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Hindawi
Mathematical Problems in Engineering
Volume 2021, Article ID 6619325, 7 pages
https://doi.org/10.1155/2021/6619325

Research Article

A Note on the Integration of Scalar Fourth-Order Ordinary Differential Equations with Four-Dimensional Symmetry Algebras

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Received 29 October 2020; Revised 25 January 2021; Accepted 2 March 2021; Published 19 March 2021

Academic Editor: Mariusz Michta

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The complete integration of scalar fourth-order ODEs with four-dimensional symmetry algebras is performed by utilizing Lie's method which was invoked to integrate scalar second-order ODEs admitting two-dimensional symmetry algebras. We obtain a complete integration of all scalar fourth-order ODEs that possess four Lie point symmetries.

1. Introduction

Lie devised a method to integrate a scalar second-order ordinary differential equation (ODE) if it admits a two-dimensional symmetry algebra by reducing the order of the equation twice and hence obtaining integrability by quadratures for the ODE (see [1, 2]). Following the same method, Ibragimov and Nucci [3] gave an algorithmic procedure to integrate scalar third-order ODEs admitting three-dimensional symmetry algebras. In their paper, Cerquetelli et al. [4] classified scalar fourth-order ODEs admitting four-dimensional symmetry algebras by first presenting a classification of realizations of all four-dimensional Lie algebras in the plane. In their work, some cases were missing which were then completed by Fatima et al. in [5]. In [4], the authors had also introduced Lie's method for integrating fourth-order equations admitting four-dimensional symmetry algebras.

In this note, we extend these works to complete the integration of scalar fourth-order ODEs admitting four-dimensional symmetry algebras. First, we compare the realizations in the plane given in [4, 5] with that of the realizations given in González-López et al. [6] and show that they are compatible. For each case, we establish the required coordinate transformations. To denote a general four-dimensional algebra, we have used the notation in [5], and for the algebras in [6], the notation (m, n), where m is the type of algebra and n is the dimension of the algebra which in our case is four, is utilized. We have then found all the scalar fourth-order ODEs with symmetry and have also used the classification given in [4, 5] whenever needed. Thereafter, we perform a complete integration of all scalar fourth-order ODEs possessing four symmetries. Note that, in [5], the authors only consider the Lie algebraic classification and not the integration of scalar fourth-order ODEs which admit four-dimensional symmetry algebras. Here, apart from the comparison of the realizations obtained in both [4, 5] with that of [6], we obtain a complete integration of all scalar fourth-order ODEs which admit four point symmetries. Also, we present applications of our results.

In Section 2, we present the integration of all scalar fourth-order ODEs that admit four Lie point symmetries. We utilise a three-dimensional ideal of the algebra to reduce the equation to order one which admits the quotient algebra

and which is hence integrable. We present the main results in tabular form: first, the correspondence of the realizations utilized for fourth-order ODEs with that of González-López et al. [6] is given in Table 1. Then, in Table 2, the canonical forms of scalar fourth-order ODEs which possess 4 symmetries are presented. In Table 3, invariant representation of 4th-order ODEs admitting 3-dimensional ideals is listed. This is followed by the integrability of 4th-order ODEs with four symmetries in Table 4. The main features of integrability are discussed thereafter. Section 3 deals with applications of scalar higher-order ODEs. These occur in practice in beam theory as well as in the reduction of partial differential equations (PDEs) which arise in several areas. A brief conclusion is provided with summary and related questions in the last section.

2. Integration of Fourth-Order ODEs Admitting Four-Dimensional Algebras

Let a scalar fourth-order ODE admit a four-dimensional symmetry algebra, say $\mathfrak{g}_4$. Such an algebra always has a three-dimensional ideal, say $\mathfrak{I}_3$. We use it to transform the fourth-order ODE into a first-order ODE in new coordinates which are precisely the differential invariants of $\mathfrak{I}_3$ of order ≤ 3. This transformed first-order ODE admits $\mathfrak{g}_4/\mathfrak{I}_3$ and hence can be integrated easily [2]. The general form of solution of this first-order ODE in original coordinates is a scalar third-order ODE admitting $\mathfrak{I}_3$ which can be integrated further by Lie's method as performed in [3]. One notes that all the four-dimensional algebras are solvable except for two cases which are (8) and (9), and thus, the corresponding equations can be integrated easily by quadratures. We follow the same steps to integrate all possible scalar fourth-order ODEs possessing four-dimensional symmetry algebras. This is the reason that we give some of the cases in detail, and all the necessary information for other cases is divided into four tables in the following pages.

