(2) (Szegő): Let $w(x)$ be a weight function on the interval $[-1,1]$, $x=\cos\theta$ such that $w(\cos\theta)|\sin\theta| = f(\theta)$ where $|f(\theta+\delta)-f(\delta)| < L|\log\delta|^{-1-\lambda}$; L and λ are fixed positive numbers, $\delta>0$ and $f(\theta)$ is a positive weight function on τ. Let sign $D(e^{i\theta}) = \dfrac{D(e^{i\theta})}{|D(e^{i\theta})|} = e^{i\tau(\theta)}$. Then uniformly on $[-1,1]$ or $0\le\theta\le\pi$, $x=\cos\theta$,

$$(1-x^2)^{1/4}\big(w(x)\big)^{1/2}p_n(x) = \left(\frac{2}{\pi}\right)^{1/2}\cos\big(n\theta+\tau(\theta)\big) + O(\log n)^{-\lambda}.$$

In fact many weights on $[-1,1]$ admit the asymptotic.

(3) $p_n(w,\cos\theta)w(\cos\theta)^{1/2}|\sin\theta|^{1/2} = \left(\frac{2}{\pi}\right)^{1/2}\cos\big(n\theta+\tau(\theta)\big) + o(1)$, $\theta\in(0,\pi)$ as $n\to\infty$ where $\tau(\theta)$ depends on w.

(4) Let $\mu(\theta) = w^2\cos\theta|\sin\theta|$, $\theta\in[-\pi,\pi]$, $w, w^2 \in L^1[-\pi,\pi]$, *supp* (w) having positive measure. Let w satisfy a Szegő condition

$$\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}dx > -\infty$$

then

$$\lim_{n\to\infty}\beta_n(w^2)z^{-n} = (2\pi)^{-1/2}D^{-1}(\mu,0)$$

and uniformly in closed subsets of $\mathbf{C}\setminus[-1,1]$,

$$\lim_{n\to\infty}\frac{p_n(w^2,z)}{\big(J(z)\big)^n} = (2\pi)^{-1/2}D^{-1}\Big(\mu,\big(2J(z)\big)^{-1}\Big).$$

(5) (Nevai et al): Let $(a_n)_{n=0}$ and $(b_n)_{n=0}$ be two sequences of real numbers such that $a_0=0$, $a_n>0$, $n=1,2,\ldots$, $\lim_{n\to\infty}a_n = \frac{1}{2}$, $\lim_{n\to\infty}b_n=0$ and $\sum_{n=0}^{\infty}\epsilon_n<\infty$, $\epsilon_n = |a_{n+1}-a_n|+|b_{n+1}-b_n|$. Also let $(p_n)_{n=0}$ be the system of polynomials generated by the three term recursion formula

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x),\ n=0,1,\ldots$$

where $p_{-1}=0$, $p_0=\beta_0>0$. Also let $d\alpha$ be the unique positive measure on $\mathbb{R}$ such that the polynomials p_n are orthonormal with respect to $d\alpha$ ([4] p.60).

Also $d\alpha$ satisfies

$$[-1,1] \subset supp(d\alpha) \subset \left[\inf_{n\geq 0} b_n - 2\sup_{n\geq 1} a_n,\ \sup b_n + 2\sup_{n\geq 1} a_n\right],$$

$\mathrm{supp}(d\alpha)\backslash[-1,1]$ is at most countable with no points of accumulation other than ± 1, $d\alpha$ absolutely continuous in $(-1,1)$ and α' positive and continuous in $(-1,1)$ ([17] p.231).

Let $J(x) = x + \sqrt{x^2-1} > 1$ if $x > 1$ and analytic in $\mathbf{C}\backslash[-1,1]$. Also define for $x \in (-1,1)$, $J(x) = \lim_{y\to 0^+} J(x+iy)$ so that $|J(x)| = 1$, $0 < \arg J(x) < \pi$. Also for a given x and n let

$$t_{1n}(x) = \sqrt{\frac{a_n}{a_{n+1}}}\,J\left(\frac{x-b_n}{2\sqrt{a_n a_{n+1}}}\right) \quad \text{and} \quad t_{2}(x) = \sqrt{\frac{a_n}{a_{n+1}}}\left[J\left(\frac{x-b_n}{2\sqrt{a_n a_{n+1}}}\right)\right]^{-1}$$

denote the zeros of $a_{n+1}t^2 + (b_n - x)t + a_n = 0$. Then the following is true.

