

https://dspace.org/

Collective field theory of schur polynomials

Smith, Stephanie

2011-10-07
http://hdl.handle.net/10539/10503

Downloaded from a DSpace repository. https://dspace.org/

M. Sc Dissertation: Collective Field Theory of Schur Polynomials

Stephanie Smith: 0601372Y

4 June 2011

Declaration

This thesis is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. I declare that this is my own unaided work and that it has not previously been submitted for any other degree or examination before now.

Stephanie Smith
11 January 2011

Abstract

We try to develop a collective field theory of single matrix models by using the formalism of Jevicki and Sakita in [1], with Schur polynomials as our collective fields. Field operators and the relation for the change of variables required to obtain the collective field Hamiltonian are found using group representation theory.

Acknowledgements

I would like to thank my supervisor Prof. Robert de Mello Koch for everything he has taught me and done for me over the time we have been working together. I also thank my family for their support.

Contents

1 Introduction **6**

2 Correlation Between QFT and Quantum Collective Field Theory **9**

2.1 Properties of N Particles in an Harmonic Oscillator Potential 9

2.2 Correlators in QM 13

2.3 Collective Field Theory 16

2.3.1 Classical Solution to the Collective Field Hamiltonian . 22

2.4 Quantum Expansion of the Classical Collective Field 24

2.5 Correlators In the Collective Field Theory 27

2.5.1 The Classical Field Correlator 28

2.5.2 Quantum Corrections to the Classical Correlator 28

2.6 Comparison 30

3 The Large N Limit of Matrix Models **32**

3.1 Correlation Functions and the Classical Limit 32

3.1.1 Calculating Correlators 33

3.1.2 The Classical Limit 37

3.1.3 Matrix Models In the Classical Limit 39

3.1.4 Eigenvalue Basis 41

3.1.5 Correlators in the Eigenvalue Basis 48

3.1.6 Where the Eigenvalue Basis Fails 49

4 Back Reaction on the Eigenvalues **51**

5 Schur Polynomials **58**

6 Collective Field Theory of Schur Polynomials **65**

6.1 Collective Field Operators 65

6.2 The Jacobian 74

7 Conclusions and Outlook **78**

1 Introduction

The ultimate goal of this work is to develop an analytical method to do calculations in multi matrix models in the large N limit. To do this, we try to develop a collective field theory of Schur Polynomials, since they provide a gauge invariant basis parametrized by N row lengths. This is useful since saddle point methods are applicable to problems phrased in terms of N variables. This will be explained fully later. To construct this theory one needs to construct the Jacobian for the change of variables from matrices to Schur polynomials. This will enable us to change from the quantum field theory path integrals of matrices to path integrals of Schur Polynomials. We use collective field theory since it is an established method to systematically change the variables of a quantum field theory. We did not achieve our goal in this project as it was highly non-trivial to obtain the Jacobian for the change of variables. However, we did succeed in finding the equation that determines the Jacobian.

We use the collective field formalism to look at the example of N bodies in a Harmonic Oscillator potential in which the initial Hamiltonian is the sum of the N individual bodies. This problem is considered as it is perhaps the simplest example with which we can demonstrate the collective field method. In this case solutions to the collective field Hamiltonian are functionals of the collective field. The systematic change of variables from individual wave functions to collective fields requires that we calculate collective field operators corresponding to the momentum term of the original N body Hamiltonian. Once these are found using the chain rule, we use them to determine the

equation for the Jacobian describing this change of variables. The collective field Hamiltonian is constructed using the collective field operators and this Jacobian. Once the collective field Hamiltonian is found, we find its classical solution by minimizing the potential term. We then add quantum corrections to the classical solution in order to demonstrate the equaivalence of the collective field theory with the original quantum mechanics. Our demonstration entails showing that the Harmonic Oscillator potential correlators calculated in quantum mechanics and collective field theory yield the same result. Note that the classical collective field theory does sum quantum effects of the original theory.

There has been extensive study of the large N limit since 't Hooft proposed that $\frac{1}{N}$ would be a good parameter in which to expand matrix model quantum field theories. This N is the dimension of the matrix valued field corresponding to the gauge group $SU(N)$. In these field theories with matrix valued fields the fluctuations are of order $\frac{1}{N^2}$. N is a better expansion parameter than other physical constants such as $\hbar$, since it is dimensionless and scale invariant. This means that if we change the dimension or scale of our field theory we do not need to account for any renormalization group flow for N. If we are working in the large N limit, fluctuations become negligible and the system approaches a classical limit. We look to Schur polynomials to provide a basis in which the large N expansion can be used to construct this classical limit. Another reason that we are attempting to define field theories in the Schur polynomial basis is to try to find a way to incorporate multiple matrices that are not necessarily simultaneously diagonalized into

our model.

Indeed, although there is no natural generalization of the eigenvalue description to the case of multi matrix models, there is a generalization of the Schur polynomial. For multi matrix models restricted Schur polynomials provide a complete basis for the local operators of the theory and they continue to diagonalize the free field theory two point function. Our hope is that the collective field theory of Schur polynomials will have a natural generalization to the collective field theory of restricted Schur polynomials.

2 Correlation Between QFT and Quantum Collective Field Theory

In this section the quantum mechanics of an N particle system will be studied. The correlation functions corresponding to the N body Harmonic Oscillator system will be calculated using an N particle description. We will then explain the change of variables from a collection of single particle wave functions to the N particle collective field theory wave functional as described by the formalism of Jevicki and Sakita in [1]. It will then be possible to calculate the same correlation functions in the collective field theory as found using quantum mechanics, and the two sets of results will be compared. We expect the two to agree since the two theories are describing the same system using different formalism.

2.1 Properties of N Particles in an Harmonic Oscillator Potential

The Hamiltonian of this N-body system is given by:

$$H = -\frac{1}{2}\sum_{i=1}^{N}(\frac{\partial^2}{\partial {x_i}^2} + \frac{\partial^2}{\partial {y_i}^2}) + \frac{\omega^2}{2}\sum_{i=1}^{N}({x_i}^2 + {y_i}^2) \tag{1}$$

It is necessary to take note of the quantum mechanical properties of the system, such as creation and annihilation operators and their commutators. This is to describe the quantum mechanics of the system, which is then adapted to collective field variables. Consider the Hamiltonian of a single oscillator in the system. This can be factorized as follows:

$$H = H_x + H_y$$

$$
\begin{aligned}
&= \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + \omega^2 x^2) + \frac{1}{2}(-\frac{\partial^2}{\partial y^2} + \omega^2 y^2) \\
&= \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial x} + \omega x)\frac{1}{\sqrt{2}}(\frac{\partial}{\partial x} + \omega x) + \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial y} + \omega y)\frac{1}{\sqrt{2}}(\frac{\partial}{\partial y} + \omega y)
\end{aligned}
$$

These factors of the Hamiltonian are in fact the following creation and annihilation operators:

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2}}(\omega x + \frac{\partial}{\partial x}) \\
a^\dagger &= \frac{1}{\sqrt{2}}(\omega x - \frac{\partial}{\partial x}) \\
b &= \frac{1}{\sqrt{2}}(\omega y + \frac{\partial}{\partial y}) \\
b^\dagger &= \frac{1}{\sqrt{2}}(\omega y - \frac{\partial}{\partial y})
\end{aligned}
$$

The Hamiltonian can be expressed in terms of these operators as:

$$H = \frac{1}{2}(aa^\dagger + a^\dagger a) + \frac{1}{2}(bb^\dagger + b^\dagger b)$$

.

The commutators between the creation and annihilation operators are found to be:

$$
\begin{aligned}
\left[a_i, a_j^\dagger\right] &= \delta_{ij}\omega \\
\left[b_i, b_j^\dagger\right] &= \delta_{ij}\omega
\end{aligned}
$$

The position operators x and y can be expressed as:

$$x = \frac{a + a^\dagger}{\omega\sqrt{2}} \tag{2}$$

$$y = \frac{b + b^\dagger}{\omega\sqrt{2}} \tag{3}$$

In this description we are using the Heisenberg picture of quantum mechanics, where states are time independent and operators are time dependent. a, $a^\dagger$, b and $b^\dagger$ are all time dependent operators. This allows us to use Wick's theorem to calculate correlators. Using the general form of the Heisenberg equations of motion,

$$i\hbar\frac{d\mathcal{O}}{dt} = [\mathcal{O}, H]$$

in units with $\hbar = 1$ we can find the time dependence of the creation and annihilation operators. We will also use $-i\frac{\partial}{\partial x} = p_x$ and $-i\frac{\partial}{\partial y} = p_y$: :

$$\begin{aligned}
\dot{a}(t) &= \frac{\omega p_x}{\sqrt{2}} - \frac{i\omega^2 x}{\sqrt{2}} = -i\omega a \\
\dot{a}^\dagger(t) &= \frac{\omega p_x}{\sqrt{2}} + \frac{i\omega^2 x}{\sqrt{2}} = i\omega a^\dagger \\
\dot{b}(t) &= \frac{\omega p_y}{\sqrt{2}} - \frac{i\omega^2 y}{\sqrt{2}} = -i\omega b \\
\dot{b}^\dagger(t) &= \frac{\omega p_y}{\sqrt{2}} - \frac{i\omega^2 y}{\sqrt{2}} = i\omega b^\dagger
\end{aligned}$$

We then introduce the initial conditions

$$a(0) = a_0 \qquad a^\dagger(0) = a^\dagger(0)$$

$$b(0) = b_0 \qquad b^\dagger(0) = b_0^\dagger$$

and solve the above equations of motion for the creation and annihilation operators:

$$\begin{aligned}
a(t) &= a_0 e^{-i\omega t} \\
a^\dagger(t) &= a_0^\dagger e^{i\omega t} \\
b(t) &= b_0 e^{-i\omega t} \\
b^\dagger(t) &= b_0^\dagger e^{i\omega t}
\end{aligned}$$

Using the properties we have explored for a single harmonic oscillator described by this Hamiltonian, it is possible to find simple correlators in the system. Correlators are the Wick contractions of terms containing operators. We wish to calculate something of the form $\overbrace{x_i x_j}$. A good example that will be used later is the equal time correlator $< z_i(t)\bar{z}_j(t) >$, where $z_i = x_i + iy_i$ and $\bar{z}_i = x_i - iy_i$. We find the correlator by expressing z_i and $\bar{z}_j$ in terms of creation and annihilation operators and allow these to act on vacuum states. $< z_i\bar{z}_j >$ is useful to calculate since it is related to the Wick contraction $\overbrace{z_i\bar{z}_j}$ as follows:

$$\begin{aligned}
\overbrace{z_i\bar{z}_j} &= < 0|\overbrace{z_i\bar{z}_j}|0 > \\
&= < 0|T(z_i\bar{z}_j)|0 > - < 0| : z_i\bar{z}_j : |0 > \\
&= < 0|T(z_i\bar{z}_j)|0 > \\
&= < 0|z_i\bar{z}_j|0 >
\end{aligned}$$

Here, the normal ordered product of z_i and $\bar{z}_j$ falls away since this product consists of annihilation operators acting on the ket and creation operators acting on the bra. The T represents the time ordered product. We will focus on equal time correlators here. It is now simple to find the expectation $< 0|z_i\bar{z}_j|0 >$:

$$\begin{aligned}
< z_i\bar{z}_j > &= < (x_i + iy_i)(x_j - iy_j). \\
&= < \frac{1}{\omega\sqrt{2}}(a_i + {a_i}^\dagger + i(b_i + {b_i}^\dagger))\frac{1}{\omega\sqrt{2}}(a_j + {a_j}^\dagger - i(b_j + {b_j}^\dagger)) > \\
&= < 0|\frac{1}{2\omega^2}(a_i {a_j}^\dagger)|0 > + < 0|\frac{1}{2\omega^2}(b_i {b_j}^\dagger)|0 >
\end{aligned}$$

The terms containing a mixture of a and b operators disappear, as do any

terms with an a or b acting on the ket or an $a^\dagger$ or $b^\dagger$ acting on the bra. It is possible to add terms containing ${a_i}^\dagger a_j$ and ${b_i}^\dagger b_j$ to the remaining terms in the correlator, since this is equivalent to adding zero. In this manner one can find $< z_i \bar{z}_j >$ in terms of the commutators of the operators found previously i.e.:

$$\begin{aligned} & < 0|\frac{1}{2\omega^2}(a_i {a_j}^\dagger)|0 > + < 0|\frac{1}{2\omega^2}(b_i {b_j}^\dagger)|0 > \\ = & < 0|\frac{1}{2\omega^2}(a_i {a_j}^\dagger - {a_i}^\dagger a_j)|0 > + < 0|\frac{1}{2\omega^2}(b_i {b_j}^\dagger - {b_i}^\dagger b_j)|0 > \\ = & < 0|\frac{\delta_{ij}\omega}{2\omega^2}|0 > + < 0|\frac{\delta_{ij}\omega}{2\omega^2}|0 > \\ = & \frac{\delta_{ij}}{\omega} \end{aligned}$$

It can be similarly shown that

$$< z_i z_j >= 0$$

$$< \bar{z}_i \bar{z}_j >= 0$$

The important result that will be used later on is:

$$< 0|\overbrace{z_i \bar{z}_j}|0 >=< 0|z_i \bar{z}_j|0 >= \frac{\delta_{ij}}{\omega} \tag{4}$$

2.2 Correlators in QM

We have explored some important properties of the Hamiltonian of the N-particle Harmonic Oscillator system that will be useful to our further calculations, so it is possible to move onto a calculation demonstrating the equivalence of the collective field and quantum mechanical methods. An example of a correlator we compute using the two methods is:

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^m > \tag{5}$$

Using equation (4) we can compute this expectation value as a sum over all possible Wick contractions of pairs of z_i's and/or $\bar{z}_j$'s, summed over i and j. This gives something of the form:

$$< 0|\overbrace{z_i z_i z_i z_i}.....\overbrace{\bar{z}_j \bar{z}_j \bar{z}_j \bar{z}_j}|0 > + < 0|\overbrace{z_i z_i z_i z_i}....\overbrace{z_i z_i \bar{z}_j \bar{z}_j}....\overbrace{\bar{z}_j \bar{z}_j}|0 > +.....$$

All possible combinations of Wick contractions must be counted. There are $(n+m-1)!!$ possible Wick contractions in total. In general there are contributions with different powers of N which must be taken into account. In order to simplify this calculation, there are a few points to take note of:

- It can be shown that if $n \neq m$ this correlator is zero.
- $< 0|z_i z_j|0 >= 0$.
- $< 0|\bar{z}_i \bar{z}_j|0 >= 0$
- The above two points mean that when the correlator is calculated only terms containing $\frac{\delta_{ij}}{\omega}$ remain.

The correlator can now be calculated. Using the properties above the order N contributions can be found. The leading at large N term corresponds to the 'classical' result, while the sub-leading in N terms are the quantum corrections. For a correlator containing k sums the leading term which gives the 'classical' result is order N^k. $< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^m >$ is calculated as follows:

- Set $m = n$, otherwise this correlator will be zero due to the factor δ_{mn}.
- Only contractions of z $\bar{z}$ pairs will give any contribution, so only terms which produce a δ_{ij} must be counted.

- This will be the sum of all permutations of possible z $\bar{z}$ pair contractions within the product $z_i^n \bar{z}_j^n$. To count all the combinations of pairs, we begin with n choices of z_i and $\bar{z}_j$ respectively, giving n possible contractions. After the first pair is chosen we have $(n-1)$ possible pair contractions left to choose from, and so on. In the end there will be n pairs contracted in each term in the sum, giving $n!$ possible pair contractions.