Remark 1. In Table 2, we have not stated the 4-dimensional algebras $(21, 4)$ and $4A_1$ of Table 1 for the reason that a 4th-order ODE admitting these can be mapped to linearizable ODEs of the same order by point transformations that admit fifth generators $\phi_1(x)\partial_y$, $\phi_1'' \neq 0$, with ϕ_1 solving the linearized ODE $y^{(4)} = (\phi^{(4)}(x)/\phi^{(2)}(x))y'' + H(x)(\phi^{(2)}y''' - \phi^{(3)}(x)y'')$ ($\phi_1$different from ϕ) and $y\partial_y$ of a linear ODE, respectively. Since such ODEs must admit 5-dimensional algebras, we exclude these. In Tables 2–4, we list the algebras of Table 1 as (1) to (23). Also, a, b are constants and H a function of its argument.

We present the salient features of integration of some of the cases as in the above tables according to the nomenclature utilized in the following tables below. The remaining cases in the tables are then easy to understand.

(1) $(2A_2, 5)$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y$.

(13, 4): with the same generators.

Fourth-order equation:

$$y^{(4)} = \frac{y''^3}{y'^2} H\left(\frac{y' y'''}{y''^2}\right). \tag{1}$$

The three-dimensional ideal of the above algebra is generated by X_1, X_2, and X_3. Invariants of this three-dimensional ideal of order ≤ 3 are $u = y'' y'^{-2}$, $v = y''' y'^{-3}$. Equation (1) in terms of u and v coordinates is

$$\left(\frac{v}{u^2} - 2\right)v' + 3\frac{v}{u} = uH\left(\frac{v}{u^2}\right). \tag{2}$$

Here, X_4 in terms of u and v coordinates is $-u\partial_u - 2v\partial_v$ which is admitted by equation (2). Defining new coordinates as $\overline{u} = \ln(1/u)$, $\overline{v} = (v/u^2)$, X_4 is transformed into $\partial_{\overline{u}}$. Note that these coordinates can be found by solving the following system of linear first-order partial differential equations by the method of characteristics:

$$\begin{aligned} -u\frac{\partial_{\overline{u}}}{\partial_u} - 2v\frac{\partial_{\overline{u}}}{\partial_v} &= 1, \\ -u\frac{\partial_{\overline{v}}}{\partial_u} - 2v\frac{\partial_{\overline{v}}}{\partial_v} &= 0. \end{aligned} \tag{3}$$

Equation (2) in terms of $\overline{u}$ and $\overline{v}$ coordinates is

$$\overline{v}' = G(\overline{v}), \tag{4}$$

where $G(\overline{v}) := ((H(\overline{v}) - 3\overline{v})/(2 - \overline{v}))$ and the equation admits $\partial_{\overline{u}}$. Integrating this equation gives $\overline{v} = F(\overline{u} + c_1)$. This in terms of the original coordinates is a third-order equation admitting the three-dimensional algebra generated by X_1, X_2, and X_3 which can be integrated as in [3].

(9) $A_{3,8} \oplus A_1$, $(\mathfrak{sl}(2, IR) \oplus A_1, 8)$: $X_1 = \partial_y, X_2 = x\partial_x + y\partial_y, X_3 = 2xy\partial_x + y^2\partial_y, X_4 = x\partial_x$.

(19, 4): $X_1 = \partial_x, X_2 = 2x\partial_x + y\partial_y, X_3 = x^2\partial_x + xy\partial_y, X_4 = y\partial_y$.

Coordinate transformations: $\overline{x} = y, \overline{y} = x$.

Change in basis: $\overline{X_1} = X_1, \overline{X_2} = (X_2/2), \overline{X_3} = X_3, \overline{X_4} = X_4$.

Fourth-order equation:

$$y^{(4)} = \frac{4}{3}\frac{y'''^2}{y''} + \frac{y''^2}{y} H(K_1), \tag{5}$$

where $K_1 = (y'''^2 y/y''^3) + 9(y'^2/yy'') + 6(y' y'''/y''^2)$.