Let $(p_n)_{n=0}$ be a system of orthonormal polynomials and let $d\alpha$ be the corresponding measure. Suppose $\lim_{n\to\infty} a_n = \frac{1}{2}$ and $\lim_{n\to\infty} b_n = 0$ and $\sum_{k=0}^{\infty} \epsilon_k < \infty$, $\epsilon_k = |a_{k+1} - a_k| + |b_{k+1} - b_k|$ where a_n and b_n are the coefficients in the three term recursion formula satisfied by p_n. Define

$$g(x) = \lim_{n\to\infty}\left(p_n(x) - (T(x))^{-1}p_{n-1}(x)\right)\prod_{k=1}^{n}\left(t_{1k}(x)\right)^{-1},\ x \neq \pm 1.$$

Then g considered as a function of a real variable is a non vanishing continuous function in $(-1,1)$ such that $g\prod_{k=1}^{n} t_{1k}$ is analytic in K, $g(x) = 0$ iff $x \in \mathrm{supp}(d\alpha)\backslash[-1,1]$ and $g'(x)$ exists and is different from zero if $g(x) = 0$. If $K \subseteq (-1,1)$ is a compact set then

$$\sqrt{\alpha'(x)(1-x^2)^{1/2}}\,p_n(x) = \left(\frac{2}{\pi}\right)^{1/2}\sin\left(\sum_{k=1}^{n}\arg t_{1k}(x) + \arg g(x)\right) +$$

$$+\,O\left(\sum_{k=n+1}^{\infty}\epsilon_k\right)$$

there exists $0 < \delta < 1$ such that

$$p_n(x)\prod_{k=1}^{n}\left(t_{1k}(x)\right)^{-1} = \frac{g(x)}{2\sqrt{x^2-1}} + O\left((\delta)^n + \sum_{k=\left[\frac{n}{2}\right]}^{\infty}\epsilon_k\right)$$

holds uniformly for $n = 1, 2, \ldots, x \in K$.

If $K \subseteq supp(d\alpha)\backslash[-1,1]$ is a compact set then

$$p_n(x)\prod_{k=1}^{n}\left(t_{1k}(x)\right)^{-1} = \frac{2}{I(x)g'(x)} + O\left(\sum_{k=\left[\frac{n}{2}\right]}^{\infty}\epsilon_k\right)$$

holds uniformly for $n = 1, 2, \ldots, x \in k$ where

$$I(x) = \lim_{x\to 0^+}\alpha(x) - \lim_{x\to 0^-}\alpha(x).$$

(6) Where Nevai considered recurrence relations, Lubinsky has considered analogues involving exponential weights for $[-1,1]$ and $\mathbb{R}$. Let $\mu(\theta) = w^2(\cos\theta)|\sin\theta|$, $\theta \in [-\pi,\pi]$, $w, w^2 \in L^1[-\pi,\pi]$, $supp(w)$ having positive measure. Also define for $1 \le p < \infty$,

$$E_{np}(w) = \inf_{\deg(P)<n}\left\|\left(x^n - P(x)\right)w(x)\right\|_{L_p[-1,1]}$$

and let $T_{np}(w,x)$ denote any monic polynomial of degree n satisfying $\left\|T_{np}(w,x)w(x)\right\|_{L_p[-1,1]} = E_{np}(w)$ and $p_{np}(w,x) = \dfrac{T_{np}(w,x)}{E_{np}(w)}$ to be the Lp extremal polynomial with $\beta_n(w^2) = \left[E_{n2}(w)\right]^{-1}$ and $p_n(w^2,x) = p_{2n}(w,x)$.

