq y$ implies $x + z \leq y + z$ for all x, y, z in $\mathcal{X}$ and
2. $\lambda \geq 0, x \geq 0$ implies $\lambda x \geq 0$ for all x in $\mathcal{X}$, when λ is a non-negative scalar,

then we say that $\mathcal{X}$ is an *ordered vector space*. The partial order is, of course, assumed to satisfy the usual axioms of reflexivity, antisymmetry, and transitivity.

Remark 2.14 If $\mathcal{X}$ is an ordered vector space, it is clear that the so-called *positive cone*

$$P = \{x \mid x \in \mathcal{X}, x \geq 0\}$$

is a salient, pointed, convex cone in $\mathcal{X}$. Conversely, given such a subset P of $\mathcal{X}$, we get a partial order by defining $x \geq y$ to signify that $x - y \in P$. This partial order turns $\mathcal{X}$ into ordered vector space, and this is the only structure of an ordered vector space on $\mathcal{X}$ for which P is precisely the set of positive elements.

Proposition 2.15 If a proper cone C is endowed with the relation $\leq$, then the relation $\leq$ defines a partial ordering on $\mathcal{X}$.

Proof:
Since $0 \in C$, we see that for any point $x \in C$, $x - x \in C$. So $x \leq x$. Hence $\leq$ is reflexive. By letting $x \leq y$ and $y \leq x$, we have both $y - x$ and $-(y - x)$ in C. Hence it follows that $y - x = 0$ and so $x = y$. This shows that $\leq$ is antisymmetric.
If $x \leq y$ and $y \leq z$ therefore we have $\{(y - x) + (z - y)\} \in C$ that implies $x \leq z$.
As we proceed with the transitivity, we can say that $\leq$ is a partial ordering. ■

The partial ordering also satisfies many other properties, which follow directly from the definition of a cone.

- Multiplication by a non-negative scalar preserves the ordering. Together $x \leq y$ and $\lambda \geq 0$ imply that $\lambda x \leq \lambda y$.
- Addition of a fixed vector preserves the ordering. The inequality $x \leq y$ implies that $x + z \leq y + z$ for any vector z.
- The ordering preserves limits. If $y_n \to y$ and each $y_n \leq x$, then $y \leq x$.

Next, we will introduce the notions of supsets and infsets of a partially ordered topological vector space. This is a generalization of the usual notions of supremum and infinimum in vector lattices.

Definition 2.16 Let $\mathcal{X}$ be a vector space over the real field. Consider a convex cone C in $\mathcal{X}$ which is generating and proper. Namely, the following two conditions are satisfied :

(a) $\mathcal{X} = C - C$.

(b) $C \cap (-C) = \{0\}$.

It is well-known that conditions (a) and (b) are equivalent to the following five conditions for a given subset C of $\mathcal{X}$.

(c) $x \geq y$ and $y \geq x \Longrightarrow x = y$.

(d) $x \geq y$ and $y \geq z \Longrightarrow x \geq z$.

(e) $x \geq y \Longrightarrow x + z \geq y + z, \forall z \in \mathcal{X}$.

(f) $x \geq 0 \Longrightarrow \alpha x \geq 0, \forall \alpha > 0$.

(g) $\forall x \in \mathcal{X}, \exists (x_1, x_2) \in \mathcal{X}^2, x_1 \geq 0, x_2 \geq 0; x = x_1 + x_2$.

$\mathcal{X}$ is called a partially ordered vector space and C is called an order of $\mathcal{X}$ provided that conditions (a) and (b) are satisfied. Elements of C are said to be positive in $\mathcal{X}$.

Define the set A as

$$A = \{a_\lambda \mid \lambda \in \Lambda\}.$$

A is a subset of a partially ordered vector space $\mathcal{X}$ with order $\geq$.

Definition 2.17 We define the *supremum* of A (or $\sup A$) to be :

$$\bigvee A = \bigvee_{\lambda\in\Lambda} a_\lambda = \{z \in \mathcal{X} \mid z \geq a_\lambda, \forall\lambda \in \Lambda, z = w \text{ whenever } z \geq w \text{ and } w \geq a_\lambda\} = \sup A$$

$\bigvee A$ is the set of all minimal elements of $\bigcup\{A\} = \{z \in \mathcal{X} \mid z \geq a, \forall a \in A\}$.
The elements of $\bigcup\{A\}$ are called upper bounds of A.

Definition 2.18 We define the *infimum* of A (or $\inf A$) to be :

$$\bigwedge A = \bigwedge_{\lambda\in\Lambda} a_\lambda = \{z \in \mathcal{X} \mid z \text{ is a maximal element of } \mathrm{L}\{A\}\} = \inf A,$$

where $\mathrm{L}\{A\} = \{z \in \mathcal{X} \mid z \leq a, \forall a \in A\}$ set of lower bounds of A.

Remark 2.19 From the assumption above, it follows that:

1. A is said to be upper bounded if $\bigcup\{A\} \neq \phi$.
2. A is said to be lower bounded if $\mathrm{L}\{A\} \neq \phi$.

Let us now state some elementary observations from [32]and [37].

Proposition 2.20 If $\sup A = \{u\}$, then u is the least upper bound of A and is called the supremum of A. If $\inf A = \{l\}$, the l is the greatest lower bound of A and is called the infimum of A.

Proposition 2.21 .

(a) $-\sup(A) = \inf(-A)$.

(b) $\forall \alpha > 0, \alpha \sup(A) = \sup(\alpha A)$.

(c) $\forall \alpha > 0, \alpha \inf(A) = \inf(\alpha A)$.

(d) $\forall b \in \mathcal{X}, \sup(A) + b = \sup(A + b)$.

(e) $\forall b \in \mathcal{X}, \inf(A) + b = \inf(A + b)$.

2.2 The Bipolar Theorem

In this section, we briefly recall some theorems which will be indispensable for the proof of the characterization of a coherent (convex) risk measure.

A pair of vector spaces (L, L') and a bilinear form $\langle ., . \rangle$ represent a *dual system* if $\langle X', X \rangle = 0$ for all $X \in L$ implies $X' = 0$ and $\langle X', X \rangle = 0$ for all $X' \in L$ implies $X = 0$. If L' is a *total subspace* of L^* (i.e. $\pi(X) = 0$ for all $\pi \in L'$ implies $X = 0$) then (L, L') is a dual system with the bilinear form $\langle \pi, X \rangle = \pi(X)$. The $\sigma(L', L)$-topology on L is the coarsest topology for which the linear functionals $X \mapsto \pi(X)$ are continuous for all $\pi \in L'$. Similarly, the $\sigma(L)$-topology on L' is the coarsest topology for which the linear functionals $X \mapsto \pi(X)$ are continuous for all $X \in L$.

Definition 2.22 Let (M, M') be a dual pair of vector spaces. For every subset $\mathcal{A} \subset M$ we define its *polar* by

$$\mathcal{A}^* \triangleq \{\phi \in M' \mid |\phi(X)| \leq 1, \quad \forall X \in \mathcal{A}\},$$

and, supposing $\mathcal{A}^* \neq \emptyset$, its *bipolar*

$$\mathcal{A}^{**} \triangleq \{X \in M \mid |\phi(X)| \leq 1, \quad \forall \phi \in \mathcal{A}^*\}.$$

Definition 2.23 The right polar set A^* of a set A is defined by:

$$A^* = \{\pi \in L' \mid -\pi(X) \leq 1, \quad \forall X \in A\}.$$

Theorem 2.24 (Bipolar Theorem) $A = (A)^{**}$ holds for a set A in the dual system (L, L') if and only if A is $\sigma(L')-$closed, convex and $0 \in A$ (see [28]).

Remark 2.25 Recall that two locally convex spaces L_1 and L_2 are in duality if there exists a non-degenerate bilinear form $\langle ., . \rangle : L_1 \times L_2 \to \mathbb{R}$ such that

$$L_1^{'} = \{\langle ., X_2 \rangle \mid X_2 \in L_2\} \quad L_2^{'} = \{\langle X_1, . \rangle \mid X_1 \in L_1\}.$$

In this case we say that the topologies on L_1 and L_2 are compatible with the duality. We denote by $\sigma(L_1, L_2)$ the weak topology in L_1; i.e. the smallest topology in L_1 that is compatible with the duality. It can be proved that $(L_1, \sigma(L_1, L_2))$ is again locally convex. In particular every locally convex space L is in natural duality with its dual L', where $\langle X, \pi \rangle = \pi(X)$ for $X \in L$ and $\pi \in L'$. Given two spaces in duality and a map $\rho : L_1 \to \mathbb{R}$, it is possible to define the conjugate ρ^* directly in L_2 and to state the *Fenchel* Theorem below in terms of this duality. Recall that $\rho : L \to \mathbb{R}$ is proper if $L - L \neq \emptyset$.

Having defined the spaces $\mathcal{X}$ and $\mathcal{X}^*$, we can return to the analysis of convex risk functions.

Definition 2.26 *The conjugate $\rho^* : \mathcal{X}^* \to \overline{\mathbb{R}}$ of a convex function $\rho : \mathcal{X} \to \overline{\mathbb{R}}$ is defined as*

$$\rho^*(X') = \sup_{X \in \mathcal{X}} \{\langle X, X' \rangle - \rho(X)\}, \tag{2.1}$$

and the conjugate of ρ^* as

$$\rho^{**}(X) = \sup_{X' \in \mathcal{X}^*} \{\langle X, X' \rangle - \rho^*(X')\}. \tag{2.2}$$

Theorem 2.27 (*Fenchel* Theorem) Let L be a locally convex space and $\rho : L \to \mathbb{R} \cup \{+\infty\}$ be proper. If ρ is convex and lower semi-continuous then ρ^{**} is well-defined and

$$\rho = \rho^{**}.$$

Conversely, if ρ^{**} is well-defined and $\rho = \rho^{**}$, then ρ is convex and lower semi-continuous.

The convexity and lower semi-continuous axioms shall be defined later. The following theorem is called the Hahn-Banach separation Theorem and it will be of great importance in this dissertation.

Theorem 2.28 Suppose A and B are disjoint, nonempty, convex sets in a topological vector space $\mathcal{X}$.

(a) If A is open there exist $\Lambda \in \mathcal{X}^*$ and $\gamma \in \mathbb{R}$ such that

$$\Lambda X < \gamma \leq \Lambda Y$$

for every $X \in A$ and for every $Y \in B$.

(b) If A is compact, B is closed, and $\mathcal{X}$ is locally convex, then there exist $\Lambda \in \mathcal{X}^*$, $\gamma_1 \in \mathbb{R}$, $\gamma_2 \in \mathbb{R}$, such that

$$\Lambda X < \gamma_1 < \gamma_2 < \Lambda Y$$

for every $X \in A$ and for every $Y \in B$.

Proof: Refer to Theorem 3.4 in [30] ■

Chapter 3

Static Risk Measures

In this chapter, we discuss the notion of static risk measure and the matter related to the risk of a financial position that is described by the corresponding payoff profile which is a real value function $X \in \mathcal{X}$ where $\mathcal{X}$ denotes some sets of possible scenarios. With the term static, we mean that the object of our own measurement is a monetary position to be liquidated at a fixed future time $T > 0$. Moreover, between the initial time (i.e. when the measurement is performed) and the time of maturity, no action such as intermediate correction or additional risk monitoring takes place. No interest rates will exist between 0 and T and we shall consider a risk measure $\rho(X)$ to be the extra minimum cash added to X that makes the position acceptable for the holder or a regulator.

Throughout the chapter, we will work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and shall consider the vector space $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ (or just L^p), for $1 \leq p < +\infty$. Even though L^p consists of equivalence classes of p-integrable random variables, we will often treat its elements as random variables. We let $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathcal{X}'$ its topological dual space, i.e the space of all continuous linear functionals $x' : \mathcal{X} \to \mathbb{R}$. $\mathcal{X}$ will be endowed with its norm topology so that $\mathcal{X}' = L^q(\Omega, \mathcal{F}, \mathbb{P})$, with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. The set $\mathcal{N} = \{x' \in \mathcal{X}' \mid x'(x) \leq 0, \text{ for all } x \leq 0,\ x'(1) = 1\}$ is the set of probability

densities in $\mathcal{X}'$ in which any element $x' \in \mathcal{X}'$ may be identified with a probability measure $Q, (Q \ll P)$ by setting $x' = \frac{dQ}{dP}$, where $\frac{dQ}{dP}$ is the the Radon-Nikodym derivative of Q with respect to P. The space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the usual L^∞ norm, is the dual space of the space of integrable (equivalence classes) random variable, $L^1(\Omega, \mathcal{F}, \mathbb{P})$. We will identify, through the Radon-Nikodym theorem, finite measures that are absolutely continuous with respect to $\mathbb{P}$ (i.e. $\mathbb{P}(A) = 0 \Longrightarrow \mu(A) = 0$, where μ is a finite measure). If Q is a probability defined on the σ-algebra $\mathcal{F}$, we will use the notation $\mathbb{E}_Q$ to denote the integral operator defined by the probability Q. Let us recall that the dual of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the Banach space $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$ of all bounded, finitely additive measures μ on $(\Omega, \mathcal{F})$. To avoid any confusion, we will abbreviate the notation to $\mathbf{ba}(\mathbb{P})$. A positive element $\mu \in \mathbf{ba}(\mathbb{P})$ such that $\mu(1) = 1$ is also called a finitely additive probability measure. In the following, we will introduce the very influential axioms which make risk measurement more consistent.

3.1 Independent Risk Measures

In this section, we are not necessarily working with risk measures which are absolutely continuous with respect to the original measure. We assume that $\mathcal{X} = L^\infty(\Omega, \mathcal{F})$ is the linear space of all bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$.

A *static risk measure* is a functional $\rho : \mathcal{X} \to \mathbb{R}$ where given a financial position X, the quantity $\rho(X)$ represents the riskiness of X and, by convention, X is *acceptable* when $\rho(X) \leq 0$, and unacceptable otherwise.

Definition 3.1 A mapping $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ is called a *monetary measure of risk* if it satisfies the following two axioms:

- Monotonicity: $X \leq Y$ a.e. $\Longrightarrow \rho(X) \geq \rho(Y)$.
- Translation invariance: $\rho(X + m) = \rho(X) - m, \quad \forall m \in \mathbb{R}$.

From a financial point of view the monotonicity shows that, if a portfolio Y is always worth at least as much as X then Y cannot be riskier than X i.e. the downside risk of a position is reduced if the payoff profile is increased. Translation invariance ensures that $\rho(X)$ is taken as a capital requirement. In a financial way the risk $\rho(X)$ decrease by m by adding a sure risk-free return m to a position X. Especially, we get $\rho(X+\rho(X)) = \rho(X) - \rho(X) = 0$ that is, when adding $\rho(X)$ to the initial position X, we obtain a *neutral* position and one can see that for all m in $\mathbb{R}$, $\rho(m) = \rho(0) - m$.

Lemma 3.2 Any monetary measure of risk ρ is *Lipschitz continuous* with respect to the supremum norm $\|.\|_\infty$:

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty.$$

Proof:
For any X and Y in $\mathcal{X}$ we have the following:

$$\begin{aligned} & X - Y \leq \|X - Y\|_\infty \\ \Longrightarrow \quad & X \leq Y + \|X - Y\|_\infty \\ \Longrightarrow \quad & \rho(X) \geq \rho(Y) - \|X - Y\|_\infty. \end{aligned}$$

$$\text{Hence} \qquad -\|X - Y\|_\infty \leq \rho(X) - \rho(Y). \tag{3.1}$$

$$\begin{aligned} \text{Also} \quad & -(X - Y) \leq \|X - Y\|_\infty \\ \Longrightarrow \quad & \rho(Y) \geq \rho(X) - \|X - Y\|_\infty. \end{aligned}$$

$$\text{Then} \qquad \rho(X) - \rho(Y) \leq \|X - Y\|_\infty. \tag{3.2}$$

Therefore (3.1) and (3.2) give that $|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty$. ■

Lemma 3.2 implies the existence of a unique extension of ρ on $\mathcal{X}$. Therefore we can define the expectation operator $\mathbb{E}_Q$ with respect to a finitely additive measure Q of total mass 1.

Two families of *static risk measures* are well known in the literature: coherent and convex. Notice that coherent risk measures correspond to the *upper expectations*. A risk measure ρ is said to be acceptable if $\rho(X) \leq 0$ and not acceptable otherwise. A monetary measure of risk ρ induces the class

$$\mathcal{A}_p := \{X \in \mathcal{X} \mid \rho(X) \leq 0\},$$

of position which is acceptable in the sense that it does not require additional capital. The class $\mathcal{A}_p$ will be called the *acceptance set* of ρ.

3.1.1 Coherent risk measure

Definition 3.3 A monetary measure of risk ρ is called a *coherent risk measure* if it satisfies the following axioms:

- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y), \quad \forall X, Y \in \mathcal{X}$.
- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X), \quad \forall \lambda \geq 0, \forall X \in \mathcal{X}$.

Subadditivity means that a merger does not create extra risk. Notice that if the subadditivity does not hold, then $\rho(X+Y) \geq \rho(X)+\rho(Y)$; therefore in order to decrease risk, a firm/company might be motivated to break up into different incorporated affiliates. From the regulatory point of view, this would allow the reduction of capital requirements. Notice that covariance is subadditive, and this property is essential in Markowitz's portfolio theory: indeed new investments increase risk. We can see that subadditivity implies $\rho(\lambda X) \leq \lambda\rho(X)$, $\forall \lambda \geq 0$, $\forall X \in \mathcal{X}$. Thus $\rho(\lambda X) \geq \lambda\rho(X)$ is imposed by the positive homogeneity axiom which is based on the assumption that risk grows in a linear way as the size of the position increases. This can be justified by liquidity considerations : a position (λX) could be less liquid, and therefore more risky than that of λ smaller position (X).

Remark 3.4 Under positive homogeneity, a monetary measure is always normalized (i.e. $\rho(0) = 0$) and using the translation invariance property for all m in $\mathbb{R}$, we have $\rho(m) = -m$.

Remark 3.5 Using the translation-invariance and the monotonicity, one can notice that $\rho(X) \leq -\inf(X)$ since for all X in $\mathcal{X}$, we have $\inf(X) \leq X$.

Remark 3.6 The homogeneity property implies that the risk grows in a linear way as the size of the position increases which may not be the case for every risk measures.

Definition 3.7 A mapping $\psi : \mathcal{X} \to \mathbb{R}$ is called *submodular* if

- $\forall X \leq 0, \quad \psi(X) \leq 0.$
- $\forall X, Y \in \mathcal{X}, \quad \psi(X+Y) \leq \psi(X) + \psi(Y).$
- $\forall \lambda \geq 0, \forall X \in \mathcal{X}, \quad \psi(\lambda X) = \lambda \psi(X).$

Moreover, the *submodular* function is called translation invariant if

- $\forall X \in \mathcal{X}, \forall a \in \mathbb{R}, \quad \psi(X+a) = \psi(X) + a.$

Definition 3.8 A mapping $\phi : \mathcal{X} \to \mathbb{R}$ is called *supermodular* if

- $\forall X \geq 0, \quad \phi(X) \geq 0.$
- $\forall X, Y \in \mathcal{X}, \quad \phi(X+Y) \geq \phi(X) + \phi(Y).$
- $\forall \lambda \geq 0, \forall X \in \mathcal{X}, \quad \phi(\lambda X) = \lambda \phi(X).$

Moreover, the *supermodular* function is called translation invariant if

- $\forall X \in \mathcal{X}, \forall a \in \mathbb{R}, \quad \phi(X+a) = \phi(X) + a.$

Remark 3.9 If we consider ρ to be a coherent risk measure and if we let $\rho(X) = \psi(-X)$, we get a translation invariant submodular functional.
If we put $\rho(X) = -\phi(X)$, we obtain a translation invariant supermodular functional.