The three-dimensional ideal: X_1, X_2, and X_3.

Invariants of order ≤ 3: $u = y'' y^3$, $v = y''' y^5 + 3y' y'' y^4$.

Equation (5) in terms of u and v coordinates is as follows:

$$v' = \frac{4}{3}\frac{v}{u} + \frac{u^2}{v}\left(3 + H\left(\frac{v^2}{u^3}\right)\right). \tag{6}$$

X_4 in terms of u and v coordinates: $4u\partial_u + 6v\partial_v$.

Table 1: Correspondence of the realizations in [5, 6].

Algebra in [5]	Algebra in [6]	Transformed generators	Coordinate transformations
$(2A_2, 5)$	(13, 4)	$X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y$	Same generators
$(2A_2, 7)$	(23, 4), $r = 2$	$X_1 = e^{-x}\partial_y, X_2 = \partial_x, X_3 = \partial_y, X_4 = y\partial_y.$	$\overline{x} = -\ln x,\ \overline{y} = y$
$A_{3,2} \oplus A_1$	(22, 4), $r = 3$	$X_1 = e^{-x}\partial_y, X_2 = -xe^{-x}\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$\overline{x} = -x,\ \overline{y} = ye^x$
$A_{3,4}^{-1} \oplus A_1$	(22, 4), $r = 3$	$X_1 = e^{-x}\partial_y, X_2 = e^x\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$\overline{x} = (\ln x/2),\ \overline{y} = y\lvert x\rvert^{(-1/2)}$
$A_{3,4}^{a} \oplus A_1$	(22, 4), $r = 3$	$X_1 = e^{-x}\partial_y, X_2 = e^{-ax}\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$\overline{x} = (\ln x/(1-a)),\ \overline{y} = y\lvert x\rvert^{(1/(a-1))}$
$A_{3,5}^{0} \oplus A_1$	(22, 4), $r = 3$	$X_1 = \sin x\partial_y, X_2 = \cos x\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$\overline{x} = \cot^{-1} x, \overline{y} = (y/(\sqrt{1+x^2}))$
$A_{3,5}^{b} \oplus A_1$	(22, 4), $r = 3$	$X_1 = e^{-bx}\sin x\partial_y, X_2 = e^{-bx}\cos x\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$\overline{x} = \cot^{-1} x, \overline{y} = (e^{-b\arctan x}/(\sqrt{1+x^2}))y$
$(\mathfrak{sl}(2, IR) \oplus A_1, 9)$	(14, 4)	$X_1 = \partial_x, X_2 = x\partial_x, X_3 = x^2\partial_x, X_4 = \partial_y$	$\overline{x} = y, \overline{y} = x$
$(\mathfrak{sl}(2, IR) \oplus A_1, 8)$	(19, 4)	$X_1 = \partial_x, X_2 = 2x\partial_x + y\partial_y, X_3 = x^2\partial_x + xy\partial_y, X_4 = y\partial_y$	Scaling X_2 by (1/2) only
$A_{4,1}$	(22, 4), $r = 3$	$X_1 = \partial_y, X_2 = x\partial_y, X_3 = (x^2/2)\partial_y, X_4 = \partial_x$	Same generators
$A_{4,2}^{b},\ b \neq 0, 1$	(22, 4), $r = 3$	$X_1 = e^{-bx}\partial_y, X_2 = e^{-x}\partial_y, X_3 = -xe^{-x}\partial_y, X_4 = \partial_x$	$\overline{x} = (\ln x/(b-1)), \overline{y} = y\lvert x\rvert^{(b/(1-b))}$
$A_{4,3}$	(22, 4), $r = 3$	$X_1 = e^{-x}\partial_y, X_2 = \partial_y, X_3 = -x\partial_y, X_4 = \partial_x$	$\overline{x} = \ln x, \overline{y} = (y/x)$
$A_{4,4}$	(22, 4), $r = 3$	$X_1 = e^{-x}\partial_y, X_2 = -xe^{-x}\partial_y, X_3 = (x^2/2)e^{-x}\partial_y, X_4 = \partial_x$	$\overline{x} = -x,\ \overline{y} = ye^x$
$A_{4,5}^{a,b,c}$	(22, 4), $r = 3$	$X_1 = e^{-ax}\partial_y, X_2 = e^{-bx}\partial_y, X_3 = e^{-x}\partial_y, X_4 = \partial_x$	$\overline{x} = x,\ \overline{y} = ye^{-ax}$
$A_{4,5}^{1,1,1}$	(21, 4), $r = 2$	$X_1 = \partial_y, X_2 = x\partial_y, X_3 = \phi(x)\partial_y, X_4 = y\partial_y$	$\overline{x} = \xi_1(x)$ and $\overline{y} = y$
$A_{4,6}^{a,b}, a > 0$	(22, 4), $r = 3$	$X_1 = e^{-ax}\partial_y, X_2 = e^{-bx}\cos x\partial_y, X_3 = e^{-bx}\sin x\partial_y, X_4 = \partial_x$	$\overline{x} = x,\ \overline{y} = ye^{-ax}$
$A_{4,7}$	(25, 4), $r = 2$	$X_1 = \partial_y, X_2 = x\partial_y, X_3 = -\partial x\,, X_4 = x\,\partial x + (2y + x^2)\partial y$	$\overline{x} = x, \overline{y} = y + x^2\ln x^{\frac{3}{2}}$
$A_{4,8}^{b},\ b = -1$	(24, 4), $\alpha = 0$	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x$	Same generators
$A_{4,8}^{b},\ \lvert b\rvert \leq 1,$ $b \neq -1$	(24, 4), $\alpha = b + 1$	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + (1+b)y\partial_y$	Same generators
$A_{4,8}^{b},\ b = 1$	(24, 4), $\alpha = 2$	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + 2y\partial_y$	Same generators
$A_{4,8}^{b},\ b = 0$	(24, 4), $\alpha = 1$	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + y\partial_y$	Same generators
$A_{4,8}^{b},\ b \neq \pm 1$	(24, 4), $\alpha = ((b+1)/b)$	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + ((1+b)/b)y\partial_y$	Scaling X_4 by $(1/b)$
$(A_{4,10}, 7)$	(23, 4), $r = 2$	$X_1 = \sin x\partial_y, X_2 = \cos x\partial_y, X_3 = y\partial_y, X_4 = \partial_x$	$\overline{x} = \cot^{-1} x, \overline{y} = (y/(1+x^2)^{(1/2)})$
$(A_{4,10}, 6^*)$	(4, 4)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_x + y\partial_y, X_4 = -y\partial_x + x\partial_y$	Same generators
$4A_1$	(20, 4), $r = 3$	$X_1 = \partial_y, X_2 = x\partial_y, X_3 = \xi_2(x)\partial_y, X_4 = \xi_3(x)\partial_y$	$\overline{x} = \xi_1(x)$ and $\overline{y} = y$