- Each of the n pair contractions picks up a $\frac{\delta_{ij}}{\omega}$, however $(\delta_{ij})^n = \delta_{ij}$ since δ_{ij} is only equal to either 1 or 0. The number of ω factors is n, so we obtain $\frac{1}{\omega^n}$

- In the end we get

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^m >= \sum_{ij}^{N} n! \frac{\delta_{ij}}{\omega^n} \delta_{mn} = \frac{n! N}{\omega^n} \delta_{mn} \tag{6}$$

Equation (6) contains only an N, indicating the N dependence of the leading term. In this case there is no further sub-leading in N correction. this term will be reproduced as a quantum effect in collective field theory. Since this correlator contains two sums, the leading 'classical' contribution would have been of order N^2. It is not surprising that equation (6) is reproduced as a quantum effect in collective field theory. A more interesting correlator to study is

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^n \sum_{k=1}^{N} z_k^p \sum_{l=1}^{N} \bar{z}_l^p >$$

The leading contribution comes from

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^n >< \sum_{k=1}^{N} z_k^p \sum_{l=1}^{N} \bar{z}_l^p >= \frac{n! p! N^2}{\omega^{n+m}}$$

where we have used equation (6). Again this would not be reproduced by classical collective field theory. The size of the classical field theory contribution to this correlator is N^4. There is a sub-leading correction arising when some of the z_i^n's are contracted with the $\bar{z}_l^p$'s. The result for this sub-leading correction is

$$\frac{[(n+p)! - n!p!]\,N}{\omega^{n+m}}$$

In total we have

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^n \sum_{k=1}^{N} z_k^p \sum_{l=1}^{N} \bar{z}_l^p >= \frac{p!n!N^2}{\omega^{n+p}} + \frac{N}{\omega^{n+p}}((n+p)! - n!p!)) \qquad (7)$$

2.3 Collective Field Theory

In order to demonstrate that the same result is obtained from the quantum collective field theory as in the previous section, some detail about the collective field method (based on [1]) will be included. The same initial Hamiltonian is used. It is useful to use z and $\bar{z}$ rather than x and y:

$$\begin{aligned} H &= -\frac{1}{2}\sum_{i=1}^{N}(\frac{\partial^2}{\partial {x_i}^2} + \frac{\partial^2}{\partial {y_i}^2}) + \frac{\omega^2}{2}\sum_{i=1}^{N}({x_i}^2 + {y_i}^2) \\ &= -2\sum_{i=1}^{N}\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i} + \frac{\omega^2}{2}\sum_{i=1}^{N} z_i\bar{z}_i \end{aligned}$$

The time independent Schrödinger equation of this system is

$$(-2\sum_{i=1}^{N}\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i} + \frac{\omega^2}{2}\sum_{i=1}^{N} z_i\bar{z}_i)\Psi(z_1, z_2, ..., z_n, \bar{z}_1, ..., \bar{z}_n) = E\Psi(z_1, z_2, ..., z_n, \bar{z}_1, ..., \bar{z}_n)$$

The quantum mechanical solution to this equation is some symmetric wave function $\Psi(z_1, z_2, ..., z_n, \bar{z}_1, ..., \bar{z}_n)$. The reason for this symmetry is that we are looking at a system of bosons. This can be rewritten as a functional

of the collective field ϕ:

$$\Psi(z_1, z_2, ..., z_n, \bar{z}_1, ..., \bar{z}_n) = \Phi\left[\phi\right]$$

The collective field is a density function of the original degrees of freedom of the system, and is given by

$$\phi(x, y) = \sum_{i=1}^{N} \delta(x - x_i)\delta(y - y_i)$$

This is a symmetric function and describes a collection of bosonic point particles in two dimensions. These collective field variables must satisfy the constraint equation

$$\int \phi(x, y)dx\, dy = N$$

This field can be defined in terms of its Fourier components:

$$\phi_k = \frac{1}{L^2}\int dx\, dy e^{-ik_x x - ik_y y}\phi(x, y) = \frac{1}{L^2}\sum_{i=1}^{N} e^{-ik_x x_i - ik_y y_i}$$

We then find ϕ in terms of z and $\bar{z}$, where

$$z = x + iy \qquad \bar{z} = x - iy$$

We also have

$$k = \frac{1}{2}(k_x + ik_y) \qquad \bar{k} = \frac{1}{2}(k_x - ik_y)$$

Then

$$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$$

$$k_x = k + \bar{k} \qquad k_y = \frac{k - \bar{k}}{i}$$

So ϕ can be written as

$$\phi_k = \frac{1}{L^2}\int dx\, dy e^{-ik\bar{z}} e^{-i\bar{k}z}\phi(x, y) = \frac{1}{L^2}\sum_{i=1}^{N} e^{-ik\bar{z}_i} e^{-i\bar{k}z_i} \tag{8}$$

The Hamiltonian after the change of variables to the collective field must be found. The identity used to change the Hamiltonian operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ into functional derivatives in terms of ϕ_k is the chain rule:

$$\begin{aligned}\frac{\partial}{\partial z_i}\Psi(z_1,z_2,...,z_n,\bar{z}_1,...,\bar{z}_n) &= \sum_k(\frac{-i\bar{k}}{L^2})e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_{\bar{k}}}\Phi[\phi]\\ \frac{\partial}{\partial \bar{z}_i}\Psi(z_1,z_2,...,z_n,\bar{z}_1,...,\bar{z}_n) &= \sum_{k'}(\frac{-ik}{L^2})e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_{k}}\Phi[\phi]\end{aligned}$$

So the kinetic term of the Hamiltonian can be found using this chain rule:

$$\begin{aligned}-2\sum_i\partial_{z_i}\partial_{\bar{z}_i} &= -2\left(\sum_k(\frac{-ik}{L^2})e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_k}\right)\left(\sum_{\bar{k}}(\frac{-i\bar{k}}{L^2})e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_{\bar{k}}}\right)\\ &= -2\left(\sum_k\frac{(-ik)^2}{L^2}\delta_{k\bar{k}}e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_k}+\sum_{k,\bar{k}}\frac{(-ik)(-i\bar{k})}{L^2}e^{-ik\bar{z}}e^{-i\bar{k}z}\frac{\partial}{\partial\phi_k}\frac{\partial}{\partial\phi_{\bar{k}}}\right)\end{aligned}$$

The following operators are defined to simplify this equation:

$$\omega(k;[\phi]) = -k^2\phi_k \tag{9}$$

$$\Omega(k,\bar{k};[\phi]) = \frac{k\bar{k}}{L}\phi_{k-\bar{k}} \tag{10}$$

$$\pi_{-k} = -i\frac{\partial}{\partial\phi_k} \tag{11}$$

After the change of variables the kinetic term of the Hamiltonian is

$$H_{Kinetic} = 2i\sum_k\omega(k;[\phi])\pi_{-k}+2\sum_{k,\bar{k}}\Omega(k,\bar{k};[\phi])\pi_{-k}\pi_{\bar{k}} \tag{12}$$

This term is not hermitian under the conditions $\phi_k^\dagger=\phi_{-k}$ and $\pi_k^\dagger=\pi_{-k}$ required for the full Hamiltonian to be hermitian. This is easily shown:

$$\begin{aligned}\left[2i\sum_k\omega(k;[\phi])\pi_{-k}\right]^\dagger &= \left[2i\sum_k k^2(\pi_{-k})^\dagger(\phi_k)^\dagger\right]\\ &= \left[2i\sum_k k^2\left(-i\frac{\partial}{\partial\phi_{-k}}\phi_{-k}\right)\right]\end{aligned}$$

$$\begin{aligned} &= 2\delta_{k\bar{k}}\sum_k k^2 + 2i\sum_k \omega(k,[\phi])\pi_{-k} \\ &\neq \left[2i\sum_k \omega(k;[\phi])\pi_{-k}\right] \end{aligned}$$

And:

$$\begin{aligned} \left[2\sum_{k,\bar{k}}\Omega(k,\bar{k};[\phi])\pi_{-k}\pi_{\bar{k}}\right]^\dagger &= \left[2\sum_{k,\bar{k}}\frac{k\bar{k}}{L}\pi^\dagger_{\bar{k}}\pi^\dagger_{-k}\phi^\dagger_{\bar{k}-k}\right] \\ &= -2\sum_{k,\bar{k}}\frac{k\bar{k}}{L}\frac{\partial}{\partial\phi_{\bar{k}}}\delta_{k-\bar{k},-k} - 2\sum_{k,\bar{k}}\frac{k\bar{k}}{L}\frac{\partial}{\partial\phi_k}\delta_{k-\bar{k},\bar{k}} \\ &- 2\sum_{k,\bar{k}}\frac{k\bar{k}}{L}\phi_{k-\bar{k}}\left(\frac{\partial}{\partial\phi_{\bar{k}}}\frac{\partial}{\partial\phi_{-k}}\right) \\ &\neq \left[2\sum_{k,\bar{k}}\Omega(k,\bar{k};[\phi])\pi_{-k}\pi_{\bar{k}}\right] \end{aligned}$$

This apparent loss of hermiticity in the Hamiltonian is not actually indicative of a non-hermitian system, since the initial quantum mechanical Hamiltonian was hermitian; rather it is merely the effect of the change of variables from the quantum theory to the collective field theory. The hermiticity of the collective field theory can be restored by finding the Jacobian of the change of variables of the system. Using the identity,

$$\int\cdots\int\prod_{k\neq 0}d\phi_k\delta\left(\phi_k - \frac{1}{L^2}\sum_{i=1}^N e^{-ik\bar{z}_i}e^{-i\bar{k}z_i}\right) = 1 \tag{13}$$

which is found using the definition of the collective field ϕ, the Jacobian of the change of variables can be defined as

$$J[\phi] = \int\cdots\int\prod_{i=1}^N dx_i\prod_{k\neq 0}\delta\left(\phi_k - \frac{1}{L^2}\sum_{i=1}^N e^{-ik\bar{z}_i}e^{-i\bar{k}z_i}\right) \tag{14}$$

This Jacobian is used to redefine the inner product of the theory according to the changed variables as follows:

$$(\psi_1,\psi_2)_{Before} = \int dx_1\ldots dx_N\psi_1^*(x_1,\ldots,x_N)\psi_2(x_1,\ldots,x_N)$$

$$\rightarrow (\psi_1, \psi_2)_{After} \;=\; \int \cdots \int \prod_{k\neq 0} d\phi_k J\left[\phi\right] \Phi_1^*\left[\phi\right] \Phi_2\left[\phi\right]$$

The Jacobian must be found to correctly define the new integration measure. In the above integral the Jacobian arises from the transformation of the coordinate system. Rescale the collective field operators under this same transformation. This is done to remove J from the inner product. The transformation of fields used to obtain an equation that can be solved for the Jacobian is:

$$\Phi\left[\phi\right] \;=\; J^{-\frac{1}{2}}\left[\phi\right] \Psi\left[\phi\right] \tag{15}$$

$$\pi_k \;\rightarrow\; J^{\frac{1}{2}}\left[\phi\right] \pi_k J^{-\frac{1}{2}}\left[\phi\right] = \pi_k + \frac{1}{2} i \frac{\partial \ln J\left[\phi\right]}{\partial \phi_k} \tag{16}$$

The inner product is now trivial and so the Hamiltonian should be manifestly hermittian. By assuming that this transformation of variables makes the Hamiltonian manifestly hermitian once again and substituting the transformation (16) into (12), it is possible to obtain an expression for $\frac{\partial \ln J[\phi]}{\partial \phi_k}$. The substitution obtains an expression for the kinetic term of the effective collective field Hamiltonian:

$$\begin{aligned} H \;&=\; \sum_k (-i2k^2 \phi_k \pi_{-k} + 2k^2 \phi_k \frac{\partial \ln J}{\partial \phi_{-k}}) + \sum_{k,\bar{k}} (2\frac{k\bar{k}}{L^2} \phi_{k-\bar{k}} \pi_{-k} \pi_{\bar{k}} \\ &+\; i\frac{k\bar{k}}{L^2} \phi_{k-\bar{k}} \pi_{-k} \frac{\partial \ln J}{\partial \phi_{\bar{k}}} + i\frac{k\bar{k}}{L^2} \phi_{k-\bar{k}} \frac{\partial \ln J}{\partial \phi_{-k}} \pi_{\bar{k}} - \frac{1}{2}\frac{k\bar{k}}{L^2} \phi_{k-\bar{k}} \frac{\partial \ln J}{\partial \phi_{-k}} \frac{\partial \ln J}{\partial \phi_{\bar{k}}}) \end{aligned}$$

Using the hermiticity conditions $\phi_k^\dagger = \phi_{-k}$ and $\pi_k^\dagger = \pi_{-k}$, the hermitian conjugate of H is found:

$$\begin{aligned} H^\dagger \;&=\; \sum_k (i2k^2 \pi_k \phi_{-k} + 2k^2 \frac{\partial \ln J}{\partial \phi_k} \phi_{-k}) + \sum_{k,\bar{k}} (2\frac{k\bar{k}}{L} \pi_{-\bar{k}} \pi_k \phi_{\bar{k}-k} \\ &+\; -i\frac{k\bar{k}}{L^2} \frac{\partial \ln J}{\partial \phi_{-\bar{k}}} \pi_k \phi_{\bar{k}-k} - i\frac{k\bar{k}}{L^2} \pi_{-\bar{k}} \frac{\partial \ln J}{\partial \phi_k} \phi_{\bar{k}-k} - \frac{1}{2}\frac{k\bar{k}}{L^2} \frac{\partial \ln J}{\partial \phi_{-\bar{k}}} \frac{\partial \ln J}{\partial \phi_k} \phi_{\bar{k}-k}) \end{aligned}$$

This is followed by subtracting H from $H^\dagger$ and allowing the π's to operate when appropriate, which yields:

$$\begin{aligned} H - H^\dagger &= \sum_{k,\bar{k}} (2i\omega\pi_{-k} + 2\Omega\pi_{-k}\pi_{\bar{k}} + 2\Omega \frac{\partial}{\partial\phi_k}\frac{\partial}{\partial\phi_{\bar{k}}} \ln J \\ &+ \frac{k\bar{k}}{L^2}\frac{\partial \ln J}{\partial\phi_{-\bar{k}}}\delta_{k,k-\bar{k}} + \frac{k\bar{k}}{L^2}\frac{\partial \ln J}{\partial' \phi_k}\delta_{\bar{k},\bar{k}-k}) = 0 \end{aligned}$$

The last two terms cancel each other out after the sum over δ is performed. After some simplification we have the following equation, from which $\frac{\partial \ln J[\phi]}{\partial\phi_k}$ is obtained:

$$0 = \omega(k;[\phi]) + \sum_{k'\neq 0} \Omega(k,k';[\phi]) \frac{\partial \ln J\,[\phi]}{\partial\phi_k} \tag{17}$$

$$\frac{\partial \ln J\,[\phi]}{\partial\phi_k} = -\sum_{k'} \Omega^{-1}(k,k';[\phi]) k'^2 \phi_{k'} \tag{18}$$

The method we used to obtain this differential equation for the Jacobian is completely general, but the equation we obtained is not. In general there is also a $\frac{\partial\Omega}{\partial\phi}$ term in this equation, see [1] for the general result.