Also let $w(x) = \exp\left(-Q(x)\right)$ $x \in (-1,1)$, Q even, convex in $(-1,1)$ and $Q(1-) = Q'(1-) = Q''(1-) = \infty$. Associated with Q is the Mhaskar–Saff number a_n defined as the positive root of

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)dt}{\sqrt{1-t^2}},\ n > 0.$$

Further for $1 \leq p \leq \infty$, $n \geq 1$, $\theta \in [-\pi, \pi]$ let

$$\mu_{n,p}(\theta) = w^2(a_n \cos\theta)|\sin\theta|^{2/p}$$

then for $1 \leq p \leq \infty$

$$\lim_{n\to\infty} p_{np}(w, a_n z)(J(z))^n \left(D\left(\mu_{np}, (J(z))^{-1}\right)^{-1}\right)^{-1} = (2\sigma_p)^{-1}$$

uniformly in closed subsets of $\mathbb{C} \setminus [-1,1]$ and for $1 \leq p < \infty$

$$\lim_{n\to\infty} E_{np}(w)(2(a_n)^{-1})^n (D(\mu_{np}, 0))^{-1} = 2\sigma_p$$

where

$$\sigma_p = \begin{cases} \left[\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p_1}{2}\right)\Gamma\left(\frac{p}{2}+1\right)^{-1}\right]^{1/p}, & 1 \leq p < \infty, \\ 1, & p = \infty, \end{cases}$$

and

$$\tau(f,\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\log f(t) - \log f(\theta)\} \cot\left(\frac{\theta - t}{2}\right) dt,$$
$$f \geq 0,\ \log f \in L^1[-\pi, \pi],$$

whenever defined.

A generalisation of a theorem of Szegö, Bernstein and Akiezer is the following Nevai conjecture:

Let $W = \exp(-|x|^{\alpha})$, $\alpha > 1$. Let $0 < \epsilon < \pi/2$. Then

$$a_n^{1/2} p_n(W^2, a_n \cos\theta) W(a_n \cos\theta)|\sin\theta|^{1/2}$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \cos(n\theta + \tau_n(\theta)) + O(n^{-1})$$

as $n \to \infty$ uniformly $\theta \in [\epsilon, \pi - \epsilon]$, τ depending on n and W. Various advances have been found including Hermite weights, differential equations and infinite–finite range inequalities. One such result is due to Lubinsky.

(7) Let $W = \exp(-Q)$, $Q\colon \mathbb{R} \to \mathbb{R}$ even, continuous, Q'' continuous in $(0,\infty)$, $Q(0) = 0$, $Q' > 0$ in $(0,\infty)$, $\alpha - 1 \le \frac{xQ''(x)}{Q'(x)} \le C$, $x \in (0,\infty)$, and some $C > 0$; $\alpha > 3$. Also let Q'' exists for large x and for some $C_1 > 0$,

$$x^2 \frac{|Q'''(x)|}{Q'(x)} \le C_1.$$

Let $S > 0$ be fixed, but large enough and let

$$C_n = a_n\left[1 + S\left[\frac{\log n}{n}\right]^{2/3}\right],\ n \ge 1$$

and for $n \ge 1$ let $f_n(\theta) = w^2(C_n\cos\theta)|\sin\theta|$, $\theta \in [-\pi,\pi]$. Then there exists $\delta > 0$ such that for n large enough and uniformly for $\theta \in [\eta^{-\delta}, \pi - \eta^{-\delta}]$,

$$\begin{aligned} &C_n^{1/2} p_n(W^2 C_n \cos\theta) W(C_n\cos\theta)|\sin\theta|^{1/2} \\ &= \left(\frac{2}{\pi}\right)^{1/2} \cos\big(n\theta + r(f_n,\theta)\big) + O(\eta^{-\delta}). \end{aligned}$$

(8) Finally we note that recalling the Erdös Turan theorem (remarks after Theorem 2.18) and using $E = [-1,1]$ it can be shown that $Cap(E) = \frac{1}{2}$ and $\lim_{n\to\infty}(\beta_n)^{1/n} = 2$ which is much weaker than our considerations above.