Table 2: Representative 4th-order ODEs with 4 symmetries.

Number	Generators of algebras in [6]	Corresponding fourth-order invariant equations
(1)	$X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y$	$y^{(4)} = (y''^3/y'^2)(H(y'y''')/y''2)$
(2)	$X_1 = e^{-x}\partial_y, X_2 = \partial_x, X_3 = \partial_y, X_4 = y\partial_y.$	$y^{(4)} = -y''' + (y'' + y''')H((y'' + y''')/(y' + y''))$
(3)	$X_1 = e^{-x}\partial_y, X_2 = -xe^{-x}\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$y^{(4)} = -y''' + (y'' + y''')H((y'' + y''')/(y' + y''))$
(4)	$X_1 = e^{-x}\partial_y, X_2 = e^x\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$y^{(4)} = y'' + H(y''' - y')$
(5)	$X_1 = e^{-x}\partial_y, X_2 = e^{-ax}\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$y^{(4)} = -(a+1)y''' - ay'' + H(y''' + (a+1)y'' + ay')$
(6)	$X_1 = \sin x\partial_y, X_2 = \cos x\partial_y, X_3 = \partial_x, X_4 = \partial_y$	$y^{(4)} = -y'' + H(y''' + y')$
(7)	$X_1 = e^{-bx}\sin x\partial_y, X_2 = e^{-bx}\cos x\partial_y$ $X_3 = \partial_x, X_4 = \partial_y$	$y^{(4)} = -2by''' - (b^2+1)y''' + H(K_1)$ $K_1 = y''' + 2by'' + (b^2+1)y'$
(8)	$X_1 = \partial_x, X_2 = x\partial_x, X_3 = x^2\partial_x, X_4 = \partial_y$	$y^{(4)} = 3(y''^3/y'^2) + 6K_1y''y'^2 + y'^4H(K_1)$ $K_1 = y'''y'^{-3} - (3/2)y''^2y'^{-4}$

Table 2: Continued.