The following theorem is an immediate application of the Bipolar Theorem (Theorem 2.22).

Theorem 3.10 Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a coherent risk measure with associated submodular (respectively, supermodular) function ψ (respectively ϕ). There is a convex $\sigma(\mathbf{ba}(\mathbb{P}), \mathcal{X})$-closed set $\mathcal{P}_{ba}$ of finitely additive probabilities, such that

$$\psi(X) = \sup_{\mu \in \mathcal{P}_{ba}} \mathbb{E}_\mu[X] \qquad \text{and} \qquad \phi(X) = \inf_{\mu \in \mathcal{P}_{ba}} \mathbb{E}_\mu[X].$$

Proof: It immediately appears that

$$\begin{aligned} \psi(X - \psi(X)) &= \psi\{-(-X + \psi(X))\} \\ &= \rho(-X + \rho(-X)) \\ &= \rho(-X) - \rho(-X) \\ &= 0. \end{aligned}$$

Since for all $X \in \mathcal{X}$, $\rho(X) = \psi(-X) = -\phi(X)$, we only need to prove one of the equalities. Consider the set $A := \{X \in \mathcal{X} \mid \psi(X) \leq 0\}$. One can notice that A is a convex closed cone in $\mathcal{X}$ such that $\mathcal{X}_+ = L^\infty_+ \subset A$.
By definition 2.21, its right polar set $A^0 := \{\mu \in \mathcal{X}' \mid \mathbb{E}_\mu[X] \leq 0, \quad \forall X \in A\}$ is also a convex cone closed for the coarsest topology on $\mathcal{X}'$. Let us observe that A^0 contains only nonnegative measures and $A^{00} = \{X \in \mathcal{X} \mid \mathbb{E}_\mu[X] \leq 0, \quad \forall \mu \in A^0\}$. This implies that for the set $\mathcal{P}_{ba} := \{\mu \in A^0 \mid \mu(1) = 1\}$, we have that $A^0 = \cup_{\lambda \geq 0} \lambda \mathcal{P}_{ba}$. By the Bipolar Theorem we have the following equivalence:

$$\psi(X) \leq 0, \quad \forall X \in \mathcal{X} \qquad \Leftrightarrow \qquad \mathbb{E}_\mu[X] \leq 0, \quad \forall \mu \in \mathcal{P}_{ba}. \tag{3.3}$$

Since $\psi(X - \psi(X)) = 0$, then $X - \psi(X) \in A$. It follows from (3.3) that we have

$$\mathbb{E}_\mu[X - \psi(X)] \leq 0, \ \ \forall \mu \in \mathcal{P}_{ba}.$$

This can be reformulated as

$$\sup_{\mu \in \mathcal{P}_{ba}} \mathbb{E}_\mu[X] \leq \psi(X). \tag{3.4}$$

Let choose an arbitrarily $\xi > 0$, then we have

$$\begin{aligned} \psi(X - \psi(X) + \xi) &= \psi\{(X - \psi(X)) + \xi\} \\ &= \psi(X - \psi(X)) + \xi \\ &= \xi > 0. \end{aligned}$$

Therefore, $X - \psi(X) + \xi \notin A$.

Hence it means that we can find μ in $\mathcal{P}_{ba}$ such that $\mathbb{E}_\mu[X - \psi(X) + \xi] > 0$. Again this can be reformulated as

$$\sup_{\mu \in \mathcal{P}_{ba}} \mathbb{E}_\mu[X] > \psi(X) - \xi. \tag{3.5}$$

Hence (3.4) and (3.5) lead us to:

$$\psi(X) = \sup_{\mu \in \mathcal{P}_{ba}} \mathbb{E}_\mu[X]. \tag{3.6}$$

■

Theorem 3.11 Typically, a *coherent risk measure* ρ can be represented by the supremum of the expected negative of final net worth for some collection of finitely additive probability measures $\mathcal{P} \in (\Omega, \mathcal{F})$ with total mass 1, such that :

$$\rho(X) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[-X], \quad X \in \mathcal{X}.$$

Moreover, $\mathcal{P}$ can be taken as a convex set for which the supremum above is attained.

Proof: The proof follows from Remark 3.9 and Theorem 3.10. ■

3.1.2 Continuity

Consider now the following continuity definitions.

Definition 3.12 We say that a map $\rho : L^\infty \to \mathbb{R}$ is:

- continuous from above if $\rho(X_n) \to \rho(X)$ provided $X_n \downarrow X$ $\mathbb{P} - a.s.$
- continuous from below if $\rho(X_n) \to \rho(X)$ provided $X_n \uparrow X$ $\mathbb{P} - a.s.$
- continuous for bounded sequences if $\rho(X_n) \to \rho(X)$ provided $X_n \to X$ $\mathbb{P} - a.s.$
- Fatou-continuous, $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$ provided $(X_n)_{n\geq 0}$ is bounded in L^∞ and $X_n \to X$ in $\mathbb{P}$-probability.

In general there are no relations among them and lower semicontinuity defined later.

Definition 3.13 A monetary measure of risk ρ is called a *distribution-invariant risk measure* if for all X, Y in $\mathcal{X}$, ρ satisfies:

- Distribution-invariance: $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1} \Rightarrow \rho(X) = \rho(Y)$.

Let now introduce a weaker notion of continuity with respect to the weak topology.

3.1.3 Continuity on compacts

Let assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with continuous distribution. We denote by $\mathcal{M}_{1,c}(K)$ the space of probability measures supported in K and consider $\mathcal{M}_{1,c} = \mathcal{M}_{1,c}(\mathbb{R})$ to be the space of Borel probability measures on $\mathbb{R}$ with compact support. Suppose we are interested in the risk of a financial position $X \in \mathcal{X}$ with distribution $\mathbb{P} \circ X^{-1} = \nu$. We endow $\mathcal{M}_{1,c}$ with the weak topology. A distribution-invariant risk measure ρ defines a functional $\rho' : \mathcal{M}_{1,c} \to \mathbb{R}$ by $\rho'(\nu) = \rho(X)$ for some $X \in \mathcal{X}$.

Definition 3.14 A distribution-invariant risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is *continuous on compacts*, if for all compact sets $K \subseteq \mathbb{R}$ the restriction $\rho' : \mathcal{M}_{1,c}(K) \to \mathbb{R}$ is continuous.

Theorem 3.15 Let ρ be a distribution-invariant risk measure. The following conditions are equivalent:

(a) ρ is continuous on compacts.

(b) ρ is both continuous from above and from below.

(c) ρ is continuous for bounded sequences.

Proof:
(a)⇒(c) Suppose that ρ is continuous on compacts. Let $(X_n)_{n\geq 0}$ be a bounded sequence converging to some X $\mathbb{P}$-a.s and $(\nu_n)_{n\geq 0}$ a sequence converging to ν in $\mathcal{M}_{1,c}(K)$. Then we can find a compact set $K \subseteq \mathbb{R}$ so that for all n in $\mathbb{N}$, we have X_n and X in K $\mathbb{P}$-a.s. Thus

$$\rho(X_n) = \rho'(\nu_n) \to \rho'(\nu) = \rho(X).$$

(c)⇒(a) Let K be a compact subset of $\mathbb{R}$ and $(\nu_n)_{n\geq 0}$ a sequence converging to ν in $\mathcal{M}_{1,c}(K)$. Denote by F_n and F the distribution functions of ν_n and ν, respectively. We can find a random variable $Z : \Omega \to \mathbb{R}$ such that $X_n := F_n^{-1}(Z)$ and $X := F^{-1}(Z)$, where F_n^{-1} and F^{-1} are the right-continuous inverses of F_n and F, respectively. Observe that $X_n \to X$ $\mathbb{P}$-a.s. as $n \to \infty$. Moreover, X_n and X are in K $\mathbb{P}$-a.s. Hence,

$$\rho'(\nu_n) = \rho(X_n) \to \rho(X) = \rho'(\nu).$$

(b)⇒(c) Let (X_n) be bounded and $X_n \to X$ $\mathbb{P} - a.s.$ Therefore $(\sup_{m\geq n} X_m)_n \downarrow X$ and $(\inf_{m\geq n} X_m)_n \uparrow X$. So we obtain that

$$\rho(X) = \lim_n \rho(\sup_{m\geq n} X_m) \leq \lim_{n\to\infty} \inf \rho(X_n) \leq \lim_{n\to\infty} \sup \rho(X_n) \leq \lim_n \rho(\inf_{m\geq n} X_m) = \rho(X).$$

(c)⇒(b) Let a bounded sequence (X_n) converges to X $\mathbb{P}$-a.s. and suppose that ρ is continuous for (X_n). Thus,

$$\begin{aligned} \lim_{n\to\infty} \rho(X_n) &= \rho(X) \\ \Rightarrow \quad \liminf_{n\to\infty} \rho(X_n) &= \limsup_{n\to\infty} \rho(X_n). \end{aligned}$$

Hence ρ is both continuous from above and from below. ■

3.1.4 Convex risk measure

Coherent risk measures were extended in general spaces by Delbaen [9] and later to convex risk measures by Föllmer and Schied [12], and independently, Frittelli and Rosazza Gianin (see [15] and [16]) proposed to relax the constraints of positive homogeneity together with subadditivity. Below, we will adopt the definition given by the last authors.

Definition 3.16 A monetary risk measure ρ is called a *convex risk measure* if for all X and Y in $\mathcal{X}$, ρ satisfies the following axiom:

- Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for any $\lambda \in (0,1)$.
- Lower semi-continuity: the set $\{X \in \mathcal{X} \mid \rho(X) \leq \lambda\}$ is closed in $\mathcal{X}$ for all $\lambda \in \mathbb{R}$.
- Normalization: $\rho(0) = 0$.

The convexity property is related to the notion of diversification in the sense that diversification in a portfolio should not increase the risk. The lower semi-continuity property guarantees that the limit position of a sequence (or net) of acceptable positions remains acceptable.

Remark 3.17 One easily notices that under the normalization assumption, convexity and subadditivity are equivalent properties.

Remark 3.18 If a risk measure ρ is coherent, then it is a convex risk measure. For the converse to hold, ρ has to be positively homogenous.

Proof:
Suppose that ρ is a coherent risk measure. Then for all X, Y in $\mathcal{X}$, $\forall\lambda \in (0,1)$, we have

$$\begin{aligned} \rho(\lambda X + (1-\lambda)Y) &\leq \rho(\lambda X) + \rho((1-\lambda)Y), \quad \text{since } \rho \text{ is subadditive} \\ &\leq \lambda\rho(X) + (1-\lambda)\rho(Y), \quad \text{since } \rho \text{ is positively homogenous.} \end{aligned}$$

If conversely ρ is positively homogenous, then by its convexity property, ρ becomes subadditive. ■

Proposition 3.19 For a risk measure $\rho : L^\infty \to \mathbb{R}$, continuity from below implies the following properties which in turn are equivalent:

(a) ρ is $\sigma(L^\infty, L^1)$-lower semicontinuous.

(b) ρ is continuous from above.

(c) ρ is F-continuous.

If ρ is a convex risk measure then (a) to (c) are equivalent to:

(d) The set $\mathcal{A}_\rho = \{X \in L^\infty \mid \rho(X) \leq 0\}$ is $\sigma(L^\infty, L^1)$-closed.

Proof: Refer to Theorem 6 in [12]. ■

Convex risk measures take into account the situations where the risk of a position increase in a nonlinear way with the size of the position. They have the corresponding structure theorem:

Theorem 3.20 There exists a convex functional $\alpha : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ called *penalty function* such that any *convex risk measure* ρ on $\mathcal{X}$ is of the form

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{\mathbb{E}_Q[-X] - \alpha(Q)\}, \quad \text{where } \inf_{Q \in \mathcal{P}} \alpha(Q) = 0.$$

Proof:

We first show that ρ is a convex risk measure.

$\rho(0) = \sup_{Q \in \mathcal{P}}(\mathbb{E}_Q[-0] - \alpha(Q)) = \sup_{Q \in \mathcal{P}}(0 - \alpha(Q)) = -\inf_{Q \in \mathcal{P}} \alpha(Q) = 0.$

In fact, it suffices to check the convexity property, since ρ is already normalized.

Consider X, Y in $\mathcal{X}$ and $\lambda \in (0,1)$. Then we have the following

$$\begin{aligned}
\rho(\lambda X + (1-\lambda)Y) &= \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-\lambda X - (1-\lambda)Y] - \alpha(Q)\} \\
&= \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-\lambda X] + \mathbb{E}_Q[-(1-\lambda)Y] - \alpha(\lambda Q + (1-\lambda)Q)\} \\
&= \sup_{Q\in\mathcal{P}}\{\lambda\mathbb{E}_Q[-X] + (1-\lambda)\mathbb{E}_Q[-Y] - (\lambda\alpha(Q) + (1-\lambda)\alpha(Q))\} \\
&= \sup_{Q\in\mathcal{P}}\{\lambda(\mathbb{E}_Q[-X] - \alpha(Q)) + (1-\lambda)(\mathbb{E}_Q[-Y] - \alpha(Q))\} \\
&\leq \sup_{Q\in\mathcal{P}}\{\lambda(\mathbb{E}_Q[-X] - \alpha(Q))\} + \sup_{Q\in\mathcal{P}}\{(1-\lambda)(\mathbb{E}_Q[-Y] - \alpha(Q))\} \\
&\leq \lambda \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-X] - \alpha(Q)\} + (1-\lambda)\sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-Y] - \alpha(Q)\} \\
&\leq \lambda\rho(X) + (1-\lambda)\rho(Y).
\end{aligned}$$

And now, let us assume that ρ is any convex risk measure defined on $\mathcal{X}$. Recall that the conjugate function ρ^* of $\rho : \mathcal{X} \to \mathbb{R}$ is defined as

$$\rho^*(Q) = \sup_{X\in\mathcal{X}'}\{\mathbb{E}_Q[-X] - \rho(X)\}, \text{ where } \mathcal{X}' \text{ is the dual space of } \mathcal{X}$$

Take $X \in \mathcal{X}$ such that $\rho(X) < +\infty$ and consider $\widetilde{X} = X + m.1$, where $m \geq 0$, is a constant term. Hence by monotonicity $\rho(\widetilde{X}) \leq \rho(X)$. Therefore we have

$$\begin{aligned}
\rho^*(Q) &\geq \sup_{\widetilde{X}\in\mathcal{X}'}\{\mathbb{E}_Q[-\widetilde{X}] - \rho(\widetilde{X})\} \\
&\geq \sup_{\widetilde{X}\in\mathcal{X}'}\{\mathbb{E}_Q[-(X+m.1)] - \rho(X+m.1)\} \\
&\geq \sup_{m\in\mathbb{R}}\{\mathbb{E}_Q[-X] - m.1 - \rho(X) + m\} \\
&\geq \mathbb{E}_Q[-X] - \rho(X).
\end{aligned}$$

Thus $\rho^*(Q)$ is finite. If we fix $X \geq 0$ and let $\lambda \geq 0$, then we get $\lambda X \geq 0$ and the monotonicity gives us that $\rho(\lambda X) \leq 0$. Henceforth

$$\rho^*(Q) \geq \sup_{\lambda\geq 0}\{\mathbb{E}_Q[-\lambda X] - \rho(\lambda X)\} \geq \sup_{\lambda\geq 0}\{\mathbb{E}_Q[-\lambda X]\}.$$

Again the conjugate $\rho^*(Q)$ can only be finite. Now it suffices to take the conjugate of ρ^* using the fact that the bidual of $\mathcal{X}$ is $\mathcal{X}$ itself, since $\mathcal{X}$ is reflexive as a Banach lattice, to deduce that

$$\rho(X) = \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-X] - \rho^*(Q)\}.$$

By defining $\alpha : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ as

$$\alpha(Q) = \rho^*(-Q), \quad \forall Q \in \mathcal{P}$$

one obtains that $\rho(X) = \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-X] - \alpha(Q)\}$ and therefore

$$\begin{aligned} -\inf_{Q\in\mathcal{P}} \alpha(Q) &= \sup_{Q\in\mathcal{P}} -\alpha(Q) \\ &= \sup_{Q\in\mathcal{P}}\{0 - \alpha(Q)\} \\ &= \sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q[-0] - \alpha(Q)\} \\ &= \rho(0) \\ &= 0. \end{aligned}$$

■

Remark 3.21 A general representation for *convex risk measure* has been obtained by Frittelli and Rosazza Gianin [15]. Note that Theorem 3.21 contains Theorem 3.11 as a particular case, since it corresponds to the penalty function

$$\alpha(Q) = \begin{cases} 0 & \text{if } Q \in \mathcal{P} \\ +\infty & \text{otherwise} \end{cases}.$$

3.1.5 Convex cones

Let L be a real vector space and $M \subseteq L$ a convex cone. Suppose also that L is endowed with a partial vector preorder $\geq$ and define $M_+ \triangleq \{X \in M : X \geq 0\}$. Note that this is the minimal algebraic framework in which the axioms of convexity, monotonicity and homogeneity for a map $\rho : M \to \mathbb{R}$ make sense.

Definition 3.22 A risk measure on M is any map $\rho : M \to \mathbb{R}$ that is convex, monotone and such that $\rho(0) = 0$. If in addition ρ satisfies the translation-invariance property, then ρ is called an *invariance risk measure*.

Definition 3.23 We say that $\mathcal{A} \subset M$ is an acceptance set if:

- $\mathcal{A}$ is solid, i.e. $X \geq Y \in \mathcal{A} \Rightarrow X \in \mathcal{A}$.
- $\mathcal{A}$ is convex.
- $\alpha\mathbf{1} \in \mathcal{A}$ if and only if $\alpha \geq 0$.
- $\forall X \in M\ ,\ \exists(\alpha, \alpha') \in \mathbb{R}^2\ ;\ X + \alpha\mathbf{1} \in \mathcal{A}\ ,\ X + \alpha'\mathbf{1} \notin \mathcal{A}$.

Definition 3.24 Given an acceptance set $\mathcal{A}$ and an element $X \in M$; we define the corresponding *capital requirement*:

$$\rho_{\mathcal{A}}(X) := \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} \in \mathcal{A}\}. \tag{3.7}$$

Definition 3.25 We say that a map $\rho : M \to \mathbb{R}$ is a *capital requirement* if $\rho = \rho_{\mathcal{A}}$ for some acceptance set $\mathcal{A}$.

Proposition 3.26 A map $\rho : M \to \mathbb{R}$ is an invariant risk measure if and only if it is a capital requirement. It is in addition homogeneous if and only if it is a capital requirement associated to a cone.