We now give the proof of Theorem 7.10:

<u>Proof:</u>

Using (7.10) and (7.11) we get for a fixed integer, k, $n + k \ge 0$,

$$\frac{p_{n,n+k+1}(x)}{p_{n,n+k}(x)} = \left[\frac{\left[1 + \Phi_{2n,2n+2k}(0)\right]^{1/2} \varphi_{2n,2n+2k+2}(w)}{w\left[1 + \Phi_{2n,2n+2k+2}(0)\right]^{1/2} \varphi_{2n,2n+2k}(w)}\right] \times$$

(7.21)
$$\times \left[\frac{1 + \varphi^*_{2n,2n+2k+2}(w)/\varphi_{2n,2n+2k+2}(w)}{1 + \varphi^*_{2n,2n+2k}(w)/\varphi_{2n,2n+2k}(w)}\right];$$

$$\frac{P_{n,n+k+1}(x)}{P_{n,n+k}(x)} = \left[\frac{[1+\Phi_{2n,2n+2k}(0)]\,[\Phi_{2n,2n+2k+2}(w)]}{2w\,[1+\Phi_{2n,2n+2k+2}(0)]\,[\Phi_{2n,2n+2k}(w)]}\right] \times$$

$$\times \left[\frac{1+\Phi^*_{2n,2n+2k+2}(w)/\Phi_{2n,2n+2k+2}(w)}{1+\Phi^*_{2n,2n+2k}(w)/\Phi^*_{2n,2n+2k}(w)}\right] \tag{7.22}$$

$x = \frac{1}{2}\left(w + \frac{1}{w}\right)$. Now the result follows using (7.21), (7.22), (3.6), (3.8) and (3.10). ■

Lemma 7.11

$$p_{n,m}(x) = \frac{\pi}{2}\left[1+\Phi_{2n,2m}(0)\right]^{-1/2} Re\left[z^{-m}\varphi_{2n,2m}(z)\right],\ (z = e^{i\theta}). \tag{7.23}$$

Let $(\alpha, (w_{2n}), 2k)$ be admissible on $[-1,1]$. Then

(7.24)

$$p_{n,n+k}(x) = \sqrt{\frac{1}{2\pi}}\,Re\left[z^{n-k}\varphi_{2n,2n+2k}(z)\right] + o\left(\left|\varphi_{2n,2n+2k}(z)\right|\right),\ z = e^{i\theta},$$

where o is uniform in θ as $n \to \infty$.

Proof:

Consider (7.11). Then if $w = e^{i\theta}$ we have that

$$\begin{aligned}
&(w)^{-m}\varphi_{2n,2m}(w) + (w)^{2m}\overline{\varphi_{2n,2m}((\bar{w})^{-1})} \\
&= (w)^{-m}\varphi_{2n,2m}(w) + w^{m}\overline{\varphi_{2n,2m}(w)} \\
&= 2Re((w)^{-m}\varphi_{2n,2m}(w)).
\end{aligned}$$

So (7.23) holds. Next by admissability $(\mu, (W_{2n}), 2k)$ is admissible on $[0, 2\pi]$. So (3.6) is valid and we obtain

$$[1+o(1)]\,p_{n,n+k}(x) = \sqrt{\frac{1}{2\pi}}\,Re\left[(z)^{-n-k}\varphi_{2n,2n+2k}(z)\right] \tag{7.25}$$

and

$$p_{n,n+k}(x) = O\left(\left|\varphi_{2n,2n+2k}(z)\right|\right) \tag{7.26}$$

where O is uniform in θ by Theorem 3.8. So by (7.25) and (7.26) the result holds. ■

We can now give the proof of Theorem 7.12:

Proof:

Since $(\alpha, (w_{2n}), 2k-2)$ is admissible on $[-1,1]$, we must have $(\alpha, (w_{2n}), 2k+2)$ is admissible on $[-1,1]$, as the zeros of w_{2n} lie in $\bar{\mathbb{C}} \setminus [-1,1]$. By Lemma 7.11, we have that

$$p_{n,n+k+1}(x) + p_{n,n+k-1}(x)$$
$$= \sqrt{\frac{1}{2\pi}} Re\left[z^{-n-k-1}\varphi_{2n,2n+2k+2}(z) + z^{-n-k+1}\varphi_{2n,2n+2k-2}(z)\right] +$$
$$+ o\left(\left|\varphi_{2n,2n+2k+2}(z)\right|\right) + o\left(\left|\varphi_{2n,2n+2k-2}(z)\right|\right). \tag{7.27}$$

Also $(\alpha, (w_{2n}), 2k)$ is admissible on $[-1,1]$. So by (3.8) for k and $k-1$ we get

$$\varphi_{2n,2n+2k\pm2}(z) = z^{\pm2}\varphi_{2n,2n+2k}(z) + o\left(\left|\varphi_{2n,2n+2k}(z)\right|\right). \tag{7.28}$$

Now put (7.28) into (7.27) to yield that

$$p_{n,n+k+1}(x) + p_{n,n+k-1}(x) = 2x\sqrt{\frac{2}{\pi}} Re\left[z^{-n-k}\varphi_{2n,2n+2k}(z)\right] +$$
$$+ o\left(\left|\varphi_{2n,2n+2k}(z)\right|\right) = 2x p_{n,n+k}(x) + o\left(\left|\varphi_{2n,2n+2k}(z)\right|\right). \tag{7.29}$$

Also recall that we have,

$$p_{n,m}(x)d\alpha_n(x) = p_{n,m}(\cos\vartheta)d\mu_{2n}(\vartheta) \tag{7.30}$$

and

$$\left(\int_0^{2\pi} \left|p_{n,m}(\cos\theta)\varphi_{2n,2n+2k}(z)\right| d\mu_{2n}(\theta)\right)^2$$
$$\leq \left(\int_0^{2\pi} \left|p_{n,m}(\cos\theta)\right|^2 d\mu_{2n}(\theta)\right)\left(\int_0^{2\pi} \left|\varphi_{2n,2n+2k}(z)\right|^2 d\mu_{2n}(\theta)\right)$$
$$\leq 2\left(\int_{-1}^{1} \left|p_{n,m}(x)\right|^2 d\alpha_n(x)\right)\left(\int_0^{2\pi} \left|\varphi_{2n,2n+2k}(z)\right|^2 d\mu_{2n}(\theta)\right)$$
$$= 4\pi.$$

So we get that

$$\left(\int_0^{2\pi} \left|p_{n,m}(\cos\theta)\varphi_{2n,2n+2k}(z)\right| d\mu_{2n}(\theta)\right)^2 \leq 4\pi. \tag{7.31}$$

So multiplying (7.28) by (7.30) and using (7.31) yields

$$\begin{aligned} &\int_{-1}^{1} 2x p_{n,n+k}(x) p_{n,m}(x) d\alpha_n(x) \\ &= \int_{-1}^{1} (p_{n,n+k+1}(x) + p_{n,n+k+1}(x)) p_{n,m}(x) d\alpha_n(x) + o(1). \end{aligned} \tag{7.32}$$

Thus we have

$$2a_{n,k,i} = \int_{-1}^{1} (p_{n,n+k+1}(x) + p_{n,n+k-1}(x)) p_{n,m}(x) d\alpha_n(x) + o(1) \tag{7.33}$$

where $m = n + k + i$.