Number	Generators of algebras in [6]	Corresponding fourth-order invariant equations
(9)	$X_1 = \partial_x, X_2 = 2x\partial_x + y\partial_y$ $X_3 = x^2\partial_x + xy\partial_y, X_4 = y\partial_y$	$y^{(4)} = (4/3)(y'''^2/y'') + (y''^2/y)H(K_1)$ $K_1 = (y'''^2y''/y''^3) + 9(y'^2/yy'') + 6(y'y'''/y''^2)$
(10)	$X_1 = \partial_y, X_2 = x\partial_y, X_3 = (x^2/2)\partial_y, X_4 = \partial_x$	$y^{(4)} = H(y''')$
(11)	$X_1 = e^{-bx}\partial_y, X_2 = e^{-x}\partial_y, X_3 = -xe^{-x}\partial_y, X_4 = \partial_x$	$y^{(4)} = -(b+2)y''' - (2b+1)y'' - by' + H(K_1)$ $K_1 = y''' + (2+b)y'' + (2b+1)y' + by$
(12)	$X_1 = e^{-x}\partial_y, X_2 = \partial_y, X_3 = -x\partial_y, X_4 = \partial_x$	$y^{(4)} = -y''' + H(y'' + y''')$
(13)	$X_1 = e^{-x}\partial_y, X_2 = -xe^{-x}\partial_y$ $X_3 = (x^2/2)e^{-x}\partial_y, X_4 = \partial_x$	$y^{(4)} = y'''H(y'''e^x)$
(14)	$X_1 = e^{-ax}\partial_y, X_2 = e^{-bx}\partial_y, X_3 = e^{-x}\partial_y, X_4 = \partial_x$	$y^{(4)} = -(a+b+1)y''' - (a+b+ab)y'' - aby' + H(K_1)$ $K_1 = y''' + (a+b+1)y'' + (a+b+ab)y' + aby$
(15)	$X_1 = e^{-ax}\partial_y, X_2 = e^{-bx}\cos x\partial_y$ $X_3 = e^{-bx}\sin x\partial_y, X_4 = \partial_x$	$y^{(4)} = -(a+2b)y''' - (b^2+2ab+1)y'' - a(b^2+1)y' + H(K_1)$ $K_1 = y''' + (a+2b)y'' + (b^2+2ab+1)y' + a(b^2+1)y$
(16)	$X_1 = \partial_y, X_2 = x\partial_y$ $X_3 = -\partial x, X_4 = x\partial x + (2y + x^2)\partial y$	$y^{(4)} = y'''^2H(y'''^2e^{y''})$
(17)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x$	$y^{(4)} = y'''^{(4/3)}H(y'''^{-2}/y''^{-3})$
(18)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + (1+b)y\partial_y$	$y^{(4)} = y'''^{((b-3)/(b-2))}H(y'''^{b-1}/y''^{b-2})$
(19)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + 2y\partial_y$	$y^{(4)} = y'''^2H(y'')$
(20)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + y\partial_y$	$y^{(4)} = y'''^{(3/2)}H(y'''^{-1}/y''^{-2})$
(21)	$X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_y, X_4 = x\partial_x + ((1+b)/b)y\partial_y$	$y^{(4)} = y'''^{((1-3b)/(1-2b))}H(y'''^{((b-1)/b)}/y''^{((b-2)/b)})$
(22)	$X_1 = \sin x\partial_y, X_2 = \cos x\partial_y, X_3 = y\partial_y, X_4 = \partial_x$	$y^{(4)} = -y'' + (y''' + y')H((y''' + y')/(y'' + y))$
(23)	$X_1 = \partial_y, X_2 = \partial_x$ $X_3 = x\partial_x + y\partial_y, X_4 = -y\partial_x + x\partial_y$	$y^{(4)} = y''^3((15y'^2 + 10y'K_1 + H(K_1))/(1+y'^2)^2)$ $K_1 = ((1+y'^2)y'''/y''^2) - 3y'$

Table 3: Invariant representation of 4th-order ODEs admitting 3-dimensional ideals.