The substitution of (18) into (16) followed by (16) into the kinetic Hamiltonian, (12), along with the addition of the Fourier transformed harmonic oscillator potential term, finally leads to the full hermitean Hamiltonian of the collective field theory:

$$\begin{aligned} H &= \sum_{kk'} \frac{\omega^2}{2} z\bar{z}\phi_{k-k'} + 2i\sum_k \left(\omega(k;[\phi])(\pi_{-k} - \frac{i}{2}\Omega^{-1}(k,k';[\phi])k'^2\phi_{-k'})\right) \\ &+ 2\sum_{k,k'} \left(\Omega(k,k';[\phi])(\pi_{-k} - \frac{i}{2}\Omega^{-1}(k,k';[\phi])k'^2\phi_{-k'})(\pi_k - \frac{i}{2}\Omega^{-1}(k,k';[\phi])k'^2\phi_{k'})\right) \\ &= \sum_{kk'} \frac{\omega^2}{2} z\bar{z}\phi_{k-k'} + 2i\sum_k \left(\omega\pi_{-k} + \frac{i}{2}\omega\Omega^{-1}k'^2\phi_{-k'}\right) + 2\sum_{kk'}(\Omega\pi_{-k}\pi_k + \frac{i}{2}\Omega(\pi_{-k}\Omega^{-1})k'^2\phi_{k'} \end{aligned}$$

$$
\begin{aligned}
- & \frac{i}{2}\Omega\Omega^{-1}\pi_{-k}(k'^2\phi_{k'}) + \frac{1}{2}\Omega\Omega^{-1}k'^2\phi_{-k}\pi_k - \frac{1}{4}k'^2\phi_{-k'}\Omega^{-1}k'^2\phi_{k'}) \\
= & \sum_{kk'}\left(\pi_{k'}\Omega(k,k';[\phi])\pi_k + \frac{1}{2}\omega(-k;[\phi])\Omega^{-1}(k,k';[\phi])\omega(k';[\phi])\right) + \sum_{kk'}\frac{\omega^2}{2}z\bar{z}\phi_{k-k'}
\end{aligned}
$$

The collective field hamiltonian can be written in integral form using the Fourier transforms of ϕ and π:

$$
\begin{aligned}
\phi(z,\bar{z}) &= \frac{1}{L}\sum_{k\neq 0} e^{ik\bar{z}}e^{i\bar{k}z}\phi_k \\
\pi(z,\bar{z}) &= \frac{1}{L}\sum_{k\neq 0} e^{ik\bar{z}}e^{i\bar{k}z}\pi_k
\end{aligned}
$$

so $\partial_z\pi(z,\bar{z}) = i\bar{k}\pi_{\bar{k}}$ and $\partial_{\bar{z}}\pi(z,\bar{z}) = -ik\pi_k$. Similarly, ϕ can be found in terms of $\partial_z\phi(z,\bar{z})$ and $\partial_{\bar{z}}\phi(z,\bar{z})$. We do not include $k=0$ terms in our theory. This is because all of the non zero momentum modes integrate to zero in a box; in this case the box is size L in the x and y directions respectively. Using this and the expansion $\phi \to \phi_0 + \phi$, with $\int \phi_0 dx = N$, it can be shown that the $k=0$ mode is zero. The effective collective field Hamiltonian in integral form:

$$
H = \iint dzd\bar{z}\left(2\partial_z\pi\phi\partial_{\bar{z}}\pi + \frac{1}{2}\frac{\partial_z\phi\partial_{\bar{z}}\phi}{\phi} + \frac{1}{2}\omega^2 z\bar{z}\phi\right) \tag{19}
$$

In this effective Hamiltonian, we define $\mathcal{V}[\phi]$ to be the second two terms in the collective field Hamiltonian, giving

$$
H = \iint dzd\bar{z}\; 2\partial_z\pi\phi\partial_{\bar{z}}\pi + \mathcal{V}[\phi] \tag{20}
$$

2.3.1 Classical Solution to the Collective Field Hamiltonian

To find a collective field, ϕ, that satisfies this theory, we expand the potential about some minimum field ϕ_0, which is a solution to

$$
\frac{\delta\mathcal{V}[\phi]}{\delta\phi(z,\bar{z})} + \lambda = 0 \tag{21}
$$

Here, λ is a Lagrange multiplier used to implement the constraint $\int \phi(x)dx = N$. To minimize the potential, its functional derivative is taken and a trial solution is placed in the above equation.

$$\begin{aligned}
\mathcal{V}[\phi] &= \iint dzd\bar{z}\left(\frac{1}{2}\frac{\partial_{\bar{z}}\phi\partial_z\phi}{\phi} + \frac{1}{2}\omega^2 z\bar{z}\phi\right) \\
\frac{\delta\mathcal{V}[\phi]}{\delta\phi} &= \iint dzd\bar{z}(-\frac{1}{2}\frac{\partial_{\bar{z}}\phi\partial_z\phi}{\phi^2}\delta(z-z')\delta(\bar{z}-\bar{z}')) + \frac{1}{2}\frac{\partial_{\bar{z}}(\delta(z-z')\delta(\bar{z}-\bar{z}')\partial_z\phi}{\phi} \\
&\quad + \frac{1}{2}\frac{\partial_{\bar{z}}\phi\partial_z(\delta(z-z')\delta(\bar{z}-\bar{z}'))}{\phi} + \frac{1}{2}\omega^2 z\bar{z}\delta(z-z')\delta(\bar{z}-\bar{z}')) \\
&= \iint dzd\bar{z}\left[-\frac{1}{2}\frac{\partial_{\bar{z}}\phi\partial_z\phi}{\phi^2} - \frac{1}{2}\partial_{\bar{z}}\left(\frac{\partial_z\phi}{\phi}\right) - \frac{1}{2}\partial_z\left(\frac{\partial_{\bar{z}}\phi}{\phi}\right) + \frac{1}{2}\omega^2 z\bar{z}\right]\delta(z-z')\delta(\bar{z}-\bar{z}')
\end{aligned}$$

After integrating over both z and $\bar{z}$, and relabeling the variables, we obtain the following equation:

$$-\frac{1}{2}\frac{\partial_{\bar{z}}\phi}{\phi^2} - \frac{1}{2}2\frac{\partial_z\partial_{\bar{z}}\phi}{\phi} + \frac{1}{2}\frac{\partial_z\phi\partial_{\bar{z}}\phi}{\phi^2} + \frac{1}{2}\omega^2 z\bar{z} + \lambda = 0$$

$$\Rightarrow \frac{\partial_z\partial_{\bar{z}}\phi}{\phi} = \frac{1}{2}\omega^2 z\bar{z} + \lambda$$

A trial solution

$$\phi = \mathcal{C}e^{-\alpha z\bar{z}} \tag{22}$$

is substituted into this equation, where $\mathcal{C}$ is a normalization constant. This enables us to find values for the Lagrange Multiplier, λ and the minimized classical field, ϕ_0. $\mathcal{C}$ is just a Gaussian normalization constant multiplied by N, the number of fields. Thus we get:

$$\lambda = -\frac{\alpha}{\sqrt{2}}$$

and

$$\phi_0 = \frac{N\omega}{\pi}e^{-\omega z\bar{z}} \tag{23}$$

.

2.4 Quantum Expansion of the Classical Collective Field

The quantum effects of the N Harmonic Oscillator system are found by expanding the collective field around the classical minimum, ϕ_0, found in the previous section. The collective field is expanded to

$$\phi = \phi_0 + \eta\sqrt{\phi_0} \tag{24}$$

The momentum term is then expanded as

$$\Pi = 0 + \frac{1}{\sqrt{\phi_0}}\tilde{\Pi} \tag{25}$$

The factors of $\sqrt{\phi_0}$ are chosen in order to preserve the commutation relation

$$[\phi(z,\bar{z}),\Pi(z,\bar{z})] = i\delta(z-z')\delta(\bar{z}-\bar{z}') \tag{26}$$

after the expansion, giving

$$\left[\eta(z,\bar{z}),\tilde{\Pi}(z,\bar{z})\right] = i\delta(z-z')\delta(\bar{z}-\bar{z}') \tag{27}$$

In order to expand η and $\tilde{\Pi}$ in terms of a set of modes, the quadratic part of the Hamiltonian describing the behavior of these quantities is found. This is done by substituting the expanded field and momentum terms into the collective field Hamiltonian and dropping all terms linear in either η or $\tilde{\Pi}$. Any terms mixing η and $\tilde{\Pi}$ are thus dropped. The remaining terms can then be integrated by parts in order to find a simplified quadratic Hamiltonian for the expanded variables. A few steps of this calculation are given. The Hamiltonian, equation (19), becomes:

$$H = \iint dzd\bar{z}2\partial_{\bar{z}}\left(\frac{\tilde{\Pi}}{\sqrt{\phi_0}}\right)\left(\phi_0+\eta\sqrt{\phi_0}\right)\partial_z\left(\frac{\tilde{\Pi}}{\sqrt{\phi_0}}\right)$$

$$
\begin{aligned}
&+\iint dzd\bar{z}\frac{1}{2}\left[\frac{\partial_{\bar{z}}(\phi_0+\eta\sqrt{\phi_0})\partial_z(\phi_0+\eta\sqrt{\phi_0})}{\phi_0+\eta\sqrt{\phi_0}}+\omega^2 z\bar{z}\right]\\
=&\iint dzd\bar{z}\left[2\partial_{\bar{z}}\left(\frac{\tilde{\Pi}}{\sqrt{\phi_0}}\right)\phi_0\partial_z\left(\frac{\tilde{\Pi}}{\sqrt{\phi_0}}\right)+\frac{1}{2}\frac{\partial_{\bar{z}}(\eta\sqrt{\phi_0})\partial_z(\eta\sqrt{\phi_0})}{\phi_0}\right]\\
=&\iint dzd\bar{z}2\left[\partial_{\bar{z}}\tilde{\Pi}\partial_z\tilde{\Pi}+\frac{\omega\bar{z}}{2}(\partial_{\bar{z}}\tilde{\Pi})\tilde{\Pi}+\frac{\omega z}{2}\tilde{\Pi}(\partial_z\tilde{\Pi})+\frac{\omega^2 z\bar{z}}{4}\tilde{\Pi}^2\right]\\
&+\iint dzd\bar{z}\frac{1}{2}\left[\partial_{\bar{z}}\eta\partial_z\eta-\frac{\omega\bar{z}}{2}(\partial_{\bar{z}}\eta)\eta-\frac{\omega z}{2}\eta(\partial_z\eta)+\frac{\omega^2 z\bar{z}}{4}\eta^2\right]\\
=&\iint dzd\bar{z}\left(\tilde{\Pi}(-2\partial_{\bar{z}}\partial_z+\frac{1}{2}\omega^2 z\bar{z}-\omega)\tilde{\Pi}+\eta(-2\partial_{\bar{z}}\partial_z+\frac{1}{2}\omega^2 z\bar{z}-\omega)\eta\right)
\end{aligned}
$$

We may then expand the expressions for η and $\tilde{\Pi}$ in terms of Harmonic oscillator eigenfunctions

$$< z,\bar{z}|J >= \psi_J = \sqrt{\frac{\omega^{J+1}}{\pi J!}}z^J e^{-\frac{\omega}{2}z\bar{z}} \tag{28}$$

$$< J|z,\bar{z} >= \bar{\psi}_J = \sqrt{\frac{\omega^{J+1}}{\pi J!}}\bar{z}^J e^{-\frac{\omega}{2}z\bar{z}} \tag{29}$$

These eigenfuntions satisfy the quadratic collective field equations of motion which are as follows:

$$(-4\partial_{\bar{z}}\partial_z+\omega^2 z\bar{z})\psi_J = 2(J+1)\omega\psi_J \tag{30}$$

$$(-4\partial_{\bar{z}}\partial_z+\omega^2 z\bar{z})\bar{\psi}_J = 2(J+1)\omega\bar{\psi}_J \tag{31}$$

This is not a complete set of eigenfunctions. Indeed, the eigenfunctions of the two dimensional harmonic oscillator are labelled by two discrete quantum numbers, which each range from 0 to ∞. Our eigenfunctions are labelled by a single discrete index. We are describing only the holomorphic sector of the theory.

We now expand:

$$\eta = \sum_{J=1}^{\infty} \left(\bar{c}_J \psi_J + c_J \bar{\psi}_J \right) \qquad \tilde{\Pi} = \sum_{J=1}^{\infty} \left(p_J \psi_J + \bar{p}_J \bar{\psi}_J \right) \tag{32}$$

In order to find the properties of $c_J, \bar{c}_J, p_J$ and $\bar{p}_J$, the orthogonality relation $< J|K >= \delta_{JK}$ is proved, using $z = re^{i\phi}$:

$$\begin{aligned} < J|K > &= \iint dz d\bar{z} \sqrt{\frac{\omega^{J+1}}{\pi J!}} \bar{z}^J \sqrt{\frac{\omega^{K+1}}{\pi K!}} z^K e^{-\omega z \bar{z}} \\ &= \int_0^{\infty} r\, dr \int_0^{2\pi} d\phi \sqrt{\frac{\omega J + K + 2}{\pi^2 J! K!}} r^{J+K} e^{i\phi(K-J)} e^{-\omega r^2} \end{aligned}$$

Integrating over ϕ obtains $< J|K >= 0$, thus the integral is only non zero if $J = K$. In this case, the ϕ integral yields a factor of 2π. So now we have:

$$\begin{aligned} < J|J > &= \int_0^{\infty} r\, dr \int_0^{2\pi} d\phi \sqrt{\frac{\omega^{2J+2}}{\pi (J!)^2}} r^{2J} e^{-\omega r^2} \\ &= \frac{2\omega^{J+1}}{J!} \int_0^{\infty} r\, dr\, r^{2J} \end{aligned}$$

Now set $x = \omega r^2$:

$$\begin{aligned} \Rightarrow \qquad < J|J > &= \frac{1}{J!} \int_0^{\infty} dx\, x^J\, e^{-x} \\ &= \frac{1}{J!} \Gamma(J+1) \\ &= \frac{J!}{J!} = 1 \\ &\Rightarrow \; < J|K >= \iint dz d\bar{z}\, \bar{\psi}_J\, \psi_K = \delta_{JK} \end{aligned}$$

Using this equation we find the c and p operators:

$$c_J = \iint dz d\bar{z}\, \psi_J \eta \qquad \bar{c}_J = \iint dz d\bar{z}\, \bar{\psi}_J \eta \tag{33}$$

$$p_J = \iint dz d\bar{z}\, \psi_J \tilde{\Pi} \qquad \bar{p}_J = \iint dz d\bar{z}\, \bar{\psi}_J \tilde{\Pi} \tag{34}$$

The equations for these operators in terms of η and $\tilde{\Pi}$ allow us to find their commutation relations. Using $\left[\eta(z,\bar{z}),\tilde{\Pi}(z,\bar{z})\right] = i\delta(z-z')\delta(\bar{z}-\bar{z}')$, we get:

$$[c_J, \bar{c}_K] = 0 \qquad [p_J, \bar{p}_K] = 0 \tag{35}$$

$$[c_J, \bar{p}_K] = i\delta_{JK} \qquad [\bar{c}_J, p_K] = i\delta_{JK} \tag{36}$$

2.5 Correlators In the Collective Field Theory

The collective field Hamiltonian has now been expanded around the classical field solution to the N body system, and quantum corrections to the model have been examined. This enables us to calculate the same correlators as those found using quantum mechanics. We shall calculate $< \sum_i z_i^n \sum_j \bar{z}_j^m >$ in the collective field theory and compare the result to the quantum mechanical result.

In collective field theory, the correlator has an integral form:

$$< \sum_{i=1}^{N} z_i^n \sum_{j=1}^{N} \bar{z}_j^m > \rightarrow \iiiint dz d\bar{z} dw d\bar{w} < \phi(z,\bar{z})\, \phi(w,\bar{w})\, z^n\, \bar{w}^m >$$

We have an integral rather than a sum because we are no longer dealing with a discrete set of particles, rather the continuous field describing their collective motion. This correlator is in fact the sum of the classical and quantum correlators, obtained by substituting $\phi = \phi_0 + \eta\sqrt{\phi_0}$ and keeping only the appropriate quadratic terms. In total, we're calculating:

$$\iiiint dz d\bar{z} dw d\bar{w} < \phi_0\, \phi_0\, z^n\, \bar{w}^m > + \iiiint dz d\bar{z} dw d\bar{w} < \eta\, \sqrt{\phi_0}\, \eta\, \sqrt{\phi_0}\, z^n\, \bar{w}^m > \tag{37}$$

2.5.1 The Classical Field Correlator

First the classical part of the correlator is calculated, using $z = re^{i\phi}$ and $w = se^{i\theta}$:

$$\begin{aligned} & \iiiint dz d\bar{z} dw d\bar{w} < \phi_0 \, \phi_0 \, z^n \, \bar{w}^m > \\ = \; & \frac{N^2\omega^2}{\pi^2} \iint dz d\bar{z} < e^{-\omega z \bar{z}} \, z^n > \iint dw d\bar{w} < e^{-\omega w \bar{w}} \bar{w}^m > \\ = \; & \frac{N^2\omega^2}{\pi^2} \int_0^\infty r \, dr \int_0^{2\pi} d\phi \, r^n e^{i\phi n} \, e^{-\omega r^2} \int_0^\infty s \, ds \int_0^{2\pi} d\theta \, s^m e^{-i\phi m} \, e^{-\omega s^2} \\ = \; & 0 \end{aligned}$$

This is because when we integrate over ϕ or θ we get zero. This is the same result that was obtained to calculate the correlator using quantum mechanics.