Proof:
$\Rightarrow$) Suppose that ρ is a capital requirement and let us assume that for any Y in $\mathcal{A}$ we have $X \geq Y$. Now since $\mathcal{A}$ is solid, we have

$$\{m \in \mathbb{R} \mid Y + m.\mathbf{1} \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \mid X + m.\mathbf{1} \in \mathcal{A}\}.$$

Thus, we get $\rho_{\mathcal{A}}(Y) \geq \rho_{\mathcal{A}}(X)$. The third property of Definition 3.23 shows that $\rho_{\mathcal{A}}(0) = \inf\{m \in \mathbb{R} \mid m.1 \in \mathcal{A}\} = 0$. Since $\mathcal{A}$ is convex, then for every $\lambda \in (0, 1)$ we have

$$\lambda\rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y) \quad = \quad \lambda \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} \in \mathcal{A}\} + (1-\lambda)\inf\{\alpha \in \mathbb{R} \mid Y + \alpha\mathbf{1} \in \mathcal{A}\}$$

$$\begin{aligned} &= \inf\{m \in \mathbb{R} \mid \lambda X + m\mathbf{1} \in \mathcal{A}_\lambda\} + \inf\{n \in \mathbb{R} \mid (1-\lambda)Y + n\mathbf{1} \in \mathcal{A}_{1-\lambda}\} \\ &= \inf\{m+n \in \mathbb{R} \mid \lambda X + m\mathbf{1} \in \mathcal{A}_\lambda, (1-\lambda)Y + n\mathbf{1} \in \mathcal{A}_{1-\lambda}\} \\ &\geq \inf\{p \in \mathbb{R} \mid \lambda X + (1-\lambda)Y + p\mathbf{1} \in \mathcal{A}\} \\ &\geq \rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y), \end{aligned}$$

where

$$\mathcal{A}_\lambda = \lambda\mathcal{A} \ , \ m = \lambda\alpha, \mathcal{A}_{1-\lambda} = (1-\lambda)\mathcal{A} \ , \ n = (1-\lambda)\alpha \ \text{ and } \ p = m+n.$$

Concerning the translation-invariance property, it is not hard to see that

$$\begin{aligned} \rho_{\mathcal{A}}(X + m\mathbf{1}) &= \inf\{\alpha \in \mathbb{R} \mid (X+m) + \alpha\mathbf{1} \in \mathcal{A}\} \\ &= \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} + m \in \mathcal{A}\} \\ &= \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} \in \mathcal{A}\} - m \\ &= \rho_{\mathcal{A}}(X) - m. \end{aligned}$$

Hence ρ is an invariant risk measure.

$\Rightarrow$) Suppose that $\rho : M \to \mathbb{R}$ is an invariance risk measure and $\mathcal{A}_\rho$ the acceptance set induced by ρ. Then

$$\begin{aligned} \rho_{\mathcal{A}_\rho}(X) &= \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} \in \mathcal{A}_\rho\} \\ &= \inf\{\alpha \in \mathbb{R} \mid \rho(X + \alpha\mathbf{1}) \leq 0\} \\ &= \inf\{\alpha \in \mathbb{R} \mid \rho(X) - \alpha \leq 0\}, \quad \text{by translation-invariance} \\ &= \inf\{\alpha \in \mathbb{R} \mid \rho(X) \leq \alpha\} \\ &= \rho(X). \end{aligned}$$

If in addition, we consider $\mathcal{A}$ to be a cone, then for any $\lambda \geq 0$, we have

$$\begin{aligned} \lambda\rho_{\mathcal{A}}(X) &= \lambda \inf\{\alpha \in \mathbb{R} \mid X + \alpha\mathbf{1} \in \mathcal{A}\} \\ &= \inf\{\lambda\alpha \in \mathbb{R} \mid \lambda X + \lambda\alpha\mathbf{1} \in \lambda\mathcal{A} = \mathcal{A}\} \\ &= \inf\{\beta \in \mathbb{R} \mid \lambda X + \beta\mathbf{1} \in \mathcal{A}\} \\ &= \rho_{\mathcal{A}}(\lambda X). \end{aligned}$$

Therefore ρ satisfies the positive homogeneity property.

Conversely set $\mathcal{A}_\rho := \{X \in M \mid \rho(X) \leq 0\}$ where ρ is an invariance risk measure. Let us prove that $\mathcal{A}_\rho$ is defined as an acceptance set induced by ρ.

$$\begin{aligned} X \geq Y \in \mathcal{A}_\rho &\Rightarrow \rho(X) \leq \rho(Y)\,, \quad \text{since } \rho \text{ is monotone} \\ &\Rightarrow \rho(X) \leq 0 \\ &\Rightarrow X \in \mathcal{A}_\rho. \end{aligned}$$

Thus $\mathcal{A}_\rho$ is solid. The convexity of ρ gives us that

$$\begin{aligned} \forall X, Y \in \mathcal{A}_\rho, \forall \lambda \in (0,1), \quad \rho(\lambda X + (1-\lambda)Y) &\leq \lambda\rho(X) + (1-\lambda)\rho(Y) \\ &\leq \lambda \cdot 0 + (1-\lambda)\cdot 0 \\ &\leq 0. \end{aligned}$$

Finally we have that $\lambda X + (1-\lambda)Y \in \mathcal{A}_\rho$ meaning that $\mathcal{A}_\rho$ is a convex set.

$$\begin{aligned} \alpha\mathbf{1} \in \mathcal{A}_\rho \Leftrightarrow \rho(\alpha\mathbf{1}) &= \alpha\rho(\mathbf{1}) \\ &= -\alpha \leq 0 \\ &\Rightarrow \alpha \geq 0. \end{aligned}$$

$$\begin{aligned} X + \alpha\mathbf{1} \in \mathcal{A}_\rho &\Leftrightarrow \rho(X + \alpha\mathbf{1}) \leq 0 \\ &\Leftrightarrow \rho(X) - \alpha \leq 0 \\ &\Leftrightarrow \alpha \geq \rho(X). \end{aligned}$$

If ρ is positively homogenous, then for any $\lambda \geq 0$, we have

$$\begin{aligned} \lambda\mathcal{A}_\rho &= \lambda\{m \in \mathbb{R} \mid \rho(X) \leq 0\} \\ &= \{\lambda m \in \mathbb{R} \mid \lambda\rho(X) \leq 0\} \\ &= \{\alpha \in \mathbb{R} \mid \rho(\lambda X) \leq 0\} \qquad \text{where } \alpha = \lambda m \\ &= \{\alpha \in \mathbb{R} \mid \rho(Z) \leq 0\} \qquad \text{where } Z = \lambda X \\ &= \mathcal{A}_\rho. \end{aligned}$$

Hence $\mathcal{A}_\rho$ is a cone and the proof is completed. ■

We provide now a representation result for such capital requirements (or equivalently for invariant risk measures) provided an "algebraic" duality with another convex cone is given. Suppose that L' is another vector space and $M' \subseteq L'$ is a convex cone and let $\langle .,. \rangle : M \times M' \to \mathbb{R} \cup \{+\infty\}$ be a map such that every section $\langle X, . \rangle$ or $\langle ., \phi \rangle$ is positively homogeneous and additive for every $X \in M$ and every $\phi \in M'$. If we set $\phi(X) \triangleq \langle X, \phi \rangle$; note that $\phi(\mathbf{0}) = 0$ for every ϕ and since $\phi(-\mathbf{1}) = -\phi(\mathbf{1})$, it follows that $\phi(X + \alpha\mathbf{1}) = \phi(X) + \alpha\phi(\mathbf{1})$ for every real number α. Define $M'_+ \triangleq \{\phi \in M' \mid \phi(X) \geq 0,\ \forall X \in M_+\}$.

Proposition 3.27 Let $\mathcal{A}$ be an acceptance set such that $\mathcal{A}^* \neq \emptyset$ and $\mathcal{A} = \mathcal{A}^{**}$ and suppose that $M'_1 \triangleq \{\phi \in M'_+ \mid \phi(\mathbf{1}) = 1\} \neq \emptyset$. Then

$$\rho_{\mathcal{A}}(X) = \sup_{\phi \in M'_1} \{-\phi(X) - \gamma(\phi)\}, \tag{3.8}$$

where $\gamma(\phi) \triangleq \sup\{-\phi(Y) \mid Y \in \mathcal{A}\}$.

Proof:

Let us fix ϕ in $M_1^{'}$ and consider $X + \rho_{\mathcal{A}}(X) \cdot \mathbf{1}$ in $\mathcal{A}$. Therefore we have

$$\begin{aligned} \gamma(\phi) &\geq -\phi(X + \rho_{\mathcal{A}}(X) \cdot \mathbf{1}) \\ &= -\phi(X) - \rho_{\mathcal{A}}(X) \cdot \phi(\mathbf{1}) \\ &= -\phi(X) - \rho_{\mathcal{A}}(X). \end{aligned}$$

So for all ϕ in $M_1^{'}$, we have

$$-\phi(X) - \gamma(\phi) \leq \rho_{\mathcal{A}}(X).$$

This shows that

$$\sup_{\phi \in M_1^{'}} \{-\phi(X) - \gamma(\phi)\} \leq \rho_{\mathcal{A}}(X). \tag{3.9}$$

Now to show the reverse inequality, it suffices to prove that $X - \phi(X) - \gamma(\phi) \in \mathcal{A}$. First of all note that $\phi(X) \leq 1, \forall X \in \mathcal{A}$ whenever $\phi \in \mathcal{A}^*$. So we easily get that

$$\inf_{X \in \mathcal{A}} \{\phi(X)\} \leq 1. \tag{3.10}$$

Hence one can see that $-\sup_{X\in\mathcal{A}}\{-\phi(X)\} \leq 1$ and get from its definition that

$$-\gamma(\phi) \leq 1, \quad \forall \phi \in \mathcal{A}^*.$$

Thus, we have that

$$\begin{aligned} \phi[X - \phi(X) - \gamma(\phi)] &= \phi(X) - \phi(X) - \gamma(\phi) \\ \Rightarrow \quad \phi[X - \phi(X) - \gamma(\phi)] &\leq 1. \end{aligned}$$

This implies that for all ϕ in $M_1^{'}$, we have $X - \phi(X) - \gamma(\phi)$ in $\mathcal{A}^{**} = \mathcal{A}$ and consequently one can deduce that

$$\begin{aligned} 0 &\geq \rho_{\mathcal{A}}[X - \phi(X) - \gamma(\phi)] \\ \Rightarrow \qquad 0 &\geq \rho_{\mathcal{A}}(X) + \phi(X) + \gamma(\phi) \\ \Rightarrow \quad \rho_{\mathcal{A}}(X) &\leq -\phi(X) - \gamma(\phi), \quad \forall \phi \in M_1^{'} \end{aligned}$$

which leads us to the following

$$\rho_{\mathcal{A}}(X) \leq \sup_{\phi\in M_1^{'}} \{-\phi(X) - \gamma(\phi)\}. \tag{3.11}$$

Finally, the inequalities (3.9) and (3.11) complete our proof. ■

3.2 Space of acceptance

In many situations, however, risk may grow in a *non-linear* way as the size of the position increases. A monetary measure of risk ρ induces the class

$$\mathcal{A} := \{X \in \mathcal{X} \mid \rho(X) \leq 0\},$$

of positions which are acceptable in the sense that they do not require additional capital. The class $\mathcal{A}_p$ will be called the *acceptance set* of ρ. The following two propositions summarize the relations between monetary measures of risk and their acceptance sets.

Definition 3.28 Given an acceptance set $\mathcal{A}$; the capital requirement associated to $\mathcal{A}$ is the map $\rho_{\mathcal{A}} : L^\infty \to \mathbb{R}$ defined by

$$\rho_{\mathcal{A}}(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}.$$

Proposition 3.29 Suppose that ρ is a monetary measure of risk with acceptance set $\mathcal{A} := \mathcal{A}_p$.

(a) $\mathcal{A}$ is non-empty, and satisfies the following two conditions:

- $\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty$.
- $X \in \mathcal{A}, Y \in \mathcal{X}, X \leq Y \Rightarrow Y \in \mathcal{A}$.

Moreover, $\mathcal{A}$ has the following closure property:
for $X \in \mathcal{A}$ and $Y \in \mathcal{X}$, $\{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathcal{A}\}$ is closed in $[0,1]$.

(b) ρ can be recovered from $\mathcal{A}$:

$$\rho(X) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\}.$$

(c) ρ is a convex risk measure if and only if $\mathcal{A}$ is convex.

Proof:

$$\begin{aligned}
(a) \qquad -\inf\{m \in \mathbb{R} | m \in \mathcal{A}\} &= \sup\{-m \in \mathbb{R} \mid m \in \mathcal{A}\} \\
&= \sup\{-m \in \mathbb{R} \mid m \in \mathcal{A}_\rho\} \\
&= \sup\{-m \in \mathbb{R} \mid \rho(m) \leq 0\} \\
&= \sup\{-m \in \mathbb{R} \mid \rho(0) - m \leq 0\}, \text{ by translation-invariance} \\
&= \sup\{-m \in \mathbb{R} \mid -m \leq -\rho(0)\} \\
&= -\rho(0) < +\infty.
\end{aligned}$$

So $\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty$.

Let $X \in \mathcal{A}$ and $Y \in \mathcal{X}$ such that $X \leq Y$, then by monotonicity, $\rho(Y) \leq \rho(X)$. But since $X \in \mathcal{A} = \mathcal{A}_\rho$, we have $\rho(Y) \leq 0$, for all Y in $\mathcal{X}$. Thus, $Y \in \mathcal{A}_\rho = \mathcal{A}$.

For the proof of the closure property, we have that

$$\begin{aligned}\{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathcal{A}\} &= \{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathcal{A}_\rho\} \\ &= \{\lambda \in [0,1] \mid \rho(\lambda X + (1-\lambda)Y) \leq 0\}.\end{aligned}$$

One can notice that using Lemma 3.2, the functional $\lambda \mapsto \rho_\lambda(X,Y) := \rho(\lambda X + (1-\lambda)Y)$ is well-defined and continuous. So one can deduce that $\rho_\lambda^{-1}((-\infty, 0])$ is closed as the inverse image of closed set. Therefore $\{\lambda \in [0,1] \mid \lambda X + (1-\lambda)Y \in \mathcal{A}\}$ is closed in [0,1].

$$\begin{aligned}(b) \quad \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\} &= \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_\rho\} \\ &= \inf\{m \in \mathbb{R} \mid \rho(X+m) \leq 0\} \\ &= \inf\{m \in \mathbb{R} \mid \rho(X) - m \leq 0\}, \text{ by translation-invariance} \\ &= \inf\{m \in \mathbb{R} \mid \rho(X) \leq m\} \\ &= \rho(X).\end{aligned}$$

(c) To show that $\rho_\mathcal{A}$ and $\mathcal{A}_\rho$ satisfy the convexity property, one has to refer to the proof of Proposition 3.26. ■

Proposition 3.30 Suppose that $\mathcal{A}$ is a non-empty set satisfying the followings:

- $\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty$.
- $X \in \mathcal{A}, Y \in \mathcal{X}, X \leq Y \Rightarrow Y \in \mathcal{A}$.

Then the functional $\rho_\mathcal{A}$ has the following properties:

(a) $\rho_\mathcal{A}$ is a monetary risk measure.

(b) ρ is positively homogeneous if and only if $\mathcal{A}$ is a cone. In particular, ρ is coherent if and only if $\mathcal{A}$ is a convex cone.

(c) If $\mathcal{A} \subseteq \mathcal{A}_{\rho_\mathcal{A}}$ satisfies the closure property, then $\mathcal{A} = \mathcal{A}_{\rho_\mathcal{A}}$.

Proof:

(a) It suffices to show that $\rho = \rho_{\mathcal{A}}$. For this, use part (b) of Proposition 3.29.

(b) For the first equivalence relation, we refer to the proof of Proposition 3.26. Let us prove the second equivalence relation.

$\Rightarrow$) Suppose that ρ is coherent. One can use Remark 3.18 and part (c) of Proposition 3.29 to deduce that $\mathcal{A}$ is convex. The first equivalence of part (b) shows that the positive homogeneity property satisfied by ρ implies that $\mathcal{A}$ is a cone. Therefore $\mathcal{A}$ is a convex cone.

$\Leftarrow$) Suppose that $\mathcal{A}$ is a convex cone. Part (c) of Proposition 3.29 shows that ρ is a convex risk measure and the first equivalence of part (b) in Proposition 3.30 implies that ρ is positively homogenous. Thus using of the converse of Remark 3.18, we can conclude that ρ is coherent.

(c) Let us take a random variable $X : \Omega \mapsto \mathbb{R}$ with $X \notin \mathcal{A}$ and consider a real value m so that

$$m > \|X\| = \sup_{\omega \in \Omega}\{X(\omega)\}.$$

Using closure property we can find $\lambda \in [0, 1]$ such that $\lambda m + (1 - \lambda)X \notin \mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$. Hence

$$\begin{aligned} 0 &< \rho_{\mathcal{A}}((1-\lambda)X + \lambda m) \\ &= \rho_{\mathcal{A}}[(1-\lambda)X] - \lambda m \end{aligned}$$

which leads us to

$$\rho_{\mathcal{A}}[(1-\lambda)X] > \lambda m. \tag{3.12}$$

Since $\rho_{\mathcal{A}}$ is a monetary risk measure, by Lemma 3.2, it follows that

$$\begin{aligned} |\rho_{\mathcal{A}}((1-\lambda)X) - \rho_{\mathcal{A}}(X)| &\leq \| - \lambda X\|_\infty \\ &\leq \lambda \|X\|_\infty \\ \text{Then} \qquad \rho_{\mathcal{A}}(X) &\geq \rho_{\mathcal{A}}[(1-\lambda)X] - \lambda\|X\| \\ &> \lambda m - \lambda\|X\|\,, \qquad \text{using equation (3.12)} \end{aligned}$$

$$\begin{aligned} &> \lambda(m - \|X\|) \\ &> 0. \end{aligned}$$

So we have $X \notin \mathcal{A}_{\rho_{\mathcal{A}}}$. Finally this gives us that $\mathcal{A}_{\rho_{\mathcal{A}}} \subseteq \mathcal{A}$ and since by assumption we have that $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$, our proof is then complete. ■

Examples 3.31 Let us consider the following two examples:

- $\mathcal{A} = L^{\infty}_{+} = \{X \in L^{\infty} \mid X \geq 0\}$. This is the smallest set of acceptance, since every acceptance set, containing 0; contains also every non-negative random variable. The corresponding capital requirement is the coherent risk measure defined by
$$\rho_{L^{\infty}_{+}}(X) = -ess\inf\{X\}.$$
- $\mathcal{A}_{\alpha} = \{X \mid \mathbb{E}_{\mathbb{P}}[X\mathbf{1}_A], \forall A \in \mathcal{F} \text{ s.t. } \mathbb{P}(A) \geq \alpha\}$, for $\alpha \in (0,1)$. The corresponding capital requirement ρ_{α} is called the Worst Conditional Measure and can be represented as follows:
$$\rho_{\alpha}(X) = WCM_{\alpha}(X) \triangleq \sup_{\mathbb{P}(A)\geq\alpha} \{-\frac{\mathbb{E}_{\mathbb{P}}[X\mathbf{1}_A]}{\mathbb{P}(A)}\}.$$

3.2.1 Comonotonic risk measures

Coherent risk measures take subadditivity as the most basic requirement for a good risk measure, i.e. in many situations, the risk of a combined position $X + Y$ will be strictly lower than the sum of the individual risks. This seems to be reasonable because we expect risks to diversify or at least remain the same when we put them together. However there are many circumstance where we might reasonably use a non-subadditive risk measure. One typical example is VaR, which is not subadditive for all risks but additive for comonotonic risks. In order to make the idea precise, we introduce the notion of comonotonicity.

Definition 3.32 Two random variables X and Y are called *comonotonic* if:

$$\forall(\omega_1, \omega_2) \in \Omega^2, \quad (X(\omega_2) - X(\omega_1))(Y(\omega_2) - Y(\omega_1)) \geq 0. \tag{3.13}$$

It is welll known that two functions X, Y are comonotonic if and only if there exist continuous, increasing functions g, h on $\mathbb{R}$ such that $g(Z) + h(Z) = Z \in \mathbb{R}$ and $X = g(X + Y), Y = h(X + Y)$.

Definition 3.33 A monetary measure of risk ρ on $\mathcal{X}$ is called *comonotonic* if:

$$\rho(X + Y) = \rho(X) + \rho(Y),$$

whenever $X, Y \in \mathcal{X}$ are comonotonic.