Algebras [6]	3-dimensional ideal	Invariants of order ≤3	Invariant equation
(1)	$\langle X_1, X_2, X_3\rangle$	$u = y''y'^{-2}, v = y'''y'^{-3}$	$((v/u^2) - 2)v' + 3(v/u) = uH(v/u^2)$
(2)	$\langle X_1, X_2, X_3\rangle$	$u = y' + y'', v = y'' + y'''$	$v' = H(v/u)$
(3)	$\langle X_1, X_2, X_3\rangle$	$u = y'' + 2y' + y$ $v = y''' + 2y'' + y'$	$v' = H(v)$
(4)	$\langle X_1, X_2, X_4\rangle$	$u = x, v = y''' - y'$	$v' = H(v)$
(5)	$\langle X_1, X_2, X_4\rangle$	$u = x$ $v = y''' + (a+1)y'' + ay'$	$v' = H(v)$
(6)	$\langle X_1, X_2, X_4\rangle$	$u = x, v = y''' + y'$	$v' = H(v)$
(7)	$\langle X_1, X_2, X_4\rangle$	$u = x$ $v = y''' + 2by'' + (b^2+1)y'$	$v' = H(v)$
(8)	$\langle X_1, X_2, X_3\rangle$	$u = y, v = y'''y'^{-3} - (3/2)y''^2y'^{-4}$	$v' = H(v)$
(9)	$\langle X_1, X_2, X_3\rangle$	$u = y''y^3, v = y'''y^5 + 3y'y''y^4$	$v' = (4/3)(v/u) + (u^2/v)(3 + H(v^2/u^3))$
(10)	$\langle X_1, X_2, X_3\rangle$	$u = x, v = y'''$	$v' = H(v)$
(11)	$\langle X_1, X_2, X_3\rangle$	$u = x$ $v = y''' + (2+b)y'' + (2b+1)y' + by$	$v' = H(v)$
(12)	$\langle X_1, X_2, X_3\rangle$	$u = x, v = y''' + y''.$	$v' = H(v)$
(13)	$\langle X_1, X_2, X_3\rangle$	$u = x, v = y'''$	$v' = vH(ve^u)$
(14)	$\langle X_1, X_2, X_3\rangle$	$u = x$ $v = y''' + (a+b+1)y'' +$ $(a+b+ab)y' + aby$	$v' = H(v)$
(15)	$\langle X_1, X_2, X_3\rangle$	$u = x$ $v = y''' + (a+2b)y'' +$ $(b^2+2ab+1)y' + a(b^2+1)y$	$v' = H(v)$
(16)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = vH(v^2e^u)$
(17)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = v^{(1/3)}H(v^{-2}/u^{-3})$
(18)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = v^{(-1/(b-2))}H(v^{b-1}/u^{b-2})$
(19)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = vH(u)$

Table 3: Continued.

Algebras [6]	3-dimensional ideal	Invariants of order ≤3	Invariant equation
(20)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = v^{(1/2)}H(v^{-1}/u^{-2})$
(21)	$\langle X_1, X_2, X_3\rangle$	$u = y'', v = y'''$	$v' = v^{(-b/(1-2b))}H(v^{((b-1)/b)}/u^{((b-2)/b)})$
(22)	$\langle X_1, X_2, X_3\rangle$	$u = y'' + y, v = y''' + y'$	$v' = H(v/u)$
(23)	$\langle X_1, X_2, X_4\rangle$	$u = y', v = y''' y''^{-2}$	$v' = -2v^2 + ((15u^2 + 10uK_1 + H(K_1))/(1+u^2)^2)$ $K_1 = (1+u^2)v - 3u$

Table 4: Integrability of 4th-order ODEs admitting 4 symmetries.