2.5.2 Quantum Corrections to the Classical Correlator

In order to determine whether collective field theory and quantum mechanical methods are in exact agreement, the collectve field theory must also produce the correct quantum corrections to the system. The second term in equation (21) is now calculated.

$$\begin{aligned} & \iiiint dz d\bar{z} dw d\bar{w} < \eta(z,\bar{z}) \, \sqrt{\phi_0} \, \eta(w,\bar{w}) \, \sqrt{\phi_0} \, z^n \, \bar{w}^m > \\ = \; & \iiiint dz d\bar{z} dw d\bar{w} < \sqrt{\phi_0(z,\bar{z})} \, \sqrt{\phi_0(w,\bar{w})} \sum_{J=1}^{\infty} (c_J \bar{\psi}_J + \bar{c}_J \psi_J) \sum_{K=1}^{\infty} (c_K \bar{\psi}_K + \bar{c}_K \psi_K) z^n \bar{w}^m > \end{aligned}$$

Using the orthogonality relation $\iint dz d\bar{z} \; \bar{\psi}_J \, \psi_K = \delta_{JK}$, we find that after multiplying out the η sums, only the terms containing combinations of both ψ and $\bar{\psi}$ remain non zero.

$$\Rightarrow \quad \iiiint dz d\bar{z} dw d\bar{w} \frac{N\omega}{\pi} e^{-\frac{\omega}{2}(z\bar{z}+w\bar{w})} \, z^n \, \bar{w}^m < \sum_{J,K} (c_J \bar{c}_K \bar{\psi}_J \psi_K + \bar{c}_J c_K \psi_J \bar{\psi}_K >$$

The c operators are defined in terms of x and y, which consist of creation and annihilation operators, as seen in equations (2) and (3). We use the commutation relations and Wick contractions in the same manner as in section 1.1, and define the c's in such a way that their expectations yield the following results:

$$< c_J \, \bar{c}_k >= \delta_{JK} \qquad < \bar{c}_J \, c_k >= \delta_{JK} \tag{38}$$

$$< c_J \, c_K >= 0 \qquad < \bar{c}_J \, \bar{c}_K >= 0 \tag{39}$$

Thus we have

$$c_J = \sqrt{\omega}(x_J + iy_J) \tag{40}$$

$$\bar{c}_J = \sqrt{\omega}(x_J - iy_J) \tag{41}$$

where

$$x_J = \frac{a_J + a_J^\dagger}{\omega\sqrt{2}} \tag{42}$$

$$y_J = \frac{b_J + b_J^\dagger}{\omega\sqrt{2}} \tag{43}$$

We now have:

$$\iiiint dz d\bar{z} dw d\bar{w} \frac{N\omega}{\pi} e^{-\frac{\omega}{2}(z\bar{z}+w\bar{w})} z^n \bar{w}^n \sum_{J,K} \delta_{JK}(\bar{\psi}_J \psi_K + \psi_J \bar{\psi}_K)$$

$$= \iiiint dz d\bar{z} dw d\bar{w} \frac{N\omega}{\pi} e^{-\frac{\omega}{2}(z\bar{z}+w\bar{w})} z^n \bar{w}^n \sum_J \sqrt{\frac{\omega^{2J+2}}{\pi^2 J!^2}}(\bar{z}^J w^J + z^J \bar{w}^J) e^{-\frac{\omega}{2}(z\bar{z}+w\bar{w})}$$

The term containing the product $\bar{w}^m \bar{w}^J$ disappears, again due to the integral over ϕ shown in the orthogonality relation of the ψ's. So the only term in the integral which must be evaluated is:

$$\sum_J \left(\frac{N\omega}{\pi} \frac{\omega^{J+1}}{\pi J!} \iint dz d\bar{z} \, z^n \bar{z}^J e^{-\omega z\bar{z}} \iint dw d\bar{w} \, \bar{w}^m w^J e^{-\omega w\bar{w}} \right)$$

$$
\begin{aligned}
&= \sum_J \left(\frac{N\omega}{\pi} \frac{\omega^{J+1}}{\pi J!} \int_0^\infty r\, dr \int_0^{2\pi} d\phi r^{n+J} e^{i\phi(n-J)} e^{-\omega r^2} \int_0^\infty s\, ds \int_0^{2\pi} d\theta s^{m+J} e^{i\theta(J-m)} e^{-\omega s^2} \right) \\
&= \delta_{nm} \frac{N\omega}{\pi} \frac{\omega^{n+1}}{\pi n!} (2\pi)^2 \int_0^\infty dr\, r^{2n+1} e^{-\omega r^2} \int_0^\infty ds\, s^{2m+1} e^{-\omega s^2} \\
&= \frac{N\omega}{\pi} \frac{\omega^{n+1}}{\pi n!} (2\pi)^2 \left[\int_0^\infty r\, dr\, r^{2n} e^{-\omega r^2} \right]^2
\end{aligned}
$$

To evaluate this integral, set $x = \omega r^2$ once again. This gives:

$$
\begin{aligned}
& \frac{4N\omega^{n+2}}{n!} \left[\int_0^\infty \frac{dx}{2\omega} \left(\frac{x}{\omega} \right)^n e^{-x} \right]^2 \\
&= \frac{N}{n!\omega^n} \left[\int_0^\infty x^n e^{-x} dx \right]^2 \\
&= \frac{N\, n!}{\omega^n}
\end{aligned}
$$

2.6 Comparison

If we now compare the correlators calculated for the sum of the classical and quantum terms in the system, we find that for both Quantum Mechanical and Collective Field Theory methods we obtain the result:

$$< z^n \bar{z}^m >= \frac{\delta_{mn} N\, n!}{\omega^n} \tag{44}$$

This allows us to conclude that either of the two methods may be used to calculate correlators of quantum systems; where the quantum mechanics of an n-body system is too difficult to use it would be possible to use quantum collective field theory instead, yielding the correct result. This is valid for general systems with general potentials other than that of the harmonic oscillator.

The reader should note that complete equality between the quantum mechanical and collective field theory methods has not been demonstrated in

full generality. Indeed, all correlators we have studied belong to the holomorphic sector of the theory.

The correlation functions calculated here are specific to the Hamiltonian of the system that we are studying, and are shown to demonstrate that we are merely performing a change of variables rather than changing any physics. Correlation functions in this formalism are specific to the Hamiltonian describing the system.

3 The Large N Limit of Matrix Models

In the previous section we calculated the classical solution to the effective collective field Hamiltonian. This gave the large N dynamics of a system of N particles. We now discuss the large N limit in theories with matrix valued fields and argue that in these models $N \to \infty$ is a classical limit of these theories in the same way that $\hbar \to 0$ is a classical limit in quantum mechanics. The term matrix model is used to describe quantum field theories in which there are matrix valued fields. These matrices are of size $N \times N$, and it is to this N that we refer when talking about the large N limit. In [2], 't Hooft proposed that fluctuations in matrix model field theories should be of order $\mathcal{O}(\frac{1}{N^2})$. This effectively replaces quantum corrections of size $\mathcal{O}(\hbar)$. In standard quantum field theories $\hbar$ can be traded for the coupling constant which runs as the scale of the quantum field is changed. In contrast to this, $\frac{1}{N^2}$ does not run. When N, the size of these matrices, is taken to be large (compared to 1), the fluctuations approach zero and we are left with a classical valued theory. The AdS/CFT correspondence tells us that the classical theory that describes the large N limit of $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ is type IIB string theory on $AdS_5 \times S^5$, [4].

3.1 Correlation Functions and the Classical Limit

Once again, methods for calculating operator expectations or correlation functions in the theory are needed. In this chapter we will explain how to calculate correlators in the large N limit and show why this is a clasical limit.

3.1.1 Calculating Correlators

It is possible to show that the large N limit is actually a classical limit. In order to do so, we will first consider calculations in the single matrix model. Suppose we have some gauge invariant operator given by $\mathcal{O}$. For example, $\mathcal{O}$ may be the trace of a product of matrices. The expectation or correlation function of this operator can be calculated using the path integral method of QFT as follows:

$$< \mathcal{O} >= \int [dX]\, \mathcal{O}\, e^{-S}$$

In this integral we have

$$[dX] = \prod_{i=1}^{N} dX_{ii} \prod_{i<j} d\Re e(X_{ij}) d\Im m(X_{ij}) \tag{45}$$

The first product represents the integrals over diagonal elements of matrix X, and the second and third represent the integrals over the real and imaginary parts of the upper right triangular matrix elements. We do not need to include those elements below the diagonal in the lower left triangle of the matrix since we have a hermitian matrix, $X = X^{\dagger}$. There are N diagonal elements, and $\frac{N(N-1)}{2}$ of both real and imaginary upper diagonal elements, giving a total of N^2 integrals contained in $[dX]$.

In single matrix models, we can use the action $S = \frac{1}{2} tr(X^2)$. This is a simple choice because we can use Gaussian integration methods in order to evaluate $< \mathcal{O} >$. For operator $\mathcal{O}$ to represent a physical observable we need it to be a trace valued operator. This is because traced matrix operators are gauge symmetry invariant, where the gauge symmetry acts as

$$X \rightarrow U^{\dagger} X U$$

Here U is a unitary matrix, and invariant operators represent physical observables.

Suppose we have $< \mathcal{O} >=< tr(X^2) >$. Using the path integral method we have:

$$\begin{aligned} < tr(X^2) > &= \frac{\int [dX]\, X_{ij} X_{ji} e^{-\frac{1}{2} tr(X^2)}}{\int [dX]\, e^{-\frac{1}{2} tr(X^2)}} \\ &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{ij}} I[J] \Big|_{J=0} \end{aligned}$$

Here, $I[J]$ is the generating functional given by

$$I[J] = \frac{\int [dX]\, e^{-\frac{1}{2} tr(X^2)} e^{tr(JX)}}{\int [dX]\, e^{-\frac{1}{2} tr(X^2)}} \tag{46}$$

J is a source coupled to the field X. To simplify I we use the method of completing the square and then calculate the integrals

$$\int [dX]\, e^{-\frac{1}{2} tr(X^2)} = \sqrt{2\pi}^{N^2} \tag{47}$$

$$\int [dX]\, e^{-\frac{1}{2} tr(X^2)} e^{tr(JX)} = e^{\frac{1}{2} tr(J^2)} \sqrt{2\pi}^{N^2}$$

This gives us

$$I[J] = e^{\frac{1}{2} tr(J^2)}$$

This factor of $\sqrt{2\pi}^{N^2}$ cancels with the normalization factor in the original correlator, so we are left with the following calculation for the expectation value:

$$\begin{aligned} < tr(X^2) > &= \frac{d}{dJ_{ji}} \frac{d}{dJ_{ij}} I[J] \Big|_{J=0} \end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}}\left[e^{\frac{1}{2}tr(J^2)}\right]\Big|_{J=0} \\
&= \frac{d}{dJ_{ji}}\left[J_{ji}e^{\frac{1}{2}tr(J^2)}\right]\Big|_{J=0} \\
&= \delta_{jj}\delta_{ii} \\
&= N^2
\end{aligned}
$$

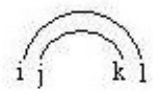
In general quantum field theory, the value of a correlation function is the classical value of the operator with quantum fluctuations added on. This is given by something of the form

$$\int [dX]\,\mathcal{O}e^{-S} = \mathcal{O}_{Classical} + (\#)\hbar + \cdots \tag{48}$$

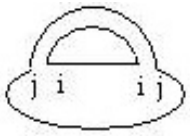
When we say that the large N limit is a classical limit with $\frac{1}{N^2}$ playing the role of $\hbar$, we mean

$$\int [dX]\,\mathcal{O}e^{-S} = \mathcal{O}_{Classical} + (\#)\frac{1}{N^2} + \cdots \tag{49}$$

To see that fluctuations are of order $\frac{1}{N^2}$ we need to consider a more complicated observable. Before doing that we will review a simpler graphical method for computing correlators, known as the ribbon diagram method. The indices of the operators are written as points and then connected in all possible ways in which the 'ribbons' do not twist. Suppose we have $< \mathcal{O} >=< X_{ij}X_{kl} >$. We write the pairs of indices as they appear in this correlator, and then join them as follows:



For a trace valued operator we would do the same thing, and also join equal indices that are traced in the definition of the operator:

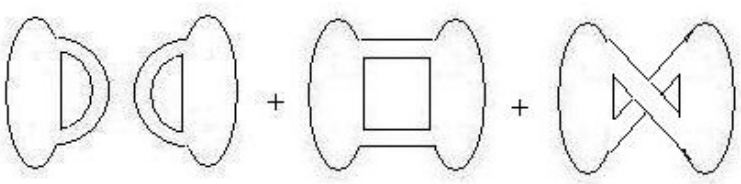


Here, each loop has a value of N, so in this case we have the correlator equal to N^2, the same as what was obtained above. When we draw the ribbons, they are not allowed to twist. Also, for a gauge invariant operator all lines must be closed loops so that any given diagram is equal to N raised to some power.

3.1.2 The Classical Limit

Now that it has been shown how to calculate correlation functions in matrix models, it is possible to show that the large N limit is a classical limit. An example will be presented, followed by a general proof. It should be noted that the large N limit is the same as the planar limit as long as we consider operators built out of $\mathcal{O}(1)$ matrices. We assume this is the case for the following discussion. Ribbon diagrams for trace valued operators which can be drawn on a planar surface, for example the surface of a sphere, are known as planar diagrams, and correspond to the classical piece of the operator. Fluctuations in the theory are represented by ribbon graphs which are not of leading order and cannot be drawn without crossing ribbons on planar surfaces. Instead, they live on toruses with one or more handles.

Suppose we wish to calculate the expectation $< tr(X^2)tr(X^2) >$. This is simplest using ribbon diagrams. To do this, draw dots corresponding to the indices of each matrix in the operator and connect each pair of dots with like indices. Next connect all possible remaining pairs of dots (indices) without twisting the ribbons. The diagrams obtained are summed. We get $< X_{ij}X_{ji}X_{kl}X_{lk} >=$



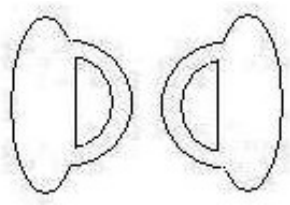
By counting the number of closed loops in these diagrams we get

$$\begin{aligned} < tr(X^2)tr(X^2) > &= N^4 + 2N^2 \\ &= N^4(1 + 2 \times \frac{1}{N^2}) \end{aligned}$$

Clearly, if we let N become very large we get

$$< tr(X^2)tr(X^2) >= N^4$$

If we now calculate the product of the expectations, $< tr(X^2) >< tr(X^2) >$, we get the following diagram:



Here there are four closed loops, giving N^4. We include this example to show that in the large N limit the expectation of a product of two operators is the same as the product of their expectations. So at large N

$$< tr(X^2)tr(X^2) >=< tr(X^2) >< tr(X^2) >$$

This is a rather general conclusion, and for a collection of n operators at large N we have:

$$< \mathcal{O}_1\mathcal{O}_2 \ldots \mathcal{O}_n >=< \mathcal{O}_1 >< \mathcal{O}_2 > \ldots < \mathcal{O}_n > \tag{50}$$

This formula assumes that n is $\mathcal{O}(1)$ and the number of X's in each operator is $\mathcal{O}(1)$. The expectation of an operator can be written as follows:

$$< \mathcal{O}_1 >= \sum_i \mu_i \mathcal{O}_1(i)$$

Here i is the index running over possible values of matrix $X = X(i)$, μ_i is the probability that X has the value $X(i)$ and $\mathcal{O}_1(i)$ is the value of operator $\mathcal{O}_1$ when $X = X(i)$. We also have $\sum_i \mu_i = 1$.

We now write the equality of the expectation of the product of operators and the product of their expectations using this notation:

$$\sum_i \mu_i \mathcal{O}_1(i)\mathcal{O}_2(i)\cdots\mathcal{O}_n(i) = \sum_{i_1} \mu_{i_1}\mathcal{O}_1(i_1)\sum_{i_2}\mu_{i_2}\mathcal{O}_2(i_2)\cdots\sum_{i_n}\mu_{i_n}\mathcal{O}_n(i_n) \quad (51)$$

This equation is only satisfied if we set one of the probabilities, for example μ_{i^*} equal to 1, implying that $\mu_i = 0$ for all $i \neq i*$. This means that we have $X = X(i^*)$ with a probability of 1, so we know the state of the operator. If we recall the difference between quantum superpositions and classical states, we know that this system is in a classical state since the superposition of states must collapse into a single state if the above equation is to be satisfied. This means that the large N limit is a classical limit.

3.1.3 Matrix Models In the Classical Limit

We now wish to perform calculations of matrix model correlation functions in the classical or large N limit. In doing so we expect to obtain results that agree with the large N limit results obtained using either the ribbon diagram or path integral methods described previously. To find the classical solution to a theory, we minimize the action and solve the resulting equation. In the model we have been working with, we use the action $S = \frac{1}{2}tr(X^2)$. This is minimized as follows:

$$\frac{dS}{dX_{ij}} = 0$$

$$\begin{aligned}\frac{d}{dX_{ij}}\frac{1}{2}tr(X^2) &= X_{ji} \\ &= 0\end{aligned}$$

So we have the solution $X_{ji} = 0$. If we were to substitute this into our matrix valued operator and try to calculate some correlators, we would get zero. This is incongruous with the expectations we have calculated, and this is the downfall of using a matrix basis in the large N limit. So why doesn't this work?