Risk measures with comonotonic subadditivity were introduced by Föllmer and Schied (2002) [11]. Independently, Heyde et al. proposed in [17], a so-called *natural risk statistic* which points out the consistency of the comonotonic subadditivity (i.e. it is better to impose preference on comonotonic random variables rather than on arbitrary random variables).
The following axioms were proposed by Song and Yan in [33]:

- Comonotonic subadditivity: If $X, Y \in \mathcal{X}$ are comonotonic, then $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- Comonotonic convexity: If $X, Y \in \mathcal{X}$ are comonotonic, then $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for any $\lambda \in (0, 1)$.

Lemma 3.34 If ρ is a comonotonic monetary risk measure on $\mathcal{X}$, then ρ is positively homogenous.

Proof:
One can notice that (X, X) is a comonotone pair and from that $\rho(2X) = 2\rho(X)$. An iteration of this argument leads to

$$\rho(rX) = r\rho(X), \ \ \forall r \in \mathbb{N}. \tag{3.14}$$

Since we can see that equation (3.14) is true for $r = 0$, let assume that it is also true for $r = n \in \mathbb{N}$. So, we have

$$\begin{aligned} \rho[(n+1)X] &= \rho(nX + X) \\ &= \rho(nX) + \rho(X) \\ &= n\rho(X) + \rho(X)\,, \quad \text{by assumption} \\ &= (n+1)\rho(X). \end{aligned}$$

Therefore ρ is positively homogenous. ■

If we add up either the subadditivity or the positive homogeneity axiom to convexity, we automatically end up into the class of sublinear risk measures.

Definition 3.35 π is called a *sublinear functional* on $\mathcal{X}$ if $\pi : \mathcal{X} \to \mathbb{R}$ and if it satisfies the positive homogeneity and the subadditivity properties.

Remark 3.36 The *monetary measure of risk* ρ is a *sublinear functional* on $\mathcal{X}$.
If π is sublinear, we then can define a risk measure ρ^π by

$$\rho^\pi(X) = \pi(-X).$$

Remark 3.37 If π is sublinear, then we have the following:

1. $\pi(0) = 0$.
2. π is convex.
3. $a\pi(X) \leq \pi(aX),\ \forall a \in \mathbb{R},\ \forall X \in \mathcal{X}$.
4. $-\pi(Y - X) \leq \pi(X) - \pi(Y) \leq \pi(X - Y), \quad \forall X, Y \in \mathcal{X}$.
   One can deduce that $|\pi(X) - \pi(Y)| \leq \max\{|\pi(X - Y)|, |\pi(Y - X)|\}$.

Remark 3.38 Since the topological structure in a topological vector space is determined by a local base at the origin, it then follows that a sublinear functional on a topological vector space $\mathcal{X}$ is continuous on $\mathcal{X}$ if and only if it is continuous at 0.

The following result proposed by Frittelli [14] is a simple application of the Hahn Banach Theorem (Extension of linear functionals).

Theorem 3.39 Let π be a sublinear functional on $\mathcal{X}$. Then

$$\pi(X) = \max_{\mu \in \mathcal{P}^\pi} \mu(X), \tag{3.15}$$

where $\mathcal{P}^\pi = \{\mu \in \mathcal{X}' \mid \mu(X) \leq \pi(X), \forall X \in \mathcal{X}\} \neq \emptyset$.

Proof:

Consider a set $M = \{X \in \mathcal{X} \mid X = \lambda Y,\ \lambda \geq 0\}$ and define a functional $g : M \to \mathbb{R}$ by $g(X) = \lambda \pi(Y)$. By Remark 3.37, we have $\lambda \pi(X) \leq \pi(\lambda X), \forall \lambda \in \mathbb{R}$ and one can simply deduce that for any $Y \in M$ such that $X = \lambda Y$, we have

$$\begin{aligned} g(X) &\leq \pi(\lambda Y) \\ &\leq \pi(X), \qquad \forall X \in M. \end{aligned}$$

So using the Hahn Banach extension theorem, one can find $f : \mathcal{X}' \to \mathbb{R}$ such that we have:

$$\begin{cases} f(X) = g(X), & \forall X \in M \\ f(X) \leq \pi(X), & \forall X \in \mathcal{X} \end{cases}. \tag{3.16}$$

Hence we have $f(Y) = g(Y) = \pi(Y)$ and we can deduce that

$$\pi(Y) = f(Y) \leq \sup_{f \in \mathcal{P}^\pi} \{f(Y)\} \leq \pi(Y). \tag{3.17}$$

This is where the supremum is attained and (3.13) follows for $f = \mu$. ■

Corollary 3.40 π is a norm continuous sublinear functional on $\mathcal{X}$ if and only if there exists a subset $\mathcal{W} \subseteq \mathcal{X}'$ such that

$$\pi(X) = \max_{\mu \in \mathcal{W}} \mu(X) < +\infty,\ \ \forall X \in \mathcal{X} \ \text{ and } \ \sup_{\mu \in \mathcal{W}} \|\mu\| < +\infty. \tag{3.18}$$

Proof:

We can find $\mathcal{W} \subseteq \mathcal{X}'$ such that π is defined by (3.15), and get that

$$\begin{aligned} |\pi(X)| &\leq \sup_{\mu \in \mathcal{W}} |\mu(X)| \\ &\leq \sup_{\mu \in \mathcal{W}} \|\mu\| \|X\|. \end{aligned}$$

Thus this suffices to say that π is continuous at 0 and using Remark 3.38 one can deduce the continuity of π. On the other hand, if π is a norm continuous sublinear functional on $\mathcal{X}$, then $\sup_{\|X\|=1} |\pi(X)|$ is finite and by Theorem 3.39, we can find a subset $\mathcal{P}^\pi \subseteq \mathcal{X}'$ such that using Remark 3.37, we have

$$|\mu(X)| \leq \max\{|\pi(X)|, |\pi(-X)|\}, \quad \forall \mu \in \mathcal{P}^\pi.$$

Hence we get that

$$\begin{aligned} |\pi(X)| &= \sup_{\mu \in \mathcal{P}^\pi} \|\mu\| \\ &= \sup_{\mu \in \mathcal{P}^\pi} \{ \sup_{\|X\|=1} |\mu(X)| \} \\ &= \sup_{\|X\|=1} \{ \sup_{\mu \in \mathcal{P}^\pi} |\mu(X)| \} \\ &\leq \sup_{\|X\|=1} |\pi(X)| \\ &< +\infty. \end{aligned}$$

■

Let $\mathcal{M}_{1,m}$ denote the family of all comonotonic set functions $\mu : \mathcal{F} \to [0,1]$ with total mass 1 and $\mathcal{P}_f$ be the set of all finitely additive measures on $(\Omega, \mathcal{F})$. We will see below that every comonotonic risk measure on $\mathcal{X}$ arises as the *Choquet integral* with respect to μ.

Definition 3.41 Let $\mu : \mathcal{F} \to [0,1]$ be any set function which is normalized and monotone. The *Choquet integral* of a bounded measurable function $X : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F})$ with respect to μ is defined as

$$\int X d\mu := \int_{-\infty}^{0} (\mu(X > x) - 1) dx + \int_{0}^{\infty} \mu(X > x) dx.$$

Let $\mu \in \mathcal{M}_{1,m}$ and $X, Y \in \mathcal{X}$. The Choquet integral has the following properties:

- monotonicity: $X \leq Y$ implies $\mu(X) \leq \mu(Y)$.
- translation-invariance: $\mu(X + m) = \mu(X) + m, \quad \forall m \in \mathbb{R}$.

- positive homogeneity: $\lambda \geq 0$ implies $\mu(\lambda X) = \lambda\mu(X)$.

One can notice that μ is a submodular.

Theorem 3.42 A monetary risk measure ρ on $\mathcal{X}$ is comonotonic if and only if there exists a normalized monotone set function μ on $(\Omega, \mathcal{F})$ such that

$$\rho(X) := \int (-X) d\mu, \quad \forall X \in \mathcal{X}.$$

Theorem 3.43 A risk measure ρ on $\mathcal{X}$ is coherent if and only if there exists a subset $\mathfrak{Q}_p$ of $\mathcal{P}_f$, such that

$$\rho(X) = \sup_{Q \in \mathfrak{Q}_p} \mathbb{E}_Q[-X].$$

Moreover, $\mathfrak{Q}_p$ can be chosen as a convex set for which the supremum above is attained. If ρ is contimuous from above, then $\mathfrak{Q}_p$ can be chosen as a convex subset of $\mathcal{P}_f$.

Proof:

One can refer to [5] and [9] for the proof of the above representation theorem for coherent independent risk measures in the finite and general case. ■

Theorem 3.44 Any risk measure π on $\mathcal{X}$ satisfying the comonotonic subadditivity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}} \mu(X),$$

where

$$\mathcal{M} = \{\mu \in \mathcal{M}_{1,m} \mid \mu(Y) \leq \pi(Y), \ \forall Y \in \mathcal{X}\}.$$

Proof:

Let consider $\mu \in \mathcal{M}_{1,m}$ such that for all $X \in \mathcal{X}$, we have $\mu(X) \leq \pi(X)$. Then by Lemma 3.34, μ is positively homogenous. One needs only to show that π is coherent

and for that, it suffices to verify its positive homogeneity. For any $\lambda \geq 0$, we have the following

$$\begin{aligned} \pi(\lambda X) &= \max_{\mu \in \mathcal{M}} \{\mu(\lambda X)\} \\ &= \max_{\mu \in \mathcal{M}} \{\lambda \mu(X)\} \\ &= \lambda \max_{\mu \in \mathcal{M}} \mu(X) \\ &= \lambda \pi(X). \end{aligned}$$

The rest of the proof follows from Theorem 3.39. ■

The following representation theorem for convex risk measures was proved by Föllmer and Schied in [12] and independently by Frittelli and Rosazza in [16].

Theorem 3.45 Any convex risk measure ρ on $\mathcal{X}$ is of the form

$$\rho(X) = \sup_{Q \in \mathcal{P}_f} \{\mathbb{E}_Q[-X] - \alpha_{min}(Q)\}, \quad \forall X \in \mathcal{X},$$

where the penalty function α_{min} is given by

$$\alpha_{min}(Q) := \sup_{\rho(X) \leq 0} \mathbb{E}[-X], \quad \forall Q \in \mathcal{P}_f.$$

Moreover, α_{min} is the minimal penalty function which represents ρ.

Remark 3.46 The representation of a coherent risk measure ρ is a particular case of the above representation theorem for convex risk measures, since it corresponds to the penalty function

$$\alpha(Q) = \begin{cases} 0 & \text{if } Q \in \mathfrak{Q}_p \\ +\infty & \text{otherwise} \end{cases},$$

where

$$\mathfrak{Q}_p = \{Q \in \mathcal{P}_f \mid \mathbb{E}_Q[-Y] \leq \rho(Y), \ \forall Y \in \mathcal{X}\}.$$

Theorem 3.47 Any risk measure π on $\mathcal{X}$ satisfying the comonotonic convexity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}_{1,m}} \{\mu(X) - \alpha(\mu)\}, \ \forall X \in \mathcal{X},$$

where

$$\alpha(\mu) = \sup_{\pi(X)\leq 0} \mu(X), \;\; \mu \in \mathcal{M}_{1,m}.$$

Proof:

Let us consider $X \in \mathcal{X}$ such that $\tilde{X} = X - \pi(X)$. By its definition above, we have

$$\begin{aligned} \alpha(\mu) &\geq \mu(\tilde{X}), \quad \forall \mu \in \mathcal{M} \\ &\geq \mu(X - \pi(X)) \\ &\geq \mu(X) - \pi(X), \quad \forall \mu \in \mathcal{M} \quad \text{since } \mu \text{ is comonotonic.} \end{aligned}$$

Then

$$\pi(X) \geq \mu(X) - \alpha(\mu), \quad \forall \mu \in \mathcal{M}.$$

Therefore we have

$$\pi(X) \geq \sup_{\mu \in \mathcal{M}} \{\mu(X) - \alpha(\mu)\} \quad \forall X \in \mathcal{X}. \tag{3.19}$$

Consider a nonempty closed set $\mathcal{C}$ defined as

$$\mathcal{C} := \{X \in \mathcal{X} \mid \pi(X) \geq 0\}.$$

Clearly one can see that for any X with $\rho(X) < 0$, X is not an element of the convex cone $\mathcal{O} := \{Y \in \mathcal{X} \mid \pi(Y) < 0\} \neq \emptyset$. So $\mathcal{O} \cap \mathcal{C} = \emptyset$ and using Hahn-Banach separation Theorem for any two distinct points $X \in \mathcal{C}$ and $Y \in \mathcal{O}$, one can find a non-zero continuous linear functional l on $\mathcal{X}$ such that $l(X) \neq l(Y)$ and

$$\sup_{X \in \mathcal{C}} l(X) \leq \inf_{Y \in \mathcal{O}} l(Y) := b. \tag{3.20}$$

Hence $l(X) \leq b$.

We claim that $Y \geq 0 \Rightarrow l(Y) \geq 0$. For any $Y \in \mathcal{O}$, and $\lambda \geq 0$, one can deduce that $1 + \lambda Y \in \mathcal{O}$, since $\mathcal{O}$ is a cone. Thus,

$$l(Y) \leq l(1 + \lambda Y) = l(1) + \lambda l(Y) \quad \forall \lambda > 0.$$

The next claim is that $l(1) > 0$. The positivity of l implies there exist some Y such that $1 = Y^+ + (1 - Y^+)$ with $l(Y^+) \geq l(Y) \geq 0$ and $l(1 - Y^+) > 0$. Therefore one

can deduce that $l(1) = l(1 - Y^+) + l(Y^+) > 0$. We claim that l is comonotonic. Let X, Y be two comonotonic random variables. Since l is a continuous linear functional, then we have $l(X + Y) = l(X) + l(Y)$ and we proved our claim. Therefore using Theorem 3.42 for all $Y \in \mathcal{X}$, we have that the integral $l(Y) = \int(-Y)d\mu$ defines a one-to-one correspondence between l and $\mu \in \mathcal{M}$. From the two preceding steps, it follows that

$$\mu(-Y) = \frac{l(Y)}{l(1)} \ , \quad \forall Y \in \mathcal{X}$$

$\mathcal{O} \subset \mathcal{A}_\pi$ and it follows that

$$\begin{aligned} \alpha(\mu) &= \sup_{\pi(Y)\leq 0} \mu(-Y) \ , \qquad \text{by its definition above} \\ &= \sup_{Y\in\mathcal{A}_\pi} \mu(-Y) \\ &\geq \sup_{Y\in\mathcal{O}} \mu(-Y) \\ &\geq -\inf_{Y\in\mathcal{O}} \mu(Y). \end{aligned}$$

Hence we have that

$$\alpha(\mu) \geq -\frac{b}{l(1)}. \tag{3.21}$$

If we choose arbitrarily $\varepsilon > 0$ such that for any $Y \in \mathcal{A}_\pi$, we have $Y + \varepsilon$ in $\mathcal{O}$, then it appears that

$$\alpha(\mu) < -\frac{b}{l(1)} - \varepsilon. \tag{3.22}$$

Thus, equations (3.21) and (3.22) give us that

$$\alpha(\mu) = -\frac{b}{l(1)}.$$

One can notice that for all $X \in \mathcal{O}$ and $\mu \in \mathcal{M}$ it follows that

$$\begin{aligned} \mu(X) - \alpha(\mu) &= \frac{l(-X)}{l(1)} + \frac{b}{l(1)} \\ &= \frac{-l(X)}{l(1)} + \frac{b}{l(1)} \\ &= \frac{b - l(X)}{l(1)}. \end{aligned}$$

Since $l(1)$ is positive, one can notice using equation (3.21) that $\mu(X) - \alpha(\mu)$ is also positive and finally deduce that

$$\mu(X) - \alpha(\mu) \geq \pi(X), \quad \forall \mu \in \mathcal{M} \tag{3.23}$$

The proof is completed using the equations (3.19) and (3.23). ■

3.3 Dependent Risk Measures

In this section we fix a probability measure $\mathbb{P}$ on $\mathcal{X}$ such that $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and consider risk measures ρ such that

$$\text{for any} \quad X, Y \in \mathcal{X} \quad \text{we have} \quad X = Y \quad \mathbb{P}\text{-a.s.} \quad \Longrightarrow \quad \rho(X) = \rho(Y).$$

Those risk measures are called dependent risk measures because we are working those which are absolutely continuous with respect to the original probability measure $\mathbb{P}$.

Define $\mathcal{P}' \triangleq \{Q \ll P\}$ to be the set of all probability measures on $\Omega, \mathcal{F}$ which are absolutely continuous with respect to $\mathbb{P}$.

3.3.1 Coherent and convex risk measures

Proposition 3.48 A map $\rho : L^\infty \to \mathbb{R}$ is a convex (resp coherent) risk measure if and only if it a capital requirement (resp associated to a cone). If this is the case and we define $\mathcal{A}_\rho \triangleq \{X : \rho(X) \leq 0\}$, then we have:

(a) $\rho(X) = \min\{m \in \mathbb{R} : X + m \in \mathcal{A}_\rho\}$.

(b) if $\mathcal{A}$ is another acceptance set for ρ, then $\mathcal{A} \subseteq \mathcal{A}_\rho$ (i.e. $\mathcal{A}_\rho$ is the maximal acceptance set for ρ).

(c) ρ is F-continuous if and only if $\mathcal{A}_\rho$ is $\sigma(L^\infty, L^1)$-closed and in this case

$$\rho(X) = \sup_{Q \ll P} \{\mathbb{E}[-X] - \alpha(Q)\}, \tag{3.24}$$

where $\alpha(Q) = \sup_{X \in \mathcal{A}} \mathbb{E}_Q[-X]$. If in addition ρ is coherent then

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X],$$

where $\mathcal{Q} = \{Q \ll P \mid \mathbb{E}_Q[X], \forall X \in \mathcal{A}_\rho\}$.

Proof:

The first two parts have already been proved using Propositions 3.29 and 3.30. Now suppose that ρ is a convex risk measure.

(a) Refer to Proposition 3.29.

(b) If $X \in \mathcal{A}$, then $\rho(X) \leq 0$. But since $\mathcal{A}_\rho = \{X : \rho(X) \leq 0\}$, thus $X \in \mathcal{A}_\rho$. Hence $\mathcal{A} \subseteq \mathcal{A}_\rho$.

(c) This equivalence follows immediately from Proposition 3.20. If in addition ρ is coherent, the equivalence relation still remains by using of Remark 3.18. ■

Definition 3.49 A risk measure is said to be F-continuous if some, hence any, of the three properties (a), (b),and (c) above is satisfied (here, F stands for Fatou).

Theorem 3.50 Suppose that $\mathcal{P}'$ is the set of all probability measures $Q \ll P$, then for any *convex risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ the following properties are equivalent:

(a) ρ is continuous from above.

(b) ρ is lower semicontinuous for the weak* topology $\sigma(L^\infty, L^1)$, i.e. the set $\{X \in L^\infty : \rho(X) \leq 0\}$ is $\sigma(L^\infty, L^1)$ -closed.

(c) ρ has the *Fatou property*: For any uniformly bounded sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ which converges P-a.s. to some $X \in \mathcal{X}$, we have

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

(d) There exists a *penalty function* $\alpha : \mathcal{P}' \to (-\infty, +\infty]$ such that

$$\rho(X) = \sup_{Q\ll P} (\mathbb{E}_Q[-X] - \alpha(Q)), \qquad \text{for all} \quad X \in \mathcal{X}.$$

(e) ρ can be represented by some penalty function on $\mathcal{P}'$.