Algebras. [6]	Symmetry	Canonical coordinates	Integrable equation
(1)	$-u\partial_u - 2v\partial_v$	$\overline{u} = \ln(1/u)$, $\overline{v} = (v/u^2)$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := ((H(\overline{v}) - 3\overline{v})/(2 - \overline{v}))$
(2)	$u\partial_u + v\partial_v$		$\overline{v}' = H(\overline{v})$
(3)	∂_u		$v' = H(v)$
(4)	∂_u		$v' = H(v)$
(5)	∂_u	$\overline{u} = \ln u$, $\overline{v} = (v/u)$	$v' = H(v)$
(6)	∂_u		$v' = H(v)$
(7)	∂_u		$v' = H(v)$
(8)	∂_u		$v' = H(v)$
(9)	$4u\partial_u + 6v\partial_v$		$v' = H(v)$ $G(\overline{v}) := (32/3)\overline{v} + 8H(\overline{v}) + 24$
(10)	∂_u	$\overline{u} = (\ln u/4)$ and $\overline{v} = (v^2/u^3)$	$v' = H(v)$
(11)	∂_u		$v' = H(v)$
(12)	∂_u		$v' = H(v)$
(13)	$-\partial_u + v\partial_v$		$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := -\overline{v}H(\overline{v})$
(14)	∂_u	$\overline{u} = -u$ and $\overline{v} = ve^u$	$v' = H(v)$
(15)	∂_u		$v' = H(v)$
(16)	$2\partial_u - v\partial_v$	$\overline{u} = (u/2)$ and $\overline{v} = v^2e^u$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := 4\overline{v}H(\overline{v})$
(17)	$-2u\partial_u - 3v\partial_v$	$\overline{u} = -(1/2)\ln u$, $\overline{v} = (v^{-2}/u^{-3})$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := 4\overline{v}^{2/3}H(\overline{v})$
(18)	$(b-1)u\partial_u + (b-2)v\partial_v$	$\overline{u} = (1/(b-1))\ln u$, $\overline{v} = (v^{b-1}/u^{b-2})$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := (b-1)^2\overline{v}^{((b-1)/(b-2))}H(\overline{v})$
(19)	$-v\partial_v$	$\overline{u} = u$, $\overline{v} = -\ln v$	$\overline{v}' = -H(\overline{u})$
(20)	$-u\partial_u - 2v\partial_v$	$\overline{u} = -\ln u$, $\overline{v} = (v^{-1}/u^{-2})$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := \overline{v}^{1/2}H(\overline{v})$
(21)	$((1-b)/b)u\partial_u + ((1-2b)/b)v\partial_v$	$\overline{u} = (b/(1-b))\ln u$, $\overline{v} = (v^{((b-1)/b)}/u^{((b-2)/b)})$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := (b^2/(1-b)^2)\overline{v}^{((1-b)/(1-2b))}H(\overline{v})$
(22)	$u\partial_u + v\partial_v$	$\overline{u} = \ln u$, $\overline{v} = (v/u)$	$\overline{v}' = H(\overline{v})$
(23)	$(1+u^2)\partial_u + (3-2uv)\partial_v$	$\overline{u} = \arctan u$, $\overline{v} = (1+u^2)v - 3u$	$\overline{v}' = G(\overline{v})$ $G(\overline{v}) := H(\overline{v}) - 2\overline{v}^2 - 3$

The coordinates which transform X_4 into $\partial_{\overline{u}}$: $\overline{u} = (\ln u/4)$ and $\overline{v} = (v^2/u^3)$.

Equation (6) in terms of $\overline{u}$ and $\overline{v}$ coordinates:

$$\overline{v}' = \frac{32}{3}\overline{v} + 8H(\overline{v}) + 24. \tag{7}$$

We define $G(\overline{v}) := (32/3)\overline{v} + 8H(\overline{v}) + 24$. Then, equation (7) becomes

$$\overline{v}' = G(\overline{v}). \tag{8}$$

Admitting $\partial_{\overline{u}}$, equation (8) can now easily be integrated.

(24) $(A_{4,10}, 6^*)$: $X_1 = \partial_y, X_2 = \partial_x, X_3 = x\partial_x + y\partial_y$, $X_4 = -y\partial_x + x\partial_y$.

(4, 4): with the same generators.