To calculate the correlation function of some operator in some matrix field theory we use the path integral method. For some matrix valued operator,

$$< \mathcal{O} > \int [dX]\, \mathcal{O} e^{-S} = \mathcal{O}_{Classical} + (\#)\frac{1}{N^2} + \cdots$$

This is the sum of the classical value of the operator and fluctuations of order $\frac{1}{N^2}$. For each integral over a matrix element in the product $[dX]$, we pick up a fluctuation term. As discussed, $[dX]$ represents integrating over N^2 matrix elements. This means that we pick up N^2 fluctuation terms when calculting $< \mathcal{O} >$, as follows:

$$\begin{aligned}\int [dX]\, \mathcal{O} e^{-S} &= \mathcal{O}_{Classical} + (N^2)\frac{\#}{N^2} + \cdots \\ &= \mathcal{O}_{Classical} + (\#)\end{aligned}$$

At large N, the fluctuation term is now given by some number, $\# \sim \mathcal{O}(1)$. This means that in large N limit, the fluctuations in the theory no longer go to zero, so we no longer have a classical limit. This behavior arises from the fact that although the fluctuation in each variable tends to zero, the number of variables tends to infinity in such a way that the product of the number

of variables with the size of the fluctuations goes to a constant. For this reason a different basis is needed in which the large N limit can be reliably computed.

3.1.4 Eigenvalue Basis

Since we cannot use a matrix basis to calculate matrix model correlation functions in the large N limit, we move to the eigenvalue basis obtained by diagonalizing matrices in the single matrix model. The eigenvalue basis is a natural choice since it represents invariant variables and allows us to calculate correlators which are classically valued in the large N limit. In order to change variables from matrices to eigenvalues, the Jacobian of change of variables, given by the Van der Monde determinant, must be included.

Previously our path integral was of the form

$$< \cdot >= \int [dX] \cdot e^{-\frac{1}{2}tr(X^2)}$$

We must now transform this to the eigenvalue basis. Summing over eigenvalues and tracing over a matrix are the same, so it is easy to translate our observables. However, the measure $[dX]$ is transformed by a Jacobian. We how have

$$< \mathcal{O} > = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \, J \, \mathcal{O} \, e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2} \tag{52}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \prod_{i\neq j} |\lambda_i - \lambda_j| \, \mathcal{O} \, e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2} \tag{53}$$

The Van der Monde Determinant

The Jacobian shown in equation (52) is known as the Van der Monde determinant. We will briefly show how this Jacobian can be derived. Suppose we have some trace valued operator, for example $< \mathcal{O} >=< tr(X^n)tr(X^4) >$. Using the Schwinger-Dyson equation,

$$0 = \int [dX] \frac{d}{dX_{ij}} \left(tr(X^n)(X^3)_{ij} e^{-S} \right)$$

we can show that

$$< tr(X^n)tr(X^4) >=< ntr(X^{n+2}) > + < 2Ntr(X^2)tr(X^n) > + < tr(X)^2 tr(X^n) >$$

We know that the trace and the sum of eigenvalues of a diagonalized matrix are equivalent, which implies that

$$< tr(X^n)tr(X^4) >= \sum_{i=1}^{N} \sum_{j=1}^{N} < \lambda_i^n \lambda_j^4 >$$

Using the above relation and equation (51), the Schwinger-Dyson equation for this operator in terms of eigenvalues is

$$0 = \int_{-\infty}^{\infty} [d\lambda] \sum_{i=1}^{N} \frac{d}{d\lambda_i} \left[J \sum_{j=1}^{N} \lambda_j^n \lambda_i^3 \, e^{-\frac{1}{2}\sum_{k=1}^{N} \lambda_k^2} \right]$$

Differentiating by λ in the above equation obtains a value for our operator in terms of eigenvalue correlation functions:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} < \lambda_j^n \lambda_i^4 >= \sum_{i=1}^{N} \sum_{j=1}^{N} < \frac{1}{J} \frac{dJ}{d\lambda_i} \lambda_j^n \lambda_i^3 > + \sum_{i=1}^{N} n < \lambda_i^{n+2} > + \sum_{i=1}^{N} \sum_{j=1}^{N} 3 < \lambda_j^n \lambda_i^2 >$$

We expect to get the same result for the expectation value of our operator whether we are working in an eigenvalue or matrix basis. This means that

we may now compare the equations for $< tr(X^n)tr(X^4) >$ and $\sum_{i=1}^{N}\sum_{j=1}^{N} < \lambda_j^n\lambda_i^4 >$ term by term to solve for the Jacobian. We end up with the following pair of equations:

$$\begin{aligned} < tr(X^n)tr(X^4) > &= < n\, tr(X^{n+2}) > + < 2N\, tr(X^2)tr(X^n) > + < tr(X)^2 tr(X^n) > \\ < tr(X^n)tr(X^4) > &= < n\, tr(X^{n+2}) > + < 3\, tr(X^2)tr(X^n) > + \sum_{i=1}^{N} < \lambda_i^3 \frac{d\, lnJ}{d\lambda_i} tr(X^n) > \end{aligned}$$

We compare the two equations to get

$$< 2N\, tr(X^2)tr(X^n) > + < tr(X)^2 tr(X^n) >= 3 < tr(X^n)tr(X^2) > + \sum_{i=1}^{N} < \lambda_i^3 \frac{d\, lnJ}{d\lambda_i} tr(X^n) >$$

Thus,

$$\sum_{i=1}^{N} \lambda_i^3 \frac{d\, lnJ}{d\lambda_i} = 2(N-1)\sum_{i=1}^{N} \lambda_i^2$$

It is simple to check that this is indeed solved by

$$J = \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|$$

This is done by differentiating and doing some algebraic manipulation as follows:

$$\begin{aligned} \sum_{i=1}^{N} \lambda_i^3 \frac{d}{d\lambda_i} \sum_{j\neq k} ln|\lambda_j - \lambda_k| &= \sum_{i=1}^{N} \lambda_i^3 \sum_{j\neq k} \left[\frac{\delta_{ij} - \delta_{ik}}{\lambda_j - \lambda_k}\right] \\ &= \sum_{i=1}^{N} \lambda_i^3 \left[\sum_{k\neq i} \frac{1}{\lambda_i - \lambda_k} - \sum_{j\neq i} \frac{1}{\lambda_j - \lambda_i}\right] \\ &= 2\sum_{i=1}^{N}\sum_{k\neq i} \frac{\lambda_i^3}{\lambda_i - \lambda_k} \\ &= \sum_{i=1}^{N}\sum_{k\neq i} \frac{\lambda_i^3}{\lambda_i - \lambda_k} + \sum_{k=1}^{N}\sum_{i\neq k} \frac{\lambda_k^3}{\lambda_k - \lambda_i} \\ &= \sum_{k=1,k\neq i,}^{N}\sum_{i=1}^{N} \frac{\lambda_i^3 - \lambda_k^3}{\lambda_i - \lambda_k} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{N}\sum_{i=1}(\lambda_i^2+\lambda_i\lambda_k+\lambda_k^2) \\ &\equiv 2N\,tr(X^2)+tr(X)^2 \end{aligned}$$

This gets the result

$$\sum_{i=1}^{N} < \lambda_i^3 \frac{d\,lnJ}{d\lambda_i} tr(x^n) > = < 2N\,tr(X^2)tr(x^n) > + < tr(X)^2 tr(X^n) >$$

which satisfies our original equation. It must be noted that the Van der Monde determinant can be written in more than one way:

$$\Delta^2(\lambda_i) = \prod_{1\le i<j\le N} |\lambda_i-\lambda_j| = \prod_{i<j} |\lambda_i-\lambda_j|^2 \tag{54}$$

Although we don't prove it, this Jacobian ensures that the complete set of Schwinger-Dyson equations derived in the trace basis agrees with the complete set of Schwinger-Dyson equations derived in the eigenvalue basis.

The Classical Limit of Eigenvalues

We can already see that this basis might be more helpful than the matrix basis since when we perform a path integral we only have the N eigenvalue variables to integrate over:

$$\begin{aligned} < \mathcal{O} > &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \prod_{i\neq j}(\lambda_i-\lambda_j)\,\mathcal{O}\,e^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2} \\ &= < \mathcal{O}_{Classical} > + N(\#)\frac{1}{N^2}\cdots \end{aligned}$$

Our fluctuations are now of order $\mathcal{O}(\frac{1}{N})$, which go to zero at large N. This allows us to predict that it will be possible to find a non zero result when we try to find a classical solution to our action.

We may now define the effective action of the path integral

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j|^2 \, e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2} \rightarrow \int \prod_{i=1}^{N} d\lambda_i \, e^{-S_e ff}$$

So the effective action is

$$S_{eff} = \frac{1}{2} \sum_{i=1}^{N} \lambda_i^2 - \sum_{i<j} ln|\lambda_i - \lambda_j|^2 \tag{55}$$

We now minimize this effective action to solve for a classical solution:

$$\begin{aligned} \frac{dS_{eff}}{d\lambda_i} &= 0 \\ &= \lambda_i - \sum_{i \neq j} \frac{2}{\lambda_i - \lambda_j} \\ \Rightarrow \frac{\lambda_i}{2} &= \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \end{aligned}$$

The solution to this equation is the classical solution for the theory. We will solve this equation in terms of the density of eigenvalues, $\phi(x) = \sum_{i=1}^{N} \delta(x - \lambda_i)$. At large N this density of eigenvalues must satisfy

$$\int dx \, \phi(x) = N \tag{56}$$

and

$$\frac{x}{2} = \fint dy \frac{\phi(y)}{x - y} \tag{57}$$

To solve this equation we introduce the function

$$G(z) = \int dy \frac{\phi(y)}{z - y} \tag{58}$$

If we let $z \rightarrow \infty$ we have

$$G(z) = \int dy \, \frac{\phi(y)}{z - y}$$

$$\begin{aligned} &= \int dy \, \frac{\phi(y)}{z} \frac{1}{1-\frac{y}{z}} \\ &= \int dy \, \frac{\phi(y)}{z} \left[1 + \frac{y}{z} + \cdots\right] \\ &= \int dy \, \frac{\phi(y)}{z} \\ &= \frac{N}{z} + \mathcal{O}(\frac{1}{z^2}) \end{aligned}$$

$G(z)$ is integrated using the Residue Theorem, since it has a singularity at $y = z$, the point analogous to $\lambda_j = \lambda_i$ that was initially excluded from our summation. We denote the maximum eigenvalue in our spectrum a, such that $\phi(\pm a) = 0$, and perform the following integral:

$$G(z \pm i\epsilon) = \int_{-a}^{a} dy \, \frac{\phi(y)}{z - y \pm i\epsilon}$$

Using the principal value prescription, and the fact that we have $-a < x < a$ we get:

$$G(x \pm i\epsilon) = \fint dy \, \frac{\phi(y)}{x - y} \pm i\pi \, \phi(x)$$

If we now go back to the original equation for the minimized action, we have

$$G(x \pm i\epsilon) = \frac{x}{2} \pm i\pi \, \phi(x)$$

Propose a general form of $G(z)$ in the following ansatz:

$$G(z) = \frac{z}{2} \pm \alpha \sqrt{z^2 - a^2}$$

We use this ansatz since we have $G(x \pm i\epsilon) = \frac{x}{2} \pm i\pi \, \phi(x)$, where the possible positive or negative values indicate a square root must be present,

and we have a term linear in x to agree with the $\frac{x}{2}$ term. α is some constant we can solve for. We also use the fact that $G(z) \to \frac{N}{z}$ as $z \to \infty$

$$\begin{aligned} \Rightarrow G(z) &= \frac{z}{2} + \alpha\, z \sqrt{1 - \frac{a^2}{z^2}} \\ &= \frac{z}{2} + \alpha\, z \left[1 - \frac{a^2}{2z^2} \cdots\right] \\ &= \left(\frac{1}{2} + \alpha\right) z + \alpha\, z \left[1 - \frac{a^2}{2z^2}\right] \end{aligned}$$

Using the value of $G(z)$ at large z allows us to solve for the unknowns α and a. We get

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \frac{a^2}{4z} &= \frac{N}{z} \\ \Rightarrow a &= 2\sqrt{N} \end{aligned}$$

We now find ϕ:

$$\phi(z) = \frac{1}{2\pi} \sqrt{a^2 - z^2} = \frac{1}{2\pi} \sqrt{4N - z^2} \tag{59}$$

This is a simple result to test; we now integrate this density to see if we get N. This gives us some more confidence in our solution.

$$\int \phi(x)\, dx = \int dx\, \frac{1}{2\pi} \sqrt{4N - x^2}$$

Let $x = \sqrt{4N}\, y \;\Rightarrow\; dx = \sqrt{4N} dy$.

$$\begin{aligned} \int_{-\sqrt{4N}}^{\sqrt{4N}} dx \frac{1}{2\pi} \sqrt{4N - x^2} &= \int \frac{1}{2\pi} \sqrt{4N} \sqrt{1 - y^2}\, \sqrt{4N} dy \\ &= \frac{4N}{2\pi} \int dy \sqrt{1 - y^2} \end{aligned}$$

Now let $y = \sin\theta$. Then $dy = \cos\theta\, d\theta$:

$$\begin{aligned}\frac{2N}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta\ \sqrt{1-sin^2\theta} &= \frac{2N}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\ &= \frac{2N}{\pi}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= N\end{aligned}$$

We have now shown that our requirement $\int \phi(x)dx = N$ is satisfied by our proposed function for the density of eigenvalues.

3.1.5 Correlators in the Eigenvalue Basis

We have shown how to calculate correlation functions in the matrix basis using ribbon diagrams and path integrals; we will now show how to calculate them in the eigenvalue basis. The results in this section should correspond to the results for the matrix basis obtained after taking the planar limit. Recall that it was not possible to minimize the action before calculating correlators in the matrix basis, as the classical solution turned out to be zero. Instead we had to perform the calculation and then let $N \to \infty$. We will now show that in the eigenvalue basis it is possible to calculate correlation functions using the classical solution obtained by minimizing the effective action.

In the large N limit, the following are equivalent:

$$< tr(X^n) >=< \sum_{i=1}^{N} \lambda_i^n >= \int dx\, x^n\, \phi(x) \tag{60}$$

In an earlier section we showed that $< tr(X^2) >= N^2$. We will now calculate this using the density of eigenvalues, $\frac{1}{2\pi}\sqrt{4N-x^2}$.

$$< tr(X^2) > \to \int dx\, x^2\, \phi(x) \;=\; \int_{-\sqrt{4N}}^{\sqrt{4N}} dx\, x^2 \frac{1}{2\pi}\sqrt{4N-x^2}$$

We now let $x = \sqrt{4N}\sin\theta$ and use our previous methods:

$$\begin{aligned}\int dx\, x^2\, \phi(x) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{4N}{2\pi}\, \cos^2\theta\, 4N\, \sin^2\theta \\ &= \frac{8N^2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{\sin^2(2\theta)}{4} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{1-\cos(4\theta)}{4.2} \\ &= N^2\end{aligned}$$

Another example is $< x^4 >$:

$$\begin{aligned}< x^4 > &= \int dx\, x^4\, \phi(x) \\ &= \int_{-\sqrt{4N}}^{\sqrt{4N}} dx\, x^4 \frac{1}{2\pi}\sqrt{4N - x^2} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{4N}{2\pi}\, Cos^2\theta\, (4N)^2\, Sin^4\theta \\ &= 2N^3\end{aligned}$$

This shows that eigenvalues are a useful basis for calculating correlation functions in the large N limit. It is possible to extend the above result to much larger traced matrix products and the leading order in N will be obtained. This corresponds to finding the ribbon diagrams of the correlator and taking the planar limit. Eigenvalues are a much more effective basis than matrices in this respect.