Proof:

(a)⇔(b)

Let $\mathcal{C}_r := \mathcal{C} \cap \{X \in L^\infty \mid \|X\|_\infty \leq r\}$ for $r > 0$. If we can find a sequence $(X_n) \in \mathcal{C}_r$ converging to some random variable $X \in L^1$, then we can find a subsequence that converges $\mathbb{P}$-a.s., and the Fatou property of ρ implies that $X \in \mathcal{C}_r$. This shows that $\mathcal{C}_r$ is closed in L^1 and using Lemma A.64 in [11], we deduce that $\mathcal{C} := \{\rho \leq c\}$ is weak* closed. The other implication is obvious.

(d)⇔(e) This follows directly from Theorem 3.21. and part (c) of Proposition 3.48.

(c)⇔(d) This obviously follows from part (c) of Proposition 3.48.

(a)⇔(c) Suppose that a convex risk measure ρ is continuous from above. Let (X_n) be a bounded sequence in $\mathcal{X}$ which converges pointwise to X. Then

$$\begin{aligned}
\rho(X) &= \lim_{n\to\infty} \rho(X_n) \\
&= \sup_{Q\ll P} [\lim_{n\to\infty} \{\mathbb{E}[-X_n] - \alpha(Q)\}] \\
&\leq \lim_{n\to\infty} \inf [\sup_{Q\ll P} \{\mathbb{E}[-X_n] - \alpha(Q)\}] \\
&= \lim_{n\to\infty} \inf \rho(X_n).
\end{aligned}$$

If we suppose that ρ is F-continuous and for each $n \in \mathbb{N}$, $X_n \searrow X$ then by monotonicity $\rho(X_n) \leq \rho(X)$ and $\rho(X_n) \nearrow \rho(X)$ follows. ■

Definition 3.51 The map $\alpha_\rho : \mathcal{P}' \to [0, +\infty]$ defined by

$$\alpha_\rho(Q) \triangleq \sup_{X\in L^\infty} \{\mathbb{E}_Q[-X] - \rho(X)\},$$

is called the *minimal penalty function for* ρ.

This terminology is justified by the following Proposition.

Proposition 3.52 The map α_ρ is a penalty function for ρ and if α is another penalty function, then $\alpha_\rho \leq \alpha$.

Proof:

From the above proposition, we can find a non-unique map α such that

$$\rho(X) = \sup_{Q \in \mathcal{P}'} \{\mathbb{E}_Q[-X] - \alpha(Q)\}.$$

Therefore,

$$\begin{aligned}
\rho(X) &\geq \mathbb{E}_Q[-X] - \alpha(Q), \quad \forall Q \in \mathcal{P}' \\
\implies \alpha(Q) &\geq \mathbb{E}_Q[-X] - \rho(X), \quad \forall X \in L^\infty, \forall Q \in \mathcal{P}' \\
\implies \alpha(Q) &\geq \sup_{X \in L^\infty} \{\mathbb{E}_Q[-X] - \rho(X)\}, \quad \forall Q \in \mathcal{P}' \\
\implies \alpha(Q) &\geq \alpha_\rho(Q), \quad \forall Q \in \mathcal{P}' \qquad \text{by Definition 3.47} \\
\implies \alpha &\geq \alpha_\rho.
\end{aligned}$$

■

The following result is useful when comparing two risk measures.

Proposition 3.53 If α and α' are penalty functions for ρ and ρ', respectively, then:

(a) $\alpha' \leq \alpha \Rightarrow \rho \leq \rho'$.

(b) $\rho \leq \rho' \Rightarrow \alpha_{\rho'} \leq \alpha$.

Proof:

(a) Let α and α' be the respective penalty functions of ρ and ρ'. Then

$$\begin{aligned}
\alpha' \leq \alpha &\implies \forall Q \in \mathcal{P}', \quad \alpha'(Q) \leq \alpha(Q) \\
&\implies \mathbb{E}_Q[-X] - \alpha(Q) \leq \mathbb{E}_Q[-X] - \alpha'(Q), \quad \forall X \in L^\infty \\
&\implies \rho(X) \leq \rho'(X), \quad \forall X \in L^\infty \\
&\implies \rho \leq \rho'.
\end{aligned}$$

(b) From Definition 3.51, $\alpha_{\rho'}$ is a penalty function for ρ'. Hence $\alpha_{\rho'} \leq \alpha'$ by Proposition 3.52. Since $\rho \leq \rho'$ we can use part (a) of proposition 3.53 to conclude that $\alpha_{\rho'} \leq \alpha$. ■

We end this section with an interesting result concerning continuity from below.

Proposition 3.54 Let ρ be a coherent risk measure. Then the following are equivalent:

(a) ρ is continuous from below.

(b) $\rho(X) = \max_{Q \in \mathcal{Q}_\rho}\{\mathbb{E}_Q[-X]$.

(c) the set $\{\frac{dQ}{dP} : Q \in \mathcal{Q}_\rho\}$ is weakly compact in L^1.

Proof:
For this proof, we refer to Corollary 4.35 in [11]. ■

Remark 3.55 These risk measures are not only comonotonically subadditive or convex, but they also respect stochastic dominance, or convex order, and consequently lead to law-invariance. In the next chapter, we will introduce similar results of dependent risk measures satisfying comonotonic subadditivity.

3.4 Some Examples

In this section we introduce some well-known examples of risk measures and the relations that hold between them. As a first example of risk measures, consider the α-quantile risk measure, often called the Value-at-Risk (VaR) at level α which at this point will start to obtain new class of risk measures.

3.4.1 Value-at-Risk (VaR)

Just a couple of years ago the lack of subadditivity of VaR was perceived by most banks as a purely mathematical question, with no practical consequences. VaR is a very easy and intuitive concept that points out how much one may lose during specified period with a given probability and how much capital should be set to

control the risk exposure of a firm. VaR serves for the determination of the capital requirements that banks have to fulfill in order to back their trading activities.

Definition 3.56 For any random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the *distribution function* F_X is defined by $F_X(t) := P[X \leq t]$, $t \in \mathbb{R}$.

Definition 3.57 Let $\alpha \in (0, 1]$ be fixed. We define

- $x_{(\alpha)} = q_\alpha(X) = \inf\{t \mid F_X(t) \geq \alpha\}$ as the lower α-quantile of X.

- $x^{(\alpha)} = q^\alpha(X) = \inf\{t \mid F_X(t) > \alpha\}$ as the upper α-quantile of X.

Definition 3.58 For any random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the *quantile function* q_X is defined by:

$$q_\alpha(X) := F_X^{-1}(t) = \inf\{x \in \mathbb{R} \mid F_X(t) \geq \alpha\}, \quad 0 < \alpha \leq 1.$$

The quantile function is the generalized inverse of the distribution function.

Definition 3.59 Given a number $\alpha \in (0, 1]$ and a position described by the random variable X, we define the *Value-at-Risk* (VaR_α) as

$$VaR_\alpha(X) = -q^\alpha(X) = q_{1-\alpha}(-X).$$

X is said to be VaR_α-*acceptable* if

$$VaR_\alpha(X) \leq 0 \text{ or, equivalently, } q^\alpha(X) \geq 0.$$

Remark 3.60 VaR can be considered as the amount of extra capital a firm needs to reduce to α the probability of bankruptcy, or the extra capital needing to be added

(as a free investment) to a given position X so that the risk X becomes acceptable to an external regulator. We can express VaR by the following

$$\begin{aligned} VaR_\alpha(X) &= q_{1-\alpha}(-X) \\ &= \inf\{m \in \mathbb{R} \mid P[-X \leq m] \geq 1-\alpha\} \\ &= \inf\{m \in \mathbb{R} \mid 1 - P[X < -m] \geq 1-\alpha\} \\ &= \inf\{m \in \mathbb{R} \mid P[X < -m] \geq \alpha\} \\ &= \inf\{m \in \mathbb{R} \mid P(X+m < 0) \geq \alpha\}. \end{aligned}$$

Lemma 3.61 Consider the risk measure VaR given by $\rho(X) = VaR_\alpha(X)$, then VaR satisfies the following conditions:

(a) $X \geq 0 \Rightarrow VaR_\alpha(X) \leq 0$.

(b) $X \geq Y \Rightarrow VaR_\alpha(X) \leq VaR_\alpha(Y)$.

(c) $\forall \lambda \geq 0,\ VaR_\lambda(\alpha X) = \lambda VaR_\alpha(X)$.

(d) $VaR_\alpha(X + c) = VaR_\alpha(X) - c,\ \forall c \in \mathbb{R}$.

Remark 3.62 The conditions (b) and (d) of the above Lemma suffice to make VaR Lipschitz-continuous with respect to the L^∞-norm. Moreover, $\rho(X) = VaR_\alpha(X)$ has the following properties:

1. Law invariance: if $P[X \leq t] = P[Y \leq t]$ for all $t \in \mathbb{R}$, then $\rho(X) = \rho(Y)$.
2. Comonotonic additivity: given two comonotonic random variables X, Y, we have $\rho(X + Y) = \rho(X) + \rho(Y)$.

Law invariance is a crucial condition when it comes to estimating a risk measure from empirical data. VaR turns out to be a model dependent risk measure since it depends, by definition, on the initial reference probability. In general VaR is not a

convex measure and in particular not subadditive even when the two random variables are independent. Therefore this is bad when it comes to practice since VaR will not be able to encourage diversification of risks. VaR becomes subadditive only when the joint distribution of return is elliptic, i.e.,

$$VaR^{\alpha}(X+Y) \leq VaR^{\alpha}(X) + VaR^{\alpha}(Y), \quad X, Y \in \mathcal{X}.$$

Definition 3.63 The *Average Value* at *Risk* at level $\lambda \in (0,1]$ of a position $X \in \mathcal{X}$ is given by

$$AVaR_{\lambda}(X) = \frac{1}{\lambda}\int_0^{\lambda} VaR_{\alpha}(X)d\alpha.$$

Sometimes, the Average Value at Risk is also called the *expected shortfall*, and one writes $ES_{\alpha}(X)$. Note that

$$AVaR_{\lambda}(X) = -\frac{1}{\lambda}\int_0^{\lambda} q^{\alpha}(X)d\alpha.$$

Let us now recall the definition of expected shortfall.

3.4.2 Expected shortfall

In simple words, ES is proposed as a efficient coherent alternative to VaR and is characterized as the smallest coherent, comonotonic additive and law invariant risk measure to dominate VaR. At a specified level α, ES is the average loss in the worst 100α percent cases.

Definition 3.64

$$ES_{\alpha}(X) = -\alpha^{-1}\int_0^{\alpha} F_X^{-1}(s)ds.$$

The accurate definition of ES_{α} is:

$$ES_{\alpha}(X) = -\alpha^{-1}(\mathbb{E}[X\mathbf{1}_{\{X\leq x_{(\alpha)}\}}] + x_{(\alpha)}(\alpha - P[X \leq \alpha])),$$

where **1** is the indicator function

$$\mathbf{1}_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}.$$

Remark 3.65 The expected shortfall becomes simple when we are dealing with a continuous distribution

$$ES_\alpha(X) = -\alpha^{-1}(\mathbb{E}[X\mathbf{1}_{\{X \le x_{(\alpha)}\}}]).$$

It can be easily checked that ES_α is a coherent risk measure and that ES_α is continuous with respect to α. ES_α is a monotonic function to α

$$i.e.\ ES_{\alpha+\varepsilon}(X) \le ES_\alpha(X), \quad \forall \alpha \in (0,1),\ \forall \varepsilon > 0,\ \text{with}\ \alpha + \varepsilon < 1.$$

Moreover ES_α is a convex function with respect to positions.

ES_α is different from the *tail conditional expectation* (TCE^α) risk measure and the *worst conditional expectation* (WCE_α) risk measure defined by the following:

$$TCE^\alpha(X) = \mathbb{E}\{X | X \le x^{(\alpha)}\},$$

$$WCE_\alpha(X) = -\inf\{\mathbb{E}[X|A] : A \in \mathcal{F}, \mathbb{P}(A) > \alpha\}.$$

The functional WCE_α is a good alternative for the VaR risk measure but it is only useful in a theoretical setting since it depends not only on the distribution of X but also on the structure of the underlying probability space. Both TCE and WCE are sensitive to small changes in the confidence level α when applied to discontinuous distributions. Comparing ES, TCE and WCE, we get

$$TCE^\alpha(X) \le WCE_\alpha(X) \le ES_\alpha(X).$$

ES is the maximum of WCEs when the underlying probability space varies. Only under suitable conditions such as the continuity of the probability distribution function (F_X) we have:

$$TCE^\alpha(X) = WCE_\alpha(X) = ES_\alpha(X).$$

The extreme cases correspond to the two following definitions.

Definition 3.66 ES_α is said to be the *very worst case scenario* when it is defined by

$$\lim_{\alpha \to 0} ES_\alpha(X) = -ess\inf(X).$$

Definition 3.67 For $\alpha = 1$, ES_α is said to be the *less pessimistic risk measure* which is defined by

$$ES_1(X) = \mathbb{E}_{\mathbb{P}}[X].$$

ES_α will serve as the basic building block for defining other coherent measures. Introducing a measure $d\mu(\alpha)$ on $\alpha \in [0,1]$, Proposition 2.2 of [10] comes with great help once we start generating the *spectral risk measure* M_ϕ. Proposition 2.2 of [10] ensures that

$$M_\mu(X) = \int_0^1 d\mu(\alpha)\alpha ES_\alpha(X) = -\int_0^1 d\mu(\alpha) \int_0^\alpha F_X^{-1}(s)ds$$

is a risk measure as long as the normalization condition

$$\int_0^1 \alpha d\mu(\alpha) = 1$$

is satisfied.

Remark 3.68 VaR is not continuous on compacts but ES is continuous on compacts.

3.4.3 Spectral risk measure

This subsection deals with a class of measures based on integrals of the quantile function. A spectral risk measure consists in a weighted average of the quantiles of the distribution using a decreasing weight function $\phi : [0,1] \to \mathbb{R}$ called *risk spectrum*.

Definition 3.69 The norm $\|\phi(.)\|$ in $L^1([0,1])$ is given by $\|\phi(s)\| = \int_0^1 \phi(s)ds$.

Definition 3.70 We will say that an element $\phi \in L^1([a,b])$ is *positive* if $\forall I \subset [a,b]$, we have

$$\int_I \phi(s)ds \geq 0.$$

We will say that an element $\phi \in L^1([a,b])$ is *decreasing* if $\forall l \in (a,b)$ and $\forall \epsilon > 0$ such that $[l-\epsilon, l+\epsilon] \subset [a,b]$, we have

$$\int_{l-\epsilon}^{l} \phi(s)ds \geq \int_{l}^{l+\epsilon} \phi(s)ds.$$

Definition 3.71 Let a spectral representation $M_\phi : \Omega \to \mathbb{R}$ of a position $X \in \mathcal{X}$ be defined by

$$M_\phi(X) = -\int_0^1 \phi(s)F_X^{-1}(s)ds, \qquad \text{where} \quad \phi \in L^1([0,1]).$$

Definition 3.72 An element $\phi \in L^1([0,1])$ is said to be an *admissible risk spectrum* if the following conditions are satisfied

(C1) Nonnegativity: $\phi(s) \geq 0$.

(C2) Normalization: $\|\phi(s)\| = 1$.

(C3) Decreasingness: $\phi'(s) \leq 0$.

Amongst all of them, the key condition is the third one.

Theorem 3.73 If ϕ is an *admissible risk spectrum* then $M_\alpha(X)$ is a risk measure called *spectral risk measure*.

Remark 3.74 The coherence of spectral risk measures comes from the assumptions made on the spectrum ϕ. All coherent risk measures cannot be written as spectral risk measures. The class of spectral risk measures is the convex hull generated by all the ES_α, but is a subset of the class of coherent risk measures. The Expected shortfall is (much) better in principle than the Value-at-Risk, but the Spectral risk measure is better in principle than the Expected shortfall.

An ordering is bound to satisfy some compatibility relations to lattice structure and slightly stronger results in risk measures can be obtained. In all the risk measures discussed so far, the concept of ordering was not taken into account. By ordering, we intuitively mean that any risk decision maker prefers a smaller risk in the given order of preferences. However, under certain assumptions and conditions, orderings could better specify the attitude of an investor toward risk. Therefore, it could be interesting to have a look at it. This will be discussed in the next chapter.

Chapter 4

Orderings in risk measures

A risk measure ρ assigns a real number to a financial position, which is described by a random variable $X \in \mathcal{X}$ with $\rho : \mathcal{X} \to \mathbb{R}$. We assume that $\rho(X)$ describes a potential loss, but we allow $\rho(X)$ to assume negative values which means that a gain occurs. Let X and Y be two random variables such that for all $t \in \mathbb{R}$ we have

$$\mathbb{P}[X \geq t] \leq \mathbb{P}[Y \geq t].$$

Then X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) and any rational decision maker would prefer X. Another important ordering to compare risks with is the convex ordering $\leq_{cx}$ which is more related to notions of risk aversion. Again if X and Y are two risks with $X \leq_{cx} Y$, then a rational decision maker would prefer X if X is risk averse.

Typically risk measures are defined by an axiomatic approach, and the most reasonable axioms used throughout the literature are: monotonicity which captures the idea that a larger loss is more risky and convexity which is always referred to diversification. The risk of a portfolio which diversifies between two risks X and Y is less than the sum of the risks in both positions. Is it true that a monotone risk measure ρ has the property that $X \leq_{st} Y$ implies $\rho(X) \leq \rho(Y)$ and that a convex risk measure has the property that $X \leq_{cx} Y$ implies $\rho(X) \leq \rho(Y)$?

The monotonocity seems to be the trivial statement, but we can show how wrong

these statements can be. The crucial point is the probability space in which the risk measures are defined. On a non-atomic probability space these implications are just fine, whereas they become wrong on certain finite probability spaces. In this section, we are more interested in consistency of risk measures with stochastic orders. By consistency, we mean that the risk measures preserve a given stochastic order $\leq_{st}$, i.e.

$$X \leq_{st} Y \Rightarrow \rho(X) \leq \rho(Y).$$

Artzner and al. [5] have established four properties to point out the notion of coherency in risk measure. Beside its axiomatic definition, a coherent risk measure can be seen as the supremum of the expected negative of final net worth for some collection of probability measures. However, coherent risk measures are not always consistent with second order stochastic dominance. We paid a particular attention to the work of E. De Giorgi [8] and we present some standard results from decision theory.

We shall let a finite set Ω to denote the state of nature in the final period $t = T$, write $\Omega = \{1, ..., S\}$ and we shall consider $\mathcal{F}$ to be a σ -algebra on Ω. We take $\mathcal{F} = 2^{\Omega}$. $(\Omega, \mathcal{F})$ define our measurable space. On $(\Omega, \mathcal{F})$ we define a probability measure $\mathbb{P}$. We call $\mathbb{P}$ the *physical probability measure* or the *objective probability measure*. We assume that $\mathbb{P}[s] > 0$ for all $s \in \Omega$.

4.1 Preferences

In this Section we will discuss some possible relationships among preferences and risk measures. Recall that a preference structure on L^{∞} is a binary relation $\preceq$ such that:

1. $\preceq$ is a preorder, i.e. it is reflexive and transitive.

2. $\preceq$ is total, i.e. either $X \preceq Y$ or $Y \preceq X$ for any $X, Y \in L^{\infty}$.