Hence for $n \geq N$ using (7.33) and orthonormality if $i = 0$, $m = n + k$ we obtain $2a_{n,k,0} = 0$ if $i = 0$, if $i = \pm 1$, $m = n + k \pm 1$, so $2a_{n,k,\pm 1} = 1 + o(1)$. ■

We give the **proof of Theorem 7.13:**

<u>Proof:</u>

We first show that (7.17) and (7.18) are equivalent. To see this we must show as in the proof of Theorem 3.10 that

$$\lim_{n\to\infty} \int_{-1}^{1} f(x) \frac{p_{n,n_k}(x) p_{n,n+k+m}(x)}{w_{2n}(x)} d\alpha_s(x) = 0. \tag{7.34}$$

To see this notice that by the Schwarz inequality it is sufficient to show

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{f(x) p_{n,n+k}^2(x)}{w_{2n}(x)} d\alpha_s(x) = 0. \tag{7.35}$$

But

$$d\mu_s(\theta) = d\alpha(x) \tag{7.36}$$

and using (7.23) with $\Phi_{2n,2m}(0) \to 0$ as $n \to \infty$ we have that there exists C_1 such that

(7.37) $$|p_{n,n+k}(x)| \le C_1 |\psi_{2n,2n+2k}(z)|, \quad \text{as } n \to \infty.$$

Now using (7.36), (7.37) and (3.14) we have (7.35) and hence (7.34).

We now proceed to prove 7.18. We first show that the result holds when f is a polynomial. Fix $\ell \in \mathbb{N}$ and apply (7.12) ℓ times. We obtain

(7.38)
$$x^\ell p_{n,n+k}(x) = \sum_{\substack{-1 \le k_i \le 1 \\ i=1,\dots,\ell}} \left(a_{n,k,k_1} \cdot a_{n,k+k_1,k_2} \cdots a_{n,k+k_1+k_2+\cdots k_{\ell-1},k_\ell}\right) \times$$
$$\times \left(p_{n,n+k+k_1+\cdots k_\ell}\right).$$

Now consider the integral of the lefthand side of (7.38) with respect to $d\alpha_n(x)$. For $n \ge N$ we notice that by the orthonormality relations, the value of the integral on the lefthand side is a number which only depends on the limit of the coefficients on the righthand side. But by Theorem 7.12, these limits do not depend on the particular choice of $(\alpha, (w_{2n}))$ that is, as long as $(\alpha, (w_{2n}), 2k - 2\ell)$ is admissible on $[-1,1]$, the result holds. So to calculate the limit, let $d\alpha = (1-x^2)^{-1/2}dx$ and $w_{2n} \equiv 1$, $n \in \mathbb{N}$. Now using the uniqueness principle Theorem 1.2 and knowing that the Chebyshev polynomials have the weight function $(1-x^2)^{-1/2}$ it follows that $p_{n,n+k}$ are the Chebyshev polynomials defined in the statement of Theorem 7.13. Thus

$$\begin{aligned} x^\ell p_{n,n+k} p_{n,n+k+m} &= (\cos\theta)^\ell \frac{2}{\pi} \cos(n+k)\theta \cos(n+k+m)(\theta) \\ &= (\cos\theta)^\ell \frac{1}{\pi}\left[\cos m\theta + \cos(2n+2k+m)\theta\right] \end{aligned}$$

so

$$\begin{aligned} &\int_{-1}^{1} x^\ell p_{n,n+k}(x) p_{n,n+k+m}(x) \frac{dx}{(1-x^2)^{1/2}} \\ &= \frac{1}{\pi}\int_{-1}^{1} (\cos\theta)^\ell \left[\cos m\theta + \cos(2n+2k+m)(\theta)\right] d\theta \\ &= \frac{1}{\pi}\int_{-1}^{1} (\cos\theta)^\ell \cos m\theta \, d\theta \quad \text{if } 2n+2k+m > \ell. \end{aligned}$$

So if $n > \dfrac{\ell - 2k - m}{2}$ we have that

$$\int_{-1}^{1} x^{\ell} \frac{p_{n,n+k}(x)p_{n,n+k+m}(x)\alpha'(x)dx}{w_{2n}(x)} = \frac{1}{\pi}\int_{-1}^{1} x^{\ell} T_m(x)\frac{dx}{(1-x^2)^{1/2}}.$$