3.1.6 Where the Eigenvalue Basis Fails

We encountered the problem that in the classical limit, fluctuations in the matrix basis do not disappear. This lead to the use of the eigenvalue basis, where we solved the equation for the minimized effective action to get the

density of eigenvalues. It was possible to calculate correlators in the planar limit in this basis, which indicates it is a more natural basis to use than matrices. However, we have only looked at single matrix models. Using an eigenvalue basis relies on the fact that it is possible to diagonalize the matrices being used. This is doable for singe matrices, but suppose we were to try to calculate some correlator that contained a trace over two matrices, for example

$$< tr(M_1 M_2) >$$

In a general theory it would not be possible to perform this calculation, since it is not necessarily always possible to simultaneously diagonalize two or more matrices.

Another problem with this basis is that if we have operators built from large numbers of X's there are many Wick contractions to be performed. If, for instance, the number of of X's in an operator is $\mathcal{O}(N)$, the number of Wick contractions overpowers the $\frac{1}{N^2}$ suppression of non-planar diagrams. This means that in this case the planar limit doesn't capture the large N limit and eigenvalues which capture the planar limit, do not capture the large N limit.

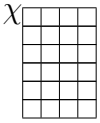
4 Back Reaction on the Eigenvalues

When the number of matrices X in an operator $\mathcal{O}$ is of order N^2 (we call this a "large operator") the planar approximation is no longer a good approximation. Because of this, we can no longer compute observables with the classical eigenvalue configuration obtained by minimizing the effective action. When operators are of this size, the large N approximation itself breaks down. In [4], correlators of large operators were studied in complex matrix models. The result in this paper was that $\frac{1}{N} \to \frac{1}{N_{eff}}$ and there is a new $\frac{1}{N}$ expansion.

Here, we want to show that this can be understood as a back reaction on the eigenvalues. This issue could not be dealt with in [4] because one dealt with a complex matrix model, restricted to the $\frac{1}{2}BPS$ sector. This made it technically cumbersome to compute the eigenvalue density. Here we will study an Hermittian matrix and explicitely compute the eigenvalue density.

Matrix Basis

We are going to study correlation functions containing $\mathcal{O}(Nm)$ operators, where m itself is $\mathcal{O}(N)$. Suppose we have a Schur polynomial of the form

χ(X)

where the Young diagram R has $\mathcal{O}(m)$ columns and $\mathcal{O}(N)$ rows. We use this polynomial to calculate correlators of the form

$$<< \mathcal{O} >>= \int [dX] \, (det(X))^{2m} \, \mathcal{O} \, e^{-S}$$

This Schur polynomial can be written in terms of matrices as

$$\chi_{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \end{array}}(X) \rightarrow (det(X))^{2m}$$

which allows us to calculate these correlation functions of large operators. Note that the factor of 2 appears if we set the number of columns in R to be even. We use Schwinger-Dyson equations with the action $S = \frac{1}{2} tr(X^2)$ to calculate the correlators as follows:

$$\begin{aligned}
0 &= \int [dX] \, \frac{d}{dX_{ij}} \left((det(X))^{2m} \, (X^n)_{ij} \, e^{-S} \right) \\
&= \int [dX] \left(\left(\frac{d}{dX_{ij}} (det(X))^{2m} \right) (X^n)_{ij} \, e^{-S} + (det(X))^{2m} \, \left(\frac{d}{dX_{ij}} (X^n)_{ij} \right) \, e^{-S} \right. \\
&\quad \left. + (det(X))^{2m} \, (X^n)_{ij} \, \left(\frac{d}{dX_{ij}} (-\frac{1}{2} tr(X^2)) \right) e^{-S} \right) \\
&= \int [dX] \left(2m (det(X))^2 m (X^n)_{ij} (X^{-1})_{ji} \, e^{-S} + (det(X))^{2m} \sum_{r=0}^{n-1} tr(X^r) tr(X^{n-1-r}) e^{-S} \right. \\
&\quad \left. - (X^n)_{ij} X_{ji} (det(X))^{2m} e^{-S} \right) \\
&= << 2m \, tr(X^{n-1}) >> + << \sum_{r=0}^{n-1} tr(X^r) tr(X^{n-1-r}) >> - << tr(X^{n+1} >> \\
\Rightarrow \qquad & << tr(X^{n+1} >> = << 2m \, tr(X^{n-1}) >> + << \sum_{r=0}^{n-1} tr(X^r) tr(X^{n-1-r}) >>
\end{aligned}$$

We will use this simple formula to calculate the correlation functions for $n = 1, 3, 5, 7$. Note that the Schwinger-Dyson equations are also used for

operators of the form

$$<< \mathcal{O} >>=<< tr(X^a)tr(X^b) >>$$

The steps to find these correlators will not be shown here as they are very similar to the above calculation. Simply note that we use Schwinger-Dyson equations of the form

$$0 = \int [dX] \frac{d}{dX_{ij}} \left((det(X))^{2m} \, tr(X^a)(X^{b-1})_{ij} \, e^{-S} \right)$$

It is necessary to mention this since terms of this form arise from

$$<< \sum_{r=0}^{n-1} tr(X^r)tr(X^{n-1-r}) >>$$

We also have that $tr(X^r) = 0$ if r is odd, due to the symmetry present where $X \to -X$. The correlation functions are now calculated using our formula:

- $n = 1$

$$<< tr(X)^2 >>= N(N + 2m)$$

- $n = 3$

$$\begin{aligned} << tr(X^4) >> &= 2m << tr(X^2) >> +2N << tr(X^2) >> \\ &= 2N(N + 2m)(m + N) \end{aligned}$$

- $n = 5$

$$\begin{aligned} << tr(X^6) >> &= 2(N+m) << tr(X^4) >> + << tr(X^2)tr(X^2) >> \\ &= 2(N+m) << tr(X^4) >> +(N^2+2mN+2) << tr(X^2) >> \\ &= 2(N+m)2N(N+2m)(m+N) + (N^2+2mN+2)N(N+2m) \\ &= 4N(N+m)^2(N+2m) + N(N+2m)(N^2+2mN+2) \end{aligned}$$

- **n=7**

$$\begin{aligned} << tr(X^8) >> &= 2(N+m) << tr(X^6) >> +2 << tr(X^4)tr(X^2) >> \\ &= 2(N+m) << tr(X^6) >> +2(N^2+2mN+3) << tr(X^4) >> \\ &= 8N(N+m)^3(N+2m) + 2N(N+m)(N+2m)(N^2+2mN+2) \\ &\quad +4N(N+m)(N^2+2mN+3)(N+2m) \end{aligned}$$

Eigenvalue Basis

We will now change to the eigenvalue basis using methods discussed in the previous section. To find the effective action we now write our initial Schur polynomial as

$$\chi_{\begin{array}{|c|c|c|c|}\hline \ \ & \ \ & \ \ & \ \ \\ \hline \ \ & \ \ & \ \ & \ \ \\ \hline \ \ & \ \ & \ \ & \ \ \\ \hline \ \ & \ \ & \ \ & \ \ \\ \hline \ \ & \ \ & \ \ & \ \ \\ \hline \ \ & \ \ & \ \ & \ \ \\ \hline \end{array}}(X) \rightarrow e^{-2m\,tr(ln\,X)}$$

Again, the Jacobian for this change of variables is the Van der Monde determinant, so we have the following effective action:

$$S_{eff} = \frac{1}{2}\sum_i \lambda_i^2 - \sum_{i \neq j} ln|\lambda_i - \lambda_j| - 2m\sum_i ln\lambda_i$$

We minimize this action to get the following equation:

$$\frac{1}{2}(\lambda_i + \frac{2m}{\lambda_i}) = \sum_{i<j} \frac{1}{\lambda_i - \lambda_j}$$

Using the methods to solve for the density of eigenvalues discussed previously and calculations outlined in [15] we can find the density of eigenvalues corresponding to this potential. In integral form, this equation is

$$\oint dy \frac{\phi(x)}{x-y} = \frac{x}{2} + \frac{m}{x}$$

We have the constraint equation

$$\int \phi(x) = N$$

and as in [15], we have a double brach cut solution. From the calculation presented in section 3 we have the function

$$G(z) = \int dy \frac{\phi(y)}{z-y}$$

and as $z \to \infty$, $G(z) \to N$. Note that this solution $G(z)$ does not have a pole at $z \to 0$. As in the previous section,

$$G(x \pm i\epsilon) = \frac{x}{2} + \frac{m}{x} \pm i\pi\phi(x)$$

We introduce an ansatz of the form

$$G(z) = \frac{z}{2} + \frac{m}{z} + \frac{a}{z}\sqrt{(z^2 - y_-^2)(z^2 - y_+^2)}$$

Here the two pairs of branch points of $G(z)$ are located at $y_\pm$ and $-y_\pm$, and we require that non-zero values of ϕ lie between the two intervals on the real axis where $y_-^2 \leq y^2 \leq y_+^2$. Again following [15], we have the normalization factor

$$a = \frac{1}{2}$$

and the values of the branch points

$$y_{\pm}^2 = 2(N+m) \pm 2\sqrt{N(N+2m)}$$

This leads us to propose that ϕ is given by

$$\phi(x) = \frac{1}{2\pi|x|}\sqrt{\left(x^2 - (2(N+m) + 2\sqrt{N(N+2m)})\right)\left(x^2 - (2(N+m) - 2\sqrt{N(N+2m)})\right)}$$

This was integrated over the two bands where ϕ is supported and it was found that it does indeed satisfy the constraint equation $\int \phi(x)dx = N$. This lead us to proceed with calculating the correlators that were found for this potential in the matrix basis. We will show the indefinite integrals obtained for the functions $< x^2 >$ and $< x^4 >$, followed by the final results found once the integration limits are included and a fair amount of simplification has occurred.

- $< x^2 >$

$$\begin{aligned}
&= \int \phi(x)x^2\,dx \\
&= \int dx\; x^2 \frac{1}{2\pi\, x}\sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)} \\
&= \frac{1}{32\pi}\sqrt{(-x^2 + y_+^2))\,(-x^2 + y_-^2)}\Bigg(- 2(y_-^2 + y_+^2 - 2x^2) \\
&\quad - \frac{(y_+^2 - y_-^2)^2\, ln\left[-(y_+^2 + y_-^2) + 2x^2 + 2\sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)}\right]}{\sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)}}\Bigg) \\
&= N(N+2m)
\end{aligned}$$

- $< x^4 >$

$$= \int \phi(x)x^4\,dx$$

$$
\begin{aligned}
&= \int dx\; x^4 \frac{1}{2\pi\, x} \sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)} \\
&= \frac{1}{64\pi} \sqrt{(-x^2 + y_+^2))\,(-x^2 + y_-^2)} \Bigg(-\frac{2}{3}(3y_+^4 + 3y_-^4 + 2y_-^2 x^2 - 8x^4 + 2y_+^2(-y_-^2 + x^2)) \\
&\quad - \frac{(y_+^2 - y_-^2)^2 (y_+^2 + y_-^2)\, ln\, \left[-(y_+^2 + y_-^2) + 2x^2 + 2\sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)} \right]}{\sqrt{(x^2 - y_+^2))\,(x^2 - y_-^2)}} \Bigg) \\
&= 2N(N + 2m)(m + N)
\end{aligned}
$$

These are the same results that were found in the matrix basis using Schwinger-Dyson equations. Once again we have an agreement between the matrix and eigenvalue bases, comparable to what was discussed in section 3. We conclude that by using back reaction on eigenvalues we can extend the eigenvalue description to situations where the number of matrices X in an operator $\mathcal{O}$ is of order N^2.

5 Schur Polynomials

In this project the ultimate goal is to employ the collective field formalism to describe a single matrix model, with Schur Polynomials as our collective fields. Schur polynomials form an invariant basis with order N labels, which may provide a useful approach to multi matrix models. We are going to try to replace the matrix basis with a Schur basis using the systematic change of variables already outlined. Some group representation theory is required to understand what a Schur polynomial is.

The symmetric group, S_n is the group of all permutations of n objects. This group contains $n!$ elements which can be represented by integer partitions. In group representation theory these partitions can be represented by Young diagrams. For example, the group S_3 is as follows:

$$S_3 = \{1, (12), (23), (13), (123), (132)\}$$

Note that there are permutation cycles of the same length in this group. Sets of these objects of the same cycle structure belong to the same conjugacy classes. Different partitions of n are represented by Young diagrams. For example, in S_3 we have the following correspondence. Denote $(1, 1, 1)$ as three one cycles, or the identity, $(2, 1)$ as the product of a two cycle and a one cycle and (3) as a three cycle. These partitions correspond to the following Young diagrams:

$$(1, 1, 1) \rightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

$$(2, 1) \rightarrow \begin{array}{|c|c|}\hline \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$$

$(3) \rightarrow$

The character of a group element is the trace over the irreducible matrix representation of that element. Using these objects we can now proceed to the Schur polynomials. The general form of a Schur polynomial is given by

$$\chi_R(X) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) X^{i_1}_{i_{\sigma(1)}} X^{i_2}_{i_{\sigma(2)}} \cdots X^{i_n}_{i_{\sigma(n)}} \tag{61}$$

Here R is the Young diagram with n boxes, indicating which representation of the group we are working with, $\chi_R(\sigma)$ is the character of the element $\sigma \in S_n$ in irreducible representation R and the matrices X have upper and lower indices corresponding to the adjoint indices of group elements. For our application X is Lie algebra valued. The indices on the X's are permuted by the group element σ. After summing over repeated indices each term in the sum is a number times a product of traces of X's.

Some Properties of Schur Polynomials and Young Diagrams

Examples

- For clarity, we will show how to obtain the Schur polynomial in S_3 for $R =$ . The character table for S_3 is:

Rep	Partition	Class		
		(1^3)	$(1,2)$	(3)
	(3)	1	1	1
	$(2,1)$	2	0	-1
	(1^3)	1	-1	1

Here, $R = \ydiagram{3}$ is known as the symmetric representation, $R = \ydiagram{1,1,1}$ is known as the completely antisymmetric representation, and $R = \ydiagram{2,1}$ is the mixed representation. So the step by step construction of $\chi_{\ydiagram{2,1}}$ is completed using this character table and equation (61). Of course for a group this size it isn't really necessary to use (61) since its straightforward to see how many elements there are of each cyclic structure.

$$\begin{aligned}
\chi_{\ydiagram{2,1}}(X) &= \frac{1}{3!}\sum_{\sigma\in S_3}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}X^{i_3}_{i_{\sigma(3)}} \\
&= \frac{1}{6}(\chi_R(1)X^{i_1}_{i_1}X^{i_2}_{i_2}X^{i_3}_{i_3} + \chi_R(12)X^{i_1}_{i_2}X^{i_2}_{i_1}X^{i_3}_{i_3} + \chi_R(13)X^{1}_{i_3}X^{i_2}_{i_2}X^{i_3}_{i_1} \\
&+ \chi_R(23)X^{i_1}_{i_1}X^{i_2}_{i_3}X^{i_3}_{i_2} + \chi_R(123)X^{i_1}_{i_2}X^{i_2}_{i_3}X^{i_3}_{i_1} + \chi_R(132)X^{i_1}_{i_3}X^{i_2}_{i_1}X^{i_3}_{i_2}) \\
&= \frac{1}{6}\left(\chi_R(1)tr(X)^3 + 3\chi_R(12)tr(X)tr(X^2) + 2\chi_R(123)tr(X^3)\right) \\
&= \frac{1}{6}\left(2tr(X)^3 - 2tr(X^3)\right) \\
&= \frac{1}{3}\left(tr(X)^3 - tr(X^3)\right)
\end{aligned}$$

The Schur polynomials for the other S_3 representations are:

$$\begin{aligned}
\chi_{\ydiagram{3}}(X) &= \frac{1}{6}\left(tr(X)^3 + 3tr(X)tr(X^2) + 2tr(X^3)\right) \\
\chi_{\ydiagram{1,1,1}}(X) &= \frac{1}{6}\left(tr(X)^3 - 3tr(X)tr(X^2) + 2tr(X^3)\right)
\end{aligned}$$

The Schur polynomials for the S_2 representations are:

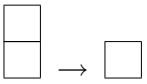
$$\begin{aligned}
\chi_{\ydiagram{2}}(X) &= \frac{1}{2}\left(tr(X)^2 + tr(X^2)\right) \\
\chi_{\ydiagram{1,1}}(X) &= \frac{1}{2}\left(tr(X)^2 - tr(X^2)\right)
\end{aligned}$$

- In our calculations we often need the derivative of a Schur polynomial. Taking the derivative of a Schur polynomial is the same as removing

a box from the Young diagram R that is used to represent it. The derivative of a general Schur polynomial is taken as follows:

$$\begin{aligned}\frac{d}{dX_i^i}\left[\chi_R(X)\right] &= \frac{d}{dX_i^i}\left[\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_n}_{i_{\sigma(n)}}\right] \\ &= \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_{n-1}}_{i_{\sigma(n-1)}}\delta^{i_n}_{i_{\sigma(n)}} \\ &= \sum_{R'}c_{RR'}\chi_{R'}(X)\end{aligned}$$

In terms of Young diagrams, this is like removing a box from the original diagram eg.