We interpret $X \preceq Y$ as "the payoff X is good at least as Y ". If $X \preceq Y$; but not $Y \preceq X$; then we say that X is strictly preferred to Y and we write $X \prec Y$: Finally, if $X \preceq Y \preceq X$; then X and Y are equivalent and we write $X \sim Y$: We say that the preference $\preceq$ is representable if there exists a functional $\rho : L^\infty \to \mathbb{R}$ called risk measure function for $\preceq$ such that

$$X \preceq Y \Leftrightarrow \rho(Y) \leq \rho(X). \tag{4.1}$$

Conversely, to every functional $\rho : L^\infty \to \mathbb{R}$ we associate the preference $\preceq_\rho$ defined through 4.1. A well-known necessary and sufficient condition for representability of a preference is Debreu-separability: there exists a sequence $(Z_n)_{n\geq 0}$ in L^∞ such that if $X \prec Y$ then $X \preceq Z_n \preceq Y$ for at least one n. In particular, this is verified if $\preceq$ satisfies the certainty-equivalent property: $\forall X \in L^\infty$, $\exists! \alpha \in \mathbb{R}$; $X \sim \alpha \mathbf{1}$.

Theorem 4.1 A preference is representable by a convex risk measure if and only if it satisfies the following properties:

(a) $X \leq Y \Rightarrow X \preceq Y$.

(b) $\alpha < \beta \Rightarrow \alpha \mathbf{1} \prec \beta \mathbf{1}$.

(c) $0 \preceq X, 0 \preceq Y \Rightarrow 0 \preceq \lambda X + (1-\lambda)Y, \forall \lambda \in [0.1]$.

(d) $X \preceq Y \Rightarrow X + \alpha \mathbf{1} \preceq Y + \alpha \mathbf{1}, \forall \alpha \in \mathbb{R}$.

(e) $X \prec Y \Rightarrow \exists \alpha > 0$; $X + \alpha \mathbf{1} \preceq Y$.

If this is the case, then it is represented by the capital requirement associated to $\mathcal{A}_\preceq \triangleq \{X \in L^\infty \mid 0 \preceq X\}$.

Proof:

Suppose that $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure and let X and Y be two random variables in L^∞.

(a) If $X \leq Y$, then by the monotonicity property, $\rho(Y) \leq \rho(X)$ and using the

equivalence relation (4.1), we easily get that $X \preceq Y$.

(b) If we consider two real values α and β such that $\alpha < \beta$, then we have the following

$$\begin{aligned} \alpha < \beta &\Rightarrow -\alpha > -\beta \\ &\Rightarrow \rho(\alpha \cdot \mathbf{1}) = -(\alpha \cdot \mathbf{1}) > -(\beta \cdot \mathbf{1}) = \rho(\beta \cdot \mathbf{1}) \\ &\Rightarrow \alpha \cdot \mathbf{1} \prec \beta \cdot \mathbf{1} \qquad \text{using (4.1).} \end{aligned}$$

(c) Let $\lambda \in [0,1]$ and suppose that $0 \preceq X$ and $0 \preceq Y$. Thus using (4.1) again, we can notice that $\rho(X) \leq 0$ and $\rho(Y) \leq 0$ and since ρ is convex, we have the following

$$\begin{aligned} \rho(\lambda X + (1-\lambda)Y) &\leq \lambda\rho(X) + (1-\lambda)\rho(Y) \\ &\leq \lambda \cdot 0 + (1-\lambda) \cdot 0 \\ &= 0. \end{aligned}$$

Hence the equivalence relation (4.1) gives us that $\lambda X + (1-\lambda)Y \preceq 0$.

(d) If we consider a real value α, then

$$\begin{aligned} X \prec Y &\Rightarrow \rho(Y) \leq \rho(X)\,, && \text{using (5.1)} \\ &\Rightarrow \rho(Y) - \alpha \cdot \mathbf{1} \geq \rho(X) - \alpha \cdot \mathbf{1}\,, && \forall \alpha \in \mathbb{R} \\ &\Rightarrow \rho(Y + \alpha \cdot \mathbf{1}) \geq \rho(X + \alpha \cdot \mathbf{1})\,, && \text{by translation-invariance} \\ &\Rightarrow X + \alpha \cdot \mathbf{1} \preceq Y + \alpha \cdot \mathbf{1}\,, && \text{using (4.1).} \end{aligned}$$

(e) For this last part, we have the following

$$\begin{aligned} X \prec Y &\Rightarrow \rho(Y) < \rho(X) \\ &\Rightarrow \exists \alpha > 0 \;;\; \rho(Y) < \rho(Y) + \alpha \cdot \mathbf{1} < \rho(X) \\ &\Rightarrow \rho(Y) \leq \rho(X) - \alpha \cdot \mathbf{1} \\ &\Rightarrow \rho(Y) \leq \rho(X + \alpha \cdot \mathbf{1})\,, && \text{by translation-invariance} \\ &\Rightarrow (X + \alpha \cdot \mathbf{1}) \preceq Y\,, && \text{using (4.1).} \end{aligned}$$

Let us assume $\preceq$ satisfy the five (a) to (e) properties above. One can notice from (4.1) that the acceptance set induced by ρ coincides with the set $\mathcal{A}_{\preceq} := \{X \in L^{\infty} \mid 0 \preceq X\}$

induced by $\preceq$. We can then define the associated risk measure $\rho_{\preceq}$ by the following

$$\rho_{\preceq}(X) = \inf\{m \in \mathbb{R} \mid 0 \preceq X + m \cdot \mathbf{1}\}.$$

The property (c) shows that $\mathcal{A}_{\preceq}$ is convex set and using Proposition 3.29, we can deduce that $\rho_{\preceq}$ is a convex risk measure. Now, it only suffices to check if $\rho_{\preceq}$ satisfies the property of monotonicity, translation-invariance. For the monotonicity, we have the following

$$\begin{aligned}
X \leq Y &\Rightarrow X + \alpha \cdot \mathbf{1} \leq Y + \alpha \cdot \mathbf{1}, \forall \alpha \in \mathbb{R} \qquad \text{using (a) and (d)} \\
&\Rightarrow \rho(Y + \alpha \cdot \mathbf{1}) \leq \rho(X + \alpha \cdot \mathbf{1}) \\
&\Rightarrow \inf\{m \in \mathbb{R} \mid Y + \alpha \cdot \mathbf{1} + m \cdot \mathbf{1} \in \mathcal{A}_\rho\} \leq \inf\{m \in \mathbb{R} \mid X + \alpha \cdot \mathbf{1} + m \cdot \mathbf{1} \in \mathcal{A}_\rho\} \\
&\Rightarrow \inf\{m \in \mathbb{R} \mid Y + \alpha \cdot \mathbf{1} \in \mathcal{A}_\rho\} - m \leq \inf\{m \in \mathbb{R} \mid X + \alpha \cdot \mathbf{1} \in \mathcal{A}_\rho\} - m \\
&\Rightarrow \inf\{m \in \mathbb{R} \mid Y + \alpha \cdot \mathbf{1} \in \mathcal{A}_{\preceq}\} \leq \inf\{m \in \mathbb{R} \mid X + \alpha \cdot \mathbf{1} \in \mathcal{A}_{\preceq}\} \\
&\Rightarrow \inf\{m \in \mathbb{R} \mid 0 \preceq Y + \alpha \cdot \mathbf{1}\} \leq \inf\{m \in \mathbb{R} \mid 0 \preceq X + \alpha \cdot \mathbf{1}\} \\
&\Rightarrow \rho_{\preceq}(Y) \leq \rho_{\preceq}(X).
\end{aligned}$$

For the translation-invariance, we have the following

$$\begin{aligned}
\rho_{\preceq}(X + m) &= \inf\{\alpha \in \mathbb{R} \mid 0 \preceq (X + m) + \alpha \cdot \mathbf{1}\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X + (m + \alpha) \cdot \mathbf{1}) \leq 0\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X) - (m + \alpha) \leq 0\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X) - m \leq \alpha\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X) \leq \alpha\} + \inf\{\alpha \in \mathbb{R} \mid -m \leq \alpha\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X) - \alpha \leq 0\} + \inf\{\alpha \in \mathbb{R} \mid -m \leq \alpha\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \rho(X + \alpha) \leq 0\} + (-m) \\
&= \inf\{\alpha \in \mathbb{R} \mid 0 \preceq X + \alpha\} - m \\
&= \rho_{\preceq}(X) - m.
\end{aligned}$$

Hence the proof is completed. ■

Let $\mathcal{G} = \{X : \Omega \to \mathbb{R} \mid X \text{ is } \mathcal{F}\text{-measurable}\}$ be the space of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that an investor have some preference $\{\succ\}$ on a subset $\mathcal{H}$ of $\mathcal{G}$. If a investor is asked to choose between two random outcomes $X, Y \in \mathcal{H}$, then the following can occur:
1. $X \succ Y$, i.e. X is preferred to Y.
2. $Y \succ X$, i.e. Y is preferred to X.
3. $X \sim Y$, i.e. investor is indifferent or indefinite between X and Y.

Let us now recall some basic definitions and results from the theory of stochastic orders.

4.2 Stochastic order

Stochastic orders aim to order distributions of random variables and random vectors, see [24] and [25]. Although throughout this section orders for random variables X and Y is discussed, stochastic orders are partial order relations for their probability distributions F_X and F_Y. Moreover, stochastic orders can be considered as integral stochastic orders. Let $\mathcal{M}_{1,m}(\mathbb{P})$ denote the set of absolutely continuous probability measures with respect to $\mathbb{P}$ and total mass 1 and $\mathcal{M}_{st}(\mathbb{P})$ (resp. $\mathcal{M}_{icx}(\mathbb{P})$, $\mathcal{M}_{cx}(\mathbb{P})$) denote the set of those $\mu \in \mathcal{M}_{1,m}(\mathbb{P})$ which satisfy the following property:

$$X \leq_{st} Y (\text{resp.} X \leq_{icx} Y, X \leq_{cx} Y) \implies \mu(X) \leq \mu(Y).$$

Two random elements X and Y are said to be ordered if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all f (for which expectations exist) from a certain family of functions. These functions are called *generators*. Particular families of generators lead to the following definition:

Definition 4.2 For given random variables X and Y, we have the order relations

- $X \leq_{st} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all coordinatewise increasing functions f.

- $X \leq_{cx} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions f.
- $X \leq_{icx} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing convex functions f.
- $X \leq_{icv} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing concave functions f.

We will hereafter refer to these orders as *usual stochastic order*, *convex order*, *increasing convex order* and *increasing concave order* respectively.

Definition 4.3 For two random variables X and Y, X is said to precede Y in first order stochastic dominance (i.e. $X \leq_{st} Y$) if and only if $F_Y(t) \leq F_X(t)$, for all $t \in \mathbb{R}$.

The following lemmas are two basic results about ordering and comonotonicity which show that the order is preserved under summing risks.

Lemma 4.4 *Assume that $X =_d X^c$ and $Y =_d Y^c$. If X^c, Y^c are comonotonic, then $X + Y \leq_{icx} X^c + Y^c$.*

Lemma 4.5 *Assume that X_1 and X_2, Y_1 and Y_2 are comonotonic, respectively. If $X_1 \leq_{st} Y_1$ and $X_2 \leq_{st} Y_2$, then $X_1 + X_2 \leq_{st} Y_1 + Y_2$; If $X_1 \leq_{icx} Y_1$ and $X_2 \leq_{icx} Y_2$, then $X_1 + X_2 \leq_{icx} Y_1 + Y_2$.*

Proof:
From our assumptions, we can by Definition 4.2 find an continuous and increasing function u on $\mathbb{R}$ such that $u(X_1+X_2) = X_1$ and $u(Y_1+Y_2) = Y_1$. But since $X_1 \leq_{st} Y_1$, we can see that

$$\begin{aligned} X_1 \leq Y_1 &\Rightarrow u(X_1 + X_2) \leq u(Y_1 + Y_2) \\ &\Rightarrow \mathbb{E}[u(X_1 + X_2)] \leq \mathbb{E}[u(Y_1 + Y_2)] \\ &\Rightarrow X_1 + X_2 \leq_{st} Y_1 + Y_2. \end{aligned}$$

An analogous proof from above with an increasing convex function u on $\mathbb{R}$ suffices to show the other implication. ■

Definition 4.6 Let $\pi : \mathcal{X} \to \mathbb{R}$. We define the following axioms about π:

- (A1)(respecting stochastic order) $X \leq_{st} Y \Rightarrow \pi(X) \leq \pi(Y)$.
- (A2)(respecting convex order) $X \leq_{cx} Y \Rightarrow \pi(X) \leq \pi(Y)$.
- (A3)(respecting increasing convex order) $X \leq_{icx} Y \Rightarrow \pi(X) \leq \pi(Y)$.

Remark 4.7 Note that since $X \leq_{st} Y \Rightarrow X \leq_{icx} Y$ and $X \leq_{cx} Y \Rightarrow X \leq_{icx} Y$, then $(A1) \Rightarrow (A3)$ and $(A2) \Rightarrow (A3)$.

4.2.1 On finite probability spaces

The following theorems give a useful characterization for stochastic orders.

Theorem 4.8 For random variables X and Y with respective distribution functions F_X and F_Y, the following statements are equivalent:

(a) $X \leq_{st} Y$.

(b) $\exists\ X', Y' \in (\Omega', \mathcal{F}', \mathbb{P}')\ ;\ X'(\omega) \leq Y'(\omega), \quad \forall \omega \in \Omega'$.

(c) $q_\alpha(X) \leq q_\alpha(Y), \quad \forall \alpha \in (0,1)$.

Proof:
$(a) \Leftrightarrow (b)$
Let us define an increasing function $f : \mathbb{R} \to \mathbb{R}$ by $f(X) = X$ Since $X \leq_{st} Y$, we can find two random variables X' and Y' in $(\Omega', \mathcal{F}', \mathbb{P}')$ such that

$$f(X(\omega)) =_d X'(\omega) \quad \text{and} \quad f(Y(\omega)) =_d Y'(\omega), \quad \forall \omega \in \Omega'.$$

Thus we have that,

$$\begin{aligned} & \mathbb{E}[f(X(\omega))] \leq \mathbb{E}[f(Y(\omega))], \quad \forall \omega \in \Omega' \\ \Rightarrow \quad & \mathbb{E}[X'(\omega)] \leq \mathbb{E}[Y'(\omega)], \quad \forall \omega \in \Omega' \\ \Rightarrow \quad & X'(\omega) \leq Y'(\omega), \quad \forall \omega \in \Omega'. \end{aligned}$$

To show the reverse implication, one has to find $X, Y \in \Omega$ such that the pairs (X, X') and (Y, Y') have the same distribution and $X'(\omega) \leq Y'(\omega),\ \forall \omega \in \Omega$. Then we have the following

$$\begin{aligned} X(\omega) \leq Y(\omega) &\Rightarrow f(X) \leq f(Y)\,, \qquad \text{for all increasing function } f \\ &\Rightarrow \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \\ &\Rightarrow X \leq_{st} Y. \end{aligned}$$

$(a) \Leftrightarrow (c)$

$$\begin{aligned} X \leq_{st} Y &\Leftrightarrow F_Y(t) \leq F_X(t)\,,\ \forall t \in \mathbb{R}, \qquad \text{by Definition 4.3} \\ &\Leftrightarrow F_X^{-1}(t) \leq F_Y^{-1}(t)\,,\ \forall t \in \mathbb{R} \\ &\Leftrightarrow q_\alpha(X) \leq q_\alpha(Y)\,,\ \forall \alpha \in (0,1), \quad \text{by Definition 3.58.} \end{aligned}$$

■

Definition 4.9 For any random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the *stop-loss transform* π_X by

$$\pi_X(t) = \int_t^\infty \overline{F}_X(s)ds,\ \ \forall t \in \mathbb{R}, \quad \text{where } \overline{F}_X(s) = 1 - F_X(s). \tag{4.2}$$

Remark 4.10 The ordering $\leq_{icx}$ is also known as *stop-loss order* because $\leq_{icx}$ holds if and only if the corresponding stop-loss transforms are ordered. The increasing concave ordering $X \leq_{icv} Y$ is the corresponding ordering for returns instead of losses. Note that $X \leq_{icv} Y$ holds if and only if $-X \geq_{icx} -Y$. Therefore presenting the subsequent results for $\leq_{icx}$ should be sufficient enough.

Definition 4.11 For two random variables X and Y, X is said to precede Y in the stop-loss order sense (i.e. $X \leq_{icx} Y$) if and only if $\pi_X(t) \leq \pi_Y(t)$, for all $t \in \mathbb{R}$.

Theorem 4.12 Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X \leq_{icx} Y$. Then there exists a random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

$$X \leq_{st} Z \leq_{cx} Y.$$

Proof:

Suppose that $X \leq_{icx} Y$. Then for any increasing convex function $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ and it follows that $X \leq_{st} Y$ holds. This shows that using Theorem 4.8 we can find two random variables X' and Y' such that for all $\omega \in \Omega$, we have $X'(\omega) \leq Y'(\omega)$. From Debreu-separability, there exists a sequence $(Z_n)_{n\geq 0} \subset L^\infty$ converging to Z such that

$$X'(\omega) \leq Z_n^{'}(\omega) \leq Y'(\omega).$$

Hence

$$X'(\omega) \leq Z'(\omega) \leq Y'(\omega) \quad \text{as } n \to \infty.$$

$$\begin{aligned} \Rightarrow \quad f(X'(\omega)) &\leq f(Z'(\omega)) \leq f(Y'(\omega)) \\ \Rightarrow \quad \mathbb{E}[f(X')] &\leq \mathbb{E}[f(Z')] \leq \mathbb{E}[f(Y')]. \end{aligned}$$

Therefore we can deduce from our assumption that $X \leq_{st} Z$ and $Z \leq_{cx} Y$ hold and complete the proof. ■

Definition 4.13 For two random variables X and Y, X is said to precede Y in convex order (i.e. $X \leq_{cx} Y$) if and only if $\pi_X(t) \leq \pi_Y(t)$, for all $t \in \mathbb{R}$ and in addition $\mathbb{E}[X] = \mathbb{E}[Y]$.

Theorem 4.14 For random variables X and Y with respective distribution functions F_X and F_Y, the following statements are equivalent:

(a) $X \leq_{cx} Y$.

(b) $\exists\ X', Y' \in (\Omega', \mathcal{F}', \mathbb{P}')\ ;\ \mathbb{E}[Y'|X'] = X'$.

(c) $\int_\alpha^1 q_s(X)ds \leq \int_\alpha^1 q_s(Y)ds,\ \ \forall \alpha \in (0,1)$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

Proof:

$(a) \Leftrightarrow (b)$

Suppose that X and Y are two random variables such that $X \leq_{cx} Y$. We can find two random variables X' and Y' in the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ such that $X =_d X'$ and $Y =_d Y'$. Then using Definition 4.13 we have the following

$$\begin{aligned} & \mathbb{E}[X'] &=& \mathbb{E}[Y'] \\ \Rightarrow \quad & \mathbb{E}[X' \mid X'] &=& \mathbb{E}[Y' \mid X'] \\ \Rightarrow \quad & X' &=& \mathbb{E}[Y' \mid X']. \end{aligned}$$

$(a) \Leftrightarrow (c)$

$$\begin{aligned} X \leq_{cx} Y \quad & \Leftrightarrow \quad \pi_X(t) \leq \pi_Y(t),\ \forall t \in \mathbb{R} \text{ and } \mathbb{E}[X] = \mathbb{E}[Y], \quad \text{by Definition 4.13} \\ & \Leftrightarrow \quad \int_t^\infty (1 - F_X(s))ds \leq \int_t^\infty (1 - F_Y(s))ds \\ & \Leftrightarrow \quad F_Y(t) \leq F_X(t) \\ & \Leftrightarrow \quad F_X^{-1}(t) \leq F_Y^{-1}(t) \\ & \Leftrightarrow \quad q_\alpha(X) \leq q_\alpha(Y),\ \forall \alpha \in (0,1), \quad \text{by Definition 3.57.} \end{aligned}$$

Hence we have that $\int_\alpha^1 q_s(X)ds \leq \int_\alpha^1 q_s(Y)ds,\ \ \forall \alpha \in (0,1)$ with $\mathbb{E}[X] = \mathbb{E}[Y]$. ■

4.2.2 On atomless probability spaces

All atomless standard probability spaces are Borel isomorph, we can assume without loss of generality that $\Omega = (0,1)$ and $\mathbb{P}$ is the Lebesgue-measure. The first two theorems follow form Theorem 4.8 and Theorem 4.14, therefore it shall not be necessary to prove them since in that case, $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ coincide.