So (7.18) holds for f a polynomial. Now let f be a bounded Borel measurable function. Then f can be approximated uniformly in L^1 by simple functions and these simple functions can be approximated uniformly in L^1 by continuous functions. These continuous functions can be approximated by Jackson polynomials with remainder g say where the difference is small in the L^1 sense. Thus assume $|f| < C_2/3$ some C_2. Let $\epsilon > 0$ and write $f = p + g$, p a polynomial, g a remainder such that we have $|g| < C_2$ and

$$\text{(7.39)} \qquad \int_{-1}^{1} |g|(x)(1-x^2)^{-1/2}dx < \epsilon,$$

thus we have that

(7.40)

$$\left|\int_{-1}^{1} g(x)T_m(x)(1-x^2)^{-1/2}dx\right| \le \epsilon\pi \max\left\{\left|T_m(x)\right| : -1 \le x \le 1\right\} \le \epsilon\pi$$

by definition of T_n. Also by the Schwarz inequality, (7.37), (7.30) and orthonormality we have that

$$\left|\int_{-1}^{1} g(x)p_{n,n+k}(x)p_{n,n+k+m}(x)d\alpha_n(x)\right|^2$$

$$\text{(7.41)} \qquad \le C_3\int_{-1}^{1} g^2(\cos\theta)\left|\varphi_{2n,2n+2k}(z)\right|^2 d\mu_{2n}(\theta) \; n \ge N.$$

Now the limit as $n \to \infty$ on the righthand side of (7.41) is by (3.14) equal to

$$C_4\int_{0}^{2\pi} g^2(\cos\theta)d\theta = C_4\int_{-1}^{1} g^2(x)(1-x^2)^{-1/2}dx \le C_5C_1$$

by (7.39) as $|g|^2 < C_2|g|$. Thus,

$$\limsup_{n\to\infty}\left|\int_{-1}^{1} g(x)p_{n,n+k}(x)p_{n,n+k+m}(x)d\alpha_n(x)\right|^2 \le C_5C_2\epsilon.$$

Now $\epsilon > 0$ is arbitrary, so using (7.40) together with the fact that the result is true for $p = f - g$ yields the result in general. ■

Proof of Theorem 7.14

Proof:

We first show that

$$\text{(7.42)} \qquad \int_{-1}^{1} \frac{p_{n,n+k}(w)p_{n,n+k}(x)}{w-x} d\alpha_n(x) = \int_{-1}^{1} \frac{p_{n,n+k}^2(x) d\alpha_n(x)}{w-x}$$

well

$$\deg\left[\frac{p_{n,n_k}(w) - p_{n,n+k}(x)}{w-x}\right] \le n+k-1$$

so

$$\int_{-1}^{1} p_{n,n+k}(x)\left[\frac{p_{n,n+k}(w) - p_{n,n_k}(x)}{w-x}\right] d\alpha_n(x) = 0$$

which shows that (7.42) is true. So

$$p_{n,n+k}(w) g_{n,n+k}(w) = \int_{-1}^{1} \frac{p_{n,n+k}^2(x) d\alpha_n(x)}{w-x}$$

by definition of $g_{n,n+k}$ and (7.42). Thus using (7.18) with $f(x) = (w-x)^{-1}$, $m = 0$ we get for $n \ge N$ using (7.43)

$$\begin{aligned} &\lim_{n\to\infty} p_{n,n+k}(w) g_{n,n+k}(w) \\ \text{(7.44)} \qquad &= \frac{1}{\pi}\int_{-1}^{1} (w-x)^{-1}(1-x^2)^{-1/2} dx = (w^2-1)^{-1/2} \end{aligned}$$

also

$$\begin{aligned} \frac{g_{n,n+k+1}}{g_{n,n+k}} &= (g_{n,n+k+1})(p_{n,n+k+1}) \cdot \left(\frac{p_{n,n_k}}{p_{n,n+k+1}}\right)(p_{n,n+k} g_{n,n+k})^{-1} \\ &\to (w^2-1)^{-1/2}\left(J(w)\right)^{-1}(w^2-1)^{1/2} = \left(J(w)\right)^{-1} \end{aligned}$$

by (7.44) and Theorem 7.10. Finally we note that the uniform convergence follows by the Vitali–Montel theorem for analytic functions as was seen in the proof of Theorem 3.13 ■

References

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