$c_{RR'}$ is the weight of the box removed when differentiating. If R is the original represention, R' is the representation after differentiating, and is the same as R with one box removed. To obtain the weight of a box in a Young diagram, N is written in the top left box, 1 is added to every box to the right in the horizontal direction and 1 is subtracted from each box as one moves down the columns, so the box removed in the above diagram would have the weight $c_{RR'} = N - 1$. $\sum_{R'} c_{RR'}\chi_{R'}(X)$ is the sum over all possible ways of removing a box from the Young diagram R, or in other words the sum over all possible representations that we could end up with by differentiating $\chi_R(X)$. Suppose we wish to differentiate $\chi_{\square\square\square}(X)$:

$$\begin{aligned}\frac{d}{dX_i^i}\left[\chi_{\square\square\square}(X)\right] &= \frac{d}{dX_i^i}\left[\frac{1}{6}\left(tr(X)^3 + 3tr(X)tr(X^2) + 2tr(X^3)\right)\right] \\ &= \frac{1}{6}\left(3Ntr(X)^2 + 3Ntr(X^2) + 6tr(X)^2 + 6tr(X^2)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{3N+6}{6}\left(tr(X)^2 + tr(X^2)\right) \\
&= (N+2)\left[\frac{1}{2}\left(tr(X)^2 + tr(X^2)\right)\right] \\
&= c_{\square\square\square\ \square\square}\,\chi_{\square\square}(X)
\end{aligned}
$$

For more uses of the theory outlined here see [5]. Also note that the derivative of a Schur polynomial where $i \neq j$ is given by

$$
\frac{d}{dX_i^j}\left[\chi_R(X)\right] = \left[\frac{1}{(n-1)!}\sum_{\sigma\in S_n}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_n-1}_{i_{\sigma(n-1)}}\delta^{i_n}_j\delta^i_{i_{\sigma(n)}}\right]
$$

- When the relevant indices are placed on the matrices X in the Schur polynomial, the trace products take on the cyclic structure of each group element. This means that there will be a certain number of trace products of the same form, corresponding to conjugacy classes of the elements. It is thus only necessary to know how many elements of each cyclic structure reside in the group, which is easy to calculate using:

$$
\frac{n!}{\prod_i i^{n_i} n_i!} \tag{62}
$$

  Here, $n!$ is the order or number of elements in the symmetric group S_n, i is the length of a particular 'i-cycle' and n_i refers to the number of times that each i-cycle appears in the element.

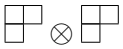
- The product of two Schur polynomials is given by

$$
\chi_R(X)\chi_S(X) = \sum_T f_{RST}\chi_T(X) \tag{63}
$$

  Schur polynomials are actually characters of $SU(N)$ when evaluated on an element of $SU(N)$. If we have the character of a tensor product of irreducible representations, we have the product of their two

characters. This is the group theoretic origin of the above formula. Along with Schur polynomials $\chi_R(U)$ as the characters of $SU(N)$, we have the characters of the group elements of S_n, $\chi_R(\sigma)$ in the Schur polynomial. The link between these two characters is known as the Schur-Weyl duality. Taking the tensor product in $SU(N)$ is Schur-Weyl dual to subducing irreducible representations in S_n Here f_{RST} denotes the Littlewood-Richardson coefficient of the product of two Young diagrams labeling $SU(N)$ irreducible representations.

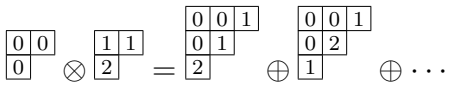
- Littlewood-Richardson coefficients are found as demonstrated in the following example. Suppose we want to calculate



This is the tensor product of irreducible representations in $SU(N)$. In the first diagram we place 0's in the boxes and in the second we place ones in the first row, twos in the second row and n's n the nth row. The labeled boxes in the second Young diagram are then added to the first, respecting a few rules. The rules for the positions of the numbers in the diagrams are:

  - The numbers must be weakly increasing along rows
  - They must be strictly increasing down columns
  - # 1's $\geq$ # 2's $\geq \cdots$ when the diagram is read from right to left starting from the first row, then moving to the second and so on.

Here is a concrete example to illustrate the procedure:



Of course the product of the original two Young diagrams will produce a sum of more diagrams than the ones we have shown here, however we are only showing how the coefficients are obtained for . There are no diagrams of this shape omitted from the right hand side. The Littlewood-Richardson coefficient for this configuration is thus

$$f_{\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}\ \begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}\ \begin{smallmatrix}\square\square\square\\\square\square\\\square\end{smallmatrix}} = 2$$

The product of the two Schur polynomials would be of the form:

$$\begin{aligned}\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(X)\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(X) &= \sum_T f_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\ \begin{smallmatrix}\square\square\\\square\end{smallmatrix}\ T}\chi_T(X)\\ &= 2\chi_{\begin{smallmatrix}\square\square\square\\\square\square\\\square\end{smallmatrix}}(X)+\cdots\end{aligned}$$

6 Collective Field Theory of Schur Polynomials

In this section the formalism of collective field theory and the properties of Schur polynomials that have been discussed in previous sections will be employed to construct our collective field theory of Schur polynomials. We will present new results we have found and describe methods that we have tried applying to works in progress. The primary aspects of collective field theory formalism needed to create a theory with Schur polynomials fields are the operators Ω and ω, and the Jacobian that arises from changing variables from matrices to Schur polynomials.

6.1 Collective Field Operators

After the background work was completed, the first thing that we looked into was calculating ω. The adapted form of the collective field theory formalism indicates that to find ω in terms of Schur polynomials we must calculate

$$\omega_R = tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X) \tag{64}$$

The derivative of the general form of Schur polynomials has been given. We now have the trace over the product of two derivatives which is performed as follows:

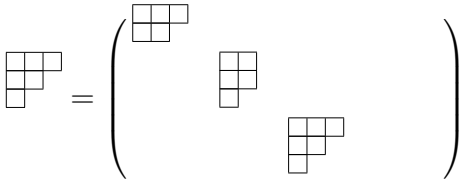
$$\begin{aligned}\omega_R &= tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X) \\ &= \frac{d}{dX^i_j}\left[\frac{1}{(n-1)!}\sum_{\sigma\in S_n}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_n-1}_{i_{\sigma(n-1)}}\delta^{i_n}_j\delta^i_{i_{\sigma(n)}}\right]\end{aligned}$$

$$= \frac{1}{(n-2)!}\sum_{\sigma\in S_n}\chi_R(\sigma)X^{i_1}_{i_{\sigma(1)}}X^{i_2}_{i_{\sigma(2)}}\cdots X^{i_n-2}_{i_{\sigma(n-2)}}\delta^{i_{n-1}}_{i_{\sigma(n-1)}}\delta^{i_n}_{i_{\sigma(n)}}$$

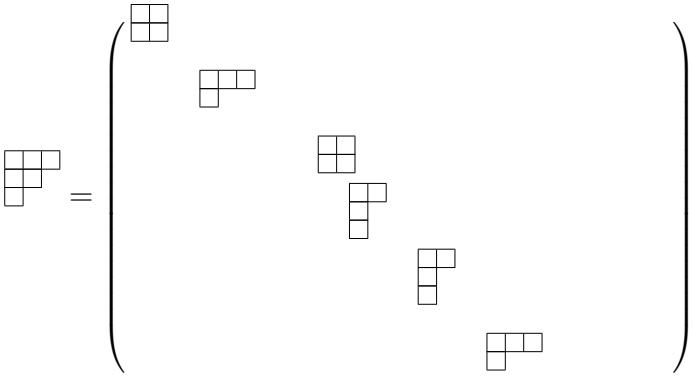
We now let $\sigma = \psi\cdot(n,n-1)$, where $\psi\in S_n$. This allows us to write

$$\begin{aligned}\omega_R &= \frac{1}{(n-2)!}\sum_{\psi\in S_n}\chi_R(\psi\cdot(n,n-1))X^{i_1}_{i_{\psi(1)}}X^{i_2}_{i_{\psi(2)}}\cdots X^{i_n-2}_{i_{\psi(n-2)}}\delta^{i_{n-1}}_{i_{\psi(n-1)}}\delta^{i_n}_{i_{\psi(n)}}\\ &= \frac{d}{dY^i_i}\frac{d}{dZ^j_j}\left[\frac{1}{(n-2)!}\sum_{\psi\in S_n}\chi_R(\psi\cdot(n,n-1))X^{i_1}_{i_{\psi(1)}}X^{i_2}_{i_{\psi(2)}}\cdots X^{i_n-2}_{i_{\psi(n-2)}}Z^{i_{n-1}}_{i_{\psi(n-1)}}Y^{i_n}_{i_{\psi(n)}}\right]\end{aligned}$$

This way of writing the derivatives corresponds to pulling two boxes off of the irreducible representation (irrep) given by Young diagram R. Irreducible representations of S_n will subduce an (in general) reducible representation of any subgroup and hence can be reduced to block diagonal matrices for elements of subgroup S_{n-2}. For example if we start with irrep R, differentiating twice takes us to the representation R''_i, were i runs over all ways of removing two blocks from R. Suppose we have $R =$  in S_6. This reduces to block diagonal matrices in for the subgroups S_{n-1} and S_{n-2}:



on S_{n-1}, and



on S_{n-2}. Here R' denotes representations of S_{n-1} and R'' denotes representations of S_{n-2}. Note that among the possible R'' the representation $R'' =$  appears twice. This is because there are two ways that boxes can be removed from $R =$  to obtain , with different boxes of weights being removed depending on the order in which this is done. The original trace over R is the trace over the representations on the diagonal in this matrix.

The character of some representation R for each element of S_n is the trace over the matrix $\Gamma_R(\sigma)$. Using Young's orthogonal representation below, we use properties of the character of R to obtain the character of the subduced representation, R''. We let $(n, n-1)$ act on some state:

$$\Gamma_{\ydiagram{4,3,2,1}}((n,n-1))\left|\begin{array}{|c|c|c|}\hline\multicolumn{3}{|c|}{}&1\\\hline\multicolumn{3}{|c|}{}\\\hline 2\\\cline{1-1}\end{array}\right\rangle = \frac{1}{c_1-c_2}\left|\begin{array}{|c|c|c|}\hline\multicolumn{3}{|c|}{}&1\\\hline\multicolumn{3}{|c|}{}\\\hline 2\\\cline{1-1}\end{array}\right\rangle + \sqrt{1-\frac{1}{(c_1-c_2)^2}}\left|\begin{array}{|c|c|c|}\hline\multicolumn{3}{|c|}{}&2\\\hline\multicolumn{3}{|c|}{}\\\hline 1\\\cline{1-1}\end{array}\right\rangle$$

Here c_1 and c_2 are the weights of the first and second respective boxes removed from the Young diagram R to reduce it to Young diagram R''. These boxes are labeled 1 and 2 on the left hand side of the above equation. The character of R'' can thus be found:

$$\begin{aligned}
& \chi_{R''}(\psi(n,n-1)) \\
= \;& tr_{R''}\left(\Gamma_R(\psi)\Gamma_R(n,n-1)\right) \\
= \;& \sum_i < i|\Gamma_R(\psi)\Gamma_R(n,n-1)|i > \\
= \;& \sum_i \left[\frac{< i|\Gamma_R(\psi)|i >}{c_1-c_2} + < i|\Gamma_R(\psi)|i' > \sqrt{1-\frac{1}{(c_1-c_2)^2}}\right]
\end{aligned}$$

The term containing $< i|\Gamma_R(\psi)|i' >$ vanishes. This is a known result for twisted restricted characters [12]. The collective field operator ω_R is thus

given by:

$$\omega_R = tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X) = \sum_{R''} \frac{c_1 c_2}{c_1 - c_2}\chi_{R''}(X) \tag{65}$$

This describes the sum over all of the ways of pulling boxes of representation R to obtain R''.

Examples of ω_R

We will now show this result applied to some Schur polynomials.

- $\omega_{\square\square\square}$

We have the equation

$$tr(\frac{d}{dX}\frac{d}{dX})\chi_{\square\square\square}(X) = \sum_{R''} \frac{c_1 c_2}{c_1 - c_2}\chi_{R''}(X)$$

On the left hand side of this equation, we simply differentiate the Schur polynomial $\chi_{\square\square\square}(X)$ as follows:

$$\begin{aligned} tr(\frac{d}{dX}\frac{d}{dX})\chi_{\square\square\square}(X) &= tr(\frac{d}{dX}\frac{d}{dX})\left[\frac{1}{6}\left(tr(X)^3 + 3tr(X)tr(X^2) + 2tr(X^3)\right)\right] \\ &= \frac{1}{6}\frac{d}{dX^i_j}\left[3tr(X)^2\delta^i_j + 3tr(X^2)\delta^i_j + 6tr(X)X^j_i + 6X^k_i X^j_k\right] \\ &= (N^2 + 3N + 2)tr(X) \\ &= \frac{(N+1)(N+2)}{N+2-(N+1)}\chi_{\square}(X) \end{aligned}$$

On the right hand side of the equation, we look at the weights of the boxes we pull off in the order they are pulled off. If we have

$$R = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \end{array}$$

the weights are given by

$$a = N \qquad b = N+1 \qquad c = N+2$$

There is only one possible way to pull off the first block, so we have $c_1 = N+2$. We are left with the representation in S_2, $R' = \boxed{a}\boxed{b}$. It is clear that $c_2 = N+1$, and that we are left with $R'' = \square$ We then have

$$\sum_{R''} \frac{c_1 c_2}{c_1 - c_2} \chi_{R''}(X) \quad = \quad \frac{(N+1)(N+2)}{N+2-(N+1)} \chi_{\square}(X)$$

This is of course the same as what we obtained for the left hand side of the equation.

- ω 

Similarly, we can show that ω_R is a simple result for more complicated Schur polynomials. We will use

$$\chi_{\begin{smallmatrix}\square\square\square\square\\ \square\square\\ \square\end{smallmatrix}}(X) = \frac{1}{6!}\left[16tr(X)^6 - 2\cdot 40 tr(X^3)tr(X)^3 + 144 tr(X^5)tr(X) - 2\cdot 40 tr(X^3)tr(X^3)\right]$$

Using matrix differentiation, and a small amount of work, we have

$$tr(\frac{d}{dX}\frac{d}{dX})\frac{1}{6!}\left[16tr(X)^6 - 2\cdot 40 tr(X^3)tr(X)^3 + 144 tr(X^5)tr(X) - 2\cdot 40 tr(X^3)tr(X^3)\right]$$
$$= \quad 0$$

We then use the formula we found for ω_R rather than straightforward differentiation. We have

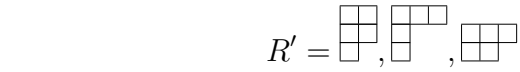
$$R = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2} f \\ \cline{1-1}\end{array}$$

with

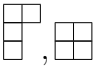
$$a = N \qquad b = N+1 \qquad c = N+2$$

$$d = N-1 \qquad e = N \qquad f = N-2$$

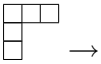
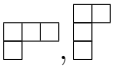
There are three choices for the first box that can be removed from $R =$ . This leaves three representations in S_5:

$$R' = \text{}$$

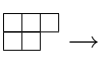
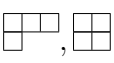
The boxes removed have weights $c_1 = N+2, N, N-2$ respectively. Each of the Young diagrams we are left with then has one or more possible ways to remove boxes, so we have the following reduced representations R'' shown with the respective weights of the second boxes removed

$$R' = \text{} \rightarrow R'' = \text{}$$

$$c_2 = N, N-2$$

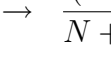
$$R' = \text{} \rightarrow R'' = \text{}$$

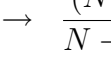
$$c_2 = N-2, N+2$$

$$R' = \text{} \rightarrow R'' = \text{}$$

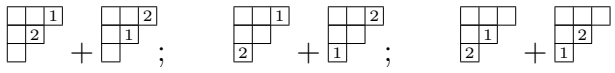
$$c_2 = N, N+2$$

We can now use $\sum_{R''} \frac{c_1 c_2}{c_1 - c_2} \chi_{R''}(X)$ as we did before, but rather than doing this we will make the following observation. Suppose we label the boxes by the order in which we remove them, so for example we would have

$$\chi_{\text{}} \rightarrow \frac{(N+2)(N-2)}{N+2-(N-2)} \chi_{\text{}}$$

$$\chi_{\text{}} \rightarrow \frac{(N-2)(N+2)}{N-2-(N+2)} \chi_{\text{}}$$

The sum of the Schur polynomials for $\chi_{R''} = \chi_{\ydiagram{2,2}}(X)$ would add up to zero when placed in the equation $\sum_{R''} \frac{c_1 c_2}{c_1 - c_2} \chi_{R''}(X)$. This can also be shown for the other possible ways of removing boxes from R, so essentially we have six different ways to remove two boxes from $R = \ydiagram{3,2,1}$, shown below in the pairs that add up to zero:



This leaves us with

$$\omega_{\ydiagram{3,2,1}} \sum_{R''} \frac{c_1 c_2}{c_1 - c_2} \chi_{R''}(X) = 0 \equiv tr(\frac{d}{dX}\frac{d}{dX}) \chi_{\ydiagram{3,2,1}}(X)$$

as expected.