Theorem 4.15 For random variables X and Y with respective distribution functions F_X and F_Y, the following statements are equivalent:

(a) $X \leq_{st} Y$.

(b) $\exists\ X', Y' \in (\Omega, \mathcal{F}, \mathbb{P})\ ;\ X'(\omega) \leq Y'(\omega),\ \forall \omega \in \Omega'$.

(c) $q_\alpha(X) \leq q_\alpha(Y)$, $\forall \alpha \in (0,1)$.

Proof:

Refer to Theorem 2.2 in [24]. ■

Theorem 4.16 For random variables X and Y with respective distribution functions F_X and F_Y, the following statements are equivalent:

(a) $X \leq_{cx} Y$.

(b) $\exists\ X', Y' \in (\Omega, \mathcal{F}, \mathbb{P})\ ;\ \mathbb{E}[Y'|X'] = X'$.

(c) $\int_\alpha^1 q_X(s)ds \leq \int_\alpha^1 q_Y(s)ds$, $\forall \alpha \in (0,1)$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

Proof:

Refer to Theorem 2.5 in [24]. ■

Corollary 4.17 If Γ is a real functional on $\mathcal{X}$ with $\Gamma(1) = 1$, monotonicity and comonotonic, then there exists $\gamma \in \mathcal{M}_{1,m}(\mathbb{P})$ representing Γ in the sense that

$$\gamma(X) = \Gamma(X), \quad \text{for all } X \in \mathcal{X}.$$

Proof:

For this proof, we refer to [34]. ■

Theorem 4.18 Any risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A1)$ and the comonotonic subadditivity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X),$$

where $\mathcal{M}(\mathbb{P}) = \{\mu \in \mathcal{M}_{st}(\mathbb{P}) \mid \mu(Y) \leq \pi(Y)\},\ \forall Y \in \mathcal{X}$.

Proof:

Suppose that $\pi : \mathcal{X} \to \mathbb{R}$ is of the form

$$\pi(X) = \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X),$$

and let verify that π satisfies $(A1)$ and the comonotonic subadditivity property. If $X \leq_{st} Y$, then for all $\mu \in \mathcal{M}_{st}(\mathbb{P})$ we have $\mu(X) \leq \mu(Y)$. Hence, we can see that

$$\max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X) \leq \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(Y),$$

and finally get

$$\pi(X) \leq \pi(Y).$$

This shows that π satisfies $(A1)$ and using Theorem 3.44, we prove that π is comonotonic subadditive.

Suppose that π satisfies $(A1)$ and the comonotonic subadditivity property. We want to show that $\pi : \mathcal{X} \to \mathbb{R}$ is of the form

$$\pi(X) = \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X). \tag{4.3}$$

Define the set $\mathfrak{B}$ by

$$\mathfrak{B} := \{Y \in \mathcal{X} \mid \exists Z \in [X]\ ,\ \pi(Z) < 1\ ;\ Y \leq Z\ a.s.\}, \tag{4.4}$$

where $[X] = \{f(X) \mid f \text{ is an increasing continuous function on } \mathbb{R}\}$.

In order to show that (4.3) holds, we will fix X and construct some risk measures $\xi_X \in \mathcal{M}(\mathbb{P})$ such that $\xi_X(X) = \pi(X)$. Now to do this, it only suffices to take $\pi(X) = 1$ for all $X \in \mathcal{X}$. Moreover, we may assume with no loss of generality that $\pi(1) = 1$. Let the unit ball be denoted by

$$B(0,1) := \{Y \in \mathcal{X} \mid \|Y\| < 1\}, \tag{4.5}$$

hence $\mathfrak{B}$ contains the unit ball $B(0,1)$ and $X \notin \mathfrak{B}$. In fact

$$\begin{aligned} X \in \mathfrak{B} \quad &\Rightarrow \quad \exists Z \in [X]\ ,\ \pi(Z) < 1\ ;\ X \leq Z\ a.s. \\ &\Rightarrow \quad \pi(X) \leq \pi(Z)\ , \quad \text{by monotoncity} \\ &\Rightarrow \quad 1 \leq \pi(Z). \end{aligned}$$

This contradicts the fact that $1 > \pi(Z)$.

We need to verify that $\mathfrak{B}$ is convex. Let $\lambda \in [0,1]$ and $Y_1, Y_2 \in \mathfrak{B}$, there exists Z_1, Z_2 in $[X]$ with $\pi(Z_1) < 1$ and $\pi(Z_2) < 1$ such that $Y_1 \leq Z_1$ *a.s.* and $Y_2 \leq Z_2$ *a.s.* This implies that $\lambda Y_1 \leq \lambda Z_1$ *a.s.* and $(1-\lambda)Y_2 \leq (1-\lambda)Z_2$ *a.s.*. Put $Z = \lambda Z_1 + (1-\lambda)Z_2$, then $Z \in [X]$ and $\lambda Y_1 + (1-\lambda)Y_2 \leq Z$ *a.s.*.

Using the Hahn-Banach separation Theorem for convex set, we can find an non-zero functional l in $ba(\Omega, \mathcal{F}, \mathbb{P})$ (i.e. the space of all finitely additive measures μ with finite total variation, which are absolutely continuous with respect to $\mathbb{P}$) such that

$$\sup_{Y \in \mathfrak{B}} l(Y) \leq l(X). \tag{4.6}$$

But since $B(0,1) \subset \mathfrak{B}$, we have that

$$0 < \|l(Y)\| = \sup_{Y \in B(0,1)} l(Y) \leq \sup_{Y \in \mathfrak{B}} l(Y). \tag{4.7}$$

Thus, we can take l such that $l(X) \in (0,1]$.

We claim that $l(Y) \geq 0$ if $Y \geq 0$.

Let $Y \in \mathfrak{B}$ such that $Y \geq 0$. Since π satisfies (A1) and the subadditivity property, then for all $\lambda \in [0,1]$ we have $\lambda Y \in \mathfrak{B}$. It follows from (4.6) and (4.7) that

$$\lambda l(Y) = l(\lambda Y) \leq l(X), \ \ \forall \lambda \geq 0,$$

which could not be true if $l(Y) < 0$. The positivity of l shows that given any $Y \geq 0$ in $\mathcal{X}$, we get that $l(Y) \geq 0$.

Our next claim is that $l(1) = 1$.

In fact if $c \leq 1$, then we can choose $c \in B(0,1)$ such that $l(c) \geq 1$. But since $X \notin \mathfrak{B}$, we obtain from equation (4.6) that

$$1 \leq l(c) \leq \sup_{c \in B(0,1)} l(c) \leq l(X).$$

By letting $X = 1$, it follows that $1 \leq l(1)$. And if $c \geq 1$, we obtain from the positivity of l that

$$\forall X \geq 0, \ \ 0 \leq l(X + c) = -c \cdot l(1) + l(X) \leq -c \cdot l(1) + 1.$$

Hence $c \cdot l(1) \leq 1$ and therefore $l(1) \leq \frac{1}{c} \leq 1$ and it follows that $l(1) = 1$.

Define $\pi^* : \mathcal{X} \to \mathbb{R}$ as

$$\pi^*(Y) = \sup\{l(Z) \mid Z \in [X], Z \leq_{st} Y \in \mathcal{X}\}. \tag{4.8}$$

We shall show that π^* is a monotone and comonotonic real functional on $\mathcal{X}$. As $l(1) = 1$ we get

$$\begin{aligned} \pi^*(1) &= \sup\{l(Z) \mid Z \in [X], Z \leq_{st} 1\} \\ &= \sup\{l(Z) \mid Z \in [X], l(Z) \leq l(1)\} \\ &= 1. \end{aligned}$$

Let verify that π^* satisfies the monotonicity property and the positive homogeneity. If we let $Z, X, Y \in \mathcal{X}$ such that $Z \leq_{st} X \leq Y$, then we can find an increasing continuous function f with $f(X) \leq f(Y)$. This implies that $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ and from using Definition 4.2, it follows that $X \leq_{st} Y$. But since l also belongs to $\mathcal{M}_{st}(\mathbb{P})$, then $l(Z) \leq l(X)$ with $Z \leq_{st} X \leq_{st} Y$ and therefore $\pi^*(X) \leq \pi^*(Y)$. This shows that π^* has the monotonicity property and π^* belongs to $\mathcal{M}_{st}(\mathbb{P})$.
For all $Y \in \mathcal{X}$ and $\lambda \in [0, 1]$, we show that $\lambda\pi^*(Y) = \pi^*(\lambda Y)$.

$$\begin{aligned} \lambda\pi^*(Y) &= \lambda \sup\{l(Z) \mid Z \in [X], Z \leq_{st} Y\} \\ &= \sup\{\lambda l(Z) \mid \lambda Z \in [X], \lambda Z \leq_{st} \lambda Y\} \\ &= \sup\{l(\lambda Z) \mid \lambda Z \in [X], \lambda Z \leq_{st} \lambda Y\} \\ &= \pi^*(\lambda Y). \end{aligned}$$

Let show that π^* satisfies the comonotonicity property.
Let $Y_1, Y_2 \in \mathcal{X}$ be two comonotonic random variables, by Definition 3.32 we can find two increasing continuous functions g and h on $\mathbb{R}$ such that for all $Y_1, Y_2, Z \in \mathbb{R}$, we have

$$g(Z) + h(Z) = Z \quad \text{with} \quad Y_1 = g(Y_1 + Y_2) \text{ and } Y_2 = h(Y_1 + Y_2).$$

So given any $Z \in [X]$ such that $Z \leq_{st} Y_1 + Y_2$, we have

$$g(Z) \leq_{st} g(Y_1 + Y_2) \quad \text{and} \quad h(Z) \leq_{st} h(Y_1 + Y_2).$$

This implies that for all $g(Z), h(Z) \in [X]$ we have $g(Z) \leq_{st} Y_1$ and $h(Z) \leq_{st} Y_2$. From the definition of π^*, we have that

$$\pi^*(Y_1) = \sup_{g(Z)\in[X]} \{l \circ g(Z) \mid g(Z) \leq_{st} Y\} \quad \text{and} \quad \pi^*(Y_2) = \sup_{h(Z)\in[X]} \{l \circ h(Z) \mid h(Z) \leq_{st} Y\}.$$

Thus, we get the following

$$\begin{aligned} \pi^*(Y_1) + \pi^*(Y_2) &\geq l \circ g(Z) + l \circ h(Z), \qquad \forall g(Z), h(Z) \in [X] \\ &= l \circ (g(Z) + h(Z)) \ , \qquad \forall Z = g(Z) + h(Z) \in [X] \\ &= \sup_{Z\in[X]} l(Z) \ , \qquad Z \leq_{st} Y_1 + Y_2. \end{aligned}$$

Therefore we have

$$\pi^*(Y_1) + \pi^*(Y_2) \geq \pi^*(Y_1 + Y_2). \tag{4.9}$$

Let us take Z_1 and Z_2 in $[X]$ so that $Z_1 \leq_{st} Y_1$ and $Z_2 \leq_{st} Y_2$. Thus, using Lemma 4.5, we obtain that $Z_1 + Z_2 \leq_{st} Y_1 + Y_2$. Therefore using π^*'s definition again, we get

$$\begin{aligned} \pi^*(Y_1 + Y_2) &\geq \pi^*(Z_1 + Z_2) \\ &\geq l(Z_1 + Z_2) \\ &= l(Z_1) + l(Z_2) \, , \qquad \forall Z_1, Z_2 \in [X] \\ &= \sup_{Z_1\in[X]} l(Z_1) + \sup_{Z_2\in[X]} l(Z_2). \end{aligned}$$

Hence we have

$$\pi^*(Y_1 + Y_2) \geq \pi^*(Y_1) + \pi^*(Y_2). \tag{4.10}$$

From equations (4.9) and (4.10), our claim is proven correct.

Since π^* is a monotonic and comonotonic real functional on $\mathcal{X}$, it follows then from Corollary 4.17 that we can find a risk measure $\xi_X \in \mathcal{M}_{1,m}(\mathbb{P})$ such that

$$\xi_X(Y) = \pi^*(Y), \quad \forall Y \in \mathcal{X}.$$

Consequently, $\xi_X \in \mathcal{M}(\mathbb{P})$ and for all $X \in \mathcal{X}$, we have $\pi^*(X) = \xi_X(X) = \pi(X)$. However, for any $Y \in [X]$, let $Y_1 - 1 = Y - \pi(Y)$. Thus, $l(Y_1) \leq 1$ and we obtain that

$$\begin{aligned} 1 &\geq l(Y - \pi(Y) + 1) \\ &\geq l(Y) - \pi(Y) + l(1), \end{aligned}$$

which implies that $l(Y) \leq \pi(Y), \forall Y \in [X]$. Then it follows that for any $\xi_X \in \mathcal{M}(\mathbb{P})$, $\pi(X) = l(X)$. The supremum is finally attained since we have obtained that

$$\pi(X) = l(X) = \pi^*(X) = \xi_X(X) = \pi(X).$$

Therefore the proof is complete. ■

Theorem 4.19 Any monetary risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A1)$ and the comonotonic convexity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \ \forall X \in \mathcal{X},$$

where

$$\alpha(\mu) = \sup_{\pi(X) \leq 0} \mu(X), \ \ \mu \in \mathcal{M}_{st}(\mathbb{P}).$$

Proof:

We want to show that π satisfies $(A1)$ and the comonotonic convexity property.

Let $\alpha : \mathcal{M}_{st}(\mathbb{P}) \to \mathbb{R}$ be any functional such $\inf_\mu \alpha(\mu)$ is finite and define $\pi : \mathcal{X} \to \mathbb{R}$ as

$$\pi(X) = \max_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \ \forall X \in \mathcal{X},$$

then for all μ in $\mathcal{M}_{st}(\mathbb{P})$ we obtain that

$$\begin{aligned} X \leq_{st} Y \ &\Rightarrow \ \mu(X) \leq \mu(Y) \\ &\Rightarrow \ \mu(X) - \alpha(\mu) \leq \mu(Y) - \alpha(\mu) \\ &\Rightarrow \ \sup_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\} \leq \sup_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(Y) - \alpha(\mu)\}. \end{aligned}$$

Hence, $\pi(X) \leq \pi(Y)$. Therefore π satisfies $(A1)$ and using Theorem 3.47, we can see that π also satisfies the comonotonic convexity property.

Assume that π satisfying $(A1)$, the comonotonic convexity property and without loss of generality let $\pi(0) = 0$. We want next to show that

$$\pi(X) = \max_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \ \forall X \in \mathcal{X}.$$

In our first step, we consider $X \in \mathcal{X}$ such that for any $\pi(X) \geq 0$, we have $\tilde{X} = X + \pi(X)$. Therefore, for all μ in $\mathcal{M}_{st}(\mathbb{P})$, we have by the definition of α that

$$\begin{aligned} \alpha(\mu) &\geq \mu(\tilde{X}) \\ &\geq \mu(X + \pi(X)) \\ &\geq \mu(X) - \pi(X) \\ \Rightarrow \quad \pi(X) &\geq \mu(X) - \alpha(\mu), \quad \forall \mu \in \mathcal{M}_{st}(\mathbb{P}). \end{aligned}$$

Thus, we get that

$$\pi(X) \geq \sup_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}. \tag{4.11}$$

Secondly, we fix X and define the set $\mathfrak{B}$ as before as

$$\mathfrak{B} = \{Y \in \mathcal{X} \mid \exists Z \in [X]\,,\ \pi(Z) < 0\,;\ Y \leq Z \ a.s.\},$$

where $[X]$ is defined as before. Clearly $\mathfrak{B}$ contains the open ball

$$B(1,1) = \{Y \in \mathcal{X} \mid \|Y - 1\| < 1\}.$$

Similarly to the proof of Theorem 4.18, X is not contained in $\mathfrak{B}$. Thus, we may use the same separation argument which yields a non-zero linear functional l on $ba(\Omega, \mathcal{F}, \mathbb{P})$ such that equation (4.6) holds. We claim that $l(1) > 0$.
Since l is non-zero, we can find some $Y \in B(0,1)$ such that $l(Y) > 0$. But since $Y^+ \geq Y$, then $l(Y^+) \geq l(Y) > 0$. On the other hand, we have $1 = Y^+ + (1 - Y^+)$ and therefore

$$l(1) = l(Y^+) + l(1 - Y^+).$$

The positivity of l implies that $l(1 - Y^+) \geq 0$ and we finally obtain that $l(1) > 0$. Consequently, we can choose l such that $l(1) = 1$. Our next claim is that $l(Y) \geq 0$ if $Y \geq 0$.

$$\begin{aligned} Y \in \mathfrak{B}\,;\ Y \geq 0 \quad &\Rightarrow \quad \exists Z \in [X]\,;\ -1 + \lambda Y \leq_{st} Z, \quad \lambda \in [0,1] \\ &\Rightarrow \quad \pi(-1 + \lambda Y) \leq \pi(Z)\,, \qquad \text{since } \pi \text{ satisfies (A1)} \\ &\Rightarrow \quad \pi(-1 + \lambda Y) < 0. \end{aligned}$$

Hence $-1 + \lambda Y \in \mathfrak{B}$. Thus from the above equation (4.6), we have

$$l(-1) + \lambda l(Y) = l(-1 + \lambda Y) \leq l(X), \quad \forall \lambda \geq 0,$$

which could not be true if $l(Y) < 0$. The positivity of l shows that given any $Y \geq 0$ in $\mathfrak{B}$, we get that $l(Y) \geq 0$. Let us recall $\pi^* : \mathcal{X} \to \mathbb{R}$ as

$$\pi^*(Y) = \sup\{l(Z) \mid Z \in [X], Z \leq_{st} Y \in \mathcal{X}\}.$$

As previously shown in Theorem 4.18, π^* satisfies monotonicity, comonotonic additivity and the following property

$$l(Y) \leq \pi^*(Y), \ \ \forall Y \in [X]. \tag{4.12}$$

Thus, using Corollary 4.17, there exists a risk measure ξ_X on $\mathcal{F}$ such that

$$\xi_X(Y) = \pi^*(Y), \ \ \forall Y \in \mathcal{X}.$$

By the definition of π^*, we obtain that $\xi_X(1) = 1$. If we choose an arbitrarily $\varepsilon > 0$ such that for any $Y, Z \in [X]$ we have $Z \leq_{st} Y - \varepsilon$, then $\pi(Y) \leq 0 \Rightarrow \pi(Y - \varepsilon) < 0$, $\forall \varepsilon > 0$. Hence we have that

$$\begin{aligned} \xi_X(Y) - \varepsilon &= \pi^*(Y) - \varepsilon \\ &= \pi^*(Y - \varepsilon) \\ &= \sup\{l(Z) \mid Z \in [X], Z \leq_{st} Y - \varepsilon \in \mathcal{X}\} \\ &\leq \sup_{Z \in \mathfrak{B}} l(Z). \end{aligned}$$

Hence for all $X \in \mathcal{X}$, we obtain that

$$\alpha(\xi_X) = \sup_{\pi(X) \leq 0} \xi_X(X) \leq \sup_{Z \in \mathfrak{B}} l(Z),$$

so that

$$\begin{aligned} \Rightarrow \quad \xi_X(X) - \alpha(\xi_X) &\geq \xi_X(X) - \sup_{Z \in \mathfrak{B}} l(Z) \\ &\geq \pi^*(X) - \sup_{Z \in \mathfrak{B}} l(Z) \\ &\geq l(X) - \sup_{Z \in \mathfrak{B}} l(Z) \quad \text{from equation (4.12)} \\ &\geq 0 \qquad\qquad\qquad\quad \text{from equation (4.6).} \end{aligned}$$

Thus, we get that

$$\pi(X) \leq \xi_X(X) - \alpha(\xi_X). \tag{4.13}$$

By letting $\xi_X = \mu$ in the set $\mathcal{M}_{st}(\mathbb{P})$, we get that

$$\pi(X) \leq \sup_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}. \tag{4.14}$$

Therefore we obtain from equations (4.11) and (4.14) that

$$\pi(X) = \sup_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \forall X \in \mathcal{X}.$$

and also that

$$\pi(X) = \xi_X(X) = \mu(X) = \pi^*(X) = l(X) = \pi(X).$$

But since the supremum is attained we can finally deduce that

$$\pi(X) = \max_{\mu \in \mathcal{M}_{st}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \forall X \in \mathcal{X}. \tag{4.15}$$

■

The following theorems give a representation of risk measures with respect to convex and stop-loss order which are not necessary to prove since, it suffices to give analogous proofs from above.