Fourth-order equation:

$$y^{(4)} = y''^3\left(\frac{15y'^2 + 10y'K_1 + H(K_1)}{\left(1+y'^2\right)^2}\right), \tag{9}$$

where $K_1 = ((1 + y'^2)y'''/y''^2) - 3y'$. The three-dimensional ideal of the above algebra is generated by X_1, X_2, and X_3. Invariants of this ideal of order ≤ 3 are $u = y'$, $v = y'''y''^{-2}$. Equation (9) in terms of u and v coordinates is

$$v' = -2v^2 + \frac{15u^2 + 10uK_1 + H(K_1)}{\left(1 + u^2\right)^2}, \tag{10}$$

where $K_1 = (1 + u^2)v - 3u$. This equation admits X_4 which in terms of u and v coordinates is $(1 + u^2)\partial_u + (3 - 2uv)\partial_v$. Defining new coordinates as $\overline{u} = \arctan u, \overline{v} = (1 + u^2)v - 3u$, X_4 transforms into $\partial_{\overline{u}}$. Equation (10) in terms of $\overline{u}$ and $\overline{v}$ coordinates then becomes

$$\overline{v}' = G(\overline{v}), \tag{11}$$

where $G(\overline{v}) := H(\overline{v}) - 2\overline{v}^2 - 3$. This equation clearly admits $\partial_{\overline{u}}$. Integrating equation (11) gives $\overline{v} = F(\overline{u} + c_1)$ which in original coordinates is a third-order equation admitting the three-dimensional algebra generated by X_1, X_2, and X_3 and therefore can be integrated further. All the cases are given in Tables 1–4.

3. Applications

Fourth-order ODEs have many applications. They occur as reductions of PDEs in, for example, fluid mechanics and dynamic beam theory (see, e.g., [7–10]). Also, they appear in static beam theory [11].

The static Euler–Bernoulli beam ODE is of order 4 and has the following form:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = g. \tag{12}$$

This ODE relates the static beam's deflection y at some point x with the applied load g which can depend upon x and y and the derivatives of the deflection y. Here, the product EI is the flexural rigidity and, in many engineering applications, is taken as constant.

Another form of a fourth-order ODE in beam-column theory is

$$EI\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f, \tag{13}$$

in which y is the deflection laterally and f the intensity of the distributed load. Also, P stands for the axial compressive force.

Interesting work on certain classes of the static beam model has been carried out in [12, 13] in which the authors utilise the Lie and Noether approaches for reductions and solutions.

Here, we show how the integration scheme obtained here for fourth-order ODEs with four symmetries can be used. If EI is constant, then (12) is a general fourth-order ODE. It can be integrated by quadrature via the approach here, provided it has 4 symmetries. There are thus many cases as stated in Table 4. We mention two other examples below for each of (12) and (13).

(1) In (6) of Table 2, we take $EI = 1$ and $P = 1$. Then, the fourth-order ODE becomes

$$y^{(4)} + y'' = H\left(y' + y'''\right). \tag{14}$$

Therefore, the intensity of the distributed load f as in (13) is H. This ODE is integrable, as shown in Tables 3 and 4.

(2) If we take $EI = 1$ in (12) and $g = y''^{\prime 2}H(y'')$, then it becomes

$$y^{(4)} = y'''^2H(y''). \tag{15}$$

This corresponds to (19) in Table 2 and is integrable by quadratures.

4. Conclusion

In this note, we firstly presented a reconciliation of the existing realizations utilized for the classification of scalar fourth-order ODEs that maximally admit four-dimensional Lie algebras. Then, we provided a complete integration algorithm for such equations thus completing the works [4, 5]. The main results are concisely presented in Section 2, wherein the correspondence of the realizations of [4, 5] is given with those of [6]. Then, the scalar fourth-order equations which possess four symmetries are listed followed by their complete integrability.

In Section 3, we presented applications of the integration results to equations in beam theory and fluids mechanics. It is worthwhile mentioning that higher-order ODEs arise in several applications, including symmetry reductions of PDEs (see [7, 14, 15]), and various integration strategies are useful, not least the symmetry approach advanced here.

Data Availability

Mathematical models and proofs are described.

Conflicts of Interest

The authors declare that there are no conflicts of interest in this work.

Acknowledgments

FM thanks the NRF of South Africa for support.

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