The Physical Interpretation of this Result

The eigenvalue dynamics of the Hermittian matrix model is known to be the dynamics of N free fermions. The fermionic wave function is antisymmetric under the exchange of any two particles. Here we see that doing the two reductions in opposite order costs a minus sign. Is this related to the fermionic statistics of the eigenvalues? This does seem to be the natural conclusion especially given the arguments in [13] and [14] which show that each row of the Young diagram corresponds to an eigenvalue. In this sense, 'row length' dynamics in the Schur polynomial description should be equivalent to eigenvalue dynamics.

Derivation of Ω_{RS}

Using the result for ω_R we can easily find Ω_{RS}. In the collective field theory formalism we have

$$\Omega_{RS} = \frac{d}{dX_i^j}\chi_R(X)\frac{d}{dX_j^i}\chi_S(X) \tag{66}$$

We started off with the product which allowed us to define ω_R and Ω_{RS}:

$$\begin{aligned} tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X)\chi_S(X) &= \left(tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X)\right)\chi_S(X) + \chi_R(X)\left(tr(\frac{d}{dX}\frac{d}{dX})\chi_S(X)\right) \\ &+ 2\frac{d}{dX_i^j}\chi_R(X)\frac{d}{dX_j^i}\chi_S(X) \end{aligned}$$

From this we get

$$\begin{aligned} \Omega_{RS} &= \frac{1}{2}\left[\frac{d}{dX_i^j}\chi_R(X)\frac{d}{dX_j^i}\chi_S(X) - \left(tr(\frac{d}{dX}\frac{d}{dX})\chi_R(X)\right)\chi_S(X) - \left(tr(\frac{d}{dX}\frac{d}{dX})\chi_S(X)\right)\right] \\ &= \frac{1}{2}\left[\sum_{T,T''} f_{RST}\frac{c_1c_2}{c_1-c_2}\chi_{T''}(X) - \sum_{R''}\frac{c_1c_2}{c_1-c_2}\chi_{R''}(X)\,\chi_S(X) - \chi_R(X)\sum_{S''}\frac{c_1c_2}{c_1-c_2}\chi_{S''}(X)\right] \end{aligned}$$

The first term arises from multiplying the representations R and S and then pulling two boxes off of the resulting Young diagrams. The sum over T indicates the sum of all the diagrams we obtain when we multiply $R \otimes S$ and the sum over T'' adds up all of the ways of pulling boxes off of the diagrams T.

In collective field theory Ω_{RS} is generically a joining operator: it 'joins' fields labelled by R with fields labelled by S to produce a new field. As a consequence of the fact that the Schur polynomials are complete Ω_{RS} can always be written as a sum of Schur polynomials, which is in harmony with a joining operation.

Example of Ω_{RS}

We will now calculate $\Omega_{\square\square\,\square}$. On the left hand side of our equation we have

$$\begin{aligned}\Omega_{\square\square\,\square} &= \left(\frac{d}{dX_i^j}\chi_{\square\square}(X)\right)\left(\frac{d}{dX_j^i}\chi_{\square}(X)\right)\\ &= \frac{d}{dX_i^j}\left(\frac{1}{2}\left[tr(X)^2+tr(X^2)\right]\right)\frac{d}{dX_j^i}\left(\frac{1}{2}\left[tr(X)^2-tr(X^2)\right]\right)\\ &= Ntr(X)^2-tr(X^2)\end{aligned}$$

We then have

$$\Omega_{\square\square\,\square} = \frac{1}{2}\left[\sum_{T,T''} f_{\square\square\,\square\,T}\frac{c_1c_2}{c_1-c_2}\chi_{T''}(X) - \sum_{R''}\frac{c_1c_2}{c_1-c_2}\chi_{R''}(X)\,\chi_{\square}(X) - \chi_{\square\square}(X)\sum_{S''}\frac{c_1c_2}{c_1-c_2}\chi_{S''}(X)\right]$$

$$\begin{aligned}R\otimes S &= \square\square\otimes\begin{smallmatrix}\square\\\square\end{smallmatrix}\\ &= \begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}\oplus\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}\\ f_{\square\square\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} &= f_{\square\square\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\,\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}} = 1\end{aligned}$$

We have

$$T = \begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}$$

If we take a block off of each of the diagrams in T, we have

$$T = \begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}\rightarrow T' = \square\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$$

$$T = \begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix}\rightarrow T' = \square\square\square,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$$

We then follow the same process to remove another block from the original diagrams T and calculate the weights of the blocks as previously described.

The same is done for R'' and S'', leaving us with the following equation:

$$\begin{aligned}
\Omega_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} &= \frac{1}{2}((N^2+3N+2)\chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(X) - (N^2-3N+2)\chi_{\square\square}(X) - (N^2+N)\chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(X) \\
&+ (N^2-N)\chi_{\square\square}(X)) \\
&= \frac{1}{2}\left((2N+2)\chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(X) + (2N-2)\chi_{\square\square}(X)\right) \\
&= Ntr(X)^2 - tr(X^2)
\end{aligned}$$

This is a simple example, so it must be noted that in general Ω_{RS} will not necessarily simply be a linear combination of Schur polynomials. This is a feature of the example with R and S having two boxes. Also note that the sums over R'' and S'' are, in general, only coefficients or numbers when the Young diagrams R or S have two boxes or less.

6.2 The Jacobian

We now have the operators that appear in the equation that must be solved to find the Jacobian for the change of variables from matrices to Schur polynomials. This equation is adapted to our bases from the original collective field formalism:

$$-\sum_R \Omega_{SR}(\partial_R \ln J) = \omega_S + \sum_R \partial_R \Omega_{SR} \tag{67}$$

In our attempts to solve this equation, we took it as far as we could with the differential operators and our formulae for collective field operators Ω and ω:

$$\begin{aligned}
- &\sum_R \Omega_{SR}(\partial_R \ln J) \\
= &\ \omega_S
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_R \partial_R \frac{1}{2}\left[\sum_{T,T''} f_{SRT}\frac{c_1c_2}{c_1-c_2}\chi_{T''}(X) - \sum_{S''}\frac{c_1c_2}{c_1-c_2}\chi_{S''}(X)\,\chi_R(X) - \chi_S(X)\sum_{R''}\frac{c_1c_2}{c_1-c_2}\chi_{R''}(X)\right] \\
&= \quad \sum_{S''}\frac{c_1c_2}{c_1-c_2}\chi_{S''} + \sum_R\sum_{T,T''}\frac{1}{2}f_{SRT}\frac{c_1c_2}{c_1-c_2}\delta_{RT''} - \frac{1}{2}\sum_{R,R''}\frac{c_1c_2}{c_1-c_2}\chi_{R''}\delta_{RS} \\
&- \quad \frac{1}{2}\sum_{RS''}\frac{c_1c_2}{c_1-c_2}\chi_R\delta_{RS''} - \frac{1}{2}\sum_{RS''}\frac{c_1c_2}{c_1-c_2}\chi_{S''}\delta_{RR} \\
&= \quad \frac{1}{2}\sum_{S''}\frac{c_1c_2}{c_1-c_2}\chi_{S''} + \sum_R\sum_{T,T''}\frac{1}{2}f_{SRT}\frac{c_1c_2}{c_1-c_2}\delta_{RT''} - \frac{1}{2}\sum_{R,R''}\frac{c_1c_2}{c_1-c_2}\chi_{R''}\delta_{RS} - \frac{1}{2}\sum_{RS''}\frac{c_1c_2}{c_1-c_2}\chi_R\delta_{RS''}
\end{aligned}
$$

There are so many possible combinations of Young diagrams that we could write as RS that despite the δ's providing us with a few conditions for this equation, this was not very useful in solving for the Jacobian. This equation is not trivial to solve since there is no general formula for the inverse of a Schur polynomial. Another problem is that there are so many representations R and S that we could have in this equation, so it would not be conceivable to try to find J for a few examples and then try to generalize.

Another method we tried to use to find this Jacobian was to try to find the transformation from the eigenvalue basis to the Schur polynomial basis in a similar way to finding the Jacobian to change basis from matrices to eigenvalues. Recall we had the matrix model generating function given by

$$\mathcal{Z} = \int dX\, dX^\dagger e^{-S}$$

where we now study a complex matrix X. We used the Schwinger-Dyson equations of some operator to find the Jacobian of the change of variables from matrices to eigenvalues to get the eigenvalues generating function

$$\mathcal{Z} = \int \prod_i d\lambda_i \bar{\lambda}_i \Delta(\lambda_i)\Delta(\bar{\lambda}_i) e^{-S}$$

If the Jacobian for the change from matrices to eigenvalues is known and we could find the Jacobian to change from eigenvalues to Schur polynomials, the transformation between matrices and Schur polynomials would be a product of the two. The generating function in the Schur polynomial basis would be

$$\mathcal{Z} = \int \prod_R d\chi_R \, d\chi_R^\dagger \, \Delta^2 \, J(\chi_R \chi_R^\dagger) \, e^{-S}$$

To do this we wanted to try to find a relation between Schur polynomials and energy eigenstates of a system such as electrons in an oscillator potential. We used the Pauli exclusion principle to label Young diagrams in terms of excitations and energy levels. The diagram is treated as the excited state of the fermions. The ground state of the system of N fermions was added as two empty spaces below the bottom left block of a Young diagram and the excitation of each state was labeled. We then labeled these states in terms of integer valued energy levels going from zero up. We added the two to get the energy eigenstate ψ of each level. The size of the matrix is given by the number of rows of R added to the two ground state spaces, so here $N = 3$. The following table illustrates this process for $R = \square\square$:

	Excitation	Level	Sum	ψ
$\square\square$	2	2	$= 4$	$z^4 \, e^{-\frac{1}{2}z^2}$
$*$	0	1	$= 1$	$z \, e^{-\frac{1}{2}z^2}$
$*$	0	0	$= 0$	$e^{-\frac{1}{2}z^2}$

We can then show that the Slater determinant of the energy eigenstates is equivalent to the product of the Van der Monde determinant and the Schur polynomial of R. For $N = 3$ the Van der Monde determinant is given by

$$\Delta(z) = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

The Schur polynomial for the Young diagram $R = \square\square$ is

$$\chi_{\square\square}(X) = \frac{1}{2}\left(tr(X)^2 + tr(X)^2\right)$$

We can rewrite this in terms of eigenvalues as follows:

$$\chi_{\square\square}(z) = \frac{1}{2}\left((z_1+z_2+z_3)^2 + (z_1^2+z_2^2+z_3^2)\right)$$

Although it is not possible to diagonalize X in general with a unitary transformation, we can bring it to upper triangular form with the eigenvalues z_i on the diagonal. This is called the Schur decomposition of X. The above formula then follows. The possible wave functions describing the excited state are given by

$$\psi = (z_1-z_2)(z_1-z_3)(z_2-z_3)\cdot\frac{1}{2}\left((z_1+z_2+z_3)^2 + (z_1^2+z_2^2+z_3^2)\right)e^{-\frac{1}{2}(z_1^2+z_2^2+z_3^2)}$$

$$\psi = det\begin{bmatrix} z_1^4 & z_2^4 & z_3^4 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{bmatrix} e^{-\frac{1}{2}(z_1^2+z_2^2+z_3^2)}$$

The first wave function here is in terms of the Schur polynomial of the excited state and the second is in terms of the energy eigenstate. We studied the two wave functions and found that they were equal, which shows that Schur polynomials are in fact proportional to energy eigenstates, with the Van der Monde determinant as a factor of proportionality dependent on the size of the matrix. We also checked this result for the representations

$$R = \begin{array}{|c|}\hline\ \\\hline\ \\\hline\end\,, \square\square\square, \begin{array}{|c|c|}\hline\ & \\\hline\ \\\cline{1-1}\end\,, \begin{array}{|c|}\hline\ \\\hline\ \\\hline\ \\\hline\end$$

and found that it held. Unfortunately this also didn't allow us to find the Jacobian in the end.

7 Conclusions and Outlook

We set out to build a collective field theory with Schur polynomials as the collective fields. The reason for doing this was to find some way of changing the variables describing a matrix model from a matrix basis to a Schur polynomial basis in order to find a way to describe multi matrix field theory models in the large N limit. We succeeded in finding the equation which determines the Jacobian of the change of variables, but we have not yet managed to solve this equation and hence find the Jacobian. Had we succeeded in doing this, the collective field theory Hamiltonian would have been relatively straight forward to write.

It is possible to conclude that using this formalism we found elegant, simple collective field operators in the Schur polynomial basis, which leads us to hope that if we manage to find the Jacobian the complete theory will be simple. Unfortunately this project has not yet been completed due to the non trivial Jacobian equation we must still solve, which means that the full physical implications of the results we did manage to get cannot be fully explored as yet. We plan to continue with this project since it would be highly valuable to find a way to do calculations in multi-matrix systems in quantum field theory and other branches of physics.

References

[1] A Jevicki, B. Sakita, *The Quantum Collective Field Method and its Application to the Planar Limit*, Nuclear Physics **B165**, 1980, 511-527.

[2] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nuclear Physics **B72**, 1974, 461-473

[3] S. R. Das and A. Jevicki,*String field Theory and Physical Interpretation of $D = 1$ Strings*, Mod. Phys. Lett. A 5, 1639 (1990).

[4] R. de Mello Koch, T. K. Dey, N. Ives and M. Stephanou, *Correlators Of Operators with a Large R-Charge*, JHEP 0908, 083 (2009) [arXiv:0905.2273 [hep-th]].

[5] R. de Mello Koch, R.Gwyn , *Giant Graviton Correlators from Dual $SU(N)$ Super Yang Mills Theory*, JHEP 0411, 081 (2009) [arXiv:0410.236v2 [hep-th]].

[6] Srednicki, M. *Quantum Field Theory.* Cambridge: Cambridge University Press, 2007

[7] Lee Tsung-dao. *Particle Physics And Introduction To Field Theory.*Chur: Harwood Academic, 1982

[8] Zwiebach, B. *A First Course in String Theory* . New York : Cambridge University Press, 2004

[9] Editors: Brezin, E. and Wadia, S.R. *The Large N expansion in quantum field theory and statistical physics: from spin systems to 2-dimensional gravity.* Singapore: World Scientific, 1993

[10] Jones, H. F. *Groups, Representations, and Physics*. Bristol : Institute of Physics, 1998

[11] Joshi, A. W .*Elements Of Group Theory For Physicists*. New Delhi: Wiley Eastern, 1977

[12] R. de Mello Koch, M. Stephanou and D. Bekker, *Giant Gravitons - with Strings Attached (III)*, JHEP 0802:029 (2008) [arXiv:0710.5372 [hep-th]]

[13] R. de Mello Koch, *Geometries from Young Diagrams*, JHEP 0811:061 (2008) [arXiv:0806.0685v2 [hep-th]]

[14] A. Okounkov, *Random Matrices and Random Permutations*, (2000) [arXiv:math/9903176v3 [math.CO]]

[15] J.P. Rodrigues, R. de Mello Koch, *The Collective Field Theory of a Singular Supersymmetric Matrix Model*, Phys.Rev. **D51** (1995) 5847-5854, [arXiv: 9410012v3[hep-th]]