Theorem 4.20 Any risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A2)$ and the comonotonic subadditivity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X),$$

where $\mathcal{M}(\mathbb{P}) = \{\mu \in \mathcal{M}_{cx}(\mathbb{P}) \mid \mu(Y) \leq \pi(Y)\}, \ \forall Y \in \mathcal{X}$.

Theorem 4.21 Any monetary risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A2)$ and the comonotonic convexity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}_{cx}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \forall X \in \mathcal{X},$$

where

$$\alpha(\mu) = \sup_{\pi(X) \leq 0} \mu(X), \ \mu \in \mathcal{M}_{cx}(\mathbb{P}).$$

The last two theorems are simple consequence of Remark 4.7.

Theorem 4.22 Any risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A3)$ and the comonotonic subadditivity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}(\mathbb{P})} \mu(X),$$

where $\mathcal{M}(\mathbb{P}) = \{\mu \in \mathcal{M}_{icx}(\mathbb{P}) \mid \mu(Y) \leq \pi(Y)\}, \ \forall Y \in \mathcal{X}$.

Theorem 4.23 Any monetary risk measure π on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $(A3)$ and the comonotonic convexity property is of the following form

$$\pi(X) = \max_{\mu \in \mathcal{M}_{icx}(\mathbb{P})} \{\mu(X) - \alpha(\mu)\}, \ \forall X \in \mathcal{X},$$

where

$$\alpha(\mu) = \sup_{\pi(X) \leq 0} \mu(X), \ \mu \in \mathcal{M}_{icx}(\mathbb{P}).$$

Remark 4.24 Any risk measure respecting the stochastic order satisfies automatically the monotonicity and law-invariant properties.

4.3 Law-invariance

This is a particular property related to the approximation of risk measure via empirical data. *Law-invariance* is usually said to be a crucial point when it comes to the application of risk measures in the financial industry. A risk measure ρ is called *law-invariant* if it gives the same value to losses with the same distribution. Coherent risk measures ρ which are not *law-invariant* on the contrary cannot have an estimator as a function of empirical data only. For such measures it may happen that two indistinguishable portfolios X and Y (i.e. two portfolios with exactly the same probability law $F_X(.) = F_Y(.)$ and therefore with indistinguishable data samples) may have different values for $\rho(X)$ and $\rho(Y)$. Law-invariant risk measures are special

class of model dependent risk measures which respect stochastic orderings. In Yan's paper [33] both stochastic dominance $\leq_{cx}$ and $\leq_{st}$ obviously imply the monotonicity and law-invariant properties.

Definition 4.25 A monetary risk measure ρ on $\mathcal{X} = L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ is called *law-invariant* if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under $\mathbb{P}$.

$$i.e. \quad F_X = F_Y \Rightarrow \rho(X) = \rho(Y).$$

Theorem 4.26 Let $\Omega = \{\omega_1, ..., \omega_n\}$ with $\mathbb{P}(\omega_i) = 1/n$, $i = 1, ..., n$, and let ρ be a monotone law-invariant risk measure on $\mathcal{X}$. Then $X \leq_{st} Y$ implies $\rho(X) \leq \rho(Y)$.

Proof:
Since $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, we can apply the Theorem 4.15 for $X \leq_{st} Y$. In fact there exist X' and Y' in $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X =_d X', Y =_d Y'$ and $X' \leq Y'$. Therefore by the use of the law-invariant and monotonicity, we have

$$\rho(X) = \rho(X') \leq \rho(Y') = \rho(Y).$$

■

Remark 4.27 Any risk measure satisfying the monotonicity and law-invariant properties respects stochastic orderings.

Theorem 4.28 Let $\Omega = \{\omega_1, ..., \omega_n\}$ with $\mathbb{P}(\omega_i) = 1/n$, $i = 1, ..., n$, and let ρ be a law-invariant risk measure on $\mathcal{X}$, having convexity property. Then $X \leq_{cx} Y$ implies $\rho(X) \leq \rho(Y)$.

Proof:
This follows by using the same previous method and applying Theorem 4.16. ■

Theorem 4.29 Let $\Omega = \{\omega_1, ..., \omega_n\}$ with $\mathbb{P}(\omega_i) = 1/n$, $i = 1, ..., n$, and let ρ be a law-invariant risk measure on $\mathcal{X}$, having the monotonicity and convexity property. Then $X \leq_{icx} Y$ implies $\rho(X) \leq \rho(Y)$.

Proof:
Again it follows by using the same previous method and applying Theorem 4.12. ■

Theorem 4.30 Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with $\rho : \mathcal{X} \to \mathbb{R}$. The following properties are equivalent :

(a) ρ is a law-invariant coherent risk measure with the Fatou property.

(b) There exists a set $\mathcal{M}_0$ of probability measure on (0,1] such that:

$$\rho(X) = \sup_{m \in \mathcal{M}_0} \int_{(0,1]} AVaR_\lambda(X) m \cdot d\lambda.$$

Proof:
(a)⇒(b) Consider $\mathfrak{N}$ to be the set of probability densities so that by the Radon Nikodym Theorem, any element of can be identify with a probability measure. Thus we can find a set $\mathcal{M}_0$ of probability measure on (0,1]. Define the set Z by

$$Z := \{m \in \mathfrak{N} \mid m \leq \frac{d\mathbb{P}}{dQ}\}$$

and fix a random variable α with continuous distribution on $(0, 1]$ so that we can define a risk measure $\rho_\alpha : \mathcal{X} \to \mathbb{R}$ as $\rho_\alpha(X) = AVaR_\alpha(X)$ and obtain a one-to-one correspondence between the laws of densities m and the probability measure μ on (0,1] (i.e. $m \cdot dQ = \mu \cdot d\alpha$ for all $\mu \in \mathcal{M}_0$). From the assumption of coherency, ρ can be represented as

$$\rho(X) = \sup_{Q \in \mathcal{M}_0} \mathbb{E}_Q[-X]$$

and one can refer to section 3.5 to notice that since the distribution is continuous

$$\rho_\alpha(X) = AVaR_\alpha(X) = \frac{1}{\alpha} \sup\{\mathbb{E}_Q[-X] \mid 0 < \alpha \leq 1, \mathbb{E}_Q[1] = 1\} \leq \rho(X).$$

Then we can deduce the following

$$\begin{aligned}
\rho(X) \geq m\rho_\alpha(X) \cdot \frac{dQ}{d\mathbb{P}} \quad &\Rightarrow \quad \rho(X) \cdot d\mathbb{P} \geq m\rho_\alpha(X) \cdot dQ \\
&\Rightarrow \quad \int_{(0,1]} \rho(X) \cdot d\mathbb{P} \geq \int_{(0,1]} m\rho_\alpha(X) \cdot dQ \\
&\Rightarrow \quad \rho(X) \geq \int_{(0,1]} m\rho_\alpha(X) \cdot dQ \\
&\Rightarrow \quad \rho(X) \geq \int_{(0,1]} \rho_\alpha(X)\mu \cdot d\alpha \ , \quad \forall \mu \in \mathcal{M}_0.
\end{aligned}$$

Hence, we have that

$$\rho(X) \geq \sup_{m \in \mathcal{M}_0} \int_{(0,1]} AVaR_\alpha(X)m \cdot d\alpha. \tag{4.16}$$

Given $\varepsilon \in (0,1)$, consider a subset $\mathfrak{S} \subset \mathcal{X}$ defined as

$$\mathfrak{S} := \{Y \in \mathcal{X} \mid AVaR_\alpha(X) \leq \beta Y \ , \ \beta < 0\}.$$

Assume that we have two uniformly bounded sequences (X_n) and (Y_n) in $\mathfrak{S}$ such that $X_n \to X, Y_n \to Y \ \ \mathbb{P}-a.s.$ and $Y_n = X_n + \varepsilon$. So since ρ is coherent, we have the following

$$\begin{aligned}
\beta\rho(Y_n) &\leq -AVaR_\alpha(X_n) \, , \quad \text{by monotonicity} \\
\Rightarrow \rho(Y_n) &\leq -\frac{1}{\beta} \, AVaR_\alpha(X_n) \\
\Rightarrow \rho(Y_n) &< \varepsilon \, AVaR_\alpha(X_n).
\end{aligned}$$

Since for all $m \in \mathcal{M}_0$, we have $\varepsilon < m \cdot \frac{dQ}{d\mathbb{P}}$, thus we obtain that

$$\begin{aligned}
\rho(Y_n) &< AVaR_\alpha(X_n)m \cdot \frac{dQ}{d\mathbb{P}} \ , \qquad \forall m \in \mathcal{M}_0 \\
\Rightarrow \quad \rho(Y_n) \cdot d\mathbb{P} &< AVaR_\alpha(X_n)m \cdot dQ \ , \qquad \forall m \in \mathcal{M}_0 \\
\Rightarrow \int_{(0,1]} \rho(Y_n) \cdot d\mathbb{P} &< \int_{(0,1]} AVaR_\alpha(X_n)m \cdot dQ \ , \quad \forall m \in \mathcal{M}_0 \\
\Rightarrow \quad \rho(Y_n) &< \int_{(0,1]} AVaR_\alpha(X_n)m \cdot dQ \ , \quad \forall m \in \mathcal{M}_0.
\end{aligned}$$

Hence for all m in $\mathcal{M}_0$, it follows that

$$\liminf_{n\to\infty} \rho(Y_n) < \liminf_{n\to\infty} \int_{(0,1]} AVaR_\alpha(X_n)m \cdot dQ = \lim_{n\to\infty} \int_{(0,1]} AVaR_\alpha(X_n)m \cdot dQ,$$

and since ρ has the Fatou-property, we can deduce that

$$\rho(X) - \varepsilon < \int_{(0,1]} AVaR_\alpha(X)m \cdot dQ, \quad \forall m \in \mathcal{M}_0. \tag{4.17}$$

Therefore, it follows from (4.16) and (4.17) that

$$\rho(X) = \sup_{m \in \mathcal{M}_0} \int_{(0,1]} AVaR_\alpha(X)m \cdot d\alpha. \tag{4.18}$$

(b)⇒(a) We refer to section 3.5 to see that $AVaR_\alpha$ is a coherent risk measure. As coherence is preserved under taking convex combinations and suprema, it follows that ρ is coherent, too. Under the same section, Remark 3.61 and Definition 3.62 give us that $AVaR_\alpha$ is law-invariant. This obviously implies that ρ is a law-invariant coherent risk measure. Let a sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ be uniformly bounded with $X_n \to X \in \mathcal{X}$ $\mathbb{P} - a.s.$ and note that that $q_\alpha(X)$ has the distribution function F_X, where α is a random variable with uniform distribution on $(0, 1]$. So it follows that $q_\alpha(X_n) \to q_\alpha(X)$ for almost all $\alpha \in (0, 1]$. But since $\{q_\alpha(X_n)\}$ is uniformly bounded and integrable on $\mathcal{M}_0$, then so be the sequence $\{AVaR_\alpha(X_n)\}$. $\mathcal{M}_0$ is a compact set and by Theorem 3.15, it follows that $AVaR_\alpha$ has the Fatou property and implies by using Definition 3.62 that ρ has also the Fatou property. ■

Remark 4.31 Any law-invariant convex risk measure satisfies the Fatou property.

The following theorem generalizes the representation above to the larger class of *law-invariant convex risk measures* satisfying some further axioms.

Theorem 4.32 Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with $\mathbb{P}$ atomless and $\rho : \mathcal{X} \to \mathbb{R}$ and let $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be endowed with the $\sigma(L^\infty, L^1)-$topology. The following properties are equivalent :

(a) ρ is a law-invariant coherent risk measure.

(b) There exists a non-empty convex set $\mathcal{P}$ of probability measures and a convex functional $\alpha : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ such that $\inf_{Q\in\mathcal{P}} \alpha(0) = 0$ and

$$\rho(X) = \sup_{Q\in\mathcal{P}} \{ \int_{[0,1]} AVaR_\lambda(X)m \cdot d\lambda - \alpha(Q)\}, \forall X \in L^\infty. \tag{4.19}$$

Let define the couple $(\rho, \preceq)$ as a risk measure induced by a functional $\rho : \mathcal{X} \to \mathbb{R}$ with respect to a specific ordering $\preceq$. The following result introduced by Yan [33] shows a strong relationship between sets of axioms for law-invariant risk measures.

Theorem 4.33 Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then we have the followings:

(a) $(\rho, \leq_{st})$ is coherent if and only if $(\rho, \leq_{icx})$ is comonotonic subadditive.

(b) $(\rho, \leq_{st})$ is convex if and only if $(\rho, \leq_{icx})$ is a comonotonic convex.

Proof:
See Theorem 5.7 in [33]. ■

Bibliography

[1] C. Acerbi, *Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion*, Journal of Banking & Finance **26**, (2002), 1505-1518.

[2] A. Adam, M. Houkari and J.P. Laurent, *Spectral risk measures and portfolio selection*, Journal of Banking & Finance **32**, (2008), 1870-1882.

[3] C.D. Aliprantis and Donald J. Brown, *Equilibria in Markets with a Riesz Space of Commodities*, Journal of Mathematical Economics **11**, (1983), 189-207.

[4] P. Artzner, *Application of Coherent Risk Measures to Capital Requirements in Insurance*, North American Actuarial Journal, Vol(3), (1999), No. 2, 11-25.

[5] P. Artzner, F. Delbaen, J.M. Eber and D. Heath, *Coherent measures of risk,* Mathematical Finance, **4** (1999), 203-228.

[6] P. Artzner, F. Delbaen, J.M. Eber, D. Heath and H. Ku, *Coherent multi-period risk adjusted values and Bellman's principle* , Annals of Operations Research (in press) (2004).

[7] S. Cheng, Y. Liu and S Wang, *Progress in Risk Measurement*, Advance Modelling and Optimization, Vol(6), (2004), No 1.

[8] E.De Giorgi, *Reward-risk portfolio selection and stochastic dominance*, Journal of Banking and Finance, **29**, (2002), 895-926.

[9] F. Delbaen, *Coherent Measures of Risk on General Probability Spaces*, Advancees in Finance and Stochastics (2002), Essays in Honour of Dieter Sondermann, Springer-Verlag, 1-37.

[10] P. Embrechts, *Integrated Risk Management for Banking and Insurance,* Latsis-Symposium, http://www.math.ethz.ch/∼embrechts/ , Septemger 2001.

[11] H.Föllmer and A Schied, *An introduction in discrete time*, Stochastic finance, Berlin, de Gruyter Studies in Mathematics (2002).

[12] H.Föllmer and A Schied, *Convex measures of risk and trading constraints*, Finance and Stochastics, **6**, (2002), 429-447.

[13] H.Föllmer and A Schied, *Robust representation of convex measures of risk*, Finance and Stochastics. Springer-Verlag (2002).

[14] M. Frittelli, *Representing Sublinear Risk Measures and Pricing Rules*, Working paper n. 10 (2000), Universitá de Milano Bicocca, Italy .

[15] M. Frittelli, E. Rosazza Gianin, *Dynamic convex risk measures,* New risk measures in the 21st Century, Szegö G.(ed.) John Wiley & Sons, (2004), 227-248.

[16] M. Frittelli, E. Rosazza Gianin, *Putting Order In Risk Measures*, Journal of Banking and Finance, Vol(26),**7** (2002), 1473-1486.

[17] C. C. Heyde, S. G. Kou and X. H. Peng, *What Is a Good Risk Measure: Bridging the Gaps between Data, Coherent Risk Measures, and Insurance Risk Measures*, Columbia University, (2006).

[18] S. Jaschke and U. Küchler, *Coherent risk measures and good deal bounds*, Finance Stochast, Vol(5), (2001), 181-200.

[19] P. Jorion, *Risk Management Lessons from LTCM,* European Financial Management, **6** (2000), 277-300.

[20] E. Jouini, M. Meddeb and N. Touzi, *Vector-valued coherent risk measures*, Finance and Stochastics, **8**, (2004), 531-552.

[21] S. Koshi and N. Komuro, *Supsets on partially ordered topological linear spaces*, Taiwanese Journal of Mathematics, Vol(4), (2000), No 2, 275-284.

[22] W-C Kuo, CCA Labuschagne, BA Watson, *Discrete-time stochastic processes in Riesz spaces*, Indagationes Mathematicae, Vol(15), (2004), 435-451.

[23] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxford university, Press, (1997).

[24] A. Müller and N. Bäuerle, *Stochastic orders and risk measures: Consistency and bounds*, Insurance: Mathematics and Economics, **38**, (2006), 132-148.

[25] A. Müller and D. Stoyan, *Comparison methods for stochastic model and risks* John Wiley & Sons Ltd, (2002), Chichester Wiley Series in Probability and Statistics.

[26] I. Penner,*Dynamic convex risk measures: time consistency, prudence, and sustainability*, PhD Thesis, (2007).

[27] F. Riedel, *Dynamic coherent risk measures*, Stochastic processes and their applications, **112**, (2004),185200.

[28] R.T. Rockafellar, S. Uryasev and M. Zabarankin, *Generalized deviations in risk analysis*, Finance and Stochastics, **10**, (2006), 51-74.

[29] E. Rosazza Gianin, *Risk Measures via g-Expectations*,
Insurance: Mathematics and Economics, **39** (2006), 19-34.

[30] W Rudin, *Functional Analysis*, 2nd Edition, McGraw-Hill, (1973).

[31] G. Scandolo, *Risk measures in a dynamic setting*, PhD Thesis, Università di Milano,(2003).

[32] H. Schaeffer, *Banach Lattices and Positive Operators*, Berlin Heidelberg New-York (1974), Springer-Verlag.

[33] Y. Song and J-A. Yan, *An Overview of Representation Theorems for Static Risk Measures*, Science in China Series A- Mathematics http://www.scichina.com, or //www.springerlink.com, (2008).

[34] Y. Song and J-A. Yan, *Risk Measures with Comonotonic Subadditivity or Convexity and Respecting Stochastic Orders*, Preprint.

[35] Y. Song and J-A. Yan, *The representations of two types of functionals on* $L^{\infty}(\Omega, \mathcal{F})$ *and* $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$, Science in China Series A-Mathematics, Vol(49), (2006), No 10, 1376-1382.

[36] K. Yosida, *Functional Analysis*, 2nd Edition, Berlin Heidelberg New-York (1968), Springer-Verlag.

[37] A. Zaanen, *Linear Analysis*, North-Holland Publishing CO-Amsterdam, (1953), Interscience Publisher Inc. New